Géométrie algorithmique non linéaire et courbes algébriques planaires

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Presentation outline

introduction
- research field
- the problem
- previous work

algorithmic issues
- overview
- details
- complexity

implementation issues
- isotop
- cgal algebraic kernel

conclusion

Where are we standing?

- exact geometric computing
  - computational geometry
  - computer algebra tools

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- a long history
  - robot motion planning
  - CAGD

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- exact geometric computing
  - computational geometry
  - computer algebra tools
- a long history
  - robot motion planning
  - CAGD
- recent advances in real solving

Exact geometric computing

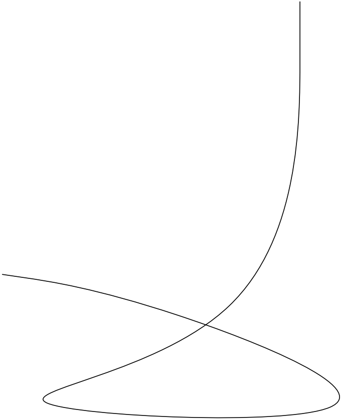
- the general problem
  - algorithms often assume that numbers are real
  - computers do not like real numbers
  - inconsistencies

Exact geometric computing

- the general problem
  - algorithms often assume that numbers are real
  - computers do not like real numbers
  - inconsistencies
- in computational geometry
  - numerical errors often lead to crash
    - exact arithmetic
    - filtered arithmetic

Topology and some geometry of real algebraic plane curves

input curve: $f(x, y) = 0$ with $f \in \mathbb{Q}[x, y]$



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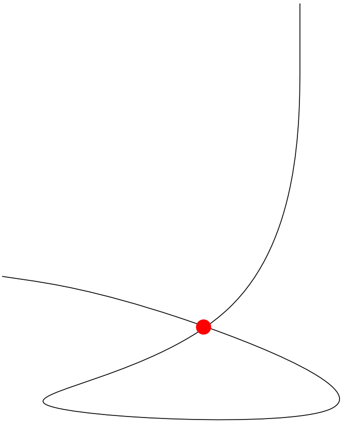

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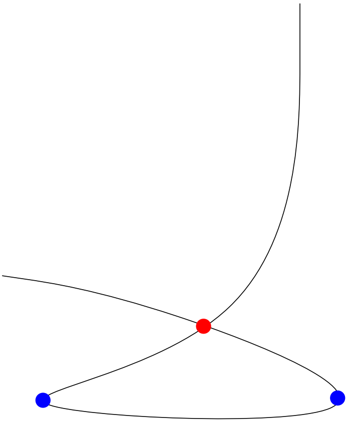

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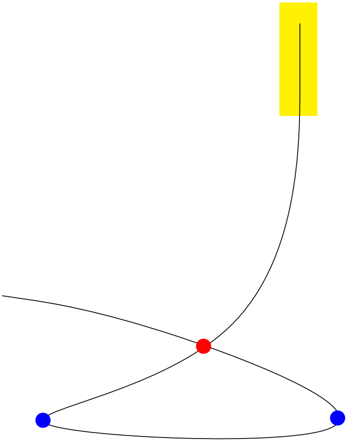

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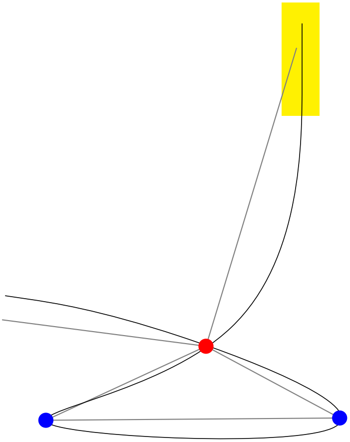

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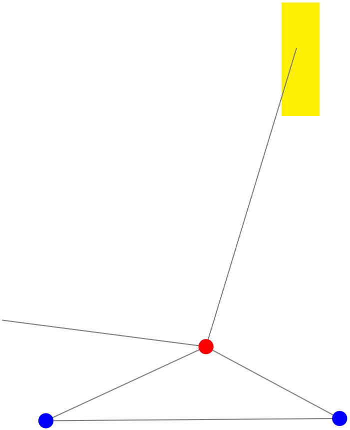
- isotopic approximation of the curve by an arrangement of polylines
- results in the original coordinate system of the plane



Applications

- curve plotting
- computing arrangements of algebraic curves

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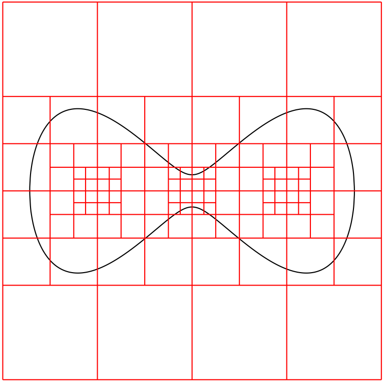
the x-extreme points of each curve have to be computed in the same coordinate system

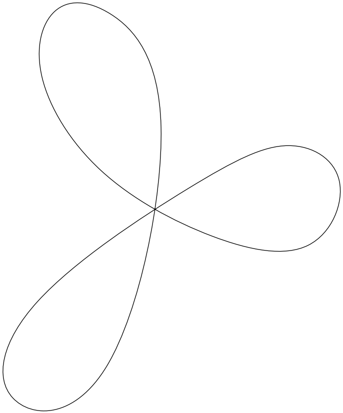
Previous work: subdivision techniques



- fast
- localizable (computation in a given box)
- not certified, unless they
- reach the separation bound

[Lorensen & Cline, 1987]
[Alberti, Mourrain & Wintz, 2008]
[Burr, Choi, Galehouse & Yap, 2008]
[Lin & Yap, 2009]

Cylindrical algebraic decomposition methods



Cylindrical algebraic decomposition methods

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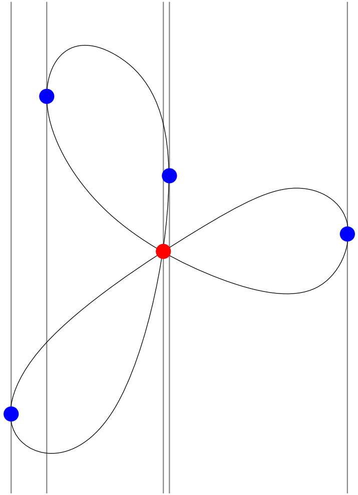

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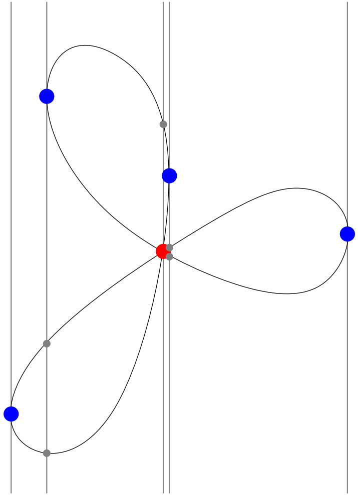

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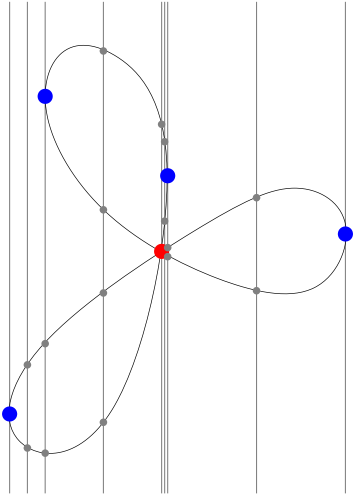

Cylindrical algebraic decomposition methods

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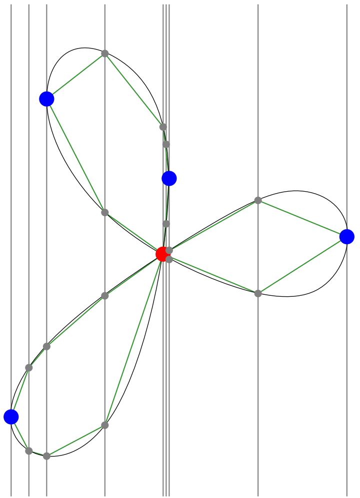

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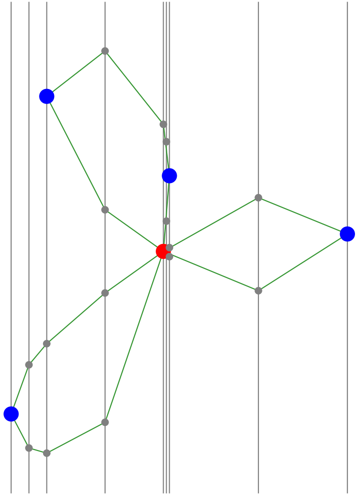

Cylindrical algebraic decomposition methods

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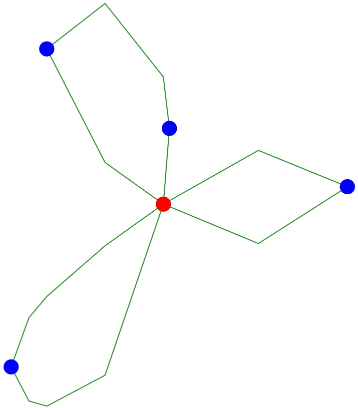
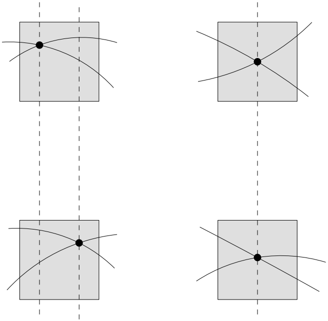
- assume generic position, or detect it, shear and shear back
- Collins, 1984
- Cad2d [Brown & al.]
  Top [Gonzalez Vega & Necula, 2002]
  Insulate [Seidel & Wolpert, 2005]
  CA [Eigenwillig, Kerber & Wolpert, 2007]



Our algorithm

- it's not a CAD
  - decomposition of the plane into rectangles: the rectangle containing each critical point may overlap in x
  - non-genericity of x-overlapping boxes is not an issue



generic non-generic

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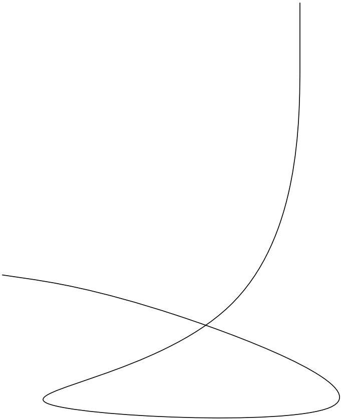
- replaces sub-resultant sequences and computations with algebraic coefficient polynomials by
  - Gröbner bases
  - Rational Univariate Representations

Algorithm outline

1. compute isolating boxes for critical points:



Algorithm outline

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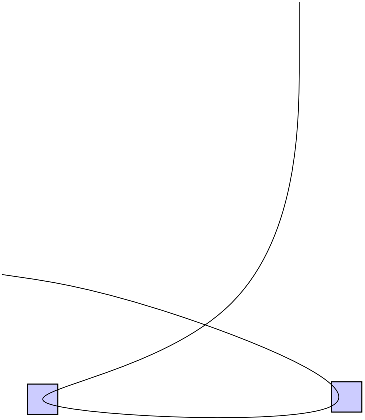

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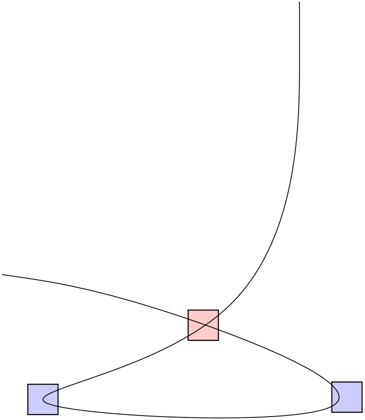

Algorithm outline

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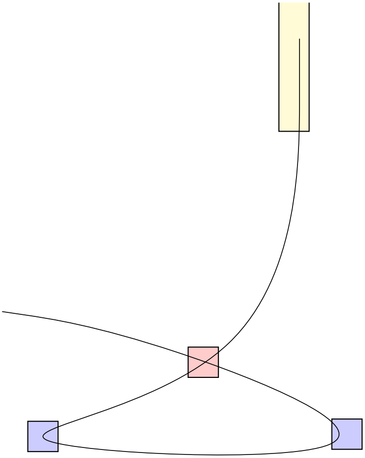

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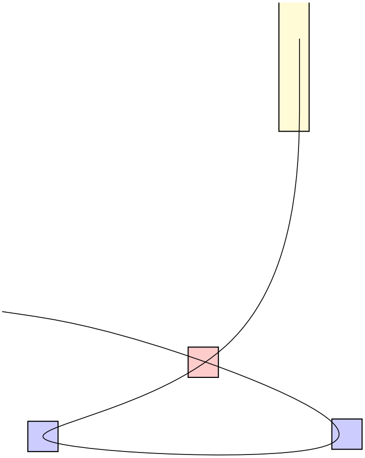

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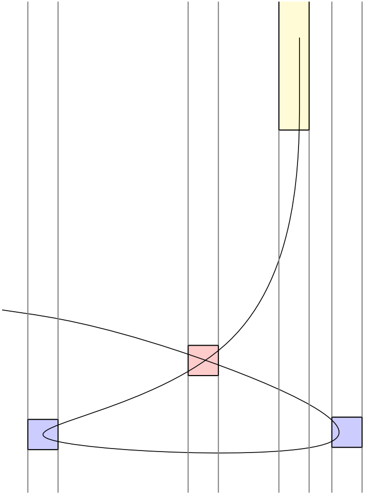
3. compute the topology in the rest of the rectangles



Notation

curve: square free polynomial $f \in \mathbb{Q}[x, y]$

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a point $\mathbf{p} = (\alpha, \beta) \in \mathbb{C}^2$ is *x-critical* if $f(\mathbf{p}) = f_y(\mathbf{p}) = 0$,

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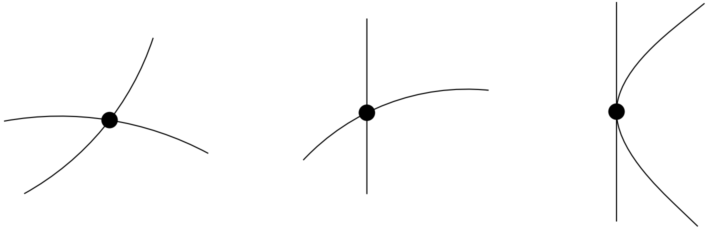

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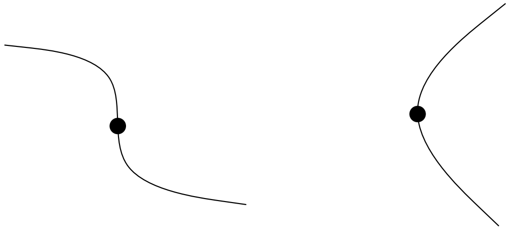
- *singular* if $f_x(\mathbf{p}) = 0$, and
- *x-extreme* if $f_x(\mathbf{p}) \neq 0$ (i.e. x-critical and non-singular)



The Rational Univariate Representation

S is a bivariate system, RUR $\rightsquigarrow$ univariate polynomial f, such that

$$t \text{ root of } f \iff \left(\frac{g_x(t)}{h(t)}, \frac{g_y(t)}{h(t)}\right) \text{ root of } S$$
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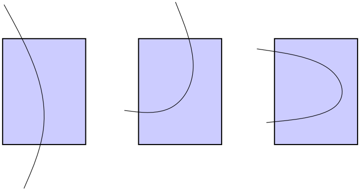
- the RUR preserves multiplicities
- we obtain the RUR from the Gröbner basis of S
- the roots of f are isolated with Descartes' method
- interval arithmetic for computing the separating boxes of the roots of the system S

Topology at extreme points

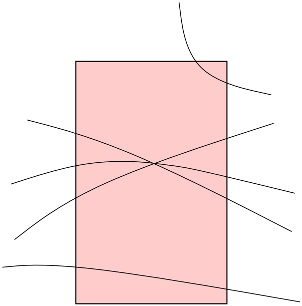
1. isolate the extreme system $S_e = \begin{cases} f(x,y) = 0 \\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial f}{\partial x} \neq 0 \end{cases}$
2. refine boxes to get only two crossings on the border



3. store the multiplicities in the system S_e for the connection step (see later)

Topology at singularities

1. isolate singular points in boxes



Topology at singularities

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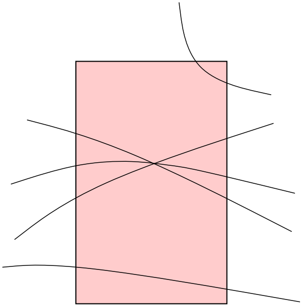

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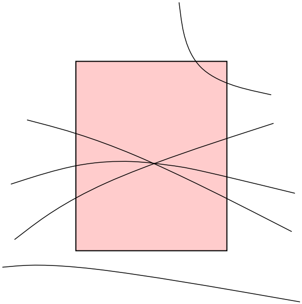

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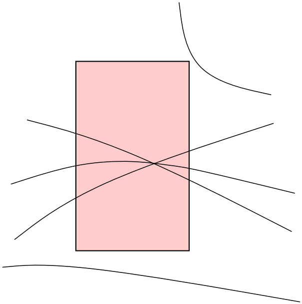
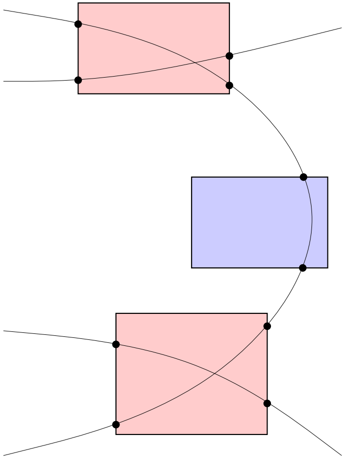


Rectangle decomposition of the plane

- the topology is known inside critical boxes



Rectangle decomposition of the plane

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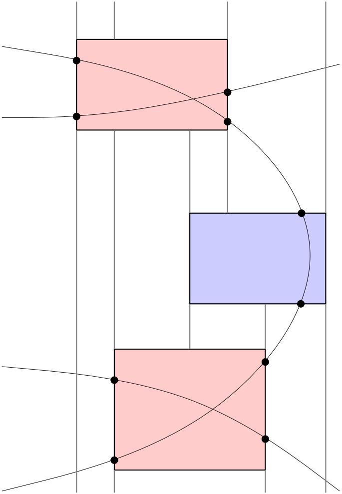
- compute a vertical decomposition of the plane with respect to these boxes



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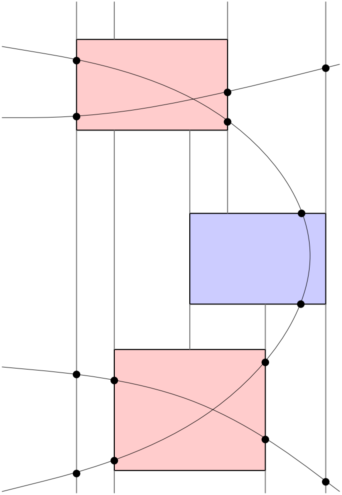

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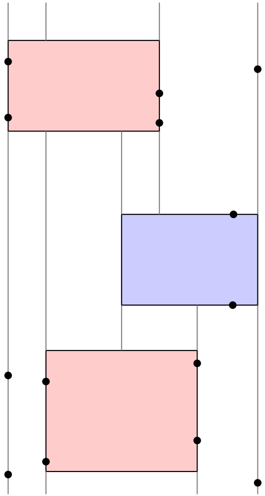
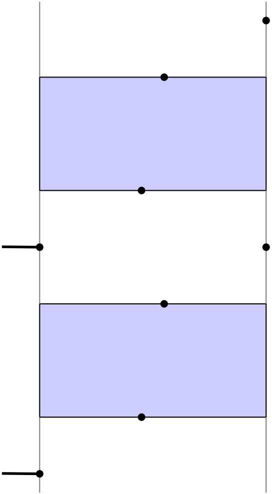


Greedy connection algorithm using multiplicities



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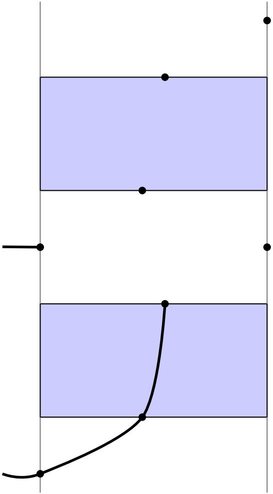

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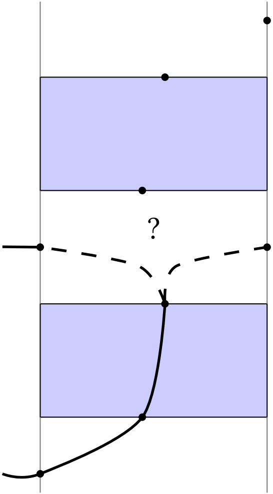

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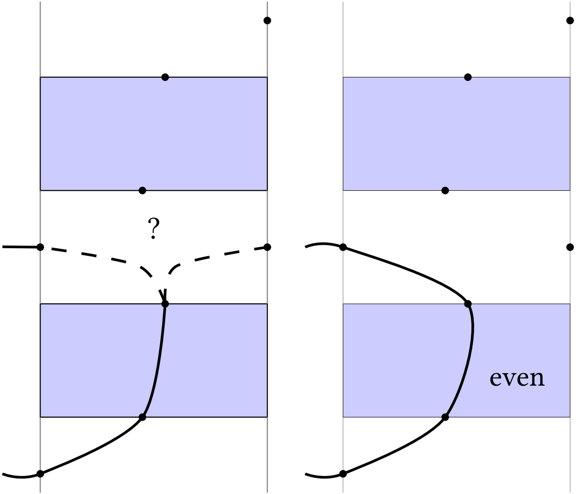

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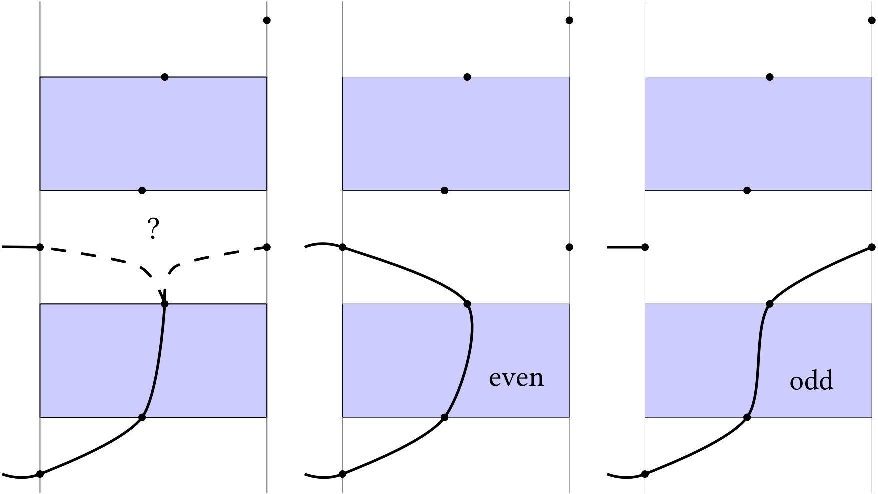

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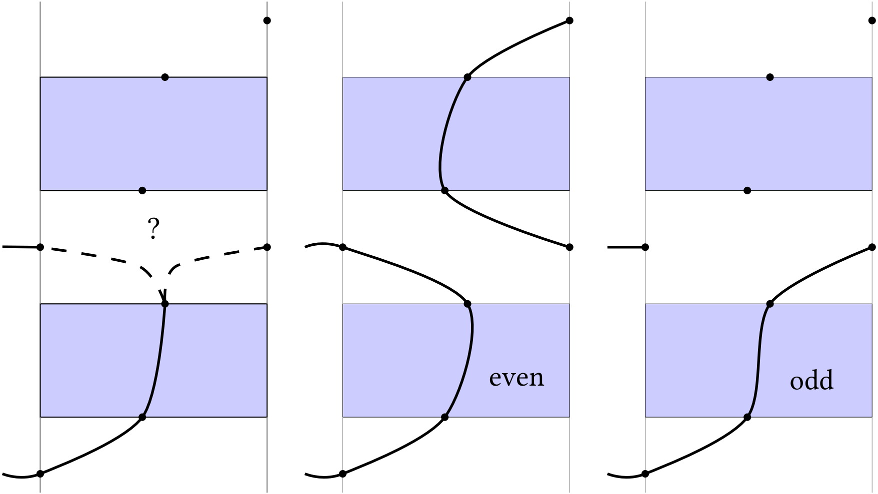

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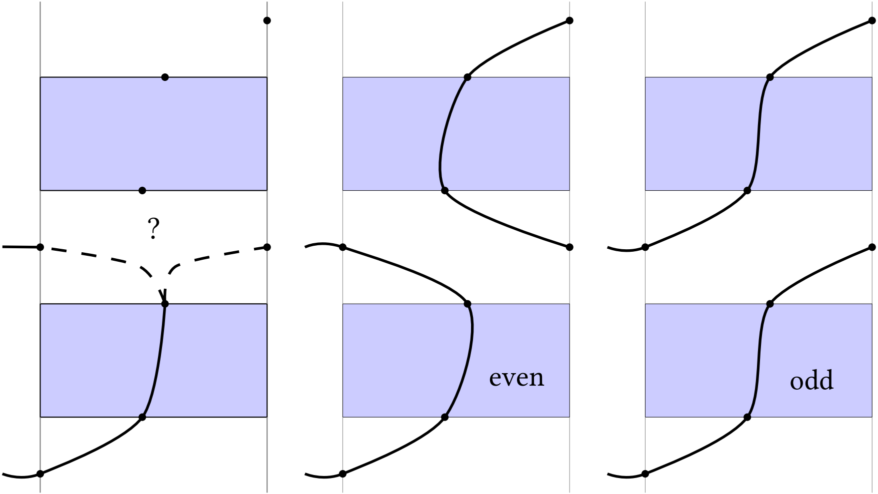

Complexity analysis

- the algorithm runs in $\widetilde{\mathcal{O}}_B(R\,d^{22}\,\tau^2)$, where
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  - d: degree of the polynomial f,
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- does not reflect practical performance

Implementation in Maple

Isotop: +7000 lines of Maple code, using

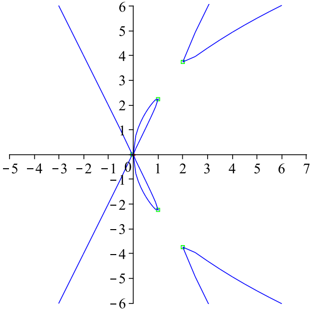
- FGB for Gröbner basis (Faugère)
- RS for RUR and isolation (Rouillier)

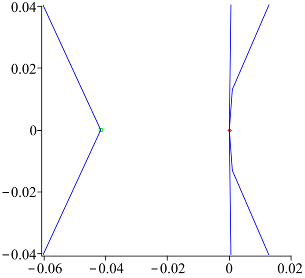
- complete: handles vertical asymptotes and vertical components
- certified

- http://vegas.loria.fr/isotop

Isotop interface

```
ISOTOP:-topology_real_curve(y^4 - 6*y^2*x + x^2 - 4*y^2*x^2 + 24*x^3,
                            verbosity=0,
                            precision=10,
                            plot_graph=true,
                            nb_splits=10);
```



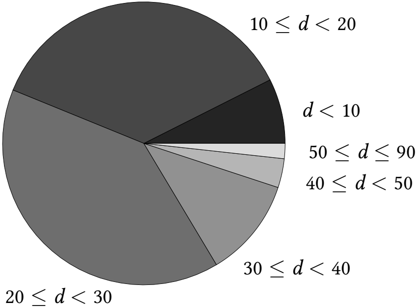


Isotop experiments

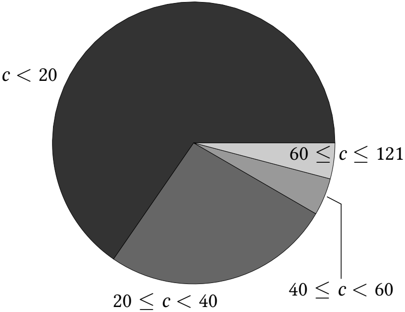
we ran large-scale tests, testing around 600 curves

- random curves
- ACS curves
- O. Labs' tough curves
- resultants of degree-3 random surfaces
- n translations $\prod_{j=0}^{n} f(x, y + j)$
- symmetric polynomials $f^2(x, y) + f^2(x, -y)$

Isotop experiments: input curves

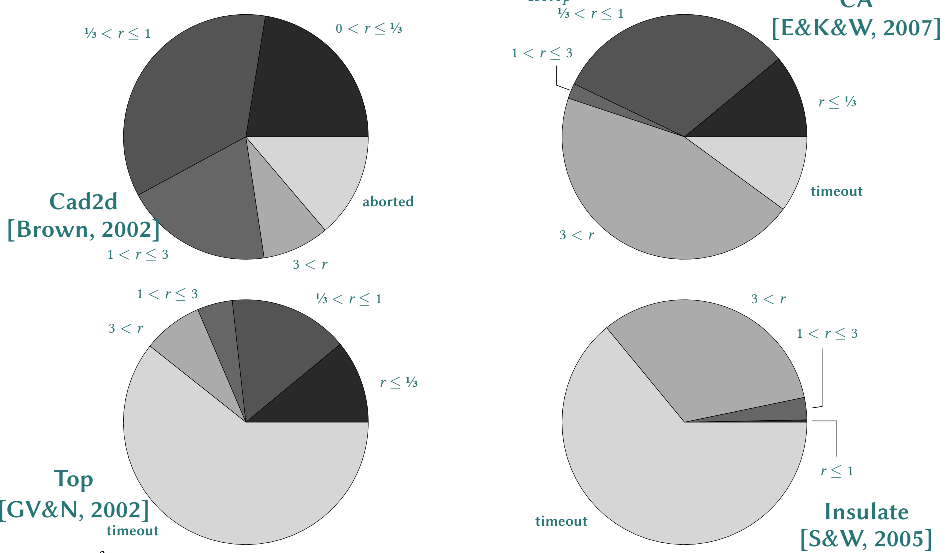


degrees



number of critical points

Experiments: results, $r = \frac{time}{time_{Isotop}}$



Experiments: conclusions

- faster for
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  - random curves
  - curves with high-tangency points

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  - high degree curves
- slower for
  - random curves
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- why?

Algebraic algorithms in CGAL

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- in general
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- CGAL
  - C++ library
  - standard in the community
  - generic programming

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  - it is not a standard in the CG community
  - Maple programs cannot be used as libraries
- in general
  - lack of algebro-geometric tools in C or C++
- CGAL
  - C++ library
  - standard in the community
  - generic programming
- equip CGAL with algebraic tools
  - also useful for future algorithms

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  - arrangements of conics

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- curves of arbitrary degree, algebraic kernels
  - univariate and bivariate
  - many variables

Algebraic Kernel

combines algebra and geometry for manipulating non-linear objects

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  - root finding
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- comparison of algebraic numbers
  - easy when intervals do not overlap
  - otherwise, test for equality
    - greatest common divisor (gcd)
    - algebraic number refinement

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- refinement
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Benchmarks

- software
  - MPII's algebraic kernel (using CORE NT)
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- software
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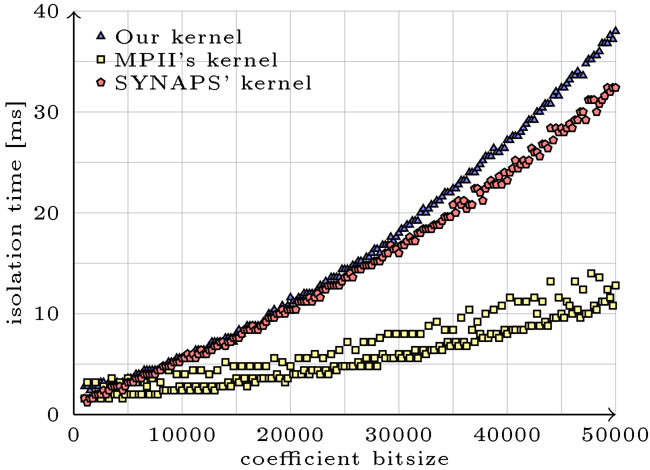
- functionalities
  - root isolation
  - algebraic number comparison
  - application: arrangement construction

Benchmark data

- first time such a big amount of data for polynomials is tested
- 60,000 polynomials (3.8 Gb)
- several weeks in total

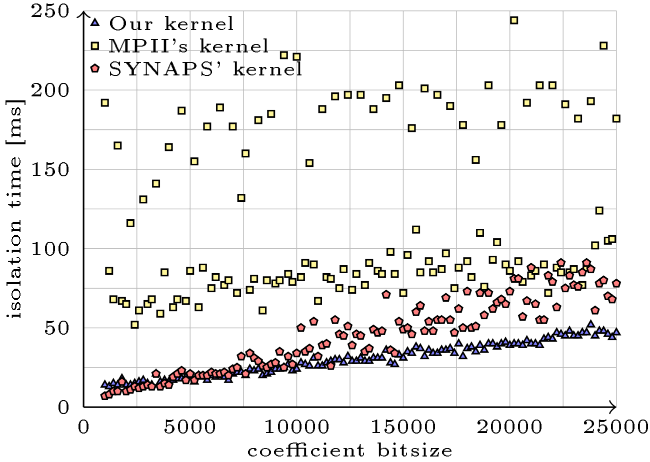
Root isolation: varying bitsize

degree-12 random polynomials



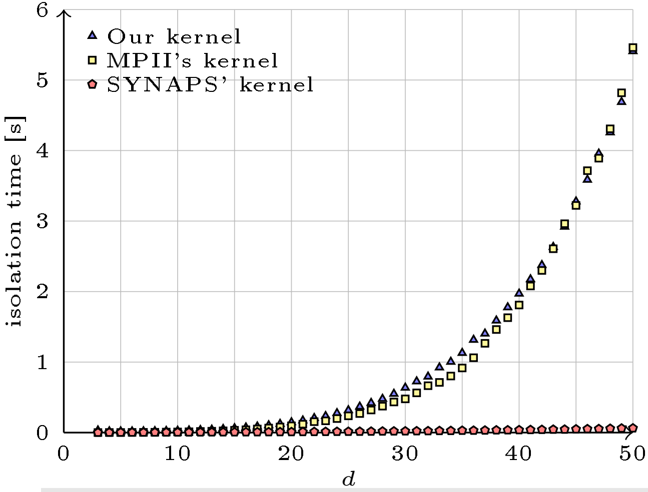
Root isolation: varying bitsize II

degree-100 random polynomials



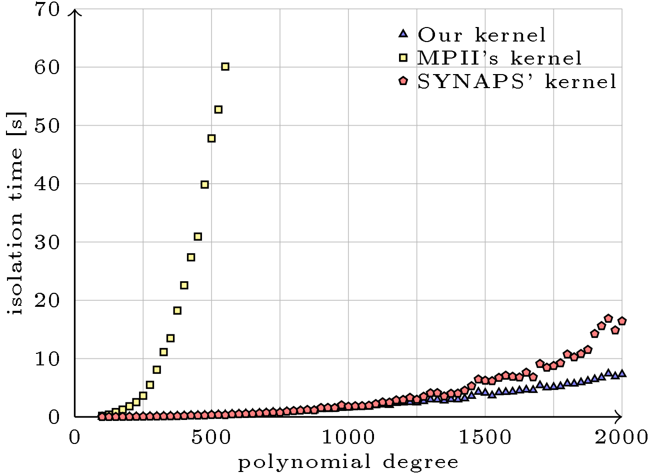
Root isolation: Mignotte polynomials

$f = x^d - 2(kx-1)^2$



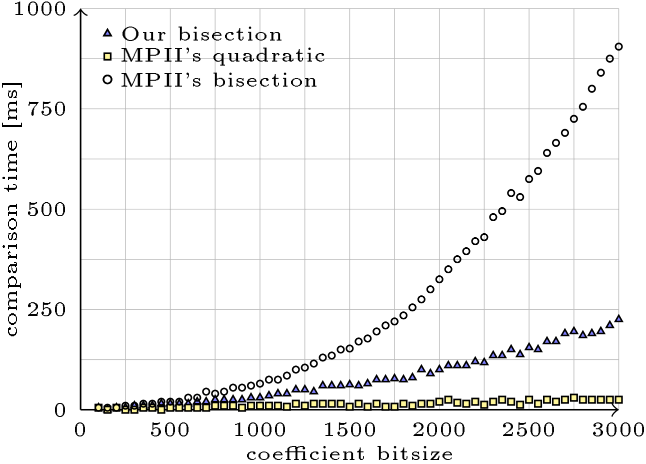
Root isolation: varying degree

bitsize-1000 random polynomials



Algebraic number comparison

almost-identical polynomials of degree 20



Arrangement benchmarks

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- CGAL's arrangement package (Tel-Aviv University)
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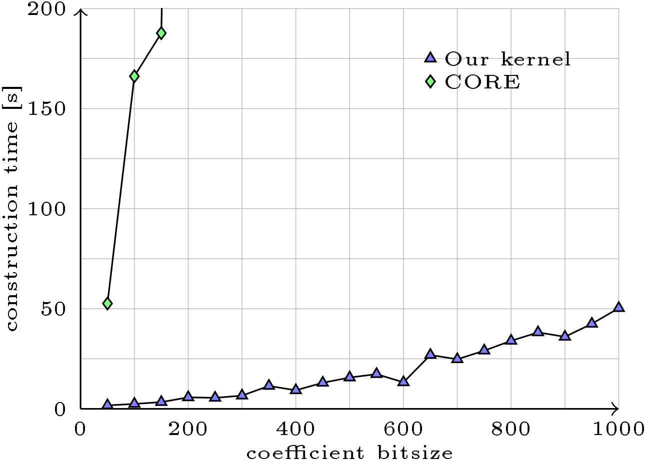
test data

- generate n random polynomials
- shift them vertically m times
- $n(m+1)$ polynomials of bitsize τ and degree d
- we fix $n = 5$ and $m = 4$ here

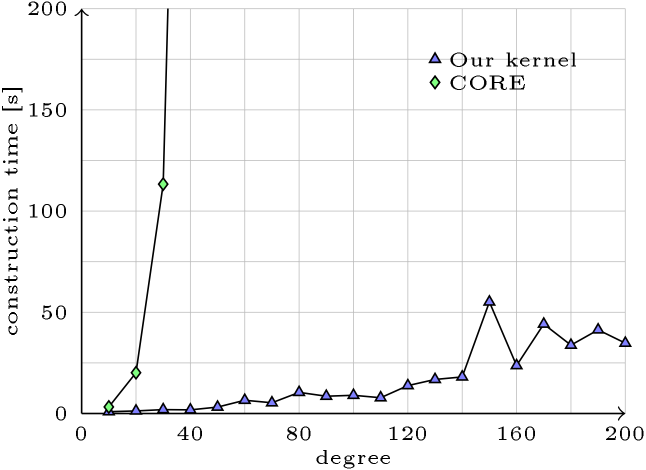
Arrangements: varying bitsize

$d = 20$



Arrangements: varying degree

$\tau = 32$



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- arrangement experiments
  - validate the algebraic kernel approach

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  - no special treatment of non-generic cases,
  - results in the original coordinate system
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- analysis of algorithms
  - output-sensitive complexity analysis

Perspectives

- improve handling of some curves
  - algebraic approach that is *always* efficient
  - arrangements of curves
- topology of surfaces, meshing
- include Isotop in Maple
- bivariate and multivariate algebraic kernel
- tighter complexity bounds