The nature of Bose-Einstein condensation enhanced by localisation

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La nature de la condensation de Bose-Einstein induite par la localisation

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Thèse présentée en vue d'obtenir le grade de

Docteur de l'Université de la Méditerranée

Spécialité: **Physique Théorique et Physique Mathématique**

Ecole doctorale: **Physique et Sciences de la Matière (ED 532)**

dans le cadre d'une co-tutelle de thèse avec University College Dublin

Soutenue le 6 Septembre 2010 devant le jury composé de Messieurs

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Résumé

Nous étudions la transition de phase survenant dans le gaz de Bose pour des systèmes *sans* invariance par translation. Bien qu'il soit prouvé depuis les années 60 que la condensation de Bose Einstein (CBE) est absente des systèmes invariants par translation en dimension 1 ou 2, on peut néanmoins déclencher cette transition de phase dans des gaz de Bose en faible dimension en ajoutant un potentiel externe approprié (et par conséquent, en perdant l'invariance par translation). Cependant, le condensat ainsi obtenu se trouve dans des états *localisés*, alors que la CBE est généralement comprise comme l'occupation macroscopique d'états cinétiques *étendus*. Il n'est pas à priori évident que cette transition de phase obtenue grâce à la localisation est de la même nature que celle reliée au concept habituel de CBE.

Dans cette thèse, nous considérons deux classes de systèmes localisés. La première est une famille de modèles *aléatoires*, pour lesquels le gaz de Bose est contenu dans un milieu désordonné, ce que nous modélisons par un potentiel externe aléatoire. La deuxième est constituée de modèles incluant un potentiel externe *faible* (d'échelle). Nous commençons par un rappel des conditions nécessaires sur ces potentiels pour obtenir une condensation dans les états localisés.

Nous montrons sous certaines hypothèses très générales que dans ces modèles, la CBE au sens habituel est aussi présente, dans un sens *généralisé*. Cela signifie que les particules sont condensées dans des états cinétiques ayant une énergie arbitrairement faible. Pour le gaz de Bose sans interactions, nous pouvons en plus prouver que les densités des deux condensats sont en fait égales.

Nous approfondissons ensuite notre étude de la CBE, en demandant si il est possible d'obtenir une condensation sur un *seul* état cinétique. Nous montrons qu'en dépit de l'existence à la fois d'une transition de phase et de la CBE généralisée, aucune condensation ne survient sur un seul état cinétique. En particulier, la fameuse *condensation sur l'état fondamental* est absente pour ces modèles localisés.

Finalement, nous établissons une généralisation possible de l'approximation de nombres complexes de Bogoliubov pour prendre en compte les propriétés très particulières de la CBE en présence de localisation, et nous discutons la façon d'interpréter le resultat du problème variationnel correspondant.

Summary

We investigate the phase transition exhibited by the Bose gas in systems which are *not* translation-invariant. Though it has been known since the sixties that Bose-Einstein condensation (BEC) cannot occur in translation invariant systems for dimension 1 or 2, one can nevertheless enhance this phase transition in low-dimensional Bose gases by the addition of suitable external potentials (thus losing translation invariance in the process). However, the resulting condensate is then found to be in *localised* states, while BEC is usually understood to be the macroscopic occupation of *extended* kinetic eigenstates. It is therefore not clear whether the phase transition obtained by means of localisation is of the same nature as the one related to the usual concept of BEC.

In this thesis, we consider two classes of localised systems. The first one is a family of *random* systems, where the Bose gas is contained in a disordered medium, which is modelled by a random external potential. Our second model consists of *weak* (scaled) external potentials. We first recall necessary conditions on these external potentials to enhance condensation in the localised states.

We then show under very general assumptions that in these models, BEC in the usual sense occurs also, in a *generalised* sense. This means that the particles condense on kinetic eigenstates with arbitrary small energy. For the non-interacting Bose gas, we can moreover show that the densities of both condensates are actually equal.

Next, we investigate BEC on a finer scale, asking whether one can obtain condensation in a *single* kinetic eigenstate. We show that in spite of the existence of a phase transition, and the occurence of generalised BEC, no condensation exists in any single kinetic eigenstate. In particular, the so-called "ground-state condensation" does *not* occur in these localised systems.

Finally, we establish a possible generalisation of the Bogoliubov c-number approximation to take into account the very specific properties of BEC in the presence of localisation, and discuss how to interpret the result of the corresponding variational problem.

Acknowledgments

I first want to express my gratitude to my two supervisors, Joseph Pulé and Valentin Zagrebnov, for their constant support and having given me the opportunity to pursue my doctoral studies in the first place. It has been a rewarding experience for me to work under their direction, and whatever may happen next, I consider myself lucky to have known them.

I also wish to thank the two reviewers of my thesis, Teunis Dorlas and Jean-Bernard Bru, for their thorough reading of my manuscript and very helpful remarks and comments.

I am grateful to all the administrative staff of the School of Mathematical Sciences for their help. In particular, I am indebted to Veronica Barker for her kind assistance at my arrival and settling in Ireland, and to Nuria García Ordiales for her help in resolving many administrative issues.

Finally, I want to thank my fellow PhD students for the friendly atmosphere that reigned in our office during my time there. I would like to say a special thanks to Gerard Boland, for having been a good comrade throughout my stay in Ireland.

I acknowledge the financial support of an UCD Ad Astra Research Scholarship.

Introduction

The first prediction of the phenomenon of Bose Einstein Condensation (BEC) goes back to an article by Einstein in 1925, [1]. In this seminal paper, he adopted a new formalism of the quantisation of photons, suggested by Bose, to treat the case of a gas of massive particles. He then predicted that there exists a critical temperature, below which a fraction of the particles condenses on the quantum ground state, essentially merging into one frozen macroscopic object, not contributing to either the total entropy or the pressure of the gas. At zero temperature, *all* the particles should fall into what is now called Bose Einstein condensate.
The class of particles having this behaviour later became known as “bosons”, as opposed to “fermions”. In view of the Pauli exclusion principle, which states that two fermions cannot be in the same quantum state, it is clear why only bosons can condense in this way. The difference between these two class of particles has to do with the value of their spin, integer for bosons and half integer for fermions. Since most elementary particles constituting matter are fermions (electrons, protons neutrons,...), it is easy to understand how features which are bosonic in nature attracted little interest in the physics community.

However, criticism of that new theory emerged soon afterward, in a paper by Uhlenbeck, [2]. This was centered on the fact that a gas formed by any finite number of particles cannot exhibit a phase transition, hence casting doubt about the new state of matter predicted by Einstein. Indeed, this opinion was correct, as Einstein failed to mention that the singularities in the thermodynamic functions of the Bose gas appear *only* in the so-called thermodynamic limit, that is the simultaneous limit of infinite volume and infinite number of particles, in such a way that the density remains constant.

No breakthrough occured until 1938, when London proposed that the then forgotten concept of BEC could help to explain some experimental feature discovered in liquid Helium. This has the surprising property of staying liquid under atmospheric pressure even at zero temperature. In addition, below a critical temperature of 2.18 K, the so-called lambda point, a range of striking features appears: the liquid becomes *superfluid*, meaning that it can flow without any viscosity, practically defeating any attempt of containment, and moreover, the heat is conducted through it at sound speed, instead of the diffusion process exhibited by any ordinary material. In his paper [3], London was the first to conjecture a link between superfluidity and Bose Einstein condensation. This was motivated in the first place by the fact that the molecules of Helium 4 are indeed bosons, but the most interesting result in that paper was that, if one considers the liquid Helium as a gas of non interacting bosons, the critical temperature turns out to be 3.13 K, which is quite close to the experimental value of the lambda point. Since then, it has been generally accepted in the physics literature that superfluidity is strongly related to the occurrence of Bose Einstein condensation.

This approach however had one weakness, which was that it did not take into account the interactions between particles. This can be seen as a reasonable assumption for dilute gases, as the particles are far away from each other, but in a liquid like Helium at those temperatures, the interactions between particles should be fairly strong, and hence, it was not clear what impact they would have on the condensation. This has turned out to be a very difficult question, and even today, a rigorous description of a bosonic system with realistic interactions seems still out of reach from the point of view of mathematical physics.

Before continuing, we shall introduce some notation. The one-particle kinetic energy operator is defined as usual as $-\frac{1}{2}\Delta$ on an open set $\Lambda_l \subset \mathbb{R}^d$, with appropriate boundary conditions. We shall denote by $\{\psi_k^l, \varepsilon_k^l\}, k \geqslant 1$ its eigenfunctions and corresponding eigenvalues. For convenience, let us order the eigenvalues in such a way that $\varepsilon_1^l \leqslant \varepsilon_2^l \leqslant \dots$. In the case of periodic boundary conditions, these eigenstates are the so-called momentum states, and it is for this reason that the phenomenon of BEC is frequently referred as condensation in the momentum space (as opposed to position space). Indeed, the particles in the condensate are in the quantum state ψ_1^l (ground state), which is constant in the periodic case. Hence, the condensate is completely delocalised in position space, and completely localised in the momentum space.

Now, we are in position to introduce the concept of *generalised* BEC. This was first proposed by Girardeau in 1960, see [4], where he studied a model of bosons with hard-core interactions in one dimension. He claimed that no condensation could occur in the ground state in that model, but he noticed that one still had condensation in an arbitrary narrow band of energy above the ground state. We call this phenomenon generalised BEC, that is with the notation that we shall use in this thesis,

$$\lim_{\delta \downarrow 0} \lim_{l \to \infty} \frac{1}{|\Lambda_l|} \sum_{k : \varepsilon_k^l \leqslant \delta} \langle N_l(\psi_k^l) \rangle_l \; > \; 0 \; ,$$

where $\langle N_l(\psi_k^l) \rangle_l$ is the mean value (at thermodynamic equilibrium) of the density of particles in the state ψ_k^l, and $|\Lambda_l|$ denotes the volume of the box Λ_l. It is clear that, if there is BEC in the ground state, that is if

$$\lim_{l \to \infty} \frac{1}{|\Lambda_l|} \langle N_l(\psi_1^l) \rangle_l \; > \; 0 \; ,$$

then there is also generalised BEC, but the other way is less trivial. The concept of generalised BEC was further developed by the Dublin School in the eighties, see e.g. [5]. They emphasised in particular that the occurrence and density of generalised BEC is a very stable phenomena, being determined only by thermodynamic properties of the system, namely the asymptotic behaviour of the density of states. On the other hand, the actual condensation in the ground state as predicted by Einstein turns out to be a very sensitive feature, depending strongly on subtle spectral properties, in particular how fast the gap between two eigenvalues vanishes in limit. Exploiting these properties, it was shown that in many systems, we can obtain generalised BEC, even though it is possible to "shift" part of the condensate outside of the ground state. In some cases, it is even possible to obtain generalised BEC *without* macroscopic occupation of the ground state. This led to the classification of generalised BEC into three types:

Type I : only a finite number of eigenstates are macroscopically occupied

Type II : an infinite number of eigenstates are macroscopically occupied

Type III : *no* eigenstates are macroscopically occupied

Note that in all three cases, the amount of *generalised* BEC is the same. Indeed, it is easy to construct models which are thermodynamically equivalent, in the sense that the pressure, density of free energy, etc ... are identical, which all exhibit a phase transition, with the same density of generalised condensate, but of different types.

One way to obtain such a situation is by considering anisotropic non interacting Bose gases, where we let the boxes Λ_l to be prisms of various shape (instead of cubes). We refer the reader to [6] where a classification of the types of generalised BEC is derived in terms of the geometry of the boxes. However, these models have in common with the usual isotropic Bose gas that the phase transition associated with Bose Einstein condensation does not occur in low dimensional systems. Indeed, it has been known since [7] that for translation invariant systems in dimensions 1 or 2, spontaneous gauge symmetry breaking, which is generaly associated with the occurrence of BEC, does not occur. This result is still true upon the introduction of a superstable interaction between particles.

On the other hand, it is known that one can trigger Bose Einstein condensation in low-dimensional systems by the addition of appropriate external potentials. In this thesis, we are concerned with two classes of these potentials, *random* potentials and *weak* (scaled) potentials. By defining the Schrödinger operator as the addition of the kinetic part and these external potentials, one can change the asymptotic behaviour of the density of states, making it vanish faster at the bottom of the spectrum than its counterpart in the *free* Bose Gas (without external potential). This is responsible for the occurrence of generalised BEC even in one- or two-dimensional systems.

In the weak potential case, the Schrödinger operator is of the form

$$-\tfrac{1}{2}\Delta + v(x/l) \tag{1}$$

where l is the length of the side of the cubic box Λ_l, and v is a non negative function defined on a unit cube. If v is chosen in a suitable way, see e.g. [8], it is known that

the *perfect* Bose gas defined by that Schrödinger operator exhibits a phase transition associated with the occurrence of generalised BEC even in low dimension. The exact expression of the density of states in terms of v has been derived in [8].
In the random case, we consider a Schrödinger operator of the form:

$$-\tfrac{1}{2}\Delta + v^{\omega}(x), \tag{2}$$

where ω denotes a particular realisation of a random field on some probability space. One may think of the random potential as a model of impurities, either distributed randomly or of random strength. The first case would correspond to light particles moving in an amorphous medium, the second to a crystal where atoms are of different species. It was pointed out in [9] that generalised BEC could be triggered in any dimension, due to the so-called *Lifshitz tails*. This behaviour, which is a fairly general feature of random Schrödinger operators, see e.g. [10], essentialy means that there are exponentially few eigenstates with energy near the bottom of spectrum. This idea was first proposed by Lifshitz himself, see [11], who noticed that, for an eigenvalue to vanish logarithmically in the thermodynamic limit, the random potential should be identically zero on a region of typical size $\ln l$, and the probability of such an "empty valley" turns out to be exponentially small.

In both the random and weak cases, the phase transition triggered by the change in the density of states has to be understood as generalised BEC. However, if one wants to obtain more knowledge about the occupation of single one-particle eigenstates, more work is required. This has been done to a large extent in the weak case by Van den Berg and Lewis, see [12], where conditions on the external potentials have been established to distinguish between the three possible types of condensation. In the random case, far less is known. To our knowledge, the exact type of generalised BEC has been determined in only one particular case, the Luttinger-Sy model, see [13].

In view of these condensates obtained in low dimensional systems by means of external potentials, one question arises. As we emphasised before, is has been rigorously established that translation invariant systems do not exhibit BEC in dimension 1 or 2. And indeed, by adding an external potential, one breaks that translation invariance. Therefore, it is not clear whether the phase transition obtained is related to Bose Einstein condensation. To see this, notice that the generalised BEC obtained in bosonic systems defined by either Schrödinger operator (1) or (2) is to be found in the eigenstates ϕ_i^l of these operators, which are clearly not the same as the kinetic eigenfunctions ψ_k^l.

It is generally believed that the fast decay of the density of states is associated with the corresponding eigenfunctions ϕ_i^l becoming *localised*. Hence, since BEC is usually associated with the macroscopic occupation of the plane waves ψ_k^l, which are spatially extended, the question of whether or not the condensation in localised eigenstates is connected with the usual BEC phenomenon arises naturally. In particular, if we consider these systems without translation invariance, can we nevertheless get some information about the occupation of the kinetic states?

This last question may be understood either from the point of view of generalised BEC or from the occupation of the kinetic ground state itself. Note that the latter is not any more the ground state in non translation invariant systems. In this thesis, we shall answer both questions.

Our first result states that, under very general assumptions on the external potentials, and for a class of Bose gas with diagonal interaction between the particles, generalised BEC in localised eigenstates occurs if and only if the same happens in the kinetic states. In the perfect Bose gas, our results are stronger, in the sense that we show that both condensate densities are actually equal.
While this may not be so surprising in the weak potential case, since the system is asymptotically translation invariant, it is less intuitive in the random case, as these systems are translation invariant in the limit only in expectation with respect to the randomness, and not for any given realisation.

Furthermore, we prove, for a class of weak potentials and a model of continuous random models that, while it is true that generalised BEC occurs in the kinetic eigenstates, it is impossible to obtain condensation in any single kinetic state. In the Dublin school classification, this means that the kinetic generalised BEC in these localised models is of type III, independently of the type of condensation in the localised states. This leads to some comments about the meaning of BEC in systems without translation invariance. The first one supports the claim that the so-called 0-mode condensation is too restrictive as a description of Bose Einstein condensation. Indeed, these condensates in low dimensional systems do exhibit a phase transition and a non-zero density of kinetic generalised BEC, but without condensation on any kinetic mode. But moreover, our proof of type III kinetic generalised BEC in these localised systems does hold in any dimension, and for arbitrary small level of randomness or arbitrary small weak potential. In particular, this means that, while a translation invariant system in dimension 3 may produce condensation in the ground state only, this cannot happen in presence of disorder or weak confinement, however small.

This then suggests that we should revisit the Bogoliubov theory, see e.g. [15]. This theory introduced by Bogoliubov in 1947 was an attempt to take into account the interactions between particles, based on the assumption that the condensate would be concentrated on the kinetic ground state. The first Ansatz was that one can then neglect the interactions between particles outside of the condensate (i.e., the ground state), and hence truncate the full interaction term by keeping only the terms with Feynman's diagrams corresponding to scattering which either make one particle leave the condensate or fall in it. The second Ansatz was to approximate the corresponding operators by complex numbers, hence making the Hamiltonian diagonalisable. The appropriate value of this complex number has then to be determined by a variational problem, with different solutions corresponding to different densities of ground state condensate. The "effective" spectrum which was derived in this way was the first to satisfy the Landau criteria of superfluidity, see [15], assuming that Bose Einstein condensation does persist even in presence of the interactions. In that model, superfluidity is therefore a consequence of BEC, giving some support to London's conjecture.

Now, in view of the likelihood of obtaining generalised BEC of type III in localised systems, the validity of the first Ansatz is put in doubt. As a first step into a generalised Bogoliubov theory, we establish the variational problem corresponding to the substitution of complex numbers for *all* eigenstates involved in the generalised condensate (that is, infinitely many). This approach has the advantage of not requiring any additional knowledge apart from the existence of generalised condensate.

Chapter 1

BEC in the eigenstates

The object of this chapter is to review the existing methods used to prove generalised Bose-Einstein condensation in the eigenstates.

We shall first introduce our models and briefly recall some standard settings of quantum statistical mechanics. Next, we turn to the basic thermodynamic quantities, in particular the pressure and mean density of particles, and show that these models exhibits a phase transition, which is due to the existence of a bounded *critical density. This is associated with the occurence of* generalised *BEC, that is the macroscopic occupation of an infinitesimal band of energy above the ground state.*

We then provide two families of external potentials, either random *or* weak *(scaled), and give technical conditions on them for the critical density to be finite, and hence enhance generalised BEC in the eigenstates.*

Finally, we briefly review what can be said about the macroscopic occupation of single eigenstates in these models.

1.1 Definitions and notation

Let $\{\Lambda_l := (-l/2, l/2)^d\}_{l\geqslant 1}$ be a sequence of hypercubes of side l in $\mathbb{R}^d$, centered at the origin of coordinates with volumes $V_l = l^d$. We consider a system of identical bosons, of mass m, contained in Λ_l. For simplicity, we use a system of units such that $\hbar = m = 1$. First we define the self-adjoint one-particle kinetic-energy operator of our system by

$$h_l^0 := -\tfrac{1}{2}\Delta_D, \tag{1.1}$$

acting in the Hilbert space $\mathscr{H}_l := L^2(\Lambda_l)$, where Δ is the usual Laplacian. The subscript D stands for *Dirichlet* boundary conditions. We denote by $\{\psi_k^l, \varepsilon_k^l\}_{k\geqslant 1}$ the set of normalised eigenfunctions and eigenvalues corresponding to h_l^0. By convention, we order the eigenvalues (counting the multiplicity) as $0 < \varepsilon_1^l \leqslant \varepsilon_2^l \leqslant \varepsilon_3^l \ldots$. Note that, since they are normalised sine waves, all kinetic states satisfy the following bound

$$|\psi_k^l(x)| \leqslant V_l^{-1/2} \tag{1.2}$$

for all $k \geqslant 1$ and all $x \in \Lambda_l$.

Next we define the Hamiltonian with an external potential

$$h_l := h_l^0 + v_l, \tag{1.3}$$

also acting in $\mathscr{H}_l$, where the potential $v_l : \Lambda_l \mapsto [0, \infty)$ is positive and bounded. We denote by $\{\phi_i^l, E_i^l\}_{i\geqslant 1}$ the set of normalised eigenfunctions and corresponding eigenvalues of h_l. Again, we order the eigenvalues (counting the multiplicity) so that $E_1^l \leqslant E_2^l \leqslant E_3^l \ldots$. Note that the *non-negativity* of the potential implies that $E_1^l > 0$.

Next, let us define the densities of states, the measures whose distributions are the integrated densities of states (IDS). For the kinetic energy operator (1.1), we use the following notation

$$\begin{aligned} \nu_l^0(A) &:= \frac{1}{V_l}\sharp\{k : \varepsilon_k^l \in A\}, \text{ for all Borel subsets } A \subset \mathbb{R} \ , \\ \nu^0(A) &:= \lim_{l\to\infty} \nu_l^0(A), \end{aligned} \tag{1.4}$$

where the limit is understood in the weak sense. Similarly for the Schrödinger operator (1.3), we let

$$\begin{aligned} \nu_l(A) &:= \frac{1}{V_l}\sharp\{k : E_i^l \in A\}, \text{ for all Borel subsets } A \subset \mathbb{R} \ , \\ \nu(A) &:= \lim_{l\to\infty} \nu_l(A) \ . \end{aligned} \tag{1.5}$$

We shall assume in the present section that the limiting measure ν exists. Later, we shall give sufficient conditions on the external potential v_l for this to be valid.

It follows from Weyl's theorem that the density of states ν^0 of the kinetic energy operator has the following form

$$\nu^0\big([0,\varepsilon]\big) \ = \ \frac{2}{d}\, C_d \ \varepsilon^{d/2}, \tag{1.6}$$

with the constant $C_d = \big((2\pi)^{d/2}\Gamma(d/2)\big)^{-1}$.

Clearly, the form for the density of state ν in general depends on the external potential v. We shall only consider external potential for which the density of states ν has the same support as the free density of states ν^0. The non-negativity of v implies that

$$\nu(A) \ = \ 0 \ , \qquad \text{for all Borel subsets } A \subset (-\infty, 0) \ ,$$

and we shall require in addition that

$$\nu\big([0,E]\big) \ > \ 0 \ , \qquad \text{for all} \quad E > 0 \ . \tag{1.7}$$

For the specific models considered in this thesis, we shall prove that this last assumption is satisfied, see Section 1.3.

Now, we turn to the many-body problem. The n-particles space $\mathscr{H}_l^{(n)}$ for bosons is defined as follows

$$\mathscr{H}_l^{(n)} \ := \ \{\psi \in \underbrace{\mathscr{H}\otimes\mathscr{H}\otimes\cdots\otimes\mathscr{H}}_{n \text{ times}} : \ \sigma_+^{(n)}\psi = \psi\} \ ,$$

where $\sigma_+^{(n)}$ is the symmetrisation operator

$$\sigma_+^{(n)}\big(\psi_1\otimes\psi_2\otimes\cdots\otimes\psi_n\big) := \frac{1}{n!}\sum_{\pi\in S_n} \psi_{\pi(1)}\otimes\psi_{\pi(2)}\otimes\cdots\otimes\psi_{\pi(n)} \ .$$

By convention, $\mathscr{H}_l^{(0)} = \mathbb{C}$ ("zero particles space").

In this thesis, we are interested in the grand canonical ensemble, where the total number of particles is not fixed. Hence, we need an appropriate Hilbert space. Let $\mathscr{F}_l = \mathscr{F}_l(\mathscr{H}_l)$ be the symmetric Fock space constructed over $\mathscr{H}_l$, that is

$$\mathscr{F}_l := \bigoplus_{n=0}^{\infty} \mathscr{H}_l^{(n)} \; . \tag{1.8}$$

We are now ready to introduce the *second quantisation* $\mathrm{d}\Gamma$. For a self-adjoint single-particle operator $A : \mathscr{D}(A) \to \mathscr{H}_l$, the map

$$\begin{aligned} \mathrm{d}\Gamma(A) : \mathscr{D}(\mathrm{d}\Gamma(A)) &\to \mathscr{F}_l \\ \big(\phi_0, \phi_1, \phi_2, \dots\big) &\mapsto \big(A^{(0)}\phi_0, A^{(1)}\phi_1, A^{(2)}\phi_2, \dots\big) \; , \end{aligned}$$

where

$$\begin{aligned} A^{(n)} \;:=\; & A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes A \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes A \\ & A^{(0)} := 0 \text{ and } A^{(1)} := A \qquad\qquad\qquad\qquad , \end{aligned}$$

is essentially self-adjoint on the set

$$\Big\{ (\phi_0, \phi_1, \dots) \in \mathscr{F}_l : \; \phi_n \in \bigotimes_{m=1}^{n} \mathscr{D}(A) \,\text{and}\, \phi_n = 0, \,\text{for}\, n \,\text{large enough} \Big\} \; .$$

The second quantisation $\mathrm{d}\Gamma(A)$ is then defined as the closure of this map.
The operators for the total number of particles, N_l, and the operator for the number of particles in a state $\varphi \in \mathscr{H}_l$, $N_l(\varphi)$, are defined as follows

$$\begin{aligned} N_l \;&:=\; \mathrm{d}\Gamma(\mathbf{1}_{\mathscr{H}}) \; , \\ N_l(\varphi) \;&:=\; \mathrm{d}\Gamma(P_\varphi) \; , \end{aligned}$$

where P_φ is the orthogonal projection onto the state φ. In terms of the creation and annihilation operators (satisfying the boson *Canonical Commutation Relations*) $a^*(\varphi), a(\varphi)$ in the state φ, one has $N_l(\varphi) = a^*(\varphi)a(\varphi)$. In particular, for any orthonormal basis $\{\varphi_i\}, i \geqslant 1$, it follows from the linearity of the second quantisation $\mathrm{d}\Gamma$ that the operator for the total number of particles N_l can be expanded as

$$N_l \;=\; \sum_{i \geqslant 1} N_l(\varphi_i) \; .$$

We denote by $H_l := \mathrm{d}\Gamma(h_l)$ the second quantisation of the *one-particle* Schrödinger operator h_l in $\mathscr{F}_l$. Note that since the set $\{\phi_i^l\}_{i \geqslant 1}$ is an orthonormal basis of $\mathscr{H}$, it follows from the spectral representation that the operator H_l acting in $\mathscr{F}_l$ has the form

$$H_l = \sum_{j\geqslant 1} E_j^l \, \mathrm{d}\Gamma(P_{\phi_i^l}) \; = \sum_{i\geqslant 1} E_i^l \, N_l(\phi_i^l),$$

Then, the grand-canonical Hamiltonian of the perfect Bose gas in an external potential is given by

$$H_l^0(\mu) \;:=\; H_l - \mu N_l \;=\; \sum_{i\geqslant 1}(E_i^l - \mu)\; N_l(\phi_i^l) \tag{1.9}$$

where μ is the chemical potential. As usual, this thermodynamic parameter will allow us to control the mean density of our models.

In addition to the perfect gas, we shall also consider the *mean field* Bose gas, which is defined by the following Hamiltonian

$$H_l^\lambda(\mu) \;:=\; H_l^0(\mu) \;+\; \frac{\lambda}{2V_l}\, N_l^2, \tag{1.10}$$

where λ is a non-negative parameter.

The thermodynamic equilibrium Gibbs state $\langle - \rangle_l$ associated with the Hamiltonian $H_l^\lambda(\mu)$ is given by

$$\langle A \rangle_l^\lambda(\beta,\mu) := \frac{1}{\Xi_l^\lambda(\beta,\mu)} \mathrm{Tr}_{\mathscr{F}_l}\{\exp(-\beta H_l^\lambda(\mu))A\} \;,$$

and the pressure is defined by

$$p_l^\lambda(\beta,\mu) \;:=\; \frac{1}{\beta V_l} \ln \Xi_l^\lambda(\beta,\mu) \;,$$

where

$$\Xi_l^\lambda(\beta,\mu) \;:=\; \mathrm{Tr}_{\mathscr{F}_l} \exp(-\beta H_l^\lambda(\mu))$$

is the corresponding partition function. The parameter β is the inverse temperature. In the rest of this thesis, we shall work at fixed, non-zero temperature, and thus, we will always omit the explicit dependence on β. For simplicity, we shall sometimes omit also the explicit mention of the dependence on μ when no confusion arises.

1.2 The pressure and mean density in the thermodynamic limit: the phase transition associated with generalised Bose-Einstein condensation

1.2.1 The perfect Bose gas

In this section, we consider a Bose gas with Hamiltonian (1.9). It is well known that, for the pressure of the perfect Bose gas to exist, the chemical potential μ must satisify the *stability condition*

$$\mu \, < \, E_1^l \; , \tag{1.11}$$

and in the stability regime, the pressure of the perfect Bose gas can be computed explicitly

$$p_l^0(\mu) \, = \, -\int_{[0,\infty)} \ln(1 - e^{-\beta(E-\mu)})\nu_l(\mathrm{d}E) \; . \tag{1.12}$$

Hence, the density of the perfect gas can also be derived exactly

$$\rho_l^0(\mu) \, := \, \partial_\mu \, p_l^0(\mu) \, = \, \int_{[0,\infty)} (e^{\beta(E-\mu)} - 1)^{-1}\nu_l(\mathrm{d}E) \; .$$

Let us introduce the sequence of occupation measure m_l^0 in the eigenstates

$$m_l^0(A) \, := \, \frac{1}{V_l} \sum_{i:E_i^l \in A} \langle N_l(\phi_i^l)\rangle_l^0(\beta,\mu) \;\text{ for all Borel subsets } A \subset [0,\infty) \; , \tag{1.13}$$

and since the mean occupations numbers $\langle N_l(\phi_i^l)\rangle_l^\lambda$ can be computed explicitly in the perfect Bose gas, one obtains the following expression

$$m_l^0(A) \, = \, \int_A (e^{\beta(E-\mu)} - 1)^{-1}\nu_l(\mathrm{d}E) \; . \tag{1.14}$$

Note that $m_l^0\big([0,\infty)\big)$ coincides by definition with the mean density $\rho_l^0(\mu)$.

Let us now introduce the *thermodynamic limit* (TL), which is the limit of infinite volume (that is, $l \to \infty$) while the density of particles remains constant. In the rest of this thesis, $\overline{\rho}$ will always denote the fixed density, which means that we require the following equation to hold for all l

$$\overline{\rho} \, = \, \rho_l^0(\mu) \; . \tag{1.15}$$

Since the occupation measures depend on the chemical potential, see (1.13), this is a condition on the chemical potential μ and one can check that, in finite volume, there always exists a solution $\mu_l = \mu_l(\overline{\rho}) < E_1^l$. This can be seen from the fact that the finite volume mean density $\rho_l^0(\mu)$ diverges when $\mu \to E_1^l$, hence allowing arbitrarily large $\overline{\rho}$. Thus, one can get an implicit expression for the finite volume pressure (1.12) as a function of the density instead of the chemical potential

$$p_l^0(\overline{\rho}) := p_l^0(\mu_l(\overline{\rho})) = -\int_{[0,\infty)} \ln(1 - e^{-\beta(E-\mu_l(\overline{\rho}))})\nu_l(\mathrm{d}E) .$$

Since $\mu_l = \mu_l(\overline{\rho}) < E_1^l$ for any $\overline{\rho} < \infty$, the finite volume pressure $p_l^0(\overline{\rho})$ is well defined. In particular, there is no phase transition, as $p_l^0(\overline{\rho})$ is continuously differentiable with respect to $\overline{\rho}$ for any finite l.

However, to study the thermodynamic limit, we must first determine whether the *critical density*

$$\rho_c := \lim_{\mu \uparrow 0} \int_{[0,\infty)} (e^{\beta(E-\mu)} - 1)^{-1}\nu(\mathrm{d}E)$$

is finite or not. Note that, since it follows from (1.7) that $E_1^l \to 0$ as $l \to \infty$, the stability condition in the thermodynamic limit becomes $\mu \leqslant 0$. If $\rho_c = +\infty$, then one gets the asymptotic behaviour for the chemical potential, see Figure 1.1

$$\lim_{l\to\infty} \mu_l(\overline{\rho}) =: \mu_\infty(\overline{\rho}) < 0.$$

In particular, in view of (1.12), the pressure becomes in the thermodynamic limit

$$p^0(\overline{\rho}) := \lim_{l\to\infty} p_l^0(\mu_l(\overline{\rho})) = -\int_{[0,\infty)} \ln(1 - e^{-\beta(E-\mu_\infty(\overline{\rho}))})\nu(\mathrm{d}E)$$

and again, since $\mu_\infty(\overline{\rho}) < 0$, there is no phase transition. One can obtain an explicit expression for the occupation measure in the thermodynamic limit

$$m^0(A) := \lim_{l\to\infty} m_l^0(A) = \int_A (e^{\beta(E-\mu_\infty(\beta,\overline{\rho}))} - 1)^{-1}\nu(\mathrm{d}E)$$

by uniform convergence. It follows immediately that the measure m is absolutely continuous on $[0, \infty)$.

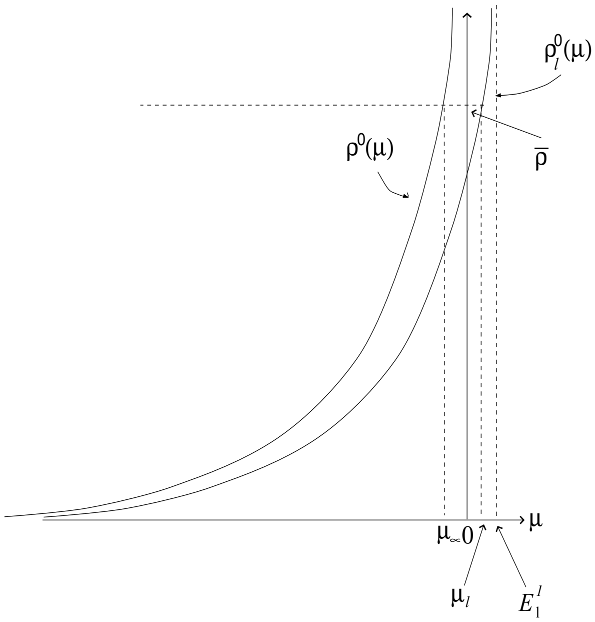

Figure 1.1: Density constraint for unbounded critical density

If the critical density ρ_c is finite, however, one needs a more careful analysis. On the one hand, if one considers the low-density regime, that is $\overline{\rho} < \rho_c$, then one can follow the same procedure as in the case where $\rho_c = +\infty$. Indeed, the density constraint (1.15) has always a unique solution in finite volume, and this remains true in the limit $l \to \infty$, cf. Figure 1.2

$$\lim_{l\to\infty} \mu_l(\overline{\rho}) \ =: \ \mu_\infty(\overline{\rho}) \ < \ 0 \ ,$$

But for the high density regime, that is $\overline{\rho} \geqslant \rho_c$, one can see that

$$\lim_{l\to\infty} \mu_l(\overline{\rho}) \ =: \ \mu_\infty(\overline{\rho}) \ = \ 0 \ ,$$

cf. Figure 1.3, and hence the limiting value of the chemical potential is independent of the density! For a rigorous version of this argument, we refer the reader to [5].

Therefore, we cannot take the limit of the fixed density constraint (1.15) directly, since it would then have *no* solution in μ. Let us rewrite this equation as follows, for some $\delta > 0$

$$\overline{\rho} \ = \ m_l^0\big([0,\delta]\big) \ + \ m_l^0\big((\delta,\infty)\big)$$

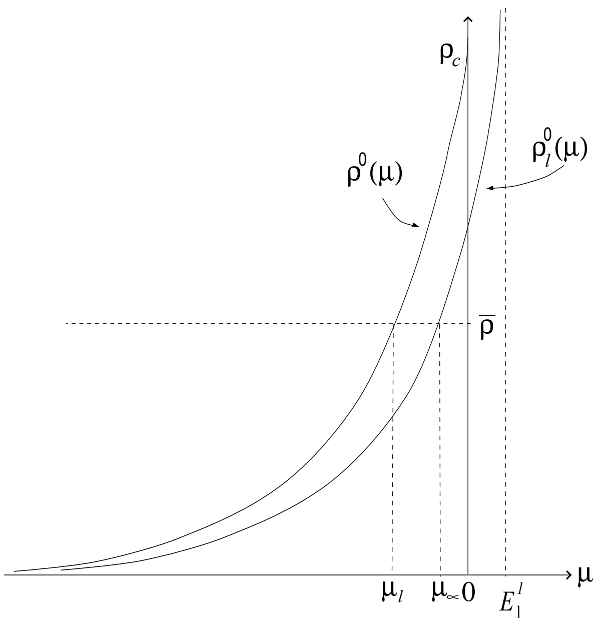

Figure 1.2: Density constraint for bounded critical density in the low density regime

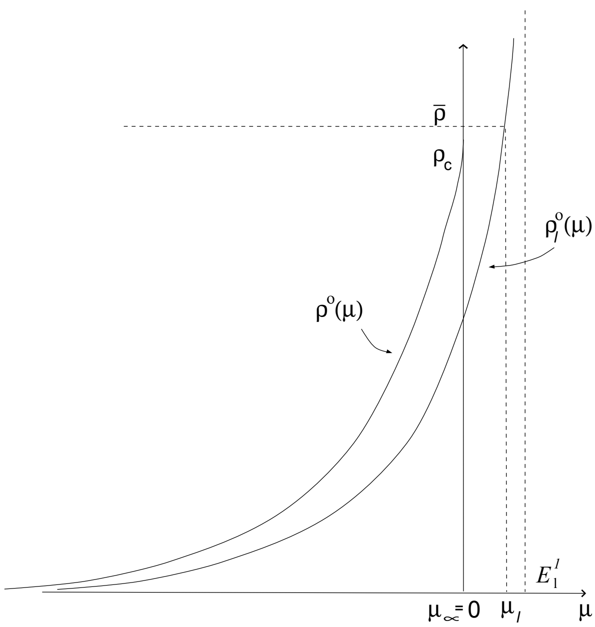

Figure 1.3: Density constraint for bounded critical density in the high density regime

and hence

$$\lim_{l\to\infty} m_l^0\big([0,\delta]\big) \;=\; \bar{\rho} - \lim_{l\to\infty} m_l^0\big((\delta,\infty)\big) \; .$$

As $\delta > 0$, we have

$$\begin{aligned}\lim_{l\to\infty} m_l^0\big((\delta,\infty)\big) &= \lim_{l\to\infty}\int_{(\delta,\infty)} (e^{\beta(E-\mu_l(\overline{\rho}))}-1)^{-1}\,\nu_l(\mathrm{d}E)\\ &= \int_{(\delta,\infty)} (e^{\beta(E-\mu_\infty(\overline{\rho}))}-1)^{-1}\,\nu(\mathrm{d}E)\\ &= \int_{(\delta,\infty)} (e^{\beta E}-1)^{-1}\,\nu(\mathrm{d}E)\ .\end{aligned}$$

Since we have assumed that $\rho_c < \infty$, it follows that

$$\lim_{\delta\downarrow 0}\lim_{l\to\infty} m_l^0\big([0,\delta)\big) = \overline{\rho} - \lim_{\delta\downarrow 0}\lim_{l\to\infty} m_l^0\big([0,\delta)\big) = \overline{\rho}-\rho_c\ ,$$

which means that the limiting occupation measure has an atom at zero-energy: this is generalised Bose-Einstein condensation. Similarly, on can obtain an explicit expression for the pressure in the high density regime, $\overline{\rho} \geqslant \rho_c$,

$$p^0(\overline{\rho}) = \lim_{l\to\infty} p_l^0(\mu_l(\overline{\rho})) = p^0(0)\ .$$

We summarise the results detailled in this section in the following proposition.

Proposition 1.2.1 *Let the critical density be defined by*

$$\rho_c := \int_{[0,\infty)} (e^{\beta E}-1)^{-1}\nu(\mathrm{d}E)\ .$$

The perfect Bose gas defined by the Hamiltonian (1.9) exhibits a phase transition in the thermodynamic limit if and only if $\rho_c < \infty$. In that case, the pressure is constant at large density

$$p^0(\overline{\rho}) = \begin{cases} p^0(\mu_\infty(\overline{\rho})) & \text{if} \quad \overline{\rho} < \rho_c \\ p^0(0) & \text{if} \quad \overline{\rho} \geqslant \rho_c \end{cases}$$

and there is generalised Bose-Einstein condensation, *i.e. the limiting occupation measure has an atom at zero energy*

$$m(\mathrm{d}E) = \begin{cases} (\overline{\rho}-\rho_c)\delta_0(\mathrm{d}E) + (e^{\beta E}-1)^{-1}\,\nu(\mathrm{d}E) & \text{if} \quad \overline{\rho} \geqslant \rho_c \\ (e^{\beta(E-\mu_\infty)}-1)^{-1}\,\nu(\mathrm{d}E) & \text{if} \quad \overline{\rho} < \rho_c\ . \end{cases}$$

1.2.2 The mean field Bose gas

In this section, we consider a simple model of interacting Bose gas, defined by the Hamiltonian (1.10). The main difference between this model and the perfect Bose gas is that the former is *superstable*, which means that the correponding pressure $p_l^\lambda(\beta,\mu)$, for any $\lambda > 0$, is well defined for all real value of the chemical potential μ. This property implies that, for any fixed mean density $\overline{\rho}$, the fixed density equation

$$\overline{\rho} = \rho_l^\lambda(\mu)$$

has not only a unique solution $\mu_l := \mu(\overline{\rho})$ for each l, but the limiting solution $\mu_\infty := \mu_\infty(\overline{\rho})$ is also unique. Hence, it is not as crucial as in the perfect Bose gas, see discussion in the previous section, to control carefully the finite-volume behaviour of μ.

Although it is not possible to compute explicitly the finite volume pressure and density of the mean field gas, there exist many ways of obtaining them in the thermodynamic limit, see e.g. [17], [18], [19], in terms of the the pressure and mean density of the perfect Bose gas. With our notation, this reads as follows.

Proposition 1.2.2 *The pressure of the mean field Bose gas is given in the thermodynamic limit by*

$$p^\lambda(\mu) := \lim_{l\to\infty} p_l^\lambda(\mu) = \begin{cases} \frac{\lambda}{2}\tilde{\rho}(\mu)^2 + p^0(\mu - \lambda\tilde{\rho}(\mu)) & \text{if} \quad \mu < \mu_c \\ \\ \dfrac{\mu^2}{2\lambda} + p^0(0) & \text{if} \quad \mu \geqslant \mu_c \end{cases}$$

and the mean density is

$$\rho^\lambda(\mu) := \partial_\mu p^\lambda(\mu) = \begin{cases} \rho^0(\mu - \lambda\tilde{\rho}(\mu)) & \text{if} \quad \mu < \mu_c \\ \\ \dfrac{\mu}{\lambda} & \text{if} \quad \mu \geqslant \mu_c \end{cases}$$

where $\mu_c = \lambda\rho_c$ *and* $\tilde{\rho}(\mu)$ *is the unique solution of the equation* $\rho = \rho^0(\mu - \lambda\rho)$. *Note that* $\rho^\lambda(\mu_c) = \rho_c$.

The pressure $p^0(\beta,\mu)$, the mean density $\rho^0(\beta,\mu)$ and the critical density ρ_c correspond to the ones established for the perfect Bose gas, see Proposition 1.2.1.

It is already clear that the mean field gas exhibits a phase transition, as the pressure is not twice differentiable at the critical value $\mu = \mu_c$. We now show briefly how this is actually due to generalised BEC.

Let us define a modified mean field Hamiltonian with a shift in a part of the spectrum

$$H_l^\lambda(\mu;\xi) := \sum_{i:E_i^l \leqslant \delta} (E_i^l - \mu) N_l(\phi_i^l) \; + \sum_{i:E_i^l > \delta} (E_i^l + \xi - \mu) N_l(\phi_i^l) \; + \; \frac{\lambda}{2V_l} N_l^2 \; .$$

and we denote by $\langle - \rangle_{l,\xi}$ its associated equlibrium state. Using the same method as in Proposition 1.2.2, one can show that its associated pressure is given by (in the thermodynamic limit)

$$p^\lambda(\mu;\xi) \; := \; \lim_{l\to\infty} p_l^\lambda(\mu;\xi) \; = \; \begin{cases} \frac{\lambda}{2}\tilde{\rho}(\mu;\xi)^2 + p^0(\mu - \lambda\tilde{\rho}(\mu;\xi);\xi) & \text{if } \mu < \mu_c \\ \\ \dfrac{\mu^2}{2\lambda} + p^0(0;\xi) & \text{if } \mu \geqslant \mu_c \end{cases}$$

where $p^0(\mu;\xi), \rho^0(\mu;\xi)$ are the pressure and mean particle-density associated with the modified perfect Bose gas Hamiltonian $H_l^0(\mu;\xi)$, $\mu_c := \lambda\rho_c$ the corresponding critical values, and $\tilde{\rho}(\mu;\xi)$ is the solution in ρ of the equation $\rho = \rho^0(\mu - \lambda\rho;\xi)$. We get the following expressions by a straightforward calculation

$$\begin{aligned} p^0(\mu;\xi) \; &= \; -\int_{[0,\delta]} \ln(1 - e^{-\beta(E-\mu)})\nu(\mathrm{d}E) \; - \int_{(\delta,\infty)} \ln(1 - e^{-\beta(E+\xi-\mu)})\nu(\mathrm{d}E), \\ \rho^0(\mu;\xi) \; &= \; \int_{[0,\delta]} \frac{1}{e^{\beta(E-\mu)} - 1}\nu(\mathrm{d}E) \; + \int_{(\delta,\infty)} \frac{1}{e^{\beta(E+\xi-\mu)} - 1}\nu(\mathrm{d}E) \; . \end{aligned} \tag{1.16}$$

We have

$$\partial_\xi \, p_l^\lambda(\mu;\xi) \; = \; -\frac{1}{V_l} \sum_{i:E_i^l > \delta} \langle N_l(\phi_i^l)\rangle_{l,\xi} \; = \; \frac{1}{V_l} \sum_{i:E_i^l \leqslant \delta} \langle N_l(\phi_i^l)\rangle_{l,\xi} \; - \; \tilde{\rho}(\mu;\xi),$$

which gives

$$\lim_{\delta\downarrow 0} \lim_{l\to\infty} \sum_{i:E_i^l \leqslant \delta} \frac{1}{V_l} \langle N_l(\phi_i^l)\rangle_{l,0} \; = \; \lim_{\xi\downarrow 0} \tilde{\rho}(\mu;\xi) \; + \; \lim_{\delta\downarrow 0} \lim_{l\to\infty} \partial_\xi \, p_l^\lambda(\mu;\xi)|_{\xi=0} \; .$$

In view of (1.16), it is easy to see that

$$\lim_{\xi\downarrow 0} \tilde{\rho}(\mu;\xi) = \begin{cases} \rho^0(\mu - \lambda\tilde{\rho}(\mu;0)) & \text{if} \quad \mu < \mu_c \\ \rho_c & \text{if} \quad \mu \geqslant \mu_c \end{cases}$$

As $p_l^\lambda(\mu;\xi)$ is a convex function of ξ, we can exchange the TL and differentiation using the Griffith lemma (see [20]). Using also the fact that $\tilde{\rho}(\mu;\xi)$ is a minimizer for $p_l^0(\mu;\xi)$, we get

$$\partial_\xi \lim_{l\to\infty} p_l^\lambda(\mu;\xi)|_{\xi=0} = \begin{cases} -\rho^0(\mu-\lambda\tilde{\rho}(\mu;0);0)) = -\rho^0(\mu-\lambda\tilde{\rho}(\mu)) & \text{if} \quad \mu<\mu_c \\ -\rho^0(0;0) = -\rho_c & \text{if} \quad \mu\geqslant\mu_c \end{cases}$$

Hence, we have proved the following:

Proposition 1.2.3 *Let the critical density be defined as in Proposition 1.2.1. The mean field Bose gas defined by the Hamiltonian (1.10) presents a phase transition in the thermodynamic limit if and only if $\rho_c<\infty$. In that case, the pressure is not twice differentiable at the critical value $\mu_c=\lambda\rho_c$, see Proposition 1.2.2, and there is* generalised Bose-Einstein condensation, *i.e. the limiting occupation measure has an atom at zero energy*

$$m^\lambda(\{0\}) = \begin{cases} \mu-\lambda\rho_c & \textit{if } \mu\geqslant\mu_c \\ 0 & \textit{if } \mu<\mu_c \end{cases}$$

1.3 The density of states for specific models

As can be seen from Propositions 1.2.1 and 1.2.3, the occurrence of generalised Bose-Einstein condensation in the perfect and mean field Bose gases is entirely controlled by the density of states ν (1.5) of the one-particle Schrödinger operator (1.3), as it determines whether a phase transition occurs or not and the density of the associated generalised condensation. More precisely, it is the ability of ν to make the critical density

$$\rho_c = \int_{[0,\infty)} (e^{\beta E}-1)^{-1}\nu(\mathrm{d}E)$$

finite that is required for the phase transition to occur. It can easily be seen from the previous integral that only the asymptotic behaviour of ν near zero energy is responsible for making the critical density finite.

Lemma 1.3.1 *The critical density ρ_c is finite if there exist constants $0\leqslant a<\infty$ and $\epsilon>0$ such that*

$$\lim_{E\downarrow 0} E^{-(1+\epsilon)}\nu([0,E]) = a$$

The proof of that lemma is elementary, using a Taylor expansion near zero energy.

If we restrict ourselves to the *free* Bose gas, the behaviour of the density of states ν^0 is known explicitly, see (1.6), and it follows that ρ_c is finite if and only if $d > 2$.

Now, one can obtain a wide range of behaviour for the density of states by the addition of external potentials. In this section, we shall review classes of external potential for which the IDS has the required behaviour to enhance generalised BEC even in low dimensional systems. In addition, we shall also prove that the required assumption on the density of states, see (1.7), holds for these models.

1.3.1 Random potentials: the Lifshitz tails

A simple model: the Luttinger-Sy model

In this subsection, we study a particular random system in dimension 1, the so-called Luttinger-Sy model with point impurities [21]. Formally, the single-particle Hamiltonian for this model is

$$h_l^\omega = -\tfrac{1}{2}\Delta + a \sum_j \delta(x - x_j^\omega) ,$$

where the x_j's are distributed according to a Poisson law and $a = +\infty$. We first recall some definitions to make sense of this formal Hamiltonian. Let $u(x) \geq 0$, $x \in \mathbb{R}$, be a continuous function with compact support called a (*repulsive*) single-impurity potential. Let $\{\mu_\lambda^\omega\}_{\omega\in\Omega}$ be the Poisson measure on $\mathbb{R}$ with intensity $\lambda > 0$,

$$\mathbb{P}\left(\{\omega \in \Omega : \mu_\lambda^\omega(\Lambda) = n\}\right) = \frac{(\lambda\,|\Lambda|)^n}{n!} e^{-\lambda|\Lambda|} \;\; , \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} , \tag{1.17}$$

for any bounded Borel set $\Lambda \subset \mathbb{R}$. Then the non-negative random potential v^ω generated by the Poisson distributed local impurities has realisations

$$v^\omega(x) := \int_{\mathbb{R}} \mu_\lambda^\omega(dy) u(x - y) = \sum_{x_j^\omega \in X^\omega} u(x - x_j^\omega) \; . \tag{1.18}$$

Here the random set X^ω corresponds to impurity positions $X^\omega = \left\{x_j^\omega\right\}_j \subset \mathbb{R}$, which are the atoms of the Poisson measure, i.e., $\sharp\{X^\omega \upharpoonright \Lambda\} = \mu_\lambda^\omega(\Lambda)$ is the number of impurities in the set Λ. Since the expectation $\mathbb{E}\left(\nu_\lambda^\omega(\Lambda)\right) = \lambda\,|\Lambda|$, the parameter λ coincides with the density of impurities on $\mathbb{R}$.

Luttinger and Sy defined their model by restriction of the single-impurity potential to the case of point δ-potential with amplitude $a \to +\infty$. Then the corresponding random potential (1.18) takes the form

$$v_a^\omega(x) := \int_{\mathbb{R}} \nu_\lambda^\omega(dy) a\delta(x-y) = a \sum_{x_j^\omega \in X^\omega} \delta(x - x_j^\omega) \ . \qquad (1.19)$$

Now the self-adjoint one-particle random Schrödinger operator $h_a^\omega := h^0 \dot{+} v_a^\omega$ is defined in the sense of the sum of quadratic forms. The strong resolvent limit $h_{LS}^\omega := s.r.\lim_{a\to+\infty} h_a^\omega$ is the Luttinger-Sy model.
Equivalently, this model can be defined by imposing a Dirichlet boundary condition at each impurity x_j^ω.

Since X^ω generates a set of intervals $\left\{I_j^\omega := (x_{j-1}^\omega, x_j^\omega)\right\}_j$ of (random) lengths $\left\{L_j^\omega := x_j^\omega - x_{j-1}^\omega\right\}_j$, one gets the decomposition of the one-particle Luttinger-Sy Hamiltonian

$$h_{LS}^\omega = \bigoplus_j h_D(I_j^\omega) \ , \quad \mathrm{dom}(h_{LS}^\omega) \subset \bigoplus_j L^2(I_j^\omega) \ \ , \ \ \omega \in \Omega \ , \qquad (1.20)$$

into random disjoint free Schrödinger operators $\left\{h_D(I_j^\omega)\right\}_{j,\omega}$ with Dirichlet boundary conditions at the end-points of intervals $\left\{I_j^\omega\right\}_j$. Then the Dirichlet restriction $h_{l,D}^\omega$ of the Hamiltonian h_{LS}^ω to a fixed interval $\Lambda_l = (-l/2, l/2)$ and the corresponding change of notation are evident: e.g., $\left\{I_j^\omega\right\}_j \mapsto \left\{I_j^\omega\right\}_{j=1}^{M^l(\omega)}$, where $M^l(\omega)$ is total number of subintervals in Λ_l corresponding to the set X^ω. For comprehensive definitions and some results concerning this model we refer the reader to [13].

The Luttinger-Sy model is special in the sense that its IDS can be computed exactly for all E, and not only near $E = 0$ as in the general case.

Lemma 1.3.2 (Lifshitz tails in the Luttinger-Sy model) *The sequence of densities of states ν_l^ω defined by the Schrödinger operator (1.20) of the Luttinger-Sy model converges a.s. (in the weak sense) in the limit $l \to \infty$ to a non-random measure ν, and*

$$\nu([0, E]) \ = \ \lambda \, \frac{e^{-c\lambda E^{-1/2}}}{1 - e^{-c\lambda E^{-1/2}}} \ ,$$

with the constant $c = \pi/\sqrt{2}$.

The proof of this lemma can be found in [13], see Proposition 3.2 in that reference. Clearly, it also implies that the density of states of this model satisfies the condition (1.7). In view of Lemma 1.3.1, we can state the following.

Corollary 1.3.1 *Let the Schrödinger operator in a random potential be defined as in (1.20). Then, the corresponding perfect and mean field Bose gases exhibits generalised BEC in the (random) eigenstates.*

A general family of random models

We define an external random potential $v^{(\cdot)}(\cdot) : \Omega \times \mathbb{R}^d \to \mathbb{R},\ x \mapsto v^\omega(x)$ as a non-negative random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Schrödinger operator in a random external potential is then given by

$$h_l^\omega := h_l^0 + v_l^\omega \ , \tag{1.21}$$

where v_l^ω denotes the restirction of the random potential v^ω to the box Λ_l. We shall assume that v_l^ω is bounded, and then the Schrödinger operator (1.21) is *almost surely* (a.s.) self-adjoint.

We shall adhere to the notation introduced in Section 1.1, adding an upper index ω to emphasise the randomness when necessary. We now turn to the density of states of these random Schrödinger operators. Although at finite-volume, the densities of states ν_l^ω defined by

$$\nu_l^\omega(A) = \frac{1}{V_l} \sharp\{k : E_i^{\omega,l} \in A\}, \text{ for all Borel subsets } A \subset \mathbb{R} \tag{1.22}$$

are random measures, one can check that for homogeneous ergodic random potentials the limiting measure ν^ω has the property of being *self-averaging*, see e.g. [10]. This means that $\nu^\omega = \nu$ is almost surely a *non-random* measure.

We shall also assume that the following technical conditions hold

1. $p := \mathbb{P}\{\omega : v^\omega(0) = 0\} < 1$

2. (a) On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ there exists a group of measure-preserving metrically transitive transformations $\{T_p\}_{p\in\mathbb{R}^d}$ of Ω, such that $v^\omega(x+p) = v^{T_p\omega}(x)$ for all $x, p \in \mathbb{R}^d$;

(b) $\mathbb{E}_\omega\{\int_{\Lambda_1} \mathrm{d}x\, |v^\omega(x)|^\kappa\} < \infty$, where $\kappa > \max(2, d/2)$, and Λ_1 the unit cube.

3. For any $\Lambda \subset \mathbb{R}^d$, let Σ_Λ be the σ-algebra generated by the random field $v^\omega(x), x \in \Lambda$. For any two arbitrary random variables on Ω, f,g satisfying (i) $|g|_\infty < \infty$, $\mathbb{E}_\omega\{|f|\} < \infty$ and (ii) the function g is Σ_{Λ_1}-measurable, the function f is Σ_{Λ_2}-measurable, where Λ_1, Λ_2 are disjoint bounded subsets of $\mathbb{R}^d$, the following holds

$$|\mathbb{E}\{|f.g|\} - \mathbb{E}\{|f|\}\mathbb{E}\{|g|\}| \leqslant |g|_\infty\, \mathbb{E}\{|f|\}\, \phi(d(\Lambda_1, \Lambda_2))$$

with $\phi(x) \to 0$ as $x \to \infty$, and $d(\Lambda_1, \Lambda_2)$ the Euclidean distance between Λ_1 and Λ_2.

Lemma 1.3.3 (Lifshitz tails) *Under the above assumptions, the following holds*

$$\text{a.s.} - \liminf_{E \to 0^+} (-E^{d/2}) \ln(\nu\big([0, E]\big)) \geqslant a > 0. \tag{1.23}$$

A proof of this form of Lifshitz tails (1.23) can be found in a paper by Kirsch and Martinelli [22], see Theorem 4 in that reference.

Corollary 1.3.2 *Let the Schrödinger operator in a random potential be defined as in (1.21), and assume the random potential satisfies the assumptions of Lemma 1.3.3. Then, the corresponding perfect and mean field Bose gases exhibit generalised BEC in the (random) eigenstates for any $d \geqslant 1$.*

Now, in order to prove our condition (1.7), we need the following additional condition

$$\mathbb{P}\Big\{\omega : \int_\Lambda v^\omega(x)\, \mathrm{d}x \;\leqslant\; t\Big\} > 0\,, \qquad \text{for all } t > 0 \text{ and for any finite } \Lambda\,. \tag{1.24}$$

We can now prove the following

Lemma 1.3.4 *Assume that a random potential satisfies the assumptions of Lemma 1.3.3 and the condition (1.24). Then, the density of states ν of the corresponding random Schrödinger operator (1.21) is such that*

$$\nu\big([0, E]\big) > 0 \quad \textit{for all } E > 0\,.$$

Proof:

Let us fix $E > 0$. We start from the following inequality, see [22],

$$\nu\big([0,E]\big) \geqslant \frac{1}{V_L}\,\mathbb{E}\big(\#\{i : E_i^{\omega,L} \leqslant E\}\big) \;, \tag{1.25}$$

which is satisfied for any $L > 0$, with V_L the volume of a cube of side L centered at some point $x \in \mathbb{R}^d$, and $E_i^{\omega,L}$ the eigenvalues corresponding to the restriction of the random Schrödinger operator to that region. Note that since we consider only expectations with respect to the random potential in (1.25), everything is actually independent of x.

By computing the expectation, one gets

$$\begin{aligned} \mathbb{E}_\omega\big(\#\{i : E_i^{\omega,L} \leqslant E\}\big) &= \sum_{n=0}^{\infty} n\mathbb{P}\{\#\{i : E_i^{\omega,L} \leqslant E\} = n\} \\ &\geqslant \mathbb{P}\big\{\#\{i : E_i^{\omega,l} \leqslant E\} \geqslant 1\big\} \end{aligned}$$

and we can then reduce the estimate in (1.25) to a condition on the first eigenvalue

$$\nu\big([0,E]\big) \geqslant L^{-d}\,\mathbb{P}\{\omega : E_1^{\omega,L} \leqslant E\}. \tag{1.26}$$

By the min-max principle, we have

$$E_1^{\omega,L} \leqslant \varepsilon_1^L + \int_{\Lambda_L} \mathrm{d}x\,|\psi_1^L(x)|^2\,v^\omega(x) \leqslant \varepsilon_1^L + L^{-d}\int_{\Lambda_L} \mathrm{d}x\,v^\omega(x) \;, \tag{1.27}$$

where ε_1^L is the first kinetic eigenvalue and ψ_1^L the corresponding eigenfunction of the kinetic energy operator $-\frac{1}{2}\Delta_D$ restricted to a cube of side L with Dirichlet boundary condition. Note that we have used the property (1.2) to estimate $|\psi_1^L(x)|$.

Let $L := \pi(E/2)^{-1/2}$, and therefore, the first kinetic eigenvalue $\varepsilon_1^L = E/2$. Hence, the inequality (1.27) becomes

$$E_1^{\omega,L} \leqslant E/2 + \pi^{-d}\Big(\frac{E}{2}\Big)^{-d/2}\int_{\Lambda_{\pi(E/2)^{-1/2}}} \mathrm{d}x\,v^\omega(x). \tag{1.28}$$

In view of (1.26), we obtain

$$\nu\big([0,E]\big) \geqslant \pi^{-d}\Big(\frac{E}{2}\Big)^{-d/2}\mathbb{P}\{\omega : \pi^{-d}\Big(\frac{E}{2}\Big)^{-d/2}\int_{\Lambda_{\pi(E/2)^{-1/2}}} \mathrm{d}x\,v^\omega(x) \leqslant E/2\} \tag{1.29}$$

and hence, it follows from (1.29) and the assumption (1.24) that

$$\nu\big([0,E]\big) \geqslant \pi^{-d}\Big(\frac{E}{2}\Big)^{-d/2}\mathbb{P}\Big\{\omega : \int_{\Lambda_{\pi(E/2)^{-1/2}}} \mathrm{d}x\,v^\omega(x) \leqslant \frac{\pi^d E^{1-d/2}}{2^{1+d/2}}\Big\} > 0\,. \quad \square$$

To finish this section, let us give some specific random potentials satisfying the assumptions that we detailed in the present section.

The Poisson potential This random potential is defined by

$$v^\omega(x) := \sum_i u(x - x_j^\omega),$$

where the x_j^ω's are the atoms of a Poisson measure, and the function $u : \mathbb{R}^d \mapsto [0, q)$ is the potential created by each impurity. In addition to the non-negativity, there are some additional conditions on the function u to ensure a well-defined random potential. For simplicity, one can assume that u has compact support, but the assumptions of Lemma 1.3.3 can also be satisfied under suitable fast decay conditions on u. The exact conditions have been the object of many studies, and we refer the interested reader to [10] for a comprehensive review.
Moreover, it is straightforward to check that the condition (1.24) is satisfied in the Poisson model with a compactly-supported function u. Indeed, it is sufficient to estimate the probability to find "empty" boxes, that is

$$\mathbb{P}\left\{\omega : \#\{j : x_j^\omega \in \Lambda_L\} = 0\right\},$$

and by the property of the Poisson distribution, this is non-zero for any finite L (although exponentially small, as expected in view of the Lifshitz tails).

Finally, we want to point out that the Lifshitz tails in the Poisson potential can be derived in a stronger form than in Lemma 1.3.3, see e.g. [10], in the sense that the limit itself is established, instead of an upper bound.

The Stollmann model The second model that we shall consider in this thesis is taken from [23], where the author calls it "the model $(P + A)$". It consists of impurities located at points of the lattice $\mathbb{Z}^d$, with appropriate assumptions over the single-impurity potential, mainly designed to obtain independence between regions which are sufficiently far away from each other. Let us make it more explicit by giving some definitions. The single-site potential f, $\Lambda_1(0) \to \mathbb{R}$ has the following properties:

1. f is bounded;

2. there exists $\sigma > 0$ such that $f(x) \geqslant \sigma > 0$ for all $x \in \Lambda_1(0)$.

The randomness in this model is given by varying the strength of each impurity. For this purpose, we define a single-site (probability) measure μ, with $\mathrm{supp}(\mu) = [0, a]$ for a finite a. We shall assume in the rest of this thesis that μ is Hölder-continuous, that is for some $\alpha > 0$,

$$\sup_{\{s,t\}} \left\{ \mu([s,t]) : 0 \leqslant t - s \leqslant \eta \right\} \leqslant \eta^{\alpha}, \quad \forall\, 0 \leqslant \eta \leqslant 1 \ . \tag{1.30}$$

The random potential is then defined by

$$v^{\omega}(x) := \sum_{k \in \mathbb{Z}^d} q^{\omega}(k)\, f(x-k), \tag{1.31}$$

where the $q^{\omega}(k)$'s are i.i.d. random variables distributed according to μ.

Since the impurities in that model are fixed on $\mathbb{Z}^d$-lattice points, each of them creating a compactly-supported potential, and the coefficients q^{ω}'s are i.i.d. random variables, it is easy to see that the assumptions of Lemma 1.23 are satisfied. For the additional condition (1.24), we can get the following estimate

$$\int_{\Lambda_L} v^{\omega}(x)\, \mathrm{d}x \leqslant \alpha_f \sum_{x_i \in I_L} q^{\omega}(x_i) \ ,$$

where the non-random, bounded constant α_f is defined by

$$\alpha_f := \int_{\Lambda_1} \mathrm{d}x\, f(x).$$

and $I_L \subset \mathbb{Z}^d$ is defined in such a way that the unions of unit cubes centered at the points $x_i \in I_L$ is the minimal cover of the box Λ_L. Hence, one needs to estimate the probability of the following set, for arbitrary $t > 0$

$$X := \{\omega : \alpha_f \sum_{x_i \in I_L} q^{\omega}(x_i) \leqslant t\} \ .$$

Using the independence of the $q^{\omega}(x_i)$'s, one can estimate the probability of the set X by considering the case where all $q^{\omega}(x_i)$ "contribute equally", so to speak, that is

$$\begin{aligned} \mathbb{P}(X) &\geqslant \mathbb{P}\Big(\bigcap_{x_i \in I_L} \{\omega : q^{\omega}(x_i) \leqslant \frac{t}{\alpha_f n_L}\} \\ &= \Big(\mathbb{P}\{\omega : q^{\omega}(0) \leqslant \frac{t}{\alpha_f n_L}\} \Big)^{n_L} \end{aligned}$$

where $n_L := |I_L|$. This can now be expressed in terms of the probability measure μ, according to which the $q^{\omega}(x_i)$'s are distributed,

$$\mathbb{P}(X) \geqslant \mu\Big([0, \frac{t}{\alpha_f n_L}] \Big)^{n_L} > 0$$

since $n_L < \infty$ for any $L < \infty$ and the measure μ has $\mathrm{supp}(\mu) = [0, a]$.

1.3.2 Weak external potentials

Let v be a non-negative, continuous real-valued function defined on the closed unit cube $\overline{\Lambda}_1 \subset \mathbb{R}^d$. We assume that the function v satisfies the two following conditions.

$$\begin{aligned} &\text{i)} \quad v(x) = 0 \quad \text{if and only if} \quad x \in \{y_j\}_{j=1}^n, \\ &\text{ii)} \quad \lim_{x\to y_j} \frac{v(x)}{|x-y_j|^{\alpha_j}} \; = \; c_j, \quad \forall j = 1,2,\ldots,n \end{aligned} \tag{1.32}$$

where $\{y_j\}_{j=1}^n$ is a sequence of points in Λ_1, and $\{\alpha_j\}$, $\{c_j\}$ corresponding sequences of positive parameters. We order the y_j in such a way that $0 < \alpha_1 \leqslant \ldots \leqslant \alpha_n$. Roughly speaking, we consider any continuous, non-negative function v that vanishes at only a finite number of points, and does so with some polynomial strength.

The *one-particle* Schrödinger operator with a *weak* external potential in a box Λ_l is defined by scaling the potential v, that is

$$h_l \; = \; -\tfrac{1}{2}\Delta_D \; + \; v(x_1/l,\ldots,x_d/l) \; . \tag{1.33}$$

The low energy behaviour of the density of states is stated in the following lemma.

Lemma 1.3.5 *Let h_l to be as above, and ν its asymptotic density of states. Then the following holds*

$$\lim_{E\downarrow 0} E^{-(d/2+d/\alpha_1)}\,\nu\big([0,E]\big) \; = \; K C_d \; .$$

The constant C_d is the same as in the Weyl formula (1.6), and K is given by

$$K \; = \; \frac{1}{c_1^{d/\alpha_1}} \int_{|z|<1} \mathrm{d}z \, (1-|z|^{\alpha_1}) \; ,$$

with c_1, α_1 as in (1.32).

Proof:

In view of the condition (1.32) satisfied by the external potential v, for some $\varepsilon > 0$ small enough, there exists $\delta_1 > 0, \ldots, \delta_n > 0$ such that for all $j = 1,\ldots,n$

$$(c_j - \varepsilon)|x-y_j|^{\alpha_j} \; \leqslant \; v(x) \; \leqslant \; (c_j+\varepsilon)|x-y_j|^{\alpha_j} \; , \tag{1.34}$$

$$\text{for all } x \text{ such that } |x-y_j| \; \leqslant \; \delta_j \; .$$

Let $\delta := \min\{\delta_j\}$, and denote by $B(y_j,\delta)$ the ball of radius δ centered at y_j. Note that by continuity, there exists a constant $\kappa > 0$ such that

$$v(x) \geqslant \kappa, \quad \text{for all } x \in \Lambda_1 \setminus \big(\bigcup_{j=1}^{n} B(y_j,\delta)\big) \ . \tag{1.35}$$

We now use a result due to Pulé, see [8],

$$\nu\big([0,E]\big) \;=\; \lim_{l\to\infty} \nu_l(E) \;=\; C_d \int_{\substack{x\in\Lambda_1 \\ v(x)<E}} \mathrm{d}x \,\big(E - v(x)\big)^{d/2}, \tag{1.36}$$

where C_d is as in the Weyl formula (1.6). It follows from (1.35) and (1.36) that, for all $E < \kappa$, we get

$$\nu\big([0,E]\big) \;=\; C_d \sum_{j=1}^{n} \int_{\substack{x\in B(y_j,\delta) \\ v(x)<E}} \mathrm{d}x \,\big(E - v(x)\big)^{d/2} \ .$$

Since we know from (1.34) that on the one hand,

$$E - v(x) \;\leqslant\; E - (c_j - \varepsilon)|x - y_j|^{\alpha_j}$$

for any $j = 1,\dots,n$ and for all $x \in B(y_j,\delta)$, and on the other hand

$$\{x \in B(y_j,\delta) : v(x) < E\} \;\subset \{x \in B(y_j,\delta) : (c_j+\varepsilon)|x-y_j|^{\alpha_j} < E\} \ ,$$

we can obtain the following upper bound from (1.36)

$$\nu\big([0,E]\big) \;\leqslant\; C_d \sum_{j=1}^{n} \int_{(c_j+\varepsilon)|x-y_j|^{\alpha_j}<E} \mathrm{d}x \,\big(E - (c_j-\varepsilon)|x-y_j|^{\alpha_j}\big)^{d/2} \ .$$

In each integral, we let $z := E^{-1/\alpha_j}(c_j+\varepsilon)^{1/\alpha j}(x-y_j)$ so that

$$\nu\big([0,E]\big) \;\leqslant\; C_d \sum_{j=1}^{n} \frac{E^{d/\alpha_j}}{(c_j+\varepsilon)^{d/\alpha_j}} \int_{|z|<1} \mathrm{d}z \, E^{d/2}\,(1 - \frac{c_j-\varepsilon}{c_j+\varepsilon}\,|z|^{\alpha_j}) \ . \tag{1.37}$$

We follow the same procedure to find the lower bound. For any $j = 1,\dots,n$, we obtain from (1.34) that

$$E - v(x) \;\geqslant\; E - (c_j + \varepsilon)|x - y_j|^{\alpha_j} \ ,$$

for all $x \in B(y_j,\delta)$, and we also have

$$\{x \in B(y_j,\delta) : v(x) < E\} \;\supset \{x \in B(y_j,\delta) : (c_j-\varepsilon)|x-y_j|^{\alpha_j} < E\} \ .$$

In a similar way as we obtained (1.37), we get the lower bound

$$\nu\big([0,E]\big) \;\geqslant\; C_d \sum_{j=1}^{n} \frac{E^{d/\alpha_j}}{(c_j-\varepsilon)^{d/\alpha_j}} \int_{|z|<1} \mathrm{d}z\, E^{d/2}\,(1-\frac{c_j+\varepsilon}{c_j-\varepsilon}\,|z|^{\alpha_j})\;. \tag{1.38}$$

Combining (1.37) and (1.38) leads to

$$C_d \sum_{j=1}^{n} E^{d/\alpha_j+d/2}\, K_1(j,\varepsilon) \;\leqslant\; \nu\big([0,E]\big) \;\leqslant\; C_d \sum_{j=1}^{n} E^{d/\alpha_j+d/2}\, K_2(j,\varepsilon)\;, \tag{1.39}$$

for any $E<\kappa$, with the constants

$$\begin{aligned} K_1(j,\varepsilon) \;&:=\; \frac{1}{(c_j-\varepsilon)^{d/\alpha_j}} \int_{|z|<1} \mathrm{d}z\,(1-\frac{c_j+\varepsilon}{c_j-\varepsilon}\,|z|^{\alpha_j})\;, \\ K_2(j,\varepsilon) \;&:=\; \frac{1}{(c_j+\varepsilon)^{d/\alpha_j}} \int_{|z|<1} \mathrm{d}z\,(1-\frac{c_j-\varepsilon}{c_j+\varepsilon}\,|z|^{\alpha_j})\;. \end{aligned} \tag{1.40}$$

Only the first terms in the sums in (1.39) contribute in the limit $E \downarrow 0$, since we have assumed that $0<\alpha_1 \leqslant \cdots \leqslant \alpha_n$, and thus

$$\begin{aligned} C_d\, K_1(1,\varepsilon) \;&\leqslant\; \liminf_{E\downarrow 0} E^{-(d/\alpha_1+d/2)}\,\nu\big([0,E]\big) \\ &\leqslant\; \limsup_{E\downarrow 0} E^{-(d/\alpha_1+d/2)}\,\nu\big([0,E]\big) \;\leqslant\; C_d\, K_2(1,\varepsilon)\;. \end{aligned}$$

As it is clear from (1.40) that we have

$$\lim_{\varepsilon\downarrow 0} K_1(1,\varepsilon) \;=\; \lim_{\varepsilon\downarrow 0} K_2(1,\varepsilon) \;=\; \frac{1}{c_1^{d/\alpha_1}} \int_{|z|<1} \mathrm{d}z\,(1-|z|^{\alpha_1})\;,$$

the lemma follows by letting $\varepsilon \to 0$. □

Note also that the proof of Lemma 1.3.5 yields directly the condition (1.7) for the weak external potential.

It is now straightforward to derive the minimal requirements on the weak potential for it to make the critical density finite, that is to satisfy the requirements of Lemma 1.3.1.

Corollary 1.3.3 *Let the Schrödinger operator in the weak external potential be defined as in (1.33). In the corresponding perfect and mean field Bose gases, the necessary and sufficient condition for generalised BEC in the eigenstates is given by*

$$\frac{d}{\alpha_1} > 1 \,-\, d/2.$$

This condition is trivial if $d > 2$, because then even the free Bose gas (without external potential) would exhibit generalised BEC. If $d = 2$, any weak potential defined by (1.32) will also satisfy it, and this comes from the fact that the density of states of the free gas is already on the edge, see (1.6), and the external potential is in this case "the straw that broke the camel's back". Things get more interesting if $d = 1$, since the weak external potential needs to be strong enough to create condensation. Recall that the function v is defined on the unit cube, and hence it is the smallest α_j that is the most important.

1.4 BEC in *single* eigenstates

In this section, we are interested in classifying the generalised BEC into the type I, II or III. As we emphasised in the introduction, one cannot deduce from the generalised BEC established in Propositions 1.2.1 and 1.2.3 in which particular eigenstate ϕ_i^l the condensate is to be found, and indeed, it does not necessarily imply condensation in the ground state ϕ_1^l. We shall give some examples of various types of generalised BEC that have been rigorously established.

Let us start with the weak external case, which has been extensively studied by the Dublin School. In a model studied in [12], the external potential was restricted to vary in one direction only, that is with our notations

$$v(x) = v(x_1) := |x_1|^\alpha, \quad \alpha > 0 .$$

While this class of weak potentials does not satisfy the technical assumptions (1.32) if $d \geqslant 2$, it is nevertheless possible to show that generalised BEC in the eigenstates occurs if the parameter α is such that

$$\frac{d}{2} + \frac{1}{\alpha} > 1 .$$

Note that this condition coincides with our result in Corollary 1.3.3 in the $d = 1$ case.

Moreover, the authors in [12] have proved that the condensate is concentrated on the ground state ϕ_1^l in $d = 1, 3, 4, \ldots$, that is the common type I generalised BEC.

However, if $d = 2$, the condensate would be spread over infinitely many eigenstates (all with arbitrary small energy), i.e. a type II condensation.

In the random case, while the occurence of generalised BEC in the eigenstates ϕ_i^l is fairly easy to prove, thanks to the Lifshitz tails, it is far more difficult to obtain more information about the actual spread of the condensate. As far as we know, it has only been done in a simple case, the Luttinger-Sy model, see Section 1.3.1. It was shown in [13] that the generalised BEC in the eigenstates is to be found in the (random) groundstate only (that is, a type I condensation)

$$\begin{aligned}
\lim_{l\to\infty} \frac{1}{l} \langle N_l(\phi_1^{\omega,l}) \rangle_l^\lambda &= \begin{cases} 0 & \text{if } \overline{\rho} < \rho_c \\ \overline{\rho} - \rho_c & \text{if } \overline{\rho} \geqslant \rho_c \end{cases} \\
\lim_{l\to\infty} \frac{1}{l} \langle N_l(\phi_i^{\omega,l}) \rangle_l^\lambda &= 0, \quad \text{for all } i > 1 \ .
\end{aligned}$$

Their proof relied on specific features of the Luttinger-Sy model, namely the fact that, due to the infinite strength of the impurities, there is no tunnelling effect between two regions separated by an impurity. One can see this as an “ideal” model from the point of view of localisation, making computation of the eigenvalues much simpler. Indeed, this is only a free Bose gas in a collection of intervals of random length, the distribution of which lengths can be deduced from the properties of the Poisson distribution. It is a very difficult problem to generalise this result to more complicated random models.

To conclude this section, let us note that, as far as we know, no generalised BEC in the eigenstates ϕ_i^l of type III has been obtained by means of an external potential. This particular type of condensation has only been shown in two models. One is the *free* Bose gas for which the thermodynamic limit is taken in a highly anisotropic way, see e.g. [6]. The other model is a Bose gas with a very specific, tailor-made, interaction between the particles to prevent accumulation of particles in any single eigenstate, [15].

Chapter 2

Generalised BEC in the kinetic states

In this chapter, we investigate whether condensation can occur in the kinetic *eigenstates ψ_k^l. Indeed, the phenomenon of Bose-Einstein condensation is generally understood in the physics literature as the so-called* 0*-mode condensation, that is, the macroscopic occupation of the kinetic eigenstate ψ_1^l in our notation. The generalised BEC established in previous chapter is however to be found in the eigenstates ϕ_i^l.*

As a first step into the understanding of this unusual condensation, especially in low dimensions where the standard BEC does not occur in translation invariant systems, we investigate the occurence of generalised condensation in the kinetic eigenstates ψ_k^l. Since these are not the eigenstates of the one-particle Schrödinger operator, the standards methods used in the previous chapter do not work, as the many particles Hamiltonians of the perfect and mean field Bose gases are not diagonal if one performs the second quantisation in the basis defined by the kinetic states.

The results of this chapter have been published in Journal of Statistical Physics [24], a copy of this article is reproduced in Appendix E.

Let us define the *kinetic* occupation measure by

$$\tilde{m}_l^\lambda(A) := \frac{1}{V_l} \sum_{k:\varepsilon_k^l \in A} \langle N_l(\psi_k^l)\rangle_l^\lambda \ , \text{ for all Borel subsets } A \subset [0,\infty) \ , \tag{2.1}$$

which is the analogue of the occupation measure (1.13) in the eigenstates, but instead measuring the occupation densities in the kinetic states ψ_k^l.

In the perfect Bose gas, we shall show that this sequence of measures, has a weak limit and we derive an explicit expression for it. In particular, we show that it has an atom at the origin, which answers the question of generalised BEC in the kinetic eigenstates. Moreover, the density of that condensate is the same as in the generalised BEC in the eigenstates.

We shall then investigate the mean field gas, and while the results that we obtain for this case are weaker than for the perfect gas, we are nevertheless able to derive lower and upper bounds for the density of kinetic condensate, in particular we shall show that it can be no less than the density of generalised BEC in the eigenstates and that it vanishes for densities below the critical value.

2.1 Some general results

We begin with some general results, the proofs of which require only the non-negativity of the external potential and a general feature of the interaction between particles (if any). The first result, though elementary, is crucial in all our analysis. It may be understood as the analogue of momentum conservation in non translation invariant systems. However, here it is not the total momentum which is conserved, since there is no momentum in the first place. Indeed, only the number of particles in each eigenstate ϕ_i^l is conserved.

Lemma 2.1.1 *Let $H_l(\mu)$ to be a many-particles Hamiltonian, and $\langle - \rangle$ its associated equilibrium state. If*

$$[H_l(\mu), N_l(\phi_i^l)] = 0, \quad \text{for all } i \ ,$$

then

$$\langle a^*(\phi_i^l)a(\phi_j^l)\rangle = 0, \quad \text{if } i \neq j \ .$$

Proof :

Let us define a unitary transformation $U_i(\theta)$ in the Fock space $\mathscr{F}_l$ by

$$U_i(\theta) \;:=\; e^{i\theta N_l(\phi_i^l)}$$

for a fixed i. Letting $\Xi := \mathrm{Tr}_{\mathscr{F}_l} \exp(-\beta H_l(\mu))$, we have

$$\begin{aligned} \langle a^*(\phi_i^l) a(\phi_j^l)\rangle &= \frac{1}{\Xi} \mathrm{Tr}_{\mathscr{F}_l}\{e^{-\beta H_l(\mu)} a^*(\phi_i^l) a(\phi_j^l)\} \\ &= \frac{1}{\Xi} \mathrm{Tr}_{\mathscr{F}_l}\{e^{-\beta H_l(\mu)} a^*(\phi_i^l) U_i(\theta) U_{-i}(\theta) a(\phi_j^l)\} \\ &= \frac{1}{\Xi} \mathrm{Tr}_{\mathscr{F}_l}\{e^{-\beta H_l(\mu)} U_{-i}(\theta) a^*(\phi_i^l) U_i(\theta) a(\phi_j^l)\} \ , \end{aligned} \tag{2.2}$$

where the last step follows from the fact that $i \neq j$, the assumption $[H_l(\mu), N_l(\phi_i^l)] = 0$ and the commutativity of the trace. Since

$$U_{-i}(\theta) a^*(\phi_i^l) U_i(\theta) \;=\; e^{i\theta} a^*(\phi_i^l) \ ,$$

the equation (2.2) becomes

$$\langle a^*(\phi_i^l) a(\phi_j^l)\rangle \;=\; e^{i\theta} \langle a^*(\phi_i^l) a(\phi_j^l)\rangle \ ,$$

and the lemma follows. □

Note that the assumption of the Lemma 2.1.1 is not only satisfied by the perfect and mean field Bose gases, but also by a class of interacting Bose gases with Hamiltonians of the form

$$H_l(\mu) \;:=\; H_l^0(\mu) \,+\, \frac{\lambda}{V_l} \sum_{i,j} b_{i,j}\, N_l(\phi_i^l) N_l(\phi_j^l) \ .$$

We shall refer to these Hamiltonians as *diagonal* models. Note that the case $b_{i,j} = 0$ corresponds to the perfect gas, and $b_{i,j} = \delta_{i,j}$ to the mean field gas (with a shift in the chemical potential). Some of the results of this section are applicable to diagonal models with non-negative external potential without further assumptions.

We can use this result to expand the measure $\tilde{m}_l^\lambda$ in terms of the equilibrium mean-values of occupation numbers in the corresponding eigenstates ϕ_i. Using the linearity (respectively conjugate linearity) of the creation and annihilation operators one obtains

$$\begin{aligned}
\tilde{m}_l^\lambda(A) &= \frac{1}{V_l}\sum_{k:\varepsilon_k^l\in A}\langle a^*(\psi_k^l)a(\psi_k^l)\rangle_l^\lambda \qquad (2.3)\\
&= \frac{1}{V_l}\sum_{i,j}\sum_{k:\varepsilon_k^l\in A}(\phi_i^l,\psi_k^l)\overline{(\phi_j^l,\psi_k^l)}\,\langle a^*(\phi_i^l)a(\phi_j^l)\rangle_l^\lambda\\
&= \frac{1}{V_l}\sum_{i}\sum_{k:\varepsilon_k^l\in A}|(\phi_i^l,\psi_k^l)|^2\,\langle a^*(\phi_i^l)a(\phi_i^l)\rangle_l^\lambda,
\end{aligned}$$

where the last equality follows from Lemma 2.1.1.

We now prove two important lemmas. Let us introduce their meaning from an heuristic point of view. In view of the generalised BEC in the eigenstates ϕ_i^l established in Propositions 1.2.1 and 1.2.3, the *total energy* of the particles in the condensate must be arbitrary low, and since the external potential is non-negative, it follows that their *kinetic energy* must also be arbitrary low. Hence, the particles involved in the generalised BEC in the eigenstates ϕ_i^l should also be condensed at low kinetic energy, which is what we shall establish in the first lemma.
On the other hand, if condensation were to occur at non-zero kinetic energy, the particles involved should have an even higher full energy. But since the condensation in the eigenstates ϕ_i^l does occur only at low full energy, it should not be possible to obtain kinetic BEC apart at zero kinetic energy, and we prove this in the second lemma.

Let us also emphasise that these two lemmas do not require the existence of a weak limit of the sequence of measures $\tilde{m}_l^\lambda$. Instead, we consider only some convergent subsequence. Note that at least one such subsequence always exists, see [25], Chapter VIII.6.

The first result states that if there is condensation in the lowest eigenstates ϕ_i^l, then there is also condensation in the lowest kinetic-energy states ψ_k^l. Moreover, the amount of the latter condensate density has to be greater than or equal to that of the former.

Lemma 2.1.2 *Let $\{\tilde{m}_{l_r}^\lambda\}_{r\geq 1}$ be a convergent subsequence. We denote by $\tilde{m}^\lambda$ its (weak) limit. For non-negative potentials, the following holds*

$$\tilde{m}^\lambda(\{0\}) \geqslant m^\lambda(\{0\}) = \begin{cases}\overline{\rho}-\rho_c & \text{if } \ \overline{\rho}\geqslant\rho_c\\ 0 & \text{if } \ \overline{\rho}<\rho_c\end{cases}.$$

Proof:

Let $\gamma > 0$. Using the expansion of the functions ψ_k^l in the basis $\{\phi_i\}_{i\geq 1}$, see (2.3), we obtain:

$$\begin{aligned}
\tilde{m}^\lambda([0,\gamma]) &= \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\leqslant\gamma} \langle N_{l_r}(\psi_k^{l_r})\rangle_{l_r}^\lambda \\
&= \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\leqslant\gamma} \sum_{i\geqslant 1} |(\phi_i^{l_r}, \psi_k^{l_r})|^2 \, \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda \\
&\geqslant \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\leqslant\gamma} \sum_{i:E_i^{l_r}\leqslant\delta} |(\phi_i^{l_r}, \psi_k^{l_r})|^2 \, \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda
\end{aligned}$$

for any $\delta > 0$. The non-negativity of the potential implies that

$$\begin{aligned}
\sum_{k:\varepsilon_k^l>\gamma} |(\phi_i^l, \psi_k^l)|^2 &\leqslant \sum_{k:\varepsilon_k^l>\gamma} \frac{\varepsilon_k^l}{\gamma} |(\phi_i^l, \psi_k^l)|^2 \leqslant \frac{1}{\gamma} \sum_{k\geqslant 1} \varepsilon_k^l |(\phi_i^l, \psi_k^l)|^2 \\
&= \frac{1}{\gamma}(\phi_i^l, h_l^0 \phi_i^l) \leqslant \frac{1}{\gamma}(\phi_i^l, h_l^\omega \phi_i^l) = \frac{E_i^l}{\gamma} \, .
\end{aligned}$$

We then obtain

$$\begin{aligned}
\tilde{m}^\lambda([0,\gamma]) &\geqslant \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{i:E_i^{l_r}\leqslant\delta} \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda \, (1 - \sum_{k:\varepsilon_k^{l_r}>\gamma} |(\phi_i^{l_r}, \psi_k^{l_r})|^2) \\
&\geqslant \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{i:E_i^{l_r}\leqslant\delta} \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda \, (1 - E_i^{l_r}/\gamma) \\
&\geqslant \lim_{r\to\infty} (1-\delta/\gamma) \frac{1}{V_{l_r}} \sum_{i:E_i^{l_r}\leqslant\delta} \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda \; = (1-\delta/\gamma)\, m([0,\delta]) \geq 0 \, .
\end{aligned}$$

But δ is arbitrary, and the lemma follows by letting $\delta \downarrow 0$. □

Remark 2.1.1 *In the perfect Bose gas, we can actually show that the limit* $\tilde{m}^\lambda$ *exists, see Section 2.2. However, even without knowing the existence of a limit, this lemma is still quite interesting, since apart the non-negativity of the external potential, its proof involves only the Lemma 2.1.1. Hence, it also applies to any diagonal model in the following form*

$$\lim_{\delta\downarrow 0} \liminf_{l\to\infty} \tilde{m}_l^\lambda([0,\delta]) \geqslant m^\lambda(\{0\}),$$

assuming that the sequence of measures m_l^λ *has a limit* m^λ*. In the mean field case, this can be shown with the techniques from Section 1.2.2 (noting that any measurable subset of the real line can be approximated by an at most countable union of intervals).*

In the next lemma, we show that the kinetic states occupation measure (2.1) can have an atom in the thermodynamic limit only at zero kinetic energy.

Lemma 2.1.3 *Assume that the occupation number* $\langle N_l(\phi_i)\rangle_l^\lambda$ *is a non-increasing function of* i*. Let* $\{\tilde{m}_{l_r}^\lambda\}_{r\geq 1}$ *be a convergent subsequence, and* $\tilde{m}^\lambda$ *be its (weak) limit. Then, for non-negative potential such that (1.7) holds,* $\tilde{m}^\lambda$ *is absolutely continuous on* $\mathbb{R}_+ := (0,\infty)$*.*

Proof :

Let A be a Borel subset of $(0,\infty)$, with Lebesgue measure 0, and let a be such that $\inf A > a > 0$. Then

$$\begin{aligned}
\tilde{m}_{l_r}^\lambda(A) &= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\in A} \langle N_{l_r}(\psi_k^{l_r})\rangle_{l_r}^\lambda \\
&= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\in A} \sum_i |(\phi_i^{l_r},\psi_k^{l_r})|^2 \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda \\
&= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\in A} \sum_{i:E_i^{l_r}\leqslant\alpha} |(\phi_i^{l_r},\psi_k^{l_r})|^2 \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda \\
&+ \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\in A} \sum_{i:E_i^{l_r}>\alpha} |(\phi_i^{l_r},\psi_k^{l_r})|^2 \langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda
\end{aligned} \tag{2.4}$$

for some $\alpha > 0$. Next, we use the non-negativity of the external potential to get the following estimate

$$E_i^l = (\phi_i^l, h_l\phi_i^l) \geqslant (\phi_i^l, h_l^0\phi_i^l) = \sum_k \varepsilon_k^l |(\phi_i^l,\psi_k^l)|^2 \geqslant a \sum_{k:\varepsilon_k\in A} |(\phi_i^l,\psi_k^l)|^2.$$

Since the equilibrium values of the occupation numbers $\langle N_l(\phi_i^l)\rangle_l^\lambda$ are decreasing with i, the estimate (2.4) implies

$$\tilde{m}_{l_r}^\lambda(A) \leqslant \frac{1}{V_{l_r}}\frac{1}{a} \sum_{i:E_i^{l_r}\leqslant\alpha} E_i^{l_r}\langle N_{l_r}(\phi_i^{l_r})\rangle_{l_r}^\lambda + \langle N_{l_r}(\phi_{i_\alpha}^{l_r})\rangle_{l_r}^\lambda \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k^{l_r}\in A} 1 \,, \tag{2.5}$$

where $\phi_{i_\alpha}^{l_r}$ denotes the eigenstate of h_{l_r} with the *smallest* eigenvalue *greater* than α. Using again the monotonicity and the finite-volume density of states ν_l, see (1.5), we can get an upper bound for the mean occupation number in the second term of (2.5), since

$$\overline{\rho} = \frac{1}{V_l}\sum_i \langle N_l(\phi_i^l)\rangle_l^\lambda \geqslant \frac{1}{V_l}\sum_{i:E_i^l\leqslant\alpha} \langle N_l(\phi_i^l)\rangle_l^\lambda \geqslant \langle N_l(\phi_{i_\alpha}^l)\rangle_l^\lambda \,\nu_l\big([0,\alpha]\big)\,. \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$\tilde{m}^{\lambda}_{l_r}(A) \leqslant \frac{\alpha\,\overline{\rho}}{a} + \frac{\overline{\rho}}{\nu^{\omega}_{l_r}([0,\alpha])} \int_A \nu^0_{l_r}(\mathrm{d}\varepsilon) \; . \tag{2.7}$$

Since the measure ν^0 (1.4) is absolutely continuous with respect to the Lebesgue measure, and $\nu\big([0,\alpha]\big)$ is strictly positive for any $\alpha > 0$ by assumption, see (1.7), the limit $r \to \infty$ in (2.7) gives:

$$\tilde{m}^{\lambda}(A) \leqslant \frac{\alpha\,\overline{\rho}}{a} \; ,$$

But $\alpha > 0$ can be chosen arbitrary small and thus $\tilde{m}(A) = 0$. To finish the proof, note that any Borel subset of $(0,\infty)$ can be expressed as a countable union of disjoint subsets with non-zero infimum. Our arguments can then be applied to each of them.

□

Remark 2.1.2 *In addition to the perfect and mean field Bose gas, this lemma is again valid for any diagonal model, with the additional assumption of monotonicity. This last property is trivial for the perfect gas, since the occupation numbers are known explicitly*

$$\langle N_l(\phi^l_i)\rangle^0_l = \frac{1}{e^{\beta(E^l_i-\mu)}-1} \; .$$

We shall show that this monotonicity condition holds also for the mean field gas, see Lemma 2.3.1.

2.2 The perfect Bose gas

In this section, we shall exploit a particular feature of the perfect Bose gas to show that the sequence of kinetic occupation measures $\{\tilde{m}^{\lambda}_l\}_l$ converges weakly in the thermodynamic limit. More precisely, we shall use the fact that the occupation numbers $\langle N_l(\phi^l_i)\rangle^0_l$ are known explicitly.

While the general scheme of the proof is the same for both the random and weak external potentials, some of the ingredients will differ substanstially.

We first decribe the proof for a general class of random potentials, and also provide a proof of finite volume Lifshitz tails, that is an estimate for the (random) finite volume densities of states ν^{ω}_l, instead of the asymptotic density of states ν as discussed in Section 1.3.1.

Finally, we shall review separately the Luttinger-Sy model at the end of that subsection, since we can actually obtain a more explicit result than in the general random case.

In the last subsection, we adapt our methods to cover the case of the weak external potential.

2.2.1 The random case

The general case

In this section, we are concerned with the general class of random Schrödinger operators satisfying the assumptions detailed in Section 1.3.1. The main result is the following.

Theorem 2.2.1 *Assume that the random potential satisfies the assumptions of Lemma 1.3.3 and the condition (1.24). Then, the sequence of measures* $\tilde{m}_l^0$ *converges a.s. in a weak sense to a non-random measure* $\tilde{m}^0$*, which is given by*

$$\tilde{m}^0(\mathrm{d}\varepsilon) \;=\; \begin{cases} (\overline{\rho}-\rho_c)\delta_0(\mathrm{d}\varepsilon) \;+\; F(\varepsilon)\mathrm{d}\varepsilon & \text{if } \overline{\rho} \geqslant \rho_c \\ F(\varepsilon)\mathrm{d}\varepsilon & \text{if } \overline{\rho} < \rho_c \end{cases}$$

with density $F(\varepsilon)$ *defined by*

$$F(\varepsilon) \;=\; (2\varepsilon)^{d/2-1}\int_{S_d^1} \mathrm{d}\sigma\, g(\sqrt{2\varepsilon}\, n_\sigma) \;.$$

Here, S_d^1 *denotes the unit sphere in* $\mathbb{R}^d$ *centered at the origin,* n_σ *the unit outward drawn normal vector, and* $\mathrm{d}\sigma$ *the surface measure of* S_d^1*. The function* g *is defined as follows*

$$g(k):\;=\;\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}x\, e^{ikx}\sum_{n\geqslant 1} e^{n\beta\mu_\infty}\,\mathbb{E}_\omega\Big(K_\omega^{n\beta}(x,0)\Big)\;, \tag{2.8}$$

where $\mathbb{E}_\omega$ *is the expectation on the probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *and* $K_\omega^t(x,x')$ *is the kernel of the operator* e^{-th^ω}*.*

Proof of Theorem 2.2.1

Before proving this theorem, we need some intermediate results. The occupation numbers in the perfect Bose gas are known explicitely

$$\langle N_l(\phi_i^l)\rangle_l^0 \;=\; \left(e^{\beta(E_i^{\omega,l}-\mu_l)}-1\right)^{-1}$$

where μ_l is the (unique) solution of the fixed-density constraint

$$\tilde{m}_l^0\big([0,\infty)\big) \;=\; \overline{\rho}. \tag{2.9}$$

It then follows from (2.3) that the kinetic occupation measure can be expressed as

$$\tilde{m}_l^0(A) = \frac{1}{V_l}\,\mathrm{Tr}\,P_A\,(e^{\beta(h_l^\omega-\mu_l)}-1)^{-1} = \sum_{n\geqslant 1}\frac{1}{V_l}\,\mathrm{Tr}\,P_A\,(e^{-n\beta(h_l^\omega-\mu_l)}), \tag{2.10}$$

where we denote by P_A the orthogonal projection onto the subspace spanned by the one-particle kinetic energy states ψ_k^l with kinetic energy ε_k^l in the set A. Now we split the measure (2.10) into two parts:

$$\begin{aligned}
\tilde{m}_l^0 \;&=\; \tilde{m}_l^{(1)} + \tilde{m}_l^{(2)} \qquad \text{with} \qquad\qquad (2.11)\\
\tilde{m}_l^{0,(1)}(A) \;&:=\; \sum_{n\geqslant 1}\frac{1}{V_l}\,\mathrm{Tr}\,P_A\,(e^{-n\beta(h_l^\omega-\mu_l)})\,\mathbf{1}(\mu_l\leqslant 1/n)\;,\\
\tilde{m}_l^{0,(2)}(A) \;&:=\; \sum_{n\geqslant 1}\frac{1}{V_l}\,\mathrm{Tr}\,P_A\,(e^{-n\beta(h_l^\omega-\mu_l)})\,\mathbf{1}(\mu_l> 1/n)\;.
\end{aligned}$$

Note that the chemical potential $\mu_l := \mu_l^\omega$ is actually a random variable. Therefore the indicator functions $\mathbf{1}(\mu_l\leqslant 1/n)$ and $\mathbf{1}(\mu_l > 1/n)$ split the range of n into the sums (2.11) in a random and volume-dependent way.

We start with the existence of a weak limit of the sequence of random measures $\tilde{m}_l^{0,(1)}$. One important ingredient of the proof is a finite volume version of the Lifshitz tails, the proof of which we postpone to the next subsection to keep this section more readable.

Theorem 2.2.2 *Assume that the random potential satisfies the assumptions of Lemma 1.3.3 and the condition (1.24). Then for any $d\geqslant 1$, the sequence of Laplace transforms of the measures $\tilde{m}_l^{(0,1)}$:*

$$f_l(t;\mu_l) \;:=\; \int_{\mathbb{R}} \tilde{m}_l^{(0,1)}(\mathrm{d}\varepsilon)\; e^{-t\varepsilon} \tag{2.12}$$

converges for any $t > 0$ *to a (non-random) limit* $f(t;\mu_\infty)$ *, which is given by:*

$$f(t;\mu_\infty) = \sum_{n\geqslant 1} e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}x \, \frac{e^{-\|x\|^2/2t}}{(4\pi^2 t)^{d/2}} \, \mathbb{E}_\omega\Big(K_\omega^{n\beta}(x,0)\Big) \, . \tag{2.13}$$

Note that the sum on the right-hand side converges for all (non-random) $\mu_\infty \geqslant 0$*, including* 0*, which corresponds to the case* $\overline{\rho} \geqslant \rho_c$*.*

Proof:

By definition of P_A the Laplace transformation (2.12) can be written as:

$$f_l(t;\mu_l) = \sum_{n\geqslant 1} \frac{1}{V_l} \,\mathrm{Tr}\, e^{-t\, h_l^0}\, (e^{-n\beta(h_l^\omega - \mu_l)})\, \mathbf{1}(\mu_l \leqslant 1/n) \ . \tag{2.14}$$

Now we have to show the uniform convergence of the sum over n to be able to take the term by term limit with respect to l. Since for any bounded operator A and for any trace-class non-negative operator B one has $\mathrm{Tr}AB \leqslant \|A\|\, \mathrm{Tr}B$, we get

$$\begin{aligned} a_l(n) \ &:= \ \frac{1}{V_l}\mathrm{Tr}\, e^{-t\, h_l^0}\, e^{-n\beta(h_l^\omega - \mu_l)}\mathbf{1}(\mu_l \leqslant 1/n) \\ &\leqslant \ \frac{1}{V_l}\mathrm{Tr}\, e^{-n\beta(h_l^\omega - \mu_l)}\mathbf{1}(\mu_l \leqslant 1/n). \end{aligned} \tag{2.15}$$

For $\overline{\rho} < \rho_c$, the uniform convergence in (2.14) is immediate. Indeed, for l large enough, the chemical potential satisfies $\mu_l < \mu_\infty/2 < 0$, and hence, we have the following a.s. estimate (2.15):

$$a_l(n) \leqslant e^{n\beta\mu_\infty/2} \int_{[0,\infty)} \nu_l^\omega(\mathrm{d}E)\, e^{-\beta E} \leqslant K_1\, e^{n\beta\mu_\infty/2}, \tag{2.16}$$

for some constant K_1.

However, for the case $\overline{\rho} \geqslant \rho_c$, this approach does not work, since, in fact, for any finite l the (random) solution $\mu_l = \mu_l^\omega$ of the constraint (2.9) could be *positive* with some probability, event though it has to *vanish* a.s. in the TL, see the discussion in Section 1.2.1. We use, therefore, the bound

$$\begin{aligned} a_l(n) \ &\leqslant \ a_l^1(n) + a_l^2(n) \ , \\ a_l^1(n) \ &:= \ \frac{1}{V_l}\, e^{\beta} \sum_{\{i:\, E_i^{\omega,l} \leqslant 1/n^{1-\eta}\}} e^{-n\beta E_i^{\omega,l}} \ , \\ a_l^2(n) \ &:= \ \frac{1}{V_l}\, e^{\beta} \sum_{\{i:\, E_i^{\omega,l} > 1/n^{1-\eta}\}} e^{-n\beta E_i^{\omega,l}} \ , \end{aligned}$$

which follows, for some $0 < \eta < 1$, from the constraint $\mu_l n \leqslant 1$ due to the indicator function in (2.15). Then the first term is bounded from above by

$$a_l^1(n) \leqslant e^{\beta}\, \nu_l^{\omega}\big([0, n^{\eta-1}]\big) \ .$$

Hence, we need to find an estimate for the *finite volume* IDS ν_l, which turns out to be a key ingredient of our proof. We obtain the required estimate in the Theorem 2.2.3 (*finite-volume* Lifshitz tails). To keep this section readable, we postpone the statement and proof of that theorem to the next section.

Using that result, it follows that for $\alpha > 0$ and $0 < \gamma < d/2$, there exists a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any $\omega \in \widetilde{\Omega}$ there exists a positive finite energy $\mathcal{E}(\omega) := \mathcal{E}_{\alpha,\gamma}(\omega) > 0$ for which one obtains

$$\nu_l^{\omega}\big([0, E]\big) \ \leqslant \ e^{-\alpha/E^{\gamma}} \ ,$$

for all $E < \mathcal{E}(\omega)$ and for any l. Therefore, for any configuration $\omega \in \widetilde{\Omega}$ (i.e. almost surely) we have the *volume independent* estimate for all $n > \mathcal{N}(\omega) := \mathcal{E}(\omega)^{1/(\eta-1)}$

$$a_l^1(n) \ \leqslant e^{\beta}\, e^{-\alpha n^{(1-\eta)\gamma}} \ . \tag{2.17}$$

To estimate the coefficients $a_l^2(n)$ from above , we use the upper bound

$$\begin{aligned} a_l^2(n) \ &\leqslant \ \int_{[1/n^{1-\eta},\infty)} \nu_l^{\omega}(\mathrm{d}E)\ e^{-n\beta E} \ \leqslant \ e^{-\beta n^{\eta}/2} \int_{[1/n^{1-\eta},\infty)} \nu_l^{\omega}(\mathrm{d}E)\ e^{-n\beta E/2} \\ &\leqslant \ e^{-\beta n^{\eta}/2} \int_{[0,\infty)} \nu_l^{\omega}(\mathrm{d}E)\ e^{-\beta E/2} \ . \end{aligned}$$

Then for some $K_2 > 0$ independent of l we obtain

$$a_l^2(n) \ \leqslant \ K_2 e^{-\beta n^{\eta}/2} \ . \tag{2.18}$$

Therefore, by (2.16) in the case $\overline{\rho} < \rho_c$, and by (2.17), (2.18) for $\overline{\rho} \geq \rho_c$, we find that there exists a sequence $a(n)$ (independent of l) such that

$$a_l(n) \leqslant a(n) \quad \text{and} \quad \sum_{n \geqslant 1} a(n) < \infty \ . \tag{2.19}$$

Thus, the series (2.14) is uniformly convergent in l, and one can exchange the sum and the limit

$$\lim_{l\to\infty} f_l(t) = \lim_{l\to\infty} \sum_{n=0}^{\infty} a_l(n) = \sum_{n=0}^{\infty} \lim_{l\to\infty} a_l(n) \ .$$

The rest of the proof is largely inspired by the paper [9]. Let

$$\Omega^T_{(x,x')} := \{\xi : \xi(0) = x,\, \xi(T) = x'\}$$

be the set of continuous trajectories (paths) $\{\xi(s)\}_{s=0}^T$ in $\mathbb{R}^d$, connecting the points x, x', and let w^T denote the conditional Wiener measure on this set. Using the Feynman-Kac representation, see e.g. [27], we obtain the following limit

$$\begin{aligned}
\lim_{l\to\infty} a_l(n) &= \lim_{l\to\infty} \frac{1}{V_l} \mathrm{Tr}\, e^{-t\,h_l^0}\, e^{-n\beta(h_l^\omega-\mu_l)} \mathbf{1}(\mu_l \leqslant 1/n) \qquad (2.20)\\
&= \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\,\mathrm{d}x'\, e^{-t\,h_l^0}(x,x')\, e^{-n\beta(h_l^\omega-\mu_l)}(x',x)\\
&= e^{n\beta\mu_\infty} \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\,\mathrm{d}x'\, \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times\\
&\quad\times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta}\mathrm{d}s\, v^\omega(\xi(s))}\, \chi_{\Lambda_l,n\beta}(\xi) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\, \chi_{\Lambda_l,t}(\xi'),
\end{aligned}$$

where we denote by $\chi_{\Lambda_l,T}(\xi)$ the characteristic function of paths ξ such that $\xi(t) \in \Lambda_l$ for all $0 < t < T$. Using Lemma A.2, see Appendix A, we can eliminate these restrictions and also extend one spatial integration over the whole space

$$\begin{aligned}
\lim_{l\to\infty} a_l(n) &= e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x\, \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x' \times \qquad (2.21)\\
&\quad\times \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta}\mathrm{d}s\, v^\omega(\xi(s))}\,.
\end{aligned}$$

Now, by the *ergodic* theorem, we obtain

$$\begin{aligned}
&\lim_{l\to\infty} a_l(n) = \lim_{l\to\infty} \frac{1}{V_l} \mathrm{Tr}\, e^{-t\,h_l^0}\, e^{-n\beta(h_l^\omega-\mu_l)} \qquad (2.22)\\
&= e^{n\beta\mu_\infty} \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x' \Big\{ \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times\\
&\times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta}\mathrm{d}s\, v^\omega(\xi(s))} \Big\}\\
&= e^{n\beta\mu_\infty} \mathbb{E}_\omega \Big\{ \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta}\mathrm{d}s\, v^\omega(\xi(s))} \Big\}\,.
\end{aligned}$$

We then get the explicit expression for the limiting Laplace transform

$$\begin{aligned}
f(t;\mu_\infty) &= \sum_{n\geqslant 1} e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times\\
&\quad\times \mathbb{E}_\omega \Big\{ \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta}\mathrm{d}s\, v^\omega(\xi(s))} \Big\}\,,
\end{aligned}$$

which finishes the proof. □

Corollary 2.2.1 *For any $\overline{\rho}$ the sequence of random measures $\tilde{m}_l^{0,(1)}$ converges a.s. in the weak sense to a bounded, absolutely continuous non-random measure $\tilde{m}^{(0,1)}$, with density $F(\varepsilon)$ given by*

$$F(\varepsilon) := (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma \, g(\sqrt{2\varepsilon}\, n_\sigma) .$$

Here, S_d^1 denotes the unit sphere in $\mathbb{R}^d$, n_σ the outward drawn normal unit vector, $\mathrm{d}\sigma$ the surface measure on S_d^1 and the function g has the form

$$g(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}x \, e^{ikx} \sum_{n\geqslant 1} e^{n\beta\mu_\infty} \, \mathbb{E}_\omega\Big(K_\omega^{n\beta}(x,0)\Big). \tag{2.23}$$

Proof:

By Theorem 2.2.2, the existence of the weak limit $\tilde{m}^{(0,1)}$ follows from the existence of the limiting Laplace transform. Moreover, we have the following explicit expression

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{m}^{0,(1)}(\mathrm{d}\varepsilon) \, e^{-t\varepsilon} &= \int_{\mathbb{R}^d} \mathrm{d}x \, \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{d/2}} \sum_{n\geqslant 1} e^{n\beta\mu} \, \frac{e^{-\|x\|^2/2n\beta}}{(2\pi n\beta)^{d/2}} \times \\
&\times \; \mathbb{E}_\omega\Big\{ \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) \, e^{-\int_0^{n\beta} \mathrm{d}s \, v^\omega(\xi(s))} \Big\} \\
&= \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{d/2}} \sum_{n\geqslant 1} e^{n\beta\mu} \mathbb{E}_\omega\Big(K_\omega^{n\beta}(x,0)\Big) \; .
\end{aligned}$$

Using the identity

$$\frac{1}{(2\pi t)^{d/2}} \, e^{-\|x\|^2/2t} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}k \, e^{ikx} \, e^{-t\|k\|^2/2} \; ,$$

we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{m}^{0,(1)}(\mathrm{d}\varepsilon) \, e^{-t\varepsilon} &= \int_{\mathbb{R}^d} \mathrm{d}x \, \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}k \, e^{-t\|k\|^2/2} \, e^{ikx} \sum_{n\geqslant 1} e^{n\beta\mu} \mathbb{E}_\omega\Big(K_\omega^{n\beta}(x,0)\Big) \\
&= \int_{\mathbb{R}^d} \mathrm{d}k \, e^{-t\|k\|^2/2} \, g(k) \\
&= \int_{[0,\infty)} \mathrm{d}r \, e^{-tr^2} \, r^{d-1} \int_{S_d^1} \mathrm{d}\sigma \, g(rn_\sigma) \\
&= \int_{[0,\infty)} \mathrm{d}\varepsilon \, e^{-t\varepsilon} \, (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma \, g(\sqrt{2\varepsilon} n_\sigma) \; ,
\end{aligned}$$

and the corollary follows. □

Corollary 2.2.2 *The measure $\tilde{m}^{0,(1)}$ satisfies the following property*

$$\int_{[0,\infty)} \tilde{m}^{0,(1)}(\mathrm{d}\varepsilon) \;=\; \begin{cases} \overline{\rho} & \text{if } \overline{\rho} < \rho_c \\ \rho_c & \text{if } \overline{\rho} \geqslant \rho_c \end{cases}$$

Proof:

By virtue of (2.14) we have

$$\int_{[0,\infty)} \tilde{m}^{0,(1)}(\mathrm{d}\varepsilon) = f(0;\beta,\mu_\infty) = \lim_{l\to\infty} \sum_{n\geqslant 1} \frac{1}{V_l}\ \mathrm{Tr}\, e^{-n\beta(h_l^\omega-\mu_l)}\, \mathbf{1}(\mu_l \leqslant 1/n)\ .$$

Note that by uniformity of convergence of the sum, see (2.17), (2.18), we can take the limit term by term (for any value of $\overline{\rho}$), and then

$$\int_{[0,\infty)} \tilde{m}^{0,(1)}(\mathrm{d}\varepsilon) = \sum_{n\geqslant 1} \lim_{l\to\infty} \frac{1}{V_l}\ \mathrm{Tr}\, e^{-n\beta(h_l^\omega-\mu_l)} =$$
$$\sum_{n\geqslant 1} \int_{[0,\infty)} \nu(\mathrm{d}E)\, e^{-n\beta(E-\mu_\infty)} = \int_{[0,\infty)} \nu(\mathrm{d}E)\, (e^{\beta(E-\mu_\infty)} - 1)^{-1}\ ,$$

where we use Fubini's theorem for the last step. □

We are now ready for the proof of the main result of this section:

Proof of Theorem 2.2.1: We first treat the case $\overline{\rho} < \rho_c$. In this situation, the measure $\tilde{m}_l^{0,(2)}$ is equal to 0 for l large enough, see (2.11), since the solution μ_l^ω of the equation (2.9) is a.s. strictly negative for l large enough. Thus, the total occupation measure $\tilde{m}_l^0$ is reduced to $\tilde{m}_l^{0,(1)}$ and the Theorem follows from Corollary 2.2.1.

Now, consider the case $\overline{\rho} \geqslant \rho_c$. Choose a subsequence l_r such that the total kinetic-energy states occupation measures $\tilde{m}_{l_r}^0$ converge weakly and a.s., and let the measure $\tilde{m}^0$ be its limit. By Corollary 2.2.1, all subsequences of measures $\tilde{m}_{l_r}^{(1)}$ converge to the limiting measure $\tilde{m}^{0,(1)}$. Therefore, by (2.11), we obtain the weak a.s. convergence:

$$\lim_{r\to\infty} \tilde{m}_{l_r}^{0,(2)} \;=:\; \tilde{m}^{0,(2)}\ .$$

By Lemma 2.1.3, we know that the measure $\tilde{m}^0$ is absolutely continuous on $(0,\infty)$, and by Corollary 2.2.1 that $\tilde{m}^{0,(1)}$ is absolutely continuous on $[0,\infty)$. Therefore we get:

$$\tilde{m}^{0,\mathrm{a.c.}} = \tilde{m}^{0,(1)} + \tilde{m}^{0,(2)\mathrm{a.c.}}\ ,$$

where *a.c.* denotes the *absolute continuous* components.

By definition of the total measure (2.11), $\tilde{m}^0([0,\infty)) = \overline{\rho}$ and by Lemma 2.1.2, $\tilde{m}(\{0\}) \geq \overline{\rho} - \rho_c$. Thus, $\tilde{m}((0,\infty)) \leq \rho_c$ and by Corollary 2.2.2, we can then deduce that the measure $\tilde{m}^{0,(2)}$ has no absolutely continuous component and therefore consists at most of an atom at $\varepsilon = 0$. Consequently, the full measure $\tilde{m}^0$ can be expressed as

$$\tilde{m}^0 \;=\; \tilde{m}^{0,\text{a.c.}} \;+\; b\delta_0 \;=\; \tilde{m}^{0,(1)} \;+\; b\delta_0 \;,$$

and since by Corollary 2.2.2

$$b \;=\; \overline{\rho} \;-\; \int_{\mathbb{R}_+} \tilde{m}_{l_r}^{0,\text{a.c.}}(\mathrm{d}\varepsilon) \;=\; \overline{\rho} \;-\; \int_{\mathbb{R}_+} \tilde{m}_{l_r}^{0,(1)}(\mathrm{d}\varepsilon) \;=\; \overline{\rho} - \rho_c$$

for the converging subsequence $\tilde{m}_{l_r}^0$, we have

$$\lim_{l_r\to\infty} \tilde{m}_{l_r}^0 \;=\; \tilde{m}^{0,(1)} \;+\; (\overline{\rho} - \rho_c)\delta_0 \;.$$

By (2.24) and Corollary 2.2.1, this limit is independent of the subsequence. Then, the limit of any convergent subsequence is the same, and therefore, the total kinetic states occupation measures $\tilde{m}_l^0$ converge weakly to this limit (see [25], Chapter VIII.6, Theorem 1). □

Finite volume Lifshitz tails

In this subsection, we give the proof of one important building block of our analysis, Theorem 2.2.3 about the *finite-volume* Lifshitz tails. Recall that this behaviour is a well-known feature of disordered systems, essentially meaning that for random Schrödinger operators which are semi-bounded from below, there are exponentially few eigenstates with energy close to the bottom of the spectrum. To our knowledge, however, this is always shown only in the *infinite-volume* limit, see e.g. [10]. Here, we derive a *finite-volume* estimate for the density of states, uniformly in l, though it could be trivial for small volumes. As one would expect our result is weaker than the asymptotic one, in the sense that we prove it for Lifshitz exponent smaller than the limiting one.

Theorem 2.2.3 *Let the random potential v^ω satisfy the assumptions of Lemma 1.3.3. Then for any $\alpha > 0$ and $0 < \gamma < d/2$, there exists a set $\widetilde{\Omega} \subset \Omega$ of full* *measure,* $\mathbb{P}(\widetilde{\Omega}) = 1$*, such that for any configuration* $\omega \in \widetilde{\Omega}$ *one can find a positive finite energy* $\mathcal{E}(\omega) := \mathcal{E}_{\alpha,\gamma}(\omega)$*, for which one has the estimate*

$$\nu_l^\omega\big([0,E]\big) \leqslant e^{-\alpha/E^\gamma}$$

for all $E < \mathcal{E}(\omega)$ *and for all* l*.*

Remark 2.2.1 *We want to stress that the statement in Theorem 2.2.3 is valid for all* l*, but of course, it can be trivial for small* l*. For example from the positivity of the potential we know that* $\nu_l^\omega(E) = 0$ *for* $E < \pi^2 d/l^2$ *and therefore the estimate is trivial for* $l < \pi/\sqrt{\mathcal{E}(\omega)}$*.*

For the proof, we first need a result from [22].

Lemma 2.2.1 *Assume that the random potential satisfy the assumptions of Lemma 1.3.3. In particular, recall that*

$$p = \mathbb{P}\Big\{\omega : v^\omega(0) = 0\Big\} < 1.$$

Let $\alpha > p/(1-p)$*,* $B = \pi/(1+\alpha)$*, and* $E_1^{\omega,l,N}$ *be the first eigenvalue of the random Schrödinger operator with Neumann (instead of Dirichlet) boundary conditions on a cube of side* l*. Then, for* l *large enough, there exists an independent of* l *constant* $A = A(\alpha)$*, such that*

$$\mathbb{P}\left\{\omega : E_1^{\omega,l,N} < B/l^2\right\} < e^{-AV_l} . \tag{2.24}$$

A sketch of the proof of this lemma is given in Appendix B. In the rest of this section, all eigenvalues will correspond to Schrödinger operators with Dirichlet boundary conditions, unless they have an upper index N in which case they are the Neumann eigenvalues. Now we use Lemma 2.2.1 to prove the following result:

Lemma 2.2.2 *Assume that the random potential satisfies the assumptions of Lemma 1.3.3. Then for any* $\alpha > 0$ *and* $0 < \gamma < d/2$*,*

$$\sum_{n\geqslant 1} \mathbb{P}\left\{\sharp\Big\{i : E_i^{\omega,l} < 1/n\Big\} > V_l\, e^{-\alpha n^\gamma} \text{ , for some } l \geqslant 1\right\} < \infty .$$

Proof: Notice that

$$\sum_{n\geqslant 1}\mathbb{P}\left\{\sharp\left\{i : E_i^{\omega,l} < 1/n\right\} > V_l\, e^{-\alpha n^\gamma}\ ,\ \text{for some } l\geqslant 1\right\}\ =\sum_{n\geqslant 1}\mathbb{P}\left\{\bigcup_{l\geqslant 1} S_l^n\right\}, \tag{2.25}$$

where S_l^n is the set

$$S_l^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\} > V_l\, e^{-\alpha n^\gamma}\right\}.$$

The right-hand side of (2.25) does not provide a very useful upper bound, since the sets S_l^n are highly overlapping. We thus need to define a new refined family of sets to avoid this difficulty.

To this end we let $[a]_+$ be the smallest integer $\geqslant a$, and we define the family of sets

$$V_k^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+} < \frac{1}{n}\right\} \geqslant k\right\}.$$

Let $k := \left[V_l e^{-\alpha n^\gamma}\right]_+$. We now use a monotonicity property associated with the Dirichlet boundary condition, namely that $h_L^\omega \geqslant h_{L'}^\omega$ whenever $L' \geqslant L$. We shall use this fact intensively in the rest of the proof. Since $V_l = l^d$, this implies that $h_l^\omega \geqslant h^\omega_{\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+}$, we get

$$\sharp\left\{i : E_i^{\omega,\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+} < \frac{1}{n}\right\} \geqslant \sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\}.$$

If now $\omega \in S_l^n$, then by the definition of k we obtain

$$\sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\} \geqslant k\ \ ,$$

since the left-hand side is itself an integer. Thus, $S_l^n \subset V_k^n$ and

$$\mathbb{P}\Big(\bigcup_{l\geqslant 1} S_l^n\Big) \leqslant \mathbb{P}\Big(\bigcup_{k\geqslant 1} V_k^n\Big). \tag{2.26}$$

We define also the sets

$$W_k^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+} < \frac{1}{n}\right\} = k\right\}. \tag{2.27}$$

Let $\omega \in (V_k^n \setminus W_k^n)$. Then by $h^\omega_{\left[((k+1)e^{\alpha n^\gamma})^{1/d}\right]_+} \leqslant h^\omega_{\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+}$ we get

$$\sharp\left\{i : E_i^{\omega,\left[((k+1)e^{\alpha n^\gamma})^{1/d}\right]_+} < \frac{1}{n}\right\} \geqslant \sharp\left\{i : E_i^{\omega,\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+} < \frac{1}{n}\right\} \geqslant k+1\ .$$

Hence, $(V_k^n \setminus W_k^n) \subset V_{k+1}^n$, and therefore we have for any fixed n and k

$$V_k^n \subset W_k^n \cup V_{k+1}^n \ . \tag{2.28}$$

Applying this inclusion M times, for $k = 1, \dots, M$, we obtain

$$\bigcup_{k=1}^{M} V_k^n \subset \Big(W_1^n \cup \bigcup_{k=2}^{M} V_k^n\Big) \subset \Big(W_1^n \cup W_2^n \cup \bigcup_{k=3}^{M} V_k^n\Big) \subset \dots \subset \left(\bigcup_{k=1}^{M} W_k^n\right) \cup V_{M+1}^n \tag{2.29}$$

Then we take the limit $M \to \infty$ to recover the infinite union that one needs in (2.26) and we use the inclusion (2.29) to find the inequality

$$\begin{aligned} \mathbb{P}\big(\bigcup_{k\geq 1} V_k^n\big) &= \lim_{M\to\infty} \mathbb{P}\big(\bigcup_{k=1}^{M} V_k^n\big) \qquad (2.30)\\ &\leqslant \lim_{M\to\infty} \Big(\sum_{k=1}^{M} \mathbb{P}\big(W_k^n\big) + \mathbb{P}\big(V_{(M+1)}^n\big)\Big) = \sum_{k=1}^{\infty} \mathbb{P}\big(W_k^n\big) + \lim_{M\to\infty} \mathbb{P}(V_M^n). \end{aligned}$$

The limit in the last term can be calculated directly

$$\begin{aligned} \lim_{M\to\infty} \mathbb{P}(V_M^n) &= \lim_{M\to\infty} \mathbb{P}\left\{\omega : \sharp\Big\{i : E_i^{\omega,[(Me^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\Big\} \geqslant M\right\} \qquad (2.31)\\ = \quad & \lim_{M\to\infty} \mathbb{P}\left\{\omega : \nu^{\omega}_{[(Me^{\alpha n^\gamma})^{1/d}]_+}\big([0,1/n]\big) \geqslant \frac{M}{[(Me^{\alpha n^\gamma})^{1/d}]_+^d}\right\} \\ = \quad & \mathbb{P}\left\{\omega : \nu\big([0,1/n]\big) \geqslant Ke^{-\alpha n^\gamma}\right\}, \end{aligned}$$

for some constant K. In the last step we used dominated convergence theorem.

Now we can use the Lifshitz tails representation for the asymptotics of the a.s. non-random limiting density of states ν, see (1.23), which implies that

$$\limsup_{n\to\infty} e^{an^{d/2}} \nu\big([0,1/n]\big) \leq 1 \ , \tag{2.32}$$

for $a > 0$. Since we assumed that $0 < \gamma < d/2$, there exists $n_0 < \infty$ such that by (2.31) and (2.32) for all $n > n_0$ we get

$$\lim_{M\to\infty} \mathbb{P}(V_M^n) = 0.$$

This last result, along with (2.26) and (2.30), implies that

$$\sum_{n>n_0} \mathbb{P}\Big(\bigcup_{l\geq l_0} S_l^n\Big) \leqslant \sum_{n>n_0} \sum_{k=1}^{\infty} \mathbb{P}\big(W_k^n\big). \tag{2.33}$$

Now, we show that the upper bound in (2.33) is finite. First we split up the box $\Lambda_{\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+}$ into $m(k,n)$ disjoints sub-cubes of the side $l(k,n)$, with the following choice of parameters

$$\begin{aligned} m(k,n) &:= [kM_n]_+ , \quad M_n := B^{-d/2} e^{\alpha n^\gamma} n^{-d/2} , \\ l(k,n) &:= \frac{\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+}{(m(k,n))^{1/d}} . \end{aligned}$$

Here B is the constant that comes from Lemma 2.2.1. Now by the Dirichlet-Neumann inequality, see e.g. [32], Chapter XIII.15, we get

$$h^D_{\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+} \geqslant h^N_{\left[(ke^{\alpha n^\gamma})^{1/d}\right]_+} \geqslant \bigoplus_{j=1}^{m(k,n)} h^{j,N}_{l(k,n)}, \tag{2.34}$$

where $h^{j,N}_{l(k,n)}$ denotes the Schrödinger operator defined in the j-th sub-cube of the side $l(k,n)$, with Neumann boundary conditions. Note that, by the *positivity* of the random potential, we obtain

$$E^{\omega,N}_{j,2} \geqslant \varepsilon^N_{j,2} \geqslant \frac{\pi}{l(k,n)^2} \geqslant \frac{1}{n} . \tag{2.35}$$

Here $E^{\omega,N}_{j,2}$ denotes the *second eigenvalue* of the operator $h^{j,N}_{l(k,n)}$, and $\varepsilon^N_{j,2}$ the *second eigenvalue* of $-\Delta^{j,N}_{l(k,n)}$, i.e. the kinetic-energy operator defined in the j-th sub-cube of the side $l(k,n)$ with the Neumann boundary conditions.

By equation (2.35), we know that to estimate the probability of the set (2.27) by using the Dirichlet-Neumann inequality (2.34), only the *ground state* of each operator $h^{j,N}_{l(k,n)}$ is relevant. Since the sub-cubes are *stochastically independent*, we have

$$\begin{aligned} \mathbb{P}\big(W^n_k\big) &\leqslant \mathbb{P}\left\{\omega : \sharp\{j : E^{\omega,N}_{j,1} < 1/n\} = k\right\} \\ &\leqslant {}^{m(k,n)}C_k\, q^k (1-q)^{m(k,n)-k} \leqslant {}^{m(k,n)}C_k\, q^k \end{aligned}$$

with q being the probability $\mathbb{P}\{\omega : E^{\omega,N}_{j,1} < 1/n\}$. The latter can be estimated by Lemma 2.2.1. So, finally we obtain the upper bound

$$\mathbb{P}\big(W^n_k\big) \leqslant {}^{m(k,n)}C_k \exp\{-kA(l(k,n))^d\} . \tag{2.36}$$

Using Stirling's inequalities, see [26], Chapter II.12

$$(2\pi)^{1/2} n^{n+1/2} e^{-n} \leqslant n! \leqslant 2(2\pi)^{1/2} n^{n+1/2} e^{-n} .$$

we can give an upper bound for the binomial coefficients ${}^{m(k,n)}C_k$ in the form

$$\frac{2(2\pi)^{\frac{1}{2}}(kM_n+\delta)^{(kM_n+\delta+1/2)}\exp(-kM_n+\delta)}{(2\pi)k^{k+\frac{1}{2}}\exp(-k)\cdot(kM_n+\delta-k)^{(kM_n+\delta-k+1/2)}\exp(-kM_n+\delta-k)}\ , \tag{2.37}$$

where $\delta \geqslant 0$ is defined by

$$m(k,n) \;=\; [kM_n]_+ \;=\; kM_n+\delta\ .$$

Then (2.37) implies the estimate

$${}^{m(k,n)}C_k \;\leqslant\; K_1\frac{(kM_n+\delta)^{kM_n+\delta+1/2}}{k^{k+\frac{1}{2}}(kM_n-k)^{kM_n+\delta-k+1/2}} \;\leqslant\; K_1(M_n)^k\Big(\frac{(1+\sigma_1)^{(kM_n+\delta+\frac{1}{2})}}{(1-\sigma_2)^{(kM_n+\delta+\frac{1}{2}-k)}}\Big),$$

for some $K_1>0$ and

$$\sigma_1 := \delta(kM_n)^{-1},\ \sigma_2 := M_n^{-1}\ .$$

Since $\delta/k<1$ and $\sigma_{1,2}\to 0$ as $n\to\infty$, and also using the fact that $x\ln(1+1/x)\to 1$ as $x\to\infty$, we can find a constant $c>0$ such that, for n large enough one gets the estimate

$${}^{m(k,n)}C_k \;\leqslant\; K_1(M_n)^k\Big(\frac{(1+M_n^{-1})^{(kM_n)}}{(1-M_n^{-1})^{(kM_n-k)}}\Big) \;\leqslant\; K_1(M_n)^k\,e^{ck}\ . \tag{2.38}$$

The side $l(k,n)$ of sub-cubes has a lower bound

$$l(k,n)=\frac{\big[(ke^{\alpha n^{\gamma}})^{1/d}\big]_+}{(m(k,n))^{1/d}} \geqslant \frac{(ke^{\alpha n^{\gamma}})^{1/d}}{(ke^{\alpha n^{\gamma}}(Bn)^{-d/2}+\delta)^{1/d}} \geqslant \Big(B^{d/2}\,n^{d/2}\,\frac{1}{1+\sigma_1}\Big)^{1/d}. \tag{2.39}$$

Combining (2.38), (2.39) and (2.36) we obtain a sufficient upper bound

$$\begin{aligned}\sum_{k\geqslant 1}\mathbb{P}\big(W_k^n\big) \;&\leqslant\; \sum_{k\geqslant 1}{}^{m(k,n)}C_k\,e^{-kAl^d(k,n)}\\ &\leqslant\; \sum_{k\geqslant 1}K_1\,(M_n)^k\,e^{ck}\,e^{-k\,A\,B^{d/2}\,n^{d/2}/(1+\sigma_1)}\\ &\leqslant\; K_2\sum_{k\geqslant 1}\exp\Big\{k\Big(\alpha n^{\gamma}-(d/2)\ln(nB)+c-AB^{d/2}n^{d/2}\Big)\Big\}\\ &\leqslant\; K_3\sum_{k\geqslant 1}\exp k\Big(\alpha n^{\gamma}-AB^{d/2}n^{d/2}+K_4\Big) \;\leqslant\; K_5\,\exp(-K_6n^{d/2})\ .\end{aligned}$$

Here the K_i's are some finite, positive constants independent of k, n, l, for any n large enough. Now the lemma immediately follows from (2.33). □

Proof of Theorem 2.2.3:

Let $A(\omega, n)$ be the event in which $\nu_l^\omega\big([0,1/n]\big) > e^{-\alpha n^\gamma}$ for some l.
By Lemma 2.2.2, we have

$$\sum_{n \geqslant 1} \mathbb{P}\{\omega : A(\omega, n) \text{ occurs}\big) \ < \ \infty \ ,$$

and therefore, by the Borel-Cantelli lemma one gets that with probability one, only a *finite* number of events $A(\omega, n)$ occur. In other words, there is a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any $\omega \in \widetilde{\Omega}$ one can find a *finite* and independent on l number $n_0(\omega) < \infty$ for which, in contrast to the event $A(\omega, n)$, we have

$$\nu_l^\omega\big([0,1/n]\big) \ \leqslant \ e^{-\alpha n^\gamma}, \quad \text{for all } \ n > n_0(\omega) \text{ and for all } l \geqslant 1.$$

Define $\mathcal{E}(\omega) := 1/n_0(\omega)$. For any $E \leqslant \mathcal{E}(\omega)$, we can find $n \geqslant n_0(\omega)$ such that

$$\frac{1}{2n} \ \leqslant \ E \ \leqslant \ \frac{1}{n},$$

and the theorem follows with the constant α modified by a factor $2^{-\gamma}$. □

The Luttinger-Sy model

In this subsection, we come back to the Luttinger-Sy model introduced in Section 1.3.1. We now consider the corresponding BEC in the kinetic-energy states.

Let us first state the equivalent of Theorem 2.2.1 for this particular model.

Theorem 2.2.4 *Theorem 2.2.1 holds with the function g defined as follows*

$$\begin{aligned} g(k) \ &= \ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}x \, e^{ikx} \sum_{n \geqslant 1} e^{n\beta\mu_\infty} \frac{e^{-\|x\|^2(1/2n\beta)}}{(2\pi n\beta)^{d/2}} \times \\ &\times \int_{\Omega_{(0,x)}^{n\beta}} w^{n\beta}(\mathrm{d}\xi) \, \exp\Big(- \lambda \big(\sup_s \xi(s) - \inf_s \xi(s) \big) \Big) \ . \end{aligned}$$

Proof:

The scheme of the proof is the same as for Theorem 2.2.1. First, we note that Lemmas 2.1.2 and 2.1.3 apply immediately. Here, the positivity of the random potential has to be understood in terms of quadratic forms, that is

$$(a) \quad Q(h_l^\omega) \subset Q(h_l^0), \quad Q \ \text{ being the quadratic form domain,} \tag{2.40}$$

$$(b) \quad (\varphi, h_l^\omega \varphi) \ \geqslant \ (\varphi, h_l^0 \varphi), \ \text{ for all } \varphi \in Q(h_l^\omega) \ .$$

Before continuing, we need to highlight a minor change concerning the *finite-volume* Lifshitz tails arguments. Although the Theorem 2.2.3 is valid for the Luttinger-Sy model, its proof requires a minor modification. Indeed, the assumption of Lemma 2.2.1 is clearly not satisfied for the case of singular potentials, since the probability of having an impurity at any given point is zero due to the properties of the Poisson distribution. However, by direct calculation we can obtain the same estimate with the constant $B = \pi^2/4$ in (2.24). First, suppose that there is at least one impurity in the box, then the eigenvalues will be of the form (for some j)

$$(n^2\pi^2)/(L_j^\omega)^2, \quad n = 1, 2, \ldots$$

if I_j^ω is an inner interval (that is, its two endpoints correspond to impurities), and

$$((n+1/2)^2\pi^2)/(L_j^\omega)^2, \quad n = 0, 1, 2, \ldots.$$

if I_j^ω is an outer interval (that is, one endpoint corresponds to an impurity, and the other one to the boundary of Λ_l). Therefore, $E_1^{\omega,l,N} \geqslant B/l^2$ since obviously $L_j^\omega < l$. Now, if there is no impurity in the box Λ_l, then $E_1^{\omega,l,N} = 0 < B/l^2$. But due to the Poisson distribution (1.17), this happens with probability $e^{-\lambda l}$, proving the same estimate as in Lemma 2.2.1.

With this last observation, the proof of the Theorem 2.2.3 can be carried out without any further changes.

Our next step is to split up the measure $\tilde{m}_l$ into two, $\tilde{m}_l^{(1)}$ and $\tilde{m}_l^{(2)}$, see (2.11), and prove the statement equivalent to the Theorem 2.2.2.

Theorem 2.2.5 *The sequence of Laplace transforms of the measures* $\tilde{m}_l^{(1)}$

$$f_l(t;\beta,\mu_l) \ := \ \int_{\mathbb{R}} \tilde{m}_l^{(1)}(\mathrm{d}\varepsilon)\ e^{-t\varepsilon}$$

converges for any $t > 0$ *to a (non-random) limit* $f(t;\beta,\mu_\infty)$ *, which is given by*

$$\begin{aligned} f(t;\beta,\mu_\infty) \ &= \ \sum_{n\geqslant 1} e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}x\, \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\ &\times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\ \exp\Big(-\lambda\big(\sup_s \xi(s) - \inf_s \xi(s)\big)\Big). \end{aligned}$$

Proof:

We follow the proof of Theorem 2.2.2, using the same notation. The uniform convergence is obtained the same way, since the bounds (2.16), (2.17), and (2.18) are also valid in this case. As in (2.22), we can use the ergodic theorem to obtain

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty}\mathbb{E}_\omega \int_{\mathbb{R}} \mathrm{d}x \, \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \sum_j \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\, \chi_{I_j{}^\omega,n\beta}(\xi) \ . \quad (2.41)$$

We have used the fact that the Dirichlet boundary conditions at the impurities split up the space $\mathcal{H}_l$ into a direct sum of Hilbert spaces (see (1.20)). This can be seen from the expression

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \ \mathbb{E}_\omega \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta}\mathrm{d}s\, a\sum_{x_j^\omega\in X^\omega}\delta(\xi(s)-x_j^\omega)}$$

by formally putting the amplitude, a, of the point impurities (1.19) equal to $+\infty$. Because of the characteristic functions $\chi_{I_j^\omega,n\beta}$, which constrain the paths ξ to remain in the interval I_j^ω in time $n\beta$, the sum in (2.41) reduces to only *one* term:

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \int_{\mathbb{R}} \mathrm{d}x \, \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \ \mathbb{E}_\omega \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\, \chi_{(a_\omega,b_\omega),n\beta}(\xi) \ , \quad (2.42)$$

where (a_ω, b_ω) is the interval among the I_j^ω's which contains 0.

The expression in (2.42) can be simplified further by computing the expectation $\mathbb{E}_\omega$ explicitly.

First, note that the Poisson impurity positions a_ω, b_ω are independent random variables and by definition, a_ω is negative while b_ω is positive. For the random variable b_ω the distribution function is

$$\mathbb{P}\ (b_\omega < b) := \mathbb{P}\{(0,b) \text{ contains at least one impurity}\} = 1 - e^{-\lambda b},$$

and therefore its probability density is $\lambda e^{-\lambda b}$ on $(0,\infty)$. Similarly for a_ω one gets

$$\mathbb{P}\ (a_\omega < a) := \mathbb{P}\{(a,0) \text{ contains no impurities}\} = e^{-\lambda|a|} \ = e^{\lambda a},$$

and thus its density is $\lambda e^{\lambda a}$ on $(-\infty, 0)$. Using these distributions in (2.42) we obtain

$$\begin{aligned}
\lim_{l\to\infty} a_l(n) &= e^{n\beta\mu_\infty}\lambda^2 \int_{-\infty}^{0} \mathrm{d}a\, e^{\lambda a} \int_{0}^{\infty} \mathrm{d}b\, e^{-\lambda b} \int_{\mathbb{R}} \mathrm{d}x\, \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
&\quad \times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\, \chi_{(a,b)}(\xi) \\
&= e^{n\beta\mu_\infty}\lambda^2 \int_{-\infty}^{0} \mathrm{d}a\, e^{\lambda a} \int_{0}^{\infty} \mathrm{d}b\, e^{-\lambda b} \int_{\mathbb{R}} \mathrm{d}x\, \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
&\quad \times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\, \mathbf{1}(\sup_s(\xi(s)) \leqslant b)\, \mathbf{1}(\inf_s(\xi(s)) \geqslant a) \\
&= e^{n\beta\mu_\infty}\lambda^2 \int_{\mathbb{R}} \mathrm{d}x\, \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
&\quad \times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) \int_{-\infty}^{\inf_s(\xi(s))} \mathrm{d}a\, e^{\lambda a} \int_{\sup_s(\xi(s))}^{\infty} \mathrm{d}b\, e^{-\lambda b}\ ,
\end{aligned}$$

and the Theorem 2.2.5 follows by explicit computation of the last two integrals. □

Proof of Theorem 2.2.4: Having proved Theorem 2.2.5, it is now straightforward to derive the analogue of Corollary 2.2.1 for the Luttinger-Sy model. Note also that the Corollary 2.2.2 remains unchanged, since only the uniform convergence was used. With these results, the proof follows in the same way as for Theorem 2.2.1. □

2.2.2 Weak external potentials

Let us recall that the Schrödinger operator with a weak external potential is of the form

$$h_l = -\tfrac{1}{2}\Delta_D + v(x_1/l, \ldots, x_d/l)\ , \tag{2.43}$$

where v is a non-negative function defined on the unit cube Λ_1. The only assumption on v that we shall make in this section is that the first eigenvalue E_1^l of the Schrödinger operator (2.43) vanishes in the thermodynamic limit. In fact, we do not even require the critical density ρ_c to be finite, hence we do not need the technical assumptions made in Section 1.3.2 to hold.

Let us state the main result of this section.

Theorem 2.2.6 *The sequence $\{\tilde{m}_l^0\}_{l\geqslant 1}$ of the one-particle kinetic states occupation measures has a weak limit $\tilde{m}^0$ given by*

$$\tilde{m}^0(\mathrm{d}\varepsilon) \;=\; \begin{cases} (\overline{\rho}-\rho_c)\delta_0(\mathrm{d}\varepsilon) \;+\; F(\varepsilon)\nu^0(\mathrm{d}\varepsilon) & , \text{ if } \overline{\rho} \geqslant \rho_c \\ F(\varepsilon)\nu^0(\mathrm{d}\varepsilon) & , \text{ if } \overline{\rho} < \rho_c \end{cases}$$

where the density $F(\varepsilon)$ is defined by

$$F(\varepsilon) \;=\; \int_{\Lambda_1} \mathrm{d}x\, \left(e^{\beta(\varepsilon+v(x)-\mu_\infty)} - 1\right)^{-1} .$$

Note the similarity of this result with the free Bose gas. Indeed, the kinetic-energy states occupation measure density is reduced to the free gas one, with the energy shifted by the external potential v and then averaged over the unit cube.

Before proceeding with the proof, we need some intermediate results. Let us first recall an estimate due to Van den Berg, [28]. For any $t > 0$, it follows from the Golden-Thompson inequality (see e.g. [16], chapter X, notes)

$$\mathrm{Tr} e^{-th_l} \;\leqslant\; \mathrm{Tr} e^{-th_l^0}\, e^{-tv} \;=\; \int_{\Lambda_l} \mathrm{d}x\,(e^{-th_l^0})(x,x)\, e^{-tv(x/l)} \;,$$

where the last step follows since the external potential is a multiplication operator. Now, since the finite-volume kernel $(e^{-th_l^0})(x,y)$ is bounded above by the infinite-volume kernel $(e^{-t\frac{1}{2}\Delta})(x,y)$, and moreover the diagonal part of the latter is independent of x, one gets

$$\begin{aligned} \frac{1}{V_l}\,\mathrm{Tr} e^{-th_l} \;&\leqslant\; \frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x\,(e^{-t\frac{1}{2}\Delta})(x,x)\, e^{-tv(x/l)} \\ &=\; \frac{1}{(2\pi t)^{d/2}}\,\frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x\, e^{-tv(x/l)} \\ &=\; \frac{1}{(2\pi t)^{d/2}}\int_{\Lambda_1} \mathrm{d}x\, e^{-tv(x)} \;. \end{aligned} \tag{2.44}$$

Next, we show that the two quantities above actually coincide in the limit $l \to \infty$, that is

Lemma 2.2.3 *Let the Schrödinger operator h_l be defined as in (2.43). Then for any $t > 0$, the following holds*

$$\lim_{l\to\infty} \frac{1}{V_l}\,\mathrm{Tr}\, e^{-th_l} \;=\; \frac{1}{(2\pi t)^{d/2}}\int_{\Lambda_1} \mathrm{d}x\, e^{-tv(x)} \;. \tag{2.45}$$

Proof:

We use the Feynman-Kac representation, see e.g. [27], to obtain

$$\begin{aligned}\lim_{l\to\infty}\frac{1}{V_l}\operatorname{Tr} e^{-th_l} &= \lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x\,\frac{1}{(2\pi t)^{d/2}}\times\\ &\times\int_{\Omega^t_{(x,x)}} w^{n\beta}(\mathrm{d}\xi)\,e^{-\int_0^t \mathrm{d}s\,v(\xi(s)/l)}\,\chi_{\Lambda_l,t}(\xi)\ ,\end{aligned}$$

where the notation for the paths and the conditional Wiener measure are the same as in the proof of Theorem 2.2.2. Since the paths ξ are closed, it is straightforward to express them in terms of the *Brownian bridge* $\alpha(\tau), 0\leqslant\tau\leqslant 1$. Let $\tilde{\Omega}$ be the set of all such bridges and D be the associated measure. We then have

$$\begin{aligned}\lim_{l\to\infty}\frac{1}{V_l}\operatorname{Tr} e^{-th_l} &= \lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x\,\frac{1}{(2\pi t)^{d/2}}\times\\ &\times\int_{\tilde{\Omega}} D(\mathrm{d}\alpha)\,\exp\Big(-\int_0^t \mathrm{d}s\,v[x/l+\frac{\sqrt{t}}{l}\alpha(s/t)]\Big)\chi_l(\alpha;x/l)\ ,\end{aligned}\tag{2.46}$$

where $\chi(\alpha;x/l)$ is the restriction on the Brownian bridge to insure that the argument of the function v is well-defined, that is for a fixed $x\in\Lambda_1$

$$\chi_l(\alpha;x) := \Big\{\alpha : \big(x+\frac{\sqrt{t}}{l}\alpha(s/t)\big)\ \in\ \Lambda_1\ ,\ \forall\, s\in[0,t]\Big\}\ .\tag{2.47}$$

Letting $y=x/l$ in (2.46), one gets

$$\begin{aligned}\lim_{l\to\infty}\frac{1}{V_l}\operatorname{Tr} e^{-th_l} &= \int_{\Lambda_1}\mathrm{d}y\,\frac{1}{(2\pi t)^{d/2}}\times\\ &\times\lim_{l\to\infty}\int_{\tilde{\Omega}} D(\mathrm{d}\alpha)\,\exp\Big(-\int_0^t \mathrm{d}s\,v[y+\frac{\sqrt{t}}{l}\alpha(s/t)]\Big)\chi_l(\alpha;y)\ ,\end{aligned}\tag{2.48}$$

and since it is clear from (2.47) that the characteristic function $\chi(\alpha;x)$ converges pointwise to 1 when $l\to\infty$, it follows from the dominated convergence theorem that

$$\lim_{l\to\infty}\frac{1}{V_l}\operatorname{Tr} e^{-th_l} = \frac{1}{(2\pi t)^{d/2}}\int_{\Lambda_1}\mathrm{d}y\,e^{-tv(y)}\ .$$

□

Remark 2.2.2 *Note that using the identity*

$$\frac{1}{t^{d/2}} = (2\pi)^{d/2}\,C_d\int_0^\infty \mathrm{d}s\,e^{-ts}\,s^{(d/2-1)}\ ,$$

where the constant C_d is as in the Weyl formula (1.6), we can obtain

$$\begin{aligned}\frac{1}{(2\pi t)^{d/2}}\int_{\Lambda_1}\mathrm{d}x\, e^{-tv(x)} &= \int_{\Lambda_1}\mathrm{d}x\, C_d\int_0^\infty \mathrm{d}s\, e^{-ts}\, s^{(d/2-1)}\, e^{-tv(x)} \\ &= \int_{\Lambda_1}\mathrm{d}x\, C_d\int_{y\geqslant v(x)}\mathrm{d}y\, e^{-ty}\,(y-v(x))^{(d/2-1)} \\ &= \int_{[0,\infty)}\mathrm{d}y\, e^{-ty}\Big(C_d\int_{x:v(x)\leqslant y}\mathrm{d}x\,(y-v(x))^{(d/2-1)}\Big)\ .\end{aligned}$$

Since the left-hand side of (2.45) is simply the sequence of Laplace transforms of the density of states ν_l, this provides an alternative method of recovering the asymptotic density of states ν of the Schrödinger operator in the weak external potential, the explicit form of which we used in (1.36). This was first derived in [8], where the author used Dirichlet-Neuman bracketing techniques.

Using the last Lemma, we are in position to establish an explicit expression for the density function of the perfect Bose gas in the weak external potential.

Corollary 2.2.3 *Consider the perfect gas (1.9) constructed from the Schrödinger operator defined as in (2.43). The density in the thermodynamic limit is given by*

$$\begin{aligned}\rho^0(\mu) &= \sum_{n\geqslant 1}\frac{1}{(2\pi n\beta)^{d/2}}\int_{\Lambda_1}\mathrm{d}x\, e^{n\beta(\mu-v(x))} \\ &= \int_{[0,\infty)}\nu^0(\mathrm{d}E)\int_{\Lambda_1}\mathrm{d}x\,\left(e^{\beta(E+v(x)-\mu)}-1\right)^{-1}\ ,\end{aligned}\tag{2.49}$$

for any $\mu < 0$. Consequently, we get the following expression for the critical density (possibly infinite)

$$\begin{aligned}\rho_c &= \sum_{n\geqslant 1}\frac{1}{(2\pi n\beta)^{d/2}}\int_{\Lambda_1}\mathrm{d}x\, e^{-n\beta v(x)} \\ &= \int_{[0,\infty)}\nu^0(\mathrm{d}E)\int_{\Lambda_1}\mathrm{d}x\,\left(e^{\beta(E+v(x))}-1\right)^{-1}\ .\end{aligned}\tag{2.50}$$

Proof:

Since the occupation numbers of the perfect gas are known explicitely

$$\langle N_l(\phi_i^l)\rangle_l^0 = \left(e^{\beta(E_i^{\omega,l}-\mu_l)}-1\right)^{-1}\ ,$$

we can express the finite-volume density as

$$\begin{aligned}\rho_l^0(\mu) := \frac{1}{V_l}\sum_{i\geqslant 1}\langle N_l(\phi_i^l)\rangle_l &= \frac{1}{V_l}\sum_{i\geqslant 1}\frac{1}{e^{\beta(E_i^l-\mu)}-1} \\ &= \frac{1}{V_l}\sum_{n\geqslant 1}\sum_{i\geqslant 1}e^{-n\beta(E_i^l-\mu)} = \sum_{n\geqslant 1}\frac{1}{V_l}\,\mathrm{Tr}\, e^{-n\beta(h_l-\mu)}\ .\end{aligned}\tag{2.51}$$

for any $\mu < E_1^l$. Since $\mu < 0$ and v is non-negative, it follows from (2.44) that the sum in the right-hand side of (2.51) is uniformly convergent with respect to l. Hence, we can take the limit term by term to obtain

$$\rho^0(\mu) := \lim_{l\to\infty} \rho_l^0(\mu) = \sum_{n\geqslant 1} e^{n\beta\mu} \lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-n\beta h_l} ,$$

and using Lemma 2.2.3 leads to

$$\rho^0(\mu) = \sum_{n\geqslant 1} e^{n\beta\mu} \frac{1}{(2\pi n\beta)^{d/2}} \int_{\Lambda_1} \mathrm{d}x \, e^{-n\beta v(x)} . \tag{2.52}$$

Now, since

$$\frac{1}{(2\pi n\beta)^{d/2}} = \frac{1}{V_l} \operatorname{Tr} e^{-n\beta h_l^0} = \int_{[0,\infty)} \nu(\mathrm{d}E) \, e^{-n\beta E} ,$$

the first statement (2.49) follows from (2.52) by Fubini's theorem. Letting $\mu \uparrow 0$, we directly recover the expression (2.50) for the critical density. □

We are now prepared to continue the proof of the main result of this section, Theorem 2.2.6. As in the random case, we split the occupation measure into two parts

$$\begin{aligned}
\tilde{m}_l^0 &= \tilde{m}_l^{0,(1)} + \tilde{m}_l^{0,(2)} \qquad \text{with} \\
\tilde{m}_l^{0,(1)}(A) &:= \sum_{n\geqslant 1} \frac{1}{V_l} \operatorname{Tr} P_A \left(e^{-n\beta(h_l-\mu_l)}\right) \mathbf{1}(\mu_l \leqslant 1/n) , \\
\tilde{m}_l^{0,(2)}(A) &:= \sum_{n\geqslant 1} \frac{1}{V_l} \operatorname{Tr} P_A \left(e^{-n\beta(h_l-\mu_l)}\right) \mathbf{1}(\mu_l > 1/n) ,
\end{aligned}$$

and we prove the following statement.

Theorem 2.2.7 *The sequence of measures $\tilde{m}_l^{0,(1)}$ converges weakly to a measure $\tilde{m}^{0,(1)}$, which is absolutely continuous with respect to ν^0 with density $F(\varepsilon)$ given by*

$$F(\varepsilon) = \int_{\Lambda_1} \mathrm{d}x \, \left(e^{\beta(\varepsilon+v(x)-\mu_\infty)} - 1\right)^{-1} .$$

Proof: We follow the line of reasoning of the proof of Theorem 2.2.2. Let $g_l(t;\beta,\mu_l)$ be the Laplace transform of the measure $\tilde{m}_l^{0,(1)}$

$$\begin{aligned}
g_l(t;\beta,\mu_l) &= \int_{\mathbb{R}} m_l^{0,(1)}(\mathrm{d}\varepsilon) \, e^{-t\varepsilon} \qquad (2.53) \\
&= \sum_{n\geqslant 1} \frac{1}{V_l} \operatorname{Tr} e^{-t\,h_l^0} \left(e^{-n\beta(h_l-\mu_l)}\right) \mathbf{1}(\mu_l \leqslant 1/n) .
\end{aligned}$$

Again, our aim is to show the uniform convergence of the sum over n with respect to l. Let

$$\begin{aligned} a_l(n) &:= \frac{1}{V_l}\mathrm{Tr}\, e^{-t\,h_l^0}\, e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \leqslant 1/n) \\ &\leqslant \frac{1}{V_l}\mathrm{Tr}\, e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \leqslant 1/n)\ . \end{aligned} \tag{2.54}$$

Then for $\overline{\rho} < \rho_c$ we can apply a similar argument as for the random case, since the estimate $\mu_l < \mu_\infty/2 < 0$ still holds, to obtain

$$a_l(n) \leqslant e^{n\beta\mu_\infty/2}\int_{[0,\infty)} e^{-\beta\varepsilon}\nu_l(\mathrm{d}\varepsilon) \leqslant K_1\, e^{n\beta\mu_\infty/2}\ .$$

If $\overline{\rho} \geqslant \rho_c$, then $\mu_l \leqslant 1/n$ in (2.54) implies that

$$a_l(n) \leqslant e^{\beta}\sum_i e^{-n\beta\, E_i^l} \leqslant \frac{e^{\beta}}{(2\pi n\beta)^{d/2}}\int_{\Lambda_1}\mathrm{d}x e^{-n\beta v(x)},$$

where the last estimate follows from (2.44). Now the uniform convergence for the sequence $a_l(n)$ follows from (2.50), since we assumed that $\rho_c < \infty$. The latter implies also that for $\overline{\rho} \geq \rho_c$, $\mu_\infty(\beta,\overline{\rho}) = 0$. Thus, we can take the limit of the Laplace transform (2.53) term by term, that is

$$\begin{aligned} \lim_{l\to\infty} a_l(n) &= \lim_{l\to\infty}\frac{1}{V_l}\mathrm{Tr}\, e^{-t\,h_l^0}\, e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \leqslant 1/n) \\ &= \lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\, e^{-t\,h_l^0}(x,x')\, e^{-n\beta(h_l-\mu_l)}(x',x) \\ &= e^{n\beta\mu_\infty}\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\ \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\times \\ &\quad\times\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\,\chi_{\Lambda_l,t}(\xi')\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta}\mathrm{d}s\, v(\xi(s)/l)}\,\chi_{\Lambda_l,n\beta}(\xi)\ . \end{aligned} \tag{2.55}$$

Here we have used the Feynman-Kac representation for free $e^{-t\,h_l^0}(x,y)$ and for non-free $e^{-\beta h_l}(x,y)$ Gibbs semi-group kernels, see e.g. [27], and w^T stands for the conditional Wiener measure on the path-space $\Omega^T_{(x,y)}$.

Note that by Lemma A.2, which demands only the *non-negativity* of the potential v, we obtain for (2.55) the representation

$$\begin{aligned} &\lim_{l\to\infty}\frac{1}{V_l}\mathrm{Tr}\, e^{-t\,h_l^0}\, e^{-n\beta(h_l-\mu_l)} \\ &= e^{n\beta\mu_\infty}\lim_{l\to\infty}\int_{\mathbb{R}^d}\mathrm{d}x\,\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x'\,\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta}\mathrm{d}s\, v(\xi(s)/l)}. \end{aligned} \tag{2.56}$$

Now we express the trajectories ξ in terms of *Brownian bridges* $\alpha(\tau) \in \tilde{\Omega}, 0 \leqslant \tau \leqslant 1$, we denote the corresponding measure by D. Letting $\tilde{x} = x'/l$, we obtain

$$
\begin{aligned}
& \lim_{l\to\infty} \frac{1}{V_l} \mathrm{Tr}\, e^{-t\, h_l^0}\, e^{-n\beta(h_l-\mu_l)} \\
= \;& e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\Lambda_1} \mathrm{d}\tilde{x}\, \frac{e^{-\|x-l\tilde{x}\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
\times \;& \int_{\tilde{\Omega}} D(\mathrm{d}\alpha)\, \exp\Big(- \int_0^{n\beta} \mathrm{d}s\, v[(1-\frac{s}{n\beta})\tilde{x} + \frac{s}{n\beta}(x/l) + \frac{\sqrt{n\beta}}{l}\alpha(s/n\beta)]\Big)\; .
\end{aligned}
$$

Since the integration with respect to x is now over the whole space, we let $y = x - l\tilde{x}$ to get

$$
\begin{aligned}
& \lim_{l\to\infty} \frac{1}{V_l} \mathrm{Tr}\, e^{-t\, h_l^0}\, e^{-n\beta(h_l-\mu_l)} \\
= \;& e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\Lambda_1} \mathrm{d}\tilde{x}\, \frac{e^{-\|y\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
\times \;& \int_{\tilde{\Omega}} D(\mathrm{d}\alpha)\, \exp\Big(- \int_0^{n\beta} \mathrm{d}s\, v\big(\tilde{x} + \frac{s}{n\beta}(y/l) + \frac{\sqrt{n\beta}}{l}\alpha(s/n\beta)\big)\Big) \\
= \;& e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}y\, \frac{e^{-\|y\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Lambda_1} \mathrm{d}\tilde{x} e^{-n\beta v(\tilde{x})}\; ,
\end{aligned}
$$

where the last step follows from dominated convergence. Therefore, we obtain by (2.53) the following expression for the limiting Laplace transform

$$
\lim_{l\to\infty} g_l(t;\beta,\mu_l) \;=\; \sum_{n\geqslant 1} e^{-n\beta(E-\mu_\infty)} \frac{1}{(2\pi(n\beta+t))^{d/2}} \int_{\Lambda_1} \mathrm{d}x e^{-n\beta v(x)}\; .
$$

It is now straightforward to invert this Laplace transform (for each term of the sum), to find that

$$
F(E)\;\nu^0(\mathrm{d}E) = \lim_{l\to\infty} \tilde{m}_l^1(\mathrm{d}E) \;=\; \sum_{n\geqslant 1} e^{-n\beta(E-\mu_\infty)} \Big(\int_{\Lambda_1} \mathrm{d}x e^{-n\beta v(x)} \Big) \nu^0(\mathrm{d}E)\; .
$$

The Theorem then follows by Fubini's theorem. □

Corollary 2.2.4 *The measure $\tilde{m}^{0,(1)}$ satisfies the following property*

$$
\int_{[0,\infty)} \tilde{m}^{0,(1)}(\mathrm{d}\varepsilon) \;=\; \begin{cases} \overline{\rho} & \text{if } \overline{\rho} < \rho_c \\ \rho_c & \text{if } \overline{\rho} \geqslant \rho_c \end{cases}
$$

The proof of that result is exactly the same as its analogue in the random case, Corollary 2.2.2.

Proof of Theorem 2.2.6:

The proof of the equivalent result in the random case, the Theorem 2.2.1, can be used without any modifications, since both Lemmas 2.1.2 and 2.1.3 are valid for any non-negative potentials, and the Corollary 2.2.4 plays the role of its analogue result in the random case, the Corollary 2.2.2. □

2.3 The mean field Bose gas

Our results for the mean field Bose gas are weaker than for the perfect gas, as we do not establish the existence of a limit for the sequence of kinetic occupation measures $\tilde{m}_l^\lambda$. However, we do prove that kinetic generalised BEC must occur if generalised condensation occurs in the generalised eigenstates, and both phenomena are absent below the critical value μ_c of the chemical potential. Note that our result does not say what happens at the critical point. The only assumption on the external potential that we shall use in this section is the non negativity.

As we emphasised in Section 1.2.2, the mean field gas is *superstable*, which imply that the pressure is well defined for any real value of the chemical potential μ. Hence, the fixed density constraint

$$\overline{\rho} = \rho_l^\lambda(\mu)$$

has always a unique solution μ_l for any given l, and this is still true in the thermodynamic limit. In particular, we can without loss of generality consider that μ is kept fixed in the thermodynamic limit, instead of fixing the mean density $\overline{\rho}$ as in the perfect gas, see Section 1.2.1.

Theorem 2.3.1 *Consider a mean field Bose gas as defined in Section 1.1. Then, the following holds*

$$i) \text{ if } \mu > \mu_c, \text{ then } \lim_{\delta \downarrow 0} \liminf_{l\to\infty} \tilde{m}_l^\lambda([0,\delta]) \geqslant m^\lambda(\{0\}) > 0,$$
$$ii) \text{ if } \mu < \mu_c, \text{ then } \lim_{\delta \downarrow 0} \limsup_{l\to\infty} \tilde{m}_l^\lambda([0,\delta]) = m^\lambda(\{0\}) = 0$$

The proof requires some intermediate results. The first one is a monotonicity result, which was first established by Fannes and Verbeure, see [29], using correlation inequalities. Here, we prove it in a simpler way, using only a convexity argument.

Lemma 2.3.1 *The function* $i \to \langle N_l(\phi_i)\rangle_l^\lambda$ *is non-increasing.*

Proof:

Let us define $f : \mathbb{R}_+ \mapsto \mathbb{R}$ by

$$\begin{aligned} f(t) &:= \ln \operatorname{Tr} \mathrm{e}^{-\beta H_l(\mu;t)}, \\ \text{where} \quad H_l(\mu;t) &:= H_l(\mu) + t(N_l(\phi_m^l) - N_l(\phi_n^l)) \ , \end{aligned}$$

for some $1 \leqslant m < n$. It follows that

$$f'(0) \;=\; \beta^{-1} \langle N_l(\phi_n^l) - N_l(\phi_m^l)\rangle_l \ ,$$

and since the function f is convex, we have the following inequality

$$\beta^{-1} \langle N_l(\phi_n^l) - N_l(\phi_m^l)\rangle_l \;\leqslant\; f'(t) \ , \tag{2.57}$$

for any $t \geqslant 0$. Now we set $t = \frac{1}{2}(E_n^l - E_m^l)$. Note that with this choice $t \geqslant 0$, since we have assumed that $m < n$. From the explicit expression (1.10) for $H_l(\mu)$, we have

$$\begin{aligned} H_l(\mu;t) \;&=\; \sum_{i \neq m,n} (E_i^l - \mu) N_l(\phi_i^l) \;+\; \frac{\lambda}{2V_l} N_l^2 \\ &+\; (\frac{E_m^l + E_n^l}{2} - \mu) N_l(\phi_m^l) \;+\; (\frac{E_m^l + E_n^l}{2} - \mu) N_l(\phi_n^l) \ . \end{aligned}$$

Since the mean-field term $\frac{\lambda}{2V_l} N_l^2$ is symmetric with respect to a permutation of any two eigenstate indices i, j, it follows that $H_l(\mu;t)$ is symmetric with respect to the exchange of m and n. Hence

$$f'(t) \;=\; \frac{\operatorname{Tr}\left(N_l(\phi_n^l) - N_l(\phi_m^l)\right) \mathrm{e}^{-\beta H_l(\mu;t)}}{\operatorname{Tr} \mathrm{e}^{-\beta H_l(\mu;t)}} \;=\; 0 \ ,$$

which in view of (2.57) gives

$$\langle N_l(\phi_n^l) - N_l(\phi_m^l)\rangle_l \;\leqslant\; 0 \ ,$$

and the lemma follows since $m < n$ are arbitrary. □

Using that monotonicity property, we can now obtain the following estimate.

Lemma 2.3.2 *If* $\mu < \mu_c$*, then there exists a constant* K *independent of* l *such that*

$$\langle N_l(\phi_i^l)\rangle_l^\lambda \;\leqslant\; K$$

for all l *large enough and for all* i.

Proof:

In view of Lemma 2.3.1, it is sufficient to find an upper bound for the occupation number of the ground state ϕ_1^l.

We define an auxiliary Hamiltonian by

$$H_l^\lambda(\mu; r, s) := H_l^0(\mu) - sN_l(\phi_1^l) + \lambda r N_l - \frac{\lambda}{2} r^2 V_l \ , \tag{2.58}$$

where r, s are two strictly positive parameters. We want to use the Bogoliubov convexity inequality

$$\frac{\operatorname{Tr} e^B (A-B)}{\operatorname{Tr} e^B} \leqslant \ln \operatorname{Tr} e^A - \ln \operatorname{Tr} e^B \leqslant \frac{\operatorname{Tr} e^A (A-B)}{\operatorname{Tr} e^A} \tag{2.59}$$

with

$$A := -\beta H_l^\lambda(\mu; r, s) \qquad \text{and} \qquad B := -\beta H_l^\lambda(\mu) \ .$$

In order to simplify the proof, we shall denote the equilibrium states corresponding to $H_l^\lambda(\mu)$ and $H_l^\lambda(\mu; r, s)$ by $\langle - \rangle_l^\lambda$ and $\langle - \rangle_{l;r,s}^\lambda$ respectively. Applying the Bogoliubov inequality (2.59) with our choice of A and B, we obtain by a straightforward computation

$$s\langle N_l(\phi_1)\rangle_l^\lambda + \frac{\lambda}{2V_l} \langle (N_l - rV_l)^2 \rangle_l^\lambda \leqslant s\langle N_l(\phi_1)\rangle_{l;r,s}^\lambda + \frac{\lambda}{2V_l} \langle (N_l - rV_l)^2 \rangle_{l;r,s}^\lambda \ .$$

As we want an upper bound for the first term, we can neglect the second term in the left hand-side (since it is positive), and we only have to compute the right-hand side. This can be done explicitly, as our auxiliary Hamiltonian is quadratic, but we first need to choose our parameters r, s. As usual with the approximating Hamiltonian technique, see e.g. [18], we want the parameter r to "play the role" of the mean density.

More precisely, let $\rho^0(x)$ be the limiting density of particles for the perfect Bose gas at (strictly negative) chemical potential x, that is:

$$\rho^0(x) = \int_{[0,\infty)} \nu(\mathrm{d}E) \frac{1}{e^{\beta(E-x)} - 1} \ ,$$

and consider the equation

$$\frac{\mu - x}{\lambda} = \rho^0(x) \ . \tag{2.60}$$

Since $\lambda\rho^0(x) + x$ increases to $\lambda\rho_c$ as $x \uparrow 0$, it follows that for any fixed $\mu < \mu_c$, there exists a solution $x_\infty < 0$ to the equation (2.60), see Figure 2.1.

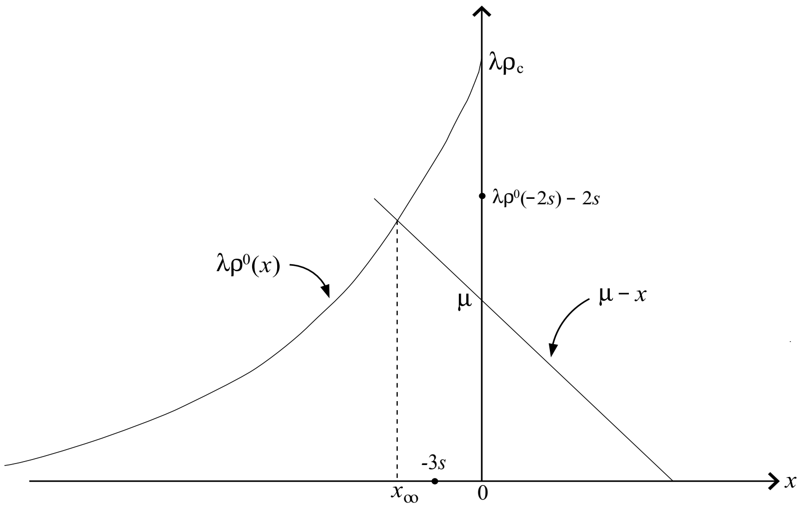

Figure 2.1: Graph of the equation 2.60

Let us fix the parameter $s > 0$ such that the following two constraints hold

$$x_\infty < -3s \quad \text{and} \quad \mu \,<\, \lambda\rho^0(-2s) - 2s \,<\, \lambda\rho_c \ . \tag{2.61}$$

Let $x := \mu - \lambda r$, and consider the finite-volume equation

$$\frac{\mu - x}{\lambda} = \frac{1}{V_l}\frac{1}{e^{\beta(E_1^l - x - s)} - 1} + \int_{[E_2^l,\infty)} \nu_l(\mathrm{d}E)\,\frac{1}{e^{\beta(E-x)} - 1} =: \frac{1}{V_l}\langle N_l\rangle_{l;r,s}^{\lambda} \ . \tag{2.62}$$

The right-hand of (2.62) side converges uniformly to $\rho^0(x)$ as $l \to \infty$ for any $x \in (-\infty, -2s)$, and hence, the equation (2.62) has a solution $x_l = \mu - \lambda r_l$ which converges to $x_\infty < -3s$ as $l \to \infty$. Hence, we have the following bound for l large enough

$$\lambda r_l - \mu \,>\, 2s \ . \tag{2.63}$$

Now, we are ready to go back to the Bogoliubov inequality (2.60), and we choose the parameters r_l, s to be as above. We then have

$$s\,\langle N_l(\phi_1^l)\rangle_l^\lambda \,\leqslant\, s\,\langle N_l(\phi_1^l)\rangle_{l;r_l,s} + \frac{\lambda}{2V_l}\,\langle\big((N_l - \overline{\rho}_l V)^2\big)\rangle_{l;r_l,s} \ . \tag{2.64}$$

First, we notice that the pressure $p_l^\lambda(\mu; r_l, s)$ associated with the auxiliary Hamiltonian $H_l^\lambda(\mu; r_l, s)$ (2.58), given by

$$p_l^\lambda(\mu; r_l, s) := \frac{1}{\beta V_l} \ln \mathrm{Tr}_{\mathscr{F}_l} \Big((E_1^l + \lambda r_l - \mu - s) a^*(\phi_1^l) a(\phi_1^l) + \sum_{i \geqslant 2} (E_i^l + \lambda r_l - \mu) N_l(\phi_i^l) - \frac{\lambda}{2} r_l^2 V_l \Big)$$

is well defined for all values of the chemical potential $\mu < \mu_c$, as the inequality (2.63) holds in this case. Then, we can get the required estimates. The first one is now straightforward

$$\langle N_l(\phi_1^l) \rangle_{l; r_l, s} = \frac{1}{e^{\beta(E_1^l + \lambda r_l - \mu - s)} - 1} \leqslant \frac{1}{e^{\beta s} - 1} . \tag{2.65}$$

For the second term, we get:

$$\begin{aligned} \frac{\lambda}{2V_l} \langle (N_l - r_l V_l)^2 \rangle_{l; r_l, s} &= \frac{\lambda}{2V_l} \langle (N_l - \langle N_l \rangle_{l; r_l, s})^2 \rangle_{l; r_l, s} \\ &= \frac{\lambda}{2} \partial_\mu^2 \, p_l^\lambda(\mu; r_l, s) . \end{aligned}$$

Note that the last step follows from the gauge invariance of the approximated Hamiltonian, that is $\left[H_l^\lambda(\mu; r_l, s), N_l \right] = 0$. Therefore, one gets:

$$\frac{\lambda}{2V_l} \langle (N_l - r_l V_l)^2 \rangle_{l; r_l, s} = \frac{\lambda}{2} \Big(\frac{1}{V_l} \frac{e^{\beta(E_1^l + \lambda r_l - \mu - s)}}{(e^{\beta(E_1^l + \lambda r_l - \mu - s)} - 1)^2} + \frac{1}{V_l} \sum_{i \geqslant 2} \frac{e^{\beta(E_i^l + \lambda r_l - \mu)}}{(e^{\beta(E_i^l + \lambda r_l - \mu)} - 1)^2} \Big) .$$

We then use the inequality

$$\frac{e^x}{e^x - 1} \leqslant 2(1 + \frac{1}{x}), \qquad x > 0$$

to get

$$\frac{\lambda}{2V_l} \langle (N_l - r_l V_l)^2 \rangle_{l; r_l, s} \leqslant \lambda (1 + \frac{1}{E_1^l + \lambda r_l - \mu - s}) \, r_l \leqslant \lambda (1 + \frac{1}{s}) r_l . \tag{2.66}$$

We can finish the proof by inserting (2.65) and (2.66) into the Bogoliubov inequality (2.64)

$$\langle N_l(\phi_1^l) \rangle_l^\lambda \leqslant \frac{1}{e^{\beta s} - 1} + \lambda (1 + \frac{1}{s}) r_l$$

where we note that $s > 0$ (and independent of l) and $r_l \to r_\infty < \infty$. Therefore, there exists a constant K, independent on l, such that $\langle N_l(\phi_1^l) \rangle_l^\lambda \leqslant K$. □

We are now prepared to prove our main result concerning kinetic generalised BEC in the mean field Bose gas.

Proof of Theorem 2.3.1:

The statement i) follows directly from Lemma 2.1.2. For the second part of that theorem, we use the expansion of the kinetic measure $\tilde{m}$ into the general eigenstates ϕ_i^l, see (2.3)

$$\frac{1}{V_l} \sum_{k:\varepsilon_k^l \leqslant \gamma} \langle N_l(\psi_k^l) \rangle_l^\lambda \;=\; \frac{1}{V_l} \sum_{k:\varepsilon_k^l \leqslant \gamma} \sum_{i \geqslant 1} |(\phi_i^l, \psi_k^l)|^2 \, \langle N_l(\phi_i^l) \rangle_l^\lambda .$$

Since we have assumed that $\mu < \mu_c$, we can use Lemma 2.3.2 and the fact that the functions ψ_k^l are normalised to obtain

$$\frac{1}{V_l} \sum_{k:\varepsilon_k^l \leqslant \gamma} \langle N_l(\psi_k^l) \rangle_l^\lambda \;\leqslant\; K \frac{1}{V_l} \sum_{k:\varepsilon_k^l \leqslant \gamma} \sum_{i \geqslant 1} |(\phi_i^l, \psi_k^l)|^2 \;\leqslant\; K \frac{1}{V_l} \sum_{k:\varepsilon_k^l \leqslant \gamma} 1 \;=\; K \, \nu_l^0\big([0,\gamma]\big).$$

Taking the thermodynamic limit leads to

$$\limsup_{l \to \infty} \tilde{m}_l^\lambda\big([0,\gamma]\big) \;\leqslant\; K \, \nu^0\big([0,\gamma]\big) \;=\; K \, \frac{2}{d} \, C_d \, \gamma^{d/2}$$

where the last step follows from the Weyl's formula (1.6). Letting $\gamma \downarrow 0$ proves the second statement of the theorem. □

Chapter 3

Localisation and BEC in *single* kinetic states

Having established the occurrence of kinetic generalised BEC in presence of an external potential in the previous chapter, the next question is to determine its type. As we discussed in Section 1.4, it is in general more difficult to find out the type of generalised BEC than to simply show the occurrence of generalised condensation, even when one considers BEC in the eigenstates.

The main idea of this chapter is to notice that in our models, the density of states is fast decreasing, which is generally believed to force the corresponding eigenstates to become localised *in the limit. However, the kinetic states are planes waves, hence delocalised in space. Hence, since these two states are "asymptotically orthogonal", this should prevent condensation to occur in any kinetic states.*

We first use the strong localisation property of the Luttinger-Sy model to prove in a simple way that the kinetic generalised BEC in this model is of the type III. We then extend that result to a more general class of localised systems, and establish that, for a more realistic random model and a general family of weak external potential, the required localisation criterion indeed holds.

The results of this chapter have been published in Journal of Mathematical Physics [30], a copy of this article is reproduced in Appendix E.

In Section 1.4, we briefly reviewed what could be rigorously proved for the condensation in single eigenstates ϕ_i^l, and in particular, we emphasised that the classification of the generalised BEC in the eigenstates required a fine knowledge of the spectrum, namely the speed at which the gap between any two eigenvalues vanishes in the limit $l \to \infty$. Clearly, this information cannot be extracted from the limiting density of states ν of the Schrödinger operator with a external potential. Hence, it is in general very complicated to classify the generalised BEC in the eigenstates, in particular in the random case where the required knowledge is quite hard to obtain even in simple examples like the Luttinger-Sy model.

In view of these difficulties, asking the same question about the generalised BEC in the kinetic states ψ_k^l could seem to be an even harder problem, since the Hamiltonians of the perfect and mean field Bose gas are not diagonal if one performs the second quantisation in the basis defined by the kinetic states, and therefore all the usuals tools for this kind of problem are unavailable. Nevertheless, in view of our basic expansion (2.3), one has the following expression for the mean occupation number in a given kinetic eigenfunction

$$\frac{1}{V_l} \langle N_l(\psi_k^l) \rangle_l^\lambda = \frac{1}{V_l} \sum_i |(\phi_i, \psi_k)|^2 \langle N_l(\phi_i^l) \rangle_l^\lambda. \tag{3.1}$$

The main idea of this chapter is to notice that the coefficients $|(\phi_i, \psi_k)|$ should be very small, since on the one hand, the kinetic state ψ_k^l is a sine-wave, hence spatially extended, while on the other hand, the eigenstates ϕ_i^l should be localised in order for the density of states ν to decrease fast enough near the bottom of the spectrum, as this is the feature of the external potential responsible for the occurrence of condensation, at least in low dimensions. Note also that apart from these coefficients, the sum in (3.1) is just the mean density, and hence as these two kinds of states are "asymptotically orthogonal", it is reasonable to expect the right-hand side in (3.1) to vanish in the limit $l \to \infty$. Of course, since the said sum is infinite, one needs to control it carefully, which is what we shall do in this section.

As an easy example, we shall first show how to make sense of this argument in the Luttinger-Sy model, making use of a specific feature of that system. Indeed, this model may be seen as "perfectly localised", in the sense that there is no tunnelling effect between two regions separated by one impurity. Hence, it is possible to establish a *uniform* (with respect to i) localisation estimate in this model, which allows us to work out the sum in (3.1), and prove that the kinetic generalised BEC is of the type III in the Luttinger-Sy model.

Since this method relies heavily on this absence of a tunnelling effect, it cannot be extended to more general models. To avoid this difficulty, we shall show that, under a relatively weak localisation property, we can deal with the sum in (3.1) and still conclude that the kinetic generalised BEC must be of the type III. The rest of this section will then be devoted to specific studies of particular models for which we can prove localisation. We shall first investigate a continuous random model, in arbitrary dimension and with a bounded potential. Hence, the simple technique that we used in the Luttinger-Sy model will not be sufficient, and we shall need more work, using multiscale analysis methods developed in the area of localisation for random Schrödinger operators. However, we note that our localisation property turns out to be quite different from the usual concept in that field, and hence, some modifications will be needed.
Finally, we shall come back to the general class of weak external potentials for which we established the asymptotic behaviour of the density of states ν in Section 1.3.2, and prove our localisation estimate in any dimension and without any conditions on the parameters α_j introduced there.

These results, along with the presence of generalised BEC in the kinetic states for any dimension, allow us to answer the main question of this thesis: what is the nature of the condensation enhanced by localisation. First, for low-dimensional systems, $d = 1, 2$, while it makes sense to speak of Bose-Einstein condensation in the generalised sense, the same mechanism that produces it, is also the one that prevents macroscopic accumulation of particles into any single kinetic state, that is, there is type III generalised BEC. But, in addition, since our results are valid even for $d \geqslant 3$, it also implies that the presence of randomness or the addition of a weak potential, however small, forces the kinetic generalised BEC to be of the type III, even if the corresponding translation invariant system (for example, the isotropic free Bose gas in $\mathbb{R}^3$) would produce condensation in the ground state only. Hence, this shows that the latter is not a reliable criterion for Bose-Einstein condensation, while on the other hand, the concept of kinetic generalised BEC is more robust.

3.1 The Luttinger-Sy model

The main result of this section is the following.

Theorem 3.1.1 *In the Luttinger-Sy model as defined in Section 1.3.1, for both the perfect and mean field gases, none of the kinetic-energy eigenstates are macroscopically occupied:*

$$\lim_{l\to\infty} \frac{1}{l} \langle N_l(\psi_k^l)\rangle_l = 0 \quad \text{for all } k \geqslant 1,$$

that is, any kinetic generalised BEC must be of type III.

Note that since we are dealing with a one-dimensional model, $V_l = l$.

Remark 3.1.1 *This result may appear quite surprising in view of the fact that, if one instead considers generalised BEC in the eigenstates, it turns out to be of the type I, see discussion in Section 1.4, and moreover, the generalised BEC in the eigenstates is entirely concentrated in the (random) groundstate. This difference in the exact type of condensation between kinetic states and eigenstates occurs even though the densities of generalised BEC in either kinetic states or (random) eigenstates are actually equal (at least in the perfect gas), see Theorem 2.2.4, which emphasises the fact that the concept of generalised condensation is a more reliable description of the phenomenon of Bose-Einstein condensation.*

Since the impurities split up the interval $\Lambda_l = (-l/2, l/2)$ into a finite number $M^l(\omega)$ of sub-intervals $\left\{I_j^\omega\right\}_{j=1}^{M^l(\omega)}$, by virtue of the corresponding orthogonal decomposition of $h_{l,D}^\omega$, cf. (1.20), the normalized random eigenfunctions $\phi_s^{\omega,l}$ are in fact sine-waves with supports in each of these sub-intervals and thus satisfy

$$|\phi_s^{\omega,l}(x)| < \sqrt{\frac{2}{L_{j_s}^\omega}}\,\mathbf{1}_{I_{j_s}^\omega}(x) \quad , \quad 1 \leq j_s \leq M^l(\omega) \ . \tag{3.2}$$

We require an estimate of the size L_j^ω of these random sub-intervals, which we obtain in the following lemma.

Lemma 3.1.1 *Let $\lambda > 0$ be a mean concentration of the point Poisson impurities on $\mathbb{R}$. Then the eigenfunctions ϕ_j^ω are localized in sub-intervals of logarithmic size, in the sense that for any $\kappa > 4$, one has a.s. the estimate*

$$\limsup_{l\to\infty} \frac{\max_{1\leqslant j\leqslant M^l(\omega)} L_j^\omega}{\ln l} \leqslant \frac{\kappa}{\lambda}.$$

Proof: Define the set

$$S_l := \big\{\omega : \max_{1\leqslant j\leqslant M^l(\omega)} L_j^\omega > \frac{\kappa}{\lambda}\ln l\big\}.$$

Let $n := \big[2\lambda l/(\kappa \ln l)\big]_+$, and define a new box

$$\widetilde{\Lambda}_l := [-\frac{n}{2}(\frac{\kappa}{2\lambda}\ln l)\,,\, \frac{n}{2}(\frac{\kappa}{2\lambda}\ln l)] \supset \Lambda_l\ .$$

Let us split up this bigger box into n identical disjoints intervals $\{I_m^l\}_{m=1}^n$ of size $\kappa(2\lambda)^{-1}\ln l$. If $\omega \in S_l$, then there exists at least one empty interval I_m^l (interval without any impurities), and therefore the set

$$S_l \subset \bigcup_{m=1}^{n} \{\omega : I_m^l \text{ is empty}\}\ .$$

For the Poisson distribution (1.17), the probability for the interval I_m^l to be empty depends only on its size, and thus

$$\mathbb{P}(S_l) \leqslant n\,\exp(-\lambda\frac{\kappa}{2\lambda}\ln l) \leqslant \left[\frac{2\lambda l}{\kappa \ln l}\right]_+ l^{-\kappa/2}\ .$$

Since we chose $\kappa > 4$, it follows that

$$\sum_{l\geqslant 1}\mathbb{P}(S_l) < \infty\ .$$

Therefore, by the Borel-Cantelli lemma, there exists a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for each $\omega \in \widetilde{\Omega}$ one can find $l_0(\omega) < \infty$ with

$$\mathbb{P}\ \{\omega : \max_{1\leqslant j\leqslant M^l(\omega)} L_j^\omega \leqslant \frac{\kappa}{\lambda}\ln l\} = 1\ ,$$

for all $l \geqslant l_0(\omega)$. □

Proof of Theorem 3.1.1:

Note that the fact that generalised BEC in the kinetic states occurs in this model has been proved for the perfect Bose gas in Theorem 2.2.4, and in a weaker form for the mean field Bose gas in Theorem 2.3.1. We are now in position to prove that it is actually of type III.

In view of our basic expansion (2.3), we have the following expression for any k

$$\begin{aligned}
\frac{1}{l}\,\langle N_l(\psi_k^l)\rangle_l^\lambda &= \frac{1}{l}\sum_i |(\phi_i^{\omega,l},\psi_k^l)|^2\langle N_l(\phi_i^{\omega,l})\rangle_l^\lambda \\
&= \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l^\lambda \left|\int_{\Lambda_l}\mathrm{d}x\;\overline{\psi}_k(x)\,\phi_i^{\omega,l}(x)\right|^2 \\
&\leqslant \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l^\lambda\;\frac{1}{l}\left(\int_{\Lambda_l}\mathrm{d}x\,|\phi_i^{\omega,l}(x)|\right)^2 ,
\end{aligned}$$

where in the last step we have used the bound $|\psi_k^l(x)| \leqslant 1/\sqrt{l}$, see (1.2) . Therefore, by (3.2) and Lemma 3.1.1, we obtain a.s. the following estimate

$$\frac{1}{l}\,\langle N_l(\psi_k^l)\rangle_l^\lambda \,\leqslant\, \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l\;\frac{1}{l}\,\frac{\kappa}{\lambda}\,\ln l\;,$$

which is valid for for large enough l and for any $\kappa > 4$. The theorem then follows by taking the thermodynamic limit. □

3.2 A general localisation criterion

As we pointed out at the beginning of this chapter, the localisation estimate that we obtained in the Luttinger-Sy model, see Lemma 3.1.1 and equation (3.2) is uniform with respect to i (the index of the eigenstates). However, it turns out that in more general models, while we still expect localisation to happen, it does not seem possible to obtain a uniform estimate. Hence, we must find a way to deal with the infinite sum in (3.1), which is the aim of this section.

Let us introduce the notation

$$\rho_i^l \,:=\, \frac{1}{V_l}\,\langle N_l(\phi_i^l)\rangle_l^\lambda\;.$$

With this notation we can write the standard fixed density condition as

$$\sum_i \rho_i^l \,=\, \overline{\rho}\;.$$

As we discussed in Section 1.2, this sum is not uniformily convergent. To avoid this difficulty, we truncate it

$$\sum_{i=1}^{N} \rho_i^l \,\leqslant\, \overline{\rho}\;,$$

for some $N < \infty$. Letting

$$\rho_i \; := \; \limsup_{l \to \infty} \; \rho_i^l \; ,$$

and taking the infinite volume limit, we then get

$$\sum_{i=1}^{N} \rho_i \; = \; \limsup_{l \to \infty} \sum_{i=1}^{N} \rho_i^l \; \leqslant \; \overline{\rho} \; .$$

Letting N tend to infinity, this gives $\sum_{i=1}^{\infty} \rho_i \leqslant \overline{\rho}$. Since this sum converges, it follows that for any $\varepsilon > 0$, there exists $i_0 < \infty$ such that $\rho_{i_0} < \varepsilon$.

Now, we can use the basic expansion (2.3) for any given kinetic state, that is with the notation introduced in the present section

$$\frac{1}{V_l} \, \langle N_l(\psi_k^l) \rangle_l^\lambda \; = \; \sum_{i \geqslant 1} \, |(\phi_i^l, \psi_k^l)|^2 \, \rho_i^l \; .$$

We then split up this sum, and use the monotonicity property (see Lemma 2.3.1), the bound (1.2) for the kinetic states and the fact that these states are normalised to obtain

$$\begin{aligned} \frac{1}{V_l} \, \langle N_l(\psi_k^l) \rangle_l^\lambda \;\; &= \;\; \sum_{i \leqslant i_0} \, |(\phi_i^l, \psi_k^l)|^2 \, \rho_i^l \, + \, \sum_{i > i_0} \, |(\phi_i^l, \psi_k^l)|^2 \, \rho_i^l \\ &\leqslant \;\; \sum_{i \leqslant i_0} \, |(\phi_i^l, \psi_k^l)|^2 \, \rho_i^l \, + \, \rho_{i_0}^l \sum_{i > i_0} \, |(\phi_i^l, \psi_k^l)|^2 \\ &\leqslant \;\; \overline{\rho} \sum_{i \leqslant i_0} \, |(\phi_i^l, \psi_k^l)|^2 \, + \, \rho_{i_0}^l \\ &\leqslant \;\; \overline{\rho} \sum_{i \leqslant i_0} \, \left(l^{-d/2} \, ||\phi_i^l||_1 \right)^2 \, + \, \rho_{i_0}^l \; . \end{aligned} \tag{3.3}$$

Since i_0 is fixed, and independent on l, it follows that if $l^{-d/2} \, ||\phi_i^l||_1 \to 0$ as $l \to \infty$ for each i, then

$$\limsup_{l \to \infty} \, \frac{1}{V_l} \, \langle N_l(\psi_k^l) \rangle_l^\lambda \; \leqslant \; \varepsilon \; ,$$

and since ε is arbitrary

$$\lim_{l \to \infty} \, \frac{1}{V_l} \, \langle N_l(\psi_k^l) \rangle_l^\lambda = 0 \; .$$

The above argument leads us to define the following localisation criterion for the absence of single mode condensation in the kinetic energy states.

Definition 3.2.1 *We call an eigenfunction ϕ_i^l localised if it satisfies the following condition*

$$\lim_{l\to\infty} \frac{1}{l^{d/2}} \int_{\Lambda_l} \mathrm{d}x \, |\phi_i^l(x)| \; = \; 0 \; . \tag{3.4}$$

Let us emphasise that this localisation property does not need to be uniform with respect to i for our argument, since we only deal with finite sums, see (3.3). Note also that this localisation condition is not as strong as the usual localisation property, in the following sense. While localisation is frequently understood to be associated with the persistence of a pure point spectrum in the limit $l \to \infty$, at least near the bottom of the spectrum, the presence of a pure point spectrum is not necessary for the condition (3.4) to hold for all eigenfunctions.

We summarise the results of this section in the following theorem.

Theorem 3.2.1 *Assume that the eigenfunctions ϕ_i^l are localized in the sense of (3.4) for all i. Then, no kinetic state ψ_k^l can be macroscopically occupied, that is*

$$\lim_{l\to\infty} \frac{1}{V_l} \langle N_l(\psi_k^l) \rangle_l^\lambda \; = \; 0 \; , \tag{3.5}$$

which implies in particular that any possible kinetic generalised BEC in these models is of type III.

The rest of this chapter is devoted to a proof of localisation in the sense of Definition 3.4 in some specific models.

In the weak external case, we are able to prove localisation for the general class of potentials defined in Section 1.3.2.

In the random case, we are so far unable to establish this localisation criterion under the sole assumptions of Lemma 1.3.3 and condition (1.24). Nevertheless, we can prove the localisation criterion (3.4) in the Stollmann model that we introduced in Section 1.3.1, and our proof holds in any dimension. Due to the fact that the random potential in this model is bounded, it is not possible as in the Luttinger-Sy model, see Section 3.1, to show that the eigenstates have support on a sufficiently small region, and hence, the simple techniques that we used for the Luttinger-Sy model are not sufficient, and we shall instead adapt the methods of multiscale analysis developed in the field of localisation in random Schrödinger operators, see e.g. [23].

3.3 Proof of localisation in some specific models

3.3.1 The Stollmann model

Let us emphasize again that the localisation property (3.4) is very different from what is usually called "exponential localisation" in the literature about random Schrödinger operators (see for example [23]). In the standard literature localisation refers to the eigenfunctions of the infinite volume Hamiltonian and requires these functions, with energies in some band, to decay very fast, in many cases exponentially. This implies that the spectrum is pure point in that band. In our case we are dealing with eigenstates in finite volume with energies tending to zero as the volume increases and so these bear no relation to the infinite volume eigenfunctions. In particular, our localisation condition (3.4) does *not* imply that the spectrum is discrete in the thermodynamic limit. While we only need the L^1 norm not to diverge too fast, because our eigenfunction depends crucially on the volume and in particular, because we do not work at a fixed energy but with volume dependent eigenvalues, we have to deal with the additional problem of controlling the finite-volume behaviour. However, we find that in fact the *multiscale analysis* developed for the infinite volume case can be adapted to establish our localisation condition.

In the present section, we shall consider the random potential v^ω from the Stollmann model, see Section 1.3.1 for detailed definitions. For convenience, let us recall some notation. The one-particle random Schrödinger operator in finite volume is then given by

$$h_l^\omega \;=\; h_l^0 + v_l^\omega \;, \tag{3.6}$$

where v_l^ω is the restriction of v^ω to the cubic box Λ_l. As usual, the eigenfunctions and eigenvalues of h_l^ω are denoted by $\phi_i^{\omega,l}$ and $E_i^{\omega,l}$ respectively. Note that there exists a non-random $M < \infty$ such that $v^\omega(x) < M$ for any x and all ω.

In the rest of this section, $h_l^\omega(x)$ will denote the restriction of the Schrödinger operator $-\frac{1}{2}\Delta + v^\omega$ to the region $\Lambda_l(x)$, with Dirichlet boundary conditions.

Before we establish the localization criterion (3.4), we first prove that any given eigenvalue of h_l^ω tends to zero as l tends to ∞.

Lemma 3.3.1 *The eigenvalues of the Schrödinger operator (3.6) vanishes with probability one, that is*

$$\text{a.s.}-\lim_{l\to\infty} E_i^{\omega,l} = 0 \ . \tag{3.7}$$

for any i.

Proof:

Recall that ν denote the limiting density of states for the Schrödinger operator h_l^ω (3.6), that is, for any Borel subset $A \subset \mathbb{R}_+$,

$$\nu(A) := \lim_{l\to\infty} \frac{1}{V_l} \#\{i \,:\, E_i^{\omega,l} \in A\}. \tag{3.8}$$

If the i-th eigenvalue E_i^l were to *not* vanish in the infinite volume limit, it would be possible to find $\delta > 0$ such that $\nu([0,\delta]) = 0$. Hence, it is clearly sufficient to prove that for every $E > 0$, $\nu([0,E]) > 0$. Since the Stollmann model satisfies (1.24), the proof follows from Lemma 1.3.4. □

We now introduce the concepts and results of multiscale analysis that we shall use to prove localisation in the sense of (3.4).
Adhering to the terminology of [23], we first define so-called "good boxes".

Definition 3.3.1 *Given* $x \in \mathbb{Z}^d$*, a scale* l*, an energy* E*, a rate of decay* $\gamma > 0$*, we call the box* $\Lambda_l(x)$ (γ, E)*-good for a particular realization* ω *of the random potential (1.31) if* $E \notin \sigma(h_l^\omega(x))$ *and*

$$||\chi_l^{out}\,(h_l^\omega(x) - E)^{-1}\,\chi_l^{int}|| \;\leqslant\; e^{-\gamma l} \ . \tag{3.9}$$

Here $\sigma(h_l^\omega(x))$ *denotes the spectrum of* $h_l^\omega(x)$*, the norm in (3.9) refers to the operator norm in* $L^2(\Lambda_l(x))$*, and* $\chi_l^{int}, \chi_l^{out}$ *are the characteristic functions of the regions* $\Lambda_l^{int}(x), \Lambda_l^{out}(x)$ *respectively, which we define as follows*

$$\Lambda_l^{int}(x) \;:=\; \Lambda_{l/3}(x)\ , \quad \Lambda_l^{out}(x) \;:=\; \Lambda_l(x) \setminus \Lambda_{l-2}(x)\ .$$

Our proof depends crucially on the following important multiscale analysis result extracted from [23], where the author used an argument originally derived in [31].

Proposition 3.3.1 *Assume that* h_l^ω *is as above with random potential given by (1.31). Then for any* $\zeta > 0$ *and any* $\alpha \in \big(1, 2 - (4d/(4d+\zeta))\big]$*, there exist a* *sequence* $\{l_k\}, k \geqslant 1$, *satisfying* $l_1 \geqslant 2$ *and* $l_{k-1}^{\alpha} \leqslant l_k \leqslant l_{k-1}^{\alpha} + 6$ *for* $k \geqslant 2$, *and constants* $r > 0$ *and* $\gamma > 0$ *such that if* $I := [0, r]$,

$$\mathbb{P}\Big\{\omega : \text{for all } E \in I, \text{either } \Lambda_{l_k}(x) \text{ or } \Lambda_{l_k}(y) \text{ is } (\gamma, E)\text{-good}\Big\} \geqslant 1 - (l_k)^{-2\zeta}, \quad (3.10)$$

for all $k \geqslant 1$ *and for all* $x, y \in \mathbb{Z}^d$, *satisfying* $|x - y| > l_k$.

As usual in the literature about localisation in random Schrödinger operators, two key ingredients are required to establish this multiscale result.

1. The *Wegner estimate*, which controls the probability of some interval I of the real line to intersect the spectrum h_l^{ω}; note that this probabilistic estimate must hold for all l.

2. The *initial scale estimate*, which establishes that the estimate (3.10) holds for a given l with large enough probability and appropriate constants.

We postpone to Appendix C a brief outline of the proof of these two assumptions in the Stollmann model.

We shall also need the so-called *Eigenfunction Decay Inequality.* We state it in a convenient form for our purpose, and give a sketch of the proof in Appendix C. Note that this inequality has to be understood for a given realisation ω.

Proposition 3.3.2 *Let* h_l^{ω} *be defined as above, and* $\phi_i^{\omega,l}$ *to be one eigenfunction with eigenvalue* $E_i^{\omega,l}$ *in some interval* $[0, s]$. *Let* $x \in \Lambda_l$, *such that* $\Lambda_{l_k}(x) \subset \Lambda_l$. *If* $E_i^{\omega,l}$ *does not belong to the spectrum of* $h_{l_k}^{\omega}(x)$, *then the following inequality holds*

$$\|\chi_{l_k}^{int}(x)\phi_i^{\omega,l}\| \leqslant \kappa\|\chi_{l_k}^{out}(x)(h_{l_k}^{\omega}(x) - E_i^{\omega,l})^{-1}\chi_{l_k}^{int}(x)\| , \quad (3.11)$$

where the norms are $L^2(\Lambda_l)$*-norm, and the constant* κ *depends only on* M *and* s.

We shall also need the following technical result.

Lemma 3.3.2 *Let* $\{l_k\}_{k \geqslant 1}$ *be a sequence satisfying* $l_1 \geqslant 2$ *and* $l_{k-1}^{\alpha} \leqslant l_k \leqslant l_{k-1}^{\alpha} + 6$ *for* $k \geqslant 2$, *with some constant* $\alpha \in (1, 2)$. *For any* $0 < \delta < 1/7$, *the intervals* $[l_k^{\frac{1}{1-\delta}}, l_k^{\frac{1}{\delta}}]$ *are a covering for the set* $[l_1^{\frac{1}{1-\delta}}, \infty)$.

Proof:

It is sufficient to show that each interval overlaps with the next one, that is,

$$l_{k+1}^{\frac{1}{1-\delta}} \leqslant l_k^{\frac{1}{\delta}} \tag{3.12}$$

for all $k \geqslant 2$. By assumption, we have the following estimates

$$l_1^{\alpha^{k-1}} \leqslant l_k \leqslant (l_1+6)^{\alpha^{k-1}} \leqslant l_1^{3\alpha^{k-1}} ,$$

where the last inequality follows from the fact that $l_1 \geqslant 2$. Hence, in order to prove (3.12), it is enough to show that

$$l_1^{\frac{3\alpha^k}{1-\delta}} \leqslant l_1^{\frac{\alpha^{k-1}}{\delta}} .$$

This last condition is equivalent to

$$\frac{3\alpha^k}{1-\delta} \leqslant \frac{\alpha^{k-1}}{\delta} ,$$

that is, we only need the condition $\delta(1+3\alpha) \leqslant 1$, which is always satisfied as we have assumed that $\alpha \in (1,2)$ and $\delta < 1/7$. □

We are now ready to prove that for our model the localisation condition (3.4) is satisfied.

Lemma 3.3.3 *Assume that h_l^ω is as in (3.6) with random potential given by (1.31). Then almost surely, for all i,*

$$\lim_{l\to\infty} \frac{1}{V_l^{1/2}} \int_{\Lambda_l} dx\, |\phi_i^{\omega,l}(x)| = 0 . \tag{3.13}$$

Proof:

We first choose $0 < \delta < 1/7$ and $\zeta > (2d+1)/2\delta$ and then we take the constants α, γ and r, and the sequence $\{l_k\}$ to be those obtained in Proposition 3.3.1 for this value of ζ.
For a given scale l large enough, it follows from Lemma 3.3.2 that there exists $k = k(l)$ such that

$$l^\delta < l_k < l^{1-\delta} . \tag{3.14}$$

We now need to define the following "good event". Roughly speaking, it consists in excluding the possibity that too many cubes of side l_k could be *not* (γ, E)-good.

Definition 3.3.2 *For a given realisation ω and a given l, let $A(\omega, l)$ be the event in which, for all $E \in I$, for all $x, y \in \Lambda_l \cap \mathbb{Z}^d$ such that $|x-y| > l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ is (γ, E)-good.*

We shall first use the Borel-Cantelli lemma to show that almost surely $A(\omega, l)$ occurs for all l large enough. Let us define

$$X_l := \Big\{\omega : A(\omega, l) \,\text{is not true at scale}\, l\Big\}.$$

Then we can write

$$\begin{aligned} X_l &:= \Big\{\omega : \exists\, E \in I,\ \exists\, x, y \in \Lambda_l \cap \mathbb{Z} \text{ with } |x-y| > l_k, \\ &\qquad \text{such that both}\, \Lambda_{l_k}(x) \text{ and } \Lambda_{l_k}(y) \,\text{are not}\, (\gamma, E)\text{-good}\Big\} \\ &= \bigcup_{\substack{x,y\in\Lambda_l\cap\mathbb{Z} \\ |x-y|>l_k}} \Big\{\omega : \exists\, E \in I, \,\text{such that both}\, \Lambda_{l_k}(x) \,\text{and}\, \Lambda_{l_k}(y) \,\text{are not}\, (\gamma, E)\text{-good}\Big\} \,, \end{aligned}$$

and by Proposition 3.3.1 we obtain

$$\mathbb{P}(X_l) \,\leqslant\, l^{2d}\,(l_k)^{-2\zeta} \,\leqslant\, l^{-2(\delta\zeta - d)} \,,$$

where the last step follows from (3.14). Since $2(\delta\zeta - d) > 1$, it follows that

$$\sum_l \mathbb{P}(X_l) \,<\, \infty \,.$$

By the Borel-Cantelli lemma, almost surely there exists $L(\omega) < \infty$ such that the event $A(\omega, l)$ occurs for all $l > L(\omega)$.

Since by Lemma 3.3.1, each eigenvalue $E_i^{\omega,l}$ a.s. tends to 0 as l tends to ∞, $E_i^{\omega,l} \in I$ for l large enough almost surely. Hence, since the good event $A(\omega, l)$ (see Definition 3.3.2) happens a.s. for all $l > L(\omega)$, there exists a subset $\tilde{\Omega} \subset \Omega$ of full probability $\mathbb{P}(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$, there is $L_1(\omega, i) < \infty$ such that for all $l > L_1(\omega, i)$ and for any $x, y \in \Lambda_l \cap \mathbb{Z}^d$ satisfying $|x-y| > l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are $(\gamma, E_i^{\omega,l})$-good.

Now we take $\omega \in \tilde{\Omega}$, $l > L_1(\omega, i)$ and partition the box $\Lambda_l(0)$ into the "interior cube" $\Lambda_l^1 := \Lambda_{l-l_k}(0)$ and the "corridor" $\Lambda_l^2 := \Lambda_l(0) \setminus \Lambda_l^1$, see figure 3.1. We then split up the integral in (3.13) accordingly

$$\int_{\Lambda_l} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \;=\; \int_{\Lambda_l^1} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \,+\, \int_{\Lambda_l^2} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \,. \tag{3.15}$$

In the second term, we can use the Schwarz inequality and the fact that the eigenfunctions are $L^2(\Lambda_l)$-normalized to obtain

$$\int_{\Lambda_l^2} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \leqslant |\Lambda_l^2|^{1/2} \leqslant 2^d l^{(d-1)/2}\, l_k^{1/2} \leqslant 2^d l^{(d-\delta)/2}\,. \tag{3.16}$$

Note that we did not use any particular property of the eigenstates ϕ_i^l in the corridor apart from the normalisation. Indeed, this estimate (3.16) does not require any kind of localisation.

For the first term in (3.15), we shall use the eigenfunction decay inequality (3.11) of Proposition 3.3.2.
We cover the "interior cube" Λ_l^1 with disjoints subcubes Λ_j of side $l_k/3$. Let us call $\{x_j\}$ their respective centers. Then for each j, the cube $\Lambda_{l_k}(x_j)$ is included in Λ_l and Λ_j coincides with $\Lambda_{l_k}^{int}(x_j)$. In Figure 3.1, we show one of the subcubes Λ_j with the corresponding $\Lambda_{l_k}(x_j)$. This makes clear why we need a corridor Λ_l^2.

Using the Schwarz inequality and Proposition 3.3.2, we obtain for any j the estimate:

$$\begin{aligned}\int_{\Lambda_j} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| &\leqslant l^{d/2}\Big(\int_{\Lambda_l} \mathrm{d}x |\chi_{l_k}^{int}(x_j)\phi_i^{\omega,l}(x)|^2\Big)^{1/2}\\ &\leqslant l^{d/2}\Big(\kappa\|\chi_{l_k}^{int}(x)(h_{l_k}^{\omega}(x_j)-E_i^{\omega,l})^{-1}\chi_{l_k}^{out}(x)\|\Big)^{1/2}.\end{aligned}$$

Hence, for any j such that $\Lambda_{l_k}(x_j)$ is $(\gamma, E_i^{\omega,l})$-good, one has the following upper bound

$$\int_{\Lambda_j} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \leqslant l^{d/2}\mathrm{e}^{-\frac{1}{2}\gamma l_k} \leqslant l^{d/2}\, \mathrm{e}^{-\frac{1}{2}\gamma l^{\delta}}. \tag{3.17}$$

Now, we distinguish two cases.

The first one corresponds to the situation where all cubes $\Lambda_{l_k}(x_j)$ are $(\gamma, E_i^{\omega,l})$-good. It then follows directly from (3.16) and (3.17) that

$$\begin{aligned} l^{-d/2}\int_{\Lambda_l} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| &\leqslant 2^d \frac{l^{(d-\delta)/2}}{l^{d/2}} + l^{-d/2}\sum_{x_j\in\Lambda_l^1} l^{d/2}\, \mathrm{e}^{-\frac{1}{2}\gamma l^{\delta}}\\ &\leqslant 2^d l^{-\delta/2} + 3^d\, \frac{(l-l^{\delta})^d}{l^{d\delta}}\, \mathrm{e}^{-\frac{1}{2}\gamma l^{\delta}}. \end{aligned} \tag{3.18}$$

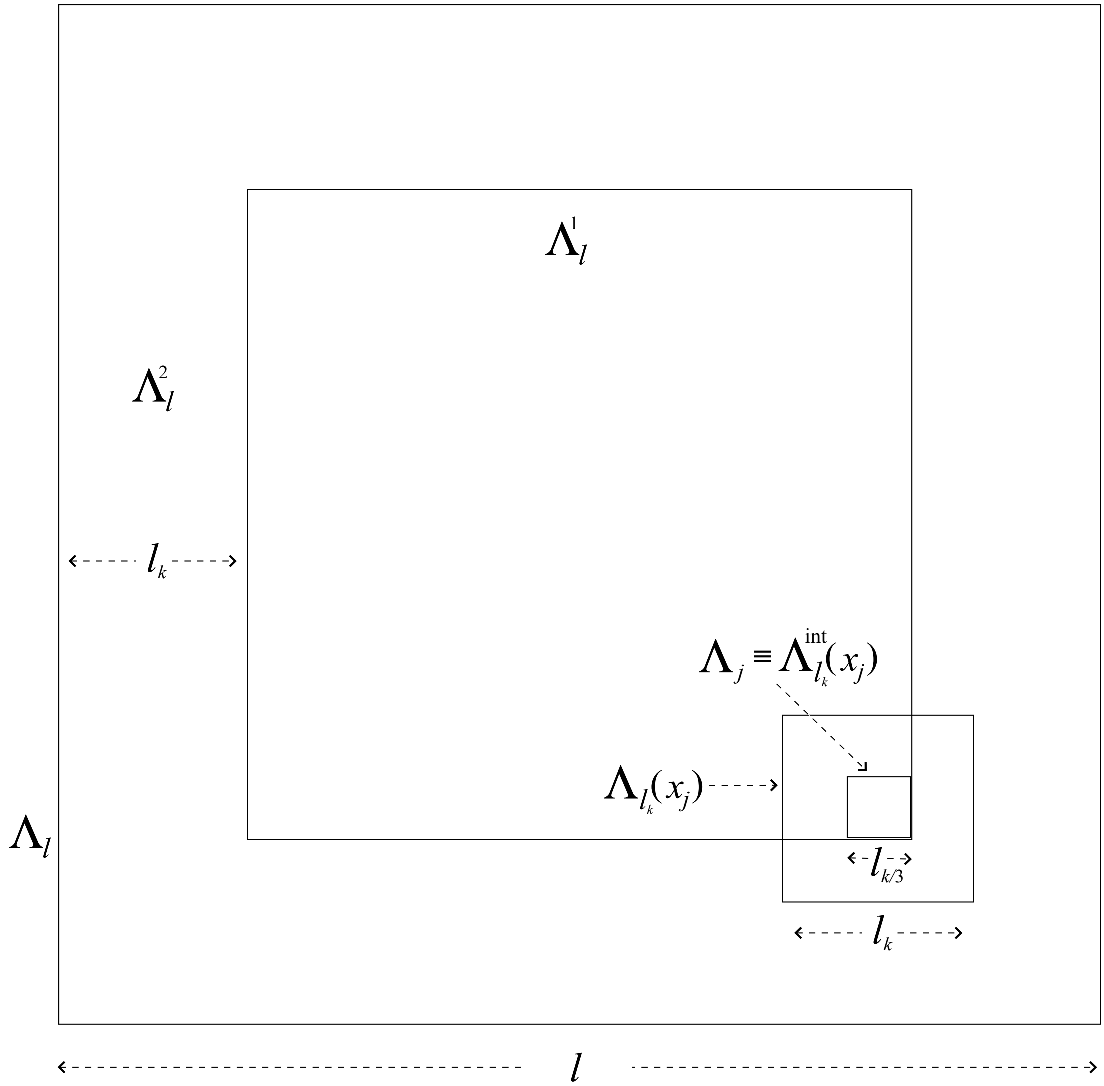

Figure 3.1: Subcubes for the multiscale analysis

The second case corresponds to the situation when there exists at least one subcube $\Lambda_{l_k}(x_j)$ which is *not* $(\gamma, E_i^{\omega,l})$-good. Let us denote by $\tilde{x}$ the center of one such bad cube. Since $\omega \in \tilde{\Omega}$ and $l > L_1(\omega, i)$, it follows from the fact that the event $A(\omega, l)$ (see Definition 3.3.2) happens that for any $x, y \in \Lambda_l \cap \mathbb{Z}^d$ satisfying $|x - y| > l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are $(\gamma, E_i^{\omega,l})$-good. We can therefore "isolate" all the possibly bad subcubes, that is there exists a box of side $2l_k$ centered at $\tilde{x}$ such that outside it, all other $\Lambda_{l_k}(x_j)$ are $(\gamma, E_i^{\omega,l})$-good. We treat the good boxes as above, and deal with $\Lambda_{2l_k}(\tilde{x})$ by using the Schwarz inequality as we did for Λ_l^2, to obtain:

$$\begin{aligned}\int_{\Lambda_l^1} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| &= \int_{\Lambda_l^1\setminus\Lambda_{2l_k}(\tilde{x})} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| + \int_{\Lambda_{2l_k}(\tilde{x})} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \\ &\leqslant \sum_{x_j\in\Lambda_l^1\setminus\Lambda_{2l_k}(\tilde{x})} l^{d/2}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta} + |\Lambda_{2l_k}(\tilde{x})|^{d/2} \\ &\leqslant l^{d/2} 3^d\, \frac{(l-l^\delta)^d}{l^{d\delta}}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta} + (2l)^{d(1-\delta)/2}.\end{aligned}$$

From that last bound and from (3.16), we get

$$l^{-d/2} \int_{\Lambda_l} \mathrm{d}x\, |\phi_i^{\omega,l}(x)| \leqslant 2^d l^{-\delta/2} + 3^d\, \frac{(l-l^\delta)^d}{l^{d\delta}}\, \mathrm{e}^{-\frac{1}{2}\gamma l^\delta} + 2^{d(1-\delta)/2} l^{-d\delta/2}. \qquad (3.19)$$

Therefore for any $\omega \in \tilde{\Omega}$ either (3.18) or (3.19) is satisfied for all l large enough and the localisation property (3.13) follows. □

Remark 3.3.1 *Let us stress that a crucial ingredient of our proof is the multiscale analysis result detailed in Proposition 3.3.1. Our methods can be extended to any model for which such a result is available, with some assumptions possibly weaker in the following sense.*

1. *The constant ζ does not need to be arbitrary, but it must be large enough in such a way that the Borel Cantelli lemma argument works (see the first part of the proof of Lemma 3.3.3).*
2. *The decay estimate for the "good" cubes, see Definition 3.3.1, could be only polynomial instead of exponential, but still strong enough in order for the upper bounds (3.18) and (3.19) to vanish in the limit $l \to \infty$. Note that we can make the "corridor" larger in order to limit the number the number of subcubes for which this rate of decay is needed (and hence, relax the minimal decay).*

We also want to point out that it is absolutely essential for our argument that the set defined in Proposition 3.3.1 has to be like

$$\mathbb{P}\Big\{\omega :\ \text{for all } E \in I,\ \text{either } \Lambda_{l_k}(x) \text{ or } \Lambda_{l_k}(y) \text{ is } (\gamma, E)\text{-good}\Big\} \geqslant 1 - (l_k)^{-2\zeta}\ ,$$

and not like

$$\text{for all } E \in I,\ \mathbb{P}\Big\{\omega :\ \text{either } \Lambda_{l_k}(x) \text{ or } \Lambda_{l_k}(y) \text{ is } (\gamma, E)\text{-good}\Big\} \geqslant 1 - (l_k)^{-2\zeta}\ ,$$

since we work for a fixed realisation ω *and a fixed* i *(the index of the eigenvalues), and then take the limit* $l \to \infty$*. This means that the eigenvalue* $E_i^{\omega,l}$ *is itself changing. We controlled that problem with the help of the Lemma 3.3.1, which guarantees that any given eigenvalue will eventually belong to the interval* $I = [0, r]$ *from Proposition 3.3.1.*

This leads us to the last assumption we could relax, that is we may allow the interval to be volume-dependent, i.e. consider the case $r = r(l)$*, as long as it does not vanish too fast. We then need to find an upper bound for any given eigenvalue, which can be done. However, due to the Lifshitz tails, it is reasonable to expect that the eigenvalues will not vanish fast (at best logarithmically), and hence, while we may allow the parameter* $r(l)$ *to vanish, it must do so "very slowly".*

3.3.2 Weak external potentials

Here, we consider the weak external potential as defined in Section 1.3.2. Recall that the Schrödinger operator with a *weak* external potential in a box Λ_l is defined by scaling the external potential v, that is

$$h_l \;=\; -\tfrac{1}{2}\Delta_D \,+\, v(x_1/l, \ldots, x_d/l)\ . \tag{3.20}$$

We recall that the eigenfunctions and eigenvalues of h_l are denoted by ϕ_i^l and E_i^l respectively. The aim of this section is to prove that our localization condition (3.4) holds for this class of weak potentials.

Lemma 3.3.4 *Let* h_l *be as in (3.20). Then, for all* i

$$\lim_{l\to\infty} \frac{1}{l^{d/2}} \int_{\Lambda_l} \mathrm{d}x\, |\phi_i^l(x)| \;=\; 0\ . \tag{3.21}$$

Proof: We start as in Lemma 1.3.5 by noting that the condition (1.32) implies that for any $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that for all $j = 1, \ldots, n$

$$(c_j - \varepsilon)|x - y_j|^{\alpha_j} \;\leqslant\; v(x) \;\leqslant\; (c_j + \varepsilon)|x - y_j|^{\alpha_j}, \tag{3.22}$$

for all $x \in B(y_j, \delta)$, the ball of radius δ centered at y_j. Note also that since the function v is continuous and vanishes only on the finite set $\{y_j\}_{j=1}^n$, there exists a constant $\kappa > 0$ such that $v(x) \geqslant \kappa$, for all $x \in \Lambda_1 \setminus \left(\bigcup_{j=1}^{n} B(y_j,\delta)\right)$. We let $K := \min(\kappa, c_1 - \varepsilon, \dots, c_n - \varepsilon)$ and $C := \max(c_1 + \varepsilon, \dots, c_n + \varepsilon)$.

The first step in our proof is to obtain an estimate for the eigenvalue E_i^l. To this end, let us denote by $h_l^{(n)}$ the restriction of the Schrödinger operator to the region $B(y_n, \delta l)$, with Dirichlet boundary conditions. Then we have

$$h_l \leqslant h_l^{(n)} \tag{3.23}$$

in quadratic form sense (cf. [32], Chapter VIII, Proposition 4). From the inequality (3.22), we obtain

$$h_l^{(n)} \leqslant \tilde{h}_l^{(n)} := \tfrac{1}{2}\Delta_D + C\left|\frac{x - y_n}{l}\right|^{\alpha_n}, \tag{3.24}$$

where the last operator acts on $L^2\big(B(y_j, \delta l)\big)$. Let $U : L^2\big(B(y_j, \delta l)\big) \mapsto L^2\big(B(0, \delta l^{1-\gamma_n})\big)$ be the unitary transformation defined by

$$(U\varphi)(x) := l^{\gamma_n/2}\,\varphi(l^{\gamma_n}(x - y_n))\ ,$$

where $\gamma_n := \alpha_n/(2 + \alpha_n)$. By direct computation, one can check that $\tilde{h}_l^{(n)} = l^{-2\gamma_n}\, U\, \hat{h}_l^{(n)}\, U^{-1}$ where

$$\hat{h}_l^{(n)} := (-\tfrac{1}{2}\Delta + C|x|^{\alpha_n})\ ,$$

acting on $L^2\big(B(0, \delta l^{1-\gamma_n})\big)$. Let $0 < D_1^l \leqslant D_2^l \leqslant \dots$ be the eigenvalues of $\hat{h}_l^{(n)}$ and $0 < D_1 \leqslant D_2 \leqslant \dots$ the eigenvalues of $\hat{h}^{(n)}$ where

$$\hat{h}^{(n)} := (-\tfrac{1}{2}\Delta + C|x|^{\alpha_n})\ ,$$

acting on $L^2(\mathbb{R}^d)$. Since for each i, $D_i^l \to D_i$ as $l \to \infty$, there are constants $\tilde{D}_i$ such that $D_i^l \leqslant \tilde{D}_i$ for all l. Using this and the operator inequalities (3.23) and (3.24) we finally get

$$E_i^l \leqslant D_i^l l^{-2\gamma_n} \leqslant \tilde{D}_i l^{-2\gamma_n}. \tag{3.25}$$

The rest of our proof relies on the methods developed in [33]. We start with some definitions. Let Ω^t, for some $t > 0$, to be the set of all continuous trajectories (paths) $\{\xi(s)\}_{s=0}^t$ in $\mathbb{R}^d$ with $\xi(0) = 0$, and let w^t denote the normalized Wiener measure on this set. For a given $x \in \mathbb{R}^d$, we define the following characteristic function

$$\chi_{x,l}(\xi) := \mathbf{1}\big\{\xi : \xi(s) \in \Lambda_l - x, \text{ for all } 0 \leqslant s \leqslant t\big\}\ .$$

We now use the following identity (cf. [34]),

$$(\mathrm{e}^{-th_l}\,\phi_i^l)(x) \;=\; \int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\,\phi_i^l(x+\xi(t))\,\chi_{x,l}(\xi)\ ,$$

from which, since E_i^l is the eigenvalue of h_l corresponding to ϕ_i^l, we get

$$|\phi_i^l(x)| \;\leqslant\; \mathrm{e}^{tE_i^l}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\,|\phi_i^l(x+\xi(t))|\,\chi_{x,l}(\xi)\ . \quad (3.26)$$

Now, we insert into the right-hand side of (3.26) the following bound proved in [35],

$$|\phi_i^l(x)| \;\leqslant\; c_d\,(E_i^l)^{d/4}\ ,$$

where $c_d := (e/\pi)^{d/4}$ and we obtain from (3.26) the following estimate

$$\begin{aligned} |\phi_i^l(x)| \;&\leqslant\; c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\,\chi_{x,l}(\xi) \\ &=\; c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\mathrm{e}^{-\frac{1}{t}\int_0^t \mathrm{d}s\, t\, v((x+\xi(s))/l)}\,\chi_{x,l}(\xi) \\ &\leqslant\; c_d\,\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\,\mathrm{e}^{-tv((x+\xi(s))/l)}\,\chi_{x,l}(\xi)\ , \end{aligned}$$

where the last step follows from Jensen's inequality. Therefore, integrating over Λ_l with respect to x, and then changing the order of integration, yields

$$\begin{aligned} l^{-d/2}\int_{\Lambda_l}\mathrm{d}x\,|\phi_i^l(x)| \;&\leqslant\; c_d\,l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Lambda_l}\mathrm{d}x\ \times \\ &\times\ \int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\,\mathrm{e}^{-tv((x+\xi(s))/l)}\,\chi_{x,l}(\xi) \\ &\leqslant\; c_d\,l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\ \times \\ &\times\ \int_{\{x\in\bigcap_{s'}(\Lambda_l-\xi(s'))\}}\mathrm{d}x\,\mathrm{e}^{-tv((x+\xi(s))/l)}\ . \end{aligned}$$

Letting $y = x+\xi(s)$ in the second integral we get

$$\begin{aligned} l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| \;&\leqslant\; c_d\,l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\,\frac{1}{t}\int_0^t \mathrm{d}s\ \times \\ &\times\ \int_{\{y-\xi(s)\in\bigcap_{s'}(\Lambda_l-\xi(s'))\}}\mathrm{d}y\,\mathrm{e}^{-tv(y/l)}\ . \end{aligned}$$

Since $\bigcap_{s'}(\Lambda_l - \xi(s') + \xi(s)) \subset \Lambda_l$ for all s, we can now extend the domain of integration over y to Λ_l and use the fact that the Wiener measure w^t is normalized to obtain

$$\begin{aligned} l^{-d/2}\int_{\Lambda_l} \mathrm{d}x|\phi_i^l(x)| &\leqslant c_d\, \mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\, l^{-d/2}\,\frac{1}{t}\int_0^t \mathrm{d}s \int_{\Lambda_l}\mathrm{d}y\, \mathrm{e}^{-tv((y)/l)} \\ &= c_d\, \mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\, l^{d/2}\int_{\Lambda_1}\mathrm{d}z\, \mathrm{e}^{-tv(z)}\ . \end{aligned} \tag{3.27}$$

Next, we obtain an upper bound for the last integral in (3.27). We have

$$\begin{aligned} \int_{\Lambda_1}\mathrm{d}z\, \mathrm{e}^{-tv(z)} &\leqslant \sum_{j=1}^{n}\int_{B(y_j,\delta)}\mathrm{d}z\, \mathrm{e}^{-tv(z)} + \int_{\Lambda_1\setminus\left(\bigcup_{i=1}^{n} B(y_j,\delta)\right)}\mathrm{d}z\, \mathrm{e}^{-tv(z)} \\ &\leqslant \mathrm{e}^{-tK} + \sum_{j=1}^{n}\int_{B(y_j,\delta)}\mathrm{d}z\, \mathrm{e}^{-tK|x-y_j|^{\alpha_j}}\ . \end{aligned} \tag{3.28}$$

For each j,

$$\int_{B(y_j,\delta)}\mathrm{d}z\, \mathrm{e}^{-tK|x-y_j|^{\alpha_j}} \leqslant t^{-d/\alpha_j}\, K^{d/\alpha_j}\int_{\mathbb{R}^d}\mathrm{d}\tilde{z}\, \mathrm{e}^{-|\tilde{z}|^{\alpha_j}} \leqslant \tilde{K}\, t^{-d/\alpha_j}\ ,$$

where $\tilde{K} := K^{d/\alpha_1}\max_j \int_{\mathbb{R}^d}\mathrm{d}\tilde{z}\, e^{-|\tilde{z}|^{\alpha_j}}$, which, in view of (3.28), gives the following bound

$$\int_{\Lambda_1}\mathrm{d}z\, \mathrm{e}^{-tv(z)} \leqslant \mathrm{e}^{-tK} + \tilde{K}\sum_{j=1}^{n} t^{-d/\alpha_j}\ .$$

Now, fixing $t = (E_i^l)^{-1}$, we get from the last inequality and (3.27)

$$l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| \leqslant c_d\, \mathrm{e}(E_i^l)^{d/4}\, l^{d/2}\left(\mathrm{e}^{-K(E_i^l)^{-1}} + \tilde{K}\sum_{j=1}^{n}(E_i^l)^{d/\alpha_j}\right)\ .$$

Since by (3.25), $E_i^l \to 0$ as $l \to \infty$, and since we have ordered the α_i's such that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, there exist new constants A_i such that the following bound holds for l large enough

$$l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| \leqslant A_i\, l^{d/2}\big(E_i^l\big)^{d(1/4+1/\alpha_n)} = A_i\, l^{d/2}\big(E_i^l\big)^{d(2-\gamma_n)/(4\gamma_n)}\ . \tag{3.29}$$

Inserting the bound (3.25), we finally obtain for l large enough

$$l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| \leqslant A_i\tilde{D}_i^{d(2-\gamma_n)/(4\gamma_n)}\, l^{-d(1-\gamma_n)/2}\ ,$$

and the lemma follows since $\gamma_n < 1$. □

Chapter 4

Generalised Bogoliubov approximation

By showing that one can correctly describe the phenomenon of condensation in the presence of suitable external potentials as one of "Bose-Einstein type", in the sense that condensation does indeed occur in the kinetic states of low-energy (see Chapter 2), we have cleared one obstacle in the way of applying the Bogoliubov approximation to these systems.

However, since the kinetic generalised BEC is of type III, this leads us to reconsider the usual one-mode *substitution. Indeed, since we anticipate that condensation does not occur in any given kinetic state, we should not expect the usual Bogoliubov approximation to give an accurate description of the fine structure of the condensate.*

We first establish a generalised Bogoliubov approximation, substituting c-numbers for all modes involved in the generalised BEC (that is, infinitely many in the limit). We show that this procedure does not affect the pressure of the system, if the complex numbers are chosen according to a suitable variational problem. As a first step in understanding the meaning of this new approach, we show by means of a very simple example why the use of external sources is able to alter drastically the fine structure of the condensate, but not the generalised condensate itself.

The results of this chapter have been accepted for publication in Journal of Mathematical Physics, [36].

4.1 Heuristic discussion

In 1947 Bogoliubov [14] proposed the Ansatz that for large Boson systems the creation and annihilation operators corresponding to zero momentum, a_0^*, a_0, can be replaced by complex numbers. This is called the Bogoliubov approximation. It is based on the idea that these creation and annihilation operators, when divided by the square root of the volume, V, of the region Λ containing the system, can be expressed as space averages

$$\frac{a_0^{\#}}{\sqrt{V}} := \frac{1}{V}\int_{\Lambda} \mathrm{d}x \, a^{\#}(x) \ ,$$

where $a^{\#}(x)$ are the usual local creation and annihilation operators. For translation invariant ergodic states these operators should converge in some weak sense to multiples of the identity, see e.g. [37]

$$\frac{a_0^{\#}}{\sqrt{V}} \to \alpha^{\#} \ .$$

These ideas were exploited by the school of Bogoliubov to construct various approximations to the full interacting boson Hamiltonian. We refer the reader to [15] for a review of these models. The most spectacular result of this Ansatz was its application to a model of a weakly interacting Bose gas [14], which gave the first microscopic theory of superfluidity and provided explicitly a spectrum of collective excitations satisfying the Landau criterion of superfluidity, see e.g. [15]. Superfluidity in these models is associated with the occurrence of Bose-Einstein condensation.

The first rigorous result concerning the Bogoliubov approximation was due to Ginibre [38]. He proved that if the Bogoliubov Ansatz is supplemented by a self-consistency equation which is obtained by maximizing the approximated pressure with respect to α, the exact pressure and the approximated one converge to the same value in the thermodynamic limit. A simpler proof of this result has recently been given by Lieb et al [39], using the Berezin-Lieb inequalities. A more delicate point is whether the value $\alpha_{\max}$ maximizing the approximated pressure corresponds to the condensate density in the ground state (or 0-mode condensation) in the thermodynamic limit. To answer this question, Bogoliubov suggested to *break* the gauge symmetry of the system, [14], by adding a source $\sqrt{V}(\eta a_0^* + \overline{\eta} a_0)$. This forces the *totality* of the condensate to be concentrated in the zero-mode (ground state). The source is then switched off ($|\eta| \to 0$ with a fixed gauge $\phi := \arg \eta$) *after* the thermodynamic limit to produce a limiting Gibbs state. The expectation defined by this state is called the Bogoliubov *quasi-average* with respect to this source, in contrast to the *average* of the gauge-invariant system. It was proven in [39] that $|\alpha_{max}|^2$ is equal to the ground state condensate density in the quasi-average sense.

In this chapter, we shall consider for simplicity the case when a random external potential is added to the system. However, similar results can be obtained for weak external potentials with only minor modifications to our arguments.

Following the general philosophy of the Bogoliubov transformation that the c-number used in the substitution corresponds to the condensate density in that mode, we would like to represent the likely presence of generalised BEC without single momentum mode macroscopic occupation by a generalised Bogoliubov approximation, in which we replace all creation/annihilation operators corresponding to momentum states with kinetic energy ε_k such that $0 < \varepsilon_k < \delta$ by complex numbers $\sqrt{V}\alpha_k^\sharp$. We show that this procedure does not affect the pressure if we maximize the approximated pressure with respect to these complex numbers, and then let $\delta \to 0$ after the thermodynamic limit.

Next, we discuss the interpretation of the variational problem established for the pressure. In particular, we highlight the fact that the link between the c-numbers that maximise the pressure and the condensate is far from straightforward. By means of a simple example, we discuss the relevance of the quasi-average method, and show that it is not very satisfactory when one suspects the generalised condensate to be of the type III (see discussion in Chapter 3) in the presence of external potentials.

4.2 Model and definitions

The one-particle operators are defined in the same way as in Section 1.1, apart from the choice of boundary conditions. In the rest of this chapter, we shall assume periodic boundary conditions. The kinetic-energy operator of our system is hence given by

$$h_l^0 := -\tfrac{1}{2}\Delta_p \ ,$$

acting in the Hilbert space $\mathscr{H}_l := L^2(\Lambda_l)$, with the subscript p denoting the choice of periodic boundary conditions. We let $\{\psi_k^l, \varepsilon_k^l\}_{k\in\Lambda^*}$ be the set of normalized eigenfunctions (that is, the *momentum states*) and eigenvalues corresponding to h_l^0

$$\psi_k^l(x) \ = \ \frac{1}{\sqrt{V_l}} e^{ik.x} \ , \quad \varepsilon_k^l \ = \ \frac{k^2}{2} \ ,$$

and Λ_l^* is the usual dual space $\{k \in \mathbb{R}^d : k^2 = \frac{n^2\pi^2}{l^2}, n \in \mathbb{N}^d\}$. As before, we denote by ν_l^0 the density of states of the kinetic-energy operator, and by ν^0 its weak limit. Note that the Weyl formula (1.6) holds in the case of periodic boundary conditions (with a modified constant).

The external potential is the family of random potentials defined as in Section 1.3.1, with the corresponding Schrödinger operator

$$h_l^\omega \ = \ h_l^0 \ + \ v_l^\omega \ ,$$

with periodic boundary conditions.

We assume that the particles interact through a two-body potential $u(x, y) := u(|x-y|)$. The second quantisation in the basis of momentum states $\{\psi_k^l\}_{k\in\Lambda^*}$ leads to the many-particles Hamiltonian

$$H_l(\mu) \ = \ \sum_{k,k'\in\Lambda_l^*} \Big(\sum_{i\geqslant 1} (\phi_i, \psi_k)(\psi_{k'}, \phi_i)(E_i^l - \mu)\Big) a_k^* a_{k'} \tag{4.1}$$

$$+ \ \frac{1}{2V_l} \sum_{q,k,k'\in\Lambda_l^*} \hat{u}_l(q) a_{k+q}^* a_{k'-q}^* a_{k'} a_k \ , \tag{4.2}$$

acting in the Fock space $\mathscr{F}_l$ (1.8). We use the notation $a_k^\# := a^\#(\psi_k^l)$ for the creation/annihilation operators in the momentum states, and the coefficients $\hat{u}_l(q)$ are defined by

$$\hat{u}_l(q) \ := \ \frac{1}{(2\pi)^{d/2}} \int_{\Lambda_l} \mathrm{d}x \, e^{ik.x} \, u(x) \ .$$

We shall assume that the function u satisfies the following conditions:

1. there exists $\gamma < \infty$ such that $|\hat{u}_l(q)| < \gamma$, for all q, l

2. u is *superstable*

3. u is *tempered*

The second and third conditions refer to the standard definitions in statistical mechanics, see e.g. [40], to ensure the existence of the thermodynamic limit of the system. If $u \in Ł^1(\mathbb{R}^d)$, the first condition is trivial, since we can take $\gamma = ||u||_1$. It was shown in [41] that, if one assumes in addition that u is positive-definite and $\hat{u}_l(0) > 0$, then the superstability condition is satisfied.

Note that the creation and annihilation operators in the interaction term of (4.1) are in the momentum eigenstates ψ_k^l, although the perfect Bose gas Hamiltonian (1.9) is not diagonal if it is expressed in the same basis.

We denote by $\langle - \rangle_l(\beta, \mu)$ the equilibrium state defined by the Hamiltonian $H_l(\mu)$

$$\langle A \rangle_l(\beta, \mu) := \frac{1}{\Xi_l(\beta, \mu)} \operatorname{Tr}_{\mathscr{F}_l} \exp(-\beta H_l(\mu)) \ ,$$

and by $p_l(\beta, \mu)$ its associated pressure

$$p_l(\beta, \mu) \ := \ \frac{1}{\beta V_l} \ln \Xi_l(\beta, \mu) \ ,$$

where

$$\Xi_l(\beta, \mu) \ := \ \operatorname{Tr}_{\mathscr{F}_l} \exp(-\beta H_l(\mu))$$

is the corresponding partition function. For simplicity, in the rest of this chapter we shall omit the explicit mention of the dependence on the temperature β.

It is known that the pressure of the corresponding non-random model (that is, $v^\omega(x) = 0$) exists and is independent of the boundary conditions for a large class of them, including the periodic case, see e.g. [42]. The proof of this statement consists essentially in showing the existence of the Dirichlet pressure using sub-additivity

$$p_\Lambda^D(\mu) \ \geqslant \ p_{\Lambda'}^D(\mu) + p_{\tau_x \Lambda''}^D(\mu) \ ,$$

where Λ', Λ'' are disjoints subsets of Λ, and τ_x denotes the translation by x. The exact value of x is chosen according to the usual tempering condition required of the two-body interaction potential u. Then, using translation invariance of the non-random model, one obtains

$$p_\Lambda^D(\mu) \ \geqslant \ p_{\Lambda'}^D(\mu) + p_{\Lambda''}^D(\mu) \ . \tag{4.3}$$

The boundeness of the pressure provided by the superstability of the system thus leads to the existence and finitness of the limiting pressure for any μ. Then, one can show using functional integration techniques, see [43], that the pressures defined with the other boundary conditions (including the periodic case) converge to the same limit.

The last part of this proof can be carried through verbatim in the presence of an external random potential. However, because of the lack of translation invariance in the random case, the inequality (4.3) for the Dirichlet pressure is modified as follows:

$$p_{\Lambda}^{D,\omega}(\mu) \geqslant p_{\Lambda'}^{D,\omega}(\mu) + p_{\tau_x \Lambda''}^{D,\omega}(\mu) = p_{\Lambda'}^{D,\omega}(\mu) + p_{\Lambda''}^{D,\tau_x\omega}(\mu) \ . \tag{4.4}$$

We have used the *stationarity* of the random potential in the last identity. To prove the existence of the thermodynamic limit one can use the Kingman sub-additive ergodic theorem, see [44]:

Proposition 4.2.1 *Let τ be a measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{g_n\}_{n\geq 1}$ be a sequence of functions $g_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the condition:*

$$g_{n+m}(\omega) \leq g_n(\omega) + g_m(\tau^n \omega) \ .$$

Then

$$a.s. - \lim_{n\to\infty} g_n(\omega)/n = g(\omega) \ ,$$

where the function $g(\omega)$ is τ-invariant: $g(\tau^s\omega) = g(\omega)$. If in addition, the functions g_n are ergodic, it follows that the limit $g(\omega)$ is a.s. non random.

4.3 The approximated pressure

4.3.1 Exactness of the generalised Bogoliubov approximation

Following Bogoliubov's approximation philosophy, we want to replace all creation and annihilation operators in momentum states ψ_k^l with kinetic energy less than some $\delta > 0$ by c-numbers. We let $I_\delta \subset \Lambda_l^*$ be the set of all *replaceable* modes, that is

$$I_\delta := \left\{ k \in \Lambda_l^* : k^2/2 \leqslant \delta \right\} ,$$

and we denote $n_\delta := \#\{k : k \in I_\delta\}$. Note that n_δ is of order V, since by definition $n_\delta = V_l \nu_l^0\big([0,\delta]\big)$. We let $\mathscr{H}_l^\delta$ to be the subspace of $\mathscr{H}_l$ spanned by the set of ψ_k^l with $k \in I_\delta$, and P_δ the projection onto this subspace. Hence, we have the natural representation for the Hilbert space and the associated symmetrised Fock space

$$\mathscr{H}_l = \mathscr{H}_l^\delta \oplus \mathscr{H}_l^\perp , \qquad \mathscr{F}_l = \mathscr{F}_l^\delta \otimes \mathscr{F}_l^\perp .$$

We then proceed to make the substitution $a_k^\sharp \to c_k$ for all $k \in I_\delta$, which provides an approximating Hamiltonian which we denote by $H_l^{Low}(\mu, \{c_k\})$. The reason for the superscript *Low* will be made clear in the next section. To keep the present chapter readable, we postpone the explicit form of this operator to Appendix D. We then obtain a new partition function and its associated pressure

$$\begin{aligned} \Xi_l^{Low}(\mu, \{c_k\}) &:= \operatorname{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{Low}(\mu,\{c_k\})} , \\ p_{l,\delta}^{Low}(\mu, \{c_k\}) &:= \frac{1}{V_l} \ln \Xi_l(\mu, \{c_k\}) . \end{aligned}$$

Our main result is the following:

Theorem 4.3.1 *The c-numbers substitution for all operators in the energy-band I_δ does not affect the pressure in the following sense*

$$\lim_{\delta\downarrow 0} \liminf_{l\to\infty} \max_{\{c_k\}} p_{l,\delta}^{Low}(\mu, \{c_k\}) = \lim_{\delta\downarrow 0} \limsup_{l\to\infty} \max_{\{c_k\}} p_{l,\delta}^{Low}(\mu, \{c_k\}) = \lim_{l\to\infty} p_l(\mu) . \tag{4.5}$$

Note that, since we let $\delta \downarrow 0$ after the thermodynamic limit, the number of substituted modes is of order of the volume.

4.3.2 The main proof

Our method is a generalisation of the one used in [39]. We postpone to the next section the proof of some technical lemmas in order to keep the main proof readable. Let us first define the normalised coherent vector

$$|c\rangle = \bigotimes_{k\in I_\delta} e^{-|c_k|^2/2 + c_k a_k^*} |0\rangle , \tag{4.6}$$

where $|0\rangle$ is the vacuum state in $\mathscr{F}_l$ and the c-numbers $\{c_k\}$ are as above. Note that $|c\rangle \in \mathscr{F}_l^\delta$. From these , we obtain the lower symbol A^{Low} for any operator A in $\mathscr{F}_l$ by the partial inner product

$$A^{Low}(\{c_k\}) := \langle c|A|c\rangle \ ,$$

which are then operators in $\mathscr{F}_l^\perp$. Next, we define the upper symbol. A^{Up} is called an upper symbol if it satisfies

$$A = \frac{1}{\pi^{n_\delta}} \int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1 \dots \mathrm{d}c_{n_\delta} \, A^{Up}(\{c_k\}) \, |c\rangle\langle c| \ .$$

Here $\mathrm{d}c_j := \mathrm{dRe}(c_j)\mathrm{dIm}(c_j)/\pi$ and $|c\rangle\langle c| := \bigotimes_{k\in I_\delta} |c_k\rangle\langle c_k|$ is the projector on the coherent vectors (4.6), with the completeness property $\int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2c_1 \dots \mathrm{d}^2c_{n_\delta} \, |c\rangle\langle c| = I$. Note that, contrary to the lower symbols, the upper symbols do not necessarily exist, and may not be unique either.

We then define two approximated Hamiltonians, that we denote $H_l^{\mathrm{Low}}(\mu, \{c_k\})$ and $H_l^{\mathrm{Up}}(\mu, \{c_k\})$, which are the lower and upper symbols of the Hamiltonian $H_l(\mu)$ (4.1). The existence of an upper symbol follows from the fact this Hamiltonian is polynomial in the creation/annihilation operators (although this does not imply unicity). We postpone to Appendix D the explicit expressions of these approximated Hamiltonians.

Note that $H_l^{\mathrm{Low}}(\mu, \{c_k\})$ is obtained simply by replacing all operators $a_k^\sharp$, $k \in I_\delta$ with the corresponding complex number $c_k^\sharp$. That is, it corresponds to the Hamiltonian obtained in the standard c-number substitution, which is the reason for using this notation in Theorem 4.3.1.

As before, we denote by $\Xi_l^{Up}(\mu, \{c_k\})$ the partition function defined by the Hamiltonian $H_l^{\mathrm{Up}}(\mu, \{c_k\})$, and $p_{l,\delta}^{Up}(\mu, \{c_k\})$ its corresponding pressure.

Finally, we define by $\langle - \rangle_{Low}$ and $\langle - \rangle_{Up}$ the equilibrium states defined by the following (integrated) partition functions

$$\begin{aligned}
\Xi_l^{Low}(\mu) &:= \frac{1}{\pi^{n_\delta}} \int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1 \dots \mathrm{d}c_{n_\delta} \mathrm{Tr}\, e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} \ , \\
\Xi_l^{Up}(\mu) &:= \frac{1}{\pi^{n_\delta}} \int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1 \dots \mathrm{d}c_{n_\delta} \mathrm{Tr}\, e^{-\beta H_l^{\mathrm{Up}}(\mu,\{c_k\})} \ ,
\end{aligned}$$

and we denote the associated pressures by $p_{l,\delta}^{Low}(\mu), p_{l,\delta}^{Up}(\mu)$.

By the Berezin-Lieb inequalities, see [39], we have

$$\Xi_l^{Low}(\mu) \leqslant \Xi_l(\mu) \leqslant \Xi_l^{Up}(\mu) \, . \tag{4.7}$$

We then relate the integrals to the maximum of their integrand. To this end, we first recall that the lower bound is fairly easy to obtain, since

$$\mathrm{Tr}\, e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} \leqslant \Xi_l^{Low}(\mu)$$

for any $\{c_k\}$, which in particular implies that

$$\max_{\{c_k\}} \mathrm{Tr}\, e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} \leqslant \Xi_l^{Low}(\mu) \, . \tag{4.8}$$

To estimate the upper bound in (4.7), we note that $H_l^{\mathrm{Low}}(\mu,\{c_k\})$ and $H_l^{\mathrm{Up}}(\mu,\{c_k\})$ are related in the following way,

$$H_l^{\mathrm{Up}}(\mu,\{c_k\}) = H_l^{\mathrm{Low}}(\mu,\{c_k\}) + \kappa(\mu,\{c_k\}) \, , \tag{4.9}$$

where the exact expression of $\kappa(\mu,\{c_k\})$ is derived in Appendix D. In view of the Bogoliubov convexity inequality

$$\begin{aligned} &\ln \frac{1}{\pi^{n_\delta}} \int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1 \ldots \mathrm{d}c_{n_\delta} \mathrm{Tr} e^{-\beta H_l^{\mathrm{Up}}(\mu,\{c_k\})} \\ - &\ln \frac{1}{\pi^{n_\delta}} \int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1 \ldots \mathrm{d}c_{n_\delta} \mathrm{Tr} e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} \\ \leqslant &\frac{\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1 \ldots \mathrm{d}c_{n_\delta} \mathrm{Tr}\big(-\kappa(\mu,\{c_k\})\big) e^{-\beta H_l^{\mathrm{Up}}(\mu,\{c_k\})}}{\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1 \ldots \mathrm{d}c_{n_\delta} \mathrm{Tr}\, e^{-\beta H_l^{\mathrm{Up}}(\mu,\{c_k\})}} \, , \end{aligned} \tag{4.10}$$

we obtain from (4.9) the following inequality

$$\ln \Xi_l^{Up}(\mu) \leqslant \ln \Xi_l^{Low}(\mu) - \langle \kappa(\mu,\{c_k\}) \rangle_{Up} \, . \tag{4.11}$$

Using the orthogonal projection $P_\delta : \mathscr{H}_l \mapsto \mathscr{H}_l^\delta$, and in view of (D.43), one can estimate the last term in (4.11) explicitly:

$$\begin{aligned} -\kappa(\mu,\{c_k\}) \leqslant\; & \mathrm{Tr}(h_l^\omega - \mu)P_\delta \\ + & -\gamma\Big(\nu_l^0\big([0,\delta]\big) + \frac{V_l}{2}\big(\nu_l^0\big([0,\delta]\big)\big)^2 + V_l \nu_l^0\big([0,\delta]\big)\nu_l^0\big([0,2\delta]\big)\Big) \\ + & \frac{\gamma}{2}\big(\frac{4}{V_l} + 2\nu_l^0\big([0,\delta]\big) + 2\nu_l^0\big([0,2\delta]\big)\big) \sum_{k\in I_\delta} |c_k|^2 \end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&+ \quad \frac{\gamma}{2}\big(2\nu_l^0\big([0,\delta]\big) + 2\nu_l^0\big([0,\delta]\big)\big) \sum_{k\in I_\delta^c} a_k^* a_k \\
&\leqslant \quad \mathrm{Tr}\big((h_l^\omega - \mu)P_\delta\big) \\
&- \quad \gamma\nu_l^0\big([0,\delta]\big)\Big(1 - 4V_l\nu_l^0\big([0,2\delta]\big) + \frac{V_l}{2}\nu_l^0\big([0,\delta]\big) + V_l\nu_l^0\big([0,2\delta]\big)\Big) \\
&+ \quad 4\gamma\nu_l^0\big([0,2\delta]\big)\Big(\sum_{k\in I_\delta}\big(|c_k|^2 - 1\big) + \sum_{k\in I_\delta^c} a_k^* a_k\Big) \ .
\end{aligned}
$$

Keeping in mind the upper symbol of the total number operator, we have the following

$$
H_l^{\mathrm{Up}}(\mu,\{c_k\}) + a\Big(\sum_{k\in I_\delta}(|c_k|^2-1) + \sum_{k\in I_\delta^c} a_k^* a_k\Big) \ = \ H_l^{\mathrm{Up}}(\mu - a,\{c_k\}) \ , \tag{4.13}
$$

which together with the equation (4.11) provides the following estimate

$$
\begin{aligned}
\Xi_l(\mu) \ &\leqslant \ \ln\frac{1}{\pi^{n_\delta}}\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\,\mathrm{Tr}\,e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} \qquad (4.14)\\
&+ \ \mathrm{Tr}\big((h_l^\omega - \mu)P_\delta\big) \\
&- \ \gamma\nu_l^0\big([0,\delta]\big)\Big(1 - 4V_l\nu_l^0\big([0,2\delta]\big) + \frac{V_l}{2}\nu_l^0\big([0,\delta]\big) + V_l\nu_l^0\big([0,2\delta]\big)\Big) \\
&+ \ 4\gamma\nu_l^0\big([0,2\delta]\big)\partial_\mu \ln\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\,\mathrm{Tr}\,e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} \ .
\end{aligned}
$$

To complete the proof, we shall need some lemmas, the proofs of which we postpone to the next section.

Lemma 4.3.1 *The physical systems described by the (integrated) partitions functions $\Xi_l^{Low}(\mu)$ and $\Xi_l^{Up}(\mu)$ have bounded mean densities for any fixed $\mu\in\mathbb{R}$ in the following sense*

$$
\begin{aligned}
\limsup_{\delta\downarrow 0}\limsup_{l\to\infty}\frac{1}{\beta}\partial_\mu p_{l,\delta}^{Low}(\mu) \ &\leqslant \ \partial_\mu p(\mu) := \rho(\mu) \ , \qquad (4.15)\\
\limsup_{\delta\downarrow 0}\limsup_{l\to\infty}\frac{1}{\beta}\partial_\mu p_{l,\delta}^{Up}(\mu) \ &\leqslant \ \partial_\mu p(\mu) := \rho(\mu) \ ,
\end{aligned}
$$

where $\rho(\mu)$, the mean density of the system without approximation, is finite for any $\mu\in\mathbb{R}$ because of the superstability of the two-body interaction potentail u.

Next, we show how to relate the integrated pressure $p_{l,\delta}^{Low}(\mu)$ to the maximum one.

Lemma 4.3.2 *For any $\alpha > 1$, the following holds*

$$\begin{aligned}
&\frac{1}{\beta V_l}\ln \Xi_l^{Low}(\mu) &(4.16)\\
\leqslant\ & \frac{1}{\beta V_l}\ln\max_{\{c_k\}}\mathrm{Tr}e^{-\beta H_l^{Low}(\mu,\{c_k\})} - \frac{1}{\beta V_l}\ln(1-1/\alpha) + \frac{\nu_l^0\big([0,\delta]\big)}{\beta}\ln(\alpha\partial_\mu p_{l,\delta}^{Low}(\mu))\\
+\ & \frac{1}{\beta}\nu_l^0\big([0,\delta]\big) - \frac{1}{2\beta}\frac{\ln V_l}{V_l} - \frac{\nu_l^0\big([0,\delta]\big)}{\beta}\ln\left(\nu_l^0\big([0,\delta]\big)\right) - \frac{1}{2\beta V_l}\ln\left(\nu_l^0\big([0,\delta]\big)\right)\ .
\end{aligned}$$

The last lemma is required because of the presence of the random external potential, and uses some ergodicity properties.

Lemma 4.3.3 *Under the assumptions on the random potential stated in Section 4.2, the following holds*

$$\limsup_{l\to\infty}\frac{1}{V_l}\mathrm{Tr}(h_l^\omega-\mu)P_\delta \leqslant \nu^0(\delta)\Big((\delta-\mu)+\mathbb{E}_\omega\big(v^\omega(0)\big)\Big)\ ,$$

where $\mathbb{E}_\omega$ denotes the expectation with respect to the probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Going back to the proof of Theorem 4.3.1, we have from (4.8) and (4.14)

$$\begin{aligned}
\max_{\{c_k\}} p_{l,\delta}^{Low}(\mu,\{c_k\}) \leqslant p_l(\mu) \ &\leqslant\ p_{l,\delta}^{Low}(\mu) + \frac{1}{\beta V_l}\Big(\mathrm{Tr}\big((h_l^\omega-\mu)P_\delta\big)\\
&-\ \gamma\nu_l^0\big([0,\delta]\big)\big(1-4V_l+\frac{V_l}{2}\nu_l^0\big([0,\delta]\big)+V_l\nu_l^0\big([0,2\delta]\big)\big)\Big)\\
&+\ 4\gamma\nu_l^0\big([0,2\delta]\big)\frac{1}{\beta V_l}\times\\
&\quad\times\partial_\mu\ \ln\int_{\mathbb{C}^{n_\delta}}\mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\ .
\end{aligned}$$

By Lemma 4.3.2, this implies

$$\max_{\{c_k\}} p_{l,\delta}^{Low}(\mu,\{c_k\}) \leqslant p_l(\mu) \leqslant \max_{\{c_k\}} p_{l,\delta}^{Low}(\mu,\{c_k\}) + K(l,\delta)\ , \tag{4.17}$$

where $K(l,\delta)$ is given by

$$\begin{aligned}
K(l,\delta) \ &=\ \frac{1}{\beta V_l}\Big(\mathrm{Tr}\big((h_l^\omega-\mu)P_\delta\big)-\gamma\nu_l^0\big([0,\delta]\big)\big(1-4V_l+\frac{V_l}{2}\nu_l^0\big([0,\delta]\big)+V_l\nu_l^0\big([0,2\delta]\big)\big)\Big)\\
&+\ 4\gamma\nu_l^0\big([0,2\delta]\big)\frac{1}{\beta V_l}\partial_\mu\ln\int_{\mathbb{C}^{n_\delta}}\mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\ .
\end{aligned}$$

Note that, by Lemmas 4.3.1 and 4.3.3, we can control this error term in the following way

$$\lim_{\delta\downarrow 0}\liminf_{l\to\infty}K(l,\delta) = \lim_{\delta\downarrow 0}\limsup_{l\to\infty}K(l,\delta) = 0\ ,$$

and hence, in view of (4.17)

$$\limsup_{l\to\infty} \max_{\{c_k\}} p_{l,\delta}^{Low}(\mu,\{c_k\}) \leqslant p(\mu) \leqslant \limsup_{l\to\infty} \max_{\{c_k\}} p_{l,\delta}^{Low}(\mu,\{c_k\}) + \liminf_{l\to\infty} K(l,\delta) .$$

Letting $\delta \downarrow 0$, one finally obtains

$$\limsup_{\delta\downarrow 0} \limsup_{l\to\infty} \max_{\{c_k\}} p_{l,\delta}^{Low}(\mu,\{c_k\}) \leqslant p(\mu) \leqslant \liminf_{\delta\downarrow 0} \limsup_{l\to\infty} \max_{\{c_k\}} p_{l,\delta}^{Low}(\mu,\{c_k\}) ,$$

which proves one of the two equalities in Theorem 4.3.1. The other is proved in the same way.

4.3.3 Some technical results

In this section, we give detailed proofs of the lemmas used in the preceding section.

Proof of Lemma 4.3.1

Notice first that, for any fixed $b \in \mathbb{R}$

$$H_l^{\mathrm{Low}}(\mu,\{c_k\}) + b\Big(\sum_{k\in I_\delta} |c_k|^2 + \sum_{k\in I_\delta^c} a_k^* a_k\Big) = H_l^{\mathrm{Low}}(\mu - b,\{c_k\}) , \tag{4.18}$$

and then, from equation (4.12), we obtain

$$\begin{aligned} H_l^{\mathrm{Up}}(\mu,\{c_k\}) &\geqslant H_l^{\mathrm{Low}}(\mu + 4\gamma\nu_l^0([0,2\delta]),\{c_k\}) \\ &- \mathrm{Tr}\big((h_l^\omega - \mu)P_\delta\big) - \gamma\nu_l^0([0,\delta])\Big(1 + \frac{V_l}{2}\nu_l^0([0,\delta]) + V_l\nu_l^0([0,2\delta])\Big) . \end{aligned}$$

By the Bogoliubov convexity inequality and the Berezin-Lieb inequalities, we get

$$p_{l,\delta}^{Low}(\mu) \leqslant p_l(\mu) \leqslant p_{l,\delta}^{Up}(\mu) \leqslant p_{l,\delta}^{Low}(\mu + 4\gamma\nu_l^0([0,2\delta])) + \frac{M(l,\delta,\mu)}{V_l} ,$$

where

$$M(l,\delta,\mu) := \mathrm{Tr}\big((h_l^\omega - \mu)P_\delta\big) + \gamma\nu_l^0([0,\delta])\Big(1 + \frac{V_l}{2}\nu_l^0([0,\delta]) + \nu_l^0([0,2\delta])\Big) . \tag{4.19}$$

Then, we have

$$\begin{aligned} \limsup_{l\to\infty} p_{l,\delta}^{Low}(\mu) \leqslant p(\mu) \leqslant \liminf_{l\to\infty} p_{l,\delta}^{Up}(\mu) &\leqslant \limsup_{l\to\infty} p_{l,\delta}^{Up}(\mu) \\ &\leqslant \liminf_{l\to\infty} p_{l,\delta}^{Low}(\mu + 4\gamma\nu_l^0([0,2\delta])) \\ &+ \liminf_{l\to\infty} \frac{M(l,\delta)}{V_l} . \end{aligned} \tag{4.20}$$

Since $p^{up}_{l,\delta}(\mu)$ is convex in μ, we have for any $t > 0$

$$\partial_\mu p^{up}_{l,\delta}(\mu) \leqslant \frac{1}{t}\big(p^{up}_{l,\delta}(\mu+t) - p^{up}_{l,\delta}(\mu)\big)$$

and thus,

$$\begin{aligned}
\limsup_{l\to\infty} \partial_\mu p^{up}_{l,\delta}(\mu) &\leqslant \frac{1}{t}\big(\limsup_{l\to\infty} p^{up}_{l,\delta}(\mu+t) - \liminf_{l\to\infty} p^{up}_{l,\delta}(\mu)\big) \\
&\leqslant \frac{1}{t}\big(\liminf_{l\to\infty} p^{low}_{l,\delta}(\mu+t+4\gamma\nu^0_l(2\delta)) + \liminf_{l\to\infty}\frac{1}{V_l}M(l,\delta,\mu) - p(\mu)\big) \\
&\leqslant \frac{1}{t}\big(p(\mu+t+4\gamma\nu^0(2\delta)) + \liminf_{l\to\infty}\frac{1}{V_l}M(l,\delta,\mu) - p(\mu)\big) \ .
\end{aligned}$$

Since it follows from (4.19) and Lemma 4.3.3 that

$$\lim_{\delta\downarrow 0}\ \liminf_{l\to\infty}\frac{1}{V_l}M(l,\delta) \ = \ 0 \ ,$$

we obtain from (4.20)

$$\limsup_{\delta\downarrow 0}\limsup_{l\to\infty}\partial_\mu p^{up}_{l,\delta}(\mu) \leqslant \frac{1}{t}\big(p(\mu+t) - p(\mu)\big) \ , \tag{4.21}$$

which is valid for any $t > 0$. Letting $t \downarrow 0$ leads to the second inequality in (4.15). The proof of the first inequality in (4.15) is similar, since we can use the fact that $p^{low}_{l,\delta}(\mu)$ is convex with respect to μ to get

$$\begin{aligned}
\limsup_{l\to\infty}\partial_\mu p^{low}_{l,\delta}(\mu) &\leqslant \frac{1}{t}\big(\limsup_{l\to\infty} p^{low}_{l,\delta}(\mu+t) - \liminf_{l\to\infty} p^{low}_{l,\delta}(\mu)\big) \\
&\leqslant \frac{1}{t}\big(\limsup_{l\to\infty} p_l(\mu+t) \\
&- \limsup_{l\to\infty} p^{up}_{l,\delta}(\mu-4\gamma\nu^0_l(2\delta)) - \liminf_{l\to\infty}\frac{1}{V_l}M(l,\delta,\mu-4\gamma\nu^0_l(2\delta))\big) \ ,
\end{aligned}$$

where we have used (4.20) twice. Using it one more time, we get

$$\limsup_{l\to\infty}\partial_\mu p^{low}_{l,\delta}(\mu) \leqslant \frac{1}{t}\big(p(\mu+t) - p(\mu-4\gamma\nu^0(2\delta)) - \liminf_{l\to\infty}\frac{1}{V_l}M(l,\delta,\mu-4\gamma\nu^0_l(2\delta))\big)$$

and in view of (4.19) and Lemma 4.3.3, the result follows by letting $\delta \downarrow 0$ and then, letting $t \downarrow 0$. □

Proof of Lemma 4.3.2

Let $\mathbb{C}^{n_\delta}_\xi := \{z \in \mathbb{C}^{n_\delta} : |z|^2 \leqslant \xi\}$, and denote the volume of this ball by

$\mathrm{Vol}(\mathbb{C}_\xi^{n_\delta}) = \pi^{n_\delta}\xi^{n_\delta}/n_\delta\Gamma(n_\delta)$. We then obtain the following bound

$$\begin{aligned}
&\Xi_l^{Low}(\mu)\\
=\;& \frac{1}{\pi^{n_\delta}}\int_{\mathbb{C}_\xi^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\,\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} + \frac{1}{\pi^{n_\delta}}\int_{\mathbb{C}^{n_\delta}\setminus\mathbb{C}_\xi^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\,\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\\
\leqslant\;& \frac{\mathrm{Vol}(\mathbb{C}_\xi^{n_\delta})}{\pi^{n_\delta}}\max_{\{c_k\}}\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\\
+\;& \frac{1}{\xi\pi^{n_\delta}}\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\big(\sum_{k\in I_\delta}|c_k|^2\big)\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\\
\leqslant\;& \frac{\mathrm{Vol}(\mathbb{C}_\xi^{n_\delta})}{\pi^{n_\delta}}\max_{\{c_k\}}\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\\
+\;& \frac{1}{\xi\pi^{n_\delta}}\,\langle\sum_{k\in I_\delta}|c_k|^2\rangle_{Low}\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\,\mathrm{Tr}\,e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}.
\end{aligned}$$

Notice than, by the form of the lower symbol for the total particle number operator, see (4.18), we can further bound the expectation value in the last term

$$\begin{aligned}
&\Xi_l^{Low}(\mu)\\
\leqslant\;& \frac{\mathrm{Vol}(\mathbb{C}_\xi^{n_\delta})}{\pi^{n_\delta}}\max_{\{c_k\}}\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\\
+\;& \frac{1}{\xi\pi^{n_\delta}}\big(V_l\partial_\mu p_{l,\delta}^{Low}(\mu)\big)\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\,\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\\
=\;& \big(\frac{\xi^{n_\delta}}{n_\delta\Gamma(n_\delta)}\big)\max_{\{c_k\}}\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\\
+\;& \frac{1}{\xi\pi^{n_\delta}}\big(V_l\partial_\mu p_{l,\delta}^{Low}(\mu)\big)\int_{\mathbb{C}^{n_\delta}} \mathrm{d}c_1\ldots\mathrm{d}c_{n_\delta}\,\mathrm{Tr}e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\;,
\end{aligned}$$

that is,

$$\big(1-\frac{V_l}{\xi}\partial_\mu p_{l,\delta}^{Low}(\mu)\big)\Xi_l^{Low}(\mu) \leqslant \big(\frac{\xi^{n_\delta}}{n_\delta\Gamma(n_\delta)}\big)\max_{\{c_k\}}\mathrm{Tr}\,e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\;.$$

Letting $\xi = \alpha V_l\partial_\mu p_{l,\delta}^{Low}(\mu)$ for some $\alpha > 1$, and using Stirling's formula, one gets

$$\frac{\xi^{n_\delta}}{n_\delta\Gamma(n_\delta)} \leqslant \frac{\big(\alpha V_l\partial_\mu p_{l,\delta}^{Low}(\mu)\big)^{n_\delta}}{n_\delta n_\delta^{n_\delta-1/2}e^{-n_\delta}} \leqslant \frac{\Big(\big(\alpha\partial_\mu p_{l,\delta}^{Low}(\mu)\big)^{\nu_l^0([0,\delta])}\Big)^{V_l}}{\big(V_l\nu_l^0([0,\delta])\big)^{V_l\nu_l^0([0,\delta])+1/2}}\,e^{-n_\delta}\;.$$

Hence, one finally obtains

$$\begin{aligned}
&\Xi_l^{Low}(\mu)\\
\leqslant\;& \frac{1}{1-\frac{1}{\alpha}}\Big(\big(\alpha\partial_\mu p_{l,\delta}^{Low}(\mu)\big)^{\nu_l^0([0,\delta])}\Big)^{V_l}V^{-1/2}\times\\
\times\;& \big(\nu_l^0([0,\delta])\big)^{-(\nu_l^0([0,\delta])V_l+1/2)}e^{\nu_l^0\big([0,\delta]\big)V_l}\max_{\{c_k\}}\mathrm{Tr}\,e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})}\;,
\end{aligned}$$

which leads to the result

$$\begin{aligned}
&\frac{1}{\beta V_l}\ln \Xi_l^{Low}(\mu) \qquad (4.22)\\
\leqslant\;& \frac{1}{\beta V_l}\ln \max_{\{c_k\}} \operatorname{Tr} e^{-\beta H_l^{\mathrm{Low}}(\mu,\{c_k\})} - \frac{1}{\beta V_l}\ln(1-1/\alpha) + \frac{\nu_l^0\big([0,\delta]\big)}{\beta}\ln(\alpha\partial_\mu p_{l,\delta}^{Low}(\mu))\\
+\;& \frac{1}{\beta}\nu_l^0\big([0,\delta]\big) - \frac{1}{2\beta}\frac{\ln V_l}{V_l} - \frac{\nu_l^0\big([0,\delta]\big)}{\beta}\ln\big(\nu_l^0\big([0,\delta]\big)\big) - \frac{1}{2\beta V_l}\ln\big(\nu_l^0\big([0,\delta]\big)\big)\ .
\end{aligned}$$

□

Proof of Lemma 4.3.3

We start with

$$\begin{aligned}
\frac{1}{V_l}\operatorname{Tr}(h_l^\omega-\mu)P_\delta &= \frac{1}{V_l}\operatorname{Tr}(h_l^0-\mu)P_\delta + \frac{1}{V_l}\operatorname{Tr}(v^\omega \restriction_{\Lambda_l})P_\delta\\
&\leqslant (\delta-\mu)\nu_l^0\big([0,\delta]\big) + \frac{1}{V_l}\sum_{k\in I_\delta}(\psi_k^l, v^\omega\psi_k^l)\ ,
\end{aligned}$$

since the projection P_δ is constructed with the basis of eigenvectors of h_l^0. We then obtain

$$\begin{aligned}
\frac{1}{V_l}\operatorname{Tr}(h_l^\omega-\mu)P_\delta &\leqslant (\delta-\mu)\nu_l^0\big([0,\delta]\big) + \int_{[0,\delta)}(\psi_k^l, v^\omega\psi_k^l)\,\nu_l^0(\mathrm{d}k)\\
&= (\delta-\mu)\nu_l^0\big([0,\delta]\big) + \int_{[0,\delta)}\nu_l^0(\mathrm{d}k)\,\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x v^\omega(x)\\
&= \nu_l^0\big([0,\delta]\big)\Big((\delta-\mu) + \frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x v^\omega(x)\Big)\ ,
\end{aligned}$$

and thus, by the ergodic theorem

$$\limsup_{l\to\infty}\frac{1}{V_l}\operatorname{Tr}(h_l^\omega-\mu)P_\delta \leqslant \nu^0(\delta)\Big((\delta-\mu) + \mathbb{E}_\omega\big(v^\omega(0)\big)\Big)\ .$$

□

4.4 From the pressure to the Bose-Einstein condensate density

In this section, we discuss the meaning of our result, in particular how one should interpret the solutions of the variational problem established in Theorem 4.3.1.

First we recall a result established in [39]. For a homogeneous system and a single-mode substitution in the mode $k=0$, the solution of the variational problem gives the *total* condensate density in the mode $k = 0$, *if* one adds to the Hamiltonian the *zero-mode* gauge-breaking term (quasi-average sources):

$$H_l(\mu;\eta) \;:=\; H_l(\mu) \;+\; \sqrt{V_l}\;\left(\overline{\eta}\, a_0 + \eta\, a_0^*\right)\;.$$

This means that after the Bogoliubov c-number substitution the solution $\alpha_l^{\max}(\beta,\mu;\eta)$ of the (finite-volume) variational problem not only provides the right pressure in the thermodynamic limit, but it also coincides with the quasi-average amount of condensate in the *zero* mode:

$$\lim_{|\eta|\downarrow 0}\;\lim_{l\to\infty}\alpha_l^{\max}(\mu;\eta) \;=\; \lim_{|\eta|\downarrow 0}\;\lim_{l\to\infty}\langle a_0^* a_0/V_l\rangle_l(\mu;\eta)\;.$$

Here $\langle - \rangle_l(\mu;\eta)$ is the equilibrium state defined by $H_l(\mu;\eta)$.

Using a simple example, we discuss the relevance of this quasi-average approach to more subtle cases in which the condensation is of type II or III. We show that the Bogoliubov quasi-average sources breaking the gauge invariance [14] are able to alter the fine structure of the condensate, reducing it to one-mode (type I).

To see this, consider the perfect Bose gas in a three-dimensional anisotropic parallelepiped $\Lambda_l := V_l^{\alpha_x}\times V_l^{\alpha_y}\times V_l^{\alpha_z}$, with periodic boundary condition and $\alpha_x \geq \alpha_y \geq \alpha_z$, $\alpha_x+\alpha_y+\alpha_z = 1$. Using our notations, the Hamiltonian is given by

$$H_l^0(\mu) \;:=\; \sum_{k\in\Lambda_l^*}(\varepsilon_k^l - \mu)a_k^* a_k\;.$$

It is known, see e.g. [5], that this system exhibits a generalised condensation of type II for $\alpha_x = 1/2$ and of type III for $\alpha_x > 1/2$ for a standard critical density ρ_c, whereas for $\alpha_x < 1/2$, the whole condensate is sitting in the mode $k = 0$, i.e, in the ground state (type I). Let us consider this model with the a quasi-average source in a single mode $\tilde{k}$

$$H_l^0(\mu;\eta) \;:=\; H_l^0(\mu) \;+\; \sqrt{V_l}\;\left(\overline{\eta}\; a_{\tilde{k}} + \eta\; a_{\tilde{k}}^*\right)\;,$$

and denote by $\langle - \rangle(\mu,\eta)$ the corresponding equilibrium state. Then for a fixed density $\overline{\rho}$, the finite-volume equation which defines the corresponding chemical potential $\mu_l(\overline{\rho},\eta)$ takes the form

$$\begin{aligned}\overline{\rho} \;&=\; \rho_l(\mu;\eta) \;:=\; \frac{1}{V_l}\sum_{k\in\Lambda_l^*}\langle a_k^* a_k\rangle_l^0(\mu,\eta) \qquad (4.23)\\ &=\; \frac{1}{V_l}(e^{\beta(\varepsilon_{\tilde{k}}-\mu)}-1)^{-1} \;+\; \frac{1}{V_l}\sum_{k\neq\tilde{k}}\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} \;+\; \frac{|\eta|^{\,2}}{(\varepsilon_{\tilde{k}}-\mu)^{\,2}}\;.\end{aligned}$$

To investigate the occurence of condensation, we must take the thermodynamic limit in the right-hand side of (4.23), and then switch off the source, that is let $|\eta| \to 0$. Let us denote by $I(\mu)$ the limit of $\rho_l(\mu, \eta = 0)$, that is the limiting density function of the *gauge-invariant* system,

$$I(\mu) \;=\; \lim_{l\to\infty} \frac{1}{V_l} \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \;=\; \int_{\mathbb{R}} \nu^0(\mathrm{d}\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} \;.$$

with critical density $\rho_c := \sup_{\mu<0} I(\mu)$.

Now we have to distinguish two cases:

(i) For any $\tilde{k}$ such that $\lim_{l\to\infty} \varepsilon_{\tilde{k}} > 0$, we obtain from (4.23)

$$\overline{\rho} \;=\; \lim_{|\eta|\to 0} \lim_{l\to\infty} \rho_l(\mu, \eta) \;=\; I(\mu) \;,$$

i.e. the quasi-average coincides with the average and we return to the analysis of the condensate equation (4.23) for $\eta = 0$. This gives again all possible types of condensation as a function of α_x.

(ii) On the other hand, if $\tilde{k}$ is such that $\lim_{l\to\infty} \varepsilon_{\tilde{k}} = 0$, then the condensate equation (4.23) yields for the quasi-average of the total particle density

$$\overline{\rho} \;=\; \lim_{|\eta|\to 0} \lim_{l\to\infty} \rho_l(\mu, \eta) \;=\; I(\mu) + \lim_{\eta\to 0} \frac{|\eta|^{\,2}}{\mu^{\,2}} \;. \tag{4.24}$$

If $\overline{\rho} \le \rho_c$, then the asymptotic solution of (4.24) is $\mu_\infty(\overline{\rho}) = \lim_{\eta\to 0} \lim_{l\to\infty} \mu_l(\overline{\rho}, \eta) < 0$ and there is no condensation in any mode.

If $\overline{\rho} > \rho_c$, then $\lim_{\eta\to 0} |\eta|^{\,2}/\mu_\infty(\overline{\rho}, \eta)^{\,2} = \overline{\rho} - \rho_c$. By explicit calculation, one also gets that only the $\tilde{k}$-mode quasi-average is non-zero

$$\lim_{\eta\to 0} \lim_{l\to\infty} \frac{1}{V_l} \;\langle a^*_{\tilde{k}} a_{\tilde{k}} \rangle^0_l(\mu_l, \eta) = \lim_{\eta\to 0} \lim_{l\to\infty} \left\{ \frac{1}{V_l} \frac{1}{e^{\beta(\varepsilon_{\tilde{k}} - \mu_l(\overline{\rho},\eta))} - 1} + \frac{|\eta|^{\,2}}{\mu_l^{\,2}} \right\} = \overline{\rho} - \rho_c \;, \tag{4.25}$$

i.e. for any α_x the condensation is of type I. Recall that the only condition on $\tilde{k}$ is that the corresponding eigenvalue $\varepsilon_{\tilde{k}}$ vanishes in the infinite volume limit. Since

$$\lim_{\eta\to 0} \lim_{l\to\infty} \frac{1}{V_l} \;\langle a^*_0 a_0 \rangle^0_l(\mu_l, \eta) = \lim_{\eta\to 0} \lim_{l\to\infty} \frac{1}{V_l} \frac{1}{e^{\beta(-\mu_l(\overline{\rho},\eta))} - 1} = 0 \;,$$

and in view of (4.25), we see that the Bogoliubov quasi-average procedure not only transforms the generalised condensates of type II or III into a one-mode condensate (i.e., type I), but this mode does not even need to be the ground-state. Therefore, using the quasi-average approach [14], one can force the condensate to be in any given mode $\tilde{k}$, as long as its energy $\varepsilon_{\tilde{k}}$ vanishes in the limit $l \to \infty$.

We want to point out that the technique of using external sources requires some *a priori* knowledge about the single modes spread of the condensate density. As it has so far remained an open problem to establish whether condensation occurs at all in a genuinely interacting Bose gas, one can at best "guess" the outcome.
This is why, for translation invariant, isotropic systems, it might be a reasonable assumption that the condensate will still be concentrated in the 0-mode, since the corresponding perfect gas does indeed exhibit ground-state condensation only. However, it turns out that the addition of an external potential (and hence, the breaking of translation invariance) seems to prevent the condensate to accumulate into any single mode, see discussion in Chapter 3. Hence, it is not clear at all why one should force the condensate in any particular mode by the addition of sources, since we have reasonable grounds to suspect that they will all be macroscopically empty.
On the other hand, if the condensation phenomena is understood from the generalised point of view, the c-numbers $\tilde{c}_k$ which solve the variational problem for the generalised Bogoliubov approximation should give the amount of generalised condensation, that is roughly speaking

$$\lim_{\delta \downarrow 0} \lim_{l \to \infty} \frac{1}{V_l} \sum_{k \in I_\delta} |\tilde{c}_k|^2 \;\approx\; \lim_{\delta \downarrow 0} \lim_{l \to \infty} \frac{1}{V_l} \sum_{k \in I_\delta} \langle a_k^* a_k \rangle_l(\mu) \; . \tag{4.26}$$

Note that, for the previous relation not to be trivial, that is to obtain a non-zero amount of generalised condensation, it is not necessary that any $|\tilde{c}_k|^2$ be of the order of the volume V_l, and therefore this approach would be more consistent with a generalised BEC of type III.

The proof of this conjecture is however out of our reach at the moment. Apart from some technical difficulties, in particular the fact that the variational problem has to be solved in finite volume, it does not follow from our result that the maximum value of the pressure depends only on the modulus of the c-number. This is due to the fact that the usual canonical gauge transformation which eliminates the phase of the c-number does not work in the generalised Bogoliubov approximation (indeed, it does not even work in the case of only two substituted modes).

Appendix A

Brownian motions

The goal of this section is to prove some technical results related to Brownian motions, which we used to establish kinetic generalised BEC, see Section 2.2.

Lemma A.1 *Let the set* $\Omega^T_{(x,x')} := \{\xi(\tau) : \xi(0) = x, \xi(T) = x'\}$ *of continuous trajectories from* x *to* x' *with the proper time* $0 \leq \tau \leq T$*, and with the conditional Wiener measure* w^T *on it. Let* x, x' *be in* Λ_l*, and* $\chi_{\Lambda_l,T}(\xi)$ *the characteristic function over* $\Omega^T_{(x,x')}$ *of trajectories* ξ *staying in* Λ_l *for all* $0 \leqslant \tau \leqslant T$*. Then one gets the estimate*

$$\int_{\Omega^T_{(x,x')}} w^T(\mathrm{d}\xi)\Big(1 - \chi_{\Lambda_l,T}(\xi)\Big) \leqslant e^{-C(T)\Big(\min\{d(x,\partial\Lambda_l),d(x',\partial\Lambda_l)\}\Big)^2} . \tag{A.1}$$

Proof:

Define a *Brownian bridge* $\alpha(s), 0 \leqslant s \leqslant 1$ by

$$\xi(t) = (1 - \tau/T)\, x + \tau/T\ x' + \sqrt{T}\,\alpha(\tau/T) .$$

Let us consider first the one dimensional case, i.e. $\Lambda_l = [-l/2, l/2]$. Without loss of generality, we can assume that

$$d(x, \partial\Lambda_l) \leqslant d(x', \partial\Lambda_l) .$$

Suppose that $x > 0$, then we have

$$-x \leqslant x' \leqslant x \quad \text{and} \quad d(x, \partial\Lambda_l) = l/2 - x .$$

Assume that the path ξ leaves the box on the right-hand side. Then, for some t, we have

$$\begin{aligned} \xi(t) &> \frac{l}{2} \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\Big(\frac{l}{2} + (t/T-1)x - \frac{t}{T}x'\Big) \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\Big(\frac{l}{2} + (t/T-1)x - \frac{t}{T}x\Big) \;=\; \frac{1}{\sqrt{T}}d(x,\partial\Lambda_l)\ . \end{aligned} \tag{A.2}$$

The case, when ξ leaves the box on the left-hand side can be treated similarly.

Let $x < 0$, then we have

$$x \leqslant x' \leqslant -x \quad \text{and} \quad d(x,\partial\Lambda_l) = l/2 + x$$

Again, assume that the path leaves the box on the right hand-side. Then, for some t, we have

$$\begin{aligned} \xi(t) &> \frac{l}{2} \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\Big(\frac{l}{2} + (t/T-1)x - \frac{t}{T}x'\Big) \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\Big(\frac{l}{2} - (t/T-1)x' - \frac{t}{T}x'\Big) \;\geqslant\; \frac{1}{\sqrt{T}}d(x,\partial\Lambda_l)\ . \end{aligned} \tag{A.3}$$

The case when ξ leaves the box on the left hand-side can be considered similarly. The relations (A.2), (A.3) imply that if ξ leaves the box Λ_l in one dimension, then the Brownian bridge α must satisfy the inequality

$$\sup_t |\alpha(t/T)| \;>\; C(T)\min\{d(x,\partial\Lambda_l), d(x',\partial\Lambda_l)\}\ , \tag{A.4}$$

for some constant $C(T)$.

This observation can easily be extended to higher dimensions, when $x := (x_1, \ldots, x_d)$ and $\alpha(s) := (\alpha_1(s), \ldots, \alpha_d(s))$. Now, if ξ leaves the (d-dimensional) box Λ_l, there exists at least one i such that similar to (A.4)

$$\sup_t |\alpha_i(t/T)| \;>\; C(T)\min\{d(x_i,\partial_i\Lambda_l)\ , d(x_i',\partial_i\Lambda_l)\},$$

where we denote $d(x_i, \partial_i\Lambda_l) := \min\{l/2 - x_i, l/2 + x_i\}$. Now, since Λ_l are cubes, we get $d(x_i, \partial_i\Lambda_l \geqslant d(x, \partial\Lambda_l$ for any $x \in \Lambda_l$. Then we obtain

$$
\begin{aligned}
\|\alpha(t/T)\| &> |\alpha_i(t/T)|, \qquad i = 1, \ldots, d \;, \\
\sup_t \|\alpha(t/T)\| &> \max_i \sup_t |\alpha_i(t/T)| \;, \\
\sup_t \|\alpha(t/T)\| &> C(T) \min\{d(x_i, \partial_i\Lambda_l), d(x_i', \partial_i\Lambda_l)\} \\
&\geqslant C(T) \min\{d(x, \partial\Lambda_l), d(x', \partial\Lambda_l)\} \;. \qquad \text{(A.5)}
\end{aligned}
$$

Therefore, the probability for the path ξ to leave the box is dominated by the probability for the one-dimensional Brownian bridge α to satisfy (A.5). The latter we can estimate using the following result from [45]

$$
\mathbb{P}\Big(\sup_s \alpha(s) \;>\; x\Big) \geqslant Ae^{-Cx^2} \;,
$$

for some positive constants A, C, which implies the bound (A.1). □

Now we establish a result that we use in the proof of Theorems 2.2.2 and 2.2.7. Let us note that the only requirement on the external potential is its non-negativity.

Lemma A.2 *Let $v : \mathbb{R}^d \to [0, \infty)$ be a non-negative external potential, and let the single-particle opertors h_l^0, h_l be defined as in Section 1.1. Let $K_l^t(x, x')$, $K_{0,l}^t(x, x')$, $K_0^t(x, x')$ be the kernels of operators* $\exp(-th_l)$, $\exp(-th_l^0)$, *and* $\exp(-t\Delta/2)$ *respectively. Then*

$$
\begin{aligned}
&\lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \int_{\Lambda_l} \mathrm{d}x\, \mathrm{d}x'\, K_{0,l}^t(x, x')\, K_l^{n\beta}(x', x) \qquad \text{(A.6)} \\
&= \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x\, \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x'\, K_0^{t+n\beta}(x, x') \int_{\Omega_{(x',x)}^{n\beta}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta} \mathrm{d}s\, v(\xi(s))} \;.
\end{aligned}
$$

Proof:

By the Feynman-Kac representation, we obtain:

$$
\begin{aligned}
&\lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \int_{\Lambda_l} \mathrm{d}x\, \mathrm{d}x'\, K_{0,l}^t(x, x')\, K_l^{n\beta}(x', x) \qquad \text{(A.7)} \\
&= \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \int_{\Lambda_l} \mathrm{d}x\, \mathrm{d}x'\, \frac{e^{-\|x-x'\|^2(1/2n\beta + 1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
&\times \int_{\Omega_{(x',x)}^{n\beta}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta} \mathrm{d}s\, v(\xi(s))}\, \chi_{\Lambda_l, n\beta}(\xi) \int_{\Omega_{(x,x')}^t} w^t(\mathrm{d}\xi')\, \chi_{\Lambda_l,t}(\xi') \;.
\end{aligned}
$$

To eliminate the characteristic functions restricting the paths ξ, ξ' in the last integral, we shall use Lemma A.1. First, we estimate the error $\gamma(d)$ when we remove the restriction on the path ξ:

$$\begin{aligned}
\gamma(d) \;:=\; & \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \, \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \qquad \text{(A.8)} \\
& \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\, e^{-\int_0^{n\beta} \mathrm{d}s\, v(\xi(s))} \big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\, \chi_{\Lambda_l,t}(\xi') \\
\leqslant\; & \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \, \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) \big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \\
\leqslant\; & \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \, \mathbf{1}\{d(x,\partial\Lambda_l) > d(x',\partial\Lambda_l)\} \, \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
& \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) \big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \\
& + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \, \mathbf{1}\{d(x,\partial\Lambda_l \leqslant d(x',\partial\Lambda_l)\}\,, \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
& \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) \big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \\
\leqslant\; & \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \, K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)d(x',\partial\Lambda_l)^2} \\
& + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)d(x,\partial\Lambda_l)^2}
\end{aligned}$$

where the last step is due to Lemma A.1. Since all integrands are positive, we can extend one of the spatial integrations to the whole space, and hence we get:

$$\begin{aligned}
\gamma(d) \;\leqslant\; & \lim_{l\to\infty} \frac{1}{V_l} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \, K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)d(x',\partial\Lambda_l)^2} \\
& + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)d(x'\partial\Lambda_l)^2} \\
\leqslant\; & \lim_{l\to\infty} \frac{1}{V_l} K_0^{t+n\beta} \int_{\Lambda_l} \mathrm{d}x' e^{-C(n\beta)d(x',\partial\Lambda_l)^2} \\
& + \lim_{l\to\infty} \frac{1}{V_l} K_0^{t+n\beta} \int_{\Lambda_l} \mathrm{d}x\, e^{-C(n\beta)d(x'\partial\Lambda_l)^2}\,,
\end{aligned}$$

where we have used the notation $K_0^{t+n\beta} := K_0^{t+n\beta}(x,x)$ since these are independent of x. Finally, using the fact that the boxes Λ_l are cubes of side l, we obtain:

$$\gamma(d) \leqslant \lim_{l\to\infty} \frac{K_0^{t+n\beta}}{l} \int_{-l/2}^{l/2} \mathrm{d}x' e^{-C(n\beta)(l/2-x')^2} + \lim_{l\to\infty} \frac{K_0^{t+n\beta}}{l} \int_{-l/2}^{l/2} \mathrm{d}x e^{-C(n\beta)(l/2-x)^2} \;=\; 0$$

We can estimate the error estimate due to the removal of the characteristic function for ξ' in (A.7) in the same way. Therefore, we get:

$$\begin{aligned}
&\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\,\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{d/2}}\times \qquad \text{(A.9)}\\
&\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\,e^{-\int_0^{n\beta}\mathrm{d}s\,v^{\omega}(\xi(s))}\,\chi_{\Lambda_l,n\beta}(\xi)\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\,\chi_{\Lambda_l,t}(\xi')\\
&=\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{d/2}}\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^{\omega}(\xi(s))}\,.
\end{aligned}$$

Now we show that one can replace the first integration over the box Λ_l by one over the whole space. Let $\tilde{\gamma}(d)$ be the error caused by this substitution. Then by the positivity of the potential we get the estimate

$$\begin{aligned}
\tilde{\gamma}(d) &:= \lim_{l\to\infty}\frac{1}{V_l}\int_{\mathbb{R}^d\setminus\Lambda_l}\mathrm{d}x\int_{\Lambda_l}\mathrm{d}x'\,\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{d/2}}\times \qquad \text{(A.10)}\\
&\quad\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\,e^{-\int_0^{n\beta}\mathrm{d}s\,v^{\omega}(\xi(s)+x')}\int_{\Omega^t_{(x,x')}} w^{n\beta}(\mathrm{d}\xi')\\
&\leqslant \lim_{l\to\infty}\frac{1}{V_l}\int_{\mathbb{R}^d\setminus\Lambda_l}\mathrm{d}x\int_{\Lambda_l}\mathrm{d}x'\,\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{d/2}}\,.
\end{aligned}$$

In the one-dimensional case the estimate of the error term (A.10) takes the form

$$\begin{aligned}
\tilde{\gamma}(1) &\leqslant \lim_{l\to\infty}\frac{1}{l}\int_{-\infty}^{-l/2}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\,\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{1/2}} \qquad \text{(A.11)}\\
&\quad+\lim_{l\to\infty}\frac{1}{l}\int_{l/2}^{\infty}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\,\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{1/2}}\,.
\end{aligned}$$

For the first term one gets

$$\begin{aligned}
&\lim_{l\to\infty}\frac{1}{l}\int_{-\infty}^{-l/2}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\,\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{1/2}}\\
&=\lim_{l\to\infty}\frac{1}{l}\int_0^l\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{1/2}}\int_{-l/2-y}^{l/2}\mathrm{d}x\\
&+\lim_{l\to\infty}\frac{1}{l}\int_l^{\infty}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2tn\beta)^{1/2}}\int_{-l/2-y}^{l/2-y}\mathrm{d}x \;=\; 0\,.
\end{aligned}$$

One obtains a similar identity for the second-term in (A.11). Direct calculation shows that, the error term for higher dimensions (A.10) reduces to a sum of products of one-dimensional terms (A.11). Then (A.9) gives

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \int_{\Lambda_l} \mathrm{d}x \, \mathrm{d}x' \, \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \\
&\times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) \, e^{-\int_0^{n\beta} \mathrm{d}s \, v^\omega(\xi(s))} \, \chi_{\Lambda_l,n\beta}(\xi) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi') \, \chi_{\Lambda_l,t}(\xi') \\
= \; &\lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x \, \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x' \, \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) \, e^{-\int_0^{n\beta} \mathrm{d}s \, v^\omega(\xi(s))} \,,
\end{aligned}$$

which finishes the proof of (A.6). □

Appendix B

Probabilistic estimates

The Lemma 2.2.1 is due to Kirsch and Martinelli [22], but its proof in this reference is embedded in a more sophisticated result. We give here the main ideas of its proof in a compact way. To avoid unnecessary complications, we shall assume without proof in this appendix a technical result about the geometric convergence of certain random quantities, see Lemma 2 in [22].

Let $h_l^{\omega,N}$ to be the Schrödinger operator (1.21), with Neumann boundary conditions instead of Dirichlet, and denote by $\{E_i^{\omega,l,N}, \phi_i^{\omega,l,N}\}_{i\geqslant 1}$ its ordered eigenvalues (including degeneracy) and the corresponding eigenvectors. Similarly we define the kinetic energy operator $h_l^{0,N}$ with the same boundary condition, and denote by $\{\varepsilon_k^{l,N}, \psi_k^{l,N}\}_{k\geqslant 1}$ its ordered eigenvalues (including degeneracy) and corresponding eigenvectors. The following result is due to Thirring, see [46].

Lemma B.1 *Let $v_{\lambda,\alpha}^{\omega} := v^{\omega} + \lambda\,\alpha$, for $\lambda, \alpha > 0$. Then*

$$E_1^{\omega,l,N} \;\geqslant\; -\lambda\alpha \;+\; \min\left\{\varepsilon_2^{l,N}, \left[\frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x\,(v_{\lambda,\alpha}^{\omega}(x))^{-1}\right]^{-1}\right\}\,.$$

Proof: Let P to be an orthogonal projection in $\mathscr{H}_l$. Then for any vector ϕ from the intersection $Q(v_{\lambda,\alpha}^{\omega})\bigcap Q((v_{\lambda,\alpha}^{\omega})^{1/2}P(v_{\lambda,\alpha}^{\omega})^{1/2})$, we have

$$\begin{aligned}
(\phi, v_{\lambda,\alpha}^{\omega}\phi) &= ((v_{\lambda,\alpha}^{\omega})^{1/2}\phi, (v_{\lambda,\alpha}^{\omega})^{1/2}\phi) \\
&= ((v_{\lambda,\alpha}^{\omega})^{1/2}\phi, P(v_{\lambda,\alpha}^{\omega})^{1/2}\phi) \;+\; ((v_{\lambda,\alpha}^{\omega})^{1/2}\phi, (1-P)(v_{\lambda,\alpha}^{\omega})^{1/2}\phi) \\
&\geqslant ((v_{\lambda,\alpha}^{\omega})^{1/2}\phi, P(v_{\lambda,\alpha}^{\omega})^{1/2}\phi)\;,
\end{aligned}$$

and therefore,

$$-\tfrac{1}{2}\Delta_N + v^{\omega}_{\lambda,\alpha} \geqslant -\tfrac{1}{2}\Delta_N + (v^{\omega}_{\lambda,\alpha})^{1/2} P (v^{\omega}_{\lambda,\alpha})^{1/2} \ , \tag{B.1}$$

in the quadratic-form sense. Let us choose

$$P := (v^{\omega}_{\lambda,\alpha})^{-1/2} \tilde{P} \left((\psi_1^{l,N}, (v^{\omega}_{\lambda,\alpha})^{-1} \psi_1^{l,N}) \right)^{-1} \tilde{P} (v^{\omega}_{\lambda,\alpha})^{-1/2} \ ,$$

where $\tilde{P}$ is the orthogonal projection onto the subspace spanned by the vector $\psi_1^{l,N}$. It can be easily checked that P is an orthogonal projection. Applying (B.1) to the function $\phi_1^{\omega,l,N}$ one gets

$$\begin{aligned} E_1^{\omega,l,N} + \lambda\alpha &\geqslant (\phi_1^{\omega,l,N}, (-\tfrac{1}{2}\Delta_N)\phi_1^{\omega,l,N}) + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \left(\psi_1^{l,N}, (v^{\omega}_{\lambda,\alpha})^{-1} \psi_1^{l,N} \right)^{-1} \\ &\geqslant \sum_{k \geqslant 1} |(\phi_1^{\omega,l,N}, \psi_k^{l,N})|^2 \, \varepsilon_k^{l,N} + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \left[\frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1} \right]^{-1} . \end{aligned}$$

But since the Neumann boundary conditions imply that $\varepsilon_1^{l,N} = 0$, we obtain

$$E_1^{\omega,l,N} + \lambda\alpha \geqslant (1 - |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2) \varepsilon_2^{l,N} + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \left[\frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1} \right]^{-1} .$$

To finish the proof, we have to study separately the two cases, namely, $\varepsilon_2^{l,N}$ less than and greater than $\left[\frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \, (v^{\omega}_{\lambda,\alpha}(x))^{-1} \right]^{-1}$. □

Proof of Lemma 2.2.1: By Lemma B.1, with $\lambda = B/l^2$ and α as defined in the assumptions, i.e. for $B = \pi/(1+\alpha)$, $\alpha > p/(1-p)$, we have

$$E_1^{\omega,l,N} \geqslant -\frac{\alpha B}{l^2} + \min(\pi/l^2, 1/X_l) \ ,$$
$$\text{where } X_l^{\omega} := \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \, \frac{1}{v^{\omega}(x) + B\alpha/l^2} \ .$$

Therefore,

$$E_1^{\omega,l,N} - \frac{B}{l^2} \geqslant -\frac{\pi}{l^2} + \min(\pi/l^2, 1/X_l^{\omega}) \ .$$

Hence, the inequality $E_1^{\omega,l,N} < B/l^2$ implies that $X_l^{\omega} > l^2/\pi$ and consequently

$$\mathbb{P}(E_1^{\omega,l,N} < B/l^2) \leqslant \mathbb{P}(X_l^{\omega} > l^2/\pi) \ . \tag{B.2}$$

Define a random variable $Y_l^{\omega}(\delta) := V_l^{-1} \int_{\Lambda_l} \mathrm{d}x \ \delta/(v^{\omega}(x) + \delta)$, which is an increasing function of δ. Then for the left-hand side of (B.2) one gets the estimate

$$\mathbb{P}(E_1^{\omega,l,N} < B/l^2) \leqslant \mathbb{P}\Big(Y_l^{\omega}(B\alpha/l^2) > \frac{\alpha}{1+\alpha}\Big) \ .$$

By Lemma 2 in [22], we know that for any positive δ, the random variables $\{Y_l^\omega(\delta)\}_l$ converges *geometrically* to a limit $Y_\infty(\delta)$ as $l \to \infty$, that is, for any $\epsilon > 0$, there exists a constant $M(\delta, \epsilon)$ such that

$$\mathbb{P}(|Y_l^\omega(\delta) - Y_\infty(\delta)| > \epsilon/2) \leqslant e^{-M(\delta,\epsilon)\, V_l} \ ,$$

for l sufficiently large. By the ergodic theorem $Y_\infty(\delta)$ is non-random and can be expressed as:

$$Y_\infty(\delta) = \mathbb{E}_\omega \left(\frac{\delta}{v^\omega(0) + \delta} \right) \ ,$$

which is again a monotonic function of $\delta \geq 0$. Note that since we have assumed that $p = \mathbb{P}\{\omega : v^\omega(0) = 0\}$, we have $\lim_{\delta \to 0} Y_\infty(\delta) = p$.

Choose $\epsilon > 0$ such that $p + \epsilon < \alpha/(1 + \alpha)$. Then we have

$$\mathbb{P}(E_1^{\omega,l,N} < \frac{B}{l^2}) \leqslant \mathbb{P}\Big(Y_l^\omega(B\alpha/l^2) > p + \epsilon\Big) \ .$$

Now we choose δ such that

$$Y_\infty(\delta) - p < \epsilon/2 \ ,$$

and let l_0 be defined by $\delta = B\alpha/l_0^2$. Then for any $l > l_0$ we have

$$\begin{aligned} \mathbb{P}(E_1^{\omega,l,N} < B/l^2) &\leqslant \mathbb{P}\Big(Y_l^\omega(B\alpha/l^2) > p + \epsilon\Big) \leqslant \mathbb{P}\Big(Y_l^\omega(\delta) - p > \epsilon\Big) \\ &\leqslant \mathbb{P}\Big(|Y_l^\omega(\delta) - Y_\infty(\delta)| > \epsilon/2\Big) \leqslant e^{-M(\delta,\epsilon)\, V_l} \ . \end{aligned}$$

□

Appendix C

Some multiscale analysis results

C.1 Sketch of the proof of Proposition 3.3.1

The object of this section is to show that the necessary conditions for the multiscale analysis result used in Section 3.3.1 are satisfied for the Stollmann model as defined in Section 1.3.1. We follow the scheme of the proof established in [23].

The following result is a combination of Theorem 3.2.2 and Corollary 3.2.6 in [23].

Proposition C.1.1 *Fix an interval $I_0 \subset \mathbb{R}$. Let $q > d$, $\zeta_0 > 0$, $\Theta \in (0, 1/2)$ and $\beta > 2\Theta$ be given. Let $\alpha \in (1, 2)$ be such that:*

$$4d\,\frac{\alpha - 1}{2 - \alpha} \leqslant \min\{\,\zeta_0, \frac{1}{4}(q - d)\,\}\ .$$

Assume that, for a given $l_0 < \infty$, the following conditions are satisfied.

i) ***The Wegner estimate***

For all $E \in I_0$, for all $l \geqslant l_0$, we have

$$\mathbb{P}\Big\{\omega :\ \mathrm{d}(\sigma(h^\omega_l), E)\ \leqslant\ e^{-l^\Theta}\Big\}\ \leqslant\ l^{-q}\ . \tag{C.1}$$

ii) ***The initial scale estimate***

For some $l^* > l_0$, and for some $\tilde{I} \subset I_0$, there exists $\gamma_0 > (l^*)^{\beta-1}$ such that for any $x, y \in \mathbb{Z}^d$, satisfying $|x - y| > l^*$,

$$\mathbb{P}\Big\{\omega :\ \forall E \in \tilde{I},\ \text{either}\ \Lambda_{l^*}(x)\ \text{or}\ \Lambda_{l^*}(y)\ \text{is}\ (\gamma_0, E)\text{-good}\Big\}\ \geqslant\ 1\ -\ (l^*)^{-2\zeta_0}\ . \tag{C.2}$$

Then, there exist a constant $\gamma > 0$ and a sequence $\{l_k\}, k \geqslant 1$, satisfying $l_1 \geqslant 2$ and $l_{k-1}^{\alpha} \leqslant l_k \leqslant l_{k-1}^{\alpha} + 6$ for $k \geqslant 2$, such that

$$\mathbb{P}\Big\{\omega: \text{ for all } E \in I^*, \text{ either } \Lambda_{l_k}(x) \text{ or } \Lambda_{l_k}(y) \text{ is } (\gamma, E)\text{-good}\Big\} \geqslant 1 - l_k^{-2\zeta} \ , \quad \text{(C.3)}$$

for all $k \geqslant 1$ and for all $x, y \in \mathbb{Z}^d$, satisfying $|x-y| > l_k$.
Note that the interval I^* is the one for which one can prove the assumption (C.2), and the constant ζ is defined by

$$\zeta \ = \ \min\{\zeta_0, \frac{1}{4}(q-d)\,\} \ . \quad \text{(C.4)}$$

Sketch of the proof for the *Wegner estimate*

As it turns out, this is the simpler of the two assumptions. Let $I_0 = [-1, 1]$. Next, we use the following result (see Theorem 2.3.2 in [23]).

Proposition C.1.2 *Fix $R > 0$. For any interval $I \subset (-R, R)$ with length $|I|$, there exists a constant C_R such that*

$$\mathbb{P}\Big\{\omega: \, \{\sigma(h_l^{\omega}) \cap I\} \neq \emptyset\Big\} \ \leqslant \ C_R \, l^{2d} \, |I|^{\alpha}$$

for all l, where α denotes the Hölder-continuity exponent of the probability distribution μ.

Let $R = 2$, and fix an energy $E \in I_0$. Define the interval $I_l(E)$, centered at E of length $e^{-l^{1/4}}$. Then, $I_l(E) \subset [-2, 2]$ for all $l > 1$. By Proposition C.1.2, one can find a constant C such that

$$\mathbb{P}\Big\{\omega: \, \{\sigma(h_l^{\omega}) \cap I_l(E)\} \neq \emptyset\Big\} \ \leqslant \ C \, l^{2d} \, |I_l(E)|^{\alpha} \ ,$$

that is,

$$\mathbb{P}\Big\{\omega: \, \mathrm{d}\big(\sigma(h_l^{\omega}), E\big) \ \leqslant \ e^{-l^{1/4}}\Big\} \ \leqslant \ C \, l^{2d} \, e^{-\alpha l^{1/4}}$$

by the definition of the interval $\tilde{I}_l(E)$. Thus, for any $q > 0$, there exists $l_0 = l_0(q)$ such that

$$\mathbb{P}\Big\{\omega: \, \mathrm{d}\big(\sigma(h_l^{\omega}), E\big) \ \leqslant \ e^{-l^{1/4}}\Big\} \ \leqslant \ l^{-q}$$

for all $E \in I_0$ and for all $l > l_0$. □

Sketch of the proof for the *initial scale estimate*

Let us start with the following result (Theorem 2.2.3 in [23]).

Proposition C.1.3 *For any $\zeta^* > 0$ and $\beta \in (0,1)$, there exists $\tilde{l} = \tilde{l}(\beta, \zeta_*)$ such that*

$$\mathbb{P}\big\{\omega : \mathrm{d}\big(\sigma(h_l^\omega, 0)\big) \leqslant l^{\beta-1})\big\} \leqslant l^{-\zeta^*}$$

for all $l > \tilde{l}$.

We use this result in conjunction with the so-called Combes-Thomas estimate (Theorem 2.4.1 in [23]). Note that this result is not probabilistic.

Proposition C.1.4 (Combes-Thomas Estimate) *Let $R > 0$, and fix a scale L. Let $A, B \subset \Lambda_L$, $\delta_0 > 0$ and $r < s$. Assume that*

1. *$\delta := \mathrm{d}(A, B) > 0$ and $\{x \in \Lambda_L : \mathrm{d}(x, \partial\Lambda_L) \leqslant \delta_0\} \subset B$,*
2. *$(r, s) \subset \big(\rho(h_l^\omega) \cap (-R, R)\big)$,*

where $\rho(.)$ denotes the resolvent set. Then there exist constants $c_1 = c_1(R)$, $c_2 = c_2(R)$ such that, for all $E \in (r, s)$ with $\eta := \mathrm{d}(E, (r,s)^c) > 0$, the following holds

$$||\chi_A(h_L^\omega - E)^{-1}\chi_B|| \leqslant c_1\eta^{-1}e^{-c_2\sqrt{s-r}\,\sqrt{\eta}\,\delta} ,$$

where $||.||$ is the $L^2(\Lambda_L)$-norm in the operator sense and χ_A, χ_B are the indicator functions of these regions.

We can now prove the initial scale estimate. For any $\zeta_0 > 0$, fix a scale $l^* > \tilde{l}$, with $\tilde{l}$ as in Proposition C.1.3. Let $\Lambda_{l^*}^1$ and $\Lambda_{l^*}^2$ be any two disjoints subcubes of side l^*. Using the same notation as in Section 3.3.1, we split up the boxes $\Lambda_{l^*}^i$ into $\Lambda_{l^*}^{i,int} := \Lambda_{l^*/3}^i$ and $\Lambda_{l^*}^{i,out} := \Lambda_{l^*}^i \setminus \Lambda_{l^*-2}^i$, where $i = 1, 2$. This implies that

$$\mathrm{d}\big(\Lambda_{l^*}^{i,int}, \Lambda_{l^*}^{i,out}\big) \geqslant \frac{2l^*}{3} - 2 \geqslant \frac{1}{3}l^* , \tag{C.5}$$

for $l^* \geqslant 6$. For any $\zeta_0 > 0$, it follows from Proposition C.1.3 that the set

$$X_{l^*} := \big\{\omega : \mathrm{d}(\sigma(h_{l^*}^\omega), 0) \geqslant (l^*)^{\beta-1}\big\} \tag{C.6}$$

has a large probability since we have assumed that $l^* > \tilde{l}$, more precisely

$$\mathbb{P}\big(X_{l^*}\big) \geqslant 1 - (l^*)^{-\zeta_0} . \tag{C.7}$$

Next, we want to apply Proposition C.1.4 for any realisation $\omega \in X_{l^*}$.
Let the regions A, B be defined by $A = \Lambda_{l^*}^{\#,int}, B = \Lambda_{l^*}^{\#,out}$. It follows from (C.5) and the definition of the region B that the first condition in Proposition C.1.4 is satisfied, with the constants $\delta = \frac{1}{3}l^*$ and $\delta_0 = 2$.
Letting the constants $r = -1, s = \frac{1}{2}(l^*)^{\beta-1}$, and choosing any realisation $\omega \in X_{l^*}$ the second assumption is satisfied in view of (C.6).
Hence, for any $E \in (-1 + \frac{1}{4}(l^*)^{\beta-1}, \frac{1}{4}(l^*)^{\beta-1})$, we can apply Proposition C.1.4 with the constant $\eta := \frac{1}{4}(l^*)^{\beta-1}$, which yields

$$\begin{aligned} ||\chi_{\Lambda_{l^*}^{\#,int}}(h_{l^*}^{\omega} - E)^{-1}\chi_{\Lambda_{l^*}^{\#,out}}|| &\leqslant c_1\eta^{-1}e^{-c_2\sqrt{r-s}\sqrt{\eta}\delta} \\ &\leqslant c_3(l^*)^{1-\beta}e^{-c_4(l^*)^{\frac{1}{2}(\beta-1)}l^*} \leqslant e^{-c_5(l^*)^{\frac{1}{2}(\beta-1)}l^*} , \end{aligned} \tag{C.8}$$

for l^* large enough. In other words, for any $\omega \in X_{l^*}$, the box $\Lambda_{l^*}^i$ is (γ_0, E)-good for all $E \in [0, \frac{1}{4}(l^*)^{\beta-1})$, with the rate of decay $\gamma_0 = c_5(l^*)^{\frac{1}{2}(\beta-1)}$ as required in (C.2), again for l^* large enough. Note that we have restricted the interval for E, since we are only interested in non-negative energy.
Therefore, since this argument works for any $\omega \in X_{l^*}$, one can estimate the probability of the set

$$Y_{l^*}^i := \big\{\omega : \forall E \in [0, \frac{1}{4}(l^*)^{\beta-1}),\, \Lambda_{l^*}^i \text{ is } (\gamma_0, E)\text{-good}\big\}$$

by (C.7), which leads to

$$\mathbb{P}\big(Y_{l^*}^i\big) \geqslant 1 - (l^*)^{-\zeta_0} .$$

Let us stress the fact that last estimate is valid for either $i = 1, 2$, and since the boxes $\Lambda_{l^*}^1$ and $\Lambda_{l^*}^2$ are disjoints, it follows that the events $Y_{l^*}^1$ and $Y_{l^*}^2$ are independent. Hence, we have

$$\begin{aligned} &\mathbb{P} \quad \big\{\omega : \exists E \in [0, \frac{1}{4}(l^*)^{\beta-1}),\ \text{both } \Lambda_{l^*}^1 \text{ and } \Lambda_{l^*}^2 \text{ are not } (\gamma_0, E)\text{-good}\big\} \\ &= \mathbb{P}\big\{(Y_{l^*}^1)^c \cap (Y_{l^*}^1)^c\big\} \leqslant (l^*)^{-2\zeta_0} . \end{aligned}$$

Note that the boxes $\Lambda_{l^*}^1$ and $\Lambda_{l^*}^1$ are arbitrary apart from the fact that they must be disjoints (hence, the distance between their centres must be at least l^*). Since the initial rate of decay γ_0 is as required in C.2, the proof of the *initial scale estimate* follows immediately. □

Remark C.1.1 *As we emphasised in the proofs, both constants q (for the Wegner estimate) and ζ_0 (for the initial scale estimate) can be chosen arbitrary large. Hence, the constant ζ, see C.4, which controls the occurrence of "bad events" can also be chosen arbitrary large. This fact played an important role in our proof of localisation in the Stollmann model, see the proof of Lemma 3.3.3.*

C.2 Sketch of the proof of the eigenfunction decay inequality

In this section, we give a sketch of the proof of the technical result that allows us to exploit the good boxes in the sense of Definition 3.3.1, roughly speaking to show that they cannot contribute much to the norm of any eigenstate.

Sketch of the proof of Proposition 3.3.2

Let us first introduce some notation.

1. $(.,.)$ denotes the usual scalar product in $L^2(\Lambda_l)$,
2. for w_1, w_2 vector-valued functions from $L^2(\Lambda_l, \mathbb{C}^d)$, with components w_1^i, w_2^i, for $i = 1, \dots, d$, we call $\langle .,. \rangle$ the associated scalar product, i.e.
$$\langle w_1, w_2 \rangle := \sum_{i=1}^{d} (w_1^i, w_2^i) \ ,$$
3. by $||.||$, we denote the norm associated with either scalar product defined above

Let $l' < l$, and assume that the cube $\Lambda_{l'}(x) \subset \Lambda_l$ for some $x \in \Lambda_l$. Let $\tilde{h}_l$ to be the quadratic form associated with h_l^ω, that is
$$\tilde{h}_l[\varphi, \psi] = \int_{\Lambda_l} \mathrm{d}x \big((\overline{\nabla\varphi}.\nabla\psi)(x) + \overline{\varphi(x)} v_l^\omega \psi(x) \big) \ ,$$
on the form domain $Q(\tilde{h}_l)$, see e.g. [32], Chapter VIII.15. Recall that $\phi_i^{\omega,l}$ denotes a normalised eigenfunction of h_l^ω with eigenvalue $E_i^{\omega,l}$, and let $u, v \in Q(\tilde{h}_l)$, with $u(x) \in \mathbb{R}$. Hence, we can write
$$\begin{aligned} & (\tilde{h}_l - E_i^{\omega,l})[u\phi_i^{\omega,l}, v] \ - \ (\tilde{h}_l - E_i^{\omega,l})[\phi_i^{\omega,l}, uv] \qquad \text{(C.9)} \\ = & \int_{\Lambda_l} \mathrm{d}x (\overline{\nabla(u\phi_i^{\omega,l})}.\nabla v)(x) \ - \ \int_{\Lambda_l} \mathrm{d}x (\overline{\nabla\phi_i^{\omega,l}}.\nabla(uv))(x) \\ = & \ \langle \phi_i^{\omega,l}\nabla u, \nabla v \rangle \ - \ \langle \nabla\phi_i^{\omega,l}, v\nabla u \rangle \ . \end{aligned}$$

Let $v = (h^\omega_{l'} - E^{\omega,l}_i)^{-1}\chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i$. It is clear that v belongs to the operator domain of $h^\omega_{l'}$, and therefore v also belongs to the form domain $Q(\tilde{h}_{l'})$. But $Q(\tilde{h}_{l'}) \subset Q(\tilde{h}_l)$ since $l' < l$, and hence $v \in Q(\tilde{h}_l)$. Note also that $(h^\omega_{l'} - E^{\omega,l}_i)^{-1}$ is well defined since we have assumed that $E^{\omega,l}_i$ is not in the spectrum $\sigma(h^\omega_l)$.

Let u be such that

$$\begin{aligned} &\mathrm{supp}(\nabla u) \subset \{\Lambda_{l'-1/2}(x) \setminus \Lambda_{l'-3/2}(x)\} \subset \Lambda^{out}_{l'}(x)\ , \qquad (C.10)\\ &||\nabla u||_\infty < K < \infty \quad \text{and} \quad \chi_{\Lambda^{int}_{l'}(x)} u = 1\ . \end{aligned}$$

Then, we have

$$(\tilde{h}_l - E^{\omega,l}_i)[u\phi^{\omega,l}_i, v] \;=\; \left(u\phi^{\omega,l}_i, (h^\omega_l - E^{\omega,l}_i)(h^\omega_{l'} - E^{\omega,l}_i)^{-1}\chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i\right)\ , \qquad (C.11)$$

and, since $h^\omega_{l'}$ and h^ω_l coincide on $\Lambda^{int}_{l'}$, it follows from the characteristic function $\chi_{\Lambda^{int}_{l'}(x)}$ in (C.11) that

$$\begin{aligned} (\tilde{h}_l - E^{\omega,l}_i)[u\phi^{\omega,l}_i, v] &= \left(\chi_{\Lambda^{int}_{l'}(x)} u\phi^{\omega,l}_i, \chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i\right) \qquad (C.12)\\ &= \left(\chi_{\Lambda^{int}_{l'}(x)}\phi^{\omega,l}_i, \chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i\right) \;=\; ||\chi_{\Lambda^{int}_{l'}(x)}\phi^{\omega,l}_i||^2\ , \end{aligned}$$

where we have used the assumption (C.10) on u. Thus, it follows from (C.9) and (C.12) that

$$\begin{aligned} ||\chi_{\Lambda^{int}_{l'}(x)}\phi^{\omega,l}_i||^2 &= (\tilde{h}_l - E^{\omega,l}_i)[\phi^{\omega,l}_i, uv] + \langle \phi^{\omega,l}_i\nabla u, \nabla v\rangle - \langle \nabla\phi^{\omega,l}_i, v\nabla u\rangle \quad (C.13)\\ &= \langle \phi^{\omega,l}_i\nabla u, \nabla v\rangle - \langle \nabla\phi^{\omega,l}_i, v\nabla u\rangle\ , \qquad (C.14) \end{aligned}$$

since $\phi^{\omega,l}_i$ is an eigenfunction of h^ω_l with eigenvalue $E^{\omega,l}_i$. Let

$$\tilde{\Lambda}(x) \;:=\; \Lambda_{l'-1/2}(x) \setminus \Lambda_{l'-3/2}(x)\ ,$$

and denote by $\tilde{\chi} = \tilde{\chi}(x)$ the corresponding characteristic function. Using (C.13) and the assumption (C.10), we obtain the estimate

$$\begin{aligned} ||\chi_{\Lambda^{int}_{l'}(x)}\phi^{\omega,l}_i||^2 &\leqslant |\langle \tilde{\chi}\phi^{\omega,l}_i\nabla u, \tilde{\chi}\nabla(h^\omega_{l'} - E^{\omega,l}_i)^{-1}\chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i\rangle| \qquad (C.15)\\ &+ |\langle \tilde{\chi}\nabla\phi^{\omega,l}_i, \tilde{\chi}(h^\omega_{l'} - E^{\omega,l}_i)^{-1}\chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i\nabla u\rangle|\\ &= K\Big(||\tilde{\chi}\phi^{\omega,l}_i||\,.\,||\tilde{\chi}\nabla(h^\omega_{l'} - E^{\omega,l}_i)^{-1}\chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i||\\ &+ ||\tilde{\chi}\nabla\phi^{\omega,l}_i||\,.\,||\tilde{\chi}(h^\omega_{l'} - E^{\omega,l}_i)^{-1}\chi_{\Lambda^{int}_{l'}(x)}\,\phi^{\omega,l}_i||\ , \end{aligned}$$

where the last step follows from the Schwarz inequality. Note that the norms are defined in $L^2(\Lambda_l)$, but because of the characteristic function $\tilde{\chi}$, we can consider them as norms in $L^2(\tilde{\Lambda}(x))$. In the rest of the proof, we shall indicate with an index the space on which the norms are defined. We now use the following result, which is a simpler version of Lemma 2.5.3 in [23].

Proposition C.2.1 *Let $\tilde{\Lambda}, \Lambda_l$ be as above. For any functions $f \in Q(\tilde{h}_l)$, $g \in L^2(\Lambda_l)$ such that*

$$\tilde{h}_l[f, h] \;=\; (g, h) \;,$$

for all $h \in C_0^\infty(\Lambda_l)$, there exists a constant $K = K(M)$, independent of f and g, such that

$$||\nabla f||_{L^2(\tilde{\Lambda})} \;\leqslant\; K\big(||\nabla f||_{L^2(\Lambda_l)} \,+\, ||\nabla g||_{L^2(\Lambda_l)}\big) \;.$$

Applying this proposition for $f = \phi_i^{\omega,l}$ and $g = E_i^{\omega,l}$ leads to

$$||\nabla \phi_i^{\omega,l}||_{L^2(\tilde{\Lambda})} \;\leqslant\; K||\phi_i^{\omega,l}||_{L^2(\Lambda_l)}(1 + E_i^{\omega,l}) \;\leqslant\; K_1||\phi_i^{\omega,l}||_{L^2(\Lambda_l)} = K_1 \;, \qquad \text{(C.16)}$$

where the new constant K_1 depends on M and the interval $[0, s]$ which contains the eigenvalue $E_i^{\omega,l}$ by assumption.
For any $h \in C_0^\infty(\Lambda_l)$, we have

$$\begin{aligned} \tilde{h}[(h_{l'}^\omega - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}, h] \;&=\; (h_{l'}^\omega - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}, h_l^\omega h) \\ &=\; E_i^{\omega,l}((h_{l'}^\omega - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}, h) \;, \end{aligned}$$

where we have used the fact that the operators h_l^ω and $(h_{l'}^\omega - E_i^{\omega,l})^{-1}$ commute. Hence, in view of Proposition C.2.1, it follows that

$$\begin{aligned} || \;\; & \nabla(h_{l'}^\omega - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}||_{L^2(\tilde{\Lambda})} \qquad \text{(C.17)} \\ \leqslant \;\; & K||(h_{l'}^\omega - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}||_{L^2(\Lambda_l)}(1 + E_i^{\omega,l}) \\ \leqslant \;\; & K_1||(h_{l'}^\omega - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}||_{L^2(\Lambda_l)} \;, \end{aligned}$$

with the same constant K_1 as in (C.16). Now, we are ready to come back to (C.15), where we recall that the norms have to be understood in $L^2(\tilde{\Lambda})$. Hence, we have the freedom to drop any $\tilde{\chi}$ at our convenience. Thus, by (C.16) and (C.17), we get for some constant K_2

$$||\chi_{\Lambda_{l'}^{int}(x)}\phi_i^{\omega,l}||^2_{L^2(\Lambda_l)} \leqslant K_2\,||\tilde{\chi}(h_{l'}^{\omega} - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}||_{L^2(\Lambda_l)}\ .$$

Since $\tilde{\Lambda} \subset \Lambda_{l'}^{out}(x)$, we have $\tilde{\chi} \leqslant \chi_{\Lambda_{l'}^{out}(x)}$ in quadratic form sense, and hence

$$||\chi_{\Lambda_{l'}^{int}(x)}\phi_i^{\omega,l}||^2_{L^2(\Lambda_l)} \leqslant K_3\,||\chi_{\Lambda_{l'}^{out}(x)}(h_{l'}^{\omega} - E_i^{\omega,l})^{-1}\chi_{\Lambda_{l'}^{int}(x)}\,\phi_i^{\omega,l}||_{L^2(\Lambda_l)}\ .$$

and the Proposition 3.3.2 follows by bounding in the right-hand side the vector-norm by the operator norm, noting that the function $\phi_i^{\omega,l}$ is normalised by definition. $\square$

Appendix D

Coherents states: lower and upper symbols

In this section, we classify the Hamiltonians in groups of terms according to their relevance to the c-numbers substitution. That is, we split up the sums in the many particles interacting Hamiltonian (4.1) according to the number of creation/annihilation operators with index $k \in I_\delta$. This lead to quite a large number of term, since, due to the generalised approximation, it is now possible to have an odd number of operators replaced, which is not the case in the standard one-mode substitution.

$$
\begin{aligned}
& H_l^0 - \mu N_l + \frac{1}{2V_l} \sum_{q,k,k' \in \Lambda_l^*} \hat{u}(q)\ a^*(\psi_{k+q}) a^*(\psi_{k'-q}) a(\psi_{k'}) a(\phi_k) \\
= & \sum_{k \in I_\delta} \big(\sum_{i \geqslant 1} |(\phi_i, \psi_k)|^2 (E_i - \mu) \big) a_k^* a_k && \text{(D.1)} \\
+ & \sum_{k \in I_\delta^c} \big(\sum_{i \geqslant 1} |(\phi_i, \psi_k)|^2 (E_i - \mu) \big) a_k^* a_k && \text{(D.2)} \\
+ & \sum_{k \in I_\delta, k' \in I_\delta^c} \big(\sum_{i \geqslant 1} (\phi_i, \psi_k)(\psi_{k'}, \phi_i)(E_i - \mu) \big) a_k^* a_{k'} && \text{(D.3)} \\
+ & \sum_{k \in I_\delta^c, k' \in I_\delta} \big(\sum_{i \geqslant 1} (\phi_i, \psi_k)(\psi_{k'}, \phi_i)(E_i - \mu) \big) a_k^* a_{k'} && \text{(D.4)} \\
+ & \sum_{k,k' \in I_\delta^c, k \neq k'} \big(\sum_{i \geqslant 1} (\phi_i, \psi_k)(\psi_{k'}, \phi_i)(E_i - \mu) \big) a_k^* a_{k'} && \text{(D.5)} \\
+ & \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta^c}} \hat{u}(q)\ a_{k+q}^* a_{k'-q}^* a_{k'} a_k && \text{(D.6)}
\end{aligned}
$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta^c}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.7}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.8}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.9}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta^c}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.10}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta^c}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.11}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.12}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.13}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta^c}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.14}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta^c}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.15}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.16}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.17}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta^c}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.18}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta^c}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.19}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta}} \hat{u}(q)\; a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.20}$$

$$+ \quad \frac{1}{2V_l}\sum_{k\in I_\delta^c}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta}}\hat{u}(q)\; a^*_{k+q}a^*_{k'-q}a_{k'}a_k \tag{D.21}$$

We now provide an explicit form of the upper and lower symbols for the Hamiltonian. We note that, as the coherent vector (4.6) is defined as a tensor product of one-mode coherent states, its effect on each creation/annihilation operator $a^\sharp_k$ is independent of all the others operators $a^\sharp_{k'}, k' \neq k$. First, we give an explicit form of the lower symbol of the full Hamiltonian, that is,

$$\begin{aligned}
H_l^{Low}(\mu,\{c_k\}) \;=\; & \sum_{k\in I_\delta}\Big(\sum_{i\geqslant 1}|(\phi_i,\psi_k)|^2(E_i-\mu)\Big)|c_k|^2 && \text{(D.22)}\\
+\; & \sum_{k\in I_\delta^c}\Big(\sum_{i\geqslant 1}|(\phi_i,\psi_k)|^2(E_i-\mu)\Big)a^*_k a_k && \text{(D.23)}\\
+\; & \sum_{k\in I_\delta,k'\in I_\delta^c}\Big(\sum_{i\geqslant 1}(\phi_i,\psi_k)(\psi_{k'},\phi_i)(E_i-\mu)\Big)\overline{c_k}a_{k'} && \text{(D.24)}\\
+\; & \sum_{k\in I_\delta^c,k'\in I_\delta}\Big(\sum_{i\geqslant 1}(\phi_i,\psi_k)(\psi_{k'},\phi_i)(E_i-\mu)\Big)c_k a^*_k && \text{(D.25)}\\
+\; & \sum_{k,k'\in I_\delta^c,k\neq k'}\Big(\sum_{i\geqslant 1}(\phi_i,\psi_k)(\psi_{k'},\phi_i)(E_i-\mu)\Big)a^*_k a_{k'} && \text{(D.26)}\\
+\; & \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta^c}}\hat{u}(q)\; c_k c_{k'}\, a^*_{k+q}a^*_{k'-q} && \text{(D.27)}\\
+\; & \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta^c}}\hat{u}(q)\; \overline{c}_{k+q}c_k c_{k'}\, a^*_{k'-q} && \text{(D.28)}\\
+\; & \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta}}\hat{u}(q)\; \overline{c}_{k'-q}c_k c_{k'}\, a^*_{k+q} && \text{(D.29)}\\
+\; & \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta}}\hat{u}(q)\; \overline{c}_{k+q}\overline{c}_{k'-q}c_k c_{k'} && \text{(D.30)}\\
+\; & \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta^c}}\hat{u}(q)\; c_{k'}\, a^*_{k+q}a^*_{k'-q}a_k && \text{(D.31)}\\
+\; & \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta^c}}\hat{u}(q)\; \overline{c}_{k+q}c_k\, a^*_{k'-q}a_{k'} && \text{(D.32)}\\
+\; & \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta}}\hat{u}(q)\; \overline{c}_{k'-q}c_k\, a^*_{k+q}a_{k'} && \text{(D.33)}
\end{aligned}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta}} \hat{u}(q)\ \bar{c}_{k+q} \bar{c}_{k'-q} c_k\, a_{k'} \tag{D.34}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta^c}} \hat{u}(q)\ c_{k'}\, a^*_{k+q} a^*_{k'-q} a_k \tag{D.35}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta^c}} \hat{u}(q)\ \bar{c}_{k+q} c_{k'}\, a^*_{k'-q} a_k \tag{D.36}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta}} \hat{u}(q)\ \bar{c}_{k'-q} c_{k'}\, a^*_{k+q} a_k \tag{D.37}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta}} \hat{u}(q)\ \bar{c}_{k+q} \bar{c}_{k'-q} c_{k'}\, a_k \tag{D.38}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta^c}} \hat{u}(q)\ a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{D.39}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta^c}} \hat{u}(q)\ \bar{c}_{k+q}\, a^*_{k'-q} a_{k'} a_k \tag{D.40}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta^c \\ k'-q \in I_\delta}} \hat{u}(q)\ \bar{c}_{k'-q}\, a^*_{k+q} a_{k'} a_k \tag{D.41}$$

$$+ \quad \frac{1}{2V_l} \sum_{k \in I_\delta^c} \sum_{k' \in I_\delta^c} \sum_{\substack{q:\, k+q \in I_\delta \\ k'-q \in I_\delta}} \hat{u}(q)\ \bar{c}_{k+q} \bar{c}_{k'-q}\, a_{k'} a_k \tag{D.42}$$

Now, we give an explicit form of the upper symbols. We recall the general form of this symbols for polynomials in the creation/annihilation operators of some mode $k \in I_\delta$

$$(a_k)^{Up} = c_k, \quad (a_k^*)^{Up} = \bar{c}_k, \quad (a_k a_k)^{Up} = (c_k)^2, \quad (a_k^* a_k^0)^{Up} = (\bar{c}_k)^2$$
$$(a_k^* a_k)^{Up} = |c_k|^2 - 1, \qquad (a_k^* a_k^* a_k a_k)^{Up} = |c_k|^4 - 4|c_k|^2 + 2$$

Note that, since the interaction term of the Hamiltonian term considered on its own does have a momentum conservation law, since it "does not see" the external potential, it is not possible to get exactly three out of four operators in the same mode k. In view of this, it can be seen that the lower and upper symbols of the Hamiltonian will differ only when two or four operators in the same mode appears, that is only terms (D.22), (D.30), (D.33), (D.34), (D.36), (D.37) and (D.38) differs in both approximated Hamiltonians.

Splitting further the sums in these terms leads finally to the final upper symbol of the Hamiltonian

$$H_l^{\mathrm{Up}}(\mu,\{c_k\}) \;=\; H_l^{\mathrm{Low}}(\mu,\{c_k\}) + \kappa(\mu,\{c_k\}) \;,$$

where

$$\begin{aligned}
\kappa(\mu,\{c_k\}) \;&=\; \sum_{k\in I_\delta}\Big(\sum_{i\leqslant 1}|(\phi_i,\psi_k)|^2(E_i-\mu)\Big) \qquad\qquad (D.43)\\
&+\; -\frac{1}{2V_l}\Big(2\sum_{k\in I_\delta}\hat{u}(0)+\sum_{\substack{k,k'\in I_\delta\\ k\neq k'}}\hat{u}(0)+\sum_{q\neq 0}\sum_{k\in I_\delta\cap I-q}\hat{u}(q)\Big)\\
&+\; \frac{1}{2V_l}\Big(4\sum_{k\in I_\delta}\hat{u}(0)|c_k|^2+\sum_{\substack{k,k'\in I_\delta\\ k\neq k'}}\hat{u}(0)(|c_k|^2+|c_{k'}|^2)+\\
&\qquad +\sum_{q\neq 0}\sum_{\substack{k\in I_\delta\cap I-q\\ k'\in I_\delta\cap I-q\\ k'-k=q}}\hat{u}(q)(|c_k|^2+|c_{k'}|^2)\Big)\\
&+\; \frac{1}{2V_l}\Big(2\sum_{k\in I_\delta}\hat{u}(0)\sum_{k'\in I_\delta^c}a_{k'}^*a_{k'}+\sum_{q\neq 0}\sum_{\substack{k\in I_\delta\\ k'\in I_\delta^c\cap I+q\\ k'-k=q}}\hat{u}(q)a_k^*a_k+\\
&\qquad +\sum_{q\neq 0}\sum_{\substack{k\in I_\delta^c\cap I-q\\ k'\in I_\delta\\ k'-k=q}}\hat{u}(q)a_k^*a_k\Big)\;.
\end{aligned}$$

Appendix E

Publications

1. **On the nature of Bose-Einstein Condensation in Disordered Systems**

   Thomas Jaeck, Joseph V. Pulé, Valentin A. Zagrebnov,

   Journal of Statistical Physics 139 (2009), pages 19-55

2. **On the nature of Bose-Einstein Condensation enhanced by Localisation**

   Thomas Jaeck, Joseph V. Pulé, Valentin A. Zagrebnov,

   Journal of Mathematical Physics 51 (2010), 103302

J Stat Phys
DOI 10.1007/s10955-009-9825-y

On the Nature of Bose-Einstein Condensation in Disordered Systems

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Received: 28 May 2009 / Accepted: 10 September 2009

Abstract We study the perfect Bose gas in *random* external potentials and show that there is generalized Bose-Einstein condensation in the random eigenstates if and only if the same occurs in the one-particle kinetic-energy eigenstates, which corresponds to the generalized condensation of the free Bose gas. Moreover, we prove that the amounts of both condensate densities are equal. Our method is based on the derivation of an explicit formula for the occupation measure in the one-body kinetic-energy eigenstates which describes the repartition of particles among these non-random states. This technique can be adapted to re-examine the properties of the perfect Bose gas in the presence of weak (scaled) *non-random* potentials, for which we establish similar results. In addition some of our results can be applied to models with diagonal interactions, that is, models which conserve the occupation density in each single particle eigenstate.

Keywords Generalized Bose-Einstein condensation · Random potentials · Integrated density of states · Lifshitz tails · Diagonal particle interactions

1 Introduction

The study of Bose-Einstein Condensation (BEC) in random media has been an important area for a long time, starting with the papers by Kac and Luttinger, see [1, 2], and then by

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Published online: 29 September 2009

Luttinger and Sy [3]. In the last reference, the authors studied a non-interacting (*perfect*) one dimensional system with point impurities distributed according to the Poisson law, the so-called *Luttinger-Sy model*. The authors conjectured a macroscopic occupation of the random ground state, but this was not rigorously proved until [5]. Although the *free* Bose gas (i.e., the *perfect* gas without external potential) does not exhibit BEC for dimension less than three, the randomness can enhance BEC even in *one dimension*, see e.g. [4]. This striking phenomenon is a consequence of the exponential decay of the one particle density of states at the bottom of the spectrum, known as *Lifshitz tail* , or "doublelogarithmic" asymptotics, which is generally believed to be associated with the existence of localized eigenstates [16].

BEC, however, is usually associated with a macroscopic occupation of the lowest one-particle kinetic-energy eigenstates, which are spatially extended (plane waves). Therefore, it is not immediately clear whether the phenomenon discovered in random boson gases, i.e. macroscopic occupations of localized one-particle states, has any relation to the standard BEC. This is of particular interest in view of the applications of the well-known Bogoliubov c-number approximation [6] to disordered boson systems, see e.g. [12, 13] where the creation/annihilation operators for the kinetic energy ground state are replaced by complex numbers. Although it has been known since the work of Ginibre [7] that this procedure gives the correct pressure in the thermodynamic limit and moreover, it does not require translation invariance, see [8], the associated variational equation (*Condensate Equation*) [9], has a trivial solution unless there is generalized condensate in the lower momentum states. Since such a condensate is not to be expected a priori in random systems, it is therefore interesting to investigate if such type of BEC occurs in some random simple models. One should note that even for translation invariant models, the relation between the solution of the condensate equation and the occupation of the kinetic energy ground state is not straightforward [10].

In this paper, we prove that for the perfect Bose gas in a general class of non-negative random potentials, BEC in the random localized one-particle states and BEC in the lowest one-particle kinetic-energy states occur simultaneously, and moreover the density of the condensate fractions are equal. Our line of reasoning is also applicable to some non-random systems, for example to the case of the perfect gas in weak (scaled) external potentials studied in [24]. We note that our proof for the fact that BEC in the random localized one-particle states *implies* BEC in the lowest one-particle kinetic-energy states holds without modification for a certain class of boson gases with *diagonal interactions* (i.e. invariant with respect to the "local" gauge transformations), while the implication in the other direction requires some additional arguments which will be given in a later work.

The structure of the paper is as follows: in Sect. 2 we describe our disordered system, and in Sect. 3, we recall standard results about the corresponding *perfect* Bose gas. The existence of *generalized* BEC in the eigenstates of the one-particle Schrödinger operator follows from the finite value of the critical density for *any dimension*, which is a consequence of the Lifshitz tail in the limiting Integrated Density of States (IDS). It is well-known that the IDS is a *non-random* quantity, see e.g. [16], and therefore the BEC density is also non-random in the thermodynamic limit. In Sect. 4, we turn to the main result of this paper: we show that this phenomenon occurs *if and only if* there is also occupation of the lowest one-particle kinetic-energy eigenstates. The latter corresponds to the usual generalized BEC in the *free* Bose gas, that is a perfect gas without external potential. To establish this we prove the existence of a non-random limiting occupation measure for kinetic energy eigenstates, and moreover, we obtain an explicit expression for it. To this end, we need some estimates for the IDS before the thermodynamic limit, namely a *finite volume* version of the Lifshitz tail estimates, which we prove in Sect. 5, using techniques developed in [14, 15]. For any *finite* but large enough system, these bounds hold almost surely with respect to random potential realizations. In Sect. 6, we look at the particular case of the Luttinger-Sy model and examine the nature of the condensate in the one-particle kinetic energy eigenstates, showing that although there is generalized BEC, *no condensation* occurs in any of them. In Sect. 7, we describe briefly how the method developed in Sect. 4 applies with minor modifications to a perfect Bose gas in a general class of weak (scaled), non-random external potentials. To make the paper more accessible and easy to read, we postpone some technical estimates concerning random potentials and Brownian motion to Appendices A and B, respectively.

2 Model, Notations and Definitions

Let $\{\Lambda_l := (-l/2, l/2)^d\}_{l\geq 1}$ be a sequence of hypercubes of side l in $\mathbb{R}^d, d \geq 1$, centered at the origin of coordinates with volumes $V_l = l^d$. We consider a system of identical bosons, of mass m, contained in Λ_l. For simplicity, we use a system of units such that $\hbar = m = 1$. First we define the self-adjoint one-particle kinetic-energy operator of our system by:

$$h_l^0 := -\frac{1}{2}\Delta_D, \tag{2.1}$$

acting in the Hilbert space $\mathscr{H}_l := L^2(\Lambda_l)$. The subscript D stands for *Dirichlet* boundary conditions. We denote by $\{\psi_k^l, \varepsilon_k^l\}_{k\geq 1}$ the set of normalized eigenfunctions and eigenvalues corresponding to h_l^0. By convention, we order the eigenvalues (counting the multiplicity) as $\varepsilon_1^l \leq \varepsilon_2^l \leq \varepsilon_3^l \leq \cdots$.

We define an external random potential $v^{(\cdot)}(\cdot) : \Omega \times \mathbb{R}^d \to \mathbb{R},\ x \mapsto v^\omega(x)$ as a random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the following conditions:

(i) $v^\omega, \omega \in \Omega$, is non-negative;
(ii) $p := \mathbb{P}\{\omega : v^\omega(0) = 0\} < 1$.

As usual, we assume that this field is *regular*, *homogeneous* and *ergodic*. These technical conditions are made more explicit in Appendix B. Then the corresponding random Schrödinger operator acting in $\mathscr{H} := L^2(\mathbb{R}^d)$ is a perturbation of the kinetic-energy operator:

$$h^\omega := -\frac{1}{2}\Delta \dot{+} v^\omega, \tag{2.2}$$

defined as a sum in the *quadratic-forms* sense. The restriction to the box Λ_l, is specified by the Dirichlet boundary conditions and for regular potentials one gets the self-adjoint operator:

$$h_l^\omega := \left(-\frac{1}{2}\Delta + v^\omega\right)_D = h_l^0 \dot{+} v^\omega, \tag{2.3}$$

acting in $\mathscr{H}_l$. We denote by $\{\phi_i^{\omega,l}, E_i^{\omega,l}\}_{i\geq 1}$ the set of normalized eigenfunctions and corresponding eigenvalues of h_l. Again, we order the eigenvalues (counting the multiplicity) so that $E_1^{\omega,l} \leq E_2^{\omega,l} \leq E_3^{\omega,l} \cdots$. Note that the *non-negativity* of the random potential implies that $E_1^{\omega,l} > 0$. So, for convenience we assume also that in the thermodynamic limit *almost surely* (a.s.) with respect to the probability $\mathbb{P}$, the lowest edge of this random one-particle spectrum is:

(iii) a.s.-$\lim_{l\to\infty} E_1^{\omega,l} = 0$.

When no confusion arises, we shall *omit* the explicit mention of l and ω dependence. Note that the non-negativity of the potential implies that:

$$\begin{aligned} &\text{(a)} \quad Q(h_l^\omega) \subset Q(h_l^0), \quad Q \text{ being the quadratic form domain,} \\ &\text{(b)} \quad (\varphi, h_l^\omega \varphi) \geq (\varphi, h_l^0 \varphi), \quad \forall \varphi \in Q(h_l^\omega). \end{aligned} \tag{2.4}$$

Now, we turn to the many-body problem. Let $\mathscr{F}_l := \mathscr{F}_l(\mathscr{H}_l)$ be the symmetric Fock space constructed over $\mathscr{H}_l$. Then $H_l := d\Gamma(h_l^\omega)$ denotes the second quantization of the *one-particle* Schrödinger operator h_l^ω in $\mathscr{F}_l$. Note that the operator H_l acting in $\mathscr{F}_l$ has the form:

$$H_l = \sum_{j \geq 1} E_j^{\omega,l} a^*(\phi_j) a(\phi_j), \tag{2.5}$$

where $a^*(\phi_i), a(\phi_i)$ are the creation and annihilation operators (satisfying the boson *Canonical Commutation Relations*) in the one-particle eigenstates $\{\phi_i := \phi_i^{\omega,l}\}_{i \geq 1}$ of h_l^ω. Then, the grand-canonical Hamiltonian of the perfect Bose gas in a random external potential is given by:

$$H_l(\mu) := H_l - \mu N_l = \sum_{i \geq 1} (E_i^{\omega,l} - \mu) N_l(\phi_i), \tag{2.6}$$

where $N_l(\phi_i) := a^*(\phi_i) a(\phi_i)$ is the operator for the number of particles in the eigenstate ϕ_i, $N_l := \sum_j N_l(\phi_j)$ is the operator for the total number of particles in Λ_l and μ is the chemical potential. Note that N_l can be expanded over *any* basis in the space $\mathscr{H}_l$, and in particular over the one defined by the free one-particle kinetic-energy eigenstates $\{\psi_k^l, \varepsilon_k\}_k$.

Although this paper is mainly devoted to the perfect Bose gas, some of our results can be extended to a class of models with "*diagonal interaction*" in addition to the random potential. By this we mean models with Hamiltonian $H_l^U(\mu) := H_l(\mu) + U_l$, where U_l is a many-body interaction, satisfying the "local" gauge invariance:

$$[H_l^U(\mu), N_l(\phi_j)] = 0 \tag{2.7}$$

for any $j \geq 1$, or equivalently:

$$e^{i\gamma_j N_l(\phi_j)} H_l^U(\mu) e^{-i\gamma_j N_l(\phi_j)} = H_l^U(\mu), \quad \gamma_j \in \mathbb{R}^1, \ j \geq 1. \tag{2.8}$$

The latter means that U_l is a function of the occupation number operators $\{N_l(\phi_j)\}_{j \geq 1}$, and for this reason it is called a "*diagonal interaction*". We shall assume that U_l is bounded from below. A well-known example is the *mean-field* interaction $U_l := \lambda N_l^2 / 2V_l$, $\lambda \geq 0$. [19, 20]. Our results for the general diagonal interaction are weaker than for the mean-field interaction, see Remarks 4.1 and 4.2.

Note that in the free Bose gas, with periodic boundary conditions the "local" gauge invariance (2.7) gives the *same* selection rule as the *momentum conservation law* which ensures that the number of particles in each momentum state is conserved. In the random model there is no such momentum selection rule but in our model it is the particle number in each random eigenstate ϕ_i that is conserved.

We denote by $\langle - \rangle_{H_l^U}$ the equilibrium quantum Gibbs state defined by the Hamiltonian $H_l^U(\mu)$:

$$\langle A \rangle_{H_l^U}(\beta, \mu) := \frac{\mathrm{Tr}_{\mathscr{F}_l}\{\exp(-\beta H_l^U(\mu)) A\}}{\mathrm{Tr}_{\mathscr{F}_l} \exp(-\beta H_l^U(\mu))},$$

and we put $\langle - \rangle_l := \langle - \rangle_{H_l^{U=0}}$. For simplicity, we shall omit in the following the explicit mention of the dependence on the thermodynamic parameters (β, μ). Finally, we define the *Thermodynamic Limit* (TL) as the limit, when $l \to \infty$.

3 Generalized BEC in One-Particle *Random* Eigenstates

In this section we consider the possibility of macroscopic occupation of the one-particle random Schrödinger operator (2.3) eigenstates $\{\phi_i\}_{i\geq 1}$. Recall that the corresponding limiting IDS, $\nu(E)$, is defined as:

$$\nu(E) := \lim_{l\to\infty} \nu_l^\omega(E) = \lim_{l\to\infty} \frac{1}{V_l} \sharp\{i : E_i^{\omega,l} \leq E\}. \tag{3.1}$$

Although the finite-volume IDS, $\nu_l^\omega(E)$, are random measures, one can check that for homogeneous ergodic random potentials the limit (3.1) has the property of *self-averaging* [16]. This means that $\nu(E)$ is almost surely (a.s.) a *non-random* measure. Let us define a (random) particle density *occupation measures* m_l by:

$$m_l(A) := \frac{1}{V_l} \sum_{i:E_i\in A} \langle N_l(\phi_i)\rangle_l, \qquad A \subset \mathbb{R}. \tag{3.2}$$

Then using standard methods, one can prove that this sequence of measures has (a.s.) a non-random weak-limit m, see (3.8) below. Moreover, if the critical density

$$\rho_c := \lim_{\mu\to 0} \int_0^\infty \frac{1}{e^{\beta(E-\mu)} - 1} \nu(\mathrm{d}E) \tag{3.3}$$

is finite, then one obtains a *generalized* Bose-Einstein condensation (g-BEC) in the sense that this measure m has an atom at the bottom of the spectrum of the random Schrödinger operator, which by (iii), Sect. 2, is assumed to be at 0:

$$m(\{0\}) = \lim_{\delta\downarrow 0} \lim_{l\to\infty} \sum_{i:E_i\leq\delta} \frac{1}{V_l} \langle N_l(\phi_i)\rangle_l = \begin{cases} 0 & \text{if } \overline{\rho} < \rho_c, \\ \overline{\rho} - \rho_c & \text{if } \overline{\rho} \geq \rho_c, \end{cases} \tag{3.4}$$

where $\overline{\rho}$ denotes a (fixed) mean density [4, 5]. Physically, this corresponds to the macroscopic occupation of the set of eigenstates ϕ_i with energy close to the ground state ϕ_1. However, we have to stress that BEC in this sense does *not* necessarily imply a macroscopic occupation of the ground state. In fact, the condensate can be spread over many (and even infinitely many) states.

These various situations correspond to classification of the g-BEC on the *types* I, II and III, introduced in the eighties by van den Berg-Lewis-Pulé, see e.g. [17] or [6, 18]. The most striking case is type III when generalized BEC occurs in the sense of (3.4) even though *none* of the eigenstates ϕ_i are macroscopically occupied. The realization of different types depends on how the relative gaps between the eigenvalues E_i at the bottom of the spectrum vanishes in the TL. To our knowledge, analysis of this behaviour in random system has only been realised in some particular cases, see [5] for a comprehensive presentation. The concept of *generalized* BEC is more stable then the standard one-mode BEC, since it depends on the global low-energy behaviour of the density of states, especially on its ability to make the critical density (3.3) finite. We note also that, since the IDS (3.1) is *not* random, the same it true for the amount of the g-BEC (3.4).

We can also obtain an explicit expression for the limiting measure m. Note that we have *fixed* the *mean density* $\overline{\rho}$, which implies that we require the chemical potential μ to satisfy the equation:

$$\overline{\rho} = \frac{1}{V_l}\langle N_l\rangle_l(\beta,\mu) = \frac{1}{V_l}\sum_{i\geq 1}\frac{1}{e^{\beta(E_i^{\omega,l}-\mu)}-1}, \tag{3.5}$$

for any l. Since the system is disordered, the unique solution $\mu_l^\omega := \mu_l^\omega(\beta,\overline{\rho})$ of this equation is a *random* variable, which is a.s. non-random in the TL [4, 5]. In the rest of this paper we denote the non-random $\mu_\infty :=$ a.s.-$\lim_{l\to\infty}\mu_l^\omega$. By condition (iii), Sect. 2, and by (3.7) it is a continuous function of $\overline{\rho}$:

$$\mu_\infty(\beta,\overline{\rho}) = \begin{cases} 0 & \text{if } \overline{\rho}\geq \rho_c, \\ \overline{\mu}<0 & \text{if } \overline{\rho}<\rho_c, \end{cases} \tag{3.6}$$

where $\overline{\mu} := \overline{\mu}(\beta,\overline{\rho})$ is a (unique) solution of the equation:

$$\overline{\rho} = \int_0^\infty \frac{1}{e^{\beta(E-\mu)}-1}\nu(\mathrm{d}E), \tag{3.7}$$

for $\overline{\rho}\leq\rho_c$.

Remark 3.1 Note that μ_∞ is non-positive (3.6), which is not true in general for the random finite-volume solution μ_l^ω. Indeed, the only restriction we have is that $\mu_l^\omega < E_1^{\omega,l}$, which is the well-known condition for the pressure of the perfect Bose gas to *exist*. We return to this question in Sect. 4 when we study BEC in the free one-particle kinetic-energy operator eigenfunctions in the presence of a random potential.

We also recall that for (3.6) the explicit expression of the weak limit for the general particle density occupation measure is:

$$m(\mathrm{d}E) = \begin{cases} (\overline{\rho}-\rho_c)\delta_0(\mathrm{d}E) + (e^{\beta E}-1)^{-1}\nu(\mathrm{d}E) & \text{if } \overline{\rho}\geq\rho_c, \\ (e^{\beta(E-\mu_\infty)}-1)^{-1}\nu(\mathrm{d}E) & \text{if } \overline{\rho}<\rho_c. \end{cases} \tag{3.8}$$

We end this section with a comment about the difference between the model of the perfect Bose gas embedded into a random potential and the *free* Bose gas. In the latter case, one should consider the IDS of the one-particle kinetic-energy operator (2.1), which is given by the *Weyl formula*:

$$\nu^0(E) = C_d E^{d/2}, \tag{3.9}$$

where is C_d is a constant term depending only on the dimensionality d. It is known that for this IDS, the critical density (3.3) is finite only when $d>2$, and hence the fact that BEC does not occur for low dimensions. On the other hand, a common feature of Schrödinger operators with regular, stationary, non-negative ergodic random potentials is the so-called *Lifshitz tails* behaviour of the IDS near the bottom of the spectrum. When the lower edge of the spectrum coincides with $E=0$ (condition (iii)), this means roughly that (see for example [16]):

$$\nu(E) \sim e^{-a/E^{d/2}} \tag{3.10}$$

for small E and $a > 0$. Hence, the critical density (3.3) is finite in any dimension, and therefore enhances BEC in the sense of (3.4) even for $d = 1, 2$. This was shown in [4, 5], where some specific examples of one-dimensional *Poisson disordered* systems exhibiting g-BEC in the sense of (3.4) were studied. In this article we require only the following rigorous upper estimate:

$$\lim_{E\to 0^+} (-E^{d/2}) \ln(\nu(E)) \geq a > 0, \tag{3.11}$$

for some constant a. This can be proved (see [14]) under the technical conditions detailed in Appendix B, which are assumed throughout this paper. In particular these conditions are satisfied in the case of Poisson random potentials with sufficiently fast decay of the potential around each impurity.

4 Generalized BEC in One-Particle *Kinetic* Energy Eigenstates

4.1 Occupation Measure for One-Particle Kinetic Energy Eigenstates

Similar to (3.2), we introduce the sequence of particle occupation measure $\tilde{m}_l$ for kinetic energy eigenfunctions $\{\psi_k := \psi_k^l\}_{k\in\Lambda_l^*}$:

$$\tilde{m}_l(A) := \frac{1}{V_l} \sum_{k:\varepsilon_k\in A} \langle N_l(\psi_k)\rangle_l, \quad A \subset \mathbb{R}, \tag{4.1}$$

but now in the *random equilibrium states* $\langle - \rangle_l$ corresponding to the perfect boson gas with Hamiltonian (2.5).

Note that, contrary to the last section, the standard arguments used to prove the existence of a limiting measure in TL are not valid for (4.1), since the kinetic energy operator (2.1) and the random Schrödinger operator (2.3) *do not commute*.

We remark also that even if we know that the measure m (3.8) has an atom at the edge of the spectrum (g-BEC), we cannot deduce that the limiting measure $\tilde{m}$ (assuming that it exists) also manifests g-BEC in the free kinetic energy eigenstates ψ_k.

Now we formulate the main result of this section.

Theorem 4.1 *The sequence of measures $\tilde{m}_l$ converges a.s. in a weak sense to a non-random measure $\tilde{m}$, which is given by*:

$$\tilde{m}(\mathrm{d}\varepsilon) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(\mathrm{d}\varepsilon) + F(\varepsilon)\mathrm{d}\varepsilon & \text{if } \overline{\rho} \geq \rho_c, \\ F(\varepsilon)\mathrm{d}\varepsilon & \text{if } \overline{\rho} < \rho_c \end{cases}$$

with density $F(\varepsilon)$ defined by:

$$F(\varepsilon) = (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma\, g(\sqrt{2\varepsilon}n_\sigma).$$

Here, S_d^1 denotes the unit sphere in $\mathbb{R}^d$ centered at the origin, n_σ the unit outward drawn normal vector, and $\mathrm{d}\sigma$ the surface measure of S_d^1. The function g is defined as follows

$$g(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x\, e^{ikx} \sum_{n\geq 1} e^{n\beta\mu_\infty} \mathbb{E}_\omega\big(K_\omega^{n\beta}(x,0)\big) \tag{4.2}$$

where $\mathbb{E}_\omega$ is the expectation on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $K^t_\omega(x, x')$ is the kernel of the operator e^{-th^ω}.

Note that since the measures $w^{n\beta}$ on $\Omega^{n\beta}_{(0,x)}$ are normalized, we recover from (4.2) the expression for the free Bose gas if we put $v^\omega = 0$.

Before proceeding with the proof, we give some comments about these results.

(a) First, the existence of a non-trivial limiting kinetic energy states occupation measure provides a rigorous basis for discussing the macroscopic occupation of the free Bose gas eigenstates.
(b) Moreover, both occupation measures (3.8) and (4.1) do not only exhibit simultaneously an atom at the bottom of the spectrum, but these atoms have the *same* non-random weights. It is quite surprising that the generalized BEC triggered by the Lifshitz tail in a low dimension disordered system produces the same value of the generalized BEC in the lowest one-particle kinetic-energy states.
(c) In addition our proofs have the following consequence for models with diagonal interaction U_l. The occurrence of generalized BEC in random one-particle states implies there is generalized BEC in the extended, i.e., kinetic-energy eigenstates and the density of the former cannot exceed the density of the latter. Our proof also shows that in spite of the lack of translation invariance in the random system, condensation always occurs in the lower kinetic energy states provided we can prove monotonicity of the finite-volume mean occupation numbers, $\langle N_l(\phi_j)\rangle_{H_l^U}$ as a function of $j \geq 1$, which can be done for the mean-field case.

4.2 Proofs

We start by expanding the measure $\tilde{m}$ in terms of the random equilibrium mean-values of occupation numbers in the corresponding eigenstates ϕ_i. Using the linearity (respectively conjugate linearity) of the creation and annihilation operators one obtains:

$$\begin{aligned}
\tilde{m}_l(A) &= \frac{1}{V_l}\sum_{k:\varepsilon_k\in A}\langle a^*(\psi_k)a(\psi_k)\rangle_l \\
&= \frac{1}{V_l}\sum_{i,j}\sum_{k:\varepsilon_k\in A}(\phi_i,\psi_k)\overline{(\phi_j,\psi_k)}\langle a^*(\phi_i)a(\phi_j)\rangle_l \\
&= \frac{1}{V_l}\sum_{i}\sum_{k:\varepsilon_k\in A}|(\phi_i,\psi_k)|^2\langle a^*(\phi_i)a(\phi_i)\rangle_l. \qquad (4.3)
\end{aligned}$$

In the last equality, we have used the "local" gauge invariance (2.7) which implies that:

$$\langle a^*(\phi_i)a(\phi_j)\rangle_l = 0 \quad \text{if } i \neq j.$$

We first prove two important lemmas.

The first result states that if there is condensation in the lowest random eigenstates $\{\phi_i\}_i$, then there is also condensation in the lowest kinetic-energy states $\{\psi_k\}_k$. Moreover, the amount of the latter condensate density has to be not *less* than the former.

Lemma 4.1 *Let* $\{\tilde{m}_{l_r}\}_{r\geq 1}$ *be a convergent subsequence. We denote by* $\tilde{m}$ *its (weak) limit. Then*:

$$\tilde{m}(\{0\}) \geq m(\{0\}) = \begin{cases} \overline{\rho} - \rho_c & \text{if } \overline{\rho} \geq \rho_c, \\ 0 & \text{if } \overline{\rho} < \rho_c. \end{cases}$$

Proof Let $\gamma > 0$. Using the expansion of the functions ψ_k in the basis $\{\phi_i\}_{i\geq 1}$, we obtain:

$$\begin{aligned} \tilde{m}([0,\gamma]) &= \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\leq\gamma} \langle N_{l_r}(\psi_k)\rangle_{l_r} \\ &= \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\leq\gamma} \sum_{i\geq 1} |(\phi_i,\psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \\ &\geq \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\leq\gamma} \sum_{i:E_i\leq\delta} |(\phi_i,\psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \end{aligned}$$

for any $\delta > 0$. The non-negativity of the random potential (2.4) implies:

$$\begin{aligned} \sum_{k:\varepsilon_k>\gamma} |(\phi_i,\psi_k)|^2 &\leq \sum_{k:\varepsilon_k>\gamma} \frac{\varepsilon_k}{\gamma} |(\phi_i,\psi_k)|^2 \leq \frac{1}{\gamma} \sum_{k\geq 1} \varepsilon_k |(\phi_i,\psi_k)|^2 = \frac{1}{\gamma}(\phi_i, h_l^0 \phi_i) \\ &\leq \frac{1}{\gamma}(\phi_i, h_l^\omega \phi_i) = \frac{E_i^\omega}{\gamma}. \end{aligned}$$

We then obtain:

$$\begin{aligned} \tilde{m}([0,\gamma]) &\geq \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{i:E_i\leq\delta} \langle N_{l_r}(\phi_i)\rangle_{l_r} \left(1 - \sum_{k:\varepsilon_k>\gamma} |(\phi_i,\psi_k)|^2\right) \\ &\geq \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{i:E_i\leq\delta} \langle N_{l_r}(\phi_i)\rangle_{l_r} (1 - E_i/\gamma) \\ &\geq \lim_{r\to\infty} (1-\delta/\gamma) \frac{1}{V_{l_r}} \sum_{i:E_i\leq\delta} \langle N_{l_r}(\phi_i)\rangle_{l_r} = (1-\delta/\gamma) m([0,\delta]) \geq 0. \end{aligned}$$

But δ is arbitrary, and the lemma follows by letting $\delta \to 0$. □

Remark 4.1 (Diagonal Interaction) The proof of Lemma 4.1 can be readily extended to a version which does not require the sequence of measures $\tilde{m}_l$ to converge. This is valid for models with Hamiltonian H_l^U, which satisfy the invariance condition (2.7) and for which the random potential is non-negative. The equivalent statement is then:

Suppose that the sequence m_l converges to m, then

$$\lim_{\delta\to 0} \liminf_{l\to\infty} \tilde{m}_l([0,\delta]) \geq m(\{0\}).$$

In the next lemma, we show that for the perfect gas the kinetic states occupation measure (4.1) can have an atom in the thermodynamic limit only at zero kinetic energy. We shall not assume that the sequence $\tilde{m}_l$ has a weak limit, instead we consider only some convergent subsequence. Note that at least one such subsequence always exists, see [21], Chap. VIII.6.

Lemma 4.2 *Let* $\{\tilde{m}_{l_r}\}_{r\geq 1}$ *be a convergent subsequence, and* $\tilde{m}$ *be its (weak) limit. Then, it is absolutely continuous on* $\mathbb{R}_+ := (0,\infty)$.

Proof Let A to be a Borel subset of $(0,\infty)$, with Lebesgue measure 0, and let a be such that $\inf A > a > 0$. Then:

$$\begin{aligned}\tilde{m}_{l_r}(A) &= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\in A} \langle N_{l_r}(\psi_k)\rangle_{l_r} \\ &= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\in A} \sum_i |(\phi_i,\psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \\ &= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\in A} \sum_{i:E_i\leq\alpha} |(\phi_i,\psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \\ &\quad + \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\in A} \sum_{i:E_i>\alpha} |(\phi_i,\psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \end{aligned} \tag{4.4}$$

for some $\alpha > 0$. Next, we use (2.4) to get the following estimate:

$$E_i^\omega = (\phi_i, h_l^\omega \phi_i) \geq (\phi_i, h_l^0 \phi_i) = \sum_k \varepsilon_k |(\phi_i,\psi_k)|^2 \geq a \sum_{k:\varepsilon_k\in A} |(\phi_i,\psi_k)|^2.$$

Since the equilibrium value of the occupation numbers $\langle N_l(\phi_i)\rangle_l = \{e^{E_i^\omega-\mu}-1\}^{-1}$ are decreasing with i, the estimate (4.4) implies:

$$\tilde{m}_{l_r}(A) \leq \frac{1}{V_{l_r}} \frac{1}{a} \sum_{i:E_i\leq\alpha} E_i^\omega \langle N_{l_r}(\phi_i)\rangle_{l_r} + \langle N_{l_r}(\phi_{i_\alpha})\rangle_{l_r} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\in A} 1, \tag{4.5}$$

where ϕ_{i_α} denotes the eigenstate of h_l^ω with the *smallest* eigenvalue *greater* than α. Using again the monotonicity and the finite-volume IDS (3.1) we can get an upper bound for the mean occupation number in the second term of (4.5), since:

$$\overline{\rho} = \frac{1}{V_l} \sum_i \langle N_l(\phi_i)\rangle_l \geq \frac{1}{V_l} \sum_{i:E_i\leq\alpha} \langle N_l(\phi_i)\rangle_l \geq \langle N_l(\phi_{i_\alpha})\rangle_l \nu_l^\omega(\alpha). \tag{4.6}$$

Combining (4.5) and (4.6) we obtain:

$$\tilde{m}_{l_r}(A) \leq \frac{\alpha\overline{\rho}}{a} + \frac{\overline{\rho}}{\nu_{l_r}^\omega(\alpha)} \int_A \nu_{l_r}^0(\mathrm{d}\varepsilon). \tag{4.7}$$

Since the measure ν^0 (3.9) is absolutely continuous with respect to the Lebesgue measure, and $\nu(\alpha)$ is strictly positive for any $\alpha > 0$ the limit $r \to \infty$ in (4.7) gives:

$$\tilde{m}(A) \leq \frac{\alpha\overline{\rho}}{a},$$

but $\alpha > 0$ can be chosen arbitrary small and thus $\tilde{m}(A) = 0$. To finish the proof, note that any Borel subset of $(0,\infty)$ can be expressed as a countable union of disjoint subsets with non-zero infimum. Our arguments than can be applied to each of them. □

Remark 4.2 (Diagonal Interaction) Lemma 4.2 can also be extended in the same way as proposed in Remark 4.1, for Lemma 4.1. Again we assume the invariance condition (2.7) for interacting bosons with Hamiltonian H_l^U and the non-negativity of the random potential, with the additional requirement that the occupation numbers $\langle N_l(\phi_i)\rangle_{H_l^U}$ are monotonic in i. This last property is valid for the Bose-gas with a mean-field interaction, see [11].

Above we exploited the fact that the sequence $\{\tilde{m}_l\}_{l\geq 1}$ has at least one accumulation point. However, to prove convergence, we need to make use of some particular and explicit features of the perfect Bose gas, as well as more detailed information about the properties of the external (random) potential. In particular, we shall need some estimates of the (random) finite volume integrated density of states, see Lemma 5.1.

To this end let us denote by P_A the orthogonal projection onto the subspace spanned by the one-particle kinetic energy states ψ_k with kinetic energy $\varepsilon(k)$ in the set A. Then using the explicit expression for the mean occupation $\langle a^*(\phi_i)a(\phi_i)\rangle_l$ and (4.3) we obtain:

$$\tilde{m}_l(A) = \frac{1}{V_l}\,\mathrm{Tr}\,P_A(e^{\beta(h_l^\omega-\mu_l)}-1)^{-1} = \sum_{n\geq 1}\frac{1}{V_l}\,\mathrm{Tr}\,P_A(e^{-n\beta(h_l^\omega-\mu_l)}). \tag{4.8}$$

Now we split the measure (4.8) into two parts:

$$\begin{aligned}\tilde{m}_l &= \tilde{m}_l^{(1)} + \tilde{m}_l^{(2)},\\ \tilde{m}_l^{(1)}(A) &:= \sum_{n\geq 1}\frac{1}{V_l}\,\mathrm{Tr}\,P_A(e^{-n\beta(h_l^\omega-\mu_l)})\mathbf{1}(\mu_l\leq 1/n),\\ \tilde{m}_l^{(2)}(A) &:= \sum_{n\geq 1}\frac{1}{V_l}\,\mathrm{Tr}\,P_A(e^{-n\beta(h_l^\omega-\mu_l)})\mathbf{1}(\mu_l > 1/n).\end{aligned} \tag{4.9}$$

Note that since the chemical potential satisfies (3.5), $\mu_l := \mu_l^\omega$, the indicator functions $\mathbf{1}(\mu_l\leq 1/n)$ and $\mathbf{1}(\mu_l > 1/n)$ split the range of n into the sums (4.9) in a random and volume-dependent way.

We start with the proof of existence of a weak limit of the sequence of random measures $\tilde{m}_l^{(1)}$:

Theorem 4.2 *Let random potential v^ω satisfy the assumptions* (i)–(iii) *of Sect.* 2. *Then for any $d\geq 1$, the sequence of Laplace transforms of the measures $\tilde{m}_l^{(1)}$*:

$$f_l(t;\beta,\mu_l) := \int_{\mathbb{R}} \tilde{m}_l^{(1)}(\mathrm{d}\varepsilon)\, e^{-t\varepsilon} \tag{4.10}$$

converges for any $t>0$ to a (*non-random*) *limit $f(t;\beta,\mu_\infty)$, which is given by*:

$$f(t;\beta,\mu_\infty) = \sum_{n\geq 1} e^{n\beta\mu_\infty}\int_{\mathbb{R}^d}\mathrm{d}x\,\frac{e^{-\|x\|^2/2t}}{(4\pi^2 t)^{d/2}}\mathbb{E}_\omega\big(K_\omega^{n\beta}(x,0)\big). \tag{4.11}$$

Here $\mathbb{E}_\omega$ denotes the expectation with respect to realizations (*configurations*) *ω of the random potential. Note that the sum on the right-hand side converges for all* (*non-random*) *$\mu_\infty\leq 0$, including* 0, *which corresponds to the case $\overline{\rho}\geq\rho_c$.*

Proof By definition of P_A the Laplace transformation (4.10) can be written as:

$$f_l(t;\beta,\mu_l) = \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0}(e^{-n\beta(h_l^\omega-\mu_l)})\mathbf{1}(\mu_l \leq 1/n). \tag{4.12}$$

Now we have to show the uniform convergence of the sum over n to be able to take the term by term limit with respect to l. Since for any bounded operator A and for any trace-class non-negative operator B one has $\operatorname{Tr} AB \leq \|A\| \operatorname{Tr} B$, we get

$$\begin{aligned} a_l(n) &:= \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l^\omega-\mu_l)}\mathbf{1}(\mu_l \leq 1/n) \\ &\leq \frac{1}{V_l} \operatorname{Tr} e^{-n\beta(h_l^\omega-\mu_l)}\mathbf{1}(\mu_l \leq 1/n). \end{aligned} \tag{4.13}$$

For $\overline{\rho} < \rho_c$, the uniform convergence in (4.11) is immediate. Indeed, for l large enough, the chemical potential satisfies $\mu_l < \mu_\infty/2 < 0$, which by (3.1) provides the following a.s. estimate for (4.13):

$$a_l(n) \leq e^{n\beta\mu_\infty/2} \int_{[0,\infty)} \nu_l^\omega(\mathrm{d}E) e^{-\beta E} \leq K_1 e^{n\beta\mu_\infty/2}, \tag{4.14}$$

with some constant K_1.

However, for the case $\overline{\rho} \geq \rho_c$, this approach does not work, since, in fact, for any finite l the solutions $\mu_l = \mu_l^\omega$ of (3.5) could be *positive* with some probability, event though by condition (iii) (see Sect. 2) it has to *vanish* a.s. in the TL. We use, therefore, the bound:

$$\begin{aligned} a_l(n) &\leq a_l^1(n) + a_l^2(n), \\ a_l^1(n) &:= \frac{1}{V_l} e^\beta \sum_{\{i:E_i^{\omega,l} \leq 1/n^{1-\eta}\}} e^{-n\beta E_i^{\omega,l}}, \\ a_l^2(n) &:= \frac{1}{V_l} e^\beta \sum_{\{i:E_i^{\omega,l} > 1/n^{1-\eta}\}} e^{-n\beta E_i^{\omega,l}}, \end{aligned}$$

which follows, for some $0 < \eta < 1$, from the constraint $\mu_l n \leq 1$ due to the indicator function in (4.13). Then the first term is bounded from above by:

$$a_l^1(n) \leq e^\beta \nu_l^\omega(n^{\eta-1}).$$

On the other hand, by Theorem 5.1 (*finite-volume* Lifshitz tails), for $\alpha > 0$ and $0 < \gamma < d/2$, there exists a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any $\omega \in \widetilde{\Omega}$ there exists a positive finite energy $\mathcal{E}(\omega) := \mathcal{E}_{\alpha,\gamma}(\omega) > 0$ for which one obtains:

$$\nu_l^\omega(E) \leq e^{-\alpha/E^\gamma},$$

for all $E < \mathcal{E}(\omega)$. Therefore, for any configuration $\omega \in \widetilde{\Omega}$ (i.e. almost surely) we have the *volume independent* estimate for all $n > \mathcal{N}(\omega) := \mathcal{E}(\omega)^{1/(\eta-1)}$:

$$a_l^1(n) \leq e^\beta e^{-\alpha n^{(1-\eta)\gamma}}. \tag{4.15}$$

To estimate the coefficients $a_l^2(n)$ from above, we use the upper bound:

$$a_l^2(n) \leq \int_{[1/n^{1-\eta},\infty)} \nu_l^{\omega}(\mathrm{d}E)\, e^{-n\beta E} \leq e^{-\beta n^{\eta}/2} \int_{[1/n^{1-\eta},\infty)} \nu_l^{\omega}(\mathrm{d}E)\, e^{-n\beta E/2}$$
$$\leq e^{-\beta n^{\eta}/2} \int_{[0,\infty)} \nu_l^{\omega}(\mathrm{d}E)\, e^{-\beta E/2}.$$

Then for some $K_2 > 0$ independent of l we obtain:

$$a_l^2(n) \leq K_2 e^{-\beta n^{\eta}/2}. \tag{4.16}$$

Therefore, by (4.14) in the case $\overline{\rho} < \rho_c$, and by (4.15), (4.16) for $\overline{\rho} \geq \rho_c$, we find that there exists a sequence $a(n)$ (independent of l) such that:

$$a_l(n) \leq a(n) \quad \text{and} \quad \sum_{n\geq 1} a(n) < \infty. \tag{4.17}$$

Thus, the series (4.12) is uniformly convergent in l, and one can exchange sum and the limit:

$$\lim_{l\to\infty} f_l(t) = \lim_{l\to\infty} \sum_{n=0}^{\infty} a_l(n) = \sum_{n=0}^{\infty} \lim_{l\to\infty} a_l(n).$$

The rest of the proof is largely inspired by the paper [4]. Let

$$\Omega^T_{(x,x')} := \{\xi : \xi(0) = x, \xi(T) = x'\}$$

be the set of continuous trajectories (paths) $\{\xi(s)\}_{s=0}^{T}$ in $\mathbb{R}^d$, connecting the points x, x', and let w^T denote the normalized Wiener measure on this set. Using the Feynman-Kac representation, we obtain the following limit:

$$\lim_{l\to\infty} a_l(n) = \lim_{l\to\infty} \frac{1}{V_l} \mathrm{Tr}\, e^{-th_l^0} e^{-n\beta(h_l^{\omega}-\mu_l)} \mathbf{1}(\mu_l \leq 1/n)$$
$$= \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\mathrm{d}x' e^{-th_l^0}(x,x') e^{-n\beta(h_l^{\omega}-\mu_l)}(x',x)$$
$$= e^{n\beta\mu_\infty} \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$
$$\times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta}\mathrm{d}s\, v^{\omega}(\xi(s))} \chi_{\Lambda_l,n\beta}(\xi) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi'), \tag{4.18}$$

where we denote by $\chi_{\Lambda_l,T}(\xi)$ the characteristic function of paths ξ such that $\xi(t) \in \Lambda_l$ for all $0 < t < T$. Using Lemma A.2, we can eliminate these restrictions, and also extend one spatial integration over the whole space:

$$\lim_{l\to\infty} \frac{1}{V_l} \mathrm{Tr}\, e^{-th_l^0} e^{-n\beta(h_l^{\omega}-\mu_l)}$$
$$= e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta}\mathrm{d}s\, v^{\omega}(\xi(s))}. \tag{4.19}$$

Now, by the *ergodic* theorem, we obtain:

$$\begin{aligned}
&\lim_{l\to\infty} a_l(n)\\
&= \lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l^\omega-\mu_l)}\\
&= e^{n\beta\mu_\infty} \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x' \left\{ \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s v^\omega(\xi(s))} \right\}\\
&= e^{n\beta\mu_\infty} \mathbb{E}_\omega \left\{ \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s v^\omega(\xi(s))} \right\}. \qquad (4.20)
\end{aligned}$$

We then get the explicit expression for the limiting Laplace transform:

$$f(t;\beta,\mu_\infty) = \sum_{n\ge 1} e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \mathbb{E}_\omega \left\{ \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s v^\omega(\xi(s))} \right\},$$

which finishes the proof. □

Corollary 4.1 *For any $\overline{\rho}$ the sequence of random measures $\tilde{m}_l^{(1)}$ converges a.s. in the weak sense to a bounded, absolutely continuous non-random measure $\tilde{m}^{(1)}$, with density $F(\varepsilon)$ given by*

$$F(\varepsilon) := (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma\, g(\sqrt{2\varepsilon} n_\sigma).$$

Here, S_d^1 denotes the unit sphere in $\mathbb{R}^d$, n_σ the outward drawn normal unit vector, $\mathrm{d}\sigma$ *the surface measure on S_d^1 and the function g has the form*

$$g(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x e^{ikx} \sum_{n\ge 1} e^{n\beta\mu_\infty} \mathbb{E}_\omega\big(K_\omega^{n\beta}(x,0)\big). \qquad (4.21)$$

Proof By Theorem 4.2, the existence of the weak limit $\tilde{m}^{(1)}$ follows from the existence of the limiting Laplace transform. Moreover, we have the following explicit expression:

$$\begin{aligned}
&\int_{\mathbb{R}} \tilde{m}^{(1)}(\mathrm{d}\varepsilon) e^{-t\varepsilon}\\
&= \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{d/2}} \sum_{n\ge 1} e^{n\beta\mu} \frac{e^{-\|x\|^2/2n\beta}}{(2\pi n\beta)^{d/2}} \mathbb{E}_\omega \left\{ \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s v^\omega(\xi(s))} \right\}\\
&= \int_{[0,\infty)} \mathrm{d}r e^{-t\|r\|^2/2} r^{d-1} \int_{S_d^1} \mathrm{d}\sigma\, g(rn_\sigma)\\
&= \int_{[0,\infty)} \mathrm{d}\varepsilon e^{-t\varepsilon} (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma\, g(\sqrt{2\varepsilon} n_\sigma),
\end{aligned}$$

which proves the corollary. □

Corollary 4.2 *The measure* $\tilde{m}^{(1)}$ *satisfies the following property*:

$$\int_{[0,\infty)} \tilde{m}^{(1)}(d\varepsilon) = \begin{cases} \overline{\rho} & \text{if } \overline{\rho} < \rho_c, \\ \rho_c & \text{if } \overline{\rho} \geq \rho_c. \end{cases}$$

Proof By virtue of (4.12) we have:

$$\int_{[0,\infty)} \tilde{m}^{(1)}(d\varepsilon) = f(0; \beta, \mu_\infty) = \lim_{l\to\infty} \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} e^{-n\beta(h_l^\omega - \mu_l)} \mathbf{1}(\mu_l \leq 1/n).$$

Note that by uniformity of convergence of the sum, see (4.15), (4.16), we can take the limit term by term (for any value of $\overline{\rho}$), and then:

$$\begin{aligned} \int_{[0,\infty)} \tilde{m}^{(1)}(d\varepsilon) &= \sum_{n\geq 1} \lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-n\beta(h_l^\omega - \mu_l)} \\ &= \sum_{n\geq 1} \int_{[0,\infty)} \nu(dE) e^{-n\beta(E-\mu_\infty)} \\ &= \int_{[0,\infty)} \nu(dE)(e^{\beta(E-\mu_\infty)} - 1)^{-1}, \end{aligned}$$

where we use Fubini's theorem for the last step. □

We are now ready for the proof of the main result of this section:

Proof of Theorem 4.1 We first treat the case $\overline{\rho} < \rho_c$. In this situation, the measure $\tilde{m}_l^{(2)}$ is equal to 0 for l large enough, see (4.9), since the solution $\lim_{l\to\infty} \mu_l^\omega$ (3.5) in the TL is a.s. strictly negative. Thus, the total occupation measure $\tilde{m}_l$ is reduced to $\tilde{m}_l^{(1)}$ and the theorem follows from Corollary 4.1.

Now, consider the case $\overline{\rho} \geq \rho_c$. Choose a subsequence l_r such that the total kinetic-energy states occupation measures $\tilde{m}_{l_r}$ converge weakly and a.s., and let the measure $\tilde{m}$ be its limit. By Corollary 4.1, all subsequences of measures $\tilde{m}_{l_r}^{(1)}$ converge to the limiting measure $\tilde{m}^{(1)}$. Therefore, by (4.9), we obtain the weak a.s. convergence:

$$\lim_{r\to\infty} \tilde{m}_{l_r}^{(2)} =: \tilde{m}^{(2)}.$$

By Lemma 4.2, we know that the measure $\tilde{m}$ is absolutely continuous on $(0,\infty)$, and by Corollary 4.1 that $\tilde{m}^{(1)}$ is absolutely continuous on $[0,\infty)$. Therefore we get:

$$\tilde{m}^{\text{a.c.}} = \tilde{m}^{(1)} + \tilde{m}^{(2)\text{a.c.}},$$

where a.c. denotes the *absolute continuous* components.

By definition of the total measure (4.9), $\tilde{m}([0,\infty)) = \overline{\rho}$ and by Lemma 4.1, $\tilde{m}(\{0\}) \geq \overline{\rho} - \rho_c$. Thus, $\tilde{m}((0,\infty)) \leq \rho_c$ and by Corollary 4.2, we can then deduce that the measure $\tilde{m}^{(2)}$ has no absolutely continuous component and therefore consists at most of an atom at $\varepsilon = 0$. Consequently, the full measure $\tilde{m}$ can be expressed as:

$$\tilde{m} = \tilde{m}^{\text{a.c.}} + b\delta_0 = \tilde{m}^{(1)} + b\delta_0,$$

and since by Corollary 4.2

$$b = \overline{\rho} - \int_{\mathbb{R}_+} \tilde{m}_{l_r}^{\text{a.c.}}(\mathrm{d}\varepsilon) = \overline{\rho} - \int_{\mathbb{R}_+} \tilde{m}_{l_r}^{(1)}(\mathrm{d}\varepsilon) = \overline{\rho} - \rho_c$$

for the converging subsequence $\tilde{m}_{l_r}$, we have:

$$\lim_{l_r \to \infty} \tilde{m}_{l_r} = \tilde{m}^{(1)} + (\overline{\rho} - \rho_c)\delta_0.$$

By (4.22) and Corollary 4.1, this limit is independent of the subsequence. Then, the limit of any convergent subsequence is the same, and therefore, using Feller's selection theorem, see [21], Chap. VIII.6, the total kinetic states occupation measures $\tilde{m}_l$ converge weakly to this limit. □

5 Finite Volume Lifshitz Tails

In this section, we give the proof of one important building block of our analysis, Theorem 5.1 about the *finite-volume* Lifshitz tails. Recall that this behaviour is a well-known feature of disordered systems, essentially meaning that for Shrödinger operators which are semi-bounded from below, there are exponentially few eigenstates with energy close to the bottom of the spectrum. To our knowledge, however, this is always shown only in the *infinite-volume* limit, see e.g. [16]. Here, we derive a *finite-volume* estimate for the density of states, uniformly in l, though it could be trivial for small volumes. As one would expect our result is weaker than the asymptotic one, in the sense that we prove it for Lifshitz exponent smaller than the limiting one.

Theorem 5.1 *Let the random potential v^ω satisfy the assumptions* (i)–(iii) *of Sect.* 2. *Then for any $\alpha > 0$ and $0 < \gamma < d/2$, there exists a set $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any configuration $\omega \in \widetilde{\Omega}$ one can find a positive finite energy $\mathcal{E}(\omega) := \mathcal{E}_{\alpha,\gamma}(\omega)$, for which one has the estimate*:

$$\nu_l^\omega(E) \le e^{-\alpha/E^\gamma}$$

for all $E < \mathcal{E}(\omega)$ and for all l.

Remark 5.1 We want to stress that the statement in Theorem 5.1 is valid for all l, but of course, it can be trivial for small l. For example from the positivity of the potential we know that $\nu_l^\omega(E) = 0$ for $E < \pi^2 d/l^2$ and therefore the estimate is trivial for $l < \pi/\sqrt{\mathcal{E}(\omega)}$.

For the proof, we first need a result from [14].

Lemma 5.1 *By assumption* (ii) (*Sect.* 2) *one has,*

$$p = \mathbb{P}\{\omega : v^\omega(0) = 0\} < 1.$$

Let $\alpha > p/(1-p)$, $B = \pi/(1+\alpha)$, and $E_1^{\omega,l,N} := E_1^{\omega,N}$ be the first eigenvalue of the random Schrödinger operator (2.3) *with Neumann* (*instead of Dirichlet*) *boundary conditions. Then, for l large enough, there exists an independent of l constant $A = A(\alpha)$, such that*

$$\mathbb{P}\{\omega : E_1^{\omega,N} < B/l^2\} < e^{-AV_l}. \tag{5.1}$$

Detailed conditions on the random potential and a sketch of the proof of this lemma are given in Appendix B. Now we use Lemma 5.1 to prove the following result:

Lemma 5.2 *Assume that the random potential satisfies the assumptions of Lemma 5.1. Then for any $\alpha > 0$ and $0 < \gamma < d/2$,*

$$\sum_{n\geq 1}\mathbb{P}\big\{\sharp\big\{i : E_i^{\omega,l} < 1/n\big\} > V_l e^{-\alpha n^\gamma},\ \textit{for some } l \geq 1\big\} < \infty.$$

Proof Notice that

$$\sum_{n\geq 1}\mathbb{P}\big\{\sharp\big\{i : E_i^{\omega,l} < 1/n\big\} > V_l e^{-\alpha n^\gamma},\ \text{for some } l \geq 1\big\} = \sum_{n\geq 1}\mathbb{P}\Big\{\bigcup_{l\geq 1} S_l^n\Big\}, \tag{5.2}$$

where S_l^n is the set

$$S_l^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\} > V_l e^{-\alpha n^\gamma}\right\}.$$

The sum in the right-hand side of (5.2) does not provide a very useful upper bound, since the sets S_l^n are highly overlapping. We thus need to define a new refined family of sets to avoid this difficulty.

To this end we let $[a]_+$ be the smallest integer $\geq a$, and we define the family of sets:

$$V_k^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq k\right\}.$$

Let $k := [V_l e^{-\alpha n^\gamma}]_+$. Since $V_l = l^d$, this implies that $h_l^\omega \geq h^\omega_{[(ke^{\alpha n^\gamma})^{1/d}]_+}$, and therefore:

$$\sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq \sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\}.$$

If now $\omega \in S_l^n$, then by the definition of k we obtain:

$$\sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\} \geq k,$$

since the left-hand side is itself an integer. Thus, $S_l^n \subset V_k^n$ and:

$$\mathbb{P}\Big(\bigcup_{l\geq 1} S_l^n\Big) \leq \mathbb{P}\Big(\bigcup_{k\geq 1} V_k^n\Big). \tag{5.3}$$

We define also the sets:

$$W_k^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} = k\right\}. \tag{5.4}$$

Let $\omega \in (V_k^n \setminus W_k^n)$. Then by $h^\omega_{[((k+1)e^{\alpha n^\gamma})^{1/d}]_+} \leq h^\omega_{[(ke^{\alpha n^\gamma})^{1/d}]_+}$ we get:

$$\sharp\left\{i : E_i^{\omega,[((k+1)e^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq \sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq k+1.$$

Hence, $(V_k^n \setminus W_k^n) \subset V_{k+1}^n$, and therefore we have for any fixed n and k:

$$V_k^n \subset W_k^n \cup V_{k+1}^n. \tag{5.5}$$

Applying this inclusion M times, for $k = 1, \dots, M$, we obtain:

$$\bigcup_{k=1}^{M} V_k^n \subset \left(W_1^n \cup \bigcup_{k=2}^{M} V_k^n\right) \subset \left(W_1^n \cup W_2^n \cup \bigcup_{k=2}^{M} V_k^n\right) \subset \cdots \subset \left(\bigcup_{k=1}^{M} W_k^n\right) \cup V_{M+1}^n. \tag{5.6}$$

Then we take the limit $M \to \infty$ to recover the infinite union that one needs in (5.3) and we use the inclusion (5.6) to find the inequality:

$$\begin{aligned}\mathbb{P}\left(\bigcup_{k\geq 1} V_k^n\right) &= \lim_{M\to\infty} \mathbb{P}\left(\bigcup_{k=1}^{M} V_k^n\right)\\ &\leq \lim_{M\to\infty}\left(\sum_{k=1}^{M} \mathbb{P}(W_k^n) + \mathbb{P}(V_{(M+1)}^n)\right) = \sum_{k=1}^{\infty} \mathbb{P}(W_k^n) + \lim_{M\to\infty} \mathbb{P}(V_M^n).\end{aligned} \tag{5.7}$$

The limit in the last term can be calculated directly:

$$\begin{aligned}\lim_{M\to\infty} \mathbb{P}(V_M^n) &= \lim_{M\to\infty} \mathbb{P}\left\{\omega : \sharp\left\{i : E_i^{\omega,[(Me^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq M\right\}\\ &= \lim_{M\to\infty} \mathbb{P}\left\{\omega : \nu^{\omega}_{[(Me^{\alpha n^\gamma})^{1/d}]_+}(1/n) \geq \frac{M}{[(Me^{\alpha n^\gamma})^{1/d}]_+^d}\right\}\\ &= \mathbb{P}\left\{\omega : \nu(1/n) \geq Ke^{-\alpha n^\gamma}\right\},\end{aligned} \tag{5.8}$$

for some constant K. In the last step we used dominated convergence theorem.

Now we can use the Lifshitz tails representation for the asymptotics of the a.s. non-random limiting IDS, $\nu(E)$, see (3.11), which implies:

$$\limsup_{n\to\infty} e^{an^{d/2}} \nu(1/n) \leq 1, \tag{5.9}$$

for $a > 0$. Since we assumed that $0 < \gamma < d/2$, there exists $n_0 < \infty$ such that by (5.8) and (5.9) for all $n > n_0$ we get:

$$\lim_{M\to\infty} \mathbb{P}(V_M^n) = 0.$$

This last result, along with (5.3) and (5.7), implies that:

$$\sum_{n>n_0} \mathbb{P}\left(\bigcup_{l\geq l_0} S_l^n\right) \leq \sum_{n>n_0} \sum_{k=1}^{\infty} \mathbb{P}(W_k^n). \tag{5.10}$$

Now, we show that the upper bound in (5.10) is finite. First we split the box $\Lambda_{[(ke^{\alpha n^\gamma})^{1/d}]_+}$ into $m(k,n)$ disjoints sub-cubes of the side $l(k,n)$, with the following choice of parameters:

$$m(k,n) := [kM_n]_+, \quad M_n := B^{-d/2} e^{\alpha n^\gamma} n^{-d/2},$$

$$l(k,n) := \frac{[(ke^{\alpha n^{\gamma}})^{1/d}]_+}{(m(k,n))^{1/d}}.$$

Here B is the constant that comes from Lemma 5.1. Now by the Dirichlet-Neumann inequality, see e.g. [22], Chap. XIII.15, we get:

$$h^D_{[(ke^{\alpha n^{\gamma}})^{1/d}]_+} \geq h^N_{[(ke^{\alpha n^{\gamma}})^{1/d}]_+} \geq \bigoplus_{j=1}^{m(k,n)} h^{j,N}_{l(k,n)}, \tag{5.11}$$

where $h^{j,N}_{l(k,n)}$ denotes the Schrödinger operator defined in the j-th sub-cube of the side $l(k,n)$, with Neumann boundary conditions. Note that, by the *positivity* of the random potential, we obtain:

$$E^{\omega,N}_{j,2} \geq \varepsilon^N_{j,2} \geq \frac{\pi}{l(k,n)^2} \geq \frac{1}{n}. \tag{5.12}$$

Here $E^{\omega,N}_{j,2}$ denotes the *second eigenvalue* of the operator $h^{j,N}_{l(k,n)}$, and $\varepsilon^N_{j,2}$ the *second eigenvalue* of $-\Delta^{j,N}_{l(k,n)}$, i.e. the kinetic-energy operator defined in the j-th sub-cube of the side $l(k,n)$ with the Neumann boundary conditions.

By (5.12), we know that to estimate the probability of the set (5.4) by using the Dirichlet-Neumann inequality (5.11), only the *ground state* of each operator $h^{j,N}_{l(k,n)}$ is relevant. Since the sub-cubes are *stochastically independent*, we have:

$$\mathbb{P}\big(W^n_k\big) \leq \mathbb{P}\big\{\omega : \sharp\big\{j : E^{\omega,N}_{j,1} < 1/n\big\} = k\big\} \leq {}^{m(k,n)}C_k q^k (1-q)^{m(k,n)-k} \leq {}^{m(k,n)}C_k q^k$$

with q being the probability $\mathbb{P}\{\omega : E^{\omega,N}_{j,1} < 1/n\}$. The latter can be estimated by Lemma 5.1. So, finally we obtain the upper bound:

$$\mathbb{P}\big(W^n_k\big) \leq {}^{m(k,n)}C_k \exp\big\{-kA(l(k,n))^d\big\}. \tag{5.13}$$

Using Stirling's inequalities, see [23], Chap. II.12:

$$(2\pi)^{1/2} n^{n+1/2} e^{-n} \leq n! \leq 2(2\pi)^{1/2} n^{n+1/2} e^{-n}$$

we can give an upper bound for the binomial coefficients ${}^{m(k,n)}C_k$ in the form:

$$\frac{2(2\pi)^{\frac{1}{2}}(kM_n+\delta)^{(kM_n+\delta+1/2)}\exp(-kM_n+\delta)}{(2\pi)k^{k+\frac{1}{2}}\exp(-k)\cdot(kM_n+\delta-k)^{(kM_n+\delta-k+1/2)}\exp(-kM_n+\delta-k)}, \tag{5.14}$$

where $\delta \geq 0$ is defined by:

$$m(k,n) = [kM_n]_+ = kM_n + \delta.$$

Then (5.14) implies the estimate:

$${}^{m(k,n)}C_k \leq K_1 \frac{(kM_n+\delta)^{kM_n+\delta+1/2}}{k^{k+\frac{1}{2}}(kM_n-k)^{kM_n+\delta-k+1/2}} \leq K_1(M_n)^k \left(\frac{(1+\sigma_1)^{(kM_n+\delta+\frac{1}{2})}}{(1-\sigma_2)^{(kM_n+\delta+\frac{1}{2}-k)}}\right),$$

for some $K_1 > 0$ and

$$\sigma_1 := \delta(kM_n)^{-1}, \qquad \sigma_2 := M_n^{-1}.$$

Since $\delta/k < 1$ and $\sigma_{1,2} \to 0$ as $n \to \infty$, and also using the fact that $x \ln(1 + 1/x) \to 1$ as $x \to \infty$, we can find a constant $c > 0$ such that, for n large enough one gets the estimate:

$$ {}^{m(k,n)}C_k \le K_1 (M_n)^k \left(\frac{(1 + M_n^{-1})^{(kM_n)}}{(1 - M_n^{-1})^{(kM_n - k)}} \right) \le K_1 (M_n)^k e^{ck}. \tag{5.15} $$

The side $l(k,n)$ of sub-cubes has a lower bound:

$$ l(k,n) = \frac{[(ke^{\alpha n^\gamma})^{1/d}]_+}{(m(k,n))^{1/d}} \ge \frac{(ke^{\alpha n^\gamma})^{1/d}}{(ke^{\alpha n^\gamma}(Bn)^{-d/2} + \delta)^{1/d}} \ge \left(B^{d/2} n^{d/2} \frac{1}{1 + \sigma_1} \right)^{1/d}. \tag{5.16} $$

Combining (5.15), (5.16) and (5.13) we obtain a sufficient upper bound:

$$ \begin{aligned} \sum_{k \ge 1} \mathbb{P}\big(W_k^n\big) &\le \sum_{k \ge 1} {}^{m(k,n)}C_k e^{-kAl^d(k,n)} \\ &\le \sum_{k \ge 1} K_1 (M_n)^k e^{ck} e^{-kAB^{d/2} n^{d/2}/(1+\sigma_1)} \\ &\le K_2 \sum_{k \ge 1} \exp\big\{k\big(\alpha n^\gamma - (d/2)\ln(nB) + c - AB^{d/2} n^{d/2}\big)\big\} \\ &\le K_3 \sum_{k \ge 1} \exp k\big(\alpha n^\gamma - AB^{d/2} n^{d/2} + K_4\big) \le K_5 \exp\big(-K_6 n^{d/2}\big). \end{aligned} $$

Here K_i are some finite, positive constants independent of k, n, l, for any n large enough. Now the lemma immediately follows from (5.10). □

Proof of Theorem 5.1 Let A_n to be the event:

$$ A_n := \{\omega : \nu_l^\omega(1/n) > e^{-\alpha n^\gamma} \text{ for some } l\}. \tag{5.17} $$

By Lemma 5.2, we have:

$$ \sum_{n \ge 1} \mathbb{P}\big(A_n\big) < \infty, $$

and therefore, by the Borel-Cantelli lemma one gets that with probability one, only a *finite* number of events A_n occur. In other words, there is a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any $\omega \in \widetilde{\Omega}$ one can find a *finite* and independent on l number $n_0(\omega) < \infty$ for which, in contrast to (5.17), we have:

$$ \nu_l^\omega(1/n) \le e^{-\alpha n^\gamma}, \quad \text{for all } n > n_0(\omega) \text{ and for all } l \ge 1. $$

Define $\mathcal{E}(\omega) := 1/n_0(\omega)$. For any $E \le \mathcal{E}(\omega)$, we can find $n \ge n_0(\omega)$ such that:

$$ \frac{1}{2n} \le E \le \frac{1}{n}, $$

and the theorem follows with the constant α modified by a factor $2^{-\gamma}$. □

6 On the Nature of the Generalized Condensates in the Luttinger-Sy Model

In this section, we study the van den Berg-Lewis-Pulé classification of generalized BE condensation (see discussion in Sect. 3) in a particular case of the so-called Luttinger-Sy model with point impurities [3]. Formally the single particle Hamiltonian for this model is

$$h_l^\omega = -\frac{1}{2}\Delta + a\sum_j \delta(x - x_j^\omega), \tag{6.1}$$

where the x_j's are distributed according to a Poisson law and $a = +\infty$.

We first recalls some definitions to make sense of this formal Hamiltonian. Let $u(x) \geq 0$, $x \in \mathbb{R}$, be a continuous function with a *compact* support called a (*repulsive*) single-impurity potential. Let $\{\mu_\lambda^\omega\}_{\omega\in\Omega}$ be the *random* Poisson measure on $\mathbb{R}$ with intensity $\lambda > 0$:

$$\mathbb{P}(\{\omega \in \Omega : \mu_\lambda^\omega(\Lambda) = n\}) = \frac{(\lambda|\Lambda|)^n}{n!} e^{-\lambda|\Lambda|}, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{6.2}$$

for any bounded Borel set $\Lambda \subset \mathbb{R}$. Then the non-negative random potential v^ω generated by the Poisson distributed local impurities has realizations

$$v^\omega(x) := \int_{\mathbb{R}} \mu_\lambda^\omega(dy) u(x - y) = \sum_{x_j^\omega \in X^\omega} u(x - x_j^\omega). \tag{6.3}$$

Here the random set X^ω corresponds to impurity positions $X^\omega = \{x_j^\omega\}_j \subset \mathbb{R}$, which are the atoms of the random point Poisson measure, i.e., $\sharp\{X^\omega \upharpoonright \Lambda\} = \mu_\lambda^\omega(\Lambda)$ is the number of impurities in the set Λ. Since the expectation $\mathbb{E}(\nu_\lambda^\omega(\Lambda)) = \lambda|\Lambda|$, the parameter λ coincides with the *density* of impurities on $\mathbb{R}$.

Luttinger and Sy defined their model by restriction of the single-impurity potential to the case of point δ-potential with amplitude $a \to +\infty$. Then the corresponding random potential (6.3) takes the form:

$$v_a^\omega(x) := \int_{\mathbb{R}} \nu_\lambda^\omega(dy) a\delta(x - y) = a \sum_{x_j^\omega \in X^\omega} \delta(x - x_j^\omega). \tag{6.4}$$

Now the self-adjoint one-particle random Schrödinger operator $h_a^\omega := h^0 \dotplus v_a^\omega$ is defined in the sense of the sum of quadratic forms (2.2). The strong resolvent limit $h_{LS}^\omega := s.r.\lim_{a\to+\infty} h_a^\omega$ is the Luttinger-Sy model.

Since X^ω generates a set of intervals $\{I_j^\omega := (x_{j-1}^\omega, x_j^\omega)\}_j$ of lengths $\{L_j^\omega := x_j^\omega - x_{j-1}^\omega\}_j$, one gets decompositions of the one-particle Luttinger-Sy Hamiltonian:

$$h_{LS}^\omega = \bigoplus_j h_D(I_j^\omega), \quad \mathrm{dom}(h_{LS}^\omega) \subset \bigoplus_j L^2(I_j^\omega), \quad \omega \in \Omega, \tag{6.5}$$

into random disjoint free Schrödinger operators $\{h_D(I_j^\omega)\}_{j,\omega}$ with *Dirichlet* boundary conditions at the end-points of intervals $\{I_j^\omega\}_j$. Then the Dirichlet restriction $h_{l,D}^\omega$ of the Hamiltonian h_{LS}^ω to a fixed interval $\Lambda_l = (-l/2, l/2)$ and the corresponding change of notations are evident: e.g., $\{I_j^\omega\}_j \mapsto \{I_j^\omega\}_{j=1}^{M^l(\omega)}$, where $M^l(\omega)$ is total number of subintervals in Λ_l corresponding to the set X^ω. For rigorous definitions and some results concerning this model we refer the reader to [5].

Since this particular choice of random potential is able to produce Lifhsitz tails in the sense of (3.11), see Proposition 3.2 in [5], it follows that such a model exhibits a generalized BEC in random eigenstates, see (3.4). In fact, it was shown in [5] that *only* the random ground state $\phi_1^{\omega,l}$ of $h_{l,D}^{\omega}$ is *macroscopically* occupied. In our notations this means that

$$\lim_{l\to\infty}\frac{1}{l}\langle N_l(\phi_1^{\omega,l})\rangle_l=\begin{cases}0 & \text{if } \overline{\rho}<\rho_c,\\ \overline{\rho}-\rho_c & \text{if } \overline{\rho}\geq\rho_c,\end{cases}$$
$$\lim_{l\to\infty}\frac{1}{l}\langle N_l(\phi_i^{\omega,l})\rangle_l=0,\quad \text{for all } i>1. \tag{6.6}$$

According to the van den Berg-Lewis-Pulé classification this corresponds to the *type I* Bose-condensation in the random eigenstates $\{\phi_i^{\omega}\}_{i\geq 1}$.

Following the line of reasoning of Sect. 4, we now consider the corresponding BEC in the kinetic-energy eigenstates. We retain the notation used in that section and explain briefly the minor changes required in the application of our method to the Luttinger-Sy model.

We first state the equivalent of Theorem 4.1 for this particular model.

Theorem 6.1 *Theorem 4.1 holds with the function g defined as follows*

$$g(k)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}x\,e^{ikx}\sum_{n\geq 1}e^{n\beta\mu_\infty}\frac{e^{-\|x\|^2(1/2n\beta)}}{(2\pi n\beta)^{d/2}}$$
$$\times\int_{\Omega_{(0,x)}^{n\beta}}w^{n\beta}(\mathrm{d}\xi)\exp\Big(-\lambda\Big(\sup_s\xi(s)-\inf_s\xi(s)\Big)\Big).$$

The scheme of the proof is the same as above, cf. Sects. 4 and 5. First, we note that Lemmas 4.1 and 4.2 apply immediately. The positivity of the random potential has to be understood in terms of quadratic forms, see (2.4).

Before continuing, we need to highlight a minor change concerning the *finite-volume* Lifshitz tails arguments. Although the Theorem 5.1 is valid for the Luttinger-Sy model, its proof (see Sect. 5) requires a minor modification, as the assumption of Lemma 5.1 is clearly not satisfied for the case of singular potentials. However, by direct calculation we can obtain the same estimate with the constant $B=\pi^2/4$ in (5.1). First, suppose that there is at least one impurity in the box, then the eigenvalues will be of the form (for some j)

$$(n^2\pi^2)/(L_j^{\omega})^2,\quad n=1,2,\ldots$$

if I_j^{ω} is an inner interval (that is, its two endpoints correspond to impurities), and

$$((n+1/2)^2\pi^2)/(L_j^{\omega})^2,\quad n=0,1,2,\ldots.$$

If I_j^{ω} is an outer interval (that is, one endpoint corresponds to an impurity, and the other one to the boundary of Λ_l). Therefore, $E_1^{\omega,l,N}\geq B/l^2$ since obviously $L_j^{\omega}<l$. Now, if there is no impurity in the box Λ_l, then $E_1^{\omega,l,N}=0<B/l^2$. But due to the Poisson distribution (6.2) this happens with probability $e^{-\lambda l}$, proving the same estimate as in Lemma 5.1.

With this last observation, the proof of the Theorem 5.1 in Sect. 5 can be carried out verbatim, without any further changes.

Our next step is to split the measure $\tilde{m}_l$ into two, $\tilde{m}_l^{(1)}$ and $\tilde{m}_l^{(2)}$, see (4.9), and prove the statement equivalent to the Theorem 4.2.

Theorem 6.2 *For any $d \geq 1$, the sequence of Laplace transforms of the measures $\tilde{m}_l^{(1)}$:*

$$f_l(t;\beta,\mu_l) := \int_{\mathbb{R}} \tilde{m}_l^{(1)}(d\varepsilon)\, e^{-t\varepsilon}$$

converges for any $t > 0$ to a (non-random) limit $f(t;\beta,\mu_\infty)$, which is given by:

$$f(t;\beta,\mu_\infty) = \sum_{n\geq 1} e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} dx \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \int_{\Omega_{(0,x)}^{n\beta}} w^{n\beta}(d\xi) \exp\Big(-\lambda\Big(\sup_s \xi(s) - \inf_s \xi(s)\Big)\Big).$$

Proof We follow the proof of Theorem 4.2, using the same notation. The uniform convergence is obtained the same way, since the bounds (4.14), (4.15), and (4.16) are also valid in this case. As in (4.20), we can use the ergodic theorem to obtain:

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty}\mathbb{E}_\omega \int_{\mathbb{R}} dx \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \sum_j \int_{\Omega_{(0,x)}^{n\beta}} w^{n\beta}(d\xi)\chi_{I_j{}^\omega,n\beta}(\xi). \tag{6.7}$$

We have used the fact that the Dirichlet boundary conditions at the impurities split up the space $\mathcal{H}_l$ into a direct sum of Hilbert spaces (see (6.5)). This can be seen from the expression

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \int_{\mathbb{R}} dx \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\mathbb{E}_\omega \int_{\Omega_{(0,x)}^{n\beta}} w^{n\beta}(d\xi) e^{-\int_0^{n\beta} ds a \sum_{x_j^\omega\in X^\omega}\delta(\xi(s)-x_j^\omega)}$$

by formally putting the amplitude, a, of the point impurities (6.4) equal to $+\infty$. Because of the characteristic functions $\chi_{I_j^\omega,n\beta}$, which constrain the paths ξ to remain in the interval I_j^ω in time $n\beta$, the sum in (6.7) reduces to only *one* term:

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \int_{\mathbb{R}} dx \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\, \mathbb{E}_\omega \int_{\Omega_{(0,x)}^{n\beta}} w^{n\beta}(d\xi)\chi_{(a_\omega,b_\omega),n\beta}(\xi), \tag{6.8}$$

where (a_ω, b_ω), is the interval among the I_j^ω's which contains 0.

The expression in (6.8) can be simplified further by computing the expectation $\mathbb{E}_\omega$ explicitly.

First, note that the Poisson impurity positions: a_ω, b_ω are independent random variables and by definition, a_ω is negative while b_ω is positive. For the random variable b_ω the distribution function is:

$$\mathbb{P}\,(b_\omega < b) := \mathbb{P}\{(0,b) \text{ contains at least one impurity}\} = 1 - e^{-\lambda b},$$

and therefore its probability density is $\lambda e^{-\lambda b}$ on $(0,\infty)$. Similarly for a_ω one gets:

$$\mathbb{P}\,(a_\omega < a) := \mathbb{P}\{(a,0) \text{ contains no impurities}\} = e^{-\lambda|a|} = e^{\lambda a},$$

and thus its density is $\lambda e^{\lambda a}$ on $(-\infty, 0)$. Using these distributions in (6.8) we obtain:

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty}\lambda^2 \int_{-\infty}^0 da e^{\lambda a} \int_0^\infty db e^{-\lambda b} \int_{\mathbb{R}} dx \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$

$$\times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\chi_{(a,b)}(\xi)$$

$$= e^{n\beta\mu_\infty}\lambda^2 \int_{-\infty}^{0} \mathrm{d}a e^{\lambda a} \int_0^\infty \mathrm{d}b e^{-\lambda b} \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$

$$\times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\mathbf{1}\Big(\sup_s(\xi(s)) \le b\Big)\mathbf{1}\Big(\inf_s(\xi(s)) \ge a\Big)$$

$$= e^{n\beta\mu_\infty}\lambda^2 \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$

$$\times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) \int_{-\infty}^{\inf_s(\xi(s))} \mathrm{d}a e^{\lambda a} \int_{\sup_s(\xi(s))}^{\infty} \mathrm{d}b e^{-\lambda b},$$

and the Theorem 6.2 follows by explicit computation of the last two integrals. □

Proof of Theorem 6.1 Having proved Theorem 6.2, it is now straightforward to derive the analogue of Corollary 4.1 for the Luttinger-Sy model. Note also that the Corollary 4.2 remains unchanged, since only the uniform convergence was used. With these results, the proof of Theorem 6.1 follows in the same way as for Theorem 4.1. □

We have proved, in Theorem 6.1, that the Luttinger-Sy model exhibits g-BEC in the kinetic energy states. But, in this particular case, we can go further and determine the particular *type* of g-BEC in the kinetic energy states. Recall that the g-BEC in the *random* eigenstates is only in the *ground* state, that is, of the *type* I, see (6.6) and [5] for a comprehensive review. Here we shall show that the g-BEC in the kinetic-energy eigenstates is in fact of the *type* III, namely:

Theorem 6.3 *In the Luttinger-Sy model none of the kinetic-energy eigenstates is macroscopically occupied*:

$$\lim_{l\to\infty} \frac{1}{l}\langle N_l(\psi_k)\rangle_l = 0 \quad \textit{for all } k \in \Lambda_l^*,$$

even though for $\overline{\rho} > \rho_c$ *there is a generalized BEC.*

To prove this theorem we shall exploit the finite-volume localization properties of the random eigenfunctions $\phi_i^{\omega,l}$ of the Hamiltonian $h_{l,D}^{\omega}$. Since the impurities split up the box Λ_l into a finite number $M^l(\omega)$ of sub-intervals $\{I_j^\omega\}_{j=1}^{M^l(\omega)}$, by virtue of the corresponding orthogonal decomposition of $h_{l,D}^{\omega}$, cf. (6.5), the normalized random eigenfunctions $\phi_s^{\omega,l}$ are in fact *sine-waves* with supports in each of these sub-intervals and thus satisfy:

$$|\phi_s^{\omega,l}(x)| < \sqrt{\frac{2}{L_{j_s}^{\omega}}}\mathbf{1}_{I_{j_s}^{\omega}}(x), \quad 1 \le j_s \le M^l(\omega). \tag{6.9}$$

We require an estimate of the size L_j^ω of these random sub-intervals, which we obtain in the following lemma.

Lemma 6.1 *Let $\lambda > 0$ be a mean concentration of the point Poisson impurities on $\mathbb{R}$. Then eigenfunctions ϕ_j^ω are localized in sub-intervals of logarithmic size, in the sense that for any $\kappa > 4$, one has a.s. the estimate*:

$$\limsup_{l\to\infty} \frac{\max_{1\le j\le M^l(\omega)} L_j^\omega}{\ln l} \le \frac{\kappa}{\lambda}.$$

Proof Define the set

$$S_l := \left\{\omega : \max_{1\le j\le M^l(\omega)} L_j^\omega > \frac{\kappa}{\lambda}\ln l\right\}.$$

Let $n := [2\lambda l/(\kappa \ln l)]_+$, and define a new box:

$$\widetilde{\Lambda}_l := \left[-\frac{n}{2}\left(\frac{\kappa}{2\lambda}\ln l\right), \frac{n}{2}\left(\frac{\kappa}{2\lambda}\ln l\right)\right] \supset \Lambda_l.$$

Split this bigger box into n identical disjoints intervals $\{I_m^l\}_{m=1}^n$ of size $\kappa(2\lambda)^{-1}\ln l$. If $\omega \in S_l$, then there exists at least one empty interval I_m^l (interval without any impurities), and therefore the set

$$S_l \subset \bigcup_{1\le m\le n} \{\omega : I_m^l \text{ is empty}\}.$$

By the Poisson distribution (6.2), the probability for the interval I_m^l to be empty depends only on its size, and thus

$$\mathbb{P}(S_l) \le n\exp\left(-\lambda\frac{\kappa}{2\lambda}\ln l\right) \le \left[\frac{2\lambda l}{\kappa\ln l}\right]_+ l^{-\kappa/2}.$$

Since we choose $\kappa > 4$, it follows that

$$\sum_{l\ge 1}\mathbb{P}(S_l) < \infty.$$

Therefore, by the Borel-Cantelli lemma, there exists a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for each $\omega \in \widetilde{\Omega}$ one can find $l_0(\omega) < \infty$ with

$$\mathbb{P}\left\{\omega : \max_{1\le j\le M^l(\omega)} L_j^\omega \le \frac{\kappa}{\lambda}\ln l\right\} = 1,$$

for all $l \ge l_0(\omega)$. □

Now we can prove the main statement of this section.

Proof of Theorem 6.3 The atom of the measure $\tilde{m}$ has already been established in Theorem 6.1. Concerning the macroscopic occupation of a single state, we have

$$\frac{1}{l}\langle N_l(\psi_k)\rangle_l = \frac{1}{l}\sum_i |(\phi_i^{\omega,l}, \psi_k)|^2 \langle N_l(\phi_i^{\omega,l})\rangle_l$$

$$= \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l \left|\int_{\Lambda_l} \mathrm{d}x \overline{\psi}_k(x)\phi_i^{\omega,l}(x)\right|^2$$

$$\leq \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l \frac{1}{l}\left(\int_{\Lambda_l} \mathrm{d}x |\phi_i^{\omega,l}(x)|\right)^2,$$

where in the last step we have used the bound $|\psi_k| \leq 1/\sqrt{l}$. Therefore, by (6.9) and Lemma 6.1, we obtain a.s. the following estimate:

$$\frac{1}{l}\langle N_l(\psi_k)\rangle_l \leq \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l \frac{1}{l}\frac{\kappa}{\lambda}\ln l,$$

which is valid for large enough l and for any $\kappa > 4$. The theorem then follows by taking the thermodynamic limit. □

7 Application to Weak (Scaled) Non-random Potentials

It is known for a long time, see e.g. [24, 25], that BEC can be enhanced in low-dimensional systems by imposing a weak (scaled) external potential. Recently this was a subject of a new approach based on the Random Boson Point Field method [26]. In this section, we show that, with some minor modifications our method can be extended to cover also the case of these scaled *non-random* potentials.

Let v be a non-negative, continuous real-valued function defined on the closed unit cube $\overline{\Lambda}_1 \subset \mathbb{R}^d$. The *one-particle* Schrödinger operator with a *weak* (*scaled*) external potential in a box Λ_l is defined by:

$$h_l = -\frac{1}{2}\Delta_D + v(x_1/l, \ldots, x_d/l). \tag{7.1}$$

Let $\{\varphi_i^l, E_i^l\}_{i\geq 1}$ be the set of orthonormal eigenvectors and corresponding eigenvalues of the operator (7.1). As usual we put $E_1 \leq E_2 \leq \cdots$ by convention. The many-body Hamiltonian for the perfect Bose gas is defined in the same way as in Sect. 2. We keep the notations m and $\tilde{m}$ for the occupation measures of the eigenstates $\{\varphi_i^l\}_{i\geq 1}$ and of the kinetic-energy states respectively. We denote the *integrated density of states* (IDS) of the Schrödinger operator (7.1) by ν_l, and by $\nu = \lim_{l\to\infty}\nu_l$ its weak limit. We assume that the first eigenvalue $E_1^l \to 0$ as $l \to \infty$, which is the case, when e.g. $v(0) = 0$. This assumption is equivalent to condition (iii), Sect. 2. It ensures that for a given mean particle density $\overline{\rho}$ the chemical potential $\mu_\infty(\beta, \overline{\rho})$ satisfies the relation (3.6), where $\overline{\mu} := \overline{\mu}(\beta, \overline{\rho})$ is a (unique) solution of the equation [24]:

$$\overline{\rho} = \sum_{n\geq 1}\frac{1}{(2\pi n\beta)^{d/2}}\int_{\Lambda_1}\mathrm{d}x e^{n\beta(\mu - v(x))} = \int_{[0,\infty)}\nu^0(\mathrm{d}E)\int_{\Lambda_1}\mathrm{d}x \left(e^{\beta(E+v(x)-\mu)} - 1\right)^{-1}, \tag{7.2}$$

for $\overline{\rho} \leq \rho_c$, where the boson critical density is given by:

$$\rho_c = \sum_{n\geq 1}\frac{1}{(2\pi n\beta)^{d/2}}\int_{\Lambda_1}\mathrm{d}x e^{-n\beta v(x)} = \int_{[0,\infty)}\nu^0(\mathrm{d}E)\int_{\Lambda_1}\mathrm{d}x \left(e^{\beta(E+v(x))} - 1\right)^{-1}. \tag{7.3}$$

Here ν^0 is the IDS (3.9) of the kinetic-energy operator (2.1). In particular the value $\rho_c = \infty$ is allowed in (7.3). If $\rho_c < \infty$, the existence of a generalized BEC in the states $\{\varphi_i^l\}_{i\geq 1}$ follows by the same arguments as in Sect. 3. For example, the choice: $v(x) = |x|$, makes the critical density finite even in dimension one, see e.g. [24].

Now, we prove the statements equivalent to the Theorem 4.1:

Theorem 7.1 *The sequence $\{\tilde{m}_l\}_{l\geq 1}$ of the one-particle kinetic states occupation measures has a weak limit $\tilde{m}$ given by*:

$$\tilde{m}(\mathrm{d}\varepsilon) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(\mathrm{d}\varepsilon) + F(\varepsilon)\nu^0(\mathrm{d}\varepsilon), & \text{if } \overline{\rho} \geq \rho_c, \\ F(\varepsilon)\nu^0(\mathrm{d}\varepsilon), & \text{if } \overline{\rho} < \rho_c, \end{cases}$$

where the density $F(\varepsilon)$ is defined by:

$$F(\varepsilon) = \int_{\Lambda_1} \mathrm{d}x \left(e^{\beta(\varepsilon + v(x) - \mu_\infty)} - 1\right)^{-1},$$

and $\mu_\infty := \mu_\infty(\beta, \overline{\rho})$ satisfies the relation (3.6).

We note the similarity of this result with the free Bose gas. Indeed, the kinetic-energy states occupation measure density is reduced to the free gas one, with the energy shifted by the external potential v and then *averaged* over the unit cube.

The proof requires the same tools as in the random case. As before, we split the occupation measure into two parts:

$$\tilde{m}_l = \tilde{m}_l^{(1)} + \tilde{m}_l^{(2)} \quad \text{with}$$
$$\tilde{m}_l^{(1)}(A) := \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} P_A(e^{-n\beta(h_l - \mu_l)})\mathbf{1}(\mu_l \leq 1/n),$$
$$\tilde{m}_l^{(2)}(A) := \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} P_A(e^{-n\beta(h_l - \mu_l)})\mathbf{1}(\mu_l > 1/n),$$

and we prove the following statement:

Theorem 7.2 *The sequence of measures $\tilde{m}_l^{(1)}$ converges weakly to a measure $\tilde{m}^{(1)}$, which is absolutely continuous with respect to ν^0 with density $F(\varepsilon)$ given by*:

$$F(\varepsilon) = \int_{\Lambda_1} \mathrm{d}x \left(e^{\beta(\varepsilon + v(x) - \mu_\infty)} - 1\right)^{-1}.$$

Proof We follow the line of reasoning of the proof of Theorem 4.2. Let $g_l(t; \beta, \mu_l)$ be the Laplace transform of the measure $\tilde{m}_l^{(1)}$:

$$\begin{aligned} g_l(t; \beta, \mu_l) &= \int_{\mathbb{R}} m_l^{(1)}(\mathrm{d}\varepsilon) e^{-t\varepsilon} \\ &= \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} \left(e^{-n\beta(h_l - \mu_l)}\right)\mathbf{1}(\mu_l \leq 1/n). \end{aligned} \tag{7.4}$$

Again, our aim is to show the uniform convergence of the sum over n with respect to l. Let

$$\begin{aligned} a_l(n) &:= \frac{1}{V_l}\,\mathrm{Tr}\, e^{-th_l^0} e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \le 1/n) \\ &\le \frac{1}{V_l}\,\mathrm{Tr}\, e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \le 1/n). \end{aligned} \tag{7.5}$$

Then for $\overline{\rho} < \rho_c$ we can apply a similar argument as for the random case, since the estimate $\mu_l < \mu_\infty/2 < 0$ still holds, to obtain:

$$a_l(n) \le e^{n\beta\mu_\infty/2}\int_{[0,\infty)} e^{-\beta\varepsilon}\nu_l(\mathrm{d}\varepsilon) \le K_1 e^{n\beta\mu_\infty/2}.$$

If $\overline{\rho} \ge \rho_c$, then $\mu_l \le 1/n$ in (7.5) implies that:

$$a_l(n) \le e^{\beta}\sum_i e^{-n\beta E_i^l} \le \frac{e^{\beta}}{(2\pi n\beta)^{d/2}}\int_{\Lambda_1} \mathrm{d}x\, e^{-n\beta v(x)},$$

where the last estimate can be found in [24] or [25]. Now the uniform convergence for the sequence $a_l(n)$ follows from (7.3), since we assumed that $\rho_c < \infty$. The latter implies also that for $\overline{\rho} \ge \rho_c$, $\mu_\infty(\beta,\overline{\rho}) = 0$. Thus, we can take the limit of the Laplace transform (7.4) term by term, that is:

$$\begin{aligned} \lim_{l\to\infty} a_l(n) &= \lim_{l\to\infty}\frac{1}{V_l}\,\mathrm{Tr}\, e^{-th_l^0} e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \le 1/n) \\ &= \lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\, e^{-th_l^0}(x,x')e^{-n\beta(h_l-\mu_l)}(x',x) \\ &= e^{n\beta\mu_\infty}\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\,\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\ &\quad\times\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi')\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\, v(\xi(s)/l)}\chi_{\Lambda_l,n\beta}(\xi). \end{aligned} \tag{7.6}$$

Here we have used the Feynman-Kac representation for free $e^{-th_l^0}(x,y)$ and for non-free $e^{-\beta h_l}(x,y)$ Gibbs semi-group kernels, where w^T stands for the *normalized* Wiener measure on the path-space $\Omega^T_{(x,y)}$, see Sect. 4.1.

Note that by Lemma A.2, which demands only the *non-negativity* of the potential v, we obtain for (7.6) the representation:

$$\begin{aligned} &\lim_{l\to\infty}\frac{1}{V_l}\,\mathrm{Tr}\, e^{-th_l^0} e^{-n\beta(h_l-\mu_l)} \\ &\quad= e^{n\beta\mu_\infty}\lim_{l\to\infty}\int_{\mathbb{R}^d}\mathrm{d}x\,\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x'\,\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\ &\quad\quad\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\, v(\xi(s)/l)}. \end{aligned} \tag{7.7}$$

Now we express the trajectories ξ in terms of *Brownian bridges* $\alpha(\tau) \in \tilde{\Omega}, 0 \leq \tau \leq 1$, we denote the corresponding measure by D. Letting $\tilde{x} = x'/l$, we obtain:

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l-\mu_l)} \\
&\quad = e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\Lambda_1} \mathrm{d}\tilde{x} \frac{e^{-\|x-l\tilde{x}\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\
&\qquad \times \int_{\tilde{\Omega}} D(\mathrm{d}\alpha) \exp\left(-\int_0^{n\beta} \mathrm{d}s\, v\left[\left(1-\frac{s}{n\beta}\right)\tilde{x} + \frac{s}{n\beta}(x/l) + \frac{\sqrt{n\beta}}{l}\alpha(s/n\beta)\right]\right).
\end{aligned}$$

Since the integration with respect to x is now over the whole space, we let $y = x - l\tilde{x}$ to get

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l-\mu_l)} \\
&\quad = e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\Lambda_1} \mathrm{d}\tilde{x} \frac{e^{-\|y\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\
&\qquad \times \int_{\tilde{\Omega}} D(\mathrm{d}\alpha) \exp\left(-\int_0^{n\beta} \mathrm{d}s\, v\left(\tilde{x} + \frac{s}{n\beta}(y/l) + \frac{\sqrt{n\beta}}{l}\alpha(s/n\beta)\right)\right) \\
&\quad = e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}y \frac{e^{-\|y\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Lambda_1} \mathrm{d}\tilde{x} e^{-n\beta v(\tilde{x})},
\end{aligned}$$

where the last step follows from dominated convergence. Therefore, we obtain by (7.4) the following expression for the limiting Laplace transform:

$$\lim_{l\to\infty} g_l(t;\beta,\mu_l) = \sum_{n\geq 1} e^{-n\beta(E-\mu_\infty)} \frac{1}{(2\pi(n\beta+t))^{d/2}} \int_{\Lambda_1} \mathrm{d}x e^{-n\beta v(x)}.$$

It is now straightforward to invert this Laplace transform (for each term of the sum), to find that:

$$F(E)\nu^0(\mathrm{d}E) = \lim_{l\to\infty} \tilde{m}_l^1(\mathrm{d}E) = \sum_{n\geq 1} e^{-n\beta(E-\mu_\infty)} \left(\int_{\Lambda_1} \mathrm{d}x e^{-n\beta v(x)}\right) \nu^0(\mathrm{d}E).$$

The theorem then follows by Fubini's theorem. □

Proof of Theorem 7.1 The proof of Theorem 4.1 can be applied directly. Note that Lemmas 4.1, 4.2 are still valid, since (as we emphasized in Remarks 4.1, 4.2), their proofs require only the non-negativity of the external potential. Similarly, Corollary 4.2 now can be used directly, since we have proved Theorem 7.2. □

Appendix A: Brownian Paths

In this section, we first give an upper estimate of the probability of a Brownian path to leave some spatial domain, cf. e.g. [27] and the references quoted therein.

Lemma A.1 *Let the set*

$$\Omega^T_{(x,x')} := \{\xi(\tau) : \xi(0) = x, \xi(T) = x'\}$$

be continuous trajectories from x to x' with the proper time $0 \le \tau \le T$, and with the normalized Wiener measure w^T on it. Let x, x' be in Λ_l, and $\chi_{\Lambda_l,T}(\xi)$ the characteristic function over $\Omega^T_{(x,x')}$ of trajectories ξ staying in Λ_l for all $0 \le \tau \le T$. Then one gets the estimate:

$$\int_{\Omega^T_{(x,x')}} w^T(\mathrm{d}\xi)\big(1 - \chi_{\Lambda_l,T}(\xi)\big) \le e^{-C(T)(\min\{d(x,\partial\Lambda_l),d(x',\partial\Lambda_l)\})^2}. \tag{A.1}$$

Proof Define a *Brownian bridge* $\alpha(s)$, $0 \le s \le 1$ by:

$$\xi(t) = (1 - \tau/T)x + \tau/T\ x' + \sqrt{T}\alpha(\tau/T).$$

Let us consider first the one dimensional case, i.e. $\Lambda_l = [-l/2, l/2]$. Without loss of generality, we can assume that:

$$d(x, \partial\Lambda_l) \le d(x', \partial\Lambda_l).$$

Suppose that $x > 0$, then we have:

$$-x \le x' \le x \quad \text{and} \quad d(x, \partial\Lambda_l) = l/2 - x.$$

Assume that the path ξ leaves the box on the right-hand side. Then, for some t, we have:

$$\begin{aligned}
\xi(t) &> \frac{l}{2}, \\
\alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} + (t/T - 1)x - \frac{t}{T}x'\right), \\
\alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} + (t/T - 1)x - \frac{t}{T}x\right) = \frac{1}{\sqrt{T}}d(x, \partial\Lambda_l).
\end{aligned} \tag{A.2}$$

The case, when ξ leaves the box on the left-hand side can be treated similarly.

Let $x < 0$, then we have:

$$x \le x' \le -x \quad \text{and} \quad d(x, \partial\Lambda_l) = l/2 + x.$$

Again, assume that the path leaves the box on the right hand-side. Then, for some t, we have:

$$\begin{aligned}
\xi(t) &> \frac{l}{2}, \\
\alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} + (t/T - 1)x - \frac{t}{T}x'\right), \\
\alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} - (t/T - 1)x' - \frac{t}{T}x'\right) \ge \frac{1}{\sqrt{T}}d(x, \partial\Lambda_l).
\end{aligned} \tag{A.3}$$

The case, when ξ leaves the box on the left hand-side can be considered similarly. The relations (A.2), (A.3) imply that if ξ leaves the box Λ_l in one dimension, then the Brownian bridge α must satisfy the inequality:

$$\sup_t |\alpha(t/T)| > C(T)\min\{d(x, \partial\Lambda_l), d(x', \partial\Lambda_l)\}, \tag{A.4}$$

for some constant $C(T)$.

This observation can easily be extended to higher dimensions, when $x := (x_1, \ldots, x_d)$ and $\alpha(s) := (\alpha_1(s), \ldots, \alpha_d(s))$. Now, if ξ leaves the (d-dimensional) box Λ_l, there exists at least one i such that similar to (A.4):

$$\sup_t |\alpha_i(t/T)| > C(T)\min\{d(x_i, \partial_i\Lambda_l), d(x_i', \partial_i\Lambda_l)\},$$

where we denote $d(x_i, \partial_i\Lambda_l) := \min\{l/2 - x_i, l/2 + x_i\}$. Now, since Λ_l are cubes, we get $d(x_i, \partial_i\Lambda_l) \geq d(x, \partial\Lambda_l)$ for any $x \in \Lambda_l$. Then we obtain:

$$\begin{aligned} &\|\alpha(t/T)\| > |\alpha_i(t/T)|, \quad i = 1, \ldots, d, \\ &\sup_t \|\alpha(t/T)\| > \max_i \sup_t |\alpha_i(t/T)|, \\ &\sup_t \|\alpha(t/T)\| > C(T)\min\{d(x_i, \partial_i\Lambda_l), d(x_i', \partial_i\Lambda_l)\} \\ &\qquad\qquad \geq C(T)\min\{d(x, \partial\Lambda_l), d(x', \partial\Lambda_l)\}. \end{aligned} \tag{A.5}$$

Therefore, the probability for the path ξ to leave the box is dominated by the probability for the one-dimensional Brownian bridge α to satisfy (A.5). The latter we can estimate using the following result from [27]:

$$\mathbb{P}\Big(\sup_s \alpha(s) > x\Big) \geq Ae^{-Cx^2}$$

valid for some positive constants A, C, which implies the bound (A.1). □

Now we establish a result, that we use in the proof of Theorem 4.2:

Lemma A.2 *Let* $K^t_{\omega,l}(x, x')$, $K^t_{0,l}(x, x')$, $K^t_0(x, x')$ *be the kernels of operators* $\exp(-th^\omega_l)$, $\exp(-th^0_l)$, *and* $\exp(-t\Delta/2)$ *respectively. Then*

$$\begin{aligned} &\lim_{l\to\infty} \frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\mathrm{d}x' K^t_{0,l}(x, x') K^{n\beta}_{\omega,l}(x', x) \\ &\quad = \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x \frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x' K_0^{t+n\beta}(x, x') \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta}\mathrm{d}s v^\omega(\xi(s))}. \end{aligned} \tag{A.6}$$

Proof By the Feynman-Kac representation, we obtain:

$$\begin{aligned} &\lim_{l\to\infty} \frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\, \mathrm{d}x' K^t_{0,l}(x, x') K^{n\beta}_{\omega,l}(x', x) \\ &\quad = \lim_{l\to\infty} \frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\, \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta + 1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\ &\qquad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta}\mathrm{d}s v^\omega(\xi(s))} \chi_{\Lambda_l, n\beta}(\xi) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi') \chi_{\Lambda_l,t}(\xi'). \end{aligned}$$

To eliminate the characteristic functions restricting the paths ξ, ξ' in the last integral, we shall use Lemma A.1. First, we estimate the error $\gamma(d)$ when we remove the restriction on the path ξ:

$$\begin{aligned}
\gamma(d) &:= \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\
&\quad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s\, v^{\omega}(\xi(s))} \big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi') \\
&\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \\
&\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \mathbb{I}\big\{d(x,\partial\Lambda_l) > d(x',\partial\Lambda_l)\big\} \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\
&\quad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \\
&\quad + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \mathbb{I}\big\{d(x,\partial\Lambda_l \le d(x',\partial\Lambda_l)\big\} \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\
&\quad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\big(1-\chi_{\Lambda_l,n\beta}(\xi)\big) \\
&\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x',\partial\Lambda_l))^2} \\
&\quad + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x,\partial\Lambda_l))^2},
\end{aligned} \tag{A.7}$$

where the last step is due to Lemma A.1. Since all integrands are positive, we can extend one of the spatial integrations to the whole space, and hence we get:

$$\begin{aligned}
\gamma(d) &\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x',\partial\Lambda_l))^2} \\
&\quad + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x'\partial\Lambda_l))^2} \\
&\le \lim_{l\to\infty} \frac{1}{V_l} K_0^{t+n\beta} \int_{\Lambda_l} \mathrm{d}x' e^{-C(n\beta)(d(x',\partial\Lambda_l))^2} + \lim_{l\to\infty} \frac{1}{V_l} K_0^{t+n\beta} \int_{\Lambda_l} \mathrm{d}x e^{-C(n\beta)(d(x'\partial\Lambda_l))^2}
\end{aligned}$$

where we have used the notation $K_0^{t+n\beta} := K_0^{t+n\beta}(x,x)$ since these are independent of x. Finally, using the fact that the boxes Λ_l are cubes of side l, we obtain:

$$\gamma(d) \le \lim_{l\to\infty} \frac{K_0^{t+n\beta}}{l} \int_{-l/2}^{l/2} \mathrm{d}x' e^{-C(n\beta)(l/2-x')^2} + \lim_{l\to\infty} \frac{K_0^{t+n\beta}}{l} \int_{-l/2}^{l/2} \mathrm{d}x e^{-C(n\beta)(l/2-x)^2} = 0.$$

We can estimate the error estimate due to the removal of the characteristic function for ξ' in (4.18) in the same way. Therefore, we get:

$$\begin{aligned}
&\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\\
&\quad\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^\omega(\xi(s))}\chi_{\Lambda_l,n\beta}(\xi)\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi')\\
&=\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\\
&\quad\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^\omega(\xi(s))}\int_{\Omega^p_{(x,x')}} w^{n\beta}(\mathrm{d}\xi'). \qquad (A.8)
\end{aligned}$$

Now we show that one can replace the first integration over the box Λ_l by one over the whole space. Let $\tilde{\gamma}(d)$ be the error caused by this substitution. Then by the positivity of the random potential we get the estimate:

$$\begin{aligned}
\tilde{\gamma}(d) &:= \lim_{l\to\infty}\frac{1}{V_l}\int_{\mathbb{R}^d\setminus\Lambda_l}\mathrm{d}x\int_{\Lambda_l}\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\\
&\quad\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^\omega(\xi(s)+x')}\int_{\Omega^t_{(x,x')}} w^{n\beta}(\mathrm{d}\xi')\\
&\le \lim_{l\to\infty}\frac{1}{V_l}\int_{\mathbb{R}^d\setminus\Lambda_l}\mathrm{d}x\int_{\Lambda_l}\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}. \qquad (A.9)
\end{aligned}$$

In the one-dimensional case the estimate of the error term (A.9) takes the form:

$$\begin{aligned}
\tilde{\gamma}(1) &\le \lim_{l\to\infty}\frac{1}{l}\int_{-\infty}^{-l/2}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\\
&\quad+\lim_{l\to\infty}\frac{1}{l}\int_{l/2}^{\infty}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}. \qquad (A.10)
\end{aligned}$$

For the first term one gets:

$$\begin{aligned}
&\lim_{l\to\infty}\frac{1}{l}\int_{-\infty}^{-l/2}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\\
&=\lim_{l\to\infty}\frac{1}{l}\int_0^l\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\int_{-l/2-y}^{l/2}\mathrm{d}x\\
&\quad+\lim_{l\to\infty}\frac{1}{l}\int_l^{\infty}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\int_{-l/2-y}^{l/2-y}\mathrm{d}x=0.
\end{aligned}$$

One obtains a similar identity for the second-term in (A.10). Direct calculation shows that, the error term for higher dimensions (A.9) reduces to a product of one-dimensional terms (A.10). Then (A.8) gives:

$$\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$
$$\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^\omega(\xi(s))}\chi_{\Lambda_l,n\beta}(\xi)\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi')$$
$$=\lim_{l\to\infty}\int_{\mathbb{R}^d}\mathrm{d}x\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^\omega(\xi(s))},\qquad\text{(A.11)}$$

which finishes the proof of (A.6). □

Appendix B: Some Probabilistic Estimates

First we recall the assumptions on the random potential v^ω used in [14], and which we also adopt in this paper:

1. (a) On the probability space $(\Omega,\mathcal{F},\mathbb{P})$ there exist a group of measure-preserving metrically transitive transformations $\{T_p\}_{p\in\mathbb{R}^d}$ of Ω, such that $v^\omega(x+p)=v^{T_p\omega}(x)$ for all $x,p\in\mathbb{R}^d$;
   (b) $\mathbb{E}_\omega\{\int_{\Lambda_1}\mathrm{d}x|v^\omega(x)|^\kappa\}<\infty$, where $\kappa>\max(2,d/2)$.
2. For any $\Lambda\subset\mathbb{R}^d$, let Σ_Λ be the σ-algebra generated by the random field $v^\omega(x)$, $x\in\Lambda$. For any two arbitrary random variables on Ω, f,g satisfying (i) $|g|_\infty<\infty$, $\mathbb{E}_\omega\{|f|\}<\infty$ and (ii) the function g is Σ_{Λ_1}-measurable, the function f is Σ_{Λ_2}-measurable, where Λ_1,Λ_2 are disjoint bounded subsets of $\mathbb{R}^d$, the following holds
$$|\mathbb{E}\{|f.g|\}-\mathbb{E}\{|f|\}\mathbb{E}\{|g|\}|\le|g|_\infty\mathbb{E}\{|f|\}\phi(d(\Lambda_1,\Lambda_2))$$
   with $\phi(x)\to 0$ as $x\to\infty$, and $d(\Lambda_1,\Lambda_2)$ the Euclidean distance between Λ_1 and Λ_2.

After recalling these conditions, we can give a sketch of the proof of Lemma 5.1.

Let $h_l^{\omega,N}$ to be the Schrödinger operator (2.3), with Neumann boundary conditions instead of Dirichlet, and denote by $\{E_i^{\omega,l,N},\phi_i^{\omega,l,N}\}_{i\ge1}$ its ordered eigenvalues (including degeneracy) and the corresponding eigenvectors. Similarly we define the kinetic energy operator $h_l^{0,N}$ with the same boundary condition, and denote by $\{\varepsilon_k^{l,N},\psi_k^{l,N}\}_{k\ge1}$ its ordered eigenvalues (including degeneracy) and corresponding eigenvectors. The following result is due to Thirring, see [28]:

Lemma B.1 *Let $v^\omega_{\lambda,\alpha}:=v^\omega+\lambda\alpha$, for $\lambda,\alpha>0$. Then,*

$$E_1^{\omega,l,N}\ge-\lambda\alpha+\min\left\{\varepsilon_2^{l,N},\left[\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x(v^\omega_{\lambda,\alpha}(x))^{-1}\right]^{-1}\right\}.$$

Proof Let P to be an orthogonal projection in $\mathscr{H}_l$. Then for any vector ϕ from the intersection $Q(v^\omega_{\lambda,\alpha})\cap Q((v^\omega_{\lambda,\alpha})^{1/2}P(v^\omega_{\lambda,\alpha})^{1/2})$, we have:

$$\begin{aligned}(\phi,v^\omega_{\lambda,\alpha}\phi)&=((v^\omega_{\lambda,\alpha})^{1/2}\phi,(v^\omega_{\lambda,\alpha})^{1/2}\phi)\\&=((v^\omega_{\lambda,\alpha})^{1/2}\phi,P(v^\omega_{\lambda,\alpha})^{1/2}\phi)+((v^\omega_{\lambda,\alpha})^{1/2}\phi,(1-P)(v^\omega_{\lambda,\alpha})^{1/2}\phi)\\&\ge((v^\omega_{\lambda,\alpha})^{1/2}\phi,P(v^\omega_{\lambda,\alpha})^{1/2}\phi),\end{aligned}$$

and therefore,

$$-\frac{1}{2}\Delta_N + v^{\omega}_{\lambda,\alpha} \geq -\frac{1}{2}\Delta_N + (v^{\omega}_{\lambda,\alpha})^{1/2} P (v^{\omega}_{\lambda,\alpha})^{1/2}, \tag{B.1}$$

in the quadratic-form sense. Let us choose:

$$P := (v^{\omega}_{\lambda,\alpha})^{-1/2} \tilde{P} \big((\psi_1^{l,N}, (v^{\omega}_{\lambda,\alpha})^{-1}\psi_1^{l,N})\big)^{-1} \tilde{P} (v^{\omega}_{\lambda,\alpha})^{-1/2},$$

where $\tilde{P}$ is the orthogonal projection onto the subspace spanned by the vector $\psi_1^{l,N}$. It can be easily checked that P is an orthogonal projection. Applying (B.1) to the function $\phi_1^{\omega,l,N}$ one gets:

$$\begin{aligned} E_1^{\omega,l,N} + \lambda\alpha &\geq \left(\phi_1^{\omega,l,N}, \left(-\frac{1}{2}\Delta_N\right)\phi_1^{\omega,l,N}\right) + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \big(\psi_1^{l,N}, (v^{\omega}_{\lambda,\alpha})^{-1}\psi_1^{l,N}\big)^{-1} \\ &\geq \sum_{k\geq 1} |(\phi_1^{\omega,l,N}, \psi_k^{l,N})|^2 \varepsilon_k^{l,N} + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \left[\frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1}\right]^{-1}. \end{aligned}$$

But since the Neumann boundary conditions imply that $\varepsilon_1^{l,N} = 0$, we obtain

$$E_1^{\omega,l,N} + \lambda\alpha \geq (1 - |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2)\varepsilon_2^{l,N} + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \left[\frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1}\right]^{-1}.$$

To finish the proof, we have to study separately the two cases, namely, $\varepsilon_2^{l,N}$ less than and greater than $[\frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1}]^{-1}$. □

Proof of Lemma 5.1 By Lemma B.1,with $\lambda = B/l^2$ and α as defined in assumptions, i.e. for $B = \pi/(1+\alpha)$, $\alpha > p/(1-p)$, we have:

$$E_1^{\omega,l,N} \geq -\frac{\alpha B}{l^2} + \min(\pi/l^2, 1/X_l),$$

where

$$X_l^{\omega} := \frac{1}{V_l}\int_{\Lambda_l} \mathrm{d}x \frac{1}{v^{\omega}(x) + B\alpha/l^2}.$$

Therefore,

$$E_1^{\omega,l,N} - \frac{B}{l^2} \geq -\frac{\pi}{l^2} + \min(\pi/l^2, 1/X_l^{\omega}).$$

Hence, the inequality $E_1^{\omega,l,N} < B/l^2$ implies that $X_l^{\omega} > l^2/\pi$ and consequently:

$$\mathbb{P}(E_1^{\omega,l,N} < B/l^2) \leq \mathbb{P}(X_l^{\omega} > l^2/\pi). \tag{B.2}$$

Define a random variable $Y_l^{\omega}(\delta) := V_l^{-1}\int_{\Lambda_l} \mathrm{d}x\ \delta/(v^{\omega}(x) + \delta)$, which is an increasing function of δ. Then for the left-hand side of (B.2) one gets the estimate:

$$\mathbb{P}(E_1^{\omega,l,N} < B/l^2) \leq \mathbb{P}\left(Y_l^{\omega}(B\alpha/l^2) > \frac{\alpha}{1+\alpha}\right).$$

By Lemma 2 in [14], we know that for any positive δ the random variables $\{Y_l^\omega(\delta)\}_l$, converges *geometrically* to a limit $Y_\infty(\delta)$ as $l \to \infty$, that is, for any $\epsilon > 0$, there exists a constant $M(\delta, \epsilon)$ such that

$$\mathbb{P}(|Y_l^\omega(\delta) - Y_\infty(\delta)| > \epsilon/2) \le e^{-M(\delta,\epsilon)V_l}$$

for l sufficiently large. By the ergodic theorem $Y_\infty(\delta)$ is non-random and can be expressed as:

$$Y_\infty(\delta) = \mathbb{E}_\omega\left(\frac{\delta}{v^\omega(0) + \delta}\right),$$

which is again a monotonic function of $\delta \ge 0$. Notice that by condition (ii), Sect. 2, we have $\lim_{\delta\to 0} Y_\infty(\delta) = p$.

Choose $\epsilon > 0$ such that $p + \epsilon < \alpha/(1+\alpha)$. Then we have

$$\mathbb{P}\left(E_1^{\omega,l,N} < \frac{B}{l^2}\right) \le \mathbb{P}\big(Y_l^\omega(B\alpha/l^2) > p + \epsilon\big).$$

Now we choose δ such that

$$Y_\infty(\delta) - p < \epsilon/2,$$

and let l_0 be defined by $\delta = B\alpha/l_0^2$. Then for any $l > l_0$ we have:

$$\begin{aligned}\mathbb{P}(E_1^{\omega,l,N} < B/l^2) &\le \mathbb{P}\big(Y_l^\omega(B\alpha/l^2) > p + \epsilon\big) \le \mathbb{P}\big(Y_l^\omega(\delta) - p > \epsilon\big)\\ &\le \mathbb{P}\big(|Y_l^\omega(\delta) - Y_\infty(\delta)| > \epsilon/2\big) \le e^{-M(\delta,\epsilon)V_l}.\end{aligned}$$ □

Acknowledgements One of the authors (Th. Jaeck) is supported by funding from the UCD Ad Astra Research Scholarship. We are grateful to the referees for some useful suggestions and remarks.

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JOURNAL OF MATHEMATICAL PHYSICS **51**, 103302 (2010)

On the nature of Bose–Einstein condensation enhanced by localization

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(Received 14 June 2010; accepted 19 August 2010; published online 13 October 2010)

In a previous paper we established that for the perfect Bose gas and the mean-field Bose gas with an external random or weak potential, whenever there is generalized Bose–Einstein condensation in the eigenstates of the single particle Hamiltonian, there is also generalized condensation in the kinetic-energy states. In these cases Bose–Einstein condensation is produced or enhanced by the external potential. In the present paper we establish a criterion for the absence of condensation in single kinetic-energy states and prove that this criterion is satisfied for a class of random potentials and weak potentials. This means that the condensate is spread over an infinite number of states with low kinetic-energy without any of them being macroscopically occupied. © *2010 American Institute of Physics.*
[doi:10.1063/1.3488965]

I. INTRODUCTION

It can be easily seen from the explicit formula for the occupation numbers in the noninteracting (*perfect*) Bose gas that, the condition for Bose–Einstein condensation (BEC) to occur, is that the density of states of the one particle Schrödinger operator decreases fast enough near the bottom of the spectrum. In the absence of any external potential, it is known that this happens only in three dimensions or higher. This is still true if one introduces a mean-field interaction between particles. It has been known for some time that the behavior of the density of states can be altered by the addition of suitable external potentials, in particular, *weak* potentials or *random* potentials. The subject of this paper is the study of models of the Bose gas in the presence of such external potentials. The first case has been extensively studied, see e.g., Refs. 1 and 2, where sufficient conditions on the external potential were derived for the occurrence of BEC. In the random case, it has been shown in Ref. 3 that the so-called *Lifshitz tails*, which are a general feature of disordered systems, see, for instance, Ref. 4, are able to produce BEC. In both cases, it is possible to obtain condensation even in dimensions 1 or 2.

While BEC has historically been associated with the macroscopic occupation of the ground-state *only*, it was pointed out in Ref. 5 that this phenomena is more thermodynamically stable if it is interpreted as the macroscopic accumulation of particles into an arbitrarily narrow band of energy above the ground-state or *generalized* BEC. While it is clear that condensation in the ground-state implies generalized BEC, there exist many situations in which the converse is not true. For instance, it was shown in Ref. 2 that in the case of the weak potential, the condensate can be in one state, in infinitely many states or even not in any state at all, depending on the external potential. These situations correspond respectively to types I, II, III generalized BEC in the classification established by Van den Berg *et al.*, see, e.g., Ref. 6. In the random case, far less is known. The only case for which a rigorous proof of the exact type of BEC has been established is

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the Luttinger–Sy model, see Ref. 7, where it was shown that the ground-state only is macroscopically occupied. As far as we know, for more complicated systems this is still an open question. The difficulty lies in the fact that the characterization of the distribution of the condensate in individual states requires much more detailed knowledge about the spectrum than the occurrence of generalized condensate. Indeed, for the latter, it is enough to know the asymptotic behavior of the density of states, while for the former, one needs in addition to know how fast the gap between two eigenvalues vanishes in limit.

In the physics literature the phenomenon of BEC is generally understood to be the macroscopic occupation of the lowest kinetic-energy (momentum) state, commonly referred as zero-mode condensation. We refer the reader to Ref. 9 for a discussion of the motivation for this type of condensation. This leads naturally to two questions in the case condensation is produced or enhanced by the addition of external potentials.

The first one comes from the fact the condensates referred to here are to be found in the eigenstates of the one-particle Schrödinger operator and not the kinetic-energy (momentum) states. Therefore, it is not immediately clear, and, in fact, counterintuitive in the random case because of the lack of translation invariance, that condensation occurs in the kinetic-energy states as well. This problem has been addressed in a previous paper, see Ref. 9, where we have shown under fairly general assumptions on the external potential (random or weak) that the amount of generalized BEC in the eigenstates in turn creates a generalized condensate in the kinetic states, and moreover in the perfect gas the densities of condensed particles are identical. These results were proven for the perfect Bose gas and can be partially extended to the mean-field Bose gas. Hence, the (generalized) condensation produced in these models by the localization property of the one-particle Schrödinger operator can be correctly described as of "Bose–Einstein" type in the traditional sense. This opens up the possibility of formulating a generalized version of the c-number Bogoliubov approximation.[9,10] In the case of the weak external potential, perhaps this result is not so surprising since the model is asymptotically translation invariant, but in the random case, it is less obvious since the system is translation invariant only in the sense that translates of the potential are equally probable, and therefore for a given configuration, the system is not translation invariant.

Having established generalized BEC in the kinetic states, the next question is about the fine structure of that condensate. In our paper,[9] we conjectured that the kinetic generalized BEC is of type III, that is, no single kinetic state is macroscopically occupied, even though the amount of generalized condensation is nonzero. Our motivation came from the fact that the fast decrease of the density of states is usually associated with the corresponding eigenstates becoming *localized* in the infinite-volume limit. Hence, since the kinetic states (plane waves) and the (localized) general eigenstates are "asymptotically orthogonal," it should follow that no condensation in any single-mode kinetic-energy state could occur, independently of whether the (localized) ground-state is macroscopically occupied or not. In Ref. 9, we were able to prove this conjecture in a simple example, the Luttinger–Sy model. Our proof in that case used the absence of tunneling effect specific to that model, which we can interpret as "perfect localization."

In this paper we give a proof of the conjecture under a fairly weak localization hypothesis and then we consider a family of continuous random models and a general class of weak external potentials for which we are able to establish this localization property. Our results hold for both the perfect and mean-field Bose gas and for any dimension. Note that, in addition to clarifying the nature of these condensates in low dimensions, we obtain an *unexpected conclusion*. Indeed, we show that the presence of randomness or a weak potential, however small, prevents condensation from occurring in any kinetic state, even if the corresponding *free* Bose gas (without external potential) exhibits zero-mode condensation (isotropic system in dimension of 3, for example). This emphasizes the importance of the concept of generalized BEC.

The structure of the paper is as follows. In Sec. II, we give the general setting for which our results are applicable and discuss generalized condensation in the kinetic-energy states, while in

Sec. III, we derive a criterion for the absence of condensation into any single kinetic-energy state. In Sec. IV, we establish that this criterion is satisfied for a class of random potentials (Sec. IV A) and for weak (scaled) potentials (Sec. IV B).

II. NOTATION AND MODELS

Let $\{\Lambda_l := (-l/2, l/2)^d\}_{l\geq 1}$ be a sequence of hypercubes of side l in $\mathbb{R}^d$, centered at the origin of coordinates with volumes $V_l = l^d$. We consider a system of identical bosons, of mass m, contained in Λ_l. For simplicity, we use a system of units such that $\hbar = m = 1$. First we define the self-adjoint one-particle kinetic-energy operator of our system by

$$h_l^0 := -\frac{1}{2}\Delta_D, \tag{2.1}$$

acting in the Hilbert space $\mathcal{H}_l := L^2(\Lambda_l)$, where Δ is the usual Laplacian. The subscript D stands for *Dirichlet* boundary conditions. We denote by $\{\psi_k^l, \varepsilon_k^l\}_{k\geq 1}$ the set of normalized eigenfunctions and eigenvalues corresponding to h_l^0. By convention, we order the eigenvalues (counting multiplicity) as $0 < \varepsilon_1^l \leq \varepsilon_2^l \leq \varepsilon_3^l \ldots$. Note that all kinetic states satisfy the following bound:

$$|\psi_k^l(x)| \leq l^{-d/2}. \tag{2.2}$$

Next we define the Hamiltonian with an external potential,

$$h_l := h_l^0 + v_l, \tag{2.3}$$

also acting in $\mathcal{H}_l$, where $v_l : \Lambda_l \mapsto [0, \infty)$ is positive and bounded. Let $\{\phi_i^l\}_{i\geq 1}$ and $\{E_i^l\}_{i\geq 1}$ be, respectively, the sets of normalized eigenfunctions and corresponding eigenvalues of h_l. Again, we order the eigenvalues (counting multiplicity) so that $E_1^l \leq E_2^l \leq E_3^l \ldots$. Note that the *non-negativity* of the potential implies that $E_1^l > 0$. We shall also assume that the lower end of the spectra of h_l^0 and h_l coincides in the limit $l \to \infty$, that is, $\lim_{l\to\infty} E_1^l = 0$. This assumption will be proven for the models considered in this paper.

Now, we turn to the many-body problem. Let $\mathcal{F}_l := \mathcal{F}_l(\mathcal{H}_l)$ be the symmetric Fock space constructed over $\mathcal{H}_l$. Then $H_l := \mathrm{d}\Gamma(h_l)$ denotes the second quantization of the *one-particle* Schrödinger operator h_l in $\mathcal{F}_l$. Note that the operator H_l acting in $\mathcal{F}_l$ has the form

$$H_l = \sum_{i\geq 1} E_i^l \, a^*(\phi_i^l) a(\phi_i^l), \tag{2.4}$$

where $a^*(\varphi), a(\varphi)$ are the creation and annihilation operators (satisfying the boson *canonical commutation relations*) for the one-particle state $\varphi \in \mathcal{H}_l$. Then, the grand-canonical Hamiltonian of the perfect Bose gas in an external potential is given by

$$H_l - \mu N_l = \sum_{i\geq 1} (E_i^l - \mu) N_l(\phi_i^l), \tag{2.5}$$

where $N_l(\phi) := a^*(\phi)a(\phi)$ is the operator for the number of particles in the normalized state ϕ, $N_l := \Sigma_i N_l(\phi_i^l)$ is the operator for the total number of particles in Λ_l, and μ is the chemical potential.

The results in this paper hold for the mean-field Bose gas whose many particle Hamiltonian $H_l(\mu)$ is obtained by adding a mean-field term to (2.5),

$$H_l(\mu) := H_l - \mu N_l + \frac{\lambda}{2V_l} N_l^2, \tag{2.6}$$

where λ is a non-negative parameter. Of course the results are valid also for the perfect Bose gas ($\lambda = 0$).

We recall that the thermodynamic equilibrium Gibbs state $\langle - \rangle_l$ associated with the Hamiltonian $H_l(\mu)$ is defined by

$$\langle A \rangle_l := \frac{\mathrm{Tr}_{\mathcal{F}_l}\{\exp(-\beta H_l(\mu_l))A\}}{\mathrm{Tr}_{\mathcal{F}_l}\exp(-\beta H_l(\mu_l))},$$

where the value of μ, μ_l is determined by fixing the mean density $\bar{\rho}>0$,

$$\frac{1}{V_l}\langle N_l \rangle_l = \bar{\rho}. \tag{2.7}$$

When referring specifically to the perfect Bose gas state, we shall use the notation $\langle - \rangle_l^0$. For simplicity, in the sequel we shall omit the explicit mention of the dependence on the thermodynamic parameters (β,μ) unless it is necessary to refer to them.

A normalized single particle state φ is macroscopically occupied if

$$\lim_{l\to\infty}\frac{1}{V_l}\langle N_l(\varphi)\rangle_l > 0,$$

and, in particular, there is condensation in the ground-state if

$$\lim_{l\to\infty}\frac{1}{V_l}\langle N_l(\phi_1^l)\rangle_l > 0.$$

The concept of generalized condensation consists in considering the possible macroscopic occupation of an arbitrary small band of energies at the bottom of the spectrum. To be more precise, we say that there is generalized condensation in the states ϕ_i^l if

$$\lim_{\delta\downarrow 0}\lim_{l\to\infty}\frac{1}{V_l}\sum_{i:E_i^l\le\delta}\langle N_l(\phi_i^l)\rangle_l > 0.$$

It is clear that the usual one-mode condensation implies generalized condensation, however, the converse is not true. Indeed, as was first established by the Dublin School in 1980s,[6] it is possible to classify generalized condensation into three types. Type I condensation, when a finite number of states are macroscopically occupied (which includes the most commonly known notion of BEC as condensation in the ground-state only), type II condensation, when condensation occurs in an infinite number of states, and finally type III, when, although the amount of generalized condensation is nonzero, *no* individual state is macroscopically occupied. One can easily show that in the perfect Bose gas, under fairly general assumptions, for both random and weak positive potentials, there is indeed generalized condensation in a suitable range of density (or temperature).

In Ref. 9 we discussed the possibility of generalized condensation not in the states ϕ_i^l but in the kinetic-energy states ψ_k^l. For both random and weak positive potentials, we established that for models which are diagonal in the occupation numbers of the eigenstates of Hamiltonian (2.3), the density of *generalized BEC* in the kinetic states is never less then that in the eigenstates of the single particle Hamiltonian. To be more precise, we proved that

$$\lim_{\delta\downarrow 0}\lim_{l\to\infty}\frac{1}{V_l}\sum_{k:\varepsilon_k^l<\delta}\langle N_l(\psi_k^l)\rangle_l \ge \lim_{\delta\downarrow 0}\lim_{l\to\infty}\frac{1}{V_l}\sum_{i:E_i^l<\delta}\langle N_l(\phi_i^l)\rangle_l.$$

We also showed that in the case of the perfect gas, the two quantities in the above inequality are equal. Here, we shall give a "localization criterion" on the states ϕ_i^l so that the condensation in the kinetic-energy states ψ_k^l is of type III, that is, no kinetic-energy state is macroscopically occupied.

It is easy to see that the mean-field gas satisfies the following commutation relation:

$$[H_l(\mu),N_l(\phi_j^l)]=0 \quad \text{for all } j. \tag{2.8}$$

This property implies that $\langle a^*(\phi_i^l)a(\phi_j^l)\rangle_l=0$ if $i\neq j$ and allows us to obtain a simple relation between the mean occupation numbers for the ψ_k^l's and ϕ_i^l's,

$$\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l=\frac{1}{V_l}\langle a^*(\psi_k^l)a(\psi_k^l)\rangle_l=\frac{1}{V_l}\sum_{i,j}(\phi_i^l,\psi_k^l)\overline{(\phi_j^l,\psi_k^l)}\langle a^*(\phi_i^l)a(\phi_j^l)\rangle_l=\frac{1}{V_l}\sum_i|(\phi_i^l,\psi_k^l)|^2\langle N_l(\phi_i^l)\rangle_l. \tag{2.9}$$

Finally, we want to point out that it may be possible to extend the results of this paper to a more general class of interacting Bose gases. More precisely, consider a class of "diagonal" interactions defined by

$$U_l := \frac{\lambda}{V_l}\sum_{i,j}a_{i,j}N_l(\phi_i)N_l(\phi_j),$$

with suitable assumptions on the coefficients $a_{i,j}$ in order to make the associated many-particle Hamiltonian well-defined, that is, self-adjoint and bounded below. Note that the mean-field gas (2.6) is a particular case of this class, in which $a_{i,j}=\delta_{i,j}$ (with a shift in the chemical potential). It is easy to see that condition (2.8) is satisfied. However, we shall also need the monotonicity of the mean occupation numbers $\langle N_l(\phi_i^l)\rangle_l$ (see Lemma 3.1), which so far we are unable to prove beyond the mean-field case.

In Sec. III we use expansion (2.9) to obtain a localization criterion for the absence of single-mode condensation in the kinetic-energy states.

III. LOCALIZATION AND KINETIC SINGLE-STATE BEC

First we shall prove the following lemma which is trivial for the perfect gas. For the mean-field Bose gas, it was proven by Fannes and Verbeure,[8] using correlations inequalities. Here we present an alternative proof, based only on a convexity argument.

Lemma 3.1: For the mean-field Bose gas, i.e., for a bosonic system with Hamiltonian (2.6), the function $i\to\langle N_l(\phi_i)\rangle_l$ is nonincreasing.

Proof: Let us define $f:\mathbb{R}_+\mapsto\mathbb{R}$ by

$$f(t) := \beta^{-1}\ln \mathrm{Tr}\, \mathrm{e}^{-\beta H_l(\mu;t)},$$

$$\text{where}\quad H_l(\mu;t) := H_l(\mu)+t(N_l(\phi_m^l)-N_l(\phi_n^l))$$

for some $1\leq m<n$. It follows that

$$f'(0)=\langle N_l(\phi_n^l)-N_l(\phi_m^l)\rangle_l,$$

and since the function f is convex, we have the following inequality:

$$\langle N_l(\phi_n^l)-N_l(\phi_m^l)\rangle_l\leq f'(t) \tag{3.1}$$

for any $t\geq 0$. Now we set $t=\frac{1}{2}(E_n^l-E_m^l)$. Note that with this choice $t\geq 0$, since we have assumed that $m<n$. From explicit expression (2.6) for $H_l(\mu)$, we have

$$H_l(\mu;t)=\sum_{i\neq m,n}(E_i^l-\mu)N_l(\phi_i^l)+\frac{\lambda}{2V_l}N_l^2+\left(\frac{E_m^l+E_n^l}{2}-\mu\right)N_l(\phi_m^l)+\left(\frac{E_m^l+E_n^l}{2}-\mu\right)N_l(\phi_n^l).$$

Since the mean-field term in (2.6) is symmetric with respect to a permutation of any two eigenstate indices i,j, it follows that $H_l(\mu;t)$ is symmetric with respect to the exchange of m and n. Hence

$$f'(t) = \frac{\mathrm{Tr}(N_l(\phi_n^l) - N_l(\phi_m^l))\mathrm{e}^{-\beta H_l(\mu;t)}}{\mathrm{Tr}\ \mathrm{e}^{-\beta H_l(\mu;t)}} = 0,$$

which in view of (3.1) gives

$$\langle N_l(\phi_n^l) - N_l(\phi_m^l)\rangle_l \leq 0,$$

and the lemma follows since $m < n$ are arbitrary. ■

Let us introduce the notation

$$\rho_i^l := \frac{1}{V_l}\langle N_l(\phi_i^l)\rangle_l.$$

With this notation we can write the standard fixed density condition (2.7) for a given density $\bar{\rho}$ as

$$\sum_i \rho_i^l = \bar{\rho},$$

and so for any $N \in \mathbb{N}$,

$$\sum_{i=1}^{N} \rho_i^l \leq \bar{\rho}.$$

Letting

$$\rho_i := \limsup_{l\to\infty} \rho_i^l,$$

and taking the infinite-volume limit, we then get

$$\sum_{i=1}^{N} \rho_i = \limsup_{l\to\infty} \sum_{i=1}^{N} \rho_i^l \leq \bar{\rho}.$$

Letting N tend to infinity, this gives $\sum_{i=1}^{\infty}\rho_i \leq \bar{\rho}$, and hence, for any $\varepsilon > 0$, there exists $i_0 < \infty$, such that $\rho_{i_0} < \varepsilon$. Splitting up the sum in (2.9) and using the monotonicity property (see Lemma 3.1), property (2.2), and the fact that the kinetic eigenfunctions ψ_k^l are normalized, we obtain

$$\begin{aligned}\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l &= \sum_{i\leq i_0} |(\phi_i^l,\psi_k^l)|^2\rho_i^l + \sum_{i>i_0} |(\phi_i^l,\psi_k^l)|^2\rho_i^l \\ &\leq \sum_{i\leq i_0} |(\phi_i^l,\psi_k^l)|^2\rho_i^l + \rho_{i_0}^l \sum_{i>i_0} |(\phi_i^l,\psi_k^l)|^2 \\ &\leq \bar{\rho}\sum_{i\leq i_0} |(\phi_i^l,\psi_k^l)|^2 + \rho_{i_0}^l \\ &\leq \bar{\rho}\sum_{i\leq i_0} (l^{-d/2}\|\phi_i^l\|_1)^2 + \rho_{i_0}^l.\end{aligned}$$

Therefore, if $l^{-d/2}\|\phi_i^l\|_1 \to 0$ as $l\to\infty$ for each i, then

$$\limsup_{l\to\infty}\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l \leq \varepsilon,$$

and since ε is arbitrary

$$\lim_{l\to\infty}\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l = 0.$$

The above argument leads us to define the following localization criterion for the absence of single mode condensation in the kinetic-energy states.

Definition 3.1: We call an eigenfunction ϕ_i^l localized if it satisfies the following condition:

$$\lim_{l\to\infty}\frac{1}{l^{d/2}}\int_{\Lambda_l} \mathrm{d}x|\phi_i^l(x)| = 0. \tag{3.2}$$

Note that this localization condition is not as strong as the usual localization property, in the following sense. While, localization is frequently understood to be associated with the persistence of a pure point spectrum in the limit $l\to\infty$, at least near the bottom of the spectrum, the presence of a pure point spectrum is not necessary for condition (3.2) to hold for all eigenfunctions. Indeed it may happen that (3.2) is satisfied and the infinite-volume Schrödinger operator has only absolutely continuous spectrum.

In Ref. 9 we conjectured that the kinetic generalized BEC observed in the random models is, in fact, of type III and gave a proof in a simple case, the Luttinger–Sy model. In the above argument we proved that our conjecture is correct under fairly weak localization hypothesis (3.2). We formulate this result in the following theorem.

Theorem 3.1: *Assume that the eigenfunctions ϕ_i^l are localized in the sense of (3.2) for all i. Then, for the mean-field Bose gas, no kinetic state ψ_k^l can be macroscopically occupied, that is,*

$$\lim_{l\to\infty}\frac{1}{V_l}\langle N_l(\psi_k^l)\rangle_l = 0, \tag{3.3}$$

which implies, in particular, that any possible kinetic generalized BEC in these models is of type III.

In this paper, we provide two classes of externals potential for which we can prove localization in the sense of (3.2). The first one is a class of *random* external potentials, the second involves *weak* external potentials.

IV. PROOF OF THE LOCALIZATION CONDITION

A. Random potentials

Before we specify the random model under consideration, let us emphasize again that the localization property (3.2) is very different from what is usually called "exponential localization" in the literature about random Schrödinger operators (see, for example, Ref. 12). In the standard literature, localization refers to the eigenfunctions of the infinite-volume Hamiltonian and requires these functions, with energies in some band, to decay very fast, in many cases exponentially. This implies that the spectrum is pure point in that band. In our case we are dealing with eigenfunctions in finite-volume with energies tending to zero as the volume increases and so these bear no relation to the infinite-volume eigenfunctions. In particular, our localization condition (3.2) does *not* imply that the spectrum is discrete in the thermodynamic limit. While we only need the L^1 norm not to diverge too fast, because our eigenfunctions depend crucially on the volume and, in particular, because we do not work at a fixed energy but with volume dependent eigenvalues, we have to deal with the additional problem of controlling the finite-volume behavior. However, we find that, in fact, the *multiscale analysis* developed for the infinite-volume case can be adapted to establish our localization condition.

The model studied in this section is taken from Ref. 12. It consists of impurities located at points of the lattice $\mathbb{Z}^d$, with appropriate assumptions over the single-impurity potential, mainly designed to obtain independence between regions which are sufficiently far away from each other.

Let us make it more explicit by giving some definitions. In the rest of this section, we shall denote by $\Lambda_l(x)$ the cubic box of side l centered at x. The single-site potential f, $\Lambda_1(0)\to\mathbb{R}$, has the following properties.

(1) f is bounded.
(2) There is $\sigma>0$ such that $f(x)\geq\sigma$ for all $x\in\Lambda_1(0)$.

The randomness in this model is given by varying the strength of each impurity. For this purpose, we define a single-site (probability) measure μ, with $\text{supp}(\mu)=[0,a]$ for a finite a. We will assume that μ is Hölder-continuous, that is, for some $\alpha>0$,

$$\sup_{\{s,t\}}\{\mu([s,t]):0\leq t-s\leq\eta\}\leq\eta^{\alpha},\quad\forall\,0\leq\eta\leq 1. \tag{4.1}$$

The random potential is then defined by

$$v^{\omega}(x):=\sum_{k\in\mathbb{Z}^d}q^{\omega}(k)f(x-k), \tag{4.2}$$

where the $q^{\omega}(k)$'s are independent and identically distributed random variables distributed according to μ. We denote by $(\Omega,\mathcal{F},\mathbb{P})$ the associated probability space and by $\omega\in\Omega$ a particular realization of the random potential. Note that by property (1) and the fact that $a<\infty$, there exists a nonrandom $M<\infty$, such that $v^{\omega}(x)<M$ for any x and all ω.

The one-particle random Schrödinger operator in finite-volume is then given as in (2.3) by

$$h_l^{\omega}=h_l^0+v_l^{\omega}, \tag{4.3}$$

where v_l^{ω} is the restriction of v^{ω} to Λ_l. The eigenfunctions and eigenvalues of h_l^{ω} are denoted by $\phi_i^{\omega,l}$ and $E_i^{\omega,l}$, respectively. We denote by $h_l^{\omega}(x)$ the restriction of the Schrödinger operator $-\frac{1}{2}\Delta+v^{\omega}$ to the region $\Lambda_l(x)$, with Dirichlet boundary conditions.

Before we establish localization criterion (3.2), we prove our assumption that the eigenvalues of h_l^{ω} tend to zero as l tends to ∞.

Lemma 4.1: With probability 1, for each i,

$$\lim_{l\to\infty}E_i^{\omega,l}=0. \tag{4.4}$$

Proof: Let ν denote the limiting density of states for the Hamiltonians h_l^{ω}, that is, for any Borel subset $A\subset\mathbb{R}_+$,

$$\nu(A):=\lim_{l\to\infty}\frac{1}{V_l}\sharp\{i:E_i^{\omega,l}\in A\}. \tag{4.5}$$

Since by ergodicity ν is nonrandom (see, for example, Theorem 5.18 in Ref. 4), it is clearly sufficient to prove that for every $E>0$, $\nu([0,E])>0$. To do this we start from the following inequality (see Eq. (4) in Ref. 16):

$$\nu([0,E])\geq\frac{1}{V_L}\mathbb{E}(\sharp\{i:E_i^{\omega,L}\leq E\})\geq\frac{1}{V_L}\mathbb{P}\{\omega:E_1^{\omega,L}\leq E\}, \tag{4.6}$$

which is satisfied for any $L>0$. From the min-max principle, we obtain

$$E_1^{\omega,L}\leq\varepsilon_1^L+\int_{\Lambda_L}\mathrm{d}x|\psi_1^L|^2v^{\omega}(x), \tag{4.7}$$

where ε_1^L is the first kinetic eigenvalue and ψ_1^L the corresponding eigenfunction. Since $|\psi_1^L(x)|^2\leq 1/V_L$, we have

$$E_1^{\omega,L} \le \varepsilon_1^L + \frac{1}{V_L}\int_{\Lambda_L} dx v^{\omega}(x) \le \varepsilon_1^L + \frac{A}{V_L}\sum_{k\in\mathbb{Z}^d\cap\Lambda_L} q^{\omega}(k), \tag{4.8}$$

where $A := \int_{\Lambda_1} dx f(x)$. Letting $L := \pi(E/2)^{-1/2}$ so that $\varepsilon_1^L = E/2$, (4.6) and (4.8) give

$$\nu([0,E]) \ge \frac{1}{V_L}\mathbb{P}\left\{\omega: \sum_{k\in\mathbb{Z}^d\cap\Lambda_L} q^{\omega}(k) \le EV_L/2A\right\}. \tag{4.9}$$

Since the right-hand side of the last inequality is strictly positive, the lemma is proven. ■

The rest of this subsection is devoted to proving that this model satisfies our localization assumption (3.2). For this purpose we need a result from multiscale analysis which exists in various forms in the literature (see references in Ref. 12). For convenience here, we follow the version in Ref. 12.

Adhering to the terminology of Ref. 12, we first define so-called "good boxes."

Definition 4.1: Given $x\in\mathbb{Z}^d$, a scale $l\in 2\mathbb{N}+1$, an energy E, and a rate of decay $\gamma>0$, we call the box $\Lambda_l(x)$ (γ,E)-good for a particular realization ω of the random potential (4.2) if $E\notin\sigma(h_l^{\omega}(x))$ and

$$\|\chi_l^{out}(h_l^{\omega}(x)-E)^{-1}\chi_l^{int}\| \le e^{-\gamma l}. \tag{4.10}$$

Here $\sigma(h_l^{\omega}(x))$ denotes the spectrum of $h_l^{\omega}(x)$, the norm in (4.10) refers to the operator norm in $L^2(\Lambda_l(x))$, and $\chi_l^{int},\chi_l^{out}$ are the characteristic functions of the regions $\Lambda_l^{int}(x),\Lambda_l^{out}(x)$, respectively, which we define as follows:

$$\Lambda_l^{int}(x) := \Lambda_{l/3}(x), \quad \Lambda_l^{out}(x) := \Lambda_l(x)\setminus\Lambda_{l-2}(x).$$

Our proof depends crucially on the following important multiscale analysis result extracted from Ref. 12. We refer the reader to Theorem 3.2.2 and Corollary 3.2.6 for the general multiscale analysis argument and to Theorems 2.3.2, 2.2.3, and 2.4.1 for proving that this particular model satisfies the necessary conditions required for multiscale analysis.

Proposition 4.1: Assume that h_l^{ω} is as above with random potential given by (4.2). Then for any $\zeta>0$ and any $\alpha\in(1,2-(4d/(4d+\zeta))]$, there exist a sequence $\{l_k\},k\ge 1$, satisfying $l_1\ge 2$ and $l_{k-1}^{\alpha}\le l_k\le l_{k-1}^{\alpha}+6$ for $k\ge 2$ and constants $r>0$ and $\gamma>0$, such that if $I:=[0,r]$,

$$\mathbb{P}\{\omega:\text{for all } E\in I, \text{ either } \Lambda_{l_k}(x) \text{ or } \Lambda_{l_k}(y) \text{ is } (\gamma,E)\text{-good}\} \ge 1-(l_k)^{-2\zeta}, \tag{4.11}$$

for all $k\ge 1$ and for all $x,y\in\mathbb{Z}^d$, satisfying $|x-y|>l_k$.

For our proof, we need also the *Eigenfunction decay inequality*. We state it in a convenient form for our purpose, and refer the reader to Ref. 12 (Lemma 3.3.2) for a detailed proof. Note that this inequality has to be understood for a given realization ω.

Proposition 4.2: Let h_l^{ω} be defined as above and $\phi_i^{\omega,l}$ to be one eigenfunction with eigenvalue $E_i^{\omega,l}$. Let $x\in\Lambda_l$, such that $\Lambda_{l_k}(x)\subset\Lambda_l$. If $E_i^{\omega,l}$ does not belong to the spectrum of $h_{l_k}^{\omega}(x)$, then the following inequality holds:

$$\|\chi_{l_k}^{int}(x)\phi_i^{\omega,l}\| \le \kappa\|\chi_{l_k}^{out}(x)(h_{l_k}^{\omega}(x)-E_i^{\omega,l})^{-1}\chi_{l_k}^{int}(x)\|, \tag{4.12}$$

where the norms are $L^2(\Lambda_l)$-norm and κ is a constant depending only on M.

We are now ready to prove that for our model localization condition (3.2) is satisfied.

Lemma 4.2: Assume that h_l^{ω} is as in (4.3) with random potential given by (4.2). Then almost surely, for all i,

$$\lim_{l\to\infty}\frac{1}{V_l^{1/2}}\int_{\Lambda_l}\mathrm{d}x|\phi_i^{\omega,l}(x)|=0. \tag{4.13}$$

Proof: We first choose $0<\delta<1/7$ and $\zeta>(2d+1)/2\delta$ and then we take the constants α, γ, and r and the sequence $\{l_k\}$ to be those obtained in Proposition 4.1 for this value of ζ. For a given scale l large enough we pick $k=k(l)$ satisfying

$$\frac{1}{\ln\,\alpha}\ln\left(\frac{\delta\ln l}{\ln l_1}\right)<k-1<\frac{1}{\ln\alpha}\ln\left(\frac{(1-\delta)\ln l}{\ln(l_1+6)}\right).$$

The fact that $\delta<1/7$ ensures that there exists such an integer k. Then, by Proposition 4.1, we have

$$l^{\delta}<l_k<l^{1-\delta}. \tag{4.14}$$

Now let us define $A(\omega,l)$ to be the event in which, for all $E\in I$, for any $x,y\in\Lambda_l\cap\mathbb{Z}^d$, such that $|x-y|>l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are (γ,E)-good.

We shall first use the Borel–Cantelli lemma to show that almost surely $A(\omega,l)$ occurs for all l large enough. Let us define

$$X_l:=\{\omega:A(\omega,l)\ \text{is not true at scale}\ l\}.$$

Then we can write

$$\begin{aligned}X_l&:=\{\omega:\exists\,E\in I,\exists\,x,y\in\Lambda_l\cap\mathbb{Z}\ \text{with}\ |x-y|>l_k,\\&\quad\text{such that both}\ \Lambda_{l_k}(x)\ \text{and}\ \Lambda_{l_k}(y)\ \text{are not}\ (\gamma,E)\text{-good}\}\\&=\bigcup_{\substack{x,y\in\Lambda_l\cap\mathbb{Z}\\|x-y|>l_k}}\{\omega:\exists\,E\in I,\ \text{such that both}\ \Lambda_{l_k}(x)\ \text{and}\ \Lambda_{l_k}(y)\ \text{are not}\ (\gamma,E)\text{-good}\},\end{aligned}$$

and by Proposition 4.1, we obtain

$$\mathbb{P}(X_l)\le l^{2d}(l_k)^{-2\zeta}\le l^{-2(\delta\zeta-d)},$$

where the last step follows from (4.14). Since $2(\delta\zeta-d)>1$, it follows that

$$\sum_l\mathbb{P}(X_l)<\infty.$$

By the Borel–Cantelli lemma, almost surely there exists $L(\omega)<\infty$, such that the event $A(\omega,l)$ occurs for all $l>L(\omega)$.

Since by Lemma 4.1 with probability 1, $E_i^{\omega,l}$ tends to 0 as l tends to ∞, $E_i^{\omega,l}\in I$ for l large enough almost surely. Therefore, there exists $\tilde{\Omega}\subset\Omega$ with $\mathbb{P}(\tilde{\Omega})=1$, such that for each $\omega\in\tilde{\Omega}$ there is $L_1(\omega)<\infty$, such that for all $l>L_1(\omega)$ and for any $x,y\in\Lambda_l\cap\mathbb{Z}^d$ satisfying $|x-y|>l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are $(\gamma,E_i^{\omega,l})$-good.

Now we take $\omega\in\tilde{\Omega}$ and $l>L_1(\omega)$ and partition the box $\Lambda_l(0)$ into $\Lambda_l^1:=\Lambda_{l-l_k}(0)$ and $\Lambda_l^2:=\Lambda_l(0)\backslash\Lambda_l^1$. We then split up the integral in (4.13) into the interior cube Λ_l^1 and the corridor Λ_l^2,

$$\int_{\Lambda_l}\mathrm{d}x|\phi_i^{\omega,l}(x)|=\int_{\Lambda_l^1}\mathrm{d}x|\phi_i^{\omega,l}(x)|+\int_{\Lambda_l^2}\mathrm{d}x|\phi_i^{\omega,l}(x)|. \tag{4.15}$$

In the second term, we can use the Schwarz inequality and the fact that the eigenfunctions are $L^2(\Lambda_l)$-normalized to obtain

$$\int_{\Lambda_l^2} \mathrm{d}x|\phi_i^{\omega,l}(x)| \le |\Lambda_l^2|^{1/2} \le 2^d l^{(d-1)/2} l_k^{1/2} \le 2^d l^{(d-\delta)/2}. \tag{4.16}$$

For the first term in (4.15), we shall use the eigenfunction decay inequality (4.12) of Proposition 4.2. We cover the "interior cube" Λ_l^1 by disjoints subcubes Λ_j of side $l_k/3$. Let us call $\{x_j\}$ their respective centers. Then for each j, the cube $\Lambda_{l_k}(x_j)$ is included in Λ_l and Λ_j coincides with $\Lambda_{l_k}^{int}(x_j)$.

Using the Schwarz inequality and Proposition 4.2, we obtain for any j the estimate

$$\int_{\Lambda_j} \mathrm{d}x|\phi_i^{\omega,l}(x)| \le l^{d/2}\left(\int_{\Lambda_l} \mathrm{d}x|\chi_{l_k}^{int}(x_j)\phi_i^{\omega,l}(x)|^2\right)^{1/2} \le l^{d/2}(\kappa\|\chi_{l_k}^{int}(x)(h_{l_k}^{\omega}(x_j)-E_i^{\omega,l})^{-1}\chi_{l_k}^{out}(x)\|)^{1/2}.$$

Hence, for any j, such that Λ_j is $(\gamma,E_i^{\omega,l})$-good, one has the following upper bound:

$$\int_{\Lambda_j} \mathrm{d}x|\phi_i^{\omega,l}(x)| \le l^{d/2}\mathrm{e}^{-(1/2)\gamma l_k} \le l^{d/2}\mathrm{e}^{-(1/2)\gamma l^{\delta}}. \tag{4.17}$$

Now, we distinguish two cases.

The first one corresponds to the situation where all cubes $\Lambda_{l_k}(x_j)$ are $(\gamma,E_i^{\omega,l})$-good. It then follows directly from (4.16) and (4.17) that

$$l^{-d/2}\int_{\Lambda_l} \mathrm{d}x|\phi_i^{\omega,l}(x)| \le 2^d\frac{l^{(d-\delta)/2}}{l^{d/2}} + l^{-d/2}\sum_{x_j\in\Lambda_1} l^{d/2}\mathrm{e}^{-(1/2)\gamma l^{\delta}} \le 2^d l^{-\delta/2} + 3^d\frac{(l-l^{\delta})^d}{l^{\delta d}}\mathrm{e}^{-(1/2)\gamma l^{\delta}}. \tag{4.18}$$

The second case corresponds to the situation when there exists at least one subcube $\Lambda_{l_k}(x_j)$ which is *not* $(\gamma,E_i^{\omega,l})$-good. Let us denote by $\tilde{x}$ the center of one such bad cube. Since $\omega\in\tilde{\Omega}$ and $l>L_1(\omega)$, for $x,y\in\Lambda_l\cap\mathbb{Z}^d$ satisfying $|x-y|>l_k$, either $\Lambda_{l_k}(x)$ or $\Lambda_{l_k}(y)$ are $(\gamma,E_i^{\omega,l})$-good. It therefore follows that, outside of a box of side $2l_k$ centered at $\tilde{x}$, all other $\Lambda_{l_k}(x_j)$ are $(\gamma,E_i^{\omega,l})$-good. We treat the good boxes as above and deal with $\Lambda_{2l_k}(\tilde{x})$ by using the Schwarz inequality as we did for Λ_l^2 to obtain

$$\begin{aligned}\int_{\Lambda_l^1} \mathrm{d}x|\phi_i^{\omega,l}(x)| &= \int_{\Lambda_l^1\setminus\Lambda_{2l_k}(\tilde{x})} \mathrm{d}x|\phi_i^{\omega,l}(x)| + \int_{\Lambda_{2l_k}(\tilde{x})} \mathrm{d}x|\phi_i^{\omega,l}(x)| \\ &\le \sum_{x_j\in\Lambda_l^1\setminus\Lambda_{2l_k}(\tilde{x})} l^{d/2}\mathrm{e}^{-(1/2)\gamma l^{\delta}} + |\Lambda_{2l_k}(\tilde{x})|^{d/2} \\ &\le l^{d/2}3^d\frac{(l-l^{\delta})^d}{l^{\delta d}}\mathrm{e}^{-(1/2)\gamma l^{\delta}} + (2l)^{d(1-\delta)/2}.\end{aligned}$$

From that last bound and from (4.16), we get

$$l^{-d/2}\int_{\Lambda_l} \mathrm{d}x|\phi_i^{\omega,l}(x)| \le 2^d l^{-\delta/2} + 3^d\frac{(l-l^{\delta})^d}{l^{\delta d}}\mathrm{e}^{-(1/2)\gamma l^{\delta}} + 2^{d(1-\delta)/2}l^{-d\delta/2}. \tag{4.19}$$

Therefore, for any $\omega\in\tilde{\Omega}$ either (4.18) or (4.19) is satisfied for all l large enough and (4.13) follows. ■

B. Weak external potentials

In this section we consider a scaled external potential. Let v be a non-negative, continuous real-valued function defined on the closed unit cube $\bar{\Lambda}_1\subset\mathbb{R}^d$ which satisfies the following two conditions.

(1) There is a finite, nonempty subset of Λ_1, $D:=\{y_j\}_{j=1}^n$, such that $v(x)=0$ if and only if $x\in D$.
(2) For each $y_j\in D$ there are strictly positive numbers $\{\alpha_j\}$, $\{c_j\}$, such that

$$\lim_{x\to y_j}\frac{v(x)}{|x-y_j|^{\alpha_j}}=c_j. \tag{4.20}$$

We order the y_j's in such a way that $0<\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_n$.

The *one-particle* Schrödinger operator with a *weak* (scaled) external potential in a box Λ_l is defined by

$$h_l=-\frac{1}{2}\Delta_D+v(x_1/l,\ldots,x_d/l). \tag{4.21}$$

We recall that the eigenfunctions and eigenvalues of h_l are denoted by ϕ_i^l and E_i^l, respectively. The aim of this section is to prove that our localization condition (3.2) holds for this class of weak potentials.

Lemma 4.3: Let h_l be as in (4.21). Then, for all i,

$$\lim_{l\to\infty}\frac{1}{l^{d/2}}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)|=0. \tag{4.22}$$

Proof: We start by noting that in view of condition (4.20), for any $\varepsilon>0$ small enough, there exists $\delta>0$, such that for all $j=1,\ldots,n$,

$$(c_j-\varepsilon)|x-y_j|^{\alpha_j}\leq v(x)\leq(c_j+\varepsilon)|x-y_j|^{\alpha_j} \tag{4.23}$$

for all $x\in B(y_j,\delta)$, the ball of radius δ centered at y_j. Note also that by continuity there exists a constant $\kappa>0$, such that $v(x)\geq\kappa$, for all $x\in\Lambda_1\backslash(\cup_{j=1}^n B(y_j,\delta))$. We let $K:=\min(\kappa,c_1-\varepsilon,\ldots,c_n-\varepsilon)$ and $C:=\max(c_1+\varepsilon,\ldots,c_n+\varepsilon)$.

The first step in our proof is to obtain an estimate for the eigenvalue E_i^l. To this end, let us denote by $h_l^{(n)}$ the restriction of the Schrödinger operator to the region $B(y_n,\delta l)$, with Dirichlet boundary conditions. Then we have

$$h_l\leq h_l^{(n)} \tag{4.24}$$

in quadratic form sense (cf. Ref. 14, Chap. VIII, Proposition 4). From inequality (4.23), we obtain

$$h_l^{(n)}\leq\tilde{h}_l^{(n)}:=\frac{1}{2}\Delta_D+C\left|\frac{x-y_n}{l}\right|^{\alpha_n}, \tag{4.25}$$

where the last operator acts on $L^2(B(y_n,\delta l))$. Let $U:L^2(B(y_n,\delta l))\mapsto L^2(B(0,\delta l^{1-\gamma_n}))$ be the unitary transformation defined by

$$(U\varphi)(x):=l^{\gamma_n/2}\varphi(l^{\gamma_n}(x-y_n)),$$

where $\gamma_n:=\alpha_n/(2+\alpha_n)$. By direct computation, one can check that $\tilde{h}_l^{(n)}=l^{-2\gamma_n}U\hat{h}_l^{(n)}U^{-1}$, where

$$\hat{h}_l^{(n)}:=\left(-\frac{1}{2}\Delta+C|x|^{\alpha_n}\right),$$

acting on $L^2(B(0,\delta l^{1-\gamma_n}))$. Let $0<D_1^l\leq D_2^l\leq\ldots$ be the eigenvalues of $\hat{h}_l^{(n)}$ and $0<D_1\leq D_2\leq\ldots$ the eigenvalues of $\hat{h}^{(n)}$, where

$$\hat{h}^{(n)} := \left(-\frac{1}{2}\Delta + C|x|^{\alpha_n}\right),$$

acting on $L^2(\mathbb{R}^d)$. Since for each i, $D_i^l \to D_i$ as $l\to\infty$, there are constants $\widetilde{D}_i$, such that $D_i^l \le \widetilde{D}_i$ for all l. Using this and operator inequalities (4.24) and (4.25), we finally get

$$E_i^l \le D_i^l l^{-2\gamma_n} \le \widetilde{D}_i l^{-2\gamma_n}. \tag{4.26}$$

The rest of our proof relies on the methods developed in Ref. 15. We start with some definitions. Let Ω^t, for some $t>0$, to be the set of all continuous trajectories (paths) $\{\xi(s)\}_{s=0}^t$ in $\mathbb{R}^d$ with $\xi(0)=0$ and let w^t denote the normalized Wiener measure on this set. For a given $x\in\mathbb{R}^d$, we define the following characteristic function:

$$\chi_{x,l}(\xi) := \mathbf{1}\{\xi:\xi(s) \in \Lambda_l - x, \ \text{for all}\ 0\le s\le t\}.$$

We now use the following identity (cf. Ref. 13):

$$(\mathrm{e}^{-th_l}\phi_i^l)(x) = \int_{\Omega^t} w^t(\mathrm{d}\xi)\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\phi_i^l(x+\xi(t))\chi_{x,l}(\xi),$$

from which, since E_i^l is the eigenvalue of h_l corresponding to ϕ_i^l, we get

$$|\phi_i^l(x)| \le \mathrm{e}^{tE_i^l}\int_{\Omega^t} w^t(\mathrm{d}\xi)\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}|\phi_i^l(x+\xi(t))|\chi_{x,l}(\xi). \tag{4.27}$$

Now, we insert into the right-hand side of (4.27) the following bound proven in Ref. 11:

$$|\phi_i^l(x)| \le c_d (E_i^l)^{d/4},$$

where $c_d := (e/\pi)^{d/4}$ and we obtain from (4.27) the following estimate:

$$\begin{aligned}
|\phi_i^l(x)| &\le c_d \mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\mathrm{e}^{-\int_0^t \mathrm{d}s v((x+\xi(s))/l)}\chi_{x,l}(\xi)\\
&= c_d \mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\mathrm{e}^{-\frac{1}{t}\int_0^t \mathrm{d}s t v((x+\xi(s))/l)}\chi_{x,l}(\xi)\\
&\le c_d \mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\frac{1}{t}\int_0^t \mathrm{d}s \mathrm{e}^{-tv((x+\xi(s))/l)}\chi_{x,l}(\xi),
\end{aligned}$$

where the last step follows from Jensen's inequality. Therefore, integrating over Λ_l with respect to x, and then changing the order of integration, yields

$$\begin{aligned}
l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| &\le c_d l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Lambda_l}\mathrm{d}x\int_{\Omega^t} w^t(\mathrm{d}\xi)\frac{1}{t}\int_0^t \mathrm{d}s \mathrm{e}^{-tv((x+\xi(s))/l)}\chi_{x,l}(\xi)\\
&\le c_d l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\frac{1}{t}\int_0^t \mathrm{d}s\int_{\{x\in\cap_{s'}(\Lambda_l-\xi(s'))\}}\mathrm{d}x \mathrm{e}^{-tv((x+\xi(s))/l)}.
\end{aligned}$$

Letting $y=x+\xi(s)$ in the second integral we get

$$l^{-d/2}\int_{\Lambda_l}\mathrm{d}x|\phi_i^l(x)| \le c_d l^{-d/2}\mathrm{e}^{tE_i^l}(E_i^l)^{d/4}\int_{\Omega^t} w^t(\mathrm{d}\xi)\frac{1}{t}\int_0^t \mathrm{d}s\,.\int_{\{y-\xi(s)\in\cap_{s'}(\Lambda_l-\xi(s'))\}}\mathrm{d}y \mathrm{e}^{-tv(y/l)}.$$

Since $\cap_{s'}(\Lambda_l-\xi(s')+\xi(s))\subset\Lambda_l$ for all s, we can now extend the domain of integration over y to Λ_l and use the fact that the Wiener measure w^t is normalized to obtain

$$l^{-d/2}\int_{\Lambda_l} \mathrm{d}x|\phi_i^l(x)| \le c_d \mathrm{e}^{tE_i^l}(E_i^l)^{d/4} l^{-d/2}\frac{1}{t}\int_0^t \mathrm{d}s \int_{\Lambda_l} \mathrm{d}y \mathrm{e}^{-tv((y)/l)} = c_d \mathrm{e}^{tE_i^l}(E_i^l)^{d/4} l^{d/2}\int_{\Lambda_1} \mathrm{d}z \mathrm{e}^{-tv(z)}. \quad (4.28)$$

Next, we obtain an upper bound for the last integral in (4.28). We have

$$\int_{\Lambda_1} \mathrm{d}z \mathrm{e}^{-tv(z)} \le \sum_{j=1}^n \int_{B(y_j,\delta)} \mathrm{d}z \mathrm{e}^{-tv(z)} + \int_{\Lambda_1 \setminus (\cup_{i=1}^n B(y_j,\delta))} \mathrm{d}z \mathrm{e}^{-tv(z)} \le \mathrm{e}^{-tK} + \sum_{j=1}^n \int_{B(y_j,\delta)} \mathrm{d}z \mathrm{e}^{-tK|x-y_j|^{\alpha_j}}. \quad (4.29)$$

For each j,

$$\int_{B(y_j,\delta)} \mathrm{d}z \mathrm{e}^{-tK|x-y_j|^{\alpha_j}} \le t^{-d/\alpha_j} K^{d/\alpha_j} \int_{\mathbb{R}^d} \mathrm{d}\tilde{z} \mathrm{e}^{-|\tilde{z}|^{\alpha_j}} \le \tilde{K} t^{-d/\alpha_j},$$

where $\tilde{K} := K^{d/\alpha_1} \max_j \int_{\mathbb{R}^d} \mathrm{d}\tilde{z} e^{-|\tilde{z}|^{\alpha_j}}$, which, in view of (4.29), gives the following bound:

$$\int_{\Lambda_1} \mathrm{d}z \mathrm{e}^{-tv(z)} \le \mathrm{e}^{-tK} + \tilde{K}\sum_{j=1}^n t^{-d/\alpha_j}.$$

Now, fixing $t=(E_i^l)^{-1}$, we get from the last inequality and (4.28)

$$l^{-d/2}\int_{\Lambda_l} \mathrm{d}x|\phi_i^l(x)| \le c_d \mathrm{e}(E_i^l)^{d/4} l^{d/2}\left(\mathrm{e}^{-K(E_i^l)^{-1}} + \tilde{K}\sum_{j=1}^n (E_i^l)^{d/\alpha_j}\right).$$

Since by (4.26), $E_i^l \to 0$ as $l\to\infty$, and since we have ordered the α_i's, such that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$, there exist new constants A_i, such that the following bound holds for l large enough:

$$l^{-d/2}\int_{\Lambda_l} \mathrm{d}x|\phi_i^l(x)| \le A_i l^{d/2}(E_i^l)^{d(1/4+1/\alpha_n)} = A_i l^{d/2}(E_i^l)^{d(2-\gamma_n)/(4\gamma_n)}. \quad (4.30)$$

Inserting bound (4.26), we finally obtain for l large enough

$$l^{-d/2}\int_{\Lambda_l} \mathrm{d}x|\phi_i^l(x)| \le A_i \tilde{D}_i^{d(2-\gamma_n)/(4\gamma_n)} l^{-d(1-\gamma_n)/2}$$

and the lemma follows since $\gamma_n < 1$. ■

ACKNOWLEDGMENTS

One of the authors (T.J.) is supported by funding from the UCD Ad Astra Research Scholarship.

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