Algèbre et opérade : cohomologie, homotopie et dualité de Koszul

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Cette thèse commence par un texte de vulgarisation expliquant les liens entre topologie et algèbre destiné à un public large. Ce texte est suivi d'une introduction en français et de trois articles de recherche en anglais. L'introduction définit le cadre de cette thèse et résume les trois articles de recherche. Elle se termine par une section d'ouverture dans laquelle nous donnons des idées pour prolonger les travaux de cette thèse.

Le plan de la thèse est le suivant :

- Préliminaires : de la topologie à l'algèbre
- Introduction
- Chapitre 1 : *André-Quillen cohomology for algebras over an operad* [Mil08],
- Chapitre 2 : *Curved Koszul duality theory* [HM10], écrit avec Joseph Hirsh (CUNY),
- Chapitre 3 et annexe A : *The Koszul complex is the cotangent complex* [Mil10].

Préliminaires : de la topologie à l'algèbre

La *topologie algébrique* consiste en l'étude des lieux ou des espaces à l'aide d'ensembles de nombres possédant une structure. On souhaite comprendre la forme globale d'un objet en autorisant les déformations continues. On cherche alors des outils pour étudier les objets à déformation près. L'*homotopie*, du grec *homós* = même, semblable et *tópos* = lieu, est une théorie mathématique qui répond à ce problème. Elle associe à un objet des *groupes d'homotopie* qui sont invariants par déformation continue de l'objet d'étude. Par exemple, un ballon de football, ou une sphère, et un ballon de rugby ont les mêmes groupes d'homotopie car il est possible de passer l'un à l'autre par une déformation continue, sans faire de découpage ou de collage.

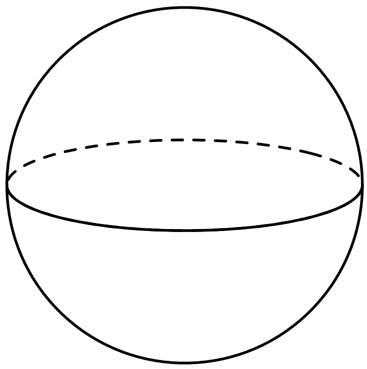


FIGURE 1 – Sphère

L'homotopie est une théorie difficile à étudier comme en témoigne le fait que le calcul des groupes d'homotopie d'objets aussi simples que les sphères n'est pas encore terminé (on peut définir une sphère pour chaque dimension : en dimension 1, la sphère est un cercle, en dimension 2, la sphère est représentée par la figure 1). Comment "simplifier" alors le calcul de ces groupes d'homotopie ? Souvent, en mathématiques, on ne simplifie pas sans efforts. Il faut donc un peu plus de travail pour définir une nouvelle théorie, appelée *cohomologie*, du grec *homós* = même, semblable et *logie* = étude. Le préfixe "co" exprime la notion duale de l'homologie, c'est-à-dire une théorie très similaire et qui est celle qui nous intéresse ici. On associe à un objet des *groupes de cohomologie* qui comptent le nombre d'espaces vides ou de trous d'un objet. Le calcul des groupes de cohomologie est souvent plus facile à effectuer que le calcul des groupes d'homotopie. Il suffit de repenser à nos sphères (de dimension $n-1$) qui possèdent chacune uniquement un espace vide, ou trou (de dimension n) : l'intérieur du cercle est un trou (de dimension 2), l'intérieur de la sphère est un trou (de dimension 3). Bien entendu, en simplifiant les calculs, on a perdu de l'information. Par exemple, les complémentaires dans l'espace des deux noeuds de la figure 2 ont les mêmes groupes de cohomologie mais ils n'ont pas les mêmes groupes d'homotopie.

FIGURE 2 – Noeud simple et noeud de trèfle

On associe de cette façon à une forme des nombres, ainsi qu'une structure algébrique sur ces nombres. Cette donnée est appelé *groupe*. La compréhension de la structure de groupe permet parfois de différencier des objets dont les nombres qui apparaissent dans les groupes d'homotopie ou de cohomologie sont les mêmes. Par exemple, les deux dessins de la figure 3 ont les mêmes groupes de cohomologie mais leur structure algébrique est différente.

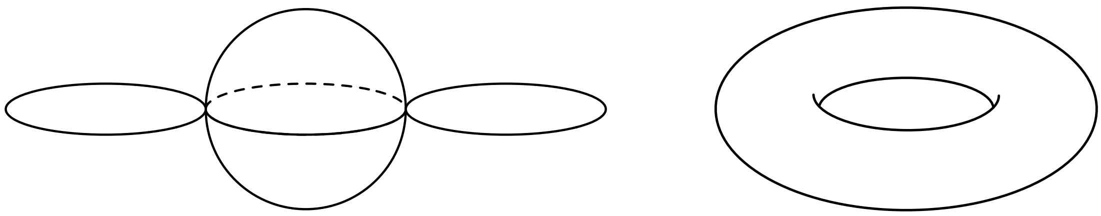


FIGURE 3 – Deux cercles collés à une sphère et un tore

Le premier groupe d'homotopie est construit à l'aide des *lacets* de notre objet d'étude (boucles dessinées sur notre objet). En parcourant deux lacets l'un après l'autre, on obtient un nouveau lacet, dit autrement, on définit le produit $a \cdot b$ de deux lacets a et b. Ce produit est défini à déformation près comme le montre la figure 4. À partir de trois lacets a, b et c, il y a deux façons de les multiplier : d'abord $a \cdot b$ puis $(a \cdot b) \cdot c$ ou bien, la multiplication de a avec $b \cdot c$ soit $a \cdot (b \cdot c)$. Comme le produit des lacets est défini à déformation près, on a $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Cette relation, appelé relation d'*associativité*, est présente un peu partout : lorsque l'on a trois choses à faire, on se demande rarement si l'on fait la première puis les deux autres ou les deux premières puis la troisième, on fait simplement la première, la seconde, puis la troisième. Cette associativité est une partie de la structure de nos groupes d'homotopie et de cohomologie et nous permet de définir la notion d'*algèbre associative*.

En mathématiques, il apparaît différentes notions d'algèbres : associatives, commuta-

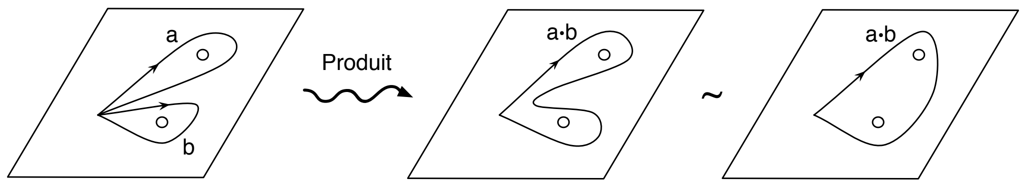


FIGURE 4 – Deux lacets dans un plan privé de deux trous

tives, de Lie... La notion d'*opérade* donne un formalisme pour étudier toutes les algèbres simultanément. Nous étudions ainsi les algèbres sur une opérade quelconque. Il existe une opérade dont les algèbres sont les algèbres associatives, une opérade dont les algèbres sont les algèbres commutatives, une opérade dont les algèbres sont les algèbres de Lie... Finalement, un résultat démontré pour les algèbres sur une opérade s'appliquera aux algèbres associatives, aux algèbres commutatives, aux algèbres de Lie...

La topologie algébrique donne des informations sur un objet ou une forme à l'aide de structure algébrique. Nous aimerions maintenant étudier les structures algébriques en elle-même. Il est possible d'abstraire les définitions de groupes d'homotopie et de cohomologie pour les appliquer aux structures algébriques. Nous obtenons donc des informations homotopiques et cohomologiques associées à des structures algébriques.

Cette thèse étudie les structures algébriques avec des méthodes topologiques telles que l'homotopie et la cohomologie. Un des outils principaux est la *dualité de Koszul*, que nous généralisons de plusieurs façons. Dans un premier temps, nous rappelons la définition de la cohomologie d'André-Quillen des algèbres sur une opérade. Nous décrivons et étudions le *complexe cotangent* représentant cette cohomologie à l'aide de la dualité de Koszul des opérades. Cependant, cette dualité de Koszul ne tient pas compte d'une éventuelle *unité* dans l'algèbre, c'est-à-dire un élément 1 tel que $1 \cdot a = a = a \cdot 1$. Nous généralisons dans un deuxième temps, la dualité de Koszul aux opérades codant des algèbres avec unités (ou algèbres *unitaires*). Nous introduisons dans ce but la notion de *courbure*. Nous obtenons ainsi la notion d'algèbre associatives unitaires à homotopie près et une théorie de cohomologie pour des algèbres associatives unitaires. Un ingrédient central de la dualité de Koszul est le *complexe de Koszul*. Dans une troisième partie, nous étendons la dualité de Koszul des algèbres associatives à tout type d'algèbres en montrant que le complexe cotangent est un bon candidat pour généraliser le complexe de Koszul.

Introduction

Cohomologie, algèbre, opérade et dualité de Koszul

La définition de l'homotopie en termes de lacets paraît essentiellement topologique. Dans l'article [Qui67] de 1967, D. Quillen présente une liste d'axiomes suffisants pour abstraire la notion d'homotopie : il introduit la notion de *catégorie de modèles*. Il est possible de munir un grand nombre de catégories d'une structure de catégorie de modèles. Une telle structure permet de faire de l'homotopie, par exemple dans des catégories d'algèbres. Nous pouvons alors construire des théories de cohomologie pour n'importe quel type d'algèbres. Nous employons ici le mot "cohomologie" et non le mot "homotopie" : Quillen a axiomatisé la notion d'homotopie mais les théories obtenues se comportent par bien des aspects comme des théories cohomologiques. La notion d'homotopie pour les algèbres sera définie plus simplement ultérieurement (pour des explications plus précises sur la question, on renvoie le lecteur à l'article de P. Goerss [Goe90]).

Une *théorie de cohomologie*, ou *cohomologie*, est la donnée d'un foncteur $\mathrm{H}^\bullet$ d'une certaine catégorie, disons la catégorie des (paires d') espaces topologiques, vers la catégorie des groupes abéliens, et d'une application bord $\partial : \mathrm{H}^\bullet(A) := \mathrm{H}^\bullet(A, \emptyset) \to \mathrm{H}^{\bullet+1}(X, A)$ où (X, A) est une paire d'espaces topologiques vérifiant $A \subseteq X$. On demande généralement à une théorie de cohomologie $\mathrm{H}^\bullet$ de vérifier les *axiomes d'Eilenberg-Steenrod* :

- l'*invariance d'homotopie* qui dit que deux applications f et g homotopes, $f \sim g$, ont la même image en cohomologie $\mathrm{H}^\bullet(f) = \mathrm{H}^\bullet(g)$,
- l'*excision* qui permet le calcul des groupes de cohomologie à l'aide de groupes de cohomologie plus simples,
- l'*axiome de la dimension* qui fixe la cohomologie du point : $\mathrm{H}^\bullet(pt) = \mathbb{Z}$,
- l'*additivité* $\mathrm{H}^\bullet(X \sqcup Y) \cong \mathrm{H}^\bullet(X) \oplus \mathrm{H}^\bullet(Y)$ qui permet de se limiter aux espaces connexes,
- la *suite exacte longue de cohomologie* qui, couplée avec l'axiome d'excision, permet de démontrer la *suite de Mayer-Vietoris* et permet le calcul de groupes de cohomologie.

Les théories de cohomologie définies en suivant les idées de D. Quillen possèdent des propriétés similaires. Dans le contexte des algèbres, il convient de dire que l'additivité est la préservation des coproduits, de remplacer l'excision par le *changement de base plat* et la suite exacte longue en cohomologie par la *transitivité*. Dans ce cas aussi, on obtient une suite de Mayer-Vietoris.

Une *algèbre associative* est la donnée (A, μ) d'un espace vectoriel A sur un corps $\mathbb{K}$ et d'une multiplication $\mu : A \otimes A \to A$ qui vérifie la relation d'associativité. On peut éventuellement demander à une algèbre associative (A, μ) de posséder une unité $u : \mathbb{K} \to A$. On transforme une algèbre non nécessairement unitaire A en une algèbre unitaire en lui adjoignant une unité, on obtient l'algèbre *augmentée* $A_+ := \mathbb{K} \oplus A$. La catégorie des algèbres associatives non nécessairement unitaires est équivalente à la catégorie des algèbres associatives unitaires augmentées. L'opération μ peut être vue comme l'image d'un arbre  à deux entrées et une sortie dans $\mathrm{Hom}(A^{\otimes 2}, A)$. Ainsi, une composée itérée de l'opération μ avec elle-même est décrite par un arbre binaire planaire et la relation d'associativité se lie comme l'image de la relation  =  dans $\mathrm{Hom}(A^{\otimes 3}, A)$. Cette description fournit une définition équivalente de la notion d'algèbre associative en terme de représentation dans $\{\mathrm{Hom}(A^{\otimes n}, A)\}_{n \geq 0}$ d'un certain objet, ici l'ensemble des arbres binaires planaires quotienté par la relation d'associativité, appelé *opérade* [May72, BV73].

Une opérade est la donnée $(\mathcal{P}, \gamma, u)$ d'un $\mathbb{S}$-*module* $\mathcal{P}$, c'est-à-dire une collection de modules $\{\mathcal{P}(n)\}_{n \geq 0}$, chacun muni d'une action du groupe symétrique $\mathbb{S}_n$ à n éléments, d'une *composition* $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$, où $\circ$ est un *produit monoidal* pour les $\mathbb{S}$-modules, et d'une *unité* $u : I \to \mathcal{P}$. La composition et l'unité satisfont des relations d'associativité et d'unitarité. Un exemple d'opérade est donné par l'opérade d'endomorphisme $\mathrm{End}_A := \{\mathrm{Hom}(A^{\otimes n}, A)\}_{n \geq 0}$, dont la composition est donnée par la composition des applications. Une représentation d'une opérade $\mathcal{P}$, c'est-à-dire un morphisme d'opérades $\mathcal{P} \to \mathrm{End}_A$, est appelée une *algèbre sur une opérade*. Il existe une opérade appelée $\mathcal{A}s$ dont les représentations sont exactement les algèbres associatives. Il est aussi possible de définir la notion de *module sur une algèbre sur une opérade*. Dans le cas de l'opérade $\mathcal{A}s$, on obtient la notion de bimodule sur une algèbre associative. Un élément de $\mathcal{P}(n)$ d'une opérade $\mathcal{P}$ code une opération à n entrées et une sortie, pour coder les opérations à plusieurs entrées et plusieurs sorties, on utilise une généralisation des opérades définie par B. Vallette [Val07] et appelée *propérade*. Dans ce cas, les $\mathbb{S}$-modules sont remplacés par des $\mathbb{S}$-*bimodules* $\mathcal{P} := \{\mathcal{P}(n, m)\}_{n, m}$.

Notons tout de même qu'une algèbre associative unitaire peut être vue comme une opérade. En effet, la catégorie monoïdale des espaces vectoriels est une sous-catégorie monoïdale de celle des $\mathbb{S}$-modules en regardant un espace vectoriel comme un $\mathbb{S}$-module concentré en arité 1. De plus, une algèbre associative est un monoïde dans la catégorie des espaces vectoriels et une opérade est un monoïde dans la catégorie des $\mathbb{S}$-modules. De la même façon, une opérade peut être vue comme une propérade. Nous pouvons ainsi étendre certaines constructions et certains résultats des algèbres associatives aux opérades et aux propérades comme les *constructions bar* B et *cobar* Ω, l'adjonction bar-cobar correspondante [GJ94, Val07]. Dans [Pri70], S. Priddy définit la dualité de *Koszul des algèbres associatives quadratiques.* Cette théorie est basée sur le complexe de chaînes *tordu* suivant

$$A \otimes_\alpha C \otimes_\alpha A := (A \otimes C \otimes A,\, d_\alpha),$$

où A est une algèbre associative, C est une *cogèbre coassociative*[1], $\alpha : C \to A$ est un *morphisme tordant* et d_α est une *différentielle tordue* qui dépend de α. E. Getzler et J. D. S. Jones généralisent la notion de morphisme tordant aux opérades [GJ94] et V. Ginzburg et M. Kapranov étendent la dualité de Koszul aux opérades quadratiques [GK94]. On peut retenir deux théorèmes importants de la dualité de Koszul. Le premier est le théorème fondamental des morphismes tordants qui s'énonce de la façon suivante :

1. Notion duale de la notion d'algèbre associative.

Théorème 1 (Théorème fondamental des morphismes tordants, [Bro59, Car55]). *Soient A une algèbre associative et C une cogèbre coassociative vérifiant de bonnes conditions de poids. Pour tout morphisme tordant $\alpha : C \to A$, les propriétés suivantes sont équivalentes :*

1. *Le morphisme tordant α est un morphisme de Koszul, c'est-à-dire $A \otimes_\alpha C \otimes_\alpha A \xrightarrow{\sim} A$,*
2. *le produit tensoriel tordu à gauche est acyclique, c'est-à-dire $A \otimes_\alpha C \xrightarrow{\sim} \mathbb{K}$,*
3. *le produit tensoriel tordu à droite est acyclique, c'est-à-dire $C \otimes_\alpha A \xrightarrow{\sim} \mathbb{K}$,*
4. *le morphisme d'algèbres associatives différentielles graduées $f_\alpha : \Omega C \to A$ est un quasi-isomorphisme,*
5. *le morphisme de cogèbres coassociatives différentielles graduées $g_\alpha : C \to \mathrm{B}A$ est un quasi-isomorphisme.*

Le second est une application du théorème fondamental des morphismes tordants au cas des algèbres quadratiques. À une algèbre quadratique A, il propose une cogèbre $C = A^{\text{¡}}$, explicitement donnée par la présentation de A. Cette cogèbre est appelée la *cogèbre duale de Koszul* de A.

Théorème 2 (Critère de Koszul, [Pri70]). *Soit (V, S) une donnée quadratique. Soient $A := A(V, S) = T(V)/(S)$ l'algèbre quadratique associée, $A^{\text{¡}} := C(sV, s^2S)$ sa cogèbre duale de Koszul et $\varkappa : A^{\text{¡}} \to A$ le morphisme tordant entre $A^{\text{¡}}$ et A. Les propriétés suivantes sont équivalentes :*

1. *Le morphisme tordant $\varkappa$ est un morphisme de Koszul, c'est-à-dire $A \otimes_\varkappa A^{\text{¡}} \otimes_\varkappa A \xrightarrow{\sim} A$,*
2. *Le complexe de Koszul $A \otimes_\varkappa A^{\text{¡}}$ est acyclique, c'est-à-dire $A \otimes_\varkappa A^{\text{¡}} \xrightarrow{\sim} \mathbb{K}$,*
3. *Le complexe de Koszul $A^{\text{¡}} \otimes_\varkappa A$ est acyclique, c'est-à-dire $A^{\text{¡}} \otimes_\varkappa A \xrightarrow{\sim} \mathbb{K}$,*
4. *Le morphisme d'algèbres associatives différentielles graduées $f_\varkappa : \Omega A^{\text{¡}} \to A$ est un quasi-isomorphisme,*
5. *Le morphisme de cogèbres coassociatives différentielles graduées $g_\varkappa : A^{\text{¡}} \to \mathrm{B}A$ est un quasi-isomorphisme.*

Le complexe tordu $A \otimes_\varkappa A^{\text{¡}} \otimes_\varkappa A$ est appelé le *complexe de Koszul*. Ces deux théorèmes admettent une version opéradique et propéradique [GK94, GJ94, Fre04, Val07, MSS02, LV].

Nous supposons à partir de maintenant que le corps $\mathbb{K}$ est de caractéristique 0. Cette hypothèse n'est pas toujours nécessaire mais elle le sera dans les exemples.

Chapitre 1 : La cohomologie d'André-Quillen des algèbres sur une opérade

M. André [And74] et D. Quillen [Qui70] définissent une cohomologie associée aux algèbres commutatives. V. Hinich [Hin97], P. Goerss et M. Hopkins [GH00] étendent cette définition aux algèbres sur une opérade. Ils dérivent, au sens de D. Quillen, le foncteur des dérivations d'une algèbre A dans un module M. Cette *cohomologie d'André-Quillen* est par définition représentée par un complexe de chaînes appelé le *complexe cotangent*. Le calcul de ce complexe cotangent nécessite la donnée d'une *résolution cofibrante* de l'algèbre de départ mais ne dépend pas de la résolution choisie. La notion de cofibrance vient du langage des catégories de modèles. Dans le cas des algèbres sur une opérade, cela correspond aux objets quasi-libres munis d'une bonne filtration.

On obtient des résolutions cofibrantes d'algèbre sur une opérade à l'aide de la dualité de Koszul des opérades. Le critère de Koszul opéradique nous assure qu'une opérade de Koszul $\mathcal{P}$ admet une résolution $\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa \mathcal{P}$, où $\mathcal{P}^{¡}$ est la *coopérade*[2] *duale de Koszul* de $\mathcal{P}$. Cette dernière fournit des résolutions fonctorielles de $\mathcal{P}$-algèbres $\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa A \xrightarrow{\sim} A$ pour toute $\mathcal{P}$-algèbre A. Il existe deux autres types de résolutions, cette fois-ci valable pour toute opérade augmentée $\mathcal{P}$ ou coopérade coaugmentée $\mathcal{C}$:

$$\begin{cases} \mathcal{P} \circ_\pi \mathrm{B}\mathcal{P} \circ_\pi A \xrightarrow{\sim} A & \text{pour toute } \mathcal{P}\text{-algèbre } A, \\ \Omega\mathcal{C} \circ_\iota \mathcal{C} \circ_\iota A \xrightarrow{\sim} A & \text{pour toute } \Omega\mathcal{C}\text{-algèbre } A, \end{cases}$$

où $\mathrm{B}\mathcal{P}$ est une coopérade appelée la *construction bar* de $\mathcal{P}$ et $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ est un morphisme tordant opéradique et $\Omega\mathcal{C}$ est une opérade appelée la *construction cobar* de $\mathcal{C}$ et $\iota : \mathcal{C} \to \Omega\mathcal{C}$ est un morphisme tordant opéradique. De façon générale, la $\mathcal{P}$-algèbre A possède une résolution du type

$$\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A \xrightarrow{\sim} A,$$

ou encore de la forme

$$\mathcal{P} \circ_\alpha C \xrightarrow{\sim} A,$$

où $\mathcal{P}$ est une opérade, $\mathcal{C}$ une coopérade, $\alpha : \mathcal{C} \to \mathcal{P}$ un morphisme tordant et C est une $\mathcal{C}$-cogèbre. Nous utilisons ce type de résolution pour décrire explicitement le complexe cotangent qui est un *A-module sur l'opérade $\mathcal{P}$*, ou en abrégé, un *A-$\mathcal{P}$-module*.

Théorème 3. *Soit $R = \mathcal{P} \circ_\alpha C$ une résolution quasi-libre de la $\mathcal{P}$-algèbre A. La représentation du complexe cotangent $\mathbb{L}_{R/A}$ correspondant à cette résolution a la forme suivante :*

$$\mathbb{L}_{R/A} \cong (A \otimes^{\mathcal{P}} C, d_\varphi = d_{A\otimes^{\mathcal{P}} C} - \delta^l_\varphi),$$

où $A \otimes^{\mathcal{P}} C$ est le A-$\mathcal{P}$-module libre sur C, $\varphi : C \to A$ est un morphisme tordant algébrique et la différentielle d_φ est une différentielle tordue.

Les opérades $\mathcal{A}s$, $\mathcal{L}ie$, $\mathcal{C}om$, $\mathcal{D}ias$, $\mathcal{L}eib$, $\mathcal{P}oiss$, $\mathcal{P}relie$ et $\mathcal{Z}inb$, entre autres, sont des opérades de Koszul. Le calcul de la représentation du complexe cotangent donnée par la résolution $\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa A \xrightarrow{\sim} A$ nous permet de retrouver les cohomologies définies par G. Hochschild [Hoc45], C. Chevalley et S. Eilenberg [CE48], D. Harrison [Har62] ou, bien entendu, M. André [And74] et D. Quillen [Qui70], A. Frabetti [Fra01], J.-L. Loday et T. Pirashvili [LP93], B. Fresse [Fre06], A. Dzhumadil'daev [Dzh99] et D. Balavoine [Bal98]. Plus généralement, dans le cas d'une opérade de Koszul, nous retrouvons le complexe de chaînes défini à la main par Balavoine [Bal98]. Nous traitons aussi le cas des algèbres sur une opérade cofibrante $\Omega\mathcal{C}$, en particulier le cas $\mathcal{P}_\infty := \Omega\mathcal{P}^{¡}$. Lorsque $\mathcal{C} = \mathcal{A}s^{¡}$, nous retrouvons la cohomologie définie par M. Markl [Mar92] pour les A_∞-algèbres et lorsque $\mathcal{C} = \mathcal{L}ie^{¡}$, nous retrouvons la cohomologie définie par V. Hinich et V. Schechtman [HS93] pour les L_∞-algèbres.

Le complexe cotangent est défini comme le foncteur dérivé du foncteur $A \otimes^{\mathcal{P}}_{-} \Omega_{\mathcal{P}}(-)$ qui à une algèbre B sur l'opérade $\mathcal{P}$ associe le A-$\mathcal{P}$-module libre (relatif) sur les *formes différentielles de Kähler* $\Omega_{\mathcal{P}}(B)$. Ainsi, à une résolution $R \xrightarrow{\sim} A$ est naturellement associé le morphisme

$$\mathbb{L}_{R/A} = A \otimes^{\mathcal{P}}_R \Omega_{\mathcal{P}}(R) \to A \otimes^{\mathcal{P}}_A \Omega_{\mathcal{P}}(A) \cong \Omega_{\mathcal{P}}(A)$$

2. La notion de coopérade est la notion duale de la notion d'opérade.

et lorsque ce morphisme est un quasi-isomorphisme, la cohomologie d'André-Quillen coïncide avec le foncteur Ext suivant :

$$\mathrm{H}^\bullet_{\mathcal{P}}(A,\, M) \cong \mathrm{Ext}^\bullet_{A\otimes^{\mathcal{P}}\mathbb{K}}(\Omega_{\mathcal{P}}(A),\, M),$$

pour toute $\mathcal{P}$-algèbre A, tout A-$\mathcal{P}$-module M et où $A \otimes^{\mathcal{P}} \mathbb{K}$ est l'algèbre enveloppante de la $\mathcal{P}$-algèbre A. Pour une opérade $\mathcal{P}$ fixée, nous étudions la propriété que la cohomologie d'André-Quillen soit un foncteur Ext pour toute $\mathcal{P}$-algèbre A et tout A-$\mathcal{P}$-module. Nous définissons un complexe cotangent fonctoriel $\mathbb{L}_{\mathcal{P}}$, c'est-à-dire ne dépendant que de l'opérade et non d'une $\mathcal{P}$-algèbre, et un module des formes différentielles de Kähler fonctoriel $\mathbf{\Omega}_{\mathcal{P}}$. Tout comme dans le cas non fonctoriel, nous avons une surjection

$$\mathbb{L}_{\mathcal{P}} \twoheadrightarrow \mathbf{\Omega}_{\mathcal{P}} \ ;$$

son noyau est appelé le *module des obstructions* et noté $\mathbb{O}_{\mathcal{P}}$. En suivant les notations de T. Pirashvili dans la critique MathSciNet de [Fra01], on définit la notion d'*opérade ayant la propriété PBW* comme étant une opérade dont les $\mathcal{P}$-algèbres vérifient un analogue du théorème de Poincaré-Birkhoff-Witt. Il ne faut pas confondre cette propriété avec la notion d'opérade de Poincaré-Birkhoff-Witt définie par E. Hoffbeck dans [Hof10]. Nous obtenons le théorème suivant :

Théorème 4. *Soit $\mathcal{P}$ une opérade ayant la propriété PBW. Les propriétés suivantes sont équivalentes :*

- (P_0) *la cohomologie d'André-Quillen est un foncteur Ext sur l'algèbre enveloppante $A\otimes^{\mathcal{P}}\mathbb{K}$ pour toute $\mathcal{P}$-algèbre A,*
- (P_1) *le complexe cotangent est quasi-isomorphe au module des formes différentielles de Kähler pour toute $\mathcal{P}$-algèbre A,*
- (P_2) *le complexe cotangent fonctoriel $\mathbb{L}_{\mathcal{P}}$ est quasi-isomorphe au module fonctoriel des formes différentielles de Kähler $\mathbf{\Omega}_{\mathcal{P}}$,*
- (P_3) *le module des obstructions $\mathbb{O}_{\mathcal{P}}$ est acyclique.*

Ce théorème nous permet de démontrer que la cohomologie d'André-Quillen pour une opérade $\mathcal{P}$ est un foncteur Ext ou non. Nous obtenons ainsi des obstructions universelles au fait que la cohomologie d'André-Quillen des algèbres commutatives, ou des algèbres permutatives, n'est pas toujours un foncteur Ext au sens défini précédemment. Nous étudions ensuite le cas des algèbres sur une opérade cofibrante de la forme $\Omega\mathcal{C}$, dont le principal exemple $\mathcal{C} = \mathcal{P}^{\text{¡}}$ concerne les algèbres à homotopie près. Dans ce cas, les obstructions s'annulent toujours et la cohomologie d'André-Quillen est toujours un foncteur Ext. Nous obtenons ainsi une nouvelle propriété homotopique de stabilité pour les algèbres sur des opérades cofibrantes après J. M. Boardman et R. M. Vogt [BV73]. Un calcul montre que la cohomologie d'André-Quillen d'une $\mathcal{P}$-algèbre vue comme une $\mathcal{P}$-algèbre ou comme une $\mathcal{P}_\infty = \Omega\mathcal{P}^{\text{¡}}$-algèbre avec des homotopies égales à 0 sont égales. Nous obtenons ainsi le théorème suivant :

Théorème 5. *Soient $\mathcal{P}$ une opérade de Koszul, A une $\mathcal{P}$-algèbre et M un A-$\mathcal{P}$-module. Nous avons*

$$\mathrm{H}^\bullet_{\mathcal{P}}(A,\, M) \cong \mathrm{Ext}^\bullet_{A\otimes^{\mathcal{P}_\infty}\mathbb{K}}(\Omega_{\mathcal{P}_\infty}(A),\, M).$$

Ainsi, même si la cohomologie d'André-Quillen des algèbres commutatives, resp. des algèbres permutatives, ne s'écrit pas comme un foncteur Ext dans la catégorie des algèbres commutatives, resp. permutatives, elle s'écrit comme une foncteur Ext dans la catégorie des algèbres commutatives à homotopie près, resp. permutatives à homotopie près.

Chapitre 2 : Dualité de Koszul à courbure

Ce deuxième chapitre continue l'étude des algèbres vue comme représentations d'une opérade, ou plus généralement, d'une propérade. L'opérade $\mathcal{A}s$ qui code les algèbres associatives est donnée par la présentation quadratique

$$\mathcal{A}s := \mathcal{F}\,(\curlyvee)\,/\big(\curlyvee\circ_1\curlyvee - \curlyvee\circ_2\curlyvee\big),$$

où $\mathcal{F}\,(\curlyvee)$ est l'*opérade libre* sur le générateur $\curlyvee$. Nous avons vu que la cohomologie d'André-Quillen dans le cas de l'opérade $\mathcal{A}s$ redonne la cohomologie de Hochschild des algèbres associatives. Cependant, dans ce cas, les algèbres ne sont pas nécessairement unitaires. Pour étudier plus spécifiquement les algèbres associatives unitaires comme des représentations d'une opérade, on est obligé de considérer une opérade $u\mathcal{A}s$ non-augmentée. Nous choisissons pour cette opérade la présentation suivante :

$$u\mathcal{A}s := \mathcal{F}\,(\overset{\bullet}{|},\, \curlyvee)\,/\big(\curlyvee\circ_1\curlyvee - \curlyvee\circ_2\curlyvee,\ \curlyvee\circ_1\overset{\bullet}{|} - |,\ \curlyvee\circ_2\overset{\bullet}{|} - |\big).$$

Comme l'opérade $u\mathcal{A}s$ n'est pas augmentée, nous étendons la définition de la *construction bar* B aux opérades, ou propérades, non-augmentées ce qui nous permet d'obtenir, pour toute algèbre unitaire associative A, une résolution fonctorielle de la forme

$$u\mathcal{A}s \circ_\pi \mathrm{B}\, u\mathcal{A}s \circ_\pi A \xrightarrow{\sim} A.$$

Le défaut d'augmentation est codé sur la construction bar par une *courbure*. Toutefois, l'espace vectoriel sous-jacent à la construction bar est très gros. À une opérade, ou propérade, définie par une présentation quadratique, linéaire et constante, nous associons une coopérade, ou *copropérade*, $\mathcal{P}^{\text{¡}}$ avec courbure, appelée la *co(pr)opérade duale de Koszul* de $\mathcal{P}$. Nous étendons la notion d'opérade ou de propérade de Koszul à de telles (pr)opérades. L'opérade $u\mathcal{A}s$ est une opérade de Koszul et nous obtenons, pour toute algèbre associative unitaire A, la résolution fonctorielle plus petite

$$u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{¡}} \circ_\kappa A \xrightarrow{\sim} A.$$

Nous étudions la cohomologie d'André-Quillen des algèbres associatives unitaires et nous montrons qu'elle correspond à la cohomologie de Hochschild $\mathrm{HH}^\bullet$ de la même algèbre dont on a oublié l'unité.

Théorème 6. *Soit A une algèbre associative unitaire différentielle graduée. Nous avons*

$$\mathrm{H}^\bullet_{u\mathcal{A}s}(A,\, M) \cong \mathrm{HH}^{\bullet+1}(A,\, M).$$

Nous étendons aussi la *construction cobar* Ω aux coopérades ou copropérades avec courbure ainsi que l'adjonction bar-cobar. Pour toute opérade ou propérade, nous obtenons ainsi une résolution cofibrante grâce à cette construction bar-cobar.

Théorème 7. *Soit $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ une propérade (*semi-augmentée*), différentielle graduée, filtrée par un poids et connexe pour ce poids. La counité de l'adjonction bar-cobar $\Omega\mathrm{B}\mathcal{P}$ est une résolution de $\mathcal{P}$, c'est-à-dire*

$$\Omega\mathrm{B}\mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Nous appliquons ce théorème à la propérade $uc\mathcal{F}rob$ qui code les algèbres de Frobenius ayant une unité et une counité.

Théorème 8. *La résolution bar-cobar, appliquée à la propérade uc$\mathcal{F}$rob, est une résolution cofibrante de la propérade uc$\mathcal{F}$rob, c'est-à-dire*

$$\Omega \mathrm{B}\, uc\mathcal{F}rob \xrightarrow{\sim} uc\mathcal{F}rob.$$

Rappelons que la donnée d'une *théorie de champs quantique topologique*, *2d-TQFT* en abrégé et en anglais, est équivalente à une structure d'algèbre de Frobenius avec unité et counité [Abr96, Koc04]. Nous obtenons donc des outils homotopiques pour étudier les théories de champs quantiques topologiques. Notre modèle permet aussi avec les méthodes de S. O. Wilson [Wil07] de mettre une structure d'algèbre de Frobenius avec unité et counité à homotopie près sur les formes différentielles $\Omega(M)$ d'une variété orientée et fermée M.

Tout comme pour les résolutions fonctorielles d'algèbres, il est possible de diminuer la taille de cette résolution pour les opérades ou propérades admettant une présentation quadratique, linéaire et constante qui sont des (pr)opérades de Koszul.

Théorème 9. *Soit $\mathcal{P}$ une propérade (semi-augmentée) de Koszul. La construction cobar sur la copropérade avec courbure duale de Koszul $\mathcal{P}^{¡}$ est une résolution cofibrante de $\mathcal{P}$, c'est-à-dire*

$$\Omega \mathcal{P}^{¡} \xrightarrow{\sim} \mathcal{P}.$$

Nous utilisons cette résolution pour étudier la théorie homotopique des algèbres associatives unitaires. En effet, d'une part, cette résolution est assez simple pour expliciter les notions d'algèbre associative unitaire à homotopie près et les ∞-morphismes entre de telles algèbres. D'autre part, le fait d'avoir une résolution cofibrante assure que nos définitions vérifient de bonnes propriétés homotopiques telles que la possibilité de rectifier la structure d'une algèbre à homotopie près en une structure d'algèbre associative unitaire et l'énoncé d'un théorème de transfert.

Théorème 10 (Théorème de rectification)**.** *Soit A une algèbre associative unitaire à homotopie près. Nous pouvons rectifier A : il existe une algèbre associative unitaire A' telle que A est ∞-quasi-isomorphe à A.*

Théorème 11 (Théorème de transfert homotopique)**.** *Soient A une algèbre associative unitaire à homotopie près et V un complexe de chaînes. Étant donné un rétract par déformation*

$$V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h\ ,$$

i.e. p et i sont des morphismes de complexes de chaînes tels que $p \circ i = id_V$ et $d_A h + h d_A = id_A - i \circ p$, il existe une structure naturelle d'algèbre à homotopie près sur V telle que i s'étende en un ∞-quasi-isomorphisme.

Différentes notions d'*algèbres associatives unitaires à homotopie près* apparaissent dans le littérature [KS06, FOOO07, LH03] en relation avec les catégories de Fukaya et la symétrie miroir. La définition proposée ici est plus générale et elle est définie à l'aide d'une opérade cofibrante. Cette dernière propriété assure de bonnes propriétés homotopiques (théorème de rectification et inversibilité des ∞-quasi-isomorphismes).

Chapitre 3 : Le complexe de Koszul est le complexe cotangent

La cohomologie d'André-Quillen associe des invariants à toute algèbre A sur une opérade. Elle est représentée par le complexe cotangent qui se calcule à l'aide d'une résolution de l'algèbre A. Nous avons vu précédemment des résolutions fonctorielles d'algèbres et nous souhaitons maintenant en réduire la taille. La dualité de Koszul des algèbres associatives permet de tester sur le complexe de Koszul la possibilité de réduire la taille des résolutions fonctorielles d'algèbres associatives. Ce chapitre propose une généralisation de la dualité de Koszul des algèbres associatives à toute algèbre sur une opérade. Nous obtiendrons alors un représentant du complexe cotangent de plus petite taille. Le problème fondamental est de trouver un bonne généralisation pour le complexe de Koszul. Nous montrons que le complexe cotangent est une bonne généralisation du complexe de Koszul du fait qu'il nous permet de généraliser les théorèmes sur lesquels s'appuie la dualité de Koszul, à savoir un théorème fondamental des morphismes tordants et un critère de Koszul.

Soient $\mathcal{P}$ une opérade, $\mathcal{C}$ une coopérade et $\alpha : \mathcal{C} \to \mathcal{P}$ un morphisme tordant opéradique entre les deux. Les constructions bar et cobar des algèbres sur une opérade, l'adjonction bar-cobar et la notion de morphisme tordant algébrique apparaissent chez E. Getzler et J. Jones [GJ94]. Pour obtenir un théorème fondamental des morphismes tordants algébriques, il faut généraliser la notion de produit tensoriel tordu. Nous démontrons que la construction du complexe cotangent $A \otimes^{\mathcal{P}} C$ convient pour définir un "produit tensoriel tordu" associé à une $\mathcal{P}$-algèbre A, une $\mathcal{C}$-cogèbre C et un morphisme tordant algébrique $\varphi : C \to A$ entre les deux. Rappelons que la notion de morphisme de Koszul opéradique $\alpha : \mathcal{C} \to \mathcal{P}$ donne des résolutions de $\mathcal{P}$-algèbres de la forme $\Omega_\alpha \mathrm{B}_\alpha A = \mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A \xrightarrow{\sim} A$.

Comme nous l'avons vu dans le chapitre 1, la cohomologie d'André-Quillen se réduit parfois à un foncteur Ext. C'est le cas lorsque la propriété suivante est satisfaite :

($\star$) Pour toute $\mathcal{P}$-algèbre A, le morphisme de A-modules $\mathbb{L}_{\Omega_\alpha \mathrm{B}_\alpha A/A} = A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$ est un quasi-isomorphisme.

Nous obtenons alors le théorème suivant :

Théorème 12 (Théorème fondamental des morphismes tordants algébriques)**.** *Soit $\alpha : \mathcal{C} \to \mathcal{P}$ un morphisme de Koszul entre une coopérade $\mathcal{C}$ et une opérade $\mathcal{P}$. Soient C une $\mathcal{C}$-cogèbre et A une $\mathcal{P}$-algèbre. On suppose que les objets $\mathcal{C}$, $\mathcal{P}$, C et A sont différentiels gradués et gradués par un poids pour lequel ils sont connexes. Soit $\varphi : sC \to A$ un morphisme tordant algébrique où sC est la suspension homologique de C et sC est une cogèbre sur la désuspension opéradique $\mathcal{S}^{-1}\mathcal{C}$ de $\mathcal{C}$. Les propriétés suivantes sont équivalentes :*

1. *le morphisme tordant φ est un morphisme de Koszul algébrique, c'est-à-dire*

$$A \otimes^{\mathcal{P}} C \xrightarrow{\sim} \mathbb{L}_{\Omega_\alpha \mathrm{B}_\alpha A/A} = A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A,$$

2. *le morphisme de $\mathcal{S}^{-1}\mathcal{C}$-cogèbres $g_\varphi : sC \xrightarrow{\sim} \mathrm{B}_\alpha A$ est un quasi-isomorphisme,*
3. *le morphisme de $\mathcal{P}$-algèbres $f_\varphi : \Omega_\alpha sC \xrightarrow{\sim} A$ est un quasi-isomorphisme.*

De plus, lorsque $\mathcal{P}$ vérifie la condition ($\star$), les propriétés précédentes sont équivalentes à

1'. l'application naturelle $A \otimes^{\mathcal{P}} C \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$ est un quasi-isomorphisme.

Dans le cas fréquent d'une opérade $\mathcal{P}$ et d'une $\mathcal{P}$-algèbre concentrées en degré homologique 0, le module des formes différentielles de Kähler $\Omega_{\mathcal{P}}(A)$ est concentré en degré homologique 0. La condition 1' se lit donc : $A \otimes^{\mathcal{P}} C$ est acyclique.

La dualité de Koszul propose une cogèbre C explicite. V. Ginzburg et M. Kapranov [GK94] étudient la notion de *$\mathcal{P}$-algèbre quadratique* et proposent une *$\mathcal{P}$-algèbre duale de Koszul $A^!$* associée à une $\mathcal{P}$-algèbre quadratique $A = A(V, S) = \mathcal{P}(V)/(S)$. Nous proposons, dans le cadre plus général des *algèbres monogènes*, une version duale et plus générale, appelée *$\mathcal{P}^{¡}$-cogèbre duale de Koszul $A^{¡}$*. Dans ce contexte, le théorème fondamental des morphismes tordants algébrique donne le critère de Koszul suivant :

Théorème 13 (Critère de Koszul)**.** *Soit (E, R) une donnée quadratique opéradique telle que $\mathcal{P} = \mathcal{P}(E, R) = \mathcal{F}(E)/(R)$ est une opérade de Koszul. Soit (V, S) une donnée monogène associée à (E, R). On note $A := \mathcal{P}(V)/(S)$. Les propriétés suivantes sont équivalentes :*

1. *le morphisme tordant κ est un morphisme de Koszul algébrique, c'est-à-dire que*

$$A \otimes^{\mathcal{P}} A^{¡} \xrightarrow{\sim} \mathbb{L}_{\Omega_\kappa \mathrm{B}_\kappa A/A} = A \otimes^{\mathcal{P}} \mathrm{B}_\kappa A$$

   est un quasi-isomorphisme,

2. *l'inclusion $g_{\varkappa} : A^{¡} \rightarrowtail \mathrm{B}_\kappa A$ est un quasi-isomorphisme,*
3. *la projection $f_{\varkappa} : \Omega_\kappa A^{¡} \twoheadrightarrow A$ est un quasi-isomorphisme.*

De plus, lorsque $\mathcal{P}$ vérifie la condition $(\star)$, les propriétés suivantes sont équivalentes à

1'. l'application naturelle $A \otimes^{\mathcal{P}} A^{¡} \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$ est un quasi-isomorphisme.

Lorsque ces propriétés sont vérifiées, la construction cobar sur $A^{¡}$ est une résolution cofibrante de la $\mathcal{P}$-algèbre A. Nous disons alors que la $\mathcal{P}$-algèbre monogène A est Koszul.

De la même façon que précédemment, lorsque l'opérade et l'algèbre sont concentrées en degré homologique 0, la propriété 1' se lit : $A \otimes^{\mathcal{P}} A^{¡}$ est acyclique. Nous appelons le complexe $A \otimes^{\mathcal{P}} A^{¡}$ le *complexe de Koszul*. Lorsque la $\mathcal{P}$-algèbre est Koszul, le complexe de Koszul est un représentant du complexe cotangent et il est un "petit" complexe qui permet de calculer la cohomologie d'André-Quillen de la $\mathcal{P}$-algèbre A.

Cette nouvelle dualité de Koszul redonne dans le cas des algèbres associatives la dualité de Koszul définie par S. Priddy [Pri70]. Dans le cas des opérades $\mathcal{C}om$ et $\mathcal{L}ie$ qui codent les algèbres commutatives et les algèbres de Lie, cette nouvelle dualité de Koszul produit des modèles minimaux de Sullivan [Sul77] et des modèles de Quillen [Qui67]. Un exemple est donné par l'algèbre commutative de cohomologie du complémentaire d'un arrangement d'hyperplan. Elle est donnée par l'algèbre de Orlik-Solomon et, dans le cas quadratique, la duale de Koszul est l'algèbre de Lie d'holonomie de Kohno [Yuz01, PY99, Koh83, Koh85]. Nous retrouvons aussi la dualité de Koszul des modules sur une algèbre associative ou commutative A, ce qui donne un bon candidat pour les syzygies d'un A-module [PP05, Eis04].

Ouverture

La théorie des morphismes tordants fournit des résolutions d'algèbres associatives. Ces morphismes sont représentés par les constructions bar et cobar et on les étudie à l'aide du produit tensoriel tordu. La dualité de Koszul permet d'appliquer cette théorie lorsqu'une algèbre associative admet une bonne présentation : quadratique (Q), quadratique-linéaire (QL), quadratique-linéaire-constant (QLC). Le cas quadratique-linéaire généralise le cas quadratique qui consiste en une partie linéaire nulle et le cas quadratique-linéaire-constant généralise le cas quadratique-linéaire qui consiste en une partie constante nulle. Nous résumons les différents cas dans le tableau suivant :

<table>
<tr><th>Relations \ Monoïdes</th><th>Algèbres associatives</th><th>Opérades</th><th>Propérades</th></tr>
<tr><td>Q</td><td rowspan="2">[Pri70]</td><td>[GJ94, GK94]</td><td>[Val07]</td></tr>
<tr><td>QL</td><td colspan="2">[GCTV09]</td></tr>
<tr><td>QLC</td><td>[Pos93, PP05]</td><td colspan="2">Chapitre 2 de cette thèse</td></tr>
<tr><th>Relations \ Représentations</th><th>Modules</th><th>$\mathcal{P}$-algèbres</th><th>$\mathcal{P}$-algèbres</th></tr>
<tr><td>Q</td><td rowspan="3">[PP05]</td><td>[GJ94, GK94] et chapitre 3 de cette thèse</td><td rowspan="3">Pas de “bigèbre” libre</td></tr>
<tr><td>QL</td><td rowspan="2">À faire...</td></tr>
<tr><td>QLC</td></tr>
</table>

Notons que le chapitre 3 utilise l'étude faite dans le chapitre 1 de la cohomologie d'André-Quillen et du complexe cotangent. La généralisation de la dualité de Koszul des algèbres sur une opérade au cas des $\mathcal{P}$-algèbre admettant une présentation quadratique-linéaire et quadratique-linéaire-constant est à faire pour compléter le tableau.

Il est aussi important d'avoir des outils pour démontrer la Koszulité d'un “monoïde” ou d'une “représentation”. De tels outils existent pour les monoïdes : bases de Poincaré-Birkhoff-Witt [Pri70, Hof10] et bases de Gröbner [Buc06, DK08, BCL09]. Il serait intéressant d'avoir les mêmes outils pour les représentations. Cela permettrait par exemple d'étudier les algèbres de Poisson quadratiques. Remarquons aussi que le tableau ci-dessus ne prend pas en compte les monoïdes et représentations admettant une présentation quadratique et ordre supérieur (ternaire...) comme celles étudiées par S. Merkulov et B. Vallette pour les monoïdes dans [MV09a], de telles présentations font apparaître des structures à homotopie près sur la duale de Koszul et le travail reste à faire pour les représentations.

Chapitre 1

André-Quillen cohomology of an algebra over an operad

Hochschild [Hoc45] introduced a chain complex which defines a cohomology theory for associative algebras. In 1948, Chevalley and Eilenberg gave a definition of a cohomology theory for Lie algebras. Both cohomology theories can be written as classical derived functors (Ext-functors). Later, Quillen [Qui70] defined a cohomology theory associated to commutative algebras with the use of model category structures. André gave similar definitions only with simplicial methods [And74]. This cohomology theory is not equal to an Ext-functor over the enveloping algebra in general.

Using conceptual model category arguments, we recall the definition of the *André-Quillen cohomology (for algebras over an operad)*, in the differential graded setting, from Hinich [Hin97] and Goerss and Hopkins [GH00]. Because we work in the differential graded setting, we use known functorial resolutions of algebras to make chain complexes which compute André-Quillen cohomology explicit. The first idea of this paper is to use Koszul duality theory of operads to provide such functorial resolutions. We can also use the simplicial bar construction, which proves that cotriple cohomology is equal to André-Quillen cohomology. The André-Quillen cohomology is represented by an object, called the *cotangent complex* which therefore plays a crucial role in this theory. The notion of *twisting morphism*, also called twisting cochain, coming from algebraic topology, has been extended to (co)operads and to (co)algebras over a (co)operad by Getzler and Jones [GJ94]. We make the differential on the cotangent complex explicit using these two notions of twisting morphisms all together. When the category of algebras is modeled by a binary Koszul operad, we give a Lie theoretic interpretation of the previous construction. In the review of [Fra01], Pirashvili asked the question of characterizing operads such that the associated André-Quillen cohomology of algebras is an Ext-functor. This paper provides a criterion to answer that question.

When the operad is Koszul, we describe the cotangent complex and the André-Quillen cohomology for the algebras over this operad using its Koszul complex. We recover the classical cohomology theories, with their underlying chain complexes, like André-Quillen cohomology for commutative algebras, Hochschild cohomology for associative algebras and Chevalley-Eilenberg cohomology for Lie algebras. We also recover cohomology theories which were defined recently like cohomology for Poisson algebras [Fre06], cohomology for Leibniz algebras

[LP93], cohomology for pre-Lie algebras [Dzh99], cohomology for diassociative algebras [Fra01] and cohomology for Zinbiel algebras [Bal98]. More generally, Balavoine introduced a chain complex when the operad is binary and quadratic [Bal98]. We show that this chain complex defines André-Quillen cohomology when the operad is Koszul. We make the new example of Perm algebras explicit. For any operad $\mathcal{P}$, we can define a relax version up to homotopy of the notion of $\mathcal{P}$-algebra as follows : we call homotopy $\mathcal{P}$-algebra any algebra over a cofibrant replacement of $\mathcal{P}$ (cf. [BV73]). Using the operadic cobar construction, we make the cotangent complex and the cohomology theories for homotopy algebras explicit. For instance, we recover the case of homotopy associative algebras [Mar92] and the case of homotopy Lie algebras [HS93].

For any algebra A, we prove that its André-Quillen cohomology is an additive derived functor, an Ext-functor, over its enveloping algebra if and only if its cotangent complex is quasi-isomorphic to its module of Kähler differential forms $\Omega_{\mathcal{P}}(A)$. We define a *functorial cotangent complex* and a *functorial module of Kähler differential forms* which depend only on the operad and we reduce the study of the quasi-isomorphisms between the cotangent complex and the module of Kähler differential forms for any algebra to the study of the quasi-isomorphisms between the cotangent complex and the module of Kähler differential forms for any chain complex, with trivial algebra structure (when $\mathcal{P}$ is an *operad satisfying the PBW property*, that is the $\mathcal{P}$-algebras satisfy an analogue of Poincaré-Birkhoff-Witt theorem). This allows us to give a uniform treatment for any algebra over an operad. Assuming that $\mathcal{P}$ is an operad satisfying the PBW property, we prove that the functorial cotangent complex is quasi-isomorphic to the functorial module of Kähler differential forms (we say sometimes concentrated in degree 0 or acyclic), if and only if the André-Quillen cohomology theory for any algebra over this operad is an Ext-functor over its enveloping algebra, so this functorial cotangent complex carries the obstructions for the André-Quillen cohomology to be an Ext-functor. For instance, we prove that the functorial cotangent complex is acyclic for the operads of associative algebras and Lie algebras. In order to control the map between the functorial cotangent complex and the functorial module of Kähler differential forms, we look at its kernel. This defines a new chain complex whose homology groups can also be interpreted as obstructions for the André-Quillen cohomology theory to be an Ext-functor. In this way, we give a new, but more conceptual proof that the cotangent complex for commutative algebras is not always acyclic. Equivalently, it means that there exist commutative algebras such that their André-Quillen cohomology is not an Ext-functor over their enveloping algebra. With the same method, we show the same result for Perm algebras. We can summarize all these properties in the following theorem (Section 4 and 5).

Theorem A. *Let $\mathcal{P}$ be an operad satisfying the PBW property. The following properties are equivalent.*

- (P_0) *The André-Quillen cohomology is an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$ for any $\mathcal{P}$-algebra A ;*
- (P_1) *the cotangent complex is quasi-isomorphic to the module of Kähler differential forms for any $\mathcal{P}$-algebra A ;*
- (P_2) *the functorial cotangent complex $\mathbb{L}_{\mathcal{P}}$ is quasi-isomorphic to the functorial module of Kähler differential forms $\boldsymbol{\Omega}_{\mathcal{P}}$;*
- (P_3) *the module of obstructions $\mathbb{O}_{\mathcal{P}}$ is acyclic.*

In the case of homotopy algebras, we prove that the obstructions for the cohomology to be an Ext-functor vanish. Moreover, any $\mathcal{P}$-algebra is also a homotopy $\mathcal{P}$-algebra. Thus we can compute its André-Quillen cohomology in two different ways. We show that the two coincide. Hence we get the following theorem.

Theorem B. *Let A be a $\mathcal{P}$-algebra and let M be an A-module over the Koszul operad $\mathcal{P}$. We have*

$$\mathrm{H}^{\bullet}_{\mathcal{P}}(A,\, M) \cong \mathrm{Ext}^{\bullet}_{A\otimes^{\mathcal{P}_\infty}\mathbb{K}}(\Omega_{\mathcal{P}_\infty}(A),\, M).$$

Therefore, even if the André-Quillen cohomology of commutative and Perm algebras cannot always be written as an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$, it is always an Ext-funtor over the enveloping algebra $A \otimes^{\mathcal{P}_\infty} \mathbb{K}$.

The paper begins with first definitions and properties of differential graded (co)operads, (co)algebras, modules and free modules over an algebra (over an operad). In Section 1, we recall the definition of the André-Quillen cohomology theory for dg algebras over a dg operad, from Hinich and Goerss-Hopkins. We introduce functorial resolutions for algebras over an operad, which allow us to make the cotangent complex and the cohomology theories explicit. Then, in Section 2, we give a Lie interpretation of the chain complex defining the André-Quillen cohomology. Using the notion of twisting morphism on the level of (co)algebra over a (co)operad, we make the differential on the cotangent complex explicit (Theorem 1.2.4.2). Section 3 is devoted to applications and examples. In Section 4, we prove that the cotangent complex is quasi-isomorphic to the module of Kähler differential forms for any algebra if and only if the André-Quillen cohomology theory is an Ext-functor over the enveloping algebra for any algebra. Moreover, we study the André-Quillen cohomology theory for operads. In Section 5, we introduce the functorial cotangent complex and the functorial module of Kähler differential forms and we finish to prove Theorem A. In Section 6, we study the André-Quillen cohomology for homotopy algebras and we prove Theorem B.

Notation and preliminary

We recall the classical notation for $\mathbb{S}$-module, composition product, (co)operad, (co)algebra over a (co)operad and module over an algebra over an operad. We refer to [GK94] and [GJ94] for a complete exposition and [Fre04] for a more modern treatment. We also refer to the books [LV] and [MSS02].

In the whole paper, we work over a field $\mathbb{K}$ of characteristic 0. In the sequel, the ground category is the category of graded modules, or *g-modules.* For a morphism $f : O_1 \to O_2$ between differential graded modules, the notation $\partial(f)$ stands for the derivative $d_{O_2} \circ f - (-1)^{|f|} f \circ d_{O_1}$. Here f is a map of graded modules and $\partial(f) = 0$ if and only if f is a map of dg-modules. Moreover, for an other morphism $g : O_1' \to O_2'$, we define a morphism $f \otimes g : O_1 \otimes O_1' \to O_2' \otimes O_2'$ using the Koszul-Quillen convention : $(f \otimes g)(o_1 \otimes o_2) := (-1)^{|g||o_1|} f(o_1) \otimes g(o_2)$, where $|e|$ denotes the degree of the element e. We denote by $g\mathcal{M}od_{\mathbb{K}}$ the category whose objects are differential graded $\mathbb{K}$-modules (and not only graded $\mathbb{K}$-modules) and morphisms are maps of graded modules. We have to be careful with this definition because it is not usual. However, we denote as usual by $dg\mathcal{M}od_{\mathbb{K}}$ the category of differential graded $\mathbb{K}$-modules. In this paper, the modules are all differential graded, except explicitly stated.

1.0.1 Differential graded $\mathbb{S}$-modules

A *dg* $\mathbb{S}$*-module* (or $\mathbb{S}$*-module* for short) M is a collection $\{M(n)\}_{n\geq 0}$ of dg modules over the symmetric group $\mathbb{S}_n$. A *morphism of dg* $\mathbb{S}$*-modules* is a collection of equivariant morphisms of chain complexes $\{f_n : M(n) \to N(n)\}_{n\geq 0}$, with respect to the action of $\mathbb{S}_n$.
We define a monoidal product on the category of dg $\mathbb{S}$-modules by

$$(M \circ N)(n) := \bigoplus_{k\geq 0} M(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1+\cdots+i_k=n} Ind^{\mathbb{S}_n}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} (N(i_1) \otimes \cdots \otimes N(i_k)) \right).$$

The unit for the monoidal product is $I := (0,\, \mathbb{K},\, 0,\, \ldots)$. Let M, N and N' be dg $\mathbb{S}$-modules. We define the right linear analog $M \circ (N,\, N')$ of the composition product by the following formula

$$[M \circ (N,\, N')](n) :=$$
$$\bigoplus_{k\geq 0} M(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1+\cdots+i_k=n} \bigoplus_{j=1}^{k} Ind^{\mathbb{S}_n}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} (N(i_1) \otimes \cdots \otimes \underbrace{N'(i_j)}_{j^{th} \text{ position}} \otimes \cdots \otimes N(i_k)) \right).$$

Let $f : M \to M'$ and $g : N \to N'$ be morphisms of dg $\mathbb{S}$-modules. We denote by $\circ'$ the *infinitesimal composite of morphisms*

$$f \circ' g : M \circ N \to M' \circ (N,\, N')$$

defined by

$$\sum_{j=1}^{k} f \otimes (id_N \otimes \cdots \otimes \underbrace{g}_{j^{th} \text{ position}} \otimes \cdots \otimes id_N).$$

Let $(M,\, d_M)$ and $(N,\, d_N)$ be two dg $\mathbb{S}$-modules. We define a grading on $M \circ N$ by

$$(M \circ N)_g(n) := \bigoplus_{\substack{k\geq 0 \\ e+g_1+\cdots+g_k=g}} M_e(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1+\cdots+i_k=n} Ind^{\mathbb{S}_n}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} (N_{g_1}(i_1) \otimes \cdots \otimes N_{g_k}(i_k)) \right).$$

The differential on $M \circ N$ is given by $d_{M\circ N} := d_M \circ id_N + id_M \circ' d_N$.
The differential on $M \circ (N,\, N')$ is given by

$$d_{M\circ(N,\,N')} := d_M \circ (id_N,\, id_{N'}) + id_M \circ' (d_N,\, id_{N'}) + id_M \circ (id_N,\, d_{N'}).$$

Moreover, for any dg $\mathbb{S}$-modules M, N, we denote by $M \circ_{(1)} N$ the dg $\mathbb{S}$-module $M \circ (I,\, N)$. When $f : M \to M'$ and $g : N \to N'$, the map $f \circ (id_I,\, g) : M \circ_{(1)} N \to M' \circ_{(1)} N'$ is denoted by $f \circ_{(1)} g$.

1.0.2 (Co)operad

An *operad* is a monoid in the monoidal category of dg $\mathbb{S}$-modules with respect to the monoidal product $\circ$. A *morphism of operads* is a morphism of dg $\mathbb{S}$-modules commuting with the operads structure. The notion of *cooperad* is the dual version, i.e. a comonoid in the category of dg $\mathbb{S}$-modules. However, we use the invariants for the diagonal actions in the definition of the monoidal product instead of the coinvariants, that is,

$$\bigoplus_{k\geq 0}\left(M(k)\otimes\left(\bigoplus_{i_1+\cdots+i_k=n}(N(i_1)\otimes\cdots\otimes N(i_k))\otimes\mathbb{K}[\mathbb{S}_n]\right)^{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}}\right)^{\mathbb{S}_k}.$$

Since we work over a field of characteristic 0, the invariants are in one-to-one correspondence with the coinvariants and both definitions are equivalent. The definition with the invariants allows to define properly the signs.

The *unit* of an operad $\mathcal{P}$ is denoted by $\iota_{\mathcal{P}}: I \to \mathcal{P}$ and the *counit* of a cooperad $\mathcal{C}$ is denoted by $\eta_{\mathcal{C}}: \mathcal{C} \to I$. Moreover when $(\mathcal{P},\,\gamma)$ is an operad, we define the *partial product* γ_p by

$$\mathcal{P}\circ_{(1)}\mathcal{P} \rightarrowtail \mathcal{P}\circ\mathcal{P} \xrightarrow{\gamma} \mathcal{P}$$

and when $(\mathcal{C},\,\Delta)$ is a cooperad, we define the *partial coproduct* Δ_p by

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C}\circ\mathcal{C} \twoheadrightarrow \mathcal{C}\circ_{(1)}\mathcal{C}.$$

Example. Let V be a dg $\mathbb{K}$-module. The dg $\mathbb{S}$-module $End(V) := \{\mathrm{Hom}(V^{\otimes n},\, V)\}_{n\geq 0}$, endowed with the composition of maps, is an operad.

1.0.3 Module over an operad and relative composition product

A *right $\mathcal{P}$-module* $(\mathcal{L},\,\rho)$ is an dg $\mathbb{S}$-module endowed with a map $\rho: \mathcal{L}\circ\mathcal{P} \to \mathcal{L}$ compatible with the product and the unit of the operad $\mathcal{P}$. We define similarly the notion of *left $\mathcal{P}$-module*.

We define the *relative composition product* $\mathcal{L}\circ_{\mathcal{P}}\mathcal{R}$ between a right $\mathcal{P}$-module $(\mathcal{L},\,\rho)$ and a left $\mathcal{P}$-module $(\mathcal{R},\,\lambda)$ by the coequalizer diagram

$$\mathcal{L}\circ\mathcal{P}\circ\mathcal{R} \underset{id_{\mathcal{L}}\circ\lambda}{\overset{\rho\circ id_{\mathcal{R}}}{\rightrightarrows}} \mathcal{L}\circ\mathcal{R} \twoheadrightarrow \mathcal{L}\circ_{\mathcal{P}}\mathcal{R}\,.$$

1.0.4 Algebra over an operad

Let $\mathcal{P}$ be an operad. An *algebra over the operad* $\mathcal{P}$, or a *$\mathcal{P}$-algebra*, is a dg $\mathbb{K}$-module V endowed with a morphism of operads $\mathcal{P} \to End(V)$.

Equivalently, a $\mathcal{P}$-algebra structure is given by a map $\gamma_V: \mathcal{P}(V) \to V$ which is compatible with the composition product and the unity, where

$$\mathcal{P}(V) := \mathcal{P}\circ(V,\,0,\,0,\,\cdots) = \bigoplus_{n\geq 0}\mathcal{P}(n)\otimes_{\mathbb{S}_n}V^{\otimes n}.$$

1.0.5 Coalgebra over a cooperad

Dually, let $\mathcal{C}$ be a cooperad. A *coalgebra over the cooperad* $\mathcal{C}$, or a *$\mathcal{C}$-coalgebra*, is a dg $\mathbb{K}$-module V endowed with a map $\delta: V \to \mathcal{C}(V) = \oplus_{n\geq 0}(\mathcal{C}(n)\otimes V^{\otimes n})^{\mathbb{S}_n}$ which satisfies compatibility properties. The notation $(-)^{\mathbb{S}_n}$ stands for the space of invariant elements.

1.0.6 Module over a $\mathcal{P}$-algebra

Let $\mathcal{P}$ be a dg $\mathbb{S}$-module and let A be a dg vector space. For a dg vector space M, we define the vector space $\mathcal{P}(A,\, M)$ by the formula

$$\mathcal{P}(A,\, M) := \mathcal{P} \circ (A,\, M) = \bigoplus_{n} \mathcal{P}(n) \otimes_{\mathbb{S}_n} \big(\overset{n}{\underset{j=1}{\oplus}} A \otimes \cdots \otimes \underbrace{M}_{j^{th}\text{ position}} \otimes \cdots \otimes A \big).$$

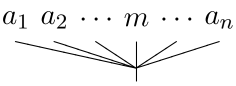
Let $(\mathcal{P},\, \gamma)$ be an operad and let $(A,\, \gamma_A)$ be a $\mathcal{P}$-algebra. An *A-module* $(M,\, \gamma_M,\ \iota_M)$, or *$A$-module over $\mathcal{P}$*, is a vector space M endowed with two maps $\gamma_M : \mathcal{P}(A,\, M) \to M$ and $\iota_M : M \to \mathcal{P}(A,\, M)$ such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(A),\, \mathcal{P}(A,\, M)) & \xrightarrow{id_{\mathcal{P}}(\gamma_A,\, \gamma_M)} & \mathcal{P}(A,\, M) \\ & & \downarrow{\scriptstyle \gamma_M} \\ \downarrow{\scriptstyle \cong} & & M \\ & & \uparrow{\scriptstyle \gamma_M} \\ (\mathcal{P} \circ \mathcal{P})(A,\, M) & \xrightarrow[\gamma(id_A,\, id_M)]{} & \mathcal{P}(A,\, M) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \xrightarrow{\iota_M} & \mathcal{P}(A,\, M) \\ & \searrow^{=} & \downarrow{\scriptstyle \gamma_M} \\ & & M. \end{array}$$

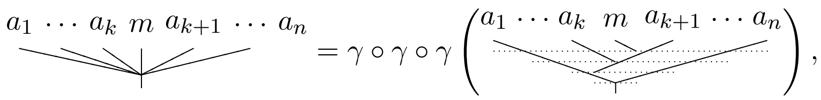
(*Associativity*) (*Unitarity*)

The category of A-modules over the operad $\mathcal{P}$ is denoted by $\mathcal{M}_A^{\mathcal{P}}$. The objects in $\mathcal{M}_A^{\mathcal{P}}$ are differential graded A-modules over $\mathcal{P}$. However, the morphisms in $\mathcal{M}_A^{\mathcal{P}}$ are only maps of graded A-modules over $\mathcal{P}$.

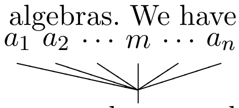
EXAMPLES.

– The operad $\mathcal{P} = \mathcal{A}s$ encodes associative algebras (not necessarily with unit). Then the map $\gamma_n : \mathcal{A}s(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \to A$ stands for the associative product of n elements, where $\mathcal{A}s(n) = \mathbb{K}[\mathbb{S}_n]$. We represent an element in $\mathcal{A}s(n)$ by a corolla with n entries. Then, an element in $\mathcal{A}s(A,\, M)$ can be represented by 

. However,



then by several uses of the associativity diagram of γ_M, we get that an A-module over the operad $\mathcal{A}s$ is given by two morphisms $A \otimes M \to M$ and $M \otimes A \to M$. Finally, we get the classical notion of dg A-bimodule.

– The operad $\mathcal{P} = \mathcal{C}om$ encodes classical associative and commutative algebras. We have $\mathcal{C}om(n) = \mathbb{K}$ and an element in $\mathcal{C}om(A,\, M)$ can be represented by 

where the corolla is non-planar. Like before, an A-module structure over the operad $\mathcal{C}om$ is given by a morphism $A \otimes M \to M$. Hence, we get the classical notion of dg A-module.

– The operad $\mathcal{P} = \mathcal{L}ie$ encodes the Lie algebras. In this case, an A-module over the operad $\mathcal{L}ie$ is actually a classical dg Lie module or equivalently a classical associative module over the universal enveloping algebra of the Lie algebra A.

Goerss and Hopkins defined in [GH00] a free A-module. We recall here the definition.

1.0.6.1 Proposition (Proposition 1.10 of [GH00])**.** *The forgetful functor* $U : \mathcal{M}_A^{\mathcal{P}} \to g\mathcal{M}od_{\mathbb{K}}$ *has a left adjoint, denoted by*

$$N \mapsto A \otimes^{\mathcal{P}} N.$$

That is we have an isomorphism of dg modules

$$(\mathrm{Hom}_{\mathcal{M}_A^{\mathcal{P}}}(A \otimes^{\mathcal{P}} N,\, M),\, \partial) \cong (\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(N,\, UM),\, \partial)$$

for all $N \in g\mathcal{M}od_{\mathbb{K}}$ *and* $M \in \mathcal{M}_A^{\mathcal{P}}$.

A description of $A \otimes^{\mathcal{P}} N$ is given by the following coequalizer diagram in $dg\mathcal{M}od_{\mathbb{K}}$

$$\mathcal{P}(\mathcal{P}(A),\, N) \underset{c_1}{\overset{c_0}{\rightrightarrows}} \mathcal{P}(A,\, N) \twoheadrightarrow A \otimes^{\mathcal{P}} N,$$

where the two first maps are given by the operad product

$$\mathcal{P}(\mathcal{P}(A),\, N) \rightarrowtail (\mathcal{P} \circ \mathcal{P})(A,\, N) \xrightarrow{\gamma(id_A,\, id_N)} \mathcal{P}(A,\, N)$$

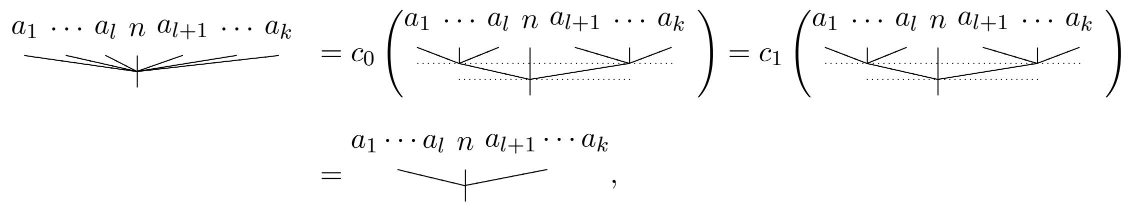
and the $\mathcal{P}$-algebra structure

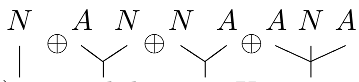
$$\mathcal{P}(\mathcal{P}(A),\, N) \xrightarrow{id_{\mathcal{P}}(\gamma_A,\, id_N)} \mathcal{P}(A,\, N).$$

Remark. We have to make note of the fact that the symbol $\otimes^{\mathcal{P}}$ is just a notation and not a classical tensor product (except in the case $\mathcal{P} = \mathcal{C}om$), as we will see in the following examples.

Examples.

– When $\mathcal{P} = \mathcal{A}s$, we can write



then we get $A \otimes^{\mathcal{A}s} N \cong$ 

$\cong N \oplus A \otimes N \oplus N \otimes A \oplus A \otimes N \otimes A \cong (\mathbb{K} \oplus A) \otimes N \otimes (\mathbb{K} \oplus A)$ as modules over $\mathbb{K}$.

– When $\mathcal{P} = \mathcal{C}om$, we get $A \otimes^{\mathcal{C}om} N \cong N \oplus A \otimes N \cong (\mathbb{K} \oplus A) \otimes N$ as modules over $\mathbb{K}$.

– When $\mathcal{P} = \mathcal{L}ie$, we get $A \otimes^{\mathcal{L}ie} N \cong U^e(A) \otimes N$ as modules over $\mathbb{K}$, where $U^e(A)$ is the enveloping algebra of the Lie algebra A.

These examples lead to the study of the A-module $A \otimes^{\mathcal{P}} \mathbb{K}$ which is the *enveloping algebra of the $\mathcal{P}$-algebra A* (defined in [HS93, GJ94]). It has a multiplication given by

$$(A \otimes^{\mathcal{P}} \mathbb{K}) \otimes (A \otimes^{\mathcal{P}} \mathbb{K}) \cong A \otimes^{\mathcal{P}} (A \otimes^{\mathcal{P}} \mathbb{K}) \to A \otimes^{\mathcal{P}} \mathbb{K},$$

where the arrow is induced by the composition γ of the operad (indeed, the kernel of the map $\mathcal{P}(A,\, \mathcal{P}(A,\, \mathbb{K})) \twoheadrightarrow A \otimes^{\mathcal{P}} (A \otimes^{\mathcal{P}} \mathbb{K})$ is sent to 0 by the map $\mathcal{P}(A,\, \mathcal{P}(A,\, \mathbb{K})) \rightarrowtail (\mathcal{P} \circ \mathcal{P})(A,\, \mathbb{K}) \xrightarrow{\gamma(id_A,\, id_{\mathbb{K}})} \mathcal{P}(A,\, \mathbb{K}) \twoheadrightarrow A \otimes^{\mathcal{P}} \mathbb{K}$). This multiplication is associative and has a unit $\mathbb{K} \to A \otimes^{\mathcal{P}} \mathbb{K}$.

1.0.6.2 Proposition (Proposition 1.14 of [GH00]). *The category $\mathcal{M}_A^{\mathcal{P}}$ of A-modules over $\mathcal{P}$ is isomorphic to the category of left unitary $A \otimes^{\mathcal{P}} \mathbb{K}$-modules $g\mathcal{M}od_{A\otimes^{\mathcal{P}}\mathbb{K}}$.*

Remark. We work in a differential graded setting. The differential on $A \otimes^{\mathcal{P}} \mathbb{K}$ is induced by the differential on $\mathcal{P}(A, \mathbb{K})$. It is easy to see that the isomorphism is compatible with the graded differential framework.

Given a map of $\mathcal{P}$-algebras $B \xrightarrow{f} A$, there exists a forgetful functor $f^* : \mathcal{M}_A^{\mathcal{P}} \to \mathcal{M}_B^{\mathcal{P}}$, whose left adjoint gives the notion of free A-module on a B-module.

1.0.6.3 Proposition (Lemma 1.16 of [GH00]). *The forgetful functor $f^* : \mathcal{M}_A^{\mathcal{P}} \to \mathcal{M}_B^{\mathcal{P}}$ has a left adjoint denoted by*

$$N \mapsto f_!(N) := A \otimes_B^{\mathcal{P}} N.$$

That is we have an isomorphism of dg modules

$$\mathrm{Hom}_{\mathcal{M}_A^{\mathcal{P}}}(f_!(N),\, M) \cong \mathrm{Hom}_{\mathcal{M}_B^{\mathcal{P}}}(N,\, f^*(M))$$

for all $M \in \mathcal{M}_A^{\mathcal{P}}$ and $N \in \mathcal{M}_B^{\mathcal{P}}$.

It is also possible to make explicit the A-module $A \otimes_B^{\mathcal{P}} N$ as the following coequalizer

$$A \otimes^{\mathcal{P}} (B \otimes^{\mathcal{P}} N) \rightrightarrows A \otimes^{\mathcal{P}} N \twoheadrightarrow A \otimes_B^{\mathcal{P}} N.$$

The module $A \otimes^{\mathcal{P}} (B \otimes^{\mathcal{P}} N)$ is a quotient of $\mathcal{P}(A,\, \mathcal{P}(B,\, N))$, then we define on $\mathcal{P}(A,\, \mathcal{P}(B,\, N))$ the composite

$$\mathcal{P}(A,\, \mathcal{P}(B,\, N)) \xrightarrow{id_{\mathcal{P}}(id_A,\, id_{\mathcal{P}}(f,\, id_N))} \mathcal{P}(A,\, \mathcal{P}(A,\, N)) \rightarrowtail (\mathcal{P} \circ \mathcal{P})(A,\, N) \xrightarrow{\gamma(id_A,\, id_N)} \mathcal{P}(A,\, N) \twoheadrightarrow A \otimes^{\mathcal{P}} N.$$

This map induced the first arrow $A \otimes^{\mathcal{P}} (B \otimes^{\mathcal{P}} N) \to A \otimes^{\mathcal{P}} N$.
Similarly, the second map is induced by the composite

$$\mathcal{P}(A,\, \mathcal{P}(B,\, N)) \xrightarrow{id_{\mathcal{P}}(id_A,\gamma_N)} \mathcal{P}(A,\, N) \twoheadrightarrow A \otimes^{\mathcal{P}} N,$$

where γ_N encodes the B-module structure on N.

Remark. The A-module $A \otimes_B^{\mathcal{P}} N$ is a quotient of the free A-module $A \otimes^{\mathcal{P}} N$. As for the notation $\otimes^{\mathcal{P}}$, we have to be careful about the notation $\otimes_B^{\mathcal{P}}$ which is not a classical tensor product over B (except for $\mathcal{P} = \mathcal{C}om$), as we see in the following examples.

Examples. Provided a morphism of algebras $B \xrightarrow{f} A$, we have the dg $\mathbb{K}$-modules isomorphisms

- $A \otimes_B^{\mathcal{A}s} N \cong (\mathbb{K} \oplus A) \otimes_B N \otimes_B (\mathbb{K} \oplus A)$, where the map $B \to \mathbb{K}$ is the zero map,
- $A \otimes_B^{\mathcal{C}om} N \cong (\mathbb{K} \oplus A) \otimes_B N$, where the map $B \to \mathbb{K}$ is the zero map,
- $A \otimes_B^{\mathcal{L}ie} N \cong U^e(A) \otimes_B N$, where $U^e(A)$ is the enveloping algebra of the Lie algebra A.

In all these examples, the notation $\otimes_B$ stands for the usual tensor product over B.

1.1 André-Quillen cohomology of algebras over an operad

First we recall the conceptual definition of André-Quillen cohomology with coefficients of an algebra over an operad from [Hin97, GH00]. Then we recall the constructions and theorems of Koszul duality theory of operads [GK94]. Finally, we recall the definition of twisting morphism given by [GJ94]. This section contains no new result but we will use these three theories throughout the text. We only want to emphasize that operadic resolutions from Koszul duality theory define functorial cofibrant resolutions on the level of algebras and then provide explicit chain complexes which compute André-Quillen cohomology.

We work with the cofibrantly generated model category of algebras over an operad and of modules over an operad given in [GJ94], [Hin97] and [BM03].

1.1.1 Derivation and cotangent complex

To study the structure of the $\mathcal{P}$-algebra A, we derive the functor of $\mathcal{P}$-derivations from A to M in the Quillen sense (non-abelian setting).

Algebras over a $\mathcal{P}$-algebra

Let A be a $\mathcal{P}$-algebra. A $\mathcal{P}$-algebra B endowed with an augmentation, that is a map of $\mathcal{P}$-algebras $B \xrightarrow{f} A$, is called a *$\mathcal{P}$-algebra over A*. We denote by $\mathcal{P}\text{-}\mathcal{A}lg/A$ the category of dg $\mathcal{P}$-algebras over A (the morphisms are given by the morphisms of graded algebras which commute with the augmentation maps).

Derivation

Let B be a $\mathcal{P}$-algebra over A and let M be an A-module. An *A-derivation from B to M* is a linear map $d : B \to M$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathcal{P}(B) = \mathcal{P} \circ B & \xrightarrow{id_{\mathcal{P}} \circ' d} & \mathcal{P}(B,\, M) & \xrightarrow{id_{\mathcal{P}} \circ (f,\, id_M)} & \mathcal{P}(A,\, M) \\ \downarrow{\scriptstyle \gamma_B} & & & & \downarrow{\scriptstyle \gamma_M} \\ B & & \xrightarrow[d]{\qquad\qquad\qquad} & & M, \end{array}$$

where the infinitesimal composite of morphisms $\circ'$ was defined in 1.0.1. We denote by $\mathrm{Der}_A(B,\, M)$ the set of A-derivations from B to M.

This functor is representable on the right by the abelian extension of A by M and on the left by the B-module $\Omega_{\mathcal{P}} B$ of Kähler differential forms as follows.

Abelian extension

Let A be a $\mathcal{P}$-algebra and let M be an A-module. The *abelian extension of A by M*, denoted by $A \ltimes M$, is the $\mathcal{P}$-algebra over A whose underlying space is $A \oplus M$ and whose algebra structure is given by

$$\mathcal{P}(A \oplus M) \twoheadrightarrow \mathcal{P}(A) \oplus \mathcal{P}(A,\, M) \xrightarrow{\gamma_A + \gamma_M} A \oplus M.$$

The morphism $A \ltimes M \to A$ is just the projection on the first summand.

1.1.1.1 Lemma (Definition 2.1 of [GH00])**.** *Let A be a $\mathcal{P}$-algebra and M be an A-module. Then there is an isomorphism of dg modules*

$$\mathrm{Der}_A(B,\, M) \cong \mathrm{Hom}_{\mathcal{P}\text{-}\mathcal{A}lg/A}(B,\, A \ltimes M).$$

PROOF. Any morphism of $\mathcal{P}$-algebras $g : B \to A \ltimes M$ is the sum of the augmentation $B \to A$ and a derivation $d : B \to M$ and vice versa. □

1.1.1.2 Lemma (Lemma 2.3 of [GH00])**.** *Let B be a $\mathcal{P}$-algebra over A and M be an A-module. There is a B-module $\Omega_{\mathcal{P}} B$ and an isomorphism of dg modules*

$$\mathrm{Der}_A(B,\, M) \cong \mathrm{Hom}_{\mathcal{M}_B^{\mathcal{P}}}(\Omega_{\mathcal{P}} B,\, f^*(M)),$$

where the forgetful functor f^* endows M with a B-module structure. Moreover, when $B = \mathcal{P}(V)$ is a free algebra, we get $\Omega_{\mathcal{P}} B \cong B \otimes^{\mathcal{P}} V$.

The second part of the lemma is given by the fact that $\mathrm{Der}_A(\mathcal{P}(V),\, M) \cong \mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(V,\, M)$, that is any derivation from a free $\mathcal{P}$-algebra is characterized by the images of its generators.

The B-module $\Omega_{\mathcal{P}} B$ is called the *module of Kähler differential forms.* It can be made explicit by the coequalizer diagram

$$B \otimes^{\mathcal{P}} \mathcal{P}(B) \rightrightarrows B \otimes^{\mathcal{P}} B \twoheadrightarrow \Omega_{\mathcal{P}} B,$$

where the first arrow is $B \otimes^{\mathcal{P}} \gamma_B$ and the map

$$\mathcal{P}(B,\, \mathcal{P}(B)) \rightarrowtail (\mathcal{P} \circ \mathcal{P})(B,\, B) \xrightarrow{\gamma(id_B,\, id_B)} \mathcal{P}(B,\, B) \twoheadrightarrow B \otimes^{\mathcal{P}} B$$

factors through $B \otimes^{\mathcal{P}} \mathcal{P}(B)$ to give the second arrow.

1.1.1.3 Corollary. *Let B be a $\mathcal{P}$-algebra over A and M be an A-module. There is an isomorphism of dg modules*

$$\mathrm{Der}_A(B,\, M) \cong \mathrm{Hom}_{\mathcal{M}_A^{\mathcal{P}}}(A \otimes_B^{\mathcal{P}} \Omega_{\mathcal{P}} B,\, M).$$

PROOF. We use Lemma 1.1.1.2 and the fact that $A \otimes_B^{\mathcal{P}} -$ is left adjoint to the forgetful functor f^* (Proposition 1.0.6.3). □

Finally, we get a pair of adjoint functors

$$A \otimes_-^{\mathcal{P}} \Omega_{\mathcal{P}} - \quad : \quad \mathcal{P}\text{-}\mathcal{A}lg/A \rightleftharpoons \mathcal{M}_A^{\mathcal{P}} \quad : \quad A \ltimes -.$$

We recall the model category structures on $\mathcal{P}\text{-}\mathcal{A}lg/A$ and $\mathcal{M}_A^{\mathcal{P}}$ given in [Hin97]. It is obtained by the following transfer principle (see also [GJ94] and [BM03]). Let $\mathcal{D}$ be a cofibrantly generated model category and let $\mathcal{E}$ be a category with small colimits and finite limits. Assume that $F : \mathcal{D} \rightleftharpoons \mathcal{E} : G$ is an adjunction with left adjoint F. Then the category $\mathcal{E}$ inherits a cofibrantly generated model category structure from $\mathcal{D}$, provided that G preserves filtered colimits and that Quillen's small object (or Quillen's path-object) argument is verified. In this model category structure, a map f in $\mathcal{E}$ is a weak equivalence (resp. fibration) if and only if $G(f)$ is a weak equivalence (resp. fibration) in $\mathcal{D}$.

In [Hin97], Hinich transfers the model category structure of the category of chain complexes over $\mathbb{K}$ to the category of $\mathcal{P}$-algebras (see Theorem 4.1.1 of [Hin97], every operad is Σ-split since $\mathbb{K}$ is of characteristic 0). Finally, we obtain a model category structure on $\mathcal{P}\text{-}\mathcal{A}lg/A$ in which $g : B \to B'$ is a weak equivalence (resp. a fibration) when the underlying map between differential graded modules is a quasi-isomorphism (resp. surjection). The category $\mathcal{M}_A^{\mathcal{P}}$ of A-modules is isomorphic to the category $g\mathcal{M}od_{A\otimes^{\mathcal{P}}\mathbb{K}}$ of differential graded module over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$ (Proposition 1.0.6.2). Then the category $\mathcal{M}_A^{\mathcal{P}}$ inherits a model category structure in which $g : M \to M'$ is a weak equivalence (resp. a fibration) when g is a quasi-isomorphism (resp. surjection) of $A \otimes^{\mathcal{P}} \mathbb{K}$-modules.

1.1.1.4 Proposition. *The pair of adjoint functors*

$$A \otimes_{-}^{\mathcal{P}} \Omega_{\mathcal{P}} - \quad : \quad \mathcal{P}\text{-}\mathcal{A}lg/A \rightleftharpoons \mathcal{M}_A^{\mathcal{P}} \quad : \quad A \ltimes -$$

forms a Quillen adjunction.

Proof. By Lemma 1.3.4 of [Hov99], it is enough to prove that $A \ltimes -$ preserves fibrations and acyclic fibrations. Let $g : M \twoheadrightarrow M'$ be a fibration (resp. acyclic fibration) between A-modules. Then g is a surjection (resp. a surjective quasi-isomorphism). The image of the map g under the functor $A \ltimes -$ is $id_A \oplus g : A \ltimes M \to A \ltimes M'$, denoted by $id_A \ltimes g$. It follows that $id_A \ltimes g$ is surjective (resp. surjective and a quasi-isomorphism), which completes the proof. □

Thus, we consider the derived functors and we get the following adjunction between the homotopy categories

$$\mathbb{L}(A \otimes_{-}^{\mathcal{P}} \Omega_{\mathcal{P}} -) \quad : \quad \mathrm{Ho}(\mathcal{P}\text{-}\mathcal{A}lg/A) \rightleftharpoons \mathrm{Ho}(\mathcal{M}_A^{\mathcal{P}}) \quad : \quad \mathbb{R}(A \ltimes -).$$

It follows that the cohomology of

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M}_A^{\mathcal{P}})}(A \otimes_R^{\mathcal{P}} \Omega_{\mathcal{P}} R,\, M) \cong \mathrm{Der}_A(R,\, M) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{P}\text{-}\mathcal{A}lg/A)}(R,\, A \ltimes M)$$

is independent of the choice of the cofibrant resolution R of A in the model category of $\mathcal{P}$-algebras over A.

André-Quillen (co)homology and cotangent complex

Let $R \xrightarrow{\sim} A$ be a cofibrant resolution of A. The *cotangent complex* is the total (left) derived functor of the previous adjunction and a representation of it is given by

$$\mathbb{L}_{R/A} := A \otimes_R^{\mathcal{P}} \Omega_{\mathcal{P}} R \in \mathrm{Ho}(\mathcal{M}_A^{\mathcal{P}}).$$

The *André-Quillen cohomology of the $\mathcal{P}$-algebra A with coefficients in an A-module M* is defined by

$$\mathrm{H}^{\bullet}_{\mathcal{P}}(A,\, M) := \mathrm{H}^{\bullet}(\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M}_A^{\mathcal{P}})}(\mathbb{L}_{R/A},\, M)).$$

The *André-Quillen homology of the $\mathcal{P}$-algebra A with coefficients in an A-module M* is defined by

$$\mathrm{H}_{\bullet}^{\mathcal{P}}(A,\, M) := \mathrm{H}_{\bullet}(M \otimes_{A\otimes^{\mathcal{P}}\mathbb{K}} \mathbb{L}_{R/A}).$$

The study of the André-Quillen homology with coefficients is analogous to the study of the André-Quillen cohomology with coefficients. In this paper, we only work with André-Quillen cohomology with coefficients.

Remark. We use the left derived functor of the adjunction to define the André-Quillen cohomology. It is equivalent to define the André-Quillen cohomology by means of the right derived functor. We make this choice here because we are interested in considering homomorphisms in a modules category.

1.1.2 Bar construction of an operad and Koszul operad

To make this cohomology theory explicit, we need a cofibrant resolution for algebras over an operad. In the model category of algebras over an operad, a cofibrant object is a retract of a quasi-free algebra endowed with a good filtration (for example, a non-negatively graded algebra). So we look for quasi-free resolutions of algebras. Operadic resolutions provide such functorial cofibrant resolutions for algebras. There are mainly three operadic resolutions : the simplicial bar construction which induces a Godement type resolution for algebras, the (co)augmented (co)bar construction on the level of (co)operads and the Koszul complex for operads. This last one induces the bar-cobar resolution (or Boardman-Vogt resolution [BV73, BM06]) on the level of algebras. The aim of the two next subsections is to recall the operadic resolutions.

Here, we briefly recall the (co)bar construction of a (co)operad and the notion of Koszul operad. We refer to [GK94, GJ94, Fre04] for a complete exposition.

Bar construction

Let $\mathcal{P}$ be an augmented operad. We denote by sV the suspension of V (that-is-to-say $(sV)_d := V_{d-1}$). The *bar construction of* $\mathcal{P}$ is the quasi-free cooperad

$$\mathrm{B}(\mathcal{P}) := (\mathcal{F}^c(s\overline{\mathcal{P}}),\, d_{\mathrm{B}(\mathcal{P})} := d_1 - d_2),$$

where the map d_1 is induced by the internal differential of the operad ($d_{s\overline{\mathcal{P}}} := id_{\mathbb{K}s} \otimes d_{\mathcal{P}}$) and the component d_2 is induced by the product of the operad by

$$\mathcal{F}_{(2)}(s\overline{\mathcal{P}}) \cong \bigoplus_{\text{2-vertices trees}} \mathbb{K}s\otimes\overline{\mathcal{P}}\otimes\mathbb{K}s\otimes\overline{\mathcal{P}} \xrightarrow{id_{\mathbb{K}s}\otimes\tau\otimes id_{\overline{\mathcal{P}}}} \bigoplus_{\text{2-vertices trees}} \mathbb{K}s\otimes\mathbb{K}s\otimes\overline{\mathcal{P}}\otimes\overline{\mathcal{P}} \xrightarrow{\Pi_s\otimes\gamma_{\mathcal{P}}} \mathbb{K}s\otimes\overline{\mathcal{P}},$$

where $\tau : \overline{\mathcal{P}} \otimes \mathbb{K}s \to \mathbb{K}s \otimes \overline{\mathcal{P}}$ is the *symmetry isomorphism* given explicitly by $\tau(o_1 \otimes o_2) := (-1)^{|o_1||o_2|} o_2 \otimes o_1$ and $\Pi_s : \mathbb{K}s \otimes \mathbb{K}s \to \mathbb{K}s$ is the morphism of degree -1 induced by $\Pi_s(s \otimes s) := s$.

Remark. Assume that $\mathcal{P}$ is weight graded. Then the bar construction is bigraded by the number (w) of non-trivial indexed vertices and by the total weight (ρ)

$$\mathrm{B}_{(w)}(\mathcal{P}) := \oplus_{\rho\in\mathbb{N}} \mathrm{B}_{(w)}(\mathcal{P})^{(\rho)}.$$

Dually, we define the *cobar construction of a coaugmented cooperad* $\mathcal{C}$ by

$$\Omega(\mathcal{C}) := (\mathcal{F}(s^{-1}\overline{\mathcal{C}}),\, d_1 - d_2).$$

From now on, we assume that $\mathcal{P}$ is an augmented operad and $\mathcal{C}$ is a coaugmented cooperad.

Quadratic operad

A operad $\mathcal{P}$ is *quadratic* when $\mathcal{P} = \mathcal{F}(V)/(R)$, where V is the $\mathbb{S}$-module of generators, $\mathcal{F}(V)$ is the free operad and the space of relations R lives in $\mathcal{F}_{(2)}(V)$, the set of trees with two vertices. We endow $\mathcal{F}(V)$ with a weight grading, which differs from the homological degree, given by the number of vertices, this induces a weight grading on each quadratic operad. In this paper, we consider only non-negatively weight graded operad and we say that a weight graded dg operad $\mathcal{P}$ is connected when $\mathcal{P} = \mathbb{K} \oplus \mathcal{P}^{(1)} \oplus \mathcal{P}^{(2)} \oplus \cdots$, where $\mathcal{P}^{(0)} = \mathbb{K}$ is concentrated in homological degree 0.

Koszul operad

We define the *Koszul dual cooperad of* $\mathcal{P}$ by the weight graded dg $\mathbb{S}$-module

$$\mathcal{P}^{\text{i}}_{(\rho)} := \mathrm{H}_\rho(\mathrm{B}_{(\bullet)}(\mathcal{P})^{(\rho)},\, d_2).$$

An operad is called a *Koszul* operad when the injection $\mathcal{P}^{\text{i}} \rightarrowtail \mathrm{B}(\mathcal{P})$ is a quasi-isomorphism.

When $\mathcal{P}$ is of finite type, that is $\mathcal{P}(n)$ is finite dimensional for each n, we can dualize linearly the cooperad $\mathcal{P}^{\text{i}}$ to get the *Koszul dual operad of* $\mathcal{P}$, denoted by $\mathcal{P}^!$. For any $\mathbb{S}_n$-module V, we denote by $V^\vee$ the $\mathbb{S}_n$-module $V^* \otimes (sgn_n)$, where (sgn_n) is the one-dimensional signature representation of $\mathbb{S}_n$. We define $\mathcal{P}^!(n) := \mathcal{P}^{\text{i}}(n)^\vee$. The product on $\mathcal{P}^!$ is given by ${}^t\Delta_{\mathcal{P}^{\text{i}}} \circ \omega$ where $\omega : \mathcal{P}^{\text{i}\vee} \circ \mathcal{P}^{\text{i}\vee} \to (\mathcal{P}^{\text{i}} \circ \mathcal{P}^{\text{i}})^\vee$.

Algebras up to homotopy

Let $\mathcal{P}$ be a Koszul operad. We define $\mathcal{P}_\infty := \Omega(\mathcal{P}^{\text{i}})$. A $\mathcal{P}_\infty$-algebra is called an *algebra up to homotopy* or *homotopy* $\mathcal{P}$*-algebra* (see [GK94]). The notion of $\mathcal{P}_\infty$-algebras is a lax version of the notion of $\mathcal{P}$-algebras.

Examples.
- When $\mathcal{P} = \mathcal{A}s$, we get the notion of A_∞-algebras ;
- when $\mathcal{P} = \mathcal{L}ie$, we get the notion of L_∞-algebras ;
- when $\mathcal{P} = \mathcal{C}om$, we get the notion of C_∞-algebras.

1.1.3 Operadic twisting morphism

We refer to [GJ94, MV09a] for a general and complete treatment. Let $\alpha,\, \beta : \mathcal{C} \to \mathcal{P}$ be morphisms of $\mathbb{S}$-modules. We define the convolution product

$$\alpha \star \beta : \mathcal{C} \xrightarrow{\Delta_p} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{\alpha \circ_{(1)} \beta} \mathcal{P} \circ_{(1)} \mathcal{P} \xrightarrow{\gamma_p} \mathcal{P}.$$

The $\mathbb{S}$-module $\mathrm{Hom}(\mathcal{C},\, \mathcal{P})$ is endowed with an operad structure. Moreover, the convolution product is a pre-Lie product on $\mathrm{Hom}(\mathcal{C},\, \mathcal{P})$, that is, it satisfies the relation

$$(\alpha \star \beta) \star \gamma - \alpha \star (\beta \star \gamma) = (-1)^{|\beta||\gamma|}[(\alpha \star \gamma) \star \beta - \alpha \star (\gamma \star \beta)] \text{ for all } \alpha,\, \beta \text{ and } \gamma \text{ in } \mathrm{Hom}(\mathcal{C},\, \mathcal{P}).$$

Definition

An *operadic twisting morphism* is a map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree -1 satisfying the *Maurer-Cartan equation*

$$\partial(\alpha) + \alpha \star \alpha = 0.$$

We denote the set of operadic twisting morphisms from $\mathcal{C}$ to $\mathcal{P}$ by $\mathrm{Tw}(\mathcal{C},\, \mathcal{P})$.

In the weight graded case, we assume that the twisting morphisms and the internal differentials preserve the weight.

1.1.3.1 Theorem (Theorem 2.17 of [GJ94]). *The functors* Ω *and* B *form a pair of adjoint functors between the category of connected coaugmented cooperads and augmented operads. The natural bijections are given by the set of operadic twisting morphisms :*

$$\mathrm{Hom}_{dg-Op}(\Omega(\mathcal{C}),\, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C},\, \mathcal{P}) \cong \mathrm{Hom}_{dg-Coop}(\mathcal{C},\, \mathrm{B}(\mathcal{P})).$$

Examples. We give examples of operadic twisting morphisms.

– When $\mathcal{C} = \mathrm{B}(\mathcal{P})$ is the bar construction on $\mathcal{P}$, the previous theorem gives a natural operadic twisting morphism $\pi : \mathrm{B}(\mathcal{P}) = \mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \xrightarrow{s^{-1}} \overline{\mathcal{P}} \rightarrowtail \mathcal{P}$. This morphism is universal in the sense that each twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factorizes uniquely through the map π

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\alpha\quad} & \mathcal{P} \\ & {\scriptstyle f_\alpha}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & \mathrm{B}(\mathcal{P}), & \end{array}$$

where f_α is a morphism of dg cooperads.

– When $\mathcal{C} = \mathcal{P}^{¡}$ is the Koszul dual cooperad of a quadratic operad $\mathcal{P}$, the map $\kappa : \mathcal{P}^{¡} \rightarrowtail \mathrm{B}(\mathcal{P}) \xrightarrow{\pi} \mathcal{P}$ is an operadic twisting morphism (the precomposition of an operadic twisting morphism by a map of dg cooperads is an operadic twisting morphism). Actually we have $\mathcal{P}^{¡} \rightarrowtail \mathcal{F}^c(sV)$ and the map κ is given by $\mathcal{P}^{¡} \twoheadrightarrow \mathcal{P}^{¡}_{(1)} \cong sV \xrightarrow{s^{-1}} V \rightarrowtail \mathcal{P}$.

– When $\mathcal{P} = \Omega(\mathcal{C})$ is the cobar construction on $\mathcal{C}$, the previous theorem gives a natural operadic twisting morphism $\iota : \mathcal{C} \twoheadrightarrow \overline{\mathcal{C}} \xrightarrow{s^{-1}} s^{-1}\overline{\mathcal{C}} \rightarrowtail \Omega(\mathcal{C}) = \mathcal{F}(s^{-1}\overline{\mathcal{C}})$. This morphism is universal in the sense that each twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factorizes uniquely through the map ι

$$\begin{array}{ccc} & \Omega(\mathcal{C}) & \\ & {\scriptstyle \iota}\nearrow \quad \searrow{\scriptstyle g_\alpha} & \\ \mathcal{C} & \xrightarrow{\quad\alpha\quad} & \mathcal{P}, \end{array}$$

where g_α is a morphism of dg operads.

Twisted composition product

Let $\mathcal{P}$ be a dg operad and let $\mathcal{C}$ be a dg cooperad. Let $\alpha : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism. The *twisted composition product* $\mathcal{P} \circ_\alpha \mathcal{C}$ is the $\mathbb{S}$-module $\mathcal{P} \circ \mathcal{C}$ endowed with a differential $d_\alpha := d_{\mathcal{P}\circ\mathcal{C}} - \delta^l_\alpha$, where δ^l_α is defined by the composite

$$\delta^l_\alpha : \mathcal{P} \circ \mathcal{C} \xrightarrow{id_{\mathcal{P}} \circ' \Delta_{\mathcal{C}}} \mathcal{P} \circ (\mathcal{C}, \mathcal{C} \circ \mathcal{C}) \xrightarrow{id_{\mathcal{P}} \circ (id_{\mathcal{C}}, \alpha \circ id_{\mathcal{C}})} \mathcal{P} \circ (\mathcal{C}, \mathcal{P} \circ \mathcal{C}) \rightarrowtail (\mathcal{P} \circ \mathcal{P}) \circ \mathcal{C} \xrightarrow{\gamma \circ id_{\mathcal{C}}} \mathcal{P} \circ \mathcal{C}.$$

Since α is an operadic twisting morphism, d_α is a differential.

When A is a $\mathcal{P}$-algebra, we denote by $\mathcal{C} \circ_\alpha A$ the chain complex $(\mathcal{C}(A),\, d_\alpha := d_{\mathcal{C}(A)} + \delta^r_\alpha)$, where δ^r_α is the composite

$$\mathcal{C}(A) \xrightarrow{\Delta_p \circ id_A} (\mathcal{C} \circ_{(1)} \mathcal{C})(A) \xrightarrow{id_{\mathcal{C}} \circ_{(1)} \alpha \circ id_A} (\mathcal{C} \circ_{(1)} \mathcal{P})(A) \xrightarrow{id_{\mathcal{C}} \circ \gamma_A} \mathcal{C}(A).$$

Finally, we denote by $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A$ the vector space $\mathcal{P} \circ \mathcal{C}(A)$ endowed with the differential

$$d_\alpha := d_{\mathcal{P}\circ\mathcal{C}(A)} - \delta^l_\alpha \circ id_A + id_{\mathcal{P}} \circ' \delta^r_\alpha = d_{\mathcal{P}\circ(\mathcal{C}\circ_\alpha A)} - \delta^l_\alpha \circ id_A.$$

The notation d_α stands for different differentials. The differential is given without ambiguity by the context.

Operadic resolutions

In [GJ94], Getzler and Jones produced functorial resolutions of algebras given by the following theorems.

1.1.3.2 Theorem (Theorem 2.19 of [GJ94])**.** *The augmented bar construction gives a resolution*

$$\mathcal{P} \circ_\pi \mathrm{B}(\mathcal{P}) \circ_\pi A \xrightarrow{\sim} A.$$

1.1.3.3 Theorem (Theorem 2.25 of [GJ94])**.** *When the operad $\mathcal{P}$ is Koszul, there is a smaller resolution of A given by the Koszul complex*

$$\mathcal{P} \circ_\kappa \mathcal{P}^{i} \circ_\kappa A \xrightarrow{\sim} A.$$

The augmented bar resolution admits a dual version.

1.1.3.4 Theorem (Theorem 4.18 of [Val07])**.** *For every weight graded coaugmented cooperad $\mathcal{C}$, there is an isomorphism*

$$\Omega(\mathcal{C}) \circ_\iota \mathcal{C} \xrightarrow{\sim} I.$$

This gives, for all $\Omega(\mathcal{C})$-algebra A, a quasi-isomorphism $\Omega(\mathcal{C}) \circ_\iota \mathcal{C} \circ_\iota A \xrightarrow{\sim} A$.

1.1.4 Description of the cotangent complex

Thanks to these resolutions, we can describe the underlying vector space of the cotangent complex.

Quasi-free resolution

Let A be a $\mathcal{P}$-algebra, let C be a $\mathcal{C}$-coalgebra endowed with a filtration F_pC such that $F_{-1}C = \{0\}$ and let $\alpha : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism. We denote by $\mathcal{P} \circ_\alpha C$ the complex $(\mathcal{P}(C),\, d_\alpha := d_{\mathcal{P}\circ C} - \delta^l_\alpha)$. The differential δ^l_α on $\mathcal{P}(C)$ is given by

$$\delta^l_\alpha : \mathcal{P}(C) \xrightarrow{id_{\mathcal{P}} \circ' \Delta} \mathcal{P} \circ (C,\, \mathcal{C}(C)) \xrightarrow{id_{\mathcal{P}} \circ (id_C,\, \alpha \circ id_C)} \mathcal{P} \circ (C,\, \mathcal{P}(C)) \rightarrowtail \mathcal{P} \circ \mathcal{P}(C) \xrightarrow{\gamma \circ id_c} \mathcal{P}(C).$$

A *quasi-free resolution of* A is a complex $\mathcal{P} \circ_\alpha C$ such that $\mathcal{P} \circ_\alpha C \xrightarrow{\sim} A$ and $\delta^l_{\alpha|F_pC} \subset \mathcal{P}(F_{p-1}C)$.

Except the normalized cotriple construction, all the previous resolutions are of this form when A is non-negatively graded. With this resolution, we make the cotangent complex explicit.

1.1.4.1 Theorem. *Let $\mathcal{P}(C)$ be a quasi-free resolution of the $\mathcal{P}$-algebra A. With this resolution, the cotangent complex has the form*

$$\mathbb{L}_{\mathcal{P}(C)/A} \cong A \otimes^{\mathcal{P}} C.$$

Proof. The cotangent complex is isomorphic to

$$\begin{aligned} A \otimes^{\mathcal{P}}_R \Omega_{\mathcal{P}} R &= A \otimes^{\mathcal{P}}_{\mathcal{P}(C)} \Omega_{\mathcal{P}}(\mathcal{P}(C)) \\ &\cong A \otimes^{\mathcal{P}}_{\mathcal{P}(C)} (\mathcal{P}(C) \otimes^{\mathcal{P}} C) \quad \text{(Lemma 1.1.1.2)} \\ &\cong A \otimes^{\mathcal{P}} C \quad \text{(Propositions 1.0.6.1 and 1.0.6.3).} \end{aligned}$$

□

When we use the augmented bar construction, we get the cotangent complex for any algebra over any operad. However this complex may be huge and it can be useful to work with smaller resolutions. When we use the Koszul resolution, we can use the Koszul complex and we get the cotangent complex of an algebra over a Koszul operad. For homotopy algebras, we use the coaugmented cobar construction. In this paper, we consider only resolutions coming from operadic resolutions. In [Mil10], we work with even smaller resolutions, but which are not functorial with respect to the algebra.

To describe completely the cotangent complex, we have to make its differential explicit. In the next section, we will trace the boundary map on $\mathrm{Der}_A(R, M)$ through the various isomorphisms.

1.2 Lie theoretic description

We endow the chain complex defining the André-Quillen cohomology with a structure of Lie algebra. The notion of twisting morphism (or twisting cochain) first appeared in [Bro59] and in [Moo71] (see also [HMS74]). It is a particular kind of maps between a coassociative coalgebra and an associative algebra. Getzler and Jones extend this definition to (co)algebras over (co)operads (see 2.3 of [GJ94]). We show that the differential on the cotangent complex $A \otimes^{\mathcal{P}} C$ is obtained by twisting the internal differential by a twisting morphism.

In the sequel, let $(\mathcal{P}, \gamma)$ denote an operad, $(\mathcal{C}, \Delta)$ denote a cooperad and (C, Δ_C) denote a $\mathcal{C}$-coalgebra.

1.2.1 A Lie algebra structure

Let $\alpha : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism. Let C be a $\mathcal{C}$-coalgebra and let A be a $\mathcal{P}$-algebra. Let M be an A-module. For all φ in $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C, A)$ and g in $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C, M)$, we define $\alpha[\varphi, g] := \sum_{n\geq 1} \alpha[\varphi, g]_n$, where $\alpha[\varphi, g]_n$ is the composite

$$C \xrightarrow{\Delta_C} \mathcal{C}(C) \twoheadrightarrow (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n} \xrightarrow{\alpha\otimes\varphi^{\otimes n-1}\otimes g} \mathcal{P}(n) \otimes A^{\otimes n-1} \otimes M \twoheadrightarrow \mathcal{P}(A, M) \xrightarrow{\gamma_M} M.$$

The notation $\otimes_H$ stands for the Hadamard product : for any $\mathbb{S}$-modules M and N, $(M \otimes_H N)(n) := M(n) \otimes N(n)$. Let $End_{s^{-1}\mathbb{K}}$ be the cooperad defined by

$$End_{s^{-1}\mathbb{K}}(n) := \mathrm{Hom}((s^{-1}\mathbb{K})^{\otimes n}, s^{-1}\mathbb{K})$$

endowed with the natural action of $\mathbb{S}_n$. When (C, Δ_C) is a $\mathcal{C}$-coalgebra, we endow $s^{-1}C := s^{-1}\mathbb{K} \otimes C$ with a structure of $End_{s^{-1}\mathbb{K}} \otimes_H \mathcal{C}$-coalgebra given by

$$\Delta_{s^{-1}C} : s^{-1}C \xrightarrow{\Delta_C(n)} s^{n-1}s^{-n}(\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n} \xrightarrow{\tau_n} \left((End_{s^{-1}\mathbb{K}}(n) \otimes \mathcal{C}(n)) \otimes (s^{-1}C)^{\otimes n}\right)^{\mathbb{S}_n},$$

where $\Delta_C(n)$ is the composite $C \xrightarrow{\Delta_C} \mathcal{C}(C) \twoheadrightarrow (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}$ and τ_n is a map which permutes components and is induced by compositions of τ (seen in Section 1.2.1). The differential on $s^{-1}C$ is given by $d_{s^{-1}C} := id_{s^{-1}\mathbb{K}} \otimes d_C$.

In the following results, the operad $\mathcal{P}$ is quadratic and binary and the cooperad $\mathcal{C} = \mathcal{P}^{¡}$ is the Koszul dual cooperad of $\mathcal{P}$. The twisting morphism $\kappa : \mathcal{P}^{¡} \to \mathcal{P}$ is defined in the examples after Section 1.1.3.1.

1.2.1.1 Theorem. *Let $\mathcal{P}$ be a quadratic binary operad and let $\mathcal{C} = \mathcal{P}^{i}$ be the Koszul dual cooperad of $\mathcal{P}$. Let A be a $\mathcal{P}$-algebra and C be a $\mathcal{P}^{i}$-coalgebra. The chain complex*

$$(\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, A),\, \kappa[-,\, -],\, \partial)$$

forms a dg Lie algebra whose bracket $\kappa[-,\, -]$ is of degree -1, that is

$$\kappa[\varphi,\, \psi] = -(-1)^{(|\varphi|-1)(|\psi|-1)}\kappa[\psi,\, \varphi].$$

Proof. There is an isomorphism of chain complexes

$$\begin{array}{ccc} \mathrm{Hom}^{\bullet}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, A) & \xrightarrow{\cong} & \mathrm{Hom}^{\bullet+1}_{g\mathcal{M}od_{\mathbb{K}}}(s^{-1}C,\, A) \\ \varphi & \mapsto & (\bar{\varphi} : s^{-1}c \mapsto \varphi(c)), \end{array}$$

since $\overline{\partial(\varphi)} = \overline{d_A \circ \varphi - (-1)^{|\varphi|}\varphi \circ d_C} = d_A \circ \bar{\varphi} - (-1)^{|\varphi|-1}\bar{\varphi} \circ d_{s^{-1}C} = \partial(\bar{\varphi})$. Moreover, we have the equality $\overline{\kappa[\varphi,\, \psi]} = (-1)^{|\bar{\varphi}|}\bar{\kappa}[\bar{\varphi},\, \bar{\psi}]$, where $\bar{\kappa}(s^{n-1}\mu^c) := \kappa(\mu^c)$ is not a map of $\mathbb{S}_n$-modules.

We show now that the dg module

$$(\mathrm{Hom}^{\bullet}_{g\mathcal{M}od_{\mathbb{K}}}(s^{-1}C,\, A),\, (-1)^{|\bar{\varphi}|}\bar{\kappa}[\bar{\varphi},\, \bar{\psi}],\, \partial)$$

forms a Lie algebra. Since C is a $\mathcal{P}^{¡}$-coalgebra, we get that $(s^{-1}C)^* \cong sC^*$ is a $\mathcal{P}^{!}$-algebra. That is, there is a morphism of operads $\mathcal{P}^{!} \to End(sC^*)$. Hence, we obtain a morphism $\mathcal{P}^{!} \otimes_H \mathcal{P} \to End(sC^*) \otimes_H End(A) \cong End(sC^* \otimes A)$. We apply Theorem 29 of [Val08] and we get that $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(s^{-1}C,\, A) \cong sC^* \otimes A$ is a Lie algebra. The Lie algebra structure is given by $(-1)^{|\bar{\varphi}|}\bar{\kappa}[\bar{\varphi},\, \bar{\psi}]$, which is of degree 0 since κ is non-zero only on $\mathcal{P}^{¡}(2)$. Therefore $\mathrm{Hom}^{\bullet+1}_{g\mathcal{M}od_{\mathbb{K}}}(s^{-1}C,\, A)$ is a Lie algebra with bracket of degree 0. □

1.2.1.2 Theorem. *Let $\mathcal{P}$ be a quadratic binary operad and take $\mathcal{C} = \mathcal{P}^{i}$. Let A be a $\mathcal{P}$-algebra, let C be a $\mathcal{C}$-coalgebra and let M be an A-module. Then the dg module*

$$(\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, M),\, \kappa[-,\, -],\, \partial)$$

is a dg Lie module over $(\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, A),\, \kappa[-,\, -],\, \partial)$.

Proof. The proof is analoguous to the proof of Theorem 1.2.1.1 in the following way. A A-module structure over the operad $\mathcal{P}$ is equivalent to a map of operads $\mathcal{P} \to End_A(M)$, where $End_A(M) := End(A) \oplus End(A,\, M)$ with

$$End(A,\, M)(n) := \bigoplus_{j=1}^{n} \mathrm{Hom}(\underbrace{A \otimes \cdots \otimes A}_{j-1 \text{ times}} \otimes M \otimes \underbrace{A \otimes \cdots \otimes A}_{n-j \text{ times}},\, M).$$

The composition product is given by the composition of maps when possible and zero otherwise. We get $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(s^{-1}C,\, M) \cong sC^* \otimes M$ and there is a map of operads $\mathcal{L}ie \to \mathcal{P}^{!} \otimes_H \mathcal{P} \to End(sC^*) \otimes End_A(M) \cong End_{sC^* \otimes A}(sC^* \otimes M)$. Therefore, $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, M)$ is a dg Lie module over $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, A)$. □

1.2.2 Algebraic twisting morphism

In this section, we define the notion of twisting morphism on the level of (co)algebras introduced in 2.3 of [GJ94]. Assume now that $\alpha : \mathcal{C} \to \mathcal{P}$ is an operadic twisting morphism. Let A be a $\mathcal{P}$-algebra and let C be a $\mathcal{C}$-coalgebra. For all φ in $\mathrm{Hom}_{\mathcal{M}od_{\mathbb{K}}}(C,\, A)$, we define the maps

$$\star_\alpha(\varphi) : C \xrightarrow{\Delta_C} \mathcal{C}(C) \xrightarrow{\alpha \circ \varphi} \mathcal{P}(A) \xrightarrow{\gamma_A} A.$$

An *algebraic twisting morphism with respect to* α is a map $\varphi : C \to A$ of degree 0 satisfying the Maurer-Cartan equation

$$\partial(\varphi) + \star_\alpha(\varphi) = 0.$$

We denote by $\mathrm{Tw}_\alpha(C, A)$ the set of algebraic twisting morphisms with respect to α.

Examples. We consider the two examples of Section 1.1.3 once again.

- The map $\eta_{\mathrm{B}(\mathcal{P})}(A) := \eta_{\mathrm{B}(\mathcal{P})} \circ id_A : \mathrm{B}(\mathcal{P})(A) \twoheadrightarrow I \circ A \cong A$ is an algebraic twisting morphism with respect to π. For simplicity, assume $d_A = 0$. We get

$$\begin{aligned} \partial(\eta_{\mathrm{B}(\mathcal{P})}(A)) &= d_A \circ \eta_{\mathrm{B}(\mathcal{P})}(A) - \eta_{\mathrm{B}(\mathcal{P})}(A) \circ d_\pi^r \\ &= -\eta_{\mathrm{B}(\mathcal{P})}(A) \circ (d_{\mathrm{B}(\mathcal{P})} \circ id_A + \delta_\pi^r) \\ &= -\eta_{\mathrm{B}(\mathcal{P})}(A) \circ \delta_\pi^r \end{aligned}$$

  since $d_{\mathrm{B}(\mathcal{P})} = 0$ on $\mathcal{F}_{(0)}(s\overline{\mathcal{P}})$. Then $\partial(\eta_{\mathrm{B}(\mathcal{P})}(A))(e)$ is non-zero if and only if $e = s\mu \otimes (a_1 \otimes \cdots \otimes a_n) \in \mathcal{F}_{(1)}(s\overline{\mathcal{P}})(A)$ and is equal to $-\mu(a_1, \cdots, a_n)$ in this case. Moreover, $\star_\pi(\eta_{\mathrm{B}(\mathcal{P})}(A))$ satisfies the same properties. So the assertion is proved.
- The map $\eta_{\mathcal{P}^{\text{¡}}}(A) : \mathcal{P}^{\text{¡}}(A) \rightarrowtail \mathrm{B}(\mathcal{P})(A) \twoheadrightarrow A$ is an algebraic twisting morphism with respect to κ.

Let us now make explicit the maps κ and $\eta_{\mathcal{P}^{\text{¡}}}(A)$ in the cases $\mathcal{P} = \mathcal{A}s$, $\mathcal{C}om$ and $\mathcal{L}ie$. We refer to [Val08] for the categorical definition of the Koszul dual cooperad.

- When $\mathcal{P} = \mathcal{A}s$, the Koszul dual $\mathcal{A}s^{\text{¡}}$ is a cooperad cogenerated by the elements $s\curlyvee$, that is the elements $\curlyvee \in \mathcal{A}s(2)$ suspended by an s of degree 1, with corelations $s\curlyvee \otimes (s\curlyvee \otimes |\,) - s\curlyvee \otimes (|\, \otimes s\curlyvee)$, that we can represent by $s^2(\curlyvee\circ_1\curlyvee - \curlyvee\circ_2\curlyvee)$. The map $\kappa : \mathcal{A}s^{\text{¡}} \to \mathcal{A}s$ sends $s\curlyvee$ onto $\curlyvee$ and is zero elsewhere. The map $\eta_{\mathcal{A}s^{\text{¡}}}(A)$ sends A onto A and is zero elsewhere.
- When $\mathcal{P} = \mathcal{C}om$, the map κ sends the cogenerator of $\mathcal{C}om^{\text{¡}}$ on the generator of $\mathcal{C}om$ and is zero outside $\mathcal{C}om^{\text{¡}}(2)$. The map $\eta_{\mathcal{C}om^{\text{¡}}}(A)$ is just the projection onto A.
- When $\mathcal{P} = \mathcal{L}ie$, the map κ sends the cogenerator of $\mathcal{L}ie^{\text{¡}}$ on the generator of $\mathcal{L}ie$ and is zero outside $\mathcal{L}ie^{\text{¡}}(2)$ and the map $\eta_{\mathcal{L}ie^{\text{¡}}}(A)$ is just the projection onto A.

When $\mathcal{P}$ is a binary quadratic operad, $\mathcal{C} = \mathcal{P}^{\text{¡}}$ is its Koszul dual cooperad and $\alpha = \kappa$, then algebraic twisting morphisms with respect to κ are in one-to-one correspondence with solutions of the Maurer-Cartan equation in the dg Lie algebra introduced in Theorem 1.2.1.1.

1.2.3 Twisted differential

Let $\alpha : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism and let $\varphi : C \to A$ be an algebraic twisting morphism with respect to α. We associate to α and φ a *twisted differential* $\partial_{\alpha,\varphi}$, denoted simply by ∂_φ, on $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, M)$ by the formula

$$\partial_\varphi(g) := \partial(g) + \alpha[\varphi,\, g].$$

1.2.3.1 Lemma. *If* $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$ *and* $\varphi \in \mathrm{Tw}_\alpha(\mathrm{C}, \mathrm{A})$*, then* ${\partial_\varphi}^2 = 0$.

Proof. We recall that $|\alpha| = -1$ and $|\varphi| = 0$. Let us modify a little bit the operator $\alpha[\varphi, g]_n$. We define for all ψ in $\mathrm{Hom}_{g\mathcal{M}od_\mathbb{K}}(C, A)$ and g in $\mathrm{Hom}_{g\mathcal{M}od_\mathbb{K}}(C, M)$ the operator $\alpha[\varphi, (\psi, g)]_n$ to be the composite :

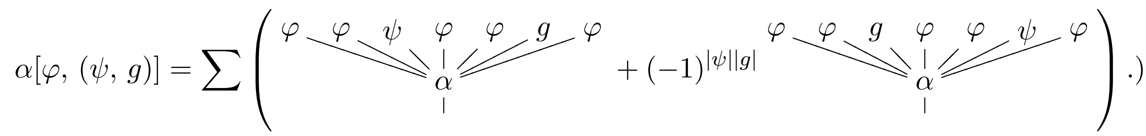
$$C \xrightarrow{\Delta_{C,\,n}} (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n} \xrightarrow{\sum_j \alpha\otimes\varphi^{j-1}\otimes\psi\otimes\varphi^{n-j-1}\otimes g} \mathcal{P}(n) \otimes A^{\otimes n-1} \otimes M \twoheadrightarrow \mathcal{P}(A, M) \xrightarrow{\gamma_M} M.$$

We define $\alpha[\varphi, (\psi, g)] := \sum_{n\geq 2} \alpha[\varphi, (\psi, g)]_n$.

(We have to pay attention to the fact that sign $(-1)^{|\psi||g|}$ may appear. The elements of $\mathcal{C}(C)$ are invariant under the action of the symmetric groups, so they are of the form $\sum_{\sigma\in\mathbb{S}_n} \varepsilon_\sigma \mu^c \cdot \sigma \otimes c_{\sigma^{-1}(1)} \otimes \cdots \otimes c_{\sigma^{-1}(n)}$, where ε_σ depends on $(-1)^{|c_i||c_j|}$. For example, $\varepsilon_{(12)} = (-1)^{|c_1||c_2|}$, $\varepsilon_{(123)} = (-1)^{|c_1||c_3|+|c_2||c_3|}$ and $\varepsilon_{(132)} = (-1)^{|c_1||c_2|+|c_1||c_3|}$. Moreover, the coinvariant elements in $\mathcal{P}(A, M)$ satisfy $\mu \otimes_{\mathbb{S}_n} (a_1 \otimes \cdots \otimes a_n) = \varepsilon_\sigma \mu \cdot \sigma \otimes_{\mathbb{S}_n} (a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)})$. The image of $(-1)^{|c_1||c_2|}\mu^c \cdot (12) \otimes c_2 \otimes c_1$ under $\gamma_M \circ (\alpha \otimes \psi \otimes g)$ in M is

$$\begin{aligned}
&(-1)^{|c_1||c_2|+|\mu^c|(|\psi|+|g|)+|g||c_2|}\alpha(\mu^c)(\psi(c_2), g(c_1)) \\
&\qquad = (-1)^{|c_1||c_2|+|\mu^c|(|\psi|+|g|)+|g||c_2|}(-1)^{|\psi(c_2)||g(c_1)|}\mu(g(c_1), \psi(c_2)) \\
&\qquad = (-1)^{|\psi||g|}(-1)^{|\mu^c|(|\psi|+|g|)+|\psi||c_1|}\mu(g(c_1), \psi(c_2)).
\end{aligned}$$

Therefore, the operator $\alpha[\varphi, (\psi, g)]$ can be understood as follows



The maps Δ_C and γ_M are maps of dg modules and we have the equality

$$\partial(\alpha \otimes \varphi^{\otimes n-1} \otimes \psi) = \partial(\alpha) \otimes \varphi^{\otimes n-1} \otimes \psi + (-1)^{|\alpha|}\alpha \otimes \partial(\varphi^{\otimes n-1}) \otimes \psi + (-1)^{|\alpha|}\alpha \otimes \varphi^{\otimes n-1} \otimes \partial(\psi),$$

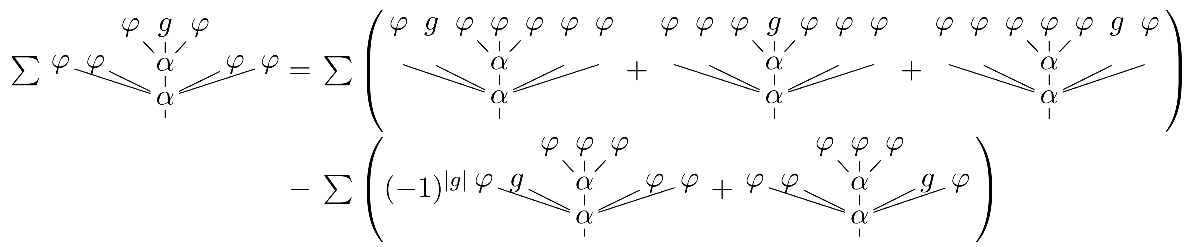
where $\partial(\varphi^{\otimes n-1}) = \sum_j \varphi^{j-1} \otimes \partial(\varphi) \otimes \varphi^{n-j-1}$. Therefore we get

$$\partial(\alpha[\varphi, g]) = \partial(\alpha)[\varphi, g] + (-1)^{|\alpha|}\alpha[\varphi, (\partial(\varphi), g)] + (-1)^{|\alpha|}\alpha[\varphi, \partial(g)].$$

It follows that

$$\begin{aligned}
{\partial_\varphi}^2(g) &= \partial_\varphi(\partial(g) + \alpha[\varphi, g]) \\
&= \partial^2(g) + \partial(\alpha[\varphi, g]) + \alpha[\varphi, \partial(g)] + \alpha[\varphi, \alpha[\varphi, g]] \\
&= \partial(\alpha)[\varphi, g] + (-1)^{|\alpha|}\alpha[\varphi, (\partial(\varphi), g)] + (-1)^{|\alpha|}\alpha[\varphi, \partial(g)] \\
&\quad +\alpha[\varphi, \partial(g)] + \alpha[\varphi, \alpha[\varphi, g]] \\
&= \partial(\alpha)[\varphi, g] - \alpha[\varphi, (\partial(\varphi), g)] + \alpha[\varphi, \alpha[\varphi, g]].
\end{aligned}$$

The following picture



models the equation

$$\alpha[\varphi,\, \alpha[\varphi,\, g]] = (\alpha \star \alpha)[\varphi,\, g] - \alpha[\varphi,\, (\star_\alpha(\varphi),\, g)]$$

(the sign $(-1)^{|g|}$ appears when we permute α and g). Thus

$$\partial_\varphi{}^2(g) = (\partial(\alpha) + \alpha \star \alpha)[\varphi,\, g] - \alpha[\varphi,\, (\partial(\varphi) + \star_\alpha(\varphi),\, g)].$$

Since α is an operadic twisting morphism and φ is an algebraic twisting morphism with respect to α, this concludes the proof. □

1.2.4 The cotangent complex of an algebra over an operad

From now on, we trace through the isomorphisms of Theorem 1.1.4.1 in order to make the differential on the cotangent complex explicit. Finally, for appropriate differentials, we obtain the isomorphism of differential graded modules

$$\mathrm{Der}_A(\mathcal{P}(C),\, M) \cong \mathrm{Hom}_{\mathcal{M}_A^{\mathcal{P}}}(A \otimes^{\mathcal{P}} C,\, M),$$

where $\mathcal{P}(C)$ is a quasi-free resolution of A.

We have in mind the resolutions obtained by means of the augmented bar construction on the level of operad, applied to an algebra, or the Koszul complex on an algebra or the coaugmented cobar construction on the level of cooperads, applied to a homotopy algebra.

The space $\mathrm{Der}_A(\mathcal{P}(C),\, M)$ is endowed with the following differential

$$\partial(f) = d_M \circ f - (-1)^{|f|} f \circ d_\alpha,$$

where d_α was defined in Section 1.1.4.

1.2.4.1 Proposition. *With the above notations, we have the following isomorphism of dg modules*

$$(\mathrm{Der}_A(\mathcal{P} \circ_\alpha C,\, M),\, \partial) \cong (\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, M),\, \partial_\varphi = \partial + \alpha[\varphi,\, -]),\ \textit{where } C = \mathcal{C}(A).$$

Proof. First, the isomorphism of $\mathbb{K}$-modules between $\mathrm{Der}_A(\mathcal{P}(C),\, M)$ and $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, M)$ is given by the restriction on the generators C.

We verify that this isomorphism commutes with the respective differentials. We fix the notations $\bar{f} := f_{|C}$ and $n := |\bar{f}| = |f|$. On the one hand, we have

$$\begin{aligned} \partial(f)_{|C} &= (d_M \circ f)_{|C} - (-1)^{|f|}(f \circ d_\alpha)_{|C} \\ &= d_M \circ \bar{f} - (-1)^n f \circ (d_{\mathcal{P}} \circ id_C + id_{\mathcal{P}} \circ' d_C - \delta^l_\alpha)_{|C}. \end{aligned}$$

Moreover, $(d_{\mathcal{P}} \circ id_C)_{|C} = 0$ since $(d_{\mathcal{P}})_{|\mathcal{P}(1)} = 0$ and $f \circ (id_{\mathcal{P}} \circ' d_C)_{|C} = \bar{f} \circ d_C$. Thus

$$\partial(f)_{|C} = d_M \circ \bar{f} - (-1)^n \bar{f} \circ d_C + (-1)^n f \circ \delta^l_{\alpha|C}.$$

On the other hand,

$$\partial_\varphi(\bar{f}) = d_M \circ \bar{f} - (-1)^n \bar{f} \circ d_C + \alpha[\varphi,\, \bar{f}].$$

With the signs $\alpha \otimes \bar{f} = (-1)^{|\alpha||\bar{f}|}(id \otimes \bar{f}) \otimes (\alpha \otimes id)$ and using the fact that f is a derivation, we verify that $(-1)^n f \circ \delta^l_{\alpha|C} = \alpha[\varphi,\, \bar{f}]$. □

Let us construct a twisted differential on the free A-module $A \otimes^{\mathcal{P}} C$ as follows. Since $A \otimes^{\mathcal{P}} C$ is a quotient of $\mathcal{P}(A,\, C)$, we define a map

$$\delta^l_1(n) : \mathcal{P}(A,\, C) \xrightarrow{id_{\mathcal{P}}(id_A,\, \Delta_C(n))} \mathcal{P}(A,\, (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}) \xrightarrow{id_{\mathcal{P}}(id_A,\, \alpha\otimes\varphi^{\otimes n-1}\otimes id_C)}$$

$$\mathcal{P}(A,\, \mathcal{P}(n) \otimes A^{\otimes n-1} \otimes C) \to (\mathcal{P} \circ \mathcal{P})(A,\, C) \xrightarrow{\gamma(id_A,\, id_C)} \mathcal{P}(A,\, C).$$

This map sends the elements $\mu \otimes \gamma_A(\nu_1 \otimes a_1 \otimes \cdots \otimes a_{i_1}) \otimes \cdots \otimes c \otimes \cdots \otimes \gamma_A(\nu_k \otimes \cdots \otimes a_n)$ and $\gamma_{\mathcal{P}}(\mu \otimes \nu_1 \otimes \cdots \otimes \nu_k) \otimes a_1 \otimes \cdots \otimes a_{i_1} \otimes \cdots \otimes c \otimes \cdots \otimes a_n$ to the same image, for $c \in C$ and $a_j \in A$ and $\mu,\, \nu_j \in \mathcal{P}$. So $\delta^l_1(n)$ induces a map on the quotient

$$\delta^l_{\alpha,\,\varphi}(n) : A \otimes^{\mathcal{P}} C \to A \otimes^{\mathcal{P}} C.$$

We write $\delta^l_1 := \sum \delta^l_1(n)$ and $\delta^l_{\alpha,\,\varphi} := \sum \delta^l_{\alpha,\,\varphi}(n)$, or simply δ^l_φ.

We define the twisted differential $\partial_{\alpha,\,\varphi}$, or simply ∂_φ on $\mathrm{Hom}_{\mathcal{M}^{\mathcal{P}}_A}(A \otimes^{\mathcal{P}} C,\, M)$ by

$$\begin{aligned} \partial_\varphi(f) :=\ & \partial(f) + (-1)^{|f|} f \circ \delta^l_\varphi \\ =\ & d_M \circ f - (-1)^{|f|} f \circ (d_{A\otimes^{\mathcal{P}}C} - \delta^l_\varphi), \end{aligned}$$

where the differential $d_{A\otimes^{\mathcal{P}}C}$ is induced by the natural differential on $\mathcal{P}(A,\, C)$. So we consider the twisted differential $d_\varphi := d_{A\otimes^{\mathcal{P}}C} - \delta^l_\varphi$ on $A \otimes^{\mathcal{P}} C$. Once again, the notation ∂_φ stands for several differentials and the relevant one is given without ambiguity by the context.

1.2.4.2 Theorem. *With the above notations, the following three dg modules are isomorphic*

$$(\mathrm{Der}_A(\mathcal{P} \circ_\alpha C,\, M),\, \partial) \cong (\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, M),\, \partial_\varphi) \cong (\mathrm{Hom}_{\mathcal{M}^{\mathcal{P}}_A}(A \otimes^{\mathcal{P}} C,\, M),\, \partial_\varphi).$$

Proof. We already know the isomorphism of $\mathbb{K}$-modules given by the restriction

$$(\mathrm{Hom}_{\mathcal{M}^{\mathcal{P}}_A}(A \otimes^{\mathcal{P}} C,\, M),\, \partial) \cong (\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(C,\, M),\, \partial)$$

from the preliminaries. We now verify that this isomorphism commutes with the differentials. With the notation $\bar{f} := f_{|C}$, we have

$$\partial_\varphi(\bar{f}) = d_M \circ \bar{f} - (-1)^{|\bar{f}|} \bar{f} \circ d_C + \alpha[\varphi,\, \bar{f}]$$

and

$$\partial_\varphi(f)_{|C} = (d_M \circ f - (-1)^{|f|} f \circ d_{A\otimes^{\mathcal{P}}C} + (-1)^{|f|} f \circ \delta^l_\varphi)_{|C}.$$

Since $(f \circ d_{A\otimes^{\mathcal{P}}C})_{|C} = \bar{f} \circ d_C$, we just need to show the equality $\alpha[\varphi,\, \bar{f}] = (-1)^{|f|}(f \circ \delta^l_\varphi)_{|C}$. This holds since $M \in \mathcal{M}^{\mathcal{P}}_A$ and f is a morphism of A-modules over $\mathcal{P}$ and the structure of A-module on C into $A \otimes^{\mathcal{P}} C$ is just the projection $\mathcal{P}(A,\, C) \twoheadrightarrow A \otimes^{\mathcal{P}} C$. □

Finally, when $\mathcal{P} \circ_\alpha C \xrightarrow{\sim} A$ is a quasi-free resolution of A, the chain complex

$$(A \otimes^{\mathcal{P}} C,\, d_\varphi = d_{A\otimes^{\mathcal{P}}C} - \delta^l_\varphi)$$

is a representation of the cotangent complex. In our cases, we have $C = \mathcal{C}(A)$. Then a representation of the cotangent complex is given by

$$(A \otimes^{\mathcal{P}} \mathcal{C}(A),\, d_\varphi = d_{A\otimes^{\mathcal{P}}\mathcal{C}(A)} - \delta^l_\varphi + \delta^r_\varphi),$$

where δ^l_φ is induced by

$$\mathcal{P}(A,\, \mathcal{C}(A)) \xrightarrow{id_{\mathcal{P}}\circ(id_A,\, \Delta_p\circ id_A)} \mathcal{P}(A,\, (\mathcal{C}\circ_{(1)}\mathcal{C})(A)) \xrightarrow{id_{\mathcal{P}}\circ(id_A,\, \alpha\circ_{(1)} id_{\mathcal{C}}\circ id_A)}$$

$$\mathcal{P}(A,\, (\mathcal{P}\circ_{(1)}\mathcal{C})(A)) \rightarrowtail (\mathcal{P}\circ\mathcal{P})(A,\, \mathcal{C}(A)) \xrightarrow{\gamma\circ(id_A,\, id_{\mathcal{C}(A)})} \mathcal{P}(A,\, \mathcal{C}(A))$$

and δ^r_φ is induced by

$$\mathcal{P}(A,\, \mathcal{C}(A)) \xrightarrow{id_{\mathcal{P}}\circ(id_A,\, \Delta_p\circ id_A)} \mathcal{P}(A,\, (\mathcal{C}\circ_{(1)}\mathcal{C})(A)) \xrightarrow{id_{\mathcal{P}}\circ(id_A,\, id_{\mathcal{C}}\circ_{(1)}\alpha\circ id_A)}$$

$$\mathcal{P}(A,\, (\mathcal{C}\circ_{(1)}\mathcal{P})(A)) \rightarrowtail \mathcal{P}(A,\, \mathcal{C}(A,\, \mathcal{P}(A))) \xrightarrow{id_{\mathcal{P}}\circ(id_A,\, id_{\mathcal{C}}\circ(id_A,\gamma_A))} \mathcal{P}(A,\, \mathcal{C}(A)).$$

Remark. Applying this description to the resolutions of algebras obtained by means of the augmented bar construction or by means of the Koszul complex, we obtain two different chain complexes which allow us to compute the André-Quillen cohomology. The one using the Koszul resolution is smaller since $\mathcal{P}^{¡} \rightarrowtail \mathrm{B}(\mathcal{P})$. However the differential on the one using the augmented bar construction is simpler as the differential strongly depends on the coproduct. The cooperad $\mathcal{P}^{¡}$ is often given up to isomorphism, therefore it is difficult to make it explicit.

1.3 Applications and new examples of cohomology theories

We apply the previous general definitions to nearly all the operads we know. We explain which resolution can be used each time. Sometimes, it corresponds to known chain complexes. We also show that the cotriple cohomology corresponds to André-Quillen cohomology. Among the new examples, we make the André-Quillen cohomology for algebras over the operad $\mathcal{P}erm$ explicit. We do the same for homotopy $\mathcal{P}$-algebras. From now on, we assume that the algebras are non-negatively graded.

1.3.1 Applications

For some operads, an explicit chain complex computing the cohomology theory for the associated algebras has already been proposed by various authors.

- When $\mathcal{P} = \mathcal{A}s$ is the operad of associative algebras, $A\otimes^{\mathcal{P}}\mathcal{P}^{¡}(A) \cong (\mathbb{K}\oplus A)\otimes B(A)\otimes(\mathbb{K}\oplus A)$ (by 1.0.6.1) is the normalized Hochschild complex (see Section 1.1.14 of [Lod98]). The André-Quillen cohomology of associative algebras is the Hochschild cohomology (see also Chapter IX, Section 6 of [CE99]).
- When $\mathcal{P} = \mathcal{L}ie$ is the operad of Lie algebras, $A\otimes^{\mathcal{P}}\mathcal{P}^{¡}(A) \cong U^e(A)\otimes\Lambda(A)$ since $\mathcal{L}ie^{¡}(A)\cong\Lambda(A)$. The André-Quillen cohomology of Lie algebras is Chevalley-Eilenberg cohomology (see Chapter XIII of [CE99]).
- When $\mathcal{P} = \mathcal{C}om$ is the operad of commutative algebras, the complex $A\otimes^{\mathcal{P}}\mathcal{P}^{¡}(A) \cong (\mathbb{K}\oplus A)\otimes\mathcal{C}om^{¡}(A)$, only valid in characteristic 0, gives the cohomology theory of commutative algebras defined by Quillen in [Qui70]. It corresponds to Harrison cohomology defined in [Har62]. We refer to [Lod98] for the relationship between the different definitions.
- When $\mathcal{P} = \mathcal{D}ias$ is the operad of diassociative algebras and with the Koszul resolution, we get the chain complex and the associated cohomology defined by Frabetti in [Fra01].

- When $\mathcal{P} = \mathcal{L}eib$ is the operad of Leibniz algebras and with the Koszul resolution, the André-Quillen cohomology of Leibniz algebras is the cohomology defined by Loday and Pirashvili in [LP93].
- For the operad $\mathcal{P} = \mathcal{P}oiss$ encoding Poisson algebras, Fresse followed, as in this paper, the ideas of Quillen to make a cohomology of Poisson algebras explicit with the Koszul resolution [Fre06].
- When $\mathcal{P} = \mathcal{P}relie$ and with the Koszul resolution, the André-Quillen cohomology of pre-Lie algebras is the one defined by Dzhumadil'daev in [Dzh99].
- When $\mathcal{P} = \mathcal{Z}inb$, or equivalently $\mathcal{L}eib^!$, and with the Koszul resolution, the André-Quillen cohomology of Zinbiel algebras is the one given in [Bal98].

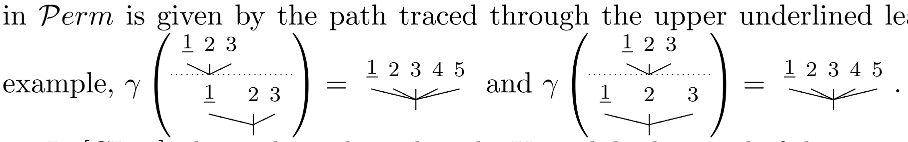
More generally,

- Balvoine introduces a chain complex in [Bal98]. When the operad $\mathcal{P}$ is a binary Koszul operad, the chain complex computing the André-Quillen cohomology obtained with the Koszul resolution corresponds to the one defined by Balavoine. Thus, the cohomology theories are the same in this case.

1.3.2 The case of Perm algebras

We denote by $\mathcal{P}erm$ the operad corresponding to Perm algebras defined in [Cha01].

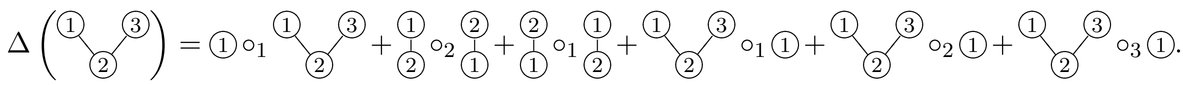
Let us recall that a basis for $\mathcal{P}erm(n)$ is given by corollas in space with n leaves labelled by 1 to n with one leaf underlined. So $\mathcal{P}erm(n)$ is of dimension n. The composition product in $\mathcal{P}erm$ is given by the path traced through the upper underlined leaf from the root. For example,



In [CL01], the authors show that the Koszul dual operad of the operad $\mathcal{P}erm$ is the operad $\mathcal{P}relie$ and that the operad $\mathcal{P}relie$ is Koszul. It follows that the operad $\mathcal{P}erm$ is Koszul (see [GK94] for general facts about Koszul duality of operads). Since $\mathcal{P}erm^{¡} \cong \mathcal{P}relie^{\vee}$, it is possible to understand the coproduct on $\mathcal{P}erm^{¡}$ if we know the product on $\mathcal{P}relie$. Chapoton and Livernet gave an explicit basis for $\mathcal{P}relie$ and made explicit the product. This basis of $\mathcal{P}relie$ is given by the rooted trees of degree n, that is with n vertices, denoted $\mathcal{RT}(n)$. Then we need to understand the coproduct on $\mathcal{P}relie^*$ which is given by

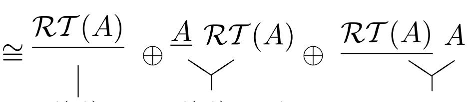
$$\Delta : \mathcal{P}relie^* \xrightarrow{{}^t\gamma} (\mathcal{P}relie \circ \mathcal{P}relie)^* \xrightarrow{\simeq} \mathcal{P}relie^* \circ \mathcal{P}relie^*,$$

where $\mathcal{P}relie^*(n) := \mathcal{P}relie(n)^*$ and ${}^t\gamma(f) := f \circ \gamma$. A rooted tree is represented as in [CL01], with its root at the bottom. We make explicit the coproduct on a particular element



Let A be a $\mathcal{P}erm$-algebra. The cotangent complex has the following form

$$A \otimes^{\mathcal{P}} \mathcal{P}^{¡}(A) = A \otimes^{\mathcal{P}} \mathcal{RT}(A) \cong$$



$$\cong \mathcal{RT}(A) \oplus A \otimes \mathcal{RT}(A) \oplus \mathcal{RT}(A) \otimes A,$$

where $\mathcal{RT}(A) = \oplus_n \mathcal{RT}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}$.

When the algebra is trivial

We assume first that A is a trivial algebra, that is $\gamma_A \equiv 0$. To make the differential on the cotangent complex explicit, we just need to describe the restriction $\mathcal{RT}(A) \to \mathcal{RT}(A) \otimes A \oplus A \otimes \mathcal{RT}(A)$ since the differential is zero on $A \otimes \mathcal{RT}(A) \oplus \mathcal{RT}(A) \otimes A$. Let T be in $\mathcal{RT}(n)$. There are several possibilities :

i) the rooted tree T has the form $\begin{smallmatrix} \textcircled{T_1} \\ | \\ \textcircled{1} \end{smallmatrix}$, where T_1 is in $\mathcal{RT}(n-1)$. In that case, the term $\begin{smallmatrix} \textcircled{2} \\ | \\ \textcircled{1} \end{smallmatrix} \circ_2 T_1$ appears in $\Delta_{\mathcal{RT}}(T)$, so the image of $T \otimes a_1 \otimes \cdots \otimes a_n$ under d_φ in $A \otimes \mathcal{RT}(A)$ contains $-a_1 \otimes (T_1 \otimes a_2 \otimes \cdots \otimes a_n)$;

ii) there exists T_2 in $\mathcal{RT}(n-1)$ such that the rooted tree T can be written $\begin{smallmatrix} \textcircled{1} \\ | \\ \textcircled{T_2} \end{smallmatrix}$. In that case, the term $\begin{smallmatrix} \textcircled{1} \\ | \\ \textcircled{2} \end{smallmatrix} \circ_2 T_2$ appears in $\Delta_{\mathcal{RT}}(T)$, so the image of $T \otimes a_1 \otimes \cdots \otimes a_n$ under d_φ in $\mathcal{RT}(A) \otimes A$ contains $-(T_2 \otimes a_2 \otimes \cdots \otimes a_n) \otimes a_1$;

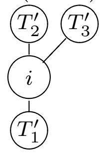
iii) the rooted tree has the form $\begin{smallmatrix} \textcircled{T_3} \\ | \\ \textcircled{n} \end{smallmatrix}$, where T_3 is in $\mathcal{RT}(n-1)$. In that case, the term $\begin{smallmatrix} \textcircled{1} \\ | \\ \textcircled{2} \end{smallmatrix} \circ_1 T_3$ appears in $\Delta_{\mathcal{RT}}(T)$, so the image of $T \otimes a_1 \otimes \cdots \otimes a_n$ under d_φ in $A \otimes \mathcal{RT}(A)$ contains $-a_n \otimes (T_3 \otimes a_1 \otimes \cdots \otimes a_{n-1})$;

iv) there exists T_4 in $\mathcal{RT}(n-1)$ such that the rooted tree can be written $\begin{smallmatrix} \textcircled{n} \\ | \\ \textcircled{T_4} \end{smallmatrix}$. In that case, the term $\begin{smallmatrix} \textcircled{2} \\ | \\ \textcircled{1} \end{smallmatrix} \circ_1 T_4$ appears in $\Delta_{\mathcal{RT}}(T)$, so the image of $T \otimes a_1 \otimes \cdots \otimes a_n$ under d_φ in $\mathcal{RT}(A) \otimes A$ contains $-(T_4 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \otimes a_n$;

A rooted tree T has the shape i) and iv), or ii) and iii), or ii) and iv), or i) only, or ii) only, or iii) only, or iv) only, or finally a shape not described in i) to iv). In this last case, the differential is 0. Otherwise, the image under the differential of an element $T \otimes a_1 \otimes \cdots \otimes a_n$ in $\mathcal{RT}(A)$ is given by the sum of the corresponding terms in i) to iv). For example, if T can be written $\begin{smallmatrix} \textcircled{T_1} \\ | \\ \textcircled{1} \end{smallmatrix}$ and $\begin{smallmatrix} \textcircled{n} \\ | \\ \textcircled{T_4} \end{smallmatrix}$, we get $d_\varphi(T \otimes a_1 \otimes \cdots \otimes a_n) = -a_1 \otimes (T_1 \otimes a_2 \otimes \cdots \otimes a_n) - (T_4 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \otimes a_n$.

For any Perm algebra

For a general $\mathcal{P}erm$-algebra A, we no longer assume a priori that the restriction of the differential $d_{A \otimes^{\mathcal{P}} \mathcal{P}^{\text{i}}(A)}$ to $\mathcal{P}^{\text{i}}(A)$, that is d_α, is zero. For a rooted tree T in $\mathcal{RT}(n)$, we define the function f by $f(T, i, j) = 1$ if $T = \begin{smallmatrix} \textcircled{T_2} & & \textcircled{T_3} \\ | & & | \\ \textcircled{i} & - & \textcircled{j} \\ | & & \\ \textcircled{T_1} & & \end{smallmatrix}$ for some rooted tree T_1 and some families of

rooted trees T_2 and T_3, and $f(T,i,j)=0$ otherwise. There exists a rooted tree T_i in $\mathcal{RT}(n-1)$ such that T appears in the product $T_i \circ_i \begin{smallmatrix} \textcircled{2} \\ | \\ \textcircled{1} \end{smallmatrix}$ if and only if $f(T,i,i+1)=1$ (take $T_i =$ 
where T'_j is the family of trees T_j with vertices $k > i$ replaced by $k+1$). Similarly there exists a rooted tree T_i in $\mathcal{RT}(n-1)$ such that T appears in the product $T_i \circ_i \begin{smallmatrix} \textcircled{1} \\ | \\ \textcircled{2} \end{smallmatrix}$ if and only if $f(T,i+1,i)=1$. We define $E^1(T) := \{i \,|\, f(T,i,i+1)=1\}$ and $E^2(T) := \{i \,|\, f(T,i+1,i)=1\}$. We obtain

$$\begin{array}{rcl} d_\alpha(T\otimes a_1\otimes\cdots\otimes a_n) & = & \sum_{i\in E^1(T)} T_i\otimes a_1\otimes\cdots\otimes\gamma_A({}^-\!\!\curlyvee\otimes a_i\otimes a_{i+1})\otimes\cdots\otimes a_n \\ & + & \sum_{i\in E^2(T)} T_i\otimes a_1\otimes\cdots\otimes\gamma_A(\curlyvee\!\!^-\otimes a_i\otimes a_{i+1})\otimes\cdots\otimes a_n, \end{array}$$

where T_i is the rooted tree such that T appears in the product $T_i \circ_i \begin{smallmatrix} \textcircled{2} \\ | \\ \textcircled{1} \end{smallmatrix}$ or $T_i \circ_i \begin{smallmatrix} \textcircled{1} \\ | \\ \textcircled{2} \end{smallmatrix}$. Finally, on $\mathcal{RT}(A)$, the differential on the cotangent complex is given by $d_\varphi = d_\alpha - \delta^l_\varphi$.

We describe now the differential δ^l_φ on $A\otimes\mathcal{RT}(A)$ thanks to the description i) - iv) of the previous section.

i)-ii) The term $\gamma_A({}^-\!\!\curlyvee\otimes a_0\otimes a_1)\otimes(T_i\otimes a_2\otimes\cdots\otimes a_n)$ appears in $\delta^l_\varphi(a_0\otimes(T\otimes a_1\otimes\cdots\otimes a_n))$ (with $i=1$ or 2);

iii)-iv) the term $\gamma_A({}^-\!\!\curlyvee\otimes a_0\otimes a_n)\otimes(T_i\otimes a_1\otimes\cdots\otimes a_{n-1})$ appears in $\delta^l_\varphi(a_0\otimes(T\otimes a_1\otimes\cdots\otimes a_n))$ (with $i=3$ or 4).

Similarly, we describe the differential δ^l_φ on $\mathcal{RT}(A)\otimes A$.

i) The term $\gamma_A({}^-\!\!\curlyvee\otimes a_1\otimes a_{n+1})\otimes(T_1\otimes a_2\otimes\cdots\otimes a_n)$ appears in $\delta^l_\varphi((T\otimes a_1\otimes\cdots\otimes a_n)\otimes a_{n+1})$;

ii) the term $(T_2\otimes a_2\otimes\cdots\otimes a_n)\otimes\gamma_A({}^-\!\!\curlyvee\otimes a_1\otimes a_{n+1})$ appears in $\delta^l_\varphi((T\otimes a_1\otimes\cdots\otimes a_n)\otimes a_{n+1})$;

iii) the term $\gamma_A({}^-\!\!\curlyvee\otimes a_n\otimes a_{n+1})\otimes(T_3\otimes a_1\otimes\cdots\otimes a_{n-1})$ appears in $\delta^l_\varphi((T\otimes a_1\otimes\cdots\otimes a_n)\otimes a_{n+1})$;

iv) the term $(T_4\otimes a_1\otimes\cdots\otimes a_{n-1})\otimes\gamma_A({}^-\!\!\curlyvee\otimes a_n\otimes a_{n+1})$ appears in $\delta^l_\varphi((T\otimes a_1\otimes\cdots\otimes a_n)\otimes a_{n+1})$.

Finally, the differential on the cotangent complex $\mathcal{RT}(A)\oplus A\otimes\mathcal{RT}(A)\oplus\mathcal{RT}(A)\otimes A$ is given by $d_\alpha + id_A\otimes d_\alpha + d_\alpha\otimes id_A - \delta^l_\varphi$.

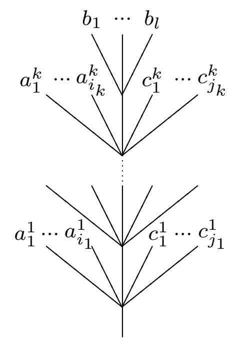
1.3.3 The case of A_∞-algebras

Markl gave in [Mar92] a definition for a cohomology theory for homotopy associative algebras. In this section, we make explicit the André-Quillen cohomology for homotopy associative algebras and we recover the complex defined by Markl.

The operad $A_\infty = \Omega(\mathcal{A}s^{\text{i}}) = \mathcal{F}(\curlyvee, \Psi, \Psi\!\!\Psi, \ldots)$ is the free operad on one generator in each degree greater than 1. We have the resolution $R := A_\infty\circ\mathcal{A}s^{\text{i}}(A) \xrightarrow{\sim} A$ and we get

$$\mathbb{L}_{R|A} = \bigoplus_{\substack{l,l_1,l_2\geq 0\\ k\geq 0}} \bigoplus_{\substack{i_1+\cdots+i_k=l_1\\ j_1+\cdots+j_k=l_2}} A^{\otimes i_1}|\cdots|A^{\otimes i_k}|A^{\otimes l}|A^{\otimes j_k}|\cdots|A^{\otimes j_1}.$$

Actually, an element in $\mathbb{L}_{R|A}$ should be seen as a planar tree



where some i_t or j_t may be 0.

An element in $\mathbb{L}_{R|A}$ is written $a_1^1 \cdots a_{i_1}^1 | \cdots | a_1^k \cdots a_{i_k}^k [b_1 \cdots b_l] c_1^k \cdots c_{j_k}^k | \cdots | c_1^1 \cdots c_{j_1}^1$.

A structure of A_∞-algebra on A is given by maps $\mu_n : A^{\otimes n} \to A$ satisfying compatibility relations and a structure of A-module over the operad A_∞ on M is given by maps $\mu_{n,i} : A^{\otimes i-1} \otimes M \otimes A^{\otimes n-i} \to M$ for $n \geq 2$ and $1 \leq i \leq n$ satisfying some compatibility relations.

In this case, the twisting morphism α is the injection $\mathcal{A}s^{\text{!`}} \rightarrowtail \Omega(\mathcal{A}s^{\text{!`}})$ and the twisting morphism on the level of (co)algebras φ is the projection $\mathcal{A}s^{\text{!`}}(A) \twoheadrightarrow A$.

When $d_A = 0$, the differential on the cotangent complex is the sum of three terms that we will make explicit. Otherwise, we have to add a term induced by d_A. The first part of the differential is $d_{A \otimes^{A_\infty} \mathcal{A}s^{\text{!`}}(A)}$ given by d_α and d_{A_∞}.

We use the fact that $\Delta_p : \mathcal{A}s^{\text{!`}} \to \mathcal{A}s^{\text{!`}} \circ_{(1)} \mathcal{A}s^{\text{!`}}$ is given by the formula

$$\Delta_p(\mu_n^c) = \sum_{\lambda,\, k} (-1)^{\lambda + k(l-\lambda+k)} \mu_{l+1-k}^c \otimes (\underbrace{id \otimes \cdots \otimes id}_{\lambda} \otimes \mu_k^c \otimes \underbrace{id \otimes \cdots \otimes id}_{l-\lambda-k})$$

to give on $\mathcal{A}s^{\text{!`}}(A)$ the differential

$$d_\alpha([b_1 \cdots b_l]) = \sum_{\lambda,\, k} (-1)^{\lambda + k(l-\lambda-k) + (|b_1| + \cdots + |b_\lambda|)(k-1)} [b_1 \cdots b_\lambda\, \mu_k(b_{\lambda+1} \cdots b_{\lambda+k}) b_{\lambda+k+1} \cdots b_l].$$

Contrary to $\mathcal{A}s$ and $\mathcal{A}s^{\text{!`}}$, A_∞ has a non-zero differential which induces a non-zero differential on $\mathbb{L}_{R|A}$ (also denoted d_{A_∞} by abuse of notations). We get
$d_{A_\infty}(a_1^1 \cdots a_{i_1}^1 | \cdots | a_1^k \cdots a_{i_k}^k [b_1 \cdots b_l] c_1^k \cdots c_{j_k}^k | \cdots | c_1^1 \cdots c_{j_1}^1) =$

$$\begin{aligned}
&- \sum \varepsilon_{\lambda,\, k,\, t}\, a_1^1 \cdots | a_1^t \cdots \mu_k(a_{\lambda+1}^t \cdots a_{\lambda+k}^t) \cdots a_{i_t}^t | \cdots [\cdots] c_1^k \cdots | \cdots | \cdots c_{j_1}^1 \\
&- \sum \varepsilon_{\lambda,\, k,\, t}\, a_1^1 \cdots | a_1^t \cdots a_\lambda^t | a_{\lambda+1}^t \cdots a_{i_t}^t | \cdots [\cdots] \cdots | c_1^t \cdots c_{k-i_t+\lambda-1}^t | c_{k-i_t+\lambda}^t \cdots c_{j_t}^t | \cdots c_{j_1}^1 \\
&- \sum \varepsilon_{\lambda,\, k,\, t}\, a_1^1 \cdots | \cdots | \cdots a_{i_k}^k [\cdots] c_1^k \cdots | c_1^t \cdots \mu_k(c_{\lambda+1}^t \cdots c_{\lambda+k}^t) \cdots c_{j_t}^t | \cdots c_{j_1}^1,
\end{aligned}$$

where $\varepsilon_{\lambda,\, k,\, t} = (-1)^{i_1 + j_1 + \cdots + i_{t-1} + j_{t-1} + \lambda + k(i_t + j_t + 1 - \lambda + k)}$.

The second part of the differential is the twisted one induced by δ_φ^l. We get
$\delta_\varphi^l(a_1^1 \cdots a_{i_1}^1 | \cdots | a_1^k \cdots a_{i_k}^k [b_1 \cdots b_l] c_1^k \cdots c_{j_k}^k | \cdots | c_1^1 \cdots c_{j_1}^1) =$

$$\sum_{\lambda,\, k} \epsilon \cdot a_1^1 \cdots | b_1 \cdots b_\lambda [b_{\lambda+1} \cdots b_{\lambda+k}] b_{\lambda+k+1} \cdots b_l | \cdots c_{j_1}^1,$$

where $\epsilon := (-1)^{i_1 + j_1 + \cdots + i_k + j_k + (|a_1^1| + \cdots + |a_{i_k}^k|)(l-k+1) + (|b_1| + \cdots + |b_\lambda|)(k-1) + \lambda + k(l-\lambda+k)}$.

1.3.4 The case of L_∞-algebras

The case of L_∞-algebras can be made explicit in the same way, with trees in space instead of planar trees. We recover then the definitions given by Hinich and Schechtman in [HS93].

1.3.5 The case of $\mathcal{P}_\infty$-algebras

The general case of homotopy $\mathcal{P}$-algebras can be treated similarly as follows. Let $\mathcal{P}$ be a Koszul operad and let $\mathcal{P}_\infty := \Omega(\mathcal{P}^{\text{i}})$ be its Koszul resolution. Any $\mathcal{P}_\infty$-algebra A admits a resolution $\mathcal{P}_\infty \circ_\iota \mathcal{P}^{\text{i}} \circ_\iota A \xrightarrow{\sim} A$, where $\iota : \mathcal{P}^{\text{i}} \to \mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}})$ is the universal twisting morphism. The cotangent complex has the same form as in the previous cases.

1.4 The cotangent complex and the module of Kähler differential forms

In this section, we show that the André-Quillen cohomology of a $\mathcal{P}$-algebra A is an Ext-functor over the enveloping algebra of A if and only if the cotangent complex of A is a resolution of the module of Kähler differential forms. Moreover, we prove that the André-Quillen cohomology theory of an operad is an Ext-functor over its enveloping algebra. We recall that we consider only non-negatively graded $\mathcal{P}$-algebras in order to have cofibrant resolutions.

1.4.1 André-Quillen cohomology as an Ext-functor

Let R be a cofibrant resolution of a $\mathcal{P}$-algebra A. Then there is a map

$$\mathbb{L}_{R/A} = A \otimes_R^{\mathcal{P}} \Omega_{\mathcal{P}}(R) \to A \otimes_A^{\mathcal{P}} \Omega_{\mathcal{P}}(A) \cong \Omega_{\mathcal{P}}(A).$$

If the functor $A \otimes_-^{\mathcal{P}} \Omega_{\mathcal{P}}(-)$ sends cofibrant resolutions to cofibrant resolutions, then the André-Quillen cohomology is the following Ext-functor

$$\mathrm{H}^\bullet_{\mathcal{P}}(A,\, M) \cong \mathrm{Ext}^\bullet_{A \otimes^{\mathcal{P}} \mathbb{K}}(\Omega_{\mathcal{P}}(A),\, M).$$

Moreover, we will see in this subsection that the reverse implication is true. Let $X_\bullet \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$ be a cofibrant resolution in $\mathcal{M}_A^{\mathcal{P}}$ and consider a quasi-free resolution $R = \mathcal{P} \circ \mathcal{C}(A)$ of A. The cotangent complex $\mathbb{L}_{R/A} \cong A \otimes^{\mathcal{P}} \mathcal{C}(A)$ is a quasi-free A-module over $\mathcal{P}$ since R is quasi-free, so this realization of the cotangent complex is a cofibrant A-module over $\mathcal{P}$. The model category structure on $\mathcal{M}_A^{\mathcal{P}}$ and the commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow \sim \\ A \otimes^{\mathcal{P}} \mathcal{C}(A) & \longrightarrow & \Omega_{\mathcal{P}}(A) \end{array}$$

give a map $A \otimes^{\mathcal{P}} \mathcal{C}(A) \to X_\bullet$. This last map induces a map

$$\mathrm{H}^\bullet_{\mathcal{P}}(A,\, M) \leftarrow \mathrm{H}^\bullet_{\mathcal{P}}(\mathrm{Hom}_{A \otimes^{\mathcal{P}} \mathbb{K}\text{-}mod}(X_\bullet,\, M)) \cong \mathrm{Ext}^\bullet_{A \otimes^{\mathcal{P}} \mathbb{K}}(\Omega_{\mathcal{P}}(A),\, M).$$

When this map is an isomorphism, the *André-Quillen cohomology is an Ext-functor over the* $\mathcal{P}$*-enveloping algebra.*

We prove the following homological lemmas.

1.4.1.1 Lemma. *Let* $\varphi : V \to W$ *be a map of dg vector spaces. If* $\varphi^* : V^* \leftarrow W^*$ *is an isomorphism then* $\varphi : V \to W$ *is an isomorphism, where* $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$.

Proof. Let $x \in V$ non zero and H be a supplementary of $\mathbb{K}x$ in $V = \mathbb{K}x \oplus H$. Since φ^* is surjective, there exists $g \in W^*$ such that $x^* = \varphi^*(g) = g \circ \varphi$, where x^* is the map in V^* which is 1 on x and 0 on H. Thus $1 = x^*(x) = g \circ \varphi(x)$, so $\varphi(x) \neq 0$ and φ is injective. Dually we show that φ is surjective. □

1.4.1.2 Lemma. *Let* S *be a dg unitary associative algebra over* $\mathbb{K}$ *and let* $\varphi : M \to N$ *be a map of dg left* S*-modules. If* $\varphi^* : \mathrm{Hom}_{S\text{-}mod}(M, M') \xleftarrow{\sim} \mathrm{Hom}_{S\text{-}mod}(N, M')$ *is a quasi-isomorphism for all dg left* S*-module* M'*, then* $\varphi : M \xrightarrow{\sim} N$.

Proof. We endow $\mathrm{Hom}_{\mathbb{K}}(S, \mathbb{K})$ with a structure of dg left S-module by $s \cdot f(x) := f(s^{-1} \cdot x)$ for $s \in S$ and $f \in \mathrm{Hom}_{\mathbb{K}}(S, \mathbb{K})$ and $x \in S$. We have the adjunction

$$\mathrm{Hom}_{S\text{-}mod}(M, \mathrm{Hom}_{\mathbb{K}}(S, \mathbb{K})) \cong \mathrm{Hom}_{\mathbb{K}}(M \otimes_S S, \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}),$$

which is an isomorphism of dg left S-modules (where $\mathbb{K}$ is endowed with a trivial structure). Thus φ^* induces a quasi-isomorphism $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}) \xleftarrow{\sim} \mathrm{Hom}_{\mathbb{K}}(N, \mathbb{K})$. Since the differential on $\mathbb{K}$ is 0, we get $\mathrm{H}_\bullet(\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})) \cong \mathrm{Hom}_{\mathbb{K}}(\mathrm{H}_\bullet(M), \mathbb{K})$. We conclude using Lemma 1.4.1.1. □

1.4.1.3 Lemma. *Let* $\mathcal{P}$ *be a dg operad and let* A *be a* $\mathcal{P}$*-algebra. Let* $\mathcal{C}$ *be a cooperad and let* $\alpha : \mathcal{C} \to \mathcal{P}$ *be an operadic twisting morphism such that* $\mathcal{P} \circ \mathcal{C}(A)$ *is a quasi-free resolution of* A*. There exists a spectral sequence which converges to the cohomology of* A *with coefficients in* M*, such that*

$$E_2^{p,\,q} \cong \mathrm{Ext}^p_{A \otimes^{\mathcal{P}} \mathbb{K}}(\mathrm{H}_q(A \otimes^{\mathcal{P}} \mathcal{C}(A)),\, M) \Rightarrow \mathrm{H}^{p+q}_{\mathcal{P}}(A,\, M).$$

Proof. The arguments of Section 5.3.1 of [Bal98] are still valid here and give the convergence of the spectral sequence. □

1.4.1.4 Theorem. *Let* $\mathcal{P}$ *be a dg operad and let* A *be a* $\mathcal{P}$*-algebra. Let* R *be a cofibrant resolution of* A*. The following properties are equivalent :*

(P_0) *the André-Quillen cohomology of* A *is an Ext-functor over the enveloping algebra* $A \otimes^{\mathcal{P}} \mathbb{K}$*, that is* $\mathrm{H}^\bullet_{\mathcal{P}}(A, M) \cong \mathrm{Ext}^\bullet_{A \otimes^{\mathcal{P}} \mathbb{K}}(\Omega_{\mathcal{P}}(A), M)$ *;*

(P_1) *the cotangent complex is quasi-isomorphic to the module of Kähler differential forms, that is* $\mathbb{L}_{R/A} \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$.

Proof. A representation of the cotangent complex is given by $A \otimes^{\mathcal{P}} \mathcal{C}(A)$, where $\mathcal{C}$ is a cooperad and $\alpha : \mathcal{C} \to \mathcal{P}$ is a Koszul morphism, e.g. $\mathcal{C} = \mathrm{B}(\mathcal{P})$ and $\alpha = \pi$. When $A \otimes^{\mathcal{P}} \mathcal{C}(A) \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$, as $A \otimes^{\mathcal{P}} \mathcal{C}(A)$ is a quasi-free $A \otimes^{\mathcal{P}} \mathbb{K}$-module, the André-Quillen cohomology is by definition an Ext-functor and the property (P_1) implies the property (P_0). Conversely, we assume that $\mathrm{H}^\bullet_{\mathcal{P}}(-, A)$ is an Ext-functor. We apply Lemma 1.4.1.2 to $S = A \otimes^{\mathcal{P}} \mathbb{K}$, to $M = A \otimes^{\mathcal{P}} \mathcal{C}(A)$ and to $N = X_\bullet$ a cofibrant resolution of $\Omega_{\mathcal{P}}(A)$. This gives that the property (P_0) implies the property (P_1). □

1.4.2 André-Quillen cohomology of operads as an Ext-functor

Rezk defined a cohomology theory for operads following the ideas of Quillen in [Rez96]. Baues, Jibladze and Tonks proposed in [BJT97] a cohomology theory for monoids in particular monoidal categories, which includes the case of operads. Later Merkulov and Vallette gave in [MV09a] the cohomology theory "à la Quillen" for properads, and so for operads. Merkulov and Vallette define the cotangent complex associated to the resolution of an operad. Let $\Omega(\mathcal{C}) \xrightarrow{\sim} \mathcal{P}$ be a cofibrant resolution of the operad $\mathcal{P}$. We get

$$\mathbb{L}_{\Omega(\mathcal{C})/\mathcal{P}} \cong \mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathcal{C}} \circ \mathcal{P}) \to \Omega_{\text{op.}}(\mathcal{P}) \cong \mathcal{P} \circ_{(1)} (\overline{\mathcal{P}} \circ \mathcal{P})/\sim \, \cong \mathcal{P} \circ_{(1)} \overline{\mathcal{P}},$$

where $\Omega_{\text{op.}}(\mathcal{P})$ is the left $\mathcal{P}$-module of Kähler differential forms (we can see $\Omega_{\text{op.}}(\mathcal{P})$ as $\Omega_S(\mathcal{P})$ with S the coloured operad whose algebras are operads). The differential on $\mathbb{L}_{\Omega(\mathcal{C})/\mathcal{P}}$ is made explicit as a truncation of the functorial cotangent complex defined in Section 1.5.1. This enables to define the *André-Quillen cohomology of an operad with coefficients in an infinitesimal $\mathcal{P}$-bimodule.*

Infinitesimal bimodule

An infinitesimal $\mathcal{P}$-bimodule is an $\mathbb{S}$-module M endowed with two degree 0 maps $\mathcal{P} \circ (\mathcal{P}, M) \to M$ and $M \circ \mathcal{P} \to M$ satisfying the commutativity of certain diagrams. We refer to Section 3 of [MV09a] for an explicit definition.

The notion of operad is a generalization of the notion of associative algebra. Thus, the following lemma can be seen as a generalization of the one in the case of associative algebra.

1.4.2.1 Lemma. *Let $\mathcal{P}$ be an augmented dg operad and $\Omega(\mathcal{C}) \xrightarrow{\sim} \mathcal{P}$ be a cofibrant resolution. The map $\mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathcal{C}} \circ \mathcal{P}) \to \Omega_{op.}(\mathcal{P}) \cong \mathcal{P} \circ_{(1)} \overline{\mathcal{P}}$ is a quasi-isomorphism.*

Proof. Since the result does not depend on the cofibrant resolution, we show it in the underlying case $\mathcal{C} = \mathrm{B}(\mathcal{P})$. We filter the complex $\mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathrm{B}(\mathcal{P})} \circ \mathcal{P})$ by the total number of elements of $\overline{\mathcal{P}}$ in $\overline{\mathrm{B}(\mathcal{P})} \circ \mathcal{P}$

$$F_p\mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathrm{B}(\mathcal{P})} \circ \mathcal{P}) := \bigoplus_{w+k\leq p} \mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathrm{B}_{(w)}(\mathcal{P})} \circ (I \oplus \underbrace{\overline{\mathcal{P}}}_{k \text{ times}})).$$

The differential in $\mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathrm{B}(\mathcal{P})} \circ \mathcal{P})$ is given by $d_{\mathcal{P}\circ_{(1)}(s^{-1}\overline{\mathrm{B}(\mathcal{P})}\circ\mathcal{P})} - \delta^l + \delta^r$. The term $-\delta^l$ decreases w and possibly k. The part of $d_{\mathcal{P}\circ_{(1)}(s^{-1}\overline{\mathrm{B}(\mathcal{P})}\circ\mathcal{P})}$ induced by $d_{\mathcal{P}}$ keeps $w+k$ constant and the part induced by d_2 of $\mathrm{B}(\mathcal{P})$ keeps $w+k$ constant when the application of γ is given by $\mathcal{P} \circ I \cong \mathcal{P} \cong I \circ \mathcal{P}$ and decreases $w+k$ by one otherwise. The term δ^r behaves as the part of the differential induced by d_2. Then, the differential respects the filtration. The filtration is bounded below and exhaustive so we can apply the classical theorem of convergence of spectral sequence (cf. Theorem 5.5.1 of [Wei94]) to obtain that the spectral sequence associated to the filtration converges to the homology of $\mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathrm{B}(\mathcal{P})} \circ \mathcal{P})$. The differential d^0 on the $E^0_{p,\bullet}$ page is given by $d_{\mathcal{P}} \circ_{(1)} id_{s^{-1}\overline{\mathrm{B}(\mathcal{P}^{triv})}\circ\mathcal{P}^{triv}} + id_{\mathcal{P}} \circ_{(1)} d_{s^{-1}\overline{\mathrm{B}(\mathcal{P}^{triv})}\circ\mathcal{P}^{triv}}$, where $\mathcal{P}^{triv}$ is the underlying dg $\mathbb{S}$-module of $\mathcal{P}$ endowed with a trivial composition structure, that is $\gamma_{\mathcal{P}^{triv}} \equiv 0$. By Maschke's theorem, since $\mathbb{K}$ is a field of characteristic 0, every $\mathbb{K}[\mathbb{S}_n]$-module is projective. Then, by the Künneth formula, we get

$$\mathrm{H}_\bullet(E^0_{p,\bullet}) = \mathrm{H}_\bullet(\mathcal{P} \circ_{(1)} (s^{-1}\overline{\mathrm{B}(\mathcal{P}^{triv})} \circ \mathcal{P}^{triv})) = \mathrm{H}_\bullet(\mathcal{P}) \circ_{(1)} \mathrm{H}_\bullet(s^{-1}\overline{\mathrm{B}(\mathcal{P}^{triv})} \circ \mathcal{P}^{triv}).$$

Similarly to the proof of $I \xrightarrow{\sim} B(\mathcal{P}) \circ \mathcal{P}$ (see Theorem 2.19 in [GJ94]), we see that $\overline{\mathcal{P}} \xrightarrow{\sim} s^{-1}\overline{B(\mathcal{P})} \circ \mathcal{P}$. Then, for $\mathcal{P}^{triv}$, we have

$$\mathrm{H}_\bullet(E^0_{p,\bullet}) = \mathrm{H}_\bullet(\mathcal{P}) \circ_{(1)} \mathrm{H}_\bullet(s^{-1}\overline{\mathrm{B}(\mathcal{P}^{triv})} \circ \mathcal{P}^{triv}) = \mathrm{H}_\bullet(\mathcal{P}) \circ_{(1)} \mathrm{H}_\bullet(\overline{\mathcal{P}}) = \mathrm{H}_\bullet(\mathcal{P} \circ_{(1)} \overline{\mathcal{P}}).$$

Finally, the spectral sequence collapses at rank 1 and the Lemma is true. □

As a corollary of the previous Lemma, we get

1.4.2.2 Theorem. *The André-Quillen cohomology of operads with coefficients in an infinitesimal $\mathcal{P}$-bimodule is the Ext-functor*

$$\mathrm{H}^\bullet(\mathcal{P},\, \mathcal{M}) \cong \mathrm{Ext}^\bullet_{\mathcal{P}\circ_{(1)}(I\circ\mathcal{P})}(\Omega_{op.}(\mathcal{P}),\, \mathcal{M}).$$

Proof. We combine Theorem 1.4.1.4 and Lemma 1.4.2.1. □

1.5 The functorial cotangent complex

In this section, we introduce a *functorial cotangent complex* and a *functorial module of Kähler differential forms*, depending only on the operad. We prove that the map between these two complexes is a quasi-isomorphism if and only if the André-Quillen cohomology is an Ext-functor. We define the module of obstructions and we show that it is acyclic if and only if the André-Quillen cohomology is an Ext-functor.

1.5.1 Definition of the functorial cotangent complex

As we explain in Section 1.1.2, the resolutions of algebras we use in this paper come from operadic resolutions. They all have the form $\mathcal{P} \circ_\alpha \mathcal{C} \xrightarrow{\sim} I$, where $\alpha : \mathcal{C} \to \mathcal{P}$ is an operadic twisting morphism. We call such a twisting morphism a *Koszul morphism*. We define (a representation of) the *functorial cotangent complex* based on such type of resolutions as follows.

We consider the dg infinitesimal $\mathcal{P}$-bimodule $L_\mathcal{P} := \mathcal{P}(I,\, \mathcal{C}\circ\mathcal{P}) = \mathcal{P} \circ_{(1)} (\mathcal{C}\circ\mathcal{P})$ endowed with the differential $d_{L_\mathcal{P}} := d_{\mathcal{P}(I,\mathcal{C}\circ\mathcal{P})} - \delta^l_{L_\mathcal{P}} + \delta^r_{L_\mathcal{P}}$, where $\delta^l_{L_\mathcal{P}}$ is defined by the composite

$$\mathcal{P} \circ_{(1)} (\mathcal{C}\circ\mathcal{P}) \xrightarrow{id_\mathcal{P}\circ_{(1)}(\Delta_p\circ id_\mathcal{P})} \mathcal{P} \circ_{(1)} ((\mathcal{C}\circ_{(1)}\mathcal{C})\circ\mathcal{P}) \xrightarrow{id_\mathcal{P}\circ_{(1)}(\alpha\circ id_\mathcal{C}\circ id_\mathcal{P})}$$

$$\mathcal{P} \circ_{(1)} ((\mathcal{P}\circ_{(1)}\mathcal{C})\circ\mathcal{P}) \rightarrowtail (\mathcal{P}\circ\mathcal{P}\circ\mathcal{P}) \circ_{(1)} (\mathcal{C}\circ\mathcal{P}) \xrightarrow{\gamma\circ\gamma\circ_{(1)} id_{\mathcal{C}\circ\mathcal{P}}} \mathcal{P} \circ_{(1)} (\mathcal{C}\circ\mathcal{P})$$

and $\delta^r_{L_\mathcal{P}}$ is defined by the composite

$$\mathcal{P} \circ_{(1)} (\mathcal{C}\circ\mathcal{P}) \xrightarrow{id_\mathcal{P}\circ_{(1)}(\Delta_p\circ id_\mathcal{P})} \mathcal{P} \circ_{(1)} ((\mathcal{C}\circ_{(1)}\mathcal{C})\circ\mathcal{P}) \xrightarrow{id_\mathcal{P}\circ_{(1)}(id_\mathcal{C}\circ\alpha\circ id_\mathcal{P})}$$

$$\mathcal{P} \circ_{(1)} ((\mathcal{C}\circ_{(1)}\mathcal{P})\circ\mathcal{P}) \rightarrowtail \mathcal{P} \circ_{(1)} (\mathcal{C}\circ\mathcal{P}\circ\mathcal{P}) \xrightarrow{id_\mathcal{P}\circ_{(1)}(id_\mathcal{C}\circ\gamma)} \mathcal{P} \circ_{(1)} (\mathcal{C}\circ\mathcal{P}).$$

The right action is given by $\mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P})\circ\mathcal{P} \rightarrowtail (\mathcal{P}\circ\mathcal{P})\circ_{(1)}(\mathcal{C}\circ\mathcal{P}\circ\mathcal{P}) \xrightarrow{\gamma\circ_{(1)} id_\mathcal{C}\circ\gamma} \mathcal{P}\circ_{(1)}(\mathcal{C}\circ\mathcal{P})$.

1.5.1.1 Proposition. *Let A be a $\mathcal{P}$-algebra. With the above notations, there is an isomorphism of chain complexes*

$$L_\mathcal{P} \circ_\mathcal{P} A \cong A \otimes^\mathcal{P} \mathcal{C}(A).$$

PROOF. We write $L_{\mathcal{P}} \circ A \cong \mathcal{P}(A, \mathcal{C} \circ \mathcal{P}(A))$. We use the description of the relative composition product $\circ_{\mathcal{P}}$ and of the description $A \otimes^{\mathcal{P}} N$ to get $L_{\mathcal{P}} \circ_{\mathcal{P}} A \cong A \otimes^{\mathcal{P}} \mathcal{C}(A)$. The equality of the differentials comes from the same descriptions. □

1.5.1.2 Corollary. *Let V be a dg trivial $\mathcal{P}$-algebra, that is $\gamma_V \equiv 0$. There is an isomorphism of chain complexes*

$$(L_{\mathcal{P}} \circ_{\mathcal{P}} I) \circ V \cong V \otimes^{\mathcal{P}} \mathcal{C}(V).$$

PROOF. When the $\mathcal{P}$-algebra V is trivial, we get the isomorphism of underlying dg modules $(L_{\mathcal{P}} \circ_{\mathcal{P}} I) \circ V \cong L_{\mathcal{P}} \circ_{\mathcal{P}} V$, where I can be seen as a left $\mathcal{P}$-module with a trivial structure. The equality of the differentials follows from their definitions. □

We denote $\overline{L}_{\mathcal{P}} := L_{\mathcal{P}} \circ_{\mathcal{P}} I$.

1.5.2 Definition of the functorial module of Kähler differential forms

Let $\mathcal{P}$ be a dg operad. We define the *functorial module of Kähler differential forms* as the following coequalizer diagram in the category of infinitesimal $\mathcal{P}$-bimodules (see 10.3 of [Fre09] for an equivalent definition)

$$\mathcal{P} \circ_{(1)} (\mathcal{P} \circ \mathcal{P}) \underset{c_2}{\overset{id_{\mathcal{P}} \circ_{(1)} \gamma}{\rightrightarrows}} \mathcal{P} \circ_{(1)} \mathcal{P} \twoheadrightarrow \Omega_{\mathcal{P}},$$

where c_2 is given by the composite

$$\mathcal{P} \circ_{(1)} (\mathcal{P} \circ \mathcal{P}) \xrightarrow{id_{\mathcal{P}} \circ_{(1)} (id_{\mathcal{P}} \circ' id_{\mathcal{P}})} \mathcal{P} \circ_{(1)} (\mathcal{P} \circ (\mathcal{P}, \mathcal{P})) \rightarrowtail (\mathcal{P} \circ \mathcal{P} \circ \mathcal{P}) \circ_{(1)} \mathcal{P} \xrightarrow{\gamma \circ \gamma \circ_{(1)} id_{\mathcal{P}}} \mathcal{P} \circ_{(1)} \mathcal{P}.$$

The right $\mathcal{P}$-module action on $\Omega_{\mathcal{P}}$ is induced by the right $\mathcal{P}$-module action on $\mathcal{P} \circ_{(1)} \mathcal{P}$ given by

$$(\mathcal{P} \circ_{(1)} \mathcal{P}) \circ \mathcal{P} \rightarrowtail (\mathcal{P} \circ \mathcal{P}) \circ_{(1)} (\mathcal{P} \circ \mathcal{P}) \xrightarrow{\gamma \circ_{(1)} \gamma} \mathcal{P} \circ_{(1)} \mathcal{P}.$$

1.5.2.1 Proposition. *Let A be a $\mathcal{P}$-algebra. There is an isomorphism of chain complexes*

$$\Omega_{\mathcal{P}} \circ_{\mathcal{P}} A \cong \Omega_{\mathcal{P}}(A).$$

PROOF. We write $A \otimes^{\mathcal{P}} A \cong (\mathcal{P} \circ_{(1)} \mathcal{P}) \circ_{\mathcal{P}} A$ and $A \otimes^{\mathcal{P}} \mathcal{P}(A) \cong (\mathcal{P} \circ_{(1)} \mathcal{P} \circ \mathcal{P}) \circ_{\mathcal{P}} A$. Thanks to the description of $\Omega_{\mathcal{P}}(A)$ given at the end of Lemma 1.1.1.2, we get the result. □

1.5.2.2 Corollary. *Let V be a dg trivial $\mathcal{P}$-algebra. There is an isomorphism of chain complexes*

$$(\Omega_{\mathcal{P}} \circ_{\mathcal{P}} I) \circ V \cong \Omega_{\mathcal{P}}(V).$$

PROOF. When the $\mathcal{P}$-algebra V is trivial, we get $(\Omega_{\mathcal{P}} \circ_{\mathcal{P}} I) \circ V \cong \Omega_{\mathcal{P}} \circ_{\mathcal{P}} \circ V$. □

We denote $\overline{\Omega}_{\mathcal{P}} := \Omega_{\mathcal{P}} \circ_{\mathcal{P}} I$.

1.5.3 Homotopy category

Let $\mathcal{P}$ be an augmented dg operad and $\mathcal{C}$ be a coaugmented dg cooperad such that $\mathcal{P}\circ_\alpha\mathcal{C} \xrightarrow{\sim} I$. We define the following surjective map of infinitesimal $\mathcal{P}$-bimodules

$$L_\mathcal{P} = \mathcal{P}(I,\, \mathcal{C}\circ\mathcal{P}) \twoheadrightarrow \mathcal{P}(I,\, I\circ\mathcal{P})/\sim\, \cong \mathcal{P}\circ_{(1)}\mathcal{P}/\sim\, \cong \Omega_\mathcal{P}.$$

This map induces a map

$$A\otimes^\mathcal{P}\mathcal{C}(A) \cong L_\mathcal{P}\circ_\mathcal{P} A \twoheadrightarrow \Omega_\mathcal{P}\circ_\mathcal{P} A \cong \Omega_\mathcal{P}(A)$$

which coincides with the map given in Section 1.4.1.

The differential on $\mathbb{L}_{\Omega(\mathcal{C})/\mathcal{P}}$ and the augmentation $\mathbb{L}_{\Omega(\mathcal{C})/\mathcal{P}} \to \mathcal{P}\circ_{(1)}(I\circ\mathcal{P})$ induce a differential on the cone $s\mathbb{L}_{\Omega(\mathcal{C})/\mathcal{P}}\oplus\mathcal{P}\circ_{(1)}(I\circ\mathcal{P})$. With this differential, we have $L_\mathcal{P}\cong s\mathbb{L}_{\Omega(\mathcal{C})/\mathcal{P}}\oplus\mathcal{P}\circ_{(1)}(I\circ\mathcal{P})$. Then, $L_\mathcal{P}$ is well-defined in the homotopy category of infinitesimal $\mathcal{P}$-bimodules. The same is true for $\overline{L}_\mathcal{P} = L_\mathcal{P}\circ_\mathcal{P} I$ and we call its image in the homotopy category of infinitesimal left $\mathcal{P}$-modules the *functorial cotangent complex*, that we denote by $\mathbb{L}_\mathcal{P}$. We denote by $\boldsymbol{\Omega}_\mathcal{P}$ the image of $\overline{\Omega}_\mathcal{P} := \Omega_\mathcal{P}\circ_\mathcal{P} I$ in the homotopy category.

1.5.4 Filtration on the cotangent complex

Let $\alpha : \mathcal{C}\to\mathcal{P}$ be a Koszul morphism between a weight graded dg cooperad and a weight graded dg operad. Let A be a dg $\mathcal{P}$-algebra. We filter $L_\mathcal{P}\circ A \cong \mathcal{P}(A,\, \mathcal{C}\circ\mathcal{P}(A))$ by the weight in the first $\mathcal{P}$ and the weight in $\mathcal{C}$:

$$F_p(L_\mathcal{P}\circ A) := \bigoplus_{m+n\leq p} \mathcal{P}^{(n)}\circ_{(1)}(\mathcal{C}^{(m)}\circ\mathcal{P})\circ A.$$

With the projection $L_\mathcal{P}\circ A \twoheadrightarrow L_\mathcal{P}\circ_\mathcal{P} A\cong A\otimes^\mathcal{P}\mathcal{C}(A)$, it induces a filtration on $A\otimes^\mathcal{P}\mathcal{C}(A)$ that we denote by $F_p(A\otimes^\mathcal{P}\mathcal{C}(A))$.

The differential on $A\otimes^\mathcal{P}\mathcal{C}(A)$ is given by $id_{L_\mathcal{P}}\circ'_\mathcal{P} d_A + d_{\mathcal{P}(I,\mathcal{C}\circ\mathcal{P})}\circ_\mathcal{P} id_A - \delta^l_{L_\mathcal{P}}\circ_\mathcal{P} id_A + \delta^r_{L_\mathcal{P}}\circ_\mathcal{P} id_A$. The part $id_{L_\mathcal{P}}\circ'_\mathcal{P} d_A + d_{\mathcal{P}(I,\mathcal{C}\circ\mathcal{P})}\circ_\mathcal{P} id_A$ keeps the sum $n+m$ constant, the part $-\delta^l_{L_\mathcal{P}}\circ_\mathcal{P} id_A$ may decrease the sum $n+m$ and the part $\delta^r_{L_\mathcal{P}}\circ_\mathcal{P} id_A$ decreases the sum $n+m$. It follows that the differential on the cotangent complex respects this filtration.

1.5.4.1 Lemma. *For any $\mathcal{P}$-algebra A, the spectral sequence associated to the filtration F_p converges to the homology of the cotangent complex*

$$E^1_{p,\,q} = \mathrm{H}_{p+q}(F_p(A\otimes^\mathcal{P}\mathcal{C}(A))/F_{p-1}(A\otimes^\mathcal{P}\mathcal{C}(A))) \Rightarrow \mathrm{H}_{p+q}(A\otimes^\mathcal{P}\mathcal{C}(A)).$$

Proof. This filtration is exhaustive and bounded below so we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain the result. □

We denote by d^0 the differential on $E^0_{p,\,\bullet}$, which depends on $d_{\mathcal{P}(I,\mathcal{C}\circ\mathcal{P})}\circ_\mathcal{P} id_A$ and on $\delta^l_{L_\mathcal{P}}\circ_\mathcal{P} id_A$. We denote by d^1 the differential on $E^1_{\bullet,\,q}$, which depends on $id_{L_\mathcal{P}}\circ'_\mathcal{P} d_A$, on $\delta^l_{L_\mathcal{P}}\circ_\mathcal{P} id_A$ and on $\delta^r_{L_\mathcal{P}}\circ_\mathcal{P} id_A$. We denote by d^r the differential on $E^r_{p,\,\bullet}$, which depends on $\delta^l_{L_\mathcal{P}}\circ_\mathcal{P} id_A$ and on $\delta^r_{L_\mathcal{P}}\circ_\mathcal{P} id_A$.

1.5.5 Filtration of the module of Kähler differential forms

Similarly, we filter $\Omega_{\mathcal{P}}(A) = \text{coequal}(\ A \otimes^{\mathcal{P}} \mathcal{P}(A) \rightrightarrows A \otimes^{\mathcal{P}} A\ )$. We filter $A \otimes^{\mathcal{P}} \mathcal{P}(A)$ and $A \otimes^{\mathcal{P}} A$ by the weight in $\mathcal{P}$. So the arrows preserve the filtration and the coequallizer is filtered. We denote by $F_p\Omega_{\mathcal{P}}(A)$ this filtration. The differential on $\Omega_{\mathcal{P}}(A)$ respects the filtration since the operad is concentrated in non-negative weight degrees. We denote by d^0 the differential on $E^0_{p,\bullet}$, which is the differential on $\Omega_{\mathcal{P}}(A)$. The differentials d^r on E^r are 0 for $r \geq 1$.

Then, for any $\mathcal{P}$-algebra A, the spectral sequence associated to the filtration F_p converges to the homology of the module of Kähler differential forms

$$E^1_{p,q} = H_{p+q}(F_p(\Omega_{\mathcal{P}}(A))/F_{p-1}(\Omega_{\mathcal{P}}(A))) \Rightarrow H_{p+q}(\Omega_{\mathcal{P}}(A)).$$

1.5.6 The cotangent complex and the module of Kähler differential forms

As in the previous sections, we endow the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$ with a filtration given by the weight in $\mathcal{P}$. Following the notations given by Pirashvili in the review of [Fra01], we say that $\mathcal{P}$ is an operad satisfying the PBW property if for any $\mathcal{P}$-algebra A, there is an isomorphism $gr(A \otimes^{\mathcal{P}} \mathbb{K}) \cong A^{\text{tr}} \otimes^{\mathcal{P}} \mathbb{K}$, where A^{tr} is the underlying space of A endowed with the trivial $\mathcal{P}$-algebra structure $\gamma_{A^{\text{tr}}} \equiv 0$. The study of the differential on the cotangent complex $A \otimes^{\mathcal{P}} \mathcal{C}(A)$ shows that $\mathcal{P}$ is an operad satisfying the PBW property if and only if for any $\mathcal{P}$-algebra A, we have the isomorphism $gr(A \otimes^{\mathcal{P}} \mathcal{C}(A)) \cong A^{\text{tr}} \otimes^{\mathcal{P}} \mathcal{C}(A^{\text{tr}})$. Moreover, the study of the coequalizer defining the module of Kähler differentials $\Omega_{\mathcal{P}}(A)$ given after Lemma 1.1.1.2 shows that when $\mathcal{P}$ is an operad satisfying the PBW property, we have the isomorphism $gr(\Omega_{\mathcal{P}}(A)) \cong \Omega_{\mathcal{P}}(A^{\text{tr}})$. This notion is different from the notion of PBW-operad defined in [Hof10].

We refine Theorem 1.4.1.4 as follows.

1.5.6.1 Theorem. *Let $\mathcal{P}$ be a weight graded operad satisfying the PBW property. The following properties are equivalent :*

- (P_0) *the André-Quillen cohomology is an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$ for any $\mathcal{P}$-algebra A ;*
- (P_1') *the cotangent complex is quasi-isomorphic to the module of Kähler differential forms for any dg vector space V, seen as an algebra with trivial structure, that is $\mathbb{L}_{R/V} \xrightarrow{\sim} \Omega_{\mathcal{P}}(V)$;*
- (P_2) *the functorial cotangent complex $\mathbb{L}_{\mathcal{P}}$ is quasi-isomorphic to the functorial module of Kähler differential forms $\mathbf{\Omega}_{\mathcal{P}}$, that is $\mathbb{L}_{\mathcal{P}} \xrightarrow{\sim} \mathbf{\Omega}_{\mathcal{P}}$.*

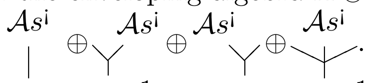
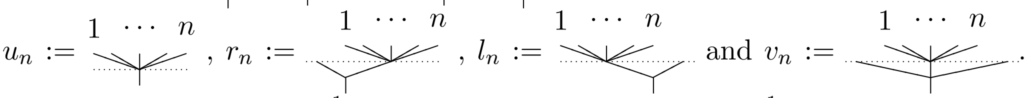
Proof. We assume the cotangent complex to be quasi-isomorphic to the module of Kähler differential forms for any dg vector space, then we show that the cotangent complex is quasi-isomorphic to the module of Kähler differential forms for any $\mathcal{P}$-algebra. Thus, the equivalence $(P_0) \Leftrightarrow (P_1')$ follows from Theorem 1.4.1.4. Let A be a $\mathcal{P}$-algebra and denote by V the underlying dg vector space of A considered as a trivial algebra. We use the filtration and the spectral sequence of the previous section. In the case of the algebra V, the differential d^1 is zero since $id_{L_{\mathcal{P}}} \circ'_{\mathcal{P}} d_A = 0$, $\delta^r_{L_{\mathcal{P}}} \circ_{\mathcal{P}} id_A = 0$ and the part induced by $\delta^l_{L_{\mathcal{P}}} \circ_{\mathcal{P}} id_A$ is 0. For any $r \geq 0$, the differential d^r is 0 since the part induced by $\delta^l_{L_{\mathcal{P}}} \circ_{\mathcal{P}} id_A$ is 0 and $\delta^r_{L_{\mathcal{P}}} \circ_{\mathcal{P}} id_A = 0$. Thus, we have $(V \otimes^{\mathcal{P}} \mathcal{C}(V),\, d_\varphi) \cong (\oplus_p E^0_{p,\bullet},\, d^0)$ as dg modules since $\mathcal{P}$ is an operad satisfying the PBW property. It follows that $\mathrm{H}_\bullet(V \otimes^{\mathcal{P}} \mathcal{C}(V)) \cong \oplus_p \mathrm{H}_\bullet(E^0_{p,\bullet}) = \oplus_p E^1_{p,\bullet}$ and the spectral sequence collapses at rank 1. Moreover, the term $E^0_{p,q}$ associated to A is equal to the one associated to V by definition and the same is true for the term $E^1_{p,q}$ since the differential d^0 does not depend on the composition product and on the differential of the algebra. Then, the page E^1 and the differentials d^r for $r \geq 1$ correspond to the page E^1 and to the differentials d_r for $r \geq 1$ associated to $\Omega_{\mathcal{P}}(A)$. This gives that the cotangent complex is quasi-isomorphic to the module of Kähler differential forms for any $\mathcal{P}$-algebra.

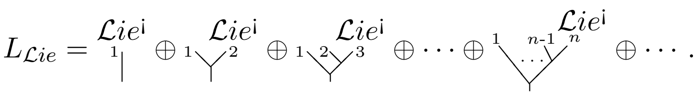
The equivalence $(P_1') \Leftrightarrow (P_2)$ follows from the equalities $(V \otimes^{\mathcal{P}} \mathcal{C}(V), d_\varphi) = (\overline{L}_{\mathcal{P}} \circ V, d_{\overline{L}_{\mathcal{P}} \circ V}) = (\overline{L}_{\mathcal{P}} \circ V, d_{\overline{L}_{\mathcal{P}}} \circ id_V)$ and $(\Omega_{\mathcal{P}}(V), d_{\Omega_{\mathcal{P}}(V)}) = (\overline{\Omega}_{\mathcal{P}} \circ V, d_{\overline{\Omega}_{\mathcal{P}}} \circ id_V)$. □

Remark. When $\mathcal{P}$ is an operad concentrated in homological degree 0, $\overline{\Omega}_{\mathcal{P}}$ is an $\mathbb{S}$-module concentrated in degree 0. In this case, we say that $\mathbb{L}_{\mathcal{P}}$ is acyclic when its homology is concentrated in degree 0 and equal to $\mathbf{\Omega}_{\mathcal{P}}$.

First applications

The operads encoding associative algebras and Lie algebras are operads satisfying the PBW property. We prove the acyclicity of the functorial cotangent complex in these cases. This gives a conceptual proof of the fact that for these operads the André-Quillen cohomology is an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$.

- We have $L_{\mathcal{A}s} =$ . Then $L_{\mathcal{A}s}(n)$ is generated by the elements . Since $d(u_n) = -l_{n-1} - (-1)^{n-1} r_{n-1}$, $d(r_n) = -v_{n-1} = (-1)^{n-1} d(l_n)$ and $d(v_n) = 0$, we define a homotopy h for d by $h(u_n) := 0$, $h(l_n) = h(r_n)(-1)^n := -\frac{1}{2} u_{n+1}$ and $h(v_n) := -\frac{1}{2}((-1)^n l_{n+1} + r_{n+1})$.
- We have



Then we can define the same homotopy as in [CE99], Theorem 7.1, Chap. XIII.
- Following Frabetti in [Fra01], we show the acyclicity of $\mathbb{L}_{\mathcal{D}ias}$.

Remark. We recall the following results.
- Loday and Pirashvili showed in [LP93] that the cohomology of Leibniz algebras can be written as an Ext-functor.
- Dzhumadil'daev showed in [Dzh99] that the cohomology of pre-Lie algebras can be written as an Ext-functor.

The module of obstructions

Let $\mathcal{P}$ be an augmented dg operad and let $\mathcal{C}$ be a coaugmented dg cooperad and let $\alpha : \mathcal{C} \to \mathcal{P}$ be a twisting morphism. The map $\overline{L}_{\mathcal{P}} \twoheadrightarrow \overline{\Omega}_{\mathcal{P}}$ is surjective and we defined

$$O_{\mathcal{P}} := \ker(\overline{L}_{\mathcal{P}} \twoheadrightarrow \overline{\Omega}_{\mathcal{P}})$$

to get the following short exact sequence of dg $\mathbb{S}$-modules

$$O_{\mathcal{P}} \rightarrowtail \overline{L}_{\mathcal{P}} \twoheadrightarrow \overline{\Omega}_{\mathcal{P}}.$$

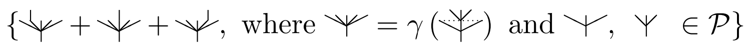
Since $\overline{L}_{\mathcal{P}}$ and $\overline{\Omega}_{\mathcal{P}}$ are well-defined in the homotopy category of infinitesimal left $\mathcal{P}$-modules, the same is true for $O_{\mathcal{P}}$. Thus we define the *module of obstructions* $\mathbb{O}_{\mathcal{P}}$ by its image in the homotopy category of infinitesimal $\mathcal{P}$-modules. We get the following short exact sequence

$$\mathbb{O}_{\mathcal{P}} \rightarrowtail \mathbb{L}_{\mathcal{P}} \twoheadrightarrow \mathbf{\Omega}_{\mathcal{P}}.$$

We compute

$$O_{\mathcal{P}} = Rel \oplus (\mathcal{P}(I,\, \mathcal{C} \circ \mathcal{P})) \circ_{\mathcal{P}} I,$$

where Rel is the image of the set of relations



in $(\mathcal{P} \circ_{(1)} I) \circ_{\mathcal{P}} I$.

We deduce the following Theorem.

1.5.6.2 Theorem. *Let $\mathcal{P}$ be a weight graded operad satisfying the PBW property. The following properties are equivalent.*

(P_0) *The André-Quillen cohomology is an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$ for any $\mathcal{P}$-algebra A;*

(P_3) *the homology of the module of obstructions $\mathbb{O}_{\mathcal{P}}$ is acyclic.*

Proof. The short exact sequence $O_{\mathcal{P}} \rightarrowtail \overline{L}_{\mathcal{P}} \twoheadrightarrow \overline{\Omega}_{\mathcal{P}}$ induces a long exact sequence in homology which gives, with the help of the equivalence $(P_0) \Leftrightarrow (P_2)$ of Theorem 1.5.6.1, the equivalence between (P_0) and (P_3). □

1.5.7 Another approach

In the parallel work [Fre09], Fresse studied the homotopy properties of modules over operads. His method applied to the present question provides the following sufficient condition for the André-Quillen cohomology to be an Ext-functor. In this section, we show the relationship between the two approaches.

Let $\mathcal{P}[1]$ be the $\mathbb{S}$-module defined in [Fre09] given by $\mathcal{P}[1](n) := \mathcal{P}(1+n)$. The $\mathbb{S}_n$-action is given by the action of $\mathbb{S}_n$ on $\{2, \ldots, n+1\} \subset \{1, \ldots, n+1\}$. Similarly to this definition, we define the $\mathbb{S}$-module $\mathcal{P}[1]_j$ by $\mathcal{P}[1]_j(n) := \mathcal{P}(n+1)$ where the $\mathbb{S}_n$-action is given by the action $\mathbb{S}_n$ on $\{1, \ldots, \hat{j}, \ldots, n+1\} \subset \{1, \ldots, n+1\}$. Thus $\mathcal{P}[1] = \mathcal{P}[1]_1$. We have $(\mathcal{P} \circ_{(1)} I)(n) \cong \underbrace{\mathcal{P}(n) \oplus \cdots \oplus \mathcal{P}(n)}_{n \text{ times}}$. As a right $\mathcal{P}$-module, we have

$$(\mathcal{P} \circ_{(1)} I)(n) \cong \underbrace{\mathcal{P}[1]_1(n-1) \oplus \cdots \oplus \mathcal{P}[1]_n(n-1)}_{n \text{ times}}.$$

When $\mathcal{P}[1]$ is a quasi-free right $\mathcal{P}$-module, that is $\mathcal{P}[1] \cong (M \circ \mathcal{P},\, d)$, we get that $\mathcal{P}[1]_j$ is a quasi-free right $\mathcal{P}$-module $(M \circ \mathcal{P},\, d)$ thanks to the isomorphism

$$\mathcal{P}[1] \to \mathcal{P}[1]_j,\ \mu \mapsto \mu \cdot (1 \cdots j).$$

We define

$$M'(n) := \oplus_{k \geq 1} \underbrace{M(n) \oplus \cdots \oplus M(n)}_{k \text{ times}}.$$

Then $\mathcal{P} \circ_{(1)} I$ is a retract of $(M' \circ \mathcal{P},\, d')$, which is quasi-free. When $\mathcal{P}[1]$ is only a retract of a quasi-free right $\mathcal{P}$-module, we get by the same argument that $\mathcal{P} \circ_{(1)} I$ is a retract of a quasi-free right $\mathcal{P}$-module. Then, $\mathcal{P}[1]$ cofibrant as a right $\mathcal{P}$-module implies that $\mathcal{P} \circ_{(1)} I$ cofibrant as a right $\mathcal{P}$-module. Finaly, when $\mathcal{P}[1]$ is a cofibrant right $\mathcal{P}$-module, $L_{\mathcal{P}} \cong (\mathcal{P} \circ_{(1)} I) \otimes (\mathcal{C} \circ \mathcal{P})$ is also cofibrant. Thus, when we assume moreover that $\Omega_{\mathcal{P}}$ is a cofibrant right $\mathcal{P}$-module, the quasi-isomorphism $L_{\mathcal{P}} \xrightarrow{\sim} \Omega_{\mathcal{P}}$ between cofibrant right $\mathcal{P}$-modules gives a quasi-isomorphism $A \otimes^{\mathcal{P}} \mathcal{C}(A) \cong L_{\mathcal{P}} \circ_{\mathcal{P}} A \xrightarrow{\sim} \Omega_{\mathcal{P}} \circ_{\mathcal{P}} A \cong \Omega_{\mathcal{P}}(A)$ (since A is cofibrant). Therefore, we have the following sufficient condition for the André-Quillen cohomology to be an Ext-functor.

1.5.7.1 Theorem (Theorem 17.3.4 in [Fre09])**.** *If $\mathcal{P}[1]$ and $\Omega_{\mathcal{P}}$ form cofibrant right $\mathcal{P}$-modules, then we have*

$$\mathrm{H}^\bullet_{\mathcal{P}}(A,\, M) \cong \mathrm{Ext}^\bullet_{A\otimes^{\mathcal{P}}\mathbb{K}}(\Omega_{\mathcal{P}}(A),\, M).$$

1.6 Is André-Quillen cohomology an Ext-functor ?

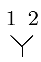
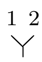
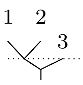
In the previous section, we showed that when $\mathcal{P}$ is an operad satisfying the PBW property, the module of obstructions $\mathbb{O}_{\mathcal{P}}$ is acyclic if and only if the André-Quillen cohomology is an Ext-functor. In this section, we apply this criterion to the operads $\mathcal{C}om$, $\mathcal{P}erm$ and to the minimal models of Koszul operads. In the case of the operads $\mathcal{C}om$ and $\mathcal{P}erm$, we provide universal obstructions for the André-Quillen cohomology to be an Ext-functor. In the case of an operad which is the cobar construction on a cooperad, we show that the obstructions always vanish. We apply this to the case of homotopy algebras.

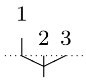
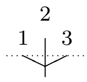
1.6.1 The case of commutative algebras

We exhibit a non-trivial element in the homology of the module of obstructions. This gives a universal obstruction for the André-Quillen cohomology of commutative algebras to be an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{C}om} \mathbb{K}$.

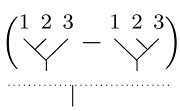
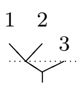
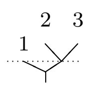
1.6.1.1 Proposition. *The module of obstructions $\mathbb{O}_{\mathcal{C}om}$ is not acyclic. More precisely, we have*

$$\mathrm{H}_1(\mathbb{O}_{\mathcal{C}om}) \neq 0.$$

Proof. Consider the element $\nu :=$  in $\mathcal{C}om^{¡} \rightarrowtail \mathrm{B}(\mathcal{C}om)$ and $\mu :=$  in $\mathcal{C}om$. The element $\mu \otimes (\nu \otimes id) =$  lives in $O_{\mathcal{C}om}$. We compute

$d_{O_{\mathcal{C}om}}(\mu \otimes (\nu \otimes id)) =$  $+$  $= 0.$

Then $\mu \otimes (\nu \otimes id)$ is a cycle in $O_{\mathcal{C}om}$. However,

$d_{O_{\mathcal{C}om}}\Bigg($  $\Bigg) =$  $-$ 

and it is impossible to obtain $\mu \otimes (\nu \otimes id)$ as a boundary of an element in $O_{\mathcal{C}om}$. Therefore, this shows that $\mathrm{H}_1(O_{\mathcal{C}om}) \neq 0$. □

Remark. The short exact sequence $O_{\mathcal{C}om} \rightarrowtail \mathbb{L}_{\mathcal{C}om} \twoheadrightarrow \Omega_{\mathcal{C}om}$ gives a long exact sequence in homology and, since $\mathrm{H}_n(\Omega_{\mathcal{C}om}) = 0$ for all $n \geq 1$, we get also $\mathrm{H}_1(\mathbb{L}_{\mathcal{C}om}) \neq 0$. It follows that there exists a commutative algebra such that the cotangent complex is not acyclic.

The operad $\mathcal{C}om$ is an operad satisfying the PBW property since $A \otimes^{\mathcal{C}om} \mathbb{K} \cong \mathbb{K} \oplus A$. Thanks to Theorem 1.5.6.2, this gives a conceptual explanation to the fact that the André-Quillen cohomology of commutative algebras cannot always be written as an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{C}om} \mathbb{K}$.

1.6.2 The case of Perm-algebras

The same argument applied to Perm algebras gives a conceptual explanation to the fact that the André-Quillen cohomology of Perm algebras cannot always be written as an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}erm} \mathbb{K}$.

1.6.2.1 Proposition. *We have*

$$\mathrm{H}_1(\mathbb{O}_{\mathcal{P}erm}) \neq 0.$$

Proof. The proof is similar to the proof of Proposition 1.6.1.1. □

1.6.3 The case of algebras up to homotopy

We show a new homotopy property for algebras over certain cofibrant operads. We apply this in the case of $\mathcal{P}$-algebras up to homotopy to prove that the André-Quillen cohomology is always an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}_\infty} \mathbb{K}$.

1.6.3.1 Theorem. *Let $\mathcal{C}$ be a coaugmented weight graded dg cooperad and $\mathcal{P} = \Omega(\mathcal{C})$ the cobar construction on it. The André-Quillen cohomology of $\Omega(\mathcal{C})$-algebras is an Ext-functor over the enveloping algebra $A \otimes^{\Omega(\mathcal{C})} \mathbb{K}$. Explicitly, for any $\Omega(\mathcal{C})$-algebra A and any A-module M, we have*

$$\mathrm{H}^\bullet_{\Omega(\mathcal{C})}(A,\, M) \cong \mathrm{Ext}^\bullet_{A \otimes^{\Omega(\mathcal{C})} \mathbb{K}}(\Omega_{\Omega(\mathcal{C})}(A),\, M).$$

Proof. As in this case of A_∞-algebras, the twisting morphism α is the map $\mathcal{C} \to \Omega(\mathcal{C})$ given in the examples after Theorem 1.1.3.1 and the twisting morphism on the level of (co)algebras φ is the projection $\mathcal{C}(A) \twoheadrightarrow A$. As a dg $\mathbb{S}$-module, the module of obstructions has the following form

$$O_{\Omega(\mathcal{C})} \cong Rel \oplus \bigoplus_{n \geq 0} \underbrace{(s^{-1}\overline{\mathcal{C}}) \circ_{(1)} \big((s^{-1}\overline{\mathcal{C}}) \circ_{(1)} \cdots \big((s^{-1}\overline{\mathcal{C}})}_{n \text{ times}} \circ_{(1)} \overline{\mathcal{C}}\big) \cdots\big)\,,$$

where $Rel \subset \bigoplus_{n\geq 0} \underbrace{(s^{-1}\overline{\mathcal{C}}) \circ_{(1)} \big((s^{-1}\overline{\mathcal{C}}) \circ_{(1)} \cdots \big((s^{-1}\overline{\mathcal{C}})}_{n \text{ times}} \circ_{(1)} I\big) \cdots\big)$ is defined in Section 1.5.6.

For any $1 \leq j \leq n$, let $\mu^c_{j,\, i_j} \in \overline{\mathcal{C}} \circ_{(1)} I$, where i_j is the emphasized entry and $\mu^c_j \in \overline{\mathcal{C}}(m_j)$. Let $\nu^c \in \overline{\mathcal{C}}(m)$. For $\sigma \in \mathbb{S}_{m_1+\cdots+m_n+m-n}$, we define the map h by

$h(s^{-1}\mu^c_{1,\, i_1} \otimes s^{-1}\mu^c_{2,\, i_2} \otimes \cdots \otimes s^{-1}\mu^c_{n,\, i_n} \otimes \nu^c \otimes \sigma) = 0$ and on Rel,
$h\left(\sum_{i_n=1}^{m_n} s^{-1}\mu^c_{1,\, i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n,\, i_n} \otimes 1^c \otimes \sigma\right) = \varepsilon_{n-1} s^{-1}\mu^c_{1,\, i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\, i_{n-1}} \otimes \mu^c_n \otimes \sigma,$

where $\varepsilon_{n-1} = (-1)^{n-1+|\mu^c_{1,\,i_1}|+\cdots+|\mu^c_{n-1,\,i_{n-1}}|}$. We compute
$(dh + hd)(s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes \nu^c \otimes \sigma) =$

$$\begin{aligned} &= 0 + h\left(\sum \cdots \otimes \nu'^c \otimes \sigma + \varepsilon_n \sum_{i=1}^m s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes s^{-1}\nu^c_i \otimes 1^c \otimes \sigma\right) \\ &= s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes \nu^c \otimes \sigma, \end{aligned}$$

where $\nu'^c \in \overline{\mathcal{C}}$ and i is the emphasized entry of ν^c_i, and
$(dh + hd)\left(\sum_{i_n=1}^{m_n} s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes 1^c \otimes \sigma\right) =$

$$\begin{aligned}
&= d(\varepsilon_{n-1}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes \mu^c_n \otimes \sigma) \\
&+ h\left(\sum_{i_n=1}^{m_n}\sum_{j=1}^{n}\sum \varepsilon_{j-1}(-1)^{|\mu'^c_{j,\,i'_j}|} s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu'^c_{j,\,i'_j} \otimes s^{-1}\mu''^c_{j,\,i''_j} \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes 1^c \otimes \sigma\right) \\
&+ h\left(\sum_{i_n=1}^{m_n}\sum_{j=1}^{n} \varepsilon_{j-1}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes (-s^{-1}d_{\mathcal{C}}(\mu_{j,\,i_j})) \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes 1^c \otimes \sigma\right) \\
&= \varepsilon_{n-1}\sum_{j=1}^{n-1} \varepsilon_{j-1}(-1)^{|\mu'^c_{j,\,i'_j}|} s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu'^c_{j,\,i'_j} \otimes s^{-1}\mu''^c_{j,\,i''_j} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes \mu^c_n \otimes \sigma \\
&+ \varepsilon_{n-1}\sum \varepsilon_{n-1}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes s^{-1}\mu'^c_{n,\,i'_n} \otimes \mu''^c_n \otimes \sigma \\
&+ \varepsilon_{n-1}\varepsilon_{n-1}\sum_{i_n=1}^{m_n} s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes 1^c \otimes \sigma \\
&- \varepsilon_{n-1}\sum_{j=1}^{n-1} \varepsilon_{j-1}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}d_{\mathcal{C}}(\mu_{j,\,i_j}) \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes \mu^c_n \otimes \sigma \\
&+ \varepsilon_{n-1}\varepsilon_{n-1}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes d_{\mathcal{C}}(\mu^c_n) \otimes \sigma \\[1ex]
&- \varepsilon_{n-1}\sum_{j=1}^{n-1} \varepsilon_{j-1}(-1)^{|\mu'^c_{j,\,i'_j}|} s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu'^c_{j,\,i'_j} \otimes s^{-1}\mu''^c_{j,\,i''_j} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes \mu^c_n \otimes \sigma \\
&- \varepsilon_{n-1}(-1)^{|\mu'^c_{n,\,i'_n}|}\sum \varepsilon_{n-1}(-1)^{|\mu'^c_{n,\,i'_n}|}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes s^{-1}\mu'^c_{n,\,i'_n} \otimes \mu''^c_n \otimes \sigma \\
&+ \varepsilon_{n-1}\sum_{j=1}^{n-1} \varepsilon_{j-1}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}d_{\mathcal{C}}(\mu_{j,\,i_j}) \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes \mu^c_n \otimes \sigma \\
&- \varepsilon_{n-1}\varepsilon_{n-1}s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n-1,\,i_{n-1}} \otimes d_{\mathcal{C}}(\mu^c_n) \otimes \sigma \\[1ex]
&= \sum_{i_n=1}^{m_n} s^{-1}\mu^c_{1,\,i_1} \otimes \cdots \otimes s^{-1}\mu^c_{n,\,i_n} \otimes 1^c \otimes \sigma.
\end{aligned}$$

Thus $dh + hd = id$ and h is a homotopy. Finally, $O_{\Omega(\mathcal{C})}$ is acyclic and Theorem 1.5.6.2 gives the theorem since any quasi-free operad $\mathcal{P} = \Omega(\mathcal{C})$ is an operad satisfying the PBW property. Indeed, a free operad has no trivial relations. □

We conjecture that this theorem is true for any cofibrant operad.

When $\mathcal{P}$ is a Koszul operad, the previous theorem applied to $\mathcal{C} = \mathcal{P}^{\text{i}}$ shows that the André-Quillen cohomology of a homotopy algebra is always an Ext-functor over its enveloping algebra.

Let A be a $\mathcal{P}$-algebra. The algebra A is a $\mathcal{P}_\infty$-algebra since there is a map of operads $\mathcal{P}_\infty = \Omega(\mathcal{P}^{\text{i}}) \to \mathcal{P}$. Similarly, an A-module over the operad $\mathcal{P}$ is also an A-module over the operad $\mathcal{P}_\infty$. This leads to the following result.

1.6.3.2 Proposition. *Let $\mathcal{P}$ be a Koszul operad and let A be a $\mathcal{P}$-algebra. The André-Quillen cohomology of the $\mathcal{P}$-algebra A is equal to the André-Quillen cohomology of the $\mathcal{P}_\infty$-algebra A. That is,*

$$\mathrm{H}^\bullet_{\mathcal{P}}(A,\, M) = \mathrm{H}^\bullet_{\mathcal{P}_\infty}(A,\, M), \ \textit{for any } A\textit{-module } M \textit{ over the operad } \mathcal{P}.$$

Proof. A resolution of A as a $\mathcal{P}$-algebra is given by $\mathcal{P} \circ \mathcal{P}^{¡}(A)$ and a resolution of A as a $\mathcal{P}_\infty$-algebra is given by $\mathcal{P}_\infty \circ \mathcal{P}^{¡}(A)$. Thus, by Theorem 1.2.4.2, we have

$$\mathrm{Hom}_{\mathcal{M}_A^{\mathcal{P}_\infty}}(A \otimes^{\mathcal{P}_\infty} \mathcal{P}^{¡}(A),\, M) = \mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(\mathcal{P}^{¡}(A),\, M) = \mathrm{Hom}_{\mathcal{M}_A^{\mathcal{P}}}(A \otimes^{\mathcal{P}} \mathcal{P}^{¡}(A),\, M).$$

Moreover, the differential on $\mathrm{Hom}_{g\mathcal{M}od_{\mathbb{K}}}(\mathcal{P}^{¡}(A),\, M)$ is the same in both cases since the higher products $\mathcal{P}^{¡}(k) \otimes_{\mathbb{S}_k} A^{\otimes k} \to A$ for $k \geq 3$ are 0. □

We showed that, for commutative algebras and Perm algebras, the André-Quillen cohomology of a $\mathcal{P}$-algebra cannot always be written as an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}} \mathbb{K}$. However, by the following theorem, it can always be written as an Ext-functor over the enveloping algebra $A \otimes^{\mathcal{P}_\infty} \mathbb{K}$.

1.6.3.3 Theorem. *Let $\mathcal{P}$ be a Koszul operad, let A be a $\mathcal{P}$-algebra and let M be an A-module over the operad $\mathcal{P}$. We have*

$$\mathrm{H}^\bullet_{\mathcal{P}}(A,\, M) \cong \mathrm{Ext}^\bullet_{A \otimes^{\mathcal{P}_\infty} \mathbb{K}}(\Omega_{\mathcal{P}_\infty}(A),\, M).$$

Proof. We make use of Theorem 1.6.3.1 and Proposition 1.6.3.2. □

Chapitre 2

Curved Koszul duality theory

In [Hoc45], Hochschild introduced a (co)homology theory for associative algebras and in [Sta63], Stasheff introduced the homotopy theory for associative algebras. Nowadays, we know how to describe these theories in operadic terms, but this approach does not encode the units in unital associative algebras. In order to define a homotopy theory and a cohomology theory for unital associative algebras, we refine the operadic theory and more precisely its Koszul duality theory.

In representation theory, an algebra A is "represented" as an algebra of operations with one input and one output on a vector space V via a representation $\mu \in \mathrm{Hom}_{Alg}(A, \mathrm{End}(V))$. To encode operations with several inputs and one output, one uses the notion of an operad [May72, BV73]. More generally, one uses the notion of properads to encode operations with several inputs and several outputs [Val07]. An associative algebra is a special kind of operad and an operad is a special kind of properad, and theories about properads generalize those of operads and associative algebras. For example a bar-cobar adjunction defined in a properadic setting generalizes one defined for operads and algebras. In [Val07], the bar construction (which we will denote by B) assigned a coaugmented dg coproperad to an augmented dg properad and the cobar construction (denoted Ω) assigns an augmented dg properad to a coaugmented dg coproperad, and the two constructions are adjoint. An important property of the adjunction is that the bar-cobar composition $\Omega \mathrm{B}\mathcal{P}$ defines a cofibrant resolution of an augmented dg properad $\mathcal{P}$.

In this paper, we extend the bar-cobar adjunction (Ω, B) to non-augmented properads. We generalize the notion of dg coproperad to involve curvature ($d^2 \neq 0$, but its deviation from 0 is controlled by a term we call "curvature"). Our bar construction assigns a coaugmented curved coproperad to a (not necessarily augmented) dg properad. We then extend the cobar construction of coaugmented coproperads to include coaugmented curved coproperads, resulting in a (not necessarily augmented) dg properad (with no curvature). The composition bar-cobar provides a cofibrant resolution $\Omega \mathrm{B}\mathcal{P}$ of a properad P. For example, we obtain a cofibrant resolution for the properad encoding unital and/or counital Frobenius algebras. Since the datum of a 2-dimensional topological quantum field theory, 2d-TQFT for short, is equivalent to a unital and counital Frobenius algebra structure [Abr96, Koc04], this provides homotopy tools to study 2d-TQFT. With our model, the methods of [Wil07] apply to show that the differential forms $\Omega_{dR}(M)$ on a closed, oriented manifold M bear a unital and counital Frobenius algebra structure up to homotopy.

The bar-cobar resolution $\Omega B\mathcal{P}$ is large and it is often desirable to have a smaller resolution. To this end, we develop a curved Koszul duality theory for properads generalizing the Koszul duality theory for properads [Val07], operads [GJ94, GK94], and associative algebras [Pri70]. One of the main object is the Koszul dual coproperad $\mathcal{P}^{¡}$, which has, here, a curvature. It applies to properads with a quadratic, linear and constant presentation. The properads for which this theory apply are called Koszul properads. In this case, the cobar construction $\Omega\mathcal{P}^{¡}$ is a resolution of $\mathcal{P}$. We summarize the different generalizations of the Koszul duality theory in the following table :

<table>
<tr><th>Relations
Monoids</th><th>Homogeneous quadratic</th><th>Quadratic and linear</th><th>Quadratic, linear and constant</th></tr>
<tr><td>Associative algebras</td><td colspan="2">[Pri70]</td><td>[Pos93, PP05]</td></tr>
<tr><td>Operads</td><td>[GJ94, GK94]</td><td rowspan="2">[GCTV09]</td><td rowspan="2">Section 2.4 of this chapter</td></tr>
<tr><td>Properads</td><td>[Val07]</td></tr>
</table>

The operad $u\mathcal{A}s$ encoding unital associative algebras is an example of an operad with quadratic, linear and constant relations. It is an inhomogeneous Koszul operad in the previous sense. Hence we get a "small" cofibrant resolution $uA_\infty := \Omega u\mathcal{A}s^{¡} \xrightarrow{\sim} u\mathcal{A}s$. This particularly simple resolution of the operad $u\mathcal{A}s$ allows us to define the notion of *homotopy unital associative algebras*. This notion corresponds to the notion of homotopy unit for A_∞-algebra which appears in [FOOO07]. However, our presentation in terms of algebras over a cofibrant operad implies good homotopy properties for these algebras. With this approach, we also obtain functorial resolutions on the level of unital associative algebras. We use these other resolutions to study the cohomology theory of unital associative algebras. After we achieved this work, we were told about the existence of the incoming paper of Lyubashenko [Lyu10] where he provides a cofibrant resolution of the operad $u\mathcal{A}s$. This resolution corresponds to the resolution presented here.

We begin the paper with a survey of the results on homotopy unital associative algebras expressed in an internal language, explained without, for example, the words "operad" or "properad." This section corresponds to the results obtained in the last section of this paper. In Section 2, we recall definitions of associative algebras, operads and properads. In Section 3, we extend the bar and the cobar construction to the non-augmented framework and we define the notion of curved twisting morphims. In Section 4, we extend the Koszul duality theory for homogeneous quadratic properads to properads with quadratic, linear and constant relations. Section 5 is devoted to resolution of non-augmented properads as bimodules over themselves and to functorial resolutions of $\mathcal{P}$-algebras. Section 6 studies the operad encoding unital associative algebras. We describe the homotopy theory and the cohomology theory for this category of algebras.

In this paper, we work over a field $\mathbb{K}$ of characteristic 0.

2.1 Results on unital associative algebras

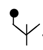
In this section, we develop the homotopy and cohomology theories of unital associative algebras. The definitions, proofs, techniques, and pictorial descriptions of the results are based on operad theory and can be found in Section 2.6. However, this section does not contain the word "operad" and can be read independently from the rest of the paper. The comparison with the work of [FOOO07] is described in Section 2.6.

2.1.1 Unital associative algebra

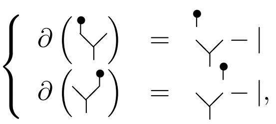
A *unital associative differential graded algebra*, or *unital dga*, is a quadruple $(A,\, \mu,\, e,\, d_A)$, where $(A,\, d_A)$ is a dg module, $\mu : A \otimes A \to A$, and $e : \mathbb{K} \to A$ are dg module maps, such that the map μ is associative and such that the element $e(1_{\mathbb{K}})$ is a left and right unit for the associative product μ.

The version of this structure "up to homotopy" is what we call a uA_∞*-algebra*, for *homotopy unital associative algebra*. Let $f : V \to W$ be a homogeneous $\mathbb{K}$-linear map of degree $|f|$. We denote its derivative by $\partial(f) := d_W \circ f - (-1)^{|f|} f \circ d_V$.

2.1.2 Homotopy unital associative algebra

A *homotopy unital associative algebra* or uA_∞*-algebra structure on a dg module* $(A,\, d_A)$ is given by a collection of maps $\{\mu_n^S : A^{\otimes(n-|S|)} \to A\}$ of degree $n - 2 + |S|$, where the set S runs over the set of subsets of $\{1,\, \dots,\, n\}$ for any integer $n \geq 2$ and where $S = \{1\}$ when $n = 1$. The μ_n^S are given pictorially by planar corollas with n entries labelled by $1, \cdots, n$ on which we put "corks" when the label is in S. For example, we have $\mu_3^{\{1\}} =$ . The maps μ_n^S satisfy the following identities :

– $\mu_2^{\{1\}}$ and $\mu_2^{\{2\}}$ are homotopies for the unit

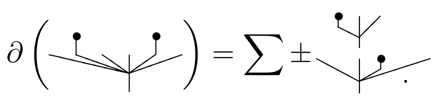


where the empty space between the corollas and the corks is the composition of operations and where $|$ is the identity of A

– for $(n,\, S) \neq (2,\, \{1\})$ and $(n,\, S) \neq (2,\, \{2\})$,

$$\partial(\mu_n^S) = \sum_{\substack{p+q+r=n \\ p+1+r=m}} (-1)^{q(r+|S_1|)+|S_2||S_1'|+p+1} \mu_m^{S_1} \circ (\underbrace{\mathrm{id}, \dots, \mathrm{id}}_{p-|S_1'|}, \mu_q^{S_2}, \underbrace{\mathrm{id}, \dots, \mathrm{id}}_{r-|S_1''|}).$$

Or, pictorially :



Examples.

1. Every unital dga $(A,\, \mu,\, e, d_A)$ naturally equips the dg module (A, d_A) with the structure of a uA_∞-algebra by

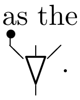
$$\mu_n^S = \begin{cases} \mu & \text{if } n = 2 \text{ and } S = \emptyset \\ e & \text{if } n = 1 \text{ and } S = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

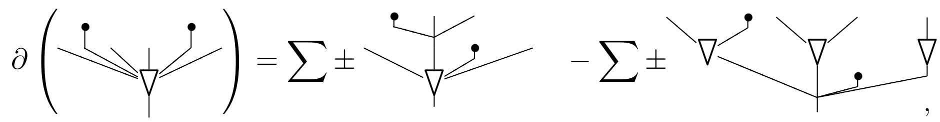
2. A *strictly unital A_∞-algebra*, or $\mathtt{su}A_\infty$-algebra is an A_∞-algebra $(A, d_A, \{\mu_n\}_{n\geq 2})$ with $e \in A$ so that e is a left and right unit for μ_2, and e annihilates μ_n for $n \geq 3$ [KS06]. Every $\mathtt{su}A_\infty$-algebra is naturally a uA_∞-algebra by

$$\mu_n^S = \begin{cases} \mu_n & \text{if } n \geq 2, S = \emptyset \\ e & \text{if } n = 1 \text{ and } S = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

Remark. Every uA_∞-algebra contains an A_∞-algebra if we take $\mu_n := \mu_n^\emptyset$ for all $n \geq 1$. The additional algebraic structure given by a uA_∞-algebra provides homotopies for the "unital relations" along with the homotopies already present for the "associative relations."

2.1.3 Infinity-morphism

We define the notion of infinity-morphism between two uA_∞-algebras A and B by a collection of maps $f_n^S : A^{\otimes(n-|S|)} \to B$ of degree $n - 1 + |S|$, represented graphically by planar trees with "corks" as the uA_∞-algebra structures but with a triangle $\bigtriangledown$ as vertex. For example, we have $f_3^{\{1\}} =$ . The f_n^S satisfy the relations :



where the planar trees with "corks" and no triangle represent the uA_∞-algebra structure of A on the top and the uA_∞-algebra structure of B on the bottom. With this definition of infinity-morphism, we prove a rectification theorem.

2.1.3.1 Theorem (Rectification Theorem, Theorem 2.6.3.2)**.** *Let A be a uA_∞-algebra. We can rectify A : there is a unital associative algebra A' such that A is uA_∞-equivalent to A'.*

Moreover, we have a transfer theorem.

2.1.3.2 Theorem (Homotopy Transfer Theorem, Theorem 2.6.4.5)**.** *Let A be a homotopy unital associative algebra and let V be a chain complex. Given a strong deformation retract*

$$V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h\ ,$$

i.e., p and i are chain maps, where $p \circ i = \mathrm{id}_V$ and $d_A h + h d_A = \mathrm{id}_A - i \circ p$, there is a natural uA_∞-algebra structure on V, and a natural extension of i to an infinity-morphism.

2.1.4 Comparison with the literature

In the literature, there are several definitions of "weakly unital" or "homotopy unital" A_∞-algebras [KS06, Lyu02, Fuk02, FOOO07]. The definitions of [KS06] and [Lyu02], describe *properties* of A_∞-algebras, while the definition presented in [FOOO07] describes a *structure* on an A_∞-algebra. In [LM06] these are compared and shown to be, in some sense, equivalent. Our notion of homotopy unital associative algebra, or uA_∞-algebra, is an A_∞-algebra with additional structure, and in fact coincides with the structure described in [FOOO07].

In [FOOO07], the authors prove (Theorem 5.4.2') that there is a (gapped, filtered) $\mathtt{su}A_\infty$ minimal model for every (gapped, filtered) uA_∞-algebra. We prove the following analogue.

2.1.4.1 Theorem (Corollary 2.6.5.3)**.** *Let A be a uA_∞-algebra. There is an $\mathtt{su}A_\infty$-algebra structure on the homology of A which is equivalent to A.*

We extend this theorem to a broad class of algebraic structures, including Batalin-Vilkovisky algebras and commutative algebras.

2.1.5 André-Quillen cohomology theory for unital associative algebra

Following the ideas of Quillen, we define a cohomology theory associated to any unital associative dga A with coefficients in a A-bimodule M, denoted $\mathrm{H}^\bullet_{u\mathcal{A}s}(A, M)$. We prove that this cohomology theory is an Ext-functor and that it is equal to the Hochschild cohomology theory of the associative algebra A.

2.1.5.1 Theorem (Theorem 2.6.6.5)**.** *Let A be a unital associative dga. We have*

$$\mathrm{H}^\bullet_{u\mathcal{A}s}(A, M) \cong \mathrm{HH}^{\bullet+1}(A, M).$$

2.2 Operads and Properads

In this section, we recall the notion of algebra, operad and properad as successive generalizations. We refer to the book of Loday and Vallette [LV] for a complete and modern exposition about algebras and operads in $\mathsf{dg\ mod}$, to the book of [MSS02] for another presentation and to the thesis of Vallette [Val07] for properads.

2.2.1 Algebras

Let $\mathbb{K}$-mod denote the monoidal category ($\mathbb{K}$-mod, $\otimes_\mathbb{K}$, $\mathbb{K}$) of $\mathbb{K}$-modules. A *unital associative algebra* is a monoid (A, μ, e) in this monoidal category. The product $\mu : A \otimes_\mathbb{K} A \to A$ is associative and $e : \mathbb{K} \to A$ is a *unit* for the product.

As in representation theory, the elements of A are seen as operations with one input and one output. Then we represent the product $a_1 \cdots a_n$ by a vertical bivalent tree whose vertices are indexed by the a_i, see Figure 2.1.

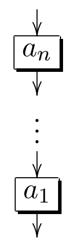


Figure 2.1 – Representation of the product $a_1 \cdots a_n$

2.2.2 Operads

An $\mathbb{S}$-*module* $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ is a collection of $\mathbb{K}$-modules $\mathcal{P}(n)$ endowed with right action of the symmetric group $\mathbb{S}_n$. One defines from [May72] the monoidal product $\circ$ on the category of $\mathbb{S}$-modules by

$$(\mathcal{P} \circ \mathcal{Q})(n) := \bigoplus_{k\geq 0} \left(\mathcal{P}(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1+\cdots+i_k=n} (\mathcal{Q}(i_1) \otimes \cdots \otimes \mathcal{Q}(i_k)) \otimes_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} \mathbb{K}[\mathbb{S}_n] \right) \right),$$

where the notation $\otimes_{\mathbb{S}_k}$ stands for the space of coinvariants under the (diagonal) action of the symmetric group $\mathbb{S}_k$:

$$(p \otimes q_1 \otimes \cdots \otimes q_k \otimes \sigma) \cdot \nu := p \cdot \nu \otimes q_{\nu^{-1}(1)} \otimes \cdots \otimes q_{\nu^{-1}(k)} \otimes \bar{\nu}^{-1} \cdot \sigma$$

for any $p \in \mathcal{P}(k)$, $q_j \in \mathcal{Q}(i_j)$, $\sigma \in \mathbb{S}_n$ and $\nu \in \mathbb{S}_k$ with $\bar{\nu} \in \mathbb{S}_n$ the induced block-wise permutation. This monoidal product encodes the composition of multilinear operations and we represent it by 2-levels trees as shown in Figure 2.2.

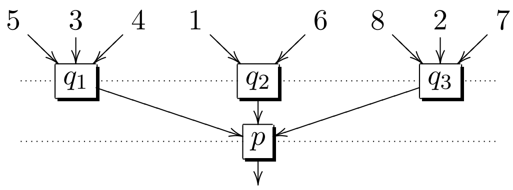


FIGURE 2.2 – An element in $(\mathcal{P} \circ \mathcal{Q})(8)$

The unit for the monoidal product is $I := (0, \mathbb{K}, 0, \ldots)$ where the $\mathbb{K}$ is in arity 1 and represent the identity element modeled by the tree $|$. It forms a monoidal category denoted by $\mathbb{S}$-Mod.

An *operad* is a monoid $(\mathcal{P}, \gamma, e)$ in the monoidal category of $\mathbb{S}$-modules $\mathbb{S}$-Mod. The associative product $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ is called the *composition product* and $e : I \to \mathcal{P}$ is the *unit* for the composition product.

EXAMPLE. A unital associative algebra induces an operad by this injective map

$$\mathsf{Unital\ associative\ algebras} \rightarrowtail \mathsf{Operads}, \quad A \mapsto (0, A, 0, \ldots)$$

2.2.3 Properads

Algebras encode operations with one input and one output. Operads encode operations with several inputs and one output. To encode operations with multiple inputs and outputs, one uses the notion of *properad*.

An $\mathbb{S}$-*bimodule* $\mathcal{P}$ is a collection $\{\mathcal{P}(m, n)\}_{m,n\geq 0}$ of $\mathbb{S}_m$-$\mathbb{S}_n$-bimodules. One recalls from [Val07] a monoidal product using 2-levels graphs as in Figure 2.3.

Let a and b the number of vertices on the first level and on the second level respectively. Let N be the number of internal edges between the two levels. We associate to an a-tuple of integers $\bar{\imath} = (i_1, \ldots, i_a)$ the sum $|\bar{\imath}| := i_1 + \cdots + i_a$. To any pair of a-tuples $\bar{\imath}$ and $\bar{\jmath}$ we denote by $\mathcal{P}(\bar{\jmath}, \bar{\imath})$ the tensor product $\mathcal{P}(j_1, i_1) \otimes \cdots \otimes \mathcal{P}(j_a, i_a)$ and by $\mathbb{S}_{\bar{\imath}}$ the image of $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_a}$ in $\mathbb{S}_{|\bar{\imath}|}$.

Let $\bar{k} = (k_1, \ldots, k_b)$ be a b-tuple and let $\bar{\jmath} = (j_1, \ldots, j_a)$ be an a-tuple such that $|\bar{k}| = |\bar{\jmath}| = N$. A $(\bar{k}, \bar{\jmath})$-*connected permutation* is a permutation σ in $\mathbb{S}_N$ such that the graph of

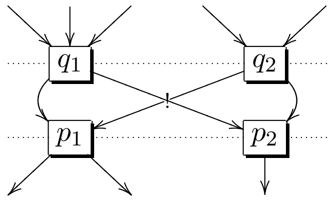


FIGURE 2.3 – An element in $(\mathcal{P} \boxtimes \mathcal{Q})(3,\, 5)$

a geometric representation of σ is connected when one connects the inputs labelled by $j_1 + \cdots + j_i + 1,\, \ldots,\, j_1 + \cdots + j_{i+1}$ for $0 \leq i \leq a-1$ and the outputs labelled by $k_1 + \cdots + k_i + 1,\, \ldots,\, k_1 + \cdots + k_{i+1}$ for $0 \leq i \leq b-1$. We denote by $\mathbb{S}^c_{\bar{k},\bar{\jmath}}$ the set of $(\bar{k},\, \bar{\jmath})$-connected permutations.

We define the monoidal product $\boxtimes$, denoted $\boxtimes_c$ in [Val07], on the category of $\mathbb{S}$-bimodules by

$$(\mathcal{P} \boxtimes \mathcal{Q})(m,\, n) := \bigoplus_{N\in\mathbb{N}} \left(\bigoplus_{\bar{l},\bar{k},\bar{\jmath},\bar{\imath}} \mathbb{K}[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} \mathcal{P}(\bar{l},\bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} \mathbb{K}[\mathbb{S}^c_{\bar{k},\bar{\jmath}}] \otimes_{\mathbb{S}_{\bar{\jmath}}} \mathcal{Q}(\bar{\jmath},\, \bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} \mathbb{K}[\mathbb{S}_n] \right)_{\mathbb{S}_b^{\mathrm{op}}\times\mathbb{S}_a},$$

where the second direct sum runs over the b-tuples $\bar{l}$, $\bar{k}$ and the a-tuples $\bar{\jmath}$, $\bar{\imath}$ such that $|\bar{l}| = m$, $|\bar{k}| = |\bar{\jmath}| = N$, $|\bar{\imath}| = n$ and we consider the module of coinvariants with respect to the $\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a$-action :

$$\rho\otimes p_1\otimes\cdots\otimes p_b\otimes\sigma\otimes q_1\otimes\cdots\otimes q_a\otimes\omega \sim \rho\cdot\tau_{\bar{l}}^{-1}\otimes p_{\tau(1)}\otimes\cdots\otimes p_{\tau(b)}\otimes\tau_{\bar{k}}\cdot\sigma\cdot\nu_{\bar{\jmath}}\otimes q_{\nu^{-1}(1)}\otimes\cdots\otimes q_{\nu^{-1}(a)}\otimes\nu_{\bar{\imath}}^{-1}\cdot\omega,$$

for $\rho \in \mathbb{S}_m$, $\omega \in \mathbb{S}_n$, $\sigma \in \mathbb{S}^c_{\bar{k},\bar{\jmath}}$ and for $\tau \in \mathbb{S}_b$ with $\tau_{\bar{k}}$ the associated block-wise permutation, $\nu \in \mathbb{S}_a$ with $\nu_{\bar{\jmath}}$ the associated block-wise permutation. We write an element in $\mathcal{P} \boxtimes \mathcal{Q}$ like this $\theta(p_1,\, \ldots,\, p_b)\sigma(q_1,\, \ldots,\, q_a)\omega$. The unit I for the monoidal product is given by

$$\begin{cases} I(1,\, 1) & := & \mathbb{K} & \text{and} \\ I(m,\, n) & := & 0 & \text{otherwise.} \end{cases}$$

The category of $\mathbb{S}-$*bimodules* with the operation $\boxtimes$ forms a monoidal category with unit I. We denote this monoidal category by $\mathbb{S}$-biMod.

A *properad* is a monoid $(\mathcal{P},\, \gamma,\, e)$ in the monoidal category $\mathbb{S}$-biMod of $\mathbb{S}$-bimodules. The associative product $\gamma : \mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$ is called the *composition product* and $e : I \to \mathcal{P}$ is the *unit* for the composition product.

EXAMPLE. An operad induces a properad as follows

$$\mathsf{Operads} \rightarrowtail \mathsf{Properads}, \quad \mathcal{P} \mapsto \widetilde{\mathcal{P}}, \text{ where } \begin{cases} \widetilde{\mathcal{P}}(1,\, n) & := & \mathcal{P}(n) & \text{and} \\ \widetilde{\mathcal{P}}(m,\, n) & := & 0 & \text{for } m \neq 1. \end{cases}$$

Finally, we have the following inclusions :

Monoidal category :	$(\mathbb{K}\text{-}\mathsf{Mod},\, \otimes_{\mathbb{K}})$	$\rightarrowtail$	$(\mathbb{S}\text{-}\mathsf{Mod},\, \circ)$	$\rightarrowtail$	$(\mathbb{S}\text{-}\mathsf{biMod},\, \boxtimes)$
Monoid :	Associative algebras	$\rightarrowtail$	Operads	$\rightarrowtail$	Properads.

The results about properads in this paper apply to algebras and operads as well by the above inclusions of categories.

One defines dually the notions of coalgebra, cooperad, coproperad. For example, a *coproperad* is a comonoid $(\mathcal{C},\, \Delta,\, \eta)$ in the monoidal category of $\mathbb{S}$-bimodules $\mathbb{S}$-biMod. The *coproduct* $\Delta : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$ is coassociative and admits a counit $\eta : \mathcal{C} \to I$. All these definitions extend to the *differential graded* setting, or *dg* setting for short. The differentials are compatible with the properad structure, resp. coproperad structure, in the sense that they are *derivations*, resp. *coderivations* (see [LV] or [Val07] for precise definitions). We will often refer to a dg "object" just as "object," for example we call dg properads "properads."

2.3 Curved twisting morphisms

In this section, we recall the notion of twisting morphisms for augmented properads and coproperads from [Val08] and [MV09a] and the associated bar-cobar adjunction. To extend these notions to the case where the properad is not augmented, we introduce the new notion of *curved coproperad* and of *curved twisting morphism* between a curved coproperad and a not necessarily augmented properad. We also extend the bar and the cobar constructions to this framework. This provides a functorial cofibrant replacement for properads. We emphasize the fact that the properad is not assumed to be augmented.

2.3.1 Twisting morphisms

We recall the theory of twisting morphisms between augmented coproperads and augmented properads from [MV09a].

Let M and N be two $\mathbb{S}$-bimodules. By abuse of notation, we will denote by $M \otimes N$ the infinitesimal composite product of one element of M with one element of N grafted above, that is the space of linear combinations of connected graphs with two vertices, the first one labelled by an element of M and the one above labelled by an element of N. This is not quite the same as $\mathcal{Q} \boxtimes_{(1,1)} \mathcal{P}$ of [MV09a], in which they define the product of augmented properads, and only take elements from the augmentation ideal. However, we write sometimes $M \boxtimes_{(1,\,1)} N$ instead of $M \otimes N$. To an operad $\mathcal{P}$, we associate the *infinitesimal composition product* $\gamma_{(1,\,1)} : \mathcal{P} \boxtimes_{(1,\,1)} \mathcal{P} \to \mathcal{P}$ with the help of e and γ. Associated to a coproperad $\mathcal{C}$, we define the *infinitesimal decomposition map* $\Delta_{(1,\,1)} : \mathcal{C} \to \mathcal{C} \boxtimes_{(1,\,1)} \mathcal{C}$ by the projection of Δ (with the help of η) on $\mathcal{C} \boxtimes_{(1,\,1)} \mathcal{C}$, or with the above notation, on $\mathcal{C} \otimes \mathcal{C}$.

We recall the convolution product $\star$ on $\mathrm{Hom}(\mathcal{C},\, \mathcal{P}) := \prod_{m,\, n \geq 0} \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}(m,\, n),\, \mathcal{P}(m,\, n))$ from [MV09a]. Let $f,\, g \in \mathrm{Hom}(\mathcal{C},\, \mathcal{P})$. We denote by $f \star g$ the composite

$$\mathcal{C} \xrightarrow{\Delta_{(1,\,1)}} \mathcal{C} \boxtimes_{(1,\,1)} \mathcal{C} \xrightarrow{f \boxtimes_{(1,\,1)} g} \mathcal{P} \boxtimes_{(1,\,1)} \mathcal{P} \xrightarrow{\gamma_{(1,\,1)}} \mathcal{P}.$$

We define the derivative ∂ of degree -1 on $\mathrm{Hom}(\mathcal{C},\, \mathcal{P})$ by

$$\partial(f) := d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ d_{\mathcal{C}}.$$

The convolution product $\star$ on $\mathrm{Hom}(\mathcal{C},\, \mathcal{P})$ is a Lie-admissible product (see [MV09a] for more details). It is stable on the space of equivariant maps from $\mathcal{C}$ to $\mathcal{P}$ denoted by $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\, \mathcal{P})$. Then the bracket $[f,\, g] := f \star g - (-1)^{|f||g|} g \star f$ is a Lie bracket on $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C},\, \mathcal{P})$.

A morphism of $\mathbb{S}$-bimodules $\alpha : (\mathcal{C}, d_{\mathcal{C}}) \to (\mathcal{P}, d_{\mathcal{P}})$ of degree -1 in the Lie algebra $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is called a *twisting morphism* if it is a solution to the *Maurer-Cartan equation*

$$\partial(\alpha) + \alpha \star \alpha = \partial(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0.$$

We denote by $\mathrm{Tw}(\mathcal{C}, \mathcal{P})$ the set of twisting morphisms in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$.

We say that an operad $\mathcal{P}$ is *augmented* when there is a morphism $\mathcal{P} \twoheadrightarrow I$ of dg properads such that $I \xrightarrow{e} \mathcal{P} \twoheadrightarrow I$ is the identity. It is equivalent to $\mathcal{P} \cong I \oplus \overline{\mathcal{P}}$ as dg properads where $\overline{\mathcal{P}} := \ker(\mathcal{P} \twoheadrightarrow I)$. Dually, we say that a coproperad $\mathcal{C}$ is *coaugmented* when there is a morphism $I \rightarrowtail \mathcal{C}$ of dg coproperads such that $I \rightarrowtail \mathcal{C} \xrightarrow{\eta} I$ is the identity. It is equivalent to $\mathcal{C} \cong I \oplus \overline{\mathcal{C}}$ as dg coproperads where $\overline{\mathcal{C}} := \mathrm{coker}(I \rightarrowtail \mathcal{C})$. When $\mathcal{P}$ is augmented and $\mathcal{C}$ is coaugmented, we require the twisting morphisms α to satisfy the compositions $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \twoheadrightarrow I$ and $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ being equal to 0. A coaugmented coproperad is called *conilpotent* when for all $x \in \overline{\mathcal{C}}$, there exists an $n > 0$ such that $\overline{\Delta}^{n}_{(1,1)}(x) = 0$, where $\overline{\Delta}_{(1,1)} : \overline{\mathcal{C}} \to \overline{\mathcal{C}} \boxtimes_{(1,1)} \overline{\mathcal{C}}$ is the primitive part of $\Delta_{(1,1)}$ and where $\overline{\Delta}^{n}_{(1,1)} = (\overline{\Delta}_{(1,1)} \otimes id_{\mathcal{C}}^{\otimes(n-1)}) \circ \overline{\Delta}^{n-1}_{(1,1)}$ (see [LV] for more details in cooperad case).

When $\mathcal{P}$ is augmented and $\mathcal{C}$ is conilpotent, we recall from [Val07] that the bifunctor $\mathrm{Tw}(-, -)$ is representable on the left by the *cobar construction* and on the right by the *bar construction*, that is we have the following adjunction

$$\Omega : \textsf{conilpotent dg coprop.} \rightleftharpoons \textsf{augmented dg prop.} : \mathrm{B}$$

and there are natural correspondences

$$\mathrm{Hom}_{\textsf{aug. dg prop.}}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{\textsf{conil. dg coprop.}}(\mathcal{C}, \mathrm{B}\mathcal{P}).$$

2.3.2 Curved twisting morphism

We refine the previous section to the case where $\mathcal{P}$ is not necessarily augmented. A *curvature* has to be introduced on the level of dg coproperads to encode the default of augmentation. The associated notion is called a *curved coproperad*. We define the notion of *curved twisting morphism* between a curved coproperad and a dg properad as a solution of the *curved Maurer-Cartan equation*.

Curved coproperad

A *curved coproperad* is a triple $(\mathcal{C}, d_{\mathcal{C}}, \theta)$, where $\mathcal{C}$ is a graded (but not dg) coproperad, the *predifferential* $d_{\mathcal{C}}$ is a coderivation of $\mathcal{C}$ of degree -1 and the *curvature* $\theta : \mathcal{C} \to I$ is a map of degree -2 such that :

a) $d_{\mathcal{C}}^2 = (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}$,

b) $\theta \circ d_{\mathcal{C}} = 0$.

A *morphism between curved coproperads* $(\mathcal{C}, d_{\mathcal{C}}, \theta) \to (\mathcal{C}', d_{\mathcal{C}'}, \theta')$ is a morphism of coproperads $f : \mathcal{C} \to \mathcal{C}'$ such that $d_{\mathcal{C}'} \circ f = f \circ d_{\mathcal{C}}$ and $\theta' \circ f = \theta$. We denote this category by curved coprop..

We prove the following technical lemma that will be useful later.

2.3.2.1 Lemma. *Let $\mathcal{C}$ be a coproperad. The cobracket $(\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}$ with a linear form $\theta : \mathcal{C} \to I$ is a coderivation.*

Proof. The coassociativity of $\Delta_{(1,1)}$ gives
$$\left[\left((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\right) \otimes id_{\mathcal{C}} + id_{\mathcal{C}} \otimes \left((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\right)\right] \circ \Delta_{(1,1)}$$

$$\begin{aligned} &= \left[\left((\theta \otimes id_{\mathcal{C}}) \circ \Delta_{(1,1)}\right) \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \left((id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\right)\right] \circ \Delta_{(1,1)} \\ &= (\theta \otimes \Delta_{(1,1)} - \Delta_{(1,1)} \otimes \theta) \circ \Delta_{(1,1)} = \Delta_{(1,1)} \circ \left((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}\right). \end{aligned}$$

□

The convolution curved Lie algebra

We define the new notion of *curved Lie algebra* generalizing the notion of dg Lie algebra. A *curved Lie algebra* is a quadruple $(\mathfrak{g}, [-,-], d_{\mathfrak{g}}, \theta)$, where $(\mathfrak{g}, [-,-])$ is a Lie algebra, the predifferential $d_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$ of degree -1 and the curvature θ is an element of $\mathfrak{g}$ (or equivalently a map $\mathbb{K} \to \mathfrak{g}$) of degree -2 such that :

a) $d_{\mathfrak{g}}^2 = [-, \theta]$;

b) $d_{\mathfrak{g}}(\theta) = 0$.

Let $(\mathcal{C}, d_{\mathcal{C}}, \theta)$ be a curved coproperad and let $(\mathcal{P}, d_{\mathcal{P}})$ be a dg properad. We fix the element

$$\Theta := e \circ \theta : \mathcal{C} \xrightarrow{\theta} I \xrightarrow{e} \mathcal{P}$$

of degree -2 in $\mathrm{Hom}(\mathcal{C}, \mathcal{P})$.

2.3.2.2 Proposition. *When $\mathcal{C}$ is a curved coproperad and $\mathcal{P}$ is a dg properad, we have on* $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P}) = \prod_{m,n \geq 0} \mathrm{Hom}_{\mathbb{S}}(\mathcal{C}(m,n), \mathcal{P}(m,n))$ *:*

$$\begin{cases} \partial^2 &= [-, \Theta] \;:=\; (\text{-} \star \Theta) - (\Theta \star \text{-}) \\ \partial(\Theta) &= 0. \end{cases}$$

Then $(\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P}), [-,-], \partial, \Theta)$ *is a curved Lie algebra, called the* convolution curved Lie algebra.

Proof. We do the computations :

$$\begin{aligned} \partial^2(f) &= d_{\mathcal{P}} \circ \partial(f) - (-1)^{|\partial(f)|} \partial(f) \circ d_{\mathcal{C}} \\ &= {d_{\mathcal{P}}}^2 \circ f - (-1)^{|f|} d_{\mathcal{P}} \circ f \circ d_{\mathcal{C}} + (-1)^{|f|}(d_{\mathcal{P}} \circ f \circ d_{\mathcal{C}} - (-1)^{|f|} f \circ {d_{\mathcal{C}}}^2) \\ &= -f \circ {d_{\mathcal{C}}}^2 = -f \circ (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)} = f \star \Theta - \Theta \star f \end{aligned}$$

and $\partial(\Theta) = d_{\mathcal{P}} \circ e \circ \theta - (-1)^{|\Theta|} e \circ \theta \circ d_{\mathcal{C}} = 0$ since $d_{\mathcal{P}} \circ e = 0$ and $\theta \circ d_{\mathcal{C}} = 0$. □

An element $\alpha : (\mathcal{C}, d_{\mathcal{C}}, \theta) \to (\mathcal{P}, d_{\mathcal{P}})$ of degree -1 in the curved Lie algebra $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$ is called a *curved twisting morphism* if it is a solution of the *curved Maurer-Cartan equation*

$$\partial(\alpha) + \alpha \star \alpha = \Theta.$$

We denote by $\mathrm{Tw}(\mathcal{C}, \mathcal{P})$ the set of curved twisting morphisms in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$.

Remark. The words "curved" and "curvature" refer to the geometric context. In that setting, the Maurer-Cartan equation applied to a connection provides the curvature form. The flat case corresponds to the curvature equal to zero, that is to the classical case.

2.3.3 Bar and cobar constructions

In this section, we extend the bar construction of augmented dg properads to a *curved bar construction* from dg properads with target in curved coproperads. In the other way round, we extend the cobar construction of coaugmented coproperads to coaugmented curved coproperads. In the algebra case, the cobar construction generalizes the bar construction of curved algebras given in [PP05] and in [Pos93] to properads, though it is not immediate that our constructions are the same, as [PP05, Pos93] do not make use of coalgebras.

Semi-augmented dg properads

A *semi-augmented dg properad*, or *sdg properad* for short, $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ is a dg properad $\mathcal{P}$ whose underlying $\mathbb{S}$-bimodule is endowed with an augmentation of $\mathbb{S}$-bimodules $\varepsilon : \mathcal{P} \twoheadrightarrow I$, not necessarily dg or of properads, called *semi-augmentation.* In other words, ε is a retraction of $\mathbb{S}$-bimodules of the unit $e : I \to \mathcal{P}$ and we have an isomorphism $e + inc : I \oplus \overline{\mathcal{P}} \xrightarrow{\cong} \mathcal{P}$ of $\mathbb{S}$-bimodules, where $\overline{\mathcal{P}} := \ker \varepsilon$ and inc is the inclusion $\overline{\mathcal{P}} \rightarrowtail \mathcal{P}$. We denote $\rho := (e + inc)^{-1|\overline{\mathcal{P}}} : \mathcal{P} \twoheadrightarrow \overline{\mathcal{P}}$. In the following, we do not write the inclusion inc in the formulae. The map $\overline{\gamma} := \rho \circ \gamma : \overline{\mathcal{P}} \boxtimes \overline{\mathcal{P}} \to \overline{\mathcal{P}}$ is not necessarily associative, even though the composition product $\gamma : \mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$ is associative.

Remark. The assumption for $\mathcal{P}$ to have a semi-augmentation ε is not restrictive since we are working over a field $\mathbb{K}$ and since we just need to fix a section of $\mathcal{P}(1,\, 1)$. When $\mathcal{P}(1,\, 1) = I$, we choose the identity map. This is often the case, as it is for the operad encoding unital associative algebras (see Section 2.6).

We define on $\overline{\mathcal{P}}$ the map $d_{\overline{\mathcal{P}}} := \rho \circ d_{\mathcal{P}}$, which is a differential since $d_{\mathcal{P}}$ is a differential and since the differential on I is 0. The differentials satisfy $\rho \circ d_{\mathcal{P}} = d_{\overline{\mathcal{P}}} \circ \rho$. However, we have $d_{\overline{\mathcal{P}}} \neq d_{\mathcal{P}}$ in general.

A *morphism between two sdg properads* $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon) \xrightarrow{f} (\mathcal{P}',\, d_{\mathcal{P}'},\, \varepsilon')$ is a morphism of dg properads $f : (\mathcal{P},\, d_{\mathcal{P}}) \to (\mathcal{P}',\, d_{\mathcal{P}'})$ such that $\varepsilon' \circ f = \varepsilon$. We define $\bar{f} := \rho' \circ f : \overline{\mathcal{P}} \to \overline{\mathcal{P}'}$ and we remark that $d_{\overline{\mathcal{P}'}} \circ \bar{f} = \bar{f} \circ d_{\overline{\mathcal{P}}}$. We denote by $\mathsf{sdg\ prop.}$ the category of semi-augmented dg properads.

Coaugmented and conilpotent curved coproperads

When $\mathcal{C}$ is coaugmented, that is, $\mathcal{C}$ has a coaugmentation $I \rightarrowtail \mathcal{C}$ so that $\mathcal{C} \cong I \oplus \overline{\mathcal{C}}$ as coproperads, we require that any twisting morphism α satisfies the compositions $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ and $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ to be zero. We denote by $\mathsf{coaug.\ curved\ coprop.}$ the category of coaugmented curved coproperads and by $\mathsf{conil.\ curved\ coprop.}$ the category of conilpotent curved coproperads (see Section 2.3.1).

We construct a pair of functors

$$\mathrm{B} : \mathsf{sdg\ prop.} \rightleftharpoons \mathsf{coaug.\ curved\ coprop.} : \Omega.$$

Let M be an $\mathbb{S}$-bimodule. The notation $\mathcal{F}(M)$, resp. $\mathcal{F}^c(M)$, stands for the *free properad* on M, resp. the *cofree coproperad* on M. A derivation on $\mathcal{F}(M)$, resp. a coderivation on $\mathcal{F}^c(M)$, is characterized by its restriction on M, resp. by its image on M. The notation sM, resp. $s^{-1}M$, stands for the *homological suspension*, resp. the *homological desuspension*, of the $\mathbb{S}$-bimodule M. We refer to [Val07] for more details.

Curved bar construction of a sdg properad

The *bar construction of the sdg properad* $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ is given by the conilpotent curved coproperad

$$\mathrm{B}\mathcal{P} := (\mathcal{F}^c(s\overline{\mathcal{P}}),\, d_{bar},\, \theta_{bar}).$$

The predifferential is defined by $d_{bar} := d_1 + d_2$, where d_2 is the unique coderivation of degree -1 which extends the map

$$\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow \mathcal{F}^c(s\overline{\mathcal{P}})^{(2)} \cong s^2(\overline{\mathcal{P}} \boxtimes_{(1,\,1)} \overline{\mathcal{P}}) \xrightarrow{s^{-1}\overline{\gamma}} s\overline{\mathcal{P}}$$

where $\overline{\gamma} := \rho \circ \gamma : \overline{\mathcal{P}} \boxtimes_{(1,\,1)} \overline{\mathcal{P}} \to \overline{\mathcal{P}}$ and d_1 is the unique coderivation of degree -1 which extends the map

$$\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \xrightarrow{id_s \otimes d_{\overline{\mathcal{P}}}} s\overline{\mathcal{P}}.$$

The curvature θ_{bar} is the map of degree -2

$$\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \oplus \mathcal{F}^c(s\overline{\mathcal{P}})^{(2)} \cong s\overline{\mathcal{P}} \oplus s^2(\overline{\mathcal{P}} \boxtimes_{(1,\,1)} \overline{\mathcal{P}}) \xrightarrow{s^{-1}d_{\mathcal{P}}+s^{-2}\gamma} \mathcal{P} \xrightarrow{\varepsilon} I.$$

2.3.3.1 Lemma. *The predifferential and the curvature satisfy*

a) ${d_{bar}}^2 = (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)}$ *;*

b) $\theta_{bar} \circ d_{bar} = 0$.

Proof. First we can restrict the proof of the equality *a*) and *b*) to $\mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 3)}$ since ${d_{bar}}^2 = \frac{1}{2}[d_{bar},\, d_{bar}]$ and $(\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)}$ are coderivations (see Lemma 2.3.2.1) and since θ_{bar} is non zero only on $\mathcal{F}^c(s\overline{\mathcal{P}})^{(2)}$.

The composite

$$\mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 3)} \xrightarrow{{d_{bar}}^{|\mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 2)}} - \left[(\theta_{bar}\otimes id - id\otimes\theta_{bar})\circ\Delta_{(1,\,1)}\right]^{|I\otimes s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}}\otimes I}} \mathcal{F}^c(s\overline{\mathcal{P}})^{(\leq 2)} \oplus$$

$$(I \otimes s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}} \otimes I) \xrightarrow{({d_{bar}}^{|s\overline{\mathcal{P}}} - \theta_{bar}) + \gamma_{|I\otimes s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}}\otimes I}} I \oplus s\overline{\mathcal{P}}$$

equals to $\left({d_{bar}}^2 - (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)} - \theta_{bar} \circ d_{bar}\right)^{|I\oplus s\overline{\mathcal{P}}}$ and to $(d_{\gamma+d_{\mathcal{P}}})^{2|I\oplus s\overline{\mathcal{P}}}$ where $d_{\gamma+d_{\mathcal{P}}}$ is the unique coderivation of degree -1 on $\mathcal{F}^c(s\mathcal{P})$ which extends the map

$$\mathcal{F}^c(s\mathcal{P}) \twoheadrightarrow \overline{\mathcal{F}^c}(s\mathcal{P})^{(\leq 2)} \cong s\mathcal{P} \oplus s^2\mathcal{P} \boxtimes_{(1,\,1)} \mathcal{P} \xrightarrow{id_s\otimes d_{\mathcal{P}} + s^{-1}\gamma} s\mathcal{P}.$$

Moreover, since γ is associative and $d_{\mathcal{P}}$ is a compatible differential, we have ${d_{\gamma+d_{\mathcal{P}}}}^2 = 0$. Thus

$${d_{bar}}^2 - (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)} - \theta_{bar} \circ d_{bar} = 0,$$

that is, due to the degree

$$\begin{cases} {d_{bar}}^2 = (\theta_{bar} \otimes id - id \otimes \theta_{bar}) \circ \Delta_{(1,\,1)} \\ \theta_{bar} \circ d_{bar} = 0. \end{cases}$$

□

2.3.3.2 Lemma. *The bar construction is a functor* B : sdg prop. → conil. curved coprop..

Proof. Let $f : (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon) \to (\mathcal{P}',\, d_{\mathcal{P}'},\, \varepsilon')$ be a morphism of sdg properads. It induces a morphism of dg $\mathbb{S}$-bimodules $\bar{f} : \overline{\mathcal{P}} \to \overline{\mathcal{P}'}$. The map $\mathcal{F}^c(\bar{f}) : \mathcal{F}^c(s\overline{\mathcal{P}}) \to \mathcal{F}^c(s\overline{\mathcal{P}'})$ is a map of coproperads by construction. The morphism $\bar{f}$ commutes with $\overline{\gamma}_{\mathcal{P}}$ and $\overline{\gamma}_{\mathcal{P}'}$, thus $\mathcal{F}^c(\bar{f})$ commutes with the predifferentials. For a similar reason $\theta'_{bar} \circ \mathcal{F}^c(\bar{f}) = \theta_{bar}$. □

Cobar construction of a coaugmented curved coproperad

The *cobar construction of the coaugmented curved coproperad* $(\mathcal{C},\, d_{\mathcal{C}}, \theta)$ is given by the sdg properad

$$\Omega\mathcal{C} := (\mathcal{F}(s^{-1}\overline{\mathcal{C}}),\, d := d_0 + d_1 - d_2,\, \varepsilon).$$

The term d_0 is the unique derivation of degree -1 which extends the map

$$s^{-1}\overline{\mathcal{C}} \xrightarrow{s\theta} I \rightarrowtail \mathcal{F}(s^{-1}\overline{\mathcal{C}}).$$

The term d_1 is the unique derivation of degree -1 which extends the map

$$s^{-1}\overline{\mathcal{C}} \xrightarrow{id_{s^{-1}}\otimes d_{\overline{\mathcal{C}}}} s^{-1}\overline{\mathcal{C}} \rightarrowtail \mathcal{F}(s^{-1}\overline{\mathcal{C}}).$$

The term d_2 is the unique derivation of degree -1 which extends the infinitesimal decomposition map of $\overline{\mathcal{C}}$, up to desuspension :

$$s^{-1}\overline{\mathcal{C}} \xrightarrow{s^{-1}\overline{\Delta}_{(1,1)}} s^{-2}\mathcal{F}^c(\overline{\mathcal{C}})^{(2)} \cong \mathcal{F}(s^{-1}\overline{\mathcal{C}})^{(2)} \rightarrowtail \mathcal{F}(s^{-1}\overline{\mathcal{C}}).$$

The semi-augmentation ε is the natural projection $\mathcal{F}(s^{-1}\overline{\mathcal{C}}) = I \oplus s^{-1}\overline{\mathcal{C}} \oplus \cdots \twoheadrightarrow I$. It is an augmentation of properads but it is not an augmentation of dg properads in general.

2.3.3.3 Lemma. *The derivation d on $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$ satisfies $d^2 = 0$.*

Proof. First of all, if we define the weight on $\mathcal{C}$ by $\mathcal{C}^{(0)} = I$, $\mathcal{C}^{(1)} = \overline{\mathcal{C}}$ and $\mathcal{C}^{(n)} = 0$ when $n \neq 0,\, 1$ and extend it to $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$, we get that the map d_0 is of weight -1, the map d_1 is of weight 0 and the map d_2 is of weight 1. Thus, the term d^2 split in the following way

$$d^2 = \underbrace{{d_0}^2}_{\text{weight}=-2} + \underbrace{d_0d_1 + d_1d_0}_{\text{weight}=-1} + \underbrace{{d_1}^2 - d_0d_2 - d_2d_0}_{\text{weight}=0} - \underbrace{(d_1d_2 + d_2d_1)}_{\text{weight}=1} + \underbrace{{d_2}^2}_{\text{weight}=2}.$$

So, we have to show that each group of terms is equal to zero. The term d_0^2 is zero because $\mathrm{im}(d_0) \subset I$, and any derivation annihilates I. The sum $d_0d_1 + d_1d_0$ is zero since $\theta \circ d_{\mathcal{C}} = 0$ and $d_{\mathcal{C}}$ is zero on I and by the Koszul sign rule. The equality $d_{\mathcal{C}}^2 = (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}$ and the Koszul sign rule give ${d_1}^2 - d_0d_2 - d_2d_0 = 0$. The equality $d_1d_2 + d_2d_1 = 0$ is due to the fact that $d_{\mathcal{C}}$ is a coderivation. Finally ${d_2}^2 = 0$ by "coassociativity" of $\overline{\Delta}_{(1,1)}$ and by the Koszul sign rule. □

2.3.3.4 Lemma. *The cobar construction is a functor* Ω : coaug. curved coprop. $\to$ sdg prop..

Proof. Let $f : (\mathcal{C},\, d_{\mathcal{C}},\, \theta) \to (\mathcal{C}',\, d_{\mathcal{C}'},\, \theta')$ be a morphism between coaugmented curved coproperads. The map $\mathcal{F}(f) : \mathcal{F}(s^{-1}\overline{\mathcal{C}}) \to \mathcal{F}(s^{-1}\overline{\mathcal{C}'})$ is a map of properads by construction and $d'_2 \circ \mathcal{F}(f) = \mathcal{F}(f) \circ d_2$ since f is a morphism of coproperads. The equality $d_{\mathcal{C}'} \circ f = f \circ d_{\mathcal{C}}$ implies $d'_1 \circ \mathcal{F}(f) = \mathcal{F}(f) \circ d_1$, the equality $\theta' \circ f = \theta$ implies $d'_0 \circ \mathcal{F}(f) = \mathcal{F}(f) \circ d_0$ and then $\mathcal{F}(f)$ commutes with the differential. □

2.3.4 Bar-cobar adjunction

The cobar construction on conilpotent curved coproperads and the bar construction on dg properads represent the bifunctor of curved twisting morphisms and form a pair of adjoint functors. The counit of adjunction provides a cofibrant replacement functor for dg properads.

2.3.4.1 Theorem. *For any conilpotent curved coproperad $\mathcal{C}$ and for any sdg properad $\mathcal{P}$, there is are natural correspondences*

$$\mathrm{Hom}_{\mathsf{sdg\ prop.}}(\Omega\mathcal{C},\, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C},\, \mathcal{P}) \cong \mathrm{Hom}_{\mathsf{coaug.\ curved\ coprop.}}(\mathcal{C},\, \mathrm{B}\mathcal{P}).$$

Proof. We make the first bijection explicit. A morphism of sdg properads $f_\alpha : \mathcal{F}(s^{-1}\overline{\mathcal{C}}) \to \mathcal{P}$ is uniquely determined by a map $s\alpha : s^{-1}\overline{\mathcal{C}} \to \mathcal{P}$ of degree 0 such that $s^{-1}\overline{\mathcal{C}} \xrightarrow{s\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ is 0, or equivalently, by a map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree -1 satisfying $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ and $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ are zero (condition for twisting morphisms when $\mathcal{C}$ is coaugmented, see 2.3.3).

Moreover, f_α commutes with the differentials if and only if the following diagram commutes

$$\begin{array}{ccccc} s^{-1}\overline{\mathcal{C}} & \xrightarrow{s\alpha} & \mathcal{P} & \xrightarrow{d_{\mathcal{P}}} & \mathcal{P} \\ {\scriptstyle d_0+d_1-d_2}\downarrow & & & & \uparrow{\scriptstyle \widetilde{\gamma}} \\ \mathcal{F}(s^{-1}\overline{\mathcal{C}}) & & \xrightarrow[\mathcal{F}(s\alpha)]{} & & \mathcal{F}(\mathcal{P}), \end{array}$$

where $\widetilde{\gamma}$ is induced by γ. We have

$$\begin{aligned} & d_{\mathcal{P}} \circ (s\alpha) = s(d_{\mathcal{P}} \circ \alpha) \\ & \widetilde{\gamma} \circ \mathcal{F}(s\alpha) \circ d_0 = e \circ (s\theta) = s(e \circ \theta) \\ & \widetilde{\gamma} \circ \mathcal{F}(s\alpha) \circ d_1 = s\alpha \circ (id_{s^{-1}} \otimes d_{\mathcal{C}}) = -s(\alpha \circ d_{\mathcal{C}}) \\ & \widetilde{\gamma} \circ \mathcal{F}(s\alpha) \circ d_2 = \gamma \circ (s\alpha \boxtimes_{(1,1)} s\alpha) \circ s^{-1}\Delta_{(1,1)} = s(\gamma \circ (\alpha \boxtimes_{(1,1)} \alpha) \circ \Delta_{(1,1)}). \end{aligned}$$

Thus the commutativity of the previous diagram is equivalent to the equality

$$e \circ \theta - \alpha \circ d - \gamma \circ (\alpha \boxtimes_{(1,1)} \alpha) \circ \Delta_{(1,1)} = d_{\mathcal{P}} \circ \alpha,$$

that is $\partial(\alpha) + \alpha \star \alpha = \Theta$.

We now make the second bijection explicit. A morphism of coaugmented coproperads $g_\alpha : \mathcal{C} \to \mathcal{F}^c(s\overline{\mathcal{P}})$ is uniquely determined by a map $s\alpha : \mathcal{C} \to s\overline{\mathcal{P}}$ which sends I to 0, that is by a map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree -1 satisfying $I \rightarrowtail \mathcal{C} \xrightarrow{\alpha} \mathcal{P}$ and $\mathcal{C} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\varepsilon} I$ are zero.

Moreover, g_α commutes with the predifferential and with the curvature if and only if the following diagrams commute

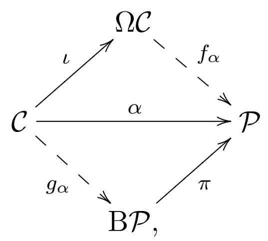
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{s\alpha + (s\alpha \otimes s\alpha)\circ\Delta_{(1,1)}} & s\overline{\mathcal{P}} \oplus s\overline{\mathcal{P}} \boxtimes_{(1,1)} s\overline{\mathcal{P}} \\ {\scriptstyle d_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle d_{bar}=d_1+d_2} \\ \mathcal{C} & \xrightarrow[s\alpha]{} & s\overline{\mathcal{P}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{g_\alpha} & \mathrm{B}\mathcal{P} \\ {\scriptstyle \theta}\downarrow & \swarrow{\scriptstyle \theta_{bar}} & \\ I. & & \end{array}$$

Since $\alpha \star \alpha = -(s^{-1}inc) \circ d_2 \circ (s\alpha \otimes s\alpha) \circ \Delta_{(1,1)} + e \circ \theta_{bar} \circ g_\alpha$, the commutativity of the diagrams gives $\partial(\alpha) + \alpha \star \alpha = \Theta$. Moreover, the projections of the curved Maurer-Cartan equation on $\overline{\mathcal{P}}$ and on I give the two commutative diagrams. This concludes the proof. $\square$

Examples.

- To the identity morphism $id_{\mathrm{B}\mathcal{P}} : \mathrm{B}\mathcal{P} \to \mathrm{B}\mathcal{P}$ of coaugmented curved coproperads corresponds the curved twisting morphism $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ defined by $\mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \cong \overline{\mathcal{P}} \rightarrowtail \mathcal{P}$.
- To the identity morphism $id_{\Omega\mathcal{C}} : \Omega\mathcal{C} \to \Omega\mathcal{C}$ of properads corresponds the curved twisting morphism $\iota : \mathcal{C} \to \Omega\mathcal{C}$ defined by $\mathcal{C} \to \overline{\mathcal{C}} \cong s^{-1}\overline{\mathcal{C}} \rightarrowtail \mathcal{F}(s\overline{\mathcal{C}})$.

2.3.4.2 Lemma. *For any conilpotent curved coproperad $\mathcal{C}$ and for any sdg properad $\mathcal{P}$, every curved twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factors through the universal curved twisting morphisms π and ι :*



where f_α is a morphism of sdg properads and g_α is a morphism of conilpotent curved coproperads.

Proof. The dashed arrows are just the images of α by the two bijections of Proposition 2.3.4.1. □

Weight filtration

We say that a dg $\mathbb{S}$-bimodule M is *weight filtered differential graded*, or *wfdg* for short, when it is endowed with a filtration of dg $\mathbb{S}$-bimodules $F_\omega M$, $\omega \in \mathbb{N}$. When M is a (co)properad, we assume that the (co)product preserves the filtration. In the weight filtered setting, we only consider those twisting morphisms that preserve the filtration. A wfdg properad $\mathcal{P}$ is called *connected* when $F_0\mathcal{P} = I$ $(= Im(e))$.

We endow any free properad $\mathcal{F}(V)$ with a weight grading given by the number of generators. This induces a weight filtration on any properad $\mathcal{F}(V)/(R)$ defined by generators and relations. Sub-coproperads of $\mathcal{F}^c(V)$ are also weight filtered by the number of generators. When $\mathcal{P}$ is a wfdg properad, $\mathrm{B}\mathcal{P}$ comes equipped with a weight filtration. An element in $\mathrm{B}\mathcal{P}$ is a connected graph whose vertices are labelled by elements μ_i of $\overline{\mathcal{P}}$. It is in the component of weight ω of $\mathrm{B}\mathcal{P}$ if there exist ω_i such that any μ_i is in the component of weight ω_i of $\overline{\mathcal{P}}$ and $\sum \omega_i \leq \omega$. Similarly, we endow $\Omega\mathcal{C}$ with a weight filtering when $\mathcal{C}$ is weight filtered.

The curved twisting morphism π preserves the weight filtration.

2.3.4.3 Theorem. *Let $(\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ be a connected wfdg semi-augmented properad. The counit of the bar-cobar adjunction is a quasi-isomorphism of wfdg semi-augmented properads, that is the* bar-cobar construction $\Omega\mathrm{B}\mathcal{P}$ *is a resolution of $\mathcal{P}$*

$$\Omega\mathrm{B}\mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

When $\mathcal{P}$ is concentrated in non-negative degree, the bar-cobar construction is a cofibrant properad for the model category defined in Appendix A of [MV09b].

Proof. We work in the model category defined in Appendix A of [MV09b]. Since $\Omega\mathrm{B}\mathcal{P}$ is quasi-free, the remark after Corollary 40 of [MV09b] gives that $\Omega\mathrm{B}\mathcal{P}$ is cofibrant when we assume that $\mathcal{P}$ is non-negatively homologically graded.

As explained in the previous section, $\Omega\mathrm{B}\mathcal{P} = (\mathcal{F}(s^{-1}\overline{\mathcal{F}}(s\overline{\mathcal{P}})),\, d = d_0 + d_1 - d_2)$ is weight filtered by F_p when $\mathcal{P}$ is weight filtered. We have

$$d_0 : F_p \to F_{p-1} \text{ and } d_1 : F_p \to F_p \text{ and } d_2 : F_p \to F_p,$$

where d_0 is induced by θ_{bar}, d_1 is induced by d_{bar} and d_2 is induced by the coproduct on $\mathcal{F}^c(s\overline{\mathcal{P}})$. So F_p is a filtration of chain complexes, it is exhaustive and bounded below and we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

$$E^{\bullet}_{p,q} \Rightarrow \mathrm{H}_{p+q}(\Omega\mathrm{B}\mathcal{P}).$$

We endow $\mathcal{P}$ with a filtration F'_p induced by the weight. This is a filtration of chain complexes since $d_{\mathcal{P}}$ preserves the weight filtration. The filtration F'_p is exhaustive and bounded below so we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

$$E'^{\bullet}_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P}).$$

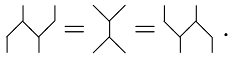
The counit of the bar-cobar adjunction preserves the filtration and induces a map of spectral sequences $E^{\bullet}_{p,q} \to E'^{\bullet}_{p,q}$. Moreover, $E^{0}_{\bullet,\bullet} = \Omega\mathrm{B}(gr\mathcal{P})$. The graded properad $gr\mathcal{P}$ associated to the filtration F'_p on $\mathcal{P}$ is always augmented and connected (in the sense of [Val07], that is $gr\mathcal{P}$ is weight graded and $gr\mathcal{P}^{(0)} = I$). However, it is not *reduced*, that is $\mathcal{P}(0,\, n)$ and $\mathcal{P}(m,\, 0)$ can be non zero. Theorem 5.8 of [Val07] applies to reduced properads for which the author provides a canonical writing of an element in $\mathcal{P} \boxtimes \mathcal{P}$ in order to define a contracting homotopy. Such a canonical writing is not possible for non reduced properads. However, it is possible to define a contracting homotopy by means of a sum over all the possibilities. This works for non reduced properads (over a field of characteristic 0) and Theorem 5.8 of [Val07] extends to non reduced properads. So we get that $E^{1}_{p,q} = gr^{(p)}(\mathrm{H}_{p+q}(gr\mathcal{P}))$. Thus the counit of the bar-cobar adjunction induces an isomorphism of spectral sequences $E^{\bullet}_{p,q} \to E'^{\bullet}_{p,q}$ when $\bullet \geq 1$. Since $E'^{\bullet}_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P})$, the same is true for $E^{\bullet}_{p,q}$ and the morphism $\Omega\mathrm{B}\mathcal{P} \xrightarrow{\sim} \mathcal{P}$ is a quasi-isomorphism. □

Remarks.

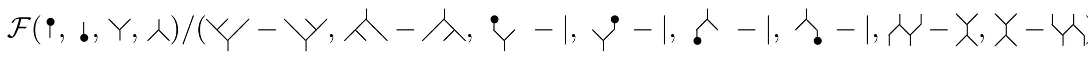
1. In [Pos09], Positselski defined a bar construction and a cobar construction between curved dg algebras and curved dg coalgebras. The curvatures on both sides encode the default of augmentation or of coaugmentation. In this paper, we are interested only in the default of augmentation and the picture becomes asymmetric. When we reduce our bar construction and our cobar construction to semi-augmented algebras and curved coalgebras, we recover the particular case of [Pos09] where the curved coalgebras are coaugmented.
2. In [Nic08], Nicolàs proved a similar bar-cobar adjunction on the level of algebras and coalgebras. But the picture is dual. The bar construction goes from *curved associative algebras* to *conilpotent graded-augmented coalgebras* (see [Nic08] for the precise definitions) and the cobar construction goes the other way around. In his case, the curvature does not control the default of augmentation with respect to the composition product and with respect to the dg setting, but only with respect to the dg setting. In the spirit of [Nic08], we should say the dual statement : the default of augmentation with respect to the dg setting measures the curvature.

Homotopy Frobenius algebras

A *unital and counital Frobenius algebra* is a quintuple $(A, \mu, \Delta, e, \eta)$ where A is a vector space, $\mu : A \otimes A \to A$ is a commutative and associative product, $\Delta : A \to A \otimes A$ is a cocommutative and coassociative coproduct, $e : \mathbb{K} \to A$ is a unit for the product and $\eta : A \to \mathbb{K}$ is a counit for the coproduct such that the product $\mu = \curlyvee$ and the coproduct $\Delta = \curlywedge$ satisfy the *Frobenius relation*



In operadic terms, we get that A is an algebra over the properad *ucFrob* :=



This properad is not augmented but Theorem 2.3.4.3 applies and we get as a corollary :

2.3.4.4 Theorem. *The bar-cobar resolution on ucFrob is a cofibrant resolution of the properad ucFrob, that is*

$$\Omega \mathrm{B}\, uc\mathcal{F}rob \xrightarrow{\sim} uc\mathcal{F}rob.$$

We define a *ucFrob-algebra up to homotopy* as an algebra over this resolution. As proved in [Abr96, Koc04], the datum of a *2-dimensional topological quantum field theory*, *2d-TQFT* for short is equivalent to a unital and counital Frobenius algebra structure. Therefore, we should be able to use this to study 2d-TQFT with homotopy methods.

There is an interesting application in differential geometry. With the present resolution of *ucFrob* and with the methods of [Wil07], one endows the differential forms $\Omega(M)$ on a closed, oriented manifold M with a structure of *ucFrob*-algebra up to homotopy, which extends the commutative algebra structure on the $\Omega(M)$ and which induces the *ucFrob*-algebra structure on the cohomology $\mathrm{H}^\bullet(M)$.

2.4 Curved Koszul duality theory

We extend the Koszul duality theory for homogeneous quadratic properads [Val07] and quadratic-linear properads [GCTV09] to *inhomogeneous quadratic properads* with a quadratic, linear and constant presentation. When the properad is inhomogeneous quadratic, it is not necessarily augmented. Therefore we introduce a Koszul dual coproperad endowed with a curvature, which measures this failure. As explained in Section 2.2, an associative algebra is a particular kind of properad. Hence this section applies to associative algebras as well to recover the construction given by [Pos93] and [PP05]. However, the presentation given here is slightly different and more general : it works without any finiteness assumption. We end the section with a Poincaré-Birkhoff-Witt theorem for properads.

2.4.1 Inhomogeneous quadratic properad

An *inhomogeneous quadratic properad* is a properad $\mathcal{P}$ which admits a presentation of the form $\mathcal{P} = \mathcal{F}(V)/(R)$, where V is a degree graded $\mathbb{S}$-bimodule and (R) is the ideal generated by a degree graded $\mathbb{S}$-bimodule $R \subset I \oplus V \oplus \mathcal{F}(V)^{(2)}$. The superscript (2) indicates the weight degree. We require that R is a direct sum of (homological) degree homogeneous subspaces. Thus the properad $\mathcal{P}$ is degree graded and has a weight filtration induced by the $\mathbb{S}$-bimodule of generators V. We assume further that the following conditions hold :

(I) The space of generators is minimal, that is $R \cap \{I \oplus V\} = \{0\}$.

(II) The space of relations is maximal, that is $(R) \cap \{I \oplus V \oplus \mathcal{F}(V)^{(2)}\} = R$.

Let $\mathrm{q} : \mathcal{F}(V) \twoheadrightarrow \mathcal{F}(V)^{(2)}$ be the canonical projection and let $\mathrm{q}R \subset \mathcal{F}(V)^{(2)}$ be the image under q of R. We consider the quadratic properad $\mathrm{q}\mathcal{P} := \mathcal{F}(V)/(\mathrm{q}R)$. Since $R \cap \{I \oplus V\} = \{0\}$, there exists a map $\varphi : \mathrm{q}R \to I \oplus V$ such that R is the graph of φ :

$$\begin{aligned} R &= \{X - \varphi(X),\, X \in \mathrm{q}R\} \\ &= \{X - \varphi_1(X) + \varphi_0(X),\, X \in \mathrm{q}R,\, \varphi_1(X) \in V,\, \varphi_0(X) \in \mathbb{K}\}. \end{aligned}$$

The weight grading on the free properad $\mathcal{F}(V)$ induces the following filtration on $\mathcal{P}$

$$F_p := \pi\left(\oplus_{\omega \leq p} \mathcal{F}(V)^{(\omega)}\right),$$

where π stands for the canonical projection $\mathcal{F}(V) \twoheadrightarrow \mathcal{P}$. We denote the associated graded properad by $gr(\mathcal{P})$. The relations $\mathrm{q}R$ hold in $gr(\mathcal{P})$. Therefore, there exists an epimorphism of graded properads

$$p : \mathrm{q}\mathcal{P} \twoheadrightarrow gr(\mathcal{P}).$$

We assume throughout that every inhomogeneous quadratic properad is semi-augmented in the sense of Section 2.3.3. We recall that sV stands for the homological suspension of V, and that the *Koszul dual coproperad of the homogeneous quadratic properad* $\mathrm{q}\mathcal{P}$ is the coproperad cogenerated by sV with corelations in $s^2\mathrm{q}R$ (see Section 2.2 of [Val08]) denoted :

$$\mathrm{q}\mathcal{P}^{\text{¡}} := \mathcal{C}(sV,\, s^2\mathrm{q}R) = I \oplus sV \oplus s^2\mathrm{q}R \oplus \cdots .$$

It is a subcoproperad of the cofree coproperad $\mathcal{F}^c(sV)$ on sV. In the cofree coproperad $\mathcal{F}^c(V)$, the weight of an element corresponds to the number of generating elements from V used to write it. There exists a unique coderivation $\tilde{d} : q\mathcal{P}^{\text{¡}} \to \mathcal{F}^c(sV)$ of degree -1 (see Section 3.2 in [MV09a]) which extends the map

$$q\mathcal{P}^{\text{¡}} \twoheadrightarrow s^2\mathrm{q}R \xrightarrow{s^{-1}\varphi_1} sV.$$

Moreover, we denote by $\theta : q\mathcal{P}^{\text{¡}} \to I$ the map of degree -2

$$q\mathcal{P}^{\text{¡}} \twoheadrightarrow s^2\mathrm{q}R \xrightarrow{s^{-2}\varphi_0} I.$$

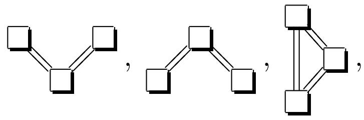
2.4.1.1 Lemma. *Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be an inhomogeneous quadratic properad. Condition (II) implies that :*

- *The coderivation $\tilde{d}$ on $\mathcal{F}^c(sV)$ restricts to a coderivation $d_{\mathcal{P}^i}$ of degree -1 on the subcoproperad $q\mathcal{P}^i = \mathcal{C}(sV,\, s^2qR)$;*
- *The coderivation $d_{\mathcal{P}^i}$ satisfies ${d_{\mathcal{P}^i}}^2 = (\theta \otimes id_{q\mathcal{P}^i} - id_{q\mathcal{P}^i} \otimes \theta) \circ \Delta_{(1,1)}$;*
- *The coderivation $d_{\mathcal{P}^i}$ satisfies $\theta \circ d_{\mathcal{P}^i} = 0$.*

Proof. We define the map

$$\begin{aligned} \psi :\ \mathrm{q}R \otimes V \oplus V \otimes \mathrm{q}R &\to \mathcal{F}(V)^{(\leq 3)} \\ r \otimes v + v' \otimes r' &\mapsto (r + \varphi_1(r) - \varphi_0(r)) \otimes v + v' \otimes (r + \varphi_1(r) - \varphi_0(r)). \end{aligned}$$

Since any kind of tree in $\mathcal{F}(V)^{(3)}$ has one of the forms



an element in $q\mathcal{P}^{\text{i}(3)}$ has two decompositions by $\overline{\Delta}_{(1,1)}$ in $s^2 qR \otimes sV \oplus sV \otimes s^2 qR \cong s^3(qR \otimes V \oplus V \otimes qR)$. Moreover, the two decompositions give the same image with an opposite sign (Koszul sign rule) under ψ. Therefore $\psi \circ (s^{-3}\Delta_{(1,1)})(q\mathcal{P}^{\text{i}(3)}) \subset \{R \otimes V \oplus V \otimes R\} \cap \{I \oplus V \oplus V^{\otimes 2}\}$.

Condition (II) implies in particular

$$\{R \otimes V + V \otimes R\} \cap \{I \oplus V \oplus V^{\otimes 2}\} \subset R.$$

Projecting on each direct summand, we can rewrite this inclusion as the system of equations

1. $(s^{-1}\varphi_1 \otimes id_{sV} + id_{sV} \otimes s^{-1}\varphi_1) \circ \Delta_{(1,1)}(q\mathcal{P}^{\text{i}(3)}) \subset q\mathcal{P}^{\text{i}(2)}$ (projection on $V^{\otimes 2}$) ;
2. $\big(s^{-1}\varphi_1 \circ (s^{-1}\varphi_1 \otimes id_{sV} + id_{sV} \otimes s^{-1}\varphi_1) - (s^{-2}\varphi_0 \otimes id_{sV} - id_{sV} \otimes s^{-2}\varphi_0)\big) \circ \Delta_{(1,1)_{|q\mathcal{P}^{\text{i}(3)}}} = 0$ (projection on V) ;
3. $s^{-2}\varphi_0 \circ (s^{-1}\varphi_1 \otimes id_{sV} + id_{sV} \otimes s^{-1}\varphi_1) \circ \Delta_{(1,1)_{|q\mathcal{P}^{\text{i}(3)}}} = 0$ (projection on I).

By the universal property which defines $q\mathcal{P}^{\text{i}} = \mathcal{C}(sV, s^2 qR)$, it is enough to check that $\tilde{d}(q\mathcal{P}^{\text{i}(3)}) \subset q\mathcal{P}^{\text{i}(2)}$ to restrict $\tilde{d}$ to a coderivation of degree -1 on $q\mathcal{P}^{\text{i}}$, this is exactly the meaning of equation (1). The equation (2) corresponds to the second point of the lemma restricted to $q\mathcal{P}^{\text{i}(3)}$. The equality extends to $q\mathcal{P}^{\text{i}}$ since ${d_{\mathcal{P}^{\text{i}}}}^2 = \frac{1}{2}[d_{\mathcal{P}^{\text{i}}}, d_{\mathcal{P}^{\text{i}}}]$ and $(\theta \otimes id_{q\mathcal{P}^{\text{i}}} - id_{q\mathcal{P}^{\text{i}}} \otimes \theta) \circ \Delta_{(1,1)}$ are coderivations (see Lemma 2.3.2.1). The equation (3) corresponds to the third point of the lemma since θ is zero outside of $q\mathcal{P}^{\text{i}(2)}$. □

2.4.2 Koszul dual coproperad

Let $\mathcal{P}$ be an inhomogeneous quadratic properad with a quadratic, linear and constant presentation $\mathcal{P} = \mathcal{F}(V)/(R)$ (such that Conditions (I) and (II) hold). The *Koszul dual coproperad* of $\mathcal{P}$ is the weight graded curved coproperad

$$\mathcal{P}^{\text{i}} := (q\mathcal{P}^{\text{i}}, d_{\mathcal{P}^{\text{i}}}, \theta).$$

2.4.3 Koszul properad

A properad is called a *Koszul properad* if it admits an inhomogeneous quadratic presentation $\mathcal{P} = \mathcal{F}(V)/(R)$ such that Conditions (I) and (II) hold and such that its associated quadratic properad $q\mathcal{P} := \mathcal{F}(V)/(qR)$ is Koszul in the classical sense.

Since the underlying $\mathbb{S}$-bimodule of $\mathcal{P}^{\text{i}}$ is $I \oplus sV \oplus s^2 qR \oplus \cdots$, we define the map of coproperads $g_\kappa : \mathcal{P}^{\text{i}} \rightarrowtail \mathcal{F}^c(sV) \rightarrowtail \mathrm{B}\mathcal{P}$. This map commutes with the predifferentials and with the curvatures, hence it is a morphism of curved coproperads. So by Lemma 2.3.4.2, there is a curved twisting morphism $\kappa : \mathcal{P}^{\text{i}} \rightarrowtail \mathrm{B}\mathcal{P} \xrightarrow{\pi} \mathcal{P}$. It is explicitly equal to $\mathcal{P}^{\text{i}} \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail \mathcal{P}$. By Theorem 2.3.4.1, we also obtain a map of dg properads $\Omega\mathcal{P}^{\text{i}} \twoheadrightarrow \Omega\mathrm{B}\mathcal{P} \to \mathcal{P}$.

2.4.3.1 Theorem. *Let $\mathcal{P}$ be a Koszul properad. The cobar construction on the Koszul dual curved coproperad $\mathcal{P}^{i}$ is a cofibrant resolution of $\mathcal{P}$:*

$$\Omega\mathcal{P}^{i} \xrightarrow{\sim} \mathcal{P}.$$

Proof. We work in the model category defined in the Appendix A of [MV09b]. Since we are working in the non-negatively graded case and $\Omega\mathcal{P}^{¡}$ is quasi-free, the remark after Corollary 40 gives that $\Omega\mathcal{P}^{¡}$ is cofibrant.

Let $\mathcal{C} := s^{-1}\overline{q\mathcal{P}^{¡}}$ be the desuspension of the augmentation coideal of the coproperad $q\mathcal{P}^{¡}$. So, the underlying $\mathbb{S}$-bimodule of $\Omega\mathcal{P}^{¡}$ is $\mathcal{F}(\mathcal{C})$. Let us consider the new "homological" degree induced by the weight of elements of $q\mathcal{P}^{¡}$, given by the weight in $\mathcal{F}^c(V)$, minus 1. As in the proof of the Appendix A of [GCTV09], Theorem 30, we call this grading the *syzygy degree*. Therefore, the syzygy degree of an element in $\mathcal{F}(\mathcal{C})$ is given by the sum of the weight of the elements which label its vertices minus the numbers of vertices. Since the weight of an element in $\mathcal{C}$ is greater than 1, the syzygy degree on $\mathcal{F}(\mathcal{C})$ is non-negative.
The term d_0, induced by θ, the term d_1, induced by $d_{\mathcal{P}^{¡}}$ and the term d_2, induced by the infinitesimal decomposition map on $\mathcal{C}$, lower the syzygy degree by 1. Hence, we get a well-defined non-negatively graded chain complex.
We endow $\Omega\mathcal{P}^{¡} = \mathcal{F}(\mathcal{C})$ with a filtration given by the total weight, that is the weight of an element in $\mathcal{F}(\mathcal{C})$ is the sum of the weight of the elements which label the vertices. We have

$$d_0 : F_p \to F_{p-2} \text{ and } d_1 : F_p \to F_{p-1} \text{ and } d_2 : F_p \to F_p.$$

This filtration is exhaustive and bounded below so we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain that

$$E^{\bullet}_{p,\,q} \Rightarrow \mathrm{H}_{p+q}(\Omega\mathcal{P}^{¡}).$$

The filtration F_p induces a filtration F_p on the homology of $\Omega\mathcal{P}^{¡}$ such that

$$E^{\infty}_{p,\,q} \cong F_p(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{¡}))/F_{p-1}(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{¡})) =: gr^{(p)}(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{¡})).$$

Moreover, we have $E^0_{p,\,q} = F_p(\mathcal{F}(\mathcal{C})_{p+q})/F_{p-1}(\mathcal{F}(\mathcal{C})_{p+q}) = \mathcal{F}(\mathcal{C})^{(p)}_{p+q}$, that is the elements of syzygy degree equal to $p+q$ and of weight p. The differential d^0 on the first term of the spectral sequence is given by d_2. Hence, since $\mathrm{q}\mathcal{P}$ is Koszul and concentrated in syzygy degree 0, we have $E^1_{p,\,q} = \mathrm{q}\mathcal{P}^{(p)}_{p+q}$ (Theorem 7.6 of [Val07] by means of the extension seen in the proof of Theorem 2.3.4.3 applies), concentrated in the line $p+q=0$ and the spectral sequence collapses at rank 1. We have

$$\begin{cases} E^1_{p,\text{-}p} & = & \mathrm{q}\mathcal{P}^{(p)} & \cong & E^{\infty}_{p,\text{-}p} & \cong & gr^{(p)}(\mathrm{H}_0(\Omega\mathcal{P}^{¡})) \\ E^1_{p,\,q} & = & 0 & = & E^{\infty}_{p,\,q} & \cong & gr^{(p)}(\mathrm{H}_{p+q}(\Omega\mathcal{P}^{¡})) \text{ when } p+q \neq 0. \end{cases}$$

For the syzygy degree, we have

$$\mathrm{H}_0(\Omega\mathcal{P}^{¡}) \cong \mathcal{F}(V)/Im(d_0 + d_1 - d_2) \cong \mathcal{P}.$$

So, the quotient $gr^{(p)}(\mathrm{H}_0(\Omega\mathcal{P}^{¡}))$ is equal to $gr^{(p)}(\mathcal{P})$. Finally, the morphism $\Omega\mathcal{P}^{¡} \xrightarrow{\sim} \mathcal{P}$ is a quasi-isomorphism. □

Remark. Following the ideas of van der Laan in [van03], we can extend this curved Koszul duality to coloured operads. Martin Doubek told us that our construction applies to the coloured operad $\mathcal{I}so$ encoding chain complexes isomorphisms to recover the resolution given by Markl in [Mar01].

2.4.3.2 Theorem (Poincaré-Birkhoff-Witt theorem)**.** *When $\mathcal{P}$ is a Koszul properad, the natural epimorphism of properads $q\mathcal{P} \twoheadrightarrow gr\mathcal{P}$ is an isomorphism of bigraded properads, with respect to the weight grading and the homological degree. Therefore, the following $\mathbb{S}$-bimodules, graded by the homological degree, are isomorphic :*

$$\mathcal{P} \cong gr(\mathcal{P}) \cong q\mathcal{P}.$$

PROOF. It is a direct corollary of the previous proof. □

To show Condition (II), that is $(R) \cap \{I \oplus V \oplus \mathcal{F}(V)^{(2)}\} = R$, can be difficult. The following proposition shows that we do not have to compute the full (R) but only the part $\{R \otimes V + V \otimes R\}$.

2.4.3.3 Proposition. *A properad $\mathcal{P}$ is Koszul if and only if it admits a presentation $\mathcal{P} = \mathcal{F}(V)/(R)$ such that $R \subset I \oplus V \oplus \mathcal{F}(V)^{(2)}$ satisfying the following conditions*

(I) $R \cap \{I \oplus V\} = \{0\}$ *;*

(II') $\{R \otimes V + V \otimes R\} \cap \{V \oplus \mathcal{F}(V)^{(2)}\} \subset R$ *;*

(III) the associated quadratic properad $q\mathcal{P} := \mathcal{F}(V)/(qR)$ is Koszul in the classical sense.

PROOF. Definition 2.4.3 always implies conditions (I), (II') and (III). First, we have to remark that the property (II') instead of (II) is enough to show Lemma 2.4.1.1 and to define $\mathcal{P}^{\text{¡}}$. Moreover, Theorem 2.4.3.1 and Theorem 2.4.3.2 are still true. Then we can apply the Poincaré-Birkhoff-Witt Theorem which gives in weight 2 that $\mathrm{q}R = \mathrm{q}((R) \cap \{I \oplus V \oplus \mathcal{F}(V)^{(2)}\})$. This last equality is equivalent to (II) under the condition (I). □

2.5 Resolution of algebras

We now give a resolution of a semi-augmented dg properad $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ as a $\mathcal{P}$-bimodule. In the operadic case, this provides functorial cofibrant resolutions for $\mathcal{P}$-algebras. We use such resolutions to define a cohomology theory associated to unital associative algebras in the next section.

2.5.1 Resolutions of properads as bimodule

We generalize the resolution given by the *bar construction with coefficients* to properads (not necessarily augmented). Moreover, for an inhomogeneous properad which is Koszul, we get a smaller resolution of it called the *Koszul complex*.

Dg composite product

Let $(M,\, d_M)$ and $(N,\, d_N)$ be two dg $\mathbb{S}$-bimodules. Recall from [MV09b] the differential on the monoidal product $\boxtimes$ of two $\mathbb{S}$-bimodules. Let $id_M \boxtimes' d_N : M \boxtimes N \to M \boxtimes N$ be the morphism of $\mathbb{S}$-bimodules defined by

$$(id_M \boxtimes' d_N)(\rho(m_1,\, \ldots,\, m_b)\sigma(n_1,\, \ldots,\, n_a)\omega) := \sum_{j=1}^{a} \pm\rho(m_1,\, \ldots,\, m_b)\sigma(n_1,\, \ldots,\, d_N(n_j),\, \ldots,\, n_a)\omega$$

and let $d_M \,{}'\!\boxtimes id_N : M \boxtimes N \to M \boxtimes N$ be the morphism of $\mathbb{S}$-bimodules defined by

$$(d_M \,{}'\!\boxtimes id_N)(\rho(m_1, \ldots, m_b)\sigma(n_1, \ldots, n_a)\omega) := \sum_{i=1}^{b} \pm \rho(m_1, \ldots, d_M(m_i), \ldots, m_b)\sigma(n_1, \ldots, n_a)\omega.$$

This gives a differential on $M \boxtimes N$ by $d_{M\boxtimes N} := d_M \,{}'\!\boxtimes id_N + id_M \boxtimes' d_N$.

Twisted composite product

In this section, we study the free dg $\mathcal{P}$-bimodules over a curved coproperad $(\mathcal{C}, d_{\mathcal{C}}, \theta)$. To any map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree $-n$, we associate the unique derivation (see Section 3.2 of [MV09b] for precise definitions) of left $\mathcal{P}$-modules $d^l_\alpha : \mathcal{P} \boxtimes \mathcal{C} \to \mathcal{P} \boxtimes \mathcal{C}$ of degree $-n$ which extends the map

$$\mathcal{C} \xrightarrow{\Delta_{(1,1)}} \mathcal{C} \boxtimes_{(1,1)} \mathcal{C} \xrightarrow{\alpha \otimes id_{\mathcal{C}}} \mathcal{P} \boxtimes_{(1,1)} \mathcal{C}.$$

By symmetry, we define also the derivation of right $\mathcal{P}$-modules $d^r_\alpha : \mathcal{C} \boxtimes \mathcal{P} \to \mathcal{C} \boxtimes \mathcal{P}$ of degree $-n$. We endow the free $\mathcal{P}$-bimodule $\mathcal{P} \boxtimes \mathcal{C} \boxtimes \mathcal{P}$ with the following derivation of $\mathcal{P}$-bimodules :

$$d_\alpha := d_{\mathcal{P}\boxtimes\mathcal{C}\boxtimes\mathcal{P}} - d^l_\alpha \,{}'\!\boxtimes id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_\alpha,$$

where $d_{\mathcal{P}\boxtimes\mathcal{C}\boxtimes\mathcal{P}} := d_{\mathcal{P}} \,{}'\!\boxtimes id_{\mathcal{C}\boxtimes\mathcal{P}} + (id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \,{}'\!\boxtimes id_{\mathcal{P}} + id_{\mathcal{P}\boxtimes\mathcal{C}} \boxtimes' d_{\mathcal{P}}$ with $(id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \,{}'\!\boxtimes id_{\mathcal{P}} = id_{\mathcal{P}} \boxtimes' (d_{\mathcal{C}} \,{}'\!\boxtimes id_{\mathcal{P}})$ by associativity of the composite product.

2.5.1.1 Lemma. *On the $\mathcal{P}$-bimodule $\mathcal{P} \boxtimes \mathcal{C} \boxtimes \mathcal{P}$, the derivation d_α satisfies*

$$d_\alpha{}^2 = -d^l_{\partial(\alpha)+\alpha\star\alpha-\Theta} \,{}'\!\boxtimes id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_{\partial(\alpha)+\alpha\star\alpha-\Theta}.$$

Thus, when $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$, we have $d_\alpha{}^2 = 0$ and the derivation d_α defines a differential on the chain complex

$$\mathcal{P} \boxtimes_\alpha \mathcal{C} \boxtimes_\alpha \mathcal{P} := (\mathcal{P} \boxtimes \mathcal{C} \boxtimes \mathcal{P}, d_\alpha = d_{\mathcal{P}\boxtimes\mathcal{C}\boxtimes\mathcal{P}} - d^l_\alpha \,{}'\!\boxtimes id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_\alpha).$$

Proof. We do the computation for $d_{\mathcal{P}} = 0$, the general case follows immediately. We have

$$\begin{aligned} d_\alpha{}^2 &= ((id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \,{}'\!\boxtimes id_{\mathcal{P}} - d^l_\alpha \,{}'\!\boxtimes id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' d^r_\alpha)^2 \\ &= (id_{\mathcal{P}} \boxtimes' d^2_{\mathcal{C}}) \,{}'\!\boxtimes id_{\mathcal{P}} + (d^l_\alpha)^2 \,{}'\!\boxtimes id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' (d^r_\alpha)^2 \\ &\quad -((id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \circ d^l_\alpha + d^l_\alpha \circ (id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}})) \,{}'\!\boxtimes id_{\mathcal{P}} + id_{\mathcal{P}} \boxtimes' ((d_{\mathcal{C}} \,{}'\!\boxtimes id_{\mathcal{P}}) \circ d^r_\alpha + d^r_\alpha \circ (d_{\mathcal{C}} \,{}'\!\boxtimes id_{\mathcal{P}})) \\ &\quad -(d^l_\alpha \,{}'\!\boxtimes id_{\mathcal{P}}) \circ (id_{\mathcal{P}} \boxtimes' d^r_\alpha) - (id_{\mathcal{P}} \boxtimes' d^r_\alpha) \circ (d^l_\alpha \,{}'\!\boxtimes id_{\mathcal{P}}). \end{aligned}$$

Since $d_{\mathcal{C}}{}^2 = (\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1,1)}$, we have $(id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}{}^2) \,{}'\!\boxtimes id_{\mathcal{P}} = d^l_\Theta \,{}'\!\boxtimes id_{\mathcal{P}} - id_{\mathcal{P}} \boxtimes' d^r_\Theta$. Moreover, the associativity of γ and the coassociativity of $\Delta_{(1,1)}$ give $(d^l_\alpha)^2 = -d^l_{\alpha\star\alpha}$ and $(d^r_\alpha)^2 = d^r_{\alpha\star\alpha}$ where the sign is given by the Koszul sign rule and the fact that α has degree -1. Then $(id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) \circ d^l_\alpha + d^l_\alpha \circ (id_{\mathcal{P}} \boxtimes' d_{\mathcal{C}}) = d^l_{\alpha\circ d_{\mathcal{C}}}$ and $(d_{\mathcal{C}} \,{}'\!\boxtimes id_{\mathcal{P}}) \circ d^r_\alpha + d^r_\alpha \circ (d_{\mathcal{C}} \,{}'\!\boxtimes id_{\mathcal{P}}) = d^r_{\alpha\circ d_{\mathcal{C}}}$ since $d_{\mathcal{C}}$ is a coderivation. Finally, $(d^l_\alpha \,{}'\!\boxtimes id_{\mathcal{P}}) \circ (id_{\mathcal{P}} \boxtimes' d^r_\alpha) + (id_{\mathcal{P}} \boxtimes' d^r_\alpha) \circ (d^l_\alpha \,{}'\!\boxtimes id_{\mathcal{P}}) = 0$ since α has degree -1. This gives the result. $\square$

Koszul morphism

A curved twisting morphism $\alpha : (\mathcal{C},\, d_{\mathcal{C}},\, \theta) \to (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ is called a *Koszul morphism* when the map ξ defined by $\mathcal{P} \boxtimes_\alpha \mathcal{C} \boxtimes_\alpha \mathcal{P} \twoheadrightarrow \mathcal{P} \boxtimes I \boxtimes \mathcal{P} \cong \mathcal{P} \boxtimes \mathcal{P} \xrightarrow{\gamma} \mathcal{P}$ is a resolution of $\mathcal{P}$, that is

$$\xi : \mathcal{P} \boxtimes_\alpha \mathcal{C} \boxtimes_\alpha \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

2.5.1.2 Proposition. *Let $\mathcal{P}$ be a wfdg semi-augmented properad. The curved twisting morphism $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ is a curved Koszul morphism, that is, the twisted composite product $\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}$ is a resolution of the properad $\mathcal{P}$ called the* augmented bar resolution

$$\xi : \mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Proof. The method is the same as in the proof of Theorem 2.3.4.3. The weight filtration on $\mathcal{P}$ induces a filtration on $\mathrm{B}\mathcal{P}$ given by the total weight. This gives a filtration F_p by the weight on $\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}$ and a filtration F'_p by the weight on $\mathcal{P}$. These filtrations are filtrations of chain complexes since the differentials either preserve or decrease the weight. The filtrations are exhaustive and bounded below and the map ξ preserves the filtrations. We apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

$$\begin{cases} E^{\bullet}_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}) \\ {E'}^{\bullet}_{p,q} \Rightarrow \mathrm{H}_{p+q}(\mathcal{P}). \end{cases}$$

Since the differential of $E^0_{\bullet,\bullet}$ is the weight preserving part of the differential of $\mathcal{P} \boxtimes_\pi \mathrm{B}\mathcal{P} \boxtimes_\pi \mathcal{P}$, the isomorphism of graded vector spaces $E^0_{\bullet,\bullet} \cong gr\mathcal{P} \boxtimes_\pi \mathrm{B}(gr\mathcal{P}) \boxtimes_\pi gr\mathcal{P}$ is an isomorphism of dg modules. Since $gr\mathcal{P}$ is an augmented properad, we can apply Theorem 4.17 of [Val07] (we use the same trick as in the proof of Theorem 2.3.4.3 for the fact that the properad is non reduced a priori) to $gr\mathcal{P}$ with $R = gr\mathcal{P}$ to get that $E^1_{p,q} = \mathrm{H}_{p+q}(gr^{(p)}\mathcal{P}) = {E'}^1_{p,q}$. Then $E^r_{p,q}$ and ${E'}^r_{p,q}$ coincide for $r \geq 1$ and ξ induces an isomorphism between $E^\infty_{p,q}$ and ${E'}^\infty_{p,q} \cong gr^{(p)}\mathrm{H}_{p+q}(\mathcal{P})$. This concludes the proof. □

Let $\mathcal{P}$ be an inhomogeneous properad, $\mathcal{P}^{\text{!`}}$ its Koszul dual cooperad and $\kappa : \mathcal{P}^{\text{!`}} \to \mathcal{P}$ the associated curved twisting morphism. The chain complex $\mathcal{P} \boxtimes_\kappa \mathcal{P}^{\text{!`}} \boxtimes_\kappa \mathcal{P}$ is called the *total Koszul complex.*

2.5.1.3 Proposition. *Let $\mathcal{P}$ be an inhomogeneous properad and $\mathcal{P}^{\text{!`}}$ be its Koszul dual coproperad. When $\mathcal{P}$ is Koszul, the curved twisting morphism $\kappa : \mathcal{P}^{\text{!`}} \to \mathcal{P}$ is a curved Koszul morphism, that is, the total Koszul complex $\mathcal{P} \boxtimes_\kappa \mathcal{P}^{\text{!`}} \boxtimes_\kappa \mathcal{P}$ is a resolution of the properad $\mathcal{P}$*

$$\xi : \mathcal{P} \boxtimes_\kappa \mathcal{P}^{\text{!`}} \boxtimes_\kappa \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Proof. The proof is similar to the proof of Proposition 2.5.1.2. The differences are the following. Since $\mathcal{P}$ is Koszul, the Poincaré-Birkhoff-Witt Theorem 2.4.3.2 gives $E^0_{\bullet,\bullet} = gr\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{\text{!`}} \boxtimes_\kappa gr\mathcal{P} \cong \mathrm{q}\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{\text{!`}} \boxtimes_\kappa \mathrm{q}\mathcal{P}$. So the Koszul criterion (Theorem 7.8 of [Val07] with the trick of the proof of Theorem 2.3.4.3 for the non reduced case) and the comparison Lemma (Theorem 5.4 of [Val07]) with $L = \mathrm{q}\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{\text{!`}} \boxtimes_\kappa \mathrm{q}\mathcal{P}$, $L' = \mathrm{q}\mathcal{P}$, $\mathcal{P}' = \mathrm{q}\mathcal{P}$, $M = \mathrm{q}\mathcal{P} \boxtimes_\kappa \mathrm{q}\mathcal{P}^{\text{!`}}$ and $M' = I$, and the Poincaré-Birkhoff-Witt Theorem 2.4.3.2 apply to give that $E^1_{p,q} = \mathrm{H}_{p+q}(\mathrm{q}\mathcal{P}^{(p)}) \cong \mathrm{H}_{p+q}(gr^{(p)}\mathcal{P})$. □

2.5.2 Resolution of algebras

From now on, we consider only operads and cooperads since there is in general no notion of free algebra over a properad. In this section, we use the resolutions of $\mathcal{P}$ as a $\mathcal{P}$-bimodule of the previous section to provide functorial resolutions for algebras over $\mathcal{P}$ as, for example, for unital associative algebras (see Section 2.6).

Coalgebra over a curved cooperad

Let $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$ be a curved cooperad. A *$(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$-coalgebra* is a triple $(C,\, \Delta_C, d_C)$ where $(\mathcal{C}, \Delta)$ is a $\mathcal{C}$-coalgebra, and a coderivation $d_C : C \to C$ of degree -1 such that :

$$d_C{}^2 = (\theta \circ id_C) \circ \Delta_C,$$

where the $\circ$ inside the parentheses is the operadic composition product and the $\circ$ outside the parentheses is the composition of morphisms.

A *morphism of $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$-coalgebras* $f : (C,\, \Delta_C, d_C) \to (C',\, \Delta_{C'}, d_{C'})$ is a morphism $f : C \to C'$ of $\mathcal{C}$-coalgebras which commutes with the predifferentials d_C and $d_{C'}$.

Relative composition product

Let $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$ be a sdg operad. A *right $\mathcal{P}$-module* $(\mathcal{L},\, \rho)$ is an $\mathbb{S}$-module endowed with a map $\rho : \mathcal{L} \circ \mathcal{P} \to \mathcal{L}$ compatible with the product and the unit of the operad. We define similarly the notion of *left $\mathcal{P}$-module*. We define the relative composite product $\mathcal{L} \circ_{\mathcal{P}} \mathcal{R}$ of a right $\mathcal{P}$-module $(\mathcal{L},\, \rho)$ and a left $\mathcal{P}$-module $(\mathcal{R},\, \lambda)$ by the coequalizer diagram

$$\mathcal{L} \circ \mathcal{P} \circ \mathcal{R} \underset{id_{\mathcal{L}} \circ \lambda}{\overset{\rho \circ id_{\mathcal{R}}}{\rightrightarrows}} \mathcal{L} \circ \mathcal{R} \twoheadrightarrow \mathcal{L} \circ_{\mathcal{P}} \mathcal{R}\ ,$$

where in the above line all $\circ$ are the operadic composition product. These definitions extend to the dg setting.

Bar construction of $\mathcal{P}$-algebras

To any curved twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ from a curved cooperad $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$ to an operad $(\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$, we associate a functor

$$\mathrm{B}_\alpha : \mathsf{dg}\ (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)\text{-}\mathsf{algebras} \to (\mathcal{C},\, d_{\mathcal{C}},\, \theta)\text{-}\mathsf{coalgebras}.$$

For a $\mathcal{P}$-algebra $(A,\, \gamma_A)$, we define on $\mathcal{C}(A) = (\mathcal{C} \circ \mathcal{P}) \circ_{\mathcal{P}} A$ the maps

$$\begin{cases} d_1 : \mathcal{C}(A) \xrightarrow{d_{\mathcal{C}} \circ id_A + id_{\mathcal{C}} \circ' d_A} \mathcal{C}(A) \\ d_2 := d^r_\alpha \circ_{\mathcal{P}} id_A : \mathcal{C}(A) \xrightarrow{\Delta_{(1)} \circ id_A} \mathcal{C} \circ_{(1)} \mathcal{C}(A) \xrightarrow{(id_{\mathcal{C}} \otimes \alpha) \circ id_A} \mathcal{C} \circ \mathcal{P}(A) \xrightarrow{id_{\mathcal{C}} \circ \gamma_A} \mathcal{C}(A), \end{cases}$$

where $(\mathcal{C} \circ \mathcal{P}) \circ A \xrightarrow{d^r_\alpha \circ id_A} (\mathcal{C} \circ \mathcal{P}) \circ A \twoheadrightarrow \mathcal{C}(A)$ factors through $\mathcal{C}(A)$ to give $d^r_\alpha \circ_{\mathcal{P}} id_A$ since γ_A is a dg map. (Here, $\Delta_{(1)}$ corresponds to the infinitesimal decomposition map $\Delta_{(1,1)}$ and $\mathcal{C} \circ_{(1)} \mathcal{C}$ corresponds to $\mathcal{C} \boxtimes_{(1,1)} \mathcal{C}$ when we restrict to cooperads.)

2.5.2.1 Lemma. *Since α is a curved twisting morphism, we have*

$$(d_1 + d_2)^2 = (\theta \circ id_{\mathcal{C}(A)}) \circ \Delta_{\mathcal{C}(A)}.$$

Proof. We compute

$$\begin{cases} {d_1}^2 & = {d_{\mathcal{C}}}^2 \circ id_A = ((\theta \otimes id_{\mathcal{C}} - id_{\mathcal{C}} \otimes \theta) \circ \Delta_{(1)}) \circ id_A \\ & = (\theta \circ id_{\mathcal{C}(A)}) \circ (\Delta_{(1)} \circ id_A) - d^r_{\Theta} \circ_{\mathcal{P}} id_A \\ & = (\theta \circ id_{\mathcal{C}(A)}) \circ \Delta_{\mathcal{C}(A)} - d^r_{\Theta} \circ_{\mathcal{P}} id_A \qquad (\theta \text{ is non-zero only on } \mathcal{P}(1)) \\ {d_2}^2 & = d^r_{\alpha\star\alpha} \circ_{\mathcal{P}} id_A \\ d_1 d_2 + d_2 d_1 & = d^r_{\partial(\alpha)} \circ_{\mathcal{P}} id_A. \end{cases}$$

Thus $(d_1 + d_2)^2 = d^r_{\partial(\alpha)+\alpha\star\alpha-\Theta} \circ_{\mathcal{P}} id_A + (\theta \circ id_{\mathcal{C}(A)}) \circ \Delta_{\mathcal{C}(A)}$ and we get the result since α is a curved twisting morphism. □

The *bar construction on A* is the $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$-coalgebra $\mathrm{B}_\alpha A := (\mathcal{C}(A),\, d := d_1 + d_2)$.

Cobar construction of a $\mathcal{C}$-coalgebra

Similarly to the previous section, to any curved twisting morphism $\alpha : (\mathcal{C},\, d_{\mathcal{C}},\, \theta) \to (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)$, we associate a functor

$$\Omega_\alpha : (\mathcal{C},\, d_{\mathcal{C}},\, \theta)\text{-coalgebras} \to \text{dg } (\mathcal{P},\, d_{\mathcal{P}},\, \varepsilon)\text{-algebras}.$$

For any $(\mathcal{C},\, d_{\mathcal{C}},\, \theta)$-coalgebra $(C,\, \Delta_C,\, d_C)$, we define on $\mathcal{P}(C)$ the maps

$$\begin{cases} d_1 : \mathcal{P}(C) \xrightarrow{d_{\mathcal{P}} \circ id_C + id_{\mathcal{P}} \circ' d_C} \mathcal{P}(C) \\ d_2 : \mathcal{P}(C) \xrightarrow{id_{\mathcal{P}} \circ' \Delta_C} \mathcal{P} \circ_{(1)} \mathcal{C}(C) \xrightarrow{(id_{\mathcal{P}} \otimes \alpha) \circ id_C} \mathcal{P} \circ \mathcal{P}(C) \xrightarrow{\gamma \circ id_C} \mathcal{P}(C). \end{cases}$$

2.5.2.2 Lemma. *Since α is a curved twisting morphism, we have*

$$(d_1 - d_2)^2 = 0.$$

Proof. We compute

$$\begin{cases} {d_1}^2 & = id_{\mathcal{P}} \circ' {d_C}^2 = id_{\mathcal{P}} \circ' ((\theta \circ id_C) \circ \Delta_C) \\ {d_2}^2 & = -(\gamma \circ id_C) \circ (id_{\mathcal{P}} \circ (\alpha \star \alpha) \circ id_C) \circ (id_{\mathcal{P}} \circ' \Delta_C) \\ -d_1 d_2 - d_2 d_1 & = -(\gamma \circ id_C) \circ (id_{\mathcal{P}} \circ \partial(\alpha) \circ id_C) \circ (id_{\mathcal{P}} \circ' \Delta_C). \end{cases}$$

Thus $(d_1 - d_2)^2 = -(\gamma \circ id_C) \circ (id_{\mathcal{P}} \circ (\partial(\alpha) + \alpha \star \alpha - \Theta) \circ id_C) \circ (id_{\mathcal{P}} \circ' \Delta_C) = 0$ since α is a curved twisting morphism. □

The *cobar construction on C* is the dg $\mathcal{P}$-algebra $\Omega_\alpha C := (\mathcal{P}(C),\, d_{\Omega_\alpha C} := d_1 - d_2)$.

The bar-cobar resolution

The bar-cobar construction on a $\mathcal{P}$-algebra provides a functorial cofibrant resolution of any $\mathcal{P}$-algebra when the curved twisting morphism α is Koszul.

2.5.2.3 Proposition. *Let $\alpha : (\mathcal{C}, d_{\mathcal{C}}, \theta) \to (\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ be a curved Koszul morphism between a curved cooperad $(\mathcal{C}, d_{\mathcal{C}}, \theta)$ and a sdg operad $(\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$. Then the bar-cobar resolution $\Omega_\alpha \mathrm{B}_\alpha A$ is a resolution of the $\mathcal{P}$-algebra A, that is,*

$$\Omega_\alpha \mathrm{B}_\alpha A = \mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A \xrightarrow{\sim} A.$$

Moreover when A is bounded below, it is a cofibrant resolution.

Proof. In the (semi-)model category structure on $\mathcal{P}$-algebras defined in [Fre09], cofibrant $\mathcal{P}$-algebras are retracts of quasi-free $\mathcal{P}$-algebras endowed with a good filtration (Proposition 12.3.8 in [Fre09]). This is the case here since the chain complexes are bounded below.

The bar-cobar construction $\Omega_\alpha \mathrm{B}_\alpha A$ is isomorphic to the relative composite product $(\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ_{\mathcal{P}} A$. So it is defined by a short exact sequence

$$0 \to (\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ \mathcal{P} \circ A \to (\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ A \to (\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ_{\mathcal{P}} A \to 0$$

which induces a long exact sequence in homology. Since $\mathbb{K}$ is a field of characteristic 0, the rings $\mathbb{K}[\mathbb{S}_n]$ are semi-simple by Maschke's Theorem, that is, every $\mathbb{K}[\mathbb{S}_n]$-module is projective. So the Künneth formula and the fact that α is Koszul imply that $\mathrm{H}_\bullet((\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ \mathcal{P} \circ A) \cong \mathrm{H}_\bullet(\mathcal{P}) \circ \mathrm{H}_\bullet(\mathcal{P}) \circ \mathrm{H}_\bullet(A)$ and that $\mathrm{H}_\bullet((\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ A) \cong \mathrm{H}_\bullet(\mathcal{P}) \circ \mathrm{H}_\bullet(A)$. Finally, this gives

$$\mathrm{H}_\bullet((\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ_{\mathcal{P}} \circ A) \cong \mathrm{H}_\bullet(\mathcal{P}) \circ_{\mathrm{H}_\bullet(\mathcal{P})} \mathrm{H}_\bullet(A) \cong \mathrm{H}_\bullet(A).$$

□

2.5.2.4 Theorem. *Let $(\mathcal{P}, d_{\mathcal{P}}, \varepsilon)$ be a sdg operad. The curved Koszul morphism $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ gives a resolution*

$$\Omega_\pi \mathrm{B}_\pi A = \mathcal{P} \circ_\pi \mathrm{B}\mathcal{P} \circ_\pi A \xrightarrow{\sim} A,$$

which is cofibrant when A is bounded below. When $\mathcal{P}$ is a Koszul operad, the total Koszul complex gives a smaller resolution

$$\Omega_\kappa \mathrm{B}_\kappa A = \mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa A \xrightarrow{\sim} A,$$

which is cofibrant when A is bounded below.

Proof. It is a direct corollary of Proposition 2.5.2.3 and Propositions 2.5.1.2 and 2.5.1.3. □

2.6 Homotopy and cohomology theories for unital associative algebras

In this section we describe a simple resolution of the operad which encodes unital associative algebras, $u\mathcal{A}s$, obtained by the methods described in section 2.4. In fact, many of the theorems in this section can be generalized in a straightforward way to any (inhomogeneous) Koszul properad. Algebras over the resolution uA_∞ are called *homotopy unital A_∞-algebras*, or *uA_∞-algebras*, for short. We use some nice properties of our resolution to prove that uA_∞-algebras may be replaced up to equivalence by strictly unital associative algebras. Using our explicit transfer formulae, we show that a unital associative algebra may be transferred to homology as a strictly unital A_∞-algebra (see Definition 2.6.5.1). This gives a proof that one may always choose a minimal model for a uA_∞-algebra which is actually a strictly unital A_∞-algebra. In this sense, it is "enough" to resolve only the associative relation of $u\mathcal{A}s$, obtaining the operad A_∞, and then adjoin a unit, giving the operad which encodes strictly unital A_∞-algebras. As a corollary of our discussion, we provide sufficient conditions so that : "When trying to find resolutions of algebraic structures with units, it is 'good enough' to resolve the structure (without its units) first, and then append the units to that resolution." The notion of uA_∞-algebras is exactly the notion of "A_∞-algebras with a homotopy unit" of [FOOO07]. Concerning the notion of ∞-morphism and the nice properties of uA_∞-algebras, we still have to compare them with the theory presented in [FOOO07].

2.6.1 Homotopy unital associative algebras

We give a presentation for the operad encoding unital associative algebras. This presentation is an inhomogeneous quadratic presentation and we can apply the theory of the previous sections to compute its Koszul dual cooperad, and hence an explicit resolution.

We use the notation $\underline{n} := \{1, \ldots, n\}$. The symbol $\overline{\mu}$ stands for an element in a cooperad and the symbol μ stands for an element in an operad.

The operad encoding unital associative algebras

We denote by $u\mathcal{A}s$ the operad whose representations in the category of dg modules are precisely differential graded unital associative algebras. We consider the following presentation

$$u\mathcal{A}s = \mathcal{F}\left(⫯, \curlyvee\right) / \left(\curlyvee\circ_1\curlyvee - \curlyvee\circ_2\curlyvee,\ \curlyvee\circ⫯ - |,\ \curlyvee\circ⫯ - |\right).$$

Remark. We fix this presentation to make our computations of the Koszul dual, $u\mathcal{A}s^{¡}$ and ultimately uA_∞. Note that this presentation for $u\mathcal{A}s$ is an inhomogeneous quadratic presentation (see 2.4.1 for a definition).

To make the Koszul dual cooperad, $u\mathcal{A}s^{¡}$ of $u\mathcal{A}s$ explicit, we compute its associated quadratic operad :

$$\mathrm{q}u\mathcal{A}s^{!} = \mathcal{F}(⫯, \curlyvee)/(\curlyvee\circ_1\curlyvee - \curlyvee\circ_2\curlyvee,\ \curlyvee\circ⫯\ ,\ \curlyvee\circ⫯\ ) = ⫯ \oplus \mathcal{A}s.$$

Let's take a moment to explain the notation on the right-hand side of the equation above.

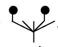
2.6.1.1 Definition. *Let $\mathcal{P}$, $\mathcal{Q}$ be augmented operads. Then the* direct sum operad $\mathcal{P} \oplus \mathcal{Q}$ *is defined to be $\mathcal{F}\left(\overline{\mathcal{P}}, \overline{\mathcal{Q}}\right) / \left(R_{\mathcal{P}}, R_{\mathcal{Q}}, R_{\mathcal{P}\mathcal{Q}}\right)$, where $R_{\mathcal{P}}, R_{\mathcal{Q}}$ are the relations in $\mathcal{P}, \mathcal{Q}$ respectively, and $R_{\mathcal{P}\mathcal{Q}}$ is the collection of all compositions of a pair of elements, one in $\overline{\mathcal{P}}$, one in $\overline{\mathcal{Q}}$.*

Remark. The direct sum operad is the product in the category of augmented operads.

2.6.1.2 Proposition. *If $\mathcal{P}$ and $\mathcal{Q}$ are both quadratic augmented operads, then $\mathcal{P} \oplus \mathcal{Q}$ is a quadratic augmented operad.*

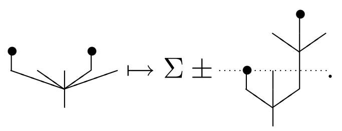
Proof. For any two presented operads, $\mathcal{P} = \mathcal{F}(V_1) / (R_1)$, $\mathcal{Q} = \mathcal{F}(V_2) / (R_2)$, the direct sum operad $\mathcal{P} \oplus \mathcal{Q}$ is naturally presented by $\mathcal{F}(V_1, V_2) / (R_1, R_2, R_{V_1 V_2})$. If (V_1, R_1) and (V_2, R_2) are both quadratic presentations, then so is the natural presentation for $\mathcal{P} \oplus \mathcal{Q}$. $\square$

We will make use of the identification $\mathrm{q}u\mathcal{A}s^{\mathrm{i}} = ⫯ \oplus \mathcal{A}s$ to compute the Koszul dual cooperad of $\mathrm{q}u\mathcal{A}s^{\mathrm{i}}$ (see 2.6.1.3). Before we compute the resulting cooperad, $\mathrm{q}u\mathcal{A}s^{\mathrm{i}}$, we first describe it.

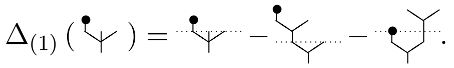
Linearly, we have an isomorphism $\mathrm{q}u\mathcal{A}s^{\mathrm{i}} \cong \mathbb{K}[\overline{\mu}_n^S]_{n\geq 1, S\subset \underline{n}}$. The element $\overline{\mu}_n^S \in \mathrm{q}u\mathcal{A}s^{\mathrm{i}}$ corresponds to a (co)operation with $n - |S|$ inputs : however, we draw this operation as a corolla with n leaves, and a *cork* covering each of the leaves in the set S. For example, $\overline{\mu}_5^{\{1,4\}}$ corresponds to . We point out here that the space of n-to-1 operations is infinite dimensional for every $n \geq 0$. To see this, note that every n-to-1 corolla is an n-ary operation, and by adding a corked leaf, we get a new n-to-1 operation. Continuing to add corked leaves gives infinitely many new n-ary operations.

Also notice that $\overline{\mu}_n^{\emptyset} =$  for $n \geq 1$ spans the subcooperad corresponding to $\mathcal{A}s^{\mathrm{i}}$ and $\{\overline{\mu}_1^{\emptyset} = |,\ \overline{\mu}_1^{\{1\}} = ⫯\}$ spans the subcooperad corresponding to $⫯^{\mathrm{i}}$ (with $\overline{\mu}_1^{\emptyset}$ corresponding to the identity cooperation in both cases).

Using this basis, the infinitesimal decomposition $\Delta_{(1)}$ is given by summing over all possible (nontrivial) ways to split the corolla into two, preserving the number of leaves and the number and positions of the corks. Pictorially :



For example,



We compute the Koszul dual cooperad, $qu\mathcal{A}s^{\mathrm{i}}$ by the following proposition.

2.6.1.3 Proposition. *Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be a quadratic operad where V is finite-dimensional. Then by Proposition 2.6.1.2 the operad $⫯ \oplus \mathcal{P}$ is given by $(⫯ \oplus \mathcal{P})(0) := (\mathbb{K}\cdot ⫯) \oplus \mathcal{P}(0)$ and $(⫯\oplus\mathcal{P})(n) := \mathcal{P}(n)$ for all $n \neq 0$ and endowed with the operadic structure given by the (trivial) structure on $⫯$, the structure on $\mathcal{P}$, and trivial composition between $⫯$ and $\mathcal{P}$. The Koszul dual cooperad of $⫯ \oplus \mathcal{P}$ is given by the coaugmented cooperad*

$$(⫯ \oplus \mathcal{P})^{i} \cong \mathbb{K}\cdot\{\overline{\mu}_S,\ \textit{where } \overline{\mu}\in\mathcal{P}^{i}(n),\ S\subset\underline{n} \textit{ and } |\overline{\mu}_S| = |\overline{\mu}| + |S|\}.$$

The set S is the set of the positions of the "corks" $⫯$. Let $\overline{\xi} \in \mathcal{P}^{i}(n)$ such that $\Delta_{(1)}(\overline{\xi}) = \sum(\overline{\mu};\underbrace{id,\ldots,id}_{p},\overline{\nu},\underbrace{id,\ldots,id}_{r})$, where $\overline{\mu}\in\mathcal{P}^{i}(m)$, $\overline{\nu}\in\mathcal{P}^{i}(q)$, $p+1+r = m$ and $p+q+r = n$. Then the infinitesimal decomposition map on $\overline{\xi}_S \in (⫯\oplus\mathcal{P})^{i}$, where $S\subset\underline{n}$, is given by

$$\Delta_{(1)}(\overline{\xi}_S) = \sum(-1)^{\epsilon}(\overline{\mu}_{S_1};\underbrace{id,\ldots,id}_{p-|S_1'|},\overline{\nu}_{S_2},\underbrace{id,\ldots,id}_{r-|S_1''|}),$$

where $\epsilon = |\overline{\nu}||S_1| + |S_2||S_1''|$,

$\overline{\mu}_{S_1}\in\mathcal{P}^{i}(m-|S_1|)$, $\overline{\nu}_{S_2}\in\mathcal{P}^{i}(q-|S_2|)$ and $\begin{cases} S_1' \subset \underline{p} \\ S_2 \subset \underline{q} \\ S_1'' \subset \{p+2,\ldots,p+1+r\}\end{cases}$ such that $S = S_1'\sqcup(S_2+p)\sqcup(S_1''+q-1)$ and $S_1 = S_1'\sqcup S_1''$.

Proof. The operad $\text{¡} \oplus \mathcal{P}$ is a quadratic operad given by $\mathcal{F}(\text{¡} \oplus V)/(R \oplus V \otimes \text{¡})$ where

$$V \otimes \text{¡} := \{\mu^{\{k\}},\ \text{with } \mu \in V(n) \text{ and } \{k\} \subset \underline{n}\}.$$

We follow Appendix B of [Lod01] defining $\mathcal{P}^! := \mathcal{F}(V^\vee)/(R^\perp)$, where $V^\vee := V^* \otimes (sgn)$ with the signature representation (sgn) and $R^\perp$ is the orthogonal space for the natural pairing $\langle -, - \rangle : V^\vee \otimes V \to \mathbb{K}$. Since $(V \otimes \text{¡})^\perp = \mathcal{F}_{(2)}(V^\vee)$, we get

$$(\text{¡} \oplus \mathcal{P})^! = \mathcal{F}(\text{¡}^\vee \oplus V^\vee)/(R^\perp \cap \mathcal{F}_{(2)}(V^\vee)) \cong \{\mu^S,\ \text{where } \mu \in \mathcal{P}^!(n) \text{ and } S \subset \underline{n}\}$$

and the composition is induced, up to signs, by the composition on $\mathcal{P}^!$.

Following [LV], we have $\mathcal{P}^{¡} := \mathcal{S}^{-1^c} \otimes_H (\mathcal{P}^!)^*$ where $\mathcal{S}^{-1^c}$ is the operadic desuspension. Then the Koszul dual cooperad of $\text{¡} \oplus \mathcal{P}$ is equal to

$$(\text{¡} \oplus \mathcal{P})^{¡} \cong \{\overline{\mu}^S,\ \text{where } \overline{\mu} \in \mathcal{P}^{¡}(n),\ S \subset \underline{n} \text{ and } |\overline{\mu}^S| = |\overline{\mu}| + |S|\}$$

and the (infinitesimal) cocomposition is given, up to signs, by the (infinitesimal) cocomposition of $\mathcal{P}^{¡}$. To compute the signs, we recall that the corks ¡ have degree -1 and we apply the Koszul rule. The sign $(-1)^{|\overline{\nu}||S_1|}$ in the formula of the proposition comes from the fact that $\overline{\nu}$ passes through the corks indexed by S_1 and the sign $(-1)^{|S_2||S_1''|}$ comes from the fact that the corks indexed by S_2 pass through the corks indexed by S_1''. □

2.6.1.4 Corollary. *The Koszul dual cooperad associated to $qu\mathcal{A}s$ is equal to*

$$qu\mathcal{A}s^{i} = (\text{¡} \oplus \mathcal{A}s)^{i} \cong \mathbb{K}\big[\overline{\mu}_n^S\big],$$

where $\overline{\mu}_n \in \mathcal{A}s^i(n)$, $S \subset \underline{n}$, so $\overline{\mu}_n^S \in u\mathcal{A}s^i(n - |S|)$ and $|\overline{\mu}_n^S| = n - 1 + |S|$. The infinitesimal decomposition map is given by

$$\Delta_{(1)}(\overline{\mu}_n^S) = \sum_{\substack{p+q+r=n \\ p+1+r=m}} (-1)^{(q+1)(r+|S_1|)+|S_2||S_1''|}(\overline{\mu}_m^{S_1}; \underbrace{id, \ldots, id}_{p-|S_1'|}, \overline{\mu}_q^{S_2}, \underbrace{id, \ldots, id}_{r-|S_1''|}),$$

where $\begin{cases} S_1' & \subset & \underline{p} \\ S_2 & \subset & \underline{q} \\ S_1'' & \subset & \{p+2, \ldots, p+1+r\} \end{cases}$ *such that $S = S_1' \sqcup (S_2 + p) \sqcup (S_1'' + q - 1)$ and $S_1 = S_1' \sqcup S_1''$.*

Moreover, the coproduct is given by

$$\Delta(\overline{\mu}_n^S) = \sum_{i_1 - |T_1| + \cdots + i_{m-|T|} - |T_{m-|T|}| = n - |S|} (-1)^\epsilon(\overline{\mu}_m^T; \overline{\mu}_{i_1}^{T_1}, \ldots, \overline{\mu}_{i_{m-|T|}}^{T_{m-|T|}}),$$

where $\begin{cases} T & \subset & \underline{m} \\ T_j & \subset & \underline{i_j} \end{cases}$ *such that $T = R_0 \sqcup \ldots \sqcup R_{m-|T|}$ and*

$$\begin{aligned} S = {} & R_0 \sqcup (T_1 + |R_0|) \sqcup (R_1 + i_1) \sqcup \ldots \sqcup \\ & (T_{m-|T|} + |R_0| + \cdots + |R_{m-|T|-1}| + i_1 + \cdots + i_{m-|T|-1}) \sqcup (R_{m-|T|} + i_1 + \cdots + i_{m-|T|}) \end{aligned}$$

and where

$$\epsilon := |T|(n-m) + \sum_{j=1}^{m-|T|} \big[(i_j - 1)(k - j + |T_1| + \cdots + |T_{j-1}|) + |R_j|(|T_1| + \cdots + |T_j|)\big].$$

Proof. Provided that the degree of $\overline{\mu}_n \in \mathcal{A}s^{¡}(n)$ is $n-1$ and provided the formula for the coproduct in $\mathcal{A}s^{¡}$ given in [LV], chapter 8, where we include the decomposition involving $\overline{\mu}_m = |$ or $\overline{\mu}_q = |$,

$$\Delta_{(1)}(\overline{\mu}_n) = \sum_{\substack{p+q+r=n \\ p+1+r=m}} (-1)^{(q+1)r} (\overline{\mu}_m; \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p}, \overline{\mu}_q, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{r}),$$

Proposition 2.6.1.3 gives the description of $\mathrm{q}u\mathcal{A}s^{¡}$ and of the infinitesimal decomposition map. The coproduct is given in the same way as explained in the proof of Proposition 2.6.1.3 thanks to the coproduct in $\mathcal{A}s^{¡}$ given in [LV] by

$$\Delta(\overline{\mu}_n) = \sum_{i_1+\cdots+i_m=n} (-1)^{\epsilon'} (\overline{\mu}_m; \overline{\mu}_{i_1}, \ldots, \overline{\mu}_{i_m}),$$

where $\epsilon' := \sum_{j=1}^{m} (i_j - 1)(k-j)$. □

2.6.1.5 Proposition. *The operad $qu\mathcal{A}s$ is Koszul, that is*

$$qu\mathcal{A}s \circ_\kappa qu\mathcal{A}s^{i} \xrightarrow{\sim} I.$$

Proof. We remark that

$$\mathrm{q}u\mathcal{A}s \circ_\kappa \mathrm{q}u\mathcal{A}s^{¡} \cong ⫯ \oplus \left\{ \bigoplus_{S \subseteq \underline{n}} (\mathcal{A}s \circ_\kappa \mathcal{A}s^{¡})(n) \right\}_{n \geq 1}.$$

Since  $= 0 =$  in $\mathrm{q}u\mathcal{A}s$, the differential on $(\mathcal{A}s \circ_\kappa \mathcal{A}s^{¡})(n)$ is given by the usual differential on $\mathcal{A}s \circ_\kappa \mathcal{A}s^{¡}$ except for $d(⫯) = ⫯$. Moreover, we know that $(\mathcal{A}s \circ_\kappa \mathcal{A}s^{¡})(n) \xrightarrow{\sim} I(n)$. Thus $\mathrm{q}u\mathcal{A}s \circ_\kappa \mathrm{q}u\mathcal{A}s^{¡} \xrightarrow{\sim} I$. □

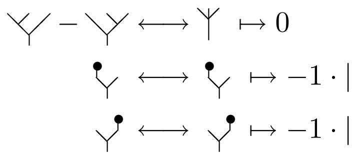
2.6.1.6 Lemma. *The curved cooperad $u\mathcal{A}s^{i}$ is equal to the curved cooperad*

$$u\mathcal{A}s^{i} = \left(qu\mathcal{A}s^{i}, \Delta_{qu\mathcal{A}s^{i}}, 0, \theta\right),$$

where $\Delta_{qu\mathcal{A}s^{i}}$ was made explicit in Corollary 2.6.1.4 and

$$\theta(\mu_n^S) = \begin{cases} -1 \cdot | & \text{if } n = 2 \text{ and, } S = \{1\} \text{ or } S = \{2\} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. For the definitions given in 2.4.1, we remark that the space of generators defining $u\mathcal{A}s$ satisfies Conditions (I) and (II) of Section 2.4.1. According to the definition 2.4.2, we just have to compute the predifferential $d_{u\mathcal{A}s^{¡}}$ and its curvature θ. Since the relations in $u\mathcal{A}s$ have no linear terms, the predifferential $d_{u\mathcal{A}s^{¡}} = 0$. To compute θ, we find the elements of weight 2, which correspond to the relations in $\mathrm{q}u\mathcal{A}s$. We identify each cooperation with the corresponding leading quadratic term of a relation in $u\mathcal{A}s$, and then assign to that operation the opposite of the corresponding constant term of the relation :



□

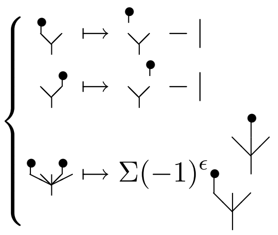
2.6.1.7 Theorem. *The cobar construction on the Koszul dual curved cooperad associated to $u\mathcal{A}s$ provides a cofibrant resolution of $u\mathcal{A}s$*

$$uA_\infty := \Omega u\mathcal{A}s^{¡} \xrightarrow{\sim} u\mathcal{A}s.$$

Proof. By Proposition 2.6.1.5, $\mathrm{q}u\mathcal{A}s$ is Koszul, and then Theorem 2.4.3.1 gives the result. □

We now make the operad uA_∞ more explicit.

The underlying operad of the dg operad $\Omega\, u\mathcal{A}s^{¡}$ is the free operad $\mathcal{F}\left(s^{-1}\overline{\mathrm{q}u\mathcal{A}s^{¡}}\right) = \mathcal{F}\left(s^{-1}\left\{\mu_n^S\right\}\right)$, $n \geq 2$, $S \subset \underline{n}$ and $n = 1, S = \{1\}$, giving a free generating set for $\Omega\, u\mathcal{A}s^{¡}$. As a derivation of the composition structure, the differential $d = d_0 + 0 - d_2$ is completely determined by its action on the generators :



(2.1)

where the last line means : for $(n,\, S) \neq (2,\, \{1\})$ and $(n,\, S) \neq (2,\, \{2\})$, we have

$$d(\overline{\mu}_n^S) = \sum_{\substack{p+q+r=n\\ p+1+r=m}} (-1)^{q(r+|S_1|)+|S_2||S_1'|+p+1} (\overline{\mu}_m^{S_1};\underbrace{\mathrm{id},\, \ldots,\, \mathrm{id}}_{p-|S_1'|},\, \overline{\mu}_q^{S_2},\, \underbrace{\mathrm{id},\, \ldots,\, \mathrm{id}}_{r-|S_1''|}),$$

Remark. On the right-hand side of equation (2.1), the two-level trees now represent the compositions in the free operad.

We obtain the following description for a uA_∞-algebra structure.

2.6.1.8 Proposition. *A* uA_∞-algebra structure *on a dg module* $(A,\, d_A)$ *is given by a collection of maps,* $\mu_1^{\{1\}}$, $\{\mu_n^S\}_{n\geq 2,\, S\subset \underline{n}}$ *where each* μ_n^S *is a map* $A^{\otimes(n-|S|)} \to A$ *of degree* $n + |S| - 2$ *which together satisfy the following identities :*

$$\begin{cases} \partial(\mu_2^{\{1\}}) &=& \mu_2^{\emptyset} \circ (\mu_1^{\{1\}}, -) - \mathrm{id}_A \\ \partial(\mu_2^{\{2\}}) &=& \mu_2^{\emptyset} \circ (-, \mu_1^{\{1\}}) - \mathrm{id}_A \end{cases}$$

and for $(n,\, S) \neq (2,\, \{1\})$ *and* $(n,\, S) \neq (2,\, \{2\})$

$$\partial(\mu_n^S) = \sum_{\substack{p+q+r=n\\ p+1+r=m}} (-1)^{q(r+|S_1|)+|S_2||S_1'|+p+1} \mu_m^{S_1} \circ (\underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{p-|S_1'|},\, \mu_q^{S_2},\, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{r-|S_1''|}).$$

Proof. Since uA_∞ is a quasi-free operad, a map $\mu_A : uA_\infty(A) \to A$ of degree 0 is determined by a collection of maps, $\mu_1^{\{1\}}$, $\{\mu_n^S\}_{n\geq 2,\, S\subset \underline{n}}$ where each μ_n^S is a map $A^{\otimes(n-|S|)} \to A$ of degree $n + |S| - 2$, defined by :

$$\mu_n^S(a_1 \otimes \cdots \otimes a_{n-|S|}) := \mu_A(\overline{\mu}_n^S \otimes a_1 \otimes \cdots \otimes a_{n-|S|}).$$

The fact that the map μ_A is a dg map gives the uA_∞ relations among the μ_n^S. □

Remark. This notion of uA_∞-algebra corresponds to the notion of homotopy unit for an A_∞-algebra given in [FOOO07].

2.6.2 Infinity-morphisms

Following the classical case, we describe the *infinity-morphisms* of algebras over the Koszul resolution of a Koszul inhomogeneous quadratic operad. We give explicit formulae for infinity-morphism of uA_∞-algebras.

Unless we indicate otherwise, for the rest of this section, $\mathcal{P}$ will denote a Koszul inhomogeneous quadratic operad, $\mathcal{P}^{\mathsf{i}}$ its curved Koszul dual cooperad and $\mathcal{P}_\infty := \Omega\mathcal{P}^{\mathsf{i}}$ denotes the Koszul resolution of $\mathcal{P}$ (see Section 2.4).

Let A be a $\mathcal{P}_\infty$-algebra, and denote its structure map by $\mu_A \in \mathrm{Hom}_{\mathsf{dg\ op}}(\mathcal{P}_\infty, \mathrm{End}_A)$. Then by the bar-cobar adjunction 2.3.4.1, we have

$$\mathrm{Hom}_{\mathsf{dg\ operads}}(\Omega\mathcal{P}^{\mathsf{i}}, \mathrm{End}_A) \cong \mathrm{Tw}(\mathcal{P}^{\mathsf{i}}, \mathrm{End}_A).$$

By classical Hom-tensor duality, we have the bijection

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{S}\text{-}\mathsf{Mod}}(\mathcal{P}^{\mathsf{i}}, \mathrm{End}_A) & \cong & \mathrm{Hom}_{\mathsf{dg\ mod}}(\mathcal{P}^{\mathsf{i}}(A), A) \\ \mu_A & \longmapsto & d_{\mu_A}. \end{array}$$

We recall the classical lemma, that we can find for example in [LV].

2.6.2.1 Lemma. *A coderivation of $\mathcal{P}^i(A)$ is completely characterized by its corestriction to the cogenerators*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{mod.}}(\mathcal{P}^i(A), A) & \cong & \mathrm{Coder}(\mathcal{P}^i(A)) \\ d_{\mu_A} & \longmapsto & D^r_{\mu_A}. \end{array}$$

We call a *curved codifferential* any coderivation D of degree -1 which satisfies

$$D^2 = (\theta \circ \mathrm{id}_{\mathcal{P}^{\mathsf{i}}(A)}) \circ \Delta_{\mathcal{P}^{\mathsf{i}}(A)}.$$

We have the following extension of a classical result about codifferentials :

2.6.2.2 Lemma. *A $\mathcal{P}_\infty$-algebra structure on A is equivalent to a codifferential on $\mathcal{P}^i(A)$*

$$\begin{array}{ccc} \mathrm{Tw}(\mathcal{P}^i, \mathrm{End}_A) & \cong & \mathrm{curCodiff}(\mathcal{P}^i(A)) \\ \mu_A & \longmapsto & D_{\mu_A} := d_{\mathcal{P}^i(A)} + D^r_{\mu_A}. \end{array}$$

Proof. The predifferential $d_{\mathcal{P}^{\mathsf{i}}}$ is a coderivation so the map $D_{\mu_A} := d_{\mathcal{P}^{\mathsf{i}}(A)} + D^r_{\mu_A}$ is a coderivation. The construction here is the same as the construction in Section 2.5.2 with $D^r_{\mu_A} = d^r_{\mu_A} \circ_{\mathcal{P}} \mathrm{id}_A$, so $\mu_A \in \mathrm{Tw}(\mathcal{P}^{\mathsf{i}}, \mathrm{End}_A)$ implies $D^2_{\mu_A} = (\theta \circ \mathrm{id}_{\mathcal{P}^{\mathsf{i}}(A)}) \circ \Delta_{\mathcal{P}^{\mathsf{i}}(A)}$.

According to the proof of Lemma 2.5.2.1, we only have to remark that $D^r_{\partial(\mu_A)+\mu_A\star\mu_A-\Theta} = D^2_{\mu_A} - (\theta \circ id_{\mathcal{P}^{\mathsf{i}}(A)}) \circ \Delta_{\mathcal{P}^{\mathsf{i}}(A)} = 0$ implies $d_A \circ d_{\mu_A} + d_{\mu_A} \circ d_{\mathcal{P}^{\mathsf{i}}(A)} + d_{\mu_A\star\mu_A} - d_\Theta = (D^r_{\partial(\mu_A)+\mu_A\star\mu_A-\Theta})^{|A} = 0$. Since $d_A \circ d_{\mu_A} + d_{\mu_A} \circ d_{\mathcal{P}^{\mathsf{i}}(A)} + d_{\mu_A\star\mu_A} - d_\Theta$ is sent to $\partial(\mu_A) + \mu_A \star \mu_A - \Theta$ and 0 is sent to 0 by reversing the bijection in the Hom-tensor duality, we get the result. ☐

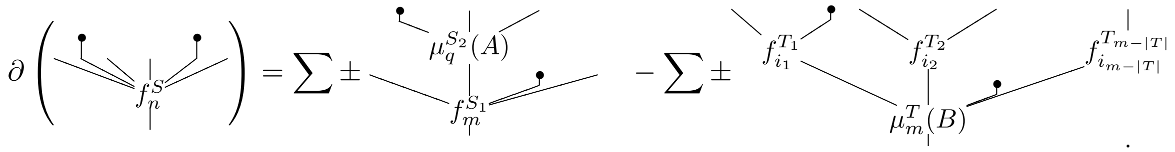
Infinity-morphism of $\mathcal{P}_\infty$-algebras

Let A and B be two $\mathcal{P}_\infty$-algebras, with structure maps μ_A and μ_B. A ∞*-morphism* $A \rightsquigarrow B$ *of $\mathcal{P}_\infty$-algebras* is a dg $\mathcal{P}^{\mathsf{i}}$-coalgebra map

$$F : (\mathcal{P}^{\mathsf{i}}(A), D_{\mu_A}) \to (\mathcal{P}^{\mathsf{i}}(B), D_{\mu_B}).$$

This description of ∞-morphisms makes it clear that $\mathcal{P}_\infty$-algebras, ∞-morphisms, and composition given by composition of dg $\mathcal{P}^{¡}$-coalgebra maps forms a category.

A $u\mathcal{A}s^{¡}$-coalgebras map $F : u\mathcal{A}s^{¡}(A) \to u\mathcal{A}s^{¡}(B)$ is characterized by its corestriction to B, that is F is determined by a collection of maps $f_n^S : A^{\otimes(n-|S|)} \to B$. The fact that F commutes with the differentials is equivalent to a family of equations on the f_n^S. Pictorially, the collection of maps f_n^S satisfy :



2.6.2.3 Proposition. *Let A, B be two uA_∞-algebras, and let $\mu_n^S(A), \mu_n^S(B)$ be the respective structure maps. An ∞-morphism between A and B is a collection of maps*

$$\{f_n^S : A^{\otimes(n-|S|)} \to B\}_{n\geq 1,\, S\subset \underline{n}} \text{ of degree } n+|S|-1,$$

satisfying : for $n=1$, $d_A \circ f_1^{\emptyset} = f_1^{\emptyset} \circ d_A$, that is $f_1^{\emptyset}$ is a chain map, and for $n+|S| \geq 2$, $\partial(f_n^S) =$

$$\sum_{\substack{p+q+r=n\\ p+1+r=m}} (-1)^{p+q(r+|S_1|)+|S_2||S_1''|} f_m^{S_1} \circ (\underbrace{id_A, \ldots, id_A}_{p-|S_1'|}, \mu_q^{S_2}(A), \underbrace{id_A, \ldots, id_A}_{r-|S_1''|})$$

$$+ \sum_{i_1-|T_1|+\cdots+i_{m-|T|}-|T_{m-|T|}|=n-|S|} -\epsilon(-1)^{(m+|T|-1)(n-m+|S|-|T|)} \mu_m^T(B) \circ \left(f_{i_1}^{T_1}, \ldots, f_{i_{m-|T|}}^{T_{m-|T|}}\right),$$

where $\begin{cases} S_1' & \subset & \underline{p} \\ S_2 & \subset & \underline{q} \\ S_1'' & \subset & \{p+2, \ldots, p+1+r\} \end{cases}$ *such that* $S = S_1' \sqcup (S_2+p) \sqcup (S_1''+q-1)$ *and* $S_1 = S_1' \sqcup S_1''$, *where* $\begin{cases} T & \subset & \underline{m} \\ T_j & \subset & \underline{i_j} \end{cases}$ *such that* $T = R_0 \sqcup \ldots \sqcup R_{m-|T|}$ *and*

$$\begin{aligned} S \;=\; & R_0 \sqcup (T_1 + |R_0|) \sqcup (R_1 + i_1) \sqcup \ldots \sqcup \\ & (T_{m-|T|} + |R_0| + \cdots + |R_{m-|T|-1}| + i_1 + \cdots + i_{m-|T|-1}) \sqcup (R_{m-|T|} + i_1 + \cdots + i_{m-|T|}) \end{aligned}$$

and where $\epsilon := |T|(n-m) + \sum_{j=1}^{m-|T|} \left[(i_j-1)(k-j+|T_1|+\cdots+|T_{j-1}|) + |R_j|(|T_1|+\cdots+|T_j|)\right]$.

Proof. An ∞-morphism $A \rightsquigarrow B$ is a $u\mathcal{A}s^{¡}$-coalgebra morphism $F : u\mathcal{A}s^{¡}(A) \to u\mathcal{A}s^{¡}(B)$. Such a morphism is completely determined by its image on the cogenerators of $u\mathcal{A}s^{¡}(B)$, that is by a map $f : u\mathcal{A}s^{¡}(A) \to B$ (of degree 0), or equivalently by a collection of maps $\{f_n^S : A^{\otimes(n-|S|)} \to B\}_{n\geq 1,\, S\subset\underline{n}}$ of degree $n+|S|-1$. The fact that F commutes with the predifferential is equivalent to the following commutative diagram

$$\begin{array}{ccccc} u\mathcal{A}s^{¡}(A) & \xrightarrow{\Delta\circ id_A} & u\mathcal{A}s^{¡} \circ u\mathcal{A}s^{¡}(A) & \xrightarrow{id\circ f} & u\mathcal{A}s^{¡}(B) \\ \downarrow{\scriptstyle d_1+d_2} & & & & \downarrow{\scriptstyle d_B + d_2^{|B}} \\ u\mathcal{A}s^{¡}(A) & & \xrightarrow{\quad f \quad} & & B. \end{array}$$

Making this diagram explicit gives exactly the formulae of the Proposition. □

Example. For $n = 1$ and $S = \{1\}$, the formula gives

$$\partial\left(f_1^{\{1\}}\right) = \begin{array}{c}\mu_1^{\{1\}}(A)\\ | \\ f_1^{\emptyset}\end{array} - \mu_1^{\{1\}}(B) \quad ,$$

that is, the element $f_1^{\{1\}}$ bounds the failure of $f_1^{\emptyset}$ to preserve the unit.

Remark. In [Lyu10], Lyubashenko proposes a definition for ∞-morphism between uA_∞-algebras as a resolution of bimodule. It would be interesting to compare his definition with our definition.

Before we end the section, we use the results above to give the following definition.

2.6.2.4 Definition. *A ∞-morphism of $\mathcal{P}_\infty$-algebras $F : A \rightsquigarrow B$ is a* quasi-isomorphism *if the chain map $f_1^{\emptyset} : A \to B$ induces an isomorphism in homology.*

2.6.3 Rectification

We now prove that for every uA_∞-algebra A there is a universal ∞-quasi-morphism I_A between A and a $u\mathcal{A}s$-algebra. This universal morphism takes the form of the unit of an adjunction. We make use of the bar and cobar constructions of algebras over Koszul operads (Sections 2.5.2, 2.5.2) for uA_∞-algebras and $u\mathcal{A}s$-algebras.

The twisting morphisms $\iota : u\mathcal{A}s^{\text{i}} \to \Omega u\mathcal{A}s^{\text{i}} = uA_\infty$ and $\kappa : u\mathcal{A}s^{\text{i}} \to u\mathcal{A}s$ are defined in Section 2.3.4 and 2.4.3.

2.6.3.1 Lemma. *Let A be uA_∞-algebra. The morphism of dg $\mathbb{S}$-modules $A \rightarrowtail \Omega_\kappa \mathrm{B}_\iota A$ is a quasi-isomorphism.*

Proof. We endow $u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota \Omega u\mathcal{A}s^{\text{i}}$ with a filtration F_p given by

$$F_p(u\mathcal{A}s \circ u\mathcal{A}s^{\text{i}} \circ \Omega u\mathcal{A}s^{\text{i}}) = \bigoplus_{\omega+m\leq p} (u\mathcal{A}s \circ u\mathcal{A}s^{\text{i}})^{(\omega)} \circ (\Omega u\mathcal{A}s^{\text{i}})_m.$$

Moreover we endow $\Omega u\mathcal{A}s^{\text{i}}$ with a filtration given by the homological degree, so that the morphism $\Omega u\mathcal{A}s^{\text{i}} \rightarrowtail u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota \Omega u\mathcal{A}s^{\text{i}}$ preserves the filtrations. Since the weight on $u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}}$ is non-negative and $\Omega u\mathcal{A}s^{\text{i}}$ is non-negatively graded, the filtrations are bounded below. Moreover, the filtrations are exhaustive. Thus, we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) to obtain

$$E^\bullet_{p,q} \Rightarrow \mathrm{H}_{p+q}(u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota \Omega u\mathcal{A}s^{\text{i}}) \text{ and } {E'}^\bullet_{p,q} \Rightarrow \mathrm{H}_{p+q}(\Omega u\mathcal{A}s^{\text{i}})$$

and an induced morphism between the spectral sequences. The differential on $E^0_{p,q}$ coincides with the differential on $\mathrm{q}u\mathcal{A}s \circ \mathrm{q}u\mathcal{A}s^{\text{i}}$, so Proposition 2.6.1.5 shows that $E^1_{p,q} \cong {E'}^1_{p,q}$. It follows that $E^r_{p,q} \cong {E'}^r_{p,q}$ for all $r \geq 1$ and we get that $\Omega u\mathcal{A}s^{\text{i}} \xrightarrow{\sim} u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota \Omega u\mathcal{A}s^{\text{i}}$.

We have $\Omega_\kappa \mathrm{B}_\iota A \cong (u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota uA_\infty) \circ_{uA_\infty} A$. The short exact sequence

$$(u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota uA_\infty) \circ uA_\infty \circ A \to (u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota uA_\infty) \circ A \to (u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{i}} \circ_\iota uA_\infty) \circ_{uA_\infty} A$$

induces a long exact sequence in homology. Since we work over a field of characteristic 0, the ring $\mathbb{K}[\mathbb{S}_n]$ is semi-simple by Maschke's theorem, that is every $\mathbb{K}[\mathbb{S}_n]$-module is projective. So the Künneth formula implies that $\mathrm{H}_\bullet((u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{¡}} \circ_\iota uA_\infty) \circ uA_\infty \circ A) \cong \mathrm{H}_\bullet(uA_\infty) \circ \mathrm{H}_\bullet(uA_\infty) \circ \mathrm{H}_\bullet(A)$ and $\mathrm{H}_\bullet((u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{¡}} \circ_\iota uA_\infty) \circ A) \cong \mathrm{H}_\bullet(uA_\infty) \circ \mathrm{H}_\bullet(A)$. Finally, this gives that $\mathrm{H}_\bullet((u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{¡}} \circ_\iota uA_\infty) \circ_{uA_\infty} A) \cong \mathrm{H}_\bullet(uA_\infty) \circ_{\mathrm{H}_\bullet(uA_\infty)} \mathrm{H}_\bullet(A) \cong \mathrm{H}_\bullet(A)$. □

2.6.3.2 Theorem (Universal rectification)**.** *Let A be a uA_∞-algebra. There is a dg $u\mathcal{A}s$-algebra, $\Omega_\kappa \mathrm{B}_\iota A$ and an ∞-quasi-isomorphism $I_A : A \overset{\sim}{\rightsquigarrow} \Omega_\kappa \mathrm{B}_\iota A$ so that for any dg $u\mathcal{A}s$-algebra B and any ∞-morphism $F : A \rightsquigarrow B$, there is a unique dg $u\mathcal{A}s$-algebras map $\tilde{f} : \Omega_\kappa \mathrm{B}_\iota A \to B$ so that $F = \tilde{f} \circ I_A$, that is the following diagram commutes :*

$$\begin{array}{ccc} \Omega_\kappa \mathrm{B}_\iota A & & \\ {\scriptstyle I_A}\uparrow & \overset{\tilde{f}}{\searrow} & \\ A & \xrightarrow[F]{} & B \end{array}$$

Proof. The map I_A is defined by

$$i_n^S(a_1, \ldots, a_{n-|S|}) = \mu_n^S(a_1, \ldots, a_{n-|S|}) \in \mathrm{B}_\iota A \hookrightarrow \Omega_\kappa \mathrm{B}_\iota A.$$

By direct computation, this map is a ∞-morphism between the uA_∞-algebras A and $\Omega_\kappa \mathrm{B}_\iota A$. To see that this map is a quasi-isomorphism, observe that $i_1^\emptyset$ is equal to the inclusion map in Lemma 2.6.3.1. To define the map $\tilde{f}$, we note that the ∞-morphism of uA_∞-algebras F is determined by the collection of maps f_n^S, or by the collection of elements $\{f_n^S(a_1, \ldots, a_{n-|S|})\}$ in B. We define the module map $\mathrm{B}_\iota A \to B$ by

$$\mu_n^S(a_1, \ldots, a_{n-|S|}) \mapsto f_n^S(a_1, \ldots, a_{n-|S|}).$$

This map is a dg module map if and only if F is a ∞-morphism. Since the $u\mathcal{A}s$-algebra $\Omega_\kappa \mathrm{B}_\iota A$ is freely generated by $\{\mu_n^S(a_1, \ldots, a_{n-|S|})\}$, we define the map $\tilde{f}$ to be the lift of the above dg map to a $u\mathcal{A}s$-algebras map $\Omega_\kappa \mathrm{B}_\iota A \to B$. By construction we have $\tilde{f} \circ I_A = F$. □

Let us interpret the result above in terms of the categories of algebras. Since we have an operad map $uA_\infty \twoheadrightarrow u\mathcal{A}s$, we have an inclusion functor (one-to-one on objects and on morphisms) $u\mathcal{A}s$-$\mathsf{alg} \hookrightarrow uA_\infty$-$\mathsf{alg}$, which we denote by i. Denote by R the assignment that takes each uA_∞-algebra A to the $u\mathcal{A}s$-algebra $R(A) = \Omega_\kappa B_\iota(A)$. Because the arrow $A \xrightarrow{I_A} iR(A)$ is universal, R can be extended to morphisms so that it becomes a functor from uA_∞-alg $\to u\mathcal{A}s$-alg :

$$\begin{array}{ccc} R(A) & \overset{R(F)}{\dashrightarrow} & R(B) \\ \uparrow & & \uparrow \\ A & \xrightarrow{F} & B. \end{array}$$

We summarize in the following proposition.

2.6.3.3 Proposition. *The functor i, the object-assignment R, and the universal morphisms $A \xrightarrow{I_A} iR(A)$ determine the extension of R to a functor $R : uA_\infty$-$\mathsf{alg} \to u\mathcal{A}s$-$\mathsf{alg}$ so that $I : \mathrm{id} \to iR$ is the unit of an adjunction :*

$$uA_\infty\text{-}\mathsf{alg} \underset{i}{\overset{R}{\rightleftarrows}} u\mathcal{A}s\text{-}\mathsf{alg}.$$

Proof. See Mac Lane [Mac98] chapter 4, theorem 1.

It is tempting to try to put a model category structure on the right-hand side so that this pair of functors becomes some kind of Quillen equivalence, as Lefevre-Hasegawa [LH03] did for A_∞-algebras and $\mathcal{A}s$-algebras. (Actually, A_∞-algebras are not quite a model category, see the referenced paper for more details). Instead we observe that each functor takes quasi-isomorphisms to quasi-isomorphisms, and so each functor induces a functor between the homotopy categories (localizations of each category by its quasi-isomorphisms). We claim these induced functors are an adjoint-equivalence of the homotopy categories.

2.6.4 Transfer formulae

In this section we provide formulae, based on labelled trees, for the pullback of a uA_∞-structure along a strong deformation retract.

For this entire section, suppose V, A are chain complexes, and

$$V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h\ ,$$

is a strong deformation retract, i.e., p and i are chain maps, where $p\circ i = \mathrm{id}_V$ and $d_A h + h d_A = \mathrm{id}_A - i\circ p$. Moreover, suppose A is a uA_∞-algebra, with structure map μ_A.

2.6.4.1 Definition. *Let $n \geq 2, S \subset \underline{n}$, we define the set $\mathcal{T}_n^S$ be the set of planar, rooted trees, with n leaves, and a cork above each ith leaf if $i \in S$ which is labelled by either the word "connected" or "disconnected." We define $\mathcal{T}_1^\emptyset = \{\,|\,\}$ and $\mathcal{T}_1^{\{1\}} = \{\,{}^{\bullet}\!\!|^{\text{connected}}\}$.*

2.6.4.2 Definition. *Let $T \in \mathcal{T}_n^S$, and let v be any internal vertex in T. We denote by $\mathrm{in}(v)$ the ordered (left-to-right) set of incoming edges to the vertex v. For each element $i \in \mathrm{in}(v)$, we define l_i and c_i as follows :*

1. *l_i is the total number of leaves without connected corks in the tree T whose (unique) path to the root passes through edge i*
2. *c_i is the total number of incoming edges to v without connected corks to the* right *of edge i.*

2.6.4.3 Definition. *For any $T \in \mathcal{T}_n^S$ and any internal vertex $v \in T$, we define*

$$\epsilon(v) = \sum_{1\leq i<j\leq|\mathrm{in}(v)|} (l_i+1)\,l_j \quad + \sum_{\substack{i\in\mathrm{in}(v)\\ \textit{with a connected}\\ \textit{cork on it}}} c_i.$$

For any tree $T \in \mathcal{T}_n^S$, we set

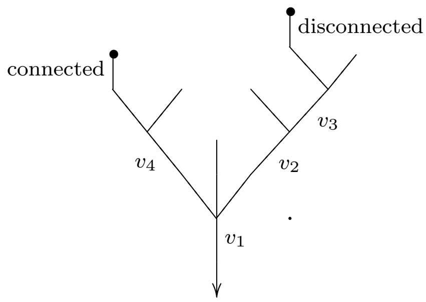
$$\epsilon(T) = \sum_{\substack{\textit{internal vertices}\\ v\in T}} \epsilon(v). \tag{2.2}$$

2.6.4.4 Definition. *Let $g_{structure} : \mathcal{T}_n^S \to \mathrm{Hom}(V^{\otimes(n-|S|)}, V)$ be the set map that takes an element $T \in \mathcal{T}_n^S$ and assigns to each vertex v the operation $\mu_{\mathrm{in}(v)}^{S(v)}(A)$ where $S(v)$ are the positions of the* connected *corks, the operation μ_1^1 to each disconnected cork, the homotopy h*

to each internal edge (that is not the outgoing edge of a connected cork), and the map i to each leaf without a cork above it, and the map p to the root of the tree. After this assignment, one composes the operations as indicated by internal edges to arrive at an operation $V^{\otimes(\text{in}(v)-|S|)} \to V$. Let $g_{morphism} : \mathcal{T}_n^S \to \text{Hom}(V^{\otimes(n-|S|)}, A)$ be the set map that takes an element $T \in \mathcal{T}_n^S$ and assigns to the tree the same element as $g_{structure}(T)$, but with the homotopy h assigned to the root, rather than the map p.

Examples. Let T be the element of $\mathcal{T}_5^{\{1,4\}}$ that looks like

$T =$

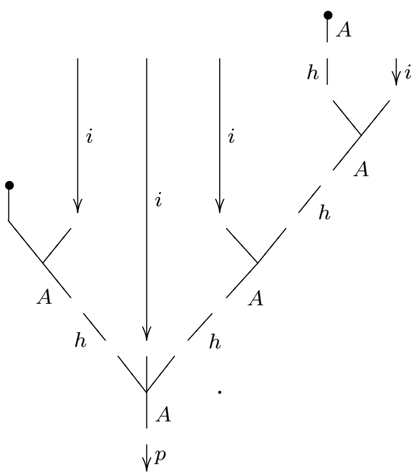


The sign $(-1)^{\epsilon(T)}$ for this tree is given by

$$\begin{aligned}
\epsilon(T) &= \epsilon(v_1) + \epsilon(v_2) + \epsilon(v_3) + \epsilon(v_4) \\
&= [(1+1)\cdot 1 + (1+1)\cdot 3 + (1+1)\cdot 3 + 0] + [(1+1)\cdot 2 + 0] + [(1+1)\cdot 1 + 0] + [(1+1)\cdot 1 + 1] \\
&= 14 + 4 + 2 + 3 \\
&= 23 \\
&\equiv 1 \bmod 2.
\end{aligned}$$

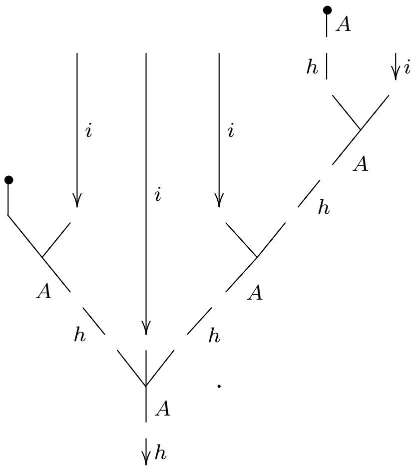
The operation assigned to the tree $T, g_{\text{structure}}(T)$, is given by the following composition of operations :

$g_{\text{structure}}(T) =$



while the morphism assigned to the tree T, $g_{\text{morphism}}(T)$ is given by :

$g_{\text{morphism}}(T) =$



2.6.4.5 Proposition. *The maps defined by*

$$\mu_n^S(V) := \sum_{T \in \mathcal{T}_n^S} (-1)^{\epsilon(T)} g_{structure}(T)\,. \tag{2.3}$$

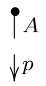
give V the structure of a uA_∞-algebra. Moreover, the maps defined by

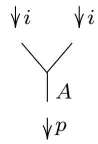
$$i_n^S := \sum_{T \in \mathcal{T}_n^S} (-1)^{\epsilon(T)} g_{morphism}(T)\,. \tag{2.4}$$

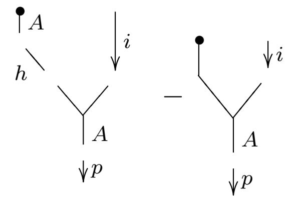
provide a ∞-quasi-isomorphism of uA_∞-algebras $I : V \stackrel{\sim}{\rightsquigarrow} A$.

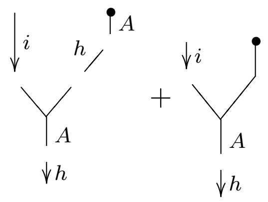
Proof. A combinatorial argument similar to the argument for transferring A_∞-structures [Mar06] will suffice. $\square$

Examples. For small values of n, the transferred structure is given by

$$\mu_1^{\{1\}}(V) := p \circ \mu_1^{\{1\}}(A) =$$


$$\mu_2^{\emptyset}(V) := p \circ \mu_2^{\emptyset}(A) \circ i^{\otimes 2} =$$


$$\mu_2^{\{1\}}(V) :=$$


$$i_2^{\{2\}}(V) :=$$


For the reader familiar with transfer of A_∞-structures, restricting attention to the operations $\mu_n^{\emptyset}(V)$ recovers the familiar transfer formulae [Kad83, Mer99, KS06, Mar06, LV].

Remark. Though our signs differ from [Mar06], we use his ideas to develop a coherent sign convention for our transfer formulae. The reader should note that our function $\epsilon(v)$ differs from the $\theta(v)$ in [Mar06] even on the operations $\mu_n^{\emptyset}(V)$, in small ways, such as right-to-left orientation of trees instead of left-to-right. Instead we choose our signs to agree with [Sta63, LV] when restricted to the classical A_∞ operations.

2.6.5 Comparing unital-(associative-infinity) and (unital-associative)-infinity

In previous sections, we have developed the definition of the operad uA_∞ whose algebras are *homotopy unital* A_∞-algebras. There have been several definitions of homotopy unital A_∞-algebras [FOOO07, KS06, Lyu02], and these notions have been compared in [LM06]. There is also a definition of *strictly unital* A_∞-algebras [KS06, FOOO07]—we will refer to these as $\mathtt{su}A_\infty$-algebras throughout this section—they may be thought of as unital-(associative-infinity) algebras as opposed to our (unital-associative)-infinity algebras. We will compare uA_∞-algebras to $\mathtt{su}A_\infty$-algebras. This comparison includes Theorem 2.6.5.3, which states that every uA_∞-algebra has an equivalent unital-A_∞-structure on its homology. We demonstrate that this theorem is fairly general, and applies to many algebraic structures with units, including unital commutative associative algebras, unital Batalin-Vilkovisky algebras, and co-algebraic versions of these structures.

First we define $\mathtt{su}A_\infty$-algebras and their ∞-morphisms.

2.6.5.1 Definition. *An* $\mathtt{su}A_\infty$-algebra $(A, \{\mu_n\}_{n\geq 1}, e)$ *is an* A_∞*-algebra* $(A, \{\mu_n\}_{n\geq 1})$ *with* $e \in A$ *such that* $d_A(e) = 0$ *and* e *is a left and right unit for* μ_2*, and* e *annihilates* μ_n *for* $n \geq 3$ *[KS06].*

Remarks. 1. There exists a dg-operad whose algebras are precisely $\mathtt{su}A_\infty$-algebras, and we denote it by $\mathtt{su}A_\infty$. Furthermore, the operad $\mathtt{su}A_\infty$ is the quotient of uA_∞ by the ideal generated by $\{\mu_n^S\}_{n\geq 2,\,|S|\geq 1}$. A quick computation yields that this map is a quasi-isomorphism.

2. The operad $\mathtt{su}A_\infty$ is not cofibrant. If it were, the lifting property would imply that it is a retract of uA_∞ by the quotient map $uA_\infty \xrightarrow{\sim} \mathtt{su}A_\infty$, which a computation shows is impossible.

We now describe a diagram of categories of algebras. We will use the following notation
- $\mathcal{A}s$-alg : the category of associative algebras with algebra homomorphisms
- $u\mathcal{A}s$-alg : the category of unital associative algebras with algebra homomorphisms that preserve the unit
- ∞-A_∞-alg : the category of A_∞-algebras with ∞-morphisms
- ∞-uA_∞-alg : the category of uA_∞ algebras with ∞-morphisms
- $\mathtt{su}A_\infty$-alg : the category of $\mathtt{su}A_\infty$-algebras with the A_∞ ∞-morphisms for which f_1 preserves the unit and f_n annihilates it (for $n \geq 2$)

First, we have the following diagram of operads :

$$\begin{array}{ccccc} u\mathcal{A}s & \overset{\sim}{\twoheadleftarrow} & \mathsf{su}A_\infty & \overset{\sim}{\twoheadleftarrow} & uA_\infty \\ \uparrow & & & & \uparrow \\ \mathcal{A}s & & \overset{\sim}{\twoheadleftarrow} & & A_\infty \end{array}$$

On the categories of algebras, the diagram becomes :

$$\begin{array}{ccccc} u\mathcal{A}s\text{-alg} & \longrightarrow & \mathsf{su}A_\infty\text{-alg} & \longrightarrow & \infty\text{-}uA_\infty\text{-alg} \\ \downarrow & & & & \downarrow \\ \mathcal{A}s\text{-alg} & & \longrightarrow & & \infty\text{-}A_\infty\text{-alg} \end{array}$$

We proved earlier (Section 2.6.3) that the first of the following composition of horizontal inclusions

$$\begin{cases} u\mathcal{A}s\text{-alg} & \to & \infty\text{-}uA_\infty\text{-alg}, \\ \mathcal{A}s\text{-alg} & \to & \infty\text{-}A_\infty\text{-alg} \end{cases}$$

has a left-adjoint, $\Omega_\iota B_\kappa$, which we called the universal rectification (it is known that the second has a similarly defined left-adjoint). Each of the inclusions,

$$\begin{cases} u\mathcal{A}s\text{-alg} & \to & \mathcal{A}s\text{-alg}, \\ \mathsf{su}\text{-}A_\infty\text{-alg} & \to & A_\infty\text{-alg} \end{cases}$$

has a left-adjoint as well, given by adjoining an element u and extending the product(s) to make u a strict unit (with appropriate annihilation conditions, in the case of $\mathtt{su}A_\infty$-algebras).

We now analyze the relationship between uA_∞ and $\mathtt{su}A_\infty$ via our transfer formulae.

2.6.5.2 Theorem. *Let* $V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h$, *be a strong deformation retract, and* $\{\dot{|}^A, \curlyvee_A\}$ *a strict* $u\mathcal{A}s$*-structure on* A*. Suppose further that* $h\left(\dot{|}^A\right) = 0$*. Then the operations* $\mu_n^S(V)$ *given by the transfer formulae (see definition in Proposition 2.6.4.5) have the property that*

$$\mu_n^S(V) = 0$$

whenever $n \geq 2$ *and* $|S| \geq 1$*. Furthermore, the* uA_∞*-morphism structure* J *on the chain map* i *has the property that whenever* $|S| \geq 1$*,*

$$J_n^S = 0,$$

even when $n = 1$*. In particular this means that the transferred* uA_∞ *structure is an* $\mathtt{su}A_\infty$*-algebra, and the* uA_∞*-*∞ *quasi-isomorphism is an* $\mathtt{su}A_\infty$*-*∞ *quasi-isomorphism.*

Proof. For $n \geq 2, |S| \geq 1$, each summand in $\mu_n^S(V)$ contains as some part of the diagram (of compositions) the following composite :

$$\begin{matrix}\dot{|}^A \\ |_h\end{matrix} = h(\dot{|}^A) = 0,$$

so each of those operations is itself 0. The same fact gives the result for J, along with the fact that

$$J_1^{\{1\}} = \begin{matrix}\dot{|}^A \\ |_h\end{matrix} = 0.$$

The vanishing of these higher operations and morphisms implies that the transferred uA_∞ structure and morphism are strictly-unital, because the operad $\mathtt{su}A_\infty$ is the quotient of uA_∞ by precisely these operations. □

Remark. We point out that since we are working over a field, and $d(\dot{|}^A) = 0$, it is always possible to choose a strong deformation retract between V and A so that $h\left(\dot{|}^A\right) = 0$ (provided, of course, V is equivalent to A).

The following corollary of Theorem 2.6.5.2 is an analogue of Theorem 5.4.2' in [FOOO07], which they prove in both the filtered and unfiltered setting.

2.6.5.3 Corollary. *Let* A *be a* uA_∞*-algebra. Then there exists a* $u\mathcal{A}s$*-algebra* R*, and an* $\mathtt{su}A_\infty$*-algebra structure on* $H_\bullet(A)$ *so that* $A \xrightarrow{\sim} R$ *and* $H_\bullet(A) \xrightarrow{\sim} R$*. That is, for an arbitrary* uA_∞*-algebra* A*, there is a minimal model for* A *which is an* $\mathtt{su}A_\infty$*-algebra.*

Proof. By Theorem 2.6.3.2, we have $I_A : A \stackrel{\sim}{\rightsquigarrow} \Omega_\kappa \mathrm{B}_\iota A = R(A)$. Note that in particular, $H_\bullet(A) \simeq_i H_\bullet(R(A))$. We will denote both by H.

Since there exist strong deformation retracts $H \underset{p}{\overset{i}{\rightleftarrows}} R(A) \circlearrowleft h$ where h annihilates the unit, transferring the $u\mathcal{A}s$ structure on $\Omega_\kappa \mathrm{B}_\iota A$ along any such strong deformation retract, by Theorem 2.6.5.2, gives an equivalent $\mathtt{su}A_\infty$-algebra structure on H. □

In what follows, we prove an analogous theorem for a wide class of properads $\mathcal{P}$. First, we must say what we mean by a "unital version" of $\mathcal{P}$.

2.6.5.4 Definition. *Let $\mathcal{P} = \mathcal{F}(V)/(R)$ be an inhomogeneous quadratic properad. We say an inhomogeneous quadratic properad $u\mathcal{P} = \mathcal{F}(⫯ \oplus V)/(R \oplus R')$ is a* unital version *of $\mathcal{P}$ if and only if*

- *the map of operads $\mathcal{P} \to u\mathcal{P}$ induced by the inclusion $V \to ⫯ \oplus V$ is injective,*
- *the induced map $q\mathcal{P} \to qu\mathcal{P}$ together with the inclusion $⫯ \rightarrowtail qu\mathcal{P}$ gives an isomorphism of operads $⫯ \oplus q\mathcal{P} \simeq qu\mathcal{P}$,*
- *the inhomogeneous quadratic relations associated to a single composition of the cork with an operation in $\mathcal{P}$ has only the leading quadratic term and a constant term.*

Remark. The name "unital version" for $u\mathcal{P}$ is not always appropriate. For example, if we take for $\mathcal{P}$ the operad $\mathcal{L}ie$, then the operad $c\mathcal{L}ie$, which governs Lie algebras with a designated central element, is a unital version of $\mathcal{L}ie$ as in the above (where the constant term is taken to be zero), though of course a central element is far from what we typically think of as a unit.

Suppose $u\mathcal{P}$ is a unital version of $\mathcal{P}$, and that both are inhomogeneous Koszul properads. Then $\Omega u\mathcal{P}^{¡} =: u\mathcal{P}_\infty \xrightarrow{\sim} u\mathcal{P}$, and by Proposition 2.6.1.3, the underlying coproperad of $u\mathcal{P}^{¡}$ is isomorphic to $⫯^{¡} * q\mathcal{P}^{¡}$. This observation allows us to define "strictly-unital $\mathcal{P}_\infty$ algebras," or $\mathtt{su}\mathcal{P}_\infty$-algebras, as we defined $\mathtt{su}A_\infty$: we can identify the "unital homotopies" as those made of a (co)operation $\mu_\alpha \in q\mathcal{P}^{¡}$ with some configuration S of corks above the leaves.

2.6.5.5 Definition. *Suppose $u\mathcal{P}$ is a unital version of $\mathcal{P}$, and that both are inhomogeneous Koszul properads. We define the properad $\mathtt{su}\mathcal{P}_\infty$ as the quotient of $u\mathcal{P}_\infty$ by the (differential) properadic ideal generated by the operations*

$$\left\{\mu_\alpha^S : \text{ for } \mu_\alpha \in q\mathcal{P}^{i} \text{ and } S \neq \emptyset\right\}.$$

That is, we quotient by all the "unital relations" and "unital homotopies."

Remark. Though it looks like we only quotient by unital homotopies in the above, taking the *differential* ideal generated by the unital homotopies means we also quotient by the image under $d_{u\mathcal{P}_\infty}$ of the unital homotopies with weight 2, which are precisely the unital relations.

If $u\mathcal{P}$ is a unital version of $\mathcal{P}$, and both are inhomogeneous Koszul properads, the quotient map $\mathtt{su}\mathcal{P}_\infty \twoheadrightarrow u\mathcal{P}$ is a quasi-isomorphism. In general, however, the operad $\mathtt{su}\mathcal{P}_\infty$ is not cofibrant. Even so, we have the following transfer theorem for $\mathtt{su}\mathcal{P}_\infty$-algebras.

2.6.5.6 Theorem. *Let $u\mathcal{P}$ be a unital version of $\mathcal{P}$, and suppose both are inhomogeneous Koszul. Then given any $u\mathcal{P}$-algebra A and a strong deformation retract $V \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h$, where the homotopy h satisfies $h(⫯^A) = 0$, the transferred $(u\mathcal{P})_\infty$-algebra is an $\mathtt{su}\mathcal{P}_\infty$-algebra structure, and the $u\mathcal{P}_\infty$ ∞-morphism structure on J is an $\mathtt{su}\mathcal{P}_\infty$ ∞-morphism.*

2.6.5.7 Corollary. *Suppose we have properads $\mathcal{P}, u\mathcal{P}$ as in Theorem 2.6.5.6, and suppose A is a $u\mathcal{P}$-algebra. Then there is an $\mathtt{su}\mathcal{P}_\infty$-algebra structure on the homology of A and an $\mathtt{su}\mathcal{P}_\infty$ ∞-quasi isomorphism $H \stackrel{\sim}{\rightsquigarrow} A$.*

Proof. It is a corollary of the proof for $u\mathcal{A}s$, given the universal rectification and transfer formulae for arbitrary Koszul inhomogeneous quadratic properads $u\mathcal{P}$ (which are not made explicit in this paper). $\square$

2.6.5.8 Corollary. *In the following list of pairs, $(\mathcal{P}, u\mathcal{P})$, $u\mathcal{P}$ is a unital model for $\mathcal{P}$ and both are inhomogeneous Koszul. In particular, each $u\mathcal{P}$-algebra structure may be transferred to an equivalent $\boldsymbol{su}\mathcal{P}_\infty$ structure on homology in the above sense.*

1. *$(\mathcal{C}om, u\mathcal{C}om)$, where $u\mathcal{C}om$ is the operad governing unital commutative associative algebras,*
2. *$(\mathcal{L}ie, c\mathcal{L}ie)$, where $c\mathcal{L}ie$ is the operad governing Lie algebras with a designated central element,*
3. *$(\mathcal{G}erst, u\mathcal{G}erst)$, where $u\mathcal{G}erst$ is the operad governing unital Gerstenhaber algebras, ie, Gerstenhaber algebras with a unit for the commutative associative product which is annihilated by the bracket,*
4. *$(\mathcal{BV}, u\mathcal{BV})$, where $u\mathcal{BV}$ is the operad governing unital $\mathcal{BV}$-algebras, ie, BV algebras with a unit for the commutative associative product which is annihilated by the bracket and the delta operator (see [GCTV09] for a treatment of $\mathcal{BV}$ as an inhomogeneous Koszul operad).*

Remark.

1. Though we have spoken only about units, counits may be treated similarly.
2. Treating $uc\mathcal{F}rob$, the properad governing Frobenius algebras with unit and counit, would be interesting to the authors.

2.6.6 Cohomology theory for unital associative algebra

In this section, we define the André-Quillen cohomology theory for unital associative algebras following the general definition of [Mil08]. We prove that the cohomology can be written as an Ext-functor and we compare this definition to the Hochschild cohomology theory.

André-Quillen cohomology theory

We consider now the operad $\mathcal{P} = u\mathcal{A}s$ and the curved cooperad $\mathcal{C} = u\mathcal{A}s^{\text{!`}} = (\mathrm{q}u\mathcal{A}s^{\text{!`}},\, 0,\, \theta)$. The Koszul morphism between $u\mathcal{A}s$ and $u\mathcal{A}s^{\text{!`}}$ is given by

$$\kappa : u\mathcal{A}s^{\text{!`}} \twoheadrightarrow \,\rotatebox{180}{$\multimap$}\, \oplus \curlyvee \rightarrowtail u\mathcal{A}s.$$

Let A be a $u\mathcal{A}s$-algebra. Following Sections 1 and 2 of [Mil08], we use the cofibrant resolution

$$\Omega_\kappa \mathrm{B}_\kappa A = u\mathcal{A}s \circ_\kappa u\mathcal{A}s^{\text{!`}}(A) \xrightarrow{\sim} A$$

of Section 2.5 to compute the André-Quillen cohomology of A thanks to the cotangent complex

$$A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{!`}}(A) \cong \underbrace{A \;\; u\mathcal{A}s^{\text{!`}}(A) \;\; A}_{} \cong A \otimes u\mathcal{A}s^{\text{!`}}(A) \otimes A.$$

We denote an element in $A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{!`}}(A)$ by $a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c$, where $a$, $b_t$ and $c$ are in $A$ and where $\overline{\mu}_n^S$ is in $u\mathcal{A}s^{\text{!`}}(n - |S|)$. Following the end of Section 2 of [Mil08], we compute the differential on $A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{!`}}(A)$, which is given by

$$d_\varphi := d_{A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{!`}}(A)} - \delta_\varphi^l + \delta_\varphi^r.$$

The differential $d_{A\otimes^{u\mathcal{A}s} u\mathcal{A}s^{¡}(A)}$ depends only on d_A (since $d_{u\mathcal{A}s} = 0$, $d_{u\mathcal{A}s^{¡}} = 0$), the map $\varphi : u\mathcal{A}s^{¡}(A) \twoheadrightarrow A$ is the projection and the terms δ^l_φ and δ^r_φ are given by the following proposition.

2.6.6.1 Proposition. *We have*
$\delta^l_\varphi(a \otimes (\overline{\mu}^S_n \otimes b_1 \cdots b_{n-|S|}) \otimes c) :=$

$$\epsilon_1 a \cdot b_1 \otimes (\overline{\mu}^{S-1}_{n-1} \otimes b_2 \cdots b_{n-|S|}) \otimes c + (-1)^n \epsilon_{n-|S|} a \otimes (\overline{\mu}^S_{n-1} \otimes b_1 \cdots b_{n-|S|-1}) \otimes b_{n-|S|} \cdot c,$$

where $\epsilon_i := \begin{cases} (-1)^{|a|+|b_i|(n-2+|S|+|b_1|+\cdots+|b_{i-1}|)} & \textit{if } i \notin S, \\ 0 & \textit{otherwise,} \end{cases}$ *and*
$\delta^r_\varphi(a \otimes (\overline{\mu}^S_n \otimes b_1 \cdots b_{n-|S|}) \otimes c) := (\delta_{\text{⫯}} + \delta_\gamma)(a \otimes (\overline{\mu}^S_n \otimes b_1 \cdots b_{n-|S|}) \otimes c) =$

$$- \sum_{S=S_1\sqcup\{u\}\sqcup S'_1} (-1)^{|a|+n+|S_1|} a \otimes (\overline{\mu}^{S\backslash u}_n \otimes b_1 \cdots 1_A \cdots b_{n-|S|}) \otimes c$$

$$- \sum_{\{t,t+1\}\sqcup S=S_2\sqcup\{t,t+1\}\sqcup S'_2} (-1)^{|a|+t+|S|} a \otimes (\overline{\mu}^{S_2\sqcup\{S'_2-1\}}_{n-1} \otimes b_1 \cdots b_t \cdot b_{t+1} \cdots b_{n-|S|}) \otimes c,$$

where $\max S_1 < u < \min S'_1$ *and* $\max S_2 < t < t+1 < \min S'_2$ *and* $\delta_{\text{⫯}}$ *holds for the first sum and* δ_γ *for the second. Moreover,* $d_\varphi(\text{⫯}) = 0$.

Proof. The differential on the cotangent complex is given following the end of Sections 2 of [Mil08]. We make the computations explicit thanks to the infinitesimal decomposition map of $u\mathcal{A}s^{¡}$, described in Corollary 2.6.1.4. □

2.6.6.2 Proposition. *The André-Quillen cohomology groups of a* $u\mathcal{A}s$*-algebra* A *with coefficients in a unital* A*-bimodule* M *are given by*

$$\mathrm{H}^\bullet_{u\mathcal{A}s}(A, M) := \mathrm{H}_\bullet(\mathrm{Hom}_{\textsf{\textit{A-bimod.}}}(A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{i}(A), M), \partial),$$

where $\partial(f) := d_M \circ f - (-1)^{|f|} f \circ d_\varphi$ *and* $\textsf{\textit{A-bimod.}}$ *is the category of unital* A*-bimodules.*

Ext-functor and comparison with the Hochschild cohomology theory

To a unital associative algebra, we can associate two abelian groups : the Hochschild cohomology groups of A (as defined in [Hoc45], or [Lod98], chap. 1, for a modern reference), that is, the André-Quillen cohomology groups of the associative algebra A (forgetting the unit), or the André-Quillen cohomology groups of A seen as a unital associative algebra (previous section). We show that the cohomology groups coincide.

2.6.6.3 Theorem. *Let* A *be a* $u\mathcal{A}s$*-algebra and let* M *be a unital* A*-bimodule. We have*

$$\mathrm{H}^\bullet_{u\mathcal{A}s}(A, M) \cong \mathrm{Ext}^\bullet_{A\otimes^{u\mathcal{A}s}\mathbb{K}}(\Omega_{u\mathcal{A}s}(A), M),$$

where $\Omega_{u\mathcal{A}s}(A)$ *is the unital* A*-bimodule of Kähler differential forms (see [Mil08] for more details).*

PROOF. Similarly to the case of Hochschild cohomology theory, we define the map h on $A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A$ by

$$h(a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c) = -(-1)^{|a|(n+|S|)} 1 \otimes (\overline{\mu}_{n+1}^{S+1} \otimes ab_1 \cdots b_{n-|S|}) \otimes c.$$

It satisfies $dh + hd = id$ on $A \otimes \overline{u\mathcal{A}s^{\text{i}}}(A) \otimes A$. Thus the chain complex

$$A \otimes \overline{u\mathcal{A}s^{\text{i}}}(A) \otimes A \xrightarrow{d_\varphi} A \otimes A \otimes A \twoheadrightarrow \Omega_{u\mathcal{A}s}(A) \to 0$$

is acyclic since we derive the left-adjoint functor of Kähler differential forms to obtain the cotangent complex, and the cohomology is an Ext-functor. □

We use this theorem to compare this cohomology theory to the Hochschild cohomology theory.

2.6.6.4 Proposition. *There is a quasi-isomorphism of unital A-bimodules*

$$A \otimes^{u\mathcal{A}s} \mathcal{A}s^{i}(A) \cong A \otimes \mathcal{A}s^{i}(A) \otimes A \xrightarrow{\sim} A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{i}(A) \cong A \otimes u\mathcal{A}s^{i}(A) \otimes A.$$

PROOF. First, we endowed $A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A$ with a filtration given by the number of corks, denoted by

$$F_p(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A) := \bigoplus_{S \subseteq \underline{n},\, |S| \leq p} A \otimes (u\mathcal{A}s^{\text{i}}(n - |S|) \otimes_{\mathbb{S}_{n-|S|}} A^{\otimes(n-|S|)}) \otimes A.$$

We have $d_{A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{i}}(A)} : F_p \to F_p$, $\delta_\varphi^l : F_p \to F_p$, $\delta_{\uparrow} : F_p \to F_{p-1}$ and $\delta_\gamma : F_p \to F_p$. Thus the filtration is a filtration of chain complexes. It is bounded below and exhaustive so we can apply the classical theorem of convergence of spectral sequences (cf. Theorem 5.5.1 of [Wei94]) and we obtain a spectral sequence $E_{p,q}^\bullet$ such that

$$E_{p,q}^\bullet \Rightarrow \mathrm{H}_{p+q}(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A).$$

The differential d^0 on $E_{p,q}^0 := F_p(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A)_{p+q} / F_{p-1}(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A)_{p+q}$ is given by $d^0 = d_{A \otimes^{u\mathcal{A}s} u\mathcal{A}s^{\text{i}}(A)} - \delta_\varphi^l + \delta_\gamma$. There is an inclusion of chain complexes

$$i : A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A \rightarrowtail \oplus_{p,q} E_{p,q}^0 \cong A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A,$$

where the last isomorphism is only of vector spaces. The projection $p : \oplus_{p,q} E_{p,q}^0 \cong A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A \oplus C_{\geq 1} \twoheadrightarrow A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A$, where $C_{\geq 1}$ is given by elements with at least one cork, is a chain complexes map. We define the map h by

$$h(a \otimes (\overline{\mu}_n^S \otimes b_1 \cdots b_{n-|S|}) \otimes c) := -(-1)^{\min S} a \otimes (\overline{\mu}_{n+1}^{S+1} \otimes b_1 \cdots b_{(\min S)-1} 1_A b_{\min S} \cdots b_{n-|S|}) \otimes c.$$

With these definitions, we have $p \circ i = id_{A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A}$ and $id_{\oplus_{p,q} E_{p,q}^0} - i \circ p = dh + hd$. Hence, we have a deformation retract

$$A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A \underset{p}{\overset{i}{\rightleftarrows}} \oplus_{p,q} E_{p,q}^0 \circlearrowright h$$

and the inclusion i is a quasi-isomorphism. It follows that $E_{p,q}^1 = 0$ when $p \neq 0$ and the spectral sequence collapses. Considering the filtration $F'_p(A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A) = A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A$ for all $p \geq 0$ (bounded below and exhaustive), the inclusion induces a map of spectral sequences which is a quasi-isomorphism on the E^1-pages and higher. Since $E'^\bullet_{p,q}$ converges to $\mathrm{H}_{p+q}(A \otimes \mathcal{A}s^{\text{i}}(A) \otimes A)$ and $E_{p,q}^\bullet$ converges to $\mathrm{H}_{p+q}(A \otimes u\mathcal{A}s^{\text{i}}(A) \otimes A)$, we get the proposition. □

2.6.6.5 Corollary. *Let A be a unital associative algebra. For $\bullet \geq 1$, we have*

$$\mathrm{H}^{\bullet}_{u\mathcal{A}s}(A,\, M) \cong \mathrm{HH}^{\bullet+1}(A,\, M).$$

Proof. The cohomology of $u\mathcal{A}s$-algebras is given by the Ext-functor $\mathrm{Ext}^{\bullet}_{A\otimes^{u\mathcal{A}s}\mathbb{K}}(\Omega_{u\mathcal{A}s}(A),\, M)$ (Theorem 2.6.6.3) and we have the projective resolution $A \otimes u\mathcal{A}s^{¡}(A) \otimes A \xrightarrow{\sim} \Omega_{u\mathcal{A}s}(A)$. By Proposition 2.6.6.4, the projective (quasi-free) A-bimodule $A \otimes \mathcal{A}s^{¡}(A) \otimes A$ is also a projective resolution of $\Omega_{u\mathcal{A}s}(A)$ and computes the Hochschild cohomology (see the definition 1.1.3 in [Lod98]). □

Chapitre 3

The Koszul complex is the cotangent complex

The Koszul duality theory of associative algebras first appeared in the work of [Pri70]. It has then been extended to operads [GJ94, GK94] and to properads [Val07]. The Koszul duality theory has a wide range of applications in mathematics : BGG correspondence [BGG78], equivariant cohomology [GKM98], and homotopy algebras [GK94, GJ94]. For even more applications, we refer to the introduction of [PP05]. Following the ideas of [Qui70], one defines a cohomology theory associated to any type of algebras [Hin97, GH00] and the Koszul duality theory of operads provides explicit chain complexes which allow us to compute it [Mil08].

The Koszul duality theory of associative algebras is based on a chain complex, called the *Koszul complex*, built on the tensor product of chain complexes. The notions of algebras, operads and properads are "associative" notions in the sense that they are all monoids in a monoidal category. This makes the generalization of the *Koszul complex* to operads and properads possible. To define a Koszul duality theory for algebras over an operad $\mathcal{P}$, we have to find a good generalization for this Koszul complex in a non-associative setting.

In [Mil08], we studied the André-Quillen cohomology theory of algebra over an operad. The latter is represented by a chain complex, called the *cotangent complex* of a $\mathcal{P}$-algebra A. Thanks to the Koszul duality theory of operads, we made a representation of the cotangent complex explicit. In this paper, we prove that the cotangent complex gives a good generalization of the Koszul complex in the sense that we get an algebraic twisting morphisms fundamental theorem (Theorem 3.2.4.1) and a Koszul criterion (Theorem 3.3.3.1).

When the $\mathcal{P}$-algebra A is *quadratic* or *monogene*, we introduce a Koszul dual coalgebra $A^{¡}$. The Koszul criterion provides a way to test whether the Koszul dual coalgebra $A^{¡}$ is a good space of *syzygies* to resolve the $\mathcal{P}$-algebra A. If applicable, the Koszul complex, which is thus a representation of the cotangent complex, is a "small" chain complex allowing to compute the cohomology theory of the $\mathcal{P}$-algebra A.

Retrospectively, the present Koszul duality theory applied to associative algebras gives the Koszul duality theory of associative algebras originally defined by Priddy [Pri70]. For commutative algebras, resp. Lie algebras, the present Koszul duality theory provides Sullivan minimal models, resp. Quillen models. An example is given by the commutative algebra of the cohomology groups of the complement of an hyperplane arrangement. It is given by the Orlik-Solomon algebra and, in the quadratic case, the Koszul dual algebra is the holonomy Lie algebra defined by Kohno [Yuz01, PY99, Koh83, Koh85]. More generally, this theory applies to give all the rational homotopy groups of formal spaces whose cohomology groups forms a Koszul (quadratic) algebra. Moreover, an associative algebra A, eventually commutative, is an example of operad and an "algebra" over this operad is a A-module. The present Koszul duality theory in this case gives the Koszul duality theory of modules which provides a good candidate for the syzygies of an A-module [PP05, Eis04].

The following table gives a summary.

	Koszul duality theory	
Monoids Section 1	Associative algebras [Pri70]	Operads [GK94, GJ94]
Representations Sections 2, 3 and 4	Modules [PP05]	$\mathcal{P}$-algebras *Goal* of the paper

We recall in the first section the Koszul duality theory for associative algebras and operads. We recall the results of [GJ94] on twisting morphisms for $\mathcal{P}$-algebras and we prove the algebraic twisting morphisms fundamental theorem in Section 2. We extend the results of [GK94] on quadratic $\mathcal{P}$-algebras and state the Koszul criterion in Section 3. The section 4 is devoted to the applications of the Koszul duality theory for $\mathcal{P}$-algebras. We prove a comparison Lemma for the twisted tensor product in the framework of $\mathcal{P}$-algebras in Appendix A.

In all this paper, $\mathbb{K}$ is a field of characteristic 0. Moreover, we assume that all the chain complexes are non-negatively graded.

3.1 Twisting morphisms for associative algebras and operads

The notion of *twisting morphism* (or twisting cochain) for associative algebras, introduced by Cartan [Car55] and Brown [Bro59], was generalized to operads by Getzler and Jones [GJ94]. We recall their definitions and the fact that the induced bifunctor is represented by the bar and the cobar constructions. We also recall the definition of the *twisted tensor product* and the *twisted composition product* and the notion of *(operadic) Koszul morphism.* We gives the twisting morphisms fundamental theorems and the Koszul criteria. We refer to the book of Loday and Vallette [LV] for a complete exposition.

3.1.1 Twisting morphisms for associative algebras

Let $(A,\, \gamma_A,\, d_A)$ be a *differential graded associative algebra*, *dga algebra* for short, and let $(C,\, \Delta_C,\, d_C)$ be a *differential graded coassociative coalgebra*, *dga coalgebra* for short. We associate to C and A the *dg convolution algebra* $(\mathrm{Hom}(C,\, A),\, \star,\, \partial)$ where $\star$ and ∂ are defined as follows

$$\begin{cases} f \star g : C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\gamma_A} A, \\ \partial(f) := d_A \circ f - (-1)^{|f|} f \circ d_C, \end{cases}$$

for $f,\, g \in \mathrm{Hom}(C,\, A)$. The associative product $\star$ on $\mathrm{Hom}(C,\, A)$ induces a Lie bracket and the solutions of degree -1 of the *Maurer-Cartan equation*

$$\partial(\alpha) + \alpha \star \alpha = \partial(\alpha) + \frac{1}{2}[\alpha,\, \alpha] = 0$$

are called *twiting morphisms*. The set of twisting morphisms is denoted by $\mathrm{Tw}(C,\, A)$.

When A is augmented, that is $A \cong \mathbb{K} \oplus \overline{A}$ as dga algebras, we recall the *bar construction on* A defined by $\mathrm{B}A := (T^c(s\overline{A}),\, d := d_1 + d_2)$, where s is the homological suspension, d_1 is induced by the differential d_A on A and d_2 is the unique coderivation which extends, up to suspension, the restriction of the product γ_A on $\overline{A}$. When C is coaugmented, that is $C \cong \mathbb{K} \oplus \overline{C}$ as coassociative coalgebras, we recall dually the *cobar construction on* C defined by $\Omega C := (T(s^{-1}\overline{C}),\, d := d_1 - d_2)$, where s^{-1} is the homological desuspension, d_1 is induced by d_C and d_2 is the unique derivation which extends, up to desuspension, the restriction of Δ_C on $\overline{C}$.

When A is augmented and C is coaugmented, we require that the composition of a twisting morphism with the augmentation map, respectively the coaugmentation map, vanishes. These constructions satisfy the following adjunction

$$\begin{array}{ccccc} \mathrm{Hom}_{\textsf{aug. dga alg.}}(\Omega C,\, A) & \cong & \mathrm{Tw}(C,\, A) & \cong & \mathrm{Hom}_{\textsf{coaug. dga coalg.}}(C,\, \mathrm{B}A) \\ f_\alpha & \longleftrightarrow & \alpha & \longleftrightarrow & g_\alpha, \end{array}$$

when the coalgebra C is *conilpotent* (see [LV] for a definition).

To a twisting morphism α between a coaugmented coalgebra C and an augmented algebra A, we associate the *left twisted tensor product*

$$A \otimes_\alpha C := (A \otimes C,\, d_\alpha := d_{A\otimes C} - d^l_\alpha),$$

where d^l_α is defined by

$$A \otimes C \xrightarrow{id_A \otimes \Delta_C} A \otimes C \otimes C \xrightarrow{id_A \otimes \alpha \otimes id_C} A \otimes A \otimes C \xrightarrow{\gamma_A \otimes id_C} A \otimes C.$$

It is a chain complex since α is a twisting morphism. We refer to [LV] for the symmetric definition of the *right twisted tensor product*

$$C \otimes_\alpha A := (C \otimes A,\, d_\alpha := d_{C\otimes A} + d^r_\alpha)$$

and one gets the *twisted tensor product*

$$A \otimes_\alpha C \otimes_\alpha A := (A \otimes C \otimes A,\, d_\alpha := d_{A\otimes C\otimes A} - d^l_\alpha \otimes id_A + id_A \otimes d^r_\alpha).$$

We say that α is a *Koszul morphism* when $A \otimes_\alpha C \otimes_\alpha A \xrightarrow{\sim} A$. We denote by $\mathrm{Kos}(C,\, A)$ the set of Koszul morphisms.

Examples. To $id_{\mathrm{B}A}$ and $id_{\Omega C}$ correspond two universal twisting morphisms $\pi : \mathrm{B}A \to A$ and $\iota : C \to \Omega C$. They are examples of Koszul morphisms, that is $A \otimes_\pi \mathrm{B}A \otimes_\pi A \xrightarrow{\sim} A$ and $\Omega C \otimes_\iota C \otimes_\iota \Omega C \xrightarrow{\sim} \Omega C$.

Later, we will need an extra grading, called *weight grading*, which differs from the homological grading. We refer to the first chapter of [LV] for more details about this and for a definition of connected wdga algebras and connected wdga coalgebras. This adjunction satisfies the following property.

3.1.1.1 Theorem (Twisting morphisms fundamental theorem)**.** *Let A be a connected wdga algebra and let C be a connected wdga coalgebra. For any twisting morphism $\alpha : C \to A$, the following assertions are equivalent :*

1. *The twisting morphism α is Koszul, that is $A \otimes_\alpha C \otimes_\alpha A \xrightarrow{\sim} A$;*
2. *The left twisted tensor product is acyclic, that is $A \otimes_\alpha C \xrightarrow{\sim} \mathbb{K}$;*
3. *The right twisted tensor product is acyclic, that is $C \otimes_\alpha A \xrightarrow{\sim} \mathbb{K}$;*
4. *The morphism of dga algebras $f_\alpha : \Omega C \to A$ is a quasi-isomorphism ;*
5. *The morphism of dga coalgebras $g_\alpha : C \to \mathrm{B}A$ is a quasi-isomorphism.*

Proof. A proof of the equivalences (2) ⇔ (3) ⇔ (4) ⇔ (5) can be found in [LV] and comes from [Bro59]. We show the equivalence (1) ⇔ (2) in the more general case of operads, see Theorem 3.1.3.3. ☐

Let (V, S) be a *quadratic data*, that is a graded vector space V and a graded subspace $R \subseteq V \otimes V$. A *quadratic algebra* is an associative algebra $A(V, S)$ of the form $T(V)/(S)$. Dually the *quadratic coalgebra* $C(V, S)$ is by definition the sub-coalgebra of the cofree coalgebra $T^c(V)$ which is *universal* among the sub-coalgebras C of $T^c(V)$ such that the composite

$$C \rightarrowtail T^c(V) \twoheadrightarrow T^c(V)^{(2)}/S$$

is 0. The word "universal" means that for any such coalgebra C, there exists a unique morphism of coalgebras $C \to C(V, S)$ such that the following diagram commutes

$$\begin{array}{ccc} C & \longrightarrow & C(V, S) \\ & \searrow & \downarrow \\ & & T^c(V). \end{array}$$

To a quadratic data (V, S), we associate the *Koszul dual coalgebra of A* given by $A^{¡} := C(sV, s^2S)$ where s is the homological suspension. We associate to the coalgebra $C(sV, s^2S)$ and to the algebra $A(V, S)$ the twisting morphism $\varkappa$ defined by

$$\varkappa : A^{¡} = C(sV, s^2S) \twoheadrightarrow sV \cong V \rightarrowtail A(V, S) = A.$$

The equality $\partial(\varkappa) + \varkappa \star \varkappa = 0$ follows from the coassociativity of $\Delta_{A^{¡}}$, the associativity of γ_A and to the fact that $A^{¡(2)} = s^2S$ and $A^{(2)} = V^{\otimes 2}/S$.

The weight grading comes from the graduation in $T(V)$ and $T^c(V)$ by the number of generators in V. The universality of the twisting morphisms π and ι provides an inclusion of coalgebras $g_\varkappa : A^{¡} \rightarrowtail \mathrm{B}A$ and a surjection of algebras $f_\varkappa : \Omega A^{¡} \twoheadrightarrow A$. The twisting morphisms fundamental theorem writes :

3.1.1.2 Theorem (Koszul criterion, [Pri70])**.** *Let (V, S) be a quadratic data. Let $A := A(V, S)$ be the associated quadratic algebra and let $A^{¡} := C(sV, s^2S)$ its Koszul dual coalgebra. The following assertions are equivalent :*

1. *The twisting morphism $\varkappa$ is Koszul, that is $A \otimes_\varkappa A^{¡} \otimes_\varkappa A \xrightarrow{\sim} A$;*
2. *The Koszul complex $A \otimes_\varkappa A^{¡}$ is acyclic, that is $A \otimes_\varkappa A^{¡} \xrightarrow{\sim} \mathbb{K}$;*
3. *The Koszul complex $A^{¡} \otimes_\varkappa A$ is acyclic, that is $A^{¡} \otimes_\varkappa A \xrightarrow{\sim} \mathbb{K}$;*

4. *The morphism of dga algebras* $f_\varkappa : \Omega A^{i} \to A$ *is a quasi-isomorphism ;*
5. *The morphism of dga coalgebras* $g_\varkappa : A^{i} \to \mathrm{B}A$ *is a quasi-isomorphism.*

Priddy [Pri70] called the chain complexes $A^{\text{i}} \otimes_\varkappa A$, resp. $A \circ_\varkappa A^{\text{i}}$, the *Koszul complexes.* We extend this definition and we call also $A \otimes_\varkappa A^{\text{i}} \otimes_\varkappa A$ the Koszul complex. An algebra is called a *Koszul algebra* when the twisting morphism $\varkappa : A^{\text{i}} \to A$ is a Koszul morphism, that is $A \circ_\varkappa A^{\text{i}} \otimes_\varkappa A \xrightarrow{\sim} A$. The previous Koszul criterion shows that this definition is equivalent to the classical one.

3.1.2 $\mathbb{S}$-module and operad

We recall the definition of an $\mathbb{S}$-*module* and of an *operad.* For a complete exposition of the concepts of this section, we refer to the books [LV] and [MSS02].

An $\mathbb{S}$-*module* M is a collection of dg modules $\{M(n)\}_{n\geq 0}$ endowed with an action of the group $\mathbb{S}_n$ of permutations on n elements. Let M, M', N and N' be $\mathbb{S}$-modules. We recall the definition of the composition product $\circ$

$$(M \circ N)(n) := \bigoplus_{k\geq 0} M(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}}^{\mathbb{S}_n} \big(N(i_1) \otimes \cdots \otimes N(i_k)\big) \right).$$

The unit for the monoidal product is $I := (0, \mathbb{K}, 0, \ldots)$. Notice that $\circ$ is not linear on the right hand side. As a consequence, we define the right linear analog $M \circ (N;\, N')$ of the composition product, linear in M and in N', by the following formula

$$M \circ (N;\, N')(n) :=$$
$$\bigoplus_{k\geq 0} M(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1+\cdots+i_k=n} \bigoplus_{j=1}^{k} \mathrm{Ind}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}}^{\mathbb{S}_n} (N(i_1) \otimes \cdots \otimes \underbrace{N'(i_j)}_{j^{th}\text{ position}} \otimes \cdots \otimes N(i_k)) \right).$$

Let $f : M \to M'$ and $g : N \to N'$ be maps of $\mathbb{S}$-modules. We denote by $\circ'$ the *infinitesimal composite of morphisms* :

$$f \circ' g : M \circ N \to M' \circ (N,\, N')$$

defined by

$$\sum_{j=1}^{k} f \otimes (id_N \otimes \cdots \otimes \underbrace{g}_{j^{th}\text{ position}} \otimes \cdots \otimes id_N).$$

The differential on $M \circ N$ is given by $d_{M\circ N} := d_M \circ id_N + id_M \circ' d_N$. The term $id_M \circ' d_N$ goes normally to $M \circ (N;\, N)$ but we assume it composed with the projection $M \circ (N;\, N) \twoheadrightarrow M \circ N$. We define the *infinitesimal composite product* $M \circ_{(1)} N$ by $M \circ (I;\, N)$, which is linear in M and in N. Moreover we denote by $f \circ_{(1)} g$ the map $f \circ (id_I,\, g) : M \circ_{(1)} N \to M' \circ_{(1)} N'$.

The category of dg $\mathbb{S}$-modules ($\mathbb{S}$-mod, $\circ$, I) is a monoidal category. A monoid $(\mathcal{P}, \gamma, u)$ in this category is called a *differential graded operad*, *dg operad* for short. Dually, a comonoid $(\mathcal{C}, \Delta, \eta)$ in this category is called a *dg cooperad.*

Let $(\mathcal{P}, \gamma, u)$ be a dg operad and $(\mathcal{C}, \Delta, \eta)$ be a dg cooperad. We denote by $\gamma_{(1)}$ the *partial product* of the operad $\mathcal{P}$

$$\mathcal{P} \circ_{(1)} \mathcal{P} \rightarrowtail \mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma} \mathcal{P}.$$

Dually, we denote by $\Delta_{(1)}$ the *partial coproduct* of the cooperad $\mathcal{C}$

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \twoheadrightarrow \mathcal{C} \circ_{(1)} \mathcal{C}.$$

The free operad $\mathcal{F}(E)$ on a $\mathbb{S}$-module E is given by all the trees whose vertices of arity n are indexed by elements in $E(n)$. The product is given by the grafting of trees. Dually, the free cooperad $\mathcal{F}^c(E)$ has the same underlying $\mathbb{S}$-module as $\mathcal{F}(E)$ and the coproduct is given by the splitting of trees.

3.1.3 Operadic twisting morphism

We recall from [GJ94, GK94] the definitions of *operadic twisting morphism*, *left twisted composition product* and *bar and cobar construction* in the setting of operads. We state the operadic twisting morphisms fundamental theorem and recall the notion of *Koszul operad*.

Let $\mathcal{P}$ be a dg operad and $\mathcal{C}$ be a dg cooperad. We recall from Chapter 6 of [LV] the *dg convolution PreLie algebra* $(\mathrm{Hom}_{\mathbb{S}\text{-mod}}(\mathcal{C},\,\mathcal{P}),\,\star,\,\partial)$ where

$$\begin{cases} f\star g : \mathcal{C} \xrightarrow{\Delta_{(1)}} \mathcal{C}\circ_{(1)}\mathcal{C} \xrightarrow{f\circ_{(1)}g} \mathcal{P}\circ_{(1)}\mathcal{P} \xrightarrow{\gamma_{(1)}} \mathcal{P}, \\ \partial(f) := d_{\mathcal{P}}\circ f - (-1)^{|f|} f\circ d_{\mathcal{C}}, \end{cases}$$

for $f,\,g \in \mathrm{Hom}(\mathcal{C},\,\mathcal{P})$. The PreLie product $\star$ induces a Lie bracket on $\mathrm{Hom}(\mathcal{C},\,\mathcal{P})$ and an *operadic twisting morphism* is a map $\alpha : \mathcal{C} \to \mathcal{P}$ of degree -1, solution of the *Maurer-Cartan equation*

$$\partial(\alpha) + \alpha\star\alpha = \partial(\alpha) + \frac{1}{2}[\alpha,\,\alpha] = 0.$$

We denote the set of operadic twisting morphisms from $\mathcal{C}$ to $\mathcal{P}$ by $\mathrm{Tw}(\mathcal{C},\,\mathcal{P})$.

In [GJ94, GK94], the authors extend the bar construction and the cobar construction of associative algebras/coalgebras to operads/cooperads. When $\mathcal{P}$ is an *augmented operad*, that is $\mathcal{P} \cong I \oplus \overline{\mathcal{P}}$, we have $\mathrm{B}\mathcal{P} := (\mathcal{F}^c(s\overline{\mathcal{P}}),\, d := d_1 + d_2)$ where $\mathcal{F}^c(s\overline{\mathcal{P}})$ is the free cooperad on the homological suspension of $\overline{\mathcal{P}}$, d_1 is the unique coderivation which extends, up to suspension, the differential $d_{\overline{\mathcal{P}}}$ and d_2 is the unique coderivation which extends, up to suspension, the restriction of the partial product $\overline{\gamma}_{(1)} : \overline{\mathcal{P}} \circ_{(1)} \overline{\mathcal{P}} \to \overline{\mathcal{P}}$. When $\mathcal{C}$ is a *coaugmented cooperad*, that is $\mathcal{C} \cong I \oplus \overline{\mathcal{C}}$ as cooperads, we have $\Omega\mathcal{C} := (\mathcal{F}(s^{-1}\overline{\mathcal{C}}),\, d := d_1 - d_2)$, where $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$ is the free operad on the homological desuspension of $\overline{\mathcal{C}}$, d_1 is the unique derivation which extends, up to desuspension, the differential $d_{\overline{\mathcal{C}}}$ and d_2 is the unique derivation which extends, up to desuspension, the partial coproduct $\overline{\Delta}_{(1)} : \overline{\mathcal{C}} \to \overline{\mathcal{C}} \circ_{(1)} \overline{\mathcal{C}}$. When $\mathcal{P}$ is an augmented operad and $\mathcal{C}$ is a coaugmented cooperad, we require that the composition of an operadic twisting morphism with the augmentation map, respectively the coaugmentation map, vanishes. As for associative algebras, these constructions satisfy the following bar-cobar adjunction.

3.1.3.1 Theorem (Theorem 2.17 of [GJ94])**.** *The functors* Ω *and* B *form a pair of adjoint functors between the category of conilpotent coaugmented dg cooperads and augmented dg operads. The natural bijections are given by the set of operadic twisting morphisms :*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathsf{dg\ op.}}(\Omega\mathcal{C},\,\mathcal{P}) & \cong & \mathrm{Tw}(\mathcal{C},\mathcal{P}) & \cong & \mathrm{Hom}_{\mathsf{dg\ coop.}}(\mathcal{C},\,\mathrm{B}\mathcal{P}) \\ f_\alpha & \longleftrightarrow & \alpha & \longleftrightarrow & g_\alpha. \end{array}$$

Examples (of operadic twisting morphisms).

- When $\mathcal{C} = \mathrm{B}\mathcal{P}$, the previous theorem gives a natural operadic twisting morphism $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$, associated to $id_{\mathrm{B}\mathcal{P}}$, which is equal to $\mathrm{B}\mathcal{P} = \mathcal{F}^c(s\overline{\mathcal{P}}) \twoheadrightarrow s\overline{\mathcal{P}} \xrightarrow{s^{-1}} \overline{\mathcal{P}} \rightarrowtail \mathcal{P}$. This morphism is universal in the sense that each operadic twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factorizes through π

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\alpha\quad} & \mathcal{P} \\ & {\scriptstyle f_\alpha} \searrow \quad \nearrow {\scriptstyle \pi} & \\ & \mathrm{B}\mathcal{P}, & \end{array}$$

where f_α is a morphism of dg cooperads.

- When $\mathcal{P} = \Omega\mathcal{C}$, the previous theorem gives a natural operadic twisting morphism $\iota : \mathcal{C} \to \Omega\mathcal{C}$, associated to $id_{\Omega\mathcal{C}}$, which is equal to $\mathcal{C} \twoheadrightarrow \overline{\mathcal{C}} \xrightarrow{s^{-1}} s^{-1}\overline{\mathcal{C}} \rightarrowtail \Omega\mathcal{C} = \mathcal{F}(s^{-1}\overline{\mathcal{C}})$. This morphism is universal in the sense that each operadic morphism $\alpha : \mathcal{C} \to \mathcal{P}$ factorizes through ι

$$\begin{array}{ccc} & \Omega\mathcal{C} & \\ & {\scriptstyle \iota} \nearrow \quad \searrow {\scriptstyle g_\alpha} & \\ \mathcal{C} & \xrightarrow{\quad\alpha\quad} & \mathcal{P}, \end{array}$$

where g_α is a morphism of dg operads.

To an operadic twisting morphism α between a coaugmented cooperad $\mathcal{C}$ and an augmented operad $\mathcal{P}$, we associate the *left twisted composition product* [Val07, LV]

$$\mathcal{P} \circ_\alpha \mathcal{C} := (\mathcal{P} \circ \mathcal{C},\, d_\alpha := d_{\mathcal{P}\circ\mathcal{C}} - d^l_\alpha),$$

where d^l_α is given by the composite

$$\mathcal{P} \circ \mathcal{C} \xrightarrow{id_\mathcal{P} \circ' \Delta} \mathcal{P} \circ (\mathcal{C};\, \mathcal{C} \circ \mathcal{C}) \xrightarrow{id_\mathcal{P} \circ (id_\mathcal{C},\, \alpha \circ id_\mathcal{C})} \mathcal{P} \circ (\mathcal{C};\, \mathcal{P} \circ \mathcal{C}) \cong (\mathcal{P} \circ_{(1)} \mathcal{P}) \circ \mathcal{C} \xrightarrow{\gamma_{(1)} \circ id_\mathcal{C}} \mathcal{P} \circ \mathcal{C}.$$

Since α is an operadic twisting morphism, d_α is a differential (see [LV]). We refer to [LV] for a definition of the *right twisted composite product*

$$\mathcal{C} \circ_\alpha \mathcal{P} := (\mathcal{C} \circ \mathcal{P},\, d_\alpha := d_{\mathcal{C}\circ\mathcal{P}} + d^r_\alpha)$$

and one gets the *twisted composite product*

$$\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P} := (\mathcal{P} \circ \mathcal{C} \circ \mathcal{P},\, d_\alpha := d_{\mathcal{P}\circ\mathcal{C}\circ\mathcal{P}} - d^l_\alpha \circ id_\mathcal{P} + id_\mathcal{P} \circ d^r_\alpha).$$

We say that α is an *operadic Koszul morphism* when $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P} \xrightarrow{\sim} \mathcal{P}$. We denote by $\mathrm{Kos}(\mathcal{C}, \mathcal{P})$ the set of operadic Koszul morphisms.

3.1.3.2 Lemma ([GJ94, Fre04, Val07]). *The twisting morphisms $\pi : \mathrm{B}\mathcal{P} \to \mathcal{P}$ and $\iota : \mathcal{C} \to \Omega\mathcal{C}$ are operadic Koszul morphisms, that is*

$$\mathcal{P} \circ_\pi \mathrm{B}\mathcal{P} \circ_\pi \mathcal{P} \xrightarrow{\sim} \mathcal{P} \text{ and } \Omega\mathcal{C} \circ_\iota \mathcal{C} \circ_\iota \Omega\mathcal{C} \xrightarrow{\sim} \Omega\mathcal{C}.$$

Sometimes, we need an extra grading, called *weight grading*, which differs from the homological degree. We say that a *weight graded dg $\mathbb{S}$-module*, *wdg $\mathbb{S}$-module* for short, M is *connected* when $M^{(0)} = I$ and $M = I \oplus M^{(1)} \oplus \cdots \oplus M^{(\omega)} \oplus \cdots$. These definitions hold for operads and cooperads. For example, the weight grading on the free operad $\mathcal{F}(E)$ or on the free cooperad $\mathcal{F}^c(E)$ is given by the number ω of vertices and denoted by $\mathcal{F}(E)^{(\omega)}$ or $\mathcal{F}^c(E)^{(\omega)}$. This induces a weight grading on each *quadratic operad* $\mathcal{P} = \mathcal{F}(E)/(R)$, where $R \subset \mathcal{F}(E)^{(2)}$ and on each sub-cooperad of a free cooperad. In the weight graded setting, we assume that the maps preserve the weight grading. For example, a twisting morphism α preserves the weight grading and when the underlying modules are connected, we have $\alpha = \alpha^{(\geq 1)}$.

As for associative algebras, the twisted composite products and the bar and cobar constructions satisfy the following property.

3.1.3.3 Theorem (Operadic twisting morphisms fundamental theorem, [LV])**.** *Let $\mathcal{P}$ be a connected wdg operad and let $\mathcal{C}$ be a connected wdg cooperad. For any operadic twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$, the following assertions are equivalent :*

1. *The twisting morphism α is Koszul, that is $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P} \xrightarrow{\sim} \mathcal{P}$;*
2. *The left twisted composite product $\mathcal{P} \circ_\alpha \mathcal{C}$ is acyclic, that is $\mathcal{P} \circ_\alpha \mathcal{C} \xrightarrow{\sim} I$;*
3. *The right twisted composite product $\mathcal{C} \circ_\alpha \mathcal{P}$ is acyclic, that is $\mathcal{C} \circ_\alpha \mathcal{P} \xrightarrow{\sim} I$;*
4. *The morphism of dg operads $f_\alpha : \Omega\mathcal{C} \to \mathcal{P}$ is a quasi-isomorphism ;*
5. *The morphism of dg cooperads $g_\alpha : \mathcal{C} \to \mathrm{B}\mathcal{P}$ is a quasi-isomorphism.*

Proof. A proof of the equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) can be find in [LV]. We show the equivalence (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2) : The Koszul complex $\mathcal{P} \circ_\alpha \mathcal{C}$ is equal to the relative composite product $(\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ_{\mathcal{P}} I$ which is defined by the short exact sequence

$$0 \to (\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ \mathcal{P} \circ I \to (\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ I \to (\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ_{\mathcal{P}} I \to 0.$$

Since we work over a field of characteristic zero, the ring $\mathbb{K}[\mathbb{S}_n]$ is semi-simple by Maschke's theorem, that is every $\mathbb{K}[\mathbb{S}_n]$-module is projective. So the Künneth formula implies $\mathrm{H}_\bullet((\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ \mathcal{P} \circ I) \cong \mathrm{H}_\bullet(\mathcal{P}) \circ \mathrm{H}_\bullet(\mathcal{P}) \circ I$. Moreover $\mathrm{H}_\bullet((\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ I) \cong \mathrm{H}_\bullet(\mathcal{P}) \circ I$. Finally, this gives $\mathrm{H}_\bullet(\mathcal{P} \circ_\alpha \mathcal{C}) \cong \mathrm{H}_\bullet((\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) \circ_{\mathcal{P}} I) \cong \mathrm{H}_\bullet(\mathcal{P}) \circ_{\mathrm{H}_\bullet(\mathcal{P})} I \cong I$.

(2) $\Rightarrow$ (1) : We define a filtration F_p on $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}$ by

$$F_p(\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}) := \oplus_{\omega \leq p} (\mathcal{P} \circ_\alpha \mathcal{C})^{(\omega)} \circ_\alpha \mathcal{P}.$$

The differential on $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}$ is given by $d_\alpha := d_{\mathcal{P} \circ \mathcal{C} \circ \mathcal{P}} - d^l_\alpha \circ id_{\mathcal{P}} + id_{\mathcal{P}} \circ d^r_\alpha$ and satisfies

$$\begin{cases} d_{\mathcal{P} \circ \mathcal{C} \circ \mathcal{P}} & : & F_p \to F_p \\ d^l_\alpha & : & F_p \to F_p \\ d^r_\alpha & : & F_p \to F_{p-1}. \end{cases}$$

So the filtration is a filtration of chain complexes. Moreover, the filtration is exhaustive and bounded below. We can apply the classical theorem of convergence of spectral sequences (Theorem 5.5.1 of [Wei94]) and we get that the induced spectral sequence $E^\bullet_{p,q}$ converges to the homology of $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P}$. We consider the trivial chain complex filtration on $\mathcal{P}$, that is $F_p \mathcal{P} := \mathcal{P}$ for all $\mathcal{P}$, so that the map $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P} \to \mathcal{P}$ respects the filtration. This last map induces a map on the E^1-pages which is an isomorphism since $E^1_{p,q} \cong \mathrm{H}_\bullet(\mathcal{P} \circ_\alpha \mathcal{C}) \circ \mathrm{H}_\bullet(\mathcal{P}) \cong \mathrm{H}_\bullet(\mathcal{P}) \cong E^1_{0,q}$ by the Künneth formula. The convergence of the spectral sequences concludes the proof. $\square$

Koszul operad

An *operadic quadratic data* $(E,\,R)$ is a graded $\mathbb{S}$-module E and a graded sub-$\mathbb{S}$-module $R \subset \mathcal{F}(E)^{(2)}$. The quotient $\mathcal{P}(E,\,R) := \mathcal{F}(E)/(R)$ is called a *quadratic operad*. Dually the *quadratic cooperad* $\mathcal{C}(E,\,R)$ is by definition the sub-cooperad of the cofree cooperad $\mathcal{F}^c(E)$ which is *universal* among the sub-cooperads $\mathcal{C}$ of $\mathcal{F}^c(E)$ such that the composite

$$\mathcal{C} \rightarrowtail \mathcal{F}^c(E) \twoheadrightarrow \mathcal{F}^c(E)^{(2)}/R$$

is 0. The word "universal" means that for any such cooperad $\mathcal{C}$, there exists a unique morphism of cooperads $\mathcal{C} \to \mathcal{C}(E,\,R)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}(E,\,R) \\ & \searrow & \downarrow \\ & & \mathcal{F}^c(E). \end{array}$$

To a quadratic data $(E,\,R)$, we associate the *Koszul dual cooperad of* $\mathcal{P}$ given by $\mathcal{P}^{¡} := \mathcal{C}(sE,\,s^2R)$ where s is the homological suspension.

Example (of operadic twisting morphism). When $\mathcal{C} = \mathcal{P}^{¡}$, the map $\kappa : \mathcal{P}^{¡} \rightarrowtail \mathrm{B}\mathcal{P} \xrightarrow{\pi} \mathcal{P}$ is an operadic twisting morphism. It is equal to $\mathcal{P}^{¡} \twoheadrightarrow sE \xrightarrow{s^{-1}} E \rightarrowtail \mathcal{P}$.

The universality of the twisting morphisms π and ι provides an inclusion of cooperads $g_\kappa : \mathcal{P}^{¡} \rightarrowtail \mathrm{B}\mathcal{P}$ and a surjection of operads $f_\kappa : \Omega\mathcal{P}^{¡} \twoheadrightarrow \mathcal{P}$. The operadic twisting morphisms fundamental theorem writes :

3.1.3.4 Theorem (Koszul criterion, [LV])**.** *Let $(E,\,R)$ be an operadic quadratic data. Let $\mathcal{P} := \mathcal{P}(E,\,R)$ be the associated quadratic operad and let $\mathcal{P}^{¡} := \mathcal{C}(sE,\,s^2R)$ its Koszul dual cooperad. The following assertions are equivalent :*

1. *The operadic twisting morphism κ is Koszul, that is $\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa \mathcal{P} \xrightarrow{\sim} \mathcal{P}$;*
2. *The Koszul complex $\mathcal{P} \circ_\kappa \mathcal{P}^{¡}$ is acyclic, that is $\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \xrightarrow{\sim} I$;*
3. *The Koszul complex $\mathcal{P}^{¡} \circ_\kappa \mathcal{P}$ is acyclic, that is $\mathcal{P}^{¡} \circ_\kappa \mathcal{P} \xrightarrow{\sim} I$;*
4. *The morphism of dg operads $f_\kappa : \Omega\mathcal{P}^{¡} \to \mathcal{P}$ is a quasi-isomorphism ;*
5. *The morphism of dg cooperads $g_\kappa : \mathcal{P}^{¡} \to \mathrm{B}\mathcal{P}$ is a quasi-isomorphism.*

The chain complexes $\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa \mathcal{P}$, resp. $\mathcal{P}^{¡} \circ_\kappa \mathcal{P}$, resp. $\mathcal{P} \circ_\kappa \mathcal{P}^{¡}$, are called the *Koszul complexes*. An operad is called a *Koszul operad* when the operadic twisting morphism $\kappa : \mathcal{P}^{¡} \to \mathcal{P}$ is an operadic Koszul morphism, that is $\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa \mathcal{P} \xrightarrow{\sim} \mathcal{P}$.

3.2 Twisting morphism for $\mathcal{P}$-algebras

In this section, we extend the Koszul duality theory for associative algebras to algebras over an operad. We recall the notions already in [GJ94] of *algebraic twisting morphism* and the *bar and the cobar constructions*. However, to describe the Koszul duality theory for $\mathcal{P}$-algebras, we need to generalize the Koszul complex. A cohomology theory associated to $\mathcal{P}$-algebras is represented by the *cotangent complex*, that we make explicit thanks to the Koszul duality theory for operads. We show that this cotangent complex generalizes the Koszul complex and we state and prove in this setting the algebraic twisting morphisms fundamental theorem.

We fix an augmented dg operad $\mathcal{P}$, a coaugmented dg cooperad $\mathcal{C}$ and an operadic twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$. From now on and until the end of the paper, we assume that $\mathcal{P}(0) = 0$ and $\mathcal{C}(0) = 0$.

3.2.1 $\mathcal{P}$-algebra

A *$\mathcal{P}$-algebra* is a dg module A endowed with a morphism of dg operads

$$\mathcal{P} \to End_A := \{\mathrm{Hom}(A^{\otimes n},\, A)\}_{n\geq 0}.$$

Equivalently, a structure of $\mathcal{P}$-algebra $(A,\, \gamma_A)$ is given by a map $\gamma_A : \mathcal{P}(A) \to A$ which is compatible with the composition product of the operad $\mathcal{P}$ and the unit of the operad $\mathcal{P}$, where

$$\mathcal{P}(A) := \mathcal{P} \circ (A,\, 0,\, 0,\, \cdots) = \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}.$$

Dually, a structure of *$\mathcal{C}$-coalgebra* $(C,\, \Delta_C)$ is a dg module C endowed with a map

$$\Delta_C : C \to \mathcal{C}(C) := \prod_{n\geq 0} (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}$$

which is compatible with the coproduct and the counit and where $(-)^{\mathbb{S}_n}$ stands for the coinvariants with respect to the diagonal action. We say that the $\mathcal{C}$-coalgebra C is *conilpotent* when the map Δ_C factors through $\oplus_{n\geq 0}(\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}$.

The notation $\otimes_H$ stands for the *Hadamard product* : for any $\mathbb{S}$-modules M and N, $(M \otimes_H N)(n) := M(n) \otimes N(n)$ with the diagonal action of $\mathbb{S}_n$. Let $\mathcal{S}^{-1}$ be the cooperad $\mathrm{End}^c_{s\mathbb{K}} := \{\mathrm{Hom}((s\mathbb{K})^{\otimes n},\, s\mathbb{K})\}_{n\geq 0}$ endowed with a natural action of $\mathbb{S}_n$ given by the signature, where s stands for the homological suspension of vector spaces. To the cooperad $\mathcal{C}$, we associate its *operadic homological desuspension* given by the cooperad $\mathcal{S}^{-1}\mathcal{C} := \mathcal{S}^{-1} \otimes_H \mathcal{C}$. A structure of $\mathcal{S}^{-1}\mathcal{C}$-coalgebra Δ_{sC} on sC is equivalent to a structure of $\mathcal{C}$-coalgebra Δ_C on C because $\mathcal{S}^{-1}\mathcal{C}(sC) \cong s\mathcal{C}(C)$.

Example. Let $\mathcal{P} = \mathcal{A}s$ be the non-symmetric operad encoding associative algebras. An $\mathcal{A}s$-algebra A is an associative algebra without unit. Moreover, a structure of $\mathcal{S}^{-1}\mathcal{A}s^{¡}$-coalgebra is exactly that of a coassociative coalgebra as in Section 3.1.1 but without counit. The category of $\mathcal{A}s$-algebras A is equivalent to the category of augmented associative algebra by adding a unit $A_+ := \mathbb{K} \oplus A$. The Koszul duality theory of $\mathcal{A}s$-algebras will be the classical Koszul duality theory of augmented associative algebras.

A *weight graded dg module*, or *wdg module* for short, is a chain complex endowed with a weight grading. We say that a *wdg $\mathcal{P}$-algebra* or *wdg $\mathcal{C}$-coalgebra V* is *connected* when it satisfies $V = V^{(1)} \oplus V^{(2)} \oplus \cdots$. Moreover, we require that the structure maps, as the composite product γ_A, preserve the weight grading.

3.2.2 Algebraic twisting morphism

From [GJ94], we recall the definition of twisting morphisms between a coalgebra over a cooperad and an algebra over an operad. We describe the bar and the cobar constructions in this setting.

Let $(A,\, \gamma_A)$ be a $\mathcal{P}$-algebra and $(C,\, \Delta_C)$ be a $\mathcal{C}$-coalgebra. Associated to $\varphi \in \mathrm{Hom}_{\mathrm{dg\,mod}}(sC,\, A)$, we define the applications

$$\begin{cases} \star_\alpha(\varphi) : sC \xrightarrow{\Delta_C} s\mathcal{C}(C) \xrightarrow{(s^{-1}\alpha)\circ(s\varphi)} \mathcal{P}(A) \xrightarrow{\gamma_A} A \\ \partial(\varphi) := d_A \circ \varphi - (-1)^{|\varphi|}\varphi \circ d_{sC}. \end{cases}$$

An *algebraic twisting morphism with respect to* α is a map $\varphi : sC \to A$ of degree -1 solution to the Maurer-Cartan equation

$$\partial(\varphi) + \star_\alpha(\varphi) = 0.$$

We denote by $\mathrm{Tw}_\alpha(C, A)$ the set of algebraic twisting morphisms with respect to α. In the weight graded setting, we require that the algebraic twisting morphisms preserve the weight grading.

Remark. We proved in [Mil08] that when $\mathcal{P}$ is binary quadratic operad and $\mathcal{C} = \mathcal{P}^{\mathrm{i}}$ is the Koszul dual cooperad, the aforementioned Maurer-Cartan equation is equal to a Maurer-Cartan equation in a dg Lie algebra.

To the operadic twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$, one associates a functor

$$\mathrm{B}_\alpha : \textsf{dg } \mathcal{P}\textsf{-algebras} \to \textsf{dg } \mathcal{S}^{-1}\mathcal{C}\textsf{-coalgebras}$$

defined by $\mathrm{B}_\alpha A = s\mathcal{C} \circ_\alpha A := (s\mathcal{C}(A),\, d_\alpha := id_s \otimes (d_{\mathcal{C}(A)} + d^r_\alpha))$, where

$$d^r_\alpha : \mathcal{C}(A) \xrightarrow{\Delta_{(1)}\circ id_A} (\mathcal{C} \circ_{(1)} \mathcal{C})(A) \xrightarrow{(id_\mathcal{C}\circ_{(1)}\alpha)\circ id_A} \mathcal{C} \circ \mathcal{P}(A) \xrightarrow{id_\mathcal{C}\circ\gamma_A} \mathcal{C}(A).$$

The coproduct defining the $\mathcal{S}^{-1}\mathcal{C}$-coalgebra structure on $s\mathcal{C}(A)$ is given by

$$s\mathcal{C}(A) \xrightarrow{s\Delta_\mathcal{C}\circ id_A} s\mathcal{C} \circ \mathcal{C}(A) \cong \mathcal{S}^{-1}\mathcal{C}(s\mathcal{C}(A)).$$

3.2.2.1 Lemma ([GJ94])**.** *The map d_α is a differential, that is $d_\alpha^2 = 0$ and $\mathrm{B}_\alpha A$ is a dg $\mathcal{S}^{-1}\mathcal{C}$-coalgebra.*

In a similar way, one associates to the operadic twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ a functor

$$\Omega_\alpha : \textsf{dg } \mathcal{S}^{-1}\mathcal{C}\textsf{-coalgebras} \to \textsf{dg } \mathcal{P}\textsf{-algebras}$$

given on a $\mathcal{S}^{-1}\mathcal{C}$-coalgebra sC by $\Omega_\alpha sC := (\mathcal{P}(C),\, d_\alpha := d_{\mathcal{P}(C)} - d^l_\alpha)$, where

$$d^l_\alpha : \mathcal{P}(C) \xrightarrow{id_\mathcal{P}\circ'\Delta_C} \mathcal{P} \circ (C,\, \mathcal{C}(C)) \xrightarrow{id_\mathcal{P}\circ(id_C,\, \alpha\circ id_C)} (\mathcal{P} \circ_{(1)} \mathcal{P})(C) \xrightarrow{\gamma_{(1)}\circ id_\mathcal{C}} \mathcal{P}(C).$$

3.2.2.2 Lemma ([GJ94])**.** *The map d_α is a differential, that is $d_\alpha^2 = 0$ and $\Omega_\alpha sC$ is a $\mathcal{P}$-algebra.*

Notice that the notation d_α stands for different differentials. The differential is given without ambiguity by the context.

Example. Assume that $\mathcal{P} = \mathcal{A}s$ is the operad encoding associative algebras, $\mathcal{C} = \mathcal{A}s^{¡}$ and $\kappa : \mathcal{A}s^{¡} \to \mathcal{A}s$. Let A be an $\mathcal{A}s$-algebra, that is an associative algebra.

The bar construction $\mathrm{B}_\kappa A$ of the $\mathcal{A}s$-algebra A is equal to the classical (reduced) bar construction $\overline{\mathrm{B}}A_+ := (\overline{T}^c(sA),\, d)$ [EML53] of the associative algebra $A_+ = \mathbb{K} \oplus A$. Moreover, let C be an $\mathcal{A}s^{¡}$-coalgebra, that is sC is a coassociative coalgebra without counit. The cobar construction $\Omega_\kappa sC$ is equal to the classical (reduced) cobar construction $\overline{\Omega} sC_+ := (\overline{T}(C),\, d)$ [Ada56] of the coaugmented coassociative coalgebras $sC_+ = \mathbb{K} \oplus sC$.

The bar construction and the cobar construction form the *bar-cobar adjunction.*

3.2.2.3 Proposition (Proposition 2.18 of [GJ94])**.** *For every conilpotent $\mathcal{C}$-coalgebra C and every $\mathcal{P}$-algebra A, there is a natural bijection*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathsf{dg}\,\mathcal{P}\text{-}\mathsf{alg.}}(\Omega_\alpha sC,\, A) & \cong & \mathrm{Tw}_\alpha(C,\, A) & \cong & \mathrm{Hom}_{\mathsf{dg}\,\mathcal{S}^{-1}\mathcal{C}\text{-}\mathsf{cog.}}(sC,\, \mathrm{B}_\alpha A) \\ f_\varphi & \longleftrightarrow & \varphi & \longleftrightarrow & g_\varphi. \end{array}$$

This adjunction produces two particular morphisms. Consider a $\mathcal{P}$-algebra A and its bar construction $sC = \mathrm{B}_\alpha A$. The morphism of dg $\mathcal{S}^{-1}\mathcal{C}$-coalgebras $id_{\mathrm{B}_\alpha A}$ gives a universal algebraic twisting morphism

$$\pi_\alpha : \mathrm{B}_\alpha A \cong s\mathcal{C}(A) \twoheadrightarrow sA \xrightarrow{s^{-1}} A$$

and the *counit* of the adjunction

$$\varepsilon_\alpha : \Omega_\alpha \mathrm{B}_\alpha A = \mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A \xrightarrow{id_{\mathcal{P}} \circ (s\pi_\alpha)} \mathcal{P}(A) \xrightarrow{\gamma_A} A.$$

Similarly, when C is a dg $\mathcal{C}$-coalgebra and $A = \Omega_\alpha sC$, the morphism $id_{\Omega_\alpha sC}$ of dg $\mathcal{P}$-algebras gives a universal algebraic twisting morphism

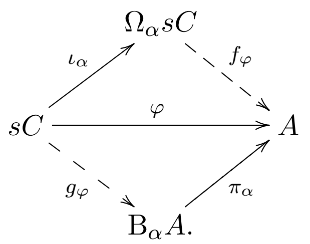
$$\iota_\alpha : sC \xrightarrow{s^{-1}} C \rightarrowtail \Omega_\alpha sC \cong \mathcal{P}(C)$$

and the *unit* of the adjunction

$$u_\alpha : sC \xrightarrow{\Delta sC} s\mathcal{C}(C) \xrightarrow{id_{s\mathcal{C}} \circ (s\iota_\alpha)} \mathrm{B}_\alpha \Omega_\alpha\, sC.$$

The morphisms π_α and ι_α are universal in the following meaning.

3.2.2.4 Lemma ([GJ94])**.** *With the above notations, any algebraic twisting morphism $\varphi : sC \to A$ with respect to α factors through the universal algebraic twisting morphisms*



Proof. The dashed arrows are just the images of φ by the two bijections of Proposition 3.2.2.3. $\square$

We now prove that the bar and the cobar construction behave well in the weight graded setting.

3.2.2.5 Lemma. *Let $\alpha : \mathcal{C} \to \mathcal{P}$ be a Koszul morphism between a wdg connected cooperad $\mathcal{C}$ and a wdg connected operad $\mathcal{P}$. The cobar construction Ω_α sends quasi-isomorphisms $g : sC \xrightarrow{\sim} sC'$ between wdg connected $\mathcal{S}^{-1}\mathcal{C}$-coalgebras to quasi-isomorphisms $\Omega_\alpha sC \xrightarrow{\sim} \Omega_\alpha sC'$ of $\mathcal{P}$-algebras.*

Similarly, the bar construction B_α sends quasi-isomorphisms $f : A \xrightarrow{\sim} A'$ between wdg connected $\mathcal{P}$-algebras to quasi-isomorphisms $\mathrm{B}_\alpha A \xrightarrow{\sim} \mathrm{B}_\alpha A'$ of $\mathcal{S}^{-1}\mathcal{C}$-coalgebras.

Proof. We show first the result for the cobar construction. Since $\mathbb{K}$ is a field of characterisic 0, every dg module is projective. Moreover, by Maschke's theorem, every $\mathbb{K}[\mathbb{S}_n]$-module is projective. So the quasi-isomorphism $sC \xrightarrow{\sim} sC'$ implies the quasi-isomorphism $C \xrightarrow{\sim} C'$, $C^{\otimes n} \xrightarrow{\sim} C'^{\otimes n}$ and $\mathcal{P}(n) \otimes_{\mathbb{S}_n} C^{\otimes n} \xrightarrow{\sim} \mathcal{P}(n) \otimes_{\mathbb{S}_n} C'^{\otimes n}$. Finally $(\mathcal{P}(C),\, d_{\mathcal{P}(C)}) \xrightarrow{\sim} (\mathcal{P}(C'),\, d_{\mathcal{P}(C')})$. We filter the chain complex $\Omega_\alpha sC = (\mathcal{P}(C),\, d_{\mathcal{P}(C)} - d^l_\alpha)$ by the total weight in C

$$F_p(\mathcal{P}(C)) := \bigoplus_{\omega_1 + \cdots + \omega_m \leq p} \bigoplus_{n \in \mathbb{N}} \mathcal{P}(n) \otimes_{\mathbb{S}_n} C^{(\omega_1)} \otimes \cdots \otimes C^{(\omega_n)}.$$

The part $d_{\mathcal{P}(C)}$ of the differential keeps the total weight in C constant. Since $\mathcal{C}$ and $\mathcal{P}$ are wdg connected, the twisting morphisms are zero on weight zero and the part d^l_α of the differential decreases the total weight at least by 1. The differential respects the filtration. This filtration is exhaustive and bounded below. So we apply the classical theorem of convergence of spectral sequences (Theorem 5.5.1 of [Wei94]) and we get that the induced spectral sequence converges to the homology of $\Omega_\alpha sC$. We consider the same filtration for C'. The terms $E^1_{p,\,q}(\mathcal{P}(C))$ are given by the homology of $(\mathcal{P}(C),\, d_{\mathcal{P}(C)})$ and are isomorphic to the terms $E^1_{p,\,q}(\mathcal{P}(C'))$, that is to the homology of $(\mathcal{P}(C'),\, d_{\mathcal{P}(C')})$. Since moreover g is a morphism of dg $\mathcal{C}$-coalgebras, the pages E^r, $r \geq 1$ are isomorphic and $\Omega_\alpha sC \xrightarrow{\sim} \Omega_\alpha sC'$ is a quasi-isomorphism.

To prove the result for the bar construction, we consider the filtration F_p on $\mathrm{B}_\alpha A$ given by

$$F_p(s\mathcal{C}(A)) := \bigoplus_{\omega \leq p} \mathcal{C}^{(\omega)}(A).$$

The rest of the proof is similar. □

3.2.3 Cotangent complex

Operadic Koszul morphisms provide functorial resolutions of $\mathcal{P}$-algebras. We use these resolutions to make the cotangent complex involved in the André-Quillen cohomology theory of $\mathcal{P}$-algebras explicit.

The *cotangent complex* associated to a $\mathcal{P}$-algebra A is a (class of) chain complexes which represents the *André-Quillen cohomology theory of A with coefficients in a module.* We make it explicit following [GH00] and [Mil08], where the reader can find complete exposition about the André-Quillen cohomology theory and the cotangent complex.

To a resolution of the $\mathcal{P}$-algebra A is associated a representation of the cotangent complex of A.

3.2.3.1 Proposition (Proposition 10.3.6 of [LV]). *The operadic twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ is Koszul, that is $\mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha \mathcal{P} \xrightarrow{\sim} \mathcal{P}$ if and only if the counit of the adjunction $\Omega_\alpha \mathrm{B}_\alpha A \xrightarrow{\sim} A$ is a quasi-isomorphism for every $\mathcal{P}$-algebra A.*

As a consequence, we recover the following theorems.

3.2.3.2 Theorem (Theorem 2.19 of [GJ94]). *For any $\mathcal{P}$-algebra A, there is a quasi-isomorphism*

$$\Omega_\pi \mathrm{B}_\pi A = \mathcal{P} \circ_\pi \mathrm{B}\mathcal{P} \circ_\pi A \xrightarrow{\sim} A.$$

3.2.3.3 Theorem (Theorem 2.25 of [GJ94]). *When the operad $\mathcal{P}$ is Koszul, there is a smaller resolution of any $\mathcal{P}$-algebra A*

$$\Omega_\kappa \mathrm{B}_\kappa A = \mathcal{P} \circ_\kappa \mathcal{P}^{¡} \circ_\kappa A \xrightarrow{\sim} A.$$

To an operadic twisting morphism $\alpha : \mathcal{C} \to \mathcal{P}$ and to an algebraic twisting morphism $\varphi : sC \to A$ with respect to α, we associate the following coequalizer $A \otimes^{\mathcal{P}} C$:

$$\mathcal{P} \circ (\mathcal{P}(A),\, C) \underset{c_1}{\overset{c_0}{\rightrightarrows}} \mathcal{P} \circ (A,\, C) \twoheadrightarrow A \otimes^{\mathcal{P}} C,$$

where $\begin{cases} c_0 : \mathcal{P} \circ (\mathcal{P}(A),\, C) \to (\mathcal{P} \circ \mathcal{P})(A,\, C) \xrightarrow{\gamma(id_A,\, id_C)} \mathcal{P}(A,\, C) \\ c_1 : \mathcal{P} \circ (\mathcal{P}(A),\, C) \xrightarrow{id_{\mathcal{P}}(\gamma_A,\, id_C)} \mathcal{P}(A,\, C) \end{cases}$.

The differential $d_\varphi := d_{A \otimes^{\mathcal{P}} C} - d^l_\varphi$ on $A \otimes^{\mathcal{P}} C$ depends on the differentials on A, $\mathcal{P}$ and C and on a twisting term d^l_φ, which is the map on the quotient $A \otimes^{\mathcal{P}} C$ induced by $d^l_1 := \sum d^l_1(n)$ where

$$d^l_1(n) : \mathcal{P}(A,\, C) \xrightarrow{id_{\mathcal{P}}(id_A,\, \Delta_C(n))} \mathcal{P}(A,\, (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}) \xrightarrow{id_{\mathcal{P}}(id_A,\, \alpha \otimes (s\varphi)^{\otimes n-1} \otimes id_C)}$$
$$\mathcal{P}(A,\, \mathcal{P}(n) \otimes A^{\otimes n-1} \otimes C) \to (\mathcal{P} \circ \mathcal{P})(A,\, C) \xrightarrow{\gamma(id_A,\, id_C)} \mathcal{P}(A,\, C),$$

with $\Delta_C(n) : C \xrightarrow{\Delta_C} \mathcal{C}(C) \twoheadrightarrow (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}$ (see Section 2 in [Mil08] for more details).

When $\Omega_\alpha sC = \mathcal{P} \circ_\alpha C \xrightarrow{\sim} A$ is a resolution of the $\mathcal{P}$-algebra A, the chain complex $A \otimes^{\mathcal{P}} C$ is a representation of the *cotangent complex*. For example, when α is a Koszul morphism, the universal twisting morphism $\pi_\alpha : \mathrm{B}_\alpha A \to A$ gives the functorial resolution $\Omega_\alpha \mathrm{B}_\alpha A \cong \mathcal{P} \circ_\alpha \mathcal{C} \circ_\alpha A \xrightarrow{\sim} A$ and a representation of the cotangent complex is given by $A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A$.

Remark. We denote it by $A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A$ instead of $A \otimes^{\mathcal{P}} s^{-1}\mathrm{B}_\alpha A$ to simplify the notation.

Example. In the case $\mathcal{P} = \mathcal{A}s$, $\mathcal{C} = \mathcal{A}s^{¡}$ and $\alpha = \kappa : \mathcal{A}s^{¡} \to \mathcal{A}s$. The cotangent complex $A \otimes^{\mathcal{A}s} \mathrm{B}_\kappa A$ is equal to the augmented bar construction $A_+ \otimes \overline{\mathrm{B}}A_+ \otimes A_+$, where $A_+ := \mathbb{K} \oplus A$ since $\mathrm{B}_\kappa A = \overline{B}A_+$ (see [Mil08] for more details).

3.2.4 Algebraic Koszul morphisms

Let $\alpha : \mathcal{C} \to \mathcal{P}$ be an operadic Koszul morphism and let $\varphi : sC \to A$ be an algebraic twisting morphism. The associated $\mathcal{S}^{-1}\mathcal{C}$-algebras morphism $g_\varphi : sC \to \mathrm{B}_\alpha A$ induces a natural morphism of dg A-modules $A \otimes^{\mathcal{P}} C \to A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A$. We say that φ is an *algebraic Koszul morphism* when the morphism $A \otimes^{\mathcal{P}} C \to A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A$ is a quasi-isomorphism. We denote by $\mathrm{Kos}_\alpha(C,\, A)$ the set of algebraic Koszul morphisms from sC to A.

There are some operads $\mathcal{P}$ such that the André-Quillen cohomology theory of any $\mathcal{P}$-algebra A is an Ext-functor over the enveloping algebra of A. These operads satisfy the following property :

$$\text{There is a quasi-isomorphism } A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A \xrightarrow{\sim} \Omega_{\mathcal{P}}(A) \text{ for any } \mathcal{P}\text{-algebra } A, \qquad (\star)$$

where $\Omega_{\mathcal{P}}(A)$ is the module of Kähler differentials forms. We refer to [Mil08] for the complete study. Hence, in this case, an algebraic twisting morphism $\varphi : sC \to A$ is an algebraic Koszul morphism if and only if the map $A \otimes^{\mathcal{P}} C \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$ is a quasi-isomorphism.

3.2.4.1 Theorem (Algebraic twisting morphisms fundamental theorem)**.** *Let $\alpha : \mathcal{C} \to \mathcal{P}$ be a Koszul morphism between a wdg connected cooperad $\mathcal{C}$ and a wdg connected operad $\mathcal{P}$. Let C be a wdg connected $\mathcal{C}$-coalgebra and A be a wdg connected $\mathcal{P}$-algebra. Let $\varphi : sC \to A$ be an algebraic twisting morphism. The following assertions are equivalent :*

1. *the twisting morphism φ is an algebraic Koszul morphism, that is $A \otimes^{\mathcal{P}} C \xrightarrow{\sim} A \otimes^{\mathcal{P}} \mathrm{B}_\alpha A$;*
2. *the map of $\mathcal{S}^{-1}\mathcal{C}$-coalgebras $g_\varphi : sC \xrightarrow{\sim} \mathrm{B}_\alpha A$ is a quasi-isomorphism ;*
3. *the map of $\mathcal{P}$-algebras $f_\varphi : \Omega_\alpha sC \xrightarrow{\sim} A$ is a quasi-isomorphism.*

Moreover, when $\mathcal{P}$ satisfies Condition $(\star)$, the previous assertions are equivalent to

(1') the natural map $A \otimes^{\mathcal{P}} C \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$ is a quasi-isomorphism.

Remark. When the operad $\mathcal{P}$ and the $\mathcal{P}$-algebra A are concentrated in homological degree 0, the module of Kähler differential forms $\Omega_{\mathcal{P}}(A)$ is concentrated in homological degree 0 and Condition $(1')$ writes : $A \otimes^{\mathcal{P}} C$ is acyclic.

Proof. We apply the comparison Lemma proved in Appendix A to sC and $sC' = \mathrm{B}_\alpha A$ to get the equivalence between (1) and (2).

To prove the equivalence (2) $\Leftrightarrow$ (3), we apply Lemma 3.2.2.5 to the quasi-isomorphism $sC \xrightarrow{\sim} \mathrm{B}_\alpha A$ to get the quasi-isomorphism $\Omega_\alpha sC \xrightarrow{\sim} \Omega_\alpha \mathrm{B}_\alpha A$. Since α is a Koszul morphism, $\Omega_\alpha \mathrm{B}_\alpha A \xrightarrow{\sim} A$ by Proposition 3.2.3.1 and we get the implication (2) $\Rightarrow$ (3).

To prove the reverse implication, we apply Lemma 3.2.2.5 to the quasi-isomorphism $\Omega_\alpha sC \xrightarrow{\sim} A$ to get the quasi-isomorphism $\mathrm{B}_\alpha \Omega_\alpha\, sC \xrightarrow{\sim} \mathrm{B}_\alpha A$. Then we just need to prove that $\mathrm{B}_\alpha \Omega_\alpha sC = s\mathcal{C} \circ_\alpha \mathcal{P} \circ_\alpha C \xrightarrow{\sim} sC$ for each $\mathcal{C}$-coalgebra C, provided that $\mathcal{C} \circ_\alpha \mathcal{P} \xrightarrow{\sim} I$ (by Theorem 3.1.3.3). To prove this, we endow $\mathcal{C} \circ_\alpha \mathcal{P} \circ_\alpha C$ with a filtration F_p given by

$$F_p(\mathcal{C} \circ_\alpha \mathcal{P} \circ_\alpha C) := \bigoplus_{\omega \leq p} \mathcal{C} \circ_\alpha \mathcal{P} \circ_\alpha \underbrace{C}_{(\omega)}.$$

The differential $d_{\mathcal{C}\circ\mathcal{P}\circ C} + d^r_\alpha \circ id_C - id_{\mathcal{C}} \circ d^l_\alpha$ satisfies

$$\begin{cases} d_{\mathcal{C}\circ\mathcal{P}\circ C} & : & F_p \to F_p \\ d^r_\alpha \circ id_C & : & F_p \to F_p \\ id_{\mathcal{C}} \circ d^l_\alpha & : & F_p \to F_{p-1}. \end{cases}$$

So the filtration is a filtration of chain complexes. Moreover, it is bounded below and exhaustive so the classical theorem of convergence of spectral sequences (Theorem 5.5.1 of [Wei94]) gives that the induced spectral sequence $E^\bullet_{p,q}$ converges to the homology of $\mathcal{C} \circ_\alpha \mathcal{P} \circ_\alpha C$. We endow similarly C with the filtration by the weight grading and we get an isomorphism between the E^1-pages, provided that $E^1_{p,q} \cong \mathrm{H}_\bullet(\mathcal{C} \circ_\alpha \mathcal{P}) \circ \mathrm{H}_\bullet(C) \cong \mathrm{H}_\bullet(C)$ by the Künneth formula. The convergence of the spectral sequences concludes the proof. $\square$

Example. When $\mathcal{P} = \mathcal{A}s$, we recover Theorem 3.1.1.1, provided that the category of $\mathcal{A}s$-algebras is equivalent to the category of augmented associative algebras and that a conipoltent $\mathcal{S}^{-1}\mathcal{A}s$-coalgebra is exactly a conilpotent coaugmented coassociative coalgebra.

3.3 Koszul duality theory for algebra over an operad

We proved in the previous section the algebraic twisting morphisms fundamental theorem. In this section, we define the notion of *monogene $\mathcal{P}$-algebra* and we associate to such a $\mathcal{P}$-algebra A its Koszul dual $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebra , which is a good candidate for the algebraic twisting morphisms fundamental theorem. We show the link with the Koszul dual $\mathcal{P}^!$-algebra defined in [GK94]. We give the Koszul criterion for $\mathcal{P}$-algebra and the definition of a Koszul $\mathcal{P}$-algebra. Thus we obtain a criterion to prove that we have a "small" resolution of A. This generalizes the Koszul duality theory for quadratic associative algebras [Pri70] and conceptually explains the form of the Koszul complex by the fact that it is a representation of the cotangent complex when the $\mathcal{P}$-algebra is Koszul.

3.3.1 Monogene $\mathcal{P}$-algebra

The notion of quadratic algebra over a quadratic operad appears in [GK94]. We extend this definition to *monogene* $\mathcal{P}$-algebra and we give the definition of *monogene* $\mathcal{C}$-coalgebra. We use the word monogene to express the fact that the relations in the $\mathcal{P}$-algebra are linearly generated by the generators of the operad.

Let $(E,\, R)$ be an operadic quadratic data (see Section 3.1.3) and $\mathcal{P} := \mathcal{F}(E)/(R)$ its associated quadratic operad. A *monogene data $(V,\, S)$ of the operadic quadratic data $(E,\, R)$* is a graded vector space V and a subspace $S \subseteq E(V)$. We associate to this monogene data the *monogene $\mathcal{P}$-algebra*

$$A(V,\, S) := \mathcal{P}(V)/(S).$$

The operad $\mathcal{P}$ is weight graded and we endow $\mathcal{P}(V)$ with a weight grading equal to the weight grading in $\mathcal{P}$. The space S is homogeneous for this grading since the weight of $E(V)$ is equal to 1, so the $\mathcal{P}$-algebra $A(V,\, S)$ is weight graded.

3.3.1.1 Proposition. *When $\mathcal{P} = \mathcal{P}(E,\, R)$ is a binary quadratic operad, that is $E = E(2) = \mathcal{P}(2)$, we have $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbb{K}$ and explicitly*

$$A(V,\, S) = \bigoplus_{n\geq 0} A(V,\, S)^{(n)} = V \oplus (E \otimes_{\mathbb{S}_2} V^{\otimes 2})/S \oplus \cdots \oplus (\mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n})/\mathrm{Im}(\psi_n) \oplus \cdots ,$$

where ψ_n is the composite

$$\bigoplus_{j=0}^{n-1} \mathcal{P}(n-1) \otimes_{\mathbb{S}_{n-1}} (V^{\otimes j} \otimes S \otimes V^{\otimes n-2-j}) \to (\mathcal{P} \circ_{(1)} E)(n) \otimes_{\mathbb{S}_n} V^{\otimes n} \rightarrowtail \mathcal{P} \circ \mathcal{P}(V) \xrightarrow{\gamma \circ id_V} \mathcal{P}(V).$$

In this case, the monogene $\mathcal{P}$-algebra is a quadratic $\mathcal{P}$-algebra *as defined in [GK94].*

Proof. The image of the map ψ_n is exactly the definition of the ideal of $\mathcal{P}(V)$ generated by $S \subseteq E(V)$. □

Example. When $\mathcal{P} = \mathcal{A}s$, we recover the notion of quadratic associative algebra $T(V)/(S)$ and the weight grading is given by the number of elements in V minus 1.

Dually, let $\mathcal{C} = \mathcal{C}(E, R)$ be the cooperad associated to the operadic quadratic data (E, R). The *monogene $\mathcal{C}$-coalgebra* associated to the monogene data (V, S) of the operadic quadratic data (E, R) is the $\mathcal{C}$-coalgebra $C(V, S)$ which is universal among the sub-$\mathcal{C}$-coalgebras C such that the composite

$$C \rightarrowtail \mathcal{C}(V) \twoheadrightarrow E(V)/S$$

is equal to 0. The word "universal" means that for any such $\mathcal{C}$-coalgebra C, there exists a unique morphism of $\mathcal{C}$-coalgebras $C \to C(V, S)$ such that the following diagram commutes

$$\begin{array}{ccc} C(V, S) & \rightarrowtail & \mathcal{C}(V). \\ \uparrow & \nearrow & \\ C & & \end{array}$$

The cooperad $\mathcal{C}$ is weight graded, so the $\mathcal{C}$-coalgebra $C(V, S)$ is weight graded by the weight on $\mathcal{C}$.

3.3.1.2 Proposition. *When $\mathcal{C} = \mathcal{C}(E, R)$ is a binary quadratic cooperad, that is $E = E(2) = \mathcal{C}(2)$, we have $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = \mathbb{K}$. Dually to the algebra case, we have explicitly*

$$C(V, S) = \bigoplus_{n \geq 0} C(V, S)^{(n)} = V \oplus S \oplus \cdots \oplus \mathrm{Ker}(\phi_n) \oplus \cdots ,$$

where ϕ_n is the composite
$$\mathcal{C}(V) \xrightarrow{\Delta \circ id_V} \mathcal{C} \circ \mathcal{C}(V) \twoheadrightarrow ((\mathcal{C} \circ_{(1)} E)(n) \otimes V^{\otimes n})^{\mathbb{S}_n} \to$$

$$((\mathcal{C} \circ_{(1)} E)(n) \otimes V^{\otimes n})^{\mathbb{S}_n} / \bigcap_{j=0}^{n-1} (\mathcal{C}(n-1) \otimes V^{\otimes j} \otimes S \otimes V^{n-2-j})^{\mathbb{S}_n}.$$

In this case, the monogene $\mathcal{C}$-coalgebra is a quadratic $\mathcal{C}$-coalgebra*.*

Proof. We dualize the map ψ_n of the previous proposition to get the map ϕ_n and the notion of "coideal" of $\mathcal{C}(V)$ "cogenerated" by S. ☐

3.3.2 Koszul dual coalgebra

We define the *Koszul dual $\mathcal{S}^{-1}\mathcal{P}^{i}$-coalgebra*, the *Koszul dual $\mathcal{P}^!$-algebra* of a monogene $\mathcal{P}$-algebra and the corresponding algebraic twisting morphism. When the operad $\mathcal{P}$ is binary and finitely generated, we recover the definition of the Koszul dual $\mathcal{P}^!$-algebra of [GK94]. We show that the Koszul dual $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebra associated to a monogene $\mathcal{P}$-algebra is the zero homology group for a certain degree of the bar construction of this $\mathcal{P}$-algebra.

To an operadic quadratic data (E, R), we associate the operad $\mathcal{P} := \mathcal{P}(E, R)$, the Koszul dual cooperad $\mathcal{P}^{¡} := \mathcal{C}(sE, s^2R)$ and its homological desuspension $\mathcal{S}^{-1}\mathcal{P}^{¡} := \mathcal{S}^{-1} \otimes_H \mathcal{P}^{¡}$. We assume that $\mathcal{P}$ is a Koszul operad (cf. 3.1.3).

Let V be a vector space and let S be a subspace of $E(V)$. We have $sS \subset sE(V) \rightarrowtail \mathcal{P}^{¡}(V)$. The *Koszul dual* $\mathcal{S}^{-1}\mathcal{P}^{i}$*-coalgebra* of the monogene $\mathcal{P}$-algebra $A(V, S)$ is the monogene $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebra

$$A^{¡} := sC(V, sS).$$

We define the morphism $\varkappa : A^{¡} \to A$ by the linear map of degree -1 :

$$A^{¡} := sC(V, sS) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail A(V, S) = A.$$

3.3.2.1 Lemma. *We have* $\star_{\kappa}(\varkappa) = 0$ *and therefore* $\varkappa$ *is an algebraic twisting morphism.*

Proof. Due to the definition of $\varkappa$, the term $\star_{\kappa}(\varkappa)$ is 0 everywhere except maybe on $s^2S \subset s^2E(V)$ where it is equal to

$$s^2S \xrightarrow{\Delta_{A^{¡}}} s^2E(V) \xrightarrow{(s^{-1}\kappa)\circ(s\varkappa)} E(V) \xrightarrow{\gamma_A} E(V)/S.$$

This last map is 0 by definition. □

Example. When $\mathcal{P} = \mathcal{A}s$, up to adding a unit, we recover the Koszul dual coalgebra defined in [Pri70].

Let $\mathcal{C} = \oplus_{n\geq 0}\mathcal{C}^{(n)}$ be a weight graded cooperad. We define the *weight-graded linear dual* of $\mathcal{C}$ by $\mathcal{C}^* := \oplus_{n\geq 0}{\mathcal{C}^{(n)}}^* = \oplus_{n\geq 0}\mathrm{Hom}_{\mathbb{K}}(\mathcal{C}^{(n)}, \mathbb{K})$. The weight-graded linear dual of $\mathcal{S}^{-1}\mathcal{P}^{¡}$ is denoted by

$$\mathcal{P}^! := (\mathcal{S}^{-1}\mathcal{P}^{¡})^* = (\mathcal{S}^{-1} \otimes_H \mathcal{P}^{¡})^*$$

and is a weight graded operad. When E is a finite dimensional $\mathbb{S}_2$-module, it corresponds to the operad defined in [GK94] by $\mathcal{P}^! = \mathcal{F}(E^{\vee})/(R^{\perp})$ (see Theorem 7.6.5 in [LV] for a proof of this fact). In the coalgebra case, let $C = \oplus_{n\geq 0}C^{(n)}$ be a weight graded $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebra. We define the *weight-graded linear dual* of C by $C^* := \oplus_{n\geq 0}{C^{(n)}}^*$. The *Koszul dual* $\mathcal{P}^!$*-algebra* $A^!$ is the following $\mathcal{P}^!$-algebra

$$A^! := \left(\bigoplus_{n\geq 0} s^{-n}{A^{¡}}^{(n)}\right)^* = \bigoplus_{n\geq 0} s^n \left({A^{¡}}^{(n)}\right)^*.$$

3.3.2.2 Proposition. *Let $\mathcal{P}$ be a finitely generated binary Koszul operad. Let (V, S) be a finitely generated monogene data. The Koszul dual $\mathcal{P}^!$-algebra $A^!$ of $A = A(V, S)$ is equal to the monogene $\mathcal{P}^!$-algebra $A^! = A(V^*, R^{\perp})$ defined in [GK94] by $V^* := \mathrm{Hom}(V, \mathbb{K})$ and $S^{\perp}$ is the annihilator of S for the natural pairing $\langle -, - \rangle : E^*(V^*) \otimes E(V) \to \mathbb{K}$.*

Proof. Since E and V are finite dimensional, we remark that

$$\left(s\bigoplus_{n\geq 0} s^{-n}{\mathcal{P}^{¡}}^{(n)}(V)\right)^* = \mathcal{P}^!(V^*).$$

After a weight graded desuspension, we linearly dualize the exact short sequence

$$0 \to A^{¡} \to s\mathcal{P}^{¡}(V) \to s^2E(V)/s^2S \to 0$$

satisfied by $A^{¡}$. We get the exact sequence

$$0 \leftarrow A^{!} \leftarrow \mathcal{P}^{!}(V^*) \leftarrow S^{\perp} \leftarrow 0,$$

where the orthogonal space $S^{\perp}$ is the annihilator S for the natural pairing $\langle -, - \rangle : E^*(V^*) \otimes E(V) \to \mathbb{K}$. Since $A^{¡}$ is universal for the first exact sequence, the dual $A^{!}$ is universal for the second one and is equal to $\mathcal{P}^{!}(V^*)/(S^{\perp})$. □

Recall that the operad $\mathcal{P}$ is weight graded by the number of elements in E. It induces a weight grading on A (the weight of V being equal to 0). Hence, we endow the bar construction $\mathrm{B}_\kappa A$ with a non-negative *weight-degree* induced only by the weight grading on A. We denote it by $\mathrm{B}_\kappa^\omega A$.

The internal differential on A is 0 since the differential on $\mathcal{P}$ and on V are 0. Thus the differential on $\mathrm{B}_\kappa A$ reduces to $d_\kappa = id_s \otimes d_\kappa^r$ (defined in Section 3.2.2). The differential d_κ raises the weight-degree by 1 and we get a cochain complex with respect to this degree. The elements of weight-degree 0 in $\mathrm{B}_\kappa A = s\mathcal{P}^{¡}(A)$ are given by $s\mathcal{P}^{¡}(V)$, then $A^{¡} \rightarrowtail \mathrm{B}_\kappa^0 A = s\mathcal{P}^{¡}(V)$.

3.3.2.3 Proposition. *Let (E, R) be an operadic quadratic data. Let (V, S) be a monogene data. The natural $\mathcal{S}^{-1}\mathcal{P}^{i}$-coalgebras inclusion $g_\varkappa : A^{i} = sC(V, sS) \rightarrowtail \mathrm{B}_\kappa A = \mathrm{B}_\kappa A(V, S)$ induces an isomorphism of graded $\mathcal{S}^{-1}\mathcal{P}^{i}$-coalgebras*

$$g_\varkappa : A^{i} \xrightarrow{\cong} \mathrm{H}^0(\mathrm{B}_\kappa^\bullet A).$$

Proof. Since there is no element in negative weight-degree, we just need to prove that the inclusion $g_\varkappa$ is exactly the kernel of the differential $id_s \otimes d^r_{\kappa|s\mathcal{P}^{¡}(V)}$. The image of $g_\varkappa$ lives in weight-degree 0. Moreover, the morphism $g_\varkappa$ commutes with the differentials $d_{A^{¡}} = 0$ and d_κ so $d_\kappa \circ g_\varkappa = g_\varkappa \circ d_{A^{¡}} = 0$ and $A^{¡} \rightarrowtail \mathrm{Ker}(d_{\kappa|weight=0}) = \mathrm{Ker}(id_s \otimes d^r_{\kappa|s\mathcal{P}^{¡}(V)}) =: K$. Since K is the kernel of a $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebras morphism, it is a $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebra. It is easy to see that the composition $K \rightarrowtail s\mathcal{P}^{¡}(V) \twoheadrightarrow s^2E(V)/s^2S$ is equal to 0 since the differential in weight 1 is the quotient map $s^2E(V) \to s(E(V)/S)$. Due to the universal property of $A^{¡}$ and since $A^{¡} \rightarrowtail K$ is a monomorphism, we get that $A^{¡} = K = \mathrm{Ker}(id_s \otimes d^r_{\kappa|s\mathcal{P}^{¡}(V)})$. □

3.3.3 Koszul criterion and Koszul $\mathcal{P}$-algebra

The previous proposition shows that the Koszul dual $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebra is a good candidate to replace the bar construction in the cotangent complex. We state the Koszul criterion shows that it is the case when the algebraic twisting morphism $\varkappa$ is Koszul. We define the notion of *Koszul $\mathcal{P}$-algebra.*

Let (E, R) be an operadic quadratic data, $\mathcal{P} := \mathcal{P}(E, R)$, $\mathcal{P}^{¡} := \mathcal{C}(sE, s^2R)$ and $\kappa : \mathcal{P}^{¡} \to \mathcal{P}$. Let (V, S) be a monogene data, $A := A(V, S) = \mathcal{P}(V)/(S)$ the associated monogene $\mathcal{P}$-algebra and $A^{¡} := sC(V, sS)$ the Koszul dual $\mathcal{S}^{-1}\mathcal{P}^{¡}$-coalgebra.

When $\mathcal{P}$ is a Koszul operad, the bar-cobar construction $\Omega_\kappa \mathrm{B}_\kappa A$ of A is a cofibrant resolution of A. To simplify this resolution, we can replace $\mathrm{B}_\kappa A$ by $A^{¡} \cong \mathrm{H}^0(\mathrm{B}_\kappa^\bullet A)$. This works when $\mathrm{H}^\bullet(\mathrm{B}_\kappa^\bullet A) = \mathrm{H}^0(\mathrm{B}_\kappa^\bullet A)$. The following Koszul criterion shows that it is the case if and only if the algebraic twisting morphism $\varkappa : A^{¡} \to A$ is Koszul.

We apply the algebraic twisting morphism fundamental theorem of the previous section (Theorem 3.2.4.1) to get the following theorems, which are the main theorems of Koszul duality for $\mathcal{P}$-algebras.

3.3.3.1 Theorem (Koszul criterion)**.** *Let (E, R) be an operadic quadratic data such that $\mathcal{P} = \mathcal{P}(E, R)$ is a Koszul operad. Let (V, S) be a monogene data associated to (E, R). The following assertions are equivalent :*

1. *the twisting morphism κ is an algebraic Koszul morphism, that is*
$$A \otimes^{\mathcal{P}} A^{¡} \xrightarrow{\sim} A \otimes^{\mathcal{P}} \mathrm{B}_{\kappa} A$$
is a quasi-isomorphism ;
2. *the inclusion $g_{\varkappa} : A^{¡} \rightarrowtail \mathrm{B}_{\kappa} A$ is a quasi-isomorphism ;*
3. *the projection $f_{\varkappa} : \Omega_{\kappa} A^{¡} \twoheadrightarrow A$ is a quasi-isomorphism.*

Moreover, when $\mathcal{P}$ satisfies Condition $(\star)$, the previous assertions are equivalent to

(1') the natural map $A \otimes^{\mathcal{P}} A^{¡} \xrightarrow{\sim} \Omega_{\mathcal{P}}(A)$ is a quasi-isomorphism.

When these assertions hold, the cobar construction on $A^{¡}$ gives a cofibrant resolution of the $\mathcal{P}$-algebra A (a minimal resolution when $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbb{K}$). A monogene $\mathcal{P}$-algebra A is called Koszul *when it satisfies the equivalent properties of this theorem.*

Proof. We apply Lemma 3.3.2.1 and Theorem 3.2.4.1 to $\mathcal{C} := \mathcal{P}^{¡}$, $A := A(V, S)$, $C := A^{¡}$ and $\varphi := \varkappa$ since the weight assumptions are satisfied.

When the assertions of the theorem hold, $\Omega_{\kappa} A^{¡}$ is a cofibrant resolution of A since it is a resolution of A and since $\Omega_{\kappa} A^{¡}$ is a quasi-free $\mathcal{P}$-algebra on the connected weight graded $\mathcal{A}s$-coalgebra $s^{-1}A^{¡}$. Moreover, when the operad satisfies $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = 0$, the differential satisfies
$$d_{\Omega_{\kappa} A}(s^{-1}A^{¡}) = -d^{l}_{\kappa}(s^{-1}A^{¡}) \subset \oplus_{n \geq 2} \mathcal{P}(n) \otimes_{\mathbb{S}_n} (s^{-1}A^{¡})^{\otimes n}$$
by construction and $\Omega_{\kappa} A$ is a minimal resolution of A. □

Remark. We recall that when the operad $\mathcal{P}$ and the $\mathcal{P}$-algebra A are concentrated in homological degree 0, the module of Kähler differential forms $\Omega_{\mathcal{P}}(A)$ is concentrated in homological degree 0 and the condition $(1')$ writes simply : $A \otimes^{\mathcal{P}} A^{¡}$ is acyclic.

The chain complex $A \otimes^{\mathcal{P}} A^{¡}$ is called the *Koszul complex*. Thus, when the algebraic twisting morphism $\varkappa : A^{¡} \to A$ is an algebraic Koszul morphism, the Koszul complex $A \otimes^{\mathcal{P}} A^{¡}$ is a representation of the cotangent complex, representing the cohomology theory of the $\mathcal{P}$-algebra A.

Example. Assume that $\mathcal{P} = \mathcal{A}s$, $\mathcal{C} = \mathcal{A}s^{¡}$ and $\kappa : \mathcal{A}s^{¡} \to \mathcal{A}s$ is the operadic twisting morphism between them. The operad $\mathcal{A}s$ satisfies the Condition $(\star)$ of Section 3.2.4, so an $\mathcal{A}s$-algebra A is Koszul if and only if for $A_+ := \mathbb{K} \oplus A$, the A-bimodules morphism $A_+ \otimes s^{-1}\overline{(A_+)^{¡}} \otimes A_+ \cong A \otimes^{\mathcal{A}s} A^{¡} \xrightarrow{\sim} \Omega_{\mathcal{A}s} A \cong A \otimes A_+$ is a quasi-isomorphism. This is equivalent to $A_+ \otimes (A_+)^{¡} \otimes A_+ \cong s(A \otimes^{\mathcal{A}s} A^{¡}) \oplus (A_+ \otimes A_+) \xrightarrow{\sim} A_+$ is a quasi-isomorphism. This last quasi-isomorphism is the classical definition for a quadratic augmented associative algebra A_+ to be Koszul (see Theorem 3.1.1.2 and the definition after).

For an associative algebra A, we already know that A is Koszul if and only if $A^{!}$ is Koszul. As an application of the Koszul criterion theorem, we obtain the same result for $\mathcal{P}$-algebras.

3.3.3.2 Theorem. *Let (E, R) be a finitely generated binary operadic quadratic data such that $\mathcal{P} = \mathcal{P}(E, R)$ is a Koszul operad. Let (V, S) be a finitely generated monogene data associated to (E, R). The $\mathcal{P}$-algebra $A := A(V, S)$ is Koszul if and only if the $\mathcal{P}^!$-algebra $A^! = \mathcal{P}^!(V^*, S^\perp)$ is Koszul.*

Proof. The Koszul criterion Theorem 3.3.3.1 (2) implies that A is a Koszul $\mathcal{P}$-algebra if and only if $A^{¡} \xrightarrow{\sim} \mathrm{B}_\kappa A = \mathcal{P}^{¡}(A)$ is a quasi-isomorphism. We consider the n-desuspension of the weight-degree n part : it is a chain complex morphism (and a quasi-isomorphism of chain complexes)

$$\bigoplus_{n\geq 0} s^{-n} A^{¡(n)} \to \bigoplus_{n\geq 0} s^{-n}(\mathcal{P}^{¡}(A))^{(n)}$$

since the differentials preserve the weight grading. We linearly dualize the quasi-isomorphisms $s^{-n}A^{¡(n)} \to s^{-n}(\mathcal{P}^{¡}(A))^{(n)}$ to get the quasi-isomorphisms $\left(s^{-n}A^{¡(n)}\right)^* \leftarrow \left(s^{-n}(\mathcal{P}^{¡}(A))^{(n)}\right)^*$ since $\mathbb{K}$ is a field. The sum of these quasi-isomorphisms gives the quasi-isomorphism $A^! \xleftarrow{\sim} \mathcal{P}^!(A^*) = \mathcal{P}^!(A^{!^{!^*}}) = \mathcal{P}^!(A^{!^{¡}})$. By the Koszul criterion Theorem 3.3.3.1 (3), this implies that $A^!$ is a Koszul $\mathcal{P}^!$-algebra. □

3.4 Links and applications

In this section, we give examples of applications of the present Koszul duality theory. We have seen the case of associative algebras, we describe now the case of commutative algebras and the case of Lie algebras. We also recover the case of modules.

3.4.1 The case of commutative algebras

Commutative algebras and Lie coalgebras

Let $\mathcal{P} = \mathcal{C}om$ be the operad encoding (non necessarily unital) associative and commutative algebras. A $\mathcal{C}om$-algebra structure on A is equivalent to commutative and associative algebra structure on A given by a commutative product $\gamma_A : A^{\otimes 2} \to A$ satisfying the associativity relation.

The Koszul dual cooperad of $\mathcal{C}om$ is $\mathcal{C}om^{¡}$, that is the suspension of the cooperad $\mathcal{L}ie^c$, which encodes Lie coalgebras. A $\mathcal{L}ie^c$-coalgebra structure on sC is equivalent to a Lie coalgebra structure on sC given by an anti-commutative coproduct $\Delta_{sC} : sC \to s(\mathcal{C}om^{¡}(2) \otimes C^{\otimes 2})^{\mathbb{S}_2} \cong \{c_1 \otimes c_2 + (-1)^{|c_1||c_2|} c_2 \otimes c_1;\ c_1, c_2 \in sC\}$ which satisfies the coJacobi relation (see [LV] for more details). We remark that $\mathrm{Im}\Delta_{sC} \subset sC \otimes sC$. When C is finitely generated and concentrated in degree 0, we have $s(\mathcal{C}om^{¡}(2) \otimes C^{\otimes 2})^{\mathbb{S}_2} \cong \Lambda^2 C$ where $|\Lambda^2| = 2$.

The cotangent complex

Let A be a $\mathcal{C}om$-algebra, let sC be a Lie coalgebra and let $\varphi : sC \to A$ be an algebraic twisting morphism. The twisted tensor product $A \otimes^{\mathcal{C}om} C$ is given by

$$A_+ \otimes_\varphi C := (A_+ \otimes C, d_\varphi := d_{A_+ \otimes C} - d^l_\varphi),$$

where $A_+ := \mathbb{K} \oplus A$ is the augmented algebra of A. The differential $d_{A_+ \otimes C}$ is equal to $d_{A_+} \otimes id_C + id_{A_+} \otimes d_C$ and the twisting differential d^l_φ is given by

$$A_+ \otimes C \xrightarrow{id_{A_+} \otimes s^{-1}\Delta_{sC}} A_+ \otimes sC \otimes C \xrightarrow{id_{A_+} \otimes \varphi \otimes id_C} A_+ \otimes A \otimes C \xrightarrow{\tilde{\gamma}_A \otimes id_C} A_+ \otimes C,$$

where $\tilde{\gamma}_A : A_+ \otimes A \cong A \oplus A \otimes A \xrightarrow{id_A + \gamma_A} A \rightarrowtail A_+$. We remark that this construction is close to the twisted tensor product of an associative algebra and a coassociative coalgebra.

The algebraic twisting morphism φ is Koszul when the A-modules morphism $A_+ \otimes_\varphi C \xrightarrow{\sim} A_+ \otimes_{\pi_\kappa} s^{-1}\mathrm{B}_\kappa A \cong A_+ \otimes_{\pi_\kappa} s^{-1}\mathcal{L}ie^c(sA)$, where $\mathcal{L}ie^c(sA)$ is the cofree Lie coalgebra on sA, is a quasi-isomorphism.

Under certain assumptions as the smoothness or the regularity of A [Qui70], the cotangent complex is quasi-isomorphic to the A-module of Kähler differential forms $\Omega^1(A)$, therefore φ is a Koszul morphism when the A-modules morphism $A_+ \otimes_\varphi C \xrightarrow{\sim} \Omega^1(A)$ is a quasi-isomorphism.

Quadratic setting

The operad $\mathcal{C}om$ is binary and therefore a monogene $\mathcal{C}om$-algebra is a quadratic commutative algebra : $A := \mathcal{S}(V)/(S)$, where $\mathcal{S}(V) := \mathcal{C}om(V)$ is the symmetric algebra and where $S \subseteq \mathcal{S}(V)^{(2)} = V \otimes_{\mathbb{S}_2} V =: V^{\odot 2}$. The Koszul dual Lie coalgebra $A^¡$ is the quadratic Lie coalgebra $sC(V,\, sS)$. When V is finite dimensional, we desuspend and linearly dualize $A^¡$ to get $A^! = \mathcal{L}ie(V^*)/(R^\perp)$, where $\mathcal{L}ie(V^*)$ is the free Lie algebra on V^*. Provided the fact that $\varkappa : A^¡ \to A$ is a Koszul morphism, the Koszul criterion gives resolutions $A^¡ \xrightarrow{\sim} \mathrm{B}_\kappa A$ and $\Omega_\kappa A^¡ \xrightarrow{\sim} A$.

Applications to rational homotopy theory

In this paper, we work in the homological setting. However, we can reverse the arrows and work in the cohomological setting. When the algebras A are generated by a finitely generated vector space V, the theory of this paper still works for commutative algebras. In this example, this condition is always verified.

The latter quasi-isomorphism $\Omega_\kappa A^¡ = \Lambda A^¡ \xrightarrow{\sim} A$ is a quadratic model in rational homotopy theory [Sul77]. Assume that X is a formal simply connected space and that the cohomology ring $A = \mathrm{H}^\bullet(X,\, \mathbb{Q})$ is a finitely generated quadratic algebra $\mathcal{S}(V)/(S)$, where V is homological graded. When A forms a Koszul algebra, the Koszul dual algebra $A^! = \mathcal{L}ie(V^*)/(S^\perp)$ is equal to the rational homotopy groups of X. Therefore the Koszul duality theory generates from the quadratic data all the syzygies ; we do not have to compute the syzygies one by one. The Lie algebra structure on $A^!$ is the Whitehead Lie bracket. By Theorem 3.3.3.2, we can either prove that A is Koszul or that $A^¡$ is Koszul. Moreover, the Koszul criterion 3.3.3.1 (1) provides a new way to prove these conditions.

The complement of a complex hyperplane arrangement is always a formal space. However, it is not necessarily simply connected. To a complex hyperplane arrangement $\mathcal{A}$, one associates the *Orlik-Solomon algebra* $A := A(\mathcal{A})$. This algebra is naturally isomorphic to the cohomology groups of the complement X of the hyperplane arrangement $\mathcal{A}$, that is $A = A(\mathcal{A}) \cong \mathrm{H}^\bullet(X,\, \mathbb{Q})$. The conditions on $\mathcal{A}$ for A to be quadratic are studied in [Yuz01]. When the Orlik-Solomon algebra A is quadratic, the Koszul dual algebra $A^!$ is the *holonomy Lie algebra* defined by

Kohno [Koh83, Koh85]. When A is a Koszul algebra, the holonomy Lie algebra $A^!$ computes the n-homotopy groups of the $\mathbb{Q}$-completion of the space X for $n \geq 2$.

Relationship with the Koszul duality theory of associative algebras

Assume that V is a finite dimensional. The free commutative algebra $\mathcal{S}(V)$ is equal to the quadratic associative algebra $T(V)/\langle v \otimes w - w \otimes v \rangle$, where $T(V)$ is the tensor algebra or the free associative algebra on V. Thus, there is a weight preserving projection $p : T(V) \twoheadrightarrow \mathcal{S}(V)$ and a functor

$$\begin{array}{rcl} \textsf{quad. comm. alg.} & \to & \textsf{quad. assoc. alg.} \\ A := \mathcal{S}(V)/(S) & \mapsto & A_{as} := T(V)/p^{-1}(S). \end{array}$$

We emphasize the fact that the Koszul complexes associated to A and A_{as} are distinct. The Koszul dual coalgebra $A^{¡}$ is a Lie coalgebra whereas the Koszul dual coalgebra ${A_{as}}^{¡}$ is a coassociative coalgebra. However, the Koszul dual algebras are linked by the equality ${A_{as}}^! = U(A^!)$, where $U(A^!)$ is the enveloping algebra of the Lie algebra $A^!$ (see [GK94] for example) :

$$\begin{array}{lcccl} \mathcal{A}s\textsf{-alg.} : & A_{as} & \xleftrightarrow{\ !\ } & {A_{as}}^! = U(A^!) & : \mathcal{A}s\textsf{-alg} \\ & \updownarrow & & \updownarrow & \\ \mathcal{C}om\textsf{-alg} : & A & \xleftrightarrow{\quad !\quad} & A^! & : \mathcal{L}ie\textsf{-alg.} \end{array}$$

A priori, the enveloping algebra of a Lie algebra has quadratic and linear relations. However, when the Lie algebra is a homogeneous quadratic Lie algebra, the enveloping algebra admits a homogeneous quadratic presentation as an associative algebra.

3.4.1.1 Theorem. *The quadratic associative algebra A_{as} is Koszul if and only if the quadratic commutative algebra A is Koszul.*

Proof. We prove the result by computing the Ext-functor $\mathrm{Ext}^\bullet_{{A_{as}}^!}(\mathbb{K}, \mathbb{K})$ in two different ways. The chain complex ${A_{as}}^! \otimes_\pi \mathrm{B}{A_{as}}^!$ is a projective resolution of $\mathbb{K}$ as an ${A_{as}}^!$-module. Moreover, since ${A_{as}}^! = U(A^!)$, the chain complex ${A_{as}}^! \otimes_\varkappa \mathrm{B}_\kappa A^! \cong {A_{as}}^! \otimes_\varkappa s\mathcal{L}ie^{¡}(A^!) \cong U(A^!) \otimes_\varkappa \Lambda^c(A^!)$ is a projective resolution of $\mathbb{K}$. The first resolution gives :

$$\mathrm{Ext}^\bullet_{{A_{as}}^!}(\mathbb{K}, \mathbb{K}) \cong \mathrm{H}^\bullet(\mathrm{Hom}_{{A_{as}}^!\textsf{-mod}}({A_{as}}^! \otimes_\pi \mathrm{B}{A_{as}}^!, \mathbb{K})) \cong \mathrm{H}^\bullet(\mathrm{Hom}(\mathrm{B}{A_{as}}^!, \mathbb{K})) \cong (\mathrm{H}^\bullet(\mathrm{B}{A_{as}}^!))^*$$

where the second isomorphism is given by the fact that $\mathbb{K}$ is a trivial ${A_{as}}^!$-module and the third isomorphism is due to the fact that $d_\mathbb{K} = 0$. Similarly, the second resolution gives :

$$\mathrm{Ext}^\bullet_{{A_{as}}^!}(\mathbb{K}, \mathbb{K}) \cong \mathrm{H}^\bullet(\mathrm{Hom}_{{A_{as}}^!\textsf{-mod}}({A_{as}}^! \otimes_\varkappa \mathrm{B}_\kappa A^!, \mathbb{K})) \cong \mathrm{H}^\bullet(\mathrm{Hom}(\mathrm{B}_\kappa A^!, \mathbb{K})) \cong (\mathrm{H}^\bullet(\mathrm{B}_\kappa A^!))^*.$$

When A_{as} is Koszul, or equivalently when ${A_{as}}^!$ is Koszul, we have $(\mathrm{H}^\bullet(\mathrm{B}{A_{as}}^!))^* \cong ({A_{as}}^!)^{¡*} \cong ({A_{as}}^!)^! \cong A_{as} \cong A$ where the last isomorphism is only an isomorphism of vector spaces. This gives that $A \cong (\mathrm{H}^\bullet(\mathrm{B}_\kappa A^!))^*$, so $A^!$ is Koszul by the Koszul criterion (Theorem 3.3.3.1) and A is Koszul by Theorem 3.3.3.2. In the other way round, when A is Koszul, or equivalently when $A^!$ is Koszul (Theorem 3.3.3.2), we have $(\mathrm{H}^\bullet(\mathrm{B}_\kappa A^!))^* \cong (A^!)^{¡*} \cong (A^!)^! \cong A \cong A_{as}$, where the last isomomorphism is only an isomorphism of vector spaces. We obtain $A_{as} \cong (\mathrm{H}^\bullet(\mathrm{B}{A_{as}}^!))^*$ and A_{as} is Koszul. $\square$

In [PY99], the authors proved this theorem (Proposition 4.4) and the fact that for a formal space X, the algebra $A = \mathrm{H}^\bullet(X, \mathbb{Q})$, or A_{as}, is Koszul if and only if the space X is a rational $K(\pi, 1)$. We emphasize however the fact that the definition for an algebra A to be Koszul in [PY99] is slightly different that the one in this paper. They require the generators of A to be in degree 1, assumptions which is not required in the present paper.

3.4.2 The case of Lie algebras

Let $\mathcal{P} = \mathcal{L}ie$ be the operad encoding Lie algebras. The Koszul dual cooperad of $\mathcal{L}ie$ is $\mathcal{L}ie^{¡}$, that is the suspension of the cooperad $\mathcal{C}om^c$ encoding co-commutative coalgebras.

The twisted tensor product and the cotangent complex

Let $\mathfrak{g}$ be a Lie algebra, let sC be a commutative coalgebra and let $\varphi : sC \to \mathfrak{g}$ be a twisting morphism. The twisted tensor product $\mathfrak{g} \otimes^{\mathcal{L}ie} C$ is given by $U(\mathfrak{g}) \otimes_\varphi C := (U(\mathfrak{g}) \otimes C, d_\varphi)$ where $U(\mathfrak{g})$ is the enveloping algebra of the Lie algebra $\mathfrak{g}$ and where the differential d_φ is obtained in the same way as the case of commutative algebras.

The cotangent complex is the Chevalley-Eilenberg complex $U(\mathfrak{g}) \otimes_{\pi_\kappa} \Lambda^\bullet(\mathfrak{g})$ and it is always quasi-isomorphic to the module of Kähler differential forms $\Omega_{\mathcal{L}ie}(\mathfrak{g})$. Therefore, the twisting morphism φ is a Koszul morphism when $U(\mathfrak{g}) \otimes_\varphi C \xrightarrow{\sim} \Omega_{\mathcal{L}ie}(\mathfrak{g})$ is a quasi-isomorphism, or equivalently, by the algebraic fundamental twisting morphism, when $\mathcal{L}ie(C) \xrightarrow{\sim} \mathfrak{g}$ is a quasi-isomorphim or when $C \xrightarrow{\sim} \Lambda^\bullet(\mathfrak{g})$ is a quasi-isomorphism.

Quadratic setting

When $\mathfrak{g}$ is a quadratic Lie algebra, its Koszul dual coalgebra $\mathfrak{g}^{¡}$ is a co-commutative coalgebra and the Koszul complex is $U(\mathfrak{g}) \otimes_\varkappa \mathfrak{g}^{¡}$, or equivalently, by the Koszul criterion, when $\mathcal{L}ie(\mathfrak{g}^{¡}) \xrightarrow{\sim} \mathfrak{g}$ is a quasi-isomorphim or when $\mathfrak{g}^{¡} \xrightarrow{\sim} \Lambda^\bullet(\mathfrak{g})$ is a quasi-isomorphism.

This provides examples of quadratic Quillen models for Lie algebras [Qui69].

3.4.3 Koszul duality theory of quadratic modules over an associative algebra

Twisted tensor product for modules

Let A be an associative algebra. Let $\mathcal{P} = A$ concentrated in arity 1, that is A is an associative algebra. A $\mathcal{P}$-algebra M is a left A-module (M, γ_M). Dually, $\mathcal{C} = C$ and a $\mathcal{C}$-coalgebra is a left C-comodule (N, Δ_N). Let $\alpha : C \to A$ be a Koszul morphism. The bar construction on M is the chain complex $C \otimes_\alpha M := (C \otimes M, d_\alpha := d_{C\otimes M} + d^r_\alpha)$ where d^r_α is the composite of

$$C \otimes M \xrightarrow{\Delta_C \otimes id_M} C \otimes C \otimes M \xrightarrow{id_C \otimes \alpha \otimes id_M} C \otimes A \otimes M \xrightarrow{id_C \otimes \gamma_M} C \otimes M.$$

The cobar construction on N is given dually by $A \otimes_\alpha N$.

The twisted tensor product is given by the cobar construction $A \otimes_\alpha N$ and a twisting morphism $\varphi : sN \to N$ is Koszul when $A \otimes_\alpha N \xrightarrow{\sim} A \otimes_\alpha C \otimes_\alpha M$ is a quasi-isomorphism. Since $\alpha : C \to A$ is a Koszul morphism, this is equivalent to $A \otimes_\alpha N \xrightarrow{\sim} M$ is a quasi-isomorphism.

Quadratic module

Assume now that $A = A(E, R)$ is a quadratic algebra $T(E)/(R)$ and $C = A^{¡}$ its Koszul dual coalgebra. We assume moreover that A is a Koszul algebra, so $\alpha = \kappa$ is a Koszul morphism. An A-module M is a *quadratic A-module* if $M = (A \otimes V)/A \cdot S$ where V is a vector space of generators, $S \subseteq A \otimes V$ is a subvector space of relations and $A \cdot S$ is the image of the map $A \otimes S \rightarrowtail A \otimes A \otimes V \xrightarrow{\gamma_A \otimes id_V} A \otimes V$. We define dually the notion of quadratic comodule and we get the Koszul dual $A^{¡}$-comodule $M^{¡}$.

In this case, the Koszul complex is equal to the cobar construction $A \otimes_\kappa M^{¡}$ and the Koszul criterion collapses to $A \otimes_\kappa M^{¡} \xrightarrow{\sim} M$ is equivalent to $M^{¡} \xrightarrow{\sim} A^{¡} \otimes_\kappa M$. Thus we recover the Koszul duality theory of A-modules given in [PP05].

Modules over a commutative algebra

Assume now that A is a commutative algebra. In this case, the Koszul duality theory provides resolutions of modules needed in algebraic geometry [Eis04]. When $A = \mathbb{K}[x_1, \ldots, x_n]$ or $A = \mathbb{K}[x_1, \ldots, x_n]/(I)$ where I is homogeneous of degree 2, the Koszul dual comodule $M^{¡}$ of a quadratic module M provides a good candidate for the *syzygies* of M.

3.4.4 Other fields of applications

It is also possible to apply the present Koszul duality theory to many other examples of the literature. For example, this Koszul duality theory for algebras applies to the operad encoding Poisson or Leibniz algebras, with possible applications in differential and Poisson geometry [KS96, Fre06], to the operad encoding PreLie algebras, with possible applications in algebraic combinatorics and links with renormalisation theory in theoretical physics [CL01, CK98], to the operad encoding hyper-commutative or gravity algebras, linked with the Gromov-Witten invariants [Get95, Man99].

There are several ways to prove Koszulity for associative algebra that we plan to extend to algebras over an operad such as Poincaré-Birkhoff-Witt bases [Pri70, Hof10] and Gröbner bases [Buc06, DK08, BCL09].

Another direction is also to extend this Koszul duality theory beyond the homogeneous quadratic case : when the algebra has quadratic and linear relations, the Koszul dual coalgebra should have an extra differential [Pri70, GCTV09], when the algebra has quadratic, linear and constant relations, the Koszul dual coalgebra should have an extra differential and a curvature [PP05, HM10] and when the algebra has quadratic and higher relations, the Koszul dual coalgebra should be a homotopy coalgebra [MV09a].

Annexe A

Comparison Lemma

In this appendix, we prove a Comparison Lemma for $\mathcal{P}$-algebras and $\mathcal{C}$-coalgebras generalizing the associative case [Car55]. We recall that all the chain complexes are non-negatively graded and we assume that $\mathcal{P}(0) = 0$ and that $\mathcal{C}(0) = 0$.

In the sequel, we consider a *bigraded module* V, that is a family of modules $\{V_d^{(n)}\}_{n,\, d \geq 0}$. The lower index is the homological degree and the upper one is the weight grading. A *weight graded dg module*, *wdg module* for short, is a bigraded module endowed with a differential which preserves the weight and lowers the homological degree by -1. We say that a wdg algebra or wdg coalgebra V is *connected* when it satisfies $V = V^{(1)} \oplus V^{(2)} \oplus \cdots$. Moreover, the structure maps, as γ_A, preserve the weight.

Theorem (Comparison Lemma). *Let $\alpha : \mathcal{C} \to \mathcal{P}$ be an operadic Koszul morphism between a wdg connected cooperad $\mathcal{C}$ and a wdg connected operad $\mathcal{P}$. Let A be a wdg connected $\mathcal{P}$-algebra and C, C' be wdg connected $\mathcal{C}$-coalgebras. Let $g : sC \to sC'$ be a morphism of wdg $\mathcal{S}^{-1}\mathcal{C}$-coalgebras. Let $\varphi : sC \to A$ and $\varphi' : sC' \to A$ be two algebraic twisting morphisms, such that $\varphi' \circ g = \varphi$. The morphism g is a quasi-isomorphism if and only if $id_A \otimes^{\mathcal{P}} g : A \otimes^{\mathcal{P}} C \xrightarrow{\sim} A \otimes^{\mathcal{P}} C'$ is a quasi-isomorphism.*

Proof. Since $\mathcal{P}$, $\mathcal{C}$, A and C are weight graded, the dg modules $\mathcal{P} \circ (\mathcal{P}(A),\, C)$ and $\mathcal{P}(A,\, C)$ are also weight graded. Moreover, the maps c_0 and c_1 (see Section 1.1.4.1) in the coequalizer

$$\mathcal{P} \circ (\mathcal{P}(A),\, C) \underset{c_1}{\overset{c_0}{\rightrightarrows}} \mathcal{P} \circ (A,\, C) \xrightarrow{proj} A \otimes^{\mathcal{P}} C$$

preserve the weight gradings, so $proj(\mathcal{P}(A,\, C)^{(n)}) =: (A \otimes^{\mathcal{P}} C)^{(n)}$ defines a weight grading on $A \otimes^{\mathcal{P}} C$. We denote by $M = \oplus_{n \geq 0} M^{(n)}$ the wdg A-module $A \otimes^{\mathcal{P}} C$ and by $M' = \oplus_{n \geq 0} M'^{(n)}$ the wdg A-module $A \otimes^{\mathcal{P}} C'$. We define a filtration F_p on $M^{(n)}$ by the formula

$$F_p(M^{(n)}) := \bigoplus_{m+r \leq p} (A \otimes^{\mathcal{P}} C_m^{(r)})^{(n)}.$$

The differential on M is given by $d_\varphi = d_{A \otimes^{\mathcal{P}} C} - d_\varphi^l = d_{A \otimes^{\mathcal{P}} \mathbb{K}} \otimes id_C + id_{A \otimes^{\mathcal{P}} \mathbb{K}} \otimes d_C - d_\varphi^l$ (see Section 1.1.4.1 for a definition of d_φ^l). We have

$$\begin{cases} d_{A \otimes^{\mathcal{P}} \mathbb{K}} \otimes id_C & : \quad F_p \to F_p \\ id_{A \otimes^{\mathcal{P}} \mathbb{K}} \otimes d_C & : \quad F_p \to F_{p-1} \\ d_\varphi^l & : \quad F_p \to F_{p-2} \quad \text{since } |\alpha| = -1 \text{ and } |\mathcal{P}| \geq 0, \text{ and } \alpha^{(0)} = 0. \end{cases}$$

Thus F_p is a filtration of the chain complex $M^{(n)}$. We denote by $E^{\bullet}_{p,\,q}$ the associated spectral sequence. We have

$$E^0_{p,\,q}(M^{(n)}) = F_p(M^{(n)})_{p+q}/F_{p-1}(M^{(n)})_{p+q} = \bigoplus_{r=0}^{n}(A\otimes^{\mathcal{P}} C^{(r)}_{p-r})^{(n)}_{p+q} = \bigoplus_{r=0}^{n}(A\otimes^{\mathcal{P}}\mathbb{K})^{(n-r)}_{q+r}\otimes C^{(r)}_{p-r}.$$

The study of the differential on M shows that $d^0 = d_{A\otimes^{\mathcal{P}}\mathbb{K}}\otimes id_C$ and $d^1 = id_{A\otimes^{\mathcal{P}}\mathbb{K}}\otimes d_C$. Hence

$$E^2_{p,\,q}(M^{(n)}) = \bigoplus_{r=0}^{n}\mathrm{H}_{q+r}((A\otimes^{\mathcal{P}}\mathbb{K})^{(n-r)}_{\bullet})\otimes \mathrm{H}_{p-r}(C^{(r)}_{\bullet}).$$

The filtration F_p being exhaustive and bounded below, we can apply the classical theorem of convergence of spectral sequences (Theorem 5.5.1 of [Wei94]) to get $E^{\infty}_{p,q}(M^{(n)}) \cong gr_p\mathrm{H}_{p+q}(M^{(n)})$.

We can define the same filtration on M' and we obtain the same result of convergence of spectral sequences.

- When g is a quasi-isomorphism, we get that $E^2_{p,\,q}(M^{(n)}) \xrightarrow{\mathrm{H}_\bullet(id_{A\otimes^{\mathcal{P}}\mathbb{K}})\otimes\mathrm{H}_\bullet(g)} E^2_{p,\,q}(M'^{(n)})$ is an isomorphism. Since $\varphi'\circ g = \varphi$, the map $\mathrm{H}_\bullet(id_{A\otimes^{\mathcal{P}}\mathbb{K}})\otimes\mathrm{H}_\bullet(g)$ is an isomorphism of chain complexes and the pages $E^r_{p,\,q}(M^{(n)})$ and $E^r_{p,\,q}(M^{(n)})$ are isomorphic for all $r\geq 2$. By the convergence's theorem of spectral sequences, we get that $gr_p\mathrm{H}_{p+q}(M^{(n)}) \cong E^{\infty}_{p,\,q}(M^{(n)}) \xrightarrow{\mathrm{H}_\bullet(id_A\otimes^{\mathcal{P}} g)} E^{\infty}_{p,\,q}(M'^{(n)}) \cong gr_p\mathrm{H}_{p+q}(M'^{n})$ is also an isomorphism.
- Assume now that $id_A\otimes^{\mathcal{P}} g$ is a quasi-isomorphism of dg A-modules. Let us work by induction on the weight n. When $n=0$, the map $g^{(0)}: 0\to 0$ is a quasi-isomorphism. Suppose now that $g^{(n-1)}$ is a quasi-isomorphism. We consider the mapping cone of $f^{(n)} := (id_A\otimes^{\mathcal{P}} g)^{(n)} : M^{(n)}\to M'^{(n)}$ defined by $cone(f^{(n)}) := s^{-1}M^{(n)}\oplus M'^{(n)}$ and the associated filtration $F_p(cone(f^{(n)})) := s^{-1}F_p(M^{(n)})\oplus F_p(M'^{(n)})$, which satisfies $E^1_{\bullet,\,q}(cone(f^{(n)})) = cone(E^1_{\bullet,\,q}(f^{(n)}))$. The mapping cone of $E^1_{\bullet,\,q}(f^{(n)})$ fits into a short exact sequence, which induces the long exact sequence

$$\cdots\to \mathrm{H}_{p+1}\left(cone(E^1_{\bullet,\,q}(f^{(n)}))\right)\to \mathrm{H}_p\left(E^1_{\bullet,\,q}(M^{(n)})\right)\xrightarrow{\mathrm{H}_p\left(E^1_{\bullet,\,q}(f^{(n)})\right)}\mathrm{H}_p\left(E^1_{\bullet,\,q}(M'^{(n)})\right)$$
$$\to \mathrm{H}_p\left(cone(E^1_{\bullet,\,q}(f^{(n)}))\right)\to\cdots.$$

This induces the long exact sequence (ξ_q)

$$(\xi_q)\ \ \cdots\to E^2_{p+1,\,q}(cone(f^{(n)}))\to E^2_{p,\,q}(M^{(n)})\xrightarrow{E^2_{p,\,q}(f^{(n)})} E^2_{p,\,q}(M'^{(n)})$$
$$\to E^2_{p,\,q}(cone(f^{(n)}))\to\cdots$$

where $E^2_{p,\,q}(f^{(n)})$ is given by $\mathrm{H}_\bullet(id_{A\otimes^{\mathcal{P}}\mathbb{K}})\otimes\mathrm{H}_\bullet(g)$. Since $\mathrm{H}_\bullet(id_{A\otimes^{\mathcal{P}}\mathbb{K}})$ is an isomorphism (it is the identity) and $(A\otimes^{\mathcal{P}}\mathbb{K})^{(0)}=\mathbb{K}$, the formula for $E^2_{p,\,q}(M^{(n)})$ given above and the induction hypothesis tells us that $\mathrm{H}_\bullet(id_{A\otimes^{\mathcal{P}}\mathbb{K}})\otimes\mathrm{H}_\bullet(g)$ is an isomorphism, except for $q=-n$ when $\mathrm{H}_{q+n}((A\otimes^{\mathcal{P}}\mathbb{K})^{(0)}) = \mathbb{K}\neq 0$. The long exact sequence (ξ_q) for $q\neq -n$ and for all p gives that $E^2_{p,q}(cone(f^{(n)})) = 0$. Thus, the spectral sequence collapses at rank 2 and $E^2_{p,\,q}(cone(f^{(n)})) = E^{\infty}_{p,\,q}(cone(f^{(n)}))$. Moreover the spectral sequence $E^{\bullet}_{p,\,q}(cone(f^{(n)}))$ converges to $\mathrm{H}_{p+q}(cone(f^{(n)}) = 0$ (since $f^{(n)}$ is a quasi-isomorphism), so $E^2_{p,\,-n}(cone(f^{(n)})) = 0$ for all p. Finally the spectral sequence (ξ_n) gives the isomorphism

$$\mathrm{H}_{p-n}(C^{(n)}_{\bullet}) = E^2_{p,\,-n}(M^{(n)}) \xrightarrow{\mathrm{H}_{\bullet}(g^{(n)})} E^2_{p,\,-n}(M'^{(n)}) = \mathrm{H}_{p-n}(C'^{(n)}_{\bullet}),$$

for every p. Hence, $g^{(n)}$ is a quasi-isomorphism. This prove the result by induction.

□

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Table des matières

Algèbre et opérade :
cohomologie, homotopie et dualité de Koszul

Joan Millès

Résumé : Nous explicitons la cohomologie d'André-Quillen des algèbres sur une opérade à l'aide de la dualité de Koszul des opérades. Cette cohomologie est représentée par le complexe cotangent. Nous donnons des critères assurant que cette cohomologie s'écrit en termes de foncteur Ext. En particulier, c'est le cas des algèbres sur des opérades cofibrantes, ce qui fournit une nouvelle propriété de stabilité homotopique de ces algèbres. Nous généralisons ensuite la dualité de Koszul des algèbres associatives dans deux directions indépendantes. D'un côté, nous étendons la dualité de Koszul aux opérades non nécessairement augmentées de façon à étudier les algèbres unitaires. La notion de courbure apparaît pour coder le défaut d'augmentation. Nous obtenons ainsi les théories homotopiques et cohomologiques des algèbres associatives unitaires ou des algèbres de Frobenius avec unité et counité. Nous détaillons le cas des algèbres associatives unitaires. D'un autre côté, nous généralisons la dualité de Koszul aux algèbres sur une opérade. Nous montrons pour cela que le complexe cotangent est la bonne généralisation du complexe de Koszul.

Mots clés : Algèbre, homologie, opérade, dualité de Koszul, algèbres à homotopie près

Algebra and operad:
cohomology, homotopy and Koszul duality theory

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Abstract: Using the Koszul duality theory of operads, we make the André-Quillen cohomology of algebras over an operad explicit. This cohomology theory is represented by a chain complex: the cotangent complex. We provide criteria for the André-Quillen cohomology theory to be an Ext-functor. In particular, this is the case for algebras over cofibrant operads and this gives a new stable homotopy property for these algebras. Then we generalize the Koszul duality theory of associative algebras in two independant directions. On the one hand, we extend the Koszul duality theory to non necessarily augmented operads in order to treat algebras with unit. The notion of curvature appears to encode the default of augmentation. As a corollary, we obtain homotopical et cohomological theories for unital associative algebras or unital and counital Frobenius algebras. We make the case of unital associative algebras explicit. On the other hand, we generalize the Koszul duality theory to algebras over an operad. To do this, we show that the cotangent complex provides the good generalization of the Koszul complex.

Keywords: Algebra, homology, operad, Koszul duality theory, homotopy algebras

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