Alternating Sign Matrices, completely Packed Loops and Plane Partitions

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Thèse presentée par

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Pour obtenir le grade de

Docteur de l'Université Pierre et Marie Curie

Spécialité : **Physique Mathématique**

Sujet :

Matrices à signes alternants, boucles denses et partitions planes

Soutenue le 24 septembre 2010 devant le jury composé de

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Remerciements

Avant tout je remercie Paul qui m'a guidé dans ma thèse et qui m'a proposé plein de problèmes intéressants, quelques uns que j'ai su résoudre et d'autres non. Pendant trois ans il a été toujours présent et disponible pour discuter et j'espère, sincèrement, qu'on pourra continuer à travailler ensemble.

Je laisse aussi un petit mot de remerciement à Jean-Bernard Zuber, Philippe Di Francesco, Philippe Nadeau, Luigi Cantini, Andrea Sportiello et Keiichi Shigechi pour toutes les discussions intéressantes sur la physique, sur les mathématiques et aussi sur les banalités de la vie sans lesquelles je n'aurais pas progressé autant. Je remercie les membres du jury pour m'avoir accordé de leur précieux temps pour lire ma thèse et venir à ma soutenance. Un merci particulier à Nikolai, Andrea, Paul, Jean-Bernard et Christian pour avoir cherché et trouvé des fautes dans mon manuscrit (surtout les deux derniers).

Je remercie aussi toutes les personnes du LPTHE (en commençant par les femmes de ménage et en finissant par notre chef Olivier) qui ont rendu le travail plus plaisant. Bien évidement, je n'oublierai jamais mes collègues de thèse : la ponctualité de Mathieu, les ragots de David, la joie de César, les discours de Clément, le métal de Quentin, les huit ans ensemble avec João, les états quadri-dimensionnels de Nicolas, le babyfoot de Benoit, le calme de Sebastian, la fuite de Camille, les 16-vertex de Demian, la détente d'Alberto et Bruno qui s'incrustait toujours avec nous pour déjeuner.

Finalement, parce que la vie ne se résume pas juste à la physique, je remercie du fond du cœur tous mes amis de Paris, de Porto, de Marco ou d'ailleurs, famille incluse.

Abstract

This thesis is devoted to the study of identities which one observes at the interface between integrable models in statistical physics and combinatorics. The story begins with Mills, Robbins and Rumsey studying the Alternating Sign Matrices (ASM). In 1982, they came out with a compact enumeration formula.

When looking for a proof of such formula they discovered the existence of other objects counted by the same formula: Totally Symmetric Self-Complementary Plane Partitions (TSSCPP).

It was only some years later that Zeilberger was able to prove this equality, proving that both objects are counted by the same formula. At the same year, Kuperberg, using quantum integrability (a concept coming from statistical physics), gave a simpler and more elegant proof.

In 2001, Razumov and Stroganov conjectured one intriguing relation between ASM and the ground state of the XXZ spin model (with $\Delta = -\frac{1}{2}$), also integrable. This conjecture was proved by Cantini and Sportiello in 2010.

The principal goal of this manuscript is to understand the role of integrability in this story, notably, the role played by the quantum Knizhnik-Zamolodchikov equation. Using this equation, we prove some combinatorial conjectures. We prove a refined version of the equality between the number of ASM and TSSCPP conjectured in 1986 by Mills, Robbins and Rumsey. We prove some conjectured properties of the components of the XXZ groundstate. Finally we present new conjectures concerning the groundstate.

Keywords

- Mathematical physics;
- Statistical physics;
- Combinatorics;
- Quantum integrability;
- Razumov–Stroganov;
- Quantum Knizhnik–Zamolodchikov equation.

Résumé court

Cette thèse est consacrée à l'étude d'identités qu'on observe à l'interface entre le domaine des modèles intégrables en physique statistique et la combinatoire. L'histoire a commencé quand Mills, Robbins et Rumsey étudiaient des Matrices à Signes Alternants (ASM). En 1982, ils proposèrent une formule d'énumération.

Pendant qu'ils cherchaient une preuve de cette formule ils découvrirent l'existence d'autres objets comptés par la même formule : les Partitions Planes Totalement Symétriques Auto-Complémentaires (TSSCPP).

C'est seulement quelques années plus tard que Zeilberger fut capable de prouver cette égalité, prouvant que les deux objets sont comptés par la même formule. La même année, Kuperberg utilise l'intégrabilité quantique (notion venue de la physique statistique) pour donner une preuve plus simple.

En 2001, Razumov et Stroganov conjecturèrent une intrigante relation entre les ASM et l'état fondamental du modèle de spins XXZ (pour $\Delta = -\frac{1}{2}$), lui aussi intégrable. Cette conjecture a été démontrée en 2010 par Cantini et Sportiello.

L'objectif principal de ce manuscrit est de comprendre le rôle de l'intégrabilité dans cette histoire, notamment le rôle joué par l'équation de Knizhnik-Zamolodchikov quantique. Grâce à cette équation nous avons été capables de démontrer plusieurs conjectures combinatoires, dont une version raffinée de l'équalité entre le nombre des TSSCPP et des ASM proposée en 1986 par Mills, Robbins et Rumsey et certains propriétés des composantes du vecteur fondamental du modèle XXZ. Est présentée aussi une série de nouvelles conjectures concernant l'état fondamental.

Mots-clés

- Physique mathématique ;
- Physique statistique ;
- Combinatoire ;
- Intégrabilité quantique ;
- Razumov–Stroganov ;
- Équation de Knizhnik–Zamolodchikov quantique.

Resumo curto

Esta tese deriva do estudo de identidades que se observam na interface entre o domínio dos modelos integráveis em física estatística e a combinatória. A história começa com o estudo das Matrizes de Sinal Alternante (ASM) por Mills, Robbins e Rumsey. Em 1982, eles propuseram uma fórmula para as contar.

Enquanto procuravam provar esta fórmula eles descobriram a existência de outros objectos contados pela mesma fórmula: as Partições Planas Totalmente Simétricas e Auto-Complementares (TSSCPP).

Foi somente alguns anos mais tarde que Zeilberger foi capaz de provar esta igualdade, provando que ambos os objectos são contados pela mesma fórmula. No mesmo ano, Kuperberg utiliza a integrabilidade quântica (noção vinda da física estatística) para dar uma prova mais simples e elegante.

Em 2001, Razumov e Stroganov conjecturaram uma intrigante relação entre as ASM e o estado fundamental do modelo de spins XXZ (para $\Delta = -\frac{1}{2}$), também ele integrável. Esta conjectura foi demonstrada em 2010 por Cantini e Sportiello.

O principal objectivo deste manuscrito é de compreender o papel da integrabilidade nesta história, nomeadamente, o papel da equação Knizhnik-Zamolodchikov quântica. Graças à qual nós fomos capazes de provar várias conjecturas combinatórias. Entre elas uma versão refinada da igualdade entre o número de TSSCPP e de ASM proposta em 1986 por Mills, Robbins e Rumsey e algumas propriedades das componentes do vector fundamental do modelo XXZ. É também apresentada uma série de novas conjecturas acerca do estado fundamental.

Palavras chave

- Física matemática;
- Física estatística;
- Combinatória;
- Integrabilidade quântica;
- Razumov–Stroganov;
- Equação Knizhnik–Zamolodchikov quântica.

Contents

Introduction

Alternating Sign Matrices (ASM) were invented by Robbins and Rumsey in their study of the "γ-determinants" in [70], which generalise determinants in a way inpired by Dodgson's condensation [19]. When Mills, Robbins and Rumsey [49, 50, 70, 51] were studying the properties of these objects, they came out with a nice compact formula for the enumeration of such matrices:

$$A_n := \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

Intrigued by this conjectural formula, they called Richard Stanley for help. That is when another combinatorial object enters: Plane Partitions (PP), in particular two subclasses of PP, the Descending Plane Partitions (DPP) and the Totally Symmetric Self-Complementary Plane Partitions (TSSCPP). In [1], Andrews proved that this product counts the total number of DPP with largest part less than or equal to n. So, if we were able to find an explicit bijection between DPP and ASM we would automatically prove the enumeration formula. However, such a bijection is not known until today.

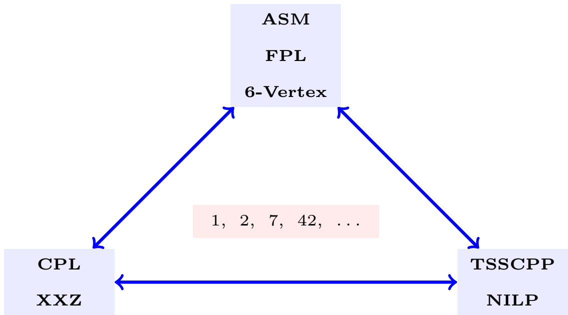
Later, Robbins discovered that there is another class of PP which are also counted by the sequence A_n, the TSSCPP. Moreover, in article [51] the authors stated, not only the number of TSSCPP is the same as the number of ASM, but also that there is a doubly refined enumeration on both sides which is equal.

In 1996, in a famous 84 pages article [80], Zeilberger proved the ASM enumeration. Some time after, Kuperberg [44] using the partition function of the 6-Vertex model, which is in bijection with the ASM, proved the same result in a more elegant and shorter fashion. His proof relies on the work of Izergin [29] and Korepin [40], and it is a consequence of the fact that the 6-Vertex model is a quantum integrable model.

The XXZ Spin Chain is an old problem of physics, and it has been studied for almost a century. In 2000, Razumov and Stroganov [68, 76] were investigating the properties of the XXZ groundstate (with anisotropic parameter $\Delta = -1/2$ and length $2n+1$), and they encountered the following surprising result: when normalized such that the smallest component is 1, the sum of the components of the groundstate is exactly A_n. Intrigued by this, physicists and mathematicians tried to prove this relation. At the same month Nienhuis, de Gier and Batchelor wrote an article [3] where they present some new conjectures and introduce a new object in the subject, the $O(n)$ loop models, here called Compact Loop Models (CPL).

Some time after, Razumov and Stroganov [65] noticed that the link between the two models is deeper. The CPL configurations can be indexed by some link patterns (we use, generally, π to designate a link pattern). The ASM are in bijection with the Fully Packed Loop (FPL) configurations which can be indexed by their connectivity at the boundary, also represented by link patterns. They discovered that when we normalize the CPL groundstate such that the smallest component is 1, the components indexed by a link pattern count the number of FPL whose connectivity is represented by the same link pattern.

The latest chapter of this story was written by Cantini and Sportiello: in 2010 they proved the Razumov–Stroganov conjecture [8]. This story can be summarized in the following graph:



Where we represent all the main objects described here, grouped by the fact that there is well known bijections between them. In the middle it appears the sequence which is omnipresent in this story, A_n. Remark that when we refer the NILP we refer those in bijection with the TSSCPP.

An important lesson to learn from the result of Kuperberg is that Quantum Integrable Models can help us to solve some difficult combinatorial problems. Furthermore, both the CPL model (or XXZ Spin Chain) and the 6-Vertex model are integrable. Thus, it seems that integrability can help us to clarify the connections between the corners of the triangle.

The quantum Knizhnik–Zamolodchikov equation (qKZ) was introduced in this context by Di Francesco and Zinn-Justin (see [16]) based on Pasquier's article [60]. In our case (level 1 and $U_q(sl(2))$), the solutions are homogeneous polynomials, and it can be shown to generate a vector space characterized by a vanishing condition, the *wheel* condition.

On the one hand, the solutions of this equation (at level 1, and $q^{2\pi i/3}$) can be identified with the CPL groundstate in its multivariate version. On the other hand, these polynomials are related with Macdonald polynomials at specialized values of their parameters $t^3 q = 1$ (see [35, 12]), and they are what Lascoux calls the Kazhdan–Lusztig polynomials [12].

It is the main purpose of this thesis to understand the role of integrability in this story. In particular, the role played by the quantum Knizhnik–Zamolodchikov equation, and its solutions.

This manuscript is organized as follows. In Chapter 1 we define the Completely Packed Loop model (CPL), and the XXZ Spin Chain, we also define a multivariate version. In order to compute the groundstate of the CPL model, we introduce the quantum Knizhnik–Zamolodchikov equation. In Section 1.4 we give an algorithm to construct its solutions. In Sections 1.5 and 1.6 we give two new methods to solve the qKZ equation, using contour integral formulæ.

In Chapter 2, we define the Alternating Sign Matrices (ASM), the 6-Vertex model and the Fully Packed Loop model. Using the fact that the 6-Vertex model is a quantum integrable model we give an explicit formula of the partition function as a determinant (the Izergin–Korepin determinant [29]) in Subsection 2.2.2. We state the Razumov–Stroganov–Cantini–Sportiello theorem.

In Chapter 3 we present the last corner of the triangle: Plane Partitions and in particular the Totally Symmetric Self Complementary Plane Partitions. We also define the Non-Intersecting Lattice Paths (NILP), and present some interesting results about them.

Chapter 4 results from the author's joint work with his supervisor [23]. In this chapter we prove the equality of doubly refined enumerations of ASM and TSSCPP, stated by Mills, Robbins and Rumsey.

In 2004, Zuber [85] conjectured that the entries of the CPL groundstate, corresponding to the link pattern with p nested arches $(\pi)_p$ are polynomials in p. In Chapter 5 we prove the polynomiality of these components. Furthermore, we study some properties of components indexed by the link patterns $(\pi)_p$ and $({}_p\alpha$ (this corresponds to link patterns which start with p openings). We also give a nice interpretation of the sum of such components in terms of punctured TSSCPP. Besides, we produce perfectly random configurations of such plane partitions (based on the guide [63]). This chapter is based (except the sampling of the random configuration) on the article [24], by the author and his supervisor.

Chapter 6 is based on the joint work of the author with Philippe Nadeau [22], only Section 6.3 is new. Its main goal is to exhibit some surprising properties of the polynomials $\psi_{(\pi)_p}$, or equivalently, the polynomials $A_{(\pi)_p}$. Although we have proven some results, almost all of the properties are conjectural.

We relegate the most technical stuff to the appendices. In Appendix A we study the dimension of the space generated by the solutions of the qKZ equation. In Appendix B we extract some properties of the basis transformation $C_{a,\pi}$. We prove two important anti-symmetrization formulæin C. We show some examples of the cited polynomials in Appendix D. And finally, in Appendix E we prove the technical result 6.2.

Chapter 1

Completely Packed Loops and the quantum Knizhnik–Zamolodchikov equation

In this chapter we briefly present some quantum integrable models, namely, the Completely Packed Loops and the XXZ Spin Chain Model.

The chapter is organized as follows. We describe the two models in Sections 1.1 and 1.2. In Section 1.3 we introduce a new equation which allows us to compute the groundstate of both models (at the special point $\Delta = -\frac{1}{2}$). In Sections 1.4, 1.5 and 1.6 we show three different methods[1] to compute the solution of such an equation. Finally, in Section 1.7 we give an example of such a computation.

1.1 Completely Packed Loops

Loop models are an important class of two-dimensional statistical lattice models. Indeed, they present a wide range of critical phenomena and many classical models can be mapped into a loop model. Here we consider the Completely Packed Loops model (CPL), also called $O(n)$ Loop Model for historical reasons[2].

Take a square grid. Each face is occupied by one of the two plaquettes shown on Figure 1.1. Naturally, we will have some closed loops, and some paths connecting boundaries. We give a weight $\tau = -q - q^{-1}$ to each closed loop. This model is known to be critical for $|\tau| \leq 2$ (see the lecture notes [57]).

Figure 1.1: The two plaquettes forming the CPL.

We can define different boundary conditions. Here we only consider CPL on a semi-infinite cylinder, where each row is made of $2n$ plaquettes which can contain the two

[1] Actually, the third method is a slight and convenient variation of the second one.

[2] This model can be seen as the high temperature's limit of the $O(n)$ model.

possible drawings, as on Figure 1.2.

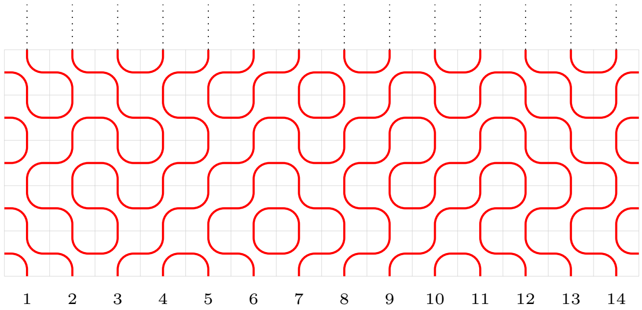


Figure 1.2: An example of a state in the Completely Packed Loop model, here with $n = 7$. Each row is filled with plaquettes that can be of two types. The left boundary is identified with the right boundary forming a cylinder. Each closed loop is given the weight τ.

1.1.1 Bond percolation

The configurations of the CPL model are in bijection with the bond configurations of a bond percolation model[3].

Consider again a square lattice. The parity of each site in the lattice is defined such that each site has all neighbors with opposite parity. For a semi-infinite cylinder (with even perimeter) there are only two choices, one corresponding to the other with inverted signs. Now, we replace each plaquette with the bond connecting its two odd sites if allowed by the loops, *i.e.* if it does not intersect the loops, as we can see in figure 1.3. We could have chosen to use the even sites instead.

Figure 1.3: A map between the CPL and bond percolation in the odd sub lattice is given by the correspondence above. Here the odd sub lattice is represented by the filled circles.

The number of loops in the CPL configuration is encoded now by the number of clusters and cycles in the percolation configuration. Let n_c be the number of clusters that do not touch the boundaries, and n_b the number of minimal cycles[4], the number of loops in the CPL configurations is equal to $n_c + n_b$. Of course this is valid for other boundary conditions, provided that we can define the parity of each site.

[3] This appears, for example in the papers [52, 53]

[4] A minimal cycle is a cycle that cannot be divided into smaller cycles. It corresponds to a cluster that does not touch the boundary in the bond percolation using even sites.

1.1.2 Connectivity

In order to set up a transfer matrix approach we introduce a state as being the connectivity of the $2n$ points at the bottom. For example the connectivity of the infinite configuration 1.2 is represented by the link pattern, sometimes also called by matching, represented on 1.4. The number of link patterns of size $2n$ is the Catalan number c_n:

$$c_n = \frac{(2n)!}{n!(n+1)!}$$

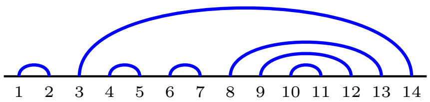


Figure 1.4: Link Pattern corresponding to the connectivity in the example 1.2

We define a formal state as being a linear combination of link patterns, $\xi = \sum_\pi \xi_\pi \pi$, where the sum runs over all c_n matchings and ξ_π are scalars. Then an operator is a linear map on vector space spanned by these formal states.

Some notation

Sometimes it is convenient to represent the link patterns by other equivalent objects:

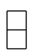
- A well-formed sequence of parentheses, also called parenthesis word. If the point is connected to the left (respectively to the right), then this is encoded by an opening parenthesis (resp. by a closing parenthesis);

   $\Leftrightarrow$ ()(())

- A Dyck Path, which is a path between $(0,0)$ and $(2n,0)$ with steps NE $(1,1)$ and SE $(1,-1)$ that never goes under the horizontal line $y=0$. An opening parenthesis corresponds to a NE step, and a closing one to a SE step;

  ()(()) $\Leftrightarrow$ 

- A Young diagram is a collection of boxes, arranged in left-justified rows, such that the size of the rows is weakly decreasing from top to bottom. Matchings with n arches are in bijection with Young diagrams such that the i^{th} row from the top has no more than $n-i$ boxes. The Young diagram can be constructed as the complement of a Dyck path, rotated 45° counterclockwise;

   $\Leftrightarrow$ 

- A sequence $a = (a_1, \ldots, a_n)$, $a_i \subseteq \{1, \ldots, 2n\}$, such that $a_{i-1} < a_i$ and $a_i \leq 2i - 1$ for all i. Here a_i is the position of the i^{th} opening parenthesis.

$$()(()) \Leftrightarrow \{1, 3, 4\}$$

We will often identify matchings under these representations, hoping that this will not confuse the reader. When needed we will use $Y(\pi)$ for the Young diagram associated to the link pattern π, $\pi(a)$ for the link pattern associated to the sequence a. Moreover we will represent p nested arches around a matching π by $(\pi)_p$, and p small arches by $()^p$. Thus for example,

$$(()^2)_3() = (((()())))()$$

Given a matching π, we define $d(\pi)$ as the total number of boxes in the Young diagram $Y(\pi)$. We also let π^* be the conjugate matching of π, defined by: $\{i, j\}$ is an arch in π^* if and only if $\{2n + 1 - j, 2n + 1 - i\}$ is an arch in π. This corresponds to a mirror symmetry of the parenthesis word, and a transposition in the Young diagram. We also define a natural *rotation* r on matchings: i, j are linked by an arch in $r(\pi)$ if and only if $i - 1, j - 1$ are linked in π (where indices are taken modulo $2n$). These last two notions are illustrated on Figure 1.5.

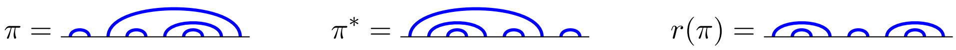


Figure 1.5: A matching, its conjugate, and the rotated matching.

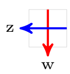
We need additional notions related to the Young diagram representation. So let Y be a Young diagram, and u one of its boxes. The *hook length* $h(u)$ is the number of boxes below u in the same column, and to its right in the same row (including the box u itself). We denote by H_Y the product of all hook lengths, i.e. $H_Y = \prod_{u \in Y} h(u)$. The *content* $c(u)$ is given by $y - x$ if u is located in the x^{th} row from the top and the y^{th} column from the left; we write $u = (x, y)$ in this case. The *rim* of Y consists of all boxes of Y which are on its southeast boundary; removing the rim of a partition leaves another partition, and repeating this operation until the partition is empty gives us the *rim decomposition* of Y.

1.1.3 Transfer Matrix and Integrability

In what follows we will ignore the loops' weights. *I.e.* we fix $q = e^{\pm 2\pi i/3}$.

The transfer matrix is defined as the addition of a new row with each plaquette chosen randomly, see Figure 1.6.

Let y_i be a parameter indexing the i^{th} column, called vertical spectral parameter, and t a parameter indexing the new row, called horizontal spectral parameter. The transfer matrix is a row of $2n$ plaquettes turning right or left with probability:



$$= \frac{qw - q^{-1}z}{qz - q^{-1}w}$$  $$+ \frac{w - z}{qz - q^{-1}w}$$  $$=: R(w, z).$$

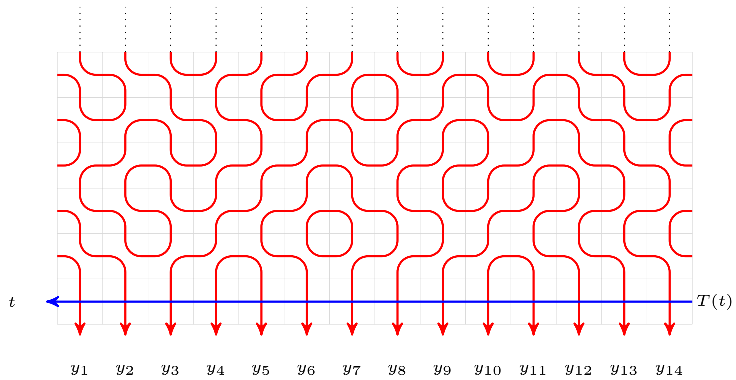


Figure 1.6: The transfer matrix $T(t)$ is defined as the addition of a new row at the bottom. We choose each plaquette randomly.

In fact, we can set q different from $e^{\pm 2\pi i/3}$, the price to pay is that now the sum of the probabilities of turning left and turning right is not 1 in general.

That is, the transfer matrix for the semi-infinite cylinder (with perimeter $2n$) is formally defined as:

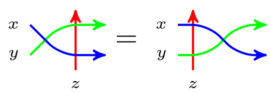
$$T(y_1, y_2, \ldots, y_{2n}; t) := Tr[R_1(y_1, t)R_2(y_2, t) \ldots R_{2n}(y_{2n}, t)],$$

where the subscript i in R indicates the position of the plaquette, and where the trace means that the first and the last plaquettes are glued, as a consequence of the periodic boundary conditions.

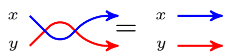
This $R(w, z)$, called the R matrix, is represented graphically by a crossing of two arrows, each arrow carrying a spectral parameter. It is the key of the integrability of our model.

Indeed,

Lemma 1.1. *The R matrix satisfies the Yang–Baxter equation:*



And also the identity equation:

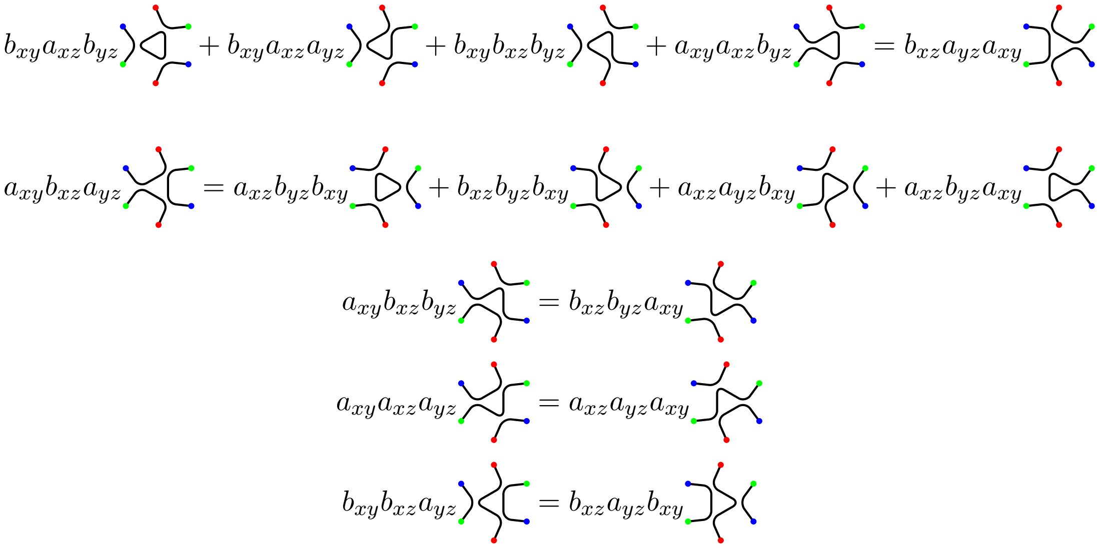


Observe that when we rotate the arrows we are, indeed, rotating also the plaquettes in the definition of the R matrix. Furthermore we ignore the box around the plaquette.

Proof. Let

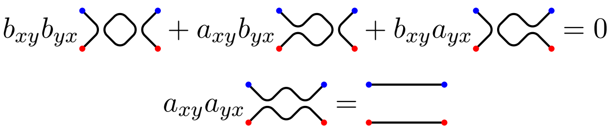
$$a_{xy} = \frac{qy - q^{-1}x}{qx - q^{-1}y}$$
$$b_{xy} = \frac{y - x}{qx - q^{-1}y}$$

The Yang–Baxter equation breaks down to five equations, corresponding to different connectivities:



Now, it is enough to compare the coefficients, not forgetting the weight τ from the closed loops.

The identity equation is simpler, we only need to check the following equations:



□

It follows from the Yang–Baxter equation that:

Proposition 1.2. *The transfer matrices commute for all t and t':*

$$[T(y_1,\ldots,y_{2n};t),T(y_1,\ldots,y_{2n};t')]=0$$

So, there is an infinite family of commuting operators, justifying the name of integrable system. Moreover, the eigenvectors of the Transfer Matrix do not depend on the parameter t.

Although this is a standard result, we repeat here the proof, as this is of major importance:

Proof. Pick two transfer matrices $T(t)T(t')$ (we omit the dependence in $y_1,\ldots,y_{2n}$). Multiply on the right by the identity, using the identity equation. Now, there is a triangle at the right, we can move the vertical line to the right, using the Yang–Baxter equation. If we apply the Yang–Baxter equation $2n$ times and use the fact that we have periodic boundary conditions we get back exactly to the same operator, but with the lines interchanged $T(t')T(t)$, as we can see in Figure 1.7. □

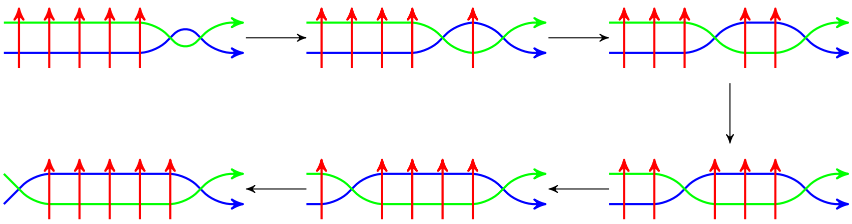
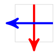


Figure 1.7: If we apply the Yang–Baxter equation consecutively we interchange both lines.

The homogeneous case

Set now $y_i = 1$ for all i, and $t = \frac{1+qp}{q+p}$, with $0 \leq p \leq 1$[5]. We obtain the new probabilities:

 $= p$  $+ (1-p)$ 

Notice that if $p = 0$ or $p = 1$, the transfer matrix will only rotate the system, and all states which are invariant by rotation are eigenvectors of the transfer matrix.

The intermediate cases $0 < p < 1$ are more interesting. Integrability says that the groundstate does not depend on p.

1.1.4 The Hamiltonian and the Temperley–Lieb algebra

In the homogeneous formulation, we can reformulate our model using a Hamiltonian. This is the formalism used, for example, in [11]. It is based on the affine Temperley–Lieb (TL) algebra.

Definition 1.3 (The Temperley–Lieb Algebra)**.** *The TL algebra is the algebra generated by the elements e_i, with i ranging from 1 to $L-1$, obeying the relations:*

$$e_i^2 = \tau e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \quad \text{if } |j-i| > 1 \tag{1.1}$$

The second equation is defined whenever it make sense.

The Temperley–Lieb algebra is a quotient of the well known Hecke algebra:

Definition 1.4 (The Hecke Algebra)**.** *The Hecke algebra is generated by the elements e_i, with i between 1 and $L-1$, obeying the relations:*

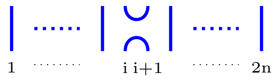
$$e_i^2 = \tau e_i \qquad e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1} \qquad e_i e_j = e_j e_i \quad \text{if } |j-i| > 1$$

The TL algebra can be extended adding a new element ρ, such that $\rho^L = Id$, which will play a role of rotation, $\rho e_i = e_{i+1}\rho$. Using this, we define a new element $e_L = \rho e_{L-1} \rho^{-1} = \rho^{-1} e_1 \rho$. Relations (1.1) can be extended to contain this new element, the

[5]We can consider the case such that p is not real, or outside $[0,1]$, losing the interpretation as probabilities.

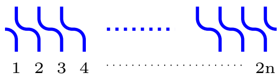
last relation being valid for $|j-i \mod L| > 1$. We could have defined e_L without defining ρ, and obtained a slightly different affine TL algebra, but this version is better for our goals.

The usual representation of the TL algebra in the link patterns can be defined purely graphically. Let $L = 2n$ be even. The operator e_i is given by the drawing:

$e_i =$ 

.

The e_{2n} operator links the first position with the last one. Furthermore the rotation operator is:

$\rho =$ 

(1.2)

We can easily verify that these operators satisfy all the rules of the affine TL algebra, if we allow for stretching the lines.

If $y_i = 1$ for all i, we can describe our model by a Hamiltonian given by the usual formula:

$$H = -\frac{d}{dt} \log T(1, \ldots 1; t)\Big|_{t=1} \tag{1.3}$$

Performing this computation we get the explicit formula,

$$H = \frac{1}{(q - q^{-1})} \sum_{i=1}^{2n} (-(q + q^{-1})e_i - Id)$$

where Id is the identity operator. In what follows we simplify the definition of the Hamiltonian to

$$H = \frac{1}{2n} \sum_{i=1}^{2n} e_i. \tag{1.4}$$

The normalization is such that it can be re-interpreted as a stochastic process for $\tau = 1$.

1.1.5 A stochastic process in the link patterns

We apply now the Hamiltonian to a random initial state, say $\xi = \sum_\pi \xi_\pi \pi$, m times

$$\xi(m) = H^m \xi,$$

where $\xi(m)$ is our state after m iterations. At the end, we will almost always obtain the stationary state given by the equation

$$\Psi = H\Psi, \tag{1.5}$$

where $\Psi = \sum_\pi \Psi_\pi \pi$.

This can be interpreted as a stochastic process. We start with a state, say π. At every iteration we randomly choose a position i with probability $\frac{1}{2n}$ and we apply e_i to π, obtaining a new state $\pi' = e_i \pi$.

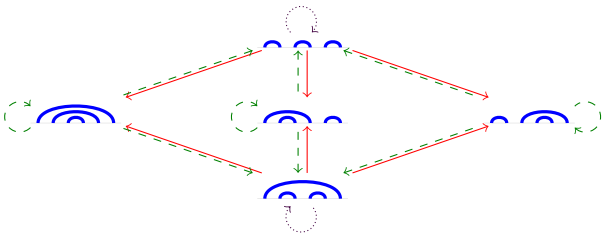


Figure 1.8: The Markov process on the link patterns. The dashed arrows correspond to a probability $\frac{1}{3}$, the normal arrows correspond to $\frac{1}{6}$ and the dotted arrows correspond to $\frac{1}{2}$.

For $n = 3$, the stochastic process can be represented by the graph 1.8. This case has one stable solution given by the node probabilities:

$$\Psi_{((()))} = \frac{1}{7} \qquad \Psi_{(())()} = \frac{1}{7} \qquad \Psi_{()(())} = \frac{1}{7}$$

$$\Psi_{()()()} = \frac{2}{7} \qquad \Psi_{(()())} = \frac{2}{7}$$

Lemma 1.5. *If we start with a state $\xi = \sum_\pi \xi_\pi \pi$, and if we evolve the system with the Hamiltonian H, we obtain in the limit:*

$$\lim_{m\to\infty} H^m \xi = \left(\sum_\pi \xi_\pi \right) \Psi,$$

where Ψ is the solution of the equation $\Psi = H\Psi$, such that $\sum_\pi \Psi_\pi = 1$.

Observe that this equation vanishes for $\sum_\pi \xi_\pi = 0$, which is not possible if we consider the ξ_π as being probabilities.

Proof. Note that, if we start in a random state, we can arrive at a given state using less then $\frac{n(n+1)}{2}$ steps, for example, to obtain $()_n$ from a random state it is enough to apply $\prod_{i=0}^{n-1} \prod_{j=0}^{i} e_{n-i+2j}$, where the product is ordered in the following way $\prod_{i=0}^{n-1} e_i = e_0 e_1 \dots e_{n-1}$.

This means that $H^{\frac{n(n+1)}{2}}$ is a matrix $c_n \times c_n$ with all entries positive. By the Perron–Frobenius theorem, there is a maximum eigenvalue, say r, such that all other eigenvalues satisfy the relation $|\lambda| < r$ and the correspondng eigenvector is the only one with all nonnegative entries.

Set $\xi(m) = H^m \xi$. We easily see that the sum of probabilities $\sum_\pi \xi_\pi(m)$, cannot vary. So, we can not have an eigenvalue bigger than 1 and, at the same time, there must be one eigenvalue which is 1. $\square$

1.2 The XXZ Spin Chain Model

Another interesting representation of the Temperley–Lieb algebra uses a spin space. Let $V = \left(\mathbb{C}^2\right)^{\otimes L}$ be the space of L spins. Furthermore define e_i as the operator that acts as the identity on all spins except the ones in positions i and $i+1$, where it is represented by the matrix:

$$e_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or, using the Pauli matrices[6]:

$$e_i = \frac{1}{2}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2}(\sigma_i^z \sigma_{i+1}^z - 1) + \frac{q-q^{-1}}{2}(\sigma_{i+1}^z - \sigma_i^z)\right).$$

If we sum over all L sites, considering periodic boundary conditions, we obtain the known XXZ Spin Chain Model in 1 dimension:

$$H = \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2}(\sigma_i^z \sigma_{i+1}^z - 1)\right).$$

This model has been heavily studied. In 1958, Orbach [59] found the exact solution of this model (for q real and positive) based on the work of Bethe for the XXX model [6]. For more details see Baxter's book [4].

1.2.1 The quantum algebra $U_q(su(2))$

The XXZ Spin Chain model can be explained using quantum algebra. Here, we will briefly introduce the quantum algebra $U_q(su(2))$, see [61]. Let q be a fixed complex number such that $q \neq 0$ and $q^2 \neq 1$.

Let the operators S^+, S^- and $q^{\pm S^z}$ obey the relations:

$$\begin{cases} q^{S^z} S^{\pm} q^{-S^z} & = q^{\pm 1} S^{\pm} \\ [S^+, S^-] & = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \end{cases}$$

Notice, that in the limit $q \to 1$ we obtain the Lie algebra $su(2)$. This algebra carries a Hopf algebra structure given by the coproduct:

$$\begin{cases} \Delta(q^{\pm S^z}) & = q^{\pm S^z} \otimes q^{\pm S^z} \\ \Delta(S^{\pm}) & = q^{-S^z} \otimes S^{\pm} + S^{\pm} \otimes q^{S^z} \end{cases}$$

[6] They are $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

the antipode:

$$\begin{cases} \mathcal{S}(q^{\pm S^z}) & = q^{\mp S^z} \\ \mathcal{S}(S^{\pm}) & = -q^{\mp 1} S^{\pm} \end{cases}$$

and the counit:

$$\begin{cases} \epsilon(q^{\pm S^z}) & = 1 \\ \epsilon(S^{\pm}) & = 0 \end{cases}$$

For a quantum spin chain of N spins (normally, we consider spins $s = \frac{1}{2}$), we have the operators:

$$\begin{cases} q^{\pm S^z} & = q^{\pm\sigma^z/2} \otimes \ldots \otimes q^{\pm\sigma^z/2} \\ S^{\pm} & = \sum_i q^{-\sigma^z/2} \otimes \ldots \otimes q^{-\sigma^z/2} \otimes \frac{\sigma_i^{\pm}}{2} \otimes q^{\sigma^z/2} \otimes \ldots \otimes q^{\sigma^z/2} \end{cases} \tag{1.6}$$

where $\sigma_i^{\pm}$ are the usual Pauli matrices, which act in the i^{th} position of the tensor product.

We can always define this operator for higher spin chains, by using this method to combine $2s$ spins and subsequently projecting onto the spin s representation.

The operator

$$S^2 = S^- S^+ + \left(\frac{q^{S^z+1/2} - q^{-S^z-1/2}}{q - q^{-1}} \right)^2 - \left(\frac{q^{1/2} - q^{-1/2}}{q - q^{-1}} \right)^2$$

generates the center of the algebra. This operator will play a role similar to the total spin in $su(2)$.

1.2.2 Translation between XXZ Spin Chain model and CPL

Let $L = 2n$. As in usual su_2, we can divide our vector space of dimension 2^{2n} into subspaces invariant under the action of the quantized algebra. We characterize our states by two quantum numbers, the total spin S and the spin in z, S_z, which commute with the XXZ Spin Chain Hamiltonian.

Take all configurations with $S = 0$ (and, consequently, $S_z = 0$). There are exactly c_n states in these conditions. As in su_2, we can build these states by putting together n singlets.

Here, we build these states doing the correspondence with the link patterns at the same time. It is a simple exercise to prove that the action of e_i for all i is equivalent on both sides.

Let $w = \sqrt{-q}$. The states we are looking for correspond to the link patterns. There we replace each arch ⌒ by a pair of spins $w{\uparrow\downarrow} + w^{-1}{\downarrow\uparrow}$. For example, for $n = 2$, there are two states with $S = 0$ and $S_z = 0$:

 $= w^2 \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow + w^{-2} \downarrow\uparrow\downarrow\uparrow$

 $= w^2 \uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\downarrow\uparrow + w^{-2} \downarrow\downarrow\uparrow\uparrow$

To check that these states belong to $S = 0$, it is enough to verify that S^+ annihilates all states:

Let π be a link pattern with arches $\{(l_1, m_1), \dots (l_n, m_n)\}$. S^+ can be seen as a sum of rising operators at the position i:

$$S^+ = \sum_{i=1}^{2n} q^{-\sigma^z/2} \otimes \dots \otimes q^{-\sigma^z/2} \otimes S_i^+ \otimes q^{\sigma^z/2} \otimes \dots \otimes q^{\sigma^z/2}$$

If we apply a pair of operators at positions l_i and m_i, we get zero.

Moreover, it can also be checked that, for generic w and n, the link patterns written in terms of spins are linearly independent, proving the correspondence.

Definition 1.6. *Let $\hat{\rho}$ be the rotation operator that picks the last spin and moves it to the first position, multiplying the configuration by $-q^{\sigma^z}$, where σ^z is applied to that spin.*

For example,

$$\hat{\rho}\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow = -q^{-1}\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow$$

It can be easily checked that this operator is equivalent to the one defined in equation (1.2), for $S^z = 0$. Thus we identify both operators.

1.3 The quantum Knizhnik–Zamolodchikov equation

As in the homogeneous case, for the CPL with vertical spectral parameters we expect that, if we apply the transfer matrix infinitely many times, the final state will converge to some $\Psi(y_1, \dots, y_{2n}) = \sum_\pi \Psi_\pi(y_1, \dots, y_{2n})\pi$. This is obviously true for the vertical spectral parameters corresponding to a Markov process.

To help us to solve this question we will introduce a tool which will give us the solution in the case $q = e^{2\pi i/3}$. Notice that for generic q things are more complicated.

1.3.1 The operator S_i

Let the operator S_i be a modified transfer matrix with horizontal spectral parameter y_i, see the example in Figure 1.9. There, the rotational symmetry is broken by multiplying the horizontal spectral parameter by s when we pass from the L^{th} to the first position. In term of the R matrices this means:

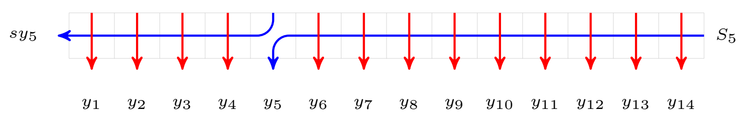


Figure 1.9: The operator S_i ($i = 5$ in this example) is like a transfer matrix with horizontal spectral parameter y_i but with the rotational symmetry broken.

$$S_i(y_1, \dots, y_{2n}) = R_1(y_1, sy_i) \dots R_{i-1}(y_{i-1}, sy_i) R_i(y_i, y_i) R_{i+1}(y_i + 1, y_i) \dots R_{2n}(y_{2n}, y_i).$$

The quantum Knizhnik–Zamolodchikov equation can be written as:

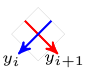
$$S_i(y_1, \ldots, y_i, \ldots, y_{2n})\Psi(y_1, \ldots, y_i, \ldots, y_{2n}) = \Psi(y_1, \ldots, sy_i, \ldots, y_{2n}). \tag{1.7}$$

This equation was introduced by Frenkel and Reshetikhin in [25] as a q-deformation of the Knizhnik–Zamolodchikov equations. s is a parameter of the equation, if we set $s = q^{2(k+l)}$ where $k = 2$ (technically this is the dual Coxeter number of the underlying quantum algebra $U_q(sl(2))$), then l is called the level of the equation. Here we set $l = 1$, so for $q = e^{2\pi i/3}$, equation (1.7) is only the application of the transfer matrix $T(y_i)$.

Another version of qKZ equation

In our work we do not use this equation directly. In its place we use a version introduced by Smirnov in [72] in the study of soliton form factors for the sine-Gordon model.

We introduce again the R matrix, but tilted by 45 degrees.



$$= \check{R}_i(y_i, y_{i+1}) = \frac{qy_{i+1} - q^{-1}y_i}{qy_i - q^{-1}y_{i+1}} Id + \frac{y_{i+1} - y_i}{qy_i - q^{-1}y_{i+1}} e_i.$$

Notice that this operator only depends on one parameter $\frac{y_{i+1}}{y_i}$, but this formulation will be useful for the rest. Normally, this operator is introduced multiplied by an overall factor $\phi(y_{i+1}/y_i)$, but for our study we will set it equal to 1.

Consider the following system of equations for homogeneous polynomials Ψ_π of the variables $y_1, \ldots, y_{2n}$ of degree δ:

- The *exchange* equation:

  $$\check{R}_i(y_i, y_{i+1})\Psi(y_1, \ldots, y_i, y_{i+1}, \ldots, y_{2n}) = \Psi(y_1, \ldots, y_{i+1}, y_i. \ldots, y_{2n}). \tag{1.8}$$

  for $i = 1, \ldots, 2n$.

- The *rotation* equation:

  $$\rho^{-1}\Psi(y_1, y_2, \ldots, y_{2n}) = \kappa\Psi(y_2, \ldots, y_{2n}, sy_1) \tag{1.9}$$

  where κ is a constant such that $\rho^{-2n} = 1$, so that $\kappa^{2n}s^\delta = 1$.

We can see that this system of equations implies equation (1.7). The converse is not true. Our goal is to find the minimal degree solutions for these equations at $s = q^6$.

1.4 Finding a solution

1.4.1 Building a solution

One way of solving this equation is to use directly the qKZ system. We start with the *exchange* equation as a system, and we re-arrange it in order to obtain a triangular system.

Partial order

First, we introduce a partial order on the link patterns, based on the Young diagrams representation. We say that one link pattern π is smaller than the other σ, we write $\pi \prec \sigma$, if and only if the corresponding Young diagrams satisfy $Y(\pi) \subset Y(\sigma)$. It can be easily checked that $\pi \prec \sigma$ and $\sigma \prec \eta$ implies $\pi \prec \eta$, as is required by the notion of partial order.

Notice that we could use other representations to define this partial order. For example:

- A Dyck path π is smaller than σ if and only if π is above σ. More precisely, let π_i be the vertical coordinate of the Dyck path at $x = i$. $\pi \preceq \sigma$ if $\pi_i \geq \sigma_i$ for all i;
- A sequence $a = (a_i, \dots, a_n)$ is smaller or equal to other b if and only if $a_i \leq b_i$ for all i.

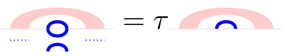
For link patterns and parentheses words, the partial order is harder to define, so we will always use one of these three representations. Note that the smallest link pattern is $()_n$ and the largest is $()^n$.

The qKZ equation as a triangular system

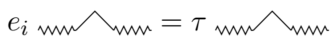
First of all, let us study the action of e_i on a generic π.

There are four different cases to consider:

- If π has a small arch between i and $i+1$, we get $e_i\pi = \tau\pi$. Graphically this means:

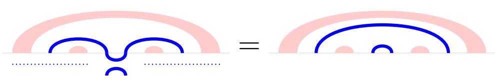
  

  or, in the Dyck path representation:

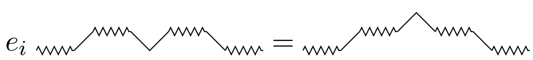
  

  The zigzag part is kept unchanged.

- If π has two arches (j, i) and $(i+1, k)$, where $j < i$ and $i + 1 < k$, $e_i\pi$ will have two arches (j, k) and $(i, i + 1)$. Let us see the picture:

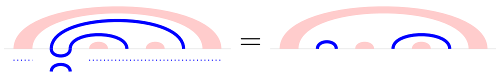
  

  or, in the Dyck path representation,

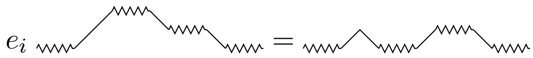
  

  Note that e_i deletes one box in the Young diagram.

- If π has two arches (i, j) and $(i+1, k)$, with $i+1 < j < k$, the action of e_i can be represented as:



In the Dyck path formalism, this translates into:



Notice that the operator e_i sends π to a bigger σ.

- Finally, if π has two arches (j, i) and $(k, i+1)$ with $k < j < i$, we obtain the symmetric case.

Notice that only in the second case $e_i\pi < \pi$. This is the central point of our method. Let us reformulate the *exchange* equation (1.8).

$$\sum_{\pi}\left(\frac{qy_{i+1} - q^{-1}y_i}{qy_i - q^{-1}y_{i+1}} - s_i\right)\Psi_\pi \pi = \sum_{\pi} \frac{y_i - y_{i+1}}{qy_i - q^{-1}y_{i+1}} \Psi_\pi e_i \pi$$

where s_i is the operator that exchanges y_i with y_{i+1}, *i.e.* $s_i f(y_i, y_{i+1}) = f(y_{i+1}, y_i)$. This is an equate between two vectors. So, we equalize the coefficients. That can be divided into two cases:

- If π does not have a little arch $(i, i+1)$:

$$\left(\frac{qy_{i+1} - q^{-1}y_i}{qy_i - q^{-1}y_{i+1}} - s_i\right)\Psi_\pi = 0 \tag{1.10}$$

But, Ψ_π is a polynomial, so, after some simple manipulation we obtain that $\Psi_\pi = (qy_i - q^{-1}y_{i+1})\Xi$, where Ξ is a polynomial symmetric in y_i and y_{i+1};

- If π has a little arch $(i, i+1)$:

$$(1 - s_i)\Psi_\pi = \sum_{\substack{\pi = e_i\pi' \\ \pi' \neq \pi}} \frac{y_i - y_{i+1}}{qy_i - q^{-1}y_{i+1}} \Psi_{\pi'}.$$

The last sum is over all π' such that $e_i\pi' = \pi$. Except one (the one when we delete a box in the Young diagram), all these π' are smaller than π. This allows us to write this system in a triangular form:

$$\Psi_{\tilde{\pi}} = \frac{qy_i - q^{-1}y_{i+1}}{y_i - y_{i+1}}(1 - s_i)\Psi_\pi - \sum_{\substack{\pi = e_i\pi' \\ \pi' < \pi}} \Psi_{\pi'}. \tag{1.11}$$

where $\tilde{\pi}$ is the Dyck path obtained from π by transforming the *mountain* at the i^{th} and $(i+1)^{\text{th}}$ steps in a *valley*.

Notice that equation (1.10) gives some constraints for the components Ψ_π. Moreover, equation (1.11) allows us to solve the qKZ equation starting from the smallest component $()_n$. Note also that the second equation transforms a polynomial with a certain degree into another polynomial with the same degree.

The process is better visualised in terms of Young diagrams. We start with the empty diagram $Y(()_n)$ and we construct $Y(\pi)$ adding boxes (using equation (1.11)) in such a way that in any intermediate step we have a Young diagram $Y(()_n) \subset Y \subset Y(\pi)$.

So, to solve the qKZ equation it is enough to find the polynomial $\Psi_{()_n}$. In order to proceed, we will need the following lemma, which generalizes equation (1.10):

Lemma 1.7. *If there are no arches between points in $i, i+1, \dots, j$ in π, then the product $\prod_{i\le k_1<k_2\le j}(qy_{k_1} - q^{-1}y_{k_2})$ divides Ψ_π.* I.e.

$$\Psi_\pi = \left(\prod_{i\le k_1<k_2\le j} (qy_{k_1} - q^{-1}y_{k_2}) \right) \phi_\pi$$

and ϕ_π is symmetric in $\{y_i, \dots, y_j\}$.

This is proved by induction on $j-i$ only using the symmetry property and equation (1.10):

Proof. The lemma is true for $j = i+1$. Assume now that it is true for $j-1$, by equation (1.10) $(qy_{j-1} - q^{-1}y_j)$ divides Ψ_π. But by symmetry in $y_1, \dots, y_{j-1}$, Ψ_π must be divisible by $\prod_{k=i}^{k=j-1}(qy_k - q^{-1}y_j)$.

The symmetry follows. □

Finally, using the fact that we want the solution with minimal degree, we get:

$$\Psi_{()_n} = \prod_{1\le i<j\le n} (qy_i - q^{-1}y_{i+1}) \prod_{n<i<j\le 2n} (qy_i - q^{-1}y_{i+1}).$$

We have chosen this normalization only for convenience. Notice that the total degree (in all y_i) is $\delta = n(n-1)$ and the partial degree (in each y_i) is $\delta_i = n-1$.

For example, the case $n = 3$:

$$\begin{aligned}
\Psi_{((()))} =& q^{-6}(q^2y_1 - y_2)(q^2y_1 - y_3)(q^2y_2 - y_3)(q^2y_4 - y_5)(q^2y_4 - y_6)(q^2y_5 - y_6)\\
\Psi_{(())()} =& q^{-8}(q^2y_1 - y_2)(q^4y_1 - y_6)(q^4y_2 - y_6)(q^2y_3 - y_4)(q^2y_3 - y_5)(q^2y_4 - y_5)\\
\Psi_{()(())} =& q^{-8}(q^4y_1 - y_5)(q^4y_1 - y_6)(q^2y_5 - y_6)(q^2y_2 - y_3)(q^2y_2 - y_4)(q^2y_3 - y_4)\\
\Psi_{()()()} =& -q^{-7}(q^4y_1 - y_6)(q^2y_2 - y_3)(q^2y_4 - y_5)\big((q^2y_1 - y_3)(q^2y_5 - y_6)(q^2y_2 - y_4)\\
&+q^{-2}(q^4y_2 - y_6)(q^4y_1 - y_4)(q^2y_3 - y_5)\big)\\
\Psi_{(()())} =& -q^{-7}(q^2y_5 - y_6)(q^2y_1 - y_2)(q^2y_3 - y_4)\big((q^4y_2 - y_6)(q^2y_4 - y_5)(q^2y_1 - y_3)\\
&+q^{-1}(q^4y_1 - y_5)(q^4y_3 - y_6)(q^2y_2 - y_4)\big)
\end{aligned}$$

It remains, however, a non trivial fact to prove, the consistency of our solution. *I.e.* if we choose such $\Psi_{()_n}$, the *rotation* equation (1.9) is well defined, and Ψ_π is independent of the order in which we add the boxes.

This can be done by checking that the Affine Temperley–Lieb algebra acts in the same way on both sides, up to duality. For this, we compute the action of the center of the Affine TL algebra on both representations (polynomials and link patterns). This will fix k and s. For the details of the proof see [84]. Here we will prove it using different methods, see section 1.5.

Wheel condition

In what follows, we prove that the polynomials Ψ_π obey an interesting condition, which allows us to characterize the vector space spanned by the Ψ_π.

Lemma 1.8. *The homogeneous polynomials Ψ_π obey the* wheel *condition,* i.e.

$$\Psi_\pi|_{y_k=q^2y_j=q^4y_i} = 0 \qquad \textit{for all } 1 \le i < j < k \le 2n$$

for all link patterns π.

Proof. Note that the rotation keeps intact this vanishing property, as $s = q^6$. It is easy to check that $\Psi_{()_n}$ obeys the *wheel* condition. So, the only thing we need to check is that the algorithm used to compute all ψ_π conserves this property. Consider the operator

$$\frac{qy_i - q^{-1}y_{i+1}}{y_i - y_{i+1}}(1 - s_i)$$

applied to a Ψ_π which obeys the *wheel* condition. Furthermore, consider the triplet $j < k < l$. Three cases can arise.

- If neither i nor $i + 1$ belong to the triplet, the result is obvious.
- If both belong to the triplet, the factor $(qy_i - q^{-1}y_{i+1})$ vanish.
- If only one belongs, say $i = j$, it's enough to use the original vanishing condition in $j + 1 < k < l$.

□

Recurrence formula

By equation (1.10), Ψ_π vanishes at $y_{i+1} = q^2y_i$ if π has a little arch $(i, i + 1)$. But, if π does not have a little arch $(i, i + 1)$, by the *wheel* condition, it factorizes into

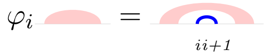
$$\prod_{j<i}(y_i - q^2y_j) \prod_{k>i+1}(y_k - q^4y_i)\phi.$$

Counting the degree of ϕ we discover that it has total degree $(n - 1)(n - 2)$ and partial degree $(n - 2)$. Moreover, it obeys the wheel condition if we ignore the variables y_i and y_{i+1}. This suggests that this is a polynomial of the same kind.

The actual relation is deeper. First, we must define the following operator:

Definition 1.9. *Let φ_i be an operator from the link patterns of size m to link pattern of size $m+1$, which acts by creating a little arch between positions i and $i+1$.*

Graphically:



Lemma 1.10. *We claim that the Ψ_π obey the following recurrence relation:*

$$\Psi_\pi\big|_{y_{i+1}=q^2y_i} = \begin{cases} 0 & \text{if} \pi \notin Im\varphi_i \\ (-q)^{-(n-1)} \prod_{j<i}(y_i - q^2y_j) \prod_{j>i+1}(q^3y_i - q^{-1}y_j)\ \Psi_{\pi'}\big|_{\hat{y}_i,\hat{y}_{i+1}} & \text{if} \pi = \varphi_i\pi' \end{cases}$$

where $\hat{y}_i$ means that this variable does not appear in the polynomial.

Proof. We will not prove it directly. Using the *rotation* equation, we see that this equation is equivalent to:

$$\Psi_{(\pi')}\big|_{y_1=q^{-4}y_{2n}} = (-1)^{n-1}q^{-4(n-1)} \prod_{j=2}^{2n-1} (y_{2n} - q^2y_j)\Psi_{\pi'}(y_2,\ldots,y_{2n-1}).$$

This is easy to check for $\pi' = ()_{n-1}$. It is enough to prove that the algorithm used to construct the solution of qKZ preserves this property. We see easily that we can construct any $\Psi_{(\pi')}$ with touching neither y_1 nor y_{2n}, as the factor $\prod_{j=2}^{2n-1}(y_{2n} - q^2y_j)$ is symmetric in y_j for $1 \leq j \leq 2n-1$. □

1.4.2 A vector space

These polynomials span a vector space $\mathcal{V}_n$ characterized by:

Definition 1.11. *The vector space $\mathcal{V}_n$ is the set of all homogeneous multivariate polynomials $P(y_1,\ldots,y_{2n})$ with total degree $\delta = n(n-1)$ and partial degree $\delta_i = n-1$, which obey the wheel condition:*

$$P\big|_{y_k=q^2y_j=q^4y_i} \quad \forall k > j > i.$$

The dimension of this vector space is the Catalan number c_n. We prove this dimension in two steps, first we find c_n linear independent polynomials and last we prove that all the polynomials in $\mathcal{V}_n$ must be linearly dependent with the former polynomials.

Call $q^\epsilon = (q^{\epsilon_1},\ldots,q^{\epsilon_{2n}})$ a specialization, where ϵ represents, also, a matching, by the following bijection:

$$\epsilon = \begin{cases} -1 & \text{if } i \text{ is an opening;} \\ 1 & \text{otherwise.} \end{cases}$$

These specializations are useful to prove the linear independence of the polynomials Ψ_π:

Lemma 1.12. *We can evaluate the c_n polynomials Ψ_π at the c_n specializations q^ϵ:*

$$\Psi_\pi(q^\epsilon) = (q - q^{-1})^{n(n-1)}\tau^{d(\pi)}\delta_{\pi,\epsilon},$$

where $d(\pi)$ is the number of boxes in the Young diagram $Y(\pi)$ and $\delta_{\pi,\epsilon}$ is one if π and ϵ represents the same matching, zero otherwise.

We use Lemma 1.10 to prove the above lemma.

Proof. Take the first small arch in ϵ, say $(i, i+1)$. We have $y_{i+1} = q^2 y_i$, so if π does not have a little arch at $(i, i+1)$ $\Psi_\pi(q^\epsilon)$ vanishes. Otherwise, we get

$$\Psi_\pi(q^\epsilon) = (q - q^{-1})^{2(n-1)} \tau^{g(\pi)} \Psi_{\pi'}(q^{\epsilon'}),$$

where $\pi = \varphi_i \pi'$ and $\epsilon = \varphi_i \epsilon'$ and $g(\pi)$ is the number of openings after the little arch. Easily, we see that $g(\pi)$ is the number of boxes in the first column of $Y(\pi)$

If we apply this procedure n times, we obtain a relation between $\Psi_\pi(q^\epsilon)$ and Ψ_0, where 0 means that there is no arches. Setting $\Psi_0 = 1$ we obtain the expected result. □

Corollary 1.13. *The polynomials Ψ_π are linearly independent.*

To complete the characterization of the space $\mathcal{V}_n$, we must prove that any polynomial in $\mathcal{V}_n$ is linearly dependent with the polynomials Ψ_π, or:

Lemma 1.14. *A polynomial in $\mathcal{V}_n$ is fully characterized by its values at the c_n specializations q^ϵ, where ϵ are all possible sets $\{\epsilon_1, \dots, \epsilon_{2n}\}$ with $\epsilon_i = \pm 1$ such that $\sum_{i=1}^{2n} \epsilon_i = 0$ and $\sum_{i=1}^{j} \epsilon_i \leq 0$ for all j between 1 and $2n$.*

In other words, if there is a polynomial independent of Ψ_π, there must exist a non zero polynomial which is zero for all specializations q^ϵ. We prove in Appendix A that such a polynomial does not exist, proving the dimension of the space. This proof is a replica of Appendix C of [23].

1.5 Contour integral formulæ

This way of computing the components of Ψ is not appropriate for our aim. Inspired by the work of Jimbo and Miwa [31], Razumov, Stroganov and Zinn-Justin proposed a multi-integral formula for Ψ in [67]. Here, we treat the even case, see [84].

At the end of this section we will prove that our former solution is consistent, *i.e.* is indeed a solution of the qKZ system.

Let $b = \{b_1, \dots, b_n\}$, with $1 \leq b_i < b_{i+1} \leq 2n$, be a spin configuration, where b_i is the position of the i^{th} spin up.

Let $\Psi = \sum_b \Psi_b b$, where Ψ_b is given by the contour integral formula:

$$\Psi_b(y_1, \dots, y_{2n}) = (-q)^{n/2} (q - q^{-1})^n \prod_{i<j}^{2n} (q y_i - q^{-1} y_j)$$
$$\oint \dots \oint \prod_{i=1}^{n} \frac{w_i dw_i}{2\pi i} \frac{\prod_{j>i}^{n} (w_j - w_i)(q w_i - q^{-1} w_j)}{\prod_{j=1}^{b_i} (w_i - y_j) \prod_{j=b_i}^{2n} (q w_i - q^{-1} y_j)} \qquad (1.12)$$

where all the variables are complex and the contours surround the poles at $w_i = y_j$ but not those at $w_i = q^{-2} y_j$ for all w_i and y_j. In practice the use of the contour integral is only formal and the real meaning is that we pick the residues at $w_i = y_j$ but not those at $w_i = q^{-2}y_j$. We claim that this vector solves the qKZ equation.

In order to prove that these formulæsolve qKZ, we introduce a new formula:

$$\bar{\Psi}_{\bar{b}}(y_1, \ldots, y_{2n}) = (-q)^{-n/2}(q - q^{-1})^n \prod_{i<j}^{2n}(qy_i - q^{-1}y_j)$$
$$\oint \ldots \oint \prod_{i=1}^{n} \frac{y_{\bar{b}_i} dw_i}{2\pi i} \frac{\prod_{j>i}^{n}(w_j - w_i)(qw_i - q^{-1}w_j)}{\prod_{j=1}^{\bar{b}_i}(w_i - y_j)\prod_{j=\bar{b}_i}^{2n}(qw_i - q^{-1}y_j)} \quad (1.13)$$

where the integral is performed around the poles $w_i = y_j$, and $\bar{b}$ are the positions of the spins down.

Here, we prove that:

- Both expressions solve the *exchange* equation (1.8);
- Indeed, they are the same: $\Psi_b = \bar{\Psi}_{\bar{b}}$;
- They satisfy the *rotation* equation (1.9);
- They belong to the vector space $\mathcal{V}_n$;
- They are exactly the same as computed by the former method.

1.5.1 The action of the qKZ equation

In this section we check the action of the qKZ equation.

Proposition 1.15. *Both expressions* $\Psi = \sum_b \Psi_b b$ *and* $\bar{\Psi} = \sum_{\bar{b}} \bar{\Psi}_{\bar{b}}\bar{b}$ *obey the* exchange *equation:*

$$\check{R}_i\Psi = s_i\Psi$$
$$\check{R}_i\bar{\Psi} = s_i\bar{\Psi}$$

Proof. Consider the first one. The equation $\check{R}_i(y_i, y_{i+1})\Psi = s_i\Psi$, can be seen as an equation in the components Ψ_b. In this perspective, we distinguish four different cases.

If both positions i and $i+1$ are occupied by a spin up, *i.e.* there is an l such that $b_l = i$ and $b_{l+1} = i+1$, the equation (1.8) says that:

$$s_i\Psi_b - \frac{qy_{i+1} - q^{-1}y_i}{qy_i - q^{-1}y_{i+1}}\Psi_b = 0. \quad (1.14)$$

Picking only the terms which depend on y_i, y_{i+1}, w_l and w_{l+1} and are not obviously symmetric, we get an integral antisymmetric between w_l and w_{l+1}, proving (1.14).

If both positions i and $i+1$ are occupied by spins down, the equation (1.14) is still valid. So, we only need to prove that $\Psi_b/(qy_i - q^{-1}y_{i+1})$ is symmetric in the exchange $y_i \leftrightarrow y_{i+1}$, which is obvious when looking at the integral formula.

For the two remaining cases, let $\hat{b} = \{b_1, \dots, b_l = i, \dots, b_n\}$ be a configuration with a spin up at the position i and a spin down at position $i+1$. Let $\tilde{b} = \{b_1, \dots, b_l = i+1, \dots, b_n\}$ be the same configuration but with spin down at i^{th} position and spin up at $(i+1)^{\text{th}}$ position. The last two cases read:

$$s_i \Psi_{\hat{b}} = \frac{qy_{i+1} - q^{-1}y_i}{qy_i - q^{-1}y_{i+1}} \Psi_{\hat{b}} - q \frac{y_{i+1} - y_i}{qy_i - q^{-1}y_{i+1}} \Psi_{\hat{b}} + \frac{y_{i+1} - y_i}{qy_i - q^{-1}y_{i+1}} \Psi_{\tilde{b}} \tag{1.15}$$

$$s_i \Psi_{\tilde{b}} = \frac{qy_{i+1} - q^{-1}y_i}{qy_i - q^{-1}y_{i+1}} \Psi_{\tilde{b}} - q^{-1} \frac{y_{i+1} - y_i}{qy_i - q^{-1}y_{i+1}} \Psi_{\tilde{b}} + \frac{y_{i+1} - y_i}{qy_i - q^{-1}y_{i+1}} \Psi_{\hat{b}} \tag{1.16}$$

After some easy and tedious computations we check that the integrand is equal on both sides. For this it is enough to look at the part which depends on y_i, y_{i+1} and w_l, proving the required result.

The case $\bar{\Psi}$ is proven exactly in the same way. □

Proposition 1.16. *The two expressions are the same:*

$$\Psi = \bar{\Psi}$$

Proof. The case corresponding to the spin chain with all spins up to the left and all spins down to the right, *i.e* $b = \{1, \dots, n\}$ and $\bar{b} = \{n+1, \dots, 2n\}$ is quite easy to compute: Ψ_b is a direct computation; for $\bar{\Psi}_{\bar{b}}$, noting that there are no poles at infinity, one can perform the integral surrounding $w_i = q^{-2}y_j$. In both computations, the result is:

$$\Psi_{\{1,\dots,n\}} = \bar{\Psi}_{\{n+1,\dots,2n\}} = (-q)^{n/2} \prod_{1 \le i < j \le n} (qy_i - q^{-1}y_j) \prod_{n < i < j \le 2n} (qy_i - q^{-1}y_j)$$

This is the same polynomial up to a multiplicative constant as in $\Psi_{()_n}$. It is easy to see that each component of Ψ can be obtained from $\Psi_{\{1,\dots,n\}}$ using the *exchange* equation (1.15):

$$\Psi_{\{\dots,i+1,\dots\}} = \frac{1}{y_{i+1} - y_i} (s_i - 1)(qy_{i+1} - q^{-1}y_i) \Psi_{\{\dots,i,\dots\}} + q\Psi_{\{\dots,i,\dots\}} \tag{1.17}$$

and equivalently for $\bar{\Psi}_{\bar{b}}$.

This proves the equality $\Psi_b = \bar{\Psi}_{\bar{b}}$.

Note that this also proves that Ψ_b are homogeneous polynomials with total degree $\delta = n(n-1)$ and partial degree $\delta_i = n-1$, because, $\Psi_{\{1,\dots,n\}}$ has these properties, and equation (1.17) preserves all these properties. □

Notice that the Ψ_b are homogeneous polynomials with total degree $\delta = n(n-1)$, so we have $\kappa = q^{-3(n-1)}$. Thus the *rotation* equation reads:

Proposition 1.17. *Equation* (1.9)*, translates in term of components as:*

$$\Psi_{\{b_1,\dots,b_{n-1},b_n\}}(y_2,\dots,y_{2n},q^6y_1) = \begin{cases} -q^{3n-4}\Psi_{\{1,b_1+1,\dots,b_{n-1}+1\}}(y_1,y_2,\dots,y_{2n}) & \text{if } b_n = 2n \\ -q^{3n-2}\Psi_{\{b_1+1,\dots,b_{n-1}+1,b_n+1\}}(y_1,y_2,\dots,y_{2n}) & \text{otherwise} \end{cases} \tag{1.18}$$

Or, using the other integral formulæ:

$$\bar\Psi_{\{\bar b_1,\dots,\bar b_{n-1},\bar b_n\}}(y_2,\dots,y_{2n},q^6y_1) = \begin{cases} -q^{3n-2}\bar\Psi_{\{1,\bar b_1+1,\dots,\bar b_{n-1}+1\}}(y_1,y_2,\dots,y_{2n}) & \text{if } \bar b_n = 2n \\ -q^{3n-4}\bar\Psi_{\{\bar b_1+1,\dots,\bar b_{n-1}+1,\bar b_n+1\}}(y_1,y_2,\dots,y_{2n}) & \text{otherwise} \end{cases} \tag{1.19}$$

Proof. The first case of (1.18) is equivalent to the second case of (1.19) and the second case of (1.18) is equivalent to the first case of (1.19). So, it is only necessary to prove either both first cases or both second cases.

Let us perform the proof of the first case of the equation (1.18). Noting that there is no pole at infinity, we can invert the domain of integration of the l.h.s. in w_n. We obtain (ignoring the terms which do not depend on y_1):

$$\Psi_{\{\dots,2n\}}(y_2,\dots,q^6y_1) = -(\dots)\prod_{i=2}^{2n}(qy_i - q^5y_1)$$
$$\oint\dots\oint\prod_{i=1}^{n-1}\frac{w_i dw_i}{2\pi i}\frac{q^3y_1\prod_{i=1}^{n-1}(q^4y_1-w_i)(qw_i-q^3y_1)(\dots)}{(q^4y_1-q^6y_1)\prod_{j=2}^{2n}(q^4y_1-y_j)\prod_{i=1}^{n-1}(qw_i-q^5y_1)(\dots)} \tag{1.20}$$

Integrating the r.h.s. around the pole $w_1 = y_1$, we get (again ignoring the terms which not depend on y_1):

$$\Psi_{\{1,\dots\}}(y_1,\dots,y_{2n}) = -q^{3n-4}(\dots)\prod_{i=2}^{2n}(qy_1 - q^{-1}y_i)$$
$$\oint\dots\oint\prod_{i=2}^{n}\frac{w_i dw_i}{2\pi i}\frac{y_1\prod_{i=2}^{n}(w_i-y_1)(qy_1-q^{-1}w_i)(\dots)}{\prod_{i=2}^{n}(w_i-y_1)\prod_{j=1}^{2n}(qy_1-q^{-1}y_j)}. \tag{1.21}$$

The parts $(\dots)$ are identical in both expressions. Note that there is a shift in the integration ($\prod_{i=1}^{n-1}$ in the l.h.s. and $\prod_{i=2}^{n}$ in the r.h.s.) which compensates the shift in the state ($\{b_1,\dots,b_{n-1},2n\}$ in the l.h.s. and $\{1,b_1+1,\dots,b_{n-1}+1\}$ in the r.h.s. Now, it is enough to check the equality of both expressions.

The same method can be applied to the first equation of (1.19), proving that these integral formulæsolve the qKZ equation. □

1.5.2 Space

A notable aspect is that:

Proposition 1.18. *The polynomials Ψ_b belong to the vector space $\mathcal{V}_n$.*

Proof. We saw in section 1.4.2 that the space $\mathcal{V}_n$ consists of homogeneous polynomials with total degree $\delta = n(n-1)$, partial degree $\delta = n-1$ which obey the wheel condition. Everything has been proven, except the wheel condition.

Being a Cauchy integral, the expression (1.12) can be easily solved by picking all possible combinations of the poles inside of the domain:

$$\Psi_b(y_1,\dots,y_{2n}) = (-q)^{n/2}(q-q^{-1})^n \prod_{i<j}^{2n}(qy_i - q^{-1}y_j) \sum_{\substack{k_1,k_2,\dots,k_n \\ k_i\neq k_j \text{ if } i\neq j \\ k_i\leq b_i}} \prod_{i=1}^{n} \frac{y_{k_i}\prod_{j>i}^{n}(y_{k_j}-y_{k_i})(qy_{k_i}-q^{-1}y_{k_j})}{\prod_{\substack{j=1\\ j\neq k_i}}^{b_i}(y_{k_i}-y_j)\prod_{j=b_i}^{2n}(qy_{k_i}-q^{-1}y_j)} \tag{1.22}$$

Consider $y_k = q^2 y_j = q^4 y_i$ with $k > j > i$. In order to have a non-vanishing result, we need to choose the terms with $(qy_i - q^{-1}y_j)$ and $(qy_j - q^{-1}y_k)$ in the denominator in the sum in expression (1.22), all other terms vanish. For this we need to choose $k_l = i$, which implies $i = k_l \leq b_l \leq j$. And also, $k_m = j$, so $j = k_m \leq b_m \leq k$. This implies $b_l \leq b_m$, *i.e.* $l \leq m$, but the two poles are distinct so $l < m$. Which will create a term in the denominator $(qw_{k_l} - q^{-1}w_{k_m}) = (qy_i - q^{-1}y_j)$. So, there is no possible choice of the poles that avoid both zeros, *i.e.* all Ψ_b vanish for $y_k = q^2 y_j = q^4 y_i$. □

1.5.3 Link Patterns

We saw in Section 1.2.2 that the link patterns π generate the subspace $S = 0$. But the spins b live in the subspace $S^z = 0$ (notice that $S = 0$ is a subspace of $q^{S^z} = 1$), which is larger. So, we need to prove that:

Proposition 1.19. *The vector $\sum_b \Psi_b(z_1,\dots,z_{2n})b$ lives in the subspace $S = 0$,* i.e. *there are some functions $\hat{\psi}_\pi(z_1,\dots,z_{2n})$ such that:*

$$\sum_b \Psi_b b = \sum_\pi \hat{\psi}_\pi \pi.$$

Proof. We know that the Ψ_b live in $\mathcal{V}_n$. Let ψ_π be a basis of $\mathcal{V}_n$ characterized by $\psi_\pi(q^\epsilon) = \delta_{\pi,\epsilon}$. So, we have:

$$\begin{aligned}\Psi &= \sum_b \sum_\pi \psi_\pi P_{\pi,b} b \\ &= \sum_\pi \psi_\pi \left(\sum_b P_{\pi,b} b\right)\end{aligned}$$

for some $P_{\pi,b}$.

If we use the specialization q^ϵ, we obtain:

$$\sum_b \Psi_b(q^\epsilon) b = \sum_b P_{\epsilon,b} b.$$

A direct computation of $\Psi_b(q^\epsilon)$ is quite tedious, instead we will use the *rotation* equation (1.9). Assume that ϵ has a little arch $(1,2)$, that is $(q^{\epsilon_1}, q^{\epsilon_2}) = (q^{-1}, q)$. There are four different cases:

- The configuration b starts with two up spins, *i.e.* $(b_1, b_2) = (1,2)$. So, there is only one choice in (1.22): $k_1 = 1$ and $k_2 = 2$. But, this case implies that there is always a term $(qy_1 - q^{-1}y_2)$ which vanishes for this specialization.

- The configuration starts with two down spins, that is $b_1 > 2$. So, in any case it is not possible to avoid the term $(qy_1 - q^{-1}y_2)$ in the numerator of (1.22). In conclusion, it vanishes.

- The configuration starts with a spin up and a spin down (in this order), that is $b_1 = 1$ and $b_2 > 2$. If we perform the integration in w_1 we obtain:

$$\Psi_{\{1,\hat{b}_1+2,\ldots,\hat{b}_{n-1}+2\}}(q^{-1}, q, y_3, \ldots, y_{2n}) = (-q)^{1/2+(n-1)} \prod_{i=3}^{2n} (q - q^{-2} y_i) \Psi_{\hat{b}}(y_3, \ldots, y_{2n}) \quad (1.23)$$

  where $\hat{b}_i$ is the positions of the i^{th} spin up in b ignoring the first one.

- The last case is when $b_1 = 2$. Here, there are two choices (in equation (1.22)), either $k_1 = 1$ or $k_1 = 2$, as the second choice (1.22) vanishes. So, we are forced to pick the pole $w_1 = y_1$, getting:

$$\Psi_{\{2,\hat{b}_1+2,\ldots,\hat{b}_{n-1}+2\}}(q^{-1}, q, y_3, \ldots, y_{2n}) = (-q)^{-1/2+(n-1)} \prod_{i=3}^{2n} (q - q^{-2} y_i) \Psi_{\hat{b}}(y_3, \ldots, y_{2n}) \quad (1.24)$$

Using the *rotation* equation (1.2) we can get a generalized recurrence:

$$\Psi_b(y_1, \ldots, y_{i-1}, q^{-1}, q, y_{i+2}, \ldots, y_{2n}) = (-1)^{i-1} \prod_{j=1}^{i-1} (q^{-2} - q y_j) \prod_{j=i+2}^{2n} (q - q^{-2} y_j)$$
$$\begin{cases} 0 & \text{if } i, i+1 \in b \\ (-q)^{1/2} \Psi_{\hat{b}}(y_1, \ldots, y_{i-1}, y_{i+2}, \ldots, y_{2n}) & \text{if } i \in b \text{ and } i+1 \notin b \\ (-q)^{-1/2} \Psi_{\hat{b}}(y_1, \ldots, y_{i-1}, y_{i+2}, \ldots, y_{2n}) & \text{if } i \notin b \text{ and } i+1 \in b \\ 0 & \text{if } i, i+1 \notin b \end{cases}$$

where $\hat{b}$ is the spin configuration without the spins in positions i and $i+1$.

Now we can compute the $P_{\epsilon,b}$. But we do not need the explicit result, instead notice that the only dependence on b is in the factor $(-q)^{\pm 1/2}$. So, by recurrence, each arch in ϵ will correspond to a singlet:

$$\frown = (-q)^{1/2} \uparrow\downarrow + (-q)^{-1/2} \downarrow\uparrow,$$

which is exactly what "defines" an $S = 0$ state. It is now clear that $\sum_b P_{\pi,b} = S_\pi \pi$, where S_π is some constant. Moreover $\hat{\psi}_\pi = S_\pi \psi_\pi$ is only a new normalization of the polynomials ψ_π. □

Finally, we can prove that:

Proposition 1.20. *The vector computed by the integral formulæ $\sum_b \Psi_b b$ is the same as computed in Section 1.4:*

$$\sum_b \Psi_b b = \sum_\pi \Psi_\pi \pi$$

Proof. The proof is straightforward. In Proposition 1.19 we proved that $\sum_b \Psi_b b = \sum_\pi \hat{\psi}_\pi \pi$. So, this set of polynomials must solve the triangular system. As they have the same degree, they must be the same up to a multiplicative factor. Computing the coefficient of $b = \{1, \dots, n\}$ we discover that the multiplicative constant is 1. □

This leads us to the important result:

Corollary 1.21. *The quantities Ψ_π computed in Section 1.4 solve the qKZ system.*

1.6 A third basis

Although we have already two ways of computing the vector $\sum_\pi \Psi_\pi \pi$, none of them is suited for the computations we want to perform. In this section we present a third method used in the article [17]. This basis is often cited as intermediate basis as it is a slightly modified version of the one presented in the previous section and it is indexed by link patterns.

1.6.1 A new integral formula

Let $a = \{a_1, \dots, a_n\}$, where $a_i \geq a_{i-1}$ for all $n \geq i > 1$, $a_n \leq 2n$ and $a_1 \geq 1$.

We slightly modify the equation (1.12), obtaining:

$$\Phi_a(y_1, \dots, y_{2n}) = \prod_{i<j}^{2n} (q y_i - q^{-1} y_j) \oint \dots \oint \prod_i \frac{dw_i}{2\pi i} \frac{\prod_{j>i}^{n} (w_j - w_i)(q w_i - q^{-1} w_j)}{\prod_{j \leq a_i} (w_i - y_j) \prod_{j > a_i} (q w_i - q^{-1} w_j)} \quad (1.25)$$

where the integration contour surrounds the poles $w_j = y_i$ but not the poles $w_j = q^{-2}y_i$ (for all pairs i, j).

We claim that these polynomials live in the space $\mathcal{V}_n$. Before proving that, we prove some propositions.

Proposition 1.22. *The quantities $\Phi_a(y_1, \ldots, y_{2n})$ are indeed polynomials.*

Proof. Using the residue techniques we obtain the expression, identical to (1.22):

$$\Phi_a(y_1, \ldots, y_{2n}) = \prod_{i<j}^{2n}(qy_i - q^{-1}y_j)$$
$$\sum_{\substack{k_1,k_2,\ldots,k_n \\ k_i \neq k_j \text{ if } i\neq j \\ k_i \leq b_i}} \prod_{i=1}^{n} \frac{\prod_{j>i}^{n}(y_{k_j} - y_{k_i})(qy_{k_i} - q^{-1}y_{k_j})}{\prod_{\substack{j\leq a_i \\ j\neq k_i}}(y_{k_i} - y_j)\prod_{j>a_i}^{2n}(qy_{k_i} - q^{-1}y_j)}. \tag{1.26}$$

To prove that this expression is a polynomial, it is enough to prove that there are no poles after performing the sum.

We start with the computation of the residue at $y_{k_i} \to y_s$, with $k_i = r$. To have a non zero residue it is necessary that $s \leq a_i$ and that $s \neq k_j$ for all k_j. Notice that $k_i \leq a_i$, so the term $k_i = s$ and $r \neq k_j$ for all j is also part of the sum. If we sum both and compute the limit $y_r \to y_s$, we get the same expression aside of the sign coming from the pole itself. *I.e.* the residue vanishes and there is no pole.

The residues at $y_{k_i} \to q^{-2}y_j$, with $j > a_i \geq k_i$ cancel with the first factor $\prod_{j>i}(qy_i - q^{-1}y_j)$. □

The second property that we need in order to proceed is the degree of the polynomial:

Proposition 1.23. *The polynomials $\Phi_a(y_1, \ldots, y_{2n})$ are homogeneous polynomials with total degree $\delta = n(n-1)$ and partial degree $\delta_i = n-1$.*

Proof. This follows directly from the integral formula, or, equivalently, from equation (1.26). □

Proposition 1.24. *They all obey the wheel condition:*

$$\Phi_a|_{y_k=q^2y_j=q^4y_i} = 0 \qquad \textit{for all } 1 \leq i < j < k \leq 2n.$$

Proof. We follow exactly the same steps as in Section 1.5.2. □

And finally, the main claim:

Lemma 1.25. *Let $a = \{a_1, \ldots, a_n\}$ be a sequence such that $a_{i+1} > a_i$ for all $1 < i \leq n$, $a_i \leq 2i-1$ and $a_1 = 1$. There are exactly c_n such integrals and they span our space $\mathcal{V}_n$.*

Proof. The proof of the first part is straightforward, as these sequences are in bijection with matchings.

The second part is harder. As they live in $\mathcal{V}_n$, we can write:

$$\Phi_a(y_1,\ldots,y_{2n}) = \sum_\pi C_{a,\pi}\Psi_\pi(y_1,\ldots,y_{2n}).$$

Where $C_{a,\pi}$ are some coefficients.

To compute $C_{a,\pi}$ we only need to evaluate Φ_a at the point q^ϵ, as:

$$\begin{aligned}\Phi_a(q^\epsilon) &= \sum_\pi C_{a,\pi}\Psi_\pi(q^\epsilon)\\ &= C_{a,\epsilon}(q-q^{-1})^{n(n-1)}\tau^{d(\epsilon)}.\end{aligned}$$

In Section 1.6.2 we compute explicitly the coefficients. Using a global order which respects the partial order defined in Section 1.4, this matrix $C_{a,\pi}$ is triangular with ones at the diagonal, so it can be inverted, which proves the lemma. □

1.6.2 Basis transformation

We assume that the Φ_a are linearly independent. So we can define the basis transformation in the other direction:

$$\Psi_\pi(y_1,\ldots,y_{2n}) = \sum_a C^{-1}_{\pi,a}\Phi_a(y_1,\ldots,y_{2n}).$$

Note that the polynomials Ψ_π and Φ_a can be multiplied by some constant without changing their characteristics. With our definitions:

$$\Psi_{()}(z_1,z_2) = \Phi_1(z_1,z_2) = 1.$$

Calculating the coefficients $C_{a,\pi}$

We know that these coefficients are fully determined by the c_n points (q^ϵ). Therefore we only need these c_n points. We will allow all $a = (a_1,\ldots,a_n)$ such that $a_i \le a_{i+1}$ with $1 \le a_i \le 2i-1$ for all i.

We pick a little arch in ϵ, say between i and $i+1$. If there are no $a_j = i$ the $\Phi_a(q^\epsilon)$ is zero (see equation (1.26)). So, a must be of the form

$$a = (a_1,\ldots,\underbrace{i,\ldots,i}_{k},\ldots,a_n)$$

After a tedious computation we obtain:

$$\Phi_a(q^\epsilon) = (-q)^{-(n-1)}\prod_{j<i}(q^{-1}-q^2q^{\epsilon_j})\prod_{j\ge i+2}(q^2-q^{-1}q^{\epsilon_j})U_{k-1}\Phi_{\hat{a}}(q^{\hat{\epsilon}}) \tag{1.27}$$

where $\hat{a} = (a_1, \ldots, \underbrace{i-1, \ldots, i-1}_{k-1}, \ldots, a_n - 2)$ and

$$U_k = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}} = q^k + q^{k-2} + \ldots + q^{-k+2} + q^{-k}$$

is a polynomial in τ of degree k.

Observe that the prefactor is exactly the same as in Lemma 1.10. So, we get:

$$C_{a,\pi} = U_{k-1} C_{\hat{a},\hat{\pi}} \tag{1.28}$$

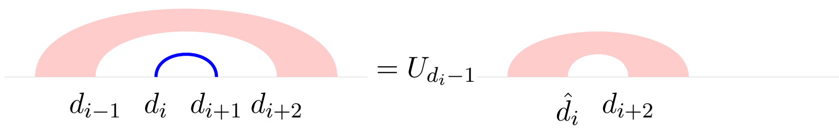
Where $\hat{\pi}$ is the link pattern π without the arch $(i, i+1)$, *i.e.* $\pi = \varphi_i \hat{\pi}$. The fact that $C_{1,()} = 1$ provides an inductive method of calculation.

There is to our knowledge no direct method to compute $C^{-1}_{\pi,a}$ explicitly. We shall use an Ansatz later for some entries of C^{-1} and confirm it by checking it at all values of the form (q^{ϵ}).

Graphical method

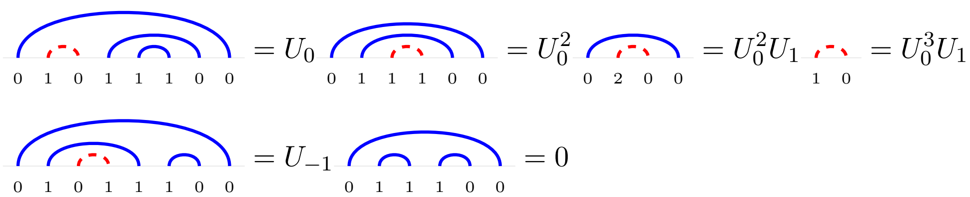
In this section we provide a graphical method to compute the coefficients $C_{a,\pi}$. This method is a simple consequence of equation (1.28) and can be seen in article [17].

Let d_i count the number of $a_j = i$ for all j, and let π be a link pattern with a little arch $(i, i+1)$. Equation (1.28) says that we can take out the little arch, at the same time as $d = (d_1, \ldots d_{i-1}, d_i, d_{i+1}, d_{i+2}, \ldots d_{2n})$ get transformed into $\hat{d} = (d_1, \ldots, d_{i-1} + d_i + d_{i+1} - 1, d_{i+2} \ldots d_{2n})$. Graphically:



where $\hat{d}_i = d_{i-1} + d_i + d_{i+1} - 1$.

Notice that $U_{-1} = 0$, so if we have a little arch $(i, i+1)$ and $d_i = 0$ the coefficient vanishes. Let us perform two examples:



where the red dashed lines mark the little arch chosen in the process.

Triangularity of the transformation of basis

Now we pick some strictly increasing set $a = (a_1, \dots, a_n)$ such that $a_i \leq 2i - 1$ and the set of openings in the matching π called $(\pi_1, \dots, \pi_n)$, in this notation, we see that the two objects are essentially the same. We define a partial order: $\pi \leq a$ if and only if $\pi_i \leq a_i$ for all i. We claim that:

Lemma 1.26. *For $C_{a,\pi}$ to be different from zero, π must be smaller or equal to a. If $\pi_i = a_i$ for all i, $C_{a,\pi} = 1$.*

Proof. In order to prove this we shall use the geometric method presented above. We pick an arch in π, going between i and j. If we use the geometric method of reduction, we see that we need at least $(j-i)/2$ a_k such that $i \leq a_k < j$, applying this in all arches in π we see that we need $\pi \leq a$.

The statement for $\pi = a$ follows easily by induction, for a reduction at a small arch of π. □

So, if we fix a total order which respects the partial order defined before, $C_{a,\pi}$ is a triangular matrix with 1's on the diagonal.

The two facts together prove that the transformation is invertible, therefore the Φ_a are linearly independent.

Coefficients as polynomials of τ

The study of these coefficients leads to the following property:

Lemma 1.27. *Let $Y(a)$ be the Young diagram $(\lambda_1, \dots, \lambda_r)$ corresponding to the sequence a with $a_i = \lambda_{n-i+1} + i$, and let $Y(\pi)$ be the Young diagram corresponding to the matching π.*

The coefficients $C_{a,\pi}$ are polynomials in τ, more precisely:

$$C_{a,\pi} = \begin{cases} 0 & \text{if } Y(\pi) \not\subseteq Y(a) \\ 1 & \text{if } Y(\pi) = Y(a) \\ P_{a,\pi}(\tau) & \text{if } Y(\pi) \subset Y(a) \end{cases} \tag{1.29}$$

where $P_{a,\pi}(\tau)$ is a polynomial of τ with degree $\delta_{a/\pi} \leq |Y(a)| - |Y(\pi)| - 2$.

Moreover, we have:

$$C_{a,\pi}(-\tau) = (-1)^{d(a)-d(\pi)} C_{a,\pi}(\tau).$$

We leave the proof of this lemma to Appendix B.

Observe that this lemma remains true for the coefficients $C^{-1}_{\pi,a}$, by using the triangularity of C and the fact that the diagonal elements are one.

1.6.3 The homogeneous limit

The limit $y_i = 1$ for all i will be of central interest in this work.

We will use lower case (ϕ_a and ψ_π) to represent the limit of the polynomials in $\{y_1, \dots, y_{2n}\}$, denoted by capital letters (Φ_a and Ψ_π, respectively).

Using the transformation of variables:

$$u_i = \frac{w_i - 1}{q w_i - q^{-1}},$$

we obtain the formula:

$$\phi_a = \oint \dots \oint \prod_i \frac{du_i}{2\pi i u_i^{a_i}} \prod_{j>i} (u_j - u_i)(1 + \tau u_j + u_i u_j),$$

where we re-normalize the state, *i.e.* divide it by $(q - q^{-1})^{n(n-1)}$, such that $\phi_{1,2,\dots,n} = 1$.

It is obvious that all ϕ_a are polynomials in τ with integer coefficients. Now, to obtain the components ψ_π we only need to know the $C^{-1}_{\pi,a}$:

$$\psi_\pi = \sum_a C^{-1}_{\pi,a} \phi_a,$$

which are also polynomials in τ with integer coefficients.

Notice that if we let $\tau = 1$ we are computing the groundstate of the CPL model without parameters. For general τ we do not have a combinatorial interpretation, besides being the solutions of the qKZ equation at level one.

1.7 An example

Even with these three methods, the computation of the groundstate of the XXZ Spin Chain model is still not trivial. Let us see an easy example.

Consider the configuration $\pi = ((\,)_\alpha (\,)_\beta)_m$, and call $n = m + \alpha + \beta$

Proposition 1.28. *For* $a = \{1, \dots, m, \underbrace{m+\alpha, \dots, m+\alpha}_{\alpha}, \underbrace{m+2\alpha+\beta, \dots, m+2\alpha+\beta}_{\beta}\}$ *we have:*

$$\Phi_a(y_1, \dots, y_{2n}) = \left(\prod_i^\alpha U_{i-1}\right)\left(\prod_i^\beta U_{i-1}\right) G \Psi_\pi(y_1, \dots, y_{2n}),$$

or, in the homogeneous limit:

$$\phi_a = \prod_i^\alpha U_{i-1} \prod_i^\beta U_{i-1} \psi_\pi$$

This proposition says that it is enough to know how to compute Φ_a in order to compute Ψ_π. Here we will only do the homogeneous case, the inhomogeneous case is probably possible, although harder.

Proof. We know that $\Phi_a = \sum_\sigma C_{a,\sigma}\Psi_\sigma$, so we only need to prove that $C_{a,\sigma}$ is zero if $\sigma \neq \pi$ and $C_{a,\pi} = \prod_i^\alpha U_{i-1} \prod_i^\beta U_{i-1}$.

Imagine that σ has a closing at a position between 1 and $m+\alpha$, this implies that there are more than β openings after position $m+\alpha$, but any opening must have at least one a_i associated and there are only β entries a_i larger than $m+\alpha$. So, we have that the first $m+\alpha$ positions are openings. There are no $a_i > m+2\alpha+\beta$, so we must have $m+\beta$ closings at the end.

Imagine now that σ has a closing at position $m+2\alpha+\beta$, this closing must link to an opening before $x < m-\beta$ in such a way that the number of $x \leq a_i < m+2\alpha+\beta$ is greater than half of the size of the arch $m+2\alpha+\beta-x$. But this implies that outside this arch there are at most $m-\beta-2$ openings for $m+\beta$ closings which is absurd. So, σ must have an opening at $m+2\alpha+\beta$. Applying the recurrence formula (1.28), we see that the only coefficient non zero is $C_{a,\pi}$.

The computation of the coefficient $C_{a,\pi}$ is straightforward. □

Call $U_{\alpha,\beta} = \prod_i^\alpha U_{i-1} \prod_i^\beta U_{i-1}$. We compute ψ_π:

$$\psi_\pi = U_{\alpha,\beta}^{-1} \oint \dots \oint \prod_i \frac{du_i}{2\pi i u_i^{a_i}} \prod_{j>i} (u_j - u_i)(1 + \tau u_j + u_i u_j)$$

We perform the integration in the first m variables:

$$\begin{aligned}\psi_\pi =& U_{\alpha,\beta}^{-1} \oint \dots \oint \prod_i^\alpha \frac{dv_i}{2\pi i v_i^\alpha}(1+\tau v_i)^m \prod_{j>i}(v_j - v_i)(1+\tau v_j + v_i v_j)\\ &\times \prod_{\hat{\imath}}^\beta \frac{dw_{\hat{\imath}}}{2\pi i w_{\hat{\imath}}^{2\alpha+\beta}}(1+w_{\hat{\imath}})^m \prod_{\hat{\jmath}>\hat{\imath}}(w_{\hat{\jmath}} - w_{\hat{\imath}})(1+\tau w_{\hat{\jmath}} + w_{\hat{\imath}} w_{\hat{\jmath}})\\ &\times \prod_{i,\hat{\imath}}(w_{\hat{\imath}} - v_i)(1+\tau w_{\hat{\imath}} + w_{\hat{\imath}} v_i)\end{aligned}$$

where we renamed the variables in a convenient way.

For what follows we need a little lemma:

Lemma 1.29. *The anti-symmetrization of* $\prod_{j>i}^r (1+\tau u_j + u_i u_j)$ *is equal to the following formula:*

$$\sum_{\sigma \in S_r} (-1)^\sigma \prod_{j>i}^r (1+\tau u_{\sigma_j} + u_{\sigma_i} u_{\sigma_j}) = (-1)^{\binom{r}{2}} \prod_i^r U_{i-1} \prod_{j>i}^r (u_j - u_i)$$

Proof. The proof is quite elementary. We perform a transformation of variables $u_i = \frac{qw_i - q^{-1}}{w_i - 1}$:

$$(q-q^{-1})^2 \prod_i^r \frac{1}{(w_i-1)^{r-1}} \prod_{j>i}(qw_j - q^{-1}w_i)$$

The anti-symmetrization of this factor is quite trivial, the only difficulty is to compute the multiplicative factor, following a technique similar to the one used in [81] we finally obtain the expression $\prod_i^r U_{i-1}$.

If we do the inverse variable transformation we get the desired result. □

Applying this formula, we obtain:

$$\begin{aligned}\psi_\pi =&(-1)^{\binom{\alpha}{2}+\binom{\beta}{2}} \frac{1}{\alpha!\beta!} \oint \dots \oint \prod_i^\alpha \frac{du_i}{2\pi i u_i^\alpha}(1+\tau u_i)^m \prod_{j>i}^\alpha (u_j-u_i)^2 \\ &\times \prod_{\hat{i}}^\beta \frac{dw_{\hat{i}}}{2\pi i w_{\hat{i}}^{2\alpha+\beta}}(1+\tau w_{\hat{i}})^m \prod_{\hat{j}>\hat{i}}^\beta (w_{\hat{j}}-w_{\hat{i}})^2 \\ &\times \prod_{i,\hat{i}} (w_{\hat{i}}-v_i)(1+\tau w_{\hat{i}}+w_{\hat{i}}v_i).\end{aligned}$$

But the integrand is symmetric in the u_i and in the $w_{\hat{i}}$ separately, so we can replace $\prod_{j>i}(u_j-u_i)$ by $\prod_i u_i^{\alpha-i}$ and analogously for $w_{\hat{i}}$:

$$\begin{aligned}\psi_\pi =&\frac{1}{\alpha!\beta!} \oint \dots \oint \prod_i^\alpha \frac{du_i}{2\pi i u_i^i}(1+\tau u_i)^m \prod_{j>i}^\alpha (u_j-u_i) \\ &\times \prod_{\hat{i}}^\beta \frac{dw_{\hat{i}}}{2\pi i w_{\hat{i}}^{2\alpha+\hat{i}}}(1+\tau w_{\hat{i}})^m \prod_{\hat{j}>\hat{i}}^\beta (w_{\hat{j}}-w_{\hat{i}}) \\ &\times \prod_{i,\hat{i}} (w_{\hat{i}}-v_i)(1+\tau w_{\hat{i}}+w_{\hat{i}}v_i).\end{aligned}$$

Computing the first α integrations, we obtain:

$$\psi_\pi = \oint \dots \oint \prod_{\hat{i}}^\beta \frac{dw_{\hat{i}}}{2\pi i w_{\hat{i}}^{\alpha+\hat{i}}}(1+\tau w_{\hat{i}})^{m+\alpha} \prod_{\hat{j}>\hat{i}} (w_{\hat{j}}-w_{\hat{i}}).$$

The expected result is achieved by performing a simple computation:

$$\begin{aligned}\psi_\pi =& \sum_{\sigma\in S_\beta} (-1)^\sigma \oint \dots \oint \prod_{\hat{i}}^\beta \frac{dw_{\hat{i}}}{2\pi i w_{\hat{i}}^{1+\alpha+\hat{i}-\sigma_i}}(1+\tau w_{\hat{i}})^{m+\alpha} \\ =& \sum_\sigma (-1)^\sigma \prod_i^\beta \tau^{\alpha+i-\sigma_i} \binom{m+\alpha}{\alpha+i-\sigma_i} \\ =& \tau^{\alpha\beta} \det \left| \binom{m+\alpha}{\alpha+i-j} \right|.\end{aligned}$$

This can be computed, we will omit here the details, and is equal to:

$$\psi_\pi = \tau^{\alpha\beta} \prod_i^\beta \prod_j^\alpha \prod_k^m \frac{i+j+k-1}{i+j+k-2}$$

which is the result conjectured by Zuber in [85].

This result was proved also in [84]. In the article [18] the authors proved that these numbers, with $\tau = 1$ count the Fully Packed Loop Configurations with π at the boundary. After the proof of the Razumov–Stroganov theorem by Cantini and Sportiello [8], which links the FPL to the XXZ model, both results are equivalent, see Section 2.3.

Chapter 2

Alternating Sign Matrices and 6-Vertex model

In this chapter we introduce a second set of models: Alternating sign matrices, Fully Packed Loops, 6-Vertex model, 2 dimensional Ice model and others. All these models are in bijection.

Alternating Sign Matrices (ASM) were invented by Robbins and Rumsey in their study of the "γ-determinants" in [70] (cleary, this work precedes [49, 50], it only appeared later), which generalise determinants in a way inspired by Dodgson's condensation [19]. In a series of articles, Mills, Robbins and Rumsey [49, 50, 70, 51] studied their properties. They counted them and noticed that there are as many ASM as Totally Symmetric Self-Complementary Plane Partitions (TSSCPP). This was the first intriguing mystery.

This fact was proved by Zeilberger [80]. Some time after, Kuperberg [44] proved exactly the same result but using some methods from quantum integrable system. His proof uses the fact that the ASM are in bijection with the 6-Vertex model with Domain Wall Boundary Conditions (DWBC). Unfortunately no bijection between TSSCPP and ASM has been found so far.

In the latest chapter of this story, Razumov and Stroganov found that there is a deep connection between the Fully Packed Loops (which are in bijection with the ASM) and the XXZ Spin Chain model. This conjecture, now a theorem, will be the subject of Section 2.3.1.

2.1 Alternating Sign Matrices

An Alternating Sign Matrix (ASM) is a square matrix made of 0s, 1s and -1s such that if one ignores the 0s, 1s and -1s alternate on each row and column starting and ending with one 1s. Here are all 3×3 ASM:

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \qquad \begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix} \qquad \begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix} \qquad \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} \qquad \begin{matrix}0&1&0\\1&-1&1\\0&1&0\end{matrix} \qquad \begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix} \qquad \begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}$$

Thus, there are exactly 7 ASM of size $n = 3$.

These matrices have been studied by Mills, Robbins and Rumsey since the early 1980s. It was then conjectured that A_n, the number of ASM of size n, is given by:

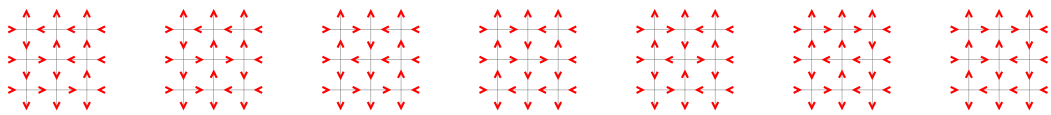
$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1,\ 2,\ 7,\ 42,\ 429,\ \ldots \tag{2.1}$$

This was subsequently proved by Zeilberger in 1996 in an 84 pages article [80]. A shorter proof was given by Kuperberg [44] in 1998. The latter is based on the equivalence to the 6-V model.

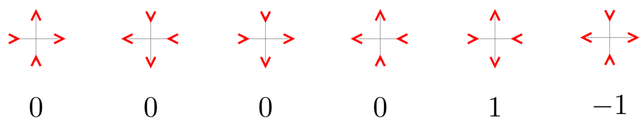
2.2 6-Vertex Model

Let us now turn to the 6-Vertex Model. The model consists of a square grid of size $n \times n$ in which each edge is given an orientation (an arrow), such that at each vertex there are two arrows pointing in and two arrows pointing out. This leaves 6 possible vertex configurations, hence the name of the model. We use here some very specific boundary conditions (Domain Wall Boundary Conditions, DWBC): all arrows at the left and the right boundaries are pointing in, and at the bottom and the top boundaries are pointing out.

For example, there are 7 possible configurations for $n = 3$:



The fact that there are again seven configurations is not a coincidence. Indeed, there is an easy bijection between ASM and 6-V configurations with DWBC, which we describe schematically:



It is clear that this transformation associates a matrix to each 6-Vertex configuration. It can be easily checked that the Domain Wall Boundary Condition and the properties of the 6-Vertex model ensure that we obtain an alternating sign matrix. Conversely, one can consistently build a 6-V configuration from an ASM starting from the fixed arrows on the boundary, continuing arrows through the $0s$ and reversing them through the ± 1.

2.2.1 Square Ice Model and a path model

There are several ways of representing the 6-Vertex Model, here we present briefly two of them:

Square Ice Model

Take a grid $n \times n$ and in each site put an oxygen atom and in the edges the hydrogen atoms. The Ice rule states that a hydrogen will be between two oxygens (with a covalent bond and one hydrogen bridge), in such a way that each oxygen is connected to 2 hydrogens by covalent bonds and have two hydrogen bridges. The DWBC are here the artificial fact that there are hydrogens at the left and right but not at the top and bottom. See Figure 2.1.

The equivalence with the 6-Vertex model is obvious, an arrow pointing in corresponds to a covalent bond, and an arrow pointing out to a hydrogen bridge.

Osculating paths

Consider a 6-Vertex configuration, and paint each edge which is occupied by a right or an up arrow. We get a set of paths going north-east, which do not cross, although they can share a site. See Figure 2.1.

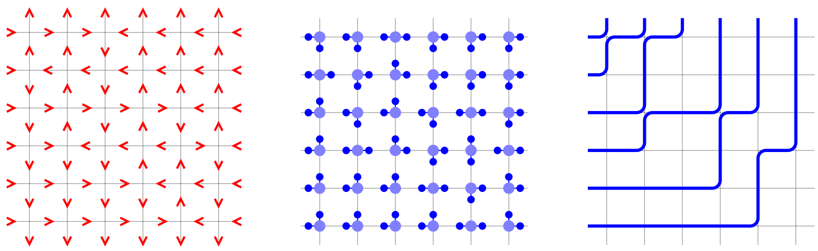


Figure 2.1: Three different representations of the same 6-Vertex configuration.

2.2.2 Partition Function

In order to solve the ASM enumeration problem, it is convenient to generalize it by considering weighted enumerations. This amounts to computing the partition function of the 6-Vertex model, that is the summation over 6-V configurations with DWBC such that to each vertex is given a statistical weight, as shown in Figure 2.2, depending on n horizontal spectral parameters (one for each row) $\{y_1, y_2, \ldots, y_n\}$, n vertical spectral parameters $\{y_{n+1}, y_{n+2}, \ldots, y_{2n}\}$ and one global parameter q. This computation was performed by Izergin [29], using recursion relations written by Korepin [40], and the result is an $n \times n$ determinant (IK determinant). It is a symmetric function of the set $\{y_1, \ldots y_n\}$ and of the set of $\{y_{n+1}, \ldots, y_{2n}\}$. Much later, it was observed by Stroganov [77] and Okada [58] that when $q = e^{2\pi i/3}$, the partition function is totally symmetric, *i.e.* symmetric in the full set $\{y_1, \ldots, y_{2n}\}$.

Here, we describe this procedure using the language of Quantum Integrable Systems.

The weights

We assign to each vertex a different weight, as shown in Figure 2.2.

Let n_i (i from 1 to 6) be the number of site configurations of each kind which appear in some configuration, then we have:

Proposition 2.1. *The quantities n_i are related by $n_1 = n_2$; $n_3 = n_4$; and, $n_5 = n_6 + n$.*

This is why we only have three different weights.

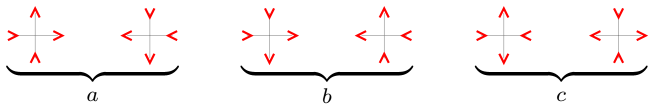


Figure 2.2: To each site configuration corresponds a statistical weight.

Proof. The first equation is obvious from the point of view of paths: it only means that the number of vertical lines is equal to the number of horizontal lines.

The second one is equivalent. If we replace the edges with a right arrow, instead of left arrow, by a path, we get the desired result.

The third one is visible on the ASM side, it only means that the number of 1s is equal to the number of -1s plus the n 1s corresponding to permutation matrices. □

The partition function is defined as usual:

$$\tilde{Z}_n = \sum_{\text{configurations}} \prod_{i,j}^{n} w_{i,j},$$

where $w_{i,j}$ is replaced by a, b or c depending on the site configuration at position (i, j).

In order to proceed the computation of the partition function it is convenient to replace the constant weights by weights that depend on the row and on the column:

$$\begin{aligned} a &= qz - q^{-1}w; \\ b &= z - w; \\ c &= (q^{-1} - q)z^{1/2}w^{1/2}, \end{aligned}$$

where w (respectively z) characterizes the column (resp. the row), known as vertical (resp. horizontal) spectral parameter. Here we will use $\{y_{n+1}, \ldots, y_{2n}\}$ (resp. $\{y_1, \ldots, y_n\}$) for the n vertical (resp. horizontal) spectral parameters. Furthermore q is a global parameter which will be eventually specialized to a cubic root of unity.

We redefine the partition function as being:

$$Z_n = (-1)^{n(n-1)/2}(q^{-1} - q)^{-n} \prod_{i=1}^{2n} y_i^{-1/2} \; \tilde{Z}_n \tag{2.2}$$

Some properties are evident from the definition:

Proposition 2.2. *The partition function Z_n is a polynomial of total degree $\delta = n(n-1)$ and partial degree $\delta_i = n - 1$.*

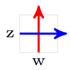
Proof. The total degree is straightforward.

For partial degree, we observe that a vertical (resp. horizontal) spectral parameter only appears in one column (resp. row), and there is at least one c. So the maximal degree is $(n - 1/2)$. Multiplying by $\prod_i^{2n} y_i^{-1/2}$ we get the desired result. □

Integrability

The interest of this model lies in the fact that it is a quantum integrable model. In this section we introduce a R matrix which satisfies the Yang–Baxter equation as in Section 1.1.3. In fact, the two R matrices have some differences: different normalization and the one defined here is invariant under reversal of every arrow. In spite of these differences, we will use the same name. This will allow us to identify some essential properties of the partition function which will be enough to compute it.

Take an $n \times n$ grid, fix the boundaries and imagine at each site an operator that takes as input the two arrows at the left and bottom edges and gives as a result the two arrows at the right and top edges. This operator is normally represented by a matrix:



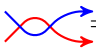
$$= \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} =: R(w,z),$$

where the entries follow the order $\{\rightarrow\uparrow, \rightarrow\downarrow, \leftarrow\uparrow, \leftarrow\downarrow\}$ and a, b and c are replaced by the values which depend on w and z. The R matrix is represented by two arrows crossing, where the arrows point on the direction of the operation.

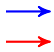
These R matrices obey the identity equation:

$$R(w,z)R(z,w) = (qz - q^{-1}w)(qw - q^{-1}z)Id.$$

Graphically:



$$= (qz - q^{-1}w)(qw - q^{-1}z)$$

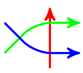


Of course we could have divided the matrix by $a = (qz - q^{-1}w)$, obtaining a simpler identity equation.

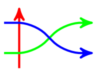
The most remarkable property of the R matrix is that it satisfies the Yang–Baxter equation:

$$R_{2,3}(y_2,y_3)R_{1,3}(y_1,y_3)R_{1,2}(y_1,y_2) = R_{1,2}(y_1,y_2)R_{1,3}(y_1,y_3)R_{2,3}(y_2,y_3),$$

where $R_{i,j}$ means that R acts in the tensor product between the i^{th} and the j^{th} vector space[1]. Graphically:



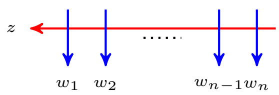
$=$



We define the transfer matrix of the model as:

$$T(z, w_1 \ldots, w_n) = R_{1,0}(w_1,z)R_{2,0}(w_2,z)\ldots R_{n-1,0}(w_{n-1},z)R_{n,0}(w_n,z)$$

$=$



where the matrices $R_{i,0}$ act in the tensor product on the i^{th} vector space and the auxiliary space (horizontal space).

[1] We consider that each line transports a vector space associated to one spin.

Korepin's recursion relations

In [40] Korepin proposed a way to compute Z_n inductively. We should repeat here the proof for the sake of completeness. The same proof appears also, for example, in [84, 41, 44].

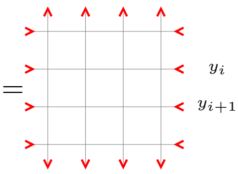
We shall construct the partition function by recursion. For this we need the following important proposition:

Proposition 2.3. *The partition function $Z_n(y_1, \ldots, y_{2n})$ is a symmetric function of the sets of variables $\{y_1, \ldots, y_n\}$ and $\{y_{n+1}, \ldots, y_{2n}\}$.*

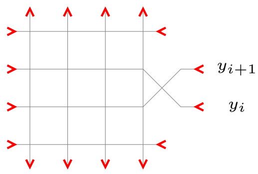
This is a simple consequence of the Yang–Baxter equation.

Proof. Let i be an integer between 1 and $n-1$. Notice that multiplying a R matrix at the right boundary at rows with spectral parameter y_i and y_{i+1} is the same as multiplying the partition function by $(qy_{i+1} - q^{-1}y_i)$. So we multiply the partition function by $R_{i,i+1}(y_i, y_{i+1})$ and divide it by $(qy_{i+1} - q^{-1}y_i)$:

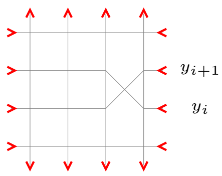
$$Z_n(y_1, \ldots, y_i, y_{i+1}, \ldots, y_{2n}) =$$



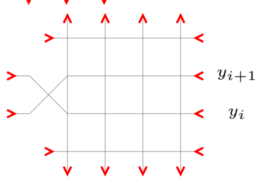
$$= \frac{1}{(qy_{i+1} - q^{-1}y_i)}$$



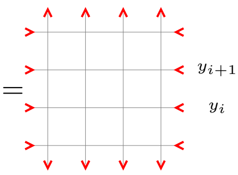
$$= \frac{1}{(qy_{i+1} - q^{-1}y_i)}$$



$$\ldots = \frac{1}{(qy_{i+1} - q^{-1}y_i)}$$



$$=$$



$$= Z_n(y_1, \ldots, y_{i+1}, y_i, \ldots, y_{2n}).$$

The proof is similar for the vertical spectral parameters. □

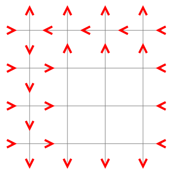
When we specialize $y_{n+1} = q^2 y_1$ we get the following recursion formula:

Proposition 2.4. *The partition function Z_n obeys the recursion relation:*

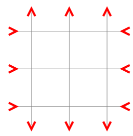
$$Z_n(y_1, \ldots, y_n, y_{n+1} = q^2 y_1, \ldots, y_{2n}) = (-1)^{n-1} \prod_{i=2}^{n} (y_1 - y_{n+i})(y_i - y_{n+1}) Z_{n-1}(y_2 \ldots, y_{n-1}, y_{n+1}, \ldots, y_{2n}). \quad (2.3)$$

Proof. The fact that $y_{n+1} = q^2 y_1$ fixes the first column and first row of a 6-Vertex configuration implies:

$$\tilde{Z}_n(y_1, \ldots, y_{n+1} = q^2 y_1, \ldots, y_{2n}) =$$



$$= (q^{-1} - q)\sqrt{y_1 y_{n+1}} \prod_{i=2}^{n} (y_1 - y_{n+i})(y_i - y_{n+1})$$



$$= (q^{-1} - q)\sqrt{y_1 y_{n+1}} \prod_{i=2}^{n} (y_1 - y_{n+i})(y_i - y_{n+1})$$

$$\times \tilde{Z}_{n-1}(y_1, \ldots, y_{n-1}, y_{n+1}, \ldots, y_{2n}),$$

where the factor comes from the fixed sites.

If we rescale $\tilde{Z}$ to Z, we obtain the desired result. □

Consider now Z_n as a polynomial in y_1. The last property together with the symmetry give us the evaluation of the polynomial for n different points of y_1. This is enough to fully determine the polynomial (notice that the degree is $n-1$ as a polynomial of y_1).

The fact that $Z_1 = 1$ completes the characterization.

Izergin's formula

In [29, 30] Izergin proposed a closed formula for the partition function:

Proposition 2.5. *The partition function of the 6-Vertex model is given by:*

$$Z_n(y_1, \ldots, y_{2n}) = \frac{\prod_{i,j}^{n} (y_i - y_{n+j})(q y_i - q^{-1} y_{n+j})}{\prod_{i<j}^{n} (y_i - y_j)(y_{n+i} - y_{n+j})} \times \det_{1 \le i,j \le n} \left| \frac{1}{(y_i - y_{n+j})(q y_i - q^{-1} y_{n+j})} \right|. \quad (2.4)$$

Proof. It is a simple exercise to prove this formula. We need only to check that the expression satisfies the following:

- It is a polynomial;
- It has the appropriate degree: total degree $\delta = n(n-1)$ and partial degree $\delta = n-1$;
- $Z_1 = 1$;
- It is symmetric in the sets $\{y_1, \ldots, y_n\}$ and $\{y_{n+1}, \ldots, y_{2n}\}$;
- When $y_{n+1} = q^2 y_1$ we obtain:

$$Z_n\big|_{y_{n+1}=q^2y_1} = (-1)^{n-1}\prod_{i=2}^{n}(y_1 - y_{n+i})(y_i - y_{n+1})Z_{n-1}.$$

□

2.2.3 The case $q^3 = 1$

Let Y_n be the Young diagram with two rows of length $n-1$, two rows of length $n-2$,…, two rows of length 1. The Schur function associated to Y_n is defined as:

$$s_{Y_n}(z_1, \ldots, z_{2n}) = \frac{\det_{1\le i,j\le 2n}\left|z_i^{2n-j+l_j}\right|}{\det_{1\le i,j\le 2n}\left|z_i^{2n-j}\right|}, \qquad (2.5)$$

where l_j is the length of the j^{th} row of Y_n: $(n-1, n-1, n-2, n-2, \ldots, 1, 1, 0, 0)$.

We claim that in the case $q = e^{2\pi i/3}$, the partition function (2.4) simplifies into:

$$Z_n(y_1, \ldots, y_n, y_{n+1}, \ldots, y_{2n}) = s_{Y_n}(q^{-1}y_1, \ldots, q^{-1}y_n, qy_{n+1}, \ldots, qy_{2n}). \qquad (2.6)$$

Thus, the partition function is symmetric in the set $\{q^{-1}y_1, \ldots, q^{-1}y_n, qy_{n+1}, \ldots, y_{2n}\}$.

The proof consists in checking that the Schur function s_{Y_n} satisfies the properties of the partition function Z_n in the case $q^3 = 1$.

- The Schur function is a polynomial;
- The Schur function is symmetric in $\{z_1, \ldots, z_{2n}\}$, in particular it is also symmetric in $\{z_1, \ldots, z_n\}$ and in $\{z_{n+1}, \ldots, z_{2n}\}$;
- The total degree is equal to the number of boxes in Y_n, so $\delta = n(n-1)$ and the partial degree is the size of the largest row, so $\delta_i = n-1$;
- A simple computation gives us $s_{Y_1} = 1$;
- s_{Y_n} obeys the relation

$$s_{Y_n}\big|_{z_j=q^2z_i} = \prod_{k\neq i,j}(q^{-2}z_i - z_k)\, s_{Y_{n-1}}\big|_{\hat{\imath},\hat{\jmath}},$$

  where $\hat{\imath}$ and $\hat{\jmath}$ means that we exclude z_i and z_j.

The last fact is not difficult to prove. Let (z_i, z_j, z_k) be such that $z_k = q^2 z_j = q^4 z_i$ for distinct i, j, k. In this case the Schur function vanishes. Indeed the three rows in the determinant of the numerator in (2.5) corresponding to z_i, z_j and z_k are linearly dependent while the denominator does not vanish.

So, we have:

$$s_{Y_n}(z_1, \dots, z_j = q^2 z_i, \dots, z_{2n}) = \prod_{k \neq i,j} (q^{-2} z_i - z_k) \tilde{s}_{n-1}(z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_{2n}).$$

where $\hat{z}_i$ means that we omit z_i.

Now set $z_i = 0$: the Schur function has $2n - 2$ remaining arguments, so the full column of length $2n - 2$ can be factored out and we are left with the Young diagram Y_{n-1}:

$$s_{Y_n}(z_1, \dots, z_i = 0, \dots, z_j = 0, \dots, z_{2n}) = \prod_{k \neq i,j} z_k s_{Y_{n-1}}(z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_{2n}).$$

By comparison, we conclude that $\tilde{s}_{n-1} = s_{Y_{n-1}}$.

If we replace y_i by $q y_i$, for $1 \leq i \leq n$ and y_i by $q^{-1} y_i$ for $n < i \leq 2n$ in Proposition 2.4 we obtain the same result. This proves the equality.

2.2.4 The homogeneous limit

We are now able to compute the number of ASM.

Put $y_i = q$ for $1 \leq i \leq n$ and $y_i = q^{-1}$ for $n < i \leq 2n$. The weights are now:

$$a = q^2 - q^{-2} = q^{-1} - q \tag{2.7}$$
$$b = q - q^{-1} \tag{2.8}$$
$$c = q^{-1} - q. \tag{2.9}$$

Notice that, in each configuration, there are an even number of sites with weight b, thus we can replace b with $-b$.

As all weights are the same, we get:

$$\begin{aligned}(q^{-1} - q)^{n(n-1)} (-1)^{\binom{n}{2}} A_n =& Z_n(q, \dots, q, q^{-1}, \dots, q^{-1}) \\ =& s_{Y_n}(1, \dots, 1).\end{aligned}$$

This means:

$$A_n = 3^{-\binom{n}{2}} s_{Y_n}(1, \dots, 1).$$

But, $s_{Y_n}(1, \dots, 1)$ is the number of Semi-Standard Young Tableaux (this is a standard result which can be found in, for example, [73]). More precisely, we have

$$s_{Y_n}(1, \dots, 1) = \prod_{u \in Y_n} \frac{c(u) + 2n}{h(u)},$$

where u runs through all boxes in Y_n, $c(u)$ is the content of box u and $h(u)$ its hook length.

After a simple computation and some manipulations of factorials, we get the formula:

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

2.3 Fully Packed Loops

A Fully Packed Loop configuration (FPL) of size n is a subgraph of the square grid with n^2 vertices, such that each vertex is connected to two edges. We furthermore impose the following boundary conditions: we select alternatively every other external edge along the boundary of the grid to be part of our FPLs. By convention, we fix that the leftmost external edge on the top boundary is part of the selected edges, which fixes the entire boundary of our FPLs. We number these external edges clockwise from 1 to $2n$, see Figure 2.3.

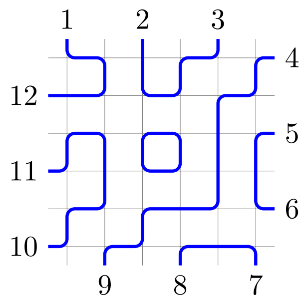
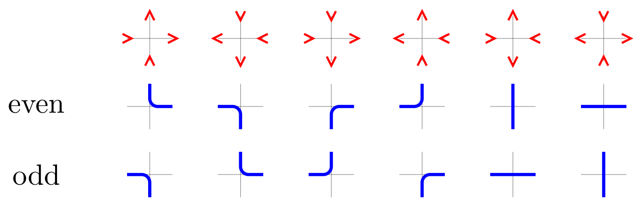


Figure 2.3: A FPL configuration example. This example is in bijection with the previous one of the 6-Vertex model, see Figure 2.1.

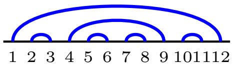
FPL and the 6-Vertex model

The FPL configurations are in bijection with the 6-Vertex configurations and, in consequence, with the ASM. We distinguish an odd and an even sub-lattice (defined by the parity of the sum of coordinates). The bijection reads:



FPL and matchings

In each FPL configuration η the chosen external edges are clearly linked by paths which do not cross each other. We define $\pi(\eta)$ as the set of pairs $\{i, j\}$ of integers in $\{1, \dots, 2n\}$ such that the external edges labeled i and j are linked by a path in η. Then $\pi(\eta)$ is a matching in the sense of Section 1.1.2; for example, the configuration of Figure 2.3, has connectivity:



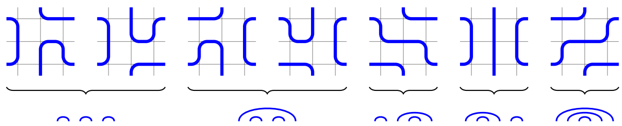
Definition 2.6 (A_π)**.** *For any matching π, we define A_π as the number of FPLs F such that $\pi(F) = \pi$.*

A result of Wieland [79] shows that the rotation on matchings leaves the numbers A_π invariant, and it is then easily seen that conjugation of matchings also leaves them invariant:

Proposition 2.7 ([79])**.** *For any matching π, we have $A_\pi = A_{r(\pi)}$ and $A_\pi = A_{\pi^*}$.*

2.3.1 The Razumov–Stroganov–Cantini–Sportiello theorem

We can easily construct the seven different configurations for $n = 3$:



We grouped the configurations by their connectivity. We can see that:

A $= 1$ A $= 1$ A $= 1$

A $= 2$ A $= 2$

Notice two things, first these numbers are invariant under conjugation and rotation of matchings, as pointed out in Proposition 2.7. Second, these numbers appear in the solution of the Completely Packed Loop model (see Section 1.1.5).

Razumov and Stroganov noticed, in 2001, that this seems to hold in at all sizes:

Theorem 2.8 ([65])**.** *The groundstate components of the Completely Packed Loops model, when normalized in such a way that the smallest component is 1, count the number of Fully Packed Loop configurations for any matching π:*

$$\Psi_\pi = A_\pi.$$

As a small historical remark, this story begun with the articles [68, 76]. The authors noticed that the groundstate of the periodic XXZ chain with $\Delta = -\frac{1}{2}$ and an odd number of sites can be normalized such a way that the smallest component is 1, has only non-negative integer components. Moreover, using this normalization, the biggest component is $1, 2, 7, 42, 429, \ldots$, the famous sequence that counts the ASM.

Later, in the article [3] the authors studied the even case conjecturing the following theorem:

Theorem 2.9 ([3]). *Let, as before, Ψ_π be the component of the grounstate of the CPL corresponding to π, normalized in such a way that the smallest component is 1.*

Let n count the number of arches. We have three important properties:

- *All components are non-negative integers;*
- *The largest component is equal to A_{n-1};*
- *The sum of all components is A_n,*

$$\sum_{\pi} \Psi_\pi = A_n. \tag{2.10}$$

The sum rule (2.10) was proved in the article [15] by Zinn-Justin and Di Francesco.

Of course that now, the first and the third points are corollaries of Theorem 2.8, which was finally proved in 2010 by Cantini and Sportiello [8].

2.3.2 Some symmetry classes

It is natural to define some subsets of the FPL (or ASM or 6-Vertex model), which are invariant under some symmetries. Here we will only refer to the one that will be needed: the subset of configurations which are invariant with respect to vertical symmetry (normally called Vertically Symmetric ASM).

Let A_n^V be the number of vertically symmetric FPLs of size n. We then have the famous product expression:

$$A_{2n+1}^V = \frac{1}{2^n} \prod_{j=1}^{n} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!}.$$

The original proof can be found in [45].

It is worth noticing that there are several variations of Theorems 2.8 and 2.9, which link different symmetry classes on the FPL side to different boundary conditions on the CPL side. Most of them are still conjectures.

For example, define $A_{2n+1,\pi}^V$ as the number of vertically symmetric fully packed loop configurations of size $2n+1 \times 2n+1$ with connectivity at the boundary represented by the n arches link pattern π (we ignore half of the FPL, since the configuration is vertically symmetric).

Let Ψ_π^o be the stationary state of the following Hamiltonian:

$$H^o = \frac{1}{2n-1} \sum_{i=1}^{2n-1} e_i,$$

which describes the CPL with open boundary conditions.

With a suitable normalization we have:

Conjecture 2.10. *Let* $\tau = 1$. *The quantities* $A^V_{2n+1,\pi}$*, which count the VSASM with boundary connectivity* π *solve the equation:*

$$\sum_{\pi} A^V_{2n+1,\pi}\pi = \sum_{\pi} H^o A^V_{2n+1,\pi}\pi.$$

We can find this example, and others, in the articles [66, 3].

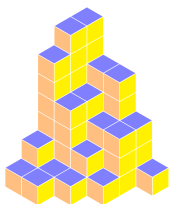
Chapter 3

Totally Symmetric Self Complementary Plane Partitions

3.1 Plane Partitions

Plane Partitions can be seen as a generalization of partitions (or number-partitions) whose study was initiated by Euler and continued by Sylvester, Frobenius and others. We describe here Plane Partitions in two different ways, either pictorially or as arrays of integers.

Pictorially, a plane partition is a stack of unit cubes pushed into a corner (gravity pushing them into the corner) and drawn in isometric perspective, as illustrated in Figure 3.1.



7	5	5	3	1
7	4	3	3	
6	4	2	1	
2	1	1		
1	1			

Figure 3.1: We can see a plane partition (PP) either as a stack of unit cubes pushed into a corner (at the left) or as an array of decreasing integers (at the right).

An equivalent way of describing these objects is to form the array of heights of each stack of cubes. In this formulation the effect of "gravity" is that each number in the array is less than or equal to the numbers immediately above and to the left. See, for example, the plane partition in Figure 3.1 represented in both ways.

Plane Partitions are counted by the following generating formula, due to MacMahon (1897) [47]:

Theorem 3.1. *Let $pp(n)$ count the number of Plane Partitions with exactly n cubes. The generating function for $pp(n)$ is given by*

$$\sum_{n=0}^{\infty} pp(n)q^n = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^j}.$$

The proof of this theorem can be found for example in Bressoud's book [7] or in MacMahon's article [48].

3.1.1 Totally Symmetric Self complementary Plane Partitions

A problem of interest is the enumeration of plane partitions that have some specific symmetries.

The Totally Symmetric Self-Complementary Plane Partitions (TSSCPP) form one of these symmetry classes. In the pictorial representation, they are Plane Partitions inside a $2n \times 2n \times 2n$ cube which are invariant under the following symmetries: all permutations of coordinates of boxes; and taking the complement, that is putting cubes where they are absent and vice versa, and flipping the resulting set of cubes to form again a Plane Partition[1]. In Figure 3.2 we can see an example of a TSSCPP.

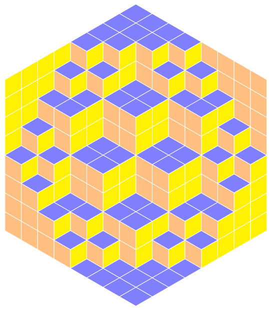


Figure 3.2: An example of Totally Symmetric Self Complementary Plane Partitions inside a $8 \times 8 \times 8$ cube.

Alternatively, they can be described as $2n \times 2n$ arrays of heights. In the $n = 3$ case, we have, once again, 7 possible configurations:

$$\begin{array}{ccccccc} 666333 & 666433 & 666433 & 666543 & 666543 & 666553 & 666553 \\ 666333 & 666333 & 666433 & 665332 & 665432 & 655331 & 655431 \\ 666333 & 665332 & 664322 & 655331 & 654321 & 655331 & 654321 \\ 333000 & 433100 & 443200 & 533110 & 543210 & 533110 & 543210 \\ 333000 & 333000 & 332000 & 433100 & 432100 & 533110 & 532110 \\ 333000 & 332000 & 332000 & 321000 & 321000 & 311000 & 311000 \end{array} \tag{3.1}$$

and more generally we obtain A_n (the number of ASM of size $n \times n$, see (2.1)) for any n:

Theorem 3.2. *The number of TSSCPP inside a $2n \times 2n \times 2n$ cube is,*

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, \ldots$$

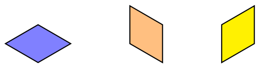
[1] In a more precise way, represent the small cubes by theirs cordinates (i, j, k), starting at $(1, 1, 1)$. Then a PP is self-complementary if and only if for all (i, j, k) belonging to the PP $(2n - i + 1, 2n - j + 1, 2n - k + 1)$ does not belong to the PP.

This result was conjectured by Mills, Robbins and Rumsey in the article [51], and it was proven only much later, in a paper by Andrews in 1994 [2] proved this theorem.

So, we have two sets counted by the same numbers. A natural problem is to find an explicit bijection between them. Until now, the mathematical community (and the physical too) is still looking for this correspondence.

TSSCPP as Tilings

Remark that in Figure 3.2 we can consider the cube faces as rhombi. In this perspective, we can consider that we have a hexagonal surface of size $2n \times 2n \times 2n$, which is tiled with three different rhombi:



This is valid for any plane partition, not only TSSCPP.

TSSCPP as a Dimer model

Perhaps more interesting, is the bijection between dimer configurations and plane partitions.

Let $\mathcal{T}$ be a honeycomb lattice in a hexagonal region of size $2n \times 2n \times 2n$. A dimer configuration is a collection of edges of $\mathcal{T}$ in such a way that each vertex is incident to exactly one of these edges.

The bijection is trivial. Take a plane partition seen as a tiling. Divide the rhombus into two equilateral triangles. Put the vertices of the grid at the center of the triangles and put edges linking neighboring centers. A dimer corresponds to a rhombus in the initial tiling configuration:

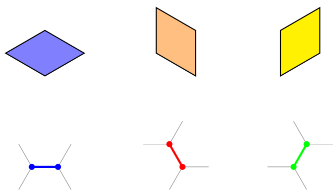


Figure 3.3 shows the translation of Figure 3.2 into a dimer configuration.

3.2 Non-Intersecting Lattice Paths

Plane partitions can also be represented as non-intersecting lattice paths. Before explaining how we can translate them, let us see briefly what they are.

Let $\mathcal{G}$ be a locally finite graph, here it will always be a lattice. Let $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_n\}$ be two collections of vertices, called initial and final points, respectively. A familly of Non-Intersecting Lattice Paths (NILP) is a set of n paths, defined in $\mathcal{G}$, going from A to B that do not touch one another.

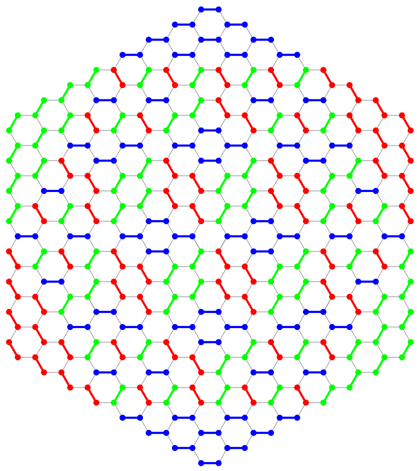


Figure 3.3: A Plane Partition can be seen as a dimer model in an hexagonal grid.

For example, take a square grid $M_1 \times M_2$, and choose n initial points $\{A_1, \ldots, A_n\}$ at the left boundary and n final points $\{B_1, \ldots, B_n\}$ at the right boundary. Now draw n non intersecting paths, whose steps can be only North or East. See the example in Figure 3.4.

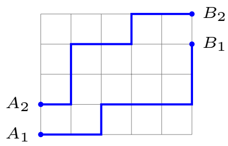


Figure 3.4: This is an example of a NILP. There are exactly 490 different NILP which start at $\{A_1, A_2\}$ and end at $\{B_1, B_2\}$.

3.2.1 The LGV formula

In [46, 27], Lindström, Gessel and Viennot proposed a formula to compute the number of NILP.

We present here a simple formulation which is sufficient for our purposes:

Lemma 3.3 ([46, 27])**.** *Let $\mathcal{G}$ be a locally finite directed acyclic graph, $U = (u_i)_{i \in [\![1,n]\!]}$ and $V = (v_i)_{i \in [\![1,n]\!]}$ be two collections of vertices of $\mathcal{G}$. A set of non-intersecting paths is a set of paths in $\mathcal{G}$ that do not share any vertex.*

Let ω_{x-y} be the weight of the oriented edge $x \to y$, and let ϵ_x be the weight of the vertex x. Let γ_{u_i,v_j} be a path from u_i to v_j. Let $\Omega(\gamma_{u_i,v_j})$ be the weight of the path, i.e. *if $\gamma_{u_i,v_j} = u_i \to x_1 \to \ldots \to x_n \to v_j$ the then weight is:*

$$\Omega(\gamma_{u_i,v_j}) = \epsilon_{u_i}\epsilon_{v_j}\omega_{u_i-x_1}\omega_{x_n-v_j}\prod_k^n \epsilon_{x_k}\prod_k^{n-1}\omega_{x_k-x_{k+1}}.$$

Let $P(u_i, v_j)$ be the sum of the weights over all paths in $\mathcal{G}$ from u_i to v_j:

$$P(u_i, v_j) = \sum_{\gamma_{u_i,v_j} \in \mathcal{G}} \Omega(\gamma_{u_i,v_j}),$$

and let $P(U, V)$ be the sum of the weights of all the sets of n non-intersecting paths from a vertex in U to a vertex in V. Assume also that paths counted by $P(U, V)$ necessarily join the vertex u_i to the vertex v_i for $i = 1 \dots n$. Then

$$P(U, V) = \det_{n \times n} |P(u_i, v_j)|$$

If we apply the above lemma to the example in Figure 3.4, we get that all weights are 1, so $P(A_i, B_j)$ counts the number of paths from A_i to B_j. $P(\{A_1, A_2\}, \{B_1, B_2\})$ is given by:

$$\begin{aligned} P(\{A_1, A_2\}, \{B_1, B_2\}) &= \det \begin{vmatrix} \binom{8}{3} & \binom{7}{2} \\ \binom{9}{4} & \binom{8}{3} \end{vmatrix} \\ &= 490 \end{aligned}$$

In [75] the author illustrates a more general framework, where different kinds of paths are counted using determinants and pfaffians.

Lemma 3.4 ([75])**.** *Assume n is even. Let $\mathcal{G}$ be a locally finite directed acyclic graph, $U = (u_i)_{i \in [\![1,n]\!]}$ and $V = (v_i)_{i \in [\![1,m]\!]}$, with $m \geq n$, be two collections of vertices of $\mathcal{G}$.*

Let γ_{u_i,v_j} be a path from u_i to v_j. Let $\Omega(\gamma_{u_i,v_j})$ be the weight of the path as in Lemma 3.3. Let $P(u_i, v_j)$ be the sum of the weights over all paths in $\mathcal{G}$ from u_i to v_j. Furthermore, let $P(U, V)$ be the sum of the weights of all sets of n non-intersecting paths from a vertex in U to a vertex in V.

Let

$$Q(u_i, u_j) = \sum_{1 \leq k < \ell \leq m} \left(P(u_i, v_k) P(u_j, v_\ell) - P(u_i, v_\ell) P(u_j, v_k) \right).$$

Assume also that paths counted by $P(U, V)$ necessarily preserve the order, i.e. *if $\gamma_{u_1,v_{j_1}}, \gamma_{u_2,v_{j_2}}, \dots, \gamma_{u_n,v_{j_n}}$ is a set of non-intersecting paths then, necessarily, $v_{j_1} < v_{j_2} < \dots < v_{j_n}$. Then*

$$P(U, V) = \mathrm{Pf}_{1 \leq i,j \leq n}(Q(u_i, u_j)).$$

If n is odd, we can add an extra path going from u_0 to v_0 with weight 1 isolated from the rest of the points in U and V, i.e. *$P(u_0, v_i) = 0$ if $i \in [\![1, n]\!]$ and $P(u_i, v_0) = 0$ if $i \in [\![1, n]\!]$ and $P(u_0, v_0) = 1$.*

Note that this is a simplified version, adapted to the purposes of this work.

3.3 TSSCPP as NILP

In order to better understand the bijection between NILP and TSSCPP, it is convenient to consider an intermediate class of objects: Non-Crossing Lattice Paths (NCLP), which are similar to NILP except for the fact the paths are allowed to share common sites, although they are still forbidden to cross each other.

We proceed with the description of the bijection between TSSCPP and a class of NCLP. Each TSSCPP is defined by a subset of numbers of the arrays of (3.1), a possible choice is the triangle at the bottom right:

0	1	2	1	2	1	2
00	00	00	10	10	11	11
000	000	000	000	000	000	000

These arrays ($a_{i,j}$ with $1 \leq i \leq n$ and $1 \leq j \leq n-i+1$) are defined by the following rules:

- $a_{i,j} \geq a_{i+1,j}$ for all $i < n$;
- $a_{i,j} \geq a_{i,j+1}$ for all $j < n-i+1$;
- $a_{i,j} \leq n-i$.

It is easy to prove that this array is enough to reconstruct the whole TSSCPP, using the symmetries which characterize the TSSCPP. Then, we draw paths separating the different numbers appearing, as explained on Figure 3.5.

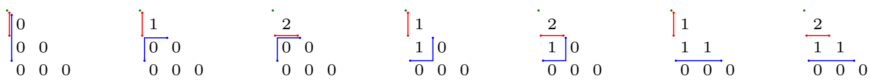


Figure 3.5: Reformulation of TSSCPP as NCLP, in the example of size $n = 3$. If the origin is at the upper right corner, then at each point $(0, -i)$, $i \in \{0, \ldots, n-1\}$, begins a path which can only go upwards or to the right, and stops when it reaches the diagonal $(j, -j)$, in such a way that the numbers below/to the right of it are exactly those less than $n - i$.

The bijection with the NILP is easily achieved by shifting the paths (NCLP) according to the following rules:

- The i^{th} path begins at $(i, -i)$;
- The vertical steps are conserved and the horizontal steps ($\rightarrow$) are replaced with diagonal steps ($\nearrow$).

An example ($n = 3$) is shown on Figure 3.6.

In our work we used others NILP which we will present below.

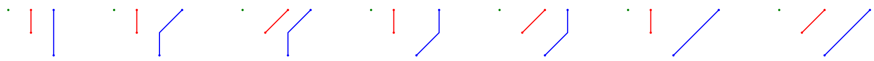


Figure 3.6: We transform our NCLP into NILP: the starting point is now shifted to the right, and the horizontal steps become diagonal steps.

3.4 Descending Plane Partitions

Although we did not work with Descending Plane Partitions (DPP) we present them here because the DPP are also counted by the sequence A_n. We start with two definitions:

A Shifted Plane Partition is an array $a_{i,j}$ of positive integers with $i \leq j \leq \mu_i$:

$$\begin{array}{cccccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & \cdots & a_{1,\mu_1} \\ & a_{2,2} & a_{2,3} & \cdots & \cdots & a_{2,\mu_2} \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & a_{r,r} & \cdots & a_{r,\mu_r} \end{array}$$

which satisfy these three properties:

- $a_{i,j} \geq a_{i,j+1}$, when both defined;
- $a_{i,j} \geq a_{i+1,j}$, when both defined;
- $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r$, when both defined.

Descending Plane Partitions are Shifted Strict Plane Partitions with two more conditions. Where strict means that they are strictly decreasing at the column $a_{i,j} > a_{i+1,j}$ (when defined).

Let $\lambda_i = \mu_i + i - 1$ be the size of a row:

- $a_{i,i} > \lambda_i$, for $1 \leq i \leq r$;
- $a_{i,i} \leq \lambda_{i-1}$, for $1 < i \leq r$.

The number of Descending Plane Partitions with no part exceeding n is A_n, the number of ASM. Moreover, there is a triple refined conjecture between the two objects. For more details see [50, 7].

CHAPTER 4

Alternating sign matrices and totally symmetric self-complementary plane partitions

It is the purpose of this chapter to solve an old problem: the doubly refined enumeration of ASM and TSSCPP, which appeared in [51]. We follow the procedure of the author's article [23]. In the literature there is another doubly refined enumeration, *e.g.* [21, 33], which we will not discuss here.

This chapter is organized as follows. In Section 4.1 we introduce the generating functions which count ASM (defined in Chapter 2) and TSSCPP (defined in Chapter 3) using two new parameters. In Section 4.2 we formulate the main theorem of the chapter: the equality of the doubly refined enumerations of ASM and TSSCPP. Section 4.3 contains the proof, based on the use of integral formulæ. Section 4.4 contain the original terms in which the theorem was conjectured by Mills, Robbins and Rumsey.

4.1 Generating functions

4.1.1 NILP

In the original formulation [51], the conjecture was in terms of totally symmetric and self complementary plane partitions, but it is convenient for us to use a NILP language (which is equivalent, recall Section 3.2). Anyway, in Section 4.4 we will translate the conjecture into the language of [51].

Take the NILP defined in Section 3.3. We add one extra step to each path, following the rule: to the first path we add a diagonal step; for the other paths the choice is made such that the difference between the final point of two consecutive paths is odd, as illustred in Figure 4.1.

NILP generating function

Let α be a NILP, we define $u_n^0(\alpha)$ as the number of vertical steps in the extra steps and $u_n^1(\alpha)$ as the number of vertical steps in the last steps (before the extra step) of each path.

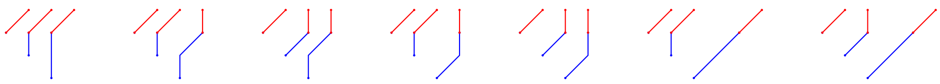


Figure 4.1: To each path we add one extra step in order that two consecutive final points differ by an odd number. The first step is chosen to be diagonal.

The generating function is:

$$U_n^{0,1}(x,y) := \sum_{\alpha} x^{u^0(\alpha)} y^{u^1(\alpha)} = \sum_{i,j} U_{n,i,j}^{0,1} x^i y^j \tag{4.1}$$

where $U_{n,i,j}^{0,1}$ is the number of NILP of size n with i vertical extra steps and j vertical last steps.

Alternative generating functions

Let us extend the definition of $u_n^k(\alpha)$ for $k > 1$:

Definition 4.1. *Let α be a NILP, the function $u_n^k(\alpha)$ counts the number of vertical steps in the extra step if $k = 0$; otherwise it counts the number of vertical steps in the* $\max\{1, t-k+1\}^{th}$ *step of the path starting at $(t, -t)$, as shown on Figure 4.2.*

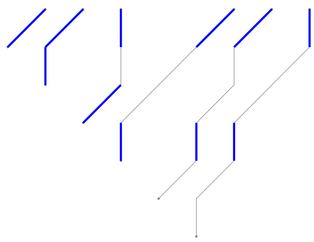


Figure 4.2: Let α be the NILP represented here. In order to calculate $u_6^0(\alpha)$ and $u_6^3(\alpha)$ we highlight the extra steps and the $\max\{1, t-3+1\}^{\text{th}}$ step of the path starting at $(t, -t)$. Here we have $u_6^0(\alpha) = 2$ and $u_6^3(\alpha) = 4$.

We can next define the function $U_n^i(x)$:

$$U_n^i(x) := \sum_k U_{n,k}^i x^k := \sum_{\alpha} x^{u_n^i(\alpha)} \tag{4.2}$$

and more complex functions $U_n^{i,j}(x,y)$:

$$U_n^{i,j}(x,y) := \sum_k U_{n,k,l}^{i,j} x^k y^l := \sum_{\alpha} x^{u_n^i(\alpha)} y^{u_n^j(\alpha)}. \tag{4.3}$$

We could generalize these even more, introducing more indices, but this is generally enough for our purposes.

These new quantities are not independent:

Proposition 4.2. *$U_n^{0,i}(x,y) = U_n^{0,j}(x,y)$ for all $1 \leq i,j \leq n$.*

Proof. For the first equality we introduce a function g_i as explained on Figure 4.3. This function interchanges the number of vertical steps in two consecutive rows leaving invariant all the other rows. It has the important property $g_i \circ g_i = Id$. So, it is straightforward from this that $U_n^{0,i} = U_n^{0,i+1}$, with i greater than 0.

In fact, we prove the more general result that $U_n^{j,i}(x,y) = U_n^{j,i'}(x,y)$ for all $i,i' > j$. □

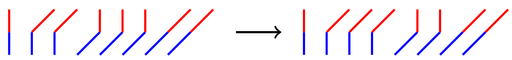


Figure 4.3: The function g_i acts in the double steps starting at $y = -i-1$ and finishing at $y = -i+1$. We can group the double steps in islands, such that all the starting points (of the double steps) are consecutive. These double steps are, necessarily, ordered in r double vertical steps, s vertical-diagonal steps, t diagonal-vertical steps and u double diagonal steps. Our function g interchanges s with t at each island, so that we interchange the number of vertical steps between the two rows.

This proposition is equivalent to the one that appears in the article [51, Theorem 2], we only use a different language here, see Section 4.4.

Proposition 4.3. *$U_{n,k,j}^{0,i} = U_{n,n-k-1,j}^{1,i}$ for all $1 < i \leq n$.*

Proof. The proof follows the same structure as the former. We construct again a function h such that $h \circ h = Id$, which interchanges the number of vertical steps at the extra steps with the number of diagonal steps at the last steps (before the extra step). This function is obviously a bijection and it leaves invariant all the rows except the last one and the extra one because it is applied at the top of the diagrams as can be seen on Figure 4.4. □

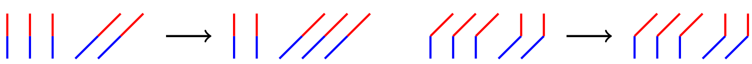


Figure 4.4: In order to satisfy the extra step rules we can only build two type of islands, one made of r double vertical steps and s double diagonal steps, and the other type made of t vertical-diagonal steps and u diagonal-vertical steps. Our function h simply interchange r with s.

4.1.2 ASM

Each ASM, as can be easily proven, has exactly a single 1 in the first row and in the last row. It is natural to classify ASM according to their positions. Therefore, we count the ASM with the first 1 in the i^{th} position and the last 1 in the j^{th} position: $\tilde{A}_{n,i,j}$.

We build the corresponding generating function:

$$\tilde{A}_n(x,y) := \sum_{i,j} \tilde{A}_{n,i,j} x^{i-1} y^{j-1}. \tag{4.4}$$

We define also $A_{n,i,j}$, which counts the ASM with the first 1 in the i^{th} column and the last 1 in the $(n-j+1)^{\mathrm{st}}$ column:

$$A_{n,i,j} = \tilde{A}_{n,i,n-j+1}. \tag{4.5}$$

We introduce the corresponding generating function:

$$A_n(x,y) := \sum_{i,j} A_{n,i,j} x^{i-1} y^{j-1}. \tag{4.6}$$

Some trivial symmetries

By reflecting the ASM horizontally and vertically one gets:

$$A_{n,i,j} = A_{n,j,i},$$

whereas by reflecting them only horizontally one gets:

$$A_{n,i,j} = A_{n,n-i+1,n-j+1}.$$

Obviously these symmetries are also valid for $\tilde{A}_{n,i,j}$.

4.2 The conjecture

We now present the conjecture, formulated by Mills, Robbins and Rumsey in a slightly different language (see Section 4.4 for a detailed translation), whose proof is the main focus of the present chapter:

Theorem 4.4. *The number of ASM of size n with the 1 of the first row in the $(i+1)^{\mathrm{st}}$ position and the 1 of the last row in the $(j+1)^{\mathrm{st}}$ position is the same as the number of NILP (corresponding to TSSCPP, and with the extra step) with i vertical extra steps and j vertical steps in the last step. Equivalently,*

$$\tilde{A}_n(x,y) = U_n^{0,1}(x,y).$$

For example, for $n = 3$, using the ASM given in Section 2.1 and the TSSCPP given in Figure 4.1, we compute:

$$\tilde{A}_3(x, y) = y^2 + y + xy^2 + x + xy + x^2y + x^2$$
$$U_n^{0,1}(x, y) = y^2 + xy + x^2 + xy^2 + x^2y + y + x$$

This is the doubly refined enumeration. Of course, by specializing one variable, one recovers the simple refined enumeration, *i.e.* that the number of ASM of size n with the 1 of the first row in the $(i + 1)^{\text{th}}$ position is the same as the number of NILP (corresponding to the TSSCPP and with the extra step) with i vertical extra steps:

$$A_n(x) := \tilde{A}_n(x, 1) = U_n^{0,1}(x, 1) := U_n^0(x).$$

and by specializing two variables, that the number of ASM of size n is the same as the number of TSSCPP of size $2n$:

$$A_n = A_n(1) = U_n^0(1).$$

Corollary 4.5. *The following identities hold:*

$$\tilde{A}_n(x, y) = U_n^{0,i}(x, y) \qquad \textit{for all } 1 \le i \le n$$
$$A_n(x, y) = U_n^{1,i}(x, y) \qquad \textit{for all } 2 \le i \le n$$

Proof. This is a simple consequence of the main theorem and Propositions 4.2 and 4.3. □

4.3 Integral formulæ

4.3.1 ASM counting as an integral formula

In this section, using the partition function obtained in Section 2.2.2, we get an integral formula for the quantity $\tilde{A}_n(x, y)$ and we prove that it is identical to a certain integral formula which counts NILP.

The first step is to modify the spectral parameters $z_i = q^{-1}y_i$ for $1 \le i \le n$ and $z_i = qy_i$ for $n < i \le 2n$. In this way, the homogeneous limit corresponds to $z_i = 1$ for all i. Moreover for $q^3 = 1$, the partition function Z_n is symmetric in the set $\{z_1, \dots, z_{2n}\}$.

The weights become

$$a = q^2 z_i - q^{-2} z_j$$
$$b = q^{-1} z_j - q z_i$$
$$c = (q^{-1} - q) z_i^{1/2} z_j^{1/2}$$

where z_i is the new horizontal spectral parameter and z_j the new vertical one. We changed the sign of b, using the fact that there is an even number of sites with weight b.

The refined counting

The case of interest to us is when all $z_i = 1$ except z_1 and z_{2n}:

$$z_1 = \frac{1+qt}{q+t}$$
$$z_{2n} = \frac{1+qu}{q+u}$$

Using the fact that $Z_n(z_1, \dots, z_{2n})$ is a symmetric function of its arguments, we have:

$$Z_n\left(z_1 = \frac{1+qt}{q+t}, 1, \dots, 1, z_{2n} = \frac{1+qu}{q+u}\right)$$
$$= Z_n\left(z_1 = \frac{1+qt}{q+t}, 1 \dots, 1, z_n = \frac{1+qu}{q+u}, 1, \dots\right).$$

The corresponding weights take the form

$$a_x = q^2\frac{1+qx}{q+x} - q^{-2} = q^2 x \frac{q-q^{-1}}{q+x};$$
$$b_x = q^{-1} - q\frac{1+qx}{q+x} = q^2\frac{q-q^{-1}}{q+x};$$
$$c_x = (q^{-1}-q)\sqrt{\frac{1+qx}{q+x}}.$$

The partition function Z_n becomes

$$Z_n = (-1)^{\binom{n}{2}}(q-q^{-1})^{n^2-3n}\sqrt{\frac{q+t}{1+qt}\frac{q+u}{1+qu}}\sum_{j,k} a_t^{j-1} b_t^{n-j} c_t a_u^{k-1} b_u^{n-k} c_u A_{n,j,k}$$
$$= 3^{\binom{n}{2}}\left(\frac{1}{q+t}\right)^{n-1}\left(\frac{1}{q+u}\right)^{n-1} q^{4(n-1)}\sum_{j,k} t^{j-1}u^{k-1}A_{n,j,k},$$

where $A_{n,j,k}$ is the number of ASM of size n such that the only 1 in the first row is in column j and the only 1 in the last row is in column $n-k+1$.

Or, explicitly:

$$\frac{(q^2(q+t)(q+u))^{n-1}}{3^{n(n-1)/2}} Z_n = \sum_{j,k} t^{j-1}u^{k-1}A_{n,j,k} = A_n(t,u). \tag{4.7}$$

Note that if one uses instead $z_{2n} = \frac{q+u}{1+qu}$, one gets the same formula, but with one index reversed

$$\frac{(q^2(q+t)(1+qu))^{n-1}}{3^{n(n-1)/2}} Z_n = \tilde{A}_n(t,u). \tag{4.8}$$

Integral formula

The traditional expression for the partition function of the 6-V model is the already mentioned IK formula. We shall not use it here. We shall only need the following facts (true at $q = e^{2\pi i/3}$):

- $Z_1 = 1$;
- $Z_n(z_1, \dots, z_{2n})$ is a polynomial of degree $n - 1$ in each variable;
- The Z_n satisfy the recursion relation for all $i, j = 1, \dots, 2n$

$$Z_n(z_1, \dots, z_j = q^2 z_i, \dots, z_{2n}) = \prod_{k \neq i,j} (q z_i - z_k) Z_{n-1}(z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_{2n}). \quad (4.9)$$

These facts were proved in Section 2.2.2. See Proposition 2.2 for the second fact. When we use the new variables z_i, Proposition 2.3 becomes (4.9), recall that $q^3 = 1$.

The strategy is now to introduce a certain integral representation of the partition function of the 6-V model with DWBC, say Z'_n

$$Z'_n := (-1)^{\binom{n}{2}} \prod_{i<j}^{2n} (q z_i - q^{-1} z_j) \times \oint \dots \oint \prod_{l}^{n} \frac{dw_l}{2\pi i} \frac{(q z_{2l-1} - q^{-1} w_l) \prod_{l<m} (w_m - w_l)(q w_l - q^{-1} w_m)}{\prod_{i \le 2l-1} (w_l - z_i) \prod_{i \ge 2l-1} (q w_l - q^{-1} z_i)}, \quad (4.10)$$

where the integration contours surround counterclockwise the z_i but not the $q^{-2} z_i$, and to show that Z_n and Z'_n are both polynomials of degree $n - 1$ in each variable which satisfy the “wheel condition” and coincide at the c_n specializations of Lemma 1.14.

Let us first check that Z_n satisfies the wheel condition. This is a direct consequence of equation (4.9) in which one sets $z_k = q^4 z_i$. It is equally straightforward to calculate Z_n at the c_n points of the lemma. The computation goes inductively using equation (4.9) and it is left to the reader to check that

$$Z_n(q^{\epsilon_1}, \dots, q^{\epsilon_{2n}}) = 3^{\binom{n}{2}}.$$

We now show that Z'_n also satisfies the hypotheses of the lemma. We proceed in steps.

Proposition 4.6. *Z'_n is a polynomial of degree $n - 1$ in each variable, and total degree $n(n - 1)$.*

Proof. By applying the residue formula to equation (4.10) we obtain

$$Z'_n = (-1)^{\binom{n}{2}} \sum_{\substack{K=(k_1,\ldots,k_n)\\ k_l\neq k_m \text{ if } l\neq m\\ k_l\leq 2l-1}} (-1)^{s(K)} \prod_{l<m} (qz_{k_l} - q^{-1}z_{k_m}) \\ \times \frac{\displaystyle\prod_{\substack{i<j\\ i\notin K \text{ or } (i=k_l \text{ and } j<2l-1)}} (qz_i - q^{-1}z_j) \prod_{2i-1\neq k_i} (qz_{2i-1} - q^{-1}z_{k_i})}{\displaystyle\prod_{\substack{i\leq 2j-1\\ i\notin K \text{ or } i>k_j}} (z_{k_j} - z_i)}, \qquad (4.11)$$

where $(-1)^{s(K)}$ is the sign of the permutation that orders the k_i. It is enough to prove that $\lim_{z_{k_j}\to z_i} Z'_n$ exists; the verification is a tedious but easy calculation (see Proposition 1.22 for a similar check).

We can now consider the leading term in each variable z_i in the summation of eq. (4.11), depending on whether $i \in K$ or not; in both cases we find a degree $n-1$.

The total degree is straightforward. □

Proposition 4.7. *Z'_n satisfies the wheel condition.*

Proof. Using the formula (4.11), we can verify that Z'_n is zero at $z_k = q^2 z_j = q^4 z_i$ for all $k > j > i$: in fact, the term $\prod_{s<r \text{ and } s\notin K}(qz_s - q^{-1}z_r)$ implies that i and $j \in K$. As a consequence of the term $\prod_{l<m}(qz_{k_l} - q^{-1}z_{k_m})$, we must have $i = k_m$ and $j = k_l$ with $l < m$, but, in this case, $j \leq 2l-1 < 2m-1$ proving that Z'_n satisfies the wheel condition. □

Proposition 4.8. *At $q = e^{2\pi i/3}$, Z'_n satisfies a weaker form of recursion relation (4.9). Let j be an integer between 1 and $2n-1$. At $z_{j+1} = q^2 z_j$, we get:*

$$Z'_n(\ldots, z_j, z_{j+1} = q^2 z_j, \ldots) = \prod_{i\neq j,j+1} (qz_j - z_i) Z'_{n-1}(z_1, \ldots, z_{j-1}, z_{j+2}, \ldots, z_{2n}). \qquad (4.12)$$

Proof. We will perform the calculation for j even.

If we look at formula (4.11) it is straightforward that all terms are zero except for $j = k_m$ and $j+1 \geq 2m-1$, *i.e.* $j = k_m = 2m-2$. Using the fact that $z_{j+1} = q^2 z_j$, we can derive

$$\begin{aligned}Z'_n\big|_{z_{j+1}=q^2z_j} =& \prod_{i<j}(qz_i - q^{-1}z_j)(qz_i - qz_j) \prod_{k>j+1}(qz_j - q^{-1}z_k)(z_j - q^{-1}z_k)(-1)^{\binom{n}{2}} \\ &\times \prod_{i<k\neq j,j+1}(qz_i - q^{-1}z_k) \oint \cdots \oint \prod_l \frac{dw_l}{2\pi i} \prod_{l\neq m}(qz_{2l-1} - q^{-1}w_l) \\ &\times \frac{\prod_{l<p\neq m}(w_p - w_l)(qw_l - q^{-1}w_p)(z_j - q^{-1}z_j)}{\prod_{l\neq m}\prod_{\substack{i\leq 2l-1\\ i\neq j,j+1}}(w_l - z_i)\prod_{\substack{i\geq 2l-1\\ i\neq j,j+1}}(qw_l - q^{-1}z_i)} \\ &\times \prod_{n>m}\frac{(w_n - z_j)(qz_j - q^{-1}w_n)}{(w_n - z_j)(w_n - q^2z_j)} \prod_{l<m}\frac{(z_j - w_l)(qw_l - q^{-1}z_j)}{(qw_l - q^{-1}z_j)(qw_l - qz_j)} \\ &\times \frac{1}{(z_j - q^2z_j)\prod_{i<j}(z_j - z_i)\prod_{k>j+1}(qz_j - q^{-1}z_k)}.\end{aligned}$$

After multiple cancellations we get the desired result, that is equation (4.12).

The formula actually holds for both parities of j; the proof for j odd is similar. □

Using the formula above, we can easily calculate Z'_n at the c_n points of the lemma.

Proposition 4.9. *Let $q^\epsilon = (q^{\epsilon_1}, \ldots, q^{\epsilon_{2n}})$ be a specialization (as in Section 1.4.2). We have:*

$$Z'_n(q^\epsilon) = 3^{\binom{n}{2}}.$$

Proof. One can always choose two consecutive variables which are (q^{-1}, q) and apply the recursion relation above:

$$\begin{aligned}Z'_n(\ldots, z_j = q^{-1}, z_{j+1} = q^2z_j = q, \ldots) &= \prod_{i\neq j,j+1}(1 - z_i)Z'_{n-1} \\ &= (1-q)^{n-1}(1-q^{-1})^{n-1}Z'_{n-1}.\end{aligned}$$

The second equality uses the fact that there is the same number of $\epsilon_i = 1$ and $\epsilon_i = -1$. Since we have $Z'_1 = 1$, we prove the proposition. □

We finally conclude, by applying Lemma 1.14, that

$$Z_n = Z'_n.$$

Starting from our new integral formula for the partition function of the 6-Vertex model (4.10), we are now in the position to calculate

$$\frac{(q^2(q+x)(1+qy))^{n-1}}{(q-q^{-1})^{n(n-1)}} Z_n\left(\frac{1+qx}{q+x}, 1, \ldots, 1, \frac{q+y}{1+qy}\right).$$

After some tedious computations and using new variables

$$u_i = \frac{w_i - 1}{qw_i - q^{-1}}$$

we obtain:

$$(y+x-yx)\oint\ldots\oint\prod_{l}^{n}\frac{du_l}{2\pi i}\frac{1}{u_l^{2l-2}}\frac{\prod_{l<m}(u_m-u_l)(1+u_m+u_m u_l)}{(1+u_l-x)(1+u_l(1-y))}\prod_{j=2}^{n}(1+u_j)\,,$$

where the integral contours surround counterclockwise $u_i=0$ and $u_i=x-1$ but not $1/(y-1)$.

To simplify our calculation we integrate with respect to u_1:

$$\tilde{A}_n(x,y)=\oint\ldots\oint\prod_{l=2}^{n}\frac{du_l}{2\pi i}\frac{(1+u_l)(1+xu_l)}{u_l^{2l-2}(1+u_l(1-y))}\prod_{l<m}(u_m-u_l)(1+u_m+u_m u_l),\quad (4.13)$$

where the contours surround the remaining poles at $u_i=0$ only.

4.3.2 NILP counting as an integral formula

We shall derive an integral formula for the generating polynomial $N'_{10}(t_0,t_1,\ldots,t_{n-1})$ of our NILP with a weight t_i per vertical step in the i^{th} slice (between $y=1-i$ and $y=-i$). We use the Lindström–Gessel–Viennot formula [46, 27] (described in 3.2.1):

$$N'_{10}(t_0,t_1,\ldots,t_{n-1})=\sum_{\substack{1=r_1<\ldots<r_{n-1}\\ r_i\le 2i+1\\ r_{i+1}-r_i\ \text{odd}}}\det[\mathcal{P}_{i,r_j}],\quad (4.14)$$

where $\mathcal{P}_{i,r}$ is the weighted sum over all possible lattice paths from $(i,-i)$ to $(r+1,1)$. Such paths counts with $r-i+1$ diagonal steps and $2i-r$ vertical ones, hence:

$$\mathcal{P}_{i,r}=\sum_{0\le i_1<\ldots<i_{2i-r}\le i}\prod_{l=1}^{2i-r}t_{i_l}=\prod_{k=0}^{i}(1+t_k u)|_{u^{2i-r}}\,,\quad (4.15)$$

where the subscript u^{2i-r} stands for the coefficient of the corresponding power of u in the polynomial.

We can reintroduce the path beginning at $(0,0)$ and rewrite the equation as a contour integral:

$$N'_{10}(t_0,t_1,\ldots,t_{n-1})=\oint\ldots\oint\prod_{i=1}^{n}\frac{du_i}{2\pi i u_i^{2i-1}}\prod_{k=0}^{i-1}(1+t_k u_i)\sum_{\substack{0=r_0<r_1<\ldots<r_{n-1}\\ r_{i+1}-r_i\ \text{odd}}}\det[u_i^{r_j-1}],$$

where the paths of integrations are small counterclockwise circles around zero.

The last sum can be evaluated as a standard result for the sum over all Schur functions corresponding to *even* partitions (see Exercise 4.3.10 in [7]):

$$\sum_{\substack{0=r_0<r_1<\ldots<r_{n-1}\\ r_{i+1}-r_i\ \text{odd}}}\det[u_i^{r_j-1}]=\frac{\prod_{j>i}(u_j-u_i)}{\prod_{j\ge i}(1-u_j u_i)},\quad (4.16)$$

where we have relaxed the condition $r_0 = 0$ into $r_0 \geq 0$ and even, since this does not affect the integral.

The integral can thus be transformed as follows:

$$N'_{10}(t_0, t_1, \dots, t_{n-1}) = \oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}} \frac{1}{1-u_i^2} \prod_{k=0}^{i-1}(1 + t_k u_i) \prod_{j>i} \frac{u_j - u_i}{1 - u_j u_i}. \quad (4.17)$$

We are mainly interested in the case where $t_0 = t$, $t_1 = s$ and all the other t_i equal to 1. In this case, we rewrite the equation:

$$\begin{aligned} N'_{10}(t, s, 1, \dots, 1) &:= U_n^{0,1}(t, s) \\ &= \oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}} \frac{1}{1-u_i^2} (1 + t u_i)(1 + s u_i)_{\hat{1}} (1 + u_i)_{\hat{1}}^{i-2} \prod_{j>i} \frac{u_j - u_i}{1 - u_j u_i}, \end{aligned} \quad (4.18)$$

where $\hat{1}$ means that we omit the term corresponding to u_1.

4.3.3 Equality between integral formulæ

At this point, we have two integral expressions, $\tilde{A}_n(x, y)$ in equation (4.13) and $U_n^{0,1}(x, y)$ in equation (4.18) and we want to prove that they are the same. The first step is to integrate over u_1 the expression (4.18):

$$U_n^{0,1}(x, y) = \oint \dots \oint \prod_{i=2}^{n} \frac{du_i}{2\pi i u_i^{2i-2}} (1 + x u_i)(1 + y u_i)(1 + u_i)^{i-2} \frac{\prod_{i<j}(u_j - u_i)}{\prod_{i \leq j}(1 - u_j u_i)}. \quad (4.19)$$

At this stage we use the following identity:

$$\begin{aligned} &\oint \dots \oint \prod_i \frac{du_i}{2\pi i} \frac{\varphi(u)}{u_i^{2i}} \prod_{i<j} (u_j - u_i)(1 + \tau u_j + u_i u_j) \\ &\qquad = \oint \dots \oint \prod_i \frac{du_i}{2\pi i} \varphi(u) \frac{(1 + \tau u_i)^{i-1}}{u_i^{2i}} \frac{\prod_{i<j}(u_j - u_i)}{\prod_{i \leq j}(1 - u_i u_j)}. \end{aligned} \quad (4.20)$$

for any $\varphi(u)$ completely symmetric in $(u_1, u_2, \dots, u_n)$ and without poles in a neighborhood of zero. This was conjectured in [17] and proved in [81]. We present in Appendix C.1 an independent proof of a stronger formula that implies eq. (4.20).

If we shift the indices $(i-1) \to i$, consider $\tau = 1$ and set $\varphi(u) = \prod_{i=1}^{n-1}(1+x u_i)(1+y u_i)$ we can apply the equality:

$$U_n^{0,1}(x, y) = \oint \dots \oint \prod_{i=2}^{n} \frac{du_i}{2\pi i u_i^{2i-2}} (1 + x u_i)(1 + y u_i) \prod_{i<j} (u_j - u_i)(1 + u_j + u_j u_i). \quad (4.21)$$

Now we observe that the two integrals are the same, except for the factors $\frac{(1+u_l)}{(1+u_l(1-y))}$ on one side, and $1 + y u_l$ on the other side. Unsurprisingly, we find that it is possible to write both integrals as special cases of the same integral:

$$I_n(x,y) = \oint \dots \oint \prod_{l=1}^{n-1} \frac{du_l}{2\pi i} \frac{(1+u_l+a_l u_l^2)(1+xu_l)}{u_l^{2l}(1+u_l(1-y))} \prod_{l<m}^{n-1} (u_m-u_l)(1+u_m+u_m u_l) \quad (4.22)$$

which takes the value of $\tilde{A}_n(x,y)$ if $a_l = 0$ for all l and takes the value of $U^{0,1}(x,y)$ if $a_l = y(1-y)$ for all l.

Surprisingly, I_n does not depend on the a_i. We shall show by induction on i that I_n is independent of a_i, noting that it is a polynomial in a_i of degree at most 1.

Let us first differentiate I_n with respect to a_1:

$$\frac{d}{da_1} I_n(x,y) = \oint \frac{du_1}{2\pi i} \frac{(1+xu_1)}{(1+u_1(1-y))}$$
$$\times \oint \dots \oint \prod_{l=2}^{n-1} \frac{du_l}{2\pi i} \frac{(1+u_l+a_l u_l^2)(1+xu_l)}{u_l^{2l}(1+u_l(1-y))} \prod_{m<l}^{n-1} (u_l-u_m)(1+u_l+u_m u_l).$$

As this integral has no poles at u_1, it vanishes.

Let us now assume by induction hypothesis that I_n does not depend on the first $a_1,\dots,a_{i-1}$, and prove that the expression (4.22) does not depend on a_i either. As the integral does not depend on a_j for all $j<i$ we can set all $a_1 = \dots = a_{i-1} = 0$.

We now differentiate with respect to a_i and analyse what happens in the integration up to u_i. We find an expression of the type:

$$J_i = \oint \frac{du_i}{2\pi i u_i^{2i-2}} \oint \cdots \oint \prod_{j=1}^{i-1} \frac{du_j}{2\pi i} \frac{1+u_j}{u_j^{2j}} \Theta_i A_i, \quad (4.23)$$

where A_i is some anti-symmetric function in the u_j for all $j \le i$ without any poles in the integration domain, and $\Theta_i = \prod_{j<i}(1+u_i+u_j u_i)$.

To prove that this integral is always zero we shall proceed once again by induction. The first one, J_1, is zero because it has no poles:

$$J_1 = \oint \frac{du_1}{2\pi i} A_1(x_1) = 0. \quad (4.24)$$

Let $J_{i-1} = 0$. All the poles are at 0, the A_i is anti-symmetric between u_i and u_{i-1}, so we can take advantage of the fact that the u_i appears with the same degree as u_{i-1} in the denominator to erase all the symmetric terms in the expression $(1+u_{i-1})(1+u_i+u_{i-1}u_1)$ and get $u_i u_{i-1}^2$:

$$J_i = \oint \frac{du_i}{2\pi i u_i^{2i-3}} \oint \frac{du_{i-1}}{2\pi i u_{i-1}^{2i-4}} \oint \dots \oint \prod_{j=1}^{i-2} \frac{du_j}{2\pi i} \frac{1+u_j}{u_j^{2j}} \hat{\Theta}_i A_i, \quad (4.25)$$

where the hat in $\hat{\Theta}_i$ means that the term $(1+u_i+u_{i-1}u_i)$ is omitted[1]. The integral does not have yet the desired form, *i.e.* J_{i-1}, it is missing the term $(1+u_i+u_{i-1}u_i)$, so we

[1] Note that $\hat{\Theta}_i$ is symmetric between u_{i-1} and u_i.

add and subtract it:

$$J_i = \oint \dots \oint \frac{du_i}{2\pi i u_i^{2i-3}} \frac{du_{i-1}}{2\pi i u_{i-1}^{2i-4}} \prod_{j=1}^{i-2} \frac{du_j}{2\pi i} \frac{1+u_j}{u_j^{2j}} (1+u_i+u_iu_{i-1}-u_i-u_iu_{i-1})\hat{\Theta}_i A_i$$
$$= \oint \dots \oint \frac{du_i}{2\pi i u_i^{2i-3}} \frac{du_{i-1}}{2\pi i u_{i-1}^{2i-4}} \prod_{j=1}^{i-2} \frac{du_j}{2\pi i} \frac{1+u_j}{u_j^{2j}} \Theta_i A_i$$
$$- \oint \dots \oint \frac{du_i}{2\pi i u_i^{2i-4}} \frac{du_{i-1}}{2\pi i u_{i-1}^{2i-4}} \prod_{j=1}^{i-2} \frac{du_j}{2\pi i} \frac{1+u_j}{u_j^{2j}} (1+u_{i-1})\hat{\Theta}_i A_i.$$

The first term is already in the form of J_{i-1}. The second term is almost symmetric in u_i and u_{i-1}. Using the same method as in (4.25) we can transform $-(1+u_{i-1})$ to $1+u_i+u_iu_{i-1}$; in this way, we recover the symmetry needed so that we can write J_i as an integral in u_i of some function multiplied by J_{i-1}, which is zero. As a consequence J_i is also zero for all i, *i.e.* $I_n(x,y)$ does not depend on any a_i. We conclude that

$$\tilde{A}_n(x,y) = U_n^{0,1}(x,y).$$

□

4.4 The original conjecture

Mills, Robbins and Rumsey conjectured this theorem by terms of TSSCPP, not NILP, but behind the different formulations lies the same result. To show that, we describe some of the content of [51] and explain the equivalence.

Recall that TSSCPP can be represented as $2n \times 2n$ matrices a, as in equation (3.1). In [51] is introduced a quantity which we shall denote by $u_n^k(a)$, for k ranging between 1 and n and which depends on the upper-left $n \times n$ submatrix of a:

$$u_n^k(a) = \sum_{t=1}^{n-k+1} (a_{t,t+k-1} - a_{t,t+k}) + \sum_{t=n-k+2}^{n} \sharp\{a_{t,n} \mid a_{t,n} > 2n-t+1\}, \qquad (4.26)$$

where $\sharp$ denotes cardinality, and where conventionally, $a_{t,n+1} := 2n-t+1$ in this equation. Also defined is the function:

$$U_n^{i,\dots k}(x,\dots,z) = \sum_a x^{u_n^i(a)} \dots z^{u_n^k(a)} \qquad \text{for all } i,\dots,k \in \{1,\dots,n+1\}. \qquad (4.27)$$

We claim that these are our functions u and U defined above. To make the connection, reexpress this function in terms of the lower-right $n \times n$ submatrix of a:

$$u_n^k(a) = \sum_{t=n+k}^{2n} (a_{t,t-k} - a_{t,t-k+1}) + \sum_{t=n+1}^{n+k-1} \sharp\{a_{t,n+1} \mid a_{t,n+1} < 2n-t\}, \qquad (4.28)$$

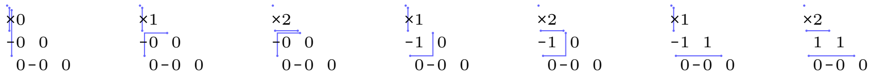


Figure 4.5: We can see on this figure what the function u_3^2 counts. The minus signs represent the parts: $a_{t,t-k} - a_{t,t-k+1}$, so they count the vertical steps, and the little crosses represent $\sharp\{a_{t,n+1} \mid a_{t,n+1} < 2n - t\}$. If we stretch our diagrams to obtain the NILP we recover our definition of u_n^k.

where we replace $a_{t,n}$ with $2n-t$. What this function counts is described on Figure 4.5. Finally, if we shift the diagrams to obtain NILP we recover our functions U_n^k as expected.

As a final remark, in the article [69] three functions are defined: f_1, f_2 and f_3 and the conjecture is stated with any two of them. In fact, f_1 is connected with the u_n^0, f_2 with the u_n^1 and f_3 with u_n^n, as can be seen using the same procedure.

Chapter 5

Completely Packed Loops and Plane Partitions

In this chapter we address a number of conjectures about the ground state of the Completely Packed Loops model, stated by Zuber in [85].

Note that thanks to the Razumov–Stroganov–Cantini–Sportiello theorem, stated in [65] and proved in [8], they can be considered as either conjectures on the FPL model (in which case several of them, including one to be discussed below, were proved in [9, 10]), or on the Completely Packed Loops model, the latter point of view being presenting here.

We shall also obtain some new results connecting the enumeration of certain classes of Plane Partitions and Non-Intersecting Lattice Paths (NILP) with matchings of the form $(\pi)_p$ and $(_p\alpha$ (see Section 5.1 for an explanation of the notation), and prove a conjecture presented in [53]. These can be thought of as a small step towards a bijection between Totally Symmetric Self–Complementary Plane Partitions (TSSCPP) and Alternating Sign Matrices (ASM), the latter being in trivial bijection with FPLs (as explained in 2.3), since they provide families of equinumerous classes of TSSCPP and ASM.

The chapter is organized as follows. In Section 5.1 we present some required notations. In the second and third sections we obtain some properties of the entries of this polynomial solution indexed by matchings of the form $(\pi)_p$ and $(_p\alpha$, respectively. In particular in each case, we describe the corresponding counting problem for NILP and TSSCPP.

5.1 Some notation

We saw in 1.1.2 that a matching can be represented by a well-formed sequence of parentheses. We use the notation $(\pi)_p$ to represent p parentheses surrounding a matching π:

$$(\pi)_p = \underbrace{(\ldots(}_{p}\pi\underbrace{)\ldots)}_{p},$$

and $()^p$ for p successive $()$:

$$()^p = \underbrace{()\ldots()}_{p}.$$

We introduce here the notation $(_p$, which represents p successive openings. Of course, these parentheses must match with some closing parentheses, for example:

$$({}_3()^2)())) = (((()())())).$$

5.2 Study of entries of the type $(\pi)_p$

In general, the computation of the polynomials Ψ_π is complicated, and there is no general closed formula. But there are some exceptions. In this section we study the polynomials indexed by configurations of the type $(\pi)_p$, *i.e.* a given configuration surrounded by p parentheses (see Figure 5.1 for an example). In the last subsection we present some properties of the polynomials for high p.

In all that follows, π is a link pattern of size $2r$, so that $(\pi)_p$ has size $2n$ with $n = r+p$.

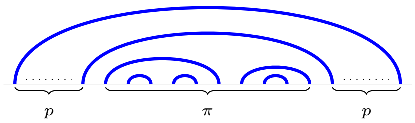


Figure 5.1: A matching $(\pi)_p$ with p arches surrounding a link pattern π.

5.2.1 a-Basis

As explained in 1.6 we decompose our polynomials in the a-basis. In this case, we find that:

Lemma 5.1. *We have the following decomposition*

$$\Psi_{(\pi)_p}(z_1,\ldots,z_{2n}) = \sum_{\substack{1\le a_1<\ldots<a_r<2r \\ a_i\le 2i-1}} C^{-1}_{\pi,a}\Phi_{1,\ldots,p,p+a_1,\ldots,p+a_r}(z_1,\ldots,z_{2n}), \tag{5.1}$$

where the coefficients are the same that occur in

$$\Psi_\pi(z_1,\ldots,z_{2r}) = \sum_{\substack{1\le a_1<\ldots<a_r<2r \\ a_i\le 2i-1}} C^{-1}_{\pi,a}\Phi_{a_1,\ldots,a_r}(z_1,\ldots,z_{2r}). \tag{5.2}$$

Thus, if we find the $C^{-1}_{\pi,a}$ which satisfy the equation (5.2), these same coefficients solve the equation (5.1).

Proof. One checks that the triangularity forces $\Phi_{1,\ldots,p,p+a_1,\ldots,p+a_r}$ to be written as a linear combination of entries of the type $\Psi_{(\pi)_p}$, and these coefficients do not depend on the value of p. When one inverts the transformation only these coefficients will matter, proving the lemma. □

5.2.2 Reduction to size r

Now, for a given link pattern π of size $2r$, we can calculate polynomials $\psi_{(\pi)_p}$ for all p. For example:

$$\begin{aligned} \psi_{(()())_p} &= (n-1)\tau \\ \psi_{(()()())_p} &= \frac{n-2}{6}\tau(2\tau^2 n^2 - 5n\tau^2 + 3\tau^2 + 6) \\ \psi_{((())())_p} &= \frac{(n-2)(n-1)}{2}\tau^2 \end{aligned} \tag{5.3}$$

where $\tau = -(q+q^{-1})$. These are the steps of the calculation:

- For given p and r we compute the $C_{a,\pi}$ for all π and a of size r;
- We invert this matrix;
- We calculate

$$\Phi_{1,\dots,p,p+a_1,\dots,p+a_r}(z_1,\dots,z_{2n}) = (-1)^{\binom{n}{2}} \prod_{i<j\leq p}(qz_i - q^{-1}z_j) \prod_{p<i<j\leq 2n}(qz_i - q^{-1}z_j)$$
$$\times \oint \cdots \oint \prod_{i=1}^{r} \frac{dw_i}{2\pi i} \frac{\prod_{1\leq i<j\leq r}(w_j - w_i)(qw_i - q^{-1}w_j)\prod_{1\leq j\leq p}(qz_j - q^{-1}w_i)}{\prod_{p<k\leq a_i+p}(w_i - z_k)\prod_{a_i+p<k\leq 2n}(qw_i - q^{-1}z_k)},$$

where the integration in the first p variables is already performed;

- We use the variable transformation

$$u_i = \frac{w_i - 1}{qw_i - q^{-1}} \tag{5.4}$$

to calculate the limit where $z_i = 1$ for all i, and to finally get:

$$\phi_{1,\dots,p,p+a_1,\dots,p+a_r} = \oint \cdots \oint \prod_{l=1}^{r} \frac{du_l}{2\pi i u_l^{a_l}} \prod_{l<m\leq r}(u_m - u_l)(1+\tau u_m + u_l u_m)(1+\tau u_m)^p.$$

In Appendix D we present a complete list of examples for $r \leq 4$.

5.2.3 Expansion for high p

Zuber conjectured the polynomial dependence in p and large p behavior of the number of Fully Packed Loop configurations with connectivity $(\pi)_p$ (Conjecture 6 in [85]). This was subsequently proved in [10]. Alternatively, due to the Razumov–Stroganov theorem, one expects the same behavior for the ground state entries of the Completely Packed Loops model.

It is important to notice that this work [24] appeared before the proof of the Razumov–Stroganov conjecture and this in conjunction with [10] provided additional support to their conjecture.

Here we generalize it to the qKZ solution for any τ:

Theorem 5.2. *For matchings of the type $(\pi)_p$, the polynomials ψ_π can be written in the following form:*

$$\psi_{(\pi)_p} = \frac{1}{|Y|!} P_Y(\tau, n),$$

where $Y = Y(\pi)$ is the Young diagram defined by π, $|Y|$ its number of boxes, and $P_Y(\tau, n)$ is a polynomial in n and τ of degree $|Y|$ in each variable with integer coefficients.

In the limit of large n, the polynomials behave like

$$\psi_{(\pi)_p} \approx \frac{\dim Y}{|Y|!} (n\tau)^{|Y|},$$

where $\dim Y$ is the dimension of the irreducible representation of the symmetric group associated to Y.

Proof. As the basis transformation is triangular, we can write:

$$\psi_{(\pi)_p} = \phi_{1,\ldots,p,p+a_1,\ldots,p+a_r} + \sum_{b<a} \tilde{C}_{\pi,b} \phi_{1,\ldots,p,p+b_1,\ldots,p+b_r},$$

where a is equivalent to π and $b < a$ means that the Young diagram of b is inside of the Young diagram of a. We denote the Young diagram corresponding to a by $Y(a)$.

Note that, by Lemma 1.27, $C^{-1}_{\pi,b}$ are polynomials of τ with integer coefficients and degree no more than $|Y(a)| - |Y(b)| - 2$.

We now prove that the integral of the first term, corresponding to the largest Young diagram in the decomposition, is a polynomial of n with the asymptotic behavior of the theorem. The other terms will possess the same polynomiality property, and they will be of lower degree in n and τ (noting that the power of τ in the non-diagonal elements is less than $|Y(a)| - |Y(b)|$ and does not affect our conclusion).

We want to calculate the integral

$$\phi_{1,\ldots,p,p+a_1,\ldots,p+a_r} = \oint \ldots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i u_i^{a_i}} \prod_{i<j\leq r} (u_j - u_i)(1 + \tau u_j + u_j u_i)(1 + \tau u_j)^p. \quad (5.5)$$

We replace the term $\prod_{i<j}(1 + \tau u_j + u_i u_j)$ with $\prod_{i<j}(1 + \tau u_j) = \prod_i (1 + \tau u_i)^{i-1}$, because any term with $u_i u_j$ in the product is formally identical to the contribution of a smaller diagram. We then compute

$$\begin{aligned}
&\oint \ldots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i u_i^{a_i}} (1 + \tau u_i)^{p+i-1} \prod_{j>i} (u_j - u_i) \\
&\qquad = \sum_\sigma (-1)^\sigma \oint \ldots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i u_i^{a_i}} (1 + \tau u_i)^{p+i-1} u_i^{\sigma_i - 1} \\
&\qquad = \sum_\sigma (-1)^\sigma \oint \ldots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i u_i^{i+\lambda_i+1-\sigma_i}} (1 + \tau u_i)^{p+i-1} \\
&\qquad = \tau^{|Y|} \sum_\sigma (-1)^\sigma \prod_{i=1}^{r} \binom{n - r + i - 1}{i + \lambda_{n-i+1} - \sigma_i},
\end{aligned}$$

where $\lambda_{n-i+1} = a_i - i$ is the size of each row in the Young diagram and $(-1)^\sigma$ is the sign of the permutation σ. We obtain that the coefficients can be written as a sum of integers divided by $\prod_i (i + \lambda_i - \sigma_i)!$ which divide $|Y|!$ as a consequence of $\sum_i (i + \lambda_i - \sigma_i) = |Y|$.

The dominant contribution as a function of n is

$$\tau^{|Y|} \sum_\sigma (-1)^\sigma \prod_{i=1}^r \frac{n^{i+\lambda_{n-i+1}-\sigma_i}}{(i + \lambda_{n-i+1} - \sigma_i)!} = (\tau n)^{|Y|} \sum_\sigma (-1)^\sigma \prod_{i=1}^r \frac{1}{(i + \lambda_{n-i+1} - \sigma_i)!}$$
$$= (\tau n)^{|Y|} \frac{\dim Y}{|Y|!},$$

where the last equality can be found, for example, in the fourth chapter of [26]. □

5.2.4 Sum rule

Consider strings a of the form $(a_1, \dots, a_r)$, with $a_i = 2i - 1$ or $a_i = 2i - 2$.[1] Let $\mathcal{L}(a)$ be the set of matchings whose openings on odd sites are exactly the odd elements in a.

From [17], Section 3.3, we know:

Lemma 5.3.

$$\Phi_a = \sum_{\pi \in \mathcal{L}(a)} \Psi_\pi.$$

Proof. We shall compute the coefficients $C_{a,\pi}$ and prove that:

$$C_{a,\pi} = \begin{cases} 1 & \text{if } \pi \in \mathcal{L}(a); \\ 0 & \text{otherwise.} \end{cases}$$

We presented in Section 1.6.2 a graphical method to compute them. Let (i,j) be an arch in π. Let $k_{i\to j}(a)$ be the number of elements of a such that $i \le a_l < j$. We easily see that we can remove the arch (i,j) if and only if $k_{i\to j}(a) \ge \frac{j-i+1}{2}$. If so, let $k = k_{i\to j}(a) - \frac{j-i+1}{2}$, we can replace the arch by U_k.

Take an odd number i and assume that it does not belong to a. An odd point i is always linked to an even point j. Two cases can arise: either $j > i$ or $j < i$. We easily see that $k_{i\to j}(a) = \frac{j-i-1}{2}$ in the first case and $k_{j\to i} = \frac{i-j+1}{2}$ in the second case. Thus we can only remove an arch if $j < i$ and in this case we replace it by $U_0 = 1$.

Assume now that the odd number i belongs to a. We do the same analysis: $k_{i\to j}(a) = \frac{j-i+1}{2}$ if $j > i$ and $k_{j\to i}(a) = \frac{i-j-1}{2}$ if $j < i$. Thus we can replace the arch (i,j) by $U_0 = 1$ if and only if $j > i$.

This completes the proof. □

It follows that, using Lemma 5.1:

$$\sum_{\substack{a=(1,\dots,p,p+a_1,\dots,p+a_r) \\ a_i=2i-1 \text{ or } 2i-2}} \tau^{r^2-\sum_i a_i} \Phi_a = \sum_\pi \tau^{o_\pi} \Psi_{(\pi)_p}, \tag{5.6}$$

[1] Observe that if $a_1 = 0$, $\Phi_a = 0$.

where o_π counts the openings in even sites, of the matching π, and $\left(r^2 - \sum_i a_i\right)$ counts the number of even a_i in a, as $o_\pi = r^2 - \sum_i a_i$ if $\pi \in \mathcal{L}(a)$.

In (5.6), at $z_i = 1$ for all i, the l.h.s. is equal to:

$$\oint \dots \oint \prod_{l=1}^{r} \frac{du_l}{2\pi i u_l^{2l-1}} \prod_{l<m\leq r} (u_m - u_l)(1 + \tau u_m + u_l u_m)(1 + \tau u_m)^{p+1},$$

but this is exactly

$$\phi_{1,\dots,p+1,p+2,p+4,\dots,p+2r} = \psi_{(()^r)_{p+1}}.$$

We can now state the main result of this section:

Theorem 5.4. *Let o_π count the number of arches of π opening at an even site, we have the following result:*

$$\sum_{\pi \text{ of size } 2r} \tau^{o_\pi} \psi_{(\pi)_p} = \psi_{(()^r)_{p+1}}. \tag{5.7}$$

At $\tau = 1$, we get the proof of [85, Conjecture 8.i]:

$$\sum_{\pi \text{ of size } 2r} \psi_{(\pi)_p} = \psi_{(()^r)_{p+1}}. \tag{5.8}$$

At $p = 0$, we can use the rotation symmetry and obtain the already known formula:

$$\psi_{(()^r)} = \psi_{()^{r+1}} = \sum_{\pi \text{ of size } 2r} \psi_\pi$$

5.2.5 A NILP formula

We can interpret the result of the previous section in terms of the NILP. We fix the p first paths as exemplified in 5.2, and using the LGV formula (see Section 3.2.1), we are able to calculate the number of different NILP. We label the paths with $i = 1, \dots, r$ and call the final locations L_i.

If we only consider one path going from i to L_i:

$$\mathcal{P}_{i,L_i} = \tau^{2i-L_i-1} \binom{p+i-1}{2i-L_i-1},$$

we require that $L_1 = 1$. We give a weight τ to each vertical step.

The LGV formula tells us that the number of paths is equal to

$$\mathcal{F}_{p,r} = \sum_{1=L_1<\dots<L_r} \det \left[\tau^{2i-L_j-1} \binom{p+i-1}{2i-L_j-1} \right]_{1\leq i,j\leq r}.$$

We can use a contour integral form for the matrix entries:

$$\mathcal{P}_{i,L_i} = \oint \frac{du}{2\pi i} \frac{(1+\tau u)^{p+i-1}}{u^{2i-L_i}}$$

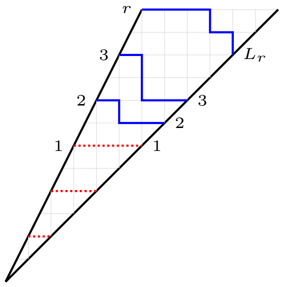


Figure 5.2: In order to apply the LGV formula we label the starting points of the paths by $1, \dots, r$ and the ending points by $L_1 = 1, \dots, L_r$. The paths can go right or down. The $p+1$ red dotted lines are fixed. As the paths do not intersect each other $L_{i+1} > L_i$.

so, the number of NILP can be expressed by:

$$\begin{aligned}
\mathcal{F}_{p,r} &= \oint \dots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \sum_{1=L_1<\dots<L_r} \det \left[\frac{(1+\tau u_k)^{p+k-1}}{u_k^{2k-L_j}} \right]_{1\le k,j\le r} \\
&= \oint \dots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{(1+\tau u_i)^{p+i-1}}{u_i^{2i-1}} \sum_{1=L_1<\dots<L_r} \det \left[u_k^{L_j-1} \right]_{1\le k,j\le r} \\
&= \oint \dots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{(1+\tau u_i)^{p+i-1}}{u_i^{2i-1}} \frac{1}{1-u_i} \prod_{i<j} \frac{u_j-u_i}{1-u_i u_j} \\
&= \oint \dots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{(1+\tau u_i)^{p+i-1}(1+u_i)}{u_i^{2i-1}} \frac{\prod_{i<j}(u_j-u_i)}{\prod_{i\le j}(1-u_i u_j)},
\end{aligned}$$

where the equality between the second and the third line can be found, for example, in the fourth chapter of [7].

Applying the identity (C.11) obtained in Section C.1, we transform the formula into:

$$\mathcal{F}_{p,r} = \oint \dots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i u_i^{2i-1}} (1+\tau u_i)^p (1+u_i) \prod_{i<j} (u_j-u_i)(1+\tau u_j + u_i u_j). \tag{5.9}$$

We recognize this integral formula:

$$\mathcal{F}_{p,r} = \sum_{\pi \text{ of size } 2r} \tau^{o_\pi} \psi_{(\pi)_p} = \psi_{(()^r)_{p+1}}.$$

There does not seem to be a simple closed formula for $\mathcal{F}_{p,r}$ even at $\tau = 1$, as suggested by the expressions of [85] for small values of p.

5.2.6 Punctured–TSSCPP

Recall that there is a bijection between NILP and TSSCPP. As is seen on Figure 5.3, fixing the first $p+1$ paths to be horizontal amounts to fixing the central hexagon of size $2p$. Observe in addition that the triangle which on the figure contains the paths is a fundamental domain *i.e.* defines the whole TSSCPP. Consequently, $\mathcal{F}_{p,r}$ also counts the number of TSSCPP with weight τ for each yellow face (the faces containing a vertical step of the paths) in the fundamental domain. See [13] for a similar interpretation of partial sums in a related model in terms of punctured plane partitions.

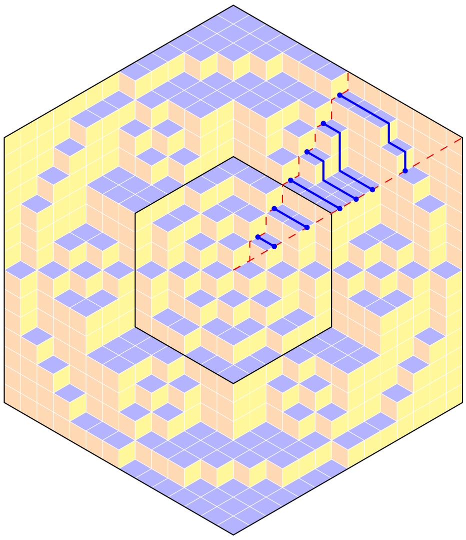


Figure 5.3: This TSSCPP with fixed central hexagon, of size $2p$, corresponds to the NILP of Figure 5.2 with $p+1$ fixed horizontal paths. The corresponding NILP are drawn in the fundamental domain (the triangle between the two blue dotted lines).

5.2.7 Limit shape

Let h_n be the stepped surface of a punctured-TSSCPP of size $2n \times 2n \times 2n$. We scale down the surface by a factor of n in order to have a constant size (when n varies). From Kenyon's article [38] we know that there must be a limit shape for n tending to infinity, which we call $\hat{h}$. For any $\epsilon > 0$, with probability tending to 1 as $n \to \infty$ a uniform random stepped surface h_n lies within ϵ of $\hat{h}$.

Thus, to see what $\hat{h}$ looks like it suffices to pick a random configuration from the huge but finite set of TSSCPP with a $2p \times 2p \times 2p$ fixed hexagon. In order to pick a random plane partition, we define an aperiodic irreducible Markov chain in the set of punctured-TSSCPP such that the stationary distribution is the uniform one. One approach is to run the Markov chain for M steps, with M sufficiently large, then the distribution will approximate the desired distribution. Unfortunately, it is difficult to know how large M must be such that the initialization bias becomes small.

Another approach is called coupling from the past. It was introduced by James Propp and David Wilson in a series of articles [62, 64, 63].

Instead of running from the present to the future, we run the Markov chain from the distant past to the present such that when we arrive at the present the initialization bias disappears. Let S be the space of configurations. Let $f_t : S \to S$ be the matrix which describes the step at time t. Let $F_{-N} = f_{-1} \circ f_{-2} \circ \ldots \circ f_{-N}$ be the composition of N steps. If F_{-N} is the constant map $F_{-N}(X) = Y$ for all X, any simulation which begins before $-N$ will converge to Y at $t = 0$. So, we only need to find N such that F_{-N} is the constant map.

There are some subtle points in their method, see [62, 64, 63] for a complete explanation. Moreover, of course, their method does not prevent the bias from the random number generator.

Preparing the algorithm

Let (i, j, k) be the coordinates of the boxes on the punctured-TSSCPP. A box is called odd if $i+j+k$ is odd, and even otherwise. Recall that the fundamental domain is defined by $n < j \leq 2n$ and $j \leq k \leq 2n$.

Let X and Y be two TSSCPP, we say that $X \preceq Y$ if all boxes in the fundamental domain of X are also in the fundamental domain of Y. There is only one maximum and one minimum configuration, call them $X_{\max}$ and $X_{\min}$ respectively.

Let f_t be a random map such that if $X \preceq Y$ then $f_t(X) \preceq f_t(Y)$. Thus, it is enough to ensure that $F_{-N}(X_{\min}) = F_{-N}(X_{\max})$, because any other configuration X will be in between $F_{-N}(X_{\min}) \preceq F_{-N}(X) \preceq F_{-N}(X_{\max})$.

The algorithm

We describe here, schematically, the program that we used to generate random configurations.

We run the algorithm for one step $F_{-1} = f_{-1}$, if F_{-1} is the constant map we stop. If not we run for 3 steps $F_{-3} = f_{-1} \circ f_{-2} \circ f_{-3}$ and we test again if F_{-3} is the constant map. We repeat the process for $N = 7,\ 15,\ \ldots,\ 2^i - 1$ until we get a constant map. In order to ensure that f_{-i} is always the same map, we keep in memory an array with the seeds to the random number generator:

```
m ← 0
Seed[0] ← random()
repeat
  begin(min, max)
  for i = m to 0 do
    seed(Seed[i])
    for j = 1 to 2^i do
      step(min, max)
    end for
  end for
```

$Seed[m+1] = \text{random}()$
$m \leftarrow m+1$
until $min = max$

where "begin" is a function which creates the minimum and the maximum configurations, and "step" apply f_t.

The map $f_t(X)$ is defined as follows:

- Consider the set of all odd boxes in the fundamental domain;
- Randomly, we distribute the boxes in two subsets $\mathcal{O}_+$ and $\mathcal{O}_-$ with equal probability;
- If a box belongs to $\mathcal{O}_+$ and can be added to the plane partition X, add it;
- Same for $\mathcal{O}_-$, if a box belongs to $\mathcal{O}_-$ and can be removed from X, remove it;
- Repeat the process with the even boxes.

Sample

In Figure 5.4, we see the result for $n = 50$ and $r = 20$ (remind that $n = p + r$). In the middle of the configuration there is a fixed hexagon (consequence of the p horizontal fixed paths). We can see the formation of facets, *i.e.* ordered regions, and a disordered region.

When n tends to infinity, the disordered face is limited by a boundary, what Kenyon called frozen boundary [36]. The exterior frozen boundary seems to be a circle for small p, which is expectable since we know that the frozen boundary of unrestricted TSSCPP is a circle. Notice that the interior frozen boundary touches the fixed hexagon and forms a cusp for each hexagon's vertex, which is also expected.

5.3 Study of entries of the type $(_p\alpha$

In this section we consider entries ψ_π of the polynomial solution of qKZ (in the homogeneous limit) which correspond to the matchings with p openings at the beginning. Explicitly, such a matching is of the form $(_p\alpha$, where α is not a state but a sequence which contains n closings and $r = n - p$ openings, see Figure 5.5 for an example. Equivalently these are the matchings for which the p first points are not connected to each other.

5.3.1 Basis transformation

We are here interested in the sum of such polynomials $\psi_{(_p\alpha}$. Recall that there is no systematic way to obtain the corresponding expression in the a-basis. However, based on some numerical experiments, the following formula can be guessed:

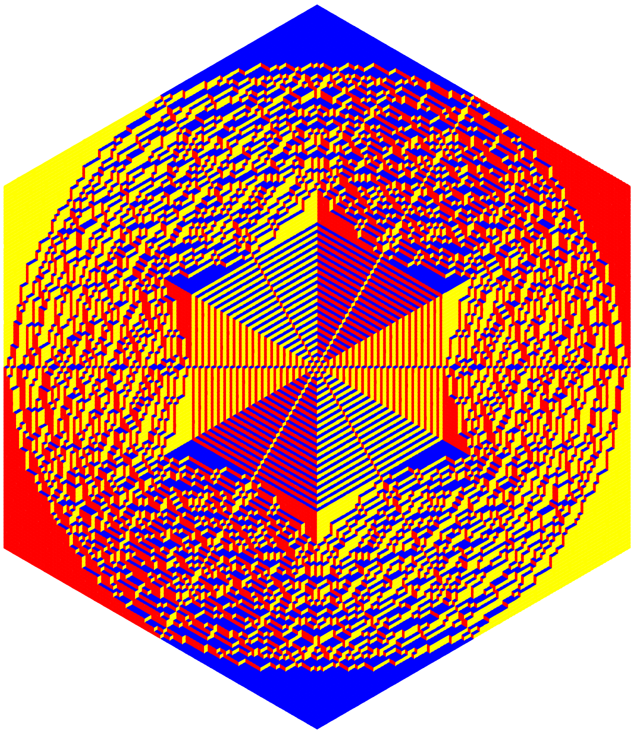


Figure 5.4: A perfectly random example of a TSSCPP with a fixed hexagon $2p \times 2p \times 2p$, with $p = 20$ and $n = 50$. The configuration was sampled using coupling from the past.

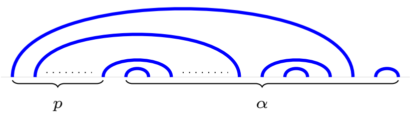


Figure 5.5: A state $(_p\alpha$ is a state that has at least p openings at the left.

Lemma 5.5. *The sum of the polynomial of the type* $\Psi_{(_p\alpha}$ *is:*

$$\sum_{\alpha} \Psi_{(_p\alpha} = \sum_{\substack{a_{i+1}>a_i \\ p<a_i\leq 2p+2i-1 \\ a_{i+1}\neq a_i+1 \, for\ all\ a_i\ even}} \Phi_{1,2,\dots,p,a_1,\dots,a_r}. \tag{5.10}$$

The proof consists in evaluating this equality at all c_n points of the type (q^ϵ).

Proof. It is obvious that the l.h.s. evaluated at (q^ϵ) such that ϵ can not be written as $(_p\alpha$ is zero.

As to the r.h.s., we integrate with respect to the first p variables:

$$\begin{aligned}
\Phi_{1,\dots,p,a_1,\dots,a_n} =&(-1)^{\binom{n}{2}} \prod_{i<j\leq 2n} (qz_i - q^{-1}z_j) \\
&\times \oint \dots \oint \prod_{i=1}^{n} \frac{dw_i}{2\pi i} \frac{\prod_{j>i}(w_j-w_i)(qw_i - q{-}1w_j)}{\prod_{b_j\geq i}(w_j - z_i)\prod_{i>b_j}(qw_j - q^{-1}z_i)} \\
=&(-1)^{\binom{n}{2}} \prod_{i<j\leq p} (qz_i - q^{-1}z_j) \prod_{p<i<j\leq 2n} (qz_i - q^{-1}z_j) \\
&\times \oint \dots \oint \prod_{i=p+1}^{n} \frac{dw_i}{2\pi i} \frac{\prod_{j>i}(w_j-w_i)(qw_i - q^{-1}w_j)\prod_{j,i}(qz_i - q^{-1}w_j)}{\prod_{a_j\geq i>p}(w_j - z_i)\prod_{i>a_j}(qw_j - q^{-1}z_i)},
\end{aligned}$$

where $\{b_1,\dots,b_p,b_{p+1},\dots,b_n\} = \{1,\dots,p,a_1,\dots,a_n\}$. $\Phi_b(q^\epsilon)$ is zero for all (q^ϵ) such that ϵ do not have p openings at the left.

We now proceed by induction on r. If $r = n - p = 0$ we find that both sides are:

$$(-1)^{\binom{n}{2}}(q - q^{-1})^{n(n-1)}. \tag{5.11}$$

We now want to show that for all ϵ of the type $(_p\alpha$ the r.h.s. satisfies the same recurrence as the l.h.s. Take an ϵ which has a pairing ("little arch") $(i, i+1)$. Using Lemma 1.10, we rewrite the l.h.s. as:

$$\sum_{\alpha} \Psi_{(_p\alpha}(q^\epsilon) = q^{-(n-1)} \prod_{j=1}^{i-1}(q^{-1} - q^2 q^{\epsilon_j}) \prod_{j=i+2}^{2n} (q^2 - q^{-1}q^{\epsilon_j}) \sum_{\hat\alpha} \Psi_{(_p\hat\alpha}(q^{\hat\epsilon}), \tag{5.12}$$

where the hat means that we remove the little arch from $(_p\alpha$ (if α does not have a little arch $(i, i+1)$, the term is zero) and ϵ. It is obvious that any matching with $n-1$ arches and p openings at the beginning is written as $(_p\hat\alpha$ for a certain $\hat\alpha$.

We proceed similarly with the r.h.s. We pick the same vector (q^ϵ). We suppose that i is even (for i odd the reasoning is analogous).

If $i \notin \{a_j\}$, the expression vanishes. So we pick $a_j = i$, and as i is even we have $a_{j+1} > i+1$. When we integrate on w_j in expression (1.25), by the rules defined in 1.6.2, we obtain a similar formula with a reduced vector ϵ of size $n-1$ that is obtained by removal of the little arch.

The integral formula is modified in the following manner:

$$\sum_{\substack{a_{i+1}>a_i \\ p<a_i\leq 2p+2i-1 \\ a_{i+1}\neq a_i+1 \text{ for all } a_i \text{ even}}} \Phi_{1,2,\ldots,p,a_1,\ldots,a_r}(q^{\epsilon}) = q^{-(n-1)}\prod_{j=1}^{i-1}(q^{-1}-q^2q^{\epsilon_j})\prod_{j=i+2}^{2n}(q^2-q^{-1}q^{\epsilon_j})$$
$$\times \sum_{\substack{\hat{a}_{i+1}>\hat{a}_i \\ p<\hat{a}_i\leq 2p+2i-1 \\ \hat{a}_{i+1}\neq \hat{a}_i+1 \text{ for all } \hat{a}_i \text{ even}}} \Phi_{1,2,\ldots,p,\hat{a}_1,\ldots,\hat{a}_{r-1}}(q^{\hat{\epsilon}}),$$

where

$$\hat{a}_i = \begin{cases} a_i & i<j; \\ a_{i+1}-2 & \text{otherwise.}\end{cases} \tag{5.13}$$

This proves the lemma. □

Now we can calculate the limit $z_i \to 1$ for all i. Using the change of variables

$$u_i = \frac{w_i - 1}{qw_i - q^{-1}}$$

and integrating with respect to the first p variables, we obtain:

$$\sum_{\alpha}\psi_{({}_p\alpha} = \sum_{\substack{a_{i+1}>a_i \\ p<a_i\leq 2i-1 \\ a_{i+1}\neq a_i+1 \text{ for all } a_i \text{ even}}} \oint\cdots\oint\prod_{m=1}^{r}\frac{du_m}{2\pi i}\frac{(1+\tau u_m)^p}{u_m^{a_m-p}}$$
$$\times\prod_{1\leq l<m\leq r}(u_m-u_l)(1+\tau u_m+u_m u_l).$$

To sum over all possible a, we can consider only the odd a_i and multiply by $(1+u_i)$, simplifying in this way the conditions:

$$\sum_{\alpha}\psi_{({}_p\alpha} = \sum_{\substack{a_{i+1}>a_i \\ p<a_i\leq 2i-1 \\ a_i \text{ odd}}} \oint\cdots\oint\prod_{m=1}^{r}\frac{du_m}{2\pi i}\frac{(1+\tau u_m)^p(1+u_m)}{u_m^{a_m-p}}$$
$$\times\prod_{1\leq l<m\leq r}(u_m-u_l)(1+\tau u_m+u_m u_l).$$

We write $b_i = p-(a_i+1)/2+i$:

$$\sum_{\alpha}\psi_{({}_p\alpha} = \sum_{0\leq b_{i+1}\leq b_i} \oint\cdots\oint\prod_{m=1}^{r}\frac{du_m}{2\pi i}\frac{(1+\tau u_m)^p(1+u_m)}{u_m^{p+2m-1}}u_m^{2b_m}$$
$$\times\prod_{1\leq l<m\leq r}(u_m-u_l)(1+\tau u_m+u_m u_l). \tag{5.14}$$

The upper bound on b_1 was relaxed, since it only excludes zero terms.

A standard calculation gives us the formula:

$$\sum_{0\leq b_{i+1}\leq b_i} \prod_i u_i^{2b_i} = \frac{1}{\prod_{m=1}^r (1-\prod_{i=1}^m u_i^2)}.$$

Substituting in (5.14), we get:

$$\sum_\alpha \psi_{({}_p\alpha} = \oint \dots \oint \prod_{m=1}^r \frac{du_m}{2\pi i} \frac{(1+\tau u_m)^p (1+u_m)}{u_m^{p+2m-1}(1-\prod_{i=1}^m u_i^2)} \prod_{1\leq l<m\leq r} (u_m - u_l)(1+\tau u_m + u_m u_l). \quad (5.15)$$

5.3.2 NILP

Instead of fixing the first p paths to be horizontal, as in Section 5.2.5, we fix them to be vertical. We count the number of paths with a weight τ for each vertical step (the first p fixed paths excluded).

We use the same method that was used for the $\Psi_{(\pi)_p}$ polynomials. In Figure 5.6 we show one example of the NILP counted.

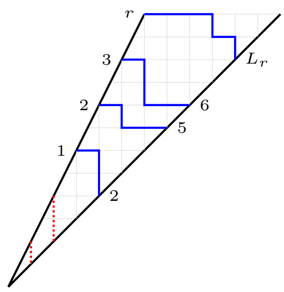


Figure 5.6: To apply the LGV formula we need to label the starting points of the paths by $1,\dots,r$ and the ending points by $1 \leq L_1,\dots,L_r$. As they do not intersect each other $L_{i+1} > L_i$.

Consider all the paths going from i to L_i, we get:

$$\mathcal{P}_{i,L_i} = \tau^{p+2i-L_i-1} \binom{p+i-1}{p+2i-L_i-1}.$$

Or, in a contour integral form:

$$\mathcal{P}_{i,L_i} = \oint \frac{du}{2\pi i} \frac{(1+\tau u)^{p+i-1}}{u^{p+2i-L_i}}.$$

We apply the LGV formula:

$$\begin{aligned}\mathcal{G}_{p,r} &= \oint \cdots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \sum_{1=L_1<\ldots<L_r} \det \left[\frac{(1+\tau u_k)^{p+k-1}}{u_k^{p+2k-L_j}} \right]_{1\le k,j\le r} \\ &= \oint \cdots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{(1+\tau u_i)^{p+i-1}}{u_i^{p+2i-1}} \sum_{1=L_1<\ldots<L_r} \det \left[u_k^{L_j-1} \right]_{1\le k,j\le r} \\ &= \oint \cdots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{(1+\tau u_i)^{p+i-1}}{u_i^{p+2i-1}} \frac{1}{1-u_i} \prod_{i<j} \frac{u_j-u_i}{1-u_iu_j} \\ &= \oint \cdots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{(1+\tau u_i)^{p+i-1}(1+u_i)}{u_i^{p+2i-1}} \frac{\prod_{i<j}(u_j-u_i)}{\prod_{i\le j}(1-u_iu_j)}.\end{aligned}$$

We now use the following identity (similar to the one formulated in [17] and proved in [81]), proved in Appendix C.2:

$$\begin{aligned}&\oint \cdots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{\prod_{i<j}(u_j-u_i)(1+\tau u_j+u_iu_j)}{u_i^{2i+p-1}(1-\prod_{j=1}^{i} u_j^2)} \Omega(u_1,\ldots,u_r) = \\ &\qquad = \oint \cdots \oint \prod_{i=1}^{r} \frac{du_i}{2\pi i} \frac{(1+\tau u_i)^{i-1} \prod_{j>i}(u_j-u_i)}{u_i^{2i+p-1} \prod_{j\ge i}(1-u_ju_i)} \Omega(u_1,\ldots,u_r), \quad (5.16)\end{aligned}$$

where $\Omega(u_1,\ldots,u_r)$ is some symmetric function. Here, $\Omega(u_1,\ldots,u_r) = \prod_{i=1}^{r}(1+\tau u_i)^p(1+u_i)$, without poles in the integration region. We thus obtain exactly equation (5.15).

In [43], Krattenthaler gave an explicit formula for $\mathcal{G}_{p,r}$ at $\tau=1$:

$$\mathcal{G}_{p,r} = \begin{cases} \prod_{i=0}^{r-1} \frac{(3p+3i+1)!}{(3p+2i+1)!(p+2i)!} \prod_{i=0}^{(r-2)/2} (2p+2i+1)!(2i)! & \text{if } r \text{ is even;} \\ 2^p \prod_{i=1}^{r-1} \frac{(3p+3i+1)!}{(3p+2i+1)!(p+2i)!} \prod_{i=1}^{(r-1)/2} (2p+2i)!(2i-1)! & \text{if } r \text{ is odd.} \end{cases} \quad (5.17)$$

Remarkably, these formulæcoincide with those conjectured in [53] for $\sum_\alpha \psi_{({}_p\alpha}$ at $\tau=1$ (more precisely, what was conjectured, *cf.* their eqs. (40–42), was the probability that p consecutive points are disconnected from each other in the $O(1)$ loop model, that is the ratio of $\sum_\alpha \psi_{({}_p\alpha)}$ by the full sum). Correcting a misprint in their eq. (42), we have

$$\mathcal{G}_{p,r} = \frac{S(2(p+r),p)}{S(2p,p)} \quad (5.18)$$

$$S(L,p) = \begin{cases} \frac{\prod_{\ell=1}^{p/2} \prod_{k=\ell}^{2\ell-1} (L^2-4k^2)}{\prod_{\ell=0}^{p/2-1} (L^2-(2\ell+1)^2)^{p/2-\ell}} & p \text{ even;} \\ \frac{\prod_{\ell=1}^{(p+1)/2} \prod_{k=\ell}^{2\ell-2} (L^2-4k^2)}{\prod_{\ell=0}^{(p-3)/2} (L^2-(2\ell+1)^2)^{(p-1)/2-\ell}} & p \text{ odd.} \end{cases} \quad (5.19)$$

The equality of (5.17) and (5.18) can be obtained by direct computation, treating separately the parities of p and of r.

5.3.3 Punctured–TSSCPP

As is seen in Figure 5.7, fixing the first p paths amounts to fixing a central hexagonal star of size p. Consequently, $\mathcal{G}_{p,r}$ also counts the number of TSSCPP with weight τ for each yellow face in the fundamental domain (the triangle outside the hexagonal star, which in the figure contains the paths).

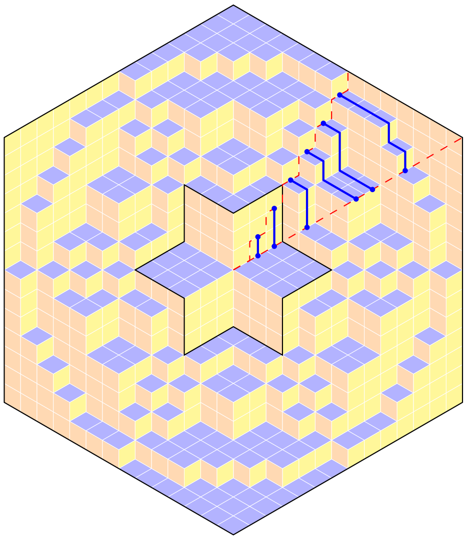


Figure 5.7: This TSSCPP with fixed central hexagonal star, of size p, corresponds to the NILP in Figure 5.6 with p fixed vertical paths.

5.3.4 Limit shape

As in the example of Section 5.2.7, we can run a coupling from the past simulation in order to get a perfectly random configuration. See in Figure 5.8 a random configuration with $n = 50$ and $p = 20$.

In the center of the configuration there is a fixed hexagonal star (consequence of the p fixed paths). As in Section 5.2.7, there is formation of facets, and the frozen boundary seems to be a circle for small p. The interior frozen boundary touches all sides of the star and forms also cusps at the vertices of the star.

5.4 Further questions

5.4.1 Zuber's conjectures

In this chapter we presented the proof of [85, Conjecture 6], on the CPL side. This conjecture states that the quantities $\psi_{(\pi)_p}$ are polynomials of degree $\delta = d(\pi)$. In the same article, Zuber conjectured that $\psi_{(\pi)_p\pi'}$ is also a polynomial in p of degree

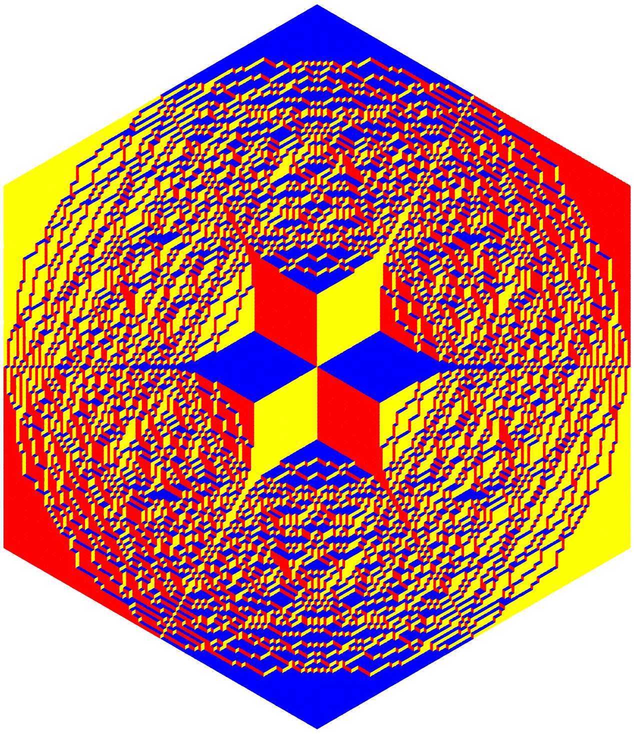


Figure 5.8: A random example of a TSSCPP with a fixed hexagonal star of size p. In this example we use $p = 20$ and $n = 50$. The configuration was picked using coupling from the past.

$\delta = d(\pi) + d(\pi')$. The difficulty here is that the basis transformation $C_{a,\pi}$ does not respect the form $(\pi)_p\pi'$.

5.4.2 Limit shape

The punctured-TSSCPP can be seen as dimer configurations in a honeycomb lattice (as we saw in Chapter 3). In a series of articles [36, 38, 39, 37], Kenyon, Okounkov and Sheffield proved that when we have a dimer model in a honeycomb lattice there is always a limit shape as the size n tends to infinity. Moreover, they reduced the problem to finding a polynomial with certain characteristics. Unfortunately, the complexity of the polynomial grows with the number of sides. This is work in progress.

Chapter 6

On the polynomials $\psi_{(\pi)_p}$: some conjectures

In Chapter 5 we proved that the quantities $\psi_{(\pi)_p}(\tau)$ are polynomials in p. Let us extend the definition of these quantities to the complex numbers. We define then the polynomial $\psi_\pi(\tau,t)$ such that $\psi_\pi(p)=\psi_{(\pi)_p}(\tau)$ when p is a nonnegative integer. We make analogous definitions for $A_\pi(t)$ and $\psi_\pi(t):=\psi_\pi(1,t)$. In particular, we have for the constant term $A_\pi(0)=A_\pi$ (and $\psi_\pi(0)=\psi_\pi$). The Razumov–Stroganov–Cantini–Sportiello theorem then states that $A_\pi(t)=\psi_\pi(t)$ for any π. The following proposition sums up some properties of the polynomials.

Proposition 6.1. *The polynomial $\psi_\pi(t)$ has degree $d(\pi)$ and leading coefficient $1/H_\pi$. Furthermore, we have $\psi_\pi(t)=\psi_{\pi^*}(t)$ (where π^* is the conjugate of π), and $\psi_{(\pi)_\ell}(t)=\psi_\pi(t+\ell)$ for any nonnegative integer ℓ*

The first part comes from Theorem 5.2, while the rest is clear when t is a nonnegative integer and thus holds in general by polynomiality in t.

It is the goal of this chapter to exhibit some surprising properties of these polynomials: the main conjectures deal with the description of real roots of the polynomials (Conjecture 6.6), their values at negative integers between $1-n$ and -1 (Conjecture 6.9), evaluations at $-n$ (Conjecture 6.12) and finally the positivity of the coefficients (Conjecture 6.13). It turns out that most of our conjectures admit a natural generalization when τ is generic.

Notice that the proof of Theorem 6.19 involves the introduction of a new multivariate integral.

The chapter is organized as follows: In the first section we present some properties of the quantities ψ_π, coming from the FPL framework, which will be useful for the rest of the chapter. In Section 6.2 we gather some conjectures about the ψ_π: with $\tau=1$ in Subsection 6.2.2 and with generic τ in Subsection 6.2.3. In Sections 6.3 and 6.4 we prove Theorem 6.16 and 6.19 respectively. The last two sections are concerned with the computation of the subleading term of polynomials as polynomials in t and the leading term as polynomials in τ.

6.1 Notes on the FPL case

If π is a matching with n arches, the polynomial $A_\pi(t)$ admits the following expression:

$$A_\pi(t) = \sum_{\sigma \leq \pi} a_\sigma^\pi \cdot S_\sigma(t - n + 1), \tag{6.1}$$

in which σ is a link pattern (cf. Section 1.1.2), the a_σ^π are the nonnegative integers denoted by $a(\sigma, \pi, \mathbf{0}_n)$ in [78], and $S_\sigma(t - n + 1)$ is the polynomial given by

$$S_\sigma(t - n + 1) = \frac{1}{H_\sigma} \prod_{u \in Y(\sigma)} (t - n + 1 + c(u)),$$

the quantities H_σ and $c(u)$ being defined in Section 1.1.2. If N denotes a nonnegative integer, $S_\sigma(N)$ enumerates semistandard Young tableaux of shape $Y(\sigma)$ with entries not larger than N: this is the *hook content formula*, cf. [73] for instance.

Equation (6.1) above can be derived from [78, Equation (4)] (itself based on the work [10]) together with Conjecture 3.4 in the same paper: this conjecture and equation (6.1) are proved in [56].

6.2 The conjectures

In this section we present several conjectures about the polynomials $\psi_\pi(t)$. For each of them, we will give strong supporting evidence. We will first give a combinatorial construction that is essential in the statement of the conjectures. In Subsection 6.2.3 we extend the conjectures to the polynomials $\psi_\pi(t)$.

6.2.1 Combinatorics

We give two rules which define certain integers attached to a matching π. It turns out that the two rules are equivalent, which is the content of Theorem 6.2.

Let π be a link pattern, and $n = |\pi|$ its number of arches. We let $Y(\pi), d(\pi)$ be the Young diagram of π and its number of boxes respectively, as defined in Section 1.1.2. We also use the notation $\widehat{x} = 2n + 1 - x$ for $x \in [\![1, 2n]\!]$.

Rule A**:** For p between 1 and $n-1$, we consider the set $\mathcal{A}_p^L(\pi)$ of arches $\{a_1, a_2\}$ such that $a_1 \leq p$ and $p < a_2 < \widehat{p}$, and the set $\mathcal{A}_p^R(\pi)$ of arches $\{a_1, a_2\}$ such that $p < a_1 < \widehat{p}$ and $\widehat{p} \leq a_2$. It is clear that $|\mathcal{A}_p^L(\pi)| + |\mathcal{A}_p^R(\pi)|$ is an even integer, and we can thus define the integer $m_p^{(A)}(\pi)$ by

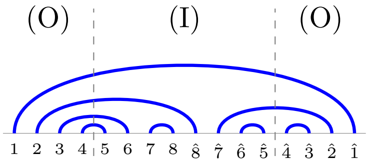
$$m_p^{(A)}(\pi) := \frac{|\mathcal{A}_p^L(\pi)| + |\mathcal{A}_p^R(\pi)|}{2}.$$

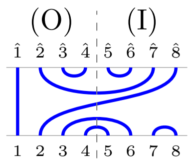
For instance, let π_0 be the matching with 8 arches represented below on the left; we give an alternative representation on the right by folding the second half of the points above the first half, so that $\widehat{x}$ and x are vertically aligned. For $p = 4$, we get

$|\mathcal{A}_p^L(\pi_0)| = 3, |\mathcal{A}_p^R(\pi_0)| = 1$, which count arches between the regions (O) and (I), and thus $m_4^{(A)}(\pi_0) = 4/2 = 2$. The reader will check that

$$m_p^{(A)}(\pi_0) = 0, 1, 2, 2, 2, 1, 1$$

for $p = 1, \ldots, 7$.





Rule B: Label the boxes of $Y(\pi)$ by associating $n+1-x-y$ to the box (x, y). Then decompose $Y(\pi)$ in rims (cf. Section 1.1.2) and let $R_1, \ldots, R_t$ be the successive rims: using the example π_0 from rule A, we represented below the $Y(\pi_0)$ with its labeling and decomposition in (three) rims. For a given rim R_ℓ, denote by i and j the labels appearing at the bottom left and top right of the rim, and by k the minimal value appearing in the rim (so that $k \leq i, j$). We define the multiset B_ℓ as

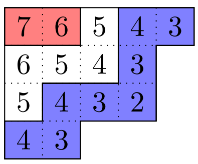
$$\{k\} \cup \{i, i-1, \ldots, k+1\} \cup \{j, j-1, \ldots k+1\},$$

and let B_π be the union of all multisets B_ℓ. Finally, we define $m_i^{(B)}(\pi)$ to be the multiplicity of the integer $i \in \{1, \ldots, n-1\}$ in B_π.

In the case of π_0, the rims give the multisets $\{2, 4, 3, 3\}$, $\{4, 5, 5\}$ and $\{6, 7\}$. Their union is $B_{\pi_0} = \{2, 3^2, 4^2, 5^2, 6, 7\}$, so that

$$m_p^{(B)}(\pi_0) = 0, 1, 2, 2, 2, 1, 1$$

for $p = 1, \ldots, 7$.



We see here that $m_p^{(A)}(\pi_0) = m_p^{(B)}(\pi_0)$ for all p, which holds in general:

Theorem 6.2. *For any matching π, and any integer p such that $1 \leq p \leq |\pi| - 1$, we have $m_p^{(A)}(\pi) = m_p^{(B)}(\pi)$.*

The proof of this theorem is a bit technical, but not difficult; it is given in Appendix E.

Definition 6.3 ($m_p(\pi)$)**.** *For any matching π and any integer p, we let $m_p(\pi)$ be the common value of $m_p^{(A)}(\pi)$ and $m_p^{(B)}(\pi)$ if $1 \leq p \leq |\pi| - 1$, and be equal to 0 otherwise.*

We have then the following result:

Proposition 6.4. *For any matching π, we have $\sum_p m_p(\pi) \leq d(\pi)$, and the difference $d(\pi) - \sum_p m_p(\pi)$ is an even integer.*

Proof. Rule B is more suited to prove this proposition. We will clearly get the result if we can prove that for each rim R_t, the number of boxes r_t in R_t is greater or equal than the cardinality b_t of the multiset B_t, and the difference between the two quantities is even. Therefore we fix a rim R_t, and we use the notations i, j, k from the definition of Rule B. We compute easily $r_t = 2n - i - j - 1$ while $b_t = i + j - 2k + 1$. The difference is thus $\delta_t := r_t - b_t = 2(k + n - 1 - (i + j))$, which is obviously even. It is also nonnegative: indeed, if c, c' are the extreme boxes with the labels i, j respectively, then the minimal value of k is obtained if the rim consists of the boxes to the right of c together with the boxes below c'. At the intersection of these two sets of boxes, the value of k is equal to $i + j - n + 1$, which shows that δ_t is nonnegative and completes the proof. □

We will use this result in Section 6.2.2.

6.2.2 The case $\tau = 1$

As we have seen before, when $\tau = 1$, the quantities $\psi_{(\pi)_p}$ have a combinatorial meaning: they count the number of FPL with connectivity $(\pi)_p$.

The rest of this section will consist of the statement of Conjectures 6.6, 6.9, 6.12 and 6.13, together with evidence in their support. The first three conjectures are related to values of the polynomials $\psi_\pi(t)$ when the argument t is a negative integer; what these conjectures imply is that some mysterious combinatorics occurs around these values $\psi_\pi(-p)$. The fourth conjecture states simply that the polynomials $\psi_\pi(t)$ have positive coefficients, and is thus slightly different in spirit from the other ones, though they are clearly related.

The principal evidence in support of the conjectures, as well as the source of their discovery, is the following result:

Fact 6.5. *Conjectures 6.6, 6.9 and 6.13 are true for all matchings π such that $\pi \leq 8$. Conjecture 6.12 is true for all $n \leq 8$.*

The corresponding polynomials $\psi_\pi(t)$ were indeed computed in Mathematica for these values of π thanks to the formula

$$\psi_\pi(\tau) = \sum_a C^{-1}_{\pi,a}(\tau)\phi_a(\tau),$$

and each conjecture was then checked from these exact expressions; note that there are 1430 matchings $|\pi|$ such that $|\pi| = 8$. In Appendix D we list the polynomials $\psi_\pi(t)$ for $|\pi| \leq 4$.

Real roots

The first conjecture gives a complete description of all real roots of the polynomials $\psi_\pi(t)$:

Conjecture 6.6. *All the real roots of the polynomials $\psi_\pi(t)$ are negative integers, and $-p$ appears with multiplicity $m_p(\pi)$. Equivalently, we have a factorization:*

$$\psi_\pi(t) = \frac{1}{|d(\pi)|!} \cdot \left(\prod_{p=1}^{|\pi|-1} (t+p)^{m_p(\pi)} \right) \cdot Q_\pi(t),$$

where $Q_\pi(t)$ is a polynomial with integer coefficients and no real roots.

We must verify first that the definition of the multiplicities is coherent with this conjecture. We know indeed by Theorem 5.2 that $\psi_\pi(t)$ has degree $d(\pi)$ in t; furthermore the degree of $Q_\pi(t)$ is necessarily even, since it is a real polynomial with no real roots. This means that the sum of the $m_p(\pi)$ should not be larger than $d(\pi)$, and should be of the same parity: this is precisely the content of Proposition 6.4.

It is also immediately checked that the conjecture is compatible with the two stability properties from Proposition 6.1, that is $\psi_\pi(t) = \psi_{\pi^*}(t)$ and $\psi_{(\pi)_\ell}(t) = \psi_\pi(t+\ell)$ for any nonnegative integer ℓ. Indeed $m_p(\pi) = m_p(\pi^*)$ is immediately seen from either one of the rules, as is $m_{p+\ell}((\pi)_\ell) = m_p(\pi)$.

As an example, the polynomial for the matching π_0 of Section 6.2.1 is:

$$\begin{aligned} A_{\pi_0}(t) = & \frac{(2+t)(3+t)^2(4+t)^2(5+t)^2(6+t)(7+t)}{145152000} \\ & \times (9t^6 + 284t^5 + 4355t^4 + 39660t^3 + 225436t^2 + 757456t + 123120), \end{aligned}$$

In Section 1.7, the following formula was established (which was first proven in the articles [18] for the FPL case, and [83] for the CPL case):

$$\psi_{()_a()_b}(t) = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{t+i+j-1}{i+j-1}.$$

This is exactly what Conjecture 6.6 predicts in this case (the constant factor is given by Theorem 5.2). This is perhaps easier to see with the definition of the $m_i(\pi)$ by rule B. Here the Young diagram is a rectangle, and it is easily seen that each box will correspond to a root of the polynomial, matching precisely the expression above.

There is an extension of this "rectangular" case in the article [9], the results of which can be reformulated as a computation of the polynomials $\psi_\pi(t)$ when the diagram $Y(\pi)$ is formed of a rectangle together with one more line consisting of one or two boxes, or two more lines with one box each. Then a simple rewriting of the formulas of Theorems 3.2 and 4.2 in [9] shows that the polynomials have indeed[1] the form predicted by Conjecture 6.6.

[1]We did not actually prove that the polynomials $Q_\pi(t)$ only have complex roots when they are of degree 4, though we tested several values; when $Q_\pi(t)$ has degree 2, then from the explicit form in [9, Theorem 3.2] one checks that it has a negative discriminant.

In Section 6.4, we will give another piece of evidence for the conjecture, by showing that -1 is a root of $A_\pi(t)$ as predicted, that is when there is no arch between 1 and $2n$ in the matching π; note though that we will not prove that we have multiplicity $m_1(\pi) = 1$ in this case.

Values for some negative parameters

We are now interested in the values of the polynomial $\psi_\pi(t)$ is, when the argument t is specialized to a negative integer which is not a root. Note first that although $\psi_\pi(t)$ does not have integer coefficients, we have the following:

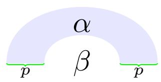
Proposition 6.7. *Let π be a matching, $p > 0$ an integer; then $\psi_\pi(-p)$ is an integer.*

Proof. This is standard: for $d = d(\pi)$, the polynomials $\binom{t+d-i}{d}, i = 0 \dots d$, form a basis of the space of complex polynomials in t of degree $\leq d$. Since $\psi_\pi(t)$ has degree d, we can write

$$\psi_\pi(t) = \sum_{i=0}^{d} c_i \binom{t+d-i}{d}. \tag{6.2}$$

Now $\psi_\pi(p) = \psi_{(\pi)_p}$ is a nonnegative integer when p is a nonnegative integer. Plugging in successively $t = 0, 1, 2, \dots, d$ in (6.2) shows then that $c_0, c_1, \dots, c_d$ are integers, which in turn implies that for negative integers $-p$ we have also that $\psi_\pi(-p)$ is an integer. $\square$

So let π be a matching, and $p \in [\![0, |\pi|]\!]$ be such that $m_p(\pi) = 0$. By Rule A in Section 6.2.1, this means that there are no arches that separate the outer part of π consisting of the first p and the last p points (denote it by α) from the inner part (denote it by β), as shown in the picture:

$\pi =$ 

Here α and β can be naturally considered as matchings in their own right (when properly relabeled), and we introduce the notation $\pi = \alpha \circ \beta$ in this situation. It turns out that the following numbers play a special role in our second conjecture:

Definition 6.8 (G_π)**.** *For any matching π we define*

$$G_\pi := \psi_\pi(-|\pi|).$$

By Proposition 6.7 above, the G_π are actually integers. For example, we can compute easily G_π, with $|\pi| = 4$. Here we index them with $Y(\pi)$ instead of π: this is well defined by the stability property $G_\pi = G_{(\pi)}$.

$G_{\cdot} = 1$	$G_{(1)} = -1$	$G_{(2)} = 1$	$G_{(2,1)} = -3$
$G_{(3)} = -1$	$G_{(2,2)} = 1$	$G_{(3,1)} = 4$	$G_{(3,1,1)} = -9$
$G_{(3,2)} = -3$	$G_{(3,2,1)} = 9$		

The next conjecture says that these numbers seem to appear naturally when evaluating our polynomials at certain negative integers:

Conjecture 6.9. *Let π be a matching and p be an integer between 1 and $|\pi|-1$ such that $m_p(\pi)=0$, and write $\pi=\alpha\circ\beta$ with $|\alpha|=p$. We then have the following factorization:*

$$\psi_\pi(-p)=G_\alpha\psi_\beta.$$

Here we need to verify a certain sign compatibility with Conjecture 6.6, which predicts that $\psi_\pi(-p)$ has sign $(-1)^{M_p}$ where $M_p=\sum_{i\leq p}m_i(\pi)$. Now for this range of i we have obviously $m_i(\pi)=m_i(\alpha)$ by rule A, so that $\psi_\pi(-p)$ has sign $(-1)^{d(\alpha)}$ by Proposition 6.4; but this is then (conjecturally) the sign of G_α (cf. Proposition 6.10 below), which is coherent with the signs in Conjecture 6.9.

Properties of the G_π

Conjecture 6.9 shows that the numbers G_π seem to play a special role in the values of $\psi_\pi(t)$ at negative integers.

Proposition 6.10. *For any matching π, $G_\pi=G_{(\pi)}$ and $G_\pi=G_{\pi^*}$. Moreover, Conjecture 6.6 implies that sign$(G_\pi)=(-1)^{d(\pi)}$.*

Proof. The first two properties are immediately derived from the polynomial identities $\psi_\pi(t+1)=\psi_{(\pi)}(t)$ and $\psi_\pi(t)=\psi_{\pi^*}(t)$ respectively, given in Proposition 6.1. Then, if all real roots of $\psi_\pi(t)$ are between -1 and $1-|\pi|$ as predicted by Conjecture 6.6, the sign of G_π must be equal to the sign of $(-1)^{d(\pi)}$, since $\psi_\pi(t)$ has leading term $t^{d(\pi)}/H_\pi$ by Theorem 5.2. □

We can compute some special cases, corresponding to $Y(\pi)$ being a rectangle, or a rectangle plus an extra row with just one box:

Proposition 6.11. *We have $G_{()_a()_b}=(-1)^{ab}$, while $G_{(()())_{a-2}()_b}=(-1)^{ab+1}(a+1)$.*

This is easily proved by using the explicit formulas for such π which were mentioned in Section 6.2.2. Finally, the most striking features about these numbers are conjectural:

Conjecture 6.12. *For any positive integer n, we have*

$$\sum_{\pi:|\pi|=n}|G_\pi|=A_n \quad \textit{and} \quad \sum_{\pi:|\pi|=n}G_\pi=(-1)^{\frac{n(n-1)}{2}}\left(A_n^V\right)^2 \tag{6.3}$$

$$G_{()^n}=\begin{cases}(-1)^{\frac{n(n-1)}{2}}\left(A_{n+1}^V\right)^2 & \textit{if } n \textit{ is even;}\\ (-1)^{\frac{n(n-1)}{2}}\left(A_n^VA_{n+2}^V\right) & \textit{if } n \textit{ is odd.}\end{cases} \tag{6.4}$$

The first equality in (6.3) is particularly interesting: it implies that the unsigned integers $|G_\pi|$, when π runs through all matchings of size n, sum up to A_n, the total number of FPL of size n. Of course the ψ_π verify exactly this also, but the properties

of G_π we have just seen show that the sets of numbers have different behaviors. For instance, the stability property $G_\pi = G_{(\pi)}$ fails for ψ_π obviously, while in general $G_{r(\pi)} \neq G_\pi$. Furthermore, $\psi_{(()())_{a-2}()_b} = a+b-1$ while $G_{(()())_{a-2}()_b} = (-1)^{ab+1}(a+1)$. This raises the problem of finding a partition of FPLs of size n –or any other combinatorial object enumerated by A_n– whose blocks $\{\mathcal{G}_\pi\}_{\pi:|\pi|=n}$ verify $|\mathcal{G}_\pi| = |G_\pi|$.
Remark: In fact, part of the conjecture is a consequence of Conjectures 6.6 and 6.9. Indeed, it was proved in [24] that, as polynomials, we have:

$$A_{()^n}(t) = \sum_{\pi:|\pi|=n} A_\pi(t-1) \tag{6.5}$$

If one evaluates this for $t = 1-n$, then two cases occur:

- *if n is even*, then we have that $1-n$ is a root of $A_{()^n}(t)$ by Conjecture 6.6, and we get from (6.5) that

$$\sum_{\pi:|\pi|=n} G_\pi = 0,$$

  which is consistent with $A_n^V = 0$ if n is even.

- *if n is odd*, then we are in the conditions of Conjecture 6.9, which tells us that $A_{()^n}(1-n) = G_{()^{n-1}} A_{()} = G_{()^{n-1}}$, and from (6.5) we have

$$\sum_{\pi:|\pi|=n} G_\pi = G_{()^{n-1}}.$$

This then proves that the second equality in (6.3) can be deduced from the first case in (6.4).

Positivity of the coefficients

Our last conjecture is a bit different from the other three ones, in that it does not deal with values of the polynomials, but their coefficients:

Conjecture 6.13. *For any π, the coefficients of $\psi_\pi(t)$ are nonnegative.*

It seems in fact to be true that the polynomials $Q_\pi(t)$ –whose existence is predicted by Conjecture 6.6– also only have nonnegative coefficients.

By Theorem 5.2, we know already that $A_\pi(t)$ is of degree $d(\pi)$ with a positive leading coefficient, so we will be interested in the *subleading* coefficient, that is, the coefficient of $t^{d(\pi)-1}$. We managed to compute this coefficient and prove that it is indeed positive: this is Theorem 6.22 in Section 6.5.

6.2.3 Generic τ

In this section we will give three conjectures extending Conjectures 6.6, 6.9 and 6.13. All of these conjectures have been verified for all $\psi_\pi(\tau,t)$ with $|\pi| \leq 8$. We introduce a contour integral expression for $G_\pi(\tau)$ which allow us to prove the conjecture mimicking Conjecture 6.12. We begin with roots:

Conjecture 6.14. *Considering $\psi_\pi(\tau,t)$ as a polynomial in t with coefficients in $\mathbb{Z}[\tau]/\mathbb{Z}$, the real roots of $\psi_\pi(\tau,t)$ are negative integers $-p$ and with multiplicity given by $m_p(\pi)$:*

$$\psi_\pi(\tau,t) = \frac{1}{|d(\pi)|!} \times \prod_{i=1}^{|\pi|} (t+i)^{m_i(\pi)} Q_\pi(t,\tau),$$

where $Q_\pi(t,\tau)$ is a polynomial in t with no real roots.

For the example π_0 of Section 6.2.1 we compute:

$$\begin{aligned}\psi_{\pi_0}(\tau,t) =& \frac{(2+t)(3+t)^2(4+t)^2(5+t)^2(6+t)(7+t)}{145152000}\tau^9 \\ &\times (84000 + 440640\tau^2 + 151440t\tau^2 + 13200t^2\tau^2 + 523680\tau^4 + 394360t\tau^4 \\ &+ 110520t^2 + \tau^4 13670t^3\tau^4 + 630t^4\tau^4 + 182880\tau^6 + 211656t\tau^6 \\ &+ 101716t^2\tau^6 + 25990t^3\tau^6 + 3725t^4\tau^6 + 284t^5\tau^6 + 9t^6\tau^6).\end{aligned}$$

We then have the natural generalization of the factorization conjecture:

Conjecture 6.15. *Let π be a matching and p be a integer between 1 and $|\pi|-1$ such that $m_p(\pi)=0$, so that $\pi = \alpha \circ \beta$ with $|\alpha| = p$; then*

$$\psi_\pi(\tau,-p) = G_\alpha(\tau)\psi_\beta(\tau).$$

Here $G_\pi(\tau)$ is naturally defined by $G_\pi(\tau) := \psi_\pi(\tau,-|\pi|)$. For example, the values for $|\pi|=4$ are:

$$\begin{array}{llll}
G_{\cdot} = 1 & G_{(1)} = -\tau & G_{(2)} = \tau^2 & G_{(1,1)} = \tau^2 \\
G_{(2,1)} = -2\tau-\tau^3 & G_{(3)} = -\tau^3 & G_{(1,1,1)} = -\tau^3 & G_{(2,2)} = \tau^4 \\
G_{(3,1)} = 3\tau^2+\tau^4 & G_{(2,1,1)} = 3\tau^2+\tau^4 & G_{(3,1,1)} = -3\tau-5\tau^3-\tau^5 & G_{(3,2)} = -2\tau^3-\tau^5 \\
G_{(2,2,1)} = -2\tau^3-\tau^5 & G_{(3,2,1)} = 3\tau^2+5\tau^4+\tau^6 & &
\end{array}$$

These $G_\pi(\tau)$ present several properties:

Theorem 6.16. *We have $G_\pi(\tau) = (-1)^{d(\pi)} g_\pi(\tau)$, where $g_\pi\tau$ is a polynomial with* non-negative *integer coefficients. Furthermore, we have the sum rule:*

$$\sum_\pi G_\pi(\tau) = \sum_\pi \psi_\pi(-\tau).$$

By Lemma 5.1, we know that

$$\psi_\pi(\tau,p) = \sum_a C^{-1}_{\pi,a}\phi_a(\tau,p) \tag{6.6}$$

where,

$$\phi_a(\tau,p) = \oint \dots \oint \prod_i^n \frac{du_i}{2\pi i u_i^{a_i}}(1+\tau u_i)^p \prod_{j>i}(u_j-u_i)(1+\tau u_j+u_iu_j)$$

For example,

$$G_{()^n} = \oint \dots \oint \prod_i^n \frac{du_i}{2\pi i u_i^{2i-1}}(1+\tau u_i)^{-n} \prod_{j>i}(u_j-u_i)(1+\tau u_j+u_iu_j).$$

In Section 6.3.3 we show that this contour integral counts a certain subset of TSSCPP.

Let $\mathcal{A}_n$ be the set of linking patterns of the form $(a_1,\dots,a_n)$, with $a_i = 2i-1$ or $a_i = 2i-2$. Using Lemma 5.3:

$$\begin{aligned}\sum_\pi G_\pi(\tau) &= \sum_{a\in\mathcal{A}_n} \phi_a(\tau,-n)\\ &= \oint \dots \oint \prod_i^n \frac{du_i}{2\pi i u_i^{a_i}}(1+u_i)(1+\tau u_i)^{-n} \prod_{j>i}(u_j-u_i)(1+\tau u_j+u_iu_j)\end{aligned}$$

Thus, we can rewrite Theorem 6.16:

$$\begin{aligned}&\oint \dots \oint \prod_i^n \frac{du_i}{2\pi i u_i^{a_i}}(1+u_i)(1+\tau u_i)^{-n} \prod_{j>i}(u_j-u_i)(1+\tau u_j+u_iu_j)\\ &\qquad = \oint \dots \oint \prod_i^n \frac{du_i}{2\pi i u_i^{a_i}}(1+u_i) \prod_{j>i}(u_j-u_i)(1-\tau u_j+u_iu_j)\end{aligned} \tag{6.7}$$

We prove this equality in Section 6.3.2, showing that both sides of equation (6.7) count TSSCPP. If we assume that $(-1)^{d(\pi)}G_\pi = |G_\pi|$ (which is a consequence of Conjecture 6.6, see Proposition 6.10), the first part of Conjecture 6.12: $\sum_\pi |G_\pi| = A_n$, in Conjecture 6.12 follows from Theorem 6.16.

Yet in Section 6.3.2 we show that the conjectured sum $\sum_\pi G_\pi = (-1)^{\binom{n}{2}}(A_n^V)^2$ is now equivalent to the minus one enumeration of TSSCPP proposed in Di Francesco's article [14].

We will show in Section 6.6 that the leading term of $g_\pi(\tau)$ is $\tau^{d(\pi)}$; we will actually compute the leading term in τ of $\psi_\pi(\tau,p)$ for various integer values of p. Another property of these $G_\pi(\tau)$ is that

$$G_\pi(\tau) = (-1)^{d(\pi)}G_\pi(-\tau),$$

so that they are odd or even polynomials depending on the parity of π. More generally, one has $\Psi_\pi(\tau,t) = (-1)^{d(\pi)}\Psi_\pi(-\tau,t)$. Indeed, this is obvious for the polynomials

$$\phi_a = \oint \dots \oint \prod_i \frac{du_i}{u_i^{a_i}}(1+\tau u_i)\prod_{j>i}(u_j-u_i)(1+\tau u_j+u_iu_j),$$

and as the basis transformation respects this parity, this holds for $\Psi_\pi(\tau,t)$ as well.

Finally, introducing a τ does not change the positivity:

Conjecture 6.17. *The bivariate polynomial $d(\pi)!P_\pi(\tau,t)$ has nonnegative integer coefficients.*

6.3 Contour integral formula for G_π

The main goal of this section is to prove Theorem 6.16. Also, we give an interpretation for $\sum_\pi G_\pi$ and $G_{()^n}$ in terms of TSSCPP.

6.3.1 Dual paths

In Section 3.3 we saw that there is a bijection between TSSCPP and a certain set of NILP. And in Section 4.3.2 we counted those NILP with a weight t_i per vertical step in the i^{th} slice.

Ignore the extra step, that is put $t_0=1$. Consider that $t_i=\tau$ for $1\le i\le n$, *i.e.* we give a weight τ per vertical step. Equation (4.18) then becomes:

$$N_{10}(\tau) := \oint \dots \oint \prod_{i=1}^n \frac{du_i}{2\pi i u_i^{2i-1}}\frac{1}{1-u_i^2}(1+u_i)(1+\tau u_i)^{i-1}\prod_{j>i}\frac{u_j-u_i}{1-u_ju_i}.$$

Using equality (C.11) we obtain:

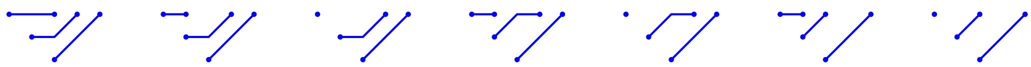
$$N_{10}(\tau) = \oint \dots \oint \prod_{i=1}^n \frac{du_i}{2\pi i u_i^{2i-1}}(1+u_i)\prod_{j>i}(u_j-u_i)(1+\tau u_j+u_iu_j)$$

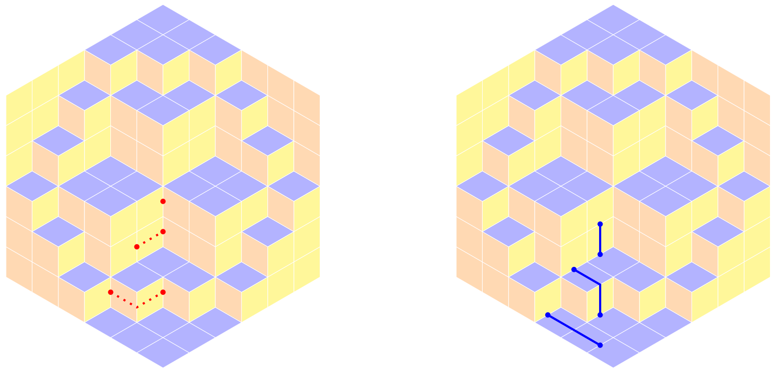
which, by Lemma 5.3, is equal to $\sum_\pi \psi_\pi(\tau)$.

Consider now the NILP defined by the following rules:

- The paths start at the points $A_i=(i-1,1-i)$ for $i=1,\dots,n$;
- The final points are of the form $E_i=(r_j,0)$ where $0\le r_j\le 2n-2$;
- Only horizontal and diagonal (NE) paths are allowed.

For example, when $n=3$ we have 7 NILP:



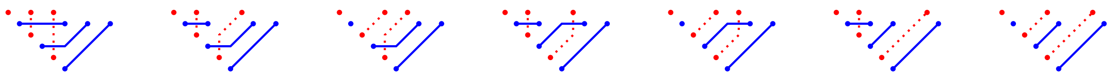


(a) Normal paths (b) Dual paths

Figure 6.1: The TSSCPP can be represented either by the normal NILPs or by the dual NILPs. The number of yellow faces is either equal to the number of vertical steps on the normal NILPs or the horizontal steps on the dual NILPs.

We claim that these NILP are in bijection with those presented in Section 3.3. Moreover, the number of horizontal steps in the first ones is equal to the number of vertical steps in the second ones. In fact, these paths are dual to the ones in Section 3.3, as we can see in Figure 6.1.

For example, the 7 examples for $n = 3$ are (the 4^{th} one corresponds to the one in Figure 6.1):



where the dual NILPs are represented by the solid blue lines and the normal ones (*i.e.* the ones defined in Section 3.3) are represented by the red dotted lines. It is easy to see that the vertical steps on the normal paths corresponds to the horizontal steps on the dual paths.

6.3.2 The sum of G_π

Next, we use the LGV formula (Section 3.2.1) to obtain an integral formula which counts the dual paths. Thus,

$$N_{10}(\tau) = \sum_{0 \leq r1 < \ldots < r_n \leq 2n-2} \det[\mathcal{P}_{i,r_j}]$$

where $\mathcal{P}_{i,r_j}$ is the weighted sum over all possible paths from $(i-1, 1-i)$ to $(r_j, 0)$. Such paths have $i-1$ diagonal steps and $r_j - 2i + 2$ horizontal steps, hence:

$$\begin{aligned}\mathcal{P}_{i,r} =& \tau^{r-2i+2}\binom{r-i+1}{r-2i+2} \\ =& (-\tau)^{r-2i+2}\binom{-i}{r-2i+2} \\ =& [u^{r-2i+2}](1-\tau u)^{-i}\end{aligned}$$

where $[u^{r-2i+2}]p(u)$ stands for the coefficient of the corresponding power of u in the polynomial $p(u)$.

We can rewrite the equation as a contour integral:

$$\begin{aligned}N_{10}(\tau) &= \oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i}(1-\tau u_i)^{-i} \sum_{0\leq r_1<\dots<r_n\leq 2n-2} \det[u_i^{-r_j+2i-3}] \qquad (6.8) \\ &= \oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}}(1-\tau u_i)^{i-1-n} \sum_{0\leq r_1<\dots<r_n} \det[u_i^{r_j}],\end{aligned}$$

where we performed the transformation $i \to n-i+1$ and $r_j \to 2n-2-r_j$. It is easily see that the terms coming from $r_j > 2n-2$ do not contribute for the contour integral.

The last sum can be evaluated by a standard result for the sum of Schur functions (see Exercise 4.3.10 in [7]):

$$\sum_{0\leq r_1<\dots<r_n} \det[u_i^{r_j}] = \prod_i \frac{1}{1-u_i} \prod_{j>i} \frac{u_j - u_i}{1-u_j u_i}.$$

The integral can thus be transformed into:

$$\oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}}(1+u_i)(1-\tau u_i)^{i-1-n} \frac{\prod_{j>i}(u_j - u_i)}{\prod_{j\geq i}(1-u_i u_j)}.$$

Finally, we use equation (C.11) in order to obtain

$$N_{10} = \oint \dots \oint \prod_{i}^{n} \frac{du_i}{2\pi i u_i^{2i-1}}(1+u_i)(1-\tau u_i)^{-n} \prod_{j>i}(u_j - u_i)(1-\tau u_j + u_i u_j),$$

which is exactly $\sum_\pi G_\pi(-\tau)$, proving Theorem 6.16.

So $\sum_\pi G_\pi$ is the minus one enumeration that appears in Di Francesco's article [14]. We proceed using Lemma 3.4. Let $Q(u_i, u_j)$ be the weighted number of non-intersecting paths from (u_i, u_j) to any arrival point (points of the form $(k, 0)$). Thus,

$$Q(u_i, u_j) = \sum_{0\leq r<s} (-1)^{r+s} \left[\binom{i-1}{r-i+1}\binom{j-1}{s-j+1} - \binom{i-1}{s-i+1}\binom{j-1}{r-j+1}\right].$$

Assume that n is even. An easy computation show that $Q(u_1, u_i) = -Q(u_i, u_1) = 0$ for all $i \in [\![1, n]\!]$. Thus by Lemma 3.4,

$$\sum_{\pi} G_\pi = \mathrm{Pf}(Q(u_i, u_j)) = 0.$$

Assume now that n is odd. Ignore the first path, and relabel all the paths $u_i \to u_{i-1}$. The next equality is not hard to prove:

$$\begin{aligned} Q(u_i, u_j) &= \sum_{1 \le r < s} (-1)^{r+s} \left[\binom{i}{r-i}\binom{j}{s-j} - \binom{i}{s-i}\binom{j}{r-j} \right] \\ &= 3(i-j)\frac{(i+j-1)!}{(2i-j)!(2j-1)!}, \end{aligned}$$

which simplifies the pfaffian in Di Francesco's article:

$$(A^V_{2n+1})^2 = \mathrm{Pf}_{1 \le i,j \le n}\left(3(j-i)\frac{(i+j-1)!}{(2i-j)!(2j-1)!} \right).$$

6.3.3 Computing $G_{()^n}$

Let $\mathcal{R}$ be the subset of dual NILPs such that $r_i - i + n$ is even for all i.

Proposition 6.18. *There are exactly $(-1)^{\binom{n}{2}} G_{()^n}$ dual NILPs in $\mathcal{R}$. Moreover, $G_{()^n}(\tau)$ is the weighted enumeration of dual paths in $\mathcal{R}$ with weight $(-\tau)$ for each horizontal step.*

Proof. We follow the same steps as in Section 6.3.2 until equation (6.8), which in this case is transformed in

$$\begin{aligned} N^{\mathcal{R}}_{10}(\tau) &:= \oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i}(1-\tau u_i)^{-i} \sum_{\substack{0 \le r_1 < \ldots < r_n \le 2n-2 \\ r_n \text{ even} \\ r_{i+1}-r_i \text{ odd}}} \det[u_i^{-r_j+2i-3}] \\ &= \oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}}(1-\tau u_i)^{i-1-n} \sum_{\substack{0 \le r_1 < \ldots < r_n \le 2n-2 \\ r_n \text{ even} \\ r_{i+1}-r_i \text{ odd}}} \det[u_i^{r_j}], \end{aligned}$$

where we performed the transformation $i \to n - i + 1$ and $r_j \to 2n - 2 - r_j$. It is easily seen that the terms coming from $r_j > 2n - 2$ do not contribute to the contour integral.

The last sum can be evaluated by a standard result for the sum of Schur functions of *even* partitions (see Exercise 4.3.10 in [7]):

$$\sum_{\substack{0 \le r_1 < \ldots < r_n \le 2n-2 \\ r_n \text{ even} \\ r_{i+1}-r_i \text{ odd}}} \det[u_i^{r_j}] = \frac{\prod_{j>i}(u_j - u_i)}{\prod_{j \ge i}(1 - u_j u_i)}.$$

The integral can thus be transformed in:

$$\oint \cdots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}} (1-\tau u_i)^{i-1-n} \frac{\prod_{j>i}(u_j-u_i)}{\prod_{j\geq i}(1-u_i u_j)}.$$

Finally, we use equation (C.11) in order to obtain

$$N_{10} = \oint \cdots \oint \prod_{i}^{n} \frac{du_i}{2\pi i u_i^{2i-1}} (1-\tau u_i)^{-n} \prod_{j>i} (u_j-u_i)(1+\tau u_j + u_i u_j),$$

which is exactly $\sum_\pi G_{()^n}(-\tau)$. □

A similar computation appears in Ishikawa's article [28][Conjecture 3.1]. In fact, it can be seen that the subset $\mathcal{R}$ is exactly the subset $\mathcal{P}_n^R$ which appears in his article (see [28][Definition 3.1]). Notice that the weights are different.

6.4 The first root

In this section we will prove the following theorem.

Theorem 6.19. *For any matching π we have*

$$\psi_\pi(\tau,-1) = \begin{cases} \psi_{\pi'}(\tau) & \text{if } \pi = (\pi'); \\ 0 & \text{otherwise.} \end{cases}$$

This is a special case of Conjecture 6.6 by setting $\tau = 1$:

Corollary 6.20. *If $m_1(\pi) = 1$, then $(t+1)$ divides the polynomial $\psi_\pi(t)$.*

Indeed $m_1(\pi) = 1$ precisely when there is no arch between 1 and $2n$ in π (cf. Rule A in Section 6.2.1), which means that π cannot be written as (π'). For the same reason, Theorem 6.19 is in general a special case of Conjecture 6.14.

To prove this theorem, we use the multiparameter version $\Psi_\pi(z_1,\dots,z_{2n})$ of the quantities ψ_π, see Chapter 1.

Recall that $\Psi_\pi(z_1,\dots,z_{2n})$ form a basis of $\mathcal{V}_n$, and that Lemma 1.12 shows that a polynomial in this space is determined by its value on these points q^ϵ. There is a small variation of this lemma, for the cases with a big arch $(1,2n)$, obtained by rotation:

$$\Psi_\pi(q^{-2},q^\epsilon,q^2) = (q-1)^{2(n-1)}(q-q^{-1})^{(n-2)(n-1)}\tau^{d(\pi)}q^{-(n-1)}\delta_{(\epsilon),\pi}.$$

In this section we will use the contour integrals $\Phi_a(z_1,\dots,z_{2n})$ (see Section 1.6). Notice that using the lemma's variation, we obtain:

$$\Phi_a(q^{-2},q^\epsilon,q^2) = \tau^{d(\epsilon)}q^{-(n-1)}\left(\frac{q-1}{q-q^{-1}}\right)^{2(n-1)} C_{a,(\epsilon)}. \tag{6.9}$$

6.4.1 The proof

By Lemma 5.1,

$$\Psi_\pi(-1) = \sum_a C^{-1}_{\pi,a}\Phi_a(-1).$$

We now introduce the following multiple integral, inspired by formula (1.25):

$$\Phi_a(z_1,\dots,z_{2n}|-1) := \frac{z_1(q-q^{-1})}{qz_1-q^{-1}z_{2n}} \prod_{1\le i<j\le 2n}(qz_i-q^{-1}z_j) \times \oint\dots\oint \prod_i \frac{dw_i}{2i\pi} \frac{\prod_{i<j}(w_j-w_i)(qw_i-q^{-1}w_j)}{\prod_{j\le a_i}(w_i-z_j)\prod_{j>a_i}(qw_i-q^{-1}z_j)} \prod_i \frac{qw_i-q^{-1}z_{2n}}{qz_1-q^{-1}w_i}. \tag{6.10}$$

The essential property of $\Phi_a(z_1,\dots,z_{2n}|-1)$ is that if all $z_i=1$, then we get $\Phi_a(-1)$; this requires the change of variables $u_i = \frac{w_i-1}{qw_i-q^{-1}}$. If we integrate in w_1, we obtain:

$$\Phi_a(z_1,\dots,z_{2n}|-1) = \prod_{i=2}^{2n-1}(qz_i-q^{-1}z_{2n}) \prod_{2\le i<j}^{2n-1}(qz_i-q^{-1}z_j) \times \oint\dots\oint \prod_{i=2} \frac{dw_i}{2i\pi} \frac{\prod_{i<j}(w_j-w_i)(qw_i-q^{-1}w_j)}{\prod_{2\le j\le a_i}(w_i-z_j)\prod_{2n>j>a_i}(qw_i-q^{-1}z_j)}.$$

The r.h.s. is now factorized in one term which depends on z_1 and z_{2n}, but not on a, and one which does not depend on z_1 and z_{2n}, and lives in the vector space $\mathcal{V}_{n-1}$ (with parameters $\{z_2,\dots,z_{2n-1}\}$). Therefore we can write $\Phi_a(z_1,\dots,z_{2n}|-1)$ as a linear combination of $\Psi_\pi(z_2,\dots,z_{2n-1})$:

$$\Phi_a(z_1,\dots,z_{2n}|-1) = \prod_{i=2}^{2n-1}(qz_i-q^{-1}z_{2n}) \times \sum_\pi \widehat{C}_{a,\pi}\Psi_\pi(z_2,\dots,z_{2n-1}). \tag{6.11}$$

We then have the following lemma:

Lemma 6.21. *For any a,ϵ we have $\widehat{C}_{a,\epsilon} = C_{a,(\epsilon)}$.*

Proof. First we integrate Formula (1.25) in w_1:

$$\Phi_a(z_1,\dots,z_{2n}) = \prod_{i=2}^{2n-1}(qz_i-q^{-1}z_{2n}) \prod_{2\le i<j<2n}(qz_i-q^{-1}z_j) \times \oint\dots\oint \prod_i \frac{dw_i}{2i\pi} \frac{\prod_{i<j}(w_j-w_i)(qw_i-q^{-1}w_j)}{\prod_{j\le a_i}(w_i-z_j)\prod_{2n>j>a_i}(qw_i-q^{-1}z_j)} \prod_{i=2}^{2n-1} \frac{qz_1-q^{-1}w_i}{qw_i-q^{-1}z_{2n}}.$$

We then make the substitutions $z_1 \mapsto q^{-2}$ and $z_{2n} \mapsto q^2$:

$$\Phi_a(q^{-2}, z_2, \ldots, z_{2n-1}, q^2) = (-1)^{n-1} \prod_{i=2}^{2n-1} (z_i - 1) \prod_{2 \le i < j < 2n} (qz_i - q^{-1}z_j)$$
$$\times \oint \cdots \oint \prod_{i=2} \frac{dw_i}{2i\pi} \frac{\prod_{i<j}(w_j - w_i)(qw_i - q^{-1}w_j)}{\prod_{2 \le j \le a_i}(w_i - z_j) \prod_{2n > j > a_i}(qw_i - q^{-1}z_j)}.$$

Comparing with the formula obtained for $\Phi_a(z_1, \ldots, z_{2n}| - 1)$, we get:

$$\Phi_a(z_1, \ldots, z_{2n}| - 1) = (-1)^{n-1} \prod_{i=2}^{2n-1} \frac{qz_i - q^{-1}z_{2n}}{z_i - 1} \Phi_a(q^{-2}, z_2, \ldots, z_{2n-1}, q^2),$$

which thanks to (6.11) becomes:

$$\sum_{\epsilon} \widehat{C}_{a,\epsilon} \Psi_\epsilon(z_2, \ldots, z_{2n-1}) = \frac{(-1)^{n-1}}{\prod_{i=2}^{2n-1} z_i - 1} \sum_{\pi} C_{a,\pi} \Psi_\pi(q^{-2}, z_2, \ldots, z_{2n-1}, q^2).$$

Now the l.h.s. lives in $\mathcal{V}_{n-1}$, so it is determined by the points (q^σ) (cf. Lemma 1.12 and its variation):

$$\sum_{\epsilon} \widehat{C}_{a,\epsilon} \delta_{\epsilon,\sigma} \tau^{d(\epsilon)} = \sum_{\pi} C_{a,\pi} \delta_{\pi,(\sigma)} \tau^{d(\pi)}.$$

This simplifies to $\widehat{C}_{a,\sigma} \tau^{d((\sigma))} = C_{a,\sigma} \tau^{d(\sigma)}$; since $d(\sigma) = d((\sigma))$, we get the expected result. □

We can now finish the proof of the theorem. In the limit $z_i = 1$ for all i, equation (6.11) becomes

$$\Phi_a(-1) = \sum_{\pi:|\pi|=n-1} \widehat{C}_{a,\pi} \Psi_\pi.$$

Using the lemma, and multiplying by $C^{-1}_{\pi,a}$, this becomes:

$$\sum_a C^{-1}_{\pi,a} \Phi_a(-1) = \sum_a \sum_\epsilon C^{-1}_{\pi,a} C_{a,(\epsilon)} \Psi_\epsilon$$
$$\Leftrightarrow \qquad \Psi_\pi(-1) = \sum_\epsilon \delta_{\pi,(\epsilon)} \Psi_\epsilon,$$

which completes the proof.

6.5 The subleading term of the polynomials

In this section we will prove the following result:

Theorem 6.22. *Given a matching π of size n, $\pi \neq ()_n$, the coefficient of $t^{d(\pi)-1}$ in $\psi_\pi(t)$ is positive.*

This is a special case of Conjecture 6.13. We will give two proofs of this theorem, one starting from the expression (6.1), the other based on the expression (6.6). As a byproduct of these proofs, we will deduce two formulas concerning products of hook lengths (Proposition 6.24).

6.5.1 First proof

We present first Nadeau's proof. We use the expression of $A_\pi(t)$ given by the sum in Equation (6.1):

$$A_\pi(t) = \sum_{\sigma \leq \pi} a_\sigma^\pi \cdot S_\sigma(t+1-n)$$

We need to gather the terms contributing to the coefficient of $t^{d(\pi)-1}$: they are of two kinds, depending on whether $S_\sigma(t+1-n)$ has degree $d(\sigma)$ equal to $d(\pi)$ or $d(\pi)-1$. Since $\sigma \leq \pi$, the first case occurs only for $\sigma = \pi$, while the second case occurs when $Y(\sigma)$ is obtained from the diagram $Y(\pi)$ by removing a *corner* from this diagram, i.e. a box of $Y(\pi)$ which has no box below it and no box to its right. We denote by $Cor(\pi)$ the set of corners of $Y(\pi)$, and we get:

$$[t^{d(\pi)-1}]A_\pi(t) = \frac{a_\pi^\pi}{H_\pi} \sum_{u \in Y(\pi)} (1 - n + c(u)) + \sum_{(x,y) \in Cor(\pi)} \frac{a_{\pi-(x,y)}^\pi}{H_{\pi-(x,y)}}.$$

It is proved in [10] that $a_\pi^\pi = 1$, and in [56] that $a_{\pi-(x,y)}^\pi = 2n-1-y$ when (x,y) belongs to $Cor(\pi)$. We can then rewrite the previous expression as follows:

$$\frac{d(\pi)(1-n)}{H_\pi} + \frac{1}{H_\pi} \sum_{u \in Y(\pi)} c(u) + \sum_{(x,y) \in Cor(\pi)} \frac{(n-1)}{H_{\pi-(x,y)}} + \sum_{(x,y) \in Cor(\pi)} \frac{(n-y)}{H_{\pi-(x,y)}}.$$

Now the first and third terms cancel each other because of the *hook length formula* (see [73] for instance), which is equivalent to

$$\frac{d(\pi)}{H_\pi} = \sum_{(x,y) \in Cor(\pi)} \frac{1}{H_{\pi-(x,y)}}.$$

Therefore we are left with

$$[t^{d(\pi)-1}]A_\pi(t) = \frac{1}{H_\pi} \sum_{u \in Y(\pi)} c(u) + \sum_{(x,y) \in Cor(\pi)} \frac{(n-y)}{H_{\pi-(x,y)}}. \tag{6.12}$$

We now wish to prove that this is positive, which is not clear since the first term can be negative. The idea is to remember that $A_\pi(t) = A_{\pi^*}(t)$ by Proposition 6.1. Now when $\pi \mapsto \pi^*$, the box (x,y) is sent to (y,x), all contents change signs, $Cor(\pi)$ is sent to $Cor(\pi^*)$, and hook lengths are preserved. From these observations we get the alternative expression:

$$[t^{d(\pi)-1}]A_\pi(t) = -\frac{1}{H_\pi} \sum_{u \in Y(\pi)} c(u) + \sum_{(x,y) \in Cor(\pi)} \frac{(n-x)}{H_{\pi-(x,y)}}. \tag{6.13}$$

Clearly in both (6.12) and (6.13) the second term is positive, since $y < n$ for all boxes (x, y) in $Y(\pi)$ (there is at least one such box because $\pi \neq ()_n$). Adding (6.12) and (6.13), and dividing by 2, we obtain that the coefficient $[t^{d(\pi)-1}]A_\pi(t)$ is positive:

$$[t^{d(\pi)-1}]A_\pi(t) = \sum_{(x,y)\in Cor(\pi)} \frac{(2n-x-y)}{H_{\pi-(x,y)}}. \qquad (6.14)$$

6.5.2 Second proof

By Lemma 1.27:

$$\phi_a(t) = \psi_\pi(t) + \sum_{\sigma<\pi} C_{\pi,\sigma}\psi_\sigma(t),$$

where $a = a(\pi)$. By Theorem 5.2, we know that $\psi_\pi(t)$ has degree $d(\pi)$. Furthermore, since $C_{\pi,\sigma}$ has degree $\leq d(\pi) - d(\sigma) - 2$ if $\sigma < \pi$, we conclude that the coefficient of $t^{d(\pi)-1}$ in $\psi_\pi(t)$ and $\phi_{a(\pi)}(t)$ is the same, so:

$$[t^{d(\pi)-1}]\psi_\pi(t) = [t^{d(\pi)-1}] \oint \dots \oint \prod_{i=1}^{|a|} \frac{du_i}{2\pi i u_i^{a_i}}(1+u_i)^t \prod_{j>i}(u_j - u_i)(1 + u_j + u_i u_j).$$

If we consider $(1 + u_j + u_i u_j) = (1 + u_j) + u_i u_j$, we notice that each time we pick the term $u_i u_j$, we decrease a_i and a_j by 1 and thus the integral corresponds formally to a diagram with two boxes less, so the degree in t decreases by 2 also; these terms can thus be ignored, which gives:

$$\begin{aligned}
[t^{d(\pi)-1}]\psi_\pi(t) =& [t^{d(\pi)-1}] \oint \dots \oint \prod_i \frac{du_i}{2\pi i u_i^{a_i}}(1+u_i)^{t+i-1} \prod_{j>i}(u_j - u_i) \\
=& [t^{d(\pi)-1}] \sum_{\sigma\in S_{|\pi|}} (-1)^\sigma \oint \dots \oint \prod_i \frac{du_i}{2\pi i a_i + 1 - \sigma_i}(1+u_i)^{t+i-1} \\
=& [t^{d(\pi)-1}] \sum_\sigma (-1)^\sigma \prod_i \binom{t+i-1}{a_i - \sigma_i} \\
=& [t^{d(\pi)-1}] \det \left| \binom{t+i-1}{a_i - j} \right|.
\end{aligned}$$

Expanding the binomial up to the second order, we get:

$$\binom{t+i-1}{a_i - j} = t^{a_i - j} \frac{1 + \frac{(a_i-j)(2i+j-a_i-1)}{t}}{(a_i - j)!} + \text{terms of lower degree.}$$

If we compute the subleading term of the determinant we get:

$$\begin{aligned}
[t^{d(\pi)-1}]\psi_\pi(t) &= [t^{-1}] \det \left| \frac{1 + \frac{(a_i-j)(2i+j-a_i-1)}{t}}{(a_i-j)!} \right| \\
&= \sum_{k=0}^{n-1} \det \left| \frac{1}{(a_i-j)!} \times \begin{cases} 1 & \text{if } i \neq k \\ (a_i-j)(2i+j-a_i-1)/2 & \text{if } i = k \end{cases} \right|. \qquad (6.15)
\end{aligned}$$

We want to show that this expression is equal to the r.h.s. of (6.13). First of all, we need to express the quantities involving hooks and contents in terms of the sequence a. Notice that the integer a_i is naturally associated to the $(n+1-i)th$ row from the top in $Y(a)$, the length of this row being given by $(a_i - i)$.

- It is well known (see for instance [71, p.132]) that

$$\frac{1}{H_{Y(a)}} = \det \left| \frac{1}{(a_i - j)!} \right|; \tag{6.16}$$

- The contents in the row indexed by a_i are given by $i-n, i-n+1, \ldots, i-n+(a_i - i - 1)$, which sum up to $\frac{1}{2}(a_i - i)(2n - a_i - i + 1)$, and therefore we get

$$\sum_{u \in Y(a)} c(u) = \sum_{i=1}^{n} \frac{1}{2}(a_i - i)(2n - a_i - i + 1);$$

- Noticing that $a_i \mapsto a_i - 1$ removes a box in $(n+1-i)th$ row, we have:

$$\sum_{(x,y)\in Cor(\pi)} \frac{n-x}{H_{\pi-(x,y)}} = \sum_{k=1}^{n} \det \left| \frac{1}{(a_i - j)!} \begin{cases} 1 & \text{if } i \neq k \\ (a_i - j)(i-1) & \text{if } i = k \end{cases} \right|. \tag{6.17}$$

  Here we can sum over all k, *i.e.* all rows, because the determinants corresponding to rows without a corner in $Y(a)$ have two equal rows and thus vanish.

Looking back at equation (6.15), we write

$$(a_i - j)(2i + j - a_i - 1)/2 = -(a_i - j)(a_i - j - 1)/2 + (a_i - j)(i - 1).$$

Next we split each determinant in two thanks to linearity in the k^{th} row. Then the expression obtained by summing the determinants corresponding to the second term is precisely (6.17); therefore all that remains to prove is the following lemma:

Lemma 6.23.

$$\sum_{k=1}^{n} \det \left| \frac{1}{(a_i - j)!} \times \begin{cases} 1 & \text{if } i \neq k \\ (a_i - j)(a_i - j - 1) & \text{if } i = k \end{cases} \right| \\ = \left(\sum_{k=1}^{n} (a_k - k)(a_k - 2n + k - 1) \right) \times \det \left| \frac{1}{(a_i - j)!} \right|. \tag{6.18}$$

Proof. We write $(a_k - k)(a_k - 2n + k - 1) = a_k(a_k - 2n - 1) + k(2n - k + 1)$ and use linearity of the determinant with respect to row (and column) k to write the r.h.s. of (6.18) as

$$\sum_{k=1}^{n} \det \left| \frac{1}{(a_i - j)!} \begin{cases} 1 & \text{if } i \neq k \\ a_i(a_i - 2n - 1) & \text{if } i = k \end{cases} \right| \\ + \sum_{k=1}^{n} \det \left| \frac{1}{(a_i - j)!} \begin{cases} 1 & \text{if } j \neq k \\ j(2n - j + 1) & \text{if } j = k \end{cases} \right|. \tag{6.19}$$

Now we notice that we have the general identity for any variables a_{ij}, b_{ij}:

$$\sum_{k=1}^{n} \det \left| a_{ij} \begin{cases} 1 & \text{if } i \neq k \\ b_{ij} & \text{if } i = k \end{cases} \right| = \sum_{k=1}^{n} \det \left| a_{ij} \begin{cases} 1 & \text{if } j \neq k \\ b_{ij} & \text{if } j = k \end{cases} \right|.$$

Indeed, both correspond to the coefficient of t^{-1} in $\det |a_{ij} + a_{ij}b_{ij}/t|$, which can be expanded using multilinearity according either to rows or to columns. We use this in the first term of (6.19) and in the l.h.s. in the lemma; putting things together, the r.h.s. of (6.18) minus the l.h.s. is equal to:

$$\sum_{k=1}^{n} \det \left| \frac{1}{(a_i - j)!} \begin{cases} 1 & \text{if } j \neq k \\ 2(n-j)(a_i - j) & \text{if } j = k \end{cases} \right|.$$

For all $k < n$ the determinants have two proportional columns (k and $k+1$), while for $k = n$ the n^{th} column of the determinant is zero. So all these determinants are zero and therefore so is their sum, which achieves the proof of the lemma. □

This completes the second proof of Theorem 6.22.

6.5.3 Application to hook length products

It turns out that some of the computations made to prove Theorem 6.22 have nice applications to certain *hook identities*. If Y is a Young diagram, let $Cor(Y)$ be its corners, and $HD(Y)$ (respectively $VD(Y)$) be the horizontal (resp. vertical) dominoes which can be removed from Y, defined as two boxes which can be removed in the same row (resp. the same column). Then we have the following identities:

Proposition 6.24. *For any Young diagram Y we have:*

$$\frac{2\sum_{u\in Y} c(u)}{H_Y} = \sum_{(x,y)\in Cor(Y)} \frac{(y-x)}{H_{Y-(x,y)}}$$

and

$$\frac{2\sum_{u\in Y} c(u)}{H_Y} = \sum_{hd\in HD(Y)} \frac{1}{H_{(Y-hd)}} - \sum_{vd\in VD(Y)} \frac{1}{H_{(Y-vd)}}.$$

Proof. We consider a, a sequence such that $Y(a) = Y$. The first formula consists simply in equating the expressions in (6.12) and (6.13).

We will see that the second formula is a reformulation of Lemma 6.23. We already identified $\frac{2}{H_Y}\sum_{u\in Y} c(u)$ as the r.h.s. of the lemma, so we want identify the sums on dominoes with the l.h.s. in Lemma 6.23. We note first that the k^{th} determinant in (6.18) is of the form (6.16) for the sequence $a^{(k)}$ which coincides with a except $a^{(k)}_k = a_k - 2$. There are three different cases to consider: firstly, if $a^{(k)}$ has two equal terms, the corresponding determinant vanishes. Then, if $a^{(k)}$ is increasing, we obtain one of the terms in the sum over $HD(Y)$. Finally, for $a^{(k)}$ to have distinct terms when it's not increasing, it is necessary and sufficient that $a_k = a_{k-1} + 1$ and $a_{k-2} < a_k - 2$. The sequence obtained by switching $a_k - 2$ and a_{k-1} is then strictly increasing; if we exchange the rows in the determinant, we will get a negative sign. It is then easy to verify that such sequences are those obtained by removing a vertical domino from Y, which achieves the proof. □

As pointed out by V. Féray [20], both formulas can in fact be deduced from the representation theory of the symmetric group, using the properties of Jucys-Murphy elements [32, 54].

6.6 The leading term of $\psi_\pi(\tau, p)$

We now consider $\psi_\pi(\tau, t)$ as a polynomial in τ, first with coefficients in $\mathbb{C}[t]$, and then with rational coefficients under the specializations $t = p$ for p an integer.

We start by deriving an expression for the leading term in τ of the polynomial $\psi_\pi(\tau, t)$. First we consider the leading term in τ of $\phi_a(\tau, t)$ for a given sequence a. We have

$$\phi_a(\tau, t) = \oint \dots \oint \prod_i \frac{du_i}{2\pi i u_i^{a_i}} (1 + \tau u_i)^t \prod_{j>i} (u_j - u_i)(1 + \tau u_j + u_i u_j).$$

It is clear that if we replace $(1 + \tau u_i + u_i u_j)$ by $(1 + \tau u_i)$ we do not change the leading term in τ(for the same reasons as in Section 6.5.2). Therefore this last expression has the same leading term in τ as

$$\begin{aligned}
&\oint \dots \oint \prod_{i=1}^n \frac{du_i}{2\pi i u_i^{a_i}} (1 + \tau u_i)^{t+i-1} \prod_{j>i} (u_j - u_i) \\
&= \sum_{\sigma \in S_n} (-1)^\sigma \oint \dots \oint \prod_{i=1}^n \frac{du_i}{2\pi i u_i^{a_i+1-\sigma_i}} (1 + \tau u_i)^{t+i-1} \\
&= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \tau^{a_i - \sigma_i} \binom{t+i-1}{a_i - \sigma_i} \\
&= \tau^{d(a)} \det_{1 \le i,j \le n} \left| \binom{t+i-1}{a_i - j} \right|.
\end{aligned}$$

So we know that the degree in τ of $\phi_a(\tau, t)$ is $d(a)$. Because of Lemma 1.27, it is clear that the leading term of $\psi_\pi(\tau, t)$ is the same as $\phi_{a(\pi)}(\tau, t)$. We have thus proved:

Proposition 6.25. *As a polynomial in τ, the leading term of $\psi_\pi(\tau, t)$ is given by $D_\pi(t)\tau^{d(\pi)}$, where for $a = a(\pi)$ we have*

$$D_\pi(t) = \det_{1 \le i,j \le n} \left| \binom{t+i-1}{a_i - j} \right|.$$

Now we turn to what happens when t is specialized to an integer p; by definition the cases $p = 0$ and $p = -|\pi|$ correspond respectively to the polynomials $\psi_\pi(\tau)$ and $G_\pi(\tau)$. Clearly if $D_\pi(p) \neq 0$ then the leading term of $\psi_\pi(\tau, p)$ is $D_\pi(p)\tau^{d(\pi)}$ by the previous proposition, while if $D_\pi(p) = 0$ the leading term is necessarily of smaller degree. Our result is the following:

Theorem 6.26. *Let π be a matching, and p be an integer; if $p < 0$, we also assume that π is not of the form $(\rho)_{|p|}$. Then $D_\pi(p) = 0$ if and only if $1-|\pi| \leq p \leq -1$. Furthermore,*

- *if $p \geq 0$ then $D_\pi(p)$ counts the number of tableaux of shape $Y(\pi)$ with entries bounded by $p + |\pi| - 1$ which are strictly increasing in rows and columns;*
- *if $p \leq -|\pi|$, then $(-1)^{d(\pi)}D_\pi(p)$ counts the number of tableaux of shape $Y(\pi)$ with entries bounded by $|p| - |\pi|$ which are weakly increasing in rows and columns;*
- *if $1 - |\pi| \leq p \leq -1$, then*
  - *if $m_{|p|}(\pi) \neq 0$, Conjecture 6.14 implies that $\psi_\pi(\tau, p)$ is the zero polynomial;*
  - *if $m_{|p|}(\pi) = 0$ and $\pi = \alpha \circ \beta$ with $|\alpha| = |p|$, Conjecture 6.15 implies that the leading term of $\psi_\pi(\tau, p)$ is given by $(-1)^{d(\alpha)}D_\beta(0)\tau^{d(\alpha)+d(\beta)}$.*

Note that the condition that π is not of the form $(\rho)_{|p|}$ is not a restriction, since in such a case $\psi_\pi(\tau, p) = \psi_\rho(\tau, 0)$.

Proof. We study separately the three cases:
Case $p \geq 0$. The determinant $D_\pi(p)$ is here a particular case of [42, Theorem 6.1], which says that indeed $D_\pi(p)$ counts tableaux of shape $Y(\pi)$ with entries bounded by $(p + |\pi| - 1)$ and increasing in both directions. For example, if $a(\pi) = \{1, 2, 4, 7\}$ and $p = 1$ we get

$$D_{\{1,2,4,7\}}(1) = \det_{1\leq i,j\leq 4} \left| \binom{i}{a_i - j} \right| = 11,$$

corresponding to the 11 tableaux:

$\begin{array}{ccc}2&3&4\\4&&\end{array}$ $\begin{array}{ccc}2&3&4\\3&&\end{array}$ $\begin{array}{ccc}1&3&4\\4&&\end{array}$ $\begin{array}{ccc}1&3&4\\3&&\end{array}$ $\begin{array}{ccc}1&3&4\\2&&\end{array}$ $\begin{array}{ccc}1&2&4\\4&&\end{array}$

$\begin{array}{ccc}1&2&4\\3&&\end{array}$ $\begin{array}{ccc}1&2&4\\2&&\end{array}$ $\begin{array}{ccc}1&2&3\\4&&\end{array}$ $\begin{array}{ccc}1&2&3\\3&&\end{array}$ $\begin{array}{ccc}1&2&3\\2&&\end{array}$

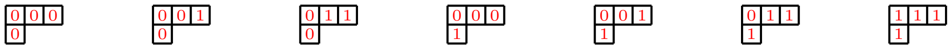
Note also that the filling of the shape $Y(\pi)$ where the cell (x, y) is labeled by $x+y-1$ is a valid tableau because $x + y \leq n$ holds for every cell, and therefore $D_\pi(p) > 0$ for $p \geq 0$.
Case $p \leq -|\pi|$. We use first the transformation $\binom{N}{k} = (-1)^k\binom{N+k-1}{k}$ for each coefficient in $D_\pi(p)$ to get:

$$D_\pi(p) = (-1)^{d(\pi)} \det_{1\leq i,j\leq n} \left| \binom{|p| + a_i - i - j}{a_i - j} \right|;$$

Here the sign comes from $(-1)^{a_i-j} = (-1)^{a_i}(-1)^{-j}$ for the coefficient (i,j), which gives the global sign $(-1)^{\sum_i a_i - \sum_j j} = (-1)^{d(\pi)}$. We can then use [42, Theorem 6.1] in this case also, which gives us that $(-1)^{d(\pi)} D_\pi(p)$ counts tableaux of shape $Y(\pi)$ with entries between 0 and $|p| - |\pi|$ which are weakly increasing in both directions. For the same partition $a(\pi) = \{1,2,4,7\}$ and $p = -5$ we get

$$|D_\pi(-5)| = \det_{1\le i,j\le 4} \left| \binom{5 + a_i - i - j}{5 - i} \right| = 7,$$

which corresponds to the 7 tableaux



Now here also $D_\pi(p) \neq 0$ because the tableau filled with zeros is valid. For $p = -|\pi|$, this is the only possible tableau and thus the leading coefficient of $G_\pi(\tau)$ is given by $D_\pi(-|\pi|) = (-1)^{d(\pi)}$.

Case $-|\pi| < p < 0$. We first want to prove that $D_\pi(p) = 0$ if π is not of the form $(\rho)_{|p|}$. We easily check that $\binom{p+i-1}{a_i-j}$ is zero unless either $i < |p+1|$ and $j < a_{|p|+1}$ or $i \geq |p+1|$ and $j \geq a_{|p|+1}$. Therefore we get a matrix which splits into two rectangular submatrices; the determinant is zero unless these submatrices are square, which means that $|p| + 1 = a_{p+1}$, and then

$$\begin{aligned} D_\pi(p) &= \det_{1\le i,j\le |p|} \left| \binom{p+i-1}{i-j} \right| \times \det_{1\le i,j\le |\pi|-|p|} \left| \binom{i-1}{\hat{a}_i - j} \right| \\ &= D_{\{1,\dots,-p\}}(p) \times D_{\hat{a}}(0), \end{aligned}$$

where $\hat{a}_i = a_{r+i} - r$. The first factor is 1, and the second is non-zero if and only if $\hat{a}$ corresponds to a matching; this is excluded because π would be of the form $(\rho)_{|p|}$, which is excluded. Therefore $D_\pi(p) = 0$ as desired.

Now Conjecture 6.14 immediately implies that if $m_{|p|}(\pi) \neq 0$, then $t = p$ is a root of $\psi_\pi(\tau, t)$, so that $\psi_\pi(\tau, p) \equiv 0$. If $m_{|p|}(\pi) = 0$, then by Conjecture 6.15, the leading term of $\psi_\pi(\tau, p)$ is equal to the product of the leading terms of $G_\alpha(\tau)$ and $\psi_\beta(\tau)$. The first one is given by $(-1)^{d(\alpha)}\tau^{d(\alpha)}$ as proved above, while the leading term of $\psi_\beta(\tau) = \psi_\beta(\tau, 0)$ is given by $D_\beta(0)\tau^{d(\beta)}$, which achieves the proof. □

6.7 Further questions

6.7.1 Solving the conjectures

Since in this chapter we present several conjectures, the most immediate problem is to solve them. We listed four conjectures in Section 6.2 which concern roots, specializations and coefficients of the polynomials $\psi_\pi(t)$. The difficulty here is that existing expressions for the polynomials $\psi_\pi(t)$ consist of certain sums of polynomials. One way to attack the conjectures would then be to find new expressions for the polynomials.

Another angle to attack some of the conjectures (namely conj. 6.6, 6.9 and their τ counterparts 6.14, 6.15) would be to extend the approach used in the proof of Theorem 6.19: one first needs to extend the multivariate integral definition (6.10) to any integer p, which can easily be done. The problem is that the expressions obtained are fairly more complicated and intricate than in the case $p = -1$. This is work in progress.

We proved that $\sum_\pi G_\pi$ and $G_{()^n}$ count some subclasses of NILP. Thus, we can easily write them as a sum of determinants and as a pfaffian, in the case of the sum. We believe that these formulæcan help solving Conjecture 6.12.

6.7.2 Combinatorial reciprocity

The idea underlying our conjectures with the exception of Conjecture 6.13 is that there should be a "combinatorial reciprocity theorem" ([74]) attached to these polynomials. That is, we believe there exist yet-to-be-discovered combinatorial objects depending on π such that $\psi_\pi(-p)$ is equal (up to sign) to the number of these objects with size p. The most well-known example in the literature of such a phenomenon concerns the *Ehrhart polynomial* $i_P(t)$ of a lattice polytope P, which counts the number of lattice points in tP when t is a positive integer: for such t, Ehrhart reciprocity then tells us that $(-1)^{\dim P} i_P(-t)$ counts lattice points strictly in tP (see [5] for instance).

Conjectures 6.6 and 6.9 tell us in particular for which values of p objects counted by $|A_\pi(-p)|$ should exist, and moreover that such objects should *split* for certain values of p. As pointed out in Section 6.2.2, Conjectures 6.9 and 6.12 make it particularly important to figure out what the numbers $G_\pi = A_\pi(-|\pi|)$ count.

6.7.3 Consequences of the conjectures

The conjectures have interesting consequences regarding the numbers a_σ^π involved in equation (6.1), since for instance Conjecture 6.6 directly implies certain linear relations among these numbers. Discovering what these numbers a_σ^π are is a step in the direction of a new proof of the Razumov–Stroganov conjecture, in the sense that it gives an expression for A_π that could be compared to the expressions for ψ_π. We note also that a conjectural expression for these numbers a_σ^π was given in [82], which if true would in fact give another proof of the Razumov–Stroganov conjecture; a special case of this expression is proven in [55].

Appendix A

The vector space of polynomials satisfying the wheel condition

This appendix is intended to prove that the dimension of the vector space $\mathcal{V}_n$, defined in 1.4.2, is the Catalan number c_n. Moreover, the specializations q^ϵ completely describe the polynomials in $\mathcal{V}_n$. Let us briefly re-introduce the concepts:

The space $\mathcal{V}_n$ consists of all the homogeneous polynomials with $2n$ variables with total degree $\delta = n(n-1)$ and partial degree $\delta = n-1$, which obey the wheel condition:

$$f(\ldots, z_i, \ldots, z_j = q^2 z_i, \ldots, z_k = q^4 z_i, \ldots) = 0 \qquad \text{for all } i < j < k.$$

In order to prove the dimension of the space, we must prove that a polynomial in $\mathcal{V}_n$ is entirely determined by its values at the following specializations: $(q^{\epsilon_1}, \ldots, q^{\epsilon_{2n}})$ for all possible choices of $\{\epsilon_i = \pm 1\}$ such that $\sum_{i=1}^{2n} \epsilon_i = 0$ and $\sum_{i=1}^{j} \epsilon_i \leq 0$ for all $j \leq 2n$ (these are just height increments of Dyck paths).

Or equivalently, if a polynomial satisfies these conditions and is zero at all the specializations, then it is identically zero. For example, at $n = 1$ the polynomial is of degree 0 *i.e.* a constant, and as it vanishes at $(z_1, z_2) = (q^{-1}, q)$ it is identically zero.

We now proceed by induction. We suppose that the lemma is true for $n < p$. Let ϕ_p be a polynomial of degree $(p-1)$ at each variable which is zero at all specializations. The polynomial satisfies the "wheel condition" at $z_{i+1} = q^2 z_i$, so we can write

$$\phi_p(z_1, \ldots, z_{2p})_{|z_{i+1}=q^2 z_i} = \prod_{j \neq i, i+1} (q z_i - z_j) \psi_{p-1}(z_1, \ldots, z_{i-1}, z_{i+2}, \ldots, z_{2p}), \tag{A.1}$$

where ψ_{p-1} is a function of degree $p-2$ in each z_j (except z_i and z_{i+1}) which still follows the "wheel condition". Furthermore, let π_p be a specialization which has $(z_i, z_{i+1}) = (q^{-1}, q)$ and π'_{p-1} the same specialization but without z_i and z_{i+1}. We apply (A.1):

$$\phi_p(\pi_p) = (1-q)^{n-1}(1-q^{-1})^{n-1}\psi_{p-1}(\pi'_{p-1}) = 0. \tag{A.2}$$

The mapping $\pi_p \mapsto \pi_{p-1}$ is a bijection from Dyck paths with (q^{-1}, q) at locations $(i, i+1)$ to all Dyck paths. Thus our induction hypothesis applies, and $\psi_{p-1} = 0$.

Therefore, one can write:

$$\phi_p = \prod_{i=1}^{2n-1} (z_{i+1} - q^2 z_i) \phi_p^{(1)}, \tag{A.3}$$

where $\phi_p^{(1)}$ is a polynomial of degree $\delta_1 = \delta_{2p} = p-2$ at z_1 and z_{2p} and $\delta_i = p-3$ at all the other variables which follows a weak version of the "wheel condition":

$$\phi^{(1)}_{p|z_k=q^2z_j=q^4z_i} = 0 \qquad \text{for all } k \geq j+2 \geq i+4.$$

This implies:

$$\phi^{(1)}_{p|z_{i+2}=q^2z_i} = \prod_{j\notin[i-1,i+3]} (qz_1 - z_j)\psi_p^{(1,i)}. \tag{A.4}$$

By degree counting in z_i we find that they are identically zero.

Now, we can write

$$\phi_p^{(1)} = \prod_{i=1}^{2n-2} (z_{i+2} - q^2 z_i)\phi_p^{(2)}, \tag{A.5}$$

where $\phi_p^{(2)}$ has degree $\delta_1 = \delta_{2p} = p-3$, $\delta_2 = \delta_{2p-1} = p-4$ and all the others $\delta_i = p-5$.

Clearly, this procedure can be repeated; at step r, $\phi_p^{(r)}$ has degree:

$$\begin{aligned}
\delta_1 &= p-r-1\\
\delta_2 &= p-r-2\\
&\vdots\\
\delta_r &= p-2r\\
&\vdots\\
\delta_i &= p-2r-1\\
&\vdots\\
\delta_{2p} &= p-r-1.
\end{aligned}$$

We write

$$\phi^{(r)}_{p|z_{i+r+1}=q^2z_i} = \prod_{j\notin[i-r,i+2r+1]} (qz_i - z_j)\psi_p^{(r,i)}.$$

Counting the degree in z_i we conclude that $\psi_p^{(r,i)} = 0$. So we can construct $\phi_p^{(r+1)}$.

When $r \geq \frac{n}{2}$ we obtain a polynomial of negative degree which implies that the polynomial is identically zero.

In other words, what this lemma shows is that the vector space of polynomials of degree at most $n-1$ in each variable satisfying the wheel condition is of dimension at most c_n. In fact it is known to be of dimension exactly c_n; the standard proof involves the fact that it is an irreducible representation of the affine Hecke algebra, see e.g. [34, 60].

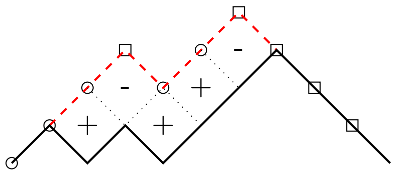
Appendix B

Proof of Lemma 1.27

In this section we sketch the proof of Lemma 1.27 about the value of $C_{a,\pi}$.

To each a and π we associate a Young diagram $Y(a)$ and $Y(\pi)$, if $Y(a)$ does not contain $Y(\pi)$, $C_{a,\pi} = 0$ as explained in section 1.6.2. If $Y(a) = Y(\pi)$ we obtain, trivially, $C_{a,\pi} = 1$. The interesting case is when $Y(\pi) \subsetneq Y(a)$.

For illustration purposes we shall use an example: let $a = (1, 3, 5, 6, 7)$ and $\pi = (((()(()))),$ and $Y(a) = (2, 2, 2, 1, 0)$ and $Y(\pi) = (1, 1, 0, 0, 0)$ the associated Young diagrams. We can also represent them in the form of Dyck paths:



The Dyck paths corresponding to $a = (1, 3, 5, 6, 7)$ is the black line (under) and $\pi = ((()(())))$ is the dashed red line (above). The openings of π are marked with blue little circles and the closings with green little squares.

As shown in section 1.6.2, each link contributes by a factor U_k. If a link starts at r and finishes at s, k is given by:

$$k = \sharp\{a_i \text{ such that } r \le a_i < s\} - \frac{s - r + 1}{2}.$$

Or, in a Dyck path representation, the k is given by a counting in the NE and SE steps in the path corresponding to a, more precisely:

$$k = \frac{\sharp\{\text{NE steps between } r \text{ and } s\} - \sharp\{\text{SE steps between } r \text{ and } s\} - 1}{2},$$

which is the same as counting squares of the skew Young Diagram $Y(a/\pi)$: those under the opening with sign plus and those under the closing with sign minus (as in the example). In the example we get: $U_1 U_0^4 = -\tau$, where the U_1 corresponds to the arch between positions 5 and 8.

If we ignore the individual arches, we note that the maximal exponent is precisely the sum of all the squares (with the sign $\pm$), in the example this number is $3 - 2 = 1$.

Knowing that there is always at least one square with a minus sign, we verify that the maximal exponent of τ is $|Y(a)| - |Y(\pi)| - 2$.

The second part of the lemma concerns the parity of the coefficient. This is obvious from the fact that $U_k(-\tau) = (-1)^k U_k(\tau)$, so the product also has a well-defined parity which is equal to the sum of all squares, but this is equal to the number of squares in $Y(a/\pi)$ minus two times the number of negative squares, proving the rest of the lemma.

Appendix C

Anti-symmetrization formulæ

The goal of this chapter is to prove two identities (4.20) and (5.16), which allows to turn equation of the type (4.19) into one of the type (4.13).

C.1 Proof of the identity (4.20)

This identity was conjectured by Di Francesco and Zinn-Justin in [17] and proved by Zeilberger [81]. Equivalently, it was proved that the integrand of the l.h.s. without the factor $\varphi(u)$, once anti-symmetrized and truncated to its negative degree part (the positive powers of the u_i cannot contribute to the integral), reduces to the integrand of the r.h.s. without the factor $\varphi(u)$. Here we prove in an independent way a much stronger statement. Indeed, here we perform the *exact* anti-symmetrization of a spectral parameter dependent generalization of the integrand.[1]

C.1.1 The general case

Let $h_q(x,y) = (qx - q^{-1}y)(qxy - q^{-1})$ (and, obviously, $h_1(x,y) = (x-y)(xy-1)$). Let us also define

$$f(w,z) = \frac{1}{z(1-q^2w^2)(q^{-2}-1)}\left(\frac{1}{h_1(w,z)} - \frac{1}{h_q(w,z)}\right) = \frac{1}{h_1(w,z)h_q(w,z)}. \tag{C.1}$$

The quantity of interest is

$$B_n(w,z) = AS\left\{\frac{\prod_{i<j}(qw_i - q^{-1}w_j)}{\prod_{i\le j} h_1(w_j,z_i)\prod_{i\ge j} h_q(w_j,z_i)}\right\}, \tag{C.2}$$

where $AS(\phi)(w_1,\dots,w_n) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{|\sigma|}\phi(w_{\sigma(1)},\dots,w_{\sigma(n)})$.

We then claim that B_n can be written as:

$$B_n(w,z) = \frac{q^{\frac{n(n-1)}{2}}\mathbf{f}_n}{\prod_{i<j} h_1(z_i,z_j)(1-q^2w_iw_j)}, \tag{C.3}$$

where $\mathbf{f}_n = \det[f(w_i,z_j)]_{i,j\le n}$.

[1] More precisely, the expression we anti-symmetrize is the integrand before the homogeneous limit in which spectral parameters come in pairs $\{z, 1/z\}$.

Again, we prove it by induction. For $n = 1$, we obtain on both sides:

$$B_1 = \frac{1}{h_1(w_1, z_1)h_q(w_1, z_1)}.$$

Let the equality of (C.2) and (C.3) hold at $n-1$. Starting from (C.2) and pushing z_n and w_j out of the anti-symmetrization we can write our equation as follows:

$$B_n(w,z) = \sum_j (-1)^{n+j} \frac{\prod_{i\neq j}(qw_i - q^{-1}w_j)}{\prod_i h_1(w_j, z_i)h_q(w_i, z_n)} AS \left\{ \frac{\prod_{i<k}(qw_l - q^{-1}w_k)}{\prod_{i\leq k} h_1(w_k, z_i) \prod_{i\geq k} h_q(w_k, z_i)} \right\}_{\hat{z}_n \hat{w}_j}$$

where the hat over $\hat{z_n}$ and $\hat{w}_j$ means that the terms that include them are absent from the anti-symmetrization. We use the hypothesis to replace the anti-symmetrization part:

$$\begin{aligned} B_n &= \sum_j (-1)^{n+j} \frac{\prod_{i\neq j}(qw_i - q^{-1}w_j)}{\prod_i h_1(w_j, z_i)h_q(w_i, z_n)} \frac{q^{\frac{(n-1)(n-2)}{2}} \mathbf{f}_{n-1,\hat{w}_j \hat{z}_n}}{\left(\prod_{i<k} h_1(z_i, z_k)(1 - q^2 w_i w_k)\right)_{\hat{w}_j \hat{z}_n}} \\ &= \sum_j (-1)^{n+j} \frac{\prod_{i\neq j} h_q(w_i, w_j) \prod_{i\neq n} h_1(z_i, z_n)}{\prod_i h_1(w_j, z_i)h_q(w_i, z_n)} \frac{(-1)^{n-1} q^{\frac{n(n-1)}{2}} \mathbf{f}_{n-1,\hat{w}_j \hat{z}_n}}{\left(\prod_{i<k} h_1(z_i, z_k)(1 - q^2 w_i w_k)\right)}. \end{aligned}$$

We rewrite this expression under the form $\sum_j (-1)^{n+j} \mathbf{f}_{n-1,\hat{w}_j \hat{z}_n} \sum_i g_i f(w_j, z_i)$ for some functions g_i. Indeed, using the fact that $\mathbf{f}_n$ is a determinant, we would get

$$\sum_j (-1)^{n+j} \mathbf{f}_{n-1,\hat{w}_j \hat{z}_n} \sum_i g_i f(w_j, z_i) = \sum_j (-1)^{n+j} \mathbf{f}_{n-1,\hat{w}_j \hat{z}_n} g_n f(w_j, z_n) = \mathbf{f}_n g_n.$$

One can guess the form of g_i:

$$g_i = \frac{\prod_{j\neq i,n} h_1(z_j, z_n) \prod_j h_q(w_j, z_i)}{\prod_{j\neq i,n} h_1(z_i, z_j) \prod_j h_q(w_j, z_n)}.$$

One can verify this decomposition directly. Equivalently, it can be written as

$$\sum_i \frac{\prod_{j\neq i,n} h_1(z_j, z_n) \prod_j h_q(w_j, z_i)}{\prod_{j\neq i,n} h_1(z_i, z_j) \prod_j h_q(w_j, z_n)} f(w_k, z_i) = \frac{\prod_{i\neq k} h_q(w_i, w_k) \prod_{i\neq n} h_1(z_i, z_n)}{\prod_i h_1(w_k, z_i)h_q(w_i, z_n)}, \quad \text{(C.4)}$$

or, by multiply both sides with $\prod_i h_1(w_k, z_i)h_q(w_i, z_n)$ to obtain polynomials of w_k of degree $2(n-1)$:

$$\sum_i \frac{\prod_{j\neq i,n} h_1(z_j, z_n)}{\prod_{j\neq i,n} h_1(z_i, z_j)} \prod_{j\neq k} h_q(w_j, z_i) \prod_{j\neq i} h_1(w_k, z_j) = \prod_{i\neq k} h_q(w_i, w_k) \prod_{i\neq n} h_1(z_i, z_n). \quad \text{(C.5)}$$

It is enough to prove that this equation is the same in all points $w_k = z_i$ and $w_k = z_i^{-1}$. In the first case we have:

$$\begin{aligned} \frac{\prod_{j\neq i,n} h_1(z_j, z_n)}{\prod_{j\neq i,n} h_1(z_i, z_j)} \prod_{j\neq k} h_q(w_j, z_i) \prod_{j\neq i} h_1(z_i, z_j) &= \prod_{j\neq k} h_q(w_j, z_i) \prod_{j\neq n} h_1(z_j, z_n) \\ \prod_{j\neq i,n} h_1(z_j, z_n) \prod_{j\neq i} h_1(z_i, z_j) &= \prod_{j\neq n} h_1(z_j, z_n) \prod_{j\neq i,n} h_1(z_i, z_j), \end{aligned}$$

which is always true. In the second case $w_k = z_i^{-1}$:

$$\frac{\prod_{j\neq i,n} h_1(z_j,z_n)}{\prod_{j\neq i,n} h_1(z_i,z_j)} \prod_{j\neq k} h_q(w_j,z_i) \prod_{j\neq i} h_1(z_i^{-1},z_j) = \prod_{j\neq k} h_q(w_j,z_i^{-1}) \prod_{j\neq n} h_1(z_i,z_n), \tag{C.6}$$

multiplying both sides by $z_i^{2(n-1)}$ and using the equalities $z_i^2 h_1(z_i^{-1},x) = h_1(z_i,x)$ and $z_i^2 h_q(x,z_i^{-1}) = h_q(x,z_i)$ we obtain the same equality.

Finally we calculate g_n:

$$g_n = \frac{\prod_{j\neq n} h_1(z_j,z_n) \prod_j h_q(w_j,z_n)}{\prod_{j\neq n} h_1(z_n,z_j) \prod_j h_q(w_j,z_n)} = (-1)^{n-1}, \tag{C.7}$$

we replace $\sum_i g_i f(w_j,z_i)$ by $g_n f(w_j,z_n)$:

$$\begin{aligned} B_n &= \sum_j (-1)^{n+j} \frac{q^{n(n-1)/2} \mathbf{f}_{n-1,\hat{w}_j\hat{z}_n} f(w_j,z_n)}{\left(\prod_{i<k} h_1(z_i,z_k)(1-q^2 w_i w_k)\right)} \\ &= q^{\frac{n(n-1)}{2}} \frac{\mathbf{f}_n}{\left(\prod_{i<k} h_1(z_i,z_k)(1-q^2 w_i w_k)\right)}. \end{aligned} \tag{C.8}$$

C.1.2 Integral version

A special case (of direct interest to us) is when the integration region only includes the poles $w_i = z_j^{\pm 1}$. Let us thus consider the following integral

$$\oint \dots \oint \prod_i \frac{dw_i}{2\pi i} \psi(w,z) B_n(w,z), \tag{C.9}$$

where $\psi(w,z)$ is an analytic function of the w in the integration region. Looking at the expression (C.1), we note that if in the calculation of $\mathbf{f}_n$ we pick a term with at least one $h_q(w_i,z_j)$ there will be fewer than n poles and the integral will be zero. This way, we can erase all the terms with $h_q(w_i,z_j)$, and form the restricted $\bar{\mathbf{f}}_n$:

$$\bar{\mathbf{f}}_n = \frac{1}{(q^{-2}-1)^n \prod_i z_i (1-q^2 w_i^2)} \det \left| \frac{1}{h_1(w_i,z_j)} \right|.$$

If we rewrite $h_1(w_i,z_j) = w_i z_j (w_i + w_i^{-1} - z_j - z_j^{-1})$ we easily identify $\bar{\mathbf{f}}$ with a Cauchy determinant, which can be evaluated:

$$\begin{aligned} \bar{\mathbf{f}}_n &= \frac{1}{(q^{-2}-1)^n \prod_i z_i^2 w_i (1-q^2 w_i^2)} \frac{\prod_{i<j} (w_i + w_i^{-1} - w_j - w_j^{-1})(z_j + z_j^{-1} - z_i - z_i^{-1})}{\prod_{i,j} (w_i + w_i^{-1} - z_j - z_j^{-1})} \\ &= \frac{1}{(q^{-2}-1)^n \prod_i z_i (1-q^2 w_i^2)} \frac{\prod_{i<j} h_1(w_i,w_j) h_1(z_j,z_i)}{\prod_{i,j} h_1(w_i,z_j)}. \end{aligned}$$

Let us now assume that ψ is of the form $\psi(w,z) = \prod_{i<j}(w_j - w_i)\,\phi(w,z)$ where ϕ is symmetric in the w_i. Then

$$\oint \ldots \oint \prod_i \frac{dw_i}{2\pi i}\psi(w,z)B_n(w,z)$$

$$= n! \oint \ldots \oint \prod_i \frac{dw_i}{2\pi i}\phi(w,z)\frac{\prod_{i<j}(w_j - w_i)(qw_i - q^{-1}w_j)}{\prod_{i\leq j} h_1(w_j,z_i)\prod_{i\geq j} h_q(w_j,z_i)}$$

$$= \frac{1}{(q^{-2}-1)^n}\oint \ldots \oint \prod_i \frac{dw_i}{z_i 2\pi i}\phi(w,z)\frac{q^{\frac{n(n-1)}{2}}\prod_{i<j}(w_j - w_i)h_1(w_j,w_i)}{\prod_{i\leq j}(1-q^2 w_i w_j)\prod_{i,j} h_1(w_i,z_j)}. \quad \text{(C.10)}$$

C.1.3 Homogeneous Limit

The case of interest to us is when we set all the $z_i = 1$. We can then use the same transformation as before:

$$u_i = \frac{w_i - 1}{qw_i - q^{-1}}$$

to deduce the desired equation from equation (C.10).

Call $\varphi(u) = \prod_i (1-qu_i)^2 \phi(w_i = \frac{1-q^{-1}u_i}{1-q\,u_i}, z_i = 1)$. The second line becomes

$$n! \oint \ldots \oint \prod_i \frac{du_i}{2\pi i u_i^{2i}} \frac{\varphi(u)}{(q-q^{-1})^{n(n+2)}} \prod_{i<j}(u_j - u_i)(1+\tau u_j + u_i u_j),$$

while the expression on the third line becomes

$$\oint \ldots \oint \prod_i \frac{du_i}{2\pi i u_i^{2n}} \frac{\varphi(u)}{(q-q^{-1})^{n(n+2)}} \frac{\prod_{i<j}(u_j - u_i)(u_i - u_j)(u_i + u_i + \tau u_i u_j)}{\prod_{i\leq j}(1-u_i u_j)}.$$

In both cases, the integrals surround zero.

In the latter, one can reinterpret some factors as a Vandermonde determinant:

$$AS\left\{\prod_i \frac{(1+\tau u_i)^{i-1}}{u_i^{2i}}\right\} = \prod_i \frac{1}{u_i^{2n}} \prod_{i<j}(u_i - u_j)(u_i + u_j + \tau u_i u_j),$$

and replace to obtain our final result:

$$\oint \ldots \oint \prod_i \frac{du_i}{2\pi i}\frac{\varphi(u)}{u_i^{2i}} \prod_{i<j}(u_j - u_i)(1+\tau u_j + u_i u_j)$$

$$= \oint \ldots \oint \prod_i \frac{du_i}{2\pi i}\varphi(u)\frac{(1+\tau u_i)^{i-1}}{u_i^{2i}}\frac{\prod_{i<j}(u_j - u_i)}{\prod_{i\leq j}(1-u_i u_j)}, \quad \text{(C.11)}$$

where we recall that $\varphi(u)$ is some analytic function in a neighborhood of zero (that is, without poles in this domain) and symmetric in the u_i.

C.2 Proof of the identity (5.16)

This identity is similar to the one proved in section C.1. To prove it we will follow the procedure of Zeilberger [81].

We recall the equality that we want to prove:

$$\mathcal{A}\left(\frac{\prod_{1\le i<j\le r}(1+\tau u_j+u_iu_j)}{\prod_{i=1}^r u_i^{2i+p-2}(1-\prod_{j=1}^i u_j^2)}\right)_{\le}=\mathcal{A}\left(\prod_{i=1}^r\frac{(1+\tau u_i)^{i-1}}{u_i^{2i+p-2}\prod_{j=i}^r(1-u_ju_i)}\right)_{\le}, \quad (C.12)$$

where $\mathcal{A}$ is the anti-symmetrization operation on the variables $u_1,\dots,u_r$, and the subscript $_\le$ means that we only are considering the monomials of the kind $\propto\prod u_i^{a_i}$ with $a_i\le 0$ for all i.

We can directly anti-symmetrize the right term:

$$\mathcal{A}\left(\prod_i\frac{(1+\tau u_i)^{i-1}}{u_i^{2i+p-2}\prod_{j\ge i}(1-u_ju_i)}\right)_{\le}=\left(\frac{\Delta(u_j^{-1})\prod_{j>i}(u_j^{-1}+u_i^{-1}+\tau)}{\prod_i u_i^p\prod_{j\ge i}(1-u_ju_i)}\right)_{\le}, \quad (C.13)$$

where $\Delta(u_i^{-1})=\prod_{j>i}(u_j^{-1}-u_i^{-1})$ is the Vandermonde determinant.

In what follows we shall call the two sides of equality (C.12) before the truncation with $_\le$, $A_{p,r}(u_1,\dots,u_r)$ and $B_{p,r}(u_1,\dots,u_r)$ respectively.

The proof will be done by induction. The first step is to calculate the case $r=1$:

$$\mathcal{A}\left(\frac{1}{u_1^p(1-u_i^2)}\right)_{\le}=\left(\frac{1}{u_1^p(1-u_1^2)}\right)_{\le}.$$

Next, we suppose that $A_{p,r-1}(u_1,\dots,u_{r-1}))_\le=B_{p,r-1}(u_1,\dots,u_{r-1})_\le$. We have

$$A_{p,r}(u_1,\dots,u_r)=\left(\sum_j(-1)^{r-j}\frac{\prod_{i\ne j}(1+\tau u_j+u_iu_j)}{u_j^{2r+p-2}(1-\prod_i^r u_i^2)}A_{p,r-1}(u_1,\dots,\hat{u}_j,\dots,u_r)\right)_{\le}.$$

Using the hypothesis and the fact that $_\le$ is a linear operator, we rewrite the conjecture as:

$$B_{p,r}(u_1,\dots,u_r)_\le=\left((-1)^{r-j}\sum_j\frac{\prod_{i\ne j}(1+\tau u_j+u_iu_j)}{u_j^{2r+p-2}(1-\prod_i^r u_i^2)}B_{p,r-1}(u_1,\dots,\hat{u}_j,\dots,u_r)\right)_{\le}.$$

Working a little bit the expression, we obtain:

$$B_{p,r\le}=\left(\sum_j\frac{\prod_i(1-u_ju_i)}{u_j^{2r-2}(1-\prod_i u_i^2)}\prod_{i\ne j}\frac{1+\tau u_j+u_iu_j}{(u_j^{-1}+u_i^{-1}+\tau)(u_j^{-1}-u_i^{-1})}B_{p,r}\right)_{\le}.$$

This equality is a consequence of the following identity, which was pointed out in [81]:

$$\sum_j\frac{\prod_i(1-u_ju_i)}{u_j^{2r-2}(1-\prod_i u_i^2)}\prod_{i\ne j}\frac{1+\tau u_j+u_iu_j}{(u_j^{-1}+u_i^{-1}+\tau)(u_j^{-1}-u_i^{-1})}=1.$$

In order to prove this identity, we replace $u_j \to u_j^{-1}$ for all j:

$$(-1)^{r-1}\sum_j \frac{\prod_i(1-u_ju_i)}{1-\prod_i u_i^2}\prod_{i\neq j}\frac{1+\tau u_i+u_iu_j}{(u_j+u_i+\tau)(u_j-u_i)}=1.$$

Or, written in another form:

$$\sum_j \frac{(\tau+2u_j)\prod_i(1-u_ju_i)(1+u_i(\tau+u_j))}{\prod_{i\neq j}(u_j-u_i)\prod_i(\tau+u_i+u_i)}=(-1)^{r-1}(1-\prod_i u_i^2). \tag{C.14}$$

We recall how to prove this identity using the Lagrange interpolation formula:

Theorem C.1. *Let $P(z)$ be a $N-1$ degree polynomial in z and let be $(w_1,\dots,w_N)$ different points. So these points define the polynomial that can be written by:*

$$P(z)=\sum_j P(w_j)\prod_{i\neq j}\frac{z-w_i}{w_j-w_i}.$$

Corollary C.2. *The maximal coefficient of $P(z)$ is:*

$$\sum_j \frac{P(w_j)}{\prod_{i\neq j}(w_j-w_i)}.$$

Let be α and β the two roots of $(1+\tau u_j+u_j^2)$. Let P be

$$P(z)=(\tau+2z)\prod_i(1+u_i(\tau+z))(1-u_iz).$$

It is a polynomial of degree $(2r+1)$, so by the Lagrange interpolation formula, and using the points $(-u_1-\tau,\dots,-u_r-\tau,u_1,\dots,u_r,\alpha,\beta)$ to describe $P(z)$, we obtain the formula:

$$\begin{aligned}2(-1)^r\prod_i u_i^2 =&\sum_j \frac{P(-\tau-u_j)}{(\tau+u_j+\alpha)(\tau+u_j+\beta)\prod_i(-\tau-u_i-u_j)\prod_{i\neq j}(-u_j-\tau+u_i+\tau)}\\ &+\sum_j\frac{P(u_j)}{(u_j-\alpha)(u_j-\beta)\prod_{i\neq j}(u_j-u_i)\prod_i(u_j+u_i+\tau)}\\ &+\frac{P(\alpha)}{(\alpha-\beta)\prod_i(\alpha-u_i)(\alpha+u_i+\tau)}\\ &+\frac{P(\beta)}{(\beta-\alpha)\prod_i(\beta-u_i)(\beta+u_i+\tau)}.\end{aligned}$$

Using $P(-\tau-u_j)=-P(u_j)$, $\alpha+\beta=-\tau$ and $\alpha\beta=1$, we observe that the first two terms are identical (and identical to the l.h.s. of (C.14)), while the sum of the last two terms simplifies to $2(-1)^r$. Thus, we get (C.14). Proving this equality.

We can rewrite the main equality as a contour integral formula:

$$\oint \dots \oint \prod_i \frac{du_i}{2\pi i} \frac{\prod_{i<j}(u_j - u_i)(1 + \tau u_j + u_i u_j)}{u_i^{2i+p-1}(1 - \prod_{j=1}^{i} u_j^2)} \Omega(u_1, \dots, u_r) =$$
$$= \oint \dots \oint \prod_i \frac{du_i}{2\pi i} \frac{(1 + \tau u_i)^{i-1} \prod_{j>i}(u_j - u_i)}{u_i^{2i+p-1} \prod_{j\geq i}(1 - u_j u_i)} \Omega(u_1, \dots, u_r), \quad \text{(C.15)}$$

where $\Omega(u_1, \dots, u_r)$ is a symmetric function in all u_i without poles in the integration region around $u_i = 0$.

Appendix D

Examples of $\Psi_{(\pi)_p}$

In this section we give some results, that confirm the theorem in 5.2.3, showing explicitly the form of the polynomials. We use $\tau = 1$ for simplicity.[1]

For $r = 2$:

$$\psi_{(()())_p} = (p+1)$$

For $r = 3$:

$$\begin{aligned}
\psi_{(()()())_p} &= \frac{(p+1)}{6}(2p^2 + 7p + 12) \\
\psi_{((())())_p} &= \frac{(p+1)(p+2)}{2}
\end{aligned}$$

For $r = 4$:

$$\begin{aligned}
\psi_{(()()()())_p} &= \frac{(p+1)(p+3)}{180}(4p^3 + 32p^3 + 155p^2 + 334p + 420) \\
\psi_{(((()))())_p} &= \frac{(p+1)(p+2)(p+3)}{6} \\
\psi_{((()())())_p} &= \frac{(p+1)(p+2)}{24}(3p^2 + 17p + 3) \\
\psi_{((())(()))_p} &= \frac{(p+1)(p+2)^2(p+3)}{12} \\
\psi_{((())()())_p} &= \frac{(p+1)(p+2)(p+3)}{24}(p^2 + 4p + 12) \\
\psi_{(()(())())_p} &= \frac{(p+1)}{20}(p^4 + 9p^3 + 36p^2 + 64p + 60)
\end{aligned}$$

[1] see (5.3) for some examples with general τ.

Appendix E

Equivalence of the definitions of root multiplicities

We will give here a proof of Theorem 6.2, which states the integers $m_i^{(A)}(\pi)$ and $m_i^{(B)}(\pi)$ defined in section 6.2.1 are equal for any matching π and any integer i with $1 \leq i \leq |\pi|-1$. This proof is borrowed from the article [22] and it was done by Nadeau.

Let π be a matching with n arches; we will prove the theorem by induction on $d(\pi)$. The theorem holds if $d(\pi) = 0$; indeed this means that $\pi = ()_n$, and clearly that $m_i^{(A)}(\pi) = m_i^{(B)}(\pi) = 0$ for all i in this case.

We now assume $d(\pi) > 0$. Let π' be the matching obtained when the external rim of π is removed. If π is represented as a parenthesis word, then π' is simply obtained by replacing the leftmost closing parenthesis of π by an opening parenthesis, and the rightmost opening parenthesis by a closing one. Let i, j, k be the indices defined in Rule B. Then in the parenthesis word representing π, the indices of the two parentheses above are respectively $i+1$ and $\widehat{\jmath}-1$. More precisely, π admits the unique factorization:

$$\pi = \left(^i\right)x_1)x_2)\cdots x_{i-k})w(y_{j-k}(\cdots(y_2(y_1()^j\,, \tag{E.1}$$

where x_t, y_t and w are (possibly empty) parenthesis words. We let $a_0 := i+1 < a_1 < \ldots < a_{i-k}$ be the indices of the closing parentheses written above and $b_{j-k} < \ldots < b_1 < b_0 = \widehat{\jmath+1}$ be the indices of opening ones.

Then by the factorization (E.1) the matching π includes the arches:

$$(k, a_{i-k}), \ldots, (i-1, a_1), (i, i+1) \quad \text{and} \quad (b_{j-k}, \widehat{k}), \ldots, (b_1, \widehat{\jmath-1}), (\widehat{\jmath+1}, \widehat{\jmath}), \tag{E.2}$$

and moreover these are exactly the arches which are modified when going from π to π'; indeed, these are replaced in π' by

$$\begin{aligned}
&(k, \widehat{k}), (k+1, \widehat{k+1}),\\
&(k+2, a_{i-k}), \ldots, (i, a_2), (i+1, a_1),\\
&(b_{j-k}, \widehat{k-2}), \ldots, (b_2, \widehat{\jmath}), (b_1, \widehat{\jmath+1}).
\end{aligned}$$

From this data we can now study the changes going from the pair $\mathcal{A}_t^L(\pi), \mathcal{A}_t^R(\pi)$ to the pair $\mathcal{A}_t^L(\pi'), \mathcal{A}_t^R(\pi')$ for any integer t between 1 and $n-1$. A case-by-case analysis shows that:

$$|\mathcal{A}_t^L(\pi)| = |\mathcal{A}_t^L(\pi')| + \delta_t, \text{ with } \delta_t = \begin{cases} 1 \text{ if } t = k; \\ 2 \text{ if } k < t \leq i; \\ 0 \text{ otherwise }, \end{cases}$$

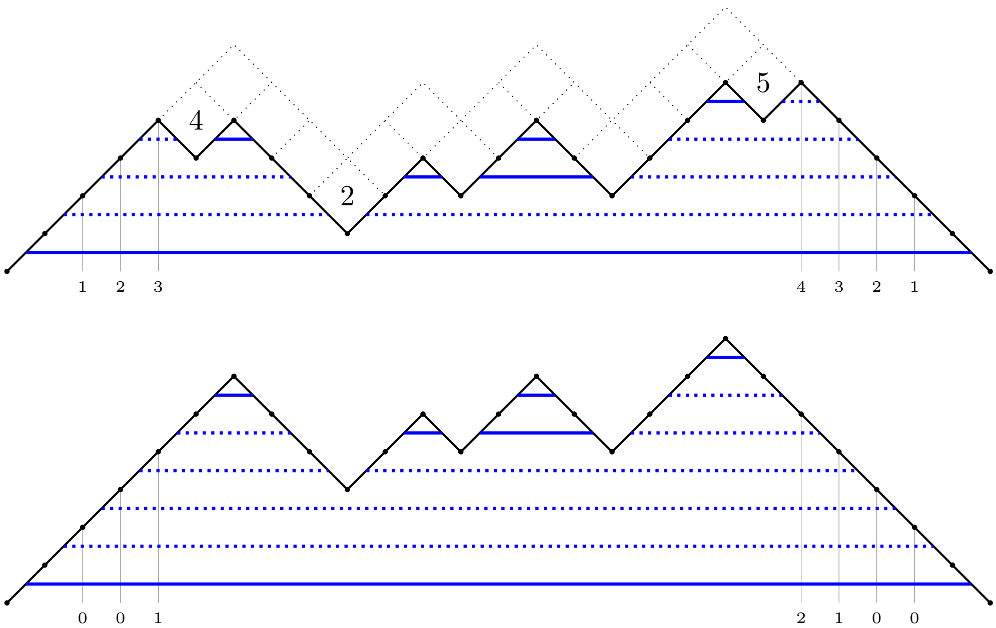


Figure E.1: In this example we have $i = 4, j = 5$ and $k = 2$, therefore the multiset attached to rim by Rule B is $\{2, 3^2, 4^2, 5\}$.

and, symmetrically:

$$|\mathcal{A}_t^R(\pi)| = |\mathcal{A}_t^R(\pi')| + \epsilon_t, \text{ with } \epsilon_t = \begin{cases} 1 \text{ if } t = k; \\ 2 \text{ if } k < t \leq j; \\ 0 \text{ otherwise }. \end{cases}$$

By definition $m_t^{(A)}(\pi) - m_t^{(A)}(\pi') = (\epsilon_t + \delta_t)/2$. From the explicit values above, this can be equivalently expressed by the fact that the multiset difference between $\{1^{m_1^{(A)}(\pi)} 2^{m_2^{(A)}(\pi)} \ldots\}$ and $\{1^{m_1^{(A)}(\pi')} 2^{m_2^{(A)}(\pi')} \ldots\}$ is:

$$\{k, i, i-1, \ldots, k+1, j, j-1, \ldots k+1\}.$$

But this is exactly the multiset associated to the rim of π in Rule B, so Theorem 6.2 is proved by induction.

Annexe F

Résumé en français

F.1 Introduction

Les matrices à signes alternants (ASM - Alternating Sign Matrices en anglais) furent inventées par Robbins et Rumsey dans leur étude des « γ-déterminants »en [70], une généralisation des déterminants inspiré par la condensation de Dodgson [19]. Mills, Robbins et Rumsey [49, 50, 70, 51] étudièrent les propriétés de ces objets et ils découvrirent une belle formule qui les compte :

$$A_n := \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

Intrigués par cette formule conjecturée, ils étudièrent la littérature. C'est à ce moment qu'un autre objet combinatoire entre en jeu : les partitions planes (PP - Plane Partitions), et en particulier deux sous-classes, les partitions planes descendantes (DPP - Descending Plane Partitions) et les partitions planes totalement symétriques et auto-complémentaires (TSSCPP - Totally Symmetric Self-Complementary Plane Partitions). En 1979, Andrews [1] prouva que ce même produit compte le nombre total de DPP de hauteur bornée par n. Donc, si on était capable de trouver une bijection explicite entre DPP et ASM on aurait automatiquement prouvé la formule d'énumération. Malheureusement, on n'a pas encore trouvé une telle bijection.

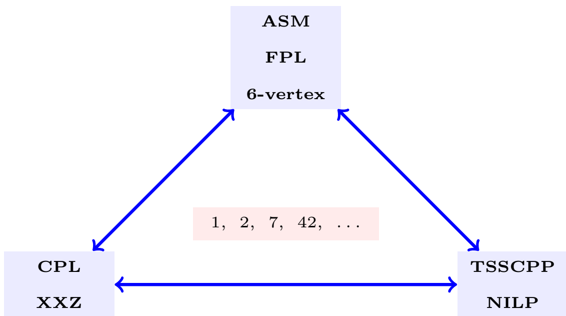
Plus tard, Robbins découvrit qu'il y a une autre sous-classe de PP qui est aussi comptée par la séquence A_n, les TSSCPP. D'ailleurs, dans l'article [51] les auteurs proposèrent que non seulement le nombre de TSSCPP est égal au nombre de ASM, mais que cette énumération peut être raffinée des deux côtés, ces raffinements donnant le même résultat.

En 1996, dans son fameux article de 84 pages [80], Zeilberger prouva finalement la formule d'énumération des ASM. Quelques mois plus tard, Kuperberg [44], utilisant la fonction de partition du modèle à 6 vertex qui est en bijection avec les ASM, prouva exactement le même résultat d'une façon plus courte et élégante. Sa preuve est basée sur le travail d'Izergin [29] et Korepin [40], et est une conséquence du fait que le modèle à 6 vertex est un modèle quantique intégrable.

La chaîne de spins XXZ est déjà un problème classique en physique, et il suscite beaucoup d'intérêt depuis quasiment un siècle. En 2000, Razumov et Stroganov [68, 76] investigaient les propriétés de l'état fondamental de ce modèle (avec paramètre anisotrope $\Delta = -1/2$), quand ils trouvèrent un résultat surprenant : lorsque normalisé tel que la composante plus petite est égale à 1, la somme des composantes du vecteur fondamental vaut exactement A_n. Intrigués par cela, physiciens et mathématiciens ont tout essayé pour prouver cette relation. Moins d'un mois après, Nienhuis, de Gier et Batchelor écrivirent un article [3] où ils présentent quelques nouvelles conjectures et introduisent un nouveau objet, le modèle à boucles $O(n)$, ici appelé de modèle à boucles denses (CPL - Compact Packed Loops).

Quelques mois plus tard, Razumov et Stroganov [65] remarquèrent que le lien entre les deux modèles est encore plus profond. Les configurations de boucles denses sont décrits par des motifs d'arches (on utilise généralement π pour désigner tels motifs). De l'autre côté, les ASM sont en bijection avec des configurations de boucles compactes (FPL - Fully Packed Loops) qui peuvent être classifiées par leur connectivité au bord également représentée par des motifs d'arches. Ils découvrirent que quand on normalise l'état fondamental du CPL de manière à ce que la composante la plus petite vaille 1, les composantes associées à un certain motif d'arches comptent les configurations FPL dont la connectivité est représentée par le même motif d'arches.

Le dernier chapitre de cette histoire fut écrit par Cantini et Sportiello. En 2010 ils prouvèrent la conjecture de Razumov et Stroganov [8]. On résume cette histoire dans le graphe :



Où on représente tous les objets principaux décrits ici, groupés par le fait qu'il y a des bijections bien connues entre eux. Au centre, il apparaît la séquence qui est omniprésente en toute cette histoire, A_n. Les NILP cités sont ces en bijection avec les TSSCPP.

On retient une importante leçon du résultat de Kuperberg : les modèles quantiques intégrables peuvent nous aider à résoudre certains problèmes difficiles la combinatoire. D'ailleurs, le modèle CPL (ou la chaîne de spins XXZ) et le modèle à 6 vertex sont tous deux intégrables. Donc, il semble que l'intégrabilité peut jouer un rôle dans la clarification des relations entre les coins du triangle.

L'équation de Knizhnik–Zamolodchikov quantique (qKZ) fut introduite dans ce contexte par Di Francesco et Zinn-Justin (dans l'article [16]). Dans notre cas (niveau 1 et $U_q(sl(2))$), ces solutions sont des polynômes homogènes, et on peut montrer qu'elles engendrent un espace vectoriel caractérisé par une condition d'annulation, la *wheel condition*.

D'un côté, les solutions de cette équation (niveau 1, et $q^{2\pi i/3}$) peuvent être identifiées avec les composantes de l'état fondamental du modèle CPL dans la version multivariables. De l'autre, ces polynômes ont un rapport avec les polynômes de Macdonald spécialisés dans les valeurs de paramètres $t^3q = 1$ (regarder [35, 12]), et ils sont ce que Lascoux appelle des polynômes de Kazhdan–Lusztig [12].

L'objectif principal de ce manuscrit est de comprendre le rôle de l'intégrabilité dans ces questions. Notamment, le rôle joué par l'équation de Knizhnik–Zamolodchikov quantique et ses solutions. Ce résumé suit l'organisation du manuscrit.

F.2 Le modèle à boucles denses et l'équation de Knizhnik–Zamolodchikov quantique

F.2.1 Boucles denses

Les modèles à boucles (Loop models en anglais) forment une classe très intéressante de modèles statistiques à deux dimensions sur réseau. En fait, ils présentent une large gamme de phénomènes critiques et plusieurs modèles classiques peuvent être vus comme des modèles à boucles. On considère ici le modèle à boucles denses (CPL - Completely Packed Loops), aussi appelé modèle à boucles $O(n)$ dans la littérature.

Prenons un réseau carré. Chaque face est occupée par une des plaquettes de la figure F.1. Il y aura des boucles fermées et aussi des chemins liant les bords. On donne un poids $\tau = -q - q^{-1}$ à chaque boucle fermée. Ce modèle est critique pour $|\tau| \leq 2$ (cf. [57]).

FIGURE F.1 – Les deux plaquettes qui forment le modèle CPL.

On considère ce modèle sur un cylindre semi-infini, où chaque ligne est composée de $2n$ plaquettes, comme le montre la figure F.2. Le bord à gauche est identifié à celui à droite.

F.2.2 Connectivité

À chaque configuration on associe un motif d'arches représentant la connectivité au bord inférieur. Par exemple, la connectivité de la configuration montré dans la figure F.2 est représenté par le motif en F.3. Le nombre de motifs de taille $2n$ est le fameux nombre de Catalan c_n :

$$c_n = \frac{(2n)!}{n!(n+1)!}$$

On définit un état formel comme une combinaison linéaire de motifs, $\xi = \sum_\pi \xi_\pi \pi$, où la somme parcourt tous les c_n motifs et ξ_π sont des scalaires. Un opérateur est alors une application linéaire dans l'espace vectoriel engendré par ces états formels.

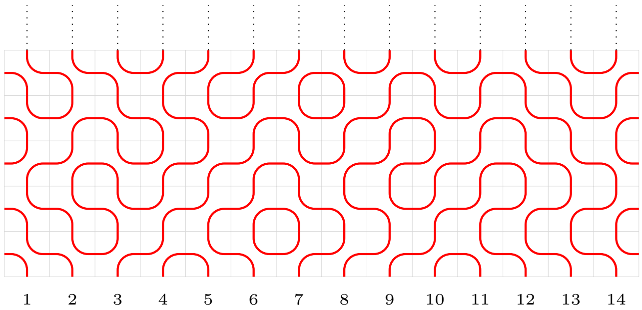


FIGURE F.2 – Un exemple d'une configuration du CPL, ici avec $n = 7$. Chaque ligne est composée de 14 plaquettes qui peuvent être de deux types. Les bords de gauche et de droite sont identifiés pour former un cylindre. À chaque boucle fermée on donne le poids τ.

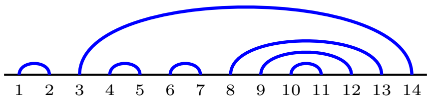


FIGURE F.3 – Motif qui correspond à la connectivité de l'exemple F.2

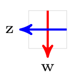
Parfois, il convient de représenter ces motifs par d'autres objets. Dans ce manuscrit on utilise souvent des mots de parenthèses, des chemins de Dyck, des diagrammes d'Young et des séquences $a = \{a_1, \ldots, a_n\} \subseteq \{1, \ldots, 2n\}$, tel que $a_{i-1} < a_i$ et $a_i \leq 2i - 1$ pour tout i. Voyons un exemple traduit en toutes ces représentations :

 $\Leftrightarrow$ ()(()) $\Leftrightarrow$  $\Leftrightarrow$  $\Leftrightarrow \{1, 3, 4\}$

F.2.3 La matrice R et l'intégrabilité

On considère maintenant une version dynamique de ce modèle où l'on insère des lignes de plaquettes dans le bord inférieur.

Prenons $q = e^{\pm 2\pi i/3}$, *i.e.* $\tau = 1$. On définit la matrice R :

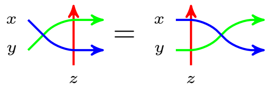


$$= \frac{qw - q^{-1}z}{qz - q^{-1}w}$$  $$+ \frac{w - z}{qz - q^{-1}w}$$  $$=: R(w, z).$$

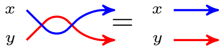
où les coefficients représentent la probabilité de choisir l'une ou l'autre plaquette. Ces probabilités dépendent de deux paramètres w et z, appelés respectivement paramètres spectraux verticaux et horizontaux, et qui varient selon la ligne e la colonne. En choisissant bien ces paramètres on peut ajuster toutes les probabilités à un demi.

Cette matrice R est la clé de l'intégrabilité :

Lemme F.1. *La matrice R obéit à l'équation de Yang–Baxter :*



et à l'équation d'identité :



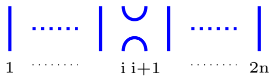
Cela nous permet d'écrire une famille infinie d'opérateurs qui commutent entre eux et de calculer analytiquement les états propres.

F.2.4 Le cas homogène

Quand tous les paramètres spectraux horizontaux sont égaux, notre système est décrit par l'hamiltonien :

$$H = \frac{1}{2n} \sum_{i=1}^{2n} e_i.$$

où

$$e_i =$$  .

Ces éléments obéissent à l'algèbre de Temperley–Lieb :

$$e_i^2 = \tau e_i \qquad e_i e_{i\pm 1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad \text{if } |j - i \text{ mod } 2n| > 1 \tag{F.1}$$

pour i et j entre 1 et $2n$. On considère aussi un élément de rotation ρ qui tourne les motifs. Évidemment, ρ^{2n} est égal à l'identité.

Une des questions qu'on essaie de résoudre dans ce travail est de trouver $\psi = \sum_\pi \psi_\pi \pi$ tel que :

$$\psi = H\psi.$$

On peut prouver que ce vecteur existe et qu'il est unique. Par exemple, pour $n = 3$, les composantes de ce vecteur sont :

ψ  $= \frac{1}{7}$ ψ  $= \frac{1}{7}$ ψ  $= \frac{1}{7}$

ψ  $= \frac{2}{7}$ ψ  $= \frac{2}{7}$

F.2.5 La chaîne de spins XXZ

Soit $V = (\mathbb{C}^2)^{\otimes 2n}$ l'espace de $2n$ spins. On définit e_i comme l'opérateur :

$$e_i = \frac{1}{2}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2}(\sigma_i^z \sigma_{i+1}^z - 1) + \frac{q-q^{-1}}{2}(\sigma_{i+1}^z - \sigma_i^z)\right)$$

où σ_i^x est la matrice de Pauli selon x et agissant dans l'i^{e} espace. Le choix du nom e_i n'est pas innocent, en fait ces opérateurs obéissent aussi à l'algèbre de Temperley–Lieb.

Si l'on somme sur tous les $2n$ sites, avec des conditions aux bords périodiques, on obtient l'hamiltonien de la chaîne de spins XXZ :

$$H = \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} (\sigma_i^z \sigma_{i+1}^z - 1) \right)$$

On cherche à calculer le vecteur fondamental d'XXZ. On peut facilement voir que les configurations CPL sont en bijection avec les configurations de spins avec $S = 0$ et $S^z = 0$[1].

F.2.6 L'équation de Knizhnik–Zamolodchikov quantique

On introduit maintenant une équation qui nous permet de calculer l'état fondamental de la chaîne de spins XXZ, ou de manière équivalente du modèle CPL, connue comme l'équation de Knizhnik–Zamolodchikov quantique :

- L'équation d'*échange* :

$$\check{R}_i(y_i, y_{i+1}) \Psi(y_1, \ldots, y_i, y_{i+1}, \ldots, y_{2n}) = \Psi(y_1, \ldots, y_{i+1}, y_i \cdot \ldots, y_{2n}). \qquad \text{(F.2)}$$

  pour $i = 1, \ldots, 2n$.

- L'équation de *rotation* :

$$\rho^{-1} \Psi(y_1, y_2, \ldots, y_{2n}) = \kappa \Psi(y_2, \ldots, y_{2n}, s y_1) \qquad \text{(F.3)}$$

  où $\kappa = q^{-3(n-1)}$ et $s = q^6$.

La lettre majuscule distingue cette version où les paramètres y_i sont distincts de la version précédente où ils sont tous égaux à 1.

Notre but est maintenant de résoudre cette équation. Il y a plusieurs façons de la résoudre. Par exemple, on peut voir cette équation comme un système d'équations triangulaire $\psi_\pi = \sum_{\sigma \prec \pi} D_{\sigma,\pi} \psi_\sigma$, où $\prec$ est un ordre partiel et $D_{\sigma,\pi}$ est un opérateur. On ne va pas décrire cette procédure ici, pour plus de détails voir la section 1.4.

On peut normaliser les solutions de cette équation, de manière à ce qu'elles soient des polynômes de degré total $n(n-1)$. On prouve aussi que ces solutions vivent dans un espace vectoriel de dimension c_n, correspondant au nombre de polynômes, caractérisé par une condition d'annulation, la *wheel condition* :

$$f(\ldots, y_i, \ldots, y_j = q^2 y_i, \ldots, y_k = q^4 y_i, \ldots) = 0 \qquad \text{pour tous } k > j > i.$$

[1]Ces nombres quantiques correspondent à l'algèbre quantique $U_q(su(2))$

F.2.7 Des intégrales de contour

Dans ce travail on a préféré utiliser une autre méthode. Soit $a = \{a_1, \ldots, a_n\}$, où $a_i \geq a_{i-1}$ pour tous $n \geq i > 1$, $a_n \leq 2n$ et $a_1 \geq 1$.

On introduit l'intégrale de contour :

$$\Phi_a(y_1, \ldots, y_{2n}) = \prod_{i<j}^{2n}(qy_i - q^{-1}y_j)$$
$$\oint \ldots \oint \prod_i \frac{dw_i}{2\pi i} \frac{\prod_{j>i}^{n}(w_j - w_i)(qw_i - q^{-1}w_j)}{\prod_{j\leq a_i}(w_i - y_j)\prod_{j>a_i}(qw_i - q^{-1}w_j)} \quad \text{(F.4)}$$

où l'intégration est faite autour des pôles $w_j = y_i$ mais pas des pôles $w_j = q^{-2}y_i$.

On prouve dans la section 1.6 que ces quantités $\Phi_a(y_1, \ldots, y_{2n})$ engendrent le même espace vectoriel que les polynômes Ψ_π. En fait il existe un changement triangulaire (et inversible) de bases :

$$\Phi_a(y_1, \ldots, y_{2n}) = \sum_{\pi \preceq a} C_{a,\pi}\Psi_\pi(y_1, \ldots, y_{2n}).$$

F.2.8 La limite homogène

Quand on a $y_i = 1$ pour tout i, les équations se simplifient. Utilisant la transformation de variables :

$$u_i = \frac{w_i - 1}{qw_i - q^{-1}}$$

on obtient facilement la formule :

$$\phi_a = \oint \ldots \oint \prod_i \frac{du_i}{2\pi i u_i^{a_i}} \prod_{j>i}(u_j - u_i)(1 + \tau u_j + u_i u_j)$$

On peut maintenant calculer le vecteur ψ. Malheureusement ces intégrales sont encore trop difficiles et la transformation de bases $C_{a,\pi}$ est compliquée.

F.3 Matrices à Signes Alternants et le modèle à 6 vertex

Dans cette section on introduit trois modèles, les Matrices à Signes Alternants (ASM), le 6-vertex et les boucles compactes (FPL - Fully Packed Loops). Tous ces modèles sont en bijection (voir chapitre 3). On calcule la fonction de partition du modèle à 6 vertex qui nous permettra de compter, par exemple, les ASM. Dans la dernière sous-section on présente le théorème de Razumov–Stroganov.

F.3.1 Matrices à Signes Alternants

Une Matrice à Signes Alternants est une matrice carrée composée uniquement de 0 et de ±1, de manière à ce que si l'on ignore les 0, les 1 et les −1 alternent sur chaque ligne ou chaque colonne, et que chaque ligne (et colonne) débute et termine par un 1. Voyons les 7 exemples du cas $n = 3$:

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\qquad\begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix}\qquad\begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix}\qquad\begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix}\qquad\begin{matrix}0&1&0\\1&-1&1\\0&1&0\end{matrix}\qquad\begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix}\qquad\begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}$$

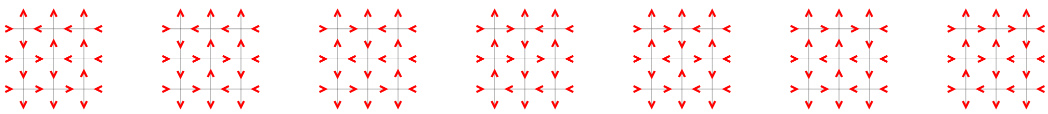
Le nombre de matrices de taille $n \times n$ est donné par la formule :

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1,\ 2,\ 7,\ 42,\ 429,\ \ldots \tag{F.5}$$

Cette expression fut prouvée par Zeilberger [80]. Mais c'est la preuve trouvé par Kuperberg [44] qui nous intéresse, celle-ci utilise un autre modèle, le 6-vertex.

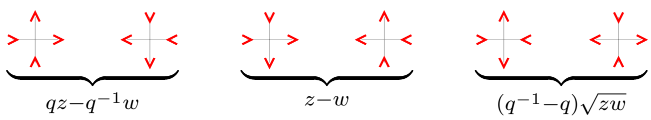
F.3.2 Modèle à 6 vertex

Prenons un réseau carré de taille $n \times n$. À chaque arête on associe une orientation (une flèche) telle que chaque sommet a deux flèches entrantes et deux flèches sortantes. Il n'y a donc que 6 configurations possibles pour un vertex (d'où le nom du modèle). Ici, on impose que toutes les flèches dans les bords gauche et droit soient entrantes et celles au-dessus et en dessous soient entrantes. On peut prouver qu'il y a une bijection entre les configurations de ce modèle et les ASM. Voyons les sept configurations de taille $n = 3$:



F.3.3 La fonction de partition

On donne à chaque vertex un poids :



où w (resp. z) caractérise les colonnes (resp. les lignes). On utilise $\{y_{n+1}, \ldots, y_{2n}\}$ (resp. $\{y_1, \ldots, y_n\}$) pour chaque colonne (resp. ligne). q est un paramètre global qui vaut, normalement, $e^{2\pi i/3}$.

La fonction de partition est définie par :

$$Z_n = (-1)^{\binom{n}{2}} (q^{-1} - q)^{-n} \prod_{i=1}^{2n} y_i^{-1/2} \sum_{\text{configurations}} \prod_{i,j} w_{i,j}$$

où $w_{i,j}$ est le poids de la configuration de chaque sommet.

Le point intéressant est que ce modèle est aussi intégrable, dans le sens que l'on peut aussi construire une matrice R, similaire à celle du modèle CPL (dans la section F.2), qui obéit aussi à l'équation de Yang–Baxter. Cela nous permet de découvrir plusieurs propriétés de la fonction de partition :

- la fonction de partition est un polynôme homogène dans les variables $\{y_1, \ldots, y_{2n}\}$;
- le degré total est $\delta = n(n-1)$ et le degré partiel est $\delta_i = n - 1$;
- la fonction de partition est un polynôme symétrique dans les ensembles $\{y_1, \ldots, y_n\}$ et $\{y_{n+1}, \ldots, y_{2n}\}$;
- la fonction de partition obéit à la relation de récurrence suivante :

$$\begin{aligned} &Z_n(y_1, \ldots, y_n, y_{n+1} = q^2 y_1, \ldots, y_{2n}) \\ &\quad = (-1)^{n-1} \prod_{i=2}^{n} (y_1 - y_{n+i})(y_i - y_{n+1}) Z_{n-1}(y_2 \ldots, y_{n-1}, y_{n+1}, \ldots, y_{2n}); \end{aligned} \tag{F.6}$$

- si $q = e^{2\pi i/3}$, la fonction de partition est un polynôme symétrique en $\{y_1, \ldots, y_{2n}\}$.

Ces propriétés permirent à Izergin de calculer la fonction de partition :

$$\begin{aligned} Z_n(y_1, \ldots, y_{2n}) = &\frac{\prod_{i,j}^{n} (y_i - y_{n+j})(qy_i - q^{-1} y_{n+j})}{\prod_{i<j}^{n} \prod (y_i - y_j)(y_{n+i} - y_{n+j})} \\ &\times \det_{1 \le i,j \le n} \left| \frac{1}{(y_i - y_{n+j})(qy_i - q^{-1} y_{n+j})} \right| \end{aligned}$$

pour q général.

Pour $q = e^{2\pi i/3}$, cette formule se simplifie. En fait on obtient une fonction de Schur :

$$Z_n(y_1, \ldots, y_n, y_{n+1}, \ldots, y_{2n}) = s_{Y_n}(q^{-1} y_1, \ldots, q^{-1} y_n, q y_{n+1}, \ldots, y_{2n}). \tag{F.7}$$

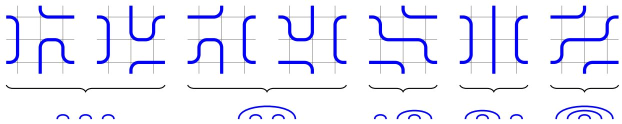
Pour prouver ces deux formules il suffit de prouver qu'elles ont toutes les propriétés citées ci-dessus.

Un exemple d'application de cette formule est le calcul du nombre d'ASM. Si l'on fixe tous les poids y_i à la valeur q pour $1 \le i \le n$ et $y_i = q^{-1}$ pour $n < i \le 2n$, on obtient que la fonction de partition est d'un côté un multiple du nombre de Matrices à Signes Alternants, et de l'autre qu'elle est égal à la fonction de Schur $s_{Y_n}(1, \ldots, 1)$. Après quelques calculs pas très difficiles on obtient le résultat désiré.

F.3.4 Boucles compactes

Il y a plusieurs modèles qui sont en bijection avec les ASM. Par exemple, le modèle de glace carré, certains modèles de chemins, ... Celui qui nous intéresse particulièrement est le modèle à boucles compactes (FPL - Fully Packed Loops).

Prenons un réseau carré $n \times n$. Par chaque sommet il y a un et seulement un chemin qui passe. Donc, ou bien les chemins commencent dans le bord et finissent au bord, au bien ce sont des chemins fermés (ils forment des boucles). On considère des conditions de bords type « parois de domaines »(DWBC - Domain Wall Boundary Condition). De toutes les arêtes qui sont au bord, une sur deux est traversée par un chemin, de manière alternée. Voyons, par exemple, toutes les sept configurations pour $n = 3$:



On a groupé les configurations par leur connectivité au bord. L'absence de croisement entre les arches est une conséquence du fait qu'en chaque sommet passe un seul un chemin.

On définit la quantité A_π comme le nombre de configurations FPL dont la connectivité est π. En 1991, Wieland [79] prouva que ces quantités sont stables par réflexion et par rotation de leur connectivité π.

Évidemment $\sum_\pi A_\pi = A_n$. Remarquez que les connectivités correspondent aux motifs définis dans la section F.2. En fait la connexion est encore plus profonde :

Théorème F.2 (Razumov–Stroganov [65]). *Les composantes ψ_π de l'état fondamental du modèles à boucles denses, normalisé de manière à ce que la composante la plus petite vaille* 1*, comptent les configurations du modèle à boucles compactes avec connectivité π :*

$$\psi_\pi = A_\pi$$

Ce théorème a été démontré par Cantini et Sportiello en 2010 [8].

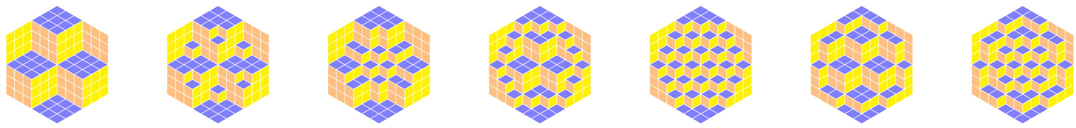
F.4 Partitions planes

F.4.1 Partitions planes totalement symétriques et auto-complémentaires

Une partition plane peut être vue comme une généralisation des partitions. Prenons un coin (d'une salle par exemple), et imaginons que la gravité tire vers ce coin. Une partition plane est donc un amas de piles de cubes que subissent cette gravité. Évidemment, on peut représenter une partition plane par les hauteurs des piles.

Maintenant on limite la partition plane à un grand cube $2n \times 2n \times 2n$ et on considère des partitions planes symétriques par échange d'axes. On peut encore rajouter une autre symétrie. On définit la partition complémentaire comme la partition formée par les cubes qui manquent pour remplir le cube $2n \times 2n \times 2n$, tourné de manière à ce qu'elle obéisse aux règles des partitions planes. Une partition plane est dite auto-complémentaire si elle coïncide avec sa partition complémentaire. Une partition plane avec toutes ces symétries s'appelle partition plane totalement symétrique et auto-complémentaire (TSSCPP de l'anglais Totally Symmetric Self-Complementary Plane Partitions).

Par exemple, pour $n = 3$, existe exactement 7 exemples :



Remarquez qu'apparaît ici le nombre de ASM de taille $n = 3$. En fait, ce n'est pas une coïncidence : on peut prouver que le nombre de TSSCPP est A_n. Ce dénombrement fut prouvé par Andrews en 1994 [2].

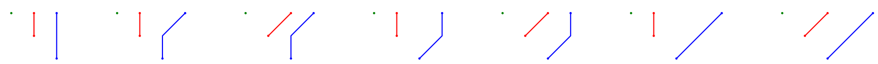
F.4.2 Chemins valués non intersectants

Les partitions planes et notamment les TSSCPP peuvent être représentées comme des pavages, des configurations de dimères ou certains ensembles de chemins sur réseau qui ne se touchent pas, dit non intersectant appelés en anglais Non-Intersecting Lattice Paths (NILP). Ces derniers sont très importants dans notre étude, ce sont eux qui nous permettent de calculer le nombre de TSSCPP.

Il n'est pas très difficile de prouver que les TSSCPP de taille $2n \times 2n \times 2n$ sont en bijection avec une sous classe de NILP définis sur un réseau carré et obéissant aux règles suivantes :

- il y a n chemins qui se touchent pas ;
- le i^{e} chemin part de la position $(i-1, 1-i)$;
- on ne permet que des pas verticaux ($\uparrow$) et des pas diagonaux ($\nearrow$) ;
- les chemins s'arrêtent quand ils arrivent à la ligne $y = 0$.

Voyons tous les cas possibles à $n = 3$:



L'intérêt de ces chemins est qu'il existe une technologie, connue comme la formule LGV (Lindström–Gessel–Viennot [46, 27]), pour les compter. D'ailleurs cette formule permet aussi des dénombrements valués dès que les poids ne dépendent que des arêtes et des sommets (et pas du chemin en soi).

F.5 Matrices à signes alternants et partitions planes totalement symétriques et auto-complémentaires

Il est connu qu'il y a autant d'ASM de taille $n \times n$ que de TSSCPP de taille $2n \times 2n \times 2n$. En fait, d'un côté on peut récrire la fonction de partition d'un modèle à 6 vertex, qui compte les ASM, sous la forme d'une multi-intégrale de contour. De l'autre, on peut utiliser la représentation des TSSCPP comme des NILP pour trouver une multi-intégrale de contour utilisant la formule LGV. Ensuite, on prouve que les deux intégrales sont égales, utilisant pour cela une égalité proposé par Di Francesco et Zinn-Justin [17] et prouvée par Zeilberger [81] (on prouve cette égalité dans l'annexe C.1). Dans l'article [23], on raffine cette procédure. En fait, on prouve une version doublement raffinée de l'égalité entre le nombre d'ASM et de TSSCPP proposée par Mills, Robbins et Rumsey [51]. Dans ce résumé on montre un esquisse de cette preuve.

F.5.1 Une conjecture raffinée

Les TSSCPP

Soit α une TSSCPP de taille $2n \times 2n \times 2n$ représentée en termes de chemins. $u^1(\alpha)$ est défini comme le nombre de chemins qui finissent avec un pas vertical et $u^n(\alpha)$ est défini comme le nombre de chemins que débutent avec un pas vertical. Par exemple les sept exemples à $n = 3$ vus dans la section ci-dessus donnent $u^1 = \{2; 1; 0; 2; 1; 1; 0\}$ et $u^n = \{2; 2; 1; 1; 0; 1; 0\}$. Remarquons que ce sont les mêmes nombres qui apparaissent mais permutés.

On définit la fonction génératrice :

$$U_n^{0n}(x, y) = \sum_{\alpha} x^{u^1(\alpha)} y^{u^n(\alpha)}$$

Par exemple, $n = 3$ donne : $U_n^{01} = x^2y^2 + xy^2 + y + x^2y + x + xy + 1$.

Les ASM

Une matrice à signes alternants possède un unique 1 dans la première ligne et un unique 1 dans la dernière ligne. Disons que le premier 1 est dans la position i et le dernier 1 est dans la position j. On définit la fonction génératrice comme étant :

$$A_n(x, y) = \sum_{ASM} x^{i-1} y^{n-j}$$

Par exemple, on a la fonction suivante pour $n = 3$: $A_n(x, y) = 1 + y + x + xy^2 + xy + x^2y + x^2y^2$.

Regardant les deux résultats, on remarque le curieux fait suivant :

Théorème F.3. *Le nombre d'ASM de taille $n \times n$ dont le 1 de la première ligne est dans la position $i - 1$ et dont le 1 de la dernière ligne est dans la position $n - j$ est égal au nombre de TSSCPP dont sa représentation en termes de chemins possède i derniers pas verticaux et j premiers pas verticaux. En termes des fonctions génératrices :*

$$A_n(x, y) = U_n^{01}(x, y)$$

F.5.2 Esquisse de la preuve

TSSCPP compté par des intégrales

On peut compter les NILP en utilisant la formule LGV [46, 27]. Cette formule permet l'utilisation de chemins avec des poids statistiques. On met un poids x par dernier pas vertical et y par premier pas vertical.

On le récrit sous la forme d'une intégrale de contour multiple. Finalement, si l'on somme sur tous les ensembles d'arrivée possibles, on obtient la formule :

$$U_n^{1n}(x,y) = \oint \ldots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}} \frac{1}{1-u_i^2} (1+xu_i)(y+u_i)_{\hat{1}} (1+u_i)_{\hat{1}}^{i-2} \prod_{j>i} \frac{u_j - u_i}{1-u_j u_i}$$

où on intègre autour de 0. La notation $\hat{1}$ signifie que l'on ignore le terme correspondant à u_1.

ASM compté par des intégrales

On part de la fonction de partition du modèle à 6 vertex introduite dans la section F.3.2 et on pose :

$$\begin{aligned} y_1 &= q\frac{1+qx}{q+x} && \\ y_i &= q && \text{if } 2 \le i \le n-1 \\ y_n &= q\frac{y+q}{1+qy} && \\ y_j &= q^{-1} && \text{if } j > n \end{aligned}$$

Avec ces paramètres, on obtient que la fonction de partition compte les ASM, en tenant compte du premier 1 et du dernier 1 : $Z_n(y_1, \ldots, y_{2n}) \propto A_n(x,y)$.

Le pas suivant consiste à trouver une formule intégrale de contour qui soit égale à Z_n. Cela est aussi connu, voir section 4.3. Après plusieurs manipulations d'intégrales, on obtient la formule :

$$(1+xy-x)y^{2(n-1)} \oint \ldots \oint \prod_{l}^{n} \frac{du_l}{2\pi i} \frac{1}{u_l^{2l-2}} \frac{\prod_{l<m}(u_m - u_l)(1+u_m+u_m u_l)}{(1+u_l-x)(y+u_l(y-1))} \prod_{j=2}^{n} (1+u_j)$$

où on intègre autour de $u_i = 0$ et $u_i = x-1$.

La preuve de l'égalité

Il reste à prouver que les deux intégrales sont égales. Le suite de la preuve est assez technique, et est basée sur plusieurs égalités entre formules intégrales.

F.6 Boucles denses et partitions planes

Dans le chapitre 5, on résout certaines conjectures sur l'état fondamental du modèle CPL. Dans l'article [85] Zuber avait conjecturé que les composantes de l'état fondamental indexées par des motifs de la forme $(\pi)_p$, *i.e.* un certain motif π entouré de p arches, sont polynomiales en p :

Théorème F.4. *Considérons un motif de la forme $(\pi)_p$, les polynômes ψ_π peuvent être écrits sous la forme suivante :*

$$\psi_{(\pi)_p} = \frac{1}{|Y|!} P_Y(\tau, n)$$

où $Y = Y(\pi)$ est le diagramme d'Young définit par π, $|Y|$ est le nombre de boîtes de Y, et $P_Y(\tau, n)$ est un polynôme en n et τ de degré $|Y|$ en chaque variable et coefficients entiers.

Dans la limite de grand n, ce polynôme se comporte comme :

$$\psi_{(\pi)_p} \approx \frac{\dim Y}{|Y|!} (n\tau)^{|Y|}$$

où $\dim Y$ est la dimension de la représentation irréductible du groupe symétrique associé à Y.

Dans cette section, on esquisse la preuve de ce théorème du côté CPL et on démontre une règle de somme pour les motifs de la forme $(\pi)_p$. On obtient aussi une règle de somme pour des motifs qui commencent avec p ouvertures, on représente ces motifs par $(_p\alpha$.

F.6.1 Polynomialité

Pour prouver que les quantités $\psi_{(\pi)_p}$ sont des polynômes, on commence par les récrire comme une somme des quantités ϕ_a définis ci-dessus dans la section F.2.8 :

$$\psi_{(\pi)_p}(\tau) = \sum_a C^{-1}_{(\pi)_p,a}(\tau) \phi_a(\tau)$$

Mais on sait que le coefficient $C^{-1}_{(\pi)_p,a}(\tau)$ est non nul uniquement si $a \preceq (\pi)_p$. En fait, si $a = (\pi)_p$ il vaut 1, et si $a \prec (\pi)_p$, il est un polynôme en τ de degré pas plus grand que $|(\pi)_p| - |a|$. D'ailleurs, si l'on note $(b)_p = \{1, 2, \ldots, p, p + b_1, p + b_n\}$, on prouve que $C^{-1}_{(\pi)_p,(b)_p} = C^{-1}_{\pi,b}$. On obtient, donc :

$$\psi_{(\pi)_p}(\tau) = \sum_b C^{-1}_{\pi,b}(\tau) \phi_{(b)_p}(\tau)$$

Prenant les expressions intégrales de $\phi_{(b)_p}(\tau)$ on conclut facilement que les $\psi_\pi(t)$ sont des polynômes avec le bon degré et le bon coefficient dominant. En fait, les termes $b \prec \pi$ contribuent uniquement aux coefficients non dominants.

F.6.2 La somme des $\psi_{(\pi)_p}$

On pose $\tau = 1$ pour simplifier. Soit a un motif de la forme $\{a_1, \dots, a_n\}$ avec $a_i = 2i - 1$ ou $a_i = 2i - 2$, le lemme 5.3 dit que :

$$\sum_{\substack{a=\{1,\dots,p,p+a_1,\dots,p+a_n\} \\ a_i=2i-1 \text{ ou } a_i=2i-2}} \phi_a = \sum_{\pi} \psi_{(\pi)_p}$$

On peut alors récrire la somme de ψ_π comme une somme d'intégrales (et simplifiant) :

$$\oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}} (1 + u_i)^{p+1} \prod_{j>i} (u_j - u_i)(1 + u_j + u_i u_j)$$

Cette intégrale peut être transformée dans une autre qui compte tous les chemins non intersectant indiqués dans l'image F.4. Ces chemins peuvent être vus comme des TSSCPP avec un hexagone fixe au milieu, comme on peut en voir dans la figure 5.3.

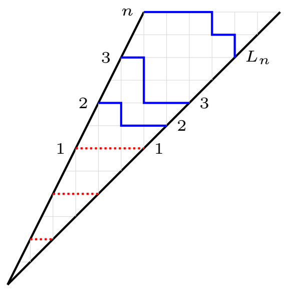


FIGURE F.4 – La somme $\sum_\pi \psi_{(\pi)_p}$ compte le nombre de NILP avec $n + p$ chemins présentés dans cette figure où les $p + 1$ premiers chemins sont fixés horizontalement.

Dans le manuscrit, on présente aussi comment générer des configurations parfaitement aléatoires (si on considère que le générateur de nombres aléatoires est parfait).

F.6.3 La somme des $\psi_{(p\alpha}$

On applique cette procédure aux composantes $\psi_{(p\alpha}$. En fait, on peut les écrire comme une combinaison linéaire des $\phi_{\{1,\dots,p,p+a_1,\dots,p+a_n\}}$ où $\{a_1, \dots, a_n\}$ ne sont pas nécessairement des motifs d'arches. Comme avant on récrit leur somme sous la forme d'une intégrale de contour multiple :

$$\oint \dots \oint \prod_{m=1}^{r} \frac{du_m}{2\pi i} \frac{(1 + \tau u_m)^p (1 + u_m)}{u_m^{p+2m-1} (1 - \prod_{i=1}^{m} u_i^2)} \prod_{1 \le l < m \le r} (u_m - u_l)(1 + \tau u_m + u_m u_l)$$

On prouve que cela compte des chemins du même genre, juste avec une petite différence (voir figure F.5). Comme ci-dessus, on peut démontrer que cela équivaut à une sous-classe des TSSCPP avec une étoile hexagonale fixe au milieu, comme le montre la figure 5.7.

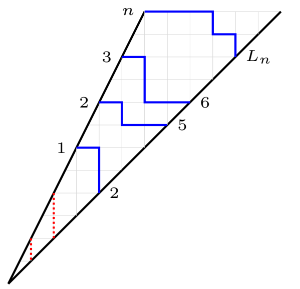


FIGURE F.5 – La somme $\sum_\alpha \psi_{(p\alpha}$ compte les NILP avec p chemins fixés verticalement, comme le montre cette figure.

F.7 Des polynômes $\psi_{(\pi)_p}$: nouvelles conjectures

Dans la section précédente on a prouvé que les quantités $\psi_{(\pi)_p}$ sont polynomiales. On peut donc définir les polynômes $\psi_\pi(t)$ tel que $\psi_\pi(p) = \psi_{(\pi)_p}$ quand p est un entier non négatif.

Dans cette section on présente quelques nouvelles conjectures sur ces polynômes, notamment sur des racines entières de ces polynômes, et l'évaluation des ce polynômes à t entier négatif. Pour simplifier, on omet les preuves et on se restreint au cas $\tau = 1$.

F.7.1 Définitions

Soit π un motif d'arches et $n = |\pi|$ le nombre d'arches de ce motif. On définit $\hat{p} = 2n + 1 - p$. Pour p entre 1 et $n - 1$ on considère l'ensemble $\mathcal{A}_p^G(\pi)$ des arches $\{a_1, a_2\}$ telle que $a_1 \leq p$ et $p < a_2 < \hat{p}$, et l'ensemble $\mathcal{A}_p^D(\pi)$ l'ensemble des arches $\{a_1, a_2\}$ telle que $p < a_1 < \hat{p}$ et $\hat{p} \leq a_2$. On définit la quantité $m_p(\pi)$ comme :

$$m_p(\pi) := \frac{|\mathcal{A}_p^G| + |\mathcal{A}_p^D|}{2}$$

Par exemple, si on considère le motif représenté dans l'image F.3 les $m_p(\pi)$ valent $\{1, 1, 1, 2, 1, 1\}$, pour $p = \{1, 2, 3, 4, 5, 6\}$.

Il existe une autre façon de obtenir ces nombres à partir des diagrammes d'Young, mais cela ce n'est pas nécessaire ici.

F.7.2 Les conjectures

Toutes les conjectures qui se suivent ont été confirmés pour $|\pi| \leq 8$ via des calculs en Mathematica.

Conjecture F.5. *Toutes les racines réelles des polynômes $\psi_\pi(t)$ sont entières négatives, et $-p$ apparaît avec la multiplicité $m_p(\pi)$. De façon équivalente, on donne la factorisa-*

tion :

$$\psi_\pi(t) = \frac{1}{|d(\pi)|!} \cdot \left(\prod_{p=1}^{|\pi|-1} (t+p)^{m_p(\pi)} \right) \cdot Q_\pi(t),$$

où $Q_\pi(t)$ est un polynôme avec coefficients entiers sans racines réelles.

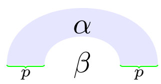
On a vérifié que cette conjecture est cohérente avec plusieurs propriétés attendues et certains cas déjà calculés dans la littérature. D'ailleurs on a prouvé que :

$$\psi_\pi(-1) = \begin{cases} \psi_{\pi'} & \text{si } \pi = (\pi') \\ 0 & \text{si } \pi \text{ n'a pas une grande arche } \{1, 2n\} \end{cases} \tag{F.8}$$

Remarquons que $m_1(\pi) = 0$ si et seulement si π peut s'écrire sous la forme $\pi = (\pi')$ pour π' quelconque. Dans le cas contraire, $m_1(\pi) = 1$.

Ensuite, on est intéressé par la valeur de $\psi_\pi(p)$ quand p est un entier négatif. Il est évident que la réponse est aussi un entier, puisque $\psi_\pi(p)$ est toujours entier quand p est un entier positif.

Soit π un motif d'arches, et $p \in [\![0, |\pi|]\!]$ tel que $m_p(\pi) = 0$. On peut donc découper la configuration π de la façon suivante :

$\pi =$ 

où α et β sont tous les deux des motifs d'arches. On notera $\pi = \alpha \circ \beta$.

On définit les quantités $G_\pi := \psi_\pi(-|\pi|)$. Ces nombres sont naturellement des entiers, par exemple :

$G_{\cdot} = 1$ $\quad$ $G_{(1)} = -1$ $\quad$ $G_{(2)} = 1$ $\quad$ $G_{(2,1)} = -3$

$G_{(3)} = -1$ $\quad$ $G_{(2,2)} = 1$ $\quad$ $G_{(3,1)} = 4$ $\quad$ $G_{(3,1,1)} = -9$

$G_{(3,2)} = -3$ $\quad$ $G_{(3,2,1)} = 9$

où on a indexé des termes par le diagramme d'Young $Y(\pi)$, car ces quantités sont stables par inclusion. Curieusement, ces quantités jouent un rôle important :

Conjecture F.6. *Soit $\pi = \alpha \circ \beta$ avec $|\alpha| = p$. On obtient la factorisation suivante :*

$$\psi_\pi(-p) = G_\alpha \psi_\beta.$$

Cela est en accord avec la formule (F.8).

Ces G_π ont des propriétés remarquables. D'abord le signe des G_π serait égal à la parité du nombre de boîtes du diagramme d'Young associé à π d'après la conjecture F.5. Quand on somme les G_π on obtiendrait encore une fois la fameuse séquence A_n. En fait, notre conjecture va un petit peu plus loin :

Conjecture F.7. *Pour un entier positif n, on a*

$$\sum_{\pi:|\pi|=n} |G_\pi| = A_n \quad et \quad \sum_{\pi:|\pi|=n} G_\pi = (-1)^{\frac{n(n-1)}{2}} \left(A_n^V\right)^2$$

$$G_{()^n} = \begin{cases} (-1)^{\frac{n(n-1)}{2}} \left(A_{n+1}^V\right)^2 & si\ n\ est\ pair; \\ (-1)^{\frac{n(n-1)}{2}} \left(A_n^V A_{n+2}^V\right) & si\ n\ est\ impair. \end{cases}$$

En fait on a prouvé que $\sum_\pi (-1)^{d(\pi)} G_\pi = A_n$ et on a prouvé aussi que la somme $\sum_\pi G_\pi$ correspond à une -1 énumération des TSSCPP déjà étudiée par Di Francesco [14] mais pas encore prouvée. Il existe donc une formule sous la forme d'un pfaffien.

Finalement, on regarde les coefficients des polynômes :

Conjecture F.8. *Pour tous π, les coefficients de $\psi_\pi(t)$ sont non négatifs.*

En fait, les polynômes $Q_\pi(t)$ introduits dans la conjecture F.5 semblent avoir la même propriété de positivité.

Il est évident que le terme dominant est positif, car $\psi_\pi(t)$ est toujours positif pour t naturel. On a calculé le terme sous-dominant et prouvé qu'il est positif. Le calcul des termes suivants s'avère bien plus difficile avec les outils que l'on connaît. En fait, toutes les méthodes que l'on connaît se basent sur des sommes de polynômes et les termes à prendre en compte deviennent de plus en plus nombreux.

F.7.3 Questions ouvertes

Évidemment, le problème le plus immédiat est de prouver les conjectures. Malheureusement ce n'est pas évident, parce que les formules que l'on connaît consistent en des sommes de polynômes, à l'exception des cas les plus simples.

Une solution serait de trouver une nouvelle formulation de ces polynômes ψ_π plus manipulable. On a introduit une nouvelle formule intégrale à plusieurs variables, semblabe à (F.4) pour calculer $\psi_\pi(-1)$. On est capable d'étendre cette définition pour tenir compte de tous les $\psi_\pi(-p)$ pour $1 \leq p \leq n$, mais les calculs se sont avérés très complexes. Finalement, on remarque l'existence d'une formule sous la forme d'un pfaffien pour la somme $\sum_\pi G_\pi$. C'est un bon début.

Une idée présente ici est qu'il doit y avoir une forme de « théorème de réciprocité combinatoire » [74] attaché à ces polynômes. C'est-à-dire que l'on pense qu'il doit exister un objet combinatoire dépendant de π, non encore découvert, tel que $|\psi_\pi(-p)|$ est le nombre de tels objets de taille p, à l'image du polynôme d'Ehrhart $i_P(t)$. On cherche donc une interprétation combinatoire des polynômes $\psi_\pi(-p)$.

On sait que les G_π sont des entiers, le problème évident est de comprendre les quantités G_π. Les G_π ont plusieurs propriétés surprenantes. La somme des $|G_\pi|$ est la séquence A_n. Par contre, les G_π sont stables par inclusion mais pas par rotation à l'instar des quantités A_π. Il est donc naturel de chercher une interprétation combinatoire de ces éléments.

Apêndice G

Resumo em português

G.1 Introdução

Matrizes de sinal alternante (ASM - Alternating Sign Matrices em inglês) foram inventadas por Robbins e Rumsey no seu estudo dos "γ-determinantes", uma generalização dos determinantes inspirada na condensação de Dodgson [19]. Mills, Robbins e Rumsey [49, 50, 70, 51] estudaram as propriedades desses objectos e descobriram uma bela fórmula que conta estas matrizes:

$$A_n := \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

Intrigados por esta conjectura, eles procuraram por pistas na literatura existente. É a esse momento que um outro objecto combinatório entra em jogo: as partições planas (PP - Plane Partitions), e em especial duas subclasses, as partições planas descendentes (DPP - Descending Plane Partitions) e as partições totalmente simétricas e auto-complementares (TSSCPP - Totally Symmetric Self-Complementary Plane Partitions). Vamos por partes, em 1979, Andrews [1] conseguiu provar que este mesmo produto conta o número total de DPP com a altura limitada a n. Logo, se fôssemos capazes de descobrir uma bijecção explícita entre as DPP e as ASM teríamos automaticamente provado a fórmula em questão. Infelizmente, tal bijecção ainda não foi encontrada.

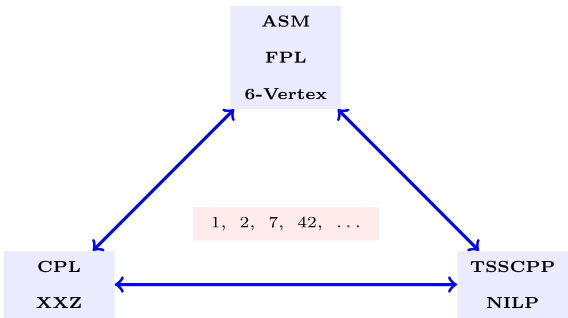
Algum tempo mais tarde, Robbins descobriu que há uma outra subclasse de PP que também é contada pela sequência A_n, as TSSCPP. Os mesmos autores propuseram no artigo [51], que não apenas o número de TSSCPP é igual ao número de ASM, mas também que esta contagem pode ser refinada de ambos os lados, refinamentos que dão o mesmo resultado.

Em 1996, num famoso artigo de 84 páginas [80], Zeilberger finalmente provou a fórmula de contagem das ASM. Alguns meses mais tarde, Kuperberg [44], utilizando a função de partição do modelo 6-vértices que está em bijecção com as ASM, provou exactamente o mesmo resultado de um modo mais conciso e elegante. Esta prova é baseada no trabalho de Izergin [29] e Korepin [40], e é uma consequência do facto que o modelo 6-vértices é um modelo quântico integrável.

A cadeia de spins XXZ é já um problema clássico em física, e suscita um grande interesse da comunidade desde há quase um século. Em 2000, Razumov e Stroganov [68, 76] investigavam as propriedades do estado fundamental deste modelo (com parâmetro anisotrópico $\Delta = -1/2$), quando descobriram um resultado surpreendente: quando normalizado tal que a componente mais pequena é 1, a soma das componentes do vector fundamental é exactamente A_n. Intrigados, físicos e matemáticos experimentaram tudo para conseguirem provar esta relação.

Alguns meses mais tarde, Razumov e Stroganov [65] notaram que a relação entre ambos os modelos é ainda mais profunda. De facto, a cadeia de spins pode ser vista como um modelo de lacetes, o modelo denso de lacetes (CPL - Compact Packed Loops). Neste modelo os estados são descritos por padrões de arcos (utiliza-se normalmente π para designar tais padrões). Por outro lado, as ASM estão em bijecção com as configurações compactas de lacetes (FPL - Fully Packed Loops) que podem ser classificadas pela sua conectividade na borda igualmente representada pelos mesmos padrões de arcos. Eles descobriram que quando se normaliza o estado fundamental do CPL tal que a componente mais pequena vale 1, as componentes associadas a um certo padrão de arcos contam as configurações FPL cuja conectividade é representada pelo mesmo padrão de arcos.

O último capítulo desta história foi escrito por Cantini e Sportiello. Em 2010 eles provaram a conjectura de Razumov e Stroganov [8]. Resumimos esta história no seguinte grafo:



Onde representamos todos os objectos principais descritos aqui, reunidos pelo facto que existe bijecções conhecidas entre eles. Ao centro a famposa sequência ominipresente em toda esta história, A_n. Há vários modelos de NILP, aqui referimo-nos aos que estão em bijecção com as TSSCPP.

Existe uma importante lição a reter do resultado de Kuperberg: os modelos quânticos integráveis podem ajudar-nos a resolver certos problemas difíceis de combinatória. Tanto o modelo CPL (ou a cadeia de spins XXZ) como o modelo 6-vértices são integráveis. Logo, a integrabilidade pode ser importante na clarificação das relações representadas pelo triângulo.

A equação Knizhnik–Zamolodchikov quântica (qKZ) foi introduzida neste contexto por Di Francesco e Zinn-Justin no artigo [16]. No nosso caso (nível 1 e $U_q(sl(2))$), as soluções são polinómios homogéneos, e pode-se mostrar que eles geram um espaço vectorial caracterizado por uma condição de anulação, a *wheel condition*.

Por um lado, as soluções desta equação (nível 1, e $q^{2\pi i/3}$) podem ser identificadas com as componentes do estado fundamental do modelo CPL na sua versão multivariáveis.

Por outro lado, estes polinómios estão relacionados com os polinómios de Macdonald respeitando a relação $t^3q = 1$ (ver [35, 12]), e são aquilo a que Lascoux chama de polinómios Kazhdan–Lusztig [12].

O objectivo principal deste manuscrito é de compreender o papel da integrabilidade nesta história. Nomeadamente, o papel da equação Knizhnik–Zamolodchikov quântica e suas soluções. Este resumo segue a organização do texto principal.

G.2 O modelo denso de lacetes e a equação Knizhnik–Zamolodchikov quântica

G.2.1 Lacetes densos

Os modelos de lacetes (Loop models em inglês) formam uma classe de modelos estatísticos a duas dimensões sobre rede bastante interessante. De facto, eles apresentam uma larga gama de fenómenos críticos e vários modelos clássicos podem ser vistos como modelos de lacetes. Consideramos aqui o modelo denso de lacetes (CPL - Completely Packed Loops), também chamado de modelo de lacetes $O(n)$ na literatura.

Tomemos uma rede quadrada. Cada face é ocupada por uma das duas plaquetas da figura G.1. Obviamente, existem caminhos fechados (lacetes) e caminhos abertos, *i.e.* unindo as bordas. Damos o peso $\tau = -q - q^{-1}$ a cada lacete. Este modelo é crítico para $|\tau| \leq 2$ (cf. [57]).

Figura G.1: As duas plaquetas que formam o modelo CPL.

Consideremos este modelo num cilindro semi infinito, onde cada linha é composta por $2n$ plaquetas, como mostra a figura G.2. Identificamos a borda da esquerda com a da direita.

G.2.2 Conectividade

A cada configuração associamos um padrão de arcos representando a conectividade na borda inferior. Por exemplo, a conectividade da configuração mostrada na figura G.2 é representada pelo padrão em G.3. O número de padrões de tamanho $2n$ é o famoso número de Catalan c_n:

$$c_n = \frac{(2n)!}{n!(n+1)!}$$

Definimos um estado formal como uma combinação linear de padrões, $\xi = \sum_\pi \xi_\pi \pi$, onde a soma percorre todos os c_n padrões e ξ_π são escalares. Um operador é então uma aplicação linear no espaço vectorial gerado por estes estados formais.

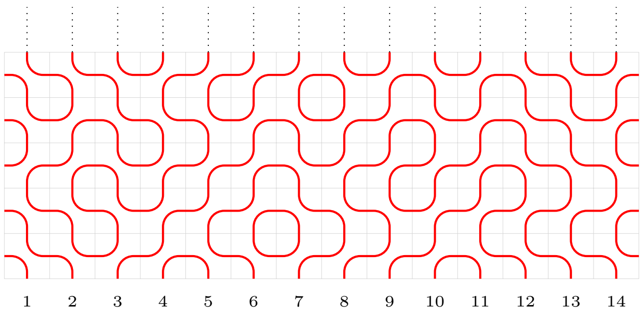


Figura G.2: Um exemplo duma configuração do CPL, aqui com $n = 7$. Cada linha é composta por 14 plaquetas que podem ser de dois tipos. As duas bordas, a da esquerda e a da direita, são identificadas de maneira a formar um cilindro. A cada lacete damos um peso τ.

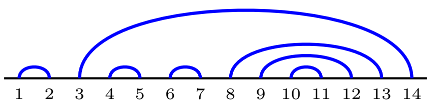


Figura G.3: Padrão que corresponde à conectividade do exemplo G.2

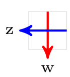
Às vezes é conveniente representar estes padrões por outros objectos. Neste trabalho utilizamos frequentemente palavras de parêntesis, caminhos de Dyck, diagramas de Young e sequências $a = \{a_1, \ldots, a_n\} \subseteq \{1, \ldots, 2n\}$, tal que $a_{i-1} < a_i$ e $a_i \leq 2i - 1$ para todo i. Vejamos um exemplo traduzido em todas estas representações:

 $\Leftrightarrow ()(()) \Leftrightarrow$  $\Leftrightarrow$  $\Leftrightarrow \{1, 3, 4\}$

G.2.3 A matriz R e a integrabilidade

Considere-se agora uma versão dinâmica deste modelo onde se inserem linhas de plaquetas na borda inferior.

Seja $q = e^{\pm 2\pi i/3}$, *i.e.* $\tau = 1$. A matriz R é definida da seguinte forma:

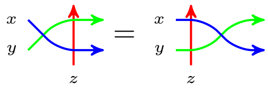


$$= \frac{qw - q^{-1}z}{qz - q^{-1}w}$$  $$+ \frac{w - z}{qz - q^{-1}w}$$  $$=: R(w, z).$$

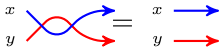
onde os coeficientes representam a probabilidade de escolher uma ou outra plaqueta. Estas probabilidades dependem de dois parâmetros w e z, chamados respectivamente de parâmetros espectrais verticais e horizontais, e que variam segundo a linha e a coluna. Escolhendo bem estes parâmetros podemos ajustar todas as probabilidades a um meio.

Esta matriz R é a chave da integrabilidade:

Lema G.1. *A matriz R obedece à equação Yanng–Baxter:*



E a equação de identidade:



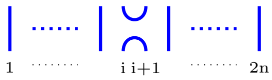
Isto permite-nos de escrever uma família infinita de operadores que comutam entre eles e de calcular analiticamente os estados próprios.

G.2.4 Caso homogéneo

Quando todos os parâmetros espectrais horizontais são iguais, o nosso sistema é descrito pelo hamiltoniano:

$$H = \frac{1}{2n}\sum_{i=1}^{2n} e_i.$$

onde

$e_i =$  .

Estes elementos obedecem à álgebra de Temperley–Lieb:

$$e_i^2 = \tau e_i \qquad e_i e_{i\pm 1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad \text{if } |j - i \bmod 2n| > 1 \tag{G.1}$$

para i e j entre 1 e $2n$. Consideremos também um elemento de rotação ρ que roda os padrões. Evidentemente, ρ^{2n} é igual à identidade.

Uma das questões que tentamos resolver neste trabalho é de encontrar $\psi = \sum_\pi \psi_\pi \pi$ tal que:

$$\psi = H\psi.$$

Podemos provar que este vector existe e que é único. Por exemplo, para $n = 3$, as componentes deste vector são:

ψ $= \frac{1}{7}$ ψ $= \frac{1}{7}$ ψ $= \frac{1}{7}$

ψ $= \frac{2}{7}$ ψ $= \frac{2}{7}$

G.2.5 A cadeia de spins XXZ

Seja $V = \left(\mathbb{C}^2\right)^{\otimes 2n}$ o espaço de $2n$ spins. Definimos e_i como o operador:

$$e_i = \frac{1}{2}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q + q^{-1}}{2}(\sigma_i^z \sigma_{i+1}^z - 1) + \frac{q - q^{-1}}{2}(\sigma_{i+1}^z - \sigma_i^z)\right)$$

onde σ_i^x é a matriz de Pauli segundo x e agindo no $i^{\underline{o}}$ espaço. A escolha do nome e_i não é inocente, de facto estes operadores obedecem também à álgebra de Temperley–Lieb.

Se somarmos em todos os $2n$ sítios, com condições de fronteira periódicas, obtemos o hamiltoniano da cadeia de spins XXZ:

$$H = \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} (\sigma_i^z \sigma_{i+1}^z - 1) \right)$$

O nosso objectivo é calcular o vector fundamental de XXZ. Vemos facilmente que as configurações CPL estão em bijecção com as configurações de spins com $S = 0$ e $S^z = 0$[1].

G.2.6 A equação Knizhnik–Zamolodchikov quântica

Introduzimos agora uma equação que nos permite calcular o estado fundamental da cadeia de spins XXZ, ou de maneira equivalente do modelo CPL, conhecida como a equação Knizhnik–Zamolodchikov quântica:

- A equação de *troca*:

$$\check{R}_i(y_i, y_{i+1})\Psi(y_1, \ldots, y_i, y_{i+1}, \ldots, y_{2n}) = \Psi(y_1, \ldots, y_{i+1}, y_i \ldots, y_{2n}). \tag{G.2}$$

  para $i = 1, \ldots, 2n$.

- A equação de *rotação*:

$$\rho^{-1}\Psi(y_1, y_2, \ldots, y_{2n}) = \kappa\Psi(y_2, \ldots, y_{2n}, sy_1) \tag{G.3}$$

  onde $\kappa = q^{-3(n-1)}$ e $s = q^6$.

A letra maiúscula distingue esta versão onde os parâmetros y_i são distintos da versão precedente onde são todos iguais a 1.

Queremos agora resolver esta equação. Há várias maneiras de a resolver. Por exemplo, podemos ver esta equação como um sistema triangular de equações $\psi_\pi = \sum_{\sigma \prec \pi} D_{\sigma,\pi}\psi_\sigma$, onde $\prec$ é uma ordem parcial e $D_{\sigma,\pi}$ é um operador. Não vamos descrever este procedimento aqui, para mais detalhes ver secção 1.4.

É possível normalizar as soluções desta equação, tal que sejam polinomiais de grau total $n(n-1)$. E é também possível provar que estas soluções vivem num espaço vectorial de dimensão c_n, correspondente ao número de polinómios, caracterizada por uma condição de anulação, a *wheel condition*:

$$f(\ldots, y_i, \ldots, y_j = q^2 y_i, \ldots, y_k = q^4 y_i, \ldots) = 0 \qquad \text{para todos } k > j > i.$$

[1]Estes números quânticos correspondem à álgebra quântica $U_q(su(2))$

G.2.7 Integrais de contorno

Neste trabalho preferimos utilizar um outro método. Seja $a = \{a_1, \ldots, a_n\}$, onde $a_i \geq a_{i-1}$ para todos $n \geq i > 1$, $a_n \leq 2n$ e $a_1 \geq 1$.

Introduzimos o integral de contorno:

$$\Phi_a(y_1, \ldots, y_{2n}) = \prod_{i<j}^{2n}(qy_i - q^{-1}y_j)$$
$$\oint \ldots \oint \prod_i \frac{dw_i}{2\pi i} \frac{\prod_{j>i}^{n}(w_j - w_i)(qw_i - q^{-1}w_j)}{\prod_{j\leq a_i}(w_i - y_j)\prod_{j>a_i}(qw_i - q^{-1}w_j)} \quad \text{(G.4)}$$

onde a integração é feita à volta dos pólos $w_j = y_i$ mas não dos pólos $w_j = q^{-2}y_i$.

Provamos na secção 1.6 que estas quantidades $\Phi_a(y_1, \ldots, y_{2n})$ geram o mesmo espaço vectorial que os polinómios Ψ_π. De facto, existe uma transformação triangular (e inversível):

$$\Phi_a(y_1, \ldots, y_{2n}) = \sum_{\pi \preceq a} C_{a,\pi}\Psi_\pi(y_1, \ldots, y_{2n}).$$

G.2.8 O limite homogéneo

Quando temos $y_i = 1$ para todos i, as equações simplificam-se. Utilizando a transformação de variáveis:

$$u_i = \frac{w_i - 1}{qw_i - q^{-1}}$$

Obtém-se facilmente a fórmula:

$$\phi_a = \oint \ldots \oint \prod_i \frac{du_i}{2\pi i u_i^{a_i}} \prod_{j>i}(u_j - u_i)(1 + \tau u_j + u_i u_j)$$

Pode-se agora calcular o vector ψ. Infelizmente estes integrais são ainda demasiado difíceis e a transformação de bases $C_{a,\pi}$ é complicada.

G.3 Matrizes de Sinal Alternante e o modelo 6-vértices

Nesta secção introduzimos três modelos, as Matrizes de Sinal Alternante (ASM), o 6-vértices e os lacetes compactos (FPL - Fully Packed Loops). Todos estes modelos estão em bijecção (ver capítulo 3). Calculamos a função de partição do modelo 6-vértices que nos permite contar, por exemplo, as ASM. Na última subsecção apresentamos o teorema Razumov–Stroganov.

G.3.1 Matrizes de Sinal Alternante

Uma Matriz de Sinal Alternante é uma matriz quadrada composta unicamente de 0 e de ± 1, tal que se ignorarmos os 0, os 1 e os -1 alternam em cada linha e cada coluna, e que cada linha (e coluna) começa e termina por um 1. Vejamos os 7 exemplos do caso $n = 3$:

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\qquad\begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix}\qquad\begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix}\qquad\begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix}\qquad\begin{matrix}0&1&0\\1&-1&1\\0&1&0\end{matrix}\qquad\begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix}\qquad\begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}$$

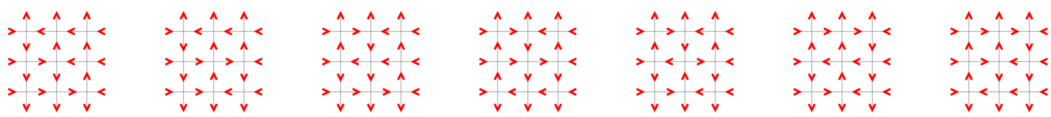
O número de matrizes de tamanho $n \times n$ é dado pela fórmula:

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1,\ 2,\ 7,\ 42,\ 429,\ \ldots \tag{G.5}$$

Esta expressão foi provada por Zeilberger [80]. Mas é a prova encontrada por Kuperberg [44] que nos interessa, esta usa um outro modelo, o 6-vértices.

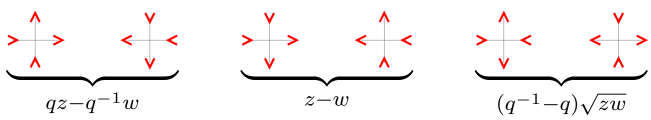
G.3.2 Modelo 6-vértices

Tomemos uma rede quadrada de tamanho $n \times n$. A cada aresta associamos uma orientação (uma flecha) tal que cada vértice tenha duas flechas que entram e duas que saiam. Há apenas 6 configurações possíveis para cada vértice (donde o nome do modelo). Aqui, impomos que todas as flechas nas bordas esquerda e direita entrem e as em baixo e no topo saiam. Podemos provar que há uma bijecção entre as configurações deste modelo e as ASM. Vejamos as sete configurações de tamanho $n = 3$:



G.3.3 A função de partição

A cada vértice damos o peso:



onde w (resp. z) caracteriza as colunas (resp. as linhas). Utilizamos $\{y_{n+1}, \ldots, y_{2n}\}$ (resp. $\{y_1, \ldots, y_n\}$) para cada coluna (resp. linha). q é um parâmetro global que vale normalmente $e^{2\pi i/3}$.

A função de partição é definida por:

$$Z_n = (-1)^{\binom{n}{2}} (q^{-1} - q)^{-n} \prod_{i=1}^{2n} y_i^{-1/2} \sum_{\text{configurações}} \prod_{i,j}^{n} w_{i,j}$$

onde $w_{i,j}$ é o peso da configuração em cada vértice.

O interesse deste modelo é que ele é integrável, no sentido em que podemos construir uma matriz R, idêntica àquela do modelo CPL (na secção G.2), que também obedece à equação Yang–Baxter. Isto permite-nos descobrir várias propriedades da função de partição:

- a função de partição é um polinómio homogéneo nas variáveis $\{y_1, \dots, y_{2n}\}$;
- o grau total é $\delta = n(n-1)$ e o grau parcial é $\delta_i = n - 1$;
- a função de partição é simétrica nos conjuntos $\{y_1, \dots, y_n\}$ e $\{y_{n+1}, \dots, y_{2n}\}$;
- a função de partição obedece à relação de recurrência seguinte:

$$\begin{aligned} &Z_n(y_1, \dots, y_n, y_{n+1} = q^2 y_1, \dots, y_{2n}) \\ &\quad = (-1)^{n-1} \prod_{i=2}^{n} (y_1 - y_{n+i})(y_i - y_{n+1}) Z_{n-1}(y_2 \dots, y_{n-1}, y_{n+1}, \dots, y_{2n}); \end{aligned} \tag{G.6}$$

- se $q = e^{2\pi i/3}$, a função de partição é um polinómio simétrico em $\{y_1, \dots, y_{2n}\}$.

Estas propriedades permitiram a Izergin de calcular a função de partição:

$$\begin{aligned} Z_n(y_1, \dots, y_{2n}) = & \frac{\prod_{i,j}^{n} (y_i - y_{n+j})(q y_i - q^{-1} y_{n+j})}{\prod_{i<j}^{n} \prod (y_i - y_j)(y_{n+i} - y_{n+j})} \\ & \times \det_{1 \le i,j \le n} \left| \frac{1}{(y_i - y_{n+j})(q y_i - q^{-1} y_{n+j})} \right| \end{aligned}$$

para q qualquer.

Para $q = e^{2\pi i/3}$, esta fórmula simplifica-se. De facto obtemos uma função de Schur:

$$Z_n(y_1, \dots, y_n, y_{n+1}, \dots, y_{2n}) = s_{Y_n}(q^{-1} y_1, \dots, q^{-1} y_n, q y_{n+1}, \dots, y_{2n}). \tag{G.7}$$

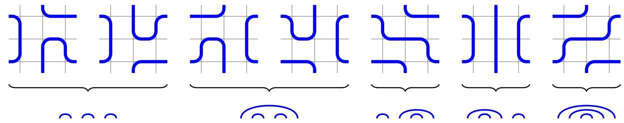
Para provar estas duas fórmulas basta provar que elas têm todas as propriedades referidas em cima.

Um exemplo duma aplicação desta fórmula é o cálculo do número de ASM. Se fixarmos todos os y_i no valor q para $1 \le i \le n$ e $y_i = q^{-1}$ para $n < i \le 2n$, obtemos que a função de partição é por um lado um múltiplo do número de Matrizes de Sinal Alternante, e por outro que ela é igual à função de Schur $s_{Y_n}(1, \dots, 1)$. Após alguns cálculos não muito difíceis obtemos o resultado desejado.

G.3.4 Lacetes compactos

Há vários modelos que estão em bijecção com as ASM. Por exemplo, o modelo de gelo quadrado, certos modelos de caminhos, ... O que nos interessa mais é o modelo compacto de lacetes (FPL - Fully Packed Loops).

Tomemos uma rede quadrada $n \times n$. Em cada vértice há um e somente um caminho que passa. Logo, ou bem que os caminhos começam numa borda e terminam noutra, ou bem que formam caminhos fechados (ditos lacetes). Nós consideramos condições de fronteira do tipo “paredes de domínio” (DWBC - Domain Wall Boundary Condition). De todas as arestas que estão na borda, uma em cada duas é atravessada por um caminho, alternadamente. Vejamos, por exemplo, todas as sete configurações para $n = 3$:



Nós agrupamos as configurações pela sua conectividade na borda. A ausência de cruzamentos entre os arcos é explicado pelo facto que em cada vértice passa um só caminho.

Definimos a quantidade A_π como o número de configurações FPL cuja conectividade é π. Em 1991, Wieland [79] provou que estas quantidades são estáveis por reflexão e por rotação da sua conectividade π.

Obviamente $\sum_\pi A_\pi = A_n$. Note que as conectividades correspondem aos padrões definidos na secção G.2. De facto a conexão é ainda mais profunda:

Teorema G.2 (Razumov–Stroganov [65])**.** *As componentes ψ_π do estado fundamental do modelo denso de lacetes, normalizado tal que a menor componente valha* 1*, contam as configurações do modelo compacto de lacetes com conectividade π:*

$$\psi_\pi = A_\pi$$

Este teorema foi demonstrado por Cantini e Sportiello em 2010 [8].

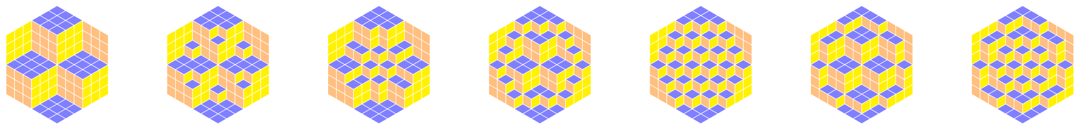
G.4 Partições planas

G.4.1 Partições planas totalmente simétricas e auto-complementares

Uma partição plana pode ser vista como una generalização das partições. Tomemos um canto (duma sala por exemplo), e imaginemos que a gravidade puxa na direcção do canto. Uma partição plana é então um conjunto de pilhas de cubos que submetidos a esta gravidade. Obviamente que podemos representar uma partição plana pela altura das pilhas.

Limitamos agora a partição plana num grande cubo $2n \times 2n \times 2n$ e consideramos as partições planas simétricas por troca dos eixos. Podemos ainda acrescentar uma outra simetria. Definimos a partição complementar como a partição formada pelos cubos que faltam para encher o cubo $2n \times 2n \times 2n$, rodado tal que obedeça às regras das partições planas. Uma partição plana é dita auto-complementar se ela coincide com a sua partição complementar. Uma partição plana com todas estas simetrias chama-se partição plana totalmente simétrica e auto-complementar (TSSCPP do inglês Totally Symmetric Self-Complementary Plane Partitions).

Por exemplo, para $n = 3$, existe exactamente 7 exemplos:



Note que aparece aqui o número de ASM de tamanho $n = 3$. De facto isto não é uma coincidência: pode-se provar que o número de TSSCPP é A_n. Este facto foi provado por Andrews em 1994 [2].

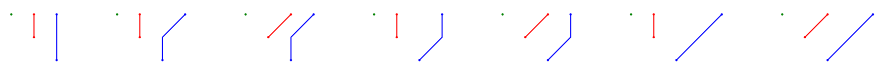
G.4.2 Caminhos ponderados não intersectantes

As partições planas e nomeadamente as TSSCPP podem ser representadas por configurações de mosaicos, configurações de dímaros ou ainda certos conjuntos de caminhos definidos numa rede com a propriedade fundamental de não se tocarem, ditos não intersectantes, chamadas em inglês Non-Intersecting Lattice Paths (NILP). Este últimos são bastante importantes no nosso estudo, são eles que nos permitem calcular o número de TSSCPP.

Não é muito difícil provar que as TSSCPP de tamanho $2n \times 2n \times 2n$ estão em bijecção com uma subclasse de NILP definidos numa rede quadrada e que obedecem às seguintes regras:

- há n caminhos que não se tocam;
- o $i^{\underline{o}}$ caminho parte da posição $(i-1, 1-i)$;
- só são permitidos passos verticais ($\uparrow$) e diagonais ($\nearrow$) ;
- os caminhos terminam na linha $y = 0$.

Vejamos todos os casos possíveis para $n = 3$:



O interesse destes caminhos é que existe uma tecnologia, conhecida como a fórmula LGV (Lindström–Gessel–Viennot [46, 27]), para os contar. Mais, esta fórmula permite também a contagem ponderada desde que os pesos apenas dependam das arestas e dos vértices (e não do caminho em si).

G.5 Matrizes de sinal alternante e partições planas totalmente simétricas e auto-complementares

É sabido que há tantas ASM de tamanho $n \times n$ como TSSCPP de tamanho $2n \times 2n \times 2n$. De facto, por um lado podemos reescrever a função de partição do modelo 6-vértices, que conta as ASM, sob a forma dum multi integral de contorno. Por outro lado, podemos utilizar a representação das TSSCPP como NILP para encontrar um multi integral de contorno utilizando a fórmula LGV. De seguida, prova-se que os dois integrais são iguais, utilizando para isso uma igualdade proposta por Di Francesco e Zinn-Justin [17] e provada por Zeilberger [81] (provada no apêndice C.1). No artigo [23], refinamos este procedimento. De facto, provamos uma versão duplamente refinada desta igualdade entre o número de ASM e de TSSCPP proposta por Mills, Robbins e Rumsey [51]. Neste resumo mostramos um esquisso dessa prova.

G.5.1 Uma conjectura refinada

As TSSCPP

Seja α uma TSSCPP de tamanho $2n \times 2n \times 2n$ representada em termos de caminhos. $u^1(\alpha)$ é definido como o número de caminhos que acabam num passo vertical e $u^n(\alpha)$ é definido como o número de caminhos que começam num passo vertical. Por exemplo os sete exemplos a $n = 3$ vistos na secção anterior dão $u^1 = \{2;1;0;2;1;1;0\}$ e $u^n = \{2;2;1;1;0;1;0\}$. Note que os números que aparecem são os mesmos, mas permutados.

Definimos a função geradora:

$$U_n^{0n}(x,y) = \sum_{\alpha} x^{u^1(\alpha)} y^{u^n(\alpha)}$$

Por exemplo $n = 3$ dá: $U_n^{01} = x^2y^2 + xy^2 + y + x^2y + x + xy + 1$.

As ASM

Uma matriz de sinal alternante tem um único 1 na primeira linha e um único 1 na última linha. Digamos que o primeiro 1 está na posição i e o último 1 está na posição j. Definimos a função gerador como sendo:

$$A_n(x,y) = \sum_{ASM} x^{i-1} y^{n-j}$$

Por exemplo, temos a seguinte função para $n = 3$: $A_n(x,y) = 1 + y + x + xy^2 + xy + x^2y + x^2y^2$.

Olhando os dois resultados, notamos o seguinte facto curioso:

Teorema G.3. *O número de ASM de tamanho $n \times n$ cujo o 1 da primeira linha está na posição $i-1$ e cujo o 1 da última linha está na posição $n-j$ é igual ao número de TSSCPP cuja sua representação através de caminhos possui i últimos passos verticais e j primeiros passo verticais. Em termos das funções geradoras:*

$$A_n(x,y) = U_n^{01}(x,y)$$

G.5.2 Esquisso da prova

TSSCPP contadas por integrais

Podemos contar os NILP através da fórmula LGV [46, 27]. Esta fórmula permite a utilização de caminhos ponderados. Pomos um peso x por cada último passo vertical e y por cada primeiro passo vertical.

Reescrevemo-la na forma dum integral de contorno múltiplo. Finalmente, se somarmos sobre todos os conjuntos de chegada possíveis, obtemos a fórmula:

$$U_n^{1n}(x,y) = \oint \dots \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i u_i^{2i-1}} \frac{1}{1-u_i^2}(1+xu_i)(y+u_i)_{\hat{1}}(1+u_i)_{\hat{1}}^{i-2} \prod_{j>i} \frac{u_j-u_i}{1-u_j u_i}$$

onde o integral contorna o 0. A notação $\hat{1}$ significa que ignoramos o termo correspondente a u_1.

ASM contadas por integrais

Partimos da função de partição do modelo 6-vértices introduzida na secção G.3.2 e metemos:

$$\begin{aligned}
y_1 &= q\frac{1+qx}{q+x} & \\
y_i &= q & \text{if } 2 \le i \le n-1 \\
y_n &= q\frac{y+q}{1+qy} & \\
y_j &= q^{-1} & \text{if } j > n
\end{aligned}$$

Com estes parâmetros, obtemos que a função de partição conta as ASM, tendo em conta o primeiro 1 e o último 1: $Z_n(y_1, \dots, y_{2n}) \propto A_n(x,y)$.

O próximo passo consiste em encontrar uma fórmula integral de contorno que seja igual a Z_n. Isto já é conhecido, ver secção 4.3. Após algumas manipulações de integrais, obtemos a fórmula:

$$(1+xy-x)y^{2(n-1)} \oint \dots \oint \prod_{l}^{n} \frac{du_l}{2\pi i} \frac{1}{u_l^{2l-2}} \frac{\prod_{l<m}(u_m-u_l)(1+u_m+u_m u_l)}{(1+u_l-x)(y+u_l(y-1))} \prod_{j=2}^{n}(1+u_j)$$

onde integramos à volta de $u_i = 0$ e $u_i = x-1$.

A prova da igualdade

Resta-nos provar que os dois integrais coincidem. Os passos seguintes são bastante técnicos, e baseiam-se em várias fórmulas integrais.

G.6 Lacetes densos e partições planas

No capítulo 5, resolvemos certas conjecturas sobre os estado fundamental do modelo CPL. No artigo [85] Zuber conjecturara que as componentes do estado fundamental indexadas por padrões da forma $(\pi)_p$, *i.e.* um certo padrão π rodeado de p arcos, são polinomiais em p:

Teorema G.4. *Consideremos um padrão da forma $(\pi)_p$, os polinómios ψ_π podem ser escritos na forma seguinte:*

$$\psi_{(\pi)_p} = \frac{1}{|Y|!} P_Y(\tau, n)$$

onde $Y = Y(\pi)$ é o diagrama de Young definido por π, $|Y|$ é o número de caixas de Y, e $P_Y(\tau, n)$ é um polinómio em n e τ de grau $|Y|$ em cada variável e coeficientes inteiros.

No limite n grande, este polinómio comporta-se como:

$$\psi_{(\pi)_p} \approx \frac{\dim Y}{|Y|!} (n\tau)^{|Y|}$$

onde $\dim Y$ *é a dimensão da representação irredutível do grupo simétrico associado a Y.*

Nesta secção, fazemos um rascunho da prova deste teorema do lado dos CPL e demonstramos uma regra de soma para os padrões da forma $(\pi)_p$. Adicionalmente, obtemos uma regra de soma para os padrões que começam com p aberturas, representados por $(_p\alpha$.

G.6.1 Polinomialidade

Para provar que as quantidades $\psi_{(\pi)_p}$ são polinomiais, começamos por reescrevê-las como a soma das quantidades ϕ_a definidas atrás na secção G.2.8:

$$\psi_{(\pi)_p}(\tau) = \sum_a C^{-1}_{(\pi)_p, a}(\tau) \phi_a(\tau)$$

Mas sabemos que o coeficiente $C^{-1}_{(\pi)_p, a}(\tau)$ é não negativo unicamente se $a \preceq (\pi)_p$. De facto, se $a = (\pi)_p$ ele vale 1, e se $a \prec (\pi)_p$, ele é um polinómio em τ de grau menor ou igual a $|(\pi)_p| - |a|$. Mais, se notarmos $(b)_p = \{1, 2, \ldots, p, p + b_1, p + b_n\}$, prova-se facilmente que $C^{-1}_{(\pi)_p, (b)_p} = C^{-1}_{\pi, b}$. Obtemos, então:

$$\psi_{(\pi)_p}(\tau) = \sum_b C^{-1}_{\pi, b}(\tau) \phi_{(b)_p}(\tau)$$

Tomando as expressões integrais de $\phi_{(b)_p}(\tau)$ concluímos facilmente que $\psi_\pi(t)$ são polinómios com o bom grau e o bom coeficiente dominante. De facto, os termos $b \prec \pi$ contribuem unicamente para os coeficientes não dominantes.

G.6.2 A soma dos $\psi_{(\pi)_p}$

Pomos $\tau = 1$ para simplificar. Seja a um padrão da forma $\{a_1, \dots, a_n\}$ com $a_i = 2i-1$ ou $a_i = 2i-2$, o lema 5.3 diz que:

$$\sum_{\substack{a=\{1,\dots,p,p+a_1,\dots,p+a_n\}\\ a_i=2i-1 \text{ ou } a_i=2i-2}} \phi_a = \sum_\pi \psi_{(\pi)_p}$$

Podemos então reescrever a soma de ψ_π como uma soma de integrais (e simplificando):

$$\oint \dots \oint \prod_{i=1}^n \frac{du_i}{2\pi i u_i^{2i-1}} (1+u_i)^{p+1} \prod_{j>i} (u_j - u_i)(1 + u_j + u_i u_j)$$

Este integral pode ser transformado num outro que conta todos os caminhos não intersectantes indicados na imagem G.4. Estes caminhos pode ser vistos como TSSCPP com um hexágono fixo ao meio, como podemos ver na figura 5.3.

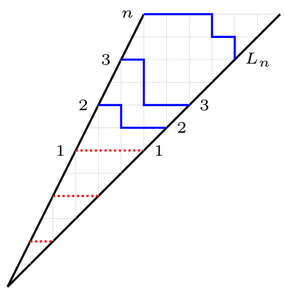


Figura G.4: A soma $\sum_\pi \psi_{(\pi)_p}$ conta o número de NILP com $n+p$ caminhos presentados nesta figura onde os $p+1$ primeiros caminhos estão fixos horizontalmente.

Neste manuscrito, apresentamos também como gerar uma configuração perfeitamente aleatória (se considerarmos que o gerador de números aleatórios é perfeito).

G.6.3 A soma dos $\psi_{(p\alpha}$

Aplicamos o mesmo procedimento às componentes $\psi_{(p\alpha}$. Da mesma maneira, podemos escrevê-las como uma combinação linear dos $\phi_{\{1,\dots,p,p+a_1,\dots,p+a_n\}}$ onde $\{a_1, \dots, a_n\}$ não são necessariamente padrões de arcos. Como antes, reescrevemos a sua soma na forma dum integral de contorno múltiplo:

$$\oint \dots \oint \prod_{m=1}^r \frac{du_m}{2\pi i} \frac{(1+\tau u_m)^p (1+u_m)}{u_m^{p+2m-1}(1 - \prod_{i=1}^m u_i^2)} \prod_{1 \le l < m \le r} (u_m - u_l)(1 + \tau u_m + u_m u_l)$$

Prova-se que conta caminhos do mesmo género, apenas com uma pequena diferença (ver figura G.5). Como atrás, podemos demonstrar que isto equivale a uma subclasse de TSSCPP com uma estrela hexagonal fixa ao meio, como mostra a figura 5.7.

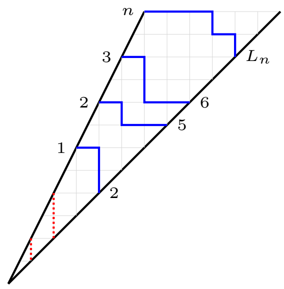


Figura G.5: A soma $\sum_\alpha \psi_{(p\alpha}$ conta os NILP com p caminhos fixos verticalmente, como mostra esta figura.

G.7 Polinómios $\psi_{(\pi)_p}$: novas conjecturas

Na secção precedente provamos que as quantidades $\psi_{(\pi)_p}$ são polinomiais. Podemos definir os polinómios $\psi_\pi(t)$ tal que $\psi_\pi(p) = \psi_{(\pi)_p}$ quando p é um inteiro não negativo.

Nesta secção apresentamos algumas novas conjecturas sobre estes polinómios, nomeadamente sobre as raízes inteiras destes polinómios, e o valor deles a t inteiro negativo. Para simplificar, omitimos as provas e restringimo-nos ao caso $\tau = 1$.

G.7.1 Algumas definições

Seja π um padrão de arcos e $n = |\pi|$ o número de arcos do padrão em causa. Definimos $\hat{p} = 2n+1-p$. Para p entre 1 e $n-1$ consideramos o conjunto $\mathcal{A}_p^G(\pi)$ dos arcos $\{a_1, a_2\}$ tal que $a_1 \leq p$ e $p < a_2 < \hat{p}$, e o conjunto $\mathcal{A}_p^D(\pi)$ o conjunto de arcos $\{a_1, a_2\}$ tal que $p < a_1 < \hat{p}$ e $\hat{p} \leq a_2$. Definimos a quantidade $m_p(\pi)$ como:

$$m_p(\pi) := \frac{|\mathcal{A}_p^G| + |\mathcal{A}_p^D|}{2}$$

Por exemplo, se considerarmos o padrão representado na imagem G.3 os $m_p(\pi)$ valem $\{1, 1, 1, 2, 1, 1\}$, para $p = \{1, 2, 3, 4, 5, 6\}$.

Existe uma outra maneira de obter estes número a partir dos diagramas de Young, mas não será necessária aqui.

G.7.2 As conjecturas

Todas as conjecturas que se seguem foram verificadas para $|\pi| \leq 8$ através de cálculos em Mathematica.

Conjectura G.5. *Todas as raízes reais dos polinómios $\psi_\pi(t)$ são inteiras e negativas, e a raiz $-p$ aparece com multiplicidade $m_p(\pi)$. De modo equivalente, damos a factorização:*

$$\psi_\pi(t) = \frac{1}{|d(\pi)|!} \cdot \left(\prod_{p=1}^{|\pi|-1} (t+p)^{m_p(\pi)} \right) \cdot Q_\pi(t),$$

onde $Q_\pi(t)$ é um polinómio com coeficientes inteiros sem raízes reais.

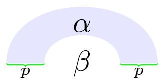
Verificámos que esta conjectura é coerente com várias propriedades esperadas e certos casos já calculados na literatura. Mais, provámos que:

$$\psi_\pi(-1) = \begin{cases} \psi_{\pi'} & \text{se } \pi = (\pi') \\ 0 & \text{se } \pi \text{ não tem um grande arco } \{1, 2n\} \end{cases} \tag{G.8}$$

Notemos que $m_1(\pi) = 0$ se e somente se π se pode escrever na forma $\pi = (\pi')$ para π' qualquer. Senão, $m_1(\pi) = 1$.

De seguida, estamos interessados no valor de $\psi_\pi(p)$ quando p é um inteiro negativo. É evidente que a resposta é também um inteiro, pois $\psi_\pi(p)$ é sempre inteiro quando p é um inteiro positivo.

Seja π um padrão de arcos, e $p \in [\![0, |\pi|]\!]$ tal que $m_p(\pi) = 0$. Podemos cortar a configuração π da forma seguinte:

$\pi =$ 

onde α e β são ambos padrões de arcos. Notaremos $\pi = \alpha \circ \beta$.

Definimos as quantidades $G_\pi := \psi_\pi(-|\pi|)$. Estes números são naturalmente inteiros, por exemplo:

$$G_{\emptyset} = 1 \qquad G_{(1)} = -1 \qquad G_{(2)} = 1 \qquad G_{(2,1)} = -3$$
$$G_{(3)} = -1 \qquad G_{(2,2)} = 1 \qquad G_{(3,1)} = 4 \qquad G_{(2,1,1)} = -9$$
$$G_{(3,2)} = -3 \qquad G_{(3,2,1)} = 9$$

onde nós indexamos os termos pelo diagrama de Young $Y(\pi)$, porque são quantidades estáveis por inclusão. Curiosamente, esta quantidades têm um papel bastante importante:

Conjectura G.6. *Seja $\pi = \alpha \circ \beta$ com $|\alpha| = p$. Obtemos a factorização seguinte:*

$$\psi_\pi(-p) = G_\alpha \psi_\beta.$$

O que está de acordo com a fórmula (G.8).

Estes G_π têm propriedades notáveis. Primeiro, o sinal dos G_π seria igual à paridade do número de caixas do diagrama de Young associado a π segundo a conjectura G.5. Quando somamos os G_π obteríamos, mais uma vez a famosa sequência A_n. De facto, a nossa conjectura vai um pouco mais longe:

Conjectura G.7. *Para um inteiro positivo n, temos*

$$\sum_{\pi:|\pi|=n} |G_\pi| = A_n \quad e \quad \sum_{\pi:|\pi|=n} G_\pi = (-1)^{\frac{n(n-1)}{2}} \left(A_n^V\right)^2$$

$$G_{()^n} = \begin{cases} (-1)^{\frac{n(n-1)}{2}} \left(A_{n+1}^V\right)^2 & \textit{se } n \textit{ for par;} \\ (-1)^{\frac{n(n-1)}{2}} \left(A_n^V A_{n+2}^V\right) & \textit{se } n \textit{ for impar.} \end{cases}$$

Provamos que $\sum_\pi (-1)^{d(\pi)} G_\pi = A_n$ e provamos também que a soma $\sum_\pi G_\pi$ corresponde a uma -1 contagem das TSSCPP já estudada por Di Francesco [14] mas ainda não provada. Existe então uma fórmula em forma de pfaffiano.

Finalmente, olhamos para os coeficientes dos polinómios:

Conjectura G.8. *Para todo π, os coeficientes de $\psi_\pi(t)$ são não negativos.*

De facto, os polinómios $Q_\pi(t)$ introduzidos na conjectura G.5 parecem ter a mesma propriedade de positividade.

É óbvio que o termo dominante é positivo, porque $\psi_\pi(t)$ é sempre positivo para t natural. Calculámos o termo sub dominante e provámos que é positivo. O cálculo dos termos seguintes mostrou-se bem mais difícil com as ferramentas que conhecemos. Infelizmente todos os métodos que conhecemos baseiam-se em somas de polinómios e os termos a ter em conta são cada vez mais numerosos.

G.7.3 Questões abertas

Obviamente, o problema mais imediato é de resolver as conjecturas. O que não parece ser fácil porque as fórmulas que conhecemos consistem em somas de polinómios, à excepção dos casos mais simples.

Uma solução seria de encontrar uma nova formulação destes polinómios ψ_π mais manipulável. Nós introduzimos uma nova formulação integral multivariáveis, similar a (G.4) para calcular $\psi_\pi(-1)$. Somos capazes de generalizar esta definição para ter em conta todos os $\psi_\pi(-p)$ para $1 \leq p \leq n$, mas os cálculos mostraram-se demasiados complicados. Finalmente, notamos a existência duma fórmula na forma dum pfaffiano para a soma $\sum_\pi G_\pi$. É um bom início.

Uma ideia presente aqui, é que deve haver uma forma de "teorema de reciprocidade combinatória" [74] para estes polinómios. Quer dizer, nós pensamos que deve existir um objecto combinatório dependente de π, ainda não descoberto, tal que $|\psi_\pi(-p)|$ é o número de tais objectos de tamanho p, à imagem do polinómio de Ehrhart $i_P(t)$. Procuramos então uma interpretação combinatória dos polinómios $\psi_\pi(-p)$.

Sabemos que os G_π são inteiros, é então natural tentar descobrir mais sobre estes G_π. Mais, os G_π têm várias propriedades surpreendentes. A soma dos $|G_\pi|$ é a sequência A_n. Ao contrário dos A_π, os G_π são estáveis por inclusão mas não por rotação. É então natural procurar uma interpretação combinatória destes elementos.

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