Modélisation mathématique pour l'environnement

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Demande d'habilitation à diriger les recherches

Outline

1. Generalized variational principle for water waves
   - Models in shallow water
   - Models in deep water
2. Velocity relaxation in two-phase flows
   - Six-equations model
   - Four-equations model
3. Powder-snow avalanche modeling
   - Incompressible model with mixing
   - Numerical results

Water wave problem

- Continuity equation

$$\Delta\varphi = 0, \quad (\vec{x}, y) \in \Omega \times [-d, \eta],$$

- Kinematic bottom condition

$$\frac{\partial\varphi}{\partial y} + \nabla\varphi \cdot \nabla d = 0, \quad y = -d,$$

- Kinematic free surface condition

$$\frac{\partial\eta}{\partial t} + \nabla\varphi \cdot \nabla\eta = \frac{\partial\varphi}{\partial y}, \quad y = \eta(\vec{x}, t),$$

  - Dynamic free surface condition

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla_{\vec{x},y}\varphi|^2 + g\eta + \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) = 0, \quad y = \eta(\vec{x}, t).$$

Hamiltonian structure

A. Petrov (1964) [Pet64] ; V. Zakharov (1968) [Zak68]

Canonical variables :

$\eta(\vec{x}, t)$: free surface elevation

$\tilde{\varphi}(\vec{x}, t)$: velocity potential at the free surface

$$\tilde{\varphi}(\vec{x}, t) := \varphi(\vec{x}, y = \eta(\vec{x}, t), t)$$

Evolution equations :

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \tilde{\varphi}}, \qquad \rho \frac{\partial \tilde{\varphi}}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta},$$

Hamiltonian :

$$\mathscr{H} = \int_{-d}^{\eta} \frac{1}{2} |\nabla_{\vec{x},y}\varphi|^2 \, dy + \frac{1}{2} g\eta^2 + \sigma\left(\sqrt{1 + |\nabla\eta|^2} - 1\right)$$

Luke's variational principles

J.C. Luke, JFM (1967) [Luk67]

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathscr{L} \, d\vec{x} \, dt, \qquad \mathscr{L} := \int_{-d}^{\eta} \Big(\frac{1}{2} |\nabla_{\vec{x},y} \varphi|^2 + \varphi_t + gy \Big) \, dy$$

$$\begin{aligned}
\delta\varphi &: \ \Delta\varphi = 0, \quad (\vec{x}, y) \in \Omega \times [-d, \eta], \\
\delta\varphi|_{y=-d} &: \ \frac{\partial\varphi}{\partial y} + \nabla\varphi \cdot \nabla d = 0, \quad y = -d, \\
\delta\varphi|_{y=\eta} &: \ \frac{\partial\eta}{\partial t} + \nabla\varphi \cdot \nabla\eta - \frac{\partial\varphi}{\partial y} = 0, \quad y = \eta(\vec{x}, t), \\
\delta\eta &: \ \frac{\partial\varphi}{\partial t} + \frac{1}{2} |\nabla\varphi|^2 + g\eta = 0, \ \ y = \eta(\vec{x}, t).
\end{aligned}$$

- We obtain the water wave problem by varying η and φ

Generalization of the Lagrangian density

D. Clamond & D. Dutykh (2010), [CD10]

$\tilde{\varphi} := \varphi(\vec{x}, y = \eta(\vec{x}, t), t)$: quantity at the free surface

$\check{\varphi} := \varphi(\vec{x}, y = -d(\vec{x}, t), t)$: value at the bottom

Equivalent form of Luke's lagrangian :

$$\mathscr{L} = \tilde{\varphi}\eta_t + \check{\varphi}d_t - \frac{1}{2}g\eta^2 + \frac{1}{2}gd^2 - \int_{-d}^{\eta} \left[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\varphi_y^2\right] dy$$

Explicitly introduce the velocity field : $\vec{u} = \nabla\varphi$, $v = \varphi_y$

$$\mathscr{L} = \tilde{\varphi}\eta_t + \check{\varphi}d_t - \frac{1}{2}g\eta^2 - \int_{-d}^{\eta} \left[\frac{1}{2}(\vec{u}^2 + v^2) + \vec{\mu}\cdot(\nabla\varphi - \vec{u}) + \nu(\varphi_y - v)\right] dy$$

$\vec{\mu}, \nu$: Lagrange multipliers or pseudo-velocity field

Generalization of the Lagrangian density

D. Clamond & D. Dutykh (2010), [CD10]

Relaxed variational principle :

$$\mathscr{L} = (\eta_t + \tilde{\mu}\cdot\nabla\eta - \tilde{\nu})\tilde{\varphi} + (d_t + \check{\mu}\cdot\nabla d + \check{\nu})\check{\varphi} - \frac{1}{2}g\eta^2 + \int_{-d}^{\eta}\Big[\vec{\mu}\cdot\vec{u} - \frac{1}{2}\vec{u}^2 + \nu v - \frac{1}{2}v^2 + (\nabla\cdot\vec{\mu} + \nu_y)\varphi\Big]\,dy$$

Classical formulation :

$$\mathscr{L} = \tilde{\varphi}\eta_t + \check{\varphi}d_t - \frac{1}{2}g\eta^2 - \int_{-d}^{\eta}\Big[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\varphi_y^2\Big]\,dy$$

Degrees of freedom : $\eta, \varphi; \vec{u}, v; \vec{\mu}, \nu$

Shallow water regime

Choice of a simple ansatz in shallow water

Ansatz :

$$\vec{u}(\vec{x},y,t) \approx \bar{u}(\vec{x},t), v(\vec{x},y,t) \approx (y+d)(\eta+d)^{-1}\ \tilde{v}(\vec{x},t)$$
$$\varphi(\vec{x},y,t) \approx \bar{\varphi}(\vec{x},t), \nu(\vec{x},y,t) \approx (y+d)(\eta+d)^{-1}\ \tilde{\nu}(\vec{x},t)$$

Lagrangian density :

$$\mathscr{L} = \bar{\varphi}\eta_t - \frac{1}{2}g\eta^2 + (\eta+d)\Big[\bar{\mu}\cdot\bar{u} - \frac{1}{2}\bar{u}^2 + \frac{1}{3}\tilde{\nu}\tilde{v} - \frac{1}{6}\tilde{v}^2 - \bar{\mu}\cdot\nabla\bar{\varphi}\Big]$$

Nonlinear Shallow Water Equations :

$$\begin{aligned} h_t + \nabla\cdot[h\bar{u}] &= 0, \\ \bar{u}_t + (\bar{u}\cdot\nabla)\bar{u} + g\nabla h &= 0. \end{aligned}$$

Constraining with free surface impermeability

Constraint :

$$\tilde{\nu} = \eta_t + \bar{\mu} \cdot \nabla \eta$$

- Generalized Serre equations :

$$\begin{aligned} h_t + \nabla \cdot [h\bar{u}] &= 0, \\ \bar{u}_t + \bar{u} \cdot \nabla \bar{u} + g\nabla h + \frac{1}{3} h^{-1} \nabla [h^2 \tilde{\gamma}] &= (\bar{u} \cdot \nabla h) \nabla (h \nabla \cdot \bar{u}) \\ & \quad - [\bar{u} \cdot \nabla (h \nabla \cdot \bar{u})] \nabla h \end{aligned}$$

$$\tilde{\gamma} = \tilde{v}_t + \bar{u} \cdot \nabla \tilde{v} = h\big((\nabla \cdot \bar{u})^2 - \nabla \cdot \bar{u}_t - \bar{u} \cdot \nabla (\nabla \cdot \bar{u})\big)$$

This model cannot be obtained from Luke's lagrangian :

$$\delta \bar{\mu} : \ \bar{u} = \nabla \bar{\varphi} - \tfrac{1}{3} \tilde{v} \nabla \eta \neq \nabla \bar{\varphi}$$

Constraining with free surface impermeability

Constraint :

$$\tilde{\nu} = \eta_t + \bar{\mu} \cdot \nabla \eta$$

- Generalized Serre equations :

$$\begin{aligned} h_t + \nabla \cdot [h\bar{u}] &= 0, \\ \bar{u}_t + \bar{u} \cdot \nabla \bar{u} + g\nabla h + \frac{1}{3} h^{-1} \nabla [h^2 \tilde{\gamma}] &= (\bar{u} \cdot \nabla h) \nabla (h \nabla \cdot \bar{u}) \\ &\quad - [\bar{u} \cdot \nabla (h \nabla \cdot \bar{u})] \nabla h \end{aligned}$$

$$\tilde{\gamma} = \tilde{v}_t + \bar{u} \cdot \nabla \tilde{v} = h\big((\nabla \cdot \bar{u})^2 - \nabla \cdot \bar{u}_t - \bar{u} \cdot \nabla (\nabla \cdot \bar{u})\big)$$

Solitary wave solution (+ cnoidal wave) :

$$\eta = a \operatorname{sech}^2 \frac{\kappa}{2}(x - ct), \quad c^2 = g(d + a), \quad (\kappa d)^2 = 3a(d + a)^{-1}$$

Incompressibility and partial potential flow

Ansatz and constraints ($v \neq \varphi_y$) :

$$\bar{\mu} = \bar{u}, \tilde{\nu} = \tilde{v}, \bar{u} = \nabla\bar{\varphi}, \tilde{v} = -(\eta + d)\nabla^2\bar{\varphi}$$

$$\mathscr{L} = \bar{\varphi}\eta_t - \frac{1}{2}g\eta^2 - \frac{1}{2}(\eta + d)(\nabla\bar{\varphi})^2 + \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\varphi})^2$$

- Generalized Kaup-Boussinesq equations :

$$\begin{aligned} \eta_t + \nabla \cdot \big[(\eta + d)\nabla\bar{\varphi}\big] + \frac{1}{3}\nabla^2\big[(\eta + d)^3(\nabla^2\bar{\varphi})\big] &= 0, \\ \bar{\varphi}_t + g\eta + \frac{1}{2}(\nabla\bar{\varphi})^2 - \frac{1}{2}(\eta + d)^2(\nabla^2\bar{\varphi})^2 &= 0. \end{aligned}$$

$$\eta = a\cos\kappa(x - ct), \quad c^2 = gd(1 - \frac{1}{3}(\kappa d)^2)$$

Incompressibility and partial potential flow

Ansatz and constraints ($v \neq \varphi_y$) :

$$\bar{\mu} = \bar{u}, \tilde{\nu} = \tilde{v}, \bar{u} = \nabla\bar{\varphi}, \tilde{v} = -(\eta + d)\nabla^2\bar{\varphi}$$

$$\mathscr{L} = \bar{\varphi}\eta_t - \frac{1}{2}g\eta^2 - \frac{1}{2}(\eta + d)(\nabla\bar{\varphi})^2 + \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\varphi})^2$$

- Generalized Kaup-Boussinesq equations :

$$\begin{aligned} \eta_t + \nabla \cdot \left[(\eta + d)\nabla\bar{\varphi}\right] + \frac{1}{3}\nabla^2\left[(\eta + d)^3(\nabla^2\bar{\varphi})\right] &= 0, \\ \bar{\varphi}_t + g\eta + \frac{1}{2}(\nabla\bar{\varphi})^2 - \frac{1}{2}(\eta + d)^2(\nabla^2\bar{\varphi})^2 &= 0. \end{aligned}$$

$$\mathscr{H} = \int_\Omega \left\{ \frac{1}{2}g\eta^2 + \frac{1}{2}(\eta + d)(\nabla\bar{\varphi})^2 - \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\varphi})^2 \right\} d\vec{x}$$

Deep water approximation

Choice of the ansatz :

$$\{\varphi; \vec{u}; v; \mu; \nu\} \approx \{\tilde{\varphi}; \tilde{u}; \tilde{v}; \tilde{\mu}; \tilde{\nu}\} e^{\kappa(y-\eta)}$$

$$2\kappa\mathscr{L} = 2\kappa\tilde{\varphi}\eta_t - g\kappa\eta^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 - \tilde{u}\cdot(\nabla\tilde{\varphi} - \kappa\tilde{\varphi}\nabla\eta) - \kappa\tilde{v}\tilde{\varphi}$$

- generalized Klein-Gordon equations :

$$\begin{aligned}
\eta_t + \frac{1}{2}\kappa^{-1}\nabla^2\tilde{\varphi} - \frac{1}{2}\kappa\tilde{\varphi} &= \frac{1}{2}\tilde{\varphi}[\nabla^2\eta + \kappa(\nabla\eta)^2] \\
\tilde{\varphi}_t + g\eta &= -\frac{1}{2}\nabla\cdot\left[\tilde{\varphi}\nabla\tilde{\varphi} - \kappa\tilde{\varphi}^2\nabla\eta\right]
\end{aligned}$$

$$\mathcal{H} = \int_\Omega \left\{\frac{1}{2}g\eta^2 + \frac{1}{4}\kappa^{-1}[\nabla\tilde{\varphi} - \kappa\tilde{\varphi}\nabla\eta]^2 + \frac{1}{4}\kappa\tilde{\varphi}^2\right\} d\vec{x}$$

Comparison with exact Stokes wave

- Cubic Zakharov Equations (CZE) :

$$\begin{aligned}\eta_t - \mathfrak{d}\tilde{\varphi} &= -\nabla\cdot(\eta\nabla\tilde{\varphi}) - \mathfrak{d}(\eta\mathfrak{d}\tilde{\varphi}) + \\ &\quad \tfrac{1}{2}\nabla^2(\eta^2\mathfrak{d}\tilde{\varphi}) + \mathfrak{d}(\eta\mathfrak{d}(\eta\mathfrak{d}\tilde{\varphi})) + \tfrac{1}{2}\mathfrak{d}(\eta^2\nabla^2\tilde{\varphi}), \\ \tilde{\varphi}_t + g\eta &= \tfrac{1}{2}(\mathfrak{d}\tilde{\varphi})^2 - \tfrac{1}{2}(\nabla\tilde{\varphi})^2 - (\eta\mathfrak{d}\tilde{\varphi})\nabla^2\tilde{\varphi} - (\mathfrak{d}\tilde{\varphi})\mathfrak{d}(\eta\mathfrak{d}\tilde{\varphi}).\end{aligned}$$

- Phase speed :

Exact : $g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^4 + \frac{707}{384}\alpha^6 + O(\alpha^8)$

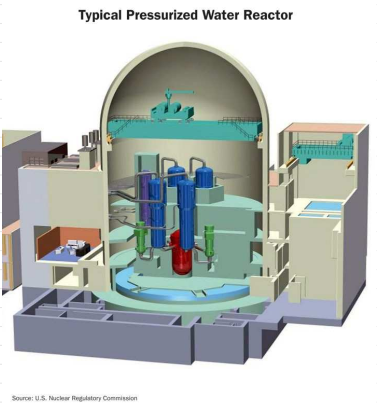
CZE : $g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \frac{1}{2}\alpha^2 + \frac{\mathbf{41}}{\mathbf{64}}\alpha^4 + \frac{\mathbf{913}}{384}\alpha^6 + O(\alpha^8)$

gKG : $g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^4 + \frac{\mathbf{899}}{384}\alpha^6 + O(\alpha^8)$

- n-th Fourier coefficient to the leading order : $\frac{n^{n-2}\alpha^n}{2^{n-1}(n-1)!}$ (the same in gKG & Stokes but not in CZE)

Industrial examples of two-phase flows

- Pressurized Water Reactors
- Liquefied Natural Gas (LNG) carriers



Two-phase flows in nature

Some typical examples

- Wave breaking phenomena
- Powder-snow avalanches

Derivation of two-phase models

Great lines of derivation procedure : Ishii (1975) [Ish75] ; Rovarch (2006) [Rov06]

- Consider governing equations in each phase (+ interface conditions) :

$$\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\vec{u}) = \nabla\cdot\sigma + \rho S, \quad \psi \in \{1, \vec{u}, e + \frac{1}{2}|\vec{u}|^2\}$$

- Introduce the characteristic function of each phase :

$$\chi_k(\vec{x}, t) = \begin{cases} 1, & \vec{x} \in \Omega_k(t) \\ 0, & \text{otherwise} \end{cases} \quad : \quad \frac{\partial\chi_k}{\partial t} + \vec{u}_i\cdot\nabla\chi_k = 0, \; k \in \{+, -\}$$

- Combination of two equations :

$$\frac{\partial(\chi_k\rho\psi)}{\partial t} + \nabla\cdot(\chi_k\rho\psi\vec{u}) = \nabla\cdot(\chi_k\sigma) + \big(\sigma - \rho\psi(\vec{u} - \vec{u}_i)\big)\cdot\vec{n}_k\delta_k + \chi_k\rho S$$

Derivation of the six-equations model

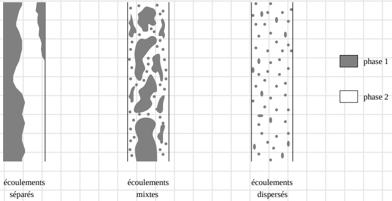
Averaging of governing equations Ishii (1975), [Ish75]

Reynolds axioms :

Linearity : $\overline{\lambda f + \mu g} = \lambda \overline{f} + \mu \overline{g}$

Idempotency : $\overline{\overline{f} g} = \overline{f}\ \overline{g}$

Commutativity : $\overline{\dfrac{\partial f}{\partial t}} = \dfrac{\partial \overline{f}}{\partial t}, \qquad \overline{\dfrac{\partial g}{\partial x}} = \dfrac{\partial \overline{g}}{\partial x}$



Volume fraction definition :

$$\alpha^{\pm} := \overline{\chi_k}, \qquad \alpha^{+}(\vec{x}, t) + \alpha^{-}(\vec{x}, t) = 1$$

- Apply this average operator to governing equations :

$$\frac{\partial \overline{(\chi_k \rho \psi)}}{\partial t} + \nabla \cdot \overline{(\chi_k \rho \psi \vec{u})} = \nabla \cdot (\overline{\chi_k \sigma}) + \overline{\big(\sigma - \rho\psi(\vec{u} - \vec{u}_i)\big) \cdot \vec{n}_k \delta_k} + \overline{\chi_k \rho S}$$

Six-equations model (in simplified form)

Governing equations are to be completed by EOS

- Mass conservation :

$$\frac{\partial(\alpha^{\pm}\rho^{\pm})}{\partial t} + \nabla\cdot(\alpha^{\pm}\rho^{\pm}\vec{u}^{\pm}) = \Gamma^{\pm}$$

- Momentum conservation :

$$\frac{\partial(\alpha^{\pm}\rho^{\pm}\vec{u}^{\pm})}{\partial t} + \nabla\cdot(\alpha^{\pm}\rho^{\pm}\vec{u}^{\pm}\otimes\vec{u}^{\pm}) + \alpha^{\pm}\nabla p = \alpha^{\pm}\rho^{\pm}\vec{g} + \nabla\cdot(\alpha^{\pm}\tau^{\pm}) + M^{\pm}$$

- Energy conservation

$$\frac{\partial(\alpha^{\pm}\rho^{\pm}E^{\pm})}{\partial t} + \nabla\cdot(\alpha^{\pm}\rho^{\pm}H^{\pm}\vec{u}^{\pm}) + p\frac{\partial\alpha^{\pm}}{\partial t} = \alpha^{\pm}\rho^{\pm}\vec{g}\cdot\vec{u}^{\pm} + \nabla\cdot(\alpha^{\pm}\tau^{\pm}\vec{u}^{\pm}) - \nabla\cdot(\alpha^{\pm}\vec{q}^{\pm}) + Q^{\pm}$$

Reduce the number of variables

Y. Meyapin *et al.* (2010), [MDG10]

Method :

- Add relaxation terms into momentum and energy conservation equations :

$$\vec{F}_d := \frac{\kappa}{\varepsilon} \frac{\alpha^+\rho^+\alpha^-\rho^-}{\alpha^+\rho^+ + \alpha^-\rho^-}(\vec{u}^+ - \vec{u}^-), \quad E_d := \vec{F}_d \cdot \bar{u}$$

- Quasilinear form :

$$\mathbb{A}(V_\varepsilon)\frac{\partial V_\varepsilon}{\partial t} + \mathbf{B}(V_\varepsilon)\nabla V_\varepsilon = \nabla \cdot \mathbf{T}(V_\varepsilon) + \mathcal{S}(V_\varepsilon) + \frac{\mathbf{R}(V_\varepsilon)}{\varepsilon}$$

- Take singular limit when $\varepsilon \to 0$ using Chapman-Enskog expansion :

$$V_\varepsilon = V + \varepsilon W + \mathcal{O}(\varepsilon^2)$$

Single velocity two-phase model

Formal justification of four-equations model proposed in F. Dias *et al* (2010), [DDG10]

After computations we get the following system :

$$\partial_t(\alpha^\pm \rho^\pm) + \nabla \cdot (\alpha^\pm \rho^\pm \vec{u}) = 0,$$
$$\partial_t(\rho\vec{u}) + \nabla \cdot (\rho\vec{u} \otimes \vec{u}) + \nabla p = \rho\vec{g} + \nabla \cdot \tau,$$
$$\partial_t(\rho E) + \nabla \cdot (\rho H\vec{u}) = \rho\vec{g} \cdot \vec{u} + \nabla \cdot (\tau\vec{u}) - \nabla \cdot \vec{q}.$$

$$p = p^\pm(\rho^\pm, e^\pm), \quad T = T^\pm(\rho^\pm, e^\pm)$$

$$\rho := \alpha^+\rho^+ + \alpha^-\rho^-, \quad \rho e := \alpha^+\rho^+e^+ + \alpha^-\rho^-e^-$$

$$\tau := \alpha^+\tau^+ + \alpha^-\tau^-, \quad H := E + \frac{p}{\rho}$$

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$$\partial_t(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \frac{1}{\mathrm{Ma}^2}\nabla p = \rho\vec{g} + \nabla\cdot\tau,$$
$$\partial_t(\rho E) + \nabla\cdot(\rho H\vec{u}) = \rho\vec{g}\cdot\vec{u} + \nabla\cdot(\tau\vec{u}) - \nabla\cdot\vec{q}.$$

$$p = p^{\pm}(\rho^{\pm}, e^{\pm}), \quad T = T^{\pm}(\rho^{\pm}, e^{\pm})$$

$$\rho := \alpha^{+}\rho^{+} + \alpha^{-}\rho^{-}, \quad \rho e := \alpha^{+}\rho^{+}e^{+} + \alpha^{-}\rho^{-}e^{-}$$

$$\tau := \alpha^{+}\tau^{+} + \alpha^{-}\tau^{-}, \quad H := E + \frac{p}{\rho}$$

Scaling parameter :

Mach number : $\mathrm{Ma} := \frac{u_0}{c_s^{+}}$ (importance of compressible effects)

Two-fluid incompressible Navier-Stokes equations

Formal low Mach number limit $\mathrm{Ma} \to 0$ of the four-equations model (see [MDG10])

- Mass conservation equation :

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

- Incompressibility condition :

$$\nabla \cdot \vec{u} = 0$$

- Momentum conservation :

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p = \nabla \cdot \big(2\mu \mathbb{D}(\vec{u})\big) + \rho \vec{g}$$

where $\rho := \alpha^+ \rho_0^+ + \alpha^- \rho_0^-, \quad \mu := \alpha^+ \rho_0^+ \nu^+ + \alpha^- \rho_0^- \nu^-$

The volume fraction is simply advected :

$$\partial_t \alpha^\pm + \vec{u} \cdot \nabla \alpha^\pm = 0$$

Two-fluid Navier-Stokes equations with mixing

In view of applications to powder-snow avalanches

Diffuse interfaces with Fick's law (A. Majda, [Maj84]) :

$$\nabla \cdot \vec{u} = -\nabla \cdot (\kappa \nabla \log \rho)$$

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

$$\partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p = \nabla \cdot \big(2\mu \mathbb{D}(\vec{u})\big) + \rho \vec{g}$$

Fluid volume velocity (after H. Brenner, [Bre06]) :

Change of variables : $\vec{v} := \vec{u} + \kappa \nabla \log \rho \quad \Rightarrow \quad \nabla \cdot \vec{v} = 0$

Modified Navier-Stokes equations

A novel model for highly inhomogeneous two-fluid flows (see DD *et al* (2010), [DARB10])

Governing equations :

$$\begin{aligned} \nabla \cdot \vec{v} &= 0 \\ \partial_t \rho + \vec{v} \cdot \nabla \rho &= \kappa \Delta \rho \\ \rho \partial_t \vec{v} + \rho(\vec{v} \cdot \nabla)\vec{v} + \nabla \pi + \kappa\, {}^t\nabla \vec{v}\, \nabla \rho - \kappa \nabla \rho \nabla \vec{v} &= \rho \vec{g} + \nabla \cdot \big(2\mu \mathbb{D}(\vec{v})\big) \end{aligned}$$

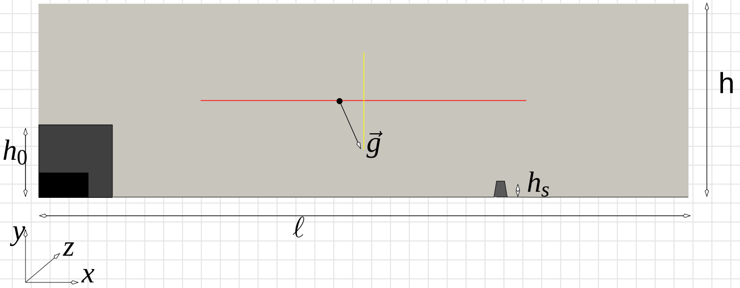
Fick's constant is chosen specifically : $\kappa = 2\bar{\nu}$

This model is energetically consistent :

$$\partial_t \int_\Omega \rho \frac{|\vec{v}|^2}{2}\, d\vec{x} = \int_\Omega \rho \vec{g} \cdot \vec{v}\, d\vec{x} - \int_\Omega 2\mu |\mathbb{D}(\vec{v})|^2\, d\vec{x} - \int_\Omega \kappa \rho |\mathbb{A}(\vec{v})|^2\, d\vec{x}.$$

Sliding mass interaction with obstacle

Some numerical illustrations



parameter	*value*
slope, θ	32°
heavy fluid density, ρ^+, kg/m^3	20
light fluid density, ρ^-, kg/m^3	1
heavy fluid kinematic viscosity, ν^+, m/s^2	4.8×10^{-4}
light fluid kinematic viscosity, ν^-, m/s^2	1.0×10^{-4}

Sliding mass interaction with obstacle

Snapshots of the volume fraction evolution, see [DARB10]

FIGURE: Avalanche at $t = 10\ s$. The color scale ranges from 0 to the maximum value 1.0.

Sliding mass interaction with obstacle

Snapshots of the volume fraction evolution, see [DARB10]

FIGURE: Avalanche at $t = 25\ s$. The color scale ranges from 0 to the maximum value 0.984.

Sliding mass interaction with obstacle

Snapshots of the volume fraction evolution, see [DARB10]

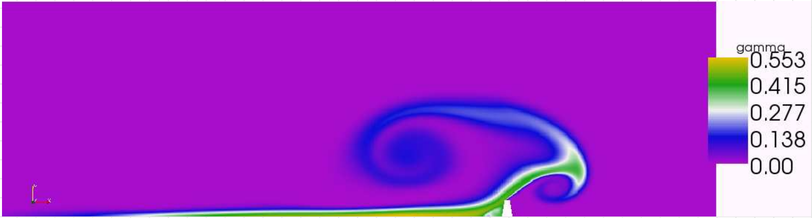


FIGURE: Avalanche at $t = 60\ s$. The color scale ranges from 0 to the maximum value 0.553.

Impact pressures on the obstacle

see DD *et al.* 2010, [DARB10]

Empiric law (Beghin & Closet, 1990, [BC90]) :

$$p_d = \frac{K}{2} K_a(z) \bar{\rho} U_f^2, \quad K_a(z) = \begin{cases} 10, & z < 0.1h, \\ 19 - 90z, & 0.1h \leq z \leq 0.2h, \\ 1, & z > 0.2h, \end{cases}$$

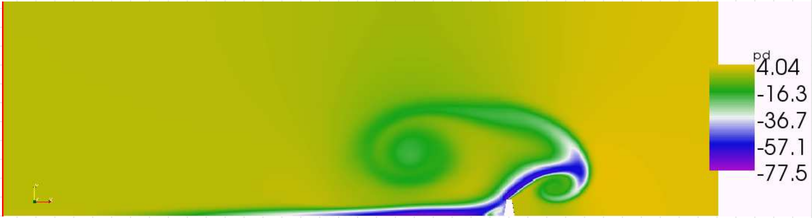


FIGURE: Dynamic pressure at $t = 60$ s.

Conclusions

Water waves :

- We presented a generalized Lagrangian variational principle
- Several novel approximate models are derived
- Analytical solutions are found in some cases

Two-phase flows :

- A hierarchy of two-phase models is presented
- Formal justification of the four-equations model is provided
- A novel model for highly-inhomogeneous incompressible flows is proposed

Recent developments

October – November 2010

Submitted papers :

- D. Dutykh, D. Mitsotakis, L. Chubarov, Yu. Shokin. *Horizontal displacements contribution to tsunami wave energy balance*. Submitted, 2010
`http://hal.archives-ouvertes.fr/hal-00530999/`
- D. Dutykh, Th. Katsaounis, D. Mitsotakis. *Finite volume methods for unidirectional dispersive wave models*. Submitted, 2010
`http://hal.archives-ouvertes.fr/hal-00538043/`

In preparation :

- D. Dutykh, D. Clamond. *Improved nonlinear shallow water equations for varying bathymetries*. In preparation, 2010

NSWE with non-zero vertical acceleration !

Thank you for your attention !

http://www.lama.univ-savoie.fr/~dutykh/

References I

- P. Beghin and J.-F. Closet, *Effet d'une digue sur l'écoulement d'une avalanche poudreuse*, Tech. report, Cemagref, 1990.
- H. Brenner, *Fluid mechanics revisited*, Physica A **370** (2006), 190–224.
- D. Clamond and D. Dutykh, *Practical use of variational principles for modeling water waves*, Submitted `http://arxiv.org/abs/1002.3019/` (2010).
- D. Dutykh, C. Acary-Robert, and D. Bresch, *Mathematical modeling of powder-snow avalanche flows*, Studies in Applied Mathematics **Accepted** (2010).
- F. Dias, D. Dutykh, and J.-M. Ghidaglia, *A two-fluid model for violent aerated flows*, Comput. & Fluids **39(2)** (2010), 283–293.

References II

M. Ishii, *Thermo-fluid dynamic theory of two-phase flow*, Eyrolles, Paris, 1975.

J.C. Luke, *A variational principle for a fluid with a free surface*, J. Fluid Mech. **27** (1967), 375–397.

A. Majda, *Compressible fluid flow and systems of conservation laws in several space dimensions*, Appl. Math. Sci. **53** (1984).

Y. Meyapin, D. Dutykh, and M. Gisclon, *Velocity and energy relaxation in two-phase flows*, Stud. Appl. Maths. **125(2)** (2010), 179–212.

A.A. Petrov, *Variational statement of the problem of liquid motion in a container of finite dimensions*, PMM **28(4)** (1964), 917–922.

References III

- J.-M. Rovarch, *Solveurs tridimensionnels pour les écoulements diphasiques avec transferts d'énergie*, Ph.D. thesis, Ecole Normale Supérieure de Cachan, 2006.
- V.E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys. **9** (1968), 1990–1994.