Goodness-of-fit test for switching ergodic diffusion process

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Outline

i.i.d. model

- Observations – $(X_1,\ldots,X_n)=X^n$–i.i.d. r. v.'s with d.f $F(x)$.
- Basic hypothesis is simple: $\mathcal{H}_0$: $F(x)=F_0(x)$, $x\in\mathbb{R}$, for known continuous function $F_0(x)$.
- Fix $\alpha\in(0,1)$ and define the class of tests of asymptotic level $1-\alpha$

$$\mathcal{K}_\alpha=\left\{\bar\Psi\ :\ \overline{\lim_{n\to\infty}}\ \mathbf{E}_0\ \bar\Psi_n(X^n)\le\alpha\right\}.$$

The Cramér-von Mises test

$$\Psi_n(X^n)=1_{\{W_n^2>c_\alpha\}},$$

where

$$W_n^2=n\int\left[\hat F_n(x)-F_0(x)\right]^2dF_0(x),$$

and

$$\hat F_n(x)=\frac{1}{n}\sum_{j=1}^n 1_{\{X_j\le x\}},\qquad x\in\mathbb{R},$$

Limit distribution under simple hypothesis, case i.i.d.

We have the convergence

$$W_n^2 \Longrightarrow W^2 = \int_0^1 B(t)^2 \, dt,$$

where $B(\cdot)$ is a standard Brownian bridge. Then the test $\Psi_n \in \mathcal{K}_\alpha$, where

$$\mathbf{P}\left\{W^2 > c_\alpha\right\} = \alpha.$$

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W^2 representation

$$B(t) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sqrt{2}\, \xi_n \sin(n\pi t), \quad W^2 = \sum_{n=1}^{\infty} \frac{\xi_n^2}{(n\pi)^2}, \quad \xi_n \sim \mathcal{N}(0,1).$$

W^2 distribution (Smirnov 1936)

$$\mathbf{P}\left(W^2 > x\right) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \int_{(2n-1)\pi}^{2n\pi} \frac{e^{-xu^2/2}}{u\sqrt{-D(u^2)}}\, du \quad \text{for} \quad x \geq 0,$$

where $D(u)$ is the *Fredholm determinant* defined, for $u \geq 0$, by

$$D(u) = \prod_{n=1}^{\infty} \left(1 - \frac{u}{(n\pi)^2}\right) = \frac{\sin(\sqrt{u})}{\sqrt{u}}.$$

Anderson-Darling test

Anderson and Darling generalized the Cramér-von Mises test by adding a weight function:

$$C_n^2 = n \int \psi\left(F_0\left(x\right)\right) \left[\hat{F}_n\left(x\right) - F_0\left(x\right)\right]^2 dF_0\left(x\right) \Longrightarrow \int_0^1 \psi(t) B\left(t\right)^2 dt,$$

where ψ is some non-negative function. Therefore the Anderson-Darling test

$$\Psi_n\left(X^n\right) = 1_{\left\{C_n^2 > c_\alpha\right\}} \in \mathcal{K}_\alpha,$$

if we choose c_α as solution of the equation

$$\mathbf{P}\left\{\int_0^1 \psi(t) B\left(t\right)^2 dt > c_\alpha\right\} = \alpha.$$

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- Deheuvels and Martynov (2003): $\psi(t) = t^{2\beta}$, $\beta = \frac{1}{2\nu} - 1 > -1$

Composite null hypothesis

Cramér (1946)

$$\mathcal{H}_0^* : F(x) = F_*(x,\vartheta), \quad \vartheta \in \Theta,$$

where ϑ is an unknown parameter.

Darling (1955)

$$C_n^2 = n \int_{\mathbb{R}} \left[\hat{F}_n(x) - F_*(x,\hat{\vartheta}_n)\right]^2 dF_*(x,\hat{\vartheta}_n),$$

where $\hat{\vartheta}_n$ is an estimate of ϑ, $\Theta = (a,b)$. Under suitable regularity conditions, if $\sqrt{n}\,(\hat{\vartheta}_n - \vartheta) \Longrightarrow \xi \sim \mathcal{N}(0,\sigma^2(\vartheta))$. We have the convergence

$$C_n^2 \Longrightarrow C^2 = \int_{\mathbb{R}} \left[B(F_*(x,\vartheta)) - \xi \frac{\partial}{\partial\vartheta} F_*(x,\vartheta)\right]^2 dF_*(x,\vartheta).$$

Note that C^2 depends on $F_*(x,\vartheta)$ and the test is no more distribution-free.
If $\mathbf{Var}(\hat{\vartheta}_n)$ goes to zero sufficiently rapidly, the limiting distributions of C_n^2 and W_n^2 are the same, and the test is distribution-free.

The observation model

The model

Suppose that we observe a trajectory $X^T = \{X_t, 0 \leq t \leq T\}$ of the ergodic diffusion process

$$dX_t = S(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where

- the conditions of the existence and uniqueness of the solution are fulfilled,
- the trend coefficient $S(\cdot)$ is unknown to the observer,
- the diffusion coefficient $\sigma(\cdot)^2$ is continuous positive function and known.

The observation model with Assumptions

$\mathcal{RP}$. The functions $S(\cdot)$ and $\sigma(\cdot)$ are such that

$$V(x) = \int_0^x \exp\left\{-2\int_0^y \frac{S(v)}{\sigma(v)^2}\,dv\right\} dy \to \pm\infty, \quad \text{as} \quad x \to \pm\infty,$$

$$G(S) = \int_{\mathbb{R}} \frac{1}{\sigma(y)^2} \exp\left\{2\int_0^y \frac{S(v)}{\sigma(v)^2}\,dv\right\} dy < \infty.$$

Under conditions $\mathcal{RP}$, the process X^T has ergodic properties with the invariant density given by

$$f_S(x) = \frac{1}{G(S)\,\sigma(x)^2} \exp\left\{2\int_0^x \frac{S(v)}{\sigma(v)^2}\,dv\right\}, \quad x \in \mathbb{R}.$$

Models under basic hypothesis $\mathcal{H}_0$

Switching diffusion

$$dX_t = -\rho\, \mathrm{sgn}(X_t - \theta)\, dt + dW_t, \quad X_0, \quad 0 \le t \le T$$

- $dX_t = -\mathrm{sgn}(X_t)\, dt + dW_t$
  - The limit distribution of the C-vM type statistics have been studied
  - Explicit representation of the limiting processes have been found
- $dX_t = -\rho\, \mathrm{sgn}(X_t - \theta)\, dt + dW_t$, ρ, θ are unknown
  - Asymptotic properties of the estimators have been studied
  - The limit distribution of the C-vM type statistics have been studied

General case

$S_0(x)$ is some known function

Explicit representations of the limit distribution of C-vM type statistics with different weighted functions have been found

Testing problem

Hypothesis

$\mathcal{H}_0$: The observed trajectory $X^T = \{X_t, 0 \leq t \leq T\}$ is solution of SDE

$$dX_t = -\mathrm{sgn}(X_t)\, dt + dW_t, \quad X_0, \quad 0 \leq t \leq T, \qquad (1)$$

where $S_0(x) = -\mathrm{sgn}(x)$ is a discontinuous function and taking just two values $+1$ and -1.

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- The conditions $\mathcal{RP}$ are fulfilled and (1) is an ergodic diffusion process.
- Its density $f_{S_0}(x)$ and invariant distribution function $F_{S_0}(x)$ are:

$$f_{S_0}(x) = e^{-2|x|}, \quad F_{S_0}(x) = 1_{\{x>0\}} - \frac{1}{2}\,\mathrm{sgn}(x)\, e^{-2|x|}, \quad x \in \mathbb{R}.$$

Tests statistics

C-vM type tests statistics

$$\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}, \quad T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dx,$$

$$\phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}}, \quad T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x).$$

These tests statistics were proposed by Dachian and Kutoyants (2007) and are based on the local time estimator estimator:

$$f_T^\circ(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon T} \int_0^T 1_{\{|X_t - x| \leq \varepsilon\}} \,\mathrm{d}t = \frac{\Lambda_T(x)}{T} \quad \text{for} \quad x \in \mathbb{R},$$

where $\Lambda_T(x)$ is the local time of the diffusion process (Revuz et Yor, 1991).

Auxiliary result

LTE

The local time estimator $f_T^\circ(x)$ is unbiased and $\sqrt{T}$ asymptotically normal

$$\eta_T(x) = \sqrt{T}\left(f_T^\circ(x) - f_S(x)\right) \Longrightarrow \mathcal{N}\left(0, R_f(x,x)\right) \quad \text{as} \quad T \to \infty.$$

Moreover, the process $(\eta_T(x), x \in \mathbb{R})$ converges weakly to the zero mean Gaussian process $(\eta_f(x), x \in \mathbb{R})$ with the covariance function

$$R_f(x,y) = 4 f_S(x) f_S(y)\, \mathbf{E}_S\left(\frac{[1_{\{\xi > x\}} - F_S(\xi)][1_{\{\xi > y\}} - F_S(\xi)]}{\sigma\left(X_t\right)^2 f_S(\xi)^2}\right).$$

Kutoyants (1997, 2004)

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Kutoyants (1997, 2004)

Davydov-Bosq(1999): LTE of the density of wide class of stationary processes.

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Remark. Note that these statistics are not distribution-free even asymptotically and the choice of thresholds c_α and d_α for these tests is much more complicate due the structure of the covariance of the process $(\eta_f(x), x \in \mathbb{R})$.

Karhunen-Loève expansion in $\mathbb{R}$

Assumptions

- $Z(t), t \in \mathbb{R}$ — centered Gaussian process
- $K(t,s) = \mathbb{E}Z(t)Z(s)$ continuous in $\mathbb{R}^2$
- $\int_{\mathbb{R}} K(s,s)\,ds < \infty$, $\lim_{|s|\to\infty} K(s,s) = 0$

The following decompositions hold a.s. (I.Novitskii, 1982, J.Buescu, 2004):

Karhunen-Loève expansion and Parseval's identity

$$Z(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,\xi_n\,e_n(t), \quad W^2 = \int_{\mathbb{R}} Z(t)^2 dt = \sum_{n=1}^{\infty} \lambda_n\,\xi_n^2,\ t \in \mathbb{R},$$

where $\{\xi_n : n \geq 1\}$ are i.i.d. $\mathcal{N}(0,1)$ and $\lambda_n, e_n, n \geq 1$ are the eigenvalues and the eigenfunctions of the Hilbert-Schmidt operator $\mathcal{K} : L_2[\mathbb{R}] \to L_2[\mathbb{R}]$:

$$\mathcal{K}(f) = \int_{\mathbb{R}} K(t,s)\,f(s)\,ds.$$

Test procedure

Let us introduce the Gaussian process $(\eta_f(x),\, x \in \mathbb{R})$ with zero mean and covariance function given by

$$\begin{aligned} R_f(x,y) &= 2\left(1_{\{x\vee y<0\}}\,e^{2(x\wedge y)} + 1_{\{x\wedge y\geq 0\}}\,e^{-2(x\vee y)}\right) \\ &\quad - \left(2(|x|+|y|) + \mathrm{sgn}(xy)\right)e^{-2(|x|+|y|)}. \end{aligned}$$

Theorem

Under hypothesis $\mathcal{H}_0$, we have the convergence

$$V_T^2 = T\int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_0}(x)\right]^2 dx \Longrightarrow \int_{\mathbb{R}} \eta_f^2(x)dx.$$

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$$\mathbf{P}_{S_0}\left\{\int_{\mathbb{R}} \eta_f^2(x)\,dx > d_\alpha\right\} = \alpha,$$

Proposition

The C-vM type test $\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}$ belongs to $\mathcal{K}_\alpha$ and is consistent.

Explicit representation of $\eta_f(\cdot)$

Theorem

The Gaussian process $(\eta_f(x), x \in \mathbb{R})$ *has a K-L expansion given by*

$$\eta_f(x) = -\sum_{n=1}^{\infty} \frac{2\,\xi_n}{z_{1,n}} \operatorname{sgn}(x)\, e^{-|x|} \frac{J_1\left(z_{1,n}\, e^{-|x|}\right)}{J_0\left(z_{1,n}\right)}$$
$$+ \sum_{n=1}^{\infty} \frac{2\,\xi'_n}{z'_n} e^{-|x|} \frac{\left(Y_2\left(z'_n\right) + \frac{4}{\pi z'^2_n}\right) J_1\left(z'_n\, e^{-|x|}\right) - J_2\left(z'_n\right)\left(Y_1\left(z'_n\, e^{-|x|}\right) + \frac{2\, e^{|x|}}{\pi z'_n}\right)}{J_1\left(z'_n\right)\left(Y_2\left(z'_n\right) + \frac{4}{\pi z'^2_n}\right) - Y_1\left(z'_n\right) J_2\left(z'_n\right)}$$

where $\{\xi_n, n \geq 1\}$, $\{\xi'_n, n \geq 1\}$ *denote two independent sequences of i.i.d.* $\mathcal{N}(0,1)$ *r.v.'s,* $\{z_{1,n}, n \geq 1\}$ *the positive zeros of* $J_1(\cdot)$ *and* $\{z'_n, n \geq 1\}$ *solutions of equation*

$$J_2(z)\left(Y_0(z) - 2\left(\ln(z/2) + \gamma\right)/\pi\right) - \left(J_0(z) - 1\right)\left(Y_2(z) + 4/(\pi z^2)\right) = 0,$$

with $J_\nu(\cdot)$ *and* $Y_\nu(\cdot)$ *the Bessel function of 1st and 2nd kind respectively.*

Parseval's identity

- Parseval's identity

$$V^2 = \int_{\mathbb{R}} \eta_f^2(x)\, dx \overset{law}{=} \sum_{n=1}^{\infty} \frac{4}{z_{1,n}^2}\, \xi_n^2 + \sum_{n=1}^{\infty} \frac{4}{z_n'^2}\, \xi_n'^2 .$$

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- Fubini's theorem

$$1 = \int_{\mathbb{R}} R_f(x,x)\, dx = \sum_{n=1}^{\infty} \frac{4}{z_{1,n}^2} + \sum_{n=1}^{\infty} \frac{4}{z_n'^2} \overset{(a)}{=} \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{z_n'^2}$$

(a) - Rayleigh's formula $\sum_{n=1}^{\infty} \frac{1}{z_{\nu,n}^2} = \frac{1}{4(\nu+1)}$.

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- Numerical simulation (for $N = 10^5$) gives

$$\left|\sum_{n=1}^{N} \frac{4}{z_n'^2} - \frac{1}{2}\right| \leq 10^{-3}$$

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Making the transformation $t = F_{S_0}(x)$, we obtain

$$\int_{\mathbb{R}} \eta_f^2(x) dF_{S_0}(x) = \int_0^1 \tilde{\eta}_f^2(t)\, dt,$$

where the Gaussian process $(\tilde{\eta}_f(t), 0 \le t \le 1)$ has covariance function given by

$$\begin{aligned} K_f(s,t) &= (1-|2s-1|)(1-|2t-1|)\ln\left[(1-|2s-1|)(1-|2t-1|)\right] \\ &\quad +4\, s\wedge t\,(1-s\vee t)\,. \end{aligned}$$

Explicit representation of $\tilde{\eta}_f(\cdot)$

Theorem

The Gaussian process $(\tilde{\eta}_f(t), 0 \leq t \leq 1)$ *has a K-L expansion given by*

$$\tilde{\eta}_f(t) = \sum_{n\geq 1} (\xi_n/(n\pi))\,\sqrt{2}\,\mathrm{sgn}(1/2 - t)\sin\left(n\,\pi(1 - |2\,t - 1|)\right)$$
$$+ \sum_{n\geq 1} (\xi'_n/\nu_n)\,\sqrt{2}[(\alpha(\nu_n)/\nu_n - \dot{\alpha}(\nu_n))\sin\left(\nu_n(1 - |2\,t - 1|)\right)$$
$$- (\sin(\nu_n)/\nu_n - \cos(\nu_n))\,\alpha\left(\nu_n(1 - |2\,t - 1|)\right)]/\mathrm{Si}(\nu_n),$$

where $\{\xi_n, n \geq 1\}$, $\{\xi'_n, n \geq 1\}$ *two independent sequences of i.i.d.* $\mathcal{N}(0, 1)$ *r.v's and* $\{\nu_n, n \geq 1\}$ *the solutions of equation*

$$G(r)\,[\sin(r) - r\cos(r)] - Si(r)\,[\alpha(r) - r\dot{\alpha}(r)] = 0,$$

with $\alpha(r) = Ci(r)\sin(r) - Si(r)\cos(r)$, $\dot{\alpha}(r) = \frac{d}{dr}\alpha(r)$, $G(r) = \int_0^r \frac{\alpha(s)}{s}\,ds$ *and* $Ci(\cdot)$, $Si(\cdot)$ *the cosine and sine integral respectively.*

Parseval's identity

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$$W^2 = \int_0^1 \tilde{\eta}_f^2(t)\,dt \overset{law}{=} \sum_{n=1}^{\infty} \frac{\xi_n^2}{n^2\pi^2} + \sum_{n=1}^{\infty} \frac{\xi_n'^2}{\nu_n^2}.$$

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- Fubini's theorem

$$\frac{4}{9} = \int_0^1 K_f(t,t)\, dt = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} + \sum_{n=1}^{\infty} \frac{1}{\nu_n^2} \overset{(a)}{=} \frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{\nu_n^2}$$

(a) - Euler's formula $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

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(a) - Euler's formula $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

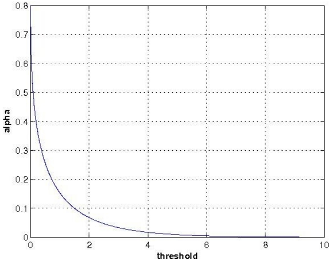
- Numerical simulation (for $N = 10^5$) gives

$$\left| \sum_{n=1}^{N} \frac{1}{\nu_n^2} - \frac{5}{18} \right| \leq 1.2 \times 10^{-3}$$

Numerical Results, I

- **Table 1**: Values of some quantiles of the random variable V^2.
- **Figure 1**: Thresholds choice of the random variable V^2.

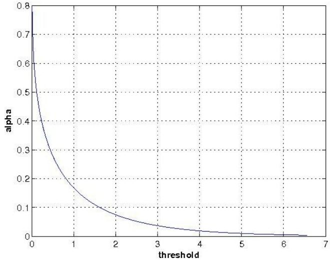
α	d_α
0.10	1.619
0.05	2.563
0.025	3.596
0.010	5.868
0.005	6.197



Numerical Results, II

- **Table 2**: Values of some quantiles of the random variable W^2.
- **Figure 2**: Thresholds choice of the random variable W^2.

α	c_α
0.10	1.501
0.05	2.420
0.025	3.433
0.010	5.050
0.005	6.004



Smirnov's formula

The approach that can be used here to calculate the quantiles for the distribution of W^2 and V^2 in Tables 1 and 2 is based on the Smirnov formula:

$$F(x) = 1 + \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \int_{\gamma_{2n-1}}^{\gamma_{2n}} \frac{e^{-xu^2/2}}{u\sqrt{|D(u^2)|}}\, du \quad \text{for} \quad x \geq 0.$$

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- For the distribution function of W^2 we have

$$D(u) = \frac{J_1(2\sqrt{u})}{\sqrt{u}} \prod_{n=1}^{\infty} \left(1 - \frac{4\,u}{z_n'^2}\right).$$

with $\gamma_n = \{z_{1,n}/2, z_n'/2\}$, for $n \geq 1$.

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  with $\gamma_n = \{z_{1,n}/2, z_n'/2\}$, for $n \geq 1$.

- For the distribution function of V^2 we have

$$D(u) = \frac{\sin(\sqrt{u})}{\sqrt{u}} \prod_{n=1}^{\infty}\left(1 - \frac{u}{\nu_n^2}\right).$$

  with $\gamma_n = \{n\pi, \nu_n\}$, for $n \geq 1$.

Parameter estimation

Let the observed process be

$$dX_t = -\rho\,\mathrm{sgn}(X_t - \theta)dt + dW_t, \quad X_0, \quad 0 \le t \le T,$$

We suppose that $\vartheta = (\rho, \theta) \in \Theta$ is unknown parameter and $\rho > 0$. The process $(X_t)_{t\ge 0}$ is an ergodic diffusion with the invariant density

$$f(\vartheta, x) = \rho\, e^{-2\,\rho|x-\theta|}, \quad \text{for} \quad x \in \mathbb{R}.$$

The likelihood ratio function is

$$L\left(\vartheta, X^T\right) = \exp\left\{-\rho\int_0^T \mathrm{sgn}(X_t - \theta)\, dX_t - \frac{\rho^2}{2}\,T\right\}.$$

The MLE $\hat{\vartheta}_T$ and BE $\tilde{\vartheta}_T$ are defined as usual by the relations

$$L\left(\hat{\vartheta}_T, X^T\right) = \sup_{\vartheta\in\Theta} L\left(\vartheta, X^T\right), \quad \tilde{\vartheta}_T = \frac{\int_\Theta y\, p\left(y\right)\, L\left(y, X^T\right)\, dy}{\int_\Theta p\left(y\right)\, L\left(y, X^T\right)\, dy}.$$

Parameter estimation

Introduce the normalized likelihood ratio process

$$Z_T(w) = \frac{L(\vartheta + \varphi_T\, w, X^T)}{L(\vartheta, X^T)}, \quad \varphi_T = \begin{pmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{pmatrix}, \quad w = (u, v) \in \mathbb{R}^2,$$

and the random vectors $\hat{w} = (\hat{v}, \hat{u})$ and $\tilde{w} = (\tilde{v}, \tilde{u})$ defined with the help of the following stochastic process:

$$Z_\vartheta(w) = Z_\rho(v)\, Z_\theta(u),$$

where

$$Z_\rho(v) = \exp\left\{ v\,\zeta - \frac{v^2}{2} \right\}, \qquad Z_\theta(u) = \exp\left\{ 2\,\rho^{3/2}\, W(u) - 2\,\rho^3\,|u| \right\},$$

as follows:

$$Z_\vartheta(\hat{w}) = \sup_{w \in \mathbb{R}^2} Z_\vartheta(w), \qquad \tilde{w} = \frac{\int_{\mathbb{R}^2} w\, Z_\vartheta(w)\, dw}{\int_{\mathbb{R}^2} Z_\vartheta(w)\, dw},$$

where $W(\cdot)$ is a two-sided Wiener process and ζ is $\mathcal{N}(0,1)$ random variable independent of $W(\cdot)$.

Parameter estimation

proposition

The MLE and BE are

- Consistent, i.e., for any $\nu > 0$

$$\mathbf{P}_{\vartheta}^{(T)}\left\{|\hat{\vartheta}_T-\vartheta|>\nu\right\}\longrightarrow 0,\qquad \mathbf{P}_{\vartheta}^{(T)}\left\{|\tilde{\vartheta}_T-\vartheta|>\nu\right\}\longrightarrow 0$$

- Have different limit distributions

$$\varphi_T^{-1}\left(\hat{\vartheta}_T-\vartheta\right)\Longrightarrow \hat{w},\qquad \varphi_T^{-1}\left(\tilde{\vartheta}_T-\vartheta\right)\Longrightarrow \tilde{w}$$

- The moments converge: for any $p > 0$

$$\mathbf{E}_{\vartheta}\left|\varphi_T^{-1}\left(\hat{\vartheta}_T-\vartheta\right)\right|^p\longrightarrow \mathbf{E}_{\vartheta}\left|\hat{w}\right|^p,\quad \mathbf{E}_{\vartheta}\left|\varphi_T^{-1}\left(\tilde{\vartheta}_T-\vartheta\right)\right|^p\longrightarrow \mathbf{E}_{\vartheta}\left|\tilde{w}\right|^p$$

The proof is based on the two remarkable Theorems 1.10.1 and 1.10.2 by Ibragimov and Khasminskii (1981).

Testing problem

Hypothesis

$\mathcal{H}_*$: The observed trajectory $X^T = \{X_t, 0 \le t \le T\}$ is solution of SDE

$$dX_t = -\rho\,\mathrm{sgn}(X_t - \theta)\,dt + dW_t, \quad X_0, \quad 0 \le t \le T,$$

where $S_*(x, \vartheta) = -\rho\,\mathrm{sgn}(x - \theta)$ and $\vartheta = (\rho, \theta) \in \Theta$ is unknown parameter.

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where $S_*(x, \vartheta) = -\rho \operatorname{sgn}(x - \theta)$ and $\vartheta = (\rho, \theta) \in \Theta$ is unknown parameter.

To test $\mathcal{H}_*$ we use these statistics constructed with the help of the MLE or BE,

$$V_T^2 = T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_*}(x, \hat{\vartheta}_T) \right]^2 dx,$$

$$W_T^2 = T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_*}(x, \hat{\vartheta}_T) \right]^2 dF_{S_0}(x, \hat{\vartheta}_T).$$

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$$W_T^2 = T \int_{\mathbb{R}} \left[f_T^\circ(x) - f_{S_*}(x, \hat{\vartheta}_T)\right]^2 dF_{S_0}(x, \hat{\vartheta}_T).$$

Let us introduce the following Gaussian process

$$\zeta(x, \vartheta) = \eta(x, \vartheta) - \xi \frac{\partial}{\partial \rho} f_{S_*}(x, \vartheta), \quad \xi \sim \mathcal{N}(0, 1).$$

Theorem

$$V_n^2 \Longrightarrow V^2 = \int_{\mathbb{R}} \zeta(x, \vartheta)^2\, dx, \quad W_n^2 \Longrightarrow W^2 = \int_{\mathbb{R}} \zeta(x, \vartheta)^2\, dF_{S_*}(x, \vartheta).$$

Testing problem

Denote by $d_\alpha(\vartheta)$ and $c_\alpha(\vartheta)$ the critical values defined by the equations

$$\mathbf{P}\left\{V^2 > d_\alpha(\vartheta)\right\} = \alpha, \quad \mathbf{P}\left\{W^2 > c_\alpha(\vartheta)\right\} = \alpha.$$

Proposition

The tests $\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha(\hat{\vartheta})\}}$ and $\phi_T(X^T) = 1_{\{W_T^2 > c_\alpha(\hat{\vartheta})\}}$ belong to $\mathcal{K}_\alpha$.

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Now, we consider the case of the one-dimensional parameter $\vartheta = \theta$, in this case, the choice of the thresholds c_α and d_α does not depend on the hypothesis $\mathcal{H}_*$ and these constants are solutions of the equation

$$\mathbf{P}_{S_0}\left\{\int_{\mathbb{R}} \eta_f^2(x)\, dx > d_\alpha\right\} = \alpha, \quad \mathbf{P}_{S_0}\left\{\int_{\mathbb{R}} \eta_f^2(x)\, dF_{S_0}(x) > c_\alpha\right\} = \alpha.$$

Proposition

The tests $\varphi_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}$ and $\phi_T(X^T) = 1_{\{W_T^2 > c_\alpha\}}$ belong to $\mathcal{K}_\alpha$.

Asymptotically distribution-free tests

Hypothesis

$\mathcal{H}_0$: The observed trajectory X^T is solution of the SDE

$$\mathrm{d}X_t = S_0\left(X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\mathrm{d}W_t, \quad 0 \leq t \leq T.$$

where $S_0(\cdot)$ is some known function.

Asymptotically distribution-free tests

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$\mathcal{H}_0$: The observed trajectory X^T is solution of the SDE

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where $S_0(\cdot)$ is some known function.

Remind here the following results by Kutoyants (2010)

$$V_T^2 = T\int_{\mu}^{\infty} h(x)\left[f_T^{\circ}(x) - f_{S_0}(x)\right]^2 dF_{S_0}(x),$$

where μ is the median of invariant law $(F_0(\mu) = 1/2)$ and

$$h\left(x\right) = \frac{2F_{S_0}\left(x\right) - 1}{4\phi\left(\mu\right)^2\sigma\left(x\right)^2 f_{S_0}\left(x\right)^4}\ \psi\left(\phi\left(x\right)/\phi\left(\mu\right)\right)\ 1_{\{x\geq\mu\}},$$

with $\psi(\cdot)$ is continuous and positive function and

$$\phi\left(x\right) = \int_{-\infty}^{\infty} \frac{\left(1_{\{y>x\}} - F_{S_0}\left(y\right)\right)^2}{\sigma\left(y\right)^2 f_{S_0}\left(y\right)}\,\mathrm{d}y.$$

Asymptotically distribution-free tests

The second statistic is based on the empirical distribution function

$$\hat{F}_T(x) = \frac{1}{T}\int_0^T 1_{\{X_t<x\}}\,\mathrm{d}t.$$

The corresponding statistic is:

$$W_T^2 = T\int_{\mathbb{R}} H(x)\left[\hat{F}_T(x) - F_{S_0}(x)\right]^2 dF_{S_0}(x),$$

where

$$H(x) = \frac{\Phi'(x)}{f_{S_0}(x)\left[F_{S_0}(x)-1\right]^2}\,\psi\left(\Phi(x)\right),$$

and

$$\Phi(x) = \int_{-\infty}^{x}\frac{F_{S_0}(y)^2}{\sigma(y)^2 f_{S_0}(y)}\mathrm{d}y + \left(\frac{F_{S_0}(x)}{F_{S_0}(x)-1}\right)^2\int_x^{\infty}\frac{\left(F_{S_0}(y)-1\right)^2}{\sigma(y)^2 f_{S_0}(y)}\mathrm{d}y.$$

Limit distributions

It is shown that

$$V_T^2 \Longrightarrow \int_0^\infty \psi(t)\, W_{t+1}^2\, dt, \qquad W_T^2 \Longrightarrow \int_0^\infty \psi(t)\, W_t^2\, dt. \qquad (2)$$

Denote by d_α and c_α the critical values defined by the equations

$$\mathbf{P}_{S_0}\left\{\int_0^\infty \psi(t)\, W_{t+1}^2\, dt > d_\alpha\right\} = \alpha, \quad \mathbf{P}_{S_0}\left\{\int_0^\infty \psi(t)\, W_t^2\, dt > c_\alpha\right\} = \alpha.$$

Hence the tests $\hat{\phi}_T(X^T) = 1_{\{V_T^2 > d_\alpha\}}$ and $\tilde{\phi}_T(X^T) = 1_{\{W_T^2 > c_\alpha\}}$ belong to $\mathcal{K}_\alpha$ and are asymptotically distribution-free.

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Objective

We provide the explicit expressions of the limit statistics in (2) via direct calculation of Laplace transforms, for the particular cases

$$\psi(t) = (t+1)^{-2\beta}, \quad \beta = \frac{1}{2\nu} + 1 > 1 \quad \text{and} \quad \psi(t) = e^{-2t}.$$

Direct calculation of the Laplace transform

Riccati-Volterra type integral equation

Let $\gamma(t,s)$ be a unique solution of the equation

$$\gamma(t,s) = K(t,s) + u\int_0^s \gamma(t,r)\,\gamma(s,r)\,dr, \quad 0 \le s \le t,$$

such that $\gamma(s,s) = \gamma(s) \ge 0$.

Laplace transform and Fredholm determinant

The following equality holds, for $u < \frac{1}{\lambda_1}$

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty Z(s)^2\,ds\right) = \exp\left(\frac{u}{2}\int_0^\infty \gamma(s,s)\,ds\right),$$

$$D(u) = \prod_{n=1}^{\infty}(1-\lambda_n u) = \exp\left(-u\int_0^\infty \gamma(s,s)\,ds\right),$$

$\lambda_n, n \ge 1$ -eigenvalues of the covariance operator of the process $\{Z(t), t \ge 0\}$.

Kleptsyna and Le Breton (2002)

Case: $\psi(t) = e^{-2t}$

Let $\{z_{0,n}, n \geq 1\}$ and $\{\delta_n, n \geq 1\}$ be respectively sequences of positive zeros of the Bessel function $J_0(\cdot)$ and solutions of equation:

$$J_0(\delta_n) - \delta_n J_1(\delta_n) = 0.$$

Theorem

The following equalities hold:

$$\mathbf{E} \exp\left(\frac{u}{2}\int_0^\infty e^{-2t} W_t^2 dt\right) = \left[J_0(\sqrt{u})\right]^{-\frac{1}{2}}, \quad u < z_{0,1}^2,$$

$$D(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{z_{0,n}^2}\right) = J_0(\sqrt{z}), \quad z \in \mathbb{C},$$

$$\mathbf{E} \exp\left(\frac{u}{2}\int_0^\infty e^{-2t} W_{t+1}^2 dt\right) = \left[J_0(\sqrt{u}) - \sqrt{u}\, J_1(\sqrt{u})\right]^{-\frac{1}{2}}, \quad u < \delta_1^2,$$

$$D(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{\delta_n^2}\right) = J_0(\sqrt{z}) - \sqrt{z}\, J_1(\sqrt{z}), \quad z \in \mathbb{C}.$$

Case: $\psi(\mathbf{t}) = \mathbf{e}^{-2\mathbf{t}}$

The direct consequence is the following:

Corollary

The following equalities hold:

$$A^2 = \int_0^\infty e^{-2t} W_t^2 \, dt \overset{law}{=} \sum_{n=1}^{\infty} \frac{\xi_n^2}{z_{0,n}^2},$$

$$B^2 = \int_0^\infty e^{-2t} W_{t+1}^2 \, dt \overset{law}{=} \sum_{n=1}^{\infty} \frac{\xi_n^2}{\delta_n^2},$$

where $\{\xi_n, n \geq 1\}$, are i.i.d. $\mathcal{N}(0,1)$ random variables.

Tables of the limit distribution

The following Tables provide some values of quantiles of distributions of A^2 and B^2. Calculations were based on the explicit form of the Smirnov formula

$$\mathbf{P}\left(A^2 > x\right) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_{z_{0,2n-1}}^{z_{0,2n}} \frac{e^{-xu^2/2}}{u\sqrt{-J_0(u)}}\,du,$$

$$\mathbf{P}\left(B^2 > x\right) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_{\delta_{2n-1}}^{\delta_{2n}} \frac{e^{-xu^2/2}}{u\sqrt{\left|J_0(u) - u\,J_1(u)\right|}}\,du.$$

$\mathbf{P}\left(A^2 > x\right)$	x
0.10	0.552
0.05	0.747
0.01	1.229
0.005	1.445
0.001	1.954

$\mathbf{P}\left(B^2 > x\right)$	x
0.10	1.832
0.05	2.552
0.01	4.323
0.005	5.113
0.001	6.982

Case: $\psi(\mathbf{t}) = (\mathbf{t}+\mathbf{1})^{-\mathbf{2}\beta}$

Let $\{z_{\nu,n}, n \geq 1\}$ the sequence of positive zeros of the Bessel function $J_\nu(\cdot)$

Theorem

The following equalities hold:

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty (t+1)^{-2\beta} W_t^2 dt\right) = \left[\Gamma(\nu+1)\,\frac{J_\nu(2\nu\sqrt{u})}{(\nu\sqrt{u})^\nu}\right]^{-\frac{1}{2}}, \quad u < \frac{z_{\nu,1}^2}{4\nu^2},$$

$$D(z) = \prod_{n=1}^{\infty}\left(1 - \frac{4\nu^2 z}{z_{\nu,n}^2}\right) = \Gamma(\nu+1)\,\frac{J_\nu(2\nu\sqrt{z})}{(\nu\sqrt{z})^\nu}, \quad z \in \mathbb{C}.$$

Corollary

The following identity holds:

$$\int_0^\infty (t+1)^{-2\beta} W_t^2\, dt \overset{law}{=} 4\nu^2 \sum_{n=1}^{\infty} \frac{\xi_n^2}{z_{\nu,n}^2},$$

where $\{\xi_n, n \geq 1\}$, are i.i.d. $\mathcal{N}(0,1)$ random variables.

Case: $\psi(\mathbf{t}) = (\mathbf{t}+\mathbf{1})^{-2\beta}$

Remark. It is shown by Deheuvels and Martynov (2003) that

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^1 t^{2(\beta-2)}B(t)^2dt\right) = \left[\Gamma(\nu+1)\,\frac{J_\nu(2\nu\sqrt{u})}{(\nu\sqrt{u})^\nu}\right]^{-\frac{1}{2}} \qquad u < \frac{z^2_{\nu,1}}{4\nu^2}.$$

Thus, the following equality holds:

$$\int_0^\infty (t+1)^{-2\beta}W_t^2\,dt \stackrel{law}{=} \int_0^1 t^{2(\beta-2)}B(t)^2\,dt.$$

Case: $\psi(t) = (t+1)^{-2\beta}$

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Thus, the following equality holds:

$$\int_0^\infty (t+1)^{-2\beta}W_t^2\,dt \overset{law}{=} \int_0^1 t^{2(\beta-2)}B(t)^2\,dt.$$

Theorem (Deheuvels and Martynov 2003)

The following equalities hold:

$$\mathbf{E}\exp\left(\frac{u}{2}\int_0^\infty (t+1)^{-2\beta}W_{t+1}^2dt\right) = \left[\Gamma(\nu)\,\frac{J_{\nu-1}(2\nu\sqrt{u})}{(\nu\sqrt{u})^{\nu-1}}\right]^{-\frac{1}{2}}, \quad u < \frac{z_{\nu-1,1}^2}{4\nu^2},$$

$$D(z) = \prod_{n=1}^\infty\left(1-\frac{4\nu^2 z}{z_{\nu-1,n}^2}\right) = \Gamma(\nu)\,\frac{J_{\nu-1}(2\nu\sqrt{z})}{(\nu\sqrt{z})^{\nu-1}}, \quad z\in\mathbb{C}.$$

Publications

1. Gassem, A., On parameter estimation for switching diffusion process, *Statistics & Probability Letters, Volume 79, Issue 24, 2484-2492, 2009*.
2. Gassem, A., Goodness-of-fit test for switching diffusion, *Statistical Inference for Stochastic processes, Volume 13, Number 2, 97-123, 2010*.
3. Gassem, A., On Cramér-von Mises type test based on local time of switching diffusion process, à paraître dans: *Journal of Statistical Planning and Inference.*
4. Gassem, A., On the goodness-of-fit testing for switching diffusion process, soumis.
5. Gassem, A., On limit distributions of some goodness-of-fit tests statistics for ergodic diffusion processes, soumis.

Open questions

- extension of the direct Laplace transform calculations method to $\mathbb{R}$
- Switching diffusion
  - explicit representation of the Fredholm determinant
  - explicit representation of weighted functions

Merci de votre attention