Typed Abstract Syntax

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Typed Abstract Syntax

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Contents

Préface

Les langages de programmation jouent un rôle important dans la science en général pas seulement en informatique. Il existe plusieures types de langages de programmation comme des langages imperatifs, foncionnels, orientés objet, etc. Dans le domaine de l'informatique théorique on étudie entre autres des modèles formels de tels langages de programmation pour spécifier leurs comportements, pour étudier leurs propriétés et afin de certifier leurs implémentations. Cette étude se divise en l'étude de la syntaxe et en l'étude de la sémantique. Pour pouvoir étudier la sémantique, un modèle formel de la syntaxe est prérequis. Cette étude est faite dans le cadre des théories mathématiques. La présente thèse est consacrée à l'étude syntactique dont le cadre est la théorie de catégories.

Le modèle formel le plus général d'un langage fonctionel est le *Lambda–Calcul* introduit par A. Church dans les années 1930. L'idée de base est de considérer tout comme une fonction. Il existe beaucoup de différentes extensions du Lambda–Calcul qui modélisent différents langages de programmation. Le Lambda–Calcul *pur* est présenté traditionellement comme suit. Prenons un ensemble X des *variables formelles* et construisons l'ensemble des *Lambda–termes* par les constructeurs *application* et *abstraction*. L'application est notée $(-,-)$ et l'abstaction par rapport à un variable $x \in X$ est notée $\lambda x.-$. Donc un Lambda–terme M sur X est de la forme

$$M ::= x|(M_1\ M_2)|\lambda x.M$$

L'abstraction $\lambda x.-$ lie la variable x. Le Lambda–Calcul pur est l'exemple typique d'une syntaxe avec liaison de variables, et il a été bien plus étudié par d'autres gens avant et après, voir par exemple [FPT99], [GP99], [HM07].

Un exemple d'une syntaxe sans liaison de variables est le calcul propositionnel. On a le droit d'utiliser la négation, conjonction, disjonction et l'implication pour former des nouveaux propositions à partir d'un ensemble d'axiomes. D'autres exemples sont des structures algébriques comme les monoïdes, les groupes et les anneaux. Cette sorte de syntaxe est consiérée en général comme plus facile à modeliser, par exemple par des théories de Lawvere, introduites dans sa thèse en 1963 (re–imprimée [Law04]) ou par la théorie des catégories, voir par exemple [Bor94b] chapitre 3.

Toutes ces syntaxes sont non–typées. Il existe également des syntaxes typées — avec et sans liaison de variables. Le cas avec liaison est considéré plus complexe. L'exemple standard d'une syntaxe typée avec liaison de variables est le Lambda–Calcul simplement typé. Il est introduit comme suit. D'abord on définit l'ensemble des types à partir d'un ensemble de types de base avec le constructeur binaire $\Rightarrow$. Etant donné une contexte Γ, c'est–à–dire un ensemble de couples de la forme $(x : t)$ qui signifie que la variable x est de type t, on a les règles de typage suivantes pour les Lambda–termes

$$\frac{(x:t)\in\Gamma}{\Gamma\vdash x:t}\,\mathrm{var}$$

$$\frac{\Gamma \vdash M : s \Rightarrow t \qquad \Gamma \vdash N : s}{\Gamma \vdash (M\ N) : t}\ \text{app}$$

$$\frac{\Gamma, (x : s) \vdash M : t}{\Gamma \vdash \lambda x : s.M : s \Rightarrow t}\ \text{abs}$$

Dans le travail présent on s'occupe de deux différents points de vue de la syntaxe abstraite. Tous les deux utilisent fortemenet le langage de la théorie des catégories. Le premier point de vue est dans le sens de l'approche catégorique aux théories algébriques. L'article de base de cet approche est celui de Fiore, Plotkin, Turi [FPT99]. Le cas simplement typé est obtenu avec une modification légère du cas non–typé. Par simplement typé on veut dire que l'ensemble de types est fixé et on s'intéresse principalement à la syntaxe sur les termes. Dans cette approche la syntaxe est caracterisée comme l'algèbre libre d'un certain foncteur.

Le deuxième point de vue est originaire de [HM07]. La différence principale est qu'il est basé sur la notion d'une monade et introduit la notion d'un *module* sur une monade qui généralise la notion d'un *module* sur un monoïde. Plus précisément, étant donné une monade R sur une catégorie $\mathcal{C}$, un R–module est un couple (M, σ) où M est un foncteur de $\mathcal{C}$ dans $\mathcal{D}$ et σ, l'*action* de module, est une transformation naturelle $M \circ R \to M$ qui est compatible avec l'unité et la multiplication de la monade R. La syntaxe est caracterisée comme l'objet initial dans la catégorie de *représentations* associée à une signature. Ces représentations sont des familles de morphismes de modules. Ce point de vue couvre les cas non–typés et simplement typés. On peut l'étendre à la syntaxe typée. Par la syntaxe typée on veut dire qu'on considère la syntaxe sur deux niveaux : sur les types et les termes et qui dépendent l'une de l'autre. Au contraire au cas simplement typé où la syntaxe des types est indépendant de celle de termes et peut être traité comme une syntaxe non–typée.

Cette thèse apporte deux contributions. Elle étudie la rélation entre les deux approches mentionnées précédemment dans les cas non–typé et simplement typé. En plus l'approche monadique est développée pour couvrir une classe plus large de signatures contenant des types ≪légèrement≫ dépendants et des liaisons de variables.

L'organisation de la thèse est comme suit. Le chapitre 2 rappelle des notions de la théorie de catégories qui sont utiles pour les chapitres suivants. Dans le chapitre 3 on décrit la partie pertinante de la théorie de monades. Le chapitre 4 détaille la rélation entre les deux approches dans le cas non–typé. Les deux chapitres suivants présentent les deux approches dans le cas simplement typé et ensuite le chapitre 7 décrit l'analogue au chapitre 4 pour le cas simplement typé. Les deux derniers chapitres 8 et 9 contiennent l'approche monadique pour des syntaxes typées, d'abord sans liaison mais avec quantification sur des variables de type et finalement l'exemple du Lambda–Calcul typé comme exemple d'une syntaxe typée avec liaison de variables.

Chapter 1

Introduction

Programming languages play an important role in many different areas of science, not only in computer science. There are various types of programming languages such as imperative, functional, object oriented, etc. In theoretical computer science one studies among other things formal models of such existing programming languages in order to specify their behaviour, to investigate their properties and to allow certification of their implementations. This study splits into syntax and semantics. In order to study semantics one needs an appropriate formal model of the syntax. This PhD thesis is dealing with the syntactical part. The study of syntax and its formal model is done in the context of mathematical theories. For the present work the mathematical tools are the ones provided by category theory.

The most general formal model of a functional programming language is the *Lambda Calculus* introduced by A. Church in the 1930's. The basic idea is to consider everything as a function. There exist many extensions of the Lambda Calculus that represent various sorts of programming languages. Traditionally the *pure* Lambda Calculus is presented the following way. One takes a set X of formal *variables* and builds then inductively the set of *Lambda–terms* with the aid of the *application* and *abstraction* constructors. The application constructor is written $(- \ -)$ and the abstraction with respect to a variable $x \in X$ is written $\lambda x.-$. So a Lambda–term M on X is of the form

$$M ::= x|(M_1 \ M_2)|\lambda x.M$$

The abstraction $\lambda x.-$ performs binding of the variable x. The pure Lambda Calculus being the standard example of a syntax with *variable binding*, it has gained much attention in research, just to mention a few, for example in [FPT99], [GP99], [HM07].

Abstract syntax without variable binding is given for example by propositional calculus, where one is allowed to use negation, conjunction, disjunction and implication to form new propositions from a given set of axioms. Other examples are algebraic structures such as monoids, groups or rings. This kind of syntax is generally considered easier to handle. That is, it can be described by mathematical theories, such as Lawvere Theories introduced in his thesis in the nineteen-sixties (reprinted version [Law04]) or by means of category theory as explained in [Bor94b] chapter 3.

All these syntaxes mentioned above are *single–sorted*. That means that all the syntactically correct terms are of the same type. There exist also *multi–sorted* syntaxes — with variable binding and without variable binding. Again the case with variable binding is generally considered to be more complex. The standard example of such a multi–sorted syntax with variable binding is the simply typed Lambda Calculus. It can be presented the following way. Let T be an inductive set of types obtained by a set of base types and a binary constructor $\Rightarrow$. Given a *context* Γ, that is, a set of pairs of the form $(x : t)$, which stand for typed variables, one has the following typing rules for Lambda–terms

$$\frac{(x : t) \in \Gamma}{\Gamma \vdash x : t}\,\text{var}$$

$$\frac{\Gamma \vdash M : s \Rightarrow t \qquad \Gamma \vdash N : s}{\Gamma \vdash (M\ N) : t}\,\text{app}$$

$$\frac{\Gamma, (x : s) \vdash M : t}{\Gamma \vdash \lambda x : s.M : s \Rightarrow t}\,\text{abs}$$

The present work is dealing mainly with two points of view of abstract syntax with variable binding. Both make extensive use of category theory. The first point of view is in the spirit of the category theoretic approach to algebraic theories. The seminal paper for untyped abstract syntax with variable binding is the one by Fiore, Plotkin, Turi [FPT99]. As the authors mention, the simply typed case can be developed along those lines. By simply typed we mean that the set of types is considered as a fixed set and we provide syntax of typed terms. In this approach the syntax associated to a signature is characterized as the free algebra of a certain signature functor. Throughout this work we refer to this approach as the *presheaf* approach.

The other point of view is a variation of the first one and originates in [HM07]. The main difference to the former approach is that it is strongly based on monads and introduces *modules* on monads as the main tool for describing syntax, so we refer to this approach as the *monadic* approach. This notion of module generalizes the well–known notion of module on a monoid. More precisely given a monad R on a category $\mathcal{C}$, an R–module is a pair (M, σ) where M is a functor $\mathcal{C} \to \mathcal{D}$ and σ, the *action*, is a natural transformation $M \circ R \to M$ satisfying compatibility conditions with the unit and multiplication of R. Syntax as a monad is characterized as the initial object in the category of *representations* for a signature. These so–called representations are collections of module morphisms. This point of view covers the untyped and simply typed cases. It can be adapted to typed syntax. By typed syntax we mean that we have syntax on two levels, on types and on terms which depend on each other. In the simply typed case however, the syntax of types is independent and can be handled as an untyped syntax before coming to the actual point of interest, the syntax of terms.

This PhD thesis has two main contributions. On the one hand the relationship between the two mentioned approaches in the untyped and simply typed cases is examined and described in detail. On the other hand the monadic approach is developed in order to cover a larger class of syntaxes. At first we consider typed signatures containing quantification over types but without variable binding. We define a notion of arity and signature in four steps. A theorem of existing initial representations is formulated and proved. Then we consider an example of such a signature and add a binding signature on top in order to prove that the typed Lambda Calculus is the initial object of a certain category of representations.

1.1 Untyped syntax with variable binding

We describe briefly the two approaches originating in [FPT99] and in [HM07]. Both have the same underlying notion of binding signature, which is a collection of binding arities. A binding arity is a finite sequence of natural numbers $(n_1, \ldots, n_p)$. Intuitively it stands for an operator of p arguments that binds n_j variables in its j-th argument.

On the one hand the presheaf approach is based on the functor category $[\mathbb{F}, \mathrm{Set}]$ from finite sets to sets. It contains the object $V = \mathbb{F}(1, -) : \mathbb{F} \hookrightarrow \mathrm{Set}$ (V for variables) which induces an endofunctor $\delta : [\mathbb{F}, \mathrm{Set}] \to [\mathbb{F}, \mathrm{Set}]$, $X \mapsto X^V \cong X(- + 1)$. By iterating δ, one associates to a binding signature the signature endofunctor $\Sigma : [\mathbb{F}, \mathrm{Set}] \to [\mathbb{F}, \mathrm{Set}]$

$$X \mapsto \sum_{i \in I} \delta^{n_{i,1}} X \times \ldots \delta^{n_{i,p_i}} X$$

where I is the index set of arities contained in the signature. One is interested in TV the free Σ–algebra on the functor V of $[\mathbb{F}, \mathrm{Set}]$, which is the set of terms modulo α–conversion. By free Σ–algebra on a functor we mean the image of the left adjoint of the forgetful functor Σ–alg $\to [\mathbb{F}, \mathrm{Set}]$. This adjunction induces a monad T on $[\mathbb{F}, \mathrm{Set}]$, that assigns to a functor X the free Σ–algebra on X. Moreover the category $[\mathbb{F}, \mathrm{Set}]$ is monoidal with respect to the following substitution monoidal product

$$X \bullet Y(n) = \int^k X(k) \times (Y(n))^k$$

and the unit being V. The signature endofunctor admits a "pointed" strength $s_{X,Y} : \Sigma(X) \bullet Y \to \Sigma(X \bullet Y)$ which extends to a strength $t_{X,Y} : T(X) \bullet Y \to T(X \bullet Y)$ for the induced monad T. Simultaneous substitution is given by a monoid in this monoidal category. The free Σ–algebra on V is a monoid

$$TV \bullet TV \to T(V \bullet TV) \cong TTV \to TV$$

One defines then the category of Σ–monoids, whose objects are Σ–algebras $\Sigma(P) \to P$ equipped with a monoidal structure and the following compatibility condition of these two structures holds, that is, the following diagram commutes

$$\begin{tikzcd} \Sigma(P) \bullet P \arrow[r] \arrow[d] & P \bullet P \arrow[dd] \\ \Sigma(P \bullet P) \arrow[d] & \\ \Sigma(P) \arrow[r] & P \end{tikzcd} \tag{1.1}$$

The main result is that TV is initial in the category of Σ–monoids.

On the other hand the monadic approach makes only use of the category of sets. An arity $(n_1, \ldots, n_p)$ is represented in a monad R on Set by a morphism of R–modules

$$R^{(n_1)} \times \ldots \times R^{(n_p)} \to R$$

where $R^{(n)}$ stands for the so–called n-th derived module. The derived module of an R–module M is defined on objects by

$$M' : A \mapsto M(A + 1)$$

and the R–action $M'RA \to M'A$ on a set A is defined by the following composite

$$M(RA + 1) \to MR(A + 1) \to M(A + 1)$$

We remark that the first arrow comes from a "pointed" strength for the endofunctor $(-)' : [\mathrm{Set}, \mathrm{Set}] \to [\mathrm{Set}, \mathrm{Set}]$. It extends to a "pointed" strength for the endofunctor $(-)^{(n_1)} \times \ldots \times$

$(-)^{(n_p)}$. Given a signature, one associates to it the category of representations whose objects are monads provided with a representation for each arity. The main theorem states that the category of representations has an initial object. It is as expected the set of terms modulo α–conversion.

Chapter 4 contains the relationship between the two approaches. There is a monoidal adjunction between the monoidal categories $[\mathbb{F}, \mathrm{Set}]$ and $[\mathrm{Set}, \mathrm{Set}]$. The left adjoint $\ell : [\mathbb{F}, \mathrm{Set}] \to [\mathrm{Set}, \mathrm{Set}]$ is given by the left Kan extension along V or equivalently by the coend formula

$$\ell(X)(A) = \int^{k \in \mathbb{F}} X(k) \times A^k$$

and the right adjoint $k : [\mathrm{Set}, \mathrm{Set}] \to [\mathbb{F}, \mathrm{Set}]$ is given by precomposition by V. Since the adjunction $\ell \dashv k$ is monoidal, it maps monoids to monoids. Moreover we consider the "large" categories of modules, that is (right) R–modules over monoids on $[\mathbb{F}, \mathrm{Set}]$ and on $[\mathrm{Set}, \mathrm{Set}]$ in the traditional sense. The monoidal adjunction maps modules to modules too. So we have the two induced adjunctions between the categories of monoids on $[\mathbb{F}, \mathrm{Set}]$ and $[\mathrm{Set}, \mathrm{Set}]$, $\ell \dashv k : \mathrm{Mon}([\mathrm{Set}, \mathrm{Set}]) \to \mathrm{Mon}([\mathbb{F}, \mathrm{Set}])$ and the categories of modules $\ell \dashv k : \mathrm{Mod}([\mathrm{Set}, \mathrm{Set}]) \to \mathrm{Mod}([\mathbb{F}, \mathrm{Set}])$. Given a signature S, it induces a strong signature endofunctor Σ, which permits to define a P–module $\Sigma(P)$ for a monoid P by

$$\Sigma(P) \bullet P \to \Sigma(P \bullet P) \to \Sigma(P)$$

and the free Σ–algebra structural map $\Sigma(P) \to P$ can be viewed as a P–module morphism by diagram (1.1). The signature S also induces an R–module morphism $M(R) \to R$ for a given monoid R on $[\mathrm{Set}, \mathrm{Set}]$ with respect to composition of functors, that is, a monad on Set. The link between Σ and M is then given by the following arrows

$$\alpha : \ell\Sigma \to M\ell \qquad\qquad \beta : \Sigma k \to kM$$

which are mates under the adjunction $\ell \dashv k : [\mathrm{Set}, \mathrm{Set}] \to [\mathbb{F}, \mathrm{Set}]$. They induce morphisms of modules $\alpha_P : \ell\Sigma(P) \to M(\ell P)$ and $\beta_R : \Sigma(kR) \to kM(R)$. In fact α_P is an isomorphism of modules. With the aid of these two module morphisms, we deduce the universal property of one approach starting with the universal property of the other one in the end of chapter 4.

1.2 Simply typed syntax with variable binding

Chapter 5 describes the presheaf approach for simply typed abstract syntax with variable binding and chapter 6 does so for the monadic approach. They are both analogous to the untyped case. For a fixed set $\mathcal{T}$ of types, we take an arity to be a collection of types written

$$(t_{1,1} \dots t_{1,m_1})t_1, \dots, (t_{n,1} \dots t_{n,m_n})t_n \to t_0$$

which stands for a binding operator with n arguments that binds m_j variables of types $t_{j,1}, \dots, t_{j,m_j}$ in its j-th argument of type t_j. It yields a term of type t_0.

The presheaf approach is based on the functor category $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ of $\mathcal{T}$–indexed presheaves from $\mathbb{F} \downarrow \mathcal{T}$ (finite sets over $\mathcal{T}$) to sets. We write $\langle t \rangle$ for the object $1 \to \mathcal{T}, 1 \mapsto t$ of $\mathbb{F} \downarrow \mathcal{T}$. The category $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ contains the collection of variables given by the Yoneda functor $\mathcal{Y}_t = \mathcal{Y}\langle t \rangle = \mathbb{F} \downarrow \mathcal{T}(\langle t \rangle, -) : \mathbb{F} \downarrow \mathcal{T} \to \mathrm{Set}$ which induces endofunctors $(-)^{\mathcal{Y}\langle t \rangle}$ on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ and

$[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$, $X \mapsto X^{\mathcal{Y}\langle t\rangle} \cong X(- + \langle t\rangle)$. They can be provided with a "pointed" strength. One associates to a binding signature the signature endofunctor $\Sigma : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$

$$(X_u)_{u\in\mathcal{T}} \mapsto \sum_{k\in I_u} \prod_{i=1}^{n_k} X_{t_i}^{\mathcal{Y}(t_{i,1}^{(k)},\dots,t_{i,m_i}^{(k)})}$$

where $I_u = \{k \in I | t_0^{(k)} = u\}$. One is interested as before in the free Σ–algebras. The forgetful functor $\Sigma\text{–alg} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ has a left adjoint, that assigns to an $X \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ the free Σ–algebra on X. This adjunction induces a monad T. The free Σ–algebra $T\mathcal{Y}$ on the functor $\mathcal{Y}$ of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ stands for the collection of typed terms modulo α–conversion. Moreover the category $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is monoidal with respect to the following substitution monoidal product

$$(X \otimes Y)_t(\Gamma) = \int^{\Delta} X_t(\Delta) \times \prod_{u\in\mathcal{T}} Y_u(\Gamma)^{\Delta^{-1}(u)}$$

and the unit being $\mathcal{Y}$. The signature endofunctor admits a "pointed" strength $s_{X,Y} : \Sigma(X)\otimes Y \to \Sigma(X \otimes Y)$ which comes from the "pointed" strength for $(-)^{\mathcal{Y}\langle u\rangle}$. It extends to a strength $t_{X,Y} : T(X) \otimes Y \to T(X \otimes Y)$ for the induced monad T. Simultaneous substitution is given by a monoid in this monoidal category. The free Σ–algebra on $\mathcal{Y}$ is a monoid

$$T\mathcal{Y} \otimes T\mathcal{Y} \to T(\mathcal{Y} \otimes T\mathcal{Y}) \cong TT\mathcal{Y} \to T\mathcal{Y}$$

The main result is again that $T\mathcal{Y}$ is initial in the category of Σ–monoids.

In the monadic approach one defines a representation of an arity

$$(t_{1,1} \dots t_{1,m_1})t_1, \dots, (t_{n,1} \dots t_{n,m_n})t_n \to t_0$$

in a monad R on $\mathrm{Set}/\mathcal{T}$. It is a morphism of R–modules of type

$$(\partial_{t_{1,m_1}} \dots \partial_{t_{1,1}} R)_{t_1} \times \dots \times (\partial_{t_{n,m_n}} \dots \partial_{t_{n,1}} R)_{t_n} \to R_{t_0}$$

where $\partial_u R$ stands for the derived module with respect to $u \in \mathcal{T}$. The derived module of an R–module M with respect to $u \in \mathcal{T}$ is defined on objects by

$$\partial_u M : \Gamma \mapsto M(\Gamma + \langle u\rangle)$$

and the R–action $\partial_u MR\Gamma \to \partial_u M\Gamma$ on an object $\Gamma \in \mathrm{Set}/\mathcal{T}$ is defined by the following composite

$$M(R\Gamma + \langle u\rangle) \to MR(\Gamma + \langle u\rangle) \to M(\Gamma + \langle u\rangle)$$

We point out that the first arrow comes from a "pointed" strength $\partial_u M \circ R \to \partial_u(M \circ R)$ for ∂_u derivation with respect to u. Moreover this "pointed" strength extends to one for the functor $(\partial_{t_{1,m_1}} \dots \partial_{t_{1,1}} -)_{t_1} \times \dots \times (\partial_{t_{n,m_n}} \dots \partial_{t_{n,1}} -)_{t_n}$ which is the underlying functor of the domain module. A new sort of modules is used in the representation of an arity, the fiber modules. The fiber module of an R–module M with respect to $u \in \mathcal{T}$ is defined as follows on objects

$$M_u(\Gamma) := (M\Gamma)^{-1}(u)$$

The main theorem states that for a given signature, the syntax of typed terms is the initial object in the category of representations.

Chapter 7 details the relationship between the two approaches in the simply typed case. There is an adjunction $\ell \dashv k : [\mathrm{Set}\,/\mathcal{T}, \mathrm{Set}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ and again a monoidal adjunction $\ell \dashv k : [\mathrm{Set}\,/\mathcal{T}, \mathrm{Set}\,/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ and thus an induced adjunction $\ell \dashv k : \mathrm{Mon}([\mathrm{Set}\,/\mathcal{T}, \mathrm{Set}\,/\mathcal{T}]) \to \mathrm{Mon}([\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}})$. Let $S = (\alpha_i)_{i \in I}$ be a signature. We write $\Sigma^{(i)}$ for the piece of the binding signature endofunctor corresponding to the arity α_i and $M^{(i)}$ for the underlying functor of the domain module of the corresponding representation. We define then two morphisms of strengths for each arity α_i of S

$$a^{(i)} : M^{(i)}\ell \to \ell\Sigma^{(i)} \qquad\qquad b^{(i)} : \Sigma^{(i)}k \to kM^{(i)}$$

where $a^{(i)}$ is an isomorphism. With the aid of $a^{(i)}$ and $b^{(i)}$ we construct the adjunction $L \dashv K : \mathrm{Rep}(S) \to \Sigma\text{–}\mathrm{Mon}$ where Σ–Mon is the category of Σ–monoids for the signature functor Σ associated to S and $\mathrm{Rep}(S)$ the category of representations of S. We cannot develop this comparison further in the simply typed case because the representations in the monadic approach cannot be "glued" together into one module. It would create a dependency of the potential domain module from the codomain module.

1.3 Typed syntax

The chapter 8 develops a theory for typed syntax without variable binding and with quantification over types. We mix arities of type constructors and term constructors in a signature and we allow dependencies between the constructors. We mean by dependency of constructors that the type of an argument of a constructor can depend on another constructor. For example the zero and succ constructors of natural numbers depend on the type constructor nat of natural numbers. So naturally one is led to a stratified signature in this sense which implies a complicated notion of signature and a definition by recursion. Our proposed solution goes as follows and tries to avoid such a recursive definition.

The base category for this setting is the category of arrows and commutative squares on sets, $\mathrm{TEns} = [2, \mathrm{Set}]$. At first we define a notion of arity and signature on an object $C : \overline{C} \to \underline{C}$ of TEns. An arity is a non–empty finite list of elements of $\underline{C}$ or an additional $\perp$. We write $\underline{\underline{C}}$ for $\underline{C} + \{\perp\}$. Intuitively this list stands for the types of each argument of the constructor and the last one for the type of the result. The additional $\perp$ is used for a type argument. Given an index set I, an I–signature on an object of TEns is a collection of arities indexed by I. There is a category $\mathrm{Sign}(I)$ whose objects are pairs (C, S) where $C \in \mathrm{TEns}$ and S is an I–signature on C. We introduce a notion of endorepresentation of an arity $a : r \to \underline{\underline{C}}$ which is a map of type

$$\prod_{j=1}^{r-1} C[a(j)] \to C[a(r)]$$

where we write

$$C[t] := \begin{cases} C^{-1}(t) & \text{if } t \in \underline{C} \\ \underline{C} & \text{if } t = \perp \end{cases}$$

An I–endorepresentation consists of an I–signature S together with an endorepresentation of each arity. We write $\mathrm{EndRep}(I)$ for the category of I–endorepresentations and morphisms of I–endorepresentations. Furthermore there is a forgetful functor $\mathrm{EndRep}(I) \to \mathrm{Sign}(I)$. We show that it has a left adjoint. This left adjoint assigns to an I–signature on C its initial representation. In order to define this adjoint and the initial representations, we define first a category of representations for a given I–signature S and we show that it has an initial object $C\langle S\rangle$.

In order to model quantification over types, we introduce a notion of arity with *degree* where arities of degree 0 are the ones introduced before. The basic idea is that a constructor using quantification over types can be thought of as a constructor depending on additional constructors representing the type variables. The degree is then just the number of these additional "local" type variables. We think that this procedure of quantification over types could be easily adapted to quantification over terms. So the next step is to define arities and signatures of higher degree on an object $C \in \mathrm{TEns}$. An arity of degree d is a map $r \to \underline{\underline{\underline{C\langle d\rangle}}}$ where d stands for the signature consisting of d equal arities $1 \mapsto \perp$. For a given weighted set I, that is, a set I together with a weight map $\mathbf{d} : I \to \mathbb{N}$, we define a category of I–signatures $\mathrm{Sign}(I)$ of higher degree with a forgetful functor to TEns and a category of I–endorepresentations $\mathrm{EndRep}(I)$. Again we show that there is an adjunction between $\mathrm{Sign}(I)$ and $\mathrm{EndRep}(I)$, with the right adjoint being the forgetful $\mathrm{EndRep}(I) \to \mathrm{Sign}(I)$.

Next we redefine all the above mentioned notions on a category of I'–endorepresentations $\mathcal{E} = \mathrm{EndRep}(I')$ for a given weighted set I'. As mentioned above it comes equipped with a forgetful functor to $\mathrm{Sign}(I')$ which can be composed with another forgetful functor $\mathrm{Sign}(I') \to \mathrm{TEns}$ to obtain $U : \mathcal{E} \to \mathrm{TEns}$. We define an arity on a $\Gamma = (C, S', \rho') \in \mathcal{E}$ and a signature on Γ. We do this in both cases of degree 0 and of higher degree. As before, in order to define the case of higher degree, we have to consider first signatures of degree 0.

An arity of degree 0 on an object $\Gamma \in \mathcal{E}$ is $a : r \to \underline{\underline{\underline{U(\Gamma)}}}$. Let I be a set. An I–signature on Γ of degree 0 is a collection of arities on Γ of degree 0 indexed by I. Similarly as before we define a category of I–signatures $\mathrm{Sign}_{\mathcal{E}}(I)$ with a forgetful functor to TEns and a category of I–endorepresentations with a forgetful functor to $\mathrm{Sign}_{\mathcal{E}}(I)$. Again our theorems state that the forgetful functor $\mathrm{EndRep}_{\mathcal{E}}(I) \to \mathrm{Sign}_{\mathcal{E}}(I)$ has a left adjoint and that there exists an initial object $\Gamma\langle S\rangle$ in the category of representations of a given I–signature S on $\Gamma \in \mathcal{E}$. The crucial point in the construction of the initial object is that one has to add constructions by arities of S' that use terms constructed by S as arguments.

Now we are in a position to define signatures of higher degrees. Let $d \in \mathbb{N}$. An arity of degree d is $a : r \to \underline{\underline{\underline{U(\Gamma\langle d\rangle)}}}$, where as before d stands for the signature of degree 0 consisting of d equal arities $1 \mapsto \perp$. For a given weighted set I, we define a category of I–signatures $\mathrm{Sign}_{\mathcal{E}}(I)$ of higher degree and a category of I–endorepresentations $\mathrm{EndRep}_{\mathcal{E}}(I)$. Again we show that there is an adjunction between $\mathrm{Sign}_{\mathcal{E}}(I)$ and $\mathrm{EndRep}_{\mathcal{E}}(I)$ with the right adjoint being the forgetful $\mathrm{EndRep}_{\mathcal{E}}(I) \to \mathrm{Sign}_{\mathcal{E}}(I)$.

Then for the next levels of signature, instead of defining arities and signatures on objects of $\mathrm{EndRep}_{\mathcal{E}}(I)$, we can re-use the above definitions since $\mathrm{EndRep}_{\mathcal{E}}(I) \cong \mathrm{EndRep}(I' + I)$. So we avoided a notion of stratified signature.

Chapter 9 treats the example of the typed Lambda Calculus. It involves variable binding, so the results of chapter 8 do not cover this case. We do not develop a general theory in this case, we only define a notion of arity and signature on top of an example of signature as in chapter 8. We think that this procedure can be generalised to any other signature with variable binding on top of any signature of chapter 8.

The base signature for the typed Lambda Calculus is an I–signature on $0 \to 0$ of TEns consisting of $I = 1, \mathbf{d} = 0$ and the only arity $3 \to \underline{0}, 1, 2, 3 \mapsto \perp$. The category of representations of this signature is written $\mathrm{TEns}^{\Rightarrow}$. An object is a pair $(\Gamma, \Rightarrow)$ where $\Gamma \in \mathrm{TEns}$ and $\Rightarrow : \underline{\Gamma} \times \underline{\Gamma} \to \underline{\Gamma}$. Then on this category we (re-)define derived modules and fibre modules which are very similar to the simply typed case. Then we define a notion of arity on $\mathrm{TEns}^{\Rightarrow}$ and we describe the signature of the typed Lambda Calculus. Finally we prove that the typed Lambda Calculus is initial in the category of representations corresponding to its signature.

1.4 Synopsis

For the present work we suppose a certain familiarity with category theory and basic notions such as categories, functors, natural transformations, monads, (co)products and adjunctions. It is organised as follows. In chapter 2 we give a short recapitulation of left Kan extensions and coends. The latter notion is extensively used in the later chapters.

In chapter 3 we summarise relevant parts of the theory of monads. We describe the construction of the free monad on an endofunctor by initial algebras of related endofunctors. Moreover we recall methods of computing those initial algebras.

Chapter 4 details the link between the two approaches for untyped syntax with variable binding. Chapter 5 presents the simply typed presheaf approach and chapter 6 the simply typed monadic approach. The relationship between these two is presented in chapter 7.

Chapter 8 and 9 develop the typed monadic approach.

Chapter 2

Generalities

2.1 Left Kan extensions

We recall some basic definitions and facts about left Kan extensions that will be used later.

Definition 2.1.1 (left Kan extension) *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three categories and $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{B}$ two functors. The* left Kan extension *of F along G is given by the pair $(\mathrm{Lan}_G(F), \alpha)$ where $\mathrm{Lan}_G(F)$ is a functor $\mathcal{B} \to \mathcal{C}$ and $\alpha : F \Rightarrow \mathrm{Lan}_G(F) \circ G$ a natural transformation, satisfying the following universal property. For all other pair (X, χ) where $X : \mathcal{B} \to \mathcal{C}$ and $\chi : F \Rightarrow X \circ G$, there exists a unique arrow $u : \mathrm{Lan}_G(F) \Rightarrow X$ such that $\chi = (u \circ G) \circ \alpha$.*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "G"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathcal{B} \arrow[dl, "\mathrm{Lan}_G(F)"'] \arrow[dl, bend left=60, "X"] \arrow[dl, phantom, bend left=30, "\Downarrow u"] \\
& \mathcal{C} &
\end{tikzcd}$$

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are small. Then the left Kan extension Lan_G exists for all $F \in [\mathcal{A}, \mathcal{C}]$ if and only if the functor $- \circ G : [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ precomposition by G has a left adjoint as explained in [Bén00].

Theorem 2.1.2 ([Bor94a] 3.7.2) *With the notations of definition 2.1.1, if $\mathcal{A}$ is small and $\mathcal{C}$ cocomplete, then the left Kan extension $\mathrm{Lan}_F(G)$ exists and is given for an object $B \in \mathcal{B}$ by a colimit, more precisely by*

$$\mathrm{Lan}_G(F)(B) = \mathrm{colim}_{(A, g : GA \to B)} F(A)$$

By this $\mathrm{colim}_{(A, g : GA \to B)} F(A)$ we take the colimit of the diagram given by the category of elements of the endofunctor $\mathcal{B}(G-, B) : \mathcal{A} \to \mathrm{Set}$.

Theorem 2.1.3 ([Bor94a] 3.7.3) *With the notations of definition 2.1.1, if $\mathcal{A}$ is small and $\mathcal{C}$ cocomplete and G full and faithful, then α is a natural isomorphism.*

In the following we consider left Kan extensions along the inclusion functor $U : \mathbb{F} \to \mathrm{Set}$. It is full and faithful.

Let us consider the left Kan extension of U along U. By definition we have a natural isomorphism $U \cong \mathrm{Lan}_U(U) \circ U$. It implies that $\mathrm{Lan}_U(U) \cong \mathrm{Id}_U$.

Let $X \in [\mathbb{F}, \mathrm{Set}]$ and $F \in [\mathrm{Set}, \mathrm{Set}]$. Now consider the left Kan extension along U of the composite FX.

$$
\begin{tikzcd}
\mathbb{F} \arrow[rr, "U"] \arrow[dr, "X"'] \arrow[r, phantom, "\beta"] & {} & \mathrm{Set} \arrow[dl, "\mathrm{Lan}_U(X)"] \arrow[dd, "\mathrm{Lan}_U(FX)"] \\
& \mathrm{Set} \arrow[dr, "F"'] \arrow[r, phantom, "\alpha"] & {} \arrow[d, phantom, "\Leftarrow h"] \\
& & \mathrm{Set}
\end{tikzcd}
$$

We have a natural isomorphism $\beta : X \Rightarrow \mathrm{Lan}_U(X) \circ U$ and a natural isomorphism $F\beta : FX \Rightarrow F \circ \mathrm{Lan}_U(F) \circ U$. By universal property of $(\mathrm{Lan}_U(FX), \alpha : FX \Rightarrow \mathrm{Lan}_U(FX) \circ U)$, there is a unique arrow $h : \mathrm{Lan}_U(FX) \Rightarrow F \circ \mathrm{Lan}_U(F)$ such that $F\beta = (hU) \circ \alpha$. Moreover by proposition 3.7.4 of [Bor94a] if F is a left adjoint, then h is a natural isomorphism.

2.2 Coends

A clean introduction to coends can be found in the lecture notes [Win05]. We give the essential extracts concerning coends.

Definition 2.2.4 (dinatural transformation) *Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ two bifunctors. A* dinatural transformation *$\alpha : F \to G$ is a collection of arrows $\alpha_C : F(C,C) \to G(C,C)$ for all $C \in \mathcal{C}$ such that the following diagram commutes for all arrow $f : C \to D$ in $\mathcal{C}$.*

$$
\begin{tikzcd}
& F(C,C) \arrow[r, "\alpha_C"] & G(C,C) \arrow[dr, "G(\mathrm{id}_C,f)"] & \\
F(D,C) \arrow[ur, "F(f,\mathrm{id}_C)"] \arrow[dr, "F(\mathrm{id}_D,f)"'] & & & G(C,D) \\
& F(D,D) \arrow[r, "\alpha_D"'] & G(D,D) \arrow[ur, "G(f,\mathrm{id}_D)"'] &
\end{tikzcd}
$$

Definition 2.2.5 (wedge) *Let $\mathcal{C}, \mathcal{D}$ be two categories, $D \in \mathcal{D}$ and $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ a bifunctor. A* wedge *from F to D is a dinatural transformation $\alpha : F \to \Delta D$ where ΔD is the constant bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ that assigns D to each pair (C', C).*

Explicitly a wedge from F to D is a collection of arrows $\alpha_C : F(C,C) \to D$ such that the following diagram commutes for all arrow $f : A \to B$ in $\mathcal{C}$.

$$\begin{tikzcd} & F(A,A) \arrow[dr, "\alpha_A"] & \\ F(B,A) \arrow[ur, "F(f{,}\mathrm{id}_A)"] \arrow[dr, "F(\mathrm{id}_B{,}f)"'] & & D \\ & F(B,B) \arrow[ur, "\alpha_B"'] & \end{tikzcd} \tag{2.1}$$

We are going to refer to this diagram (2.1) as the wedge condition.

Definition 2.2.6 (coend) *Let $\mathcal{C},\mathcal{D}$ be two categories, $D \in \mathcal{D}$ and $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ a bifunctor. The* coend *of F is a pair $(K,(\alpha_C)_{C\in\mathcal{C}})$ consisting of an object K of $\mathcal{D}$ and a wedge $(\alpha_C)_{C\in\mathcal{C}}$ from F to K such that for all pair $(X,(\chi_C)_{C\in\mathcal{C}})$ there is a unique arrow $u : K \to X$ such that $\chi_C = u \circ \alpha_C$.*

We write $\int^C F(C,C)$ for the object K. Therefore for all arrow $f : C \to D$ in $\mathcal{C}$ we have the following commutative diagram

$$\begin{tikzcd} & F(C,C) \arrow[dr, "\alpha_C"] \arrow[drr, bend left, "\chi_C"] & & \\ F(D,C) \arrow[ur, "F(f{,}\mathrm{id}_C)"] \arrow[dr, "F(\mathrm{id}_D{,}f)"'] & & \int^C F(C,C) \arrow[r, "u"] & X \\ & F(D,D) \arrow[ur, "\alpha_D"'] \arrow[urr, bend right, "\chi_D"'] & & \end{tikzcd}$$

So to give an arrow $\int^C F(C,C) \to X$ is by universal property of the coend equivalent to give a wedge from F to X. That is, a collection of arrows $\chi_C : F(C,C) \to X$ satisfying the wedge condition (2.1).

In the following we often use the following reasoning. If we wish to show that two elements $a \in F(A,A)$ and $b \in F(B,B)$ are sent by the corresponding coprojections to the same element in $\int^C F(C,C)$, it suffices to find an arrow $h : A \to B$ and an element $x \in F(B,A)$ such that $F(h,\mathrm{id}_A)(x) = a$ and $F(\mathrm{id}_B,h)(x) = b$. Since the coprojections of $\int^C F(C,C)$ satisfy the wedge condition, the following diagram commutes

$$\begin{tikzcd} & F(A,A) \arrow[dr] & \\ F(B,A) \arrow[ur, "F(h{,}\mathrm{id}_A)"] \arrow[dr, "F(\mathrm{id}_B{,}h)"'] & & \int^C F(C,C) \\ & F(B,B) \arrow[ur] & \end{tikzcd}$$

and this shows that a and b are identical.

To finish this section we recall the *Fubini theorem for coends*, a theorem that will be used later on.

Theorem 2.2.7 ([Win05] or [ML98] IX. 8) *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ be categories and $F : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{D}$ such that $\int^A F(A, A, B, B')$ exists for all $B, B' \in \mathcal{B}$ and $\int^B F(A, A', B, B)$ exists for all $A, A' \in \mathcal{A}$ then*

$$\int^A \int^B F(A, A, B, B) \to \int^B \int^A F(A, A, B, B)$$

is a natural isomorphism and if one side exists so does the other.

2.3 Left Kan extensions as Coends

We recall that the left Kan extension can be expressed as a coend formula. This is explained in [ML98], chapter X section 4. The main theorem is the following

Theorem 2.3.8 ([ML98] X.4.1) *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three categories and $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{B}$ two functors such that for all $A, A' \in \mathcal{A}$ and all $B \in \mathcal{B}$ the copowers $\mathcal{B}(GA, B) \bullet FA'$ exist in $\mathcal{C}$, then F has a left Kan extension $\mathrm{Lan}_G(F)$ along G if for every $B \in \mathcal{B}$ the following coend exists, and when this is the case, the object functor of $\mathrm{Lan}_G(F)$ is this coend*

$$\mathrm{Lan}_G(F)(B) = \int^A \mathcal{B}(GA, B) \bullet FA$$

If we consider the left Kan extension of X along $U : \mathbb{F} \to \mathrm{Set}$, then we have the follwing formula

$$\mathrm{Lan}_U(X)(A) = \int^{n \in \mathbb{F}} \mathrm{Set}(U(n), A) \times X(n) = \int^{n \in \mathbb{F}} X(n) \times A^n$$

since the copower in Set is given by the product.

2.4 Ends

The dual notion of coend is the notion of end.

Let $\mathcal{C}, \mathcal{D}$ be two categories, $D \in \mathcal{D}$ and $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ a bifunctor. A wedge from D to F is a dinatural transformation $\alpha : \Delta D \to F$ where ΔD is the constant bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ that assigns D to each pair (C', C).

So explicitly a wedge from D to F is a collection of arrows $\alpha_C : D \to F(C, C)$ such that the following diagram commutes for all arrow $f : A \to B$ in $\mathcal{C}$.

$$\begin{tikzcd}
 & F(A,A) \arrow[dr, "F(\mathrm{id}_A, f)"] & \\
D \arrow[ur, "\alpha_A"] \arrow[dr, "\alpha_B"'] & & F(A,B) \\
 & F(B,B) \arrow[ur, "F(f, \mathrm{id}_B)"'] &
\end{tikzcd}$$

Definition 2.4.9 (end) *Let $\mathcal{C}, \mathcal{D}$ be two categories, $D \in \mathcal{D}$ and $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ a bifunctor. The* end *of F is a pair $(E, (\alpha_C)_{C \in \mathcal{C}})$ consisting of an object E of $\mathcal{D}$ and a wedge $(\alpha_C)_{C \in \mathcal{C}}$ from E to F such that for all pair $(X, (\chi_C)_{C \in \mathcal{C}})$ there is a unique arrow $u : X \to E$ such that $\chi_C = \alpha_C \circ u$.*

We write usually $\int_C F(C,C)$ for E. We recall the *Naturality Formula* and the *Fubini Formula for Ends.*

Proposition 2.4.10 ([Win05]) *Let $\mathcal{C},\mathcal{D}$ be two categories and $F,G:\mathcal{C}\to\mathcal{D}$ be two functors. Then*

$$[\mathcal{C},\mathcal{D}](F,G)=\int_C \mathcal{D}(F(C),G(C))$$

Theorem 2.4.11 ([Win05]) *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ be categories and $F:\mathcal{A}^{\mathrm{op}}\times\mathcal{A}\times\mathcal{B}^{\mathrm{op}}\times\mathcal{B}\to\mathcal{D}$ such that $\int_A F(A,A,B,B')$ exists for all $B,B'\in\mathcal{B}$ and $\int_B F(A,A',B,B)$ exists for all $A,A'\in\mathcal{A}$ then*

$$\int_A\int_B F(A,A,B,B)\to\int_B\int_A F(A,A,B,B)$$

is a natural isomorphism and if one side exists so does the other.

Chapter 3

Monads and their algebras

3.1 Free monads

Let $\mathcal{C}$ be a category and G an endofunctor on $\mathcal{C}$. Let us suppose that $\mathcal{C}$ is such that initial algebras for all functors G_X of the form $Y \mapsto G_X(Y) = X + G(Y)$ with $X \in \mathcal{C}$ exist.

Notation 3.1.12 *We write T for the functor $\mathcal{C} \to \mathcal{C}$ assigning $X \in \mathcal{C}$ the initial G_X–algebra.*

So TX is the initial G_X–algebra. By the following lemma due to Lambek the structural map of the initial algebra is an isomorphism.

Lemma 3.1.13 (Lambek) *Let $\mathbb{C}$ be a category and $F : \mathbb{C} \to \mathbb{C}$. The structural morphism a of the initial F–algebra $(A, a : FA \to A)$ is an isomorphism (if it exists).*

Proof. The pair $(FA, Fa : FFA \to FA)$ is as well an F–algebra. By initiality of (A, a), there exists a unique morphism of F–algebras $h : A \to FA$ such that

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "a"'] & FFA \arrow[d, "Fa"] \\ A \arrow[r, "h"'] & FA \end{tikzcd}$$

commutes. Now let us consider the following diagram

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "a"'] & FFA \arrow[r, "Fa"] \arrow[d, "Fa"] & FA \arrow[d, "a"] \\ A \arrow[r, "h"'] & FA \arrow[r, "a"'] & A \end{tikzcd}$$

The composite $a \circ h$ along the bottom is identity on A by initiality of (A, a), so the composite $Fa \circ Fh$ along the top is identity on FA. By commutativity of the left–hand square $h \circ a = Fa \circ Fh$, so $h \circ a = \mathrm{id}_{FA}$ which concludes the proof. $\square$

Proposition 3.1.14 *The functor T can be provided with the structure of a monad on $\mathcal{C}$.*

Proof. Since TX is the initial G_X–algebra, we write for the following isomorphism

$$[\eta_X, \sigma_X] : X + GTX \xrightarrow{\cong} TX$$

Functoriality of T

To define the image of an arrow $f : X \to Y$ under T, we provide TY with a G_X–algebra structure. By initiality of TX, we take Tf to be the unique morphism of G_X–algebras from TX to TY. Indeed we have $[\eta_Y \circ f, \sigma_Y] : X + GTY \to TY$.

Unit of T

We define the unit η of T to be η_X for all $X \in \mathcal{C}$. To check the naturality of η in X, we consider the definition of the arrow Tf for a given $f : X \to Y$ in $\mathcal{C}$.

$$\begin{tikzcd}
X + GTX \arrow[r, "\mathrm{id}_X + GTf"] \arrow[dd, "{[\eta_X, \sigma_X]}"'] & X + GTY \arrow[d, "f + \mathrm{id}"] \\
& Y + GTY \arrow[d, "{[\eta_Y, \sigma_Y]}"] \\
TX \arrow[r, "Tf"'] & TY
\end{tikzcd}$$

This diagram implies the commutativity of the two following diagrams

$$\begin{tikzcd}
X \arrow[d, "\eta_X"'] \arrow[dr, "\eta_Y"] & \\
TX \arrow[r, "Tf"'] & TY
\end{tikzcd}
\qquad
\begin{tikzcd}
GTX \arrow[r, "GTf"] \arrow[d, "\sigma_X"'] & GTY \arrow[d, "\sigma_Y"] \\
TX \arrow[r, "Tf"'] & TY
\end{tikzcd}$$

which prove the naturality of η and σ.

Multiplication of T

To define the composition $\mu_X : T^2X \to TX$ of T for all $X \in \mathcal{C}$, we provide TX with a G_{TX}–algebra structure. Then by initiality of TTX, we have a unique morphism $TTX \to TX$ which we take for μ_X. The arrow $TX + GTX \to TX$ is given by $[\mathrm{id}_{TX}, \sigma_X]$.

Next we check the naturality of μ. Let $f : X \to Y$ be an arrow in $\mathcal{C}$. We have to check the commutativity of the following square.

$$\begin{tikzcd}
TTX \arrow[r, "\mu_X"] \arrow[d, "TTf"'] & TX \arrow[d, "Tf"] \\
TTY \arrow[r, "\mu_Y"'] & TY
\end{tikzcd}$$

We provide TY with a G_{TX}–algebra structure and show that Tf and μ_Y are morphisms of G_{TX}–algebras. Since μ_X and TTf are by definition morphisms of G_{TX}–algebras, we can conclude by initiality of TTX that $\mu_Y \circ TTf$ and $Tf \circ \mu_X$ are equal.

- TY is a G_{TX}–algebra:

$$TX + GTY \xrightarrow{Tf + \mathrm{id}} TY + GTY \xrightarrow{[\mathrm{id}, \sigma_Y]} TY$$

- Tf is a morphism of G_{TX}–algebras:

$$\begin{tikzcd} TX + GTX \arrow[r, "\mathrm{id} + GTf"] \arrow[dd, "{[\mathrm{id}, \sigma_X]}"'] & TX + GTY \arrow[d, "Tf + \mathrm{id}"] \\ & TY + GTY \arrow[d, "{[\mathrm{id}, \sigma_Y]}"] \\ TX \arrow[r, "Tf"'] & TY \end{tikzcd}$$

This diagram commutes by naturality of σ.

- μ_Y is a morphism of G_{TX}–algebras: We have to check the commutativity of the following diagram

$$\begin{tikzcd} TX + GTTY \arrow[r, "\mathrm{id}_{TX} + G\mu_Y"] \arrow[d, "Tf + \mathrm{id}_{GTTY}"'] & TX + GTY \arrow[d, "Tf + \mathrm{id}"] \\ TY + GTTY \arrow[r, "\mathrm{id}_{TY} + G\mu_Y"] \arrow[d, "{[\eta^T_{TY}, \sigma_{TY}]}"'] & TY + GTY \arrow[d, "{[\mathrm{id}_{TY}, \sigma_Y]}"] \\ TTY \arrow[r, "\mu_Y"'] & TY \end{tikzcd}$$

The top square commutes obviously, the bottom square commutes in its first component by one of the monad axioms and in its second component by definition of μ_Y being a morphism of G_{TY}–algebras.

Monad axioms
See appendix A.1.

□

Notation 3.1.15 *We write* $\mathrm{End}(\mathcal{C})$ *for the category of endofunctors on* $\mathcal{C}$ *and* $\mathrm{Mon}(\mathcal{C})$ *for the category of monads on* $\mathcal{C}$*. We write* Ψ *for the forgetful functor* $\mathrm{Mon}(\mathcal{C}) \to \mathrm{End}(\mathcal{C})$ *which forgets the monad structure:* $(T, \eta, \mu) \mapsto T$*.*

Proposition 3.1.16 *The forgetful functor* $\Psi : \mathrm{Mon}_{\mathcal{C}} \to \mathrm{End}_{\mathcal{C}}$ *has a left adjoint* $\Phi : \mathrm{End}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{C})$*. It is given on objects by the construction of proposition 3.1.14 and its proof.*

Proof. See appendix A.2. □

3.2 Universal property for free monads

The construction of T admits a natural transformation $G \to T$. Its component at $X \in \mathcal{C}$ is given by $GX \xrightarrow{G\eta_X} GTX \xrightarrow{\sigma_X} TX$. It is natural in X since η and σ both are.

Consider the following category. Its objects are pairs (M, ρ) where M is a monad on $\mathcal{C}$ and ρ a natural transformation $G \to M$. Its arrows are morphisms of monads $\phi : M \to N$ such that

$$\begin{tikzcd} & G \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ M \arrow[rr, "\phi"'] & & N \end{tikzcd}$$

commutes. We write Mon^G for this category.

Proposition 3.2.17 *T is the initial object in the category Mon^G.*

Proof. We have seen that $(T, \sigma \circ G\eta)$ is an object in Mon^G. Let (M, ρ) be another object of Mon^G. We construct a monad morphism $\phi : T \to M$ componentwise. It will be unique by initiality of TX for each $X \in \mathcal{C}$. We provide MX with a G_X–algebra structure and by initiality of TX we take the unique morphism of G_X–algebras to be ϕ_X. Indeed we have $X + GMX \xrightarrow{\mathrm{id}_X + \rho_{MX}} X + MMX \xrightarrow{[\eta_X^M, \mu_X^M]} MX$.

For the remaining verifications of naturality, monad morphism and morphism of Mon^G, see appendix A.3. □

In [BW85] chapter 9 section 4 this property of T proved in 3.2.17 serves as the definition of the free monad generated by the endofunctor G.

3.3 Free algebras and initial algebras

Let $\mathcal{C}$ be a category and G an endofunctor on $\mathcal{C}$. We write G–alg for the category of G–algebras. Explicitly objects are pairs (C, c) where $C \in \mathcal{C}$ and $c : GC \to C$ is a map in $\mathcal{C}$. A morphism from (C, c) to (D, d) in G–alg is an arrow $f : C \to D$ of $\mathcal{C}$ such that

$$\begin{tikzcd} GC \arrow[r, "Gf"] \arrow[d, "c"'] & GD \arrow[d, "d"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

commutes. The category of G–algebras comes equipped with a forgetful functor $U : G\text{–alg} \to \mathcal{C}$, $(C, c) \mapsto C$.

Suppose this forgetful functor U has a left adjoint $F : \mathcal{C} \to G$–alg. Then this adjunction induces a monad on $\mathcal{C}$. Moreover this monad is the free monad generated by G by the following theorem 4.4 of chapter 9 in [BW85].

Theorem 3.3.18 ([BW85] 9.4.4) *If $U : G\text{–alg} \to \mathcal{C}$ has a left adjoint F, then the resulting monad is the free monad generated by G.*

The inverse implication is true if $\mathcal{C}$ is complete. And the categories of G–algebras and T–algebras are equivalent. Let us recall that an algebra for a monad T on $\mathcal{C}$ is a pair (C, c) consisting of an object $C \in \mathcal{C}$ and an arrow $c : TC \to C$ such that the following diagrams commute.

$$\begin{tikzcd} TTC \arrow[r, "\mu_C"] \arrow[d, "Tc"'] & TC \arrow[d, "c"] \\ TC \arrow[r, "c"'] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\mathrm{id}_C"'] & TC \arrow[d, "c"] \\ & C \end{tikzcd}$$

A morphism of T–agebras from (C, c) to (D, d) is an arrow $f : C \to D$ in $\mathcal{C}$ such that

$$\begin{tikzcd} TC \arrow[r, "Tf"] \arrow[d, "c"'] & TD \arrow[d, "d"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

commutes. T–algebras and morphisms of T–algebras form a category T–Alg, the Eilenberg–Moore category.

Proposition 3.3.19 **([BW85] 9.4.5)** *Let $\mathcal{C}$ be a complete category and G an endofunctor on $\mathcal{C}$ which generates a free monad T. Then $U : G\text{–alg} \to \mathcal{C}$ has a left adjoint and T–Alg is equivalent to G–alg.*

The free monad generated by G exists if U has a left adjoint F. This left adjoint can be computed as the colimit of a countable chain.

Proposition 3.3.20 **([BW85] 9.4.7)** *Let $\mathcal{C}$ be complete and have finite colimits and colimits of countable chains. Let G be an endofunctor on $\mathcal{C}$ which commutes with colimits of countable chains. Then G generates a free monad.*

In the proof of this proposition, the free G–algebra is computed as the colimit of a countable chain. Let $X \in \mathcal{C}$. We define the countable chain

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

and by the proof of the above proposition the underlying object of the free G–algebra on X is given by $F(X) = \operatorname{colim} X_n$. We set $X_0 := X$ and $X_1 := X + G(X)$. The objects X_i for $i \geq 2$ are defined by the following diagrams

$$\begin{tikzcd}
 & G(X_0) \arrow[r, "Ge_0"] \arrow[d, "c_1"'] & G(X_1) \arrow[r, "Ge_1"] \arrow[d, "c_2"'] & G(X_2) \arrow[r, "Re_2"] \arrow[d, "c_3"'] & \cdots \\
X_0 \arrow[r, "e_0"'] & X_1 \arrow[r, "e_1"'] & X_2 \arrow[r, "e_2"'] & X_3 \arrow[r, "e_3"'] & \cdots
\end{tikzcd}$$

where each square

$$\begin{tikzcd}
G(X_{i-1}) \arrow[r, "Ge_{i-1}"] \arrow[d, "c_i"'] & G(X_i) \arrow[d, "c_{i+1}"] \\
X_i \arrow[r, "e_i"'] & X_{i+1}
\end{tikzcd}$$

is a pushout square. Remark that these squares are of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_A"'] & B \arrow[d] \\
A + C \arrow[r] & D
\end{tikzcd}$$

so $D \cong B + C$ by universal properties of sums and pushouts. So the chain

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

is

$$X \longrightarrow X + G(X) \longrightarrow X + G(X + G(X)) \longrightarrow \cdots \tag{3.1}$$

and the free G–algebra on X is computed as the colimit of this countable chain.

Now we show that this colimit is equivalent to another colimit of a countable chain. Consider the following chain

$$0 \longrightarrow G_X(0) \longrightarrow G_X^2(0) \longrightarrow \cdots \tag{3.2}$$

where 0 is the initial object of $\mathcal{C}$. Its colimit is equivalent to the colimit of (3.1). Consider the following diagram connecting the chains (3.1) and (3.2)

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "i"] & G_X(X) \arrow[r] \arrow[dr] & G_X^2(X) \arrow[r] \arrow[dr] & \cdots \\ 0 \arrow[u, "u"] \arrow[r] & G_X(0) \arrow[u] \arrow[r] & G_X^2(0) \arrow[u] \arrow[r] & \cdots \end{tikzcd}$$

The first arrow $u : 0 \to X$ is the initial morphism and the other vertical arrows are $G_X(u)$, $G_X^2(u)$, etc. The first map $i : X \to G_X(0) = X + G(0)$ is the inclusion map and the other diagonal arrows are $G_X(i)$, $G_X^2(i)$, etc. Note that all the triangles commute. Let us write C for the colimit of (3.1) and C' for the colimit of (3.2). The chain (3.1) forms a cocone for C' and the chain (3.2) forms a cocone for C. The cocone conditions are satisfied since all triangles commute. So there are two arrows $C \to C'$ and $C' \to C$ which are inverse to each other by universal properties of the two colimits. To summarise this we showed that the free algebra can be computed as well as $\mathrm{colim}_n\, G_X^n(0)$.

Now we drop the completeness condition of $\mathcal{C}$.

Proposition 3.3.21 *Suppose that $\mathcal{C}$ has coproducts and initial algebras for functors G_X for all $X \in \mathcal{C}$. Then the forgetful functor $U : G\text{–alg} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to G\text{–alg}$. The free G–algebra FX on $X \in \mathcal{C}$ is given by the initial G_X–algebra TX.*

Proof. The initial G_X–algebra TX is also a G–algebra with $\sigma_X : GTX \to TX$ as the structural map.

Let $(Y, y) \in G$–alg and $X \in \mathcal{C}$. We are going to show the following isomorphism of hom–sets

$$G\text{–}\mathrm{alg}((TX, \sigma_X), (Y, y)) \cong \mathcal{C}(X, Y)$$

and its naturality in (Y, y) and X.

Suppose given a morphism h of G–algebras $TX \to Y$. It makes commute the following square by definition

$$\begin{tikzcd} GTX \arrow[r, "Gh"] \arrow[d, "\sigma_X"'] & GY \arrow[d, "y"] \\ TX \arrow[r, "h"'] & Y \end{tikzcd}$$

Since TX is the initial G_X–algebra we have the arrow $\eta_X : X \to TX$ and by composition with h we obtain the desired arrow $X \to TX \to Y$.

Suppose given a morphism $f : X \to Y$ in $\mathcal{C}$. Since Y is a G–algebra, we have the structural map $y : GY \to Y$ and together $[f, y] : X + GY \to Y$ which means that Y is a G_X–algebra. By initiality of TX there is a unique map of G_X–algebras $h : TX \to Y$ such that

$$\begin{tikzcd} X + GTX \arrow[r, "\mathrm{id}_X + Gh"] \arrow[d, "{[\eta_X, \sigma_X]}"'] & X + GY \arrow[d, "{[f, y]}"] \\ TX \arrow[r, "h"'] & Y \end{tikzcd} \tag{3.3}$$

commutes. This implies that

$$\begin{tikzcd} GTX \arrow[r, "Gh"] \arrow[d, "\sigma_X"'] & GY \arrow[d, "y"] \\ TX \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes as well.

Now we check that the above constructed assignations are inverse to each other. Starting with an $f : X \to Y$, we assign the unique map of G_X–algebras $h : TX \to Y$. Then we precompose it with η_X. By commutativity of (3.3) $h \circ \eta_X = f$.

Starting with a morphism h' of G–algebras, we precompose by η_X to obtain $h' \circ \eta_X : X \to TX \to Y$. Then we assign the unique map of G_X–algebras $h : TX \to Y$ such that (3.3) commutes. In fact h' is also a map of G_X–algebras since

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\eta_X"'] & GY \arrow[d, "\eta_X \circ h"] \\ TX \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes. Since h is the unique map of G_X–algebras $TX \to Y$, it follows that $h' = h$.

Next we check naturalities in (Y, y) and X. Let $f : (Y, y) \to (Z, z)$ in G–alg. The naturality square

$$\begin{tikzcd} G\text{–}\mathrm{alg}((TX, \sigma_X), (Y, y)) \arrow[r] \arrow[d] & \mathcal{C}(X, Y) \arrow[d] \\ G\text{–}\mathrm{alg}((TX, \sigma_X), (Z, z)) \arrow[r] & \mathcal{C}(X, Z) \end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd} TX \xrightarrow{h} Y \arrow[r, mapsto] \arrow[d, mapsto] & X \xrightarrow{\eta_X} TX \xrightarrow{h} Y \arrow[d, mapsto] \\ TX \xrightarrow{h} Y \xrightarrow{f} Z \arrow[r, mapsto] & X \xrightarrow{\eta_X} TX \xrightarrow{h} Y \xrightarrow{f} Z \end{tikzcd}$$

Now let $f : X \to Z$ in $\mathcal{C}$. The naturality square

$$\begin{tikzcd} G\text{–}\mathrm{alg}((TZ, \sigma_X), (Y, y)) \arrow[r] \arrow[d] & \mathcal{C}(Z, Y) \arrow[d] \\ G\text{–}\mathrm{alg}((TX, \sigma_X), (Y, y)) \arrow[r] & \mathcal{C}(X, Y) \end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd} TZ \xrightarrow{h} Y \arrow[r, mapsto] \arrow[d, mapsto] & Z \xrightarrow{\eta_Z} TZ \xrightarrow{h} Y \arrow[d, mapsto] \\ TX \xrightarrow{Tf} TZ \xrightarrow{h} Y \arrow[r, mapsto] & \begin{array}{l} X \xrightarrow{f} Z \xrightarrow{\eta_Z} TZ \xrightarrow{h} Y \\ = X \xrightarrow{\eta_X} TX \xrightarrow{Tf} TZ \xrightarrow{h} Y \end{array} \end{tikzcd}$$

by naturality of η, we have $\eta_Z \circ f = Tf \circ \eta_X$. $\square$

Proposition 3.3.22 *Let F be a functor $\mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ with initial object and colimits of the countable chain*

$$0 \to F(0) \to F^2(0) \to \ldots \to F^n(0) \to \ldots \tag{3.4}$$

If F preserves the colimit of the above countable chain then F has an initial algebra, which is given by the colimit of (3.4).

Proof. We write $C := \operatorname{colim} F^n(0)$. First we construct the structure map $c : F(C) \to C$. Since F preserves the colimit of (3.4), $F(C) = F(\operatorname{colim} F^n(0)) \cong \operatorname{colim} F^{n+1}(0)$. So $F(C)$ is the colimit of the subchain

$$F(0) \to F^2(0) \to \ldots \to F^n(0) \to \ldots$$

Since C and its coprojection maps $F^i(0) \to C$ for $i \geq 1$ are a cocone for this subchain, there exists a unique map $\operatorname{colim} F^{n+1}(0) \to C$ which we take to be the structure map c.

Next we show initiality of (C, c). Let $(B, b : F(B) \to B)$ be another F–algebra. We construct $h : C \to B$ by universal property of the colimit C. There is a unique map $u : 0 \to B$ because 0 is initial. Then $b \circ F(u) : F(0) \to F(B) \to B$ and $b \circ F(b \circ F(u)) : F^2(0) \to F(B) \to B$ and by iterating this procedure we obtain a cocone $(F^i(0) \to B)_{i \in \mathbb{N}_0}$. It is indeed a cocone since for all arrow $F^i(0) \to F^{i+1}(0)$ the triangle

$$\begin{tikzcd} F^i(0) \arrow[rr] \arrow[dr] & & F^{i+1}(0) \arrow[dl] \\ & B & \end{tikzcd}$$

commutes for all $i \in \mathbb{N}_0$ because the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "u"'] & F(0) \arrow[d, "F(u)"] \\ B & F(B) \arrow[l, "b"] \end{tikzcd}$$

commutes by initiality of 0 and thus all squares

$$\begin{tikzcd} F^i(0) \arrow[r] \arrow[d, "F^i(u)"'] & F^{i+1}(0) \arrow[d, "F^{i+1}(u)"] \\ B & F(B) \arrow[l, "b"] \end{tikzcd}$$

commute as well. The algebra morphism axiom

$$\begin{tikzcd} F(C) \arrow[r, "Fh"] \arrow[d, "c"'] & F(B) \arrow[d, "b"] \\ C \arrow[r, "h"'] & B \end{tikzcd}$$

commutes by universal property of the colimit $F(C) = \operatorname{colim} F^{i+1}(0)$. $\square$

Proposition 3.3.23 *Let $\mathcal{C}$ be a category with finite coproducts and colimits of countable chains. If G preserves colimits of countable chains, then $U : G\text{–alg} \to \mathcal{C}$ has a left adjoint.*

Proof. Let $X \in \mathcal{C}$. We apply proposition 3.3.22 with $F = G_X$. Then by proposition 3.3.21, U has a left adjoint. So it suffices to show that G_X preserves the colimit of

$$0 \to G_X 0 \to G_X^2 0 \to \ldots \to G_X^n 0 \to \ldots$$

Since G preserves colimits of countable chains

$$G(\operatorname{colim} G_X^i 0) \cong \operatorname{colim}(G \circ G_X^i 0)$$

and thus

$$\begin{aligned} X + G(\operatorname{colim} G_X^i 0) &\cong X + \operatorname{colim}(G \circ G_X^i 0) \\ &\cong \operatorname{colim}((X + G) \circ G_X^i 0) \\ &\cong \operatorname{colim} G_X^{i+1} 0 \end{aligned}$$

□

Chapter 4

Comparison untyped syntax

We suppose familiarity with the two approaches to (untyped) abstract syntax with variable binding. The first one that we are referring to as the *presheaf approach* originates in [FPT99]. The other approach that we are referring to as the *monadic approach* originates in [HM07]. For a short summary of these see section 1.1.

4.1 Base categories

The base category of the presheaf approach is $\mathcal{F} := [\mathbb{F}, \mathrm{Set}]$. We write U for the representable functor $\mathbb{F}(1, -)$ which is the inclusion $\mathbb{F} \hookrightarrow \mathrm{Set}$. This inclusion is full and faithful. The category $[\mathbb{F}, \mathrm{Set}]$ is monoidal with respect to the following monoidal product

$$X \otimes Y(n) = \int^{k \in \mathbb{F}} X(k) \times Y^k(n)$$

This monoidal product is also given by left Kan extension along U

$$X \otimes Y = \mathrm{Lan}_U(X) \circ Y$$

In this category the left Kan extension is given by a colimit formula

$$\mathrm{Lan}_U\, X(A) = \mathrm{colim}_{k \to A}\, X(k)$$

or by the following equivalent coend formula

$$\mathrm{Lan}_U\, X(A) = \int^{k \in \mathbb{F}} X(k) \times A^k$$

The unit for this monoidal product is U.

Lemma 4.1.24 *The category $\mathcal{F}$ with $\otimes$ and U as defined above is monoidal.*

Proof. See appendix B.1. □

Lemma 4.1.25 *Let $X, Y, Z \in \mathcal{F}$. Then*

$$(X \times Y) \otimes Z \to (X \otimes Z) \times (Y \otimes Z)$$

is a natural isomorphism.

Proof. See appendix B.2. □

Lemma 4.1.26 *Let $X, Y, Z \in \mathcal{F}$. Then*

$$(X + Y) \otimes Z \to (X \otimes Z) + (Y \otimes Z)$$

is a natural isomorphism.

Proof. See appendix B.3. □

The base category of the monadic approach is the functor category $\mathcal{E} := [\mathrm{Set}, \mathrm{Set}]$. It is monoidal with respect to composition of functors and the unit is given by the identity functor. The axioms of a monoidal category are obviously satisfied.

4.2 Monoidal functors

We have the following adjunction between $\mathcal{E}$ and $\mathcal{F}$. The right adjoint $k : \mathcal{E} \to \mathcal{F}$ is given by precomposition with U

$$k : F \mapsto F \circ U$$

and the left adjoint $\ell : \mathcal{F} \to \mathcal{E}$ is given by left Kan extension along U

$$\ell : X \mapsto \mathrm{Lan}_U\, X$$

The left Kan extension along U is also given by the following coend formula as explained in [ML98] (or see section 2.3)

$$\ell(X)(A) = \int^k X(k) \times A^k$$

Lemma 4.2.27 *The unit η of the above adjunction $\ell \dashv k$ is a natural isomorphism.*

Proof. The arrow $\eta_X : X \to \mathrm{Lan}_U(X) \circ U$ is an isomorphism for each $X \in [\mathbb{F}, \mathrm{Set}]$, see for example [Bor94a]. So $\eta : \mathrm{Id} \to k \circ \ell$ is a natural isomorphism. □

In the following we will need the following fact.

Lemma 4.2.28 *Let $X, Y \in \mathcal{F}$. Then*

$$\ell(X \times Y) \to \ell X \times \ell Y$$

is a natural isomorphism.

Proof. See appendix B.4. □

In the remaining part of this section we characterise k and ℓ as monoidal functors.

Definition 4.2.29 (monoidal functor) *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories. A* monoidal functor *$F : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation*

$$\phi_{A,B} : FA \bullet FB \to F(A \otimes B)$$

and a morphism

$$\phi : J \to FI$$

such that the following diagrams commute

$$\begin{tikzcd}
(FA \bullet FB) \bullet FC \arrow[r, "\alpha_{\mathcal{D}}"] \arrow[d, "\phi_{A,B} \bullet 1"'] & FA \bullet (FB \bullet FC) \arrow[d, "1 \bullet \phi_{B,C}"] \\
F(A \otimes B) \bullet FC \arrow[d, "\phi_{A \otimes B, C}"'] & FA \bullet F(B \otimes C) \arrow[d, "\phi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F\alpha_{\mathcal{C}}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

and

$$\begin{tikzcd}
FA \bullet J \arrow[r, "1 \bullet \phi"] \arrow[d, "\rho_{\mathcal{D}}"'] & FA \bullet FI \arrow[d, "\phi_{A,I}"] \\
FA & F(A \otimes I) \arrow[l, "F\rho_{\mathcal{C}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
J \bullet FB \arrow[r, "\phi \bullet 1"] \arrow[d, "\lambda_{\mathcal{D}}"'] & FI \bullet FB \arrow[d, "\phi_{I,B}"] \\
FB & F(I \otimes B) \arrow[l, "F\lambda_{\mathcal{C}}"]
\end{tikzcd}$$

The arrows ϕ are called structure morphisms.

Proposition 4.2.30 *The functor k is monoidal.*

Proof. The arrow $\psi : U \to k(\mathrm{Id}) = \mathrm{Id} \circ U = U$ is given by the identity on U. The natural transformation

$$\psi_{F,G} : kF \otimes kG \to k(F \circ G)$$

is explicitly

$$\mathrm{Lan}_U(F \circ U) \circ G \circ U \to F \circ G \circ U$$

It is given by the counit $\varepsilon : \ell k \to \mathrm{Id}$ of the adjunction $\ell \dashv k$

$$\varepsilon_F \circ G \circ U : \mathrm{Lan}_U(F \circ U) \circ G \circ U \to F \circ G \circ U$$

For the remaining verifications see appendix B.5. □

Proposition 4.2.31 *The functor ℓ is monoidal.*

Proof. The arrow $\phi_A : \mathrm{Id}(A) \to \ell(U)(A) = \int^n n \times A^n$ is given by the following mapping composed with the corresponding coprojection

$$a \quad \mapsto \quad (1, a) \in 1 \times A^1$$

Let $f : A \to B$. The naturality square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \ell U(A) \arrow[d] \\
B \arrow[r] & \ell(U)(B)
\end{tikzcd}$$

commutes because we have on elements

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, mapsto] & (1, a) \arrow[d, mapsto] \\
f(a) \arrow[r, mapsto] & (1, f(a))
\end{tikzcd}$$

Next we define a natural transformation

$$\phi_{X,Y} : \ell X \circ \ell Y \to \ell(X \otimes Y)$$

Let A be a set. We rewrite the domain and codomain using the coend notation

$$\begin{aligned}
\ell X(\ell Y(A)) &= \int^n X(n) \times \Big(\int^m Y(m) \times A^m\Big)^n \\
&= \int^n \int^{m_1} \dots \int^{m_n} X(n) \times Y(m_1) \times \dots \times Y(m_n) \times A^{\sum_{i=1}^n m_i} \\
\ell(X \otimes Y)(A) &= \int^r \int^n X(n) \times Y(r)^n \times A^r
\end{aligned}$$

By universal property of the coend it suffices to give for all $n, m_1, \dots, m_n \in \mathbb{F}$ an arrow

$$X(n) \times Y(m_1) \times \dots \times Y(m_n) \times A^{\sum_{i=1}^n m_i} \to \ell(X \otimes Y)(A)$$

satisfying the wedge condition. We take the following composite

$$\begin{array}{c}
X(n) \times Y(m_1) \times \dots \times Y(m_n) \times A^{\sum_{i=1}^n m_i} \\
\downarrow \\
X(n) \times Y(\sum_{i=1}^n m_i) \times \dots \times Y(\sum_{i=1}^n m_i) \times A^{\sum_{i=1}^n m_i} \\
\downarrow \\
X(n) \times \int^r Y(r)^n \times A^r \\
\downarrow \\
\int^n X(n) \times \int^r Y(r)^n \times A^r
\end{array}$$

For the remaining verifications see appendix B.6. □

Lemma 4.2.32 *Let $(F, \phi) : (\mathcal{A}, \otimes, I) \to (\mathcal{B}, \odot, J)$ and $(G, \psi) : (\mathcal{B}, \odot, J) \to (\mathcal{C}, \bullet, K)$ be two monoidal functors. Then their composite $G \circ F : \mathcal{A} \to \mathcal{C}$ is monoidal.*

Proof. The structural morphisms are given by the composites of ϕ and ψ:

$$K \xrightarrow{\psi} GJ \xrightarrow{G\phi} GFI$$

and

$$GFA \bullet GFA' \xrightarrow{\psi_{FA,FA'}} G(FA \odot FA') \xrightarrow{G\phi_{A,A'}} GF(A \otimes A')$$

We check the monoidal functor axioms.

$$\begin{tikzcd}
(GFA \bullet GFB) \bullet GFC \arrow[rr] \arrow[d] & & GFA \bullet (GFB \bullet GFC) \arrow[d] \\
G(FA \odot FB) \bullet GFC \arrow[r] \arrow[d] \arrow[dr, phantom, "II."] & G((FA \odot FB) \odot FC) \arrow[d] \arrow[ddl] \arrow[ur, phantom, "I."] & GFA \bullet G(FB \odot FC) \arrow[dl] \arrow[d] \\
GF(A \otimes B) \bullet GFC \arrow[d] & G(FA \odot (FB \odot FC)) \arrow[dr] \arrow[r, phantom, "III."] & GFA \bullet GF(B \otimes C) \arrow[d] \\
G(F(A \otimes B) \odot FC) \arrow[d] \arrow[rr, phantom, "IV."] & & G(FA \odot F(B \otimes C)) \arrow[d] \\
GF((A \otimes B) \otimes C) \arrow[rr] & & GF(A \otimes (B \otimes C))
\end{tikzcd}$$

Diagrams I. and IV. commute because G and F are monoidal, diagrams II. and III. are naturality squares of ψ.

$$\begin{tikzcd}
K \bullet GFA \arrow[r] \arrow[dd] \arrow[dr, phantom, "I."] & GJ \bullet GFA \arrow[r] \arrow[d] \arrow[dr, phantom, "II."] & GFI \bullet GFA \arrow[d] \\
& G(J \odot FA) \arrow[r] \arrow[dl] \arrow[d, phantom, "III."] & G(FI \odot FA) \arrow[d] \\
GFA & & GF(I \otimes A) \arrow[ll]
\end{tikzcd}$$

where the diagram I. commutes because G is monoidal and the diagram III. because F is monoidal. The square II. is a naturality square of ψ.

$$\begin{tikzcd}
GFA \bullet K \arrow[r] \arrow[dd] \arrow[dr, phantom, "I."] & GFA \bullet GJ \arrow[r] \arrow[d] \arrow[dr, phantom, "II."] & GFA \bullet GFI \arrow[d] \\
& G(FA \odot J) \arrow[r] \arrow[dl] \arrow[d, phantom, "III."] & G(FA \odot FI) \arrow[d] \\
GFA & & GF(A \otimes I) \arrow[ll]
\end{tikzcd}$$

where the diagram I. commutes because G is monoidal and the diagram III. because F is monoidal. The square II. is a naturality square of ψ. □

4.3 Monoidal natural transformations

In this section we are going to characterise the unit η and the counit ε as monoidal natural transformations

Definition 4.3.33 (monoidal natural transformation) *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories and (F, ϕ), (G, ψ) two monoidal funtors $(\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$. A* monoidal natural transformation *is a natural transformation $\alpha : F \to G$ such that the following diagrams commute for all $A, B \in \mathcal{C}$*

$$\begin{tikzcd} FA \bullet FB \arrow[r, "\alpha_A \bullet \alpha_B"] \arrow[d, "\phi_{A,B}"'] & GA \bullet GB \arrow[d, "\psi_{A,B}"] \\ F(A \otimes B) \arrow[r, "\alpha_{A\otimes B}"'] & G(A \otimes B) \end{tikzcd}$$

and

$$\begin{tikzcd} & J \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ FI \arrow[rr, "\alpha_I"'] & & GI \end{tikzcd}$$

Proposition 4.3.34 *The unit $\eta : \mathrm{Id}_{\mathcal{F}} \to k\ell$ of the adjunction $\ell \dashv k$ is a monoidal natural transformation.*

Proof. See appendix B.7. □

Proposition 4.3.35 *The counit $\varepsilon : \ell k \to \mathrm{Id}_{\mathcal{E}}$ of the adjunction $\ell \dashv k$ is a monoidal natural transformation.*

Proof. See appendix B.8. □

4.4 Monoidal adjunctions

Definition 4.4.36 (monoidal adjunction) *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories. An adjunction between the two monoidal functors $(F, \phi) : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$ and $(G, \psi) : (\mathcal{D}, \bullet, J) \to (\mathcal{C}, \otimes, I)$ is an adjunction such that its unit and counit are monoidal natural transformations.*

We showed above that the adjunction $\ell \dashv k$ is monoidal. We recall a theorem due to Brian Day about monoidal adjunctions.

Theorem 4.4.37 (Day) *Let $(F, \phi) \dashv (G, \psi) : (\mathcal{D}, \bullet, J) \to (\mathcal{C}, \otimes, I)$ be a monoidal adjunction. Then F is a strong monoidal functor, that is, its structure morphisms ϕ are isomorphisms.*

Proof. We have to show that $\phi : J \to F(I)$ has an inverse $F(I) \to J$. Since F is a left adjoint, it suffices to give its transpose $I \to G(J)$. Indeed we have $\psi : I \to G(J)$ because G is monoidal. Explicitly ϕ^{-1} is given by the composite

$$FI \xrightarrow{F\psi} FGJ \xrightarrow{\varepsilon_J} J$$

Consider the composite $\phi \circ \phi^{-1}$

$$\begin{tikzcd} FI \arrow[r, "F\psi"] \arrow[dr, "F\eta_I"'] \arrow[ddrr, bend right, "\mathrm{id}"'] & FGJ \arrow[r, "\varepsilon_J"] \arrow[d, "FG\phi"] & J \arrow[d, "\phi"] \\ & FGFI \arrow[r, "\varepsilon_{FI}"] \arrow[dr, "\varepsilon_{FI}"'] & FI \arrow[d, "\mathrm{id}"] \\ & & FI \end{tikzcd}$$

The upper left triangle commutes because η is monoidal, the square is a naturality square of ε and the left curved triangle is one of the triangle identities for the unit and the counit of the adjunction. Now consider the composite $\phi^{-1} \circ \phi$.

$$\begin{tikzcd}
& & J \arrow[dl, "\phi"'] \arrow[dd, "\mathrm{id}"] \\
& FI \arrow[dl, "F\psi"'] & \\
FGJ \arrow[rr, "\varepsilon_J"'] & & J
\end{tikzcd}$$

This diagram commutes because ε is monoidal.

Next we have to show that $\phi_{A,B} : FA \bullet FB \to F(A \otimes B)$ has an inverse $\phi^{-1}_{A,B} : F(A \otimes B) \to FA \bullet FB$. Since F is a left adjoint, it suffices to define its transpose $A \otimes B \to G(FA \bullet FB)$. We define it to be the following composite

$$A \otimes B \xrightarrow{\eta_A \otimes \eta_B} GFA \otimes GFB \xrightarrow{\psi_{FA,FB}} G(FA \bullet FB)$$

where η is the unit of the adjunction. Explicitly $\phi^{-1}_{A,B}$ is given by the composite

$$F(A \otimes B) \xrightarrow{F(\eta_A \otimes \eta_B)} F(GFA \otimes GFB) \xrightarrow{F\psi_{FA,FB}} FG(FA \bullet FB) \xrightarrow{\varepsilon_{FA \bullet FB}} FA \bullet FB$$

Consider the composite $\phi_{A,B} \circ \phi^{-1}_{A,B}$

$$\begin{tikzcd}
F(A \otimes B) \arrow[r, "F(\eta_A \otimes \eta_B)"] \arrow[drr, "F\eta_{A \otimes B}"] \arrow[ddrrr, bend right, "\mathrm{id}"'] & F(GFA \otimes GFB) \arrow[r, "F\psi_{FA,FB}"] & FG(FA \bullet FB) \arrow[r, "\varepsilon_{FA \bullet FB}"] \arrow[d, "FG\phi_{A,B}"] & FA \bullet FB \arrow[dd, "\phi_{A,B}"] \\
& & FGF(A \otimes B) \arrow[dr, "\varepsilon_{F(A \otimes B)}"] & \\
& & & F(A \otimes B)
\end{tikzcd}$$

The upper left triangle commutes because η is monoidal, the right square is a naturality square of ε and the bottom triangle is one of the triangle identities for the unit and the counit of the adjunction. Now consider the other composite $\phi^{-1}_{A,B} \circ \phi_{A,B}$

$$\begin{tikzcd}
FA \bullet FB \arrow[rr, "\phi_{A,B}"] \arrow[dr, "F\eta_A \bullet F\eta_B"] \arrow[dddrr, bend right, "\mathrm{id} \bullet \mathrm{id}"'] & & F(A \otimes B) \arrow[d, "F(\eta_A \otimes \eta_B)"] \\
& FGFA \bullet FGFB \arrow[r, "\phi_{GFA,GFB}"] \arrow[ddr, "\varepsilon_A \bullet \varepsilon_B"'] & F(GFA \otimes GFB) \arrow[d, "F\phi_{FA,FB}"] \\
& & FG(FA \bullet FB) \arrow[d, "\varepsilon_{FA \bullet FB}"] \\
& & FA \bullet FB
\end{tikzcd}$$

The upper square is a naturality square of ϕ, the bottom triangle commutes because ε is monoidal and the curved left triangle is one of the triangle identities for the unit and counit of the adjunction. $\square$

Applying this result to our adjunction $\ell \dashv k : \mathcal{E} \to \mathcal{F}$ means that we have natural isomorphisms

$$\phi_{X,Y} : \ell(X) \circ \ell(Y) \xrightarrow{\cong} \ell(X \otimes Y)$$

and

$$\phi : \mathrm{Id} \xrightarrow{\cong} \ell(U)$$

4.5 Category of monoids

Definition 4.5.38 (monoid) *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A* monoid *consists an object C of $\mathcal{C}$ together with two arrows*

$$m : C \otimes C \to C$$

and

$$e : I \to C$$

such that the following diagrams commute

$$\begin{tikzcd}
(C \otimes C) \otimes C \arrow[r, "\alpha"] \arrow[d, "m\otimes C"'] & C \otimes (C \otimes C) \arrow[r, "C\otimes m"] & C \otimes C \arrow[d, "m"] \\
C \otimes C \arrow[rr, "m"'] & & C
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \otimes C \arrow[r, "e\otimes C"] \arrow[dr, "\lambda"'] & C \otimes C \arrow[d, "m"] & C \otimes I \arrow[l, "C\otimes e"'] \arrow[dl, "\rho"] \\
& C &
\end{tikzcd}$$

A morphism of monoids $(C, m, e) \to (C', m', e')$ is an arrow $f : C \to C'$ such that the following diagrams commute

$$\begin{tikzcd}
C \otimes C \arrow[r, "f\otimes f"] \arrow[d, "m"'] & C' \otimes C' \arrow[d, "m'"] \\
C \arrow[r, "f"'] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
C \arrow[rr, "f"'] & & C'
\end{tikzcd}$$

Monoids in a monoidal category $(\mathcal{C}, \otimes, I)$ and morphisms of monoids form a category. We write $\mathrm{Mon}(\mathcal{C})$ for it.

A monoid in $(\mathcal{E}, \circ, \mathrm{Id})$ is a monad on Set. The category of monoids in $\mathcal{E}$ is the category of monads on Set.

We are going to extend the adjunction $\ell \dashv k : \mathcal{E} \to \mathcal{F}$ to the categories of monoids $\mathrm{Mon}(\mathcal{F})$ and $\mathrm{Mon}(\mathcal{E})$.

Proposition 4.5.39 *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories and $(F, \phi) : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$ a monoidal functor. It induces a functor $F : \mathrm{Mon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{D})$.*

Proof. Let (C, m, e) be a monoid of $\mathcal{C}$. We are going to show that FC is a monoid in $\mathcal{D}$. The multiplication is given by the composite

$$FC \bullet FC \xrightarrow{\phi_{C,C}} F(C \otimes C) \xrightarrow{Fm} FC$$

and the unit is given by

$$J \xrightarrow{\phi} FI \xrightarrow{Fe} FC$$

We check the commutativity of the three monoid axiom diagrams.

$$\begin{tikzcd}
J \bullet FC \arrow[r, "\phi \bullet FC"] \arrow[ddrr, bend right, "\lambda^{\mathcal{D}}_{FC}"'] & FI \bullet FC \arrow[r, "Fe \bullet FC"] \arrow[d, "\phi_{I,C}"'] & FC \bullet FC \arrow[d, "\phi_{C,C}"] & FC \bullet FI \arrow[l, "FC \bullet Fe"'] \arrow[d, "\phi_{C,I}"] & FC \bullet J \arrow[l, "FC \bullet \phi"'] \arrow[ddll, bend left, "\rho^{\mathcal{D}}_{FC}"] \\
& F(I \otimes C) \arrow[r, "F(e \otimes C)"] \arrow[dr, "F(\lambda^{\mathcal{C}}_{C})"'] & F(C \otimes C) \arrow[d] & F(C \otimes I) \arrow[l, "F(C \otimes e)"'] \arrow[dl, "F(\rho^{\mathcal{C}}_{D})"] & \\
& & FC & &
\end{tikzcd}$$

The two upper squares are naturality squares of ϕ, the bottom triangles are monoid axioms for C and the two remaining diagrams on the sides commute because F is monoidal.

$$\begin{tikzcd}
(FC \bullet FC) \bullet FC \arrow[r, "\alpha^{\mathcal{D}}"] \arrow[d, "\phi_{C,C} \bullet FC"'] & FC \bullet (FC \bullet FC) \arrow[r, "FC \bullet \phi_{C,C}"] & FC \bullet F(C \otimes C) \arrow[r, "FC \bullet Fm"] \arrow[d, "\phi_{C,C \otimes C}"'] & FC \bullet FC \arrow[d, "\phi_{C,C}"'] \\
F(C \otimes C) \bullet FC \arrow[r, "\phi_{C \otimes C,C}"] \arrow[d, "Fm \bullet FC"'] & F((C \otimes C) \otimes C) \arrow[r, "F\alpha^{\mathcal{C}}"] \arrow[d, "F(m \otimes C)"] & F(C \otimes (C \otimes C)) \arrow[r, "F(C \otimes m)"] & F(C \otimes C) \arrow[d, "Fm"] \\
FC \bullet FC \arrow[r, "\phi_{C,C}"'] & F(C \otimes C) \arrow[rr, "Fm"'] & & FC
\end{tikzcd}$$

The upper left rectangle commutes because F is monoidal, the two squares are naturality squares of ϕ and the bottom right rectangle is one of the monoid axioms for C.

Now let $f : (C, m, e) \to (C', m', e')$ be a morphism of monoids in $\mathcal{C}$. We have to show that $Ff : FC \to FC'$ is a morphism of monoids in $\mathcal{D}$.

$$\begin{tikzcd}
J \arrow[r, "\phi"] & FI \arrow[r, "Fe"] \arrow[dr, "Fe'"'] & FC \arrow[d, "Ff"] \\
& & FC'
\end{tikzcd}$$

commutes by one of the monoid morphism axioms.

$$\begin{tikzcd}
FC \bullet FC \arrow[r, "\phi_{C,C}"] \arrow[d, "Ff \bullet Ff"'] & F(C \otimes C) \arrow[r, "Fm"] \arrow[d, "F(f \otimes f)"'] & FC \arrow[d, "Ff"] \\
FC' \bullet FC' \arrow[r, "\phi_{C',C'}"'] & F(C' \otimes C') \arrow[r, "Fm'"'] & FC'
\end{tikzcd}$$

The left square is a naturality square of ϕ and the right square commutes by one of the monoid morphism axioms. $\square$

Corollary 4.5.40 *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories and let $F \dashv G$ be a monoidal adjunction between two monoidal functors $(F, \phi) : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$ and $(G, \psi) : (\mathcal{D}, \bullet, J) \to (\mathcal{C}, \otimes, I)$. Then there is an adjunction $F \dashv G : \mathrm{Mon}(\mathcal{D}) \to \mathrm{Mon}(\mathcal{C})$.*

Proof. We have the induced functors $F : \mathrm{Mon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{D})$ and $G : \mathrm{Mon}(\mathcal{D}) \to \mathrm{Mon}(\mathcal{C})$. It suffices to check that for all monoid (S, m, e) in $\mathcal{C}$ the unit $\eta_S : S \to GFS$ and for all $(R, m,', e')$ in $\mathcal{D}$ the counit $\varepsilon_R : FGR \to R$ is a monoid morphisms. We check the monoid morphism axioms.

$$\begin{tikzcd}
S \otimes S \arrow[r, "\eta_S \otimes \eta_S"] \arrow[d, "m"'] \arrow[ddr, "\eta_{S\otimes S}"] & GFS \otimes GFS \arrow[d, "\psi_{FS,FS}"] \\
S \arrow[ddr, "\eta_S"'] & G(FS \bullet FS) \arrow[d, "G(\phi_{S,S})"] \\
& GF(S \otimes S) \arrow[d, "GFm"] \\
& GFS
\end{tikzcd}$$

The top triangle commutes because $\eta : \mathrm{Id} \to GF$ is a monoidal natural transformation and the bottom square is a naturality square of η.

$$\begin{tikzcd}
I \arrow[r, "\psi"] \arrow[d, "\mathrm{Id}"'] & GJ \arrow[d, "G\phi"] \\
I \arrow[r, "\eta_I"'] & GF(I)
\end{tikzcd}$$

This square commutes because η is a monoidal natural transformation.

$$\begin{tikzcd}
FGR \bullet FGR \arrow[r, "\varepsilon_R \otimes \varepsilon_R"] \arrow[d, "\phi_{GR,GR}"'] & R \bullet R \arrow[d, "m"] \\
G(FR \otimes FR) \arrow[d, "F(\psi_{R,R})"'] & R \\
FG(R \bullet R) \arrow[uur, "\varepsilon_{R \bullet R}"] \arrow[d, "GFm"'] & \\
FGR \arrow[uur, "\varepsilon_R"'] &
\end{tikzcd}$$

The top triangle commutes because $\varepsilon : GF \to \mathrm{Id}$ is a monoidal natural transformation and the bottom square is a naturality square of ε.

$$\begin{tikzcd}
FI \arrow[d, "F\psi"'] & J \arrow[l, "\phi"'] \arrow[d, "\mathrm{Id}"] \\
FGJ \arrow[r, "\varepsilon_J"'] & J
\end{tikzcd}$$

This square commutes because ε is a monoidal natural transformation. $\square$

So we have the induced adjunction $\ell \dashv k : \mathrm{Mon}(\mathcal{E}) \to \mathrm{Mon}(\mathcal{F})$.

4.6 Category of modules

Definition 4.6.41 (module) *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and (R, m, e) be a monoid in this category. A* right R–module *is an object M of $\mathcal{C}$ together with an arrow $s : M \otimes R \to M$ called* action *such that the following diagrams commute*

$$\begin{tikzcd}
(M \otimes R) \otimes R \arrow[r, "\alpha"] \arrow[d, "s\otimes R"'] & M \otimes (R \otimes R) \arrow[r, "M\otimes m"] & M \otimes R \arrow[d, "s"] \\
M \otimes R \arrow[rr, "s"'] & & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \otimes I \arrow[r, "M\otimes e"] \arrow[dr, "\rho^{-1}"'] & M \otimes R \arrow[d, "s"] \\
& M
\end{tikzcd}$$

One can define a left R–module in a similar way, but we are only concerned with right modules in this work, so we drop the word right and call a right module simply a module.

Definition 4.6.42 (morphism of R–modules) *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and (R, m, e) be a monoid in this category. Let (M, s) and (N, t) be two R–modules. A morphism of R–modules from (M, s) to (N, t) is an arrow $\tau : M \to N$ of $\mathcal{C}$ such that the following diagram commutes*

$$\begin{tikzcd}
M \otimes R \arrow[r, "\tau\otimes R"] \arrow[d, "s"'] & N \otimes R \arrow[d, "t"] \\
M \arrow[r, "\tau"'] & N
\end{tikzcd}$$

Notation 4.6.43 *R–modules and morphisms of R–modules form a category, written* $\mathrm{Mod}_R(\mathcal{C})$.

Definition 4.6.44 (large category of modules) *We also have a large category of modules* $\mathrm{Mod}(\mathcal{C})$*. An object (R, M) consists of a monoid R and an R–module M. An arrow (f, τ) from (R, M) to (S, N) consists of a morphism of monoids $f : R \to S$ and a morphism of R–modules $M \to f^*N$. The R–module f^*N is based on the S–module N and its action is given by the composite*

$$N \otimes R \xrightarrow{N\otimes f} N \otimes S \to N$$

Proposition 4.6.45 *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories and $(F, \phi) : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$ a monoidal functor. Let (S, m, e) be a monoid in $(\mathcal{C}, \otimes, I)$. It induces a functor $F : \mathrm{Mod}_S(\mathcal{C}) \to \mathrm{Mod}_{FS}(\mathcal{D})$.*

Proof. Let (M, r) be an S–module. Then FM is an FS–module. The action is given by the composite

$$FM \bullet FS \xrightarrow{\phi_{M,S}} F(M \otimes S) \xrightarrow{Fr} FM$$

It satisfies the module axioms:

$$\begin{tikzcd}
(FM \bullet FS) \bullet FS \arrow[r, "\alpha^{\mathcal{D}}"] \arrow[d, "\phi_{M,S} \bullet FS"'] & FM \bullet (FS \bullet FS) \arrow[r, "FM \bullet Fm"] & FM \bullet F(S \otimes S) \arrow[r, "FM \bullet Fm"] \arrow[d, "\phi_{M,S\otimes S}"'] & FM \bullet FS \arrow[d, "\phi_{M,S}"] \\
F(M \otimes S) \bullet FS \arrow[r, "\phi_{M\otimes S,S}"] \arrow[d, "Fr \bullet FS"'] & F((M \otimes S) \otimes S) \arrow[r, "F\alpha^{\mathcal{C}}"] \arrow[d, "F(r\otimes S)"] & F(M \otimes (S \otimes S)) \arrow[r, "F(S\otimes m)"] & F(M \otimes S) \arrow[d, "Fr"] \\
FM \bullet FS \arrow[r, "\phi_{M,S}"'] & F(M \otimes S) \arrow[rr, "Fr"'] & & FM
\end{tikzcd}$$

The upper right rectangle commutes because F is monoidal, the upper right and bottom left squares are naturality squares of ϕ and the bottom right rectangle is one of the module axioms for M.

$$\begin{tikzcd}
FM \bullet J \arrow[r, "FM\bullet\phi"] \arrow[rrdd, bend right, "\rho^{-1}"'] & FM \bullet FI \arrow[r, "FM\bullet Fe"] \arrow[d, "\phi_{M,I}"'] & FM \bullet FS \arrow[d, "\phi_{M,S}"] \\
& F(M \otimes I) \arrow[r, "F(M\otimes e)"] \arrow[dr, "F\rho^{-1}"'] & F(M \otimes S) \arrow[d, "Fr"] \\
& & FM
\end{tikzcd}$$

The upper right square is a naturality square of ϕ, the bottom triangle is one of the module axioms for M and the remaining part on the left commutes because F is monoidal.

Now let (M, r) and (N, q) be two S–modules and τ a module morphism $M \to N$. Then $F\tau$ is a morphism of FS–modules. We check the module morphism axiom

$$\begin{tikzcd}
FM \bullet FS \arrow[r, "F\tau\bullet FS"] \arrow[d, "\phi_{M,S}"'] & FN \bullet FS \arrow[d, "\phi_{N,S}"] \\
F(M \otimes S) \arrow[r, "F(\tau\otimes S)"] \arrow[d, "Fr"'] & F(N \otimes S) \arrow[d, "Fq"] \\
FM \arrow[r, "F\tau"'] & FN
\end{tikzcd}$$

The upper square is a naturality square of ϕ and the bottom square is the module morphism axiom for τ. $\square$

Corollary 4.6.46 *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories and let $F \dashv G$ be a monoidal adjunction between two monoidal functors $(F, \phi) : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$ and $(G, \psi) : (\mathcal{D}, \bullet, J) \to (\mathcal{C}, \otimes, I)$. Then there is an adjunction $F \dashv G : \mathrm{Mod}(\mathcal{D}) \to \mathrm{Mod}(\mathcal{C})$.*

Proof. By the previous proposition we have $F : \mathrm{Mod}_{\mathcal{C}} \to \mathrm{Mod}(\mathcal{D})$, $(S, P) \mapsto (FS, FP)$ and $G : \mathrm{Mod}(\mathcal{D}) \to \mathrm{Mod}(\mathcal{C})$, $(R, M) \mapsto (GR, GM)$. We are going to check that the unit η and the counit ε are module morphisms. Let (S, m, e) be a monoid in $\mathcal{C}$ and (P, r) be an S–module. To show that the arrow $\eta_P : P \to GFP$ is also a morphism $(S, P) \to (GFS, GFP)$ in $\mathrm{Mod}(\mathcal{C})$ we have to check the commutativity of the following diagram

$$\begin{tikzcd}
P \otimes S \arrow[r, "\eta_P \otimes S"] \arrow[dr, "\eta_P \otimes \eta_S"'] \arrow[dddd, "r"'] \arrow[dddr, bend right, "\eta_{P \otimes S}"'] & GFP \otimes S \arrow[d, "GFP \otimes \eta_S"] \\
& GFP \otimes GFS \arrow[d, "\psi_{FP,FS}"] \\
& G(FP \bullet FS) \arrow[d, "G(\phi_{P,S})"] \\
& GF(P \otimes S) \arrow[d, "GFr"] \\
P \arrow[r, "\eta_P"'] & GFP
\end{tikzcd}$$

The bottom square is a naturality square for η, the diagram in the middle commutes because η is monoidal and the top triangle commutes obviously. Now let (R, m', e') be a monoid in $\mathcal{D}$. We have to check the commutativity of the following diagram

$$\begin{tikzcd}
FGR \bullet FGR \arrow[r, "\varepsilon_R \bullet FGR"] \arrow[d, "\phi_{GR,GR}"'] \arrow[dr, "\varepsilon_R \bullet \varepsilon_R"'] & R \bullet FGR \arrow[d, "R \bullet \varepsilon_R"] \\
F(GR \otimes GR) \arrow[d, "F(\psi_{R,R})"'] & R \bullet R \arrow[dd, "m'"] \\
FG(R \bullet R) \arrow[ur, "\varepsilon_{R \bullet R}"'] \arrow[d, "FGm'"'] & \\
FGR \arrow[r, "\varepsilon_R"'] & R
\end{tikzcd}$$

The bottom square is a naturality square of ε, the diagram in the middle commutes because ε is monoidal and the top triangle commutes obviously. $\square$

So we have the induced adjunction $\ell \dashv k : \mathrm{Mod}(\mathcal{E}) \to \mathrm{Mod}(\mathcal{F})$.

4.7 Strengths

The definitions of this section and the following come mainly from [Fio08].

Definition 4.7.47 (right $\mathcal{V}$–action) *Let $(\mathcal{V}, \otimes, I)$ be a monoidal category and $\mathcal{C}$ a category. A $\mathcal{V}$–*action *on $\mathcal{C}$ is a bifunctor*

$$* : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$$

together with a natural isomorphism whose components are

$$a_{A,B,C} : (A * B) * C \to A * (B \otimes C)$$

and another natural isomorphism whose components are

$$r_A : A * I \to A$$

such that

$$\begin{tikzcd}
((A*B)*C)*D \arrow[r] \arrow[d] & (A*B)*(C\otimes D) \arrow[dd] \\
(A*(B\otimes C))*D \arrow[d] & \\
A*((B\otimes C)\otimes D) \arrow[r] & A*(B\otimes(C\otimes D))
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A*B)*I \arrow[r, "a_{A,B,I}"] \arrow[d, "r_{A*B}"'] & A*(B\otimes I) \arrow[dl, "\rho_B\otimes I"] \\
A*B &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A*I)*B \arrow[r, "a_{A,I,B}"] \arrow[d, "r_A\otimes B"'] & A*(I\otimes B) \arrow[dl, "A*\lambda_B"] \\
A*B &
\end{tikzcd}$$

commute.

Lemma 4.7.48 *Let $(\mathcal{V},\otimes,I)$ be a monoidal category. Then the category $I\downarrow\mathcal{V}$ of elements over I is monoidal too.*

Proof. We write for the monoidal category isomorphisms of $\mathcal{V}$

$$\alpha_{A,B,C} : (A\otimes B)\otimes C \to A\otimes(B\otimes C)$$

$$\lambda_A : I\otimes A \to A$$

and

$$\rho_A : I\otimes A \to A$$

Now we show that they are as well the monoidal category isomorphisms of $I\downarrow\mathcal{V}$. Let $a : I\to A$ be an object of $I\downarrow\mathcal{V}$.

$$\begin{tikzcd}
& & I \arrow[dl, "\lambda_A^{-1}"'] \arrow[ddr, "a"] & \\
& I\otimes I \arrow[dl, "\mathrm{id}\otimes a"'] & & \\
I\otimes A \arrow[rrr, "\lambda_A"'] & & & A
\end{tikzcd}$$

commutes since it is one of the monoidal category axioms of $\mathcal{V}$.

$$\begin{tikzcd}
& & I \arrow[dl, "\lambda_A^{-1}"'] \arrow[ddr, "a"] & \\
& I\otimes I \arrow[dl, "a\otimes\mathrm{id}"'] & & \\
A\otimes I \arrow[rrr, "\rho_A"'] & & & A
\end{tikzcd}$$

commutes since $\lambda_I = \rho_I$ and so it is one of the monoidal category axioms of $\mathcal{V}$. Let $b : I \to B$ and $c : I \to C$ be two objects of $I \downarrow \mathcal{V}$.

$$\begin{tikzcd}
 & I \arrow[d, "\lambda_I^{-1}"'] & \\
 & I \otimes I \arrow[dl, "\mathrm{id} \otimes \lambda_I^{-1}"'] \arrow[dr, "\lambda_I^{-1} \otimes \mathrm{id}"] & \\
I \otimes (I \otimes I) \arrow[rr, "\alpha_{I,I,I}"'] \arrow[d, "a \otimes (b \otimes c)"'] & & (I \otimes I) \otimes I \arrow[d, "(a \otimes b) \otimes c"] \\
A \otimes (B \otimes C) \arrow[rr, "\alpha_{A,B,C}"'] & & A \otimes (B \otimes C)
\end{tikzcd}$$

The top triagle is one of the monoidal category axioms and the bottom square commutes by naturality for α. The monoidal category axioms for $I \downarrow \mathcal{V}$ are satisfied since α, λ and ρ satisfy the axioms for $\mathcal{V}$. $\square$

We are interested in two cases. The category $U \downarrow \mathcal{F}$ is monoidal and acts on $\mathcal{F}$ by

$$(X, U \to Y) \quad \mapsto \quad X \otimes Y$$

The category $\mathrm{Id}_{\mathrm{Set}} \downarrow \mathcal{E}$ id monoidal and acts on $\mathcal{E}$ by

$$(F, \mathrm{Id} \to G) \quad \mapsto \quad F \circ G$$

The action axioms are satisfied in both cases since the categories $\mathcal{F}$ and $\mathcal{E}$ are monoidal. To lighten up notation we are going to write simply $\mathrm{Id} \downarrow \mathcal{E}$ for $\mathrm{Id}_{\mathrm{Set}} \downarrow \mathcal{E}$.

Definition 4.7.49 ($\mathcal{V}$–strength) *Let $(\mathcal{V}, \otimes, I)$ be a monoidal category, $\mathcal{C}$ a category provided with a $\mathcal{V}$–action $*$ and $\mathcal{C}'$ a category provided with a $\mathcal{V}$–action $*'$. Let F be a functor $\mathcal{C} \to \mathcal{C}'$. We say that F is $\mathcal{V}$–*strong *if there exists a natural transformation f whose components are*

$$f_{A,B} : FA *' B \to F(A * B)$$

such that

$$\begin{tikzcd}
(FA *' B) *' C \arrow[r, "a'_{FA,B,C}"] \arrow[d, "f_{A,B} * C"'] & FA *' (B \otimes C) \arrow[dd, "f_{A,B \otimes C}"] \\
F(A * B) *' C \arrow[d, "f_{A * B, C}"'] & \\
F((A * B) * C) \arrow[r, "Fa_{A,B,C}"'] & F(A * (B \otimes C))
\end{tikzcd}
\qquad
\begin{tikzcd}
FA *' I \arrow[r, "f_{A,I}"] \arrow[d, "r_{FA}"'] & F(A * I) \arrow[dl, "Fr_A"] \\
FA &
\end{tikzcd}$$

*commute. We call f a $\mathcal{V}$–*strength*.*

We are interested in the following two cases of $U \downarrow \mathcal{F}$–strong and $\mathrm{Id} \downarrow \mathcal{E}$–strong endofunctors.

Proposition 4.7.50 *The endofunctor δ on $\mathcal{F}$ is $U \downarrow \mathcal{F}$–strong.*

Proof. First we recall the construction of the $U \downarrow \mathcal{F}$–strength of δ

$$s_{X,Y} : \delta X \otimes Y \to \delta(X \otimes Y)$$

where $X \in \mathcal{F}$ and $(Y, y) \in U \downarrow \mathcal{F}$. We rewrite the domain and codomain at the component $n \in \mathbb{F}$ by using the coend notation

$$\int^{m} X(m+1) \times Y(n)^{m} \to \int^{r} X(r) \times Y(n+1)^{r}$$

By universal property of the coend it suffices to give a collection of arrows

$$X(m+1) \times Y(n)^{m} \to \int^{r} X(r) \times Y(n+1)^{r}$$

for all $m \in \mathbb{F}$ that satisfies the wedge condition. We take the following composite

$$\begin{array}{c} X(m+1) \times Y(n)^{m} \\ \downarrow \\ X(m+1) \times Y(n+1)^{m} \times 1 \\ \downarrow \\ X(m+1) \times Y(n+1)^{m} \times Y(n+1) \\ \downarrow \\ \int^{r} X(r) \times Y(n+1)^{r} \end{array}$$

where we used the transpose $\overline{y} : 1 \to Y^{U}$ of the arrow $y : U \to Y$. For the remaining verifications see appendix B.9. $\square$

Proposition 4.7.51 *The endofunctor* $(-)'$ *on* $\mathcal{E}$ *is* $\mathrm{Id} \downarrow \mathcal{E}$*–strong.*

Proof. We recall the $\mathrm{Id} \downarrow \mathcal{E}$–strength for $F \in \mathcal{E}$ and $(G, g) \in \mathrm{Id} \downarrow \mathcal{E}$

$$\sigma_{F,G} : F' \circ G \to (F \circ G)'$$

its component at $A \in \mathrm{Set}$ is

$$\sigma_{F,G,A} : F(GA+1) \to FG(A+1)$$

it is given by the arrow

$$F([Gi_{l}, g_{A+1} \circ i_{r}])$$

where $i_l : A \to A+1$ and $i_r : 1 \to A+1$. For the remaining verification see appendix B.10. $\square$

Lemma 4.7.52 *Let* $(\mathcal{V}, \otimes, I)$ *be a monoidal category,* $\mathcal{C}$ *a category provided with a* $\mathcal{V}$*–action* $*$*,* $\mathcal{C}'$ *a category provided with a* $\mathcal{V}$*–action* $*'$ *and* $\mathcal{C}''$ *a category provided with a* $\mathcal{V}$*–action* $*''$*. Let* (F, f) *and* (F', f') *be two composable* $\mathcal{V}$*–strong functors* $\mathcal{C} \to \mathcal{C}'$ *and* $\mathcal{C}' \to \mathcal{C}''$*. Their composite* $F'F$ *is as well* $\mathcal{V}$*–strong.*

Proof. The $\mathcal{V}$–strength of $F'F$ is given by the following composite

$$F'f_{A,B} \circ f'_{FA,B} : F'FA *'' B \to F'(FA *' B) \to F'F(A * B)$$

Let us check the $\mathcal{V}$–strong functor axioms.

$$\begin{tikzcd}
(F'FA *'' B) *'' C \arrow[rr, "a''"] \arrow[d, "f' *'' C"'] & & F'FA *'' (B \otimes C) \arrow[dd, "f'"] \\
F'(FA *' B) *'' C \arrow[d, "F'f *'' C"'] \arrow[dr, "f'"] \arrow[dr, phantom, "III." near start, swap] & I. & \\
F'F(A * B) *'' C \arrow[d, "f'"'] & F'((FA *' B) *' C) \arrow[r, "Fa'"] \arrow[dl, "F'(f *' C)"] & F'(FA *' (B \otimes C)) \arrow[dd, "F'f"] \\
F'(F(A * B) *' C) \arrow[d, "F'f"'] & II. & \\
F'F((A * B) * C) \arrow[rr, "F'Fa"'] & & F'F(A * (B \otimes C))
\end{tikzcd}$$

The diagrams I. and II. commute because f' and f are $\mathcal{V}$–strengths and the diagram III. is a naturality square of f'.

$$\begin{tikzcd}
F'FA *'' I \arrow[r, "f'"] \arrow[dr, "r''"'] & F'(FA *' I) \arrow[r, "F'f"] \arrow[d, "F'r'"'] & F'F(A * I) \arrow[dl, "F'Fr"] \\
& F'FA &
\end{tikzcd}$$

The two triangles commute because f and f' are $\mathcal{V}$–strengths. $\square$

Applying this result to δ, we have that δ^n is $U \downarrow \mathcal{F}$–strong for all $n \in \mathbb{N}$. In a similar way, $(-)^{(n)}$ is $\mathrm{Id} \downarrow \mathcal{E}$–strong for all $n \in \mathbb{N}$.

Lemma 4.7.53 *Let $(\mathcal{V}, \otimes, I)$ be a monoidal category, $\mathcal{C}$ and $\mathcal{C}'$ two categories with finite products, $\mathcal{C}$ provided with a $\mathcal{V}$–action $*$ and $\mathcal{C}'$ provided with a $\mathcal{V}$–action $*'$ such that*

$$(A \times B) *' C \cong (A *' C) \times (B *' C)$$

Let (F, f) and (F', f') be two $\mathcal{V}$–strong functors $\mathcal{C} \to \mathcal{C}'$. Their product $F \times F'$ is $\mathcal{V}$–strong.

Proof. We take

$$(f \times f')_{A,B} : (FA \times F'A) *' B \cong (FA *' B) \times (F'A *' B) \xrightarrow{f_{A,B} \times f'_{A,B}} F(A * B) \times F'(A * B)$$

to be the $\mathcal{V}$–strength of $F \times F'$. The commutativity of the $\mathcal{V}$–strong functor axiom follows directly from the commutativity of the respective axioms of f and f'. $\square$

Given an arity $(n_1, \ldots, n_p)$ the associated endofunctor $\delta^{n_1} \times \ldots \times \delta^{n_p}$ on $\mathcal{F}$ is $U \downarrow \mathcal{F}$–strong since $(X \times Y) \otimes Z \cong (X \otimes Z) \times (Y \otimes Z)$ by proposition 4.1.25. The associated endofunctor $(-)^{(n_1)} \times \ldots \times (-)^{(n_p)}$ on $\mathcal{E}$ is $\mathrm{Id} \downarrow \mathcal{E}$–strong since $(F \times G) \circ H = (F \circ H) \times (G \circ H)$.

Lemma 4.7.54 *Let $(\mathcal{V}, \otimes, I)$ be a monoidal category, $\mathcal{C}$ and $\mathcal{C}'$ two categories with coproducts, $\mathcal{C}$ provided with a $\mathcal{V}$–action $*$ and $\mathcal{C}'$ provided with a $\mathcal{V}$–action $*'$ such that*

$$(A + B) *' C \cong (A *' C) + (B *' C)$$

Let (F, f) and (F', f') be two $\mathcal{V}$–strong endofunctors on $\mathcal{C}$. Their sum $F + F'$ is $\mathcal{V}$–strong.

Proof. We take

$$(f + f')_{A,B} : (FA + F'A) * B \cong (FA * B) + (F'A * B) \xrightarrow{f_{A,B}+f'_{A,B}} F(A * B) + F'(A * B)$$

to be the $\mathcal{V}$–strength of $F + F'$. The commutativity of the $\mathcal{V}$–strong functor axiom follows directly from the commutativity of the respective axioms of f and f'. □

Given a signature, that is, a collection of arities, where the i–th arity is of the form

$$(n_{i,1}, \dots, n_{i,p_i})$$

the induced binding signature functor

$$\sum_{i \in I} \prod_{j=1}^{p_i} \delta^{n_{i,j}} : \mathcal{F} \to \mathcal{F}$$

is $U \downarrow \mathcal{F}$–strong.

4.8 Morphisms of strengths

Definition 4.8.55 (morphism of actions) *Let $(\mathcal{V}, \otimes, I)$ and $(\mathcal{V}', \bullet, J)$ be two monoidal categories and $\mathcal{C}$ and $\mathcal{D}$ two categories such that $\mathcal{V}$ acts on $\mathcal{C}$ by $*$ and $\mathcal{V}'$ acts on $\mathcal{D}$ by $\star$. A* morphism of actions *is a triple $((M, \phi), N, \chi)$ consisting of a monoidal functor $(M, \phi) : \mathcal{V} \to \mathcal{V}'$, a functor $N : \mathcal{C} \to \mathcal{D}$ and a natural transformation whose components are*

$$\chi_{A,B} : NA \star MB \to N(A * B)$$

such that

$$\begin{tikzcd}
(NA \star MB) \star MC \arrow[r, "a_{NA,MB,MC}"] \arrow[d, "\chi_{A,B} \star MC"'] & NA \star (MB \bullet MC) \arrow[d, "NA \star \phi_{B,C}"] \\
N(A * B) \star MC \arrow[d, "\chi_{A*B,C}"'] & NA \star M(B \otimes C) \arrow[d, "\chi_{A,B\otimes C}"] \\
N((A * B) * C) \arrow[r, "Na_{A,B,C}"'] & N(A * (B \otimes C))
\end{tikzcd}$$

and

$$\begin{tikzcd}
NA \star J \arrow[r, "NA \star \phi"] \arrow[d, "r_{NA}"'] & NA \star MI \arrow[d, "\chi_{A,I}"] \\
NA & N(A * I) \arrow[l, "Nr_A"]
\end{tikzcd}$$

commute.

Lemma 4.8.56 *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \bullet, J)$ be two monoidal categories and $(F, \phi) : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \bullet, J)$ be a monoidal functor. Then F induces a monoidal functor $I \downarrow \mathcal{C} \to J \downarrow \mathcal{D}$.*

Proof. By lemma 4.7.48 the categories $I \downarrow \mathcal{C}$ and $J \downarrow \mathcal{D}$ are monoidal as well. We show that the structure morphisms ϕ of F are morphisms in $J \downarrow \mathcal{D}$. Let $a : I \to A$ and $b : I \to B$ two objects of $I \downarrow \mathcal{C}$. We check that the following diagrams commute

$$\begin{tikzcd}
& J \arrow[dl] \arrow[dr] & \\
FA \bullet FB \arrow[rr] & & F(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
& J \arrow[dl] \arrow[dr] & \\
J \arrow[rr] & & FI
\end{tikzcd}$$

The left diagram is explicitly

$$\begin{tikzcd}
& J \arrow[dl] \arrow[dr, "\phi"] & \\
J \bullet J \arrow[ur, bend left, "\lambda_J"] \arrow[r, "J\bullet\phi"] \arrow[d, "\phi\bullet\phi"'] & J \bullet FI \arrow[r, "\lambda_{FI}"] \arrow[dl, "\phi\bullet FI"] & FI \arrow[d] \\
FI \bullet FI \arrow[rr, "\phi_{I,I}"'] \arrow[d, "Fa\bullet Fb"'] & & F(I \otimes I) \arrow[u, bend right, "F\lambda_I"'] \arrow[d, "F(a\otimes b)"] \\
FA \bullet FB \arrow[rr, "\phi_{A,B}"'] & & F(A \otimes B)
\end{tikzcd}$$

The bottom square is a naturality square of ϕ the right–hand side square with the curved inverse arrow is a monoidal functor axiom of F. The left triangle commutes trivially and the top triangle with the curved inverse arrow is a naturality square of λ of $\mathcal{D}$.

The right diagram commutes trivially.

Moreover the axioms for the monoidal functor $I \downarrow \mathcal{C} \to J \downarrow \mathcal{D}$ are satisfied since this is the case for the ones for $F : \mathcal{C} \to \mathcal{D}$. □

So the monoidal functor $(\ell, \phi) : \mathcal{F} \to \mathcal{E}$ induces a monoidal functor $(\ell, \phi) : U \downarrow \mathcal{F} \to \mathrm{Id} \downarrow \mathcal{E}$. Therefore $((\ell, \phi), \ell, \phi)$ is a morphism of actions. The axioms are satisfied by the monoidal functor axioms. In a similar way $((k, \psi), k, \psi)$ is a morphism of actions.

Definition 4.8.57 (morphism of strengths) *Suppose given the following*

- *monoidal category $(\mathcal{V}, \otimes, I)$*
- *monoidal category $(\mathcal{U}, \bullet, J)$*
- *category $\mathcal{C}$ provided with a $\mathcal{V}$–action $*$*
- *category $\mathcal{C}'$ provided with a $\mathcal{V}$–action $*'$*
- *category $\mathcal{D}$ provided with a $\mathcal{U}$–action $\star$*
- *category $\mathcal{D}'$ provided with a $\mathcal{U}$–action $\star'$*
- *$\mathcal{V}$–strong functor $(F, f) : (\mathcal{C}, *) \to (\mathcal{C}', *')$*
- *$\mathcal{U}$–strong functor $(G, g) : (\mathcal{D}, \star) \to (\mathcal{D}', \star')$*

- *morphism of actions* $((M,\phi),N,\chi)$ *from* $(\mathcal{C},*)$ *to* $(\mathcal{D},\star)$
- *morphism of actions* $((M,\phi),N',\chi')$ *from* $(\mathcal{C}',*')$ *to* $(\mathcal{D}',\star')$

A morphism of strengths *is a natural transformation* $\alpha : GN \to N'F$ *such that*

$$\begin{tikzcd} GNA \star' MB \arrow[r, "\alpha_{A \star' MB}"] \arrow[d, "g_{NA,MB}"'] & N'FA \star' MB \arrow[d, "\chi'_{FA,B}"] \\ G(NA \star MB) \arrow[d, "G\chi_{A*B}"'] & N'(FA *' B) \arrow[d, "N'f_{A,B}"] \\ GN(A*B) \arrow[r, "\alpha_{A*B}"'] & N'F(A*B) \end{tikzcd}$$

commutes.

In the following we define two morphisms of strengths.

Definition 4.8.58 *We define a morphism of strengths* $\alpha_1^{-1} : (\ell(-))' \to \ell(\delta -)$.

Let $X \in \mathcal{F}$. *At first let us construct the arrow* $\alpha_{1,X}^{-1} : (\ell X)' \to \ell(\delta X)$ *for all* $A \in \mathrm{Set}$. *Recall that*

$$(\ell X)'(A) = \int^n X(n) \times (A+1)^n$$

and

$$\ell(\delta X)(A) = \int^m X(m+1) \times A^m$$

To define an arrow $(\ell X)'(A) \to \ell(\delta X)(A)$, *by universal property of the coend it suffices to give an arrow for all* $n \in \mathbb{F}$

$$X(n) \times (A+1)^n \to \int^m X(m+1) \times A^m$$

that satisfies the wedge condition. We take the following composite

$$X(n) \times (A+1)^n \to X(m+1) \times A^m \to \int^m X(m+1) \times A^m$$

where we use the following maps. Given a map $f : n \to A+1$, *we have a subset* m *of* n *defined as the set* $\{i \in n \text{ s.t. } f(i) \in A\}$. *This defines obviously a map* $\overline{f} : m \to A$. *Moreover we can define a map* $\phi : n \to m+1$ *by*

$$i \mapsto \begin{cases} i & \text{if } i \in m \\ 1 & \text{if } i \notin m \end{cases}$$

For the remaining verifications see appendix B.11.

Definition 4.8.59 *We define a morphism of strengths* $\beta_1 : \delta(k-) \to k((-)')$.

Let $F \in \mathcal{E}$ *and* $n \in \mathbb{F}$, *then we have explicitly*

$$\begin{matrix} (F \circ U)(n+1) \\ \cong F(n+1) \end{matrix} \to \begin{matrix} F' \circ U(n) \\ \cong F(n+1) \end{matrix}$$

which is given by the identity on $F(n+1)$. *For the remaining verifications see appendix B.12.*

Lemma 4.8.60 *Suppose given the following*

- *monoidal category* $(\mathcal{V}, \otimes, I)$
- *monoidal category* $(\mathcal{U}, \bullet, J)$
- *category* $\mathcal{C}$ *provided with a* $\mathcal{V}$*–action* $*$
- *category* $\mathcal{C}'$ *provided with a* $\mathcal{V}$*–action* $*'$
- *category* $\mathcal{C}''$ *provided with a* $\mathcal{V}$*–action* $*''$
- *category* $\mathcal{D}$ *provided with a* $\mathcal{U}$*–action* $\star$
- *category* $\mathcal{D}'$ *provided with a* $\mathcal{U}$*–action* $\star'$
- *category* $\mathcal{D}''$ *provided with a* $\mathcal{U}$*–action* $\star''$
- $\mathcal{V}$*–strong functor* $(F, f) : (\mathcal{C}, *) \to (\mathcal{C}', *')$
- $\mathcal{V}$*–strong functor* $(F', f') : (\mathcal{C}', *') \to (\mathcal{C}'', *'')$
- $\mathcal{U}$*–strong functor* $(G, g) : (\mathcal{D}, \star) \to (\mathcal{D}', \star')$
- $\mathcal{U}$*–strong functor* $(G', g') : (\mathcal{D}', \star') \to (\mathcal{D}'', \star'')$
- *morphism of actions* $((M, \phi), N, \chi)$ *from* $(\mathcal{C}, *)$ *to* $(\mathcal{D}, \star)$
- *morphism of actions* $((M, \phi), N', \chi')$ *from* $(\mathcal{C}', *')$ *to* $(\mathcal{D}', \star')$
- *morphism of actions* $((M, \phi), N'', \chi'')$ *from* $(\mathcal{C}'', *'')$ *to* $(\mathcal{D}'', \star'')$
- *morphism of strengths* $\alpha : GN \to N'F$
- *morphism of strengths* $\beta : G'N' \to N''F'$

Then the composite $\beta F \circ G'\alpha : G'GN \to G'N'F \to N''F'F$ *is a morphism of strengths.*

Proof. Recall that by lemma 4.7.52, $F'F$ and $G'G$ are $\mathcal{V}$–strong and $\mathcal{U}$–strong respectively. Their strengths are given by

$$F'f_{A,B} \circ f'_{FA,B} : F'FA *'' B \to F'(FA *' B) \to F'F(A * B)$$

$$G'g_{C,D} \circ g'_{GC,D} : G'GC \star'' D \to G'(GC \star' D) \to G'G(C \star D)$$

Now let us check that $\beta F \circ G'\alpha$ is a morphism of strengths.

$$\begin{tikzcd}
G'GNA \star'' MB \arrow[r, "G'\alpha_A \star'' MB"] \arrow[d, "g'_{GNA,MB}"'] & G'N'FA \star'' MB \arrow[r, "\beta_{FA} \star'' MB"] \arrow[d, "g'_{N'FA,MB}"'] & N''F'FA \star'' MB \arrow[d, "\chi''_{F'FA,B}"] \\
G'(GNA \star' MB) \arrow[r, "G'(\alpha_A \star' MB)"] \arrow[d, "G'g_{NA,MB}"'] & G'(N'FA \star' MB) \arrow[d, "G'\chi'_{FA,B}"] & N''(F'FA *'' B) \arrow[d, "N''f'_{FA,B}"] \\
G'G(NA \star MB) \arrow[d, "G'G\chi_{A,B}"'] & G'N'(FA *' B) \arrow[r, "\beta_{FA *' B}"] \arrow[d, "G'N'(f_{A,B})"] & N''F'(FA *' B) \arrow[d, "N''F'f_{A,B}"] \\
G'GN(A * B) \arrow[r, "G'\alpha_{A * B}"'] & G'N'F(A * B) \arrow[r, "\beta_{F(A * B)}"'] & N''F'F(A * B)
\end{tikzcd}$$

The top square on the left is a naturality square of g', the bottom diagram on the left commutes because β is a morphism of strengths. The top diagram on the right commutes because α is a morphism of strengths and the bottom square on the right is a naturality square of β. $\square$

Applying this result to our previously constructed morphisms of strengths, we have strength morphisms

$$\alpha_n^{-1} : (\ell -)^{(n)} \to \ell\delta^n(-)$$

and

$$\beta_n : \delta^n k(-) \to k((-)^{(n)})$$

for all $n \in \mathbb{N}$.

Lemma 4.8.61 *Suppose given the following*

- *monoidal category* $(\mathcal{V}, \otimes, I)$
- *monoidal category* $(\mathcal{U}, \bullet, J)$
- *category* $\mathcal{C}$ *provided with a* $\mathcal{V}$*–action* $*$
- *category* $\mathcal{C}'$ *with finite products provided with a* $\mathcal{V}$*–action* $*'$ *such that*

  $$(A \times B) *' C \cong (A *' C) \times (B *' C)$$

  for all $A, B \in \mathcal{C}'$ *and* $C \in \mathcal{V}$
- *category* $\mathcal{D}$ *provided with a* $\mathcal{U}$*–action* $\star$
- *category* $\mathcal{D}'$ *with finite products provided with a* $\mathcal{U}$*–action* $\star'$ *such that*

  $$(C \times D) \star' E \cong (C \star' E) \times (D \star' E)$$

  for all $C, D \in \mathcal{D}'$ *and* $D \in \mathcal{U}$
- $\mathcal{V}$*–strong functors* $(F, f), (F', f') : (\mathcal{C}, *) \to (\mathcal{C}', *')$
- $\mathcal{U}$*–strong functors* $(G, g), (G', g') : (\mathcal{D}, \star) \to (\mathcal{D}', \star')$
- *morphism of actions* $((M, \phi), N, \chi)$ *from* $(\mathcal{C}, *)$ *to* $(\mathcal{D}, \star)$
- *morphism of actions* $((M, \phi), N', \chi')$ *from* $(\mathcal{C}', *')$ *to* $(\mathcal{D}', \star')$ *such that* N' *preserves finite products*
- *morphism of strengths* $\alpha : GN \to N'F$
- *morphism of strengths* $\beta : G'N \to N'F'$

Then $\alpha \times \beta : (G \times G')N \to N'(F \times F')$ *is a morphism of strengths.*

Proof. $F \times F'$ is $\mathcal{V}$–strong and $G \times G'$ is $\mathcal{U}$–strong by lemma 4.7.53. We take $\alpha \times \beta$ to be the following composite

$$(G \times G')N = GN \times G'N \to N'F \times N'F' \cong N'(F \times F')$$

The strength morphism axiom for $\alpha \times \beta$ follows directly from the two corresponding axioms for α and β. $\square$

Applying this result to our case, for a given arity $(n_1, \ldots, n_p)$ we have the following strength morphisms

$$\prod_{i=1}^{p} (\ell -)^{(n_i)} \to \ell\big(\prod_{i=1}^{p} \delta^{n_i}(-)\big)$$

and

$$\prod_{i=1}^{p} \delta^{n_i} k(-) \to k\big(\prod_{i=1}^{p} (-)^{(n_i)}\big)$$

Notation 4.8.62 *Given an arity $(n_1, \ldots, n_p)$, we write α^{-1} for the arrow*

$$\alpha^{-1} : \prod_{i=1}^{p} (\ell -)^{(n_i)} \to \ell\big(\prod_{i=1}^{p} \delta^{n_i}(-)\big)$$

and β for the arrow

$$\beta : \prod_{i=1}^{p} \delta^{n_i} k(-) \to k\big(\prod_{i=1}^{p} (-)^{(n_i)}\big)$$

Lemma 4.8.63 *Suppose given the following*

- *monoidal category $(\mathcal{V}, \otimes, I)$*
- *monoidal category $(\mathcal{U}, \bullet, J)$*
- *category $\mathcal{C}$ provided with a $\mathcal{V}$–action $*$*
- *category $\mathcal{C}'$ with coproducts provided with a $\mathcal{V}$–action $*'$ such that*

$$(A + B) *' C \cong (A *' C) + (B *' C)$$

  for all $A, B \in \mathcal{C}'$ and $C \in \mathcal{V}$

- *category $\mathcal{D}$ provided with a $\mathcal{U}$–action $\star$*
- *category $\mathcal{D}'$ with coproducts provided with a $\mathcal{U}$–action $\star'$ such that*

$$(C + D) \star' E \cong (C \star' E) + (D \star' E)$$

  for all $C, D \in \mathcal{D}'$ and $D \in \mathcal{U}$

- *$\mathcal{V}$–strong functors $(F, f), (F', f') : (\mathcal{C}, *) \to (\mathcal{C}', *')$*
- *$\mathcal{U}$–strong functors $(G, g), (G', g') : (\mathcal{D}, \star) \to (\mathcal{D}', \star')$*
- *morphism of actions $((M, \phi), N, \chi)$ from $(\mathcal{C}, *)$ to $(\mathcal{D}, \star)$*
- *morphism of actions $((M, \phi), N', \chi')$ from $(\mathcal{C}', *')$ to $(\mathcal{D}', \star')$ such that N' preserves coproducts*
- *morphism of strengths $\alpha : GN \to N'F$*

- *morphism of strengths* $\beta : G'N \to N'F'$

Then $\alpha + \beta : (G + G')N \to N'(F + F')$ *is a morphism of strengths.*

Proof. $F + F'$ is $\mathcal{V}$–strong and $G + G'$ is $\mathcal{U}$–strong by lemma 4.7.54. We take $\alpha + \beta$ to be the following composite

$$(G + G')N = GN + G'N \to N'F + N'F' \cong N'(F + F')$$

The strength morphism axiom for $\alpha + \beta$ follows directly from the two corresponding axioms for α and β. □

Given a signature S where we write $(n_{i,1}, \ldots, n_{i,p_i})$ for the i-th arity for all $i \in I$. We have the following strength morphisms

$$\sum_{i \in I} \prod_{j=1}^{p_i} (\ell-)^{(n_{i,j})} \to \ell(\sum_{i \in I} \prod_{j=1}^{p_i} \delta^{n_{i,j}}(-))$$

and

$$\sum_{i \in I} \prod_{j=1}^{p_i} \delta^{n_{i,j}} k(-) \to k(\sum_{i \in I} \prod_{j=1}^{p_i} (-)^{(n_{i,j})})$$

4.9 Mateness

In this section we show that α^{-1} is a natural isomorphism and we characterise its inverse α and β as mates over the adjunction $\ell \vdash k$.

Proposition 4.9.64 *The arrow*

$$\alpha_1^{-1} : (\ell-)' \to \ell(\delta-)$$

constructed in definition 4.8.58 is an isomorphism.

Proof. First we construct the inverse arrow $\alpha_1 : \ell(\delta-) \to (\ell-)'$. To define an arrow $\ell(\delta X)(A) \to (\ell X)'(A)$, by universal property of the coend, it suffices to give for all $m \in \mathbb{F}$ an arrow

$$X(m + 1) \times A^m \to \int^n X(n) \times (A + 1)^n$$

satisfying the wedge condition. We take it to be the composite

$$X(m + 1) \times A^m \to X(m + 1) \times (A + 1)^{m+1} \to \int^n X(n) \times (A + 1)^n$$

For the remaining verifications see appendix B.13. □

It follows that the arrow $\alpha_n^{-1} : (\ell-)^{(n)} \to \ell(\delta^n-)$ is also an isomorphism for all $n \in \mathbb{N}$ since it is a composite of n arrows of the form α_1^{-1}. The inverse is given by the composite of the inverses. Given an arity $(n_1, \ldots, n_p)$, each $\alpha_{n_i}^{-1}$ is an isomorphism, so their product $\prod_{i=1}^{p} \alpha_{n_i}^{-1}$ is as well an isomorphism.

Notation 4.9.65 *Given an arity $(n_1, \dots, n_p)$, we write α for the arrow*

$$\alpha : \ell\Big(\prod_{i=1}^{p} \delta^{n_i}(-)\Big) \to \prod_{i=1}^{p} (\ell-)^{(n_i)}$$

Definition 4.9.66 (mates under adjunctions) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be two adjunctions. Let*

$$\alpha : F'H \to KF$$

and

$$\beta : HG \to G'K$$

be two natural transformations for functors $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to \mathcal{D}'$. Diagrammatically

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \mathcal{C}' \arrow[d, bend right, "F'"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{D} \arrow[u, bend right, "G"'] \arrow[r, "K"'] \arrow[r, phantom, shift left=3, "\Searrow\beta \quad \Swarrow\alpha"] & \mathcal{D}' \arrow[u, bend right, "G'"']
\end{tikzcd}$$

α and β are mates under the adjunctions *$F \dashv G$ and $F' \dashv G'$ if β is the composite*

$$HG \xrightarrow{\eta' HG} G'F'HG \xrightarrow{G'\alpha G} G'KFG \xrightarrow{G'K\varepsilon} G'K$$

and α is the composite

$$F'H \xrightarrow{F'H\eta} F'HGF \xrightarrow{F'\beta F} F'G'KF \xrightarrow{\varepsilon' KF} KF$$

Proposition 4.9.67 *The arrows α_1 and β_1 are mates under the adjunction $\ell \dashv k : \mathcal{E} \to \mathcal{F}$.*

Proof. Let X be a functor in $\mathcal{F}$. We show that $\alpha_{1,X} = \varepsilon_{(\ell X)'} \circ \ell\beta_{1,\ell X} \circ \ell\delta\eta_X$.

$$\begin{tikzcd}
\int^r X(r+1) \times A^r \arrow[d, "\ell\delta\eta_{X,A}"'] & (s, f : r \to A) \arrow[d, mapsto] \\
\int^r \int^m X(m) \times (r+1)^m \times A^r \arrow[d, "\ell\beta_{1,\ell X,A}"'] & (s, \mathrm{id}_{r+1}, f) \arrow[d, mapsto] \\
\int^r \int^m X(m) \times (r+1)^m \times A^r \arrow[d, "\varepsilon_{(\ell X)',A}"'] & (s, \mathrm{id}_{r+1}, f) \arrow[d, mapsto] \\
\int^r X(r) \times (A+1)^r & \ell X(f+1)(s, \mathrm{id}_{r+1}) = (s, f+1)
\end{tikzcd}$$

This is indeed $\alpha_{1,X,A}$. Let F be a functor in $\mathcal{E}$. Now we show that $\beta_{1,F} = k(\varepsilon_F)' \circ k\alpha_{1,kF} \circ \eta_{\delta kF}$.

$$
\begin{tikzcd}
\delta kF(n) = F(n+1) \arrow[d, "\eta_{\delta kF,n}"'] & x \arrow[d, mapsto] \\
\int^m F(m+1) \times n^m \arrow[d, "k\alpha_{1,kF,n}"'] & (x, \mathrm{id}_n) \arrow[d, mapsto] \\
\int^m F(m) \times (n+1)^m \arrow[d, "k(\varepsilon_F)'"'] & (x, \mathrm{id}_n + 1) \arrow[d, mapsto] \\
F(n+1) = (kF)'(n) & F(\mathrm{id}_{n+1})(x) = x
\end{tikzcd}
$$

This composite is indeed $\beta_{1,F,n}$. □

Proposition 4.9.68 *The arrows α_n and β_n are mates under the adjunction $\ell \dashv k : \mathcal{E} \to \mathcal{F}$ for all $n \in \mathbb{N}$.*

Proof. The assertion is true for $n = 1$ by proposition 4.9.67. Now suppose that the assertion is true for an $n \in \mathbb{N}$, that is,

$$\alpha_{n,X} = \varepsilon_{(\ell X)^{(n)}} \circ \ell\beta_{n,\ell X} \circ \ell\delta^n \eta_X$$

and

$$\beta_{n,F} = k(\varepsilon_F)^{(n)} \circ k(\alpha_{n,kF}) \circ \eta_{\delta^n kF}$$

for all functor $X \in \mathcal{F}$ and $F \in \mathcal{E}$.

We are going to show that the assertion is true for $n+1$. Explicitly we wish to show that

$$\alpha_{n+1,X} = \varepsilon_{(\ell X)^{(n+1)}} \circ \ell\beta_{n+1,\ell X} \circ \ell\delta^{n+1}\eta_X$$

But by definition of α_{n+1}

$$\alpha_{n+1,X} = \alpha_{1,X}^{(n)} \circ \alpha_{n,\delta X}$$

So we show that

$$\varepsilon_{(\ell X)^{(n+1)}} \circ \ell\beta_{n+1,\ell X} \circ \ell\delta^{n+1}\eta_X = \varepsilon^{(n)}_{(\ell X)'} \circ (\ell\beta_{1,\ell X})^{(n)} \circ (\ell\delta\eta_X)^{(n)} \circ \varepsilon_{(\ell\delta X)^{(n)}} \circ \ell\beta_{n,\ell\delta X} \circ \ell\delta^n\eta_{\delta X}$$

for all functor $X \in \mathcal{F}$.

$$
\begin{tikzcd}
\ell\delta^n\delta X \arrow[r, "\ell\delta^n\eta_{\delta X}"] \arrow[d, "\ell\delta^n\delta\eta_X"'] \arrow[rrr, bend left, "\alpha_{n,\delta X}"] & \ell\delta^n k\ell\delta X \arrow[r, "\ell\beta_{n,\ell\delta X}"] \arrow[d, "\ell\delta^n k\ell\delta\eta_X"] & \ell k(\ell\delta X)^{(n)} \arrow[r, "\varepsilon_{(\ell\delta X)^{(n)}}"] \arrow[d, "\ell k(\ell\delta\eta_X)^{(n)}"] & (\ell\delta X)^{(n)} \arrow[d, "(\ell\delta\eta_X)^{(n)}"] \arrow[ddd, bend left=60, "(\alpha_{1,X})^{(n)}"] \\
\ell\delta^n\delta k\ell X \arrow[r, "\ell\delta^n\eta_{\delta k\ell X}"] \arrow[d, "\ell\delta^n\beta_{1,\ell X}"'] & \ell\delta^n k\ell\delta k\ell X \arrow[r, "\ell\beta_{n,\ell\delta k\ell X}"] \arrow[d, "\ell\delta^n k\ell\beta_{1,\ell X}"] & \ell k(\ell\delta k\ell X)^{(n)} \arrow[r, "\varepsilon_{(\ell\delta k\ell X)^{(n)}}"] \arrow[d, "\ell k(\ell\beta_{1,\ell X})^{(n)}"] & (\ell\delta\ell X)^{(n)} \arrow[d, "(\ell\beta_{1,\ell X})^{(n)}"] \\
\ell\delta^n k(\ell X)' \arrow[r, "\ell\delta^n\eta_{k(\ell X)'}"] \arrow[dr, "\ell\delta^n \mathrm{id}"'] & \ell\delta^n k\ell k(\ell X)' \arrow[r, "\ell\beta_{n,\ell k(\ell X)'}"] \arrow[d, "\ell\delta^n k\varepsilon_{(\ell X)'}"] & \ell k(\ell k(\ell X)')^{(n)} \arrow[r, "\varepsilon_{(\ell k(\ell X)')^{(n)}}"] \arrow[d, "\ell k(\varepsilon_{(\ell X)'})^{(n)}"] & (\ell k(\ell X)')^{(n)} \arrow[d, "(\varepsilon_{(\ell X)'})^{(n)}"] \\
 & \ell\delta^n k(\ell X)' \arrow[r, "\ell\beta_{n,(\ell X)'}"'] & \ell k((\ell X)')^{(n)} \arrow[r, "\varepsilon_{(\ell X)^{(n+1)}}"'] & (\ell X)^{(n+1)}
\end{tikzcd}
$$

The three squares on the right are naturality squares of ε, the three square in the middle are naturality squares of β and the two squares on the right are naturality squares of η. The bottom left triangle is one of the triangle identities for η and ε.

We also wish to show that

$$\beta_{n+1,F} = k(\varepsilon_F)^{(n+1)} \circ k(\alpha_{n+1,kF}) \circ \eta_{\delta^{n+1}kF}$$

But by definition of β_{n+1}

$$\beta_{n+1,F} = \beta_{1,F^{(n)}} \circ \delta\beta_{n,F}$$

So we show that

$$\begin{aligned} k(\varepsilon_F)^{(n+1)} \circ k(\alpha_{n+1,kr}) \circ \eta_{\delta kF^{(n)}} &= \\ &= k(\varepsilon_{F^{(n)}})' \circ k(\alpha_{1,k(F^{(n)})}) \circ \eta_{\delta k(F^{(n)})} \circ \delta k(\varepsilon_F)^{(n)} \circ \delta k(\alpha_{n,kF}) \circ \delta\eta_{\delta^n kF} \end{aligned}$$

$$\begin{tikzcd}
\delta\delta^n kF \arrow[r, "\delta\eta_{\delta^n kF}"] \arrow[d, "\eta_{\delta\delta^n kF}"'] \arrow[rrr, bend left, "\delta\beta_{n,F}"] & \delta k\ell\delta^n kF \arrow[r, "\delta k\alpha_{n,kF}"] \arrow[d, "\eta_{\delta k\ell\delta^n kF}"] & \delta k(\ell kF)^{(n)} \arrow[r, "\delta k(\varepsilon_F)^{(n)}"] \arrow[d, "\eta_{\delta k(\ell kF)^{(n)}}"] & \delta k(F^{(n)}) \arrow[d, "\eta_{\delta k(F^{(n)})}"] \arrow[ddd, bend left, "\beta_{1,F^{(n)}}"] \\
k\ell\delta\delta^n kF \arrow[r, "k\ell\delta\eta_{\delta^n kF}"] \arrow[d, "k\alpha_{1,\delta^n kF}"'] & k\ell\delta k\ell\delta^n kF \arrow[r, "k\ell\delta k\alpha_{n,kF}"] \arrow[d, "k\alpha_{1,k\ell\delta^n kF}"] & k\ell\delta k(\ell kF)^{(n)} \arrow[r, "k\ell\delta k(\varepsilon_F)^{(n)}"] \arrow[d, "k\alpha_{1,k(\ell kF)^{(n)}}"] & k\ell\delta k(F^{(n)}) \arrow[d, "k\alpha_{1,k(F^{(n)})}"] \\
k(\ell\delta^n kF)' \arrow[r, "k(\ell\eta_{\delta^n kF})'"] \arrow[dr, "k(\mathrm{id}_{\ell\delta^n kF})'"'] & k(\ell k\ell\delta^n kF)' \arrow[r, "k(\ell k\alpha_{n,kF})'"] \arrow[d, "k(\varepsilon_{\ell\delta kF})'"] & k(\ell k(\ell kF)^{(n)})' \arrow[r, "k(\ell k(\varepsilon_F)^{(n)})'"] \arrow[d, "k(\varepsilon_{(\ell kF)^{(n)}})'"] & k(\ell kF^{(n)})' \arrow[d, "k(\varepsilon_{F^{(n)}})'"] \\
& k(\ell\delta^n kF)' \arrow[r, "k(\alpha_{n,kF})'"'] & k((\ell kF)^{(n)})' \arrow[r, "k(\varepsilon_F)^{(n+1)}"'] & kF^{(n+1)}
\end{tikzcd}$$

The top three squares are naturality squares of η, the three squares in the middle row are naturality squares of α and the bottom two squares are naturality squares of ε. The bottom left triangle is one of the triangle identities for η and ε. □

Proposition 4.9.69 *Given an arity $(n_1, \ldots, n_p)$, the arrows*

$$\alpha : \ell\big(\prod_{i=1}^{p} \delta^{n_i}(-)\big) \to \prod_{i=1}^{p}(\ell-)^{(n_i)}$$

and

$$\beta : \prod_{i=1}^{p} \delta^{n_i} k(-) \to k\big(\prod_{i=1}^{p}(-)^{(n_i)}\big)$$

are mates under the adjunction $\ell \dashv k : \mathcal{E} \to \mathcal{F}$.

Proof. Let $F \in \mathcal{E}$. We check first that $\beta_F = k\prod_{i=1}^{p}(\varepsilon_F)^{(n_i)} \circ k\alpha_{kF} \circ \eta_{\prod_{i=1}^{p}\delta^{n_i}kF}$

$$\begin{tikzcd}
\prod_{i=1}^{p}\delta^{n_i}kF \arrow[r, "\eta"] \arrow[d, "\prod\eta"] \arrow[ddd, bend right, "\prod\beta_{n_i}"'] & k\ell\prod_{i=1}^{p}\delta^{n_i}kF \arrow[d, "\cong"] \\
\prod_{i=1}^{p}k\ell\delta^{n_i}kF \arrow[r, equal] \arrow[d] & k\prod_{i=1}^{p}\ell\delta^{n_i}kF \arrow[d, "k\prod\alpha_{n_i}"] \\
\prod_{i=1}^{p}k(\ell kF)^{(n_i)} \arrow[r, equal] \arrow[d] & k\prod_{i=1}^{p}(\ell kF)^{(n_i)} \arrow[d, "k\prod\varepsilon"] \\
\prod_{i=1}^{p}kF^{(n_i)} \arrow[r, equal] & k\prod_{i=1}^{p}F^{(n_i)}
\end{tikzcd}$$

The two bottom squares commute since $k(F \times G) = kF \times kG$. The top square commutes because squares of the following form commute

$$\begin{tikzcd}
X \times Y \arrow[r, "\eta_{X\times Y}"] \arrow[d, "\eta_X\times\eta_Y"'] & k\ell(X \times Y) \arrow[d, "\cong"] \\
k\ell(X) \times k\ell(Y) \arrow[r, equal] & k(\ell(X) \times \ell(Y))
\end{tikzcd}$$

since we have on elements

$$\begin{tikzcd}
(x, y) \arrow[r, mapsto] \arrow[d, mapsto] & (x, y, \mathrm{id}_n) \arrow[d, mapsto] \\
(x, \mathrm{id}_n, y, \mathrm{id}_n) \arrow[r, equal] & (x, \mathrm{id}_n, y, \mathrm{id}_n)
\end{tikzcd}$$

Let $X \in \mathcal{F}$. Now we check that $\alpha_X = \ell\prod_{i=1}^{p}\delta^{n_i}\eta_X \circ \ell\beta_{\ell X} \circ \varepsilon_{\prod_{i=1}^{p}(\ell X)^{(n_i)}}$

$$\begin{tikzcd}
\ell\prod_{i=1}^{p}\delta^{n_i}X \arrow[r, "\cong"] \arrow[d, "\ell\prod\delta^{n_i}\eta"'] & \prod_{i=1}^{p}\ell\delta^{n_i}X \arrow[d] \arrow[ddd, bend left, "\prod\alpha_{n_i}"] \\
\ell\prod_{i=1}^{p}\delta^{n_i}k\ell X \arrow[r, "\cong"] \arrow[d, "\ell\prod\beta_{n_i}"'] & \prod_{i=1}^{p}\ell\delta^{n_i}k\ell X \arrow[d] \\
\ell\prod_{i=1}^{p}k(\ell X)^{(n_i)} \arrow[r, "\cong"] \arrow[d, equal] & \prod_{i=1}^{p}\ell k(\ell X)^{(n_i)} \arrow[d] \\
\ell k\prod_{i=1}^{p}(\ell X)^{(n_i)} \arrow[r, "\varepsilon"'] & \prod_{i=1}^{p}(\ell X)^{(n_i)}
\end{tikzcd}$$

The two top squares are naturality square of the isomorphism $\ell(X \times Y) \xrightarrow{\cong} \ell(X) \times \ell(Y)$. The bottom square commutes because it is of the form

$$\begin{tikzcd}
\ell k(X \times Y) \arrow[r, equal] \arrow[d, "\varepsilon_{X\times Y}"'] & \ell(kX \times kY) \arrow[d, "\cong"] \\
X \times Y & \ell kX \times \ell kY \arrow[l, "\varepsilon_X \times \varepsilon_Y"]
\end{tikzcd}$$

which commutes because we have on elements

$$\begin{tikzcd}
(x, y, f) \arrow[r, equal] \arrow[d, mapsto] & (x, y, f) \arrow[d, mapsto] \\
\begin{array}{c}(X \times Y)(f)(x, y) \\ = (X(f)(x), Y(f)(y))\end{array} & (x, f, y, f) \arrow[l, mapsto]
\end{tikzcd}$$

□

Proposition 4.9.70 *Given a signature S where we write $(n_{i,1}, \ldots, n_{i,p_i})$ for the i-th arity ($i \in I$), the arrows*

$$\alpha : \ell\big(\sum_{i\in I} \prod_{j=1}^{p_i} \delta^{n_{i,j}}(-)\big) \to \sum_{i\in I} \prod_{j=1}^{p_i} (\ell-)^{(n_{i,j})}$$

and

$$\beta : \sum_{i\in I} \prod_{j=1}^{p_i} \delta^{n_{i,j}} k(-) \to k\big(\sum_{i\in I} \prod_{j=1}^{p_i} (-)^{(n_{i,j})}\big)$$

are mates under the adjunction $\ell \dashv k : \mathcal{E} \to \mathcal{F}$.

Proof. Let $F \in \mathcal{E}$. We show that $\beta_F = k \sum\limits_{i\in I} \prod\limits_{j=1}^{p_i} \varepsilon_F^{(n_j)} \circ k\alpha_{kF} \circ \eta_{\sum_{i\in I} \prod_{j=1}^{p_i} \delta^{n_j} kF}$

$$\begin{tikzcd}
\sum\limits_{i\in I} \prod\limits_{j=1}^{p_i} \delta^{n_j} kF \arrow[r, "\eta"] \arrow[d, "\sum \eta"] \arrow[ddd, bend right, "\sum \beta_i"'] & k\ell \sum\limits_{i\in I} \prod\limits_{j=1}^{p_i} \delta^{n_j} kF \arrow[d, "\cong"] \\
\sum\limits_{i\in I} k\ell \prod\limits_{j=1}^{p_i} \delta^{n_j} kF \arrow[r, equal] \arrow[d] & k \sum\limits_{i\in I} \ell \prod\limits_{j=1}^{p_i} \delta^{n_j} kF \arrow[d, "k \sum \alpha_i"] \\
\sum\limits_{i\in I} k \prod\limits_{j=1}^{p_i} (\ell kF)^{(n_j)} \arrow[r, equal] \arrow[d] & k \sum\limits_{i\in I} \prod\limits_{j=1}^{p_i} (\ell kF)^{(n_j)} \arrow[d, "k \sum \prod \varepsilon"] \\
\sum\limits_{i\in I} k \prod\limits_{j=1}^{p_i} F^{(n_j)} \arrow[r, equal] & k \sum\limits_{i\in I} \prod\limits_{j=1}^{p_i} F^{(n_j)}
\end{tikzcd}$$

The two bottom squares commute because $k \sum\limits_{i\in I} X_i = \sum\limits_{i\in I} kX_i$. The top square is a square of the form

$$\begin{tikzcd}
X+Y \arrow[r, "\eta_{X+Y}"] \arrow[d, "\eta_X+\eta_Y"'] & k\ell(X+Y) \arrow[d, "\cong"] \\
k\ell X + k\ell Y \arrow[r, equal] & k(\ell X + \ell Y)
\end{tikzcd}$$

which commutes because we have on elements

$$\begin{tikzcd}
z \arrow[r, mapsto] \arrow[d, mapsto] & (z, \mathrm{id}_n) \arrow[d, mapsto] \\
(z, \mathrm{id}_n) \arrow[r, equal] & (z, \mathrm{id}_n)
\end{tikzcd}$$

Let $X \in \mathcal{F}$. We show that $\alpha_X = \ell \sum\limits_{i\in I} \prod\limits_{j=1}^{p_i} \delta^{n_j}\eta_X \circ \ell\beta_{\ell X} \circ \varepsilon_{\sum_{i\in I}\prod_{j=1}^{p_i}(\ell X)^{(n_j)}}$

$$\begin{tikzcd}
\ell\big(\sum\limits_{i\in I}\prod\limits_{j=1}^{p_i}\delta^{n_j}X\big) \arrow[r, "\cong"] \arrow[d, "\ell\sum\prod\delta^{n_j}\eta"'] & \sum\limits_{i\in I}\ell\big(\prod\limits_{j=1}^{p_i}\delta^{n_j}X\big) \arrow[d] \arrow[dddd, bend left, "\sum\alpha_i"] \\
\ell\big(\sum\limits_{i\in I}\prod\limits_{j=1}^{p_i}\delta^{n_j}k\ell X\big) \arrow[r, "\cong"] \arrow[d, "\ell\sum\beta_i\ell"'] & \sum\limits_{i\in I}\ell\big(\prod\limits_{j=1}^{p_i}\delta^{n_j}k\ell X\big) \arrow[d] \\
\ell\big(\sum\limits_{i\in I}k\prod\limits_{j=1}^{p_i}(\ell X)^{(n_j)}\big) \arrow[r, "\cong"] \arrow[d, equal] & \sum\limits_{i\in I}\ell k\big(\prod\limits_{j=1}^{p_i}(\ell X)^{(n_j)}\big) \arrow[d, "\sum\varepsilon"'] \\
\ell k\big(\sum\limits_{i\in I}\prod\limits_{j=1}^{p_i}(\ell X)^{(n_j)}\big) \arrow[r, "\varepsilon"'] & \sum\limits_{i\in I}\prod\limits_{j=1}^{p_i}(\ell X)^{(n_j)}
\end{tikzcd}$$

The top two squares are naturality squares of $\ell(\sum X_i) \xrightarrow{\cong} \sum \ell(X_i)$. The bottom square commutes because it is of the form

$$\begin{tikzcd}
\ell k(X+Y) \arrow[r, equal] \arrow[d, "\varepsilon_{X+Y}"'] & \ell(kX+kY) \arrow[d, "\cong"] \\
X+Y & \ell kX + \ell kY \arrow[l, "\varepsilon_X+\varepsilon_Y"]
\end{tikzcd}$$

and we have on elements

$$\begin{tikzcd}
(z,f) \arrow[r, equal] \arrow[d, mapsto] & (z,f) \arrow[d, mapsto] \\
\begin{array}{c}(X+Y)(f)(z)\\ = X(f)(z)+Y(f)(z)\end{array} & (z,f) \arrow[l, mapsto]
\end{tikzcd}$$

$\square$

4.10 Modules associated to a signature

Let $(n_1, \ldots, n_p)$ be an arity. By the presheaf approach we associate the $U \downarrow \mathcal{F}$–strong enodfunctor $\delta^{n_1} \times \ldots \times \delta^{n_p}$ on $\mathcal{F}$ to it. For a signature consisting of a collection of arities of the above form, we associate the $U \downarrow \mathcal{F}$–strong endofunctor $\sum_{i \in I} \delta^{n_{i,1}} \times \ldots \times \delta^{n_{i,p_i}}$ on $\mathcal{F}$ to it. We show in this section that this endofunctor can be provided with the structure of a module.

By the monadic approach we associate to an arity a morphism of modules and to a signature as many morphisms of modules as it has arities. We show that the source of each module morphism comes from a $\mathrm{Id} \downarrow \mathcal{E}$–strong endofunctor $(-)^{(n_1)} \times \ldots \times (-)^{(n_p)}$ on $\mathcal{E}$.

Moreover we are going to show that the constructed morphisms of strengths α^{-1} and β are module morphisms.

Proposition 4.10.71 *The $U \downarrow \mathcal{F}$–strong endofunctor $\sum\limits_{i \in I} \prod\limits_{j=1}^{p_i} \delta^{n_{i,j}}(-)$ on $\mathcal{F}$ induces a functor $P : \mathrm{Mon}(\mathcal{F}) \to \mathrm{Mod}(\mathcal{F})$.*

Proof. Let (S, m, e) be a monoid in $\mathrm{Mon}(\mathcal{F})$. We define

$$P(S) = \sum_{i \in I} \prod_{j=1}^{p_i} \delta^{n_{i,j}} S$$

The functor $P(S)$ can be provided with the following action

$$P(S) \otimes S \xrightarrow{s_{S,S}} P(S \otimes S) \xrightarrow{P(m)} P(S)$$

where the first arrow is given by the $U \downarrow \mathcal{F}$–strength for $\sum_{i \in I} \prod_{j=1}^{p_i} \delta^{n_{i,j}}(-)$ and the second is given by the monoid multiplication m. We check the module axioms.

$$\begin{tikzcd}
(P(S) \otimes S) \otimes S \arrow[r, "\alpha"] \arrow[d, "s_{S,S} \otimes S"'] \arrow[dr, phantom, "I."] & P(S) \otimes (S \otimes S) \arrow[r, "s_{S,S \otimes S}"] & P((S \otimes (S \otimes S))) \arrow[d, "P(S \otimes m)"] \\
P(S \otimes S) \otimes S \arrow[r, "s_{S \otimes S,S}"] \arrow[d, "P(m) \otimes S"'] \arrow[dr, phantom, "II."] & P((S \otimes S) \otimes S) \arrow[ur, "P(\alpha)"] \arrow[d, "P(m \otimes S)"] \arrow[dr, phantom, "III."] & P(S \otimes S) \arrow[d, "P(m)"] \\
P(S) \otimes S \arrow[r, "s_{S,S}"'] & P(S \otimes S) \arrow[r, "P(m)"'] & P(S)
\end{tikzcd}$$

Diagram I. commutes because it is one of the $U \downarrow \mathcal{F}$–strength axioms, diagram III. is one of the monoid axioms and square II. is a naturality square of s.

$$\begin{tikzcd}
P(S) \otimes I \arrow[rr, "P(S) \otimes e"] \arrow[dr, "s_{S,I}"] \arrow[ddr, bend right, "\rho^{-1}"'] & & P(S) \otimes S \arrow[d, "s_{S,S}"] \\
& P(S \otimes I) \arrow[r, "P(S \otimes e)"] \arrow[dr, "P(\rho^{-1})"] & P(S \otimes S) \arrow[d, "P(m)"] \\
& & P(S)
\end{tikzcd}$$

The top square is a naturality square of s, the bottom triangle is a monoid axiom and the left curved diagram is a $U \downarrow \mathcal{F}$–strength axiom.

Let $h : (S, m, e) \to (S', m', e')$ be a morphism of monoids. The arrow $P(h) : P(S) \to P(S')$ is an S–module morphism since it satisfies the axiom

$$
\begin{tikzcd}
P(S) \otimes S \arrow[r, "P(h)\otimes S"] \arrow[dd, "s_{S,S}"'] \arrow[dr, "P(h)\otimes h"'] & P(S') \otimes S \arrow[d, "P(S')\otimes h"] \\
 & P(S') \otimes S' \arrow[d, "s_{S',S'}"'] \\
P(S \otimes S) \arrow[r, "P(h\otimes h)"'] \arrow[d, "P(m)"'] & P(S' \otimes S') \arrow[d, "P(m')"] \\
P(S) \arrow[r, "P(h)"'] & P(S')
\end{tikzcd}
$$

The top triangle commutes trivially, the middle square is a naturality square of s and the bottom square is a monoid morphism axiom of h. □

Proposition 4.10.72 *The* Id $\downarrow \mathcal{E}$*–strong endofunctor* $\prod_{i=1}^{p} (-)^{(n_i)}$ *on* $\mathcal{E}$ *induces a functor* $M : \mathrm{Mon}(\mathcal{E}) \to \mathrm{Mod}(\mathcal{E})$.

Proof. Let (R, μ, η) be a monoid in $\mathrm{Mon}(\mathcal{E})$. We define

$$M(R) = \prod_{i=1}^{p} R^{(n_i)}$$

The functor $M(R)$ can be provided with the following action

$$M(R) \circ R \xrightarrow{\sigma_{R,R}} M(R \circ R) \xrightarrow{M(\mu)} M(R)$$

where the first arrow is given by the Id $\downarrow \mathcal{E}$–strength for $\prod_{i=1}^{p} (-)^{(n_i)}$ and the second is given by the monoid multiplication μ. We check the module axioms.

$$
\begin{tikzcd}
(M(R) \circ R) \circ R \arrow[r, "\alpha"] \arrow[d, "\sigma_{R,R} \circ R"'] \arrow[dr, phantom, "I."] & M(R) \circ (R \circ R) \arrow[r, "\sigma_{R,R\circ R}"] & M(R \circ (R \circ R)) \arrow[d, "M(R\circ\mu)"] \\
M(R \circ R) \circ R \arrow[r, "\sigma_{R\circ R,R}"] \arrow[d, "M(\mu)\circ R"'] \arrow[dr, phantom, "II."] & M((R \circ R) \circ R) \arrow[ur, "M(\alpha)"] \arrow[d, "M(\mu\circ R)"'] \arrow[dr, phantom, "III."] & M(R \circ R) \arrow[d, "M(\mu)"] \\
M(R) \circ R \arrow[r, "\sigma_{R,R}"'] & M(R \circ R) \arrow[r, "M(\mu)"'] & M(R)
\end{tikzcd}
$$

Diagram I. commutes because it is one of the Id $\downarrow \mathcal{E}$–strength axioms, diagram III. is one of the monoid axioms and square II. is a naturality square of σ.

$$\begin{tikzcd}
M(R)\circ \mathrm{Id} \arrow[rr, "M(R)\circ\eta"] \arrow[dr, "\sigma_{R,\mathrm{Id}}"] \arrow[ddrr, bend right, "\rho^{-1}"'] & & M(R)\circ R \arrow[d, "\sigma_{R,R}"] \\
& M(R\circ \mathrm{Id}) \arrow[r, "M(R\circ\eta)"] \arrow[dr, "M(\rho^{-1})"] & M(R\circ R) \arrow[d, "M(\mu)"] \\
& & M(R)
\end{tikzcd}$$

The top square is a naturality square of σ, the bottom triangle is a monoid axiom and the left curved diagram is a $\mathrm{Id} \downarrow \mathcal{E}$–strength axiom.

Let $h : (R_1, \mu_1, \eta_1) \to (R_2, \mu_2, \eta_2)$ be a morphism of monoids. The arrow $M(h) : M(R_1) \to M(R_2)$ is an R_1–module morphism since it satisfies the axiom

$$\begin{tikzcd}
M(R_1)\circ R_1 \arrow[r, "M(h)\circ R_1"] \arrow[dd, "\sigma_{R_1,R_1}"'] \arrow[dr, "M(h)\circ h"'] & M(R_2)\circ R_1 \arrow[d, "M(R_2)\circ h"] \\
& M(R_2)\circ R_2 \arrow[d, "\sigma_{R_2,R_2}"'] \\
M(R_1\circ R_1) \arrow[r, "M(h\circ h)"'] \arrow[d, "M(\mu_1)"'] & M(R_2\circ R_2) \arrow[d, "M(\mu_2)"] \\
M(R_1) \arrow[r, "M(h)"'] & M(R_2)
\end{tikzcd}$$

The top triangle commutes trivially, the middle square is a naturality square of σ and the bottom square is a monoid morphism axiom of h. $\square$

Let us write $M_i : \mathrm{Mon}(\mathcal{E}) \to \mathrm{Mod}(\mathcal{E})$ for the induced functor by the i–th arity. Now we consider $M := \sum_{i\in I} M_i$.

Proposition 4.10.73 *Let (S, m, e) be a monoid of $\mathrm{Mon}(\mathcal{F})$. The morphism of strengths $\alpha_S^{-1} : M(\ell S) \to \ell P(S)$ induces a morphism of ℓS–modules.*

Proof. We check the module morphism axiom

$$\begin{tikzcd}
M(\ell S)\circ \ell S \arrow[r, "\alpha_S^{-1}\circ \ell S"] \arrow[d, "\sigma_{\ell S,\ell S}"'] & \ell P(S)\circ \ell S \arrow[d, "\phi_{P(S),S}"] \\
M(\ell S\circ \ell S) \arrow[d, "M(\phi_{S,S})"'] & \ell(P(S)\otimes S) \arrow[d, "\ell s_{S,S}"] \\
M(\ell(S\otimes S)) \arrow[r, "\alpha_{S\otimes S}^{-1}"'] \arrow[d, "M(\ell m)"'] & \ell P(S\otimes S) \arrow[d, "\ell P(m)"] \\
M(\ell S) \arrow[r, "\alpha_S^{-1}"'] & \ell P(S)
\end{tikzcd}$$

The top diagram commutes because α^{-1} is a morphism of strengths and the bottom square is a naturality square of α^{-1}. $\square$

Corollary 4.10.74 *Let (S, m, e) be a monoid of* $\mathrm{Mon}(\mathcal{F})$*. The arrow $\alpha_S^{-1} : M(\ell S) \to \ell P(S)$ is an isomorphism of ℓS–modules.*

Proof. Its inverse is given by α_S. $\square$

Proposition 4.10.75 *Let (R, μ, η) be a monoid of* $\mathrm{Mon}(\mathcal{E})$*. The morphism of strengths $\beta_R : P(kR) \to kM(R)$ induces a morphism of kR–modules.*

Proof. We check the module morphism axiom

$$\begin{tikzcd}
P(kR) \otimes kR \arrow[r, "\beta_R \otimes kR"] \arrow[d, "s_{kR,kR}"'] & kM(R) \otimes kR \arrow[d, "\psi_{M(R),R}"] \\
P(kR \otimes kR) \arrow[d, "P(\psi_{R,R})"'] & k(M(R) \circ R) \arrow[d, "k\sigma_{R,R}"] \\
P(k(R \circ R)) \arrow[r, "\beta_{R \circ R}"'] \arrow[d, "P(k\mu)"'] & kM(R \circ R) \arrow[d, "kM(\mu)"] \\
P(kR) \arrow[r, "\beta_R"'] & kM(R)
\end{tikzcd}$$

The top diagram commutes because β is a morphism of strengths and the bottom square is a naturality square of β. $\square$

4.11 Universal properties

For this section we suppose given a signature and we consider the associated functors $P : \mathrm{Mon}(\mathcal{F}) \to \mathrm{Mod}(\mathcal{F})$ of the presheaf approach and $M : \mathrm{Mon}(\mathcal{E}) \to \mathrm{Mod}(\mathcal{E})$ of the monadic approach. The object of interest from the former point of view is the initial monoid S with a map of S–module morphisms $P(S) \to S$. The object of interest from the latter point of view is the initial monoid R with a map of R–modules $M(R) \to R$. The aim of this section is to deduce the initiality of the monad ℓS from the initiality of S and inversely to deduce the initiality of the monoid kR from the initiality of R.

4.11.1 From presheaf to monadic

Suppose that S is the initial monoid in $\mathcal{F}$ together with a map of S–modules $s : P(S) \to S$. We wish show that $\ell(S)$ is the initial monoid in $\mathcal{E}$ together with a map of ℓS–modules $M(\ell S) \cong \ell P(S) \to \ell S$.

Suppose we have a monoid R in $\mathcal{E}$ together with a map of R–modules $r : M(R) \to R$. Its image under k is the monoid kR in $\mathcal{F}$ together with the map of kR–modules $kr : kM(R) \to kR$, precomposed with β_R, we get the module morphism $kr \circ \beta_R : P(kR) \to kR$. By initiality of S, there exists a unique morphism of monoids $u : S \to kR$ such that

$$\begin{tikzcd} P(S) \arrow[r, "Pu"] \arrow[d, "s"'] & P(kR) \arrow[d, "kr\circ\beta_R"] \\ S \arrow[r, "u"'] & kR \end{tikzcd} \tag{4.1}$$

commutes. The transpose of u across the adjunction $\ell \dashv k$ is a morphism of monoids $\overline{u} : \ell S \to R$. Now we are going to calculate the transpose of (4.1) across the adjunction. The transpose of

$$P(S) \xrightarrow{s} S \xrightarrow{u} kR$$

is

$$\ell P(S) \xrightarrow{\ell s} \ell S \xrightarrow{\overline{u}} R$$

The transpose of

$$P(S) \xrightarrow{Pu} P(kR) \xrightarrow{\beta_R} kM(R) \xrightarrow{kr} kR$$

is

$$\ell P(S) \xrightarrow{\ell Pu} \ell PkR \xrightarrow{\overline{\beta}_R} MR \xrightarrow{r} R$$

where $\overline{\beta}R$ is the composite

$$\ell P(kR) \xrightarrow{\ell\beta_R} \ell kMR \xrightarrow{\varepsilon_{MR}} MR$$

Next we remark that

$$\begin{tikzcd} \ell P(S) \arrow[r, "\ell Pu"] \arrow[d, "\alpha_S"'] & \ell P(kR) \arrow[r, "\ell\beta_R"] \arrow[d, "\alpha_{kR}"] & \ell kM(R) \arrow[d, "\varepsilon_{MR}"] \\ M(\ell S) \arrow[r, "M\ell u"'] \arrow[rr, bend right, "M\overline{u}"'] & M(\ell kR) \arrow[r, "M\varepsilon_R"'] & M(R) \end{tikzcd}$$

commutes because the left square is a naturality square of α and the right square commutes because the following diagram commutes

$$\begin{tikzcd} \ell P(kR) \arrow[r, "\ell\eta_{P(kR)}"] \arrow[d, "\mathrm{id}"'] & \ell k\ell P(kR) \arrow[r, "\ell k\alpha_{kR}"] \arrow[dl, "\varepsilon_{\ell P(kR)}"] & \ell kM(\ell kR) \arrow[d, "\ell kM\varepsilon_R"] \arrow[ddll, "\varepsilon_{M(\ell kR)}"] \\ \ell P(kR) \arrow[d, "\alpha_{kR}"'] & & \ell kM(R) \arrow[d, "\varepsilon_{M(R)}"] \\ M(\ell kR) \arrow[rr, "M\varepsilon_R"'] & & M(R) \end{tikzcd}$$

the right and middle diagrams are naturality diagrams of ε and the upper triangle is one of the triangle identities the unit and counit of the adjunction. We used the fact that β is the mate of α.

Summarizing we found that the transpose of (4.1) is the following commutative square

$$\begin{tikzcd}
\ell P(S) \arrow[r, "\alpha_S"] \arrow[d, "\ell s"'] & M\ell(S) \arrow[r, "M\overline{u}"] & MR \arrow[d, "r"] \\
\ell S \arrow[rr, "\overline{u}"'] & & R
\end{tikzcd}$$

which means that

$$\begin{tikzcd}
\ell P(S) \arrow[d, "\ell s"'] & M\ell(S) \arrow[l, "\alpha_S^{-1}"'] \arrow[r, "M\overline{u}"] & MR \arrow[d, "r"] \\
\ell S \arrow[rr, "\overline{u}"'] & & R
\end{tikzcd}$$

commutes and this shows that $\overline{u}$ is the unique morphism of monoids $\ell(S) \to R$ such that

$$\begin{tikzcd}
M(\ell S) \arrow[r, "M(\overline{u})"] \arrow[d] & M(R) \arrow[d] \\
\ell S \arrow[r, "\overline{u}"'] & R
\end{tikzcd}$$

commutes.

4.11.2 From monadic to presheaf

Suppose that R is the initial monoid in $\mathcal{E}$ with a map of R–modules $r : M(R) \to R$. By applying k and precomposing with β_R we obtain

$$P(kR) \xrightarrow{\beta_R} kM(R) \xrightarrow{kr} kR$$

We wish to show that kR is the initial monoid in $\mathcal{F}$ together with the map of kR–modules $\beta_R \circ kr$. Given another monoid S in $\mathcal{F}$ together with a map of S–modules $Ps : P(S) \to S$, by applying ℓ and precomposing by α^{-1}

$$M(\ell S) \xrightarrow{\alpha_S^{-1}} \ell P(S) \xrightarrow{\ell s} \ell S$$

we have a monoid in $\mathcal{E}$ together with a map of ℓS-modules $M(\ell S) \to \ell S$. By initiality of R there exists a unique monoid morphism $u : R \to \ell S$ such that the following square commutes

$$\begin{tikzcd}
M(R) \arrow[r, "Mu"] \arrow[dd, "r"'] & M(\ell S) \arrow[d, "\alpha_S^{-1}"] \\
 & \ell P(S) \arrow[d, "\ell s"] \\
R \arrow[r, "u"'] & \ell S
\end{tikzcd} \tag{4.2}$$

We set

$$v : kR \xrightarrow{ku} k\ell S \xrightarrow{\eta_S^{-1}} S$$

and we wish to show that v makes the following square commute

$$\begin{tikzcd}
P(kR) \arrow[r, "Pv"] \arrow[d, "kr\circ\beta_R"'] & PS \arrow[d, "s"] \\
kR \arrow[r, "v"'] & S
\end{tikzcd}$$

which implies that v is the unique monoid morphism $kR \to S$. By replacing v by its definition we have the following diagram

$$\begin{tikzcd}
P(kR) \arrow[r, "Pku"] \arrow[d, "\beta_R"'] & P(k\ell S) \arrow[rr, "P\eta_S^{-1}"] \arrow[d, "\beta_{\ell S}"'] & & PS \arrow[dd, "s"] \\
kM(R) \arrow[r, "kMu"] \arrow[d, "kr"'] & kM(\ell S) \arrow[r, "k\alpha_S^{-1}"] & k\ell P(S) \arrow[ur, "\eta_{P(S)}^{-1}"] \arrow[d, "k\ell s"'] & \\
kR \arrow[rr, "ku"'] & & k\ell S \arrow[r, "\eta_S^{-1}"'] & S
\end{tikzcd}$$

The upper left square is a naturality square of β, the lower left diagram commutes because (4.2) does and the bottom right square is a naturality square of η. It remains to show the commutativity of

$$\begin{tikzcd}
P(S) \arrow[r, "P\eta_S"] \arrow[d, "\eta_{P(S)}"'] & P(k\ell S) \arrow[d, "\beta_{\ell S}"] \\
k\ell P(S) \arrow[r, "k\alpha_S"'] & kM(\ell S)
\end{tikzcd}$$

Since α and β are mates under the adjunction $\ell \dashv k$, we can express β by α and we obtain the following diagram

$$\begin{tikzcd}
P(S) \arrow[r, "P(\eta_S)"] \arrow[d, "\eta_{P(S)}"'] & P(k\ell S) \arrow[d, "\eta_{P(k\ell S)}"] \\
k\ell P(S) \arrow[r, "k\ell P(\eta_S)"] \arrow[d, "k\alpha_S"'] & k\ell P(k\ell S) \arrow[d, "k\alpha_{k\ell S}"] \\
kM(\ell S) \arrow[r, "kM(\ell\eta_S)"] \arrow[dr, "kM\,\mathrm{id}_{\ell S}"'] & kM(\ell k\ell S) \arrow[d, "kM\varepsilon_{\ell S}"] \\
& kM(\ell S)
\end{tikzcd}$$

the upper square is a naturality square of η, the square in the middle is a naturality square of α and the bottom triangle is one of the triangle identities for the unit and the counit. $\square$

Chapter 5

The presheaf approach

This chapter details the theory of simply typed abstract syntax and variable binding in the sense of [FPT99]. Some parts of it have been developed for example in [Fio02] and [Fio05], here we give a complete description. We fix a set of types $\mathcal{T}$ for this chapter.

5.1 Category of contexts

We take the free cocartesian category generated by $\mathcal{T}$ to be the category of contexts. We use the notation from [Fio02] for this category which is $\mathbb{F} \downarrow \mathcal{T}$. We describe this category explicitly. Its objects are arrows $n \xrightarrow{\Gamma} \mathcal{T}$ where n stands for an object of the category of finite sets $\mathbb{F}$.

An arrow from $n \xrightarrow{\Gamma} \mathcal{T}$ to $m \xrightarrow{\Delta} \mathcal{T}$ is a map $h : n \to m$ of $\mathbb{F}$ such that

$$\begin{tikzcd} n \arrow[rr, "h"] \arrow[dr, "\Gamma"'] & & m \arrow[dl, "\Delta"] \\ & \mathcal{T} & \end{tikzcd}$$

commutes in the category of sets.

The coproduct $\Gamma + \Delta$ of two objects $\Gamma : n \to \mathcal{T}$ and $\Delta : m \to \mathcal{T}$ in $\mathbb{F} \downarrow \mathcal{T}$ is given by the unique map $[\Gamma, \Delta] : n + m \to \mathcal{T}$ and the coprojection maps $\Gamma \to \Gamma + \Delta$ and $\Delta \to \Gamma + \Delta$ are given by the coprojection maps $n \to n + m$ and $m \to n + m$ in $\mathbb{F}$.

The initial object of $\mathbb{F} \downarrow \mathcal{T}$ is given by the unique map $0 \to \mathcal{T}$.

Notation 5.1.76 *For each element $t \in \mathcal{T}$ we have an object of $\mathbb{F} \downarrow \mathcal{T}$, which is the arrow $\langle t \rangle : 1 \to \mathcal{T}$ that maps 1 to t.*

5.2 The structure of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$

Similar to the untyped setting in [FPT99] we consider the presheaf category on the category of contexts in order to define later the signature endofunctor. Throughout this section we fix a set of types $\mathcal{T}$.

The objects of the presheaf category $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ are functors $P : \mathbb{F} \downarrow \mathcal{T} \to \mathrm{Set}$ and an arrow $P \to Q$ is a natural transformation $\rho : P \to Q$. It is complete and cocomplete and limits and colimits are computed pointwise.

The Yoneda embedding $\mathcal{Y} : (\mathbb{F} \downarrow \mathcal{T})^{\mathrm{op}} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ sends an object Δ of $(\mathbb{F} \downarrow \mathcal{T})^{\mathrm{op}}$ to the representable presheaf $\mathcal{Y}(\Delta) = \mathbb{F} \downarrow \mathcal{T}(\Delta, -)$, in particular we have $\mathcal{Y}(\langle t \rangle)(\Gamma) = \mathbb{F} \downarrow \mathcal{T}(\langle t \rangle, \Gamma) =$

$\Gamma^{-1}(t)$ for a type $t \in \mathcal{T}$ and a context Γ. We think of the set $\Gamma^{-1}(t)$ as the set of variables of type t in the context Γ.

Notation 5.2.77 *The presheaf of variables of type t is the Yoneda embedding $\mathcal{Y}(\langle t\rangle)$. We omit the parentheses and write $\mathcal{Y}\langle t\rangle$.*

We show now how to model variable binding in our typed setting. Let P be a functor in $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ and let us calculate the exponential functor $P^{\mathcal{Y}\langle t\rangle} = P^{\mathbb{F}\downarrow\mathcal{T}(\langle t\rangle,-)}$ for a type $t \in \mathcal{T}$:

$$\begin{aligned} P^{\mathbb{F}\downarrow\mathcal{T}(\langle t\rangle,-)}(\Gamma) &\cong [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}](\mathbb{F}\downarrow\mathcal{T}(\Gamma,-), P^{\mathbb{F}\downarrow\mathcal{T}(\langle t\rangle,-)}) \\ &\cong [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}](\mathbb{F}\downarrow\mathcal{T}(\Gamma,-)\times\mathbb{F}\downarrow\mathcal{T}(\langle t\rangle,-), P) \\ &\cong [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}](\mathbb{F}\downarrow\mathcal{T}(\Gamma+\langle t\rangle,-), P) \\ &\cong P(\Gamma+\langle t\rangle) \end{aligned}$$

So we have

$$P^{\mathcal{Y}\langle t\rangle} \cong P(-+\langle t\rangle)$$

Let $t_1,\ldots,t_n \in \mathcal{T}$. Since the Yoneda embedding preserves products we have

$$\mathcal{Y}\langle t_1\rangle \times \ldots \times \mathcal{Y}\langle t_n\rangle \cong \mathcal{Y}(\langle t_1\rangle + \ldots + \langle t_n\rangle)$$

We write shortly $\mathcal{Y}(t_1,\ldots,t_n)$ for $\mathcal{Y}(\langle t_1\rangle+\ldots+\langle t_n\rangle)$. Now we calculate the exponential functor $P^{\mathcal{Y}(t_1,\ldots,t_n)} \cong P^{\prod_{i=1}^{n}\mathbb{F}\downarrow\mathcal{T}(\langle t_i\rangle,-)}$:

$$\begin{aligned} P^{\mathbb{F}\downarrow\mathcal{T}(\langle t_1\rangle,-)\times\ldots\times\mathbb{F}\downarrow\mathcal{T}(\langle t_n\rangle,-)}(\Gamma) &\cong P^{\mathbb{F}\downarrow\mathcal{T}(\langle t_1\rangle+\ldots+\langle t_n\rangle,-)}(\Gamma) \\ &\cong [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}](\mathbb{F}\downarrow\mathcal{T}(\Gamma,-), P^{\mathbb{F}\downarrow\mathcal{T}(\langle t_1\rangle+\ldots+\langle t_n\rangle,-)}) \\ &\cong [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}](\mathbb{F}\downarrow\mathcal{T}(\Gamma,-)\times\mathbb{F}\downarrow\mathcal{T}(\langle t_1\rangle+\ldots+\langle t_n\rangle,-), P) \\ &\cong [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}](\mathbb{F}\downarrow\mathcal{T}(\Gamma+\langle t_1\rangle+\ldots+\langle t_n\rangle,-), P) \\ &\cong P(\Gamma+\langle t_1\rangle+\ldots+\langle t_n\rangle) \end{aligned}$$

So we have

$$P^{\mathcal{Y}(t_1,\ldots,t_n)} \cong P(-+\langle t_1\rangle+\ldots+\langle t_n\rangle)$$

5.3 Notion of signature

In this section we recall the definition of the notion of simply typed signature with variable binding which is a collection of simply typed arities with variable binding. We also define the binding signature endofunctor associated to a signature. We keep in mind our leading example, the simply typed Lambda Calculus.

Definition 5.3.78 (arity) *An* arity *is a collection of types consisting of t_i for $i=0,\ldots,n$ and $(t_{i,j})_{j=1,\ldots,m_i}$ for all $i=1,\ldots,n$ written*

$$(t_{1,1}\ldots t_{1,m_1})t_1,\ldots,(t_{n,1}\ldots t_{n,m_n})t_n \to t_0$$

Intuitively this stands for an operator that binds m_k variables of types $t_{k,1},\ldots,t_{k,m_k}$ in its k–th argument of type t_k. It yields a term of type t_0.

Example 5.3.79 (Abstraction of the simply typed Lambda Calculus) *Let $\mathcal{T}$ be the inductive set of types defined by*

$$\mathcal{T} \quad ::= \quad T \mid \mathcal{T} \Rightarrow \mathcal{T}$$

where T is a set of base types. Let s, t be two types of $\mathcal{T}$. The arity of the abstraction that binds a variable of type s in a term of type t is

$$(s)t \to s \Rightarrow t$$

Definition 5.3.80 (signature) *A* signature *is a collection of arities.*

Now let us take a look at some examples of signatures.

Example 5.3.81 (Lambda Calculus without product type) *Let $\mathcal{T}$ be the inductive set of types defined by*

$$\mathcal{T} \quad ::= \quad T \mid \mathcal{T} \Rightarrow \mathcal{T}$$

where T is a set of base types. The signature of the simply typed Lambda Calculus is given by two collections of arities $S = (\alpha_{s,t}, \beta_{s,t})_{s,t \in \mathcal{T}}$ where

$$\alpha_{s,t} \quad = \quad (s)t \to s \Rightarrow t$$

and

$$\beta_{s,t} \quad = \quad s \Rightarrow t, s \to t$$

Example 5.3.82 (Lambda Calculus with product type) *Let $\mathcal{T}$ be the inductive set of types defined by*

$$\mathcal{T} \quad ::= \quad T \mid 1 \mid \mathcal{T} \Rightarrow \mathcal{T} \mid \mathcal{T} \times \mathcal{T}$$

where T is a set of base types. The signature of the simply typed Lambda Calculus is given by the following collection of arities

$$\forall s, t \in \mathcal{T}, \quad s, t \to s \times t$$

$$\forall s, t \in \mathcal{T}, \quad s \times t \to s$$

$$\forall s, t \in \mathcal{T}, \quad s \times t \to t$$

$$\forall s, t \in \mathcal{T}, \quad (s)t \to s \Rightarrow t$$

$$\forall s, t \in \mathcal{T}, \quad s \Rightarrow t, s \to t$$

Notation 5.3.83 *The collections of $\mathcal{T}$–indexed presheaves of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ form a category which we call $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$.*

We interpret an arity such as in definition 5.3.78 in a $\mathcal{T}$–indexed collection of presheaves $P = (P_t)_{t \in \mathcal{T}} \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ as an arrow

$$\prod_{i=1}^{n} P_{t_i}^{\mathcal{Y}(t_{i,1}, \ldots, t_{i,m_i})} \to P_{t_0}$$

which is at the component $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$

$$P_{t_1}(\Gamma + (t_{1,1}, \ldots, t_{1,m_1})) \times \ldots \times P_{t_n}(\Gamma + (t_{n,1}, \ldots, t_{n,m_n})) \to P_{t_0}(\Gamma)$$

To a signature we associate the following signature functor.

Definition 5.3.84 (signature functor) *Let $S = (\alpha_k)_{k\in I}$ be a signature and α_k an arity for each k as in definition 5.3.78*

$$(t^{(k)}_{1,1}\dots t^{(k)}_{1,m_1})t^{(k)}_1,\dots,(t^{(k)}_{n_k,1}\dots t^{(k)}_{n_k,m_{n_k}})t_{n_k}\to t^{(k)}_0$$

We associate to S the signature endofunctor $\Sigma : [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$, $(P_t)_{t\in\mathcal{T}} \mapsto (Q_u)_{u\in\mathcal{T}}$ where

$$Q_u = \sum_{k\in I_u}\prod_{i=1}^{n_k} P_{t_i}^{(t^{(k)}_{i,1},\dots,t^{(k)}_{i,m_i})}$$

and $I_u = \{k\in I | t^{(k)}_0 = u\}$ for all $u\in\mathcal{T}$.

Example 5.3.85 (Lambda Calculus without product type) *The signature functor Σ_λ of the simply typed Lambda Calculus without product type is given by the collection $(Q_u)_{u\in\mathcal{T}}$ where*

$$Q_u = \begin{cases} \sum\limits_{v\in\mathcal{T}} P_{v\Rightarrow(s\Rightarrow t)}\times P_v + P_t^{\mathcal{Y}\langle s\rangle} & \textit{if } u = s\Rightarrow t\\ \sum\limits_{v\in\mathcal{T}} P_{v\Rightarrow u}\times P_v & \textit{else}\end{cases}$$

Example 5.3.86 (Lambda Calculus with product type) *The signature functor $\Sigma_{\lambda\times}$ of the simply typed lambda calculus with product type is given by the collection $(Q_u)_{u\in\mathcal{T}}$ where*

$$Q_u = \begin{cases} \sum\limits_{v\in\mathcal{T}} (P_{v\Rightarrow 1}\times P_v + P_{1\times v} + P_{v\times 1}) + 1 & \textit{if } u = 1\\ \sum\limits_{v\in\mathcal{T}} (P_{v\Rightarrow(s\times t)}\times P_v + P_{(s\times t)\times v} + P_{v\times(s\times t)}) + P_s\times P_t & \textit{if } u = s\times t\\ \sum\limits_{v\in\mathcal{T}} (P_{v\Rightarrow(s\Rightarrow t)}\times P_v + P_{(s\Rightarrow t)\times v} + P_{v\times(s\Rightarrow r)}) + P_t^{\mathcal{Y}\langle s\rangle} & \textit{if } u = s\Rightarrow t\\ \sum\limits_{v\in\mathcal{T}} (P_{v\Rightarrow u}\times P_v + P_{u\times v} + P_{v\times u}) & \textit{else}\end{cases}$$

5.4 Free algebras

Given a signature, we consider free algebras for the signature functor $\Sigma : [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ associated to the signature.

By chapter 3, the forgetful functor $U : \Sigma\text{–alg} \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ has a left adjoint F if the initial $X+\Sigma$–algebra TX exists (since $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ has finite coproducts) and F maps X to TX. By Lambek's Lemma the structure map of the initial algebra is an isomorphism, so $X+\Sigma TX \cong TX$. Furthermore $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ has an initial object 0 and colimits of countable chains, so TX can be computed as the colimit of the following countable chain

$$0\to X+\Sigma(0)\to X+\Sigma(X+\Sigma(0))\to\dots$$

if Σ preserves colimits of countable chains. So we chow now that Σ preserves such colimits. Let

$$Z_0\to Z_1\to Z_2\to\dots\to Z_n\to\dots$$

be a countable chain of presheaves $Z_i\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$. We show first that for all $u\in\mathcal{T}$, the endofunctor $(-)^{\mathcal{Y}\langle u\rangle}$ preserves such colimits. We compute

$$\begin{aligned}(\mathrm{colim}\, Z_n)^{\mathcal{Y}\langle u\rangle}(\Gamma) &= (\mathrm{colim}\, Z_n)(\Gamma+\langle u\rangle)\\ &= \mathrm{colim}(Z_n(\Gamma+\langle u\rangle))\end{aligned}$$

and

$$\begin{aligned}(\mathrm{colim}(Z_n^{\mathcal{Y}\langle u\rangle}))(\Gamma) &= \mathrm{colim}(Z_n^{\mathcal{Y}\langle u\rangle}(\Gamma)) \\ &= \mathrm{colim}(Z_n(\Gamma + \langle u\rangle))\end{aligned}$$

so we have an isomorphism $(\mathrm{colim}\, Z_n)^{\mathcal{Y}\langle u\rangle}(\Gamma) \to (\mathrm{colim}(Z_n^{\mathcal{Y}\langle u\rangle}))(\Gamma)$ for all $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$ and thus an isomorphism $(\mathrm{colim}\, Z_n)^{\mathcal{Y}\langle u\rangle} \to \mathrm{colim}(Z_n^{\mathcal{Y}\langle u\rangle})$. Now let $u_1, \ldots, u_m \in \mathcal{T}$. We compute

$$\begin{aligned}(\mathrm{colim}\, Z_n)^{\mathcal{Y}(u_1,\ldots,u_m)}(\Gamma) &= (\mathrm{colim}\, Z_n)(\Gamma + \mathcal{Y}\langle u\rangle) \\ &= \mathrm{colim}(Z_n(\Gamma + \langle u_1\rangle + \ldots + \langle u_m\rangle))\end{aligned}$$

and

$$\begin{aligned}(\mathrm{colim}(Z_n^{\mathcal{Y}(u_1,\ldots,u_m)}))(\Gamma) &= \mathrm{colim}(Z_n^{\mathcal{Y}(u_1,\ldots,u_m)}(\Gamma)) \\ &= \mathrm{colim}(Z_n(\Gamma + \langle u_1\rangle + \ldots + \langle u_m\rangle))\end{aligned}$$

So we have an isomorphism $(\mathrm{colim}\, Z_n)^{\mathcal{Y}(u_1,\ldots,u_m)} \to \mathrm{colim}(Z_n^{\mathcal{Y}(u_1,\ldots,u_m)})$. Now let Σ be a signature endofunctor and

$$Z_0 \to Z_1 \to Z_2 \to \ldots \to Z_n \to \ldots$$

be a countable chain of presheaves $Z_i \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. We compute

$$\begin{aligned}\Sigma(\mathrm{colim}\, Z_n)_u(\Gamma) &= \sum_{i\in I_u} \prod_{j=1}^{n^i} (\mathrm{colim}\, Z_n)_{t_j}^{\mathcal{Y}(t_{j,1},\ldots,t_{j,m_j^i})}(\Gamma) \\ &= \sum_{i\in I_u} \prod_{j=1}^{n^i} (\mathrm{colim}\, Z_n)_{t_j}(\Gamma + \langle t_{j,1}\rangle + \ldots + \langle t_{j,m_j^i}\rangle) \\ &= \sum_{i\in I_u} \prod_{j=1}^{n^i} \mathrm{colim}(Z_{n,t_j}(\Gamma + \langle t_{j,1}\rangle + \ldots + \langle t_{j,m_j^i}\rangle))\end{aligned}$$

and

$$\begin{aligned}(\mathrm{colim}(\Sigma Z_n))_u(\Gamma) &= \mathrm{colim}((\Sigma Z_n)_u(\Gamma)) \\ &= \mathrm{colim} \sum_{i\in I_u} \prod_{j=1}^{n^i} Z_{n,t_j}(\Gamma + \langle t_{j,1}\rangle + \ldots + \langle t_{j,m_j^i}\rangle) \\ &\cong \sum_{i\in I_u} \prod_{j=1}^{n^i} \mathrm{colim}(Z_{n,t_j}(\Gamma + \langle t_{j,1}\rangle + \ldots + \langle t_{j,m_j^i}\rangle))\end{aligned}$$

where the last isomorphism comes from commutation of colimits with colimits and limits with filtered colimits. So we have an isomorphism $\Sigma(\mathrm{colim}\, Z_n) \to \mathrm{colim}(\Sigma Z_n)$.

Example 5.4.87 (Lambda Calculus with product type) *The free algebra $(\mathfrak{L}_u)_{u\in\mathcal{T}}$ on the presheaf of variables $\mathcal{Y} := (\mathcal{Y}\langle t\rangle)_{t\in\mathcal{T}}$, that is, the initial $\mathcal{Y}+\Sigma_{\lambda\times}$– algebra for the signature functor $\Sigma_{\lambda\times}$ from the example 5.3.86 is the collection of arrows $(\phi_u)_{u\in\mathcal{T}}$, $\phi_u : \mathcal{Y}\langle u\rangle + (\Sigma_{\lambda\times}\mathfrak{L})_u \to \mathfrak{L}_u$.*

Explicitly we have the following arrows

$$\begin{array}{ll}
\mathcal{Y}\langle 1\rangle + \sum\limits_{v\in\mathcal{T}} \left(\mathfrak{L}_{v\Rightarrow 1}\times\mathfrak{L}_v + \mathfrak{L}_{1\times v} + \mathfrak{L}_{v\times 1}\right) + 1 \to \mathfrak{L}_1 & \textit{if } u = 1\\
\mathcal{Y}\langle s\times t\rangle + \sum\limits_{v\in\mathcal{T}} \left(\mathfrak{L}_{v\Rightarrow(s\times t)}\times\mathfrak{L}_v + \mathfrak{L}_{(s\times t)\times v} + \mathfrak{L}_{v\times(s\times t)}\right) + \mathfrak{L}_s\times\mathfrak{L}_t \to \mathfrak{L}_{s\times t} & \textit{if } u = s\times t\\
\mathcal{Y}\langle s\Rightarrow t\rangle + \sum\limits_{v\in\mathcal{T}} \left(\mathfrak{L}_{v\Rightarrow(s\Rightarrow t)}\times\mathfrak{L}_v + \mathfrak{L}_{(s\Rightarrow t)\times v} + \mathfrak{L}_{v\times(s\Rightarrow r)}\right) + \mathfrak{L}_t^{\mathcal{Y}\langle s\rangle} \to \mathfrak{L}_{s\Rightarrow t} & \textit{if } u = s\Rightarrow t\\
\mathcal{Y}\langle u\rangle + \sum\limits_{v\in\mathcal{T}} \left(\mathfrak{L}_{v\Rightarrow u}\times\mathfrak{L}_v + \mathfrak{L}_{u\times v} + \mathfrak{L}_{v\times u}\right) \to \mathfrak{L}_u & \textit{else}
\end{array}$$

Just like in the untyped case we have the following theorem.

Theorem 5.4.88 *The terms of the simply typed Lambda Calculus modulo α–equivalence form a free algebra on $(\mathcal{Y}\langle t\rangle)_{t\in\mathcal{T}}$ for the signature functor $\Sigma_{\lambda\times}$ from the example 5.3.86.*

We do not give here the proof since it does not reflect the complete characterisation of the object of interest for this particular signature $\Sigma_{\lambda\times}$.

5.5 Monoidal structure on $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$

We recall the notations for the substitution monoidal structure introduced in [Fio05]. Let $P = (P_t)_{t\in\mathcal{T}} \in [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ and $\Delta : \overline{\Delta} \to \mathcal{T} \in (\mathbb{F}\downarrow\mathcal{T})^{\mathrm{op}}$. The Δ–fold product of the components of P is given by the formula

$$P^{\times\Delta} := \prod_{k\in\overline{\Delta}} P_{\Delta(k)}$$

Remark that $P^{\times\Delta} \in [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$. If $\Delta = (u_1,\ldots,u_m)$, then

$$P^{\times\Delta} = P_{u_1}\times\ldots\times P_{u_m}$$

or equivalently for all $\Gamma\in\mathbb{F}\downarrow\mathcal{T}$

$$P^{\times\Delta}(\Gamma) = \prod_{u\in\mathcal{T}} P_u(\Gamma)^{\Delta^{-1}(u)}$$

Let $R\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ and $\Gamma\in\mathbb{F}\downarrow\mathcal{T}$. We write

$$(R\bullet P)(\Gamma) := \int^{\Delta\in(\mathbb{F}\downarrow\mathcal{T})^{\mathrm{op}}} R(\Delta)\times P^{\times\Delta}(\Gamma)$$

Now let $Q = (Q_t)_{t\in\mathcal{T}} \in [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$. The monoidal product of Q and P at the component $u\in\mathcal{T}$ is given by

$$(Q\otimes P)_u(\Gamma) := (Q_u\bullet P)(\Gamma) = \int^{\Delta\in(\mathbb{F}\downarrow\mathcal{T})^{\mathrm{op}}} Q_u(\Delta)\times P^{\times\Delta}(\Gamma)$$

The unit for this monoidal product is given by the collection of variables $\mathcal{Y} := (\mathcal{Y}\langle t\rangle)_{t\in\mathcal{T}}$. For the construction of the monoidal category isomorphisms

$$\begin{aligned}
\alpha_{P,Q,R} &: ((P\otimes Q)\otimes R)_u(\Gamma) \to (P\otimes(Q\otimes R))_u(\Gamma)\\
\lambda_P &: (\mathcal{Y}\otimes P)_u(\Gamma) \to P_u(\Gamma)\\
\rho_Q &: (Q\otimes\mathcal{Y})_u(\Gamma) \to Q_u(\Gamma)
\end{aligned}$$

see appendix C.1.

Proposition 5.5.89 *The above defined* $\bullet : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}] \times \mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$

$$(P, \mathcal{Y} \to Q) \quad \mapsto \quad P \bullet Q$$

is a right $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$*–action.*

Proof. Let $P \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ and $Q, R \in \mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. We define the two natural isomorphisms

$$a_{P,Q,R} : (P \bullet Q) \bullet R \to P \bullet (Q \otimes R)$$

and

$$r_P : P \bullet \mathcal{Y} \to P$$

Let $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$. We rewrite the domain and the codomain of $a_{P,Q,R,\Gamma}$ using the coend notation

$$((P \bullet Q) \bullet R)(\Gamma) = \int^{\Delta} \int^{\Delta'} P(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$$

and

$$\begin{aligned}(P \bullet (Q \otimes R))(\Gamma) = \int^{\Delta'} \int^{\Delta_1} \ldots \int^{\Delta_m} P(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times Q_{u_m}(\Delta_m) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}\end{aligned}$$

where $\Delta' = (u_1, \ldots, u_m)$. To define a map $((P \bullet Q) \bullet R)(\Gamma) \to (P \bullet (Q \otimes R))(\Gamma)$, it suffices to give a collection of arrows

$$P(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \to (P \bullet (Q \otimes R))(\Gamma)$$

for all $\Delta, \Delta' \in \mathbb{F} \downarrow \mathcal{T}$ that satisfies the wedge condition. We take the following composite

$$\begin{array}{c}
P(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \\
\downarrow \\
\begin{array}{l}
P(\Delta') \times Q_{u_1}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \\
\quad \times \ldots \\
\quad \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}
\end{array} \\
\downarrow \\
\begin{array}{l}
P(\Delta') \times \int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\
\quad \times \ldots \\
\quad \times \int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}
\end{array} \\
\downarrow \\
\begin{array}{l}
\int^{\Delta'} P(\Delta') \times \int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\
\quad \times \ldots \\
\quad \times \int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}
\end{array}
\end{array}$$

For the remaining verifications see appendix C.2.

Next we define r_P. Let $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$. To define an arrow

$$\int^{\Delta} P(\Delta) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)} \to P(\Gamma)$$

is equivalent by universal property of the coend to give a collection of arrows

$$P(\Delta) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)} \to P(\Gamma)$$

for all $\Delta \in \mathbb{F} \downarrow \mathcal{T}$ that satisfies the wedge condition. We take the following mapping

$$p, (h_t)_{t\in\mathcal{T}} \quad \mapsto \quad P\big(\sum_{t\in\mathcal{T}} h_t\big)(p)$$

For the remaining verifications see appendix C.2. □

Lemma 5.5.90 *Let $P, Q \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ and $R \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. Then*

$$(P \times Q) \bullet R \to (P \bullet R) \times (Q \bullet R)$$

is a natural isomorphism.

Proof. See appendix C.3. □

Corollary 5.5.91 *Let $P, Q, R \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. Then*

$$(P \times Q) \otimes R \to (P \otimes R) \times (Q \otimes R)$$

is a natural isomorphism.

Proof. The above bijection is componentwise the assertion of the previous lemma

$$((P_t \times Q_t) \bullet R)(\Gamma) \to (P_t \bullet R)(\Gamma) \times (Q_t \bullet R)(\Gamma)$$

is a natural isomorphism for all $t \in \mathcal{T}$ and $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$. □

Lemma 5.5.92 *Let $P, Q \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ and $R \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. Then*

$$(P + Q) \bullet R \to (P \bullet R) + (Q \bullet R)$$

is a natural isomorphism.

Proof. See appendix C.4. □

Corollary 5.5.93 *Let $P, Q, R \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. Then*

$$(P + Q) \otimes R \to (P + R) \otimes (Q + R)$$

is a natural isomorphism.

Proof. The above bijection is componentwise the assertion of the previous lemma:

$$((P + Q)_t \bullet R)(\Gamma) \to (P_t \bullet R)(\Gamma) + (Q_t \bullet R)(\Gamma)$$

is a natural isomorphism for all $t \in \mathcal{T}$ and $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$. □

5.6 Monoidal closed structure on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$

Let $Y \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. It induces a functor $- \bullet Y : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ by $X \mapsto X \bullet Y$ and a functor $- \otimes Y : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$.

We check functoriality. Let $f : X \to Z$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ and $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$. To give an arrow $(X \bullet Y)(\Gamma) \to (Z \bullet Y)(\Gamma)$ is equivalent to give a collection of arrows

$$X(\Delta) \times \prod_{t \in \mathcal{T}} Y_t(\Gamma)^{\Delta^{-1}(t)} \to (Z \bullet Y)(\Gamma)$$

for all $\Delta \in \mathbb{F} \downarrow \mathcal{T}$ that satisfies the wedge condition. We take the following composite

$$\begin{array}{c} X(\Delta) \times \prod_{t \in \mathcal{T}} Y_t(\Gamma)^{\Delta^{-1}(t)} \\ \downarrow \\ Z(\Delta) \times \prod_{t \in \mathcal{T}} Y_t(\Gamma)^{\Delta^{-1}(t)} \\ \downarrow \\ \int^{\Delta} Z(\Delta) \times \prod_{t \in \mathcal{T}} Y_t(\Gamma)^{\Delta^{-1}(t)} \end{array}$$

We check the wedge condition. Let $g : \Delta_1 \to \Delta_2$. The diagram

$$\begin{tikzcd}
 & X(\Delta_1) \times \prod_{t\in\mathcal{T}} Y_t(\Gamma)^{\Delta_1^{-1}(t)} \arrow[dr] & \\
X(\Delta_1) \times \prod_{t\in\mathcal{T}} Y_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur, "\mathrm{id}\times(-\circ g_t)_{t\in\mathcal{T}}"] \arrow[dr, "X(g)\times\mathrm{id}"'] & & (Z \bullet Y)(\Gamma) \\
 & X(\Delta_2) \times \prod_{t\in\mathcal{T}} Y_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
 & x, (h_t \circ g_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\
x, (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} f_{\Delta_1}(x), (h_t \circ g_t)_{t\in\mathcal{T}} \\ = f_{\Delta_2}(X(g)(x)), (h_t)_{t\in\mathcal{T}} \end{array} \\
 & X(g)(x), (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}$$

By naturality of f we have $f_{\Delta_2} \circ X(g) = Z(g) \circ f_{\Delta_1}$. The elements $f_{\Delta_1}(x), (h_t \circ g_t)_{t\in\mathcal{T}}$ and $Z(g)(f_{\Delta_1}(x)), (h_t)_{t\in\mathcal{T}}$ are equal since they come from $f_{\Delta_1}(x), (h_t)_{t\in\mathcal{T}} \in Z(\Delta_1) \times \prod_{t\in\mathcal{T}} Y_t(\Gamma)^{\Delta_2^{-1}(t)}$ with the arrow g

$$\begin{tikzcd}
 & Z(\Delta_1) \times \prod_{t\in\mathcal{T}} Y_t(\Gamma)^{\Delta_1^{-1}(t)} \arrow[dr] & \\
Z(\Delta_1) \times \prod_{t\in\mathcal{T}} Y_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur, "\mathrm{id}\times(-\circ g_t)_{t\in\mathcal{T}}"] \arrow[dr, "Z(g)\times\mathrm{id}"'] & & (Z \bullet Y)(\Gamma) \\
 & Z(\Delta_2) \times \prod_{t\in\mathcal{T}} Y_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

Now we check that $f \bullet Y$ is natural in Γ. Let $g : \Gamma_1 \to \Gamma_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
(X \bullet Y)(\Gamma_1) \arrow[r] \arrow[d] & (Z \bullet Y)(\Gamma_1) \arrow[d] \\
(X \bullet Y)(\Gamma_2) \arrow[r] & (Z \bullet Y)(\Gamma_2)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
x, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & f_\Delta(x), (h_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
x, (Y_t(g) \circ h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & f_\Delta(x), (Y_t(g) \circ h_t)_{t\in\mathcal{T}}
\end{tikzcd}$$

The functoriality of $-\otimes Y : [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ follows from the functoriality of $-\bullet Y : [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}] \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$.

Proposition 5.6.94 *The functor* $-\bullet Y$ *on* $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ *has a right adjoint.*

Proof. Let $Z \in [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ and $\Gamma \in \mathbb{F}\downarrow\mathcal{T}$. We set

$$Y \multimap Z(\Gamma) = \int_{\Delta\in\mathbb{F}\downarrow\mathcal{T}} \mathrm{Set}(Y^{\times\Gamma}(\Delta), Z(\Delta))$$

and $Y \multimap -$ is a functor $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}] \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$, $X \mapsto Y \multimap X$.

We check functoriality. Let $f : X \to Z$ in $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ and $\Gamma \in \mathbb{F}\downarrow\mathcal{T}$. To give an arrow $(Y \multimap X)(\Gamma) \to (Y \multimap Z)(\Gamma)$ is equivalent to give a collection of arrows

$$(Y \multimap X)(\Gamma) \to \mathrm{Set}(Y^{\times\Gamma}(\Delta), Z(\Delta))$$

for all $\Delta \in \mathbb{F}\downarrow\mathcal{T}$ that satisfies the wedge condition. We take the following composite

$$\begin{tikzcd}
(Y \multimap X)(\Gamma) \arrow[d] \\
\mathrm{Set}(Y^{\times\Gamma}(\Delta), X(\Delta)) \arrow[d] \\
\mathrm{Set}(Y^{\times\Gamma}(\Delta), Z(\Delta))
\end{tikzcd}$$

We check the wedge condition. Let $g : \Delta_1 \to \Delta_2$. The diagram

$$\begin{tikzcd}
& \mathrm{Set}(Y^{\times\Gamma}(\Delta_1), X(\Delta_1)) \arrow[dr] & \\
(Y \multimap X)(\Gamma) \arrow[ur] \arrow[dr] & & \mathrm{Set}(Y^{\times\Gamma}(\Delta_1), Z(\Delta_2)) \\
& \mathrm{Set}(Y^{\times\Gamma}(\Delta_2), X(\Delta_2)) \arrow[ur] &
\end{tikzcd}$$

commutes because on elements we have the following assignations

$$\begin{tikzcd}
& h_1 \arrow[dr, mapsto] & \\
\bullet \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} Z(g)\circ f_{\Delta_1}\circ h_1 \\ = f_{\Delta_2}\circ h_2\circ Y^{\times\Gamma}(g) \end{array} \\
& h_2 \arrow[ur, mapsto] &
\end{tikzcd}$$

The composites $Z(g)\circ f_{\Delta_1}\circ h_1 = f_{\Delta_1}\circ X(g)\circ h_1$ and $f_{\Delta_2}\circ h_2 \circ Y^{\times\Gamma}(g)$ are equal since $(Y \multimap X)(\Gamma)$ is an end and thus

$$\begin{tikzcd}
 & \mathrm{Set}(Y^{\times\Gamma}(\Delta_1), X(\Delta_1)) \arrow[dr] & \\
(Y \multimap X)(\Gamma) \arrow[ur] \arrow[dr] & & \mathrm{Set}(Y^{\times\Gamma}(\Delta_1), X(\Delta_2)) \\
 & \mathrm{Set}(Y^{\times\Gamma}(\Delta_2), X(\Delta_2)) \arrow[ur] &
\end{tikzcd}$$

commutes which means that $X(g)\circ h_1 = h_2 \circ Y^{\times\Gamma}(g)$.

Now we check that $Y \multimap f$ is natural in Γ. Let $g : \Gamma_1 \to \Gamma_2$ in $\mathbb{F}\downarrow\mathcal{T}$. The naturality square

$$\begin{tikzcd}
(Y \multimap X)(\Gamma_1) \arrow[r] \arrow[d] & (Y \multimap Z)(\Gamma_1) \arrow[d] \\
(Y \multimap X)(\Gamma_2) \arrow[r] & (Y \multimap Z)(\Gamma_2)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & f\circ h \arrow[d, mapsto] \\
h\circ Y^{\times g} \arrow[r, mapsto] & f\circ h\circ Y^{\times g}
\end{tikzcd}$$

Next we show that the following Hom–sets are isomorphic

$$[\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}](X \bullet Y, Z) \cong [\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}](X, Y \multimap Z)$$

and thus $Y \multimap -$ is the right adjoint of $Y \bullet -$. We compute

$$\begin{aligned}
[\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}](X \bullet Y, Z) &\cong \int_\Delta \mathrm{Set}(X \bullet Y(\Delta), Z(\Delta)) \\
&= \int_\Delta \mathrm{Set}(\int^\Gamma X(\Gamma)\times Y^{\times\Gamma}(\Delta), Z(\Delta)) \\
&\cong \int_\Delta\int_\Gamma \mathrm{Set}(X(\Gamma)\times Y^{\times\Gamma}(\Delta), Z(\Delta)) \\
&\cong \int_\Gamma\int_\Delta \mathrm{Set}(X(\Gamma)\times Y^{\times\Gamma}(\Delta), Z(\Delta)) \\
&\cong \int_\Gamma\int_\Delta \mathrm{Set}(X(\Gamma), Z(\Delta)^{Y^{\times\Gamma}(\Delta)}) \\
&\cong \int_\Gamma \mathrm{Set}(X(\Gamma), \int_\Delta Z(\Delta)^{Y^{\times\Gamma}(\Delta)}) \\
&\cong \int_\Gamma \mathrm{Set}(X(\Gamma), \int_\Delta \mathrm{Set}(Y^{\times\Gamma}(\Delta), Z(\Delta))) \\
&= \int_\Gamma \mathrm{Set}(X(\Gamma), Y \multimap Z(\Gamma)) \\
&\cong [\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}](X, Y \multimap Z)
\end{aligned}$$

where we used the naturality formula, universal properties of colimits and limits, the Fubini formula for ends and cartesian closedness of Set. $\square$

Corollary 5.6.95 *The functor* $-\otimes Y$ *on* $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ *is a left adjoint.*

Proof. Let $X\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ and $t\in\mathcal{T}$. We set

$$(Y\multimap X)_t=(Y\multimap X_t)$$

where $\multimap$ on the right–hand side stands for the right adjoint defined in the proof of proposition 5.6.94. $\square$

5.7 Strength for Σ

Let $P=(P_t)_{t\in\mathcal{T}}\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ and $u\in\mathcal{T}$. We write $P^{\mathcal{Y}\langle u\rangle}$ for the collection of presheaves $(P_t^{\mathcal{Y}\langle u\rangle})_{t\in\mathcal{T}}$.

Proposition 5.7.96 *Let* $u\in\mathcal{T}$. *The endofunctor* $(-)^{\mathcal{Y}\langle u\rangle}$ *on* $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ *is* $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$*–strong.*

Proof. First we define the $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strength

$$s^{(u)}_{P,Q}:P^{\mathcal{Y}\langle u\rangle}\bullet Q\to(P\bullet Q)^{\mathcal{Y}\langle u\rangle}$$

Let $\Gamma\in\mathbb{F}\downarrow\mathcal{T}$. We rewrite the domain and codomain of $s^{(u)}_{P,Q}$ using the coend notation

$$P^{\mathcal{Y}\langle u\rangle}\bullet Q=\int^{\Delta=(u_1,\ldots,u_m)}P(\Delta+\langle u\rangle)\times Q_{u_1}(\Gamma)\times\ldots\times Q_{u_m}(\Gamma)$$

and

$$(P\bullet Q)^{\mathcal{Y}\langle u\rangle}(\Gamma)=\int^{\Delta'=(t_1,\ldots,t_n)}P(\Delta')\times Q_{t_1}(\Gamma+\langle u\rangle)\times\ldots\times Q_{t_n}(\Gamma+\langle u\rangle)$$

To define an arrow $(P^{\mathcal{Y}\langle u\rangle}\bullet Q)(\Gamma)\to(P\bullet Q)(\Gamma+\langle u\rangle)$ it suffices to give a collection of arrows

$$P(\Delta+\langle u\rangle)\times Q_{u_1}(\Gamma)\times\ldots\times Q_{u_m}(\Gamma)\to(P\bullet Q)(\Gamma+\langle u\rangle)$$

for all Δ that satisfies the wedge condition. We take the following composite

$$\begin{array}{c}
P(\Delta+\langle u\rangle)\times Q_{u_1}(\Gamma)\times\ldots\times Q_{u_m}(\Gamma)\\
\downarrow\\
P(\Delta+\langle u\rangle)\times Q_{u_1}(\Gamma+\langle u\rangle)\times\ldots\times Q_{u_m}(\Gamma+\langle u\rangle)\times 1\\
\downarrow\\
P(\Delta+\langle u\rangle)\times Q_{u_1}(\Gamma+\langle u\rangle)\times\ldots\times Q_{u_m}(\Gamma+\langle u\rangle)\times Q_u(\Gamma+\langle u\rangle)\\
\downarrow\\
\int^{\Delta'=(t_1,\ldots,t_n)}P(\Delta')\times Q_{t_1}(\Gamma+\langle u\rangle)\times\ldots\times Q_{t_n}(\Gamma+\langle u\rangle)
\end{array}$$

where we used $\overline{q_u}:1\to Q_u^{\mathcal{Y}\langle u\rangle}$ the transpose of $q_u:\mathcal{Y}\langle u\rangle\to Q_u$. For the remaining verifications see appendix C.5. $\square$

Corollary 5.7.97 *Let $u \in \mathcal{T}$. The endofunctor $(-)^{\mathcal{Y}\langle u\rangle}$ on $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ is $U\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strong.*

Proof. Let $P \in [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ and $Q \in \mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$. Then by the previous proposition, there is a $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strength

$$P_t^{\mathcal{Y}\langle u\rangle} \bullet Q \to (P_t \bullet Q)^{\mathcal{Y}\langle u\rangle}$$

for all $t \in \mathcal{T}$. So we take them as the fibres to define a $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strength

$$P^{\mathcal{Y}\langle u\rangle} \otimes Q \to (P \otimes Q)^{\mathcal{Y}\langle u\rangle}$$

The commutativity of the following diagrams

$$\begin{tikzcd}
(P^{\mathcal{Y}\langle u\rangle}\otimes Q)\otimes R \arrow[r, "\alpha"] \arrow[d, "s^{(u)}\otimes R"'] & P^{\mathcal{Y}\langle u\rangle}\otimes(Q\otimes R) \arrow[dd, "s^{(u)}"] \\
(P\otimes Q)^{\mathcal{Y}\langle u\rangle}\otimes R \arrow[d, "s^{(u)}"'] & \\
((P\otimes Q)\otimes R)^{\mathcal{Y}\langle u\rangle} \arrow[r, "\alpha^{\mathcal{Y}\langle u\rangle}"'] & (P\otimes(Q\otimes R))^{\mathcal{Y}\langle u\rangle}
\end{tikzcd}
\qquad
\begin{tikzcd}
P^{\mathcal{Y}\langle u\rangle}\otimes\mathcal{Y} \arrow[r, "s^{(u)}"] \arrow[d, "\rho"'] & (P\otimes\mathcal{Y})^{\mathcal{Y}\langle u\rangle} \arrow[dl, "\rho^{\mathcal{Y}\langle u\rangle}"] \\
P^{\mathcal{Y}\langle u\rangle} &
\end{tikzcd}$$

follows from the commutativity of the respective diagrams in each component $t \in \mathcal{T}$. □

Proposition 5.7.98 *Let $S = (\alpha_k)_{k\in I}$ be a signature and α_k an arity for each k as in definition 5.3.78*

$$(t_{1,1}^{(k)}\dots t_{1,m_1}^{(k)})t_1^{(k)},\dots,(t_{n_k,1}^{(k)}\dots t_{n_k,m_{n_k}}^{(k)})t_{n_k}^{(k)} \to t_0^{(k)}$$

Let Σ be the binding signature functor associated to S. Then Σ is $U\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strong.

Proof. By proposition 4.7.52, for a collection of types $u_1,\dots,u_m \in \mathcal{T}$, the functor $(-)^{\mathcal{Y}(u_1,\dots,u_m)}$ is $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strong on $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ and $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$ as well. By propositions 4.7.53 and 4.7.54, finite products and coproducts of such functors are $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strong as well. A binding signature endofunctor is made up of such ingredients, so it is $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$–strong. □

5.8 Strength for T

Definition 5.8.99 (strength for a monad) *Let $(\mathcal{C},\otimes,I)$ be a monoidal category and (T,η,μ) a monad on $\mathcal{C}$. A strength for the monad T is a strength for the endofunctor T*

$$t_{A,B} : TA\otimes B \to T(A\otimes B)$$

such that the following diagrams commute

$$\begin{tikzcd}
A\otimes B \arrow[r, "\eta_A\otimes B"] \arrow[dr, "\eta_{A\otimes B}"'] & TA\otimes B \arrow[d, "t_{A,B}"] \\
& T(A\otimes B)
\end{tikzcd}$$

and

$$\begin{tikzcd} TTA \otimes B \arrow[r, "t_{TA,B}"] \arrow[d, "\mu_A \otimes B"'] & T(TA \otimes B) \arrow[r, "Tt_{A,B}"] & TT(A \otimes B) \arrow[d, "\mu A \otimes B"] \\ TA \otimes B \arrow[rr, "t_{A,B}"'] & & T(A \otimes B) \end{tikzcd}$$

Let $\Sigma : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ be a binding signature functor and $s_{P,Q} : \Sigma P \otimes Q \to \Sigma(P \otimes Q)$ a $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$–strength for Σ.

Proposition 5.8.100 *The free monad T generated by Σ is $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$–strong.*

Proof. We are going to construct a natural transformation $t_{X,Y} : TX \otimes Y \to T(X \otimes Y)$ for $X \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ and $Y \in \mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ and show then the monad strength axioms. By the adjunction $- \otimes Y \dashv Y \multimap -$ explained in section 5.6, we define the transpose $\widehat{t}_{X,Y} : TX \to Y \multimap T(X \otimes Y)$ by initiality of TX. We provide $Y \multimap T(X \otimes Y)$ with a Σ_X–algebra structure and take $\widehat{t}_{X,Y}$ to be the unique Σ_X–algebra morphism from TX to $Y \multimap T(X \otimes Y)$.

To give an arrow $X + \Sigma(Y \multimap T(X \otimes Y)) \to Y \multimap T(X \otimes Y)$, is equivalent to giving two arrows

$$X \to Y \multimap T(X \otimes Y)$$

and

$$\Sigma(Y \multimap T(X \otimes Y)) \to Y \multimap T(X \otimes Y)$$

We have $\eta_{X \otimes Y} : X \otimes Y \to T(X \otimes Y)$ and by the above adjunction

$$\widehat{\eta}_{X \otimes Y} : X \to Y \multimap T(X \otimes Y)$$

Next we consider the composition of the following arrows

$$\begin{tikzcd} \Sigma(Y \multimap T(X \otimes Y)) \odot Y \arrow[d, "s_{Y \multimap T(X \otimes Y), Y}"] \\ \Sigma((Y \multimap T(X \otimes Y)) \odot Y) \arrow[d, "\Sigma\varepsilon_{T(X \otimes Y)}"] \\ \Sigma T(X \otimes Y) \arrow[d, "\sigma_{X \otimes Y}"] \\ T(X \otimes Y) \end{tikzcd}$$

where ε is the counit of the above adjunction. The transpose of $\sigma_{X \otimes Y} \circ \Sigma\varepsilon_{T(X \otimes Y)} \circ s_{Y \multimap T(X \otimes Y), Y}$, denoted α, is of type $\Sigma(Y \multimap T(X \otimes Y)) \to Y \multimap T(X \otimes Y)$. For the remaining verifications see appendix C.6. $\square$

5.9 Substitution

Simultaneous substitution on a presheaf of terms is modeled by a monoid on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ with the above definition of substitution monoidal product.

Proposition 5.9.101 *Let* $\Sigma : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ *be a binding signature functor and* $T\mathcal{Y}$ *the free* Σ*–algebra on* $\mathcal{Y}$*. Then* $T\mathcal{Y}$ *can be provided with a monoidal structure in* $([\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}, \otimes, \mathcal{Y})$*.*

Proof. The multiplication $m : T\mathcal{Y} \otimes T\mathcal{Y} \to T\mathcal{Y}$ is given by the following composite:

$$T\mathcal{Y} \otimes T\mathcal{Y} \xrightarrow{t_{\mathcal{Y},T\mathcal{Y}}} T(\mathcal{Y} \otimes T\mathcal{Y}) \xrightarrow{T\lambda} TT\mathcal{Y} \xrightarrow{\mu_{\mathcal{Y}}} T\mathcal{Y}$$

where t denotes the strength of T. The unit is given by

$$\mathcal{Y} \xrightarrow{\eta_{\mathcal{Y}}} T\mathcal{Y}$$

We check the monoid axioms. The first one

$$\begin{tikzcd}
(T\mathcal{Y} \otimes T\mathcal{Y}) \otimes T\mathcal{Y} \arrow[r, "\alpha"] \arrow[d, "m \otimes T\mathcal{Y}"'] & T\mathcal{Y} \otimes (T\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r, "T\mathcal{Y} \otimes m"] & T\mathcal{Y} \otimes T\mathcal{Y} \arrow[d, "m"] \\
T\mathcal{Y} \otimes T\mathcal{Y} \arrow[rr, "m"'] & & T\mathcal{Y}
\end{tikzcd}$$

becomes the following when we unfold the definition of m

$$\begin{tikzcd}
(T\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r] \arrow[d] & T\mathcal{Y} \otimes (T\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r] \arrow[dr] \arrow[d, phantom, "I."] & T\mathcal{Y} \otimes T(\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r] \arrow[dr] \arrow[d, phantom, "II."] & T\mathcal{Y} \otimes TT\mathcal{Y} \arrow[r] \arrow[dr] \arrow[d, phantom, "II."] & T\mathcal{Y} \otimes T\mathcal{Y} \arrow[dr] \arrow[d, phantom, "II."] & \\
T(\mathcal{Y} \otimes T\mathcal{Y}) \otimes T\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "II."] & T((\mathcal{Y} \otimes T\mathcal{Y}) \otimes T\mathcal{Y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "III."] & T(\mathcal{Y} \otimes (T\mathcal{Y} \otimes T\mathcal{Y})) \arrow[r] \arrow[dl] \arrow[d, phantom, "IV."] & T(\mathcal{Y} \otimes T(\mathcal{Y} \otimes T\mathcal{Y})) \arrow[r] \arrow[dl] \arrow[d, phantom, "IV."] & T(\mathcal{Y} \otimes TT\mathcal{Y}) \arrow[r] \arrow[dl] \arrow[d, phantom, "IV."] & T(\mathcal{Y} \otimes T\mathcal{Y}) \arrow[dl] \\
TT\mathcal{Y} \otimes T\mathcal{Y} \arrow[r] \arrow[d] & T(T\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r] \arrow[d, phantom, "V."] & TT(\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "VI."] & TTT\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "VII."] & TT\mathcal{Y} \arrow[d] & \\
T\mathcal{Y} \otimes T\mathcal{Y} \arrow[rr] & & T(\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r] & TT\mathcal{Y} \arrow[r] & T\mathcal{Y} &
\end{tikzcd}$$

The diagram I. is a strength axiom, the squares II. are naturality squares of t, the triangle III. is a monoidal category axiom, the squares IV. are naturality squares of λ, diagram V. is a monad strength axiom, the square VI. is a naturality square of μ and square VII. is a monad axiom.

The unit axioms

$$\begin{tikzcd}
\mathcal{Y} \otimes T\mathcal{Y} \arrow[r, "\eta \otimes T\mathcal{Y}"] \arrow[dr, "\lambda"'] & T\mathcal{Y} \otimes T\mathcal{Y} \arrow[d, "m"] & T\mathcal{Y} \otimes \mathcal{Y} \arrow[l, "T\mathcal{Y} \otimes \eta"'] \arrow[dl, "\rho"] \\
& T\mathcal{Y} &
\end{tikzcd}$$

become the following when we unfold m

$$\begin{tikzcd}
\mathcal{Y} \otimes T\mathcal{Y} \arrow[r, "\eta \otimes T\mathcal{Y}"] \arrow[dr, "T\eta"'] \arrow[dddr, "\lambda"'] & T\mathcal{Y} \otimes T\mathcal{Y} \arrow[d, "t"] & & T\mathcal{Y} \otimes \mathcal{Y} \arrow[ll, "T\mathcal{Y} \otimes \eta"'] \arrow[dl, "t"'] \arrow[ddl, "\rho"] \\
& T(\mathcal{Y} \otimes T\mathcal{Y}) \arrow[d, "T\lambda"', "T\rho"] & T(\mathcal{Y} \otimes \mathcal{Y}) \arrow[l, "T(\mathcal{Y} \otimes \eta)"'] \arrow[d, "T\rho"'] & \\
& TT\mathcal{Y} \arrow[d, "\mu"] & T\mathcal{Y} \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}"] & \\
& T\mathcal{Y} & &
\end{tikzcd}$$

The left upper triangle is an axiom for strength of monads and the bottom square is a naturality square of λ. The right upper square is a naturality square of t and the right bottom square is a naturality square of ρ. The upper right triangle is an axiom for strengths and the bottom triangle is a monad axiom. $\square$

5.10 Initial algebra semantics

The notion that combines algebra structure for a strong endofunctor F and substitution is the notion of F–monoid. We recall the definition.

Let (F, f) be a strong endofunctor on a monoidal category $(\mathcal{C}, \otimes, I)$. An F–monoid is a quadruple (A, m, e, a) consisting of an object A of $\mathcal{C}$ such that (A, m, e) is a monoid in $(\mathcal{C}, \otimes, I)$ and (A, a) is an F–algebra such that

$$\begin{tikzcd} FA \otimes A \arrow[r, "f_{A,A}"] \arrow[d, "a \otimes A"'] & F(A \otimes A) \arrow[r, "Fm"] & FA \arrow[d, "a"] \\ A \otimes A \arrow[rr, "m"'] & & A \end{tikzcd}$$

commutes.

A morphism of F–algebras is a morphism of $\mathcal{C}$ such that it is a morphism of F–algebras and a monoid morphism.

F–monoids and morphisms of F–monoids form a category, we write F–Mon for this category.

Theorem 5.10.102 *Let $\Sigma : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ be a binding signature functor and $T\mathcal{Y}$ the free Σ–algebra on $\mathcal{Y}$. Then $T\mathcal{Y}$ is an initial Σ–monoid.*

Proof. We have seen that $T\mathcal{Y}$ can be provided with a monoid structure. In order to show that it is an object of Σ–Mon, we check the compatibility of the monoidal and the Σ–algebra structures, that is, that the following diagram commutes.

$$\begin{tikzcd} \Sigma T\mathcal{Y} \otimes T\mathcal{Y} \arrow[r] \arrow[d] & \Sigma(T\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r] & \Sigma T\mathcal{Y} \arrow[d] \\ T\mathcal{Y} \otimes T\mathcal{Y} \arrow[rr] & & T\mathcal{Y} \end{tikzcd}$$

we unfold the arrows and we find

$$\begin{tikzcd} \Sigma T\mathcal{Y} \otimes T\mathcal{Y} \arrow[r, "s"] \arrow[d, "\sigma \otimes T\mathcal{Y}"'] & \Sigma(T\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r, "\Sigma t"] & \Sigma T(\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r, "\cong"] \arrow[d, "\sigma"] & \Sigma TT\mathcal{Y} \arrow[r, "\Sigma\mu"] \arrow[d, "\sigma"] & \Sigma T\mathcal{Y} \arrow[d, "\sigma"] \\ T\mathcal{Y} \otimes T\mathcal{Y} \arrow[rr, "t"'] & & T(\mathcal{Y} \otimes T\mathcal{Y}) \arrow[r, "\cong"'] & TT\mathcal{Y} \arrow[r, "\mu"'] & T\mathcal{Y} \end{tikzcd}$$

The diagram on the left commutes because of (C.7), the square in the middle is a naturality square of σ and the square on the right commutes by definition of $\mu_{\mathcal{Y}}$ being the initial $T\mathcal{Y} + \Sigma$–algebra.

In order to check initiality suppose that (X, m, e, x) is another object of Σ–Mon. It is in particular a $\mathcal{Y} + \Sigma$–algebra by

$$[e, x] : \mathcal{Y} + \Sigma X \to X$$

By initiality of $T\mathcal{Y}$ there exists a unique morphism h of Σ–algebras from $T\mathcal{Y}$ to X, so the following square commutes

$$\begin{tikzcd} \mathcal{Y}+\Sigma TX \arrow[r, "\mathcal{Y}+\Sigma h"] \arrow[d, "{[\eta_{\mathcal{Y}},\sigma_{\mathcal{Y}}]}"'] & \mathcal{Y}+\Sigma X \arrow[d, "{[e,x]}"] \\ T\mathcal{Y} \arrow[r, "h"'] & X \end{tikzcd} \tag{5.1}$$

It remains to check that h is as well a morphism of monoids and thus a morphism of Σ–monoids. The diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\eta_{\mathcal{Y}}"] \arrow[dr, "e"'] & T\mathcal{Y} \arrow[d, "h"] \\ & X \end{tikzcd}$$

commutes by (5.1). The diagram

$$\begin{tikzcd} T\mathcal{Y}\otimes T\mathcal{Y} \arrow[r, "h\otimes h"] \arrow[d] & X\otimes X \arrow[d, "m"] \\ T\mathcal{Y} \arrow[r, "h"'] & X \end{tikzcd}$$

becomes the following when we unfold the arrows

$$\begin{tikzcd} T\mathcal{Y}\otimes T\mathcal{Y} \arrow[r, "T\mathcal{Y}\otimes h"] \arrow[d, "t"'] & T\mathcal{Y}\otimes X \arrow[r, "h\otimes X"] & X\otimes X \arrow[ddd, "m"] \\ T(\mathcal{Y}\otimes T\mathcal{Y}) \arrow[d, "\cong"'] & & \\ TT\mathcal{Y} \arrow[d, "\mu"'] & & \\ T\mathcal{Y} \arrow[rr, "h"'] & & X \end{tikzcd} \tag{5.2}$$

First remark that there is a unique arrow u of $X+\Sigma$–algebras $TX \to X$, defined by initiality of TX since X can be provided with a $X+\Sigma$–algebra structure

$$[\mathrm{id}_X, x] : X+\Sigma X \to X$$

Next remark that the following diagram commutes

$$\begin{tikzcd} T\mathcal{Y} \arrow[rr, "Te"] \arrow[dr, "h"'] & & TX \arrow[dl, "u"] \\ & X & \end{tikzcd}$$

Te is by definition a morphism of $\mathcal{Y}+\Sigma$–algebras. We show that so is u and then by initiality of $T\mathcal{Y}$ we can conclude that the two morphisms of $\mathcal{Y}+\Sigma$–algebras h and $u\circ Te$ are equal. We check that the following diagram commutes

$$
\begin{tikzcd}
\mathcal{Y}+\Sigma TX \arrow[r, "\mathcal{Y}+\Sigma u"] \arrow[d, "e+\mathrm{id}"'] & \mathcal{Y}+\Sigma X \arrow[d, "e+\mathrm{id}"] \\
X+\Sigma TX \arrow[r, "X+\Sigma u"] \arrow[d, "{[\eta_X,\sigma_X]}"'] & X+\Sigma X \arrow[d, "{[\mathrm{id}_X,x]}"] \\
TX \arrow[r, "u"'] & X
\end{tikzcd}
$$

The top square commutes trivially and the bottom square by definition of u being the unique morphism of $X+\Sigma$–algebras from TX to X.

Now we rewrite the diagram (5.2) using the factorisation $h = u \circ Te$.

$$
\begin{tikzcd}
T\mathcal{Y}\otimes T\mathcal{Y} \arrow[r, "T\mathcal{Y}\otimes h"] \arrow[d, "t"'] \arrow[dr, phantom, "I."] & T\mathcal{Y}\otimes X \arrow[r, "Te\otimes X"] \arrow[d, "t"'] \arrow[dr, phantom, "III."] & TX\otimes X \arrow[r, "u\otimes X"] \arrow[d, "t"'] & X\otimes X \arrow[ddd, "m"] \\
T(\mathcal{Y}\otimes T\mathcal{Y}) \arrow[r, "T(\mathcal{Y}\otimes h)"] \arrow[d, "\cong"'] \arrow[dr, phantom, "II."] & T(\mathcal{Y}\otimes X) \arrow[r, "T(e\otimes X)"] \arrow[d, "\cong"'] \arrow[dr, phantom, "IV."] & T(X\otimes X) \arrow[dl, "Tm"] \arrow[dr, phantom, "VI."] & \\
TT\mathcal{Y} \arrow[r, "Th"] \arrow[d, "\mu_{\mathcal{Y}}"'] & TX \arrow[drr, "u"] \arrow[d, phantom, "V."] & & \\
T\mathcal{Y} \arrow[rrr, "h"'] & & & X
\end{tikzcd}
$$

The squares I. and III. are naturality squares of t, square II. is a naturality square of λ and triangle IV. is one of the monoid axioms. It remains to check the commutativity of diagrams V. and VI.

First we check the commutativity of V. We show that u and h are morphisms of $T\mathcal{Y}+\Sigma$–algebras. By definition $\mu_{\mathcal{Y}}$ and Th morphisms of $T\mathcal{Y}+\Sigma$–algebras and by initiality of $TT\mathcal{Y}$ we can conclude that the two morphisms of $T\mathcal{Y}+\Sigma$–algebras $h \circ \mu_{\mathcal{Y}}$ and $u \circ Th$ are equal.

- X is a $T\mathcal{Y}+\Sigma$–algebra:

$$[h, x] : T\mathcal{Y}+\Sigma X \to X$$

- $h : T\mathcal{Y} \to X$ is a morphism of $T\mathcal{Y}+\Sigma$–algebras:

$$
\begin{tikzcd}
T\mathcal{Y}+\Sigma T\mathcal{Y} \arrow[r, "T\mathcal{Y}+\Sigma h"] \arrow[d, "{[\mathrm{id},\sigma_X]}"'] & T\mathcal{Y}+\Sigma X \arrow[d, "{[h,x]}"] \\
T\mathcal{Y} \arrow[r, "h"'] & X
\end{tikzcd}
$$

  This square commutes by definition of h being the unique morphism of $\mathcal{Y}+\Sigma$–algebras $T\mathcal{Y} \to X$.

- $u : TX \to X$ is a morphism of $T\mathcal{Y}+\Sigma$–algebras:

$$
\begin{tikzcd}
T\mathcal{Y}+\Sigma TX \arrow[r, "T\mathcal{Y}+\Sigma u"] \arrow[d, "{[Te,\sigma_X]}"'] & T\mathcal{Y}+\Sigma X \arrow[d, "{[h,x]}"] \\
TX \arrow[r, "u"'] & X
\end{tikzcd}
$$

This square commutes by $h = Te \circ u$ and by definition of u being the unique morphism of $X + \Sigma$–algebras $TX \to X$.

To show the commutativity of VI., we show the commutativity of its transpose

$$\begin{tikzcd}
TX \arrow[rr, "\overline{u \otimes X}"] \arrow[d, "\widehat{t}"'] & & X \multimap (X \otimes X) \arrow[dd, "X \multimap m"] \\
X \multimap T(X \otimes X) \arrow[d, "X \multimap Tm"'] & & \\
X \multimap TX \arrow[rr, "X \multimap u"'] & & X \multimap X
\end{tikzcd}$$

where $\overline{u \otimes X}$ is the transpose of $u \otimes X$. By definition

$$\overline{u \otimes X} = (X \multimap (u \otimes X)) \circ \eta_{TX}$$

where η is the unit of the adjunction $- \otimes X \dashv X \multimap -$. By naturality of η

$$(X \multimap (u \otimes X)) \circ \eta_{TX} = \eta_X \circ u$$

So we are going to show the commutativity of

$$\begin{tikzcd}
TX \arrow[r, "u"] \arrow[d, "\widehat{t}"'] & X \arrow[r, "\eta_X"] & X \multimap (X \otimes X) \arrow[dd, "X \multimap m"] \\
X \multimap T(X \otimes X) \arrow[d, "X \multimap Tm"'] & & \\
X \multimap TX \arrow[rr, "X \multimap u"'] & & X \multimap X
\end{tikzcd}$$

We check that $X \multimap TX$ and $X \multimap X$ are $X + \Sigma$–algebras and that $X \multimap Tm$, $X \multimap u$ and $X \multimap m \circ \eta_X$ are morphisms of $X + \Sigma$–algebras. Then by initiality of TX we can conclude that the composites $X \multimap u \circ X \multimap Tm \circ \widehat{t}$ and $X \multimap m \circ \eta_X \circ u$ are equal.

- $X \multimap TX$ is a $X + \Sigma$–algebra: The arrow $X + \Sigma(X \multimap TX) \to X \multimap TX$ is given by

$$\frac{X \to X \multimap TX}{X \otimes X \to TX}$$

  where $X \otimes X \xrightarrow{m} X \xrightarrow{\eta_X} TX$ and

$$\frac{\Sigma(X \multimap TX) \to X \multimap TX}{\Sigma(X \multimap TX) \otimes X \to TX}$$

  where

$$\begin{tikzcd}
\Sigma(X \multimap TX) \otimes X \arrow[d, "s"] \\
\Sigma((X \multimap TX) \otimes X) \arrow[d, "\Sigma\varepsilon"] \\
\Sigma TX \arrow[d, "\sigma_X"] \\
TX
\end{tikzcd}$$

- $X \multimap X$ is a $X + \Sigma$–algebra: The arrow $X + \Sigma(X \multimap X) \to X \multimap X$ is given by

$$\frac{X \to X \multimap X}{X \otimes X \to X \otimes X}$$

and

$$\frac{\Sigma(X \multimap X) \to X \multimap X}{\Sigma(X \multimap X) \otimes X \to X}$$

where where

$$\begin{tikzcd}
\Sigma(X \multimap X) \otimes X \arrow[d, "s"] \\
\Sigma((X \multimap X) \otimes X) \arrow[d, "\Sigma\varepsilon"] \\
\Sigma X \arrow[d, "x"] \\
X
\end{tikzcd}$$

- $X \multimap Tm$ is a morphism of $X+\Sigma$–algebras: We show that the following diagram commutes

$$\begin{tikzcd}
X + \Sigma(X \multimap T(X \otimes X)) \arrow[r, "X+\Sigma(X\multimap Tm)"] \arrow[d] & X + \Sigma(X \multimap TX) \arrow[d] \\
X \multimap T(X \otimes X) \arrow[r, "X\multimap Tm"'] & X \multimap TX
\end{tikzcd}$$

It commutes because the two following diagrams commute

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d] \\
X \multimap T(X \otimes X) \arrow[r, "X\multimap Tm"'] & X \multimap TX
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Sigma(X \multimap T(X \otimes X)) \arrow[r, "\Sigma(X\multimap Tm)"] \arrow[d] & \Sigma(X \multimap TX) \arrow[d] \\
X \multimap T(X \otimes X) \arrow[r, "X\multimap Tm"'] & X \multimap TX
\end{tikzcd}$$

They commute because their transposes commute

$$\begin{tikzcd}
X \otimes X \arrow[r, "m"] \arrow[d, "\eta_{X\otimes X}"] & X \arrow[d, "\eta_X"] \\
T(X \otimes X) \arrow[r, "Tm"'] & TX
\end{tikzcd}$$

which is a naturality square of η and

$$\begin{tikzcd}
\Sigma(X \multimap T(X \otimes X)) \otimes X \arrow[r, "\Sigma(X \multimap Tm) \otimes X"] \arrow[d, "s"'] & \Sigma(X \multimap TX) \otimes X \arrow[d, "s"] \\
\Sigma((X \multimap T(X \otimes X)) \otimes X) \arrow[r, "\Sigma((X \multimap Tm) \otimes X)"] \arrow[d, "\Sigma\varepsilon_{T(X \otimes X)}"'] & \Sigma((X \multimap TX) \otimes X) \arrow[d, "\Sigma\varepsilon_{TX}"] \\
\Sigma T(X \otimes X) \arrow[r, "\Sigma Tm"] \arrow[d, "\sigma_{X \otimes X}"'] & \Sigma TX \arrow[d, "\sigma_X"] \\
T(X \otimes X) \arrow[r, "Tm"'] & TX
\end{tikzcd}$$

the top square is a naturality square of s, the middle one a naturality square of ε and the bottom one a naturality square of σ.

- $X \multimap u$ is a morphism of $X + \Sigma$–algebras: We show that the following diagram commutes

$$\begin{tikzcd}
X + \Sigma(X \multimap TX) \arrow[r, "X + \Sigma(X \multimap u)"] \arrow[d] & X + \Sigma(X \multimap X) \arrow[d] \\
X \multimap TX \arrow[r, "X \multimap u"'] & X \multimap X
\end{tikzcd}$$

It commutes because the two following diagrams commute

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "\overline{\eta_X \circ m}"'] & X \arrow[d, "\overline{m}"] \\
X \multimap TX \arrow[r, "X \multimap u"'] & X \multimap X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Sigma(X \multimap TX) \arrow[r, "\Sigma(X \multimap u)"] \arrow[d, "\overline{\sigma_X \circ \Sigma\varepsilon_{TX} \circ s}"'] & \Sigma(X \multimap X) \arrow[d, "\overline{x \circ \Sigma\varepsilon_X \circ s}"] \\
X \multimap TX \arrow[r, "X \multimap u"'] & X \multimap X
\end{tikzcd}$$

They commute because their transposes commute

$$\begin{tikzcd}
X \otimes X \arrow[d, "m"'] \arrow[ddr, "m"] & \\
X \arrow[d, "\eta_X"'] & \\
TX \arrow[r, "u"'] & X
\end{tikzcd}$$

commutes since by definition of u being the unique morphism of $X + \Sigma$–algebras from TX

to X, we have $u \circ \eta_X = \mathrm{id}_X$.

$$\begin{tikzcd}
\Sigma(X \multimap TX) \otimes X \arrow[r, "\Sigma(X \multimap u)\otimes X"] \arrow[d, "s"'] & \Sigma(X \multimap X) \otimes X \arrow[d, "s"] \\
\Sigma((X \multimap TX) \otimes X) \arrow[r, "\Sigma((X \multimap u)\otimes X)"] \arrow[d, "\Sigma\varepsilon_{TX}"'] & \Sigma((X \multimap X) \otimes X) \arrow[d, "\Sigma\varepsilon_X"] \\
\Sigma TX \arrow[r, "\Sigma u"] \arrow[d, "\sigma_X"'] & \Sigma X \arrow[d, "x"] \\
TX \arrow[r, "u"'] & X
\end{tikzcd}$$

The top square is a naturality square of s, the middle one a naturality square of ε and the bottom square commutes by definition of u being the unique morphism of $X + \Sigma$–algebras from TX to X.

- $(X \multimap m) \circ \eta_X$ is a morphism of $X + \Sigma$–algebras: We show that the following diagram commutes

$$\begin{tikzcd}
X + \Sigma X \arrow[r, "X+\Sigma((X \multimap m)\circ\eta_X)"] \arrow[d, "{[\mathrm{id}_X, x]}"'] & X + \Sigma(X \multimap X) \arrow[d, "{[\overline{m}, \overline{s\circ\Sigma\varepsilon_X\circ x}]}"] \\
X \arrow[r, "(X \multimap m)\circ\eta_X"'] & X \multimap X
\end{tikzcd}$$

It commutes because the following two diagrams commute

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[d, "\overline{m}"] \\
X \arrow[r, "(X \multimap m)\circ\eta_X"'] & X \multimap X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma((X \multimap m)\circ\eta_X)"] \arrow[d, "x"'] & \Sigma(X \multimap X) \arrow[d, "\overline{s\circ\Sigma\varepsilon_X\circ x}"] \\
X \arrow[r, "(X \multimap m)\circ\eta_X"'] & X \multimap X
\end{tikzcd}$$

The first diagram commutes trivially since by definition of the transpose $\overline{m} = (X \multimap m) \circ \eta_X$. The second diagram commutes because its transpose commutes

$$\begin{tikzcd}
\Sigma X \otimes X \arrow[rr, "\Sigma\eta_X\otimes X"] \arrow[dr, "s"'] \arrow[ddd, "x\otimes X"'] \arrow[ddr, phantom, "III."] & & \Sigma(X \multimap (X\otimes X))\otimes X \arrow[r, "\Sigma(X\multimap m)\otimes X"] \arrow[d, "s"] \arrow[dr, phantom, "II."] & \Sigma(X \multimap X)\otimes X \arrow[d, "s"] \\
 & \Sigma(X\otimes X) \arrow[r, "\Sigma(\eta_X\otimes X)"] \arrow[dr, "\mathrm{id}"'] \arrow[ur, phantom, "I."] & \Sigma((X \multimap (X\otimes X))\otimes X) \arrow[r, "\Sigma((X\multimap m)\otimes X)"] \arrow[d, "\Sigma\varepsilon_{X\otimes X}"] \arrow[dl, phantom, "IV."] \arrow[dr, phantom, "V."] & \Sigma((X \multimap X)\otimes X) \arrow[d, "\Sigma\varepsilon_X"] \\
 & & \Sigma(X\otimes X) \arrow[r, "\Sigma m"'] & \Sigma X \arrow[d, "x"] \\
X\otimes X \arrow[rrr, "m"'] & & & X
\end{tikzcd}$$

The squares I. and II. are naturality squares of s, diagram III. is a Σ–monoid axiom, triangle IV. is one of the triangle identities of the adjunction and V. is a naturality square of ε.

$\square$

Chapter 6

The monadic approach

In this chapter we are going to focus on the point of view of Hirschowitz and Maggesi. It is strongly based on monads which encompass variables and capture avoiding (simultaneous) substitution.

6.1 Modules on monads

First we give the key notion of a module on a monad and we give some general constructions of modules. This notion of module is more general then the classical notion of module on a monoid such as in definition 4.6.41.

Definition 6.1.103 (module) *Let (R, η, μ) be a monad on a category $\mathcal{C}$. A* module *on R (or an R–module) is a pair (M, σ) where M is a functor $\mathcal{C} \to \mathcal{D}$ and σ a natural transformation $M \circ R \to M$ such that*

$$
\begin{tikzcd}
MRR \arrow[r, "M\mu"] \arrow[d, "\sigma R"'] & MR \arrow[d, "\sigma"] \\
MR \arrow[r, "\sigma"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M\,\mathrm{Id} \arrow[r, "M\eta"] \arrow[dr] & MR \arrow[d, "\sigma"] \\
 & M
\end{tikzcd}
$$

commute.

We call σ the module action of M.

Definition 6.1.104 (module morphism) *Let (R, η, μ) be a monad on a category $\mathcal{C}$. A* module morphism *from $(M, \sigma^{(M)})$ to $(N, \sigma^{(N)})$ is a natural transformation $\tau : M \to N$ such that*

$$
\begin{tikzcd}
MR \arrow[r, "\tau R"] \arrow[d, "\sigma^{(M)}"'] & NR \arrow[d, "\sigma^{(N)}"] \\
M \arrow[r, "\tau"'] & N
\end{tikzcd}
$$

commutes.

Notation 6.1.105 *Modules on R with common codomain $\mathcal{D}$ and module morphisms between them form a category* $\mathrm{Mod}^R_{\mathcal{D}}$.

Example 6.1.106 *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three categories.*

- *A monad R on $\mathcal{C}$ is an R–module.*
- *Let R be a monad on $\mathcal{C}$, M an R–module with codomain $\mathcal{D}$ and F a functor $\mathcal{D} \to \mathcal{E}$. Then $F \circ M$ is also an R–module.*
- *Let $D \in \mathcal{D}$ be an object. The constant functor $\mathcal{C} \to \mathcal{D}$, $C \mapsto D$ for all C is an R–module for every monad R on $\mathcal{C}$.*
- *Let R be a monad, M, N two R–modules with codomain $\mathcal{D}$ and suppose that $\mathcal{D}$ is complete. Then the pointwise product $M \times N$ is an R–module.*

Example 6.1.107 (pull–back module) *Let R, S be two monads on a category $\mathcal{C}$ and ρ be a monad morphism $R \to S$. Let (M, σ) be an S–module. Then $(M, \sigma \circ M\rho)$ is an R–module, the pull–back module along ρ. We write $\rho^* M$ for it.*

Example 6.1.108 *Let R, S be two monads and ρ a monad morphism $R \to S$. It induces an R–module morphism $R \to \rho^* S$ still denoted ρ.*

There a large category of modules where modules based on different monads are mixed together.

Notation 6.1.109 *The objects are pairs (R, M) where R is a monad and M is an R–module. An arrow from (R, M) to (S, N) is a pair (ρ, τ) where $\rho : R \to S$ is a monad morphism and τ is an R–module morphism $M \to \rho^* N$. We write* $\mathrm{Mod}_{\mathcal{D}}$ *for this category.*

6.2 Modules on $\mathrm{Set}/\mathcal{T}$

In this section we introduce special kinds of modules on the slice category $\mathrm{Set}/\mathcal{T}$ for a given set $\mathcal{T}$: the derived modules and the fibre modules. At first we give some notations on the slice category $\mathrm{Set}/\mathcal{T}$.

The category $\mathrm{Set}/\mathcal{T}$ has finite coproducts. The coproduct of $\Gamma : \overline{\Gamma} \to \mathcal{T}$ and $\Delta : \overline{\Delta} \to \mathcal{T}$ is given by the unique arrow $[\Gamma, \Delta] : \overline{\Gamma} + \overline{\Delta} \to \mathcal{T}$ in Set.

Moreover an element t of $\mathcal{T}$ induces an object in $\mathrm{Set}/\mathcal{T}$: $\langle t \rangle : 1 \to \mathcal{T}, 1 \mapsto t$.

6.2.1 Derived module

Definition 6.2.110 (derived module) *Let R be a monad on* $\mathrm{Set}/\mathcal{T}$*, M an R–module with codomain $\mathcal{C}$ and $u \in \mathcal{T}$. The* derived module *of M with respect to u is given on objects by*

$$\partial_u M(\Gamma) := M(\Gamma + \langle u \rangle)$$

We have to check that this definition is correct, that is, that $\partial_u M$ is an R–module.

First let us check the functoriality. Let $h : \Gamma \to \Delta$ be an arrow in $\mathrm{Set}/\mathcal{T}$. We write $h + \langle u \rangle$ for the unique arrow $\Gamma + \langle u \rangle \to \Delta + \langle u \rangle$ given by the universal property of the coproduct and the composites $\Gamma \xrightarrow{h} \Delta \xrightarrow{incl_\ell} \Delta + \langle u \rangle$ and $\langle u \rangle \xrightarrow{incl_r} \Delta + \langle u \rangle$. We define $\partial_u M(h)$ to be the arrow

$$M(h + \langle u \rangle) : M(\Gamma + \langle u \rangle) \to M(\Delta + \langle u \rangle)$$

Next we provide $\partial_u M$ with a module action $\sigma' : \partial_u M \circ R \to \partial_u M$. We construct it componentwise: $\sigma'_\Gamma : M(R\Gamma + \langle u \rangle) \to M(R(\Gamma + \langle u \rangle))$. The arrow

$$\alpha_\Gamma : R\Gamma + \langle u \rangle \to R(\Gamma + \langle u \rangle)$$

is given by the two arrows

$$R\Gamma \xrightarrow{Rincl_\ell} R(\Gamma + \langle u \rangle)$$

and

$$\langle u \rangle \xrightarrow{incl_r} \Gamma + \langle u \rangle \xrightarrow{\eta_{\Gamma+\langle u \rangle}} R(\Gamma + \langle u \rangle)$$

We define σ'_Γ to be the composite

$$M(R\Gamma + \langle u \rangle) \xrightarrow{M\alpha_\Gamma} MR(\Gamma + \langle u \rangle) \xrightarrow{\sigma_{\Gamma+\langle u \rangle}} M(\Gamma + \langle u \rangle)$$

σ' is natural in Γ since α and σ are.

At last we check the module axioms for σ'.

$$\begin{tikzcd}
M(\Gamma + \langle u \rangle) \arrow[r, "M(\eta_\Gamma+\langle u \rangle)"] \arrow[dr, "M\eta_{\Gamma+\langle u \rangle}"'] \arrow[ddr, bend right, "\mathrm{id}_{M(\Gamma+\langle u \rangle)}"'] & M(R\Gamma + \langle u \rangle) \arrow[d, "M\alpha_\Gamma"] \\
& MR(\Gamma + \langle u \rangle) \arrow[d, "\sigma_{\Gamma+\langle u \rangle}"] \\
& M(\Gamma + \langle u \rangle)
\end{tikzcd}$$

The bottom triangle commutes because it is the module axiom for σ at the component $\Gamma + \langle u \rangle$ and the upper triangle commutes because the following triangle commutes

$$\begin{tikzcd}
\Gamma + \langle u \rangle \arrow[r, "\eta_\Gamma+\langle u \rangle"] \arrow[dr, "\eta_{\Gamma+\langle u \rangle}"'] & R\Gamma + \langle u \rangle \arrow[d, "{[Rincl_\ell, \eta_{\Gamma+\langle u \rangle} \circ incl_r]}"] \\
& R(\Gamma + \langle u \rangle)
\end{tikzcd}$$

where we unfold the definition of α_Γ and by naturality of η we have $incl_\ell \circ \eta_{\Gamma+\langle u \rangle} = Rincl_\ell \circ \eta_\Gamma$

The other module axiom becomes the following diagram when we unfold σ'.

$$\begin{tikzcd}
M(RR\Gamma + \langle u \rangle) \arrow[rr, "M(\mu_\Gamma+\langle u \rangle)"] \arrow[d, "M\alpha_{R\Gamma}"'] & & M(R\Gamma + \langle u \rangle) \arrow[d, "M\alpha_\Gamma"] \\
MR(R\Gamma + \langle u \rangle) \arrow[r, "MR\alpha_\Gamma"] \arrow[d, "\sigma_{R\Gamma+\langle u \rangle}"'] & MRR(\Gamma + \langle u \rangle) \arrow[r, "M\mu_{\Gamma+\langle u \rangle}"] \arrow[d, "\sigma_{R(\Gamma+\langle u \rangle)}"'] & MR(\Gamma + \langle u \rangle) \arrow[d, "\sigma_{\Gamma+\langle u \rangle}"] \\
M(R\Gamma + \langle u \rangle) \arrow[r, "M\alpha_\Gamma"'] & MR(\Gamma + \langle u \rangle) \arrow[r, "\sigma_{\Gamma+\langle u \rangle}"'] & M(\Gamma + \langle u \rangle)
\end{tikzcd}$$

The bottom right square is a module axiom for σ at the component $\Gamma + \langle u \rangle$. The bottom left square is a naturality square of σ. The top square commutes because the two following diagrams commute.

$$\begin{tikzcd}
RR\Gamma \arrow[rr, "\mu_\Gamma"] \arrow[d, "Rincl_\ell"'] \arrow[dr, "RRincl_\ell"] & & R\Gamma \arrow[d, "Rincl_\ell"] \\
R(R\Gamma + \langle u \rangle) \arrow[r, "R\alpha_\Gamma"'] & RR(\Gamma + \langle u \rangle) \arrow[r, "\mu_{\Gamma+\langle u \rangle}"'] & R(\Gamma + \langle u \rangle)
\end{tikzcd}$$

the triangle commutes by definition of α and the square is a naturality square of μ.

$$\begin{tikzcd}
\langle u \rangle \arrow[r, "\mathrm{id}"] \arrow[d, "incl_r"'] & \langle u \rangle \arrow[d, "incl_r"] \\
R\Gamma + \langle u \rangle \arrow[d, "\eta_{R\Gamma + \langle u \rangle}"'] & \Gamma + \langle u \rangle \arrow[l, "\eta_{\Gamma+\langle u \rangle}"'] \arrow[d, "\eta_{\Gamma+\langle u \rangle}"] \\
R(R\Gamma + \langle u \rangle) \arrow[d, "R\alpha_\Gamma"'] & R(\Gamma + \langle u \rangle) \arrow[l, "R(\eta_{\Gamma+\langle u \rangle})"'] \arrow[d, "\mathrm{id}"] \arrow[dl, "R(\eta_{\Gamma+\langle u \rangle})"] \\
RR(\Gamma + \langle u \rangle) \arrow[r, "\mu_{\Gamma+\langle u \rangle}"'] & R(\Gamma + \langle u \rangle)
\end{tikzcd}$$

the top square commutes obviously, the square in the middle is a naturality square of η, the upper triangle commutes by definition of α and the bottom triangle is a monad axiom.

Lemma 6.2.111 *For each $u \in \mathcal{T}$, derivation with respect to u yields an endofunctor $\partial_u : \mathrm{Mod}^R_{\mathcal{C}} \to \mathrm{Mod}^R_{\mathcal{C}}$.*

Proof. The functor ∂_u is given on objects by the above construction. Let $\tau : M \to N$ be an R–module morphism for a monad R. The arrow $\partial_u \tau : \partial_u M \to \partial_u N$ is given by the components

$$\tau_{\Gamma+\langle u \rangle} : M(\Gamma + \langle u \rangle) \to N(\Gamma + \langle u \rangle)$$

The module axioms for $\partial_u \tau$ follow immediately from the ones for τ. □

Remark 6.2.112 *More generally the endofunctor $\partial_u : [\mathrm{Set}/\mathcal{T}, \mathcal{C}] \to [\mathrm{Set}/\mathcal{T}, \mathcal{C}]$ can be provided with an $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strength whose components are*

$$p^{(u)}_{F,G} : \partial_u F \circ G \to \partial_u (F \circ G)$$

Proof. Let $\Gamma \in \mathrm{Set}/\mathcal{T}$. The arrow $p_{F,G,\Gamma}$ is given by

$$F[G(i_\Gamma), g_{\Gamma+\langle u \rangle} \circ j_\Gamma] : F(G\Gamma + \langle u \rangle) \to F(G(\Gamma + \langle u \rangle))$$

where $i_\Gamma : \Gamma \to \Gamma + \langle u \rangle$, $j_\Gamma : \langle u \rangle \to \Gamma + \langle u \rangle$. Naturality in Γ is given since all components are natural in Γ. We check naturality in F and G. Let $f : F \to H$ be an arrow in $[\mathrm{Set}/\mathcal{T}, \mathcal{C}]$. The naturality square

$$\begin{tikzcd}
\partial_u F \circ G \arrow[r] \arrow[d] & \partial_u (F \circ G) \arrow[d] \\
\partial_u H \circ G \arrow[r] & \partial_u (H \circ G)
\end{tikzcd}$$

commutes since it is a naturality square of f

$$\begin{tikzcd}
F(G\Gamma + \langle u \rangle) \arrow[r] \arrow[d, "f"'] & FG(\Gamma + \langle u \rangle) \arrow[d, "f"] \\
H(G\Gamma + \langle u \rangle) \arrow[r] & HG(\Gamma + \langle u \rangle)
\end{tikzcd}$$

Let $f : G \to H$ in $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$. The naturality square

$$\begin{tikzcd} \partial_u F \circ G \arrow[r] \arrow[d] & \partial_u(F \circ G) \arrow[d] \\ \partial_u F \circ H \arrow[r] & \partial_u(F \circ H) \end{tikzcd}$$

since the following two diagrams commute

$$\begin{tikzcd} G\Gamma \arrow[r] \arrow[d, "f"'] & G(\Gamma + \langle u \rangle) \arrow[d, "f"] \\ H\Gamma \arrow[r] & H(\Gamma + \langle u \rangle) \end{tikzcd} \qquad \begin{tikzcd} & & G(\Gamma + \langle u \rangle) \arrow[d, "f"] \\ \langle u \rangle \arrow[r] & \Gamma + \langle u \rangle \arrow[ur, "g"] \arrow[r, "h"'] & H(\Gamma + \langle u \rangle) \end{tikzcd}$$

We check the commutativity of the $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strength axioms

$$\begin{tikzcd} (\partial_u F \circ G) \circ H \arrow[r, equal] \arrow[d] & \partial_u F \circ (G \circ H) \arrow[dd] \\ \partial_u(F \circ G) \circ H \arrow[d] & \\ \partial_u((F \circ G) \circ H) \arrow[r, equal] & \partial_u(F \circ (G \circ H)) \end{tikzcd} \qquad \begin{tikzcd} \partial_u F \circ \mathrm{Id} \arrow[r, equal] \arrow[d, equal] & \partial_u(F \circ \mathrm{Id}) \arrow[dl, equal] \\ \partial_u F & \end{tikzcd}$$

The triangle commutes trivially. As for the rectangular diagram along the left–hand side we have the following composite

$$FG[H(i_\Gamma), h_{\Gamma+\langle u \rangle} \circ j_\Gamma] \circ F[G(i_{H\Gamma}), g_{H\Gamma+\langle u \rangle} \circ j_{H\Gamma}]$$

and along the right–hand side

$$F[GH(i_\Gamma), gh_{\Gamma+\langle u \rangle} \circ j_\Gamma]$$

where $gh : \mathrm{Id} \to G \circ H$ is the following composite

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "g"] \arrow[d, "h"'] & G \arrow[d, "Gh"] \\ H \arrow[r, "gH"'] & GH \end{tikzcd}$$

which are the same since the following diagrams commute

$$\begin{tikzcd} H\Gamma \arrow[r, "i_{H\Gamma}"] \arrow[dr, "H(i_\Gamma)"'] & H\Gamma + \langle u \rangle \arrow[d, "{[H(i_\Gamma), h_{\Gamma+\langle u \rangle} \circ j_\Gamma]}"] \\ & H(\Gamma + \langle u \rangle) \end{tikzcd}$$

and

$$\begin{tikzcd} \langle u \rangle \arrow[r, "j_{H\Gamma}"] \arrow[d, "j_\Gamma"'] & H\Gamma + \langle u \rangle \arrow[r, "g_{H\Gamma+\langle u \rangle}"] \arrow[d, "{[H(i_\Gamma), h_{\Gamma+\langle u \rangle} \circ j_\Gamma]}"] & G(H\Gamma + \langle u \rangle) \arrow[d, "{G[H(i_\Gamma), h_{\Gamma+\langle u \rangle} \circ j_\Gamma]}"] \\ \Gamma + \langle u \rangle \arrow[r, "h_{\Gamma+\langle u \rangle}"'] \arrow[rr, bend right, "gh_{\Gamma+\langle u \rangle}"'] & H(\Gamma + \langle u \rangle) \arrow[r, "g_{H(\Gamma+\langle u \rangle)}"'] & GH(\Gamma + \langle u \rangle) \end{tikzcd}$$

□

Because of this Id $\downarrow$ [Set $/\mathcal{T}$, Set $/\mathcal{T}$]–strength for ∂_u, we can "upgrade" ∂_u into a functor $\mathrm{Mod}^R_{\mathcal{C}} \to \mathrm{Mod}^R_{\mathcal{C}}$ as stated in lemma 6.2.111.

6.2.2 Fibre module

Next we introduce another kind of modules.

Definition 6.2.113 (fibre module) *Let R be a monad on* Set $/\mathcal{T}$*, M an R–module with codomain* Set $/\mathcal{T}$ *and $u \in \mathcal{T}$. The* fibre module *of M with respect to u is given on objects by*

$$M_u(\Gamma) := (M\Gamma)^{-1}(u)$$

We have to check that this notion is well–defined, that is, that M_u is an R–module with codomain Set.

First we check the functoriality. Let $h : \Gamma \to \Delta$ be an arrow in Set $/\mathcal{T}$. We define $M_u(h)$ to be the restriction of $M(h)$ on the subset $(M\Gamma)^{-1}(u)$:

$$M(h)|_{(M\Gamma)^{-1}(u)} : (M\Gamma)^{-1}(u) \to (M\Delta)^{-1}(u)$$

Next we provide M_u with a module action $\sigma'' : M_u \circ R \to M_u$. We define it componentwise for all $\Gamma \in$ Set $/\mathcal{T}$ to be the restriction of the module action σ of M on the subset $(MR\Gamma)^{-1}(u)$:

$$\sigma_\Gamma\ |_{(MR\Gamma)^{-1}(u)} : (MR\Gamma)^{-1}(u) \to (M\Gamma)^{-1}(u)$$

The module axioms for σ'' are obviously satisfied. This follows immediately from the module axioms for M and σ.

Lemma 6.2.114 *For each $u \in \mathcal{T}$, fibre modules extend to a functor $(-)_u : \mathrm{Mod}^R_{\mathrm{Set}\,/\mathcal{T}} \to \mathrm{Mod}^R_{\mathrm{Set}}$.*

Proof. The assignation on objects is given by the above construction. Let $\tau : M \to N$ be an R–module morphism. We construct an R–module morphism $\tau_u : M_u \to N_u$. We take the component in $\Gamma \in$ Set $/\mathcal{T}$ to be the fibre of $\tau_\Gamma : M\Gamma \to N\Gamma$ in u, which is $(M\Gamma)^{-1}(u) \to (N\Gamma)^{-1}(u)$. Naturality and the module morphism axioms follow directly from the ones for τ. □

Proposition 6.2.115 *Let $\mathcal{C}$ be a category and $\mathcal{D}$ a category with finite products. Let R, S be two monads on a category $\mathcal{C}$, $\rho : R \to S$ a monad morphism and M, N two S–modules with codomain $\mathcal{D}$. Then we have the following isomorphisms*

1. *if $\mathcal{D}$ has finite products then*

$$\rho^*(M \times N) = (\rho^* M) \times (\rho^* N)$$

2. *if $\mathcal{C} =$ Set $/\mathcal{T}$ and $u \in \mathcal{T}$ then*

$$\rho^*(\partial_u M) = \partial_u(\rho^* M)$$

3. *if $\mathcal{C} =$ Set $/\mathcal{T}$ and $u, v \in \mathcal{T}$ then*

$$\partial_v \partial_u M \cong \partial_u \partial_v M$$

4. *if* $\mathcal{C} = \mathcal{D} = \mathrm{Set}/\mathcal{T}$ *and* $t, u \in \mathcal{T}$ *then*

$$\partial_u(M_t) = (\partial_u M)_t$$

Proof.

1. Since this is the pointwise product of modules, the action is given by the pointwise product of the respective actions

$$(M \times N) \circ R = MR \times NR \to MS \times NS \to M \times N$$

2. The action of $\rho^*(\partial_u M)$ at the component $\Gamma \in \mathrm{Set}/\mathcal{T}$ is given by the composite

$$M(R\Gamma + \langle u \rangle) \to M(S\Gamma + \langle u \rangle) \to MS(\Gamma + \langle u \rangle) \to M(\Gamma + \langle u \rangle)$$

and the action of $\partial_u(\rho^* M)$ at the component Γ is given by

$$M(R\Gamma + \langle u \rangle) \to MR(\Gamma + \langle u \rangle) \to MS(\Gamma + \langle u \rangle) \to M(\Gamma + \langle u \rangle)$$

They are the same since the following square commutes

$$\begin{tikzcd} R\Gamma + \langle u \rangle \arrow[r] \arrow[d] & S\Gamma + \langle u \rangle \arrow[d] \\ R(\Gamma + \langle u \rangle) \arrow[r] & S(\Gamma + \langle u \rangle) \end{tikzcd}$$

and this is the case because the following two diagrams commmute

$$\begin{tikzcd} R\Gamma \arrow[r, "\rho"] \arrow[d] & S\Gamma \arrow[d] \\ R(\Gamma + \langle u \rangle) \arrow[r, "\rho"'] & S(\Gamma + \langle u \rangle) \end{tikzcd} \qquad \begin{tikzcd} \langle u \rangle \arrow[d] & \\ \Gamma + \langle u \rangle \arrow[d] \arrow[dr] & \\ R(\Gamma + \langle u \rangle) \arrow[r, "\rho"'] & S(\Gamma + \langle u \rangle) \end{tikzcd}$$

The square on the left is a naturality square of ρ and the right triangle is a monad morphism axiom.

3. Consider the two modules at the component $\Gamma \in \mathrm{Set}/\mathcal{T}$.

$$M(\Gamma + \langle u \rangle + \langle v \rangle) \cong M(\Gamma + \langle v \rangle + \langle u \rangle)$$

by universal property of the coproduct.

4. In this equation ∂_u on the left–hand side is an endofunctor $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$ and on the right–hand side an endofunctor on $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$. So if we consider the two modules at the component $\Gamma \in \mathrm{Set}/\mathcal{T}$, we find

$$M(\Gamma + \langle u \rangle)^{-1}(u) = M(\Gamma + \langle u \rangle)^{-1}(u)$$

$\square$

6.3 Signatures and their representations

Again we fix a set of types $\mathcal{T}$ and we use the same notions of arity and signature as introduced in the previous chapter, in definitions 5.3.78 and 5.3.80. We call these arities and signatures $\mathcal{T}$–arities and $\mathcal{T}$–signatures now.

Definition 6.3.116 (representation) *Let α be a $\mathcal{T}$–arity*

$$(t_{1,1}\dots t_{1,m_1})t_1,\dots,(t_{n,1}\dots t_{n,m_n})t_n \to t_0$$

and R a monad on $\operatorname{Set}/\mathcal{T}$. A representation *of α in the monad R is an R–module morphism*

$$r^{(R)} : (\partial_{t_{1,m_1}}\dots\partial_{t_{1,1}}R)_{t_1} \times \dots \times (\partial_{t_{n,m_n}}\dots\partial_{t_{n,1}}R)_{t_n} \to R_{t_0}$$

Definition 6.3.117 *Let $S = (\alpha_k)_{k\in I}$ be a $\mathcal{T}$–signature, R a monad on $\operatorname{Set}/\mathcal{T}$. A* representation *of S in R consists of a representation of each $\mathcal{T}$–arity α_k in R.*

Example 6.3.118 (simply typed Lambda Calculus) *The signature of the simply typed Lambda Calculus without product type is given by the signature in the example 5.3.81. A representation in a monad R on $\operatorname{Set}/\mathcal{T}$ is given by two collections of R–module morphisms*

$$\forall s,t\in\mathcal{T},\quad (\partial_s R)_t \to R_{s\Rightarrow t}$$

$$\forall s,t\in\mathcal{T},\quad R_{s\Rightarrow t}\times R_s \to R_t$$

Definition 6.3.119 (morphism of representations) *Let $S = (\alpha_k)_{k\in I}$ be a $\mathcal{T}$–signature and $(P,(p_k)_{k\in I})$ a representation of S in the monad P on $\operatorname{Set}/\mathcal{T}$ and $(R,(r_k)_{k\in I})$ a representation of S in the monad R on $\operatorname{Set}/\mathcal{T}$. A morphism of representations from $(P,(p_k)_{k\in I})$ to $(R,(r_k)_{k\in I})$ is a morphism of monads $\rho : P \to R$ such that*

$$\begin{tikzcd}
\prod_{i=1}^{n}(\partial_{t_{i,m_i}}\dots\partial_{t_{i,1}}P)_{t_i} \arrow[r, "p"] \arrow[d, "\prod_{i=1}^{n}(\partial_{t_{i,m_i}}\dots\partial_{t_{i,1}}\rho)_{t_i}"'] & P_{t_0} \arrow[d, "\rho_{t_0}"] \\
\prod_{i=1}^{n}(\partial_{t_{i,m_i}}\dots\partial_{t_{i,1}}(\rho^*R))_{t_i} \arrow[r, "\rho^* r"'] & (\rho^*R)_{t_0}
\end{tikzcd}$$

commutes in $\operatorname{Mod}^P_{\operatorname{Set}}$ for each arity $\alpha = (t_{1,1}\dots t_{1,m_1})t_1,\dots,(t_{n,1}\dots t_{n,m_n})t_n \to t_0$ of S.

Notation 6.3.120 *Let S be a $\mathcal{T}$–signature. Representations of S and morphisms of representations form a category. We write $\operatorname{Rep}(S)$ for this category.*

6.4 Initial representations

The object of interest of this approach is the initial object of $\operatorname{Rep}(S)$.

Theorem 6.4.121 *Let S be a $\mathcal{T}$–signature. Then the category $\operatorname{Rep}(S)$ has an initial object.*

Proof. We write $S = (\alpha_k)_{k\in I}$ and for each $\mathcal{T}$–arity

$$(t^{(k)}_{1,1}\dots t^{(k)}_{1,m_1})t^{(k)}_1, \dots, (t^{(k)}_{n_k,1}\dots t^{(k)}_{n_k,m_{n_k}})t_{n_k} \to t^{(k)}_0$$

In this proof we write shortly $s^{(k)}_i$ for the list of types $(t^{(k)}_{i,1}\dots t^{(k)}_{i,m_i})$ and $\langle s^{(k)}_i\rangle$ for the corresponding object of $\mathrm{Set}/\mathcal{T}$.

Definition of STS

We define an object $\mathsf{STS}(\Gamma) \in \mathrm{Set}/\mathcal{T}$ for all $\Gamma \in \mathrm{Set}/\mathcal{T}$ the following way. First we define by recursion the objects $\mathsf{STS}'_n(\Gamma)$ for all $n \in \mathbb{N}$. Intuitively the set $(\mathsf{STS}'_n)_t(\Gamma) := (\mathsf{STS}'_n)(\Gamma)^{-1}(t)$ for a $t \in \mathcal{T}$ is the set of trees of type t and of depth n. We write shortly $\mathsf{STS}'_{n,t}$ for $(\mathsf{STS}'_n)_t$. We set for all $\Gamma \in \mathrm{Set}/\mathcal{T}$ and $t \in \mathcal{T}$

$$\mathsf{STS}'_{0,t}(\Gamma) := \Gamma^{-1}(t)$$

We write $I_t := \{k \in I \text{ such that } t^{(k)}_0 = t\}$ for the index set of arities of S that yield a term of type t. Let $\Theta_{p,n}$ be the set of functions $h : \{1,\dots,p\} \to \{1,\dots,n\}$ such that there exists a $j_h \in \{1,\dots,n\}$ such that $h(j_h) = n$. Then we define

$$\mathsf{STS}'_{n+1,t}(\Gamma) := \sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t^{(k)}_i}(\Gamma + \langle s^{(k)}_i\rangle)$$

and we set

$$\mathsf{STS}_t(\Gamma) := \sum_{n\in\mathbb{N}}\mathsf{STS}'_{n,t}(\Gamma)$$

By setting $\sum\limits_{t\in\mathcal{T}}\mathsf{STS}_t(\Gamma) \to \mathcal{T}$, we obtain an object of $\mathrm{Set}/\mathcal{T}$.

Functoriality of STS

We check functoriality of STS. For this we check functoriality of STS'_n for all $n \in \mathbb{N}$. Let $f : \Gamma \to \Delta$.

$$\mathsf{STS}'_{0,t}(\Gamma) \to \mathsf{STS}'_{0,t}(\Delta)$$

is given by $f_t : \Gamma^{-1}(t) \to \Delta^{-1}(t)$ for all $t \in \mathcal{T}$. Which shows the functoriality of STS'_0. Suppose that STS'_m is functorial for all $m = 0,\dots,n$, then we construct

$$\mathsf{STS}'_{n+1,t}(\Gamma) \to \mathsf{STS}'_{n+1,t}(\Delta)$$

which is explicitly

$$\sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t^{(k)}_i}(\Gamma + \langle s^{(k)}_i\rangle) \to \sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t^{(k)}_i}(\Delta + \langle s^{(k)}_i\rangle)$$

it is given by $\sum\limits_{k\in I_t}\sum\limits_{h\in\Theta_{n_k,n}}\prod\limits_{i=1}^{n_k}(\mathsf{STS}'_{h(i),t^{(k)}_i}(f + \langle s^{(k)}_i\rangle))$ where $f + \langle s^{(k)}_i\rangle : \Gamma + \langle s^{(k)}_i\rangle \to \Delta + \langle s^{(k)}_i\rangle$.

We define the arrow $\mathsf{STS}(\Gamma) \to \mathsf{STS}(\Delta)$ or equivalently the arrow $\sum\limits_{n\in\mathbb{N}}\mathsf{STS}'_{n,t}(\Gamma) \to \mathsf{STS}_t(\Delta)$ for all $t \in \mathcal{T}$ by universal property of the coproduct. We define a cocone $(\mathsf{STS}'_{n,t}(\Gamma) \to \mathsf{STS}_t(\Delta))_{n\in\mathbb{N}}$:

$$\mathsf{STS}'_{n,t}(\Gamma) \to \mathsf{STS}'_{n,t}(\Delta) \to \mathsf{STS}_t(\Delta)$$

Construction of ψ

Next we construct for all $n \in \mathbb{N}$ and all $u \in \mathcal{T}$ an arrow

$$\psi^{\langle u\rangle}_{n,\Gamma,t} : \mathsf{STS}'_{n,t}(\Gamma) + \langle u\rangle_t \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$$

For $n = 0$, the arrow $\psi^{\langle u\rangle}_{0,\Gamma,t} : \mathsf{STS}'_{0,t}(\Gamma) + \langle u\rangle_t \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$ is given by the composite of the identity on $(\Gamma + \langle u\rangle)_t$ with the inclusion $c_{0,\Gamma+\langle u\rangle,t} : \mathsf{STS}'_{0,t}(\Gamma + \langle u\rangle) \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$. We construct

$$\psi^{\langle u\rangle}_{n+1,\Gamma,t} : \mathsf{STS}'_{n+1,t}(\Gamma) + \langle u\rangle_t \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$$

It is explicitly

$$\sum_{k\in I_t} \sum_{h\in\Theta_{n_k,n}} \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)}\rangle) + \langle u\rangle_t \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$$

this arrow is given by the two following arrows

$$\langle u\rangle_t \to (\Gamma + \langle u\rangle)_t = \mathsf{STS}'_{0,t}(\Gamma + \langle u\rangle) \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$$

and

$$\sum_{k\in I_t} \sum_{h\in\Theta_{n_k,n}} \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)}\rangle) \to$$

$$\to \sum_{k\in I_t} \sum_{h\in\Theta_{n_k,n}} \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle u\rangle + \langle s_i^{(k)}\rangle) \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$$

So by universal property of the coproduct, we have the arrow

$$\psi^{\langle u\rangle}_{\Gamma,t} : \mathsf{STS}_t(\Gamma) + \langle u\rangle_t \to \mathsf{STS}_t(\Gamma + \langle u\rangle)$$

Given a finite set of types $u_1, \ldots, u_r$, by iterating the above constructed arrow, we can define

$$\psi^{\langle u_i\rangle}_{\Gamma,t} : \mathsf{STS}_t(\Gamma) + \langle u_1, \ldots, u_r\rangle_t \to \mathsf{STS}_t(\Gamma + \langle u_1, \ldots, u_r\rangle) \tag{6.1}$$

We check naturality of $\psi^{\langle u\rangle}_{n,\Gamma,t}$ in Γ. So let $f : \Gamma \to \Delta$ in $\mathrm{Set}/\mathcal{T}$. We check the commutativity of the following square

$$\begin{tikzcd}
\mathsf{STS}'_{n,t}(\Gamma) + \langle u\rangle_t \arrow[r] \arrow[d] & \mathsf{STS}_t(\Gamma + \langle u\rangle) \arrow[d] \\
\mathsf{STS}'_{n,t}(\Delta) + \langle u\rangle_t \arrow[r] & \mathsf{STS}_t(\Delta + \langle u\rangle)
\end{tikzcd}$$

by induction. For $n = 0$ we find

$$\begin{tikzcd}
\Gamma^{-1}(t) + \langle u\rangle_t \arrow[r, "c_{0,\Gamma+\langle u\rangle,t}"] \arrow[d, "f_t+\langle u\rangle_t"'] & \mathsf{STS}_t(\Gamma + \langle u\rangle) \arrow[d, "\mathsf{STS}_t(f+\langle u\rangle)"] \\
\Delta^{-1}(t) + \langle u\rangle_t \arrow[r, "c_{0,\Delta+\langle u\rangle,t}"'] & \mathsf{STS}_t(\Delta + \langle u\rangle)
\end{tikzcd}$$

which commutes by naturality of c_0. For $n+1$ we have to show that

$$\begin{array}{ccc}
\sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle)+\langle u\rangle_t & \longrightarrow & \mathsf{STS}_t(\Gamma+\langle u\rangle) \\
\downarrow & & \downarrow \\
\sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\Delta+\langle s_i^{(k)}\rangle)+\langle u\rangle_t & \longrightarrow & \mathsf{STS}_t(\Delta+\langle u\rangle)
\end{array}$$

commutes. It commutes because the two following diagrams commute

$$\begin{array}{ccccc}
\langle u\rangle_t & \xrightarrow{i_{\langle u\rangle}} & (\Gamma+\langle u\rangle)_t & \xrightarrow{c_{0,\Gamma+\langle u\rangle,t}} & \mathsf{STS}_t(\Gamma+\langle u\rangle) \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle (f+\langle u\rangle)_t} & & \Big\downarrow{\scriptstyle \mathsf{STS}_t(f+\langle u\rangle)} \\
\langle u\rangle_t & \xrightarrow[i_{\langle u\rangle}]{} & (\Delta+\langle u\rangle)_t & \xrightarrow[c_{0,\Delta+\langle u\rangle,t}]{} & \mathsf{STS}_t(\Delta+\langle u\rangle)
\end{array}$$

and

$$\begin{array}{ccc}
\sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle) & \longrightarrow & \sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\Delta+\langle s_i^{(k)}\rangle) \\
\downarrow & & \downarrow \\
\sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle+\langle u\rangle) & \longrightarrow & \sum_{k\in I_t}\sum_{h\in\Theta_{n_k,n}}\prod_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\Delta+\langle s_i^{(k)}\rangle+\langle u\rangle) \\
\downarrow & & \downarrow \\
\mathsf{STS}_t(\Gamma+\langle u\rangle) & \longrightarrow & \mathsf{STS}_t(\Delta+\langle u\rangle)
\end{array}$$

which commutes by naturality of the inclusions $\Gamma+\langle s_i^{(k)}\rangle\to\Gamma+\langle s_i^{(k)}\rangle+\langle u\rangle$ and c_{n+1}.

Naturality of $\psi^{\langle u\rangle}_{\Gamma,t}$ in Γ follows from the naturality of $\psi^{\langle u\rangle}_{n,\Gamma,t}$. And this implies naturality of (6.1) in Γ.

Definition of ρ

Let $\alpha=(s_1)t_1,\ldots,(s_n)t_n\to t_0$ be one of the arities of S. We construct an arrow

$$\rho^\alpha_\Gamma:\prod_{i=1}^{n}\mathsf{STS}_{t_i}(\Gamma+\langle s_i\rangle)\to\mathsf{STS}_{t_0}(\Gamma) \tag{6.2}$$

By definition of STS

$$\prod_{i=1}^{n}\mathsf{STS}_{t_i}(\Gamma+\langle s_i\rangle)=\prod_{i=1}^{n}\sum_{n\in\mathbb{N}}\mathsf{STS}'_{n,t_i}(\Gamma+\langle s_i\rangle)$$

By distributivity of the category of sets

$$\prod_{i=1}^{n}\sum_{n\in\mathbb{N}}\mathsf{STS}'_{n,t_i}(\Gamma+\langle s_i\rangle)\cong\sum_{g\in\mathbb{N}^n}\prod_{i=1}^{n}\mathsf{STS}'_{g(i),t_i}(\Gamma+\langle s_i\rangle)$$

Remark that to each $g \in \mathbb{N}^n$ we can associate a function $h_g \in \Theta_{n,m_g}$ where $m_g = \max(g(i))$, so for a fixed g we have the inclusion

$$\prod_{i=1}^{n} \mathsf{STS}'_{g(i),t_i}(\Gamma + \langle s_i \rangle) \to \sum_{k \in I_{t_0}} \sum_{h \in \Theta_{n_k,m_g}} \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}} = \mathsf{STS}'_{m_g+1,t_0}(\Gamma)$$

and thus

$$\sum_{g \in \mathbb{N}^n} \prod_{i=1}^{n} \mathsf{STS}'_{g(i),t_i}(\Gamma + \langle s_i \rangle) \to \mathsf{STS}_{t_0}(\Gamma)$$

So we take the composite of the above arrows for (6.2).

We check naturality of ρ^{α}_{Γ} in Γ. Let $f : \Gamma \to \Delta$ in $\mathrm{Set}/\mathcal{T}$. The naturality square

$$\begin{tikzcd}
\prod\limits_{i=1}^{n} \mathsf{STS}_{t_i}(\Gamma + \langle s_i \rangle) \arrow[r] \arrow[d] & \prod\limits_{i=1}^{n} \mathsf{STS}_{t_i}(\Delta + \langle s_i \rangle) \arrow[d] \\
\mathsf{STS}_{t_0}(\Gamma) \arrow[r] & \mathsf{STS}_{t_0}(\Delta)
\end{tikzcd}$$

commutes because if we unfold the arrows we find

$$\begin{tikzcd}
\prod\limits_{i=1}^{n} \sum\limits_{n \in \mathbb{N}} \mathsf{STS}'_{n,t_i}(\Gamma + \langle s_i \rangle) \arrow[r] \arrow[d] & \prod\limits_{i=1}^{n} \sum\limits_{n \in \mathbb{N}} \mathsf{STS}'_{n,t_i}(\Delta + \langle s_i \rangle) \arrow[d] \\
\sum\limits_{g \in \mathbb{N}^n} \prod\limits_{i=1}^{n} \mathsf{STS}'_{g(i),t_i}(\Gamma + \langle s_i \rangle) \arrow[r] \arrow[d] & \sum\limits_{g \in \mathbb{N}^n} \prod\limits_{i=1}^{n} \mathsf{STS}'_{g(i),t_i}(\Delta + \langle s_i \rangle) \arrow[d] \\
\sum\limits_{g \in \mathbb{N}^n} \mathsf{STS}'_{m_g+1,t_0}(\Gamma) \arrow[r] \arrow[d] & \sum\limits_{g \in \mathbb{N}^n} \mathsf{STS}'_{m_g+1,t_0}(\Delta) \arrow[d] \\
\mathsf{STS}_{t_0}(\Gamma) \arrow[r] & \mathsf{STS}_{t_0}(\Delta)
\end{tikzcd}$$

where the top square is a naturality square of the canonical isomorphism, the middle square is a naturality square of inclusions and the bottom square is a naturality square of $\sum\limits_{g \in \mathbb{N}^n} c_{m_g+1}$.

Unit of STS

Now we provide the functor STS with a monad structure. The unit $\eta_{\Gamma,t} : \Gamma^{-1}(t) \to \mathsf{STS}_t(\Gamma)$ is given by the inclusion $c_{0,\Gamma,t} : \mathsf{STS}'_{0,t}(\Gamma) \to \sum\limits_{n \in \mathbb{N}} \mathsf{STS}'_{n,t}(\Gamma)$. Its naturality follows from the naturality of the inclusion.

Multiplication of STS

Next we define the multiplication $\mu_{\Gamma,t} : \mathsf{STS}_t(\mathsf{STS}(\Gamma)) \to \mathsf{STS}_t(\Gamma)$ by universal property of the coproduct. We define a cocone $(\mathsf{STS}'_{n,t}(\mathsf{STS}(\Gamma)) \to \mathsf{STS}_t(\Gamma))_{n \in \mathbb{N}}$. For $n = 0$, the arrow

$$\mathsf{STS}'_{0,t}(\mathsf{STS}(\Gamma)) \to \mathsf{STS}_t(\Gamma)$$

is given by the identity on $\mathsf{STS}_t(\Gamma)$ since $\mathsf{STS}'_{0,t}(\mathsf{STS}(\Gamma)) = \mathsf{STS}_t(\Gamma)$. Suppose given an arrow $\mathsf{STS}'_{m,t}(\mathsf{STS}(\Gamma)) \to \mathsf{STS}_t(\Gamma)$ for all $m = 0, \ldots, n$, we construct

$$\mathsf{STS}'_{n+1,t}(\mathsf{STS}(\Gamma)) \to \mathsf{STS}_t(\Gamma)$$

by definition of $\mathsf{STS}'_{n+1,t}$ we construct the arrow

$$\sum_{k \in I_t} \sum_{h \in \Theta_{n_k,n}} \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}\big(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle\big) \to \mathsf{STS}_t(\Gamma)$$

By universal property of the coproducts we give an arrow for all $k \in I_t$ and $h \in \Theta_{n_k,n}$

$$\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}\big(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle\big) \to \mathsf{STS}_t(\Gamma)$$

We take the following composite

$$\begin{array}{c}
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}\big(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle\big) \\
\downarrow \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}\, \mathsf{STS}(\Gamma + \langle s_i^{(k)} \rangle) \\
\downarrow \\
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \\
\downarrow \\
\mathsf{STS}_t(\Gamma)
\end{array}$$

where the first arrow is $\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\psi_\Gamma^{\langle s_i^{(k)} \rangle})$, the second comes from the induction hypotheses and the third is $\rho_\Gamma^{\alpha_k}$. We check naturality of $\mu_{\Gamma,t}$ in Γ. So let $f : \Gamma \to \Delta$ in $\mathrm{Set}/\mathcal{T}$. We check the commutativity of the following square

$$\begin{tikzcd}
\mathsf{STS}(\mathsf{STS}(\Gamma)) \arrow[r] \arrow[d] & \mathsf{STS}(\mathsf{STS}(\Delta)) \arrow[d] \\
\mathsf{STS}(\Gamma) \arrow[r] & \mathsf{STS}(\Delta)
\end{tikzcd}$$

by induction. For $n = 0$

$$\begin{tikzcd}
\mathsf{STS}'_{0,t}(\mathsf{STS}(\Gamma)) \arrow[r] \arrow[d] & \mathsf{STS}'_{0,t}(\mathsf{STS}(\Delta)) \arrow[d] \\
\mathsf{STS}_t(\Gamma) \arrow[r] & \mathsf{STS}_t(\Delta)
\end{tikzcd}$$

commutes since $\mathsf{STS}'_{0,t}(\mathsf{STS}(\Gamma)) = \mathsf{STS}_t(\Gamma)$ and $\mathsf{STS}'_{0,t}(\mathsf{STS}(\Delta)) = \mathsf{STS}_t(\Delta)$, the vertical arrows are identities on $\mathsf{STS}_t(\Gamma)$ and $\mathsf{STS}_t(\Delta)$ and the horizontal arrows are both $\mathsf{STS}_t(f)$. Suppose that

$$\begin{tikzcd}
\mathsf{STS}'_{m,t}(\mathsf{STS}(\Gamma)) \arrow[r] \arrow[d] & \mathsf{STS}'_{m,t}(\mathsf{STS}(\Delta)) \arrow[d] \\
\mathsf{STS}_t(\Gamma) \arrow[r] & \mathsf{STS}_t(\Delta)
\end{tikzcd}$$

commutes for all $m \leq n$. We show the commutativity of the square for $n+1$. So explicitly we show that for all $k \in I_t$ and $h \in \Theta_{n_k,n}$, the following diagram commutes

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle) \arrow[r] \arrow[d] & \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Delta) + \langle s_i^{(k)} \rangle) \arrow[d] \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma + \langle s_i^{(k)} \rangle)) \arrow[r] \arrow[d] & \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Delta + \langle s_i^{(k)} \rangle)) \arrow[d] \\
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r] \arrow[d] & \prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Delta + \langle s_i^{(k)} \rangle) \arrow[d] \\
\mathsf{STS}_t(\Gamma) \arrow[r] & \mathsf{STS}_t(\Delta)
\end{tikzcd}$$

The top square is a naturality square of $\psi^{\langle s_i^{(k)} \rangle}$, the middle square commutes by induction hypotheses and the bottom square is a naturality square of ρ^{α_k}. Then by universal property of the coproducts

$$\begin{tikzcd}
\mathsf{STS}'_{n+1,t}(\mathsf{STS}(\Gamma)) \arrow[r] \arrow[d] & \mathsf{STS}'_{n+1,t}(\mathsf{STS}(\Delta)) \arrow[d] \\
\mathsf{STS}_t(\Gamma) \arrow[r] & \mathsf{STS}_t(\Delta)
\end{tikzcd}$$

commutes.

First monad axiom

We check the monad axioms. First we check the commutativity of

$$\begin{tikzcd}
\mathsf{STS}(\Gamma) \arrow[r, "\eta_{\mathsf{STS}(\Gamma)}"] \arrow[dr, "\mathrm{id}"'] & \mathsf{STS}(\mathsf{STS}(\Gamma)) \arrow[d, "\mu_\Gamma"] \\
& \mathsf{STS}(\Gamma)
\end{tikzcd}$$

for all Γ. We show that

$$\begin{tikzcd}
\mathsf{STS}_t(\Gamma) \arrow[r, "\eta_{\mathsf{STS}(\Gamma),t}"] \arrow[dr, "\mathrm{id}_{\mathsf{STS}_t(\Gamma)}"'] & \mathsf{STS}_t(\mathsf{STS}(\Gamma)) \arrow[d, "\mu_{\Gamma,t}"] \\
& \mathsf{STS}_t(\Gamma)
\end{tikzcd}$$

commutes for all Γ and t. By definition of $\eta_{\mathsf{STS}(\Gamma),t}$ being the coprojection $c_{0,\mathsf{STS}(\Gamma),t}$: $\mathsf{STS}_{0,t}(\mathsf{STS}(\Gamma)) \to \sum_{n\in\mathbb{N}} \mathsf{STS}'_{n,t}(\mathsf{STS}(\Gamma))$ and the component of μ at 0 being the identity on $\mathsf{STS}_t(\Gamma)$, the above triangle commutes.

Second monad axiom

Next we check the commutativity of

$$\begin{tikzcd}
\mathsf{STS}(\Gamma) \arrow[r, "\mathsf{STS}\,\eta_\Gamma"] \arrow[dr, "\mathrm{id}"'] & \mathsf{STS}(\mathsf{STS}(\Gamma)) \arrow[d, "\mu_\Gamma"] \\
& \mathsf{STS}(\Gamma)
\end{tikzcd}$$

for all Γ or equivalently

$$\begin{tikzcd}
\mathsf{STS}'_{n,t}(\Gamma) \arrow[r, "\mathsf{STS}'_{n,t}\,\eta_\Gamma"] \arrow[dr, "c_{n,\Gamma,t}"'] & \mathsf{STS}'_{n,t}(\mathsf{STS}(\Gamma)) \arrow[d, "\mu_{n,\Gamma,t}"] \\
& \mathsf{STS}_t(\Gamma)
\end{tikzcd} \tag{6.3}$$

for all Γ, t and $n \in \mathbb{N}$. For $n = 0$, we have

$$\begin{tikzcd}
\mathsf{STS}'_{0,t}(\Gamma) \arrow[r, "c_{0,\Gamma,t}"] \arrow[dr, "c_{0,\Gamma,t}"'] & \sum_{n\in\mathbb{N}} \mathsf{STS}'_{n,t}(\Gamma) \arrow[d, "\mathrm{id}"] \\
& \mathsf{STS}_t(\Gamma)
\end{tikzcd}$$

Now suppose that (6.3) commutes for all $m \leq n$, t and Γ and we show the commutativity of (6.3) for $n+1$. We replace STS'_{n+1} by its definition and unfold the arrows. Then we show that the following diagram

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)}\rangle) \arrow[r] \arrow[dr, bend right=10] \arrow[ddr, bend right=30] \arrow[dddr, bend right=45] \arrow[dr, phantom, shift left=4ex, "I."] & \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)}\rangle) \arrow[d] \\
& \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma + \langle s_i^{(k)}\rangle)) \arrow[d] \arrow[l, phantom, "II."] \\
& \prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)}\rangle) \arrow[d] \arrow[l, phantom, "III."] \\
& \mathsf{STS}_t(\Gamma)
\end{tikzcd}$$

commutes for all $k \in I_t$ and $h \in \Theta_{n_k,n}$. The triangle I. commutes because the following diagram commutes

$$\begin{tikzcd}
\Gamma^{-1}(t) + \langle u \rangle_t \arrow[r, "c_{0,\Gamma,t}+\langle u \rangle_t"] \arrow[dr, "c_{0,\Gamma+\langle u \rangle,t}"'] & \sum\limits_{n\in\mathbb{N}} \mathsf{STS}'_{n,t}(\Gamma) + \langle u \rangle_t \arrow[d, "{[\sum\limits_{n\in\mathbb{N}} \mathsf{STS}'_{n,t}(i_\Gamma), c_{0,\Gamma+\langle u \rangle,t} \circ i_{\langle u \rangle}]}"] \\
 & \sum\limits_{n\in\mathbb{N}} \mathsf{STS}'_{n,t}(\Gamma + \langle u \rangle)
\end{tikzcd}$$

for all t and Γ where the vertical arrow comes from (6.1). It commutes since $\sum\limits_{n\in\mathbb{N}} \mathsf{STS}'_{n,t}(i_\Gamma) \circ c_{0,\Gamma,t} = c_{0,\Gamma+\langle u \rangle,t} \circ i_\Gamma$ by naturality of the inclusion c_0.

Triangle II. commutes by induction hypotheses. Triangle III. commutes because by construction of (6.2), $\max(h(i)) = n$ and thus

$$\begin{tikzcd}
\prod\limits_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r, "\prod\limits_{i=1}^{n_k} c_{h(i)}"] \arrow[dr, "c_{n+1,t}"'] & \prod\limits_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[d] \\
 & \mathsf{STS}_t(\Gamma)
\end{tikzcd}$$

commutes.

Third monad axiom

Next we check the commutativity of

$$\begin{tikzcd}
\mathsf{STS}(\mathsf{STS}(\mathsf{STS}(\Gamma))) \arrow[r, "\mathsf{STS}\,\mu_\Gamma"] \arrow[d, "\mu_{\mathsf{STS}(\Gamma)}"'] & \mathsf{STS}(\mathsf{STS}(\Gamma)) \arrow[d, "\mu_\Gamma"] \\
\mathsf{STS}(\mathsf{STS}(\Gamma)) \arrow[r, "\mu_\Gamma"'] & \mathsf{STS}(\Gamma)
\end{tikzcd}$$

for all Γ or equivalently

$$\begin{tikzcd}
\mathsf{STS}'_{n,t}(\mathsf{STS}(\mathsf{STS}(\Gamma))) \arrow[r, "\mathsf{STS}'_{n,t}\,\mu_\Gamma"] \arrow[d, "\mu_{n,\mathsf{STS}(\Gamma),t}"'] & \mathsf{STS}'_{n,t}(\mathsf{STS}(\Gamma)) \arrow[d, "\mu_{\Gamma,t}"] \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma)) \arrow[r, "\mu_{\Gamma,t}"'] & \mathsf{STS}_t(\Gamma)
\end{tikzcd} \tag{6.4}$$

for all Γ, n and t. For $n = 0$ the square (6.4) becomes

$$\begin{tikzcd}
\mathsf{STS}_t(\mathsf{STS}(\Gamma)) \arrow[r, "\mu_{\Gamma,t}"] \arrow[d, "\mathrm{id}_{\mathsf{STS}_t(\mathsf{STS}(\Gamma))}"'] & \mathsf{STS}_t(\Gamma) \arrow[d, "\mathrm{id}_{\mathsf{STS}_t(\Gamma)}"] \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma)) \arrow[r, "\mu_{\Gamma,t}"'] & \mathsf{STS}_t(\Gamma)
\end{tikzcd}$$

which commutes. Suppose that (6.4) commutes for all $m \le n$; we show the commutativity for $n+1$. Let $k \in I_t$ and $h \in \Theta_{n_k,n}$. We have to show that

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\mathsf{STS}(\Gamma)) + \langle s_i^{(k)} \rangle) \arrow[r] \arrow[d] \arrow[dr, phantom, "I."] & \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle)) \arrow[d] \arrow[dr] \arrow[r, phantom, "II."] & \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle) \arrow[d] & \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\mathsf{STS}(\Gamma + \langle s_i^{(k)} \rangle))) \arrow[dl] \arrow[d] & \prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle) \arrow[dl] \arrow[d] \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma + \langle s_i^{(k)} \rangle)) \arrow[d] \arrow[r, phantom, "III."] & \prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\mathsf{STS}(\Gamma + \langle s_i^{(k)} \rangle)) \arrow[dl] \arrow[r, phantom, "IV."] & \mathsf{STS}_t(\mathsf{STS}(\Gamma)) \arrow[dl] \\
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r] & \mathsf{STS}_t(\Gamma) &
\end{tikzcd}$$

commutes. Diagram III. commutes by induction hypotheses, diagram II. commutes by naturalities of $\mu_{h(i)}$. It remains to check the commutativity of I. and IV. The diagram I. commutes since the following diagram commutes

$$\begin{tikzcd}
\mathsf{STS}(\mathsf{STS}(\Gamma)) + \langle u \rangle \arrow[r, "\mu_\Gamma + \mathrm{id}"] \arrow[d, "\psi^{\langle u \rangle}_{\mathsf{STS}(\Gamma)}"'] & \mathsf{STS}(\Gamma) + \langle u \rangle \arrow[dd, "\psi^{\langle u \rangle}_{\Gamma}"] \\
\mathsf{STS}(\mathsf{STS}(\Gamma) + \langle u \rangle) \arrow[d, "\mathsf{STS}(\psi^{\langle u \rangle}_{\Gamma})"'] & \\
\mathsf{STS}(\mathsf{STS}(\Gamma + \langle u \rangle)) \arrow[r, "\mu_{\Gamma + \langle u \rangle}"'] & \mathsf{STS}(\Gamma + \langle u \rangle)
\end{tikzcd}$$

We show its commutativity by induction for all n

$$\begin{tikzcd}
\mathsf{STS}'_{n,t}(\mathsf{STS}(\Gamma)) + \langle u \rangle_t \arrow[r] \arrow[d] & \mathsf{STS}_t(\Gamma) + \langle u \rangle_t \arrow[dd] \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma) + \langle u \rangle) \arrow[d] & \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma + \langle u \rangle)) \arrow[r] & \mathsf{STS}_t(\Gamma + \langle u \rangle)
\end{tikzcd} \tag{6.5}$$

For $n = 0$

$$\begin{tikzcd}
\mathsf{STS}_t(\Gamma) + \langle u \rangle_t \arrow[rr, "\mathrm{id} + \mathrm{id}"] \arrow[d, "c_{0,\mathsf{STS}(\Gamma)+\langle u \rangle,t}"'] \arrow[dr, "\psi^{\langle u \rangle}_{\Gamma,t}"'] & & \mathsf{STS}_t(\Gamma) + \langle u \rangle_t \arrow[dd, "\psi^{\langle u \rangle}_{\Gamma,t}"] \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma) + \langle u \rangle) \arrow[d, "\sum_{n \in \mathbb{N}} \mathsf{STS}'_{n,t}(\psi^{\langle u \rangle}_{\Gamma})"'] & \mathsf{STS}'_{0,t}(\mathsf{STS}(\Gamma + \langle u \rangle)) \arrow[dl, "c_{0,t,\mathsf{STS}(\Gamma+\langle u \rangle)}"'] \arrow[dr, "\mathrm{id}_{\mathsf{STS}_t(\Gamma+\langle u \rangle)}"] & \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma + \langle u \rangle)) \arrow[rr, "\mu_{\Gamma+\langle u \rangle,t}"'] & & \mathsf{STS}_t(\Gamma + \langle u \rangle)
\end{tikzcd}$$

since $\psi^{\langle u\rangle}_{n,\Gamma,t} = [c_{n,t}(\mathsf{STS}'_{n,t}(i_\Gamma)), c_{0,t}(i_{\langle u\rangle})]$ and $\psi^{\langle u\rangle}_{0,\Gamma,t} = [c_{0,t}(i_\Gamma), c_{0,t}(i_{\langle u\rangle})] = c_{0,\Gamma+\langle u\rangle,t}$. The triangle on the left is a naturality square of c_0, the bottom triangle commutes by definition of μ and the diagram on the right commutes obviously.

Now suppose that (6.5) commutes for all $m \leq n$. We show the commutativity for $n+1$.

$$\begin{tikzcd}
\langle u\rangle_t \arrow[rr, "\mathrm{id}"] \arrow[d, "i_{\langle u\rangle}"'] & & \langle u\rangle_t \arrow[d, "i'_{\langle u\rangle}"] \\
\mathsf{STS}_t(\Gamma)+\langle u\rangle_t \arrow[d, "c_0"'] \arrow[dr, "\psi^{\langle u\rangle}_{\Gamma,t}"] & & (\Gamma+\langle u\rangle)_t \arrow[dl, "c_0"'] \arrow[dd, "c_0"] \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma)+\langle u\rangle) \arrow[d, "\sum\limits_{n\in\mathbb{N}}\mathsf{STS}'_{n,t}(\psi^{\langle u\rangle}_\Gamma)"'] & \mathsf{STS}'_{0,t}(\mathsf{STS}(\Gamma+\langle u\rangle)) \arrow[dl, "c_0"] \arrow[dr, "\mathrm{id}_{\mathsf{STS}_t(\Gamma+\langle u\rangle)}"] & \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma+\langle u\rangle)) \arrow[rr, "\mu_{\Gamma+\langle u\rangle,t}"'] & & \mathsf{STS}_t(\Gamma+\langle u\rangle)
\end{tikzcd}$$

The top diagram commutes by definition of $\psi^{\langle u\rangle}_\Gamma$, the left triangle by naturality of c_0, the right triangle commutes obviously and the bottom one by definition of μ. To show the commutativity of

$$\begin{tikzcd}
\mathsf{STS}'_{n+1,t}(\mathsf{STS}(\Gamma)) \arrow[r] \arrow[d] & \mathsf{STS}_t(\Gamma) \arrow[dd] \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma)+\langle u\rangle) \arrow[d] & \\
\mathsf{STS}_t(\mathsf{STS}(\Gamma+\langle u\rangle)) \arrow[r] & \mathsf{STS}_t(\Gamma+\langle u\rangle)
\end{tikzcd}$$

we replace STS'_{n+1} by its definition and chose a $k \in I_t$ and a $h \in \Theta_{n_k,n}$. So it remains to show the commutativity of the following diagram

$$\begin{tikzcd}
\prod\limits_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma)+\langle s_i^{(k)}\rangle) \arrow[d] \arrow[dd, bend left] \arrow[dr] & \\
\prod\limits_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma)+\langle u\rangle+\langle s_i^{(k)}\rangle) \arrow[d] & \prod\limits_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma+\langle s_i^{(k)}\rangle)) \arrow[d] \arrow[dl] \\
\prod\limits_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma+\langle u\rangle)+\langle s_i^{(k)}\rangle) \arrow[d] & \prod\limits_{i=1}^{n_k}\mathsf{STS}_{t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle) \arrow[d] \arrow[dl, "\prod\limits_{i=1}^{n_k}\mathsf{STS}_{t_i^{(k)}}(i_\Gamma+\langle s_i^{(k)}\rangle)"] \\
\prod\limits_{i=1}^{n_k}\mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma+\langle u\rangle+\langle s_i^{(k)}\rangle)) \arrow[d] & \mathsf{STS}_t(\Gamma) \arrow[d, "\mathsf{STS}_t(i_\Gamma)"] \\
\prod\limits_{i=1}^{n_k}\mathsf{STS}'_{t_i^{(k)}}(\Gamma+\langle u\rangle+\langle s_i^{(k)}\rangle) \arrow[r] & \mathsf{STS}_t(\Gamma+\langle u\rangle)
\end{tikzcd}$$

The bottom three diagrams are naturality squares of i_Γ and the top left curved diagram commutes because

$$\begin{tikzcd}
\mathsf{STS}(\Gamma) \arrow[d] \arrow[dr] & \\
\mathsf{STS}(\Gamma) + \langle u \rangle \arrow[r] & \mathsf{STS}(\Gamma + \langle u \rangle)
\end{tikzcd}$$

commutes by definition of $\psi_\Gamma^{\langle u \rangle}$.

It remains to show the commutativity of the diagram IV. which is explicitly

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle) \arrow[r, "\rho^{\alpha_k}_{\mathsf{STS}(\Gamma)}"] \arrow[d, "\prod \mathsf{STS}_{t_i^{(k)}}(\psi_\Gamma^{\langle s_i \rangle})"'] & \mathsf{STS}_t(\mathsf{STS}(\Gamma)) \arrow[dd, "\mu_{\Gamma,t}"] \\
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\mathsf{STS}(\Gamma + \langle s_i^{(k)} \rangle)) \arrow[d, "\prod \mu_{\Gamma+\langle s_i \rangle, t_i}"'] & \\
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r, "\rho_\Gamma^{\alpha_k}"'] & \mathsf{STS}_t(\Gamma)
\end{tikzcd} \tag{6.6}$$

Since

$$\begin{aligned}
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle) &= \prod_{i=1}^{n_k} \sum_{n \in \mathbb{N}} \mathsf{STS}'_{n,t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle) \\
&\cong \sum_{g \in \mathbb{N}^n} \prod_{i=1}^{n_k} \mathsf{STS}'_{g(i),t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle)
\end{aligned}$$

(6.6) becomes the following

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}'_{g(i)t_i^{(k)}}(\mathsf{STS}(\Gamma) + \langle s_i^{(k)} \rangle) \arrow[r] \arrow[d, "\prod \mathsf{STS}'_{g(i),t_i}(\psi_\Gamma^{\langle s_i \rangle})"'] & \mathsf{STS}'_{m_g+1,t}(\mathsf{STS}(\Gamma)) \arrow[dd, "\mu_{m_g+1,\Gamma,t}"] \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{g(i),t_i^{(k)}}(\mathsf{STS}(\Gamma + \langle s_i^{(k)} \rangle)) \arrow[d, "\prod \mu_{g(i),\Gamma+\langle s_i \rangle, t_i}"'] & \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{g(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r, "\rho_\Gamma^{\alpha_k}"'] & \mathsf{STS}_t(\Gamma)
\end{tikzcd}$$

for all $g \in \mathbb{N}^n$ where $m_g = \max(g(i))$. It commutes by definition of $\mu_{m_g+1,\Gamma,t}$.

Representations

The arrows ρ^{α_k} together with commutative diagrams such as in (6.6) define a module morphism for each arity α_k.

So we showed that STS is an object of the category of representations of S.

Construction φ

Next we show that STS is initial, that is, given another object $((R, \eta^R, \mu^R), (r^{\alpha_k})_{k \in I})$, we construct a morphism of representations $\varphi : \mathsf{STS} \to R$ and show then its uniqueness.

First we construct $\varphi_{\Gamma,t} : \mathsf{STS}_t(\Gamma) \to R_t(\Gamma)$ for all $t \in \mathcal{T}$ and $\Gamma \in \mathrm{Set}/\mathcal{T}$. By universal property of the coproduct we give an arrow $\varphi_{n,\Gamma,t} : \mathsf{STS}'_{n,t}(\Gamma) \to R_t(\Gamma)$ for all $n \in \mathbb{N}$. For $n = 0$, we take $\eta^R_\Gamma : \Gamma^{-1}(t) \to R_t(\Gamma)$ to be $\varphi_{0,n,\Gamma}$. Suppose given arrows $\varphi_{m,\Gamma,t} : \mathsf{STS}'_{m,t}(\Gamma) \to R_t(\Gamma)$ for all $m \leq n$, t and Γ, we define $\varphi_{n+1,\Gamma,t}$. By universal properties of the coproducts we give an arrow

$$\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \to R_t(\Gamma)$$

for all $k \in I_t$ and $h \in \Theta_{n_k,n}$. We take the following composite

$$\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \to \prod_{i=1}^{n_k} R_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \to R_t(\Gamma)$$

where the first arrow comes from the induction hypotheses and the last one is $r^{\alpha_k}_\Gamma$.

Naturality of φ

We check naturality of $\varphi_{\Gamma,t}$ in Γ. So let $f : \Gamma \to \Delta$ in $\mathrm{Set}/\mathcal{T}$. The naturality square

$$\begin{tikzcd}
\mathsf{STS}(\Gamma) \arrow[r] \arrow[d] & R(\Gamma) \arrow[d] \\
\mathsf{STS}(\Delta) \arrow[r] & R(\Delta)
\end{tikzcd}$$

commutes since

$$\begin{tikzcd}
\mathsf{STS}'_{n,t}(\Gamma) \arrow[r] \arrow[d] & R_t(\Gamma) \arrow[d] \\
\mathsf{STS}'_{n,t}(\Delta) \arrow[r] & R_t(\Delta)
\end{tikzcd} \tag{6.7}$$

commutes for all $n \in \mathbb{N}$. For $n = 0$

$$\begin{tikzcd}
\Gamma^{-1}(t) \arrow[r, "\eta^R_{\Gamma,r}"] \arrow[d, "f_t"'] & R_t(\Gamma) \arrow[d, "R_t(f)"] \\
\Delta^{-1}(t) \arrow[r, "\eta^R_{\Delta,t}"'] & R_t(\Delta)
\end{tikzcd}$$

commutes by naturality of η^R. Suppose that (6.7) commutes for all $m \leq n$ and we show that it commutes for $n + 1$.

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r] \arrow[d] & \prod_{i=1}^{n_k} R_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r] \arrow[d] & R_t(\Gamma) \arrow[d] \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\Delta + \langle s_i^{(k)} \rangle) \arrow[r] & \prod_{i=1}^{n_k} R_{t_i^{(k)}}(\Delta + \langle s_i^{(k)} \rangle) \arrow[r] & R_t(\Delta)
\end{tikzcd}$$

commutes for all $k \in I_t$ and $h \in \Theta_{n_k,n}$ since the left square commutes by induction hypotheses and the right square by naturality of r^{α_k}.

φ is a monad morphism

Next we check that φ is a morphism of monads $\mathsf{STS} \to R$. We check the commutativity of

$$\begin{tikzcd} & \Gamma \arrow[dl, "\eta_\Gamma"'] \arrow[dr, "\eta^R_\Gamma"] & \\ \mathsf{STS}(\Gamma) \arrow[rr, "\varphi_\Gamma"'] & & R(\Gamma) \end{tikzcd}$$

By definition of η and φ_0 we have that

$$\begin{tikzcd} & \Gamma^{-1}(t) \arrow[dl, "c_{0,\Gamma,t}"'] \arrow[dr, "\eta^R_{\Gamma,t}"] & \\ \mathsf{STS}_t(\Gamma) \arrow[rr, "\varphi_{\Gamma,t}"'] & & R_t(\Gamma) \end{tikzcd}$$

commutes. Next we check that

$$\begin{tikzcd} \mathsf{STS}(\mathsf{STS}(\Gamma)) \arrow[r, "\mathsf{STS}(\varphi_\Gamma)"] \arrow[d, "\mu_\Gamma"'] & \mathsf{STS}(R(\Gamma)) \arrow[r, "\varphi_{R(\Gamma)}"] & R(R(\Gamma)) \arrow[d, "\mu^R_\Gamma"] \\ \mathsf{STS}(\Gamma) \arrow[rr, "\varphi_\Gamma"'] & & R(\Gamma) \end{tikzcd}$$

commutes. By universal property of the coproduct, we show that

$$\begin{tikzcd} \mathsf{STS}'_{n,t}(\mathsf{STS}(\Gamma)) \arrow[r, "\mathsf{STS}'_{n,t}(\varphi_\Gamma)"] \arrow[d, "\mu_{n,\Gamma,t}"'] & \mathsf{STS}'_{n,t}(R(\Gamma)) \arrow[r, "\varphi_{n,R(\Gamma),t}"] & R_t(R(\Gamma)) \arrow[d, "\mu^R_{\Gamma,t}"] \\ \mathsf{STS}_t(\Gamma) \arrow[rr, "\varphi_{\Gamma,t}"'] & & R_t(\Gamma) \end{tikzcd} \tag{6.8}$$

commutes for all $n \in \mathbb{N}$. For $n = 0$

$$\begin{tikzcd} \mathsf{STS}_t(\Gamma) \arrow[r, "\varphi_{\Gamma,t}"] \arrow[d, "\mathrm{id}"'] & R_t(\Gamma) \arrow[r, "\eta^R_{R(\Gamma),t}"] \arrow[dr, "\mathrm{id}"'] & R_t(R(\Gamma)) \arrow[d, "\mu^R_{\Gamma,t}"] \\ \mathsf{STS}_t(\Gamma) \arrow[rr, "\varphi_{\Gamma,t}"'] & & R_t(\Gamma) \end{tikzcd}$$

commutes by one of the monad axioms of R. Suppose that (6.8) commutes for all $m \leq n$, Γ and t and we show the commutativity for $n+1$ for all $k \in I_t$ and $h \in \Theta_{n_k,n}$

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma)+\langle s_i^{(k)}\rangle) \arrow[r] \arrow[d] \arrow[dr, phantom, "I."] & \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(R(\Gamma)+\langle s_i^{(k)}\rangle) \arrow[r] \arrow[d] \arrow[dr, phantom, "II."] & \prod_{i=1}^{n_k} R_{t_i^{(k)}}(R(\Gamma)+\langle s_i^{(k)}\rangle) \arrow[r] \arrow[d] & R_t(R(\Gamma)) \arrow[ddd] \\
\prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(\mathsf{STS}(\Gamma+\langle s_i^{(k)}\rangle)) \arrow[r] \arrow[d] & \prod_{i=1}^{n_k} \mathsf{STS}'_{h(i),t_i^{(k)}}(R(\Gamma+\langle s_i^{(k)}\rangle)) \arrow[r] \arrow[d, phantom, "IV."] & \prod_{i=1}^{n_k} R_{t_i^{(k)}}(R(\Gamma+\langle s_i^{(k)}\rangle)) \arrow[d] \arrow[r, phantom, "III."] & \\
\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle) \arrow[rr] \arrow[d] & \arrow[d, phantom, "V."] & \prod_{i=1}^{n_k} R_{t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle) \arrow[dr] & \\
\mathsf{STS}_t(\Gamma) \arrow[rrr] & & & R_t(\Gamma)
\end{tikzcd}$$

Diagram II. commutes by naturality of φ, diagram III. commutes because r^{α_k} is a module morphism and the commutativity of IV. is given by the induction hypotheses. It remains to check the commutativity of I. and V. Diagram I. commutes since the following diagram commutes

$$\begin{tikzcd}
\mathsf{STS}(\Gamma)+\langle u\rangle \arrow[r, "\varphi_\Gamma+\mathrm{id}"] \arrow[d, "\psi_\Gamma^{\langle u\rangle}"'] & R(\Gamma)+\langle u\rangle \arrow[d, "{[R(i_\Gamma),\eta^R_{\Gamma+\langle u\rangle}(i_{\langle u\rangle})]}"] \\
\mathsf{STS}(\Gamma+\langle u\rangle) \arrow[r, "\varphi_{\Gamma+\langle u\rangle}"'] & R(\Gamma+\langle u\rangle)
\end{tikzcd}$$

for all Γ and u. In its second component we have

$$\begin{tikzcd}
\langle u\rangle_t \arrow[r, "\mathrm{id}"] \arrow[d, "i_{\langle u\rangle}"'] & \langle u\rangle_t \arrow[d, "i_{\langle u\rangle}"] \\
(\Gamma+\langle u\rangle)_t \arrow[d, "\mathrm{id}"'] & (\Gamma+\langle u\rangle)_t \arrow[dd, "\eta^R_{\Gamma+\langle u\rangle}"] \\
\mathsf{STS}_{0,t}(\Gamma+\langle u\rangle) \arrow[d, "c_{0,t}"'] \arrow[dr, "\eta^R_{\Gamma+\langle u\rangle}"] & \\
\mathsf{STS}_t(\Gamma+\langle u\rangle) \arrow[r, "\varphi_{\Gamma+\langle u\rangle,t}"'] & R_t(\Gamma+\langle u\rangle)
\end{tikzcd}$$

and in the first component

$$\begin{tikzcd}
\mathsf{STS}(\Gamma) \arrow[r, "\varphi_\Gamma"] \arrow[d, "\mathsf{STS}(i_\Gamma)"'] & R(\Gamma) \arrow[d, "R(i_\Gamma)"] \\
\mathsf{STS}(\Gamma+\langle u\rangle) \arrow[r, "\varphi_{\Gamma+\langle u\rangle}"'] & R(\Gamma+\langle u\rangle)
\end{tikzcd}$$

which commutes by naturality of φ. Next we check the commutativity of V. Since

$$\prod_{i=1}^{n_k} \mathsf{STS}_{t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle) = \prod_{i=1}^{n_k}\sum_{n\in\mathbb{N}} \mathsf{STS}'_{n,t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle) \cong \sum_{g\in\mathbb{N}^n}\prod_{i=1}^{n_k} \mathsf{STS}'_{g(i),t_i^{(k)}}(\Gamma+\langle s_i^{(k)}\rangle)$$

We show the commutativity of

$$\begin{tikzcd}
\prod_{i=1}^{n_k} \mathsf{STS}'_{g(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[r, "\prod \varphi_{g(i)}"] \arrow[d] & \prod_{i=1}^{n_k} R_{t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle) \arrow[dd, "r_\Gamma^{\alpha_k}"] \\
\mathsf{STS}'_{m_g+1,t}(\Gamma) \arrow[d, "c_{m_g+1,\Gamma,t}"'] \arrow[dr, "\varphi_{m_g+1,\Gamma,t}"] & \\
\mathsf{STS}_t(\Gamma) \arrow[r, "\varphi_{\Gamma,t}"'] & R_t(\Gamma)
\end{tikzcd}$$

for all $g \in \mathbb{N}^n$. It commutes by definition of $\varphi_{m_g+1,\Gamma,t}$ where $m_g = \max(g(i))$.

So φ is a monad morphism $\mathsf{STS} \to R$.

φ as morphism of representations

Moreover since the square V. commutes for all arity α_k, it shows the compatibility of φ with the representations and thus φ is a morphism of representations.

Uniqueness of φ

In order to prove the initiality of STS in $\mathrm{Rep}(S)$, we show that φ is the unique morphism of representations $\mathsf{STS} \to R$. So suppose given another morphism of representations $\chi : \mathsf{STS} \to R$, we show that $\chi = \varphi$ or equivalently that $\chi_{\Gamma,t}(M) = \varphi_{\Gamma,t}(M)$ for all Γ, t and $M \in \mathsf{STS}_t(\Gamma) = \sum_{n\in\mathbb{N}} \mathsf{STS}'_{n,t}(\Gamma)$. Suppose $M = c_{0,\Gamma,t}(x)$ with $x \in \mathsf{STS}'_{0,t}(\Gamma) = \Gamma^{-1}(t)$, then by definition

$$\varphi_{\Gamma,t}(M) = \eta^R_{\Gamma,t}(x)$$

and since χ is a monad morphism and compatible with the units η and η^R

$$\chi_{\Gamma,t}(M) = \chi_{\Gamma,t}(c_{0,\Gamma,t}(x)) = \chi_{\Gamma,t}(\eta_{\Gamma,t}(x)) = \eta^R_{\Gamma,t}(x)$$

Now suppose that $\chi_{m,\Gamma,t}(N) = \varphi_{m,\Gamma,t}(N)$ for all $m \leq n$, Γ, t and $N \in \mathsf{STS}'_{m,t}(\Gamma)$. Then we show that for $M = c_{n+1,\Gamma,t}(M')$ with $M' \in \mathsf{STS}'_{n+1,t}(\Gamma)$ we have also $\chi_{\Gamma,t}(M) = \varphi_{\Gamma,t}(M)$ or equivalently $\chi_{n+1,\Gamma,t}(M') = \varphi_{n+1,\Gamma,t}$. By definition $M' = \rho_\Gamma^{\alpha_k}(N_1, \ldots, N_{n_k})$ for a certain $k \in I_t$ and $h \in \Theta_{n_k,n}$ with $N_i \in \mathsf{STS}'_{h(i),t_i^{(k)}}(\Gamma + \langle s_i^{(k)} \rangle)$ for all $i = 1, \ldots, n_k$.

$$\begin{aligned}
\chi_{n+1,\Gamma,t}(M') &= \chi_{n+1,\Gamma,t}(\rho_\Gamma^{\alpha_k}(N_1, \ldots, N_{n_k})) \\
&= r_\Gamma^{\alpha_k}(\chi_{h(1),\Gamma+\langle s_i^{(k)} \rangle, t_i^{(k)}}(N_1), \ldots, \chi_{h(n_k),\Gamma+\langle s_{n_k}^{(k)} \rangle, t_{n_k}^{(k)}}(N_{n_k})) \\
&= r_\Gamma^{\alpha_k}(\varphi_{h(1),\Gamma+\langle s_i^{(k)} \rangle, t_i^{(k)}}(N_1), \ldots, \varphi_{h(n_k),\Gamma+\langle s_{n_k}^{(k)} \rangle, t_{n_k}^{(k)}}(N_{n_k})) \\
&= \varphi_{n+1,\Gamma,t}(\rho_\Gamma^{\alpha_k}(N_1, \ldots, N_{n_k})) \\
&= \varphi_{n+1,\Gamma,t}(M')
\end{aligned}$$

□

Example 6.4.122 *The initial object of the category of representations of the signature of example 5.3.81 is the simply typed Lambda Calculus. We write STLC_T for the monad on Set/T and we have the following two collections of STLC–module morphisms*

$$\mathrm{app}_{s,t} : (\mathsf{STLC})_{s\Rightarrow t} \times (\mathsf{STLC})_s \to (\mathsf{STLC})_t$$

and

$$\mathrm{abs}_{s,t} : (\mathsf{STLC}^s)_t \to (\mathsf{STLC})_{s \Rightarrow t}$$

Chapter 7

Comparison simply typed syntax

7.1 Two equivalences of categories

We have the following equivalences of categories.

Lemma 7.1.123 *The category* $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ *is equivalent to the functor category* $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}/\mathcal{T}]$.

Proof. Let $(X_t)_{t\in\mathcal{T}}$ be an object of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. We associate to it the functor $Y \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}/\mathcal{T}]$ that assigns an object $\Gamma : n \to \mathcal{T}$ of $\mathbb{F} \downarrow \mathcal{T}$ the object $\sum_{t\in\mathcal{T}} X_t(\Gamma) \to \mathcal{T}$ of $\mathrm{Set}/\mathcal{T}$. The functor Y assigns to a morphism $h : \Gamma \to \Delta$ the arrow $\sum_{t\in\mathcal{T}} X_t(h)$.

Inversely let Y be a functor $\mathbb{F} \downarrow \mathcal{T} \to \mathrm{Set}/\mathcal{T}$. We assign to it the object $(\lambda\Gamma.Y(\Gamma)^{-1}(t))_{t\in\mathcal{T}}$ of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$.

These two assignations are inverse to each other. Starting with an object $(X_t)_{t\in\mathcal{T}}$ of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$, we get $(\lambda\Gamma.X_t(\Gamma))_{t\in\mathcal{T}}$, which is just $(X_t)_{t\in\mathcal{T}}$.

Starting with a functor $Y : \mathbb{F} \downarrow \mathcal{T} \to \mathrm{Set}/\mathcal{T}$, we get a functor Z that assigns to an object $\Gamma : n \to \mathcal{T}$ of $\mathbb{F} \downarrow \mathcal{T}$ the arrow

$$\sum_{t\in\mathcal{T}}(\lambda\Delta.Y(\Delta)^{-1}(t))(\Gamma) \to \mathcal{T}$$

which is

$$\sum_{t\in\mathcal{T}}(Y(\Gamma)^{-1}(t)) \to \mathcal{T}$$

and this is $Y(\Gamma)$. □

The cellection of variables $\mathcal{Y} = (\mathcal{Y}\langle t\rangle)_{t\in\mathcal{T}}$ becomes the functor that assigns Γ of $\mathbb{F} \downarrow \mathcal{T}$ the object Γ of $\mathrm{Set}/\mathcal{T}$. We write $\mathcal{Y}$ as well for this functor $\mathbb{F} \downarrow \mathcal{T} \to \mathrm{Set}/\mathcal{T}$.

In a similar way we can show the following equivalence.

Lemma 7.1.124 *The category* $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ *is equivalent to the functor category* $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$.

7.2 From $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ to $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$

In this section we define first a functor $\ell : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}] \to [\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$.

Definition 7.2.125 *Let X be a functor of $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$. For each $\Gamma \in \mathrm{Set}/\mathcal{T}$, we define*

$$\begin{aligned}\ell(X)(\Gamma) &:= \int^{\Gamma'=(u_1,\ldots,u_m)\in\mathbb{F}\downarrow\mathcal{T}} X(\Gamma')\times\Gamma^{-1}(u_1)\times\ldots\times\Gamma^{-1}(u_m)\\ &= \int^{\Gamma'\in\mathbb{F}\downarrow\mathcal{T}} X(\Gamma')\times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Gamma'^{-1}(t)}\end{aligned}$$

Now we define ℓ on morphisms. Let $f : X \to Y$ be an arrow in $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$. We construct the arrow $\ell(X)(\Gamma)\to\ell(Y)(\Gamma)$. By universal property, it suffices to give a collection of arrows

$$X(\Gamma')\times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \to \int^{\Gamma'\in\mathbb{F}\downarrow\mathcal{T}} Y(\Gamma')\times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Gamma'^{-1}(t)}$$

for all $\Gamma'\in\mathbb{F}\downarrow\mathcal{T}$ satisfying the wedge condition. We take the following mapping composed with the corresponding coprojection

$$(x,(h_t)_{t\in\mathcal{T}}) \quad\mapsto\quad (f_{\Gamma'}(x),(h_t)_{t\in\mathcal{T}})$$

For the remaining verifications see appendix D.1.

Now we define another functor $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}\to[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$ still called ℓ.

Definition 7.2.126 *We define the functor $\ell : [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}\to[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$ by*

$$(X_t)_{t\in\mathcal{T}} \quad\mapsto\quad (\ell(X_t))_{t\in\mathcal{T}}$$

The functor ℓ preserves finitie products by the following lemma and corollary.

Lemma 7.2.127 *Let $P,Q\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$. Then*

$$(\ell P)\times(\ell Q)\to\ell(P\times Q)$$

is a natural isomorphism.

Proof. See appendix D.2. □

Corollary 7.2.128 *Let $P,Q\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$. Then*

$$(\ell P)\times(\ell Q)\to\ell(P\times Q)$$

is a natural isomorphism.

Proof. Let $\Gamma\in\mathrm{Set}/\mathcal{T}$ and $t\in\mathcal{T}$. The fibre in t

$$(\ell P)(\Gamma)^{-1}(t)\times(\ell Q)(\Gamma)^{-1}(t)\to\ell(P\times Q)(\Gamma)^{-1}(t)$$

is given by the construction in the proof of lemma 7.2.127. The naturality in P and in Q follows from the naturalities of each fibre in t. □

7.3 From $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ to $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$

We define the functor $k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ which is precomposition with $\mathcal{Y}$,

$$k : F \mapsto F \circ \mathcal{Y}$$

Later we will also use the notation k for the functor $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ which is precomposition by $\mathcal{Y}$ as well.

Lemma 7.3.129 *Let* $F, G \in [\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$. *Then we have the following formulas*

$$k(F \times G) = kF \times kG \qquad \textit{and} \qquad k(F + G) = kF + kG$$

Proof. Let $\Delta \in \mathbb{F} \downarrow \mathcal{T}$.

$$k(F \times G)(\Delta) = (F \times G)(\mathcal{Y}(\Delta)) = F(\mathcal{Y}(\Delta)) \times G(\mathcal{Y}(\Delta)) = kF(\Delta) \times kG(\Delta)$$

and

$$k(F + G)(\Delta) = (F + G)(\mathcal{Y}(\Delta)) = F(\mathcal{Y}(\Delta)) + G(\mathcal{Y}(\Delta)) = kF(\Delta) + kG(\Delta)$$

□

Corollary 7.3.130 *Let* $F, G \in [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$. *Then we have the following formulas*

$$k(F \times G) = kF \times kG \qquad \textit{and} \qquad k(F + G) = kF + kG$$

Proof. Let $\Delta \in \mathbb{F} \downarrow \mathcal{T}$.

$$k(F \times G)(\Delta) = (F \times G)(\mathcal{Y}(\Delta)) = F(\mathcal{Y}(\Delta)) \times G(\mathcal{Y}(\Delta)) = kF(\Delta) \times kG(\Delta)$$

and

$$k(F + G)(\Delta) = (F + G)(\mathcal{Y}(\Delta)) = F(\mathcal{Y}(\Delta)) + G(\mathcal{Y}(\Delta)) = kF(\Delta) + kG(\Delta)$$

□

7.4 Adjunction between $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ and $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$

The functors ℓ and k define an adjunction between $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$ and $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ on the one hand and between $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ and $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ on the other hand. In both cases ℓ is the left and k the right adjoint.

At first we consider the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The unit $\eta : \mathrm{Id}_{[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]} \to k\ell$ is in the component $X \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ at $\Delta \in \mathbb{F} \downarrow \mathcal{T}$

$$X(\Delta) \to \int^{\Gamma'} X(\Gamma') \times \prod_{t \in \mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t')}$$

It is given by the following mapping composed with the corresponding coprojection

$$x \quad \mapsto \quad (x, (\mathrm{id}_{\Delta^{-1}(t)})_{t \in \mathcal{T}})$$

For the remaining verifications see appendix D.3.

We can also define an inverse for $\eta_{X,\Delta}$. By universal property it suffices to give a collection of arrows

$$X(\Gamma') \times \prod_{t \in \mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)} \to X(\Delta)$$

for all $\Gamma' \in \mathbb{F} \downarrow \mathcal{T}$ that satisfies the wedge condition. We take the following mapping

$$(x, (h_t)_{t \in \mathcal{T}}) \quad \mapsto \quad X(\sum_{t \in \mathcal{T}} h_t)(x)$$

We check the wedge condition. Let $g : \Gamma' \to \Gamma''$ in $\mathbb{F} \downarrow \mathcal{T}$. The diagram

$$\begin{tikzcd}
 & X(\Gamma') \times \prod_{t \in \mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)} \arrow[dr] & \\
X(\Gamma') \times \prod_{t \in \mathcal{T}} \Delta^{-1}(t)^{\Gamma''^{-1}(t)} \arrow[ur, "\mathrm{id} \times (- \circ g_t)"] \arrow[dr, "X(g) \times \mathrm{id}"'] & & X(\Delta) \\
 & X(\Gamma'') \times \prod_{t \in \mathcal{T}} \Delta^{-1}(t)^{\Gamma''^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & (x, (h_t \circ g_t)_{t \in \mathcal{T}}) \arrow[dr, mapsto] & \\
(x, (h_t)_{t \in \mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & X\big(\sum_{t \in \mathcal{T}} h_t \circ g\big)(x) \\
 & (X(g)(x), (h_t)_{t \in \mathcal{T}}) \arrow[ur, mapsto] &
\end{tikzcd}$$

So we have the following lemma.

Lemma 7.4.131 *The unit of the adjunction* $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ *is a natural isomorphism.*

Proof. See appendix D.4. □

Now we turn to the unit $\eta : \mathrm{Id}_{[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}} \to k\ell$ of the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. Its component at $t \in \mathcal{T}$ is given by the above construction. Moreover we have an analogous lemma to 7.4.131.

Lemma 7.4.132 *The unit of the adjunction* $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ *is a natural isomorphism.*

Now we define the counit $\varepsilon : \ell k \to \mathrm{Id}_{[\mathrm{Set}/\mathcal{T},\mathrm{Set}]}$ of the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T},\mathrm{Set}] \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$. Its component at $F \in [\mathrm{Set}/\mathcal{T},\mathrm{Set}]$ is

$$\int^{\Gamma'} (F\circ\mathcal{Y})(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \to F(\Gamma)$$

By universal property of the coend it suffices to give a collection of maps

$$F(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \to F(\Gamma)$$

for all $\Gamma' \in \mathbb{F}\downarrow\mathcal{T}$ that satisfies the wedge condition. We take the following mapping

$$(x,(h_t)_{t\in\mathcal{T}}) \quad\mapsto\quad (F(\sum_{t\in\mathcal{T}} h_t))(x)$$

For the remaining verifications see appendix D.5.

Next we define the counit $\varepsilon : \ell k \to \mathrm{Id}_{[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]}$ of the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}] \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$. Its component at $F \in [\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$ in the fibre $t\in\mathcal{T}$ is

$$\int^{\Gamma'} (F\circ\mathcal{Y})(\Gamma')^{-1}(t) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \to F(\Gamma)^{-1}(t)$$

It is given by the above construction.

We have to check the the triangle identities for our definitions of $\eta : \mathrm{Id}_{[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]} \to k\ell$ and $\varepsilon : \ell k \to \mathrm{Id}_{[\mathrm{Set}/\mathcal{T},\mathrm{Set}]}$. Let $X \in [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ and $\Gamma \in \mathrm{Set}/\mathcal{T}$.

$$\begin{array}{ccc}
\int^{\Gamma'} X(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} & & (x,(f_t)_{t\in\mathcal{T}}) \\
\big\downarrow \ell\eta_X & & \big\downarrow \\
\int^{\Gamma'}\int^{\Delta'} X(\Delta') \times \prod_{u\in\mathcal{T}} (\mathcal{Y}(\Gamma'))^{-1}(u)^{\Delta'^{-1}(u)} \times & & \\
\times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} & & ((x,(\mathrm{id}_{\Gamma'^{-1}(u)})_{u\in\mathcal{T}}),(f_t)_{t\in\mathcal{T}}) \\
\big\downarrow \varepsilon_{\ell X} & & \big\downarrow \\
\int^{\Gamma'} X(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} & & \ell X(\sum_{t\in\mathcal{T}} f_t)(x,(\mathrm{id}_{\Gamma'^{-1}(u)})_{u\in\mathcal{T}}) \\
 & & = (x,(f_t\circ \mathrm{id}_{\Gamma'^{-1}(t)})_{t\in\mathcal{T}})
\end{array}$$

This composite is the identity on $(x,(f_t)_{t\in\mathcal{T}})$. Now let $F \in [\mathrm{Set}/\mathcal{T},\mathrm{Set}]$ and $\Delta \in \mathbb{F}\downarrow\mathcal{T}$.

$$\begin{array}{ccc}
F(\mathcal{Y}(\Delta)) & & x \\
\big\downarrow \eta_{kF} & & \big\downarrow \\
\int^{\Gamma'} F(\mathcal{Y}(\Gamma')) \times \prod_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)} & & (x,(\mathrm{id}_{\Delta^{-1}(t)})_{u\in\mathcal{T}}) \\
\big\downarrow k\varepsilon_F & & \big\downarrow \\
F(\mathcal{Y}(\Delta)) & & \sum_{t\in\mathcal{T}} \mathrm{id}_{\Delta^{-1}(t)}(x) = x
\end{array}$$

This composite is the identity on x.

The triangle identities for $\eta : \mathrm{Id}_{[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}} \to k\ell$ and $\varepsilon : \ell k \to \mathrm{Id}_{[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]}$ follow from the triangle identities for $\eta : \mathrm{Id}_{[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]} \to k\ell$ and $\varepsilon : \ell k \to \mathrm{Id}_{[\mathrm{Set}/\mathcal{T},\mathrm{Set}]}$.

So we have the adjunctions $\ell \dashv k : [\mathrm{Set}/\mathcal{T},\mathrm{Set}] \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$ and $\ell \dashv k : [\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}] \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$.

7.5 Monoidal adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}] \to [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$

We are going to show in the following that the constructed adjunction is monoidal. By theorem 4.4.37, it follows that the structure morphisms of ℓ are isomorphisms.

Proposition 7.5.133 *The functor $\ell : [\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}} \to [\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$ is monoidal.*

Proof. Let $X = (X_t)_{t\in\mathcal{T}}$ and $Y = (Y_t)_{t\in\mathcal{T}}$. We are going to construct an arrow

$$\phi_{X,Y} : \ell(X)\circ\ell(Y) \to \ell(X\otimes Y)$$

in $[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$, natural in X and Y. Let $\Gamma \in \mathrm{Set}/\mathcal{T}$ and $t \in \mathcal{T}$. Let us rewrite the domain and codomain using the coend notation.

$$\begin{aligned}
&\ell(X)(\ell(Y)(\Gamma))^{-1}(t) = \\
&\quad = \int^{\Delta=(u_1,\ldots,u_m)} X_t(\Delta)\times(\ell Y)(\Gamma)^{-1}(u_1)\times\ldots\times(\ell Y)(\Gamma)^{-1}(u_m)\\
&\quad = \int^{\Delta=(u_1,\ldots,u_m)} X_t(\Delta)\times\Big(\int^{\Delta_1} Y_{u_1}(\Delta_1)\times\prod_{u_1\in\mathcal{T}}\Gamma^{-1}(u_1)^{\Delta_1^{-1}(u_1)}\Big)\times\\
&\qquad\qquad \times\ldots\times\\
&\qquad\qquad \times\Big(\int^{\Delta_m} Y_{u_m}(\Delta_m)\times\prod_{u_m\in\mathcal{T}}\Gamma^{-1}(u_m)^{\Delta_1^{-1}(u_m)}\Big)\\
&\quad \cong \int^{\Delta=(u_1,\ldots,u_m)} \int^{\Delta_1}\ldots\int^{\Delta_m} X_t(\Delta)\times Y_{u_1}(\Delta_1)\times\ldots\times Y_{u_m}(\Delta_m)\times\\
&\qquad\qquad \times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Delta_1+\ldots+\Delta_m)^{-1}(t')}
\end{aligned}$$

and

$$\begin{aligned}
&\ell(X\otimes Y)(\Gamma)^{-1}(t) = \\
&\quad = \int^{\Gamma'} (X\otimes Y)_t(\Gamma')\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma'^{-1}(t')}\\
&\quad = \int^{\Gamma'}\int^{\Delta} X_t(\Delta)\times\prod_{v\in\mathcal{T}} Y(\Gamma')^{-1}(v)^{\Delta^{-1}(v)}\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma'^{-1}(t')}\\
&\quad = \int^{\Delta=(u_1,\ldots,u_m)}\int^{\Gamma'} X_t(\Delta)\times Y_{u_1}(\Gamma')\times\ldots\times Y_{u_m}(\Gamma')\times\\
&\qquad\qquad \times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma'^{-1}(t')}
\end{aligned}$$

By universal property of the coends, it suffices to give a collection of arrows

$$X_t(\Delta) \times Y_{u_1}(\Delta_1) \times \ldots \times Y_{u_m}(\Delta_m) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta_1 + \ldots + \Delta_m)^{-1}(t')} \to \ell(X \otimes Y)(\Gamma)^{-1}(t)$$

for all $\Delta, \Delta_1, \ldots, \Delta_m \in \mathbb{F} \downarrow \mathcal{T}$ that satisfies the wedge condition. We define it to be the following mapping composed with the corresponding coprojections

$$(x, y_1, \ldots, y_m, (h_{t'})_{t' \in \mathcal{T}}) \quad \mapsto \quad (x, Y_{u_1}(i_1)(y_1), \ldots, Y_{u_m}(i_m)(y_m), (h_{t'})_{t' \in \mathcal{T}})$$

where $i_j : \Delta_j \to \Delta_1 + \ldots + \Delta_m$ is the j-th inclusion.

For the remaining verifications see appendix D.6.

Now we construct the arrow $\phi : \mathrm{Id}_{\mathrm{Set}\,/\mathcal{T}} \to \ell(\mathcal{Y})$. The component in $\Gamma \in \mathrm{Set}\,/\mathcal{T}$ at the fibre $t \in \mathcal{T}$ is

$$\phi : \Gamma^{-1}(t) \to \int^{\Gamma'} \Gamma'^{-1}(t) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Gamma'^{-1}(t')}$$

we take the following mapping composed with the $\langle t \rangle$-th coprojection

$$x \quad \mapsto \quad 1 \times x^1 \in \langle t \rangle^{-1}(t) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\langle t \rangle^{-1}(t')}$$

For the remaining verifications see appendix D.6.

□

Proposition 7.5.134 *The functor $k : [\mathrm{Set}\,/\mathcal{T}, \mathrm{Set}\,/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is monoidal.*

Proof. Let F and $G \in [\mathrm{Set}\,/\mathcal{T}, \mathrm{Set}\,/\mathcal{T}]$, we construct an arrow $\psi_{F,G} : k(F) \otimes k(G) \to k(F \circ G)$ for all $\Delta \in \mathbb{F} \downarrow \mathcal{T}$ and $t \in \mathcal{T}$. Let us rewrite the domain and the codomain using the coend notation.

$$\begin{aligned}(k(F) \otimes k(G))(\Delta)^{-1}(t) &= \int^{\Gamma'} (F(\mathcal{Y}(\Gamma')))^{-1}(t) \times \prod_{u \in \mathcal{T}} (G(\mathcal{Y}(\Delta)))^{-1}(u)^{\Gamma'^{-1}(u)} \\ &= \int^{\Gamma'} (F(\Gamma'))^{-1}(t) \times \prod_{u \in \mathcal{T}} (G(\Delta))^{-1}(u)^{\Gamma'^{-1}(u)}\end{aligned}$$

and

$$\begin{aligned}k(F \circ G)(\Delta)^{-1}(t) &= (F \circ G \circ \mathcal{Y})(\Delta)^{-1}(t) \\ &= (F \circ G)(\Delta)^{-1}(t)\end{aligned}$$

By universal property of the coend, it suffices to give a collection of arrows

$$F(\Gamma')^{-1}(t) \times \prod_{u \in \mathcal{T}} G(\Delta)^{-1}(u)^{\Gamma'^{-1}(u)} \to (F \circ G)(\Delta)^{-1}(t)$$

for all $\Gamma' \in \mathbb{F} \downarrow \mathcal{T}$ that satisfies the wedge condition. We take the mapping

$$(x, (h_u)_{u \in \mathcal{T}}) \quad \mapsto \quad F\Big(\sum_{u \in \mathcal{T}} h_u\Big)_t(x)$$

where we write $F(\sum_{u \in \mathcal{T}} h_u)_t(x)$ for the restriction of $F(\sum_{u \in \mathcal{T}} h_u)$ to the fibre over t.

The arrow $\psi : \mathcal{Y} \to k(\mathrm{Id})$ is given by the identity on $\mathcal{Y}$. For the remaining verifications see appendix D.7. □

Proposition 7.5.135 *The unit $\eta : \mathrm{Id} \to k\ell$ of the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is monoidal.*

Proof. See appendix D.8. $\square$

Proposition 7.5.136 *The counit $\varepsilon : \ell k \to \mathrm{Id}$ of the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is monoidal.*

Proof. See appendix D.9. $\square$

The monoidal adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ extends to the adjunction $\ell \dashv k : \mathrm{Mon}([\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]) \to \mathrm{Mon}([\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}})$ by 4.5.40.

7.6 Strengths and morphisms of strengths

In this section we are concerned with the compatibility of the two "pointed" strengths for $(-)^{\mathcal{Y}\langle u\rangle}$ on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ and $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ on the one hand and for ∂_u on $[\mathrm{Set}/\mathcal{T}, \mathcal{C}]$, where in our cases of interest $\mathcal{C}$ is Set or $\mathrm{Set}/\mathcal{T}$, on the other hand. The former induces a "pointed" strength for the binding signature endofunctor on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ associated to a signature. The latter strength is necessary to provide $\partial_u R$ on $[\mathrm{Set}/\mathcal{T}, \mathcal{C}]$ with a module structure for an monoid R on $[\mathrm{Set}/\mathcal{T}, \mathcal{C}]$.

We show that the link between these strengths involved in the presheaf and in the monadic approach is given by two morphisms of strengths $\partial_u \ell \to \ell(-^{\mathcal{Y}\langle u\rangle})$ and $k\partial_u \to (k-)^{\mathcal{Y}\langle u\rangle}$. These morphisms of strengths extend then to the strengths of the signature endofunctor and the strength necessary for the module structure of the monadic approach.

Consider the category $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$, the category of objects under $\mathcal{Y}$. It is monoidal by lemma 4.7.48 since $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is monoidal. The category $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ acts on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ by the following action

$$(P, \mathcal{Y} \to Q) \quad \mapsto \quad P \otimes Q$$

where $\otimes$ is the monoidal product of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The action axioms are satisfied since $([\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}, \otimes, \mathcal{Y})$ is a monoidal category.

Next consider $\mathrm{Id}_{\mathrm{Set}/\mathcal{T}} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$, the category of objects under $\mathrm{Id}_{\mathrm{Set}/\mathcal{T}}$. To lighten up notation we write simply $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$. It is monoidal by lemma 4.7.48 since $([\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}], \circ, \mathrm{Id}_{\mathrm{Set}/\mathcal{T}})$ is monoidal. The category $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ acts on $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ by the following action

$$(F, \mathrm{Id} \to G) \quad \mapsto \quad F \circ G$$

where $\circ$ is the composition of endofunctors on $\mathrm{Set}/\mathcal{T}$. The action axioms are satisfied since $([\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}], \circ, \mathrm{Id})$ is a monoidal category.

For all $u \in \mathcal{T}$, the endofunctor $(-)^{\mathcal{Y}\langle u\rangle}$ on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$–strong by lemma 5.7.97. By proposition 5.7.98, the signature endofunctor associated to a signature is also $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$–strong.

For all $u \in \mathcal{T}$, derivation ∂_u on $[\mathrm{Set}/\mathcal{T}, \mathcal{C}]$ is $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strong by remark 6.2.112. For all $u_1, \ldots, u_m \in \mathcal{T}$, iterated derivation

$$\partial_{u_m} \ldots \partial_{u_1}$$

on $[\mathrm{Set}/\mathcal{T}, \mathcal{C}]$ is $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strong by proposition 4.7.52. Let $t \in \mathcal{T}$. We write $(-)_t$ for the fibre functor $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$. The fibre

$$(\partial_{u_m} \ldots \partial_{u_1})_t$$

is as well $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strong since each fibre of the $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strength

$$\partial_{u_m} \dots \partial_{u_1} F \circ G \to \partial_{u_m} \dots \partial_{u_1} (F \circ G)$$

is as well an $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strength.

Given a $\mathcal{T}$–arity

$$(t_{1,1} \dots t_{1,m_1})t_1, \dots, (t_{n,1} \dots t_{n,m_n})t_n \to t_0$$

the functor

$$\prod_{i=1}^{n} (\partial_{t_{i,m_i}} \dots \partial_{t_{i,1}})_{t_i}$$

is $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$–strong by proposition 4.7.53.

The monoidal functor $(\ell, \phi) : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ extends to the monoidal functor $(\ell, \phi) : \mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to \mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ by lemma 4.8.56. Therefore $((\ell, \phi), \ell, \phi)$ is a morphism of actions. The axioms are satisfied since ℓ is monoidal.

The monoidal functor $(k, \psi) : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ extends to the monoidal functor $\mathrm{Id} \downarrow [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to \mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ by lemma 4.8.56. Therefore $((k, \psi), k, \psi)$ is a morphism of actions. The axioms are satisfied since k is monoidal.

We construct two morphisms of strengths in the following.

Definition 7.6.137 *Let $u \in \mathcal{T}$. We construct a morphism of strengths $b^{(u)} : (k(-))^{\mathcal{Y}\langle u \rangle} \to k(\partial_u(-))$. Let $R \in [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$ and $\Delta \in \mathbb{F} \downarrow \mathcal{T}$. The component $b^{(u)}_{R,\Delta}$ is explicitly*

$$b^{(u)}_{R,\Delta} : \begin{array}{c} R(\mathcal{Y}(\Delta + \langle u \rangle)) \\ \cong R(\Delta + \langle u \rangle) \end{array} \to \begin{array}{c} R(\mathcal{Y}(\Delta) + \langle u \rangle) \\ R(\Delta + \langle u \rangle) \end{array}$$

which is given by the identity on $R(\Delta + \langle u \rangle)$. For the remaining verifications see appendix D.10.

Definition 7.6.138 *Let $u \in \mathcal{T}$. We construct a morphism of strengths $a^{(u)} : \partial_u(\ell(-)) \to \ell((-)^{\mathcal{Y}\langle u \rangle})$.*

Let $P \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ and $\Gamma \in \mathrm{Set}/\mathcal{T}$. We construct the following arrow

$$a^{(u)}_{P,\Gamma} : (\ell P)(\Gamma + \langle u \rangle)^{-1}(t) \to (\ell(P^{\mathcal{Y}\langle u \rangle}))(\Gamma)^{-1}(t)$$

Using the coend notation we rewrite the domain and codomain

$$(\ell P)(\Gamma + \langle u \rangle)^{-1}(t) = \int^{\Delta} P_t(\Delta) \times \prod_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{\Delta^{-1}(t')}$$

and

$$(\ell(P^{\mathcal{Y}\langle u \rangle}))(\Gamma)^{-1}(t) = \int^{\Delta'} P_t(\Delta' + \langle u \rangle) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')}$$

By universal property of the coend, it suffices to give a collection of arrows

$$P_t(\Delta) \times \prod_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{\Delta^{-1}(t')} \to (\ell(P^{\mathcal{Y}\langle u \rangle}))(\Gamma)^{-1}(t)$$

for all Δ satisfying the wedge property. We take the following arrow

$$P_t(\Delta) \times \prod_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{\Delta^{-1}(t')} \to P_t(\Delta' + \langle u \rangle) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')}$$

$$p, (h_{t'})_{t' \in \mathcal{T}} \mapsto P_t(f^{\Delta})(p), (\widehat{h}_{t'})_{t' \in \mathcal{T}}$$

composed with the corresponding coprojection, where for each $(h_{t'} : \Delta^{-1}(t') \to (\Gamma + \langle u \rangle)^{-1}(t'))_{t' \in \mathcal{T}}$, we set $\Delta' \subseteq \Delta$ such that

$$\Delta'^{-1}(t') = \begin{cases} \Delta^{-1}(t') & \text{if } t' \neq u \\ \{x \in \Delta^{-1}(u) \text{ such that } h_u(x) \in \Gamma^{-1}(u)\} & \text{if } t' = u \end{cases}$$

This defines $(\widehat{h}_{t'} : \Delta'^{-1}(t') \to \Gamma^{-1}(t'))_{t' \in \mathcal{T}}$ where

$$\widehat{h}_{t'} = \begin{cases} h_{t'} : \Delta^{-1}(t') \to \Gamma^{-1}(t') & \text{if } t' \neq u \\ h_u : \Delta'^{-1}(u) \to \Gamma^{-1}(u) & \text{if } t' = u \end{cases}$$

Then we can define $f^{\Delta} : \Delta \to \Delta' + \langle u \rangle$ where the fibre in t' is given by

$$f^{\Delta}_{t'}(x) = \begin{cases} x & \text{if } t' \neq u \\ x & \text{if } t' = u \text{ and } x \in \Delta'^{-1}(u) \\ 1 & \text{if } t' = u \text{ and } x \notin \Delta'^{-1}(u) \end{cases}$$

This construction of f is natural in Δ. For the remaining verifications see appendix D.11.

Let $u_1, \ldots, u_m \in \mathcal{T}$. Then by lemma 4.8.60 the arrows

$$a^{(u_1,\ldots,u_m)} : \partial_{u_m} \ldots \partial_{u_1}(\ell(-)) \to \ell((-)^{\mathcal{Y}(u_1,\ldots,u_m)}$$

and

$$b^{(u_1,\ldots,u_m)} : (k(-))^{\mathcal{Y}(u_1,\ldots,u_m)} \to k(\partial_{u_m} \ldots \partial_{u_1}(-))$$

are morphisms of strengths.

Let $t \in \mathcal{T}$. The fibres

$$(\partial_{u_m} \ldots \partial_{u_1}(\ell(-)))_t \to \ell((-)_t^{\mathcal{Y}(u_1,\ldots,u_m)}$$

and

$$(k(-))_t^{\mathcal{Y}(u_1,\ldots,u_m)} \to (k(\partial_{u_m} \ldots \partial_{u_1}(-)))_t$$

are as well morphisms of strengths.

Notation 7.6.139 *Given a $\mathcal{T}$–arity*

$$\alpha = (t_{1,1} \ldots t_{1,m_1})t_1, \ldots, (t_{n,1} \ldots t_{n,m_n})t_n \to t_0$$

by lemma 4.8.61 the arrows a^{α} and b^{α} are morphisms of strengths, where write

$$a^{\alpha} : \prod_{i=1}^{n} \partial_{t_{i,m_i}} \ldots \partial_{t_{i,1}}(\ell(-))_{t_i} \to \ell(\prod_{i=1}^{n}(-)_{t_i}^{\mathcal{Y}(t_{i,1},\ldots,t_{i,m_i})})$$

and

$$b^{\alpha} : \prod_{i=1}^{n}(k(-))_{t_i}^{\mathcal{Y}(t_{i,1},\ldots,t_{i,m_i})} \to k(\prod_{i=1}^{n} \partial_{t_{i,m_i}} \ldots \partial_{t_{i,1}}(-))_{t_i}$$

Notation 7.6.140 *Given a $\mathcal{T}$–signature $S = (\alpha_i)_{i\in I}$ where we write*

$$\alpha = (t^{(i)}_{1,1} \dots t^{(i)}_{1,m_1})t^{(i)}_1, \dots, (t^{(i)}_{n,1} \dots t^{(i)}_{n,m_n})t^{(i)}_n \to t^{(i)}_0$$

for the arity α_i. We have the following morphisms of strengths by lemma 4.8.63

$$a^S : \sum_{i\in I}\prod_{j=1}^{n_i} \partial_{t^{(i)}_{j,m_j}} \dots \partial_{t^{(i)}_{j,1}}(\ell(-))_{t_j} \to \ell(\sum_{i\in I}\prod_{j=1}^{n_i}(-)_{t_j}^{\mathcal{Y}(t^{(i)}_{j,1},\dots,t^{(i)}_{j,m_j})})$$

and

$$b^S : \sum_{i\in I}\prod_{j=1}^{n_i}(k(-))_{t^{(i)}_j}^{\mathcal{Y}(t^{(i)}_{j,1},\dots,t^{(i)}_{j,m_j})} \to k(\sum_{i\in I}\prod_{j=1}^{n_i} \partial_{t^{(i)}_{j,m_j}} \dots \partial_{t^{(i)}_{j,1}}(-))_{t^{(i)}_j}$$

Proposition 7.6.141 *The arrow $a^{(u)} : \partial_u(\ell(-)) \to \ell((-)^{\mathcal{Y}\langle u\rangle})$ is an isomorphism.*

Proof. Let $P \in [\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. $a^{(u)}_P$ has an inverse, $d^{(u)}_P : \ell(P^{\mathcal{Y}\langle u\rangle}) \to \partial_u(\ell P)$. Recall that

$$(\ell(P^{\mathcal{Y}\langle u\rangle}))(\Gamma)^{-1}(t) = \int^{\Delta'} P_t(\Delta' + \langle u\rangle) \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')}$$

and

$$(\ell P)(\Gamma + \langle u\rangle)^{-1}(t) = \int^{\Delta} P_t(\Delta) \times \prod_{t'\in\mathcal{T}} (\Gamma + \langle u\rangle)^{-1}(t')^{\Delta^{-1}(t')}$$

so by universal property of the coend, to define $d^{(u)}_{P,\Gamma}$ it is equivalent to give a collection of arrows

$$P_t(\Delta' + \langle u\rangle) \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')} \to (\ell P)(\Gamma + \langle u\rangle)^{-1}(t)$$

for all $\Delta' \in \mathbb{F}\downarrow\mathcal{T}$ that satisfies the wedge condition. We take the composite of the following mapping with the corresponding coprojection.

$$(x, (h_{t'})_{t'\in\mathcal{T}}) \quad\mapsto\quad (x, (h_{t'} + \langle u\rangle_{t'})_{t'\in\mathcal{T}})$$

For the remaining verifications see appendix D.12. □

It follows that $a^{(u_1,\dots,u_m)}$ is as well an isomorphism for all $u_1, \dots, u_m \in \mathcal{T}$ and we write $d^{(u_1,\dots,u_m)}$ for its inverse. Let $t \in \mathcal{T}$, the fibre of $a^{(u_1,\dots,u_m)}$ in t is an isomorphism too.

Notation 7.6.142 *Let α be a $\mathcal{T}$–arity. We write d^α for the inverse of a^α.*

$$d^\alpha : \ell(\prod_{i=1}^{n}(-)_{t_i}^{\mathcal{Y}(t_{i,1},\dots,t_{i,m_i})}) \to \prod_{i=1}^{n}\partial_{t_{i,m_i}} \dots \partial_{t_{i,1}}(\ell(-))_{t_i}$$

Now we investigate the link between d and b.

Lemma 7.6.143 *Let $u \in \mathcal{T}$ and $R \in [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$. We have the following formula*

$$b^{(u)}_R = k\partial_u\varepsilon_R \circ kd^{(u)}_{kR} \circ \eta_{k(R^{\mathcal{Y}\langle u\rangle})}$$

where η and ε are unit and counit of the adjunction $\ell \dashv k$.

Proof. Let $\Delta \in \mathbb{F} \downarrow \mathcal{T}$ and $t \in \mathcal{T}$. Along the composite we have

$$\begin{array}{ccc}
R(\Delta + \langle u \rangle)^{-1}(t) & & x \\
\downarrow & & \big\downarrow{\scriptstyle \Delta'=\Delta} \\
\int^{\Delta'} R(\Delta' + \langle u \rangle)^{-1}(t) \times \prod_{t' \in \mathcal{T}} \Delta^{-1}(t')^{\Delta'^{-1}(t)} & & (x, (\mathrm{id}_{\Delta^{-1}(t')})_{t'}) \\
\downarrow & & \big\downarrow{\scriptstyle \Delta''=\Delta'+\langle u \rangle} \\
\int^{\Delta''} R(\Delta'')^{-1}(t) \times \prod_{t' \in \mathcal{T}} (\Delta + \langle u \rangle)^{-1}(t')^{\Delta''^{-1}(t)} & & \begin{array}{c}(x, (\mathrm{id}_{\Delta^{-1}(t')} + \langle u \rangle^{-1}(t'))_{t'}) \\ (x, (\mathrm{id}_{(\Delta+\langle u \rangle)^{-1}(t')})_{t'})\end{array} \\
\downarrow & & \big\downarrow \\
R(\Delta + \langle u \rangle)^{-1}(t) & & R(\sum_{t' \in \mathcal{T}} \mathrm{id})_t(x) = x
\end{array}$$

which is the fibre of $b^{(u)}_{R,\Delta}$ in t. □

Let $u_1, \ldots, u_m \in \mathcal{T}$. By recursion as in the proof of proposition 4.9.68 we have as well the following formula

$$b_R^{(u_1,\ldots,u_m)} = k\partial_{u_m} \ldots \partial_{u_1} \varepsilon_R \circ kd_{kR}^{(u_1,\ldots,u_m)} \circ \eta_{k(R^{\mathcal{Y}(u_1,\ldots,u_m)})}$$

Then for a given arity α we have the formula

$$b_R^{\alpha} = k\prod_{i=1}^{n} \partial_{t_{i,m_i}} \ldots \partial_{t_{i,1}} (\varepsilon_R)_{t_i} \circ kd_{kR}^{\alpha} \circ \eta_{\prod_{i=1}^{n} (kR)_{t_i}^{\mathcal{Y}(t_{i,1},\ldots,t_{i,m_i})}}$$

by a similar proof as the one of proposition 4.9.69. And finally for a signature S we have the formula

$$b_R^{S} = k\sum_{i \in I} \prod_{j=1}^{n} \partial_{t^{(i)}_{j,m_j}} \ldots \partial_{t^{(i)}_{j,1}} (\varepsilon_R)_{t^{(i)}_j} \circ kd_{kR}^{S} \circ \eta_{\sum_{i \in I} \prod_{j=1}^{n} (kR)_{t^{(i)}_j}^{\mathcal{Y}(t^{(i)}_{j,1},\ldots,t^{(i)}_{j,m_j})}}$$

by a similar proof as the one of proposition 4.9.70.

7.7 From Σ–Mon to $\mathrm{Rep}(S)$

Given a signature $S = (\alpha_i)_{i \in I}$, by the presheaf approach we associate to it a signature endofunctor Σ. In this section we start with an object of the category of Σ–monoids (P, p, m, e) where (P, p) is a Σ–algebra and where (P, m, e) is a monoid in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The aim is to construct an object of the category of representations $\mathrm{Rep}(S)$.

First we introduce the following notations.

Notation 7.7.144 *If we write for the arity* α_i

$$\alpha_i = (t^{(i)}_{1,1} \ldots t^{(i)}_{1,m_1}) t^{(i)}_1, \ldots, (t^{(i)}_{n_i,1} \ldots t^{(i)}_{n_i,m_{n_i}}) t^{(i)}_{n_i} \to t^{(i)}_0$$

then we write $\Sigma^{(i)}$ *for the corresponding piece of the signature functor; so explicitly*

$$\Sigma^{(i)} = \prod_{j=1}^{n_i} (-)_{t^{(i)}_j}^{\mathcal{Y}(t^{(i)}_{j,1} \ldots t^{(i)}_{j,m_j})}$$

and we write $M^{(i)}$ *for the underlying functor of the associated module of the monadic approach; explicitly*

$$M^{(i)} = \prod_{j=1}^{n_i} \partial_{t^{(i)}_{j,m_j}} \dots \partial_{t^{(i)}_{n_i,m_{n_i}}} (-)_{t^{(i)}_j}$$

Using these notations recall that we have morphisms of strengths

$$a^{(i)} : M^{(i)}\ell \to \ell\Sigma^{(i)}$$

(with their inverses $d^{(i)} : \ell\Sigma^{(i)} \to M^{(i)}\ell$).

Recall that the fibre of ΣP in $u \in \mathcal{T}$ is by definition

$$(\Sigma P)_u = \sum_{i \in I_u} \Sigma^{(i)}(P)$$

and the fibre in u of the algebra structure map $p : \Sigma P \to P$ is by definition

$$p_u : \sum_{i \in I_u} \Sigma^{(i)}(P) \to P_u$$

By universal property of the coproduct the arrow p_u is equivalent to $[p^{(i)}_u]_{i \in I_u}$ with $p^{(i)}_u : \Sigma^{(i)}(P) \to P_u$.

Since the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is monoidal, the object ℓP is a monoid in $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$, that is a monad on $\mathrm{Set}/\mathcal{T}$. We construct now a morphism of ℓP–modules

$$M^{(i)}(\ell P) \to (\ell P)_{t^{(i)}_0}$$

for all $i \in I$ in the sense of the monadic approach. We define it to be the following composite

$$M^{(i)}(\ell P) \xrightarrow{a^{(i)}_P} \ell(\Sigma^{(i)} P) \xrightarrow{\ell(p^{(i)}_{t_0})} \ell(P)_{t^{(i)}_0} = (\ell P)_{t^{(i)}_0}$$

Note that we use the same notation ℓ for the functor $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}] \to [\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$ and the functor $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}} \to [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$. In order to show that this is a morphism of ℓP–modules, we have to verify that the following diagram commutes

$$\begin{tikzcd}
M^{(i)}(\ell P) \circ (\ell P) \arrow[r] \arrow[d] & (\ell P)_{t^{(i)}_0} \circ (\ell P) \arrow[d] \\
M^{(i)}(\ell P) \arrow[r] & (\ell P)_{t^{(i)}_0}
\end{tikzcd}$$

for all $i \in I$. This square is explicitly the following diagram

$$\begin{tikzcd}
M^{(i)}(\ell P) \circ (\ell P) \arrow[r] \arrow[d] & \ell(\Sigma^{(i)} P) \circ (\ell P) \arrow[r] \arrow[d] \arrow[dr, phantom, "II."] & \ell P_{t^{(i)}_0} \circ (\ell P) \arrow[d] \\
M^{(i)}((\ell P) \circ (\ell P)) \arrow[r, phantom, "I."] \arrow[d] & \ell(\Sigma^{(i)} P \bullet P) \arrow[r] \arrow[d] & \ell(P \otimes P)_{t^{(i)}_0} \arrow[dd] \\
M^{(i)}(\ell(P \otimes P)) \arrow[r] \arrow[d] \arrow[dr, phantom, "III."] & \ell(\Sigma^{(i)}(P \otimes P)) \arrow[d] \arrow[dr, phantom, "IV."] & \\
M^{(i)}(\ell P) \arrow[r] & \ell(\Sigma^{(i)} P) \arrow[r] & (\ell P)_{t_0}(\Gamma)
\end{tikzcd}$$

The square II. is a naturality square of ϕ, the square III. is a naturality square of $a^{(i)}$. The diagram IV. commutes because P is a Σ–monoid and the diagram I. commutes because $a^{(i)}$ is a morphism of strengths.

7.8 From $\mathrm{Rep}(S)$ to Σ–Mon

As in section 7.7 we suppose given a signature $S = (\alpha_i)$ and we use notations of 7.7.144. In this section we start with an object of $\mathrm{Rep}(S)$ and construct a Σ–monoid.

Let (R, η, μ) be a monad on $\mathrm{Set}/\mathcal{T}$ together with a representation of each arity α_i

$$r^{(i)} : M^{(i)}(R) \to R_{t_0^{(i)}}$$

Since $k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is monoidal, kR is a monoid on $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. We show that kR is an object of the category Σ–Mon, where Σ is the associated signature functor to S.

Recall that we have the following morphisms of strengths

$$b^{(i)} : \Sigma^{(i)}k \to kM^{(i)}$$

By proposition 4.8.63, we also have the following morphisms of strengths

$$b = \sum_{i \in I_u} b^{(i)} : \sum_{i \in I_u} \Sigma^{(i)}k \to k\big(\sum_{i \in I_u} M^{(i)}\big)$$

First we provide kR with a Σ–algebra structure map $p : \Sigma kR \to kR$. Let $u \in \mathcal{T}$ such that $I_u = \{i \in I \text{ such that } t_0^{(i)} = u\} \neq \emptyset$. By definition the fibre of ΣkR in $u \in \mathcal{T}$ is

$$(\Sigma kR)_u = \sum_{i \in I_u} \Sigma^{(i)}(kR)$$

We define p_u to be the following composite

$$\begin{tikzcd}
(\Sigma kR)_u = \sum_{i \in I_u} \Sigma^{(i)}(kR) \arrow[d, "b_R"] \\
k\big(\sum_{i \in I_u} M^{(i)}(R)\big) \arrow[d, "k[r^{(i)}]_{i \in I_u}"] \\
k(R_u) = (kR)_u
\end{tikzcd}$$

Next we have to show the commutativity of the following diagram

$$\begin{tikzcd}
\Sigma kR \otimes kR \arrow[r] \arrow[d] & kR \otimes kR \arrow[dd] \\
\Sigma(kR \otimes kR) \arrow[d] & \\
\Sigma kR \arrow[r] & kR
\end{tikzcd}$$

This diagram in the fibre u is explicitly the following

$$\begin{tikzcd}
\sum\limits_{i\in I_u} \Sigma^{(i)}(kR) \bullet kR \arrow[r] \arrow[d] & k(\sum\limits_{i\in I_u} M^{(i)}(R)) \bullet kR \arrow[r] \arrow[d] \arrow[dr, phantom, "II."] & (kR)_{t_0} \bullet kR \arrow[d] \\
\sum\limits_{i\in I_u} \Sigma^{(i)}(kR \otimes kR) \arrow[d] \arrow[r, phantom, "I."] & k(\sum\limits_{i\in I_u} M^{(i)}(R) \circ R) \arrow[r] \arrow[d] & k(R \circ R)_{t_0} \arrow[dd] \\
\sum\limits_{i\in I_u} \Sigma^{(i)}(k(R \circ R)) \arrow[r] \arrow[d] \arrow[dr, phantom, "III."] & k(\sum\limits_{i\in I_u} M^{(i)}(R \circ R)) \arrow[d] \arrow[r, phantom, "IV."] & \\
\sum\limits_{i\in I_u} \Sigma^{(i)}(kR) \arrow[r] & k(\sum\limits_{i\in I_u} M^{(i)}(R)) \arrow[r] & (kR)_{t_0}
\end{tikzcd}$$

The diagram I. commutes because b is a morphism of strengths, square III. is a naturality square of b. The square II. is a naturality square of ψ and the diagram IV. commutes because it is the R–module morphism axiom of the $r^{(i)}$'s.

7.9 Adjunction between Rep(S) and Σ–Mon

In this section we prove the following theorem

Theorem 7.9.145 *Let $S = (\alpha_i)_{i\in I}$ be a binding signature as defined in . There is an adjunction*

$$L \dashv K : \mathrm{Rep}(S) \to \Sigma\text{–}\mathrm{Mon}$$

Proof. Again we write

$$\alpha_i = (t^{(i)}_{1,1} \dots t^{(i)}_{1,m_1})t^{(i)}_1, \dots, (t^{(i)}_{n_i,1} \dots t^{(i)}_{n_i,m_{n_i}})t^{(i)}_{n_i} \to t^{(i)}_0$$

Functor $K : \mathrm{Rep}(S) \to \Sigma\text{–}\mathrm{Mon}$

In section 7.8 we constructed the assignation on objects of the functor

$$K : \mathrm{Rep}(S) \to \Sigma\text{–}\mathrm{Mon}$$

Now we construct the assignation on arrows. Let $\rho : ((R, \eta^R, \mu^R), r) \to ((Q, \eta^Q, \mu^Q), q)$ be a morphism of representations. We show that $k\rho$ is a morphism of Σ–monoids $kR \to kQ$. Since k is a monoidal functor, $k\rho$ is a morphism of monoids. It remains to show that $k\rho$ is a morphism of Σ–algebras, that is, that the following square commutes

$$\begin{tikzcd}
\Sigma kR \arrow[r] \arrow[d] & \Sigma kQ \arrow[d] \\
kR \arrow[r] & kQ
\end{tikzcd}$$

this square is explicitly in the fibre $u \in \mathcal{T}$

$$\begin{tikzcd}
\sum_{i\in I_u} \Sigma^{(i)}(kR) \arrow[r] \arrow[d] & \sum_{i\in I_u} \Sigma^{(i)}(kQ) \arrow[d] \\
k \sum_{i\in I_u} M^{(i)}(R) \arrow[r] \arrow[d, "{k[r^{(i)}]_{i\in I_u}}"'] & k \sum_{i\in I_u} M^{(i)}(Q) \arrow[d, "{k[q^{(i)}]_{i\in I_u}}"] \\
kR_u \arrow[r] & kQ_u
\end{tikzcd}$$

which commutes by naturality of the representations and by naturality of b_R in R.

Functor $L : \Sigma\text{– Mon} \to \text{Rep}(S)$

In section 7.7 we constructed the assignation on objects of the functor

$$L : \Sigma\text{– Mon} \to \text{Rep}(S)$$

Now we construct the assignation on arrows. Let $f : (P, m, e, p) \to (Q, n, i, q)$ be a morphism of Σ–monoids. Since ℓ is monoidal, $\ell(f) : \ell P \to \ell Q$ is a monoid morphism. It remains to show that $\ell(f)$ is a morphism of representations, that is, that the following square commutes for each arity α_i

$$\begin{tikzcd}
M^{(i)}(\ell P) \arrow[r] \arrow[d] & M^{(i)}(\ell Q) \arrow[d] \\
(\ell P)_{t_0^{(i)}} \arrow[r] & (\ell Q)_{t_0^{(i)}}
\end{tikzcd}$$

we unfold the vertical arrows and obtain

$$\begin{tikzcd}
M^{(i)}(\ell P) \arrow[r] \arrow[d] & M^{(i)}(\ell Q) \arrow[d] \\
\ell(\Sigma^{(i)}(P)) \arrow[r] \arrow[d, "\ell(p^{(i)})"'] & \ell(\Sigma^{(i)}(Q)) \arrow[d, "\ell(q^{(i)})"] \\
\ell(P)_{t_0^{(i)}} \arrow[r] & \ell(Q)_{t_0^{(i)}}
\end{tikzcd}$$

The bottom square commutes because f is a morphism of Σ–algebras and the top square is a naturality square of $a^{(i)}$.

Unit

Next we show that the unit η of the adjunction $\ell \dashv k : [\text{Set}/\mathcal{T}, \text{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \text{Set}]^{\mathcal{T}}$ is a morphism of Σ–monoids. The component η_P in a Σ–monoid (P, m, e, p) is clearly a monoid morphism since the adjunction is monoidal. It remains to show that η_P is a morphism of Σ–algebras, that is, that the following square commutes

$$\begin{tikzcd}
\Sigma P \arrow[r, "\Sigma\eta_P"] \arrow[d, "p"'] & \Sigma k\ell P \arrow[d] \\
P \arrow[r, "\eta_P"'] & k\ell P
\end{tikzcd}$$

Remark that by proposition 4.8.63, we have the following morphisms of strengths

$$a = \sum_{i \in I_u} a^{(i)} : \sum_{i \in I_u} M^{(i)} \ell \to \ell\big(\sum_{i \in I_u} \Sigma^{(i)}\big)$$

and their inverses

$$d = \sum_{i \in I_u} d^{(i)} : \ell\big(\sum_{i \in I_u} \Sigma^{(i)}\big) \to \sum_{i \in I_u} M^{(i)} \ell$$

Let $u \in \mathcal{T}$ such that $I_u \neq \emptyset$. The diagram in the fibre in u is explicitly the following

$$\begin{tikzcd}
\sum_{i \in I_u} \Sigma^{(i)} P \arrow[r, "(\Sigma \eta_P)_u"] \arrow[drr, "(\eta_{\Sigma P})_u"'] \arrow[dd, "p_u"'] & \sum_{i \in I_u} \Sigma^{(i)} k\ell P \arrow[r, "b_{\ell P}"] & k(\sum_{i \in I_u} M^{(i)} \ell P) \arrow[d, "ka_P"] \\
& & k\ell(\sum_{i \in I_u} \Sigma^{(i)} P) \arrow[d, "kl(p)_u"] \\
P_u \arrow[rr, "(\eta_P)_u"'] & & (k\ell P)_u
\end{tikzcd}$$

The bottom diagram is a naturality square of η. It remains to show that the top right triangle commutes. We use the formula

$$b = kM\varepsilon \circ kdk \circ \eta \Sigma k$$

where we write $M = \sum_{i \in I_u} M^{(i)}$ and it becomes explicitly

$$\begin{tikzcd}
\sum_{i \in I_u} \Sigma^{(i)} P \arrow[rr, "\Sigma\eta"] \arrow[d, "\eta\Sigma"'] \arrow[drr, phantom, "I."] & & \sum_{i \in I_u} \Sigma^{(i)} k\ell P \arrow[d, "\eta\Sigma k\ell"] \\
k\ell \sum_{i \in I_u} \Sigma^{(i)} P \arrow[rr, "k\ell\Sigma\eta"] \arrow[dr, "kd"] \arrow[ddddrr, bend right, "\mathrm{id}"'] \arrow[drr, phantom, "II."] & & k\ell \sum_{i \in I_u} \Sigma^{(i)} k\ell P \arrow[d, "kdk\ell"] \\
& k \sum_{i \in I_u} M^{(i)} \ell P \arrow[r, "kM\ell\eta"] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "IV."] & k \sum_{i \in I_u} M^{(i)} \ell k\ell P \arrow[d, "kM\varepsilon\ell"] \arrow[dl, phantom, "III."] \\
& \, & k \sum_{i \in I_u} M^{(i)} \ell P \arrow[d, "ka"] \\
& & k\ell \sum_{i \in I_u} \Sigma^{(i)} P
\end{tikzcd}$$

Square I. is a naturality square of η. Square II. is a naturality square of d. Triangle III. is one of the triangular identities and diagram IV. commutes since a and d are inverse.

Counit

We show that the counit ε of the adjunction $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ is a morphism of representations. The component ε_R in a monad (R, μ, η) is clearly a monad morphism since the adjunction is monoidal. It remains to show that it is a morphism of representations. That is that the following diagram commutes for each arity α_i of S.

$$\begin{tikzcd}
M^{(i)}(\ell kR) \arrow[r, "r^{(i)}_{\ell kR}"] \arrow[d, "M^{(i)}\varepsilon_R"'] & (\ell kR)_{t_0} \arrow[d, "(\varepsilon_R)_{t_0^{(i)}}"] \\
M^{(i)}(\varepsilon^* R) \arrow[r, "\varepsilon^* r^{(i)}_R"'] & (\varepsilon^* R)_{t_0^{(i)}}
\end{tikzcd}$$

This is explicitly

$$\begin{tikzcd}
M^{(i)}(\ell kR) \arrow[rr, "r^{(i)}_{\ell kR}"] \arrow[d, "a^{(i)}_{kR}"'] \arrow[dr, "M^{(i)}(\varepsilon_R)"] & & (\ell kR)_{t_0} \arrow[dd, "(\varepsilon_R)_{t_0^{(i)}}"] \\
\ell\Sigma^{(i)}(kR) \arrow[d, "\ell b^{(i)}_R"'] & & \\
\ell kM^{(i)}(R) \arrow[r, "\varepsilon_{M^{(i)}(R)}"'] & M^{(i)}(R) \arrow[r, "r^{(i)}_R"'] & R_{t_0^{(i)}}
\end{tikzcd}$$

The diagram on the right is a naturality square of $r^{(i)}$, so it remains to check the commutativity of the left bottom diagram.

$$\begin{tikzcd}
M^{(i)}\ell kR \arrow[d, "a^{(i)}_{kR}"'] \arrow[rrddd, bend left, "\mathrm{id}"] & & \\
\ell\Sigma^{(i)}kR \arrow[d, "\ell\eta_{\Sigma^{(i)}kR}"'] \arrow[dr, "\mathrm{id}"] \arrow[r, phantom, "IV."] & & \\
\ell k\ell\Sigma^{(i)}kR \arrow[r, "\varepsilon_{\ell\Sigma^{(i)}kR}"] \arrow[d, "\ell kd^{(i)}_{kR}"'] \arrow[u, phantom, "III."] & \ell\Sigma^{(i)}kR \arrow[dr, "d^{(i)}_{kR}"] \arrow[d, phantom, "II."] & \\
\ell kM^{(i)}\ell kR \arrow[rr, "\varepsilon_{M^{(i)}\ell kR}"] \arrow[d, "\ell kM^{(i)}\varepsilon_R"'] \arrow[drr, phantom, "I."] & & M^{(i)}\ell kR \arrow[d, "M^{(i)}\varepsilon_R"] \\
\ell kM^{(i)}R \arrow[rr, "\varepsilon_{M^{(i)}R}"'] & & M^{(i)}R
\end{tikzcd}$$

The two squares I. and II. are naturality squares of ε, the triangle III. is one of the triangle identities for the unit and counit and the diagram IV. commutes because $a^{(i)}_{kR}$ and $d^{(i)}_{kR}$ are inverse to each other. $\square$

Chapter 8

Typed syntax without variable binding

In this chapter we develop the monadic approach further. The class of signatures we consider is typed signature with quantification over types only. We consider type and term constructors and so we gather arities of both such constructors in a signature. We allow dependencies between the constructors and thus between the arities.

First we give an intuitive description of typical signatures in pseudo-code coq. Then in the following sections we introduce our notions of signature and prove the theorems stating that initial representations exist.

8.1 Intuitive description

We consider the following example that corresponds to a sequence of declaration of variables using the syntax of the Coq Proof Assistant ([INR]).

Example 8.1.146 (Lists of integers with length)

$$\begin{array}{lll}
\mathrm{tip} & : & \mathrm{Type};\\
\mathrm{term} & : & \mathrm{tip} \to \mathrm{Type};\\
\mathrm{nat} & : & \mathrm{tip};\\
\mathrm{zero} & : & \mathrm{term}(\mathrm{nat});\\
\mathrm{succ} & : & \mathrm{term}(\mathrm{nat}) \to \mathrm{term}(\mathrm{nat});\\
\mathrm{list} & : & \mathrm{term}(\mathrm{nat}) \to \mathrm{tip} \to \mathrm{tip};\\
\mathrm{nil} & : & \forall t : \mathrm{tip}, \mathrm{term}(\mathrm{list}(\mathrm{zero}, t));\\
\mathrm{cons}_k & : & \forall t : \mathrm{tip}, \mathrm{term}(\mathrm{list}(\underline{k}, t)) \to \mathrm{term}(t) \to \mathrm{term}(\mathrm{list}(\mathrm{succ}(\underline{k}), t));
\end{array}$$

where we write $\underline{k}$ *for the term* $\mathrm{succ}^k(\mathrm{zero})$ *of type* $\mathrm{term}(\mathrm{nat})$.

More generally let Γ_0 be the list of variable declarations

$$\begin{array}{lll}
\mathrm{tip} & : & \mathrm{Type};\\
\mathrm{term} & : & \mathrm{tip} \to \mathrm{Type};
\end{array}$$

and consider the following typing rules

$$\frac{}{\vdash_{FO} \Gamma_0}(1)$$

$$\frac{\vdash_{FO} \Gamma \qquad \Gamma \vdash_T t}{\vdash_{FO} \Gamma, (x : t)}(2)$$

$$\frac{\vdash_{FO} \Gamma}{\Gamma \vdash_T \ \mathrm{tip}}(3)$$

$$\frac{\vdash_{FO} \Gamma \qquad \Gamma \vdash_T Y}{\Gamma \vdash_T \mathrm{tip} \to Y}(4)$$

$$\frac{\vdash_{FO} \Gamma \qquad \Gamma \vdash_F t : \mathrm{tip} \qquad \Gamma \vdash_T X}{\Gamma \vdash_T \mathrm{term}(t) \to X}(5)$$

$$\frac{\vdash_{FO} \Gamma \qquad (x : t) \in \Gamma}{\Gamma \vdash_F x : t}(6)$$

$$\frac{\vdash_{FO} \Gamma \qquad \Gamma \vdash_F t : \mathrm{tip}}{\Gamma \vdash_T \mathrm{term}(t)}(7)$$

$$\frac{\vdash_{FO} \Gamma \qquad \Gamma \vdash_F f : \mathrm{tip} \to X \qquad \Gamma \vdash_F x : \mathrm{tip}}{\Gamma \vdash_F (fx) : X}(8)$$

$$\frac{\vdash_{FO} \Gamma \qquad \Gamma \vdash_F f : \mathrm{term}(t) \to X \qquad \Gamma \vdash_F x : \mathrm{term}(t)}{\Gamma \vdash_F (fx) : X}(9)$$

$$\frac{\vdash_{FO} \Gamma \qquad \Gamma, t : \mathrm{tip} \vdash_T X}{\Gamma \vdash_T \forall t : \mathrm{tip}, X}(10)$$

The signatures we consider in this chapter are equivalent of a list of variable declarations Γ in coq satisfying the judgment $\vdash_{FO} \Gamma$ by the above rules.

8.2 Base category

We take the category TEns := [2, Set] to be the base category for the typed case, where 2 stands for the category with two objects and exactly one non–identity arrow. We think of an object of TEns as an environment $\Gamma : X \to T$ where X is a set of variables or terms and T the set of types. The category TEns is complete and cocomplete and comes equipped with two forgetful functors to Set

Notation 8.2.147 *We write* $\overline{(-)}$ *for the forgetful functor* TEns $\to$ Set, $X \to T \mapsto X$. *We write* $\underline{(-)}$ *for the forgetful functor* TEns $\to$ Set, $X \to T \mapsto T$.

Notation 8.2.148 *Let* $\Gamma \in \mathrm{TEns}$. *We write* $\underline{\Gamma}$ *for its set of types and* $\underline{\underline{\Gamma}}$ *for the set* $\underline{\Gamma} + \{\bot\}$.
Let $f : \Gamma \to \Delta$. *We write* $\underline{\underline{f}} : \underline{\underline{\Gamma}} \to \underline{\underline{\Delta}}$ *where* $\underline{\underline{f}} = \underline{f} + \bot$.

Notation 8.2.149 *Let* $\Gamma \in \mathrm{TEns}$ *and* $t \in \underline{\underline{\Gamma}}$. *We write*

$$\Gamma[t] := \begin{cases} \Gamma^{-1}(t) & \textit{if } t \in \underline{\Gamma} \\ \underline{\Gamma} & \textit{if } t = \bot \end{cases}$$

Let $f : \Gamma \to \Delta$ *and* $t \in \underline{\underline{\Gamma}}$. *We write*

$$f[t] := \begin{cases} f_t : \Gamma^{-1}(t) \to \Delta^{-1}(\underline{f}(t)) & \textit{if } t \in \underline{\Gamma} \\ \underline{f} : \underline{\Gamma} \to \underline{\Delta} & \textit{if } t = \bot \end{cases}$$

8.3 Signature of degree 0

In this section we define the notion of arity and signature of degree 0, that is without quantification over types.

Definition 8.3.150 (arity) *Let $C \in \mathrm{TEns}$. An* arity *on C of length r is an arrow $a : r \to \underline{\underline{C}}$.*

Definition 8.3.151 (I–signature) *Let $I \in \mathrm{Set}$, $C \in \mathrm{TEns}$. An* I–signature *S on C is a collection of arities $(S_i)_{i\in I}$ on C.*

Definition 8.3.152 (direct image) *Let $I \in \mathrm{Set}$, $C, D \in \mathrm{TEns}$ and S an I–signature on C. Given a morphism $f : C \to D$ we define f_*S the* direct image *of S along f, it is an I–signature on D consisting of the following arities*

$$r_i \xrightarrow{S_i} \underline{\underline{C}} \xrightarrow{\underline{\underline{f}}} \underline{\underline{D}}$$

Definition 8.3.153 (category of I–signatures) *Let $I \in \mathrm{Set}$, $C, D \in \mathrm{TEns}$. A* morphism of I–signatures *from an I–signature S on C to an I–signature T on D is a morphism $f : C \to D$ in* TEns *such that $T = f_*S$. I–Signatures and morphisms of I–signatures form a category,* $\mathrm{Sign}(I)$*. It comes equipped with a forgetful functor* $\mathrm{Sign}(I) \to \mathrm{TEns}$.

Definition 8.3.154 (category of I–endorepresentations) *Let $I \in \mathrm{Set}$. An object of the category of I–endorepresentations is an I–signature $S = (S_i)_{i\in I}$ on C together with a map*

$$\rho_i : \prod_{j=1}^{r_i-1} C[S_i(j)] \to C[S_i(r_i)]$$

for all $i \in I$.

A morphism from (C, S, ρ) to (D, T, ξ) is a morphism of I–signatures $f : (C, S) \to (D, T)$ such that

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} C[S_i(j)] \arrow[r, "\rho_i"] \arrow[d, "\prod f[S_i(j)]"'] & C[S_i(r_i)] \arrow[d, "f[S_i(r_i)]"] \\
\prod_{j=1}^{r_i-1} D[T_i(j)] \arrow[r, "\xi_i"'] & D[T_i(r_i)]
\end{tikzcd}$$

commutes for all $i \in I$.

I–Endorepresentations and morphisms of I–endorepresentations form a category written $\mathrm{EndRep}(I)$*. It comes equipped with a forgetful functor* $\mathrm{EndRep}(I) \to \mathrm{Sign}(I)$.

Definition 8.3.155 (representation) *Let $I \in \mathrm{Set}$, $C \in \mathrm{TEns}$ and S be an I–signature on C. A* representation *of S in an object $D \in \mathrm{TEns}$ is a morphism $f : C \to D$ and an I–endorepresentation $\xi = (\xi_i)_{i\in I}$ of the I–signature f_*S on D.*

Definition 8.3.156 (category of representations) *Let $I \in \mathrm{Set}$, $C \in \mathrm{TEns}$ and $S = (S_i)_{i\in I}$ be an I–signature on C. We build* $\mathrm{Rep}(S)$ *the* category of representations *of S. An object is a representation (D, f, ξ) of S in an object D.*

A morphism of representations *from* (D, f, ξ) *to* (D', f', ξ') *is a morphism* $\varphi : D \to D'$ *of* TEns *such that*

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ D \arrow[rr, "\varphi"'] & & D' \end{tikzcd}$$

commutes and which induces a morphism of I–endorepresentations from (D, f_*S, ξ) *to* (D', f'_*S, ξ').

Proposition 8.3.157 *Let* $I \in \mathrm{Set}$, $C \in \mathrm{TEns}$ *and* $S = (a_i)_{i \in I}$ *be an* I*–signature on* C. *The category* $\mathrm{Rep}(S)$ *has an initial object. We write* $C\langle S\rangle$ *for the underlying object in* TEns *of the initial representation.*

Proof. To each arity $a_i : r_i \to \underline{\underline{C}}$, we associate the $\underline{\underline{C}}$–arity (without variable binding)

$$\alpha_i : a_i(1), \ldots, a_i(r_i - 1) \to a_i(r_i)$$

Then we construct the initial object $C\langle S\rangle'$ of the category of representations of the $\underline{\underline{C}}$–signature $(\alpha_i)_{i \in I}$ as in the proof of theorem 6.4.121 with $\mathcal{T}$ being $\underline{\underline{C}}$ and the starting object C'. This object C' of $\mathrm{Set}/\underline{\underline{C}}$ is obtained from C by adding $\perp$ to $\underline{C}$ and adding $\underline{C}$ as the fibre of $\perp$.

Construction of the initial object $C\langle S\rangle$

Then we construct $C\langle S\rangle$ by adding the fibre of $\perp$ to $\underline{C\langle S\rangle'}$ and erasing $\perp$

$$\underline{C\langle S\rangle} = \underline{C\langle S\rangle'} \cup C\langle S\rangle'^{-1}(\perp) \setminus \{\perp\}$$

and

$$C\langle S\rangle^{-1}(c) = C\langle S\rangle'^{-1}(c)$$

for all $c \in \underline{C}$.

The arrow $\imath : C \to C\langle S\rangle$ is given by the inclusion $C \hookrightarrow C\langle S\rangle$ which comes from the inclusion $C' \hookrightarrow C\langle S\rangle'$. We construct the I–endorepresentation of S in $C\langle S\rangle$. By construction of $C\langle S\rangle'$ we have for each arity α_i an arrow

$$\prod_{j=1}^{r_i - 1} C\langle S\rangle'^{-1}(a_i(j)) \to C\langle S\rangle'^{-1}(a_i(r_i))$$

by our construction of $C\langle S\rangle$ this amounts to an arrow

$$\rho_i : \prod_{j=1}^{r_i - 1} C\langle S\rangle[\underline{\underline{\imath}}(a_i(j))] \to C\langle S\rangle[\underline{\underline{\imath}}(a_i(r_i))]$$

as desired for an I–endorepresentation of $\imath_* S$.

Construction of the arrow $h : C\langle S\rangle \to D$

Let $(D, f, (\xi_i)_{i \in I})$ be another representation. Explicitly ξ_i is of the form

$$\prod_{j=1}^{r_i - 1} D[\underline{\underline{f}} \circ a_i(j)] \to D[\underline{\underline{f}} \circ a_i(r_i)]$$

We associate the object D' to D such that ξ_i is now of type

$$\prod_{j=1}^{r_i-1} D'^{-1}(\underline{\underline{f}} \circ a_i(j)) \to D'^{-1}(\underline{\underline{f}} \circ a_i(r_i))$$

for all $i \in I$.

We construct a morphism of I–endorepresentations h from $(C\langle S\rangle, \imath_* S, \rho)$ to $(D, f_* S, \xi_i)$. First we construct an arrow $h' : C\langle S\rangle' \to D'$. We define it recursively by giving an arrow $h'_n : C\langle S\rangle'_n \to D'$ for all n.

For $n = 0$ we have $C\langle S\rangle'_0 = C'$, so the arrow h'_0 is given by $f' : C' \to D'$. Suppose given an arrow $h'_{m,t} : C\langle S\rangle'_{m,t} \to D'^{-1}(\underline{\underline{f}}(t))$ for all $m \leq n$ and $t \in \underline{\underline{C}}$, then we construct the arrow $h'_{n+1,t}$ for all $t \in \underline{\underline{C}}$.

By definition $C\langle S\rangle'_{n+1,t} = \sum_{i \in I_t} \sum_{g \in \Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)}$. So by universal property of the sums, it suffices to give an arrow $\prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)} \to D'^{-1}(\underline{\underline{f}}(t))$ for all $i \in I_t$ and $g \in \Theta_{r_i-1,n}$. We take the following composite

$$\prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)} \to \prod_{j=1}^{r_i-1} D'^{-1}(\underline{\underline{f}}(a_i(j))) \to D'^{-1}(\underline{\underline{f}}(t))$$

since $t = a_i(r_i)$ for all $a_i \in I_t$.

Then we set $h' = [h'_n]_{n \in \mathbb{N}} : C\langle S\rangle' \to D'$. By undoing the operation $'$, we obtain $h : C\langle S\rangle \to D$. By undoing the operation $'$ we mean the following. The object $C\langle S\rangle'$ is of the form

$$\begin{tikzcd}
(\overline{C} + A) + (\underline{C} + B) \arrow[d, "C\langle S\rangle'"] \\
\underline{C} + \{\perp\}
\end{tikzcd}$$

where $\underline{C} + B$ is in the fibre of $\perp$ and h' is explicitly

$$\begin{tikzcd}
(\overline{C} + A) + (\underline{C} + B) \arrow[r, "\overline{h'}"] \arrow[d, "C\langle S\rangle'"'] & \overline{D} + \underline{D} \arrow[d, "D'"] \\
\underline{C} + \{\perp\} \arrow[r, "\underline{h'}"'] & \underline{D} + \{\perp\}
\end{tikzcd}$$

since we have $\underline{C} + B \to \underline{D}$, the following square commutes

$$\begin{tikzcd}
\overline{C} + A \arrow[r] \arrow[d, "C\langle S\rangle"'] & \overline{D} \arrow[d, "D"] \\
\underline{C} + B \arrow[r] & \underline{D}
\end{tikzcd}$$

which we take to be h. The fibre of B along $C\langle S\rangle$ is empty.

$h : C\langle S\rangle \to D$ as a morphism of representations

The triangle

$$\begin{tikzcd} & C \arrow[dl, "\imath"'] \arrow[dr, "f"] & \\ C\langle S\rangle \arrow[rr, "h"'] & & D \end{tikzcd}$$

commutes by construction of $C\langle S\rangle$ and $\imath$ being the inclusion. The following diagram

$$\begin{tikzcd} \prod_{j=1}^{r_i-1} C\langle S\rangle[\underline{\imath}(a_i(j))] \arrow[r, "\rho_i"] \arrow[d, "\prod h[\underline{\imath}(a_i(j))]"'] & C\langle S\rangle[\underline{\imath}(a_i(r_i))] \arrow[d, "h[\underline{\imath}(a_i(r_i))]"] \\ \prod_{j=1}^{r_i-1} D[\underline{\underline{f}}(a_i(j))] \arrow[r, "\xi_i"'] & D[\underline{\underline{f}}(a_i(r_i))] \end{tikzcd}$$

commutes since

$$\begin{tikzcd} \prod_{j=1}^{r_i-1} C\langle S\rangle'^{-1}(a_i(j)) \arrow[r, "\rho_i"] \arrow[d, "\prod h'_{a_i(j)}"'] & C\langle S\rangle'^{-1}(a_i(r_i)) \arrow[d, "h'_{a_i(r_i)}"] \\ \prod_{j=1}^{r_i-1} D'^{-1}(\underline{\underline{f}}(a_i(j))) \arrow[r, "\xi_i"'] & D'^{-1}(\underline{\underline{f}}(a_i(r_i))) \end{tikzcd}$$

commutes by definition of h'.

Uniqueness $h : C\langle S\rangle \to D$

Suppose given another morphism of I–endorepresentations k from $(C\langle S\rangle, \imath_* S, \rho)$ to $(D, f_* S, \xi)$. We show that $k = h$ and we can conclude then that $C\langle S\rangle$ is initial in $\mathrm{Rep}(S)$. We write k'_n for the composite

$$C\langle S\rangle'_n \hookrightarrow C\langle S\rangle' \xrightarrow{k'} D'$$

and $k'_{n,t}$ for ts fibre in t. We show by induction that $k'_{n,t} = h'_{n,t}$ for all $n \in \mathbb{N}$ and $t \in \underline{\underline{C}}$.

If $n = 0$, we have

$$k'_{0,t}(x) = g'_t(x) = h'_{0,t}(x)$$

since both h' and k' make the triangle

$$\begin{tikzcd} & C' \arrow[dl, "\imath'"'] \arrow[dr, "f'"] & \\ C\langle S\rangle' \arrow[rr] & & D' \end{tikzcd}$$

commute. Suppose that $k'_{m,t} = h'_{m,t}$ for all $m \leq n$ and $t \in \underline{\underline{C}}$. We show that $k'_{n+1,t} = h'_{n+1,t}$. Let $M \in C\langle S\rangle'_{n+1,t}$. By definition there exists a $i \in I$ and a $g \in \Theta_{r_i-1,n}$ such that $M =$

$(N_1, \dots, N_{r_i-1}) \in \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)}$. We compute

$$\begin{aligned} h'_{n+1,t}(M) &= h'_{n+1,t}(N_1, \dots, N_{r_i-1}) \\ &= \xi_i(h'_{g(1),a_i(1)}(N_1), \dots, h'_{g(r_i-1),a_i(r_i-1)}(N_{r_i-1})) \\ &= \xi_i(k'_{g(1),a_i(1)}(N_1), \dots, k'_{g(r_i-1),a_i(r_i-1)}(N_{r_i-1})) \\ &= k'_{n+1,t}(N_1, \dots, N_{r_i-1}) \\ &= k'_{n+1,t}(M) \end{aligned}$$

since

$$\begin{tikzcd} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)} \arrow[r] \arrow[d, "\prod k'"'] & C\langle S\rangle'_{n+1,t} \arrow[d, "k'_{n+1,t}"] \\ \prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{f}}(a_i(j))) \arrow[r, "\xi_i"'] & (D')^{-1}(\underline{\underline{f}}(t)) \end{tikzcd}$$

commutes by definition of k. □

Proposition 8.3.158 *Using the notations of the proof of proposition 8.3.157 the assignation*

$$(C, S) \mapsto (C\langle S\rangle, \imath_* S, \rho)$$

induces a functor $\mathrm{Sign}(I) \to \mathrm{EndRep}(I)$ *.*

Proof. The construction on objects is given by the proof of proposition 8.3.157.

Construction $f\langle S\rangle : C\langle S\rangle \to D\langle f_* S\rangle$

Let $f : C \to D$ in TEns. We construct the arrow $f\langle S\rangle : C\langle S\rangle \to D\langle f_* S\rangle$ induced by f.

We write C' for the object obtained by adding $\perp$ to $\underline{C}$ and adding $\underline{C}$ as the fiber of $\perp$. By a similar construction we obtain D' and also $f' : C' \to D'$.

First we build the arrow $f\langle S\rangle' : C\langle S\rangle' \to D\langle f_* S\rangle'$. By definition $C\langle S\rangle' = \sum_{n\in\mathbb{N}} C\langle S\rangle'_n$ and $D\langle f_* S\rangle' = \sum_{n\in\mathbb{N}} D\langle f_* S\rangle'_n$. So we define recursively $f\langle S\rangle'_n$ for all $n \in \mathbb{N}$.

For $n = 0$ the arrow $f\langle S\rangle'_0 : C\langle S\rangle'_0 \to D\langle f_* S\rangle'_0$ is given by $f' : C' \to D'$ since $C\langle S\rangle'_0 = C'$ and $D\langle f_* S\rangle'_0 = D'$ by definition.

Suppose given an arrow $f\langle S\rangle'_{m,t} : C\langle S\rangle'_{m,t} \to D\langle f_* S\rangle'_{m,\underline{f'}(t)}$ for all $m \leq n$ and $t \in \underline{\underline{C}} = \underline{C'}$. By definition

$$C\langle S\rangle'_{n+1,t} = \sum_{i\in I_t} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)}$$

So by applying the induction hypotheses, we obtain an arrow

$$\begin{tikzcd} \sum_{i\in I_t} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)} \arrow[d] \\ \sum_{i\in I_{\underline{f'}(t)}} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),\underline{f'}(a_i(j))} = D\langle f_* S\rangle'_{n+1,\underline{f'}(t)} \end{tikzcd}$$

We set $f\langle S\rangle'_t = \sum\limits_{n\in\mathbb{N}} f\langle S\rangle'_{n,t}$. Then we undo $'$ of $f\langle S\rangle'$ to obtain $f\langle S\rangle$.

We write ξ for the 0–endorepresentation of f_*S on $D\langle f_*S\rangle$, defined as in theorem 8.3.157.

$f\langle S\rangle : C\langle S\rangle \to D\langle f_*S\rangle$ as a morphism of I–endorepresentations

We check that $f\langle S\rangle$ is a morphism of I–endorepresentations $(C\langle S\rangle, \imath_*S, \rho) \to (D\langle f_*S\rangle, f_*S, \xi)$, that is, that the following square

$$\begin{tikzcd}
\prod\limits_{j=1}^{r_i-1} (C\langle S\rangle')^{-1}(\underline{\underline{\imath}}(a_i(j))) \arrow[r, "\rho_i"] \arrow[d, "\prod f\langle S\rangle'"'] & (C\langle S\rangle')^{-1}(\underline{\underline{\imath}}(a_i(r_i))) \arrow[d, "f\langle S\rangle'"] \\
\prod\limits_{j=1}^{r_i-1} (D\langle f_*S\rangle')^{-1}(\underline{\underline{f}}(a_i(j))) \arrow[r, "\xi_i"'] & (D\langle f_*S\rangle')^{-1}(\underline{\underline{f}}(a_i(r_i)))
\end{tikzcd}$$

commutes for all $i \in I$ or equivalently that the following square

$$\begin{tikzcd}
\prod\limits_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)} \arrow[r] \arrow[d, "\prod f\langle S\rangle'_{g(j)}"'] & (C\langle S\rangle')^{-1}(a_i(r_i)) \arrow[d, "f\langle S\rangle'"] \\
\prod\limits_{j=1}^{r_i-1} D\langle f_*S\rangle'_{g(j),\underline{\underline{f}}(a_i(j))} \arrow[r] & (D\langle f_*S\rangle')^{-1}(\underline{\underline{f}}(a_i(r_i)))
\end{tikzcd}$$

commutes for all $i \in I$ and $g : r_i - 1 \to \mathbb{N}$. If we unfold the horizontal arrows (which are vertical in the following diagram), we obtain the following diagram

$$\begin{tikzcd}
\prod\limits_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)} \arrow[r, "\prod f\langle S\rangle'_{g(j)}"] \arrow[d] & \prod\limits_{j=1}^{r_i-1} D\langle f_*S\rangle'_{g(j),\underline{\underline{f}}(a_i(j))} \arrow[d] \\
\sum\limits_{i\in I_{a_i(r_i)}} \sum\limits_{g\in\Theta_{r_i-1,n}} \prod\limits_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)} \arrow[r, "\sum\sum\prod f\langle S\rangle'_{g(j)}"] \arrow[d, equal] & \sum\limits_{i\in I_{\underline{\underline{f}}(a_i(r_i))}} \sum\limits_{g\in\Theta_{r_i-1,n}} \prod\limits_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),\underline{\underline{f}}(a_i(j))} \arrow[d, equal] \\
C\langle S\rangle'_{m,a_i(r_i)} \arrow[r, "f\langle S\rangle'_m"] \arrow[d] & D\langle f_*S\rangle'_{m,\underline{\underline{f}}(a_i(r_i))} \arrow[d] \\
(C\langle S\rangle')^{-1}(a_i(r_i)) \arrow[r, "f\langle S\rangle'"'] & (D\langle f_*S\rangle')^{-1}(\underline{\underline{f}}(a_i(r_i)))
\end{tikzcd}$$

where we write $m = \max\limits_j g(j) + 1$. The first and third arrows on the right– and left–hand side are inclusions, so all three squares commute by definition of $f\langle S\rangle'$. $\square$

Corollary 8.3.159 *The forgetful functor $U : \mathrm{EndRep}(I) \to \mathrm{Sign}(I)$ has a left adjoint, which is given by the functor $F : \mathrm{Sign}(I) \to \mathrm{EndRep}(I)$ of proposition 8.3.158.*

Proof. We define unit and counit and show then the triangle identities. The the unit $\eta : \mathrm{Id} \to UF$ at the component $(C,S) \in \mathrm{Sign}(I)$ is $(C,S) \to (C\langle S\rangle, \imath_* S)$, which is a morphism of I–signatures induced by $\imath_C : C \hookrightarrow C\langle S\rangle$ of TEns.

The counit ε is given by initiality of $(C\langle S\rangle, \imath_* S, \rho)$ in the category of representations of S, its component $(C\langle S\rangle, \imath_* S, \rho) \to (C, S, \rho')$ at $(C, S, \rho') \in \mathrm{EndRep}(I)$ is given by the unique morphism of representations by 8.3.157.

We check the triangle identities. First we check the commutativity of

$$\begin{tikzcd} U(C,S,\rho') \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & UFU(C,S,\rho') \arrow[d, "U\varepsilon"] \\ & U(C,S,\rho') \end{tikzcd}$$

it becomes

$$\begin{tikzcd} (C,S) \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & (C\langle S\rangle, \imath_* S) \arrow[d, "U\varepsilon"] \\ & (C,S) \end{tikzcd}$$

which commutes since η is induced by the inclusion $\imath : C \hookrightarrow C\langle S\rangle$. Next we check the commutativity of

$$\begin{tikzcd} F(C,S) \arrow[r, "F\eta"] \arrow[dr, "\mathrm{Id}"'] & FUF(C,S) \arrow[d, "\varepsilon F"] \\ & F(C,S) \end{tikzcd}$$

it becomes

$$\begin{tikzcd} (C\langle S\rangle, \imath_* S, \rho) \arrow[r, "\eta_{(C,S)}\langle S\rangle"] \arrow[dr, "\mathrm{Id}"'] & ((C\langle S\rangle)\langle \imath_* S\rangle, \jmath_*(\imath_* S), \zeta) \arrow[d, "\varepsilon_{(C\langle S\rangle, \imath_* S, \rho)}"] \\ & (C\langle S\rangle, \imath_* S, \rho) \end{tikzcd} \tag{8.1}$$

where we write $\jmath$ for the morphism induced by the inclusion $C\langle S\rangle \hookrightarrow (C\langle S\rangle)\langle \imath_* S\rangle$. The vertical arrow is by definition the unique morphism $((C\langle S\rangle)\langle \imath_* S\rangle, \jmath_*(\imath_* S), \zeta) \to (C\langle S\rangle, \imath_* S, \rho)$ in $\mathrm{Rep}(\imath_* S)$.

But an object of $\mathrm{Rep}(\imath_* S)$ is as well an object of $\mathrm{Rep}(S)$ and a morphism in $\mathrm{Rep}(\imath_* S)$ is as well a morphism of $\mathrm{Rep}(S)$ for the following reason. Let $f : C\langle S\rangle \to D$ and ζ an endorepresentation of $f_*(\imath_* S)$. If we write f_0 for the composite

$$C \xrightarrow{\imath} C\langle S\rangle \xrightarrow{f} D$$

then $(f_0, (f_0)_* S, \zeta)$ is an object of $\mathrm{Rep}(S)$. Let φ be a morphism $(D, f_*(\imath_* S), \zeta) \to (D', f'_*(\imath_* S), \zeta')$ of $\mathrm{Rep}(\imath_* S)$. It is a morphism $(D, (f_0)_* S, \zeta) \to (D', (f'_0)_* S, \zeta')$ of $\mathrm{Rep}(S)$ as well since the following diagram commutes

$$\begin{tikzcd} & C \arrow[d, "\imath"] \arrow[ddl, bend right, "f_0"'] \arrow[ddr, bend left, "f'_0"] & \\ & C\langle S\rangle \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ D \arrow[rr, "\varphi"'] & & D' \end{tikzcd}$$

So the triangle (8.1) commutes by initiality of $(C\langle S\rangle, \imath_* S, \rho)$ in $\mathrm{Rep}(S)$. $\square$

8.4 Signature of higher degree

In this section we define the notion of arity and signature of higher degree, that is signature containing arities with quantification over types.

Definition 8.4.160 (arity) *Let $C \in \mathrm{TEns}$ and $d \in \mathbb{N}$. An* arity of degree d *on C is an arity (of degree 0) on $C\langle d\rangle$, where d stands for the signature on C consisting of d equal arities on C $1 \to \underline{\underline{C}}$, $1 \mapsto \bot$.*

Remark 8.4.161 *Let $C \in \mathrm{TEns}$ and $d \in \mathbb{N}$. The object $C\langle d\rangle$ of* TEns *is just an arrow $\overline{C} \to (\underline{C} + \{1, \ldots, d\})$ where the fibres of the d additional types are empty.*

Notation 8.4.162 (weighted set) *We call a pair $(I, \mathbf{d})$ a* weighted set *where $I \in \mathrm{Set}$ and $\mathbf{d} : I \to \mathbb{N}$. By abuse of notation, we write I as well for the underlying set and d_i for $\mathbf{d}(i)$ for an $i \in I$.*

For the remaining part of this section we fix a weighted set I.

Definition 8.4.163 (I–signature) *Let $C \in \mathrm{TEns}$. An* I–signature *on C is a collection $(S_i)_{i\in I}$ of arities of degrees d_i on C.*

Definition 8.4.164 (direct image) *Let $C, D \in \mathrm{TEns}$, $f : C \to D$ and $S = (S_i)_{i\in I}$ an I–signature. The* direct image $f_* S$ *of S along f is the I–signature on D consisting of the following arities on D*

$$r_i \to \underline{\underline{C\langle d_i\rangle}} \to \underline{\underline{D\langle d_i\rangle}}$$

Here we have written $\underline{\underline{f\langle d_i\rangle}}$ for the induced arrow $\underline{\underline{C\langle d_i\rangle}} \to \underline{\underline{D\langle d_i\rangle}}$ by f as explained in 8.3.158.

Definition 8.4.165 (category of I–signatures) *Let $C, D \in \mathrm{TEns}$, S an I–signature on C and T an I–signature on D. A* morphism of I–signatures *from (C, S) to (D, T) is an arrow $f : C \to D$ such that $T = f_* S$. I–Signatures and morphisms of I–signatures form a category,* $\mathrm{Sign}(I)$*. It comes equipped with a forgetful functor* $\mathrm{Sign}(I) \to \mathrm{TEns}$.

Definition 8.4.166 (category of I–endorepresentations) *An object of the category of I–endorepresentations is an I–signature $S = (S_i)_{i\in I}$ on C and for all $i \in I$ and for all $t : d_i \to \underline{C}$ a map $\rho_{i,t}$ of type*

$$\rho_{i,t} : \prod_{j=1}^{r_i - 1} C[S_{i,t}(j)] \to C[S_{i,t}(r_i)]$$

Here we have written $S_{i,t} : r_i \to \underline{\underline{C}}$ for the composite

$$r_i \xrightarrow{S_i} \underline{\underline{C\langle d_i\rangle}} = \underline{\underline{C}} + d_i \xrightarrow{[\mathrm{id},t]} \underline{\underline{C}}$$

A morphism from (C,S,ρ) to (D,T,ξ) is a morphism of I–signatures $f:(C,S)\to(D,T)$ such that

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} C[S_{i,t}(j)] \arrow[r, "\rho_{i,t}"] \arrow[d, "\prod f[S_{i,t}(j)]"'] & C[S_{i,t}(r_i)] \arrow[d, "f[S_{i,t}(r_i)]"] \\
\prod_{j=1}^{r_i-1} D[T_{i,\underline{f}\circ t}(j)] \arrow[r, "\xi_{i,\underline{f}\circ t}"'] & D[T_{i,\underline{f}\circ t}(r_i)]
\end{tikzcd}$$

commutes for all $t:d_i\to\underline{C}$ and all $i\in I$, since

$$\begin{tikzcd}
r_i \arrow[r, "S_i"] \arrow[dr, "T_i"'] & \underline{\underline{C}}+d_i \arrow[r, "{[\mathrm{id},t]}"] \arrow[d, "\underline{\underline{f}}+d_i"] & \underline{\underline{C}} \arrow[d, "\underline{\underline{f}}"] \\
& \underline{\underline{D}}+d_i \arrow[r, "{[\mathrm{id},\underline{f}\circ t]}"'] & \underline{\underline{D}}
\end{tikzcd}$$

commutes.

I–Endorepresentations and morphisms of I–endorepresentations form a category written $\mathrm{EndRep}(I)$*. It comes equipped with a forgetful functor* $\mathrm{EndRep}(I)\to\mathrm{Sign}(I)$*.*

Definition 8.4.167 (representation) *Let $C\in$* TEns *and S an I–signature on C. A* representation *of S in $D\in$* TEns *consists of an arrow $f:C\to D$ of* TEns *and an I–endorepresentation of f_*S.*

Definition 8.4.168 (category of representations) *Let $C,D,D'\in$* TEns*, S an I–signature on C. An object of the category of representations of S is a representation (D,f,ξ).*

A morphism of representations *from (D,f,ξ) to (D',f',ξ') consists of a morphism $\varphi:D\to D'$ such that*

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
D \arrow[rr, "\varphi"'] & & D'
\end{tikzcd}$$

commutes and which induces a morphism of I–endorepresentations from (D,f_*S,ξ) to (D',f'_*S,ξ').

Representations of S and morphisms of representations form a category $\mathrm{Rep}(S)$*.*

Proposition 8.4.169 *Let $C\in$* TEns *and $S=(a_i)_{i\in I}$ an I–signature on C. The category* $\mathrm{Rep}(S)$ *has an initial object, its underlying object of* TEns *is written $C\langle S\rangle$.*

Proof. We associate to each arity $a_i:r_i\to\underline{\underline{C\langle d_i\rangle}}=\underline{\underline{C}}+d_i$ a collection of $\underline{\underline{C}}$–arities $\alpha_{i,t}$ indexed by $t:d_i\to\underline{C}$. First we set

$$a_{i,t}:r_i\xrightarrow{a_i}\underline{\underline{C}}+d_i\xrightarrow{[\mathrm{id},t]}\underline{\underline{C}}$$

and then we define

$$\alpha_{i,t}:a_{i,t}(1),\ldots,a_{i,t}(r_i-1)\to a_{i,t}(r_i)$$

Construction of the initial object $C\langle S\rangle$

We construct the initial object $C\langle S\rangle'$ of the category of representations of the $\underline{\underline{C}}$–signature $(\alpha_{i,t})_{t,i}$ as in the proof of theorem 6.4.121 with $\mathcal{T}$ being $\underline{\underline{C}}$ and the starting object C'. This object C' of Set $/\underline{\underline{C}}$ is obtained from C by adding $\perp$ to $\underline{C}$ and adding $\underline{C}$ as the fibre of $\perp$.

Then we construct $C\langle S\rangle$ by adding the fibre of $\perp$ to $\underline{C\langle S\rangle'}$ and erasing $\perp$

$$\underline{C\langle S\rangle} = \underline{C\langle S\rangle'} \cup C\langle S\rangle'^{-1}(\perp) \setminus \{\perp\}$$

and

$$C\langle S\rangle^{-1}(c) = C\langle S\rangle'^{-1}(c)$$

for all $c \in \underline{C}$.

The arrow $\imath : C \to C\langle S\rangle$ is given by the inclusion $C \hookrightarrow C\langle S\rangle$. We construct the I–endorepresentation of S in $C\langle S\rangle$. By construction of $C\langle S\rangle'$ we have for each arity $\alpha_{i,t}$ an arrow

$$\prod_{j=1}^{r_i-1} C\langle S\rangle'^{-1}(a_{i,t}(j)) \to C\langle S\rangle'^{-1}(a_{i,t}(r_i))$$

by our construction of $C\langle S\rangle$ this amounts to an arrow

$$\rho_{i,t} : \prod_{j=1}^{r_i-1} C\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(j))] \to C\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(r_i))]$$

as desired for an I–endorepresentation of $\imath_* S$.

Construction of the arrow $h : C\langle S\rangle \to D$

Let (D, f, ξ) be another representation. Explicitly $\xi_{i,t}$ is of the form

$$\prod_{j=1}^{r_i-1} D[\underline{\underline{f}} \circ a_{i,t}(j)] \to D[\underline{\underline{f}} \circ a_{i,t}(r_i)]$$

We associate the object D' to D such that $\xi_{i,t}$ is now of type

$$\prod_{j=1}^{r_i-1} D'^{-1}(\underline{\underline{f}} \circ a_{i,t}(j)) \to D'^{-1}(\underline{\underline{f}} \circ a_{i,t}(r_i))$$

for all $t : d_i \to \underline{C}$ and all $i \in I$.

We construct a morphism of I–endorepresentations h from $(C\langle S\rangle, \imath, \rho)$ to (D, f, ξ). First we construct an arrow $h' : C\langle S\rangle' \to D'$. We define it recursively by giving an arrow $h'_n : C\langle S\rangle'_n \to D'$ for all n.

For $n = 0$ we have $C\langle S\rangle'_0 = C'$, so the arrow h'_0 is given by $f' : C' \to D'$. Suppose given an arrow $h'_{m,\tau} : C\langle S\rangle'_{m,\tau} \to D'^{-1}(\underline{\underline{f}}(\tau))$ for all $m \le n$ and $\tau \in \underline{\underline{C}}$, then we construct the arrow $h'_{n+1,\tau}$ for a $\tau \in \underline{\underline{C}}$.

By definition $C\langle S\rangle'_{n+1,\tau} = \sum_{i\in I_\tau} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)}$. So by universal property of the sums, it suffices to give an arrow $\prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)} \to D'^{-1}(\underline{\underline{f}}(\tau))$ for all $i \in I_\tau$ and $g \in \Theta_{r_i-1,n}$. We take the following composite

$$\prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)} \to \prod_{j=1}^{r_i-1} D'^{-1}(\underline{\underline{f}}(a_{i,t}(j))) \to D'^{-1}(\underline{\underline{f}}(\tau))$$

since $a_{i,t}(r_i) = \tau$ for all $(i,t) \in I_\tau$.

Then we set $h' = [h'_n]_{n\in\mathbb{N}} : C\langle S\rangle' \to D'$. By undoing the operation $'$, we obtain $h : C\langle S\rangle \to D$. By undoing the operation $'$ we mean the following. The object $C\langle S\rangle'$ is of the form

$$\begin{tikzcd}
(\overline{C} + A) + (\underline{C} + B) \arrow[d, "C\langle S\rangle'"] \\
\underline{C} + \{\perp\}
\end{tikzcd}$$

where $\underline{C} + B$ is in the fibre of $\perp$ and h' is explicitly

$$\begin{tikzcd}
(\overline{C} + A) + (\underline{C} + B) \arrow[r, "\overline{h'}"] \arrow[d, "C\langle S\rangle'"'] & \overline{D} + \underline{D} \arrow[d, "D'"] \\
\underline{C} + \{\perp\} \arrow[r, "\underline{h'}"'] & \underline{D} + \{\perp\}
\end{tikzcd}$$

since we have $\underline{C} + B \to \underline{D}$, the following square commutes

$$\begin{tikzcd}
\overline{C} + A \arrow[r] \arrow[d, "C\langle S\rangle"'] & \overline{D} \arrow[d, "D"] \\
\underline{C} + B \arrow[r] & \underline{D}
\end{tikzcd}$$

which we take to be h. The fibre of B along $C\langle S\rangle$ is empty.

$h : C\langle S\rangle \to D$ as a morphism of representations

The triangle

$$\begin{tikzcd}
& C \arrow[dl, "\imath"'] \arrow[dr, "f"] & \\
C\langle S\rangle \arrow[rr, "h"'] & & D
\end{tikzcd}$$

commutes by construction of h and $\imath$ being the inclusion. The following diagram

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} C\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(j))] \arrow[r, "\rho_{i,t}"] \arrow[d, "\prod h[\underline{\underline{\imath}}(a_{i,t}(j))]"'] & C\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(r_i))] \arrow[d, "h[\underline{\underline{\imath}}(a_{i,t}(r_i))]"] \\
\prod_{j=1}^{r_i-1} D[\underline{\underline{f}}(a_{i,t}(j))] \arrow[r, "\xi_{i,t}"'] & D[\underline{\underline{f}}(a_{i,t}(r_i))]
\end{tikzcd}$$

commutes since

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} C\langle S\rangle'^{-1}(a_{i,t}(j)) \arrow[r, "\rho_{i,t}"] \arrow[d, "\prod h'_{a_{i,t}(j)}"'] & C\langle S\rangle'^{-1}(a_{i,t}(r_i)) \arrow[d, "h'_{a_{i,t}(r_i)}"] \\
\prod_{j=1}^{r_i-1} D'^{-1}(\underline{\underline{f}}(a_{i,t}(j))) \arrow[r, "\xi_{i,t}"'] & D'^{-1}(\underline{\underline{f}}(a_{i,t}(r_i)))
\end{tikzcd}$$

commutes by definition of h'.

Uniqueness of $h : C\langle S\rangle \to D$

Suppose given another morphism of I–endorepresentations k from $(C\langle S\rangle, \imath, \rho)$ to (D, f, ξ). We show that $k = h$ and we can conclude then that $(C\langle S\rangle, \imath, \rho)$ is initial in $\mathrm{Rep}(S)$. We write k'_n for the composite

$$C\langle S\rangle'_n \hookrightarrow C\langle S\rangle' \xrightarrow{k'} D'$$

and $k'_{n,t}$ for its fibre in t. We show by induction that $k'_{n,t} = h'_{n,t}$ for all $n \in \mathbb{N}$ and $t \in \underline{\underline{C}}$.

If $n = 0$, we have

$$k'_{0,t}(x) = g'_t(x) = h'_{0,t}(x)$$

since both h' and k' make the triangle

$$\begin{tikzcd} & C' \arrow[dl, "\imath'"'] \arrow[dr, "f'"] & \\ C\langle S\rangle' \arrow[rr] & & D' \end{tikzcd}$$

commute. Suppose that $k'_{m,t} = h'_{m,t}$ for all $m \leq n$ and $t \in \underline{\underline{C}}$. We show that $k'_{n+1,\tau} = h'_{n+1,\tau}$. Let $M \in C\langle S\rangle'_{n+1,\tau}$. By definition there exists a $(i,t) \in I$ and a $g \in \Theta_{r_i-1,n}$ such that $M = (N_1, \ldots, N_{r_i-1}) \in \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_i(j)}$. We compute

$$\begin{aligned} h'_{n+1,\tau}(M) &= h'_{n+1,\tau}(N_1, \ldots, N_{r_i-1}) \\ &= \xi_{i,t}(h'_{g(1),a_{i,t}(1)}(N_1), \ldots, h'_{g(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\ &= \xi_{i,t}(k'_{g(1),a_{i,t}(1)}(N_1), \ldots, k'_{g(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\ &= k'_{n+1,t}(N_1, \ldots, N_{r_i-1}) \\ &= k'_{n+1,t}(M) \end{aligned}$$

since

$$\begin{tikzcd} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod k'"'] & C\langle S\rangle'_{n+1,t} \arrow[d, "k'_{n+1,t}"] \\ \prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{f}}(a_{i,t}(j))) \arrow[r, "\xi_{i,t}"'] & (D')^{-1}(\underline{\underline{f}}(t)) \end{tikzcd}$$

commutes by definition of k. □

Proposition 8.4.170 *With the notations of the proof of proposition 8.4.169, the assignation*

$$(C, S) \mapsto (C\langle S\rangle, \imath_* S, \rho)$$

induces a functor $\mathrm{Sign}(I) \to \mathrm{EndRep}(I)$*.*

Proof. The construction on objects is given by the proof of proposition 8.4.169. Let $f : C \to D$ in TEns.

Construction of $f\langle S\rangle : C\langle S\rangle \to D\langle f_* S\rangle$

We construct the arrow $f\langle S\rangle : C\langle S\rangle \to D\langle f_* S\rangle$ induced by f. We write C' for the object obtained by adding $\perp$ to $\underline{C}$ and adding $\underline{C}$ as the fiber of $\perp$. By a similar construction we obtain D' and also $f' : C' \to D'$.

First we build the arrow $f\langle S\rangle' : C\langle S\rangle' \to D\langle f_* S\rangle'$. By definition $C\langle S\rangle' = \sum_{n\in\mathbb{N}} C\langle S\rangle'_n$ and $D\langle f_* S\rangle' = \sum_{n\in\mathbb{N}} D\langle f_* S\rangle'_n$. So we define recursively $f\langle S\rangle'_n$ for all $n \in \mathbb{N}$.

For $n = 0$ the arrow $f\langle S\rangle'_0 : C\langle S\rangle'_0 \to D\langle f_* S\rangle'_0$ is given by $f' : C' \to D'$ since $C\langle S\rangle'_0 = C'$ and $D\langle f_* S\rangle'_0 = D'$ by definition.

Suppose given an arrow $f\langle S\rangle'_{m,\tau} : C\langle S\rangle'_{m,t} \to D\langle f_* S\rangle'_{m,\underline{f'}(\tau)}$ for all $m \leq n$ and $\tau \in \underline{\underline{C}} = \underline{C'}$. By definition

$$C\langle S\rangle'_{n+1,\tau} = \sum_{(i,t)\in I_\tau} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)}$$

So by applying the induction hypotheses, we obtain an arrow

$$\begin{array}{c} \displaystyle\sum_{(i,t)\in I_\tau} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)} \\ \big\downarrow \\ \displaystyle\sum_{(i,t)\in I_{\underline{f'}(\tau)}} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),\underline{f'}(a_{i,t}(j))} = D\langle f_* S\rangle'_{n+1,\underline{f'}(\tau)} \end{array}$$

We set $f\langle S\rangle'_\tau = \sum_{n\in\mathbb{N}} f\langle S\rangle'_{n,\tau}$. Then we undo $'$ of $f\langle S\rangle'$ to obtain $f\langle S\rangle$.

We write ξ for the I–endorepresentation of $f_* S$ on $D\langle f_* S\rangle$, defined as in theorem 8.4.169.

$f\langle S\rangle : C\langle S\rangle \to D\langle f_* S\rangle$ as a morphism of I–endorepresentations

We check that $f\langle S\rangle$ is a morphism of I–endorepresentations $(C\langle S\rangle, \imath_* S, \rho) \to (D\langle f_* S\rangle, f_* S, \xi)$, that is, that the following square

$$\begin{tikzcd} \prod_{j=1}^{r_i-1} (C\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\rho_{i,t}"] \arrow[d, "\prod f\langle S\rangle'"'] & (C\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "f\langle S\rangle'"] \\ \prod_{j=1}^{r_i-1} (D\langle f_* S\rangle')^{-1}(\underline{\underline{f}}(a_{i,t}(j))) \arrow[r, "\xi_{i,t}"'] & (D\langle f_* S\rangle')^{-1}(\underline{\underline{f}}(a_{i,t}(r_i))) \end{tikzcd}$$

commutes for all $i \in I$ and $t : d_i \to \underline{C}$ or equivalently that the following square

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod f\langle S\rangle'_{g(j)}"'] & (C\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "f\langle S\rangle'"] \\
\prod_{j=1}^{r_i-1} D\langle f_*S\rangle'_{g(j),\underline{\underline{f}}(a_{i,t}(j))} \arrow[r] & (D\langle f_*S\rangle')^{-1}(\underline{\underline{f}}(a_{i,t}(r_i)))
\end{tikzcd}$$

commutes for all $i \in I$, $t : d_i \to \underline{C}$ and $g : r_i - 1 \to \mathbb{N}$. If we unfold the horizontal arrows (which are vertical in the following diagram) we obtain the following diagram

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)} \arrow[r, "\prod f\langle S\rangle'_{g(j)}"] \arrow[d] & \prod_{j=1}^{r_i-1} D\langle f_*S\rangle'_{g(j),\underline{\underline{f}}(a_{i,t}(j))} \arrow[d] \\
\sum_{(i,t)\in I_{a_{i,t}(r_i)}} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),a_{i,t}(j)} \arrow[r, "\sum\sum\prod f\langle S\rangle'_{g(j)}"] \arrow[d, equal] & \sum_{(i,t)\in I_{\underline{\underline{f}}(a_{i,t}(r_i))}} \sum_{g\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S\rangle'_{g(j),\underline{\underline{f}}(a_{i,t}(j))} \arrow[d, equal] \\
C\langle S\rangle'_{m,a_{i,t}(r_i)} \arrow[r, "f\langle S\rangle'_m"] \arrow[d] & D\langle f_*S\rangle'_{m,\underline{\underline{f}}(a_{i,t}(r_i))} \arrow[d] \\
(C\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[r, "f\langle S\rangle'"'] & (D\langle f_*S\rangle')^{-1}(\underline{\underline{f}}(a_{i,t}(r_i)))
\end{tikzcd}$$

where we write $m = \max_j g(j) + 1$. The first and third arrows on the right– and left–hand side are inclusions, so all three squares commute by definition of $f\langle S\rangle'$. $\square$

Corollary 8.4.171 *The forgetful functor $U : \mathrm{EndRep}(I) \to \mathrm{Sign}(I)$ has a left adjoint, which is given by the functor $F : \mathrm{Sign}(I) \to \mathrm{EndRep}(I)$ of proposition 8.4.170.*

Proof. We define unit and counit and show then the triangle identities. The unit $\eta : \mathrm{Id} \to UF$ is the morphism of I–signatures induced by the inclusion $\imath_C : C \hookrightarrow C\langle S\rangle$

$$(C, S) \to (C\langle S\rangle, \imath_* S)$$

at the component $(C, S) \in \mathrm{Sign}(I)$. The counit ε is given by initiality of $(C\langle S\rangle, \imath_* S, \rho)$ in the category of representations of S, the component at $(C, S, \rho') \in \mathrm{EndRep}(I)$ is the unique morphism

$$(C\langle S\rangle, \imath_* S, \rho) \to (C, S, \rho')$$

We check the triangle identities. First we check the commutativity of

$$\begin{tikzcd}
U(C,S,\rho') \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & UFU(C,S,\rho') \arrow[d, "U\varepsilon"] \\
& U(C,S,\rho')
\end{tikzcd}$$

it becomes

$$\begin{tikzcd}
(C,S) \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & (C\langle S\rangle, \imath_* S) \arrow[d, "U\varepsilon"] \\
& (C,S)
\end{tikzcd}$$

which commutes since η is the inclusion $C \hookrightarrow C\langle S\rangle$. Next we check the commutativity of

$$\begin{tikzcd}
F(C,S) \arrow[r, "F\eta"] \arrow[dr, "\mathrm{Id}"'] & FUF(C,S) \arrow[d, "\varepsilon F"] \\
& F(C,S)
\end{tikzcd}$$

it becomes

$$\begin{tikzcd}
(C\langle S\rangle, \imath_* S, \rho) \arrow[r, "\eta_{(C,S)}\langle S\rangle"] \arrow[dr, "\mathrm{Id}"'] & ((C\langle S\rangle)\langle \imath_* S\rangle, \jmath_* S, \zeta) \arrow[d, "\varepsilon_{(C\langle S\rangle, \imath_* S, \rho)}"] \\
& (C\langle S\rangle, \imath_* S, \rho)
\end{tikzcd} \tag{8.2}$$

where we write $\jmath$ for the morphism induced by the inclusion $C\langle S\rangle \hookrightarrow (C\langle S\rangle)\langle \imath_* S\rangle$. The vertical arrow is by definition the unique morphism $((C\langle S\rangle)\langle \imath_* S\rangle, \jmath_*(\imath_* S), \zeta) \to (C\langle S\rangle, \imath_* S, \rho)$ in $\mathrm{Rep}(\imath_* S)$.

But an object of $\mathrm{Rep}(\imath_* S)$ is as well an object of $\mathrm{Rep}(S)$ and a morphism in $\mathrm{Rep}(\imath_* S)$ is as well a morphism of $\mathrm{Rep}(S)$ for the following reason. Let $f : C\langle S\rangle \to D$ and ζ an endorepresentation of $f_*(\imath_* S)$. If we write f_0 for the composite

$$C \xrightarrow{\imath} C\langle S\rangle \xrightarrow{f} D$$

then $(f_0, (f_0)_* S, \zeta)$ is an object of $\mathrm{Rep}(S)$. Let φ be a morphism $(D, f_*(\imath_* S), \zeta) \to (D', f'_*(\imath_* S), \zeta')$ of $\mathrm{Rep}(\imath_* S)$. It is a morphism $(D, (f_0)_* S, \zeta) \to (D', (f'_0)_* S, \zeta')$ of $\mathrm{Rep}(S)$ as well since the following diagram commutes

$$\begin{tikzcd}
& C \arrow[d, "\imath"] \arrow[ddl, bend right, "f_0"'] \arrow[ddr, bend left, "f'_0"] & \\
& C\langle S\rangle \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
D \arrow[rr, "\varphi"'] & & D'
\end{tikzcd}$$

So the triangle (8.2) commutes by initiality of $(C\langle S\rangle, \imath_* S, \rho)$ in $\mathrm{Rep}(S)$. □

8.5 Mixed signature

In this section we define a notion of arity and signature on an object of a category of representations of another signature.

For this section suppose given a weighted set I'. We write short $\mathcal{E}$ for the category of endorepresentations $\mathrm{EndRep}(I')$ and $U : \mathcal{E} \to \mathrm{TEns}$.

Definition 8.5.172 (arity) *Let $\Gamma \in \mathcal{E}$. An* arity *on Γ of degree 0 and of length r is an arrow $a : r \to \underline{\underline{U(\Gamma)}}$ in* Set*.*

Such an arity a on Γ is an arity on $U(\Gamma)$ of degree 0 as defined previously. We write only $a : r \to \underline{\underline{\Gamma}}$ instead.

In the following we fix a weighted set I with all weights equal to 0, so I is just an ordinary set.

Definition 8.5.173 (I–signature) *Let $\Gamma \in \mathcal{E}$. An I–*signature *S of degree 0 on Γ is a collection $(S_i)_{i \in I}$ of arities on Γ of degrees 0.*

Definition 8.5.174 (direct image) *Let $\Gamma, \Delta \in \mathcal{E}$, $f : \Gamma \to \Delta$ and S an I–signature on Γ. The* direct image *of S along f is an I–signature on Δ whose arities are the following ones*

$$r_i \to \underline{\underline{\Gamma}} \to \underline{\underline{\Delta}}$$

We write $f_* S$ for this signature on Δ.

Definition 8.5.175 (category of I–signatures) *Let $\Gamma, \Delta \in \mathcal{E}$. A* morphism of I–signatures *from an I–signature S on Γ to an I–signature T on Δ is an arrow $f : \Gamma \to \Delta$ of $\mathcal{E}$ such that $T = f_* S$. I–Signatures and morphisms of I–signatures form a category,* $\mathrm{Sign}_{\mathcal{E}}(I)$*.*

Definition 8.5.176 (category of I–endorepresentations) *An object of the category of I–endorepresentations is an I–signature S on Γ together with a map*

$$\rho_i : \prod_{j=1}^{r_i - 1} \Gamma[S_i(j)] \to \Gamma[S_i(r_i)]$$

for all $i \in I$.

A morphism from (Γ, S, ρ) to (Δ, T, ξ) is a morphism of I–signatures $f : (\Gamma, S) \to (\Delta, T)$ such that

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma[S_i(j)] \arrow[r, "\rho_i"] \arrow[d, "\prod f[S_i(j)]"'] & \Gamma[S_i(r_i)] \arrow[d, "f[S_i(r_i)]"] \\
\prod_{j=1}^{r_i-1} \Delta[T_i(j)] \arrow[r, "\xi_i"'] & \Delta[T_i(r_i)]
\end{tikzcd}$$

commutes for all $i \in I$.

I–Endorepresentations and morphisms of I–endorepresentations form a category written $\mathrm{EndRep}_{\mathcal{E}}(I)$*. It comes equipped with a forgetful functor* $\mathrm{EndRep}_{\mathcal{E}}(I) \to \mathrm{Sign}_{\mathcal{E}}(I)$*.*

Definition 8.5.177 (representation) *Let $\Gamma \in \mathcal{E}$ and S an I–signature on Γ. A* representation *of S in an object Δ of $\mathcal{E}$ is a morphism $f : \Gamma \to \Delta$ of $\mathcal{E}$ together with an I–endorepresentation of $f_* S$ in Δ.*

Definition 8.5.178 (category of representations) *Let $\Gamma \in \mathcal{E}$ and S an I–signature on Γ. We build* $\mathrm{Rep}(S)$ *the* category of representations *of S. An object of* $\mathrm{Rep}(S)$ *is a triple (Δ, f, ξ) where Δ is an object of $\mathcal{E}$, $f : \Gamma \to \Delta$ a morphism of $\mathcal{E}$ and ξ an I–endorepresentation of $f_* S$ in Δ.*

A morphism of representations *from a representation* (Δ, f, ξ) *to a representation* (Δ', f', ξ') *is a morphism* $\varphi : \Delta \to \Delta'$ *of* $\mathcal{E}$ *such that*

$$\begin{tikzcd} & \Gamma \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \Delta \arrow[rr, "\varphi"'] & & \Delta' \end{tikzcd}$$

commutes and such that φ *is a morphism of* I*–endorepresentations* $(\Delta, f_* S, \xi) \to (\Delta', f'_* S, \xi')$. *We write* $\mathrm{Rep}(S)$ *for this category.*

Proposition 8.5.179 *Let* $\Gamma \in \mathcal{E}$ *and* S *an* I*–signature on* Γ. *The category of representations* $\mathrm{Rep}(S)$ *has an initial object. We write* $\Gamma\langle S\rangle$ *for the underlying object of* TEns *of the initial representation.*

Proof. We write $C = U(\Gamma)$, S' for the I'–signature on C of Γ and ρ' for its I'–endorepresentation.

Construction of the initial object $\Gamma\langle S\rangle$

We concatenate the signatures S' and S by setting $J = I' + I$ and $\mathbf{n} = [\mathbf{d}', 0] : J \to \mathbb{N}$ where 0 is the constant function $I \to \mathbb{N}$, $i \mapsto 0$ and we write $S' + S$ for this J–signature of degree $\mathbf{n}$. By proposition 8.4.169 the category of representations of $S' + S$ has an initial object, its underlying object of TEns is written $C\langle S' + S\rangle$.

In the construction of $C\langle S' + S\rangle$ we use the related object $C\langle S' + S\rangle'$ that has an additional type $\perp$ and $\overline{C\langle S' + S\rangle}$ as its fibre. The object $C\langle S' + S\rangle'$ in the fibre $\tau \in \underline{\underline{C}}$ is constructed as the coproduct $\sum_{n\in\mathbb{N}} C\langle S' + S\rangle'_{n,\tau}$.

We define a subset $\mathsf{TO}_{n,\tau}$ of each $C\langle S' + S\rangle'_{n,\tau}$. Intuitively the set of trees TO stands for trees having a subtree constructed by S'. We do not want these trees in $\Gamma\langle S\rangle$ since there is already an element of C' that corresponds to such a subtree.

For $n = 0$ we set $\mathsf{TO}_{0,\tau} = 0$. For $n = 1$, we have $C\langle S' + S\rangle'_{1,\tau} = \sum_{i\in J_\tau} \sum_{g\in\Theta_{0,r_i-1}} \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{g(j),a_i(j)}$ where $g \in \Theta_{0,r_i-1}$ is the constant function 0 and we chose the constructions of indexes $i \in I'_\tau$ to be in $\mathsf{TO}_{1,\tau}$. More precisely if $M \in C\langle S' + S\rangle'_{1,\tau}$, there exists $i \in J_\tau$, $t : n_i \to \underline{C}$ such that $M \in \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{0,a_{i,t}(j)}$ and we set $M \in \mathsf{TO}_{1,\tau}$ if $i \in I'_\tau$.

Suppose given $\mathsf{TO}_{m,\tau} \subseteq C\langle S' + S\rangle_{m,\tau}$ for all $m \le n$ and all $\tau \in \underline{\underline{C}}$, we construct $\mathsf{TO}_{n+1,\tau}$. Let $M \in C\langle S' + S\rangle_{n+1,\tau}$, by definition there exists $i \in J_\tau$, $t : n_i \to \underline{\underline{C}}$, $g \in \Theta_{r_i-1,n}$ such that $M \in \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{g(j),a_{i,t}(j)}$ and we write $M = (N_1, \ldots, N_{r_i-1})$. We take $M \in \mathsf{TO}_{n+1,\tau}$ if there exists a $j \in \{1, \ldots, r_i - 1\}$ such that $N_j \in \mathsf{TO}_{g(j),a_{i,t}(j)}$.

Then we take $\Gamma\langle S\rangle'_{n,\tau} = C\langle S' + S\rangle'_{n,\tau} \setminus \mathsf{TO}_{n,\tau}$ for all τ and $\Gamma\langle S\rangle'_\tau = \sum_{n\in\mathbb{N}} \Gamma\langle S\rangle'_{n,\tau}$. By undoing $'$ of $\Gamma\langle S\rangle'$ we obtain $\Gamma\langle S\rangle$.

Construction $p : C\langle S' + S\rangle' \to \Gamma\langle S\rangle'$

Intuitively this function p sends a tree of $C\langle S' + S\rangle'$ containing a subtree constructed by S' to the same tree where we replace the subtree by the corresponding element of C'.

First we construct a collection of arrows $s_{n,\tau} : \mathsf{TO}_{n,\tau} \to (\Gamma\langle S\rangle')^{-1}(\tau)$ induced by the d'–endorepresentation ρ'. We construct $s_{n,\tau}$ for all $n \in \mathbb{N}$ and $\tau \in \underline{\underline{C}}$ by recursion.

For $n = 0$ we have $\mathsf{TO}_{n,\tau} = 0$, so the arrow $s_{0,\tau}$ is given by initiality of $0 \in$ Set. For $n = 1$ let $M \in \mathsf{TO}_{1,\tau}$, by definition $M \in \sum_{(i,t)\in I'} \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{0,a_{i,t}(j)}$, so there exists a $(i,t) \in I'$ such that $M \in \prod_{j=1}^{r_i-1} C'^{-1}(a_{i,t}(j))$. By $\rho'_{i,t}$ we have

$$\prod_{j=1}^{r_i-1} C'^{-1}(a_{i,t}(j)) \to C'^{-1}(\tau) = \Gamma\langle S\rangle'_{0,\tau}$$

and then by inclusion

$$\Gamma\langle S\rangle'_{0,\tau} \to (\Gamma\langle S\rangle')^{-1}(\tau)$$

Suppose given $s_{q,\tau} : \mathsf{TO}_{q,\tau} \to (\Gamma\langle S\rangle')^{-1}(\tau)$ for all $q \leq n$ and $\tau \in \underline{\underline{C}}$ then we construct $s_{n+1,\tau}$. Let $M \in \mathsf{TO}_{n+1,\tau}$ then by definition there exists an $i \in J$ and $h \in \Theta_{r_i-1,n}$ such that $M = (N_1, \ldots, N_{r_i-1}) \in \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{h(j),a_i(j)}$ where $N_j \in \mathsf{TO}_{h(j),a_i(j)}$ or $N_j \in \Gamma\langle S\rangle'_{h(j),a_i(j)}$ for all $j = 1, \ldots, r_i - 1$. Applying the recursion hypotheses to $N_j \in \mathsf{TO}_{h(j),a_i(j)}$ we obtain

$$s_{h(j),a_i(j)}(N_j) \in (\Gamma\langle S\rangle')^{-1}(a_i(j))$$

Then we set

$$s_{n+1,\tau}(M) = (\widetilde{N}_1, \ldots, \widetilde{N}_{r_i-1}) \in (\Gamma\langle S\rangle')^{-1}(\tau)$$

where $\widetilde{N}_j = N_j$ if $N_j \in \Gamma\langle S\rangle'_{h(j),a_i(j)}$ and $\widetilde{N}_j = s_{h(j),a_i(j)}(N_j)$ if $N_j \in \mathsf{TO}_{h(j),a_i(j)}$.

So we take $s_\tau = [s_{n,\tau}]_{n\in\mathbb{N}}$. The arrow $p'_{n,\tau}$ is given by

$$C\langle S' + S\rangle'_{n,\tau} = \Gamma\langle S\rangle'_{n,\tau} + \mathsf{TO}_{n,\tau} \xrightarrow{[incl, s_{n,\tau}]} \Gamma\langle S\rangle'_\tau$$

I–Endorepresentation of S

We construct an I–endorepresentation $(\Gamma\langle S\rangle, \imath_* S, \xi)$ of $\imath_* S$ on $\Gamma\langle S\rangle$ along the inclusion $\imath : C \to \Gamma\langle S\rangle$. Let $i \in I$, we have to construct an arrow

$$\xi_i : \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_i(j))] \to \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_i(r_i))]$$

or equivalently

$$\xi_i : \prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_i(j)) \to (\Gamma\langle S\rangle')^{-1}(a_i(r_i))$$

By distributivity of Set we have

$$\prod_{j=1}^{r_i-1} \sum_{n\in\mathbb{N}} \Gamma\langle S\rangle'_{n,a_i(j)} \cong \sum_{h:\mathbb{N}^{r_i-1}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)}$$

So for all $h : r_i - 1 \to \mathbb{N}$, we construct an arrow

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \to (\Gamma\langle S\rangle')^{-1}(a_i(r_i))$$

Since $\Gamma\langle S\rangle' \subseteq C\langle S'+S\rangle'$, we have

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \to \prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_i(j)}$$

and by definition of $C\langle S'+S\rangle'_{m,a_i(r_i)}$

$$\prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_i(j)} \to C\langle S'+S\rangle'_{m,a_i(r_i)}$$

where $m = \max_j(h(j)) + 1$ then by p'

$$C\langle S'+S\rangle'_{m,a_i(r_i)} \to (\Gamma\langle S\rangle')^{-1}(a_i(r_i))$$

Morphism of I'–endorepresentations

Next we construct an I'–endorepresentation ξ' of $\imath_* S'$ along the inclusion $\imath : C \to \Gamma\langle S\rangle$. Let $i \in I'$ and $t : d_i \to \underline{C}$, we have to construct an arrow

$$\xi'_{i,t} : \prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

By distributivity of Set we have

$$\prod_{j=1}^{r_i-1} \sum_{n\in\mathbb{N}} \Gamma\langle S\rangle'_{n,a_{i,t}(j)} \cong \sum_{h:\mathbb{N}^{r_i-1}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)}$$

So for all $h : r_i - 1 \to \mathbb{N}$, we construct an arrow

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

Since $\Gamma\langle S\rangle' \subseteq C\langle S'+S\rangle'$, we have

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \to \prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_{i,t}(j)}$$

and by definition of $C\langle S'+S\rangle'_{m,a_{i,t}(r_i)}$

$$\prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_{i,t}(j)} \to C\langle S'+S\rangle'_{m,a_{i,t}(r_i)}$$

where $m = \max_j(h(j)) + 1$, then by p'

$$C\langle S'+S\rangle'_{m,a_{i,t}(r_i)} \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

Next we have to check that

$$\begin{tikzcd} \prod_{j=1}^{r_i} C[a_{i,t}(j)] \arrow[r] \arrow[d] & C[a_{i,t}(r_i)] \arrow[d] \\ \prod_{j=1}^{r_i} \Gamma\langle S\rangle[\underline{\imath}(a_{i,t}(j))] \arrow[r] & \Gamma\langle S\rangle[\underline{\imath}(a_{i,t}(r_i))] \end{tikzcd}$$

commutes for all $t : d_i \to \underline{C}$ and $i \in I'$ or equivalently that

$$\begin{tikzcd} \prod_{j=1}^{r_i} (C')^{-1}(a_{i,t}(j)) \arrow[r, "\rho'_{i,t}"] \arrow[d] & (C')^{-1}(a_{i,t}(r_i)) \arrow[d] \\ \prod_{j=1}^{r_i} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\xi'_{i,t}"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \end{tikzcd}$$

commutes. It does by construction of $\xi'_{i,t}$ since $\imath$ comes from the inclusion $\Gamma\langle S\rangle'_0 \to \Gamma\langle S\rangle'$.

Construction $\varphi : \Gamma\langle S\rangle \to D$

Let (Δ, g, ζ) be another object of $\mathrm{Rep}(S)$, that is, $\Delta = (D, g_*S', \zeta') \in \mathcal{E}$, $g : \Gamma \to \Delta$ a morphism of $\mathcal{E}$ and (D, g_*S, ζ) an I–endorepresentation of g_*S in Δ. We construct a morphism $\varphi : (\Gamma\langle S\rangle, \imath, \xi) \to (\Delta, g, \zeta)$ of $\mathrm{Rep}(S)$.

First we construct the underlying arrow $\varphi : \Gamma\langle S\rangle \to D$ of TEns or equivalently the arrow $\varphi' : \Gamma\langle S\rangle' \to D'$ by recursion. We construct an arrow $\varphi'_{n,\tau} : \Gamma\langle S\rangle'_{n,\tau} \to (D')^{-1}(\underline{\underline{g}}(\tau))$ for all $n \in \mathbb{N}$ and $\tau \in \underline{\underline{C}}$.

For $n = 0$ the arrow $\varphi'_{0,\tau} : (C')^{-1}(\tau) \to (D')^{-1}(\underline{\underline{g}}(\tau))$ is given by the fibre in τ of $g' : C' \to D'$. For $n = 1$ an element $M \in C\langle S' + S\rangle'_{1,\tau} = \sum_{(i,t)\in(I'+I)_\tau} \prod_{j=1}^{r_i-1} (C')^{-1}(a_{i,t}(j))$ is in $\Gamma\langle S\rangle'_{1,\tau}$ if the corresponding $(i,t) \in (I' + I)_\tau$ is actually $i \in I_\tau$. Then we take the following composite

$$\begin{tikzcd} \sum_{i\in(I)_\tau} \prod_{j=1}^{r_i-1} (C')^{-1}(a_i(j)) \arrow[d, "\sum\prod g'"] \\ \sum_{i\in(I)_\tau} \prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_i(j))) \arrow[d, "{[\zeta_i]_{i\in I_\tau}}"] \\ (D')^{-1}(\underline{\underline{g}}(\tau)) \end{tikzcd}$$

Suppose given $\varphi'_{q,\tau} : \Gamma\langle S\rangle'_{q,\tau} \to (D')^{-1}(\underline{\underline{g}}(\tau))$ for all $q \le n$ and all $\tau \in \underline{\underline{C}}$. We construct $\varphi'_{n+1,\tau}$. An element $M = (N_1, \ldots, N_{r_i-1})$ of $C\langle S'+S\rangle'_{n+1,\tau} = \sum_{(i,t)\in(I'+I)_\tau} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_{i,t}(j)}$

is in $\Gamma\langle S\rangle'_{n+1,\tau}$ if all $N_j \in \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)}$. Let $(i,t) \in I'_\tau$ and $h \in \Theta_{r_i-1,n}$ and we take the composite

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[d, "\prod \varphi'_{h(j)}"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[d, "\zeta'_{i,t}"] \\
(D')^{-1}(\underline{\underline{g}}(\tau))
\end{tikzcd}$$

Let $i \in I_\tau$ and $h \in \Theta_{r_i-1,n}$ and we take the composite

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \arrow[d, "\prod \varphi'_{h(j)}"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_i(j))) \arrow[d, "\zeta_{i,t}"] \\
(D')^{-1}(\underline{\underline{g}}(\tau))
\end{tikzcd}$$

Then we set $\varphi'_\tau = [\varphi'_{n,\tau}]_{n\in\mathbb{N}}$. By undoing $'$ of φ'_τ we obtain $\varphi_\tau : \Gamma\langle S\rangle^{-1}(\tau) \to D^{-1}(\underline{\underline{g}}(\tau))$. Note that by definition of φ' the following diagram commutes

$$\begin{tikzcd}
 & C' \arrow[dl, "\imath'"'] \arrow[dr, "g'"] & \\
\Gamma\langle S\rangle' \arrow[rr, "\varphi'"'] & & D'
\end{tikzcd}$$

$\varphi : \Gamma\langle S\rangle \to D$ as a morphism of I–endorepresentations

We have to check that φ is a morphism of I–endorepresentations $(\Gamma\langle S\rangle, \imath_* S, \xi) \to (D, g_* S, \zeta)$, that is, that the following square commutes

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_i(j)) \arrow[r, "\xi_i"] \arrow[d, "\prod \varphi'"'] & (\Gamma\langle S\rangle')^{-1}(a_i(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_i(j))) \arrow[r, "\zeta_i"'] & (D')^{-1}(\underline{\underline{g}}(a_i(r_i)))
\end{tikzcd}$$

for all $i \in I$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \arrow[r, "\xi_i"] \arrow[d, "\prod \varphi'_{h(j)}"'] & (\Gamma\langle S\rangle')^{-1}(a_i(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_i(j))) \arrow[r, "\zeta_i"'] & (D')^{-1}(\underline{\underline{g}}(a_i(r_i)))
\end{tikzcd}$$

for all $i \in I$ and $h : r_i - 1 \to \mathbb{N}$. When we unfold the above constructed arrow ξ and φ' on the right–hand side we obtain the following diagram

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \arrow[r] \arrow[d, "\prod \varphi'"'] & \sum_{i\in I_{a_i(r_i)}} \sum_{h\in\Theta_{r_i-1,m}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \arrow[d, "\sum\sum\prod \varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_i(j))) \arrow[dr, "\zeta_i"'] & \sum_{i\in I_{a_i(r_i)}} \sum_{h\in\Theta_{r_i-1,m}} \prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_i(j))) \arrow[d, "{[\zeta_i]_{h,i}}"] \\
 & (D')^{-1}(\underline{\underline{g}}(a_i(r_i)))
\end{tikzcd}$$

It commutes since the horizontal arrow on the top is an inclusion.

$\varphi : \Gamma\langle S\rangle \to D$ as a morphism of I'–endorepresentations

We have to check that φ is a morphism of I'–endorepresentations $(\Gamma\langle S\rangle, \imath_* S', \xi') \to (D, g_* S', \zeta')$ such that

$$\begin{tikzcd}
 & (C, S', \rho') \arrow[dl, "\imath'"'] \arrow[dr, "g"] & \\
(\Gamma\langle S\rangle, \imath_* S', \xi') \arrow[rr, "\varphi"'] & & (D, g_* S', \zeta')
\end{tikzcd} \tag{8.3}$$

commutes. First we check the commutativity of

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\xi'_{i,t}"] \arrow[d, "\prod \varphi'"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

for all $(i, t) \in I'$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r, "\xi'_{i,t}"] \arrow[d, "\prod \varphi'_{h(j)}"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

for all $(i,t) \in I'$ and $h : r_i - 1 \to \mathbb{N}$. When we unfold the above constructed arrow ξ' and φ' on the right–hand side we obtain the following diagram

$$\begin{tikzcd}
\prod\limits_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod \varphi'"'] & \sum\limits_{i\in I_{a_{i,t}(r_i)}} \sum\limits_{h\in\Theta_{r_i-1,m}} \prod\limits_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[d, "\sum\sum\prod \varphi'"] \\
\prod\limits_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[dr, "\zeta'_{i,t}"'] & \sum\limits_{i\in I_{a_{i,t}(r_i)}} \sum\limits_{h\in\Theta_{r_i-1,m}} \prod\limits_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[d, "{[\zeta'_{i,t}]_{h,(i,t)}}"] \\
 & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

It commutes since the horizontal arrow on the top is an inclusion.

Next we check the commutativity of (8.3) in $\mathcal{E}$. By definition of $\imath'$ being the inclusion $C' \to \Gamma\langle S\rangle'$, the following diagram commutes

$$\begin{tikzcd}
\prod\limits_{j=1}^{r_i-1} (C')^{-1}(a_{i,t}(j)) \arrow[r, "\rho'_{i,t}"] \arrow[d, "\prod \imath'"] \arrow[dd, bend right=60, "\prod g"'] & (C')^{-1}(a_{i,t}(r_i)) \arrow[d, "\imath'"] \arrow[dd, bend left=60, "g"] \\
\prod\limits_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\xi'_{i,t}"] \arrow[d, "\prod \varphi'"] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod\limits_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

Uniqueness of $\varphi : \Gamma\langle S\rangle \to D$

Let ψ be another morphism $(\Gamma\langle S\rangle, \imath, \xi) \to (\Delta, g, \zeta)$ of $\mathrm{Rep}(S)$. We show that $\psi = \varphi$. We show that the underlying arrows on TEns are equal. We write ψ'_n for the composite $\Gamma\langle S\rangle'_n \hookrightarrow \Gamma\langle S\rangle \xrightarrow{\psi} D'$ for all $n \in \mathbb{N}$ and we write $\psi'_{n,\tau}$ for its fibre in $\tau \in \underline{\underline{C}}$.

We show by recursion on n that $\varphi'_{n,\tau} = \psi'_{n,\tau}$. For $n = 0$, we have

$$\varphi'_{n,\tau}(x) = g'_\tau(x) = \psi'_{n,\tau}(x)$$

since both φ' and ψ' make the following triangle commute

$$\begin{tikzcd}
 & C' \arrow[dl, "\imath'"'] \arrow[dr, "g'"] & \\
\Gamma\langle S\rangle' \arrow[rr] & & D'
\end{tikzcd}$$

Suppose that $\varphi'_{m,\tau} = \psi'_{m,\tau}$ is true for all $m \le n$. We show that $\varphi'_{n+1,\tau} = \psi'_{n+1,\tau}$. Let $i \in I_\tau$ and

$h \in \Theta_{r_i-1,n}$ and $M = (N_1, \ldots, N_{r_i-1}) \in \Gamma\langle S\rangle'_{n+1,\tau}$. We compute

$$\begin{aligned}
\psi'_{n+1,\tau}(M) &= \psi'_{n+1,\tau}(N_1, \ldots, N_{r_i-1}) \\
&= \zeta_i(\psi'_{h(1),a_i(1)}(N_1), \ldots, \psi'_{h(r_i-1),a_i(r_i-1)}(N_{r_i-1})) \\
&= \zeta_i(\varphi'_{h(1),a_i(1)}(N_1), \ldots, \varphi'_{h(r_i-1),a_i(r_i-1)}(N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(\xi_i(N_1, \ldots, N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(M)
\end{aligned}$$

since

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \arrow[r] \arrow[d, "\prod \varphi'"'] & \Gamma\langle S\rangle'_{n+1,t} \arrow[d, "\varphi'_{n+1,\tau}"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_i(j))) \arrow[r, "\zeta_i"'] & (D')^{-1}(\underline{\underline{g}}(\tau))
\end{tikzcd}$$

commutes by definition of φ'. Let $(i,t) \in I'_\tau$ and $h \in \Theta_{r_i-1,n}$ and $M = (N_1, \ldots, N_{r_i-1}) \in \Gamma\langle S\rangle'_{n+1,\tau}$. We compute

$$\begin{aligned}
\psi'_{n+1,\tau}(M) &= \psi'_{n+1,\tau}(N_1, \ldots, N_{r_i-1}) \\
&= \zeta_i(\psi'_{h(1),a_{i,t}(1)}(N_1), \ldots, \psi'_{h(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\
&= \zeta_i(\varphi'_{h(1),a_{i,t}(1)}(N_1), \ldots, \varphi'_{h(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(\xi_{i,t}(N_1, \ldots, N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(M)
\end{aligned}$$

since

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod \varphi'"'] & \Gamma\langle S\rangle'_{n+1,\tau} \arrow[d, "\varphi'_{n+1,t}"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(\tau))
\end{tikzcd}$$

commutes by definition of φ'. □

Proposition 8.5.180 *Using notations of the proof of proposition 8.5.179, the assignation*

$$(\Gamma, S) \mapsto (\Gamma\langle S\rangle, \imath_* S, \xi)$$

induces a functor $\mathrm{Sign}_{\mathcal{E}}(I) \to \mathrm{EndRep}_{\mathcal{E}}(I)$.

Proof. Let $g : (\Gamma, S) \to (\Delta, T)$ be a morphism of signatures. By definition there is an underlying arrow $g : \Gamma \to \Delta$ of $\mathcal{E}$ and $T = g_* S$. By definition of the underlying arrow $g : \Gamma = (C, S', \rho') \to \Delta = (D, T', \zeta')$ of $\mathcal{E}$, it consists of an underlying arrow $g : C \to D$ of TEns and $T' = g_* S'$.

Construction $g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$

First we construct an arrow $g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$ or equivalently an arrow $g\langle S\rangle' : \Gamma\langle S\rangle' \to \Delta\langle g_* S\rangle'$ by recursion. For $n = 0$ the arrow $g\langle S\rangle'_0$ is given by $g' : C' \to D'$. Suppose given an arrow $g\langle S\rangle'_{m,\tau} : \Gamma\langle S\rangle'_{m,\tau} \to \Delta\langle g_* S\rangle'_{m,\underline{\underline{g}}(\tau)}$ for all $m \leq n$ and $\tau \in \underline{\underline{C}}$, we construct $g\langle S\rangle'_{n+1}$. By definition $\Gamma\langle S\rangle_{n+1,\tau} = \sum_{i\in(I'+I)_\tau} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle_{h(j),a_i(j)}$. We take the following composite

$$\begin{array}{c}
\sum_{i\in(I'+I)_\tau} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle_{h(j),a_i(j)} \\
\downarrow \\
\sum_{i\in(I'+I)_{\underline{\underline{g}}(\tau)}} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle_{h(j),\underline{\underline{g}}(a_i(j))}
\end{array}$$

Then we set $g\langle S\rangle'_\tau = [g\langle S\rangle'_{n,\tau}]_{n\in\mathbb{N}}$ and by undoing $'$ of $g\langle S\rangle'$ we obtain $g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$.

$g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$ as a morphism of $\mathcal{E}$

We check that $g\langle S\rangle$ is an arrow in $\mathcal{E}$, that is, that the following diagram commutes

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(j))] \arrow[r, "\xi'_{i,t}"] \arrow[d] & \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(r_i))] \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle[\underline{\underline{g}}(a_{i,t}(j))] \arrow[r] & \Delta\langle S\rangle[\underline{\underline{g}}(a_{i,t}(r_i))]
\end{tikzcd}$$

commutes for all $(i,t) \in I'$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \arrow[r] & (\Delta\langle S\rangle')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

commutes for all $h : r_i - 1 \to \mathbb{N}$ and $(i,t) \in I'$. It does by definition of the horizontal arrows

(that become the vertical ones in the following)

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d] & \prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \arrow[d] \\
\prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d] & \prod_{j=1}^{r_i-1} D\langle g_* S' + g_* S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \arrow[d] \\
C\langle S' + S\rangle'_{m,a_{i,t}(r_i)} \arrow[r] \arrow[d] & D\langle g_* S' + g_* S\rangle'_{m,\underline{\underline{g}}(a_{i,t}(r_i))} \arrow[d] \\
(\Gamma\langle S\rangle)^{-1}(a_{i,t}(r_i)) \arrow[r] & (\Delta\langle g_* S\rangle)^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

where $m = \max_j h(j)$.

$g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$ as a morphism of representations

We check that $g\langle S\rangle$ is an arrow in $\mathrm{EndRep}_{\mathcal{E}}(I)$, that is, that the following diagram commutes

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle[\underline{\imath}(a_i(j))] \arrow[r, "\xi_i"] \arrow[d] & \Gamma\langle S\rangle[\underline{\imath}(a_i(r_i))] \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle[\underline{\underline{g}}(a_i(j))] \arrow[r] & \Delta\langle S\rangle[\underline{\underline{g}}(a_i(r_i))]
\end{tikzcd}$$

commutes for all $i \in I$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \arrow[r] \arrow[d] & (\Gamma\langle S\rangle')^{-1}(a_i(r_i)) \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle'_{h(j),\underline{\underline{g}}(a_i(j))} \arrow[r] & (\Delta\langle S\rangle')^{-1}(\underline{\underline{g}}(a_i(r_i)))
\end{tikzcd}$$

commutes for all $h : r_i - 1 \to \mathbb{N}$ and $i \in I$. It does by definition of the horizontal arrows (that become the vertical ones in the following)

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_i(j)} \arrow[r] \arrow[d] & \prod_{j=1}^{r_i-1} \Delta\langle g_*S\rangle'_{h(j),\underline{\underline{g}}(a_i(j))} \arrow[d] \\
\prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_i(j)} \arrow[r] \arrow[d] & \prod_{j=1}^{r_i-1} D\langle g_*S'+g_*S\rangle'_{h(j),\underline{\underline{g}}(a_i(j))} \arrow[d] \\
C\langle S'+S\rangle'_{m,a_i(r_i)} \arrow[r] \arrow[d] & D\langle g_*S'+g_*S\rangle'_{m,\underline{\underline{g}}(a_i(r_i))} \arrow[d] \\
(\Gamma\langle S\rangle)^{-1}(a_i(r_i)) \arrow[r] & (\Delta\langle g_*S\rangle)^{-1}(\underline{\underline{g}}(a_i(r_i)))
\end{tikzcd}$$

where $m = \max_j h(j)$. $\square$

Corollary 8.5.181 *The forgetful functor* $U : \mathrm{EndRep}_{\mathcal{E}}(I) \to \mathrm{Sign}_{\mathcal{E}}(I)$ *has a left adjoint, which is given by the functor* $F : \mathrm{Sign}_{\mathcal{E}}(I) \to \mathrm{EndRep}_{\mathcal{E}}(I)$ *of proposition 8.5.180.*

Proof. We define unit and counit and show then the triangle identities. The the unit $\eta : \mathrm{Id} \to UF$ at the component $(\Gamma, S) \in \mathrm{Sign}_{\mathcal{E}}(I)$ is $(\Gamma, S) \to (\Gamma\langle S\rangle, \imath_*S)$, which is a morphism of I–signatures induced by $\imath : \Gamma \hookrightarrow \Gamma\langle S\rangle$ of $\mathcal{E}$.

The counit ε is given by initiality of $(\Gamma\langle S\rangle, \imath_*S, \xi)$ in the category of representations of S, its component $(\Gamma\langle S\rangle, \imath_*S, \xi) \to (\Gamma, S, \zeta')$ at $(\Gamma, S, \zeta') \in \mathrm{EndRep}_{\mathcal{E}}(I)$ is given by the unique morphism of representations by 8.5.179.

We check the triangle identities. First we check the commutativity of

$$\begin{tikzcd}
U(\Gamma, S, \zeta') \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & UFU(\Gamma, S, \zeta') \arrow[d, "U\varepsilon"] \\
 & U(\Gamma, S, \zeta')
\end{tikzcd}$$

it becomes

$$\begin{tikzcd}
(\Gamma, S) \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & (\Gamma\langle S\rangle, \imath_*S) \arrow[d, "U\varepsilon"] \\
 & (\Gamma, S)
\end{tikzcd}$$

which commutes since η is induced by the inclusion $\imath : \Gamma \hookrightarrow \Gamma\langle S\rangle$. Next we check the commutativity of

$$\begin{tikzcd}
F(\Gamma, S) \arrow[r, "F\eta"] \arrow[dr, "\mathrm{Id}"'] & FUF(\Gamma, S) \arrow[d, "\varepsilon F"] \\
 & F(\Gamma, S)
\end{tikzcd}$$

it becomes

$$\begin{tikzcd}
(\Gamma\langle S\rangle, \imath_* S, \xi) \arrow[r, "\eta_{(\Gamma,S)}\langle S\rangle"] \arrow[dr, "\mathrm{Id}"'] & ((\Gamma\langle S\rangle)\langle \imath_* S\rangle, \jmath_*(\imath_* S), \zeta) \arrow[d, "\varepsilon_{(\Gamma\langle S\rangle, \imath_* S, \xi)}"] \\
& (\Gamma\langle S\rangle, \imath_* S, \xi)
\end{tikzcd} \tag{8.4}$$

where we write $\jmath$ for the morphism induced by the inclusion $\Gamma\langle S\rangle \hookrightarrow (\Gamma\langle S\rangle)\langle \imath_* S\rangle$. The vertical arrow is by definition the unique morphism $((\Gamma\langle S\rangle)\langle \imath_* S\rangle, \jmath_*(\imath_* S), \zeta) \to (\Gamma\langle S\rangle, \imath_* S, \xi)$ in $\mathrm{Rep}(\imath_* S)$.

But an object of $\mathrm{Rep}(\imath_* S)$ is as well an object of $\mathrm{Rep}(S)$ and a morphism in $\mathrm{Rep}(\imath_* S)$ is as well a morphism of $\mathrm{Rep}(S)$ for the following reason. Let $f : \Gamma\langle S\rangle \to \Delta$ and ζ an endorepresentation of $f_*(\imath_* S)$. If we write f_0 for the composite

$$\Gamma \xrightarrow{\imath} \Gamma\langle S\rangle \xrightarrow{f} \Delta$$

then $(f_0, (f_0)_* S, \zeta)$ is an object of $\mathrm{Rep}(S)$. Let φ be a morphism $(\Delta, f_*(\imath_* S), \zeta) \to (\Delta', f'_*(\imath_* S), \zeta')$ of $\mathrm{Rep}(\imath_* S)$. It is a morphism $(\Delta, (f_0)_* S, \zeta) \to (\Delta', (f'_0)_* S, \zeta')$ of $\mathrm{Rep}(S)$ as well since the following diagram commutes

$$\begin{tikzcd}
& \Gamma \arrow[d, "\imath"] \arrow[ddl, bend right, "f_0"'] \arrow[ddr, bend left, "f'_0"] & \\
& \Gamma\langle S\rangle \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
\Delta \arrow[rr, "\varphi"'] & & \Delta'
\end{tikzcd}$$

So the triangle (8.4) commutes by initiality of $(\Gamma\langle S\rangle, \imath_* S, \xi)$ in $\mathrm{Rep}(S)$. □

8.6 Mixed signature of higher degree

In this section we define a notion of arity and signature of higher degree on an object of a category of representations of another signature.

For this section too suppose given a weighted set I'. We write short $\mathcal{E}$ for the category of endorepresentations $\mathrm{EndRep}(I')$ and $U : \mathcal{E} \to \mathrm{TEns}$.

Definition 8.6.182 (arity) *Let $\Gamma \in \mathcal{E}$ and $d \in \mathbb{N}$. An* arity *on Γ of degree d and of length r is an arrow $a : r \to \underline{\underline{\Gamma\langle d\rangle}}$ in* Set. *Here we refer to d as the signature consisting of d equal arities $1 \mapsto \perp$.*

For the remaining part of this section we fix a weighted set I of arbitrary weights $\mathbf{d}$.

Definition 8.6.183 (I–signature) *Let $\Gamma \in \mathcal{E}$. An I–*signature *S on Γ is a collection $(S_i)_{i\in I}$ of arities on Γ of degrees d_i.*

Definition 8.6.184 (direct image) *Let $\Gamma, \Delta \in \mathcal{E}$, $f : \Gamma \to \Delta$ and S an I–signature on Γ. The* direct image *of S along f is an I–signature on Δ whose arities are the following ones*

$$r_i \to \underline{\underline{\Gamma\langle d_i\rangle}} \to \underline{\underline{\Delta\langle d_i\rangle}}$$

where the second arrow is given by functoriality as explained in proposition 8.5.180. We write $f_* S$ for this signature on Δ.

Definition 8.6.185 (category of I–signatures) *Let $\Gamma, \Delta \in \mathcal{E}$. A* morphism of I–signatures *from an I–signature S on Γ to an I–signature T on Δ is an arrow $f : \Gamma \to \Delta$ of $\mathcal{E}$ such that $T = f_*S$. I–Signatures and morphisms of I–signatures form a category,* $\mathrm{Sign}_{\mathcal{E}}(I)$.

Notation 8.6.186 *Let $\Gamma \in \mathcal{E}$ and $t : d \to \underline{\Gamma}$. The arrow t is a d–endorepresentation in Γ of the signature $S = d$ on Γ consisting of d equal arities $1 \mapsto \bot$.*

So together with the identity arrow of d'–endorepresentations $\Gamma \to \Gamma$, the pair (id_Γ, t) forms a representation of d on Γ. By initiality of $\Gamma\langle d\rangle$ in $\mathrm{Rep}(d)$, there exists a unique morphism $\widetilde{t} : \Gamma\langle d\rangle \to \Gamma$.

Definition 8.6.187 (category of I–endorepresentations) *An object of the category of I–endorepresentations is an I–signature S on Γ together with a map*

$$\rho_{i,t} : \prod_{j=1}^{r_i-1} \Gamma[S_{i,t}(j)] \to \Gamma[S_{i,t}(r_i)]$$

for all $t : d_i \to \underline{\Gamma}$ and all $i \in I$ where we have written

$$S_{i,t} : r_i \xrightarrow{S_i} \underline{\underline{\Gamma\langle d_i\rangle}} \xrightarrow{\widetilde{t}} \underline{\underline{\Gamma}}$$

the second arrow is the the unique arrow of notation 8.6.186.

A morphism from (Γ, S, ρ) to (Δ, T, ξ) is a morphism of I–signatures $f : (\Gamma, S) \to (\Delta, T)$ such that

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma[S_{i,t}(j)] \arrow[r, "\rho_{i,t}"] \arrow[d, "\prod f[S_{i,t}(j)]"'] & \Gamma[S_{i,t}(r_i)] \arrow[d, "f[S_{i,t}(r_i)]"] \\
\prod_{j=1}^{r_i-1} \Delta[T_{i,\underline{f}\circ t}(j)] \arrow[r, "\xi_{i,\underline{f}\circ t}"'] & \Delta[T_{i,\underline{f}\circ t}(r_i)]
\end{tikzcd}$$

commutes for all $t : d_i \to \underline{\Gamma}$ and all $i \in I$.

I–Endorepresentations and morphisms of I–endorepresentations form a category written $\mathrm{EndRep}_{\mathcal{E}}(I)$. *It comes equipped with a forgetful functor* $\mathrm{EndRep}_{\mathcal{E}}(I) \to \mathrm{Sign}_{\mathcal{E}}(I)$.

Definition 8.6.188 (representation) *Let $\Gamma \in \mathcal{E}$ and S an I–signature on Γ. A* representation *of S in an object Δ of $\mathcal{E}$ is a morphism $f : \Gamma \to \Delta$ of $\mathcal{E}$ together with an I–endorepresentation of f_*S in Δ.*

Definition 8.6.189 (category of representations) *Let $\Gamma \in \mathcal{E}$ and S an I–signature on Γ. We build* $\mathrm{Rep}(S)$ *the* category of representations *of S. An object of* $\mathrm{Rep}(S)$ *is a triple (Δ, f, ξ) where Δ is an object of $\mathcal{E}$, $f : \Gamma \to \Delta$ a morphism of $\mathcal{E}$ and ξ an I–endorepresentation of f_*S in Δ.*

A morphism of representations *from a representation (Δ, f, ξ) to a representation (Δ', f', ξ') is a morphism $\varphi : \Delta \to \Delta'$ of $\mathcal{E}$ such that*

$$\begin{tikzcd}
& \Gamma \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
\Delta \arrow[rr, "\varphi"'] & & \Delta'
\end{tikzcd}$$

commutes and such that φ is a morphism of endorepresentations $(\Delta, f_*S, \xi) \to (\Delta', f'_*S, \xi')$. We write $\mathrm{Rep}(S)$ *for this category.*

Proposition 8.6.190 *Let $\Gamma \in \mathcal{E}$ and S an I–signature on Γ. The category of representations* $\mathrm{Rep}(S)$ *has an initial object. We write $\Gamma\langle S\rangle$ for the underlying object of* TEns *of the initial representation.*

Proof. We write $C = U(\Gamma)$, S' the I'–signature of Γ and ρ' for its I'–endorepresentation.

Construction of the initial object $\Gamma\langle S\rangle$

We concatenate the signatures S' and S by setting $J = I' + I$ and $\mathbf{n} = [\mathbf{d}', \mathbf{d}] : J \to \mathbb{N}$ and we write $S' + S$ for this J–signature of degree $\mathbf{n}$. By proposition 8.4.169 the category of representations of $S'+S$ has an initial object, its underlying object of TEns is written $C\langle S'+S\rangle$.

In the construction of $C\langle S' + S\rangle$ we use the related object $C\langle S' + S\rangle'$ that has an additional type $\perp$ and $\underline{C\langle S' + S\rangle}$ as its fibre. The object $C\langle S' + S\rangle'$ in the fibre $\tau \in \underline{\underline{C}}$ is constructed as the coproduct $\sum\limits_{n\in\mathbb{N}} C\langle S' + S\rangle'_{n,\tau}$.

We define a subset $\mathsf{TO}_{n,\tau}$ of each $C\langle S'+S\rangle'_{n,\tau}$. Intuitively the set of trees TO stands for trees having a subtree constructed by S'. We do not want these trees in $\Gamma\langle S\rangle$ since there is already an element of C' that corresponds to such a subtree.

For $n = 0$ we set $\mathsf{TO}_{0,\tau} = 0$. For $n = 1$, we have $C\langle S' + S\rangle'_{1,\tau} = \sum\limits_{(i,t)\in J_\tau} \sum\limits_{g\in\Theta_{0,r_i-1}} \prod\limits_{j=1}^{r_i-1} C\langle S' + S\rangle'_{g(j),a_{i,t}(j)}$ where $g \in \Theta_{0,r_i-1}$ is the constant function 0 and we chose the constructions of indexes $(i,t) \in I'_\tau$ to be in $\mathsf{TO}_{1,\tau}$. More precisely if $M \in C\langle S' + S\rangle'_{1,\tau}$, there exists $(i,t) \in J_\tau$ such that $M \in \prod\limits_{j=1}^{r_i-1} C\langle S' + S\rangle'_{0,a_{i,t}(j)}$ and we set $M \in \mathsf{TO}_{1,\tau}$ if $(i,t) \in I'_\tau$.

Suppose given $\mathsf{TO}_{m,\tau} \subseteq C\langle S' + S\rangle_{m,\tau}$ for all $m \le n$ and all $\tau \in \underline{\underline{C}}$, we construct $\mathsf{TO}_{n+1,\tau}$. Let $M \in C\langle S' + S\rangle_{n+1,\tau}$, by definition there exists $(i,t) \in J_\tau$, $t : n_i \to \underline{\underline{C}}$, $g \in \Theta_{r_i-1,n}$ such that $M \in \prod\limits_{j=1}^{r_i-1} C\langle S' + S\rangle'_{g(j),a_{i,t}(j)}$ and we write $M = (N_1, \ldots, N_{r_i-1})$. We take $M \in \mathsf{TO}_{n+1,\tau}$ if there exists a $j \in \{1, \ldots, r_i - 1\}$ such that $N_j \in \mathsf{TO}_{g(j),a_{i,t}(j)}$.

Then we take $\Gamma\langle S\rangle'_{n,\tau} = C\langle S' + S\rangle'_{n,\tau} \setminus \mathsf{TO}_{n,\tau}$ for all τ and the $\Gamma\langle S\rangle'_\tau = \sum\limits_{n\in\mathbb{N}} \Gamma\langle S\rangle'_{n,\tau}$. By undoing $'$ of $\Gamma\langle S\rangle'$ we obtain $\Gamma\langle S\rangle$.

Construction $p : C\langle S' + S\rangle' \to \Gamma\langle S\rangle'$

First we construct a collection of arrows $s_{n,\tau} : \mathsf{TO}_{n,\tau} \to (\Gamma\langle S\rangle')^{-1}(\tau)$ induced by the d'–endorepresentation ρ'. We construct $s_{n,\tau}$ for all $n \in \mathbb{N}$ and $\tau \in \underline{\underline{C}}$ by recursion.

For $n = 0$ we have $\mathsf{TO}_{n,\tau} = 0$, so the arrow $s_{0,\tau}$ is given by initiality of $0 \in \mathrm{Set}$. For $n = 1$ let $M \in \mathsf{TO}_{1,\tau}$, by definition $M \in \sum\limits_{(i,t)\in I'} \prod\limits_{j=1}^{r_i-1} C\langle S' + S\rangle'_{0,a_{i,t}(j)}$, so there exists a $(i,t) \in I'$ such that $M \in \prod\limits_{j=1}^{r_i-1} C'^{-1}(a_{i,t}(j))$. By $\rho'_{i,t}$ we have

$$\prod_{j=1}^{r_i-1} C'^{-1}(a_{i,t}(j)) \to C'^{-1}(\tau) = \Gamma\langle S\rangle'_{0,\tau}$$

and then by inclusion

$$\Gamma\langle S\rangle'_{0,\tau} \to (\Gamma\langle S\rangle')^{-1}(\tau)$$

Suppose given $s_{q,\tau} : \mathsf{TO}_{q,\tau} \to (\Gamma\langle S\rangle')^{-1}(\tau)$ for all $q \le n$ and $\tau \in \underline{\underline{C}}$ then we construct $s_{n+1,\tau}$. Let $M \in \mathsf{TO}_{n+1,\tau}$ then by definition there exists an $(i,t) \in J_\tau$ and $h \in \Theta_{r_i-1,n}$ such that

$M = (N_1, \ldots, N_{r_i-1}) \in \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{h(j),a_{i,t}(j)}$ where $N_j \in \mathsf{TO}_{h(j),a_{i,t}(j)}$ or $N_j \in \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)}$ for all $j = 1, \ldots, r_i - 1$. Applying the recursion hypotheses to $N_j \in \mathsf{TO}_{h(j),a_{i,t}(j)}$ we obtain

$$s_{h(j),a_{i,t}(j)}(N_j) \in (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j))$$

Then we set

$$s_{n+1,\tau}(M) = (\widetilde{N}_1, \ldots, \widetilde{N}_{r_i-1})$$

where $\widetilde{N}_j = N_j$ if $N_j \in \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)}$ and $\widetilde{N}_j = s_{h(j),a_{i,t}(j)}(N_j)$ if $N_j \in \mathsf{TO}_{h(j),a_{i,t}(j)}$.

So we take $s_\tau = [s_{n,\tau}]_{n\in\mathbb{N}}$. The arrow $p'_{n,\tau}$ is given by

$$C\langle S' + S\rangle'_{n,\tau} = \Gamma\langle S\rangle'_{n,\tau} + \mathsf{TO}_{n,\tau} \xrightarrow{[incl, s_{n,\tau}]} \Gamma\langle S\rangle'_\tau$$

I–**Endorepresentation of** S

We construct an I–endorepresentation $(\Gamma\langle S\rangle, \imath_* S, \xi)$ of S on $\Gamma\langle S\rangle$. Let $(i,t) \in I$, we have to construct an arrow

$$\xi_{i,t} : \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle[\underline{\imath}(a_{i,t}(j)))] \to \Gamma\langle S\rangle[\underline{\imath}(a_{i,t}(r_i))]$$

or equivalently

$$\xi_{i,t} : \prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

By distributivity of Set we have

$$\prod_{j=1}^{r_i-1} \sum_{n\in\mathbb{N}} \Gamma\langle S\rangle'_{n,a_{i,t}(j)} \cong \sum_{h:\mathbb{N}^{r_i-1}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)}$$

So for all $h : r_i - 1 \to \mathbb{N}$, we construct an arrow

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

Since $\Gamma\langle S\rangle' \subseteq C\langle S' + S\rangle'$, we have

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \to \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{h(j),a_{i,t}(j)}$$

and by definition of $C\langle S' + S\rangle'_{m,a_{i,t}(r_i)}$

$$\prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{h(j),a_{i,t}(j)} \to C\langle S' + S\rangle'_{m,a_{i,t}(r_i)}$$

where $m = \max_j(h(j)) + 1$ then by p'

$$C\langle S' + S\rangle'_{m,a_{i,t}(r_i)} \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

Morphism of I'–endorepresentations

Next we construct an I'–endorepresentation of $\imath_* S'$. First we construct an I'–endorepresentation $(\Gamma\langle S\rangle', \imath_* S', \xi')$ of $\imath_* S'$ on $\Gamma\langle S\rangle'$. Let $(i,t) \in I'$, we have to construct an arrow

$$\xi'_{i,t} : \prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

By distributivity of Set we have

$$\prod_{j=1}^{r_i-1} \sum_{n\in\mathbb{N}} \Gamma\langle S\rangle'_{n,a_{i,t}(j)} \cong \sum_{h:\mathbb{N}^{r_i-1}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)}$$

So for all $h : r_i - 1 \to \mathbb{N}$, we construct an arrow

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

Since $\Gamma\langle S\rangle' \subseteq C\langle S' + S\rangle'$, we have

$$\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \to \prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{h(j),a_{i,t}(j)}$$

and by definition of $C\langle S' + S\rangle'_{m,a_{i,t}(r_i)}$

$$\prod_{j=1}^{r_i-1} C\langle S' + S\rangle'_{h(j),a_{i,t}(j)} \to C\langle S' + S\rangle'_{m,a_{i,t}(r_i)}$$

where $m = \max_j(h(j)) + 1$, then by p'

$$C\langle S' + S\rangle'_{m,a_{i,t}(r_i)} \to (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))$$

Next we have to check that

$$\begin{tikzcd}
\prod_{j=1}^{r_i} C[a_{i,t}(j)] \arrow[r] \arrow[d] & C[a_{i,t}(r_i)] \arrow[d] \\
\prod_{j=1}^{r_i} \Gamma\langle S\rangle[\underline{\imath}(a_{i,t}(j))] \arrow[r] & \Gamma\langle S\rangle[\underline{\imath}(a_{i,t}(r_i))]
\end{tikzcd}$$

commutes for all $(i,t) \in I'$ or equivalently that

$$\begin{tikzcd}
\prod_{j=1}^{r_i} (C')^{-1}(a_{i,t}(j)) \arrow[r, "\rho'_{i,t}"] \arrow[d] & (C')^{-1}(a_{i,t}(r_i)) \arrow[d] \\
\prod_{j=1}^{r_i} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\xi'_{i,t}"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i))
\end{tikzcd}$$

commutes. It does by construction of $\xi'_{i,t}$ since $\imath$ comes from the inclusion $\Gamma\langle S\rangle'_0 \to \Gamma\langle S\rangle'$.

Construction $\varphi : \Gamma\langle S\rangle \to D$

Let (Δ, g, ζ) be another object of $\mathrm{Rep}(S)$, that is, $\Delta = (D, g_*S', \zeta') \in \mathcal{E}$, $g : \Gamma \to \Delta$ a morphism of $\mathcal{E}$ and (D, g_*S, ζ) a 0–endorepresentation of g_*S in Δ. We construct a morphism $\varphi : (\Gamma\langle S\rangle, \imath, \xi) \to (\Delta, g, \zeta)$ of $\mathrm{Rep}(S)$.

First we construct the underlying arrow $\varphi : \Gamma\langle S\rangle \to D$ of TEns or equivalently the arrow $\varphi' : \Gamma\langle S\rangle' \to D'$ by recursion. We construct an arrow $\varphi'_{n,\tau} : \Gamma\langle S\rangle'_{n,\tau} \to (D')^{-1}(\underline{\underline{g}}(\tau))$ for all $n \in \mathbb{N}$ and $\tau \in \underline{\underline{C}}$.

For $n = 0$ the arrow $\varphi'_{0,\tau} : (C')^{-1}(\tau) \to (D')^{-1}(\underline{\underline{g}}(\tau))$ is given by the fibre in τ of $g' : C' \to D'$. For $n = 1$ an element $M \in C\langle S' + S\rangle'_{1,\tau} = \sum_{(i,t)\in(I'+I)_\tau} \prod_{j=1}^{r_i-1} (C')^{-1}(a_{i,t}(j))$ is in $\Gamma\langle S\rangle'_{1,\tau}$ if the corresponding $(i,t) \in (I' + I)_\tau$ is actually $(i,t) \in I_\tau$. Then we take the following composite

$$\begin{array}{c}
\sum_{(i,t)\in(I)_\tau} \prod_{j=1}^{r_i-1} (C')^{-1}(a_{i,t}(j)) \\
\Big\downarrow{\scriptstyle \sum\prod g'} \\
\sum_{i\in(I)_\tau} \prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \\
\Big\downarrow{\scriptstyle [\zeta_{i,t}]_{(i,t)\in I_\tau}} \\
(D')^{-1}(\underline{\underline{g}}(\tau))
\end{array}$$

Suppose given $\varphi'_{q,\tau} : \Gamma\langle S\rangle'_{q,\tau} \to (D')^{-1}(\underline{\underline{g}}(\tau))$ for all $q \le n$ and all $\tau \in \underline{\underline{C}}$. We construct $\varphi'_{n+1,\tau}$. An element $M = (N_1, \ldots, N_{r_i-1})$ of $C\langle S'+S\rangle'_{n+1,\tau} = \sum_{(i,t)\in(I'+I)_\tau} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_{i,t}(j)}$ is in $\Gamma\langle S\rangle'_{n+1,\tau}$ if all $N_j \in \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)}$. Let $(i,t) \in I'_\tau$ and $h \in \Theta_{r_i-1,n}$ and we take the composite

$$\begin{array}{c}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \\
\Big\downarrow{\scriptstyle \prod \varphi'_{h(j)}} \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \\
\Big\downarrow{\scriptstyle \zeta'_{i,t}} \\
(D')^{-1}(\underline{\underline{g}}(\tau))
\end{array}$$

Let $(i,t) \in I_\tau$ and $h \in \Theta_{r_i-1,n}$ and we take the composite

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[d, "\prod \varphi'_{h(j)}"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[d, "\zeta_{i,t}"] \\
(D')^{-1}(\underline{\underline{g}}(\tau))
\end{tikzcd}$$

Then we set $\varphi'_\tau = [\varphi'_{n,\tau}]_{n\in\mathbb{N}}$. By undoing $'$ of φ'_τ we obtain $\varphi_\tau : \Gamma\langle S\rangle^{-1}(\tau) \to D^{-1}(\underline{\underline{g}}(\tau))$. Note that by definition of φ' the following diagram commutes

$$\begin{tikzcd}
& C' \arrow[dl, "\imath'"'] \arrow[dr, "g'"] & \\
\Gamma\langle S\rangle' \arrow[rr, "\varphi'"'] & & D'
\end{tikzcd}$$

since $\imath$ comes from the inclusion $C' \hookrightarrow \Gamma\langle S\rangle'$.

$\varphi : \Gamma\langle S\rangle \to D$ as a morphism of I–endorepresentations

We have to check that φ is a morphism of I–endorepresentations $(\Gamma\langle S\rangle, \imath_* S, \xi) \to (D, g_* S, \zeta)$, that is, that the following square commutes

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\xi_{i,t}"] \arrow[d, "\prod \varphi'"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

for all $(i,t) \in I$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r, "\xi_{i,t}"] \arrow[d, "\prod \varphi'_{h(j)}"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

for all $(i,t) \in I$ and $h : r_i - 1 \to \mathbb{N}$. When we unfold the above constructed arrow ξ and φ' on the right–hand side we obtain the following diagram

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1}\Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod\varphi'"'] & \sum_{i\in I_{a_{i,t}(r_i)}}\sum_{h\in\Theta_{r_i-1,m}}\prod_{j=1}^{r_i-1}\Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[d, "\sum\sum\prod\varphi'"] \\
\prod_{j=1}^{r_i-1}(D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[dr, "\zeta_{i,t}"'] & \sum_{i\in I_{a_{i,t}(r_i)}}\sum_{h\in\Theta_{r_i-1,m}}\prod_{j=1}^{r_i-1}(D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[d, "{[\zeta_{i,t}]_{h,(i,t)}}"] \\
 & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

It commutes since the horizontal arrow on the top is an inclusion.

$\varphi:\Gamma\langle S\rangle\to D$ as a morphism of I'–endorepresentations

We have to check that φ is a morphism of I'–endorepresentations $(\Gamma\langle S\rangle,\imath_*S',\xi')\to(D,g_*S',\zeta')$ such that

$$\begin{tikzcd}
 & (C,S',\rho') \arrow[dl, "\imath'"'] \arrow[dr, "g"] & \\
(\Gamma\langle S\rangle,\imath_*S',\xi') \arrow[rr, "\varphi"'] & & (D,g_*S',\zeta')
\end{tikzcd} \tag{8.5}$$

commutes. First we check the commutativity of

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1}(\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\xi'_{i,t}"] \arrow[d, "\prod\varphi'"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1}(D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

for all $(i,t)\in I'$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1}\Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r, "\xi'_{i,t}"] \arrow[d, "\prod\varphi'_{h(j)}"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1}(D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

for all $(i,t)\in I'$ and $h:r_i-1\to\mathbb{N}$. When we unfold the above constructed arrow ξ' and φ' on the right–hand side we obtain the following diagram

$$
\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod \varphi'"'] & \sum_{i\in I_{a_{i,t}(r_i)}} \sum_{h\in\Theta_{r_i-1,m}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[d, "\sum\sum\prod\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[dr, "\zeta'_{i,t}"'] & \sum_{i\in I_{a_{i,t}(r_i)}} \sum_{h\in\Theta_{r_i-1,m}} \prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[d, "{[\zeta'_{i,t}]_{h,(i,t)}}"] \\
 & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}
$$

It commutes since the horizontal arrow on the top is an inclusion.

Next we check the commutativity of (8.5) in $\mathcal{E}$. By definition of $\imath'$ being the inclusion $C' \to \Gamma\langle S\rangle'$, the following diagram commutes

$$
\begin{tikzcd}
\prod_{j=1}^{r_i-1} (C')^{-1}(a_{i,t}(j)) \arrow[r, "\rho'_{i,t}"] \arrow[d, "\prod \imath'"'] \arrow[dd, bend right, "\prod g"'] & (C')^{-1}(a_{i,t}(r_i)) \arrow[d, "\imath'"] \arrow[dd, bend left, "g"] \\
\prod_{j=1}^{r_i-1} (\Gamma\langle S\rangle')^{-1}(a_{i,t}(j)) \arrow[r, "\xi'_{i,t}"] \arrow[d, "\prod \varphi'"'] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d, "\varphi'"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}
$$

Uniqueness of $\varphi : \Gamma\langle S\rangle \to D$

Let ψ be another morphism $(\Gamma\langle S\rangle, \imath, \xi) \to (\Delta, g, \zeta)$ of $\mathrm{Rep}(S)$. We show that $\psi = \varphi$. We show that the underlying arrows on TEns are equal. We write ψ'_n for the composite $\Gamma\langle S\rangle'_n \hookrightarrow \Gamma\langle S\rangle \xrightarrow{\psi} D'$ for all $n \in \mathbb{N}$ and we write $\psi'_{n,\tau}$ for its fibre in $\tau \in \underline{\underline{C}}$.

We show by recursion on n that $\varphi'_{n,\tau} = \psi'_{n,\tau}$. For $n = 0$, we have

$$\varphi'_{n,\tau}(x) = g'_\tau(x) = \psi'_{n,\tau}(x)$$

since both φ' and ψ' make the following triangle commute

$$
\begin{tikzcd}
 & C' \arrow[dl, "\imath'"'] \arrow[dr, "g'"] & \\
\Gamma\langle S\rangle' \arrow[rr] & & D'
\end{tikzcd}
$$

Suppose that $\varphi'_{m,\tau} = \psi'_{m,\tau}$ is true for all $m \leq n$. We show that $\varphi'_{n+1,\tau} = \psi'_{n+1,\tau}$. Let $(i,t) \in I_\tau$

and $h \in \Theta_{r_i-1,n}$ and $M = (N_1, \ldots, N_{r_i-1}) \in \Gamma\langle S\rangle'_{n+1,\tau}$. We compute

$$\begin{aligned}
\psi'_{n+1,\tau}(M) &= \psi'_{n+1,\tau}(N_1, \ldots, N_{r_i-1}) \\
&= \zeta_{i,t}(\psi'_{h(1),a_{i,t}(1)}(N_1), \ldots, \psi'_{h(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\
&= \zeta_{i,t}(\varphi'_{h(1),a_{i,t}(1)}(N_1), \ldots, \varphi'_{h(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(\xi_{i,t}(N_1, \ldots, N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(M)
\end{aligned}$$

since

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod \varphi'"'] & \Gamma\langle S\rangle'_{n+1,t} \arrow[d, "\varphi'_{n+1,\tau}"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(\tau))
\end{tikzcd}$$

commutes by definition of φ'. Let $(i,t) \in I'_\tau$ and $h \in \Theta_{r_i-1,n}$ and $M = (N_1, \ldots, N_{r_i-1}) \in \Gamma\langle S\rangle'_{n+1,\tau}$. We compute

$$\begin{aligned}
\psi'_{n+1,\tau}(M) &= \psi'_{n+1,\tau}(N_1, \ldots, N_{r_i-1}) \\
&= \zeta_{i}(\psi'_{h(1),a_{i,t}(1)}(N_1), \ldots, \psi'_{h(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\
&= \zeta_{i}(\varphi'_{h(1),a_{i,t}(1)}(N_1), \ldots, \varphi'_{h(r_i-1),a_{i,t}(r_i-1)}(N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(\xi_{i,t}(N_1, \ldots, N_{r_i-1})) \\
&= \varphi'_{n+1,\tau}(M)
\end{aligned}$$

since

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d, "\prod \varphi'"'] & \Gamma\langle S\rangle'_{n+1,\tau} \arrow[d, "\varphi'_{n+1,t}"] \\
\prod_{j=1}^{r_i-1} (D')^{-1}(\underline{\underline{g}}(a_{i,t}(j))) \arrow[r, "\zeta'_{i,t}"'] & (D')^{-1}(\underline{\underline{g}}(\tau))
\end{tikzcd}$$

commutes by definition of φ'. $\square$

Proposition 8.6.191 *Using notations of the proof of proposition 8.6.190, the assignation*

$$(\Gamma, S) \mapsto (\Gamma\langle S\rangle, \imath_* S, \xi)$$

induces a functor $\mathrm{Sign}_{\mathcal{E}}(I) \to \mathrm{EndRep}_{\mathcal{E}}(I)$.

Proof. Let $g : (\Gamma, S) \to (\Delta, T)$ be a morphism of signatures. By definition there is an underlying arrow $g : \Gamma \to \Delta$ of $\mathcal{E}$ and $T = g_* S$. By definition of the underlying arrow $g : \Gamma = (C, S', \rho') \to \Delta = (D, T', \zeta')$ of $\mathcal{E}$, it consists of an underlying arrow $g : C \to D$ of TEns and $T' = g_* S'$.

Construction of $g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$

First we construct an arrow $g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$ or equivalently an arrow $g\langle S\rangle' : \Gamma\langle S\rangle' \to \Delta\langle g_* S\rangle'$ by recursion. For $n = 0$ the arrow $g\langle S\rangle'_0$ is given by $g' : C' \to D'$. Suppose given an arrow $g\langle S\rangle'_{m,\tau} : \Gamma\langle S\rangle'_{m,\tau} \to \Delta\langle g_* S\rangle'_{m,\underline{\underline{g}}(\tau)}$ for all $m \leq n$ and $\tau \in \underline{\underline{C}}$, we construct $g\langle S\rangle'_{n+1}$. By definition $\Gamma\langle S\rangle_{n+1,\tau} = \sum_{(i,t)\in(I'+I)_\tau} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle_{h(j),a_{i,t}(j)}$. We take the following composite

$$\begin{array}{c}
\displaystyle\sum_{(i,t)\in(I'+I)_\tau} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} \Gamma\langle S\rangle_{h(j),a_{i,t}(j)} \\
\big\downarrow \\
\displaystyle\sum_{(i,t)\in(I'+I)_{\underline{\underline{g}}(\tau)}} \sum_{h\in\Theta_{r_i-1,n}} \prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle_{h(j),\underline{\underline{g}}(a_{i,t}(j))}
\end{array}$$

Then we set $g\langle S\rangle'_\tau = [g\langle S\rangle'_{n,\tau}]_{n\in\mathbb{N}}$ and by undoing $'$ of $g\langle S\rangle'$ we obtain $g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$.

$g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_* S\rangle$ as a morphism of $\mathcal{E}$

We check that $g\langle S\rangle$ is an arrow in $\mathcal{E}$, that is, that the following diagram commutes

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(j))] \arrow[r, "\xi'_{i,t}"] \arrow[d] & \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(r_i))] \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle[\underline{\underline{g}}(a_{i,t}(j))] \arrow[r] & \Delta\langle S\rangle[\underline{\underline{g}}(a_{i,t}(r_i))]
\end{tikzcd}$$

commutes for all $(i,t) \in I'$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \arrow[r] & (\Delta\langle S\rangle')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

commutes for all $h : r_i - 1 \to \mathbb{N}$ and $(i,t) \in I'$. It does by definition of the horizontal arrows

(that become the vertical ones in the following)

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d] & \prod_{j=1}^{r_i-1} \Delta\langle g_*S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \arrow[d] \\
\prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d] & \prod_{j=1}^{r_i-1} D\langle g_*S'+g_*S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \arrow[d] \\
C\langle S'+S\rangle'_{m,a_{i,t}(r_i)} \arrow[r] \arrow[d] & D\langle g_*S'+g_*S\rangle'_{m,\underline{\underline{g}}(a_{i,t}(r_i))} \arrow[d] \\
(\Gamma\langle S\rangle)^{-1}(a_{i,t}(r_i)) \arrow[r] & (\Delta\langle g_*S\rangle)^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

where $m = \max_j h(j)$.

$g\langle S\rangle : \Gamma\langle S\rangle \to \Delta\langle g_*S\rangle$ as a morphism of representations

We check that $g\langle S\rangle$ is an arrow in $\mathrm{EndRep}_{\mathcal{E}}(I)$, that is, that the following diagram commutes

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(j))] \arrow[r, "\xi_{i,t}"] \arrow[d] & \Gamma\langle S\rangle[\underline{\underline{\imath}}(a_{i,t}(r_i))] \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_*S\rangle[\underline{\underline{g}}(a_{i,t}(j))] \arrow[r] & \Delta\langle S\rangle[\underline{\underline{g}}(a_{i,t}(r_i))]
\end{tikzcd}$$

commutes for all $(i,t) \in I$ or equivalently

$$\begin{tikzcd}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} \arrow[r] \arrow[d] & (\Gamma\langle S\rangle')^{-1}(a_{i,t}(r_i)) \arrow[d] \\
\prod_{j=1}^{r_i-1} \Delta\langle g_*S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \arrow[r] & (\Delta\langle S\rangle')^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{tikzcd}$$

commutes for all $h : r_i - 1 \to \mathbb{N}$ and $(i,t) \in I$. It does by definition of the horizontal arrows

(that become the vertical ones in the following)

$$\begin{array}{ccc}
\prod_{j=1}^{r_i-1} \Gamma\langle S\rangle'_{h(j),a_{i,t}(j)} & \longrightarrow & \prod_{j=1}^{r_i-1} \Delta\langle g_* S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \\
\downarrow & & \downarrow \\
\prod_{j=1}^{r_i-1} C\langle S'+S\rangle'_{h(j),a_{i,t}(j)} & \longrightarrow & \prod_{j=1}^{r_i-1} D\langle g_* S'+g_* S\rangle'_{h(j),\underline{\underline{g}}(a_{i,t}(j))} \\
\downarrow & & \downarrow \\
C\langle S'+S\rangle'_{m,a_{i,t}(r_i)} & \longrightarrow & D\langle g_* S'+g_* S\rangle'_{m,\underline{\underline{g}}(a_{i,t}(r_i))} \\
\downarrow & & \downarrow \\
(\Gamma\langle S\rangle)^{-1}(a_{i,t}(r_i)) & \longrightarrow & (\Delta\langle g_* S\rangle)^{-1}(\underline{\underline{g}}(a_{i,t}(r_i)))
\end{array}$$

where $m = \max_j h(j)$. □

Corollary 8.6.192 *The forgetful functor $U : \mathrm{EndRep}_{\mathcal{E}}(I) \to \mathrm{Sign}_{\mathcal{E}}(I)$ has a left adjoint, which is given by the functor $F : \mathrm{Sign}_{\mathcal{E}}(I) \to \mathrm{EndRep}_{\mathcal{E}}(I)$ of proposition 8.6.191.*

Proof. We define unit and counit and show then the triangle identities. The the unit $\eta : \mathrm{Id} \to UF$ at the component $(\Gamma, S) \in \mathrm{Sign}_{\mathcal{E}}(I)$ is $(\Gamma, S) \to (\Gamma\langle S\rangle, \imath_* S)$, which is a morphism of I–signatures induced by $\imath : \Gamma \hookrightarrow \Gamma\langle S\rangle$ of $\mathcal{E}$.

The counit ε is given by initiality of $(\Gamma\langle S\rangle, \imath_* S, \xi)$ in the category of representations of S, its component $(\Gamma\langle S\rangle, \imath_* S, \xi) \to (\Gamma, S, \zeta')$ at $(\Gamma, S, \zeta') \in \mathrm{EndRep}_{\mathcal{E}}(I)$ is given by the unique morphism of representations by 8.6.190.

We check the triangle identities. First we check the commutativity of

$$\begin{tikzcd}
U(\Gamma, S, \zeta') \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & UFU(\Gamma, S, \zeta') \arrow[d, "U\varepsilon"] \\
& U(\Gamma, S, \zeta')
\end{tikzcd}$$

it becomes

$$\begin{tikzcd}
(\Gamma, S) \arrow[r, "\eta U"] \arrow[dr, "\mathrm{Id}"'] & (\Gamma\langle S\rangle, \imath_* S) \arrow[d, "U\varepsilon"] \\
& (\Gamma, S)
\end{tikzcd}$$

which commutes since η is induced by the inclusion $\imath : \Gamma \hookrightarrow \Gamma\langle S\rangle$. Next we check the commutativity of

$$\begin{tikzcd}
F(\Gamma, S) \arrow[r, "F\eta"] \arrow[dr, "\mathrm{Id}"'] & FUF(\Gamma, S) \arrow[d, "\varepsilon F"] \\
& F(\Gamma, S)
\end{tikzcd}$$

it becomes

$$\begin{tikzcd}
(\Gamma\langle S\rangle, \imath_* S, \xi) \arrow[r, "\eta_{(\Gamma,S)}\langle S\rangle"] \arrow[dr, "\mathrm{Id}"'] & ((\Gamma\langle S\rangle)\langle \imath_* S\rangle, \jmath_*(\imath_* S), \zeta) \arrow[d, "\varepsilon_{(\Gamma\langle S\rangle, \imath_* S, \xi)}"] \\
 & (\Gamma\langle S\rangle, \imath_* S, \xi)
\end{tikzcd} \tag{8.6}$$

where we write $\jmath$ for the morphism induced by the inclusion $\Gamma\langle S\rangle \hookrightarrow (\Gamma\langle S\rangle)\langle \imath_* S\rangle$. The vertical arrow is by definition the unique morphism $((\Gamma\langle S\rangle)\langle \imath_* S\rangle, \jmath_*(\imath_* S), \zeta) \to (\Gamma\langle S\rangle, \imath_* S, \xi)$ in $\mathrm{Rep}(\imath_* S)$.

But an object of $\mathrm{Rep}(\imath_* S)$ is as well an object of $\mathrm{Rep}(S)$ and a morphism in $\mathrm{Rep}(\imath_* S)$ is as well a morphism of $\mathrm{Rep}(S)$ for the following reason. Let $f : \Gamma\langle S\rangle \to \Delta$ and ζ an endorepresentation of $f_*(\imath_* S)$. If we write f_0 for the composite

$$\Gamma \xrightarrow{\imath} \Gamma\langle S\rangle \xrightarrow{f} \Delta$$

then $(f_0, (f_0)_* S, \zeta)$ is an object of $\mathrm{Rep}(S)$. Let φ be a morphism $(\Delta, f_*(\imath_* S), \zeta) \to (\Delta', f'_*(\imath_* S), \zeta')$ of $\mathrm{Rep}(\imath_* S)$. It is a morphism $(\Delta, (f_0)_* S, \zeta) \to (\Delta', (f'_0)_* S, \zeta')$ of $\mathrm{Rep}(S)$ as well since the following diagram commutes

$$\begin{tikzcd}
 & \Gamma \arrow[d, "\imath"] \arrow[ddl, bend right, "f_0"'] \arrow[ddr, bend left, "f'_0"] & \\
 & \Gamma\langle S\rangle \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
\Delta \arrow[rr, "\varphi"'] & & \Delta'
\end{tikzcd}$$

So the triangle (8.6) commutes by initiality of $(\Gamma\langle S\rangle, \imath_* S, \xi)$ in $\mathrm{Rep}(S)$. $\square$

Example 8.6.193 *We describe the example 8.1.146 with our introduced notions. We write 0 for the initial object* $0 \to 0$ *of* TEns.

We set $I_1 = (1, \mathbf{d}_1)$ *and* $\mathbf{d}_1 : 1 \mapsto 0$. *The first layer of* I_1*–signature* S^1 *consists of one* 0*–arity*

$$\mathrm{nat} : 1 \to \underline{0}, 1 \mapsto \bot$$

Then we set $C_1 := 0\langle S^0\rangle$ *which is* $0 \to \{\mathrm{nat}\}$. *Together with the obvious representation of* S^1 *in* C_1*, it forms an object* Γ_1 *of the category of representations of* S^1.

We set $I_2 = (3, \mathbf{d}_2)$ *and* $\mathbf{d}_2 : 1, 2, 3 \mapsto 0$. *The next layer of* I_2*–signature* S^2 *on* Γ_1 *consists of the following arities*

$$\mathrm{zero} : 1 \to \underline{\underline{\Gamma_1}}, 1 \mapsto \mathrm{nat}$$

$$\mathrm{succ} : 2 \to \underline{\underline{\Gamma_1}}, 1, 2 \mapsto \mathrm{nat}$$

$$\mathrm{list} : 3 \to \underline{\underline{\Gamma_1}}, 1 \mapsto \mathrm{nat}, 2, 3 \mapsto \bot$$

We set $C_2 := \Gamma_1\langle S^2\rangle$ *which is the following object of* TEns

$$\begin{array}{c}
\{\mathrm{zero}, \mathrm{succ}^k(\mathrm{zero}) | k \in \mathbb{N}\} \\
\Big\downarrow \\
\{\mathrm{nat}, \\
\mathrm{list}(\mathrm{succ}^{i_1}(\mathrm{zero}), \mathrm{nat}), \\
\mathrm{list}(\mathrm{succ}^{i_2}(\mathrm{zero}), \mathrm{list}(\mathrm{succ}^{i_1}(\mathrm{zero}), \mathrm{nat})), \\
\ldots | j \in \mathbb{N}, i_j \in \mathbb{N}_0\}
\end{array}$$

where only the fibre of nat *is not empty. Together with the obvious representation it forms an object* Γ_2 *of the category of representations of* S^2*.*

We set $I_3 = (2, \mathbf{d}_3)$ *and* $\mathbf{d}_3 : 1, 2 \mapsto 1$*. The third layer is the following* I_3*–signature* S^3 *consisting of the following arities*

$$\mathrm{nil} : 1 \to \underline{\underline{\Gamma_2\langle 1\rangle}}, 1 \mapsto \mathrm{list}(\mathrm{zero}, 1)$$

$$\mathrm{cons}_k : 1 \to \underline{\underline{\Gamma_2\langle 1\rangle}}, 1 \mapsto \mathrm{list}(\mathrm{succ}^k(\mathrm{zero}), 1), 2 \mapsto 1, 3 \mapsto \mathrm{list}(\mathrm{succ}^{k+1}(\mathrm{zero}), 1)$$

We set $C_2 := \Gamma_2\langle S^3\rangle$*. We do not describe it here explicitly. It can be provided obviously with representations of* S^3 *and it forms the initial object in the category of representations of* S^3*.*

Chapter 9

Typed Lambda Calculus

The aim of this chapter is to characterise the typed Lambda Calculus as the initial object of a certain category. We do not develop a whole theory of signatures and their representations for describing typed syntax with variable binding more generally, we consider only the particular signature of the Lambda Calculus.

Example 9.0.194 *We write 0 for the initial object* $0 \to 0$ *of* TEns*. Consider the 0–signature* S *consisting of the arity* $\Rightarrow: 3 \to \underline{\underline{0}},\ 1,2,3 \mapsto \bot$*. The category of representations has objects* $(\Gamma, \Rightarrow)$ *where* $\Gamma \in \mathrm{TEns}$ *and* $\Rightarrow: \underline{\Gamma} \times \underline{\Gamma} \to \underline{\Gamma}$*. We write* $\mathrm{TEns}^{\Rightarrow}$ *for this category. We write* U *for the forgetful functor* $\mathrm{TEns}^{\Rightarrow} \to \mathrm{TEns}$*.*

For this chapter we consider monads R on $\mathrm{TEns}^{\Rightarrow}$ and derived monads of R: monads on the comma categories $(0 \to n) \downarrow U$.

Notation 9.0.195 *We write* ${}_n\mathrm{TEns}^{\Rightarrow}$ *for the comma category* $(0 \to n) \downarrow U$*.*

9.1 Monads on comma categories

For our definition of arity and signature, we recall first the following fact.

Proposition 9.1.196 *Let* $\mathcal{C}$*,* $\mathcal{D}$ *be two categories,* $F : \mathcal{C} \to \mathcal{D}$*,* D *an object of* $\mathcal{D}$ *and* (R, η, μ) *be a monad on* $\mathcal{C}$*. It induces a monad* R^D *on the comma category* $D \downarrow F$*.*

Proof. We set

$$R^D : D \xrightarrow{f} FC \quad \mapsto \quad D \xrightarrow{f} FC \xrightarrow{F\eta_C} FRC$$

We check that this definition is correct, that is, that this assignment on objects defines indeed a monad R^D on $D \downarrow F$. We check first the functoriality. Let

$$\begin{tikzcd} & D \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ FC_1 \arrow[rr, "Fh"'] & & FC_2 \end{tikzcd}$$

be an arrow of $D \downarrow F$. Its image under R^D is given by

$$\begin{tikzcd} & D \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ FC_1 \arrow[rr, "Fh"'] \arrow[d, "F\eta_{C_1}"'] & & FC_2 \arrow[d, "F\eta_{C_2}"] \\ FRC_1 \arrow[rr, "FRh"'] & & FRC_2 \end{tikzcd}$$

We set

$$\eta^D = F\eta \circ - \qquad \text{and} \qquad \mu^D = F\mu \circ -$$

Let $A \xrightarrow{f} C$ be an object of $A \downarrow \mathcal{C}$. The diagram

$$\begin{tikzcd} D \arrow[dr, "f"] \arrow[dd, "f"'] & \\ & FC \arrow[d, "F\eta_C"] \\ C \arrow[r, "F\eta_C"'] & FRC \end{tikzcd}$$

commutes and the diagram

$$\begin{tikzcd} & D \arrow[dl, "f"'] \arrow[dr, "f"] & \\ FC \arrow[rr, "\mathrm{id}"] \arrow[d, "F\eta_C"'] & & FC \arrow[d, "F\eta_C"] \\ FRC \arrow[rr, "\mathrm{id}"] \arrow[d, "F\eta_{RC}"'] & & FRC \\ FRRC \arrow[urr, "F\mu_C"'] & & \end{tikzcd}$$

commutes as well. $\square$

Remark that the above construction is functorial. Let $\rho : R \to P$ be a morphism of monads on $\mathcal{C}$. It induces a morphism of monads $\rho^D : R^D \to P^D$. We define its component at $f : D \to FC \in D \downarrow F$ by

$$\begin{tikzcd} & D \arrow[dl, "f"'] \arrow[dr, "f"] & \\ FC \arrow[rr, "\mathrm{id}"] \arrow[d, "F\eta^R_C"'] & & FC \arrow[d, "F\eta^P_C"] \\ FRC \arrow[rr, "F\rho_C"'] & & FPC \end{tikzcd}$$

since ρ is compatible with the units.

9.2 Modules on derived monads

Similarly to the simply typed case, we use modules on monads to describe syntax. More precisely we use again derived and fibre modules and a new type of modules, the underlined modules. Instead of an arbitrary category we define those modules on $\mathrm{TEns}^{\Rightarrow}$ only.

Definition 9.2.197 (derived module) *Let R be a monad on ${}_n\mathrm{TEns}^{\Rightarrow}$, M an R–module and τ a natural transformation $1 \to \underline{U}$. We define the* derived module *of M with respect to τ on objects by*

$$M^{\tau} : (\Gamma, t_1, \ldots, t_n, \Rightarrow) \mapsto M([\Gamma, \tau], t_1, \ldots, t_n, \Rightarrow)$$

where we write τ as well for the image of 1 at the component $(\Gamma, t_1, \ldots, t_n, \Rightarrow)$.

Definition 9.2.198 (fibre module) *Let R be a monad on ${}_n\mathrm{TEns}^{\Rightarrow}$, M an R–module with codomain ${}_n\mathrm{TEns}^{\Rightarrow}$ and σ a natural transformation $1 \to \underline{U \circ M}$. We define the* fibre module *of M with respect to σ on objects by*

$$M_{\sigma} : (\Gamma, t_1, \ldots, t_n, \Rightarrow) \mapsto M(\Gamma, t_1, \ldots, t_n, \Rightarrow)^{-1}(\sigma)$$

where we write σ as well for the image of 1 at the component $(\Gamma, t_1, \ldots, t_n, \Rightarrow)$.

Notation 9.2.199 *We write $\mathcal{R}$ for the category of monads on $\mathrm{TEns}^{\Rightarrow}$. We write ${}_n\mathcal{R}$ for the category of monads on ${}_n\mathrm{TEns}^{\Rightarrow}$. We write ${}_n\mathrm{Mod}_{\mathcal{D}}$ for the large category of modules on monads of ${}_n\mathcal{R}$ with codomain $\mathcal{D}$. If $\mathcal{D} = \mathrm{Set}$ we write only ${}_n\mathrm{Mod}$.*

We upgrade the modules into functors from the category of monads $\mathcal{R}$ into the large category of modules ${}_n\mathrm{Mod}_{\mathcal{D}}$.

Notation 9.2.200 *We write*

- ${}_n\Theta : \mathcal{R} \to {}_n\mathrm{Mod}_{{}_n\mathrm{TEns}^{\Rightarrow}}$, $R \mapsto ({}_nR, {}_nR)$ *for the tautological modules*
- $\underline{{}_n\Theta} : \mathcal{R} \to {}_n\mathrm{Mod}$, $R \mapsto ({}_nR, \underline{{}_nR})$ *for the underlined module where $\underline{{}_nR}$ is the composite of ${}_nR$ and $\underline{(-)}$*
- $M^{\tau} : {}_n\mathrm{Mod}_{\mathcal{D}} \to {}_n\mathrm{Mod}_{\mathcal{D}}$, $({}_nR, M) \mapsto ({}_nR, M^{\tau})$ *for the derived module of M with respect to $\tau : 1 \to \underline{U}$*
- $[M]_{\sigma} : {}_n\mathrm{Mod}_{{}_n\mathrm{TEns}^{\Rightarrow}} \to {}_n\mathrm{Mod}$, $({}_nR, M) \mapsto ({}_nR, M_{\sigma})$ *for the fibre module of M with respect to $\sigma : 1 \to \underline{M}$*

9.3 Signature

We define the notion of arity, signature and representation that goes with in the particular case of the category $\mathrm{TEns}^{\Rightarrow}$.

Definition 9.3.201 *(*$\mathrm{TEns}^{\Rightarrow}$*–arity) A* $\mathrm{TEns}^{\Rightarrow}$*–arity is a pair (M, N) of functors $\mathcal{R} \to {}_n\mathrm{Mod}$, written $M \to N$ such that*

- *M is either the constant funcor 1 or can be expressed as*

$$M = \prod_{i=1}^{p} M_i$$

such that there exists a permutation σ of $\{1, \ldots, p\}$ and two integers $1 \leq \ell \leq k \leq p$ such that

$$\prod_{i=1}^{p} M_{\sigma(i)} = \prod_{i=1}^{\ell} \underline{{}_n\Theta} \times \prod_{i=\ell+1}^{k} [{}_n\Theta]_{t_i} \times \prod_{i=k+1}^{p} [{}_n\Theta^{s_i}]_{t_i}$$

where t_i is a natural transformation $1 \to \underline{\Theta}$ for all $i = \ell + 1, \ldots, p$ and s_i is a natural transformation $1 \to \underline{U}$ for $i = k + 1, \ldots, p$.

- *N is either $\underline{{}_n\Theta}$ or $[{}_n\Theta]_{t_0}$ with $t_0 : 1 \to \underline{\Theta}$.*

Definition 9.3.202 *(*$\mathrm{TEns}^{\Rightarrow}$*–signature) A* $\mathrm{TEns}^{\Rightarrow}$*–signature is a collection of* $\mathrm{TEns}^{\Rightarrow}$*–arities.*

Example 9.3.203 (typed Lambda Calculus) *The signature of the typed Lambda Calculus consists of the following arities on* $\mathrm{TEns}^{\Rightarrow}$

$$\mathrm{abs} : [\mathbf{2}\Theta^1]_2 \to [\mathbf{2}\Theta]_{1\Rightarrow 2}$$

and

$$\mathrm{app} : [\mathbf{2}\Theta]_{1\Rightarrow 2} \times [\mathbf{2}\Theta]_1 \to [\mathbf{2}\Theta]_2$$

The module $[\mathbf{2}\Theta^1]_2$ assigns a monad R on $\mathrm{TEns}^{\Rightarrow}$*, the ${}_2R$–module $[\mathbf{2}R^1]_2$. This module assigns an object $(\Gamma, s, t, \Rightarrow)$ the set ${}_2R([\Gamma, \langle s\rangle], s, t, \Rightarrow])^{-1}(\underline{\eta}_{(\Gamma,\Rightarrow)}(t))$.*

The module $[\mathbf{2}\Theta]_{1\Rightarrow 2}$ assigns a monad R on $\mathrm{TEns}^{\Rightarrow}$*, the ${}_2R$–module $[\mathbf{2}R]_{1\Rightarrow 2}$. This module assigns an object $(\Gamma, s, t, \Rightarrow)$ the set ${}_2R(\Gamma, s, t, \Rightarrow])^{-1}(\underline{\eta}_{(\Gamma,\Rightarrow)}(s \Rightarrow t))$.*

Definition 9.3.204 (representation in a monad) *Let $\alpha = M \to N$ a* $\mathrm{TEns}^{\Rightarrow}$*–arity. A* representation *of α in the monad R on* $\mathrm{TEns}^{\Rightarrow}$ *is a morphism of ${}_nR$–modules $M(R) \to N(R)$.*

Let S be a $\mathrm{TEns}^{\Rightarrow}$*–signature. A* representation *of S in the monad R consists of a representation of each arity of S in the monad R.*

Definition 9.3.205 (category of representations) *Let $S = (\alpha_i)_{i \in I}$ be a* $\mathrm{TEns}^{\Rightarrow}$*–signature. We define the category of representations,* $\mathrm{Rep}(S)$*. An object consists of a monad R on* $\mathrm{TEns}^{\Rightarrow}$ *and a representation of each α_i of S. A morphism from $(R, (r_i)_{i \in I})$ to $(Q, (q_i)_{i \in I})$ is a morphism of monads $\rho : R \to Q$ such that*

$$\begin{tikzcd} M_i(R) \arrow[r] \arrow[d] & N_i(R) \arrow[d] \\ M_i(Q) \arrow[r] & N_i(Q) \end{tikzcd}$$

commutes for all $\alpha_i = M_i \to N_i$.

9.4 Typed Lambda Calculus

In this section we prove the following theorem

Theorem 9.4.206 *The typed Lambda Calculus is the initial object in the category of representations* $\mathrm{Rep}(S)$ *of the signature of example 9.3.203.*

Proof.

Definition of the initial object TLC

Let $(\Gamma : X \to T, \Rightarrow : T \times T \to T)$ be an object of $\mathrm{TEns}^{\Rightarrow}$. Remark that $\Gamma : X \to T$ is an object of $\mathrm{Set}\,/T$. By chapter 6 the simply typed Lambda–Calculus is the initial object of a certain category of representations. We write STLC_T for this object. So we define

$$\mathsf{TLC}(\Gamma, \Rightarrow) := (\mathsf{STLC}_T(\Gamma), \Rightarrow)$$

Remark that $\underline{\mathsf{TLC}}(\Gamma) = T$.

We check the functoriality of TLC. Let $h : (\Gamma, \Rightarrow) \to (\Gamma', \Rightarrow')$ be a morphism of $\mathrm{TEns}^{\Rightarrow}$. We define $\mathsf{TLC}(h)$ by defining $\overline{\mathsf{TLC}}(h) : \overline{\mathsf{TLC}}(\Gamma, \Rightarrow) \to \overline{\mathsf{TLC}}(\Gamma', \Rightarrow')$ and $\underline{\mathsf{TLC}}(h) : \underline{\mathsf{TLC}}(\Gamma, \Rightarrow) \to \underline{\mathsf{TLC}}(\Gamma', \Rightarrow')$ separately.

Since $\underline{\mathsf{TLC}}(\Gamma, \Rightarrow) = \underline{\Gamma}$ and $\underline{\mathsf{TLC}}(\Gamma', \Rightarrow') = \underline{\Gamma'}$, we set $\underline{\mathsf{TLC}}(h) = \underline{h}$.

We define $\overline{\mathsf{TLC}}(h)$ by induction. Remark that $\overline{\mathsf{TLC}}(\Gamma, \Rightarrow) = \overline{\mathsf{STLC}_T}(\Gamma) = \sum_{t \in T} [\mathsf{STLC}_T]_t$. Let $t \in T$ and $P \in [\mathsf{STLC}_T]_t$.

- If $P = \mathrm{var}_\Gamma(x)$ where $x \in \overline{\Gamma}$, we set

$$\overline{\mathsf{TLC}}(h)(P) := \overline{\mathrm{var}}_{\Gamma'}(\overline{h}(x))$$

- If $P = \mathrm{app}_{u,t,\Gamma}(M, N)$ where $M \in [\mathsf{STLC}_T(\Gamma)]_{u \Rightarrow t}$ and $N \in [\mathsf{STLC}_T(\Gamma)]_u$, we set

$$\overline{\mathsf{TLC}}(h)(P) = \overline{\mathsf{TLC}}(h)(\mathrm{app}_{u,t,\Gamma}(M, N)) := \mathrm{app}_{\underline{h}(u), \underline{h}(t), \Gamma'}(\overline{\mathsf{TLC}}(h)(M), \overline{\mathsf{TLC}}(h)(N))$$

- If $t = t_1 \Rightarrow t_2$ and $P = \mathrm{abs}_{t_1, t_2, \Gamma}(M)$ where $M \in [\mathsf{STLC}_T([\Gamma, \langle t_1 \rangle])]_{t_2}$, we set

$$\overline{\mathsf{TLC}}(h)(P) = \overline{\mathsf{TLC}}(h)(\mathrm{abs}_{t_1, t_2, \Gamma}(M)) := \mathrm{abs}_{\underline{h}(t_1), \underline{h}(t_2), \Gamma'}(\overline{\mathsf{TLC}}(h+1)(M))$$

  where we write $h + 1 : [\Gamma, \langle t_1 \rangle] \to [\Gamma', \langle \underline{h}(t_1) \rangle]$ for

$$\begin{tikzcd}
X + 1 \arrow[r, "\overline{h}+1"] \arrow[d, "{[\Gamma, \langle t_1 \rangle]}"'] & X' + 1 \arrow[d, "{[\Gamma', \langle \underline{h}(t_1) \rangle]}"] \\
T \arrow[r, "\underline{h}"'] & T'
\end{tikzcd}$$

TLC **as a monad on** $\mathrm{TEns}^{\Rightarrow}$

First we show that TLC is a monad on $\mathrm{TEns}^{\Rightarrow}$. We set

$$\mathrm{var}^{\mathsf{TLC}}_{(\Gamma, \Rightarrow)} := (\mathrm{var}^{\mathsf{STLC}_T}_{\Gamma}, \mathrm{id}_T)$$

and

$$\mathrm{subst}^{\mathsf{TLC}}_{(\Gamma, \Rightarrow)} := (\mathrm{subst}^{\mathsf{STLC}_T}_{\Gamma}, \mathrm{id}_T)$$

Let $(\Gamma, \Rightarrow)$ be an object of $\mathrm{TEns}^{\Rightarrow}$ where $\Gamma : X \to T$. We have to check the commutativity of the following diagrams

$$\begin{tikzcd}
\mathsf{TLC}(\Gamma,\Rightarrow) \arrow[r, "\mathsf{TLC}(\mathrm{var}_{(\Gamma,\Rightarrow)})"] \arrow[dr, "\mathrm{id}_{\mathsf{TLC}(\Gamma,\Rightarrow)}"'] & \mathsf{TLC}(\mathsf{TLC}(\Gamma,\Rightarrow)) \arrow[d, "\mathrm{subst}_{(\Gamma,\Rightarrow)}"] & \mathsf{TLC}(\Gamma,\Rightarrow) \arrow[l, "\mathrm{var}_{\mathsf{TLC}(\Gamma,\Rightarrow)}"'] \arrow[dl, "\mathrm{id}_{\mathsf{TLC}(\Gamma,\Rightarrow)}"] \\
& \mathsf{TLC}(\Gamma,\Rightarrow) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{TLC}(\mathsf{TLC}(\mathsf{TLC}(\Gamma,\Rightarrow))) \arrow[r, "\mathsf{TLC}\,\mathrm{subst}_{(\Gamma,\Rightarrow)}"] \arrow[d, "\mathrm{subst}_{\mathsf{TLC}(\Gamma,\Rightarrow)}"'] & \mathsf{TLC}(\mathsf{TLC}(\Gamma,\Rightarrow)) \arrow[d, "\mathrm{subst}_{(\Gamma,\Rightarrow)}"] \\
\mathsf{TLC}(\mathsf{TLC}(\Gamma,\Rightarrow)) \arrow[r, "\mathrm{subst}_{(\Gamma,\Rightarrow)}"'] & \mathsf{TLC}(\Gamma,\Rightarrow)
\end{tikzcd}$$

We check these diagrams separately in the overlined and underlined parts. In the underlined parts we have only identity arrows, so the diagrams commute trivially. In the overlined parts we have the following diagrams

$$\begin{tikzcd}
\overline{\mathsf{STLC}_T(\Gamma)} \arrow[r, "\mathsf{STLC}_T(\mathrm{var}_\Gamma)"] \arrow[dr, "\mathrm{id}_{\mathsf{STLC}_T(\Gamma)}"'] & \overline{\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))} \arrow[d, "\mathrm{subst}_\Gamma"] & \overline{\mathsf{STLC}_T(\Gamma)} \arrow[l, "\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}"'] \arrow[dl, "\mathrm{id}_{\mathsf{STLC}_T(\Gamma)}"] \\
& \overline{\mathsf{STLC}_T(\Gamma)} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\overline{\mathsf{STLC}_T(\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma)))} \arrow[r, "\mathsf{STLC}_T\,\mathrm{subst}_\Gamma"] \arrow[d, "\mathrm{subst}_{\mathsf{STLC}_T(\Gamma)}"'] & \overline{\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))} \arrow[d, "\mathrm{subst}_\Gamma"] \\
\overline{\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))} \arrow[r, "\mathrm{subst}_\Gamma"'] & \overline{\mathsf{STLC}_T(\Gamma)}
\end{tikzcd}$$

These diagrams commute since STLC_T is a monad on Set/T.

TLC as an object of $\mathrm{Rep}(S)$

Next we provide TLC with a representation of the signature of example 9.3.203.

$$[_2\mathsf{TLC}^1]_2 \to [_2\mathsf{TLC}]_{1\Rightarrow 2}$$

and

$$[_2\mathsf{TLC}]_{1\Rightarrow 2} \times [_2\mathsf{TLC}]_1 \to [_2\mathsf{TLC}]_2$$

Let $(\Gamma, s, t, \Rightarrow)$ be an object of $\mathrm{TEns}_2^{\Rightarrow}$ where $\Gamma : X \to T$ and $v \in T$. Remark that

$$[_2\mathsf{TLC}(\Gamma, s, t, \Rightarrow)]_v = [\mathsf{STLC}_T(\Gamma)]_v \tag{9.1}$$

So we set

$$\mathrm{app}^{\mathsf{TLC}}_{(\Gamma,s,t,\Rightarrow)} := \mathrm{app}^{\mathsf{STLC}_T}_{s,t,\Gamma}$$

and

$$\mathrm{abs}^{\mathsf{TLC}}_{(\Gamma,s,t,\Rightarrow)} := \mathrm{abs}^{\mathsf{STLC}_T}_{s,t,\Gamma}$$

where

$$\mathrm{app}_{s,t}^{\mathsf{STLC}_T} : [\mathsf{STLC}_T]_{s\Rightarrow t} \times [\mathsf{STLC}_T]_s \to [\mathsf{STLC}_T]_t$$

and

$$\mathrm{abs}_{s,t}^{\mathsf{STLC}_T} : [\mathsf{STLC}_T^s]_t \to [\mathsf{STLC}_T]_{s\Rightarrow t}$$

are the STLC_T–module morphisms of application and abstraction of the simply typed Lambda–Calculus of chapter 6.

We have to check that the following diagrams commute

$$\begin{tikzcd}
[{}_2\mathsf{TLC}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_{s\Rightarrow t} \times [{}_2\mathsf{TLC}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_s \arrow[r, "\mathrm{app}_{2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)}"] \arrow[d, "\overline{\mathrm{subst}}^{\mathsf{TLC}}_{(\Gamma,\Rightarrow)} \times \overline{\mathrm{subst}}^{\mathsf{TLC}}_{(\Gamma,\Rightarrow)}"'] & [{}_2\mathsf{TLC}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_t \arrow[d, "\overline{\mathrm{subst}}^{\mathsf{TLC}}_{(\Gamma,\Rightarrow)}"] \\
[{}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_{s\Rightarrow t} \times [{}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_s \arrow[r, "\mathrm{app}_{(\Gamma,s,t,\Rightarrow)}"'] & [{}_2\mathsf{TLC}((\Gamma,s,t,\Rightarrow))]_t
\end{tikzcd}$$

and

$$\begin{tikzcd}
[{}_2\mathsf{TLC}^1{}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_t \arrow[r, "\mathrm{abs}_{2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)}"] \arrow[d, "\overline{\mathrm{subst}}^{2\mathsf{TLC}^1}_{(\Gamma,s,t,\Rightarrow)}"'] & [{}_2\mathsf{TLC}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_{s\Rightarrow t)} \arrow[d, "\overline{\mathrm{subst}}^{\mathsf{TLC}}_{(\Gamma,s,t,\Rightarrow)}"] \\
[{}_2\mathsf{TLC}^1(\Gamma,s,t,\Rightarrow)]_t \arrow[r, "\mathrm{abs}_{(\Gamma,s,t,\Rightarrow)}"'] & [{}_2\mathsf{TLC}(\Gamma,s,t,\Rightarrow)]_{s\Rightarrow t)}
\end{tikzcd}$$

By replacing the definitions and by using formula 9.1 we obtain

$$\begin{tikzcd}
[\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_{s\Rightarrow t} \times [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_s \arrow[r, "\mathrm{app}_{s,t,\mathsf{STLC}_T(\Gamma)}"] \arrow[d, "\mathrm{subst}_\Gamma \times \mathrm{subst}_\Gamma"'] & [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_t \arrow[d, "\mathrm{subst}_\Gamma"] \\
[\mathsf{STLC}_T(\Gamma)]_{s\Rightarrow t} \times [\mathsf{STLC}_T(\Gamma)]_s \arrow[r, "\mathrm{app}_{s,t,\Gamma}"'] & [\mathsf{STLC}_T(\Gamma)]_t
\end{tikzcd}$$

and

$$\begin{tikzcd}
[\mathsf{STLC}_T([\mathsf{STLC}_T(\Gamma),\langle s\rangle])]_t \arrow[r, "\mathrm{abs}_{s,t,\mathsf{STLC}(\Gamma)}"] \arrow[d, "\sigma"'] & [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_{s\Rightarrow t} \arrow[dd, "\mathrm{subst}_\Gamma"] \\
[\mathsf{STLC}_T(\mathsf{STLC}_T([\Gamma,\langle s\rangle]))]_t \arrow[d, "\mathrm{subst}_{[\Gamma,\langle s\rangle]}"'] & \\
[\mathsf{STLC}_T([\Gamma,\langle s\rangle])]_t \arrow[r, "\mathrm{abs}_{s,t,\Gamma}"'] & [\mathsf{STLC}_T(\Gamma)]_{s\Rightarrow t}
\end{tikzcd}$$

These last two diagrams commute since app and abs are STLC_T–module morphisms.

Construction of the arrow $\varphi : \mathsf{TLC} \to R$

Let $((R,\eta^R,\mu^R),\mathrm{app}^R,\mathrm{abs}^R)$ be another object in the category of representations $\mathrm{Rep}(S_1)$ of the signature of example 9.3.203. In this section we construct a morphism of representations $\varphi : ((\mathsf{TLC},\mathrm{var},\mathrm{subst}),\mathrm{app},\mathrm{abs}) \to ((R,\eta^R,\mu^R),\mathrm{app}^R,\mathrm{abs}^R)$ and show then its uniqueness.

We construct separately the underlined and overlined parts. Let $(\Gamma,\Rightarrow)$ be an object of $\mathrm{TEns}^{\Rightarrow}$.

First we construct $\underline{\varphi}_{(\Gamma,\Rightarrow)} : \underline{\mathsf{TLC}}(\Gamma,\Rightarrow) \to \underline{R}(\Gamma,\Rightarrow)$. Since $\underline{\mathsf{TLC}}(\Gamma,\Rightarrow) = (\underline{\Gamma},\Rightarrow)$, we set

$$\underline{\varphi}_{(\Gamma,\Rightarrow)} := \underline{\eta}^R_{(\Gamma,\Rightarrow)}$$

Next we construct $\overline{\varphi}_{(\Gamma,\Rightarrow)} : \overline{\mathsf{TLC}}(\Gamma,\Rightarrow) \to \overline{R}(\Gamma,\Rightarrow)$ or equivalently $\overline{\mathsf{STLC}_T}(\Gamma) \to \overline{R}(\Gamma,\Rightarrow)$ by structural induction. Note that $\overline{\mathsf{STLC}_T}(\Gamma) = \sum_{t\in T}[\mathsf{STLC}_T(\Gamma)]_t$, so let $t \in T$ and $P \in [\mathsf{STLC}_T(\Gamma)]_t$.

- If $P = \overline{\mathrm{var}}^{\mathsf{STLC}_T}_{\Gamma}(x)$ where $x \in \overline{\Gamma}$, we set

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(P) = \overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{var}}^{\mathsf{STLC}_T}(x)) := \overline{\eta}^R_{(\Gamma,\Rightarrow)}(x)$$

- If $P = \mathrm{app}^{\mathsf{STLC}_T}_{u,t,\Gamma}(M,N)$ with $u \in T$, $M \in [\mathsf{STLC}_T(\Gamma)]_{u\Rightarrow t}$ and $N \in [\mathsf{STLC}_T(\Gamma)]_u$, by definition of TLC we have that

$$M \in [\mathsf{TLC}(\Gamma,\Rightarrow)]_{u\Rightarrow v} \qquad \text{and} \qquad N \in [\mathsf{TLC}(\Gamma,\Rightarrow)]_u$$

  By induction hypothesis

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(M) \in [R(\Gamma,\Rightarrow)]_{\underline{\eta}^R_{(\Gamma,\Rightarrow)}(u)\Rightarrow^R\underline{\eta}^R_{(\Gamma,\Rightarrow)}(t)}$$

  and

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(N) \in [R(\Gamma,\Rightarrow)]_{\underline{\eta}^R_{(\Gamma,\Rightarrow)}(u)}$$

  We also have

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(M) = {}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}(M) \in [{}_2R(\Gamma,u,t,\Rightarrow)]_{\underline{\eta}^R_{(\Gamma,\Rightarrow)}(u)\Rightarrow^R\underline{\eta}^R_{(\Gamma,\Rightarrow)}(v)}$$

  and

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(N) = {}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}(N) \in [{}_2R(\Gamma,u,t,\Rightarrow)]_{\underline{\eta}^R_{(\Gamma,\Rightarrow)}(u)}$$

  So we set

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(P) = \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{app}^{\mathsf{STLC}_T}_{u,v,\Gamma}(M,N)) := \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}({}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}(M), {}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}(N)) \tag{9.2}$$

- If $t = t_1 \Rightarrow t_2$ and $P = \mathrm{abs}^{\mathsf{STLC}_T}_{t_1,t_2,\Gamma}(M)$ with $M \in [\mathsf{STLC}_T([\Gamma,\langle t_1\rangle])]_{t_2}$, by definition of TLC we have

$$M \in [\mathsf{TLC}([\Gamma,\langle t_1\rangle],\Rightarrow)]_{t_2}$$

  where $[\Gamma,\langle t_1\rangle] : X+1 \to T$. By induction hypothesis

$$\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M) \in [R([\Gamma,\langle t_1\rangle],\Rightarrow)]_{\underline{\eta}^R_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(t_2)}$$

  and equally

$$\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M) = {}_2\overline{\varphi}_{([\Gamma,\langle t_1\rangle],t_1,t_2,\Rightarrow)}(M) \in [{}_2R([\Gamma,\langle t_1\rangle],t_1,t_2,\Rightarrow)]_{\underline{\eta}^R_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(t_2)}$$

  So we set

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(P) = \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}^{\mathsf{STLC}_T}_{t_1,t_2,\Gamma}(M)) := \mathrm{abs}^R_{(\Gamma,t_1,t_2,\Rightarrow)}({}_2\overline{\varphi}_{([\Gamma,t_1],t_1,t_2,\Rightarrow)}(M)) \tag{9.3}$$

We check naturality in $(\Gamma, \Rightarrow)$ of the above defined φ. Let $h : (\Gamma, \Rightarrow) \to (\Gamma', \Rightarrow')$ be a morphism of $\mathrm{TEns}^{\Rightarrow}$ where $\Gamma' : Y \to U$. We check that the following square commutes

$$\begin{tikzcd}
\mathsf{TLC}(\Gamma, \Rightarrow) \arrow[r, "\mathsf{TLC}\,h"] \arrow[d, "\varphi_{(\Gamma,\Rightarrow)}"'] & \mathsf{TLC}(\Gamma', \Rightarrow') \arrow[d, "\varphi_{(\Gamma',\Rightarrow')}"] \\
R(\Gamma, \Rightarrow) \arrow[r, "Rh"'] & R(\Gamma', \Rightarrow')
\end{tikzcd}$$

In the underlined part we have the following square

$$\begin{tikzcd}
\underline{\Gamma} \arrow[r, "\underline{h}"] \arrow[d, "\underline{\eta}_{\Gamma}"'] & \underline{\Gamma'} \arrow[d, "\underline{\eta}_{\Gamma'}"] \\
\underline{R}(\Gamma, \Rightarrow) \arrow[r, "\underline{Rh}"'] & \underline{R}(\Gamma', \Rightarrow')
\end{tikzcd}$$

which commutes by naturality of η^R. In the overlined part we have the following square

$$\begin{tikzcd}
\overline{\mathsf{STLC}_T}(\Gamma) \arrow[r, "\overline{\mathsf{TLC}}h"] \arrow[d, "\overline{\varphi}_{(\Gamma,\Rightarrow)}"'] & \overline{\mathsf{STLC}_U}(\Gamma') \arrow[d, "\overline{\varphi}_{(\Gamma',\Rightarrow')}"] \\
\overline{R}(\Gamma, \Rightarrow) \arrow[r, "\overline{Rh}"'] & \overline{R}(\Gamma', \Rightarrow')
\end{tikzcd}$$

We check the commutativity in the three possible cases of $P \in \overline{\mathsf{STLC}_T}(\Gamma)$.

- If $P = \mathrm{var}_{\Gamma}(x)$ with $x \in [\Gamma]_t$ where $t \in \underline{\Gamma}$, we have

$$\begin{aligned}
\overline{\varphi}_{(\Gamma',\Rightarrow')} \circ \overline{\mathsf{TLC}}h(P) &= \overline{\varphi}_{(\Gamma',\Rightarrow')}(\mathrm{var}_{(\Gamma',\Rightarrow')}(\overline{h}(x))) \\
&= \overline{\eta}^R_{(\Gamma',\Rightarrow')}(\overline{h}(x)) \\
&= \overline{Rh}(\overline{\eta}^R_{(\Gamma,\Rightarrow)}(x)) \\
&= \overline{Rh}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{var}_{\Gamma}(x))) \\
&= \overline{Rh}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(P))
\end{aligned}$$

- If $P = \mathrm{app}_{u,t,\Gamma}(M, N)$ where $M \in [\mathsf{STLC}_T(\Gamma)]_{u \Rightarrow t}$ and $N \in [\mathsf{STLC}_T(\Gamma)]_u$, by induction hypothesis we have $\overline{Rh} \circ \overline{\varphi}_{(\Gamma,\Rightarrow)}(M) = \overline{\varphi}_{(\Gamma',\Rightarrow')} \circ \overline{\mathsf{TLC}}h(M)$ and $\overline{R}h \circ \overline{\varphi}_{(\Gamma,\Rightarrow)}(N) = \overline{\varphi}_{(\Gamma',\Rightarrow')} \circ \overline{\mathsf{TLC}}h(N)$.

$$\begin{aligned}
\overline{\varphi}_{(\Gamma',\Rightarrow')} \circ \overline{\mathsf{TLC}}h(P) &= \overline{\varphi}_{(\Gamma',\Rightarrow')}(\overline{\mathsf{TLC}}h(\mathrm{app}_{\Gamma,t,u}(M, N))) \\
&= \overline{\varphi}_{(\Gamma',\Rightarrow')}(\mathrm{app}_{\underline{h}(u),\underline{h}(t),\Gamma'}(\overline{\mathsf{TLC}}h(M), \overline{\mathsf{TLC}}h(N))) \\
&= \mathrm{app}^R_{(\Gamma',\underline{h}(u),\underline{h}(t),\Rightarrow')}(\overline{\varphi}_{(\Gamma',\Rightarrow')}(\overline{\mathsf{TLC}}h(M), \overline{\mathsf{TLC}}h(N))) \\
&= \mathrm{app}^R_{(\Gamma',\underline{h}(u),\underline{h}(t),\Rightarrow')}(\overline{R}h(\overline{\varphi}_{(\Gamma,\Rightarrow)}(M)), \overline{R}h(\overline{\varphi}_{(\Gamma,\Rightarrow)}(N))) \\
&= \overline{R}h(\mathrm{app}^R_{\Gamma,u,t,\Rightarrow}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(M), \overline{\varphi}_{(\Gamma,\Rightarrow)}(N))) \\
&= \overline{R}h(\overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{app}_{\Gamma,u,t}(M, N))) \\
&= \overline{R}h \circ \overline{\varphi}_{(\Gamma,\Rightarrow)}(P)
\end{aligned}$$

- If $P = \mathrm{abs}_{u,t,\Gamma}(M)$ where $M \in [\mathsf{STLC}_T([\Gamma, \langle u \rangle])]_t$, by induction hypothesis we have $\overline{R}(h+1) \circ \overline{\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(M) = \overline{\varphi}_{([\Gamma',\langle \underline{h}(u)\rangle],\Rightarrow')} \circ \overline{\mathsf{TLC}}(h+1)(M)$.

$$\begin{aligned}
\overline{\varphi}_{(\Gamma',\Rightarrow')} \circ \overline{\mathsf{TLC}}h(P) &= \overline{\varphi}_{(\Gamma',\Rightarrow')}(\overline{\mathsf{TLC}}h(\mathrm{abs}_{u,t,\Gamma}(M))) \\
&= \overline{\varphi}_{(\Gamma',\Rightarrow')}(\mathrm{abs}_{\underline{h}(u),\underline{h}(v),\Gamma'}(\overline{\mathsf{TLC}}(h+1)(M))) \\
&= \mathrm{abs}^R_{(\Gamma'_{\underline{h}(u),\underline{h}(v)},\Rightarrow')}(\overline{\varphi}_{([\Gamma',\langle \underline{h}(u)\rangle],\Rightarrow')}(\overline{\mathsf{TLC}}(h+1)(M))) \\
&= \mathrm{abs}^R_{(\Gamma',\underline{h}(u),\underline{h}(t),\Rightarrow')}(\overline{R}(h+1)(\overline{\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(M))) \\
&= \overline{R}h(\mathrm{abs}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(M))) \\
&= \overline{R}h(\overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{u,t,\Gamma}(M))) \\
&= \overline{R}h \circ \overline{\varphi}_{(\Gamma,\Rightarrow)}(P)
\end{aligned}$$

$\varphi : \mathsf{TLC} \to R$ as morphism of monads

Next we check that φ is a morphism of monads $(\mathsf{TLC}, \mathrm{var}, \mathrm{subst}) \to (R, \eta^R, \mu^R)$. The diagram

$$\begin{tikzcd}
(\Gamma,\Rightarrow) \arrow[r, "\mathrm{var}_{(\Gamma,\Rightarrow)}"] \arrow[dr, "\eta^R_{(\Gamma,\Rightarrow)}"'] & \mathsf{TLC}(\Gamma,\Rightarrow) \arrow[d, "\varphi_{(\Gamma,\Rightarrow)}"] \\
& R(\Gamma,\Rightarrow)
\end{tikzcd}$$

commutes because we have in the overlined and underlined parts

$$\begin{tikzcd}
X \arrow[r, "\overline{\mathrm{var}}^{\mathsf{STLC}_T}_{\Gamma}"] \arrow[dr, "\overline{\eta}^R_{(\Gamma,\Rightarrow)}"'] & \overline{\mathsf{STLC}_T}(\Gamma) \arrow[d, "\overline{\varphi}_{(\Gamma,\Rightarrow)}"] \\
& \overline{R}(\Gamma,\Rightarrow)
\end{tikzcd}$$

commutes by definition of $\overline{\varphi}$ and

$$\begin{tikzcd}
T \arrow[r, "\mathrm{id}_T"] \arrow[dr, "\underline{\eta}^R_{(\Gamma,\Rightarrow)}"'] & T \arrow[d, "\underline{\eta}^R_{(\Gamma,\Rightarrow)}"] \\
& \underline{R}(\Gamma,\Rightarrow)
\end{tikzcd}$$

commutes by definition of $\underline{\varphi}$. The second monad morphism axiom is the following

$$\begin{tikzcd}
\mathsf{TLC}(\mathsf{TLC}(\Gamma,\Rightarrow)) \arrow[r, "\varphi_{\mathsf{TLC}(\Gamma,\Rightarrow)}"] \arrow[d, "\mathrm{subst}_{(\Gamma,\Rightarrow)}"'] & R(\mathsf{TLC}(\Gamma,\Rightarrow)) \arrow[r, "R\varphi_{(\Gamma,\Rightarrow)}"] & R(R(\Gamma,\Rightarrow)) \arrow[d, "\mu^R_{(\Gamma,\Rightarrow)}"] \\
\mathsf{TLC}(\Gamma,\Rightarrow) \arrow[rr, "\varphi_{(\Gamma,\Rightarrow)}"'] & & R(\Gamma,\Rightarrow)
\end{tikzcd}$$

We have in the underlined part

$$\begin{tikzcd}
(\underline{\Gamma},\Rightarrow) \arrow[r, "\underline{\eta}^R_{(\Gamma,\Rightarrow)}"] \arrow[d, "\mathrm{id}_{\underline{\Gamma}}"'] & \underline{R(\Gamma,\Rightarrow)} \arrow[r, "\underline{R\eta^R_{(\Gamma,\Rightarrow)}}"] & \underline{R(R(\Gamma,\Rightarrow))} \arrow[d, "\underline{\mu}^R_{(\Gamma,\Rightarrow)}"] \\
(\underline{\Gamma},\Rightarrow) \arrow[rr, "\underline{\eta}^R_{(\Gamma,\Rightarrow)}"'] & & \underline{R(\Gamma,\Rightarrow)}
\end{tikzcd}$$

which commutes by one of the monad axioms for R. In the overlined part we have

$$\begin{tikzcd}
\overline{(\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma)),\Rightarrow)} \arrow[r, "\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}"] \arrow[d, "\overline{\mathrm{subst}}_\Gamma"'] & \overline{R(\mathsf{STLC}_T(\Gamma),\Rightarrow)} \arrow[r, "\overline{R\varphi}_{(\Gamma,\Rightarrow)}"] & \overline{R(R(\Gamma,\Rightarrow))} \arrow[d, "\overline{\mu}^R_{(\Gamma,\Rightarrow)}"] \\
\overline{(\mathsf{STLC}_T(\Gamma),\Rightarrow)} \arrow[rr, "\overline{\varphi}_{(\Gamma,\Rightarrow)}"'] & & \overline{R(\Gamma,\Rightarrow)}
\end{tikzcd}$$

We check the commutativity by induction.

- If $\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(P) \in \overline{(\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma)),\Rightarrow)}$,
 more precisely $\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(P) \in [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_t$ for a $t \in \underline{\Gamma}$ with $P \in [\mathsf{STLC}_T(\Gamma)]_t$, we have three possible subcases.
  - If $P = \mathrm{var}_\Gamma(x)$ with $x \in [\Gamma]_t$, we set
  $$\begin{aligned}\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(\mathrm{var}_\Gamma(x)))) &= \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{var}_\Gamma(x))\\ &= \overline{\eta}^R_{(\Gamma,\Rightarrow)}(x)\end{aligned}$$
  and
  $$\begin{aligned}\overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(\mathrm{var}_\Gamma(x))))) &=\\ &= \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\eta}^R_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{var}_\Gamma(x))))\\ &= \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{\eta}^R_{R(\Gamma,\Rightarrow)}(\overline{\eta}^R_{(\Gamma,\Rightarrow)}(x)))\\ &= \overline{\eta}^R_{(\Gamma,\Rightarrow)}(x)\end{aligned}$$
  - If $P = \mathrm{app}_{u,t,\Gamma}(M,N)$ with $M \in [\mathsf{STLC}_T(\Gamma)]_{u\Rightarrow t}$ and $N \in [\mathsf{STLC}_T(\Gamma)]_u$, we have
  $$\begin{aligned}\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(\mathrm{app}_{u,t,\Gamma}(M,N)))) &= \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{app}_{u,t,\Gamma}(M,N))\\ &= \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(M),\overline{\varphi}_{(\Gamma,\Rightarrow)}(N))\end{aligned}$$
  and
  $$\begin{aligned}\overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(\mathrm{app}_{u,t,\Gamma}(M,N))))) &=\\ &= \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\eta}^R_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{app}_{u,t,\Gamma}(M,N))))\\ &= \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{\eta}^R_{R(\Gamma,\Rightarrow)}(\mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(M),\overline{\varphi}_{(\Gamma,\Rightarrow)}(N))))\\ &= \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(M),\overline{\varphi}_{(\Gamma,\Rightarrow)}(N))\end{aligned}$$
  - If $t = t_1 \Rightarrow t_2$ and $P = \mathrm{abs}_{t_1,t_2,\Gamma}(M)$ with $M \in [\mathsf{STLC}_T([\Gamma,\langle t_1\rangle])]_{t_2}$, then we have
  $$\begin{aligned}\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(\mathrm{abs}_{t_1,t_2,\Gamma}(M)))) &= \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{t_1,t_2,\Gamma}(M))\\ &= \mathrm{abs}^R_{(\Gamma,t_1,t_2,\Rightarrow)}(\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))\end{aligned}$$
  and
  $$\begin{aligned}\overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{var}_{\mathsf{STLC}_T(\Gamma)}(\mathrm{abs}_{t_1,t_2,\Gamma}(M))))) &=\\ &= \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\eta}^R_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{abs}_{t_1,t_2,\Gamma}(M))))\\ &= \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{\eta}^R_{R(\Gamma,\Rightarrow)}(\mathrm{abs}^R_{(\Gamma_{t_1,t_2},\Rightarrow)}(\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))))\\ &= \mathrm{abs}^R_{(\Gamma_{t_1,t_2},\Rightarrow)}(\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))\end{aligned}$$

- If $\mathrm{app}_{u,t,\mathsf{STLC}_T(\Gamma)}(M,N) \in \overline{(\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma)),\Rightarrow)}$, more precisely $\mathrm{app}_{u,t,\mathsf{STLC}_T(\Gamma)}(M,N) \in [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_t$ where $u,t \in \underline{\Gamma}$ with $M \in [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_{u\Rightarrow t}$ and $N \in [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_u$, by induction hypothesis we have

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(M)) = \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(M)))$$

and

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(N)) = \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(N)))$$

We find

$$\begin{aligned}
&\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(\mathrm{app}_{u,t,\mathsf{STLC}_T(\Gamma)}(M,N))) = \\
&\qquad = \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{app}_{u,t,\Gamma}(\overline{\mathrm{subst}}_\Gamma(M),\overline{\mathrm{subst}}_\Gamma(N))) \\
&\qquad = \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(M)),\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(N))) \\
&\qquad = \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(M))), \\
&\qquad\qquad \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(N))))
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{app}_{u,t,\mathsf{STLC}_T(\Gamma)}(M,N)))) = \\
&\quad = \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{app}^R_{(\mathsf{STLC}_T(\Gamma),u,t,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(M),\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(N)))) \\
&\quad = \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\mathrm{app}^R_{R(\Gamma,u,t,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(M)),\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(N)))) \\
&\quad = \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(M))), \\
&\qquad\qquad \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(N))))
\end{aligned}$$

- If $\mathrm{abs}_{u,t,\mathsf{STLC}_T(\Gamma)}(M,N) \in \overline{(\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma)),\Rightarrow)}$, more precisely $\mathrm{abs}_{u,t,\mathsf{STLC}_T(\Gamma)}(M) \in [\mathsf{STLC}_T(\mathsf{STLC}_T(\Gamma))]_{u\Rightarrow t}$ where $u,t \in \underline{\Gamma}$ with $M \in [\mathsf{STLC}_T([\mathsf{STLC}_T(\Gamma),\langle u\rangle])]_t$, by induction hypothesis we have

$$\begin{aligned}
&\overline{\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(\overline{\mathrm{subst}}_{[\Gamma,\langle u\rangle]}(\sigma^{\mathsf{STLC}_T}_{\Gamma,u}(M))) \\
&\qquad\qquad = \overline{\mu}^R_{([\Gamma,\langle u\rangle],\Rightarrow)}(\overline{R\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T([\Gamma,\langle u\rangle]),\Rightarrow)}(\sigma^{\mathsf{STLC}_T}_{\Gamma,u}(M))))
\end{aligned}$$

So we find

$$\begin{aligned}
&\overline{\varphi}_{(\Gamma,\Rightarrow)}(\overline{\mathrm{subst}}_\Gamma(\mathrm{abs}_{u,t,\mathsf{STLC}_T(\Gamma)}(M))) = \\
&\qquad = \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{u,t,\Gamma}(\overline{\mathrm{subst}}_{[\Gamma,\langle u\rangle]}(\sigma^{\mathsf{STLC}_T}_{\Gamma,u}(M)))) \\
&\qquad = \mathrm{abs}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(\overline{\mathrm{subst}}_{[\Gamma,\langle u\rangle]}(\sigma^{\mathsf{STLC}_T}_{\Gamma,u}(M)))) \\
&\qquad = \mathrm{abs}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\mu}^R_{([\Gamma,\langle u\rangle],\Rightarrow)}(\overline{R\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T([\Gamma,\langle u\rangle]),\Rightarrow)}(\sigma^{\mathsf{STLC}_T}_{\Gamma,u}(M)))))
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T(\Gamma),\Rightarrow)}(\mathrm{abs}_{u,t,\mathsf{STLC}_T(\Gamma)}(M)))) = \\
&\qquad = \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\overline{R\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}^R_{(\mathsf{STLC}_T(\Gamma),u,t,\Rightarrow)}(\overline{\varphi}_{([\mathsf{STLC}_T(\Gamma),\langle u\rangle],\Rightarrow)}(M)))) \\
&\qquad = \overline{\mu}^R_{(\Gamma,\Rightarrow)}(\mathrm{abs}^R_{R(\Gamma,u,t,\Rightarrow)}(\overline{R(\varphi_{(\Gamma,\Rightarrow)}+1)}(\overline{\varphi}_{([\mathsf{STLC}_T(\Gamma),\langle u\rangle],\Rightarrow)}(M)))) \\
&\qquad = \mathrm{abs}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\mu}^R_{([\Gamma,\langle u\rangle],\Rightarrow)}(\sigma^R_{\Gamma,u}(\overline{R(\varphi_{(\Gamma,\Rightarrow)}+1)}(\overline{\varphi}_{([\mathsf{STLC}_T(\Gamma),\langle u\rangle],\Rightarrow)}(M)))))
\end{aligned}$$

It remains to check

$\sigma^R_{\Gamma,u}(\overline{R(\varphi_{(\Gamma,\Rightarrow)}+1)}(\overline{\varphi}_{([\mathsf{STLC}_T(\Gamma),\langle u\rangle],\Rightarrow)}(M))) = \overline{R\varphi}_{([\Gamma,\langle u\rangle],\Rightarrow)}(\overline{\varphi}_{(\mathsf{STLC}_T([\Gamma,\langle u\rangle]),\Rightarrow)}(\sigma^{\mathsf{STLC}_T}_{\Gamma,u}(M)))$

This is true since the following squares commute

$$\begin{tikzcd}
\mathsf{STLC}_T([\mathsf{STLC}_T,\langle u\rangle]) \arrow[r, "\mathsf{STLC}_T(\sigma^{\mathsf{STLC}_T}_{\Gamma,u})"] \arrow[d, "\varphi_{[\mathsf{STLC}_T(\Gamma),\langle u\rangle]}"'] & \mathsf{STLC}_T(\mathsf{STLC}_T([\Gamma,\langle u\rangle])) \arrow[d, "\varphi_{\mathsf{STLC}_T([\Gamma,\langle u\rangle])}"] \\
R([\mathsf{STLC}_T,\langle u\rangle]) \arrow[r, "R(\sigma^{\mathsf{STLC}_T}_{\Gamma,u})"] \arrow[d, "R(\varphi_\Gamma+1)"'] & R(\mathsf{STLC}_T([\Gamma,\langle u\rangle])) \arrow[d, "R(\varphi_{[\Gamma,\langle u\rangle]})"] \\
R([R\Gamma,\langle \underline{\eta}^R_1(u)\rangle]) \arrow[r, "R(\sigma^R_{\Gamma,u})"'] & R(R([\Gamma,\langle u\rangle]))
\end{tikzcd}$$

The top square is a naturality square of φ and the bottom one is a naturality square of σ.

$\varphi : \mathsf{TLC} \to R$ as morphism of representations

We check the compatibility of φ with app and abs. By formula (9.2) we have

$$\begin{tikzcd}
[_2\mathsf{TLC}(\Gamma,u,t,\Rightarrow)]_{u\Rightarrow t} \times [_2\mathsf{TLC}(\Gamma,u,t,\Rightarrow)]_u \arrow[r, "\mathrm{app}^{\mathsf{TLC}}_{(\Gamma,u,t,\Rightarrow)}"] \arrow[d, "{}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}\times {}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}"'] & [_2\mathsf{TLC}(\Gamma,u,t,\Rightarrow)]_t \arrow[d, "{}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}"] \\
[_2R(\Gamma,u,t,\Rightarrow)]_{\underline{\eta}^R_\Gamma(u\Rightarrow t)} \times [_2R(\Gamma,u,t,\Rightarrow)]_{\underline{\eta}^R_\Gamma(u)} \arrow[r, "\mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}"'] & [_2R(\Gamma,u,t,\Rightarrow)]_{\underline{\eta}^R_\Gamma(t)}
\end{tikzcd}$$

since $[_2\mathsf{TLC}(\Gamma,u,t,\Rightarrow)]_{u\Rightarrow t} = [\mathsf{STLC}_T(\Gamma)]_{u\Rightarrow t}$, $[_2\mathsf{TLC}(\Gamma,u,t,\Rightarrow)]_u = [\mathsf{STLC}_T(\Gamma)]_u$, $[_2\mathsf{TLC}(\Gamma,u,t,\Rightarrow)]_t = [\mathsf{STLC}_T(\Gamma)]_t$, $\mathrm{app}^{\mathsf{TLC}}_{(\Gamma,u,t,\Rightarrow)} = \mathrm{app}^{\mathsf{STLC}_T}_{u,t,\Gamma}$ and ${}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)} = \overline{\varphi}_{(\Gamma,\Rightarrow)}$.

By formula (9.3) we have

$$\begin{tikzcd}
[_2\mathsf{TLC}([\Gamma,\langle u\rangle],u,t,\Rightarrow)]_t \arrow[r, "\mathrm{abs}^{\mathsf{TLC}}_{(\Gamma,u,t,\Rightarrow)}"] \arrow[d, "{}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}"'] & [_2\mathsf{TLC}(\Gamma,u,t,\Rightarrow)]_{u\Rightarrow t} \arrow[d, "{}_2\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}"] \\
[_2R([\Gamma,\langle u\rangle],u,t,\Rightarrow)]_{\underline{\eta}^R_\Gamma(t)} \arrow[r, "\mathrm{abs}^R_{(\Gamma,u,t,\Rightarrow)}"'] & [_2R(\Gamma,u,t,\Rightarrow)]_{\underline{\eta}^R_\Gamma(u\Rightarrow t)}
\end{tikzcd}$$

Uniqueness of the previously defined $\varphi : \mathsf{TLC} \to R$

Let ψ be a morphism of representations $((\mathsf{TLC},\mathrm{var},\mathrm{subst}),\mathrm{app},\mathrm{abs}) \to ((R,\eta^R,\mu^R),\mathrm{app}^R,\mathrm{abs}^R)$. Let $(\Gamma,\Rightarrow) \in \mathrm{TEns}^{\Rightarrow}$ where $\Gamma : X \to T$, $t \in \underline{\mathsf{TLC}}(\Gamma,\Rightarrow)$ and $P \in \overline{\mathsf{TLC}}(\Gamma,\Rightarrow)$. We are going to show

$$\underline{\varphi}_{(\Gamma,\Rightarrow)}(t) = \underline{\psi}_{(\Gamma,\Rightarrow)}(t) \qquad \text{and} \qquad \overline{\varphi}_{(\Gamma,\Rightarrow)}(P) = \overline{\psi}_{(\Gamma,\Rightarrow)}(P)$$

By definition $\underline{\varphi}_{(\Gamma,\Rightarrow)}(t) = \underline{\eta}^R_{(\Gamma,\Rightarrow)}(t)$. The morphism of representations ψ is in particular morphism of monads $(\mathsf{TLC},\mathrm{var},\mathrm{subst}) \to (R,\eta^R,\mu^R)$ thus compatible with the units var and η^R.

$$\begin{aligned}
\underline{\psi}_{(\Gamma,\Rightarrow)}(t) &= \underline{\psi}_{(\Gamma,\Rightarrow)} \circ \mathrm{id}_T(t) \\
&= \underline{\psi}_{(\Gamma,\Rightarrow)} \circ \underline{\mathrm{var}}_{(\Gamma,\Rightarrow)}(t) \\
&= \underline{\eta}^R_{(\Gamma,\Rightarrow)}(t)
\end{aligned}$$

Let $P \in [\mathsf{TLC}(\Gamma, \Rightarrow)]_t = [\mathsf{STLC}_T(\Gamma)]_t$ for a $t \in T$. We show $\overline{\varphi}_{(\Gamma,\Rightarrow)}(P) = \overline{\psi}_{(\Gamma,\Rightarrow)}(P)$ by induction.

- If $P = \mathrm{var}_\Gamma(x)$ where $x \in [\Gamma]_t$, by definition of φ we have

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{var}_\Gamma(x)) = \overline{\eta}^R_{(\Gamma,\Rightarrow)}(x)$$

by compatibility of ψ with the units η^R and var

$$\overline{\psi}_{(\Gamma,\Rightarrow)}(\mathrm{var}_\Gamma(x)) = \overline{\psi}_{(\Gamma,\Rightarrow)} \circ \overline{\mathrm{var}}_{(\Gamma,\Rightarrow)}(x) = \underline{\eta}^R_{(\Gamma,\Rightarrow)}(t)$$

- If $P = \mathrm{app}_{(\Gamma,u,t,\Rightarrow)}(M,N) = \mathrm{app}_{u,t,\Gamma}(M,N)$ with $M \in [\mathsf{STLC}_T(\Gamma)]_{u \Rightarrow t}$ and $N \in [\mathsf{STLC}_T(\Gamma)]_u$, by induction hypothesis we have $\overline{\varphi}_{(\Gamma,\Rightarrow)}(M) = \overline{\psi}_{(\Gamma,\Rightarrow)}(M)$ and $\overline{\varphi}_{(\Gamma,\Rightarrow)}(N) = \overline{\psi}_{(\Gamma,\Rightarrow)}(N)$. By definition of φ

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{app}_{(\Gamma,u,t,\Rightarrow)}(M,N)) = \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{(\Gamma,\Rightarrow)}(M), \overline{\varphi}_{(\Gamma,\Rightarrow)}(N))$$

Since ψ is a morphism of representations and compatible with app and app^R, we have

$$\begin{aligned}\overline{\psi}_{(\Gamma,\Rightarrow)}(\mathrm{app}_{(\Gamma,u,t,\Rightarrow)}(M,N)) &= \overline{{}_2\psi}_{(\Gamma,u,t,\Rightarrow)}(\mathrm{app}_{(\Gamma,u,t,\Rightarrow)}(M,N))\\ &= \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{{}_2\psi}_{(\Gamma,u,t,\Rightarrow)}(M), \overline{{}_2\psi}_{(\Gamma,u,t,\Rightarrow)}(N))\\ &= \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\psi}_{(\Gamma,u,t,\Rightarrow)}(M), \overline{\psi}_{(\Gamma,u,t,\Rightarrow)}(N))\end{aligned}$$

Applying the induction hypothesis

$$\begin{aligned}\overline{\psi}_{(\Gamma,\Rightarrow)}(\mathrm{app}_{(\Gamma,u,t,\Rightarrow)}(M,N)) &= \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\psi}_{(\Gamma,u,t,\Rightarrow)}(M), \overline{\psi}_{(\Gamma,u,t,\Rightarrow)}(N))\\ &= \mathrm{app}^R_{(\Gamma,u,t,\Rightarrow)}(\overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}(M), \overline{\varphi}_{(\Gamma,u,t,\Rightarrow)}(N))\\ &= \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{app}_{(\Gamma,u,t,\Rightarrow)}(M,N))\end{aligned}$$

- If $t = t_1 \Rightarrow t_2$ and $P = \mathrm{abs}_{(\Gamma,t_1,t_2,\Rightarrow)}(M) = \mathrm{abs}_{t_1,t_2,\Gamma}(M)$ where $M \in [\mathsf{STLC}_T([\Gamma, \langle t_1 \rangle])]_{t_2}$, we have the following induction hypothesis $\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M) = \overline{\psi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M)$. By definition of φ

$$\overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{(\Gamma,t_1,t_2,\Rightarrow)}(M)) = \mathrm{abs}^R_{(\Gamma,t_1,t_2,\Rightarrow)}(\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))$$

Since ψ is a morphism of representations and thus compatible with abs and abs^R

$$\begin{aligned}\overline{\psi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{(\Gamma,t_1,t_2,\Rightarrow)}(M)) &= \overline{{}_2\psi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{(\Gamma,t_1,t_2,\Rightarrow)}(M))\\ &= \mathrm{abs}^R_{(\Gamma,t_1,t_2,\Rightarrow)}(\overline{{}_2\psi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))\\ &= \mathrm{abs}^R_{(\Gamma,t_1,t_2,\Rightarrow)}(\overline{\psi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))\end{aligned}$$

Applying the induction hypothesis

$$\begin{aligned}\overline{\psi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{(\Gamma,t_1,t_2,\Rightarrow)}(M)) &= \mathrm{abs}^R_{(\Gamma,t_1,t_2,\Rightarrow)}(\overline{\psi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))\\ &= \mathrm{abs}^R_{(\Gamma,t_1,t_2,\Rightarrow)}(\overline{\varphi}_{([\Gamma,\langle t_1\rangle],\Rightarrow)}(M))\\ &= \overline{\varphi}_{(\Gamma,\Rightarrow)}(\mathrm{abs}_{(\Gamma,t_1,t_2,\Rightarrow)}(M))\end{aligned}$$

$\square$

Appendix A

Proofs of chapter 3

A.1 Proof of proposition 3.1.14

We have to verify the monad axioms with our definitions of μ and η. To verify that the triangle

$$\begin{tikzcd}
TX \arrow[r, "T\eta_X"] \arrow[dr, "\mathrm{id}_{TX}"'] & TTX \arrow[d, "\mu_X"] \\
 & TX
\end{tikzcd} \tag{A.1}$$

commutes for all $X \in \mathcal{C}$, we have to provide TTX with a G_X–algebra structure:

$$X + GTTX \longrightarrow X + GTX + GTTX \xrightarrow{\cong} TX + GTTX \xrightarrow{\cong} TTX$$

and check that μ_X is a G_X–algebra morphism

$$\begin{tikzcd}
X + GTTX \arrow[rr, "\mathrm{id}_X + G\mu_X"] \arrow[d] & & X + GTX \arrow[dl, "\eta_X + \mathrm{id}_{GTX}"'] \arrow[ddd, "{[\eta_X,\sigma_X]}"] \\
X + GTX + GTTX \arrow[d, "{[\eta_X,\sigma_X] + \mathrm{id}_{GTTX}}"'] \arrow[r, phantom, "\mathrm{I.}"] & & \\
TX + GTTX \arrow[r, "\mathrm{id}_{TX} + G\mu_X"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\mathrm{II.}"] & TX + GTX \arrow[ddr, "{[\mathrm{id}_{TX},\sigma_X]}"] \arrow[ur, phantom, "\mathrm{III.}"] & \\
TTX \arrow[rr, "\mu_X"'] & & TX
\end{tikzcd}$$

Square II. commutes because μ_X is a G_X–algebra morphism by definition. Triangle III. commutes obviously. Square I. commutes because the following two squares commute

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[dd, "\eta_X"] \\
X \arrow[d, "\eta_X"'] & \\
TX \arrow[r, "\mathrm{id}_{TX}"'] & TX
\end{tikzcd}
\qquad
\begin{tikzcd}
GTTX \arrow[r, "G\mu_X"] \arrow[d, "\mathrm{id}_{GTTX}"'] & GTX \arrow[dd, "\mathrm{id}_{GTX}"] \\
GTTX \arrow[d, "\mathrm{id}_{GTTX}"'] & \\
GTTX \arrow[r, "G\mu_X"'] & GTX
\end{tikzcd}$$

By definition $T\eta_X$ is the unique G_X–algebra morphism $TX \to TTX$ and consequently the composition $\mu_X \circ T\eta_X : TX \to TX$ is a morphism of G_X–algebras. So by initiality of TX, we have $\mathrm{id}_{TX} = \mu_X \circ T\eta_X$, which shows (A.1).

In a similar way we show that the square

$$\begin{tikzcd} T^3X \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"'] & TX \end{tikzcd} \tag{A.2}$$

commutes for all $X \in \mathcal{C}$. We provide TTX and TX with a structure of G_{TTX}–algebras

$$TTX + GTTX \xrightarrow{[\mathrm{id}_{TTX}, \sigma_{TX}]} TTX$$

and

$$TTX + GTX \xrightarrow{[\mu_X, \sigma_X]} TX$$

and we check that μ_X is a morphism of G_{TTX}–algebras

$$\begin{tikzcd} TTX + FTTX \arrow[r, "\mathrm{id}_{TTX} + G\mu_X"] \arrow[d, "{[\mathrm{id}, \sigma_{TX}]}"'] & TTX + GTX \arrow[d, "{[\mu_X, \sigma_X]}"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

commutes for all $X \in \mathcal{C}$ because the following two diagrams commute

$$\begin{tikzcd} TTX \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd} \qquad \begin{tikzcd} GTTX \arrow[r, "G\mu_X"] \arrow[d, "\sigma_{TX}"'] & GTX \arrow[d, "\sigma_X"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

where the second square commutes because μ_X is by definition a morphism of G_{TX}–algebras. Consequently we have two morphisms of G_{TTX}–algebras : $\mu_X \circ T\mu_X : T^3X \to TX$ and $\mu_X \circ \mu_{TX} : T^3X \to TX$, by initiality of T^3X we have $\mu_X \circ T\mu_X = \mu_X \circ \mu_{TX}$, which shows (A.2).

The third monad axiom

$$\begin{tikzcd} TTX \arrow[d, "\mu_X"'] & TX \arrow[l, "\eta_{TX}"'] \arrow[dl, "\mathrm{id}_{TX}"] \\ TX & \end{tikzcd}$$

for all $X \in \mathcal{C}$, is a direct consequence of the definition of μ_X.

A.2 Proof of proposition 3.1.16

A.2.1 Functoriality of Φ

At first let us define Φ on arrows. Let $\rho : P \to Q$ be an arrow in $\mathrm{End}_{\mathcal{C}}$. We denote the corresponding monads $S := \Phi(P)$ and $T := \Phi(Q)$. We want to define an arrow $\Phi(\rho) : S \to T$ in $\mathrm{Mon}_{\mathcal{C}}$, that is a monad morphism $\Phi(\rho) : S \to T$. Let X be an object of $\mathcal{C}$. We define $\Phi(\rho)$ componentwise by initiality of SX, which is the initial P_X–algebra. We provide TX with a P_X–algebra structure and by initiality of SX we take the unique P_X–algebra morphism for $\Phi(\rho)_X$. Indeed we have $X + PTX \xrightarrow{\mathrm{id} + \rho_{TX}} X + QTX \xrightarrow{\cong} TX$.

Next we check that $\Phi(\rho)$ is natural in X. Let $f : X \to Y$ be a morphism of $\mathrm{End}_{\mathcal{C}}$ then we need to check that

$$\begin{tikzcd} SX \arrow[r, "\Phi(\rho)_X"] \arrow[d, "Sf"'] & TX \arrow[d, "Tf"] \\ SY \arrow[r, "\Phi(\rho)_Y"'] & TY \end{tikzcd}$$

commutes. We are going to provide TY with a P_X–algebra structure and check that Tf and $\Phi(\rho)_Y$ are morphisms of P_X–algebras. By definition Sf and $\Phi(\rho)_X$ are morphisms of P_X–algebras. Then by initiality of SX, we conclude that the two composites $\Phi(\rho)Y \circ Sf$ and $Tf \circ \Phi(\rho)_X$ are equal.

- TY is a P_X–algebra: $X + PTY \xrightarrow{f+\rho_{TY}} Y + QTY \xrightarrow{\cong} TY$
- $\Phi(\rho)_Y$ is a morphism of P_X–algebras:

$$\begin{tikzcd} X + PSY \arrow[r, "\mathrm{id} + P\Phi(\rho)_Y"] \arrow[d, "f+\mathrm{id}"'] & X + PTY \arrow[d, "f+\mathrm{id}"] \\ Y + PSY \arrow[r, "\mathrm{id} + P\Phi(\rho)_Y"] \arrow[ddd, "\cong"'] & Y + PTY \arrow[d, "\mathrm{id} + \rho_{TY}"] \\ & Y + QTY \arrow[dd, "\cong"] \\ & \\ SY \arrow[r, "\Phi(\rho)_Y"'] & TY \end{tikzcd}$$

  The bottom square commutes by definition of $\Phi(\rho)_Y$ being the unique morphism of P_Y–algebras. The top square commutes evidently.
- Tf is a morphism of P_X–algebras:

$$\begin{tikzcd} X + PTX \arrow[r, "\mathrm{id} + PTf"] \arrow[d, "\mathrm{id} + \rho_{TX}"'] & X + PTY \arrow[d, "\mathrm{id} + \rho_{TY}"] \\ X + QTX \arrow[r, "\mathrm{id} + QTf"] \arrow[ddd, "\cong"'] & X + QTY \arrow[d, "f+\mathrm{id}"] \\ & Y + QTY \arrow[dd, "\cong"] \\ & \\ TX \arrow[r, "Tf"'] & TY \end{tikzcd}$$

  The bottom square commutes by definition of Tf being the unique morphism of Q_X–algebras $TX \to TY$. The top square commutes by naturality of ρ.

Next let us check that $\Phi(\rho)$ is a monad morphism $S \to T$. By its definition at the component X the following square commutes:

$$\begin{tikzcd} X + PSX \arrow[r, "\mathrm{id} + P\Phi(\rho)_X"] \arrow[dd, "{[\eta^S_X, \sigma^S_X]}"'] & X + PTX \arrow[d, "\mathrm{id} + \rho_{TX}"] \\ & X + QTX \arrow[d, "{[\eta^T_X, \sigma^T_X]}"] \\ SX \arrow[r, "\Phi(\rho)_X"'] & TX \end{tikzcd}$$

this implies that the following triangle commutes

$$\begin{tikzcd} X \arrow[d, "\eta^S_X"'] \arrow[dr, "\eta^T_X"] & \\ SX \arrow[r, "\Phi(\rho)_X"'] & TX \end{tikzcd}$$

which proves the compatibility of $\Phi(\rho)$ with η. In order to check the compatibility of $\Phi(\rho)$ with μ we have to check that the following diagram commutes for all object X of $\mathcal{C}$

$$\begin{tikzcd} SSX \arrow[r, "\mu^S_X"] \arrow[d, "S(\Phi(\rho)_X)"'] & SX \arrow[dd, "\Phi(\rho)_X"] \\ STX \arrow[d, "\Phi(\rho)_{TX}"'] & \\ TTX \arrow[r, "\mu^T_X"'] & TX \end{tikzcd}$$

We are going to provide STX, TTX, TX with a structure of P_{SX}–algebras and check that $\Phi(\rho)_X$, $\Phi(\rho)_{TX}$ and μ^T_X are morphisms of P_{SX}–algebras. Then by initiality of SSX we can conclude that $\Phi(\rho)_X \circ \mu^S_X$ and $\mu^T_X \circ T(\Phi(\rho)_X) \circ \Phi(\rho)_{SX}$ are equal.

- STX is a P_{SX}–algebra:
  $SX + PSTX \xrightarrow{\Phi(\rho)_X + \mathrm{id}} TX + PSTX \xrightarrow{[\eta^S_{TX}, \sigma^S_{TX}]} STX$
- TTX is a P_{SX}–algebra:
  $SX + PTTX \xrightarrow{[\Phi(\rho)_X, \rho_{TTX}]} TX + QTTX \xrightarrow{[\eta^T_X, \sigma^T_X]} TTX$
- TX is a P_{SX}–algebra:
  $SX + PTX \xrightarrow{[\Phi(\rho)_X, \rho_{TX}]} TX + QTX \xrightarrow{[\mathrm{id}, \sigma^T_X]} TX$
- $\Phi(\rho)_X$ is a morphism of P_{SX}–algebras:

$$\begin{tikzcd} SX + PSX \arrow[r, "\mathrm{id} + P\Phi(\rho)_X"] \arrow[dd, "{[\mathrm{id}, \sigma^S_X]}"'] & SX + PTX \arrow[d, "\Phi(\rho)_X + \rho_{TX}"] \\ & TX + QTX \arrow[d, "{[\mathrm{id}, \sigma^T_X]}"] \\ SX \arrow[r, "\Phi(\rho)_X"'] & TX \end{tikzcd}$$

  commutes because the following diagram commutes by definition of $\Phi(\rho)_X$

$$\begin{tikzcd} PSX \arrow[r, "P\Phi(\rho)_X"] \arrow[dd, "\sigma^S_X"'] & PTX \arrow[d, "\rho_{TX}"] \\ & QTX \arrow[d, "\sigma^T_X"] \\ SX \arrow[r, "\Phi(\rho)_X"'] & TX \end{tikzcd}$$

- $\Phi(\rho)_{TX}$ is a morphism of P_{SX}–algebras:

$$\begin{tikzcd}
SX + PSTX \arrow[r, "\mathrm{id} + P\Phi(\rho)_{TX}"] \arrow[d, "\Phi(\rho)_X + \mathrm{id}"'] & SX + PTTX \arrow[d, "\Phi(\rho)_X + \mathrm{id}"] \\
TX + PSTX \arrow[r, "\mathrm{id} + P\Phi(\rho)_{TX}"] \arrow[dd, "{[\eta^S_{TX}, \sigma^S_{TX}]}"'] & TX + PTTX \arrow[d, "\mathrm{id} + \rho_{TTX}"] \\
 & TX + QTTX \arrow[d, "{[\eta^T_{TX}, \sigma^T_{TX}]}"] \\
STX \arrow[r, "\Phi(\rho)_{TX}"'] & TTX
\end{tikzcd}$$

The top square commutes obviously and the bottom square commutes by definition of $\Phi(\rho)_{TX}$ being a morphism of P_{TX}–algebras.

- μ^T_X is a morphism of P_{SX}–algebras:

$$\begin{tikzcd}
SX + PTTX \arrow[r, "\mathrm{id} + P\mu^T_X"] \arrow[d, "\Phi(\rho)_X + \mathrm{id}"'] & SX + PTX \arrow[d, "\Phi(\rho)_X + \mathrm{id}"] \\
TX + PTTX \arrow[r, "\mathrm{id} + P\mu^T_X"] \arrow[d, "\mathrm{id} + \rho_{TTX}"'] & TX + PTX \arrow[d, "\mathrm{id} + \rho_{TX}"] \\
TX + QTTX \arrow[r, "\mathrm{id} + Q\mu^T_X"] \arrow[d, "{[\eta^T_{TX}, \sigma^T_{TX}]}"'] & TX + QTX \arrow[d, "{[\mathrm{id}, \sigma^T_X]}"] \\
TTX \arrow[r, "\mu^T_X"'] & TX
\end{tikzcd}$$

The top square commutes obviously, the middle square is a naturality square of ρ and the bottom square commutes by definition of μ^T_X being a morphism of Q_{TX}–algebras.

We define now the counit and the unit of the adjunction. Let (S, η^S, μ^S) be a monad. We denote (T, η^T, μ^T) the monad $\Phi(S)$.

A.2.2 Counit

At first we define the monad morphism $\varepsilon_{(S,\eta^S,\mu^S)} : (T, \eta^T, \mu^T) \to (S, \eta^S, \mu^S)$ componentwise. Let X be an object of $\mathcal{C}$. We provide SX with a S_X–algebra structure and by initiality of TX we take the unique S_X–algebra morphism to be $\varepsilon_{(S,\eta^S,\mu^S),X}$. Indeed we have $X + SSX \xrightarrow{[\eta^S_X, \mu^S_X]} SX$.

Now we check the naturality of $\varepsilon_{(S,\eta^S,\mu^S),X}$ in X. Let $f : X \to Y$ be a morphism of $\mathcal{C}$. We have to check the commutativity of the following square.

$$\begin{tikzcd}
TX \arrow[r, "\varepsilon_{(S,\eta^S,\mu^S),X}"] \arrow[d, "Tf"'] & SX \arrow[d, "Sf"] \\
TY \arrow[r, "\varepsilon_{(S,\eta^S,\mu^S),Y}"'] & SY
\end{tikzcd}$$

We provide SY with a S_X–algebra structure and check that Sf and $\varepsilon_{(S,\eta^S,\mu^S),Y}$ are morphisms of S_X–algebras. By definition $\varepsilon_{(S,\eta^S,\mu^S),X}$ and Tf are morphisms of S_X–algebras, then by initiality of TX, we can conclude that $Sf \circ \varepsilon_{(S,\eta^S,\mu^S),X}$ and $\varepsilon_{(S,\eta^S,\mu^S),Y} \circ Tf$ are equal.

- SY is a S_X–algebra: $X + SSY \xrightarrow{f+\mathrm{id}} Y + SSY \xrightarrow{[\eta^S_Y,\mu^S_Y]} SY$
- Sf is a morphism of S_X–algebras:

$$\begin{array}{ccc} X + SSX & \xrightarrow{\mathrm{id}+SSf} & X + SSY \\ & & \downarrow{\scriptstyle f+\mathrm{id}} \\ \downarrow{\scriptstyle [\eta^S_X,\mu^S_X]} & & Y + SSY \\ & & \downarrow{\scriptstyle [\eta^S_Y,\mu^S_Y]} \\ SX & \xrightarrow[Sf]{} & SY \end{array}$$

  commutes by naturality of η^S and μ^S.

- $\varepsilon_{(S,\eta^S,\mu^S),Y}$ is a morphism of S_X–algebras:

$$\begin{array}{ccc} X + STY & \xrightarrow{\mathrm{id}+S\varepsilon_{(S,\eta^S,\mu^S),Y}} & X + SSY \\ \downarrow{\scriptstyle f+\mathrm{id}} & & \downarrow{\scriptstyle f+\mathrm{id}} \\ Y + STY & \xrightarrow{\mathrm{id}+S\varepsilon_{(S,\eta^S,\mu^S),Y}} & Y + SSY \\ \downarrow{\scriptstyle [\eta^T_Y,\sigma^T_Y]} & & \downarrow{\scriptstyle [\eta^S_Y,\mu^S_Y]} \\ TY & \xrightarrow[\varepsilon_{(S,\eta^S,\mu^S),Y}]{} & SY \end{array}$$

Next we check the naturality of $\varepsilon_{(S,\eta^S,\mu^S)}$ in (S,η^S,μ^S). Let $\varphi : (S_1,\eta^{S_1},\mu^{S_1}) \to (S_2,\eta^{S_2},\mu^{S_2})$ be a monad morphism. We denote the corresponding monads $(T_1,\eta^{T_1},\mu^{T_1}) := \Phi(S_1)$ and $(T_2,\eta^{T_2},\mu^{T_2}) := \Phi(S_2)$. We have to check the commutativity of the following square

$$\begin{array}{ccc} (T_1,\eta^{T_1},\mu^{T_1}) & \xrightarrow{\varepsilon_{(S_1,\eta^{S_1},\mu^{S_1})}} & (S_1,\eta^{S_1},\mu^{S_1}) \\ \downarrow{\scriptstyle \Phi(\Psi(\varphi))} & & \downarrow{\scriptstyle \varphi} \\ (T_2,\eta^{T_2},\mu^{T_2}) & \xrightarrow[\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2})}]{} & (S_2,\eta^{S_2},\mu^{S_2}) \end{array}$$

componentwise for all $X \in \mathcal{C}$

$$\begin{array}{ccc} T_1X & \xrightarrow{\varepsilon_{(S_1,\eta^{S_1},\mu^{S_1}),X}} & S_1X \\ \downarrow{\scriptstyle \Phi(\Psi(\varphi))_X} & & \downarrow{\scriptstyle \varphi_X} \\ T_2X & \xrightarrow[\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X}]{} & S_2X \end{array}$$

We provide S_2X with a S_{1X}–algebra structure and check that φ_X and $\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X}$ are morphisms of S_{1X}–algebras. By definition $\varepsilon_{(S_1,\eta^{S_1},\mu^{S_1}),X}$ and $\Phi(\Psi(\varphi))_X$ are morphisms of S_{1X}–algebras. By initiality of T_1X we can conclude that $\varphi_X \circ \varepsilon_{(S_1,\eta^{S_1},\mu^{S_1}),X}$ and $\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X} \circ \Phi(\Psi(\varphi))_X$ are equal.

- S_2X is a S_{1X}–algebra: $X + S_1S_2X \xrightarrow{\mathrm{id}+\varphi_{S_2X}} X + S_2S_2X \xrightarrow{[\eta^{S_2}_X,\mu^{S_2}_X]} S_2X$

- φ_X is a morphism of ${S_1}_X$–algebras:

$$\begin{tikzcd}
X + S_1S_1X \arrow[r, "\mathrm{id} + S_1\varphi_X"] \arrow[dd, "{[\eta^{S_1}_X, \mu^{S_1}_X]}"'] & X + S_1S_2X \arrow[d, "\mathrm{id} + \varphi_{S_2X}"] \\
 & X + S_2S_2X \arrow[d, "{[\eta^{S_2}_X, \mu^{S_2}_X]}"] \\
S_1 \arrow[r, "\varphi_X"'] & S_2
\end{tikzcd}$$

This diagram commutes because φ is a monad morphism.

- $\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X}$ is a morphism of ${S_1}_X$–algebras:

$$\begin{tikzcd}
X + S_1T_2X \arrow[r, "\mathrm{id} + S_1\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X}"] \arrow[d, "\mathrm{id} + \varphi_{T_2X}"'] & X + S_1S_2X \arrow[d, "\mathrm{id} + \varphi_{S_2X}"] \\
X + S_2T_2X \arrow[r, "\mathrm{id} + S_2\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X}"] \arrow[d, "{[\eta^{T_2}_X, \sigma^{T_2}_X]}"'] & X + S_2S_2X \arrow[d, "{[\eta^{S_2}_X, \mu^{S_2}_X]}"] \\
T_2 \arrow[r, "\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X}"'] & S_2
\end{tikzcd}$$

The top square is a naturality square of φ and the bottom square commutes by definition of $\varepsilon_{(S_2,\eta^{S_2},\mu^{S_2}),X}$ being a morphism of ${S_2}_X$–algebras.

A.2.3 Unit

Next we define the unit $\eta : \mathrm{Id}_{\mathrm{End}_{\mathcal{C}}} \to \Psi\Phi$ of the adjunction componentwise. Let P be an endofunctor on $\mathcal{C}$ and X be an object of $\mathcal{C}$. We define $\eta_{P,X}$ to be the composite

$$PX \xrightarrow{P\eta^T_X} PTX \xrightarrow{\sigma^T_X} TX$$

where T denotes the underlying functor of the monad $\Phi(P)$. The naturality of $\eta_{P,X}$ in X follows from the naturalities of η^T_X and of σ^T_X in X. We check the naturality of η in P. Let $\rho : P \to Q$ be a morphism of $\mathrm{End}_{\mathcal{C}}$ and we denote the corresponding monads $S := \Phi(P)$ and $T := \Phi(Q)$. We have to check the commutativity of the following diagram.

$$\begin{tikzcd}
PX \arrow[r, "\eta_{P,X}"] \arrow[d, "\rho_X"'] & SX \arrow[d, "\Phi(\rho)_X"] \\
QX \arrow[r, "\eta_{Q,X}"'] & TX
\end{tikzcd}$$

It commutes because of the following diagram commutes

$$\begin{tikzcd}
PX \arrow[r, "P\eta^S_X"] \arrow[dr, "P\eta^T_X"'] \arrow[dd, "\rho_X"'] & PSX \arrow[r, "\sigma^S_X"] \arrow[d, "P\Phi(\rho)_X"] & SX \arrow[dd, "\Phi(\rho)_X"] \\
 & PTX \arrow[d, "\rho_{TX}"] & \\
QX \arrow[r, "Q\eta^T_X"'] & QTX \arrow[r, "\sigma^T_X"'] & TX
\end{tikzcd}$$

The triangle commutes because $\Phi(\rho)$ is a monad morphism. The left square is a naturality square of ρ and the right square commutes by definition of $\Phi(\rho)_X$ being a morphism of P_X–algebras.

A.2.4 Triangle identities

We have defined the counit and the unit of the adjunction, it remains to check the two triangle identities. Let (S, η^S, μ^S) be a monad on $\mathcal{C}$. We have to check the commutativity of the following triangle.

$$\begin{tikzcd} \Psi(S,\eta^S,\mu^S) \arrow[r, "\eta_{\Psi(S,\eta^S,\mu^S)}"] \arrow[dr, "\mathrm{Id}"'] & \Psi\Phi\Psi(S,\eta^S,\mu^S) \arrow[d, "\Psi\varepsilon_{(S,\eta^S,\mu^S)}"] \\ & \Psi(S,\eta^S,\mu^S) \end{tikzcd}$$

Let X be an object of $\mathcal{C}$. This diagram is componentwise

$$\begin{tikzcd} SX \arrow[r, "S\eta^T_X"] \arrow[dr, "S\eta^S_X"'] \arrow[drr, bend right=60, "\mathrm{id}"'] & STX \arrow[r, "\sigma^T_X"] \arrow[d, "S\varepsilon_{(S,\eta^S,\mu^S),X}"] & TX \arrow[d, "\varepsilon_{(S,\eta^S,\mu^S),X}"] \\ & SSX \arrow[r, "\mu^S_X"'] & SX \end{tikzcd}$$

where T denotes the monad $\Phi(S)$. The triangle commutes because $\varepsilon_{(S,\eta^S,\mu^S),X}$ is a monad morphism and the square commutes by definition of $\varepsilon_{(S,\eta^S,\mu^S),X}$ being a morphism of S_X–algebras.

Next we have to check the commutativity of the following diagram for all $P \in \mathrm{End}_{\mathcal{C}}$.

$$\begin{tikzcd} \Phi(P) \arrow[r, "\Phi(\eta_P)"] \arrow[dr, "\mathrm{Id}"'] & \Phi(\Psi(\Phi(P))) \arrow[d, "\varepsilon_{\Phi P}"] \\ & \Phi(P) \end{tikzcd}$$

Let X be an object of $\mathcal{C}$. This diagram's component at X is the following diagram

$$\begin{tikzcd} \Phi(P)(X) \arrow[r, "\Phi(P\eta^T)_X"] \arrow[drr, "\mathrm{Id}_{\Phi(P)(X)}"'] & \Phi(PT)(X) \arrow[r, "\Phi(\sigma^T)_X"] & \Phi(T)(X) \arrow[d, "\varepsilon_{(T,\eta^T,\mu^T),X}"] \\ & & \Phi(P)(X) \end{tikzcd}$$

where (T, η^T, μ^T) denotes the monad $\Phi(P)$. We are going to check that $\Phi(\sigma^T)_X$ and $\varepsilon_{(T,\eta^T,\mu^T),X}$ are morphisms of P_X–algebras. The arrow $\Phi(P\eta^T)_X$ is by definition a morphism of P_X–algebras, then by initiality of $TX = \Phi(P)(X)$ we can conclude that $\varepsilon_{(T,\eta^T,\mu^T),X} \circ \Phi(\sigma^T)_X \circ \Phi(P\eta^T)_X$ and $\mathrm{Id}_{\Phi(P)(X)}$ are equal.

- $\Phi(\sigma^T)_X$ is a morphism of P_X–algebras:

$$\begin{tikzcd}
X + P\Phi(PT)(X) \arrow[r, "\mathrm{id} + P\Phi(\sigma^T)_X"] \arrow[d, "\mathrm{id} + P\eta^T_{\Phi(PT)(X)}"'] & X + P\Phi(T)(X) \arrow[d, "\mathrm{id} + P\eta^T_{\Phi(T)(X)}"] \\
X + PT\Phi(PT)(X) \arrow[r, "\mathrm{id} + PT\Phi(\sigma^T)_X"] \arrow[dd, "\cong"'] & X + PT\Phi(T)(X) \arrow[d, "\mathrm{id}_X + \sigma^T_{\Phi(T)(X)}"] \\
 & X + T\Phi(T)(X) \arrow[d, "\cong"] \\
\Phi(PT)(X) \arrow[r, "\Phi(\sigma^T)_X"'] & \Phi(T)(X)
\end{tikzcd}$$

The top square is a naturality square of η^T and the bottom square commutes by definition of $\Phi(\sigma^T)_X$ being a morphism of PT_X–algebras.

- $\varepsilon_{(T,\eta^T,\mu^T),X}$ is a morphisms of P_X–algebras:

$$\begin{tikzcd}
X + P\Phi(T)(X) \arrow[r, "\mathrm{id} + P\varepsilon_{(T,\eta^T,\mu^T),X}"] \arrow[d, "\mathrm{id} + P\eta^T_{\Phi(T)(X)}"'] & X + PT(X) \arrow[d, "\mathrm{id} + P\eta^T_{T(X)}"] \\
X + PT\Phi(T)(X) \arrow[r, "\mathrm{id} + PT\varepsilon_{(T,\eta^T,\mu^T),X}"] \arrow[d, "\mathrm{id} + \sigma^T_{\Phi(T)(X)}"'] & X + PTT(X) \arrow[d, "\mathrm{id} + \sigma^T_{T(X)}"] \\
X + T\Phi(T)(X) \arrow[r, "\mathrm{id} + T\varepsilon_{(T,\eta^T,\mu^T),X}"] \arrow[d, "\cong"'] & X + TT(X) \arrow[d, "{[\eta^T_X, \mu^T_X]}"] \\
\Phi(T)(X) \arrow[r, "\varepsilon_{(T,\eta^T,\mu^T),X}"'] & T(X)
\end{tikzcd}$$

The top square is a naturality square of η^T, the middle square is a naturality square of σ^T and the bottom square commutes by definition of $\varepsilon_{(T,\eta^T,\mu^T),X}$ being a morphism of T_X–algebras.

A.3 Proof of proposition 3.2.17

Next we check naturality of ϕ. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. To check that

$$\begin{tikzcd}
TX \arrow[r, "\phi_X"] \arrow[d, "Tf"'] & MX \arrow[d, "Mf"] \\
TY \arrow[r, "\phi_Y"'] & MY
\end{tikzcd}$$

commutes, we check that TY and MY are G_X–algebras and that Mf and ϕ_Y are morphisms of G_X–algebras. The two arrows ϕ_X and Tf are by definition morphisms of G_X–algebras. Then by initiality of TX we can conclude that $Mf \circ \phi_X$ and $\phi_Y \circ Tf$ are equal.

- TY is a G_X–algebra: $X + GTY \xrightarrow{f+\mathrm{id}} Y + GTY \xrightarrow{[\eta_Y, \sigma_Y]} TY$

- MY is a G_X–algebra: $X + GMY \xrightarrow{f+\rho_{MY}} Y + MMY \xrightarrow{[\eta^M_Y, \mu^M_Y]} MY$

- Mf is a morphism of G_X–algebras:

$$\begin{tikzcd}
X + GMX \arrow[r, "\mathrm{id} + GMf"] \arrow[d, "\mathrm{id} + \rho_{MX}"'] & X + GMY \arrow[d, "\mathrm{id} + \rho_{MY}"] \\
X + MMX \arrow[r, "\mathrm{id} + MMf"] \arrow[dd, "{[\eta^M_X, \mu^M_X]}"'] & X + MMY \arrow[d, "f + \mathrm{id}"] \\
 & Y + MMY \arrow[d, "{[\eta^M_Y, \mu^M_Y]}"] \\
MX \arrow[r, "Mf"'] & MY
\end{tikzcd}$$

The top square is a naturality square of ρ and the bottom square commutes because of the naturalities of η^M and μ^M of M.

- ϕ_Y is a morphism of G_X–algebras:

$$\begin{tikzcd}
X + GTY \arrow[r, "\mathrm{id}_X + G\phi_Y"] \arrow[d, "f + \mathrm{id}"'] & X + GMY \arrow[d, "f + \mathrm{id}"] \\
Y + GTY \arrow[r, "\mathrm{id}_Y + G\phi_Y"] \arrow[dd, "{[\eta_Y, \sigma_Y]}"'] & Y + GMY \arrow[d, "\mathrm{id}_Y + \rho_{MY}"] \\
 & Y + MMY \arrow[d, "{[\eta^M_Y, \mu^M_Y]}"] \\
TY \arrow[r, "\phi_Y"'] & MY
\end{tikzcd}$$

The top square commutes obviously and the bottom square commutes because ϕ_Y is by definition the unique morphism of G_Y–algebras $TY \to MY$.

Next we check that ϕ is a monad morphism. By definition of ϕ_X, it is the unique morphism of G_X–algebras such that

$$\begin{tikzcd}
X + GTX \arrow[r, "\mathrm{id} + G\phi_X"] \arrow[dd, "{[\eta_X, \sigma_X]}"'] & X + GMX \arrow[d, "\mathrm{id} + \rho_{MX}"] \\
 & X + MMX \arrow[d, "{[\eta^M_X, \mu^M_X]}"] \\
TX \arrow[r, "\phi_X"'] & MX
\end{tikzcd}$$

commutes. This implies the commutativity of the following triangle

$$\begin{tikzcd}
X \arrow[d, "\eta_X"'] \arrow[dr, "\eta^M_X"] & \\
TX \arrow[r, "\phi_X"'] & MX
\end{tikzcd}$$

which is one of the monad morphism axioms. In order to check the other monad morphism axiom

$$\begin{tikzcd}
TTX \arrow[r, "\mu_X"] \arrow[d, "T\phi_X"'] & TX \arrow[dd, "\phi_X"] \\
TMX \arrow[d, "\phi_{MX}"'] & \\
MMX \arrow[r, "\mu^M_X"'] & MX
\end{tikzcd}$$

we are going to check that MMX and MX are G_{TX}–algebras and that ϕ_{MX}, μ^M_X and ϕ_X are G_{TX}–algebra morphisms. By definition $T\phi_X$ and μ_X are morphisms of G_{TX}–algebras, so by initiality of TTX we can conclude that the two composites $\phi_X \circ \mu_X$ and $\mu^M_X \circ \phi_{MX} \circ T\phi_X$ are equal.

- MMX is a G_{TX}–algebra: $TX + GMMX \xrightarrow{\phi_X + \mathrm{id}} MX + GMMX \xrightarrow{\mathrm{id} + \rho_{MMX}} MX + MMMX \xrightarrow{[\eta^M_X, \mu^M_X]} MMX$
- MX is a G_{TX}–algebra: $TX + GMX \xrightarrow{\phi_X + \mathrm{id}} MX + GMX \xrightarrow{\mathrm{id} + \rho_{MX}} MX + MMX \xrightarrow{[\mathrm{id}, \mu^M_X]} MX$
- ϕ_{MX} is a morphism of G_{TX}–algebras:

$$\begin{tikzcd}
TX + GTMX \arrow[r, "\mathrm{id} + G\phi_{MX}"] \arrow[d, "\phi_X + \mathrm{id}"'] & TX + GMMX \arrow[d, "\phi_X + \mathrm{id}"] \\
MX + GTMX \arrow[r, "\mathrm{id} + G\phi_{MX}"] \arrow[ddd, "{[\eta_{MX}, \sigma_{MX}]}"'] & MX + GMMX \arrow[d, "\mathrm{id} + \rho_{MMX}"] \\
 & MX + MMMX \arrow[dd, "{[\eta^M_{MX}, \mu^M_{MX}]}"] \\
 & \\
TMX \arrow[r, "\phi_{MX}"'] & MMX
\end{tikzcd}$$

  The top square commutes obviously and the bottom square commutes by definition of ϕ_{MX} being the unique morphism of G_{MX}–algebras $TMX \to MMX$.

- μ^M_X is a morphism of G_{TX}–algebras:

$$\begin{tikzcd}
TX + GMMX \arrow[r, "\mathrm{id} + G\mu^M_X"] \arrow[d, "\phi_X + \mathrm{id}"'] & TX + GMX \arrow[d, "\phi_X + \mathrm{id}"] \\
MX + GMMX \arrow[r, "\mathrm{id} + G\mu^M_X"] \arrow[d, "\mathrm{id} + \rho_{MMX}"'] & MX + GMX \arrow[d, "\mathrm{id} + \rho_{MX}"] \\
MX + MMMX \arrow[r, "\mathrm{id} + M\mu^M_X"] \arrow[d, "{[\eta^M_{MX}, \mu^M_{MX}]}"'] & MX + MMX \arrow[d, "{[\mathrm{id}, \mu^M_X]}"] \\
MMX \arrow[r, "\mu^M_X"'] & MX
\end{tikzcd}$$

  The top square commutes obviously, the middle square is a naturality square of ρ and the bottom square commutes because of the monad axioms of M.

- ϕ_X is a morphism of G_{TX}–algebras:

$$\begin{tikzcd} TX + GTX \arrow[r, "\mathrm{id} + G\phi_X"] \arrow[dd, "{[\mathrm{id}, \sigma_X]}"'] & TX + GMX \arrow[d, "\phi_X + \rho_{MX}"] \\ & MX + MMX \arrow[d, "{[\mathrm{id}, \mu^M_X]}"] \\ TX \arrow[r, "\phi_X"'] & MX \end{tikzcd}$$

The above square commutes because the following square

$$\begin{tikzcd} GTX \arrow[r, "G\phi_X"] \arrow[dd, "\sigma_X"'] & GMX \arrow[d, "\rho_{MX}"] \\ & MMX \arrow[d, "\mu^M_X"] \\ TX \arrow[r, "\phi_X"'] & MX \end{tikzcd}$$

commutes by definition of ϕ_X being the unique morphism of G_X–algebras $TX \to MX$.

To prove that ϕ is a morphism of Mon^G, we check the commutativity of the following diagram for all $X \in \mathcal{C}$

$$\begin{tikzcd} & & GX \arrow[dll, "G\eta_X"'] \arrow[ddr, "\rho_X"] & \\ GTX \arrow[d, "\sigma_X"'] & & & \\ TX \arrow[rrr, "\phi_X"'] & & & MX \end{tikzcd}$$

this diagram becomes

$$\begin{tikzcd} GX \arrow[rr, "\rho_X"] \arrow[d, "G\eta_X"'] \arrow[dr, "G\eta^M_X"] & & MX \arrow[d, "M\eta^M_X"] \arrow[ddr, bend left, "\mathrm{id}_{MX}"] & \\ GTX \arrow[r, "G\phi_X"] \arrow[d, "\sigma_X"'] & GMX \arrow[r, "\rho_{MX}"] & MMX \arrow[dr, "\mu^M_X"] & \\ TX \arrow[rrr, "\phi_X"'] & & & MX \end{tikzcd}$$

Triangle I. commutes because ϕ is a monad morphism, square II. is a naturality square of ρ, triangle III. is a monad axiom and diagram IV. commutes by definition of ϕ_X being a morphism of G_X–algebras.

Appendix B

Proofs of chapter 4

B.1 Proof of lemma 4.1.24

We are going to construct the three natural isomorphisms

$$\begin{aligned}\alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \\ \rho_X &: X \otimes U \to X \\ \lambda_X &: U \otimes X \to X\end{aligned}$$

- We write the domain and codomain of $\alpha_{X,Y,Z}$ using the coend notation

$$\begin{aligned}(X \otimes Y) \otimes Z(n) &= \int^k (X \otimes Y)(k) \times Z(n)^k \\ &= \int^k \int^r X(r) \times Y(k)^r \times Z(n)^k\end{aligned}$$

and

$$\begin{aligned}X \otimes (Y \otimes Z)(n) &= \int^r X(r) \times (Y \otimes Z)^r(n) \\ &= \int^r X(r) \times \left(\int^p Y(p) \times Z(r)^p\right)^r \\ &= \int^r \int^{p_1} \ldots \int^{p_r} X(r) \times Y(p_1) \times \ldots \times Y(p_r) \times Z(n)^{p_1+\ldots+p_r}\end{aligned}$$

To define an arrow $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, by universal property of the coends it suffices to give a family of arrows

$$X(r) \times Y(k)^r \times Z(n)^k \to X \otimes (Y \otimes Z)$$

for all $r, k \in \mathbb{F}$ that satisfies the wedge condition. We take the following composite

$$\begin{array}{c} X(r) \times Y(k)^r \times Z(n)^k \\ \downarrow \\ X(r) \times Y(k) \times \ldots \times Y(k) \times Z(n)^{r \times k} \\ \downarrow \\ X(r) \times \left(\int^{p_1} Y(p_1) \times Z(n)^{p_1}\right) \times \ldots \times \left(\int^{p_r} Y(p_r) \times Z(n)^{p_r}\right) \\ \downarrow \\ \int^r X(r) \times \left(\int^p Y(p) \times Z(n)^p\right)^r \end{array}$$

by using $r \times k \to k$. We check the wedge conditions. Let $h : k \to \ell$. The following diagram commutes

$$\begin{tikzcd}
 & X(r) \times Y(k)^r \times Z(n)^k \arrow[dr] & \\
X(r) \times Y(k)^r \times Z(n)^\ell \arrow[ur, "\mathrm{id} \times (- \circ h)"] \arrow[dr, "\mathrm{id} \times Y(h)^r \times \mathrm{id}"'] & & (X \otimes (Y \otimes Z))(n) \\
 & X(r) \times Y(\ell)^r \times Z(n)^\ell \arrow[ur] &
\end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd}
 & (x, y_1, \dots, y_r, f \circ h) \arrow[dr, mapsto] & \\
(x, y_1, \dots, y_r, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}(x, (y_1, f \circ h), \dots, (y_r, f \circ h)) \\ = (x, (Y(h)(y_1), f), \dots, (Y(h)(y_r), f))\end{array} \\
 & (x, Y(h)(y_1), \dots, Y(h)(y_r), f) \arrow[ur, mapsto] &
\end{tikzcd}$$

and by universal property of the coend $(y_i, f \circ h) = (Y(h)(y_i), f)$ since they come from $(y_i, f) \in Y(k) \times Z(n)^\ell$ with the arrow h

$$\begin{tikzcd}
 & Y(k) \times Z(n)^k \arrow[dr] & \\
Y(k) \times Z(n)^\ell \arrow[ur, "\mathrm{id} \times (- \circ h)"] \arrow[dr, "Y(h) \times \mathrm{id}"'] & & \int^k Y(k) \times Z(n)^k \\
 & Y(\ell) \times Z(n)^\ell \arrow[ur] &
\end{tikzcd}$$

Let $g : r \to p$. The following diagram commutes

$$\begin{tikzcd}
 & X(r) \times \int^k Y(k)^r \times Z(n)^k \arrow[dr] & \\
X(r) \times \int^k Y(k)^p \times Z(n)^k \arrow[ur] \arrow[dr] & & (X \otimes (Y \otimes Z))(n) \\
 & X(p) \times \int^k Y(k)^p \times Z(n)^k \arrow[ur] &
\end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd}
 & (x, y_{g(1)}, \dots, y_{g(r)}, f) \arrow[dr, mapsto] & \\
(x, y_1, \dots, y_r, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}(x, (y_{g(1)}, f), \dots (y_{g(r)}, f)) \\ = (X(g)(x), (y_1, f), \dots, (y_p, f))\end{array} \\
 & (X(g)(x), y_1, \dots, y_r, f) \arrow[ur, mapsto] &
\end{tikzcd}$$

The elements $(x, (y_{g(1)}, f), \dots (y_{g(r)}, f))$ and $(X(g)(x), y_1, \dots, y_r, f)$ are equal by universal property of the coend since they come from $(x, (y_1, f), \dots, (y_r, f)) \in X(r) \times (\int^k Y(k) \times Z(n)^k)^p$ with the arrow g

$$\begin{tikzcd}
 & X(r) \times \left(\int^k Y(k) \times Z(n)^k\right)^r \arrow[dr] & \\
X(r) \times \left(\int^k Y(k) \times Z(n)^k\right)^p \arrow[ur] \arrow[dr] & & (X \otimes (Y \otimes Z))(n) \\
 & X(p) \times \left(\int^k Y(k) \times Z(n)^k\right)^p \arrow[ur] &
\end{tikzcd}$$

Inversely to define an arrow $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, by universal property of the coends it suffices to give a family of arrows

$$X(r) \times Y(p_1) \times \dots \times Y(p_r) \times Z(n)^{p_1 + \dots + p_r} \to (X \otimes Y) \otimes Z$$

for all $r, p_1, \dots, p_r \in \mathbb{F}$ that satisfies the wedge condition. We take the following composite

$$\begin{tikzcd}
X(r) \times Y(p_1) \times \dots \times Y(p_r) \times Z(n)^{p_1 + \dots + p_r} \arrow[d] \\
X(r) \times Y(\sum_{i=1}^{r} p_i) \times \dots \times Y(\sum_{i=1}^{r} p_i) \times Z(n)^{p_1 + \dots + p_r} \arrow[d] \\
X(r) \times \int^k Y(k)^r \times Z(n)^k \arrow[d] \\
\int^r X(r) \times \int^k Y(k)^r \times Z(n)^k
\end{tikzcd}$$

We check the wedge conditions. Let $h_i : p_i \to q_i$ for $i = 1, \dots, r$. The following diagram commutes

$$\begin{tikzcd}
 & \begin{array}{l} X(r) \\ \times Y(p_1) \times \dots \times Y(p_r) \\ \times Z(n)^{p_1 + \dots + p_r} \end{array} \arrow[dr] & \\
\begin{array}{l} X(r) \\ \times Y(p_1) \times \dots \times Y(p_r) \\ \times Z(n)^{q_1 + \dots + q_r} \end{array} \arrow[ur] \arrow[dr] & & ((X \otimes Y) \otimes Z)(n) \\
 & \begin{array}{l} X(r) \\ \times Y(q_1) \times \dots \times Y(q_r) \\ \times Z(n)^{q_1 + \dots + q_r} \end{array} \arrow[ur] &
\end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd}
 & \begin{array}{c} x, \\ y_1,\dots,y_r, \\ f\circ\sum h_i \end{array} \arrow[dr, mapsto] & \\
\begin{array}{c} x, \\ y_1,\dots,y_r, \\ f \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} x, Y(c_1)(y_1),\dots,Y(c_r)(y_r), \\ f\circ\sum h_i \\ = x, Y(c'_1\circ h_1)(y_1),\dots,Y(c'_r\circ h_r)(y_r), \\ f \end{array} \\
 & \begin{array}{c} x, \\ Y(h_1)(y_1),\dots,Y(h_r)(y_r), \\ f \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

where $c_i : p_i \to \sum_{j=1}^{r} p_j$ and $c'_i : q_i \to \sum_{j=1}^{r} q_j$ and by naturality of these inclusions $\sum_{j=1}^{r} h_j \circ c_i = c'_i \circ h_i$ for all $i = 1,\dots,r$. By universal property of the coend, the elements $x, Y(c_1)(y_1),\dots,Y(c_r)(y_r), f\circ \sum_{i=1}^{r} h_i$ and $x, Y(c'_1\circ h_1)(y_1),\dots,Y(c'_r\circ h_r)(y_r), f$ are equal since they come from $x, Y(c_1)(y_1),\dots,Y(c_r)(y_r), f \in X(r)\times Y(\sum_{i=1}^{r} p_i)\times Z(n)^{q_1+\dots+q_r}$ with the arrow $\sum_{i=1}^{r} h_i$

$$\begin{tikzcd}
 & \begin{array}{c} X(r)\times Y(\sum_{i=1}^{r} p_i)^r \\ \times Z(n)^{p_1+\dots+p_r} \end{array} \arrow[dr] & \\
\begin{array}{c} X(r)\times Y(\sum_{i=1}^{r} p_i) \\ \times Z(n)^{q_1+\dots+q_r} \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} X(r)\times \int^k X(k)^r \\ \times Z(n)^k \end{array} \\
 & \begin{array}{c} X(r)\times Y(\sum_{i=1}^{r} q_i)^r \\ \times Z(n)^{q_1+\dots+q_r} \end{array} \arrow[ur] &
\end{tikzcd}$$

Let $g : r \to s$. The following diagram commutes

$$\begin{tikzcd}
 & \begin{array}{c} X(r) \\ \times\big(\int^k Y(k)\times Z(n)^k\big)^r \end{array} \arrow[dr] & \\
\begin{array}{c} X(r) \\ \times\big(\int^k Y(k)\times Z(n)^k\big)^s \end{array} \arrow[ur] \arrow[dr] & & ((X\otimes Y)\otimes Z)(n) \\
 & \begin{array}{c} X(s) \\ \times\big(\int^k Y(k)\times Z(n)^k\big)^s \end{array} \arrow[ur] &
\end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd}[column sep=small]
& \begin{matrix} x \\ (y_{g(1)}, f_{g(1)}), \dots, (y_{g(r)}, f_{g(r)}) \end{matrix} \arrow[dr, mapsto] & \\
\begin{matrix} x, \\ (y_1, f_1), \dots, (y_s, f_s) \end{matrix} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{matrix} x, Y(c'_1)(y_{g(1)}), \dots, \\ Y(c'_r)(y_{g(r)}), [f_{g(1)}, \dots, f_{g(r)}] \\ = (X(g)(x), Y(c_1)(y_1), \dots, \\ Y(c_s)(y_s), [f_1, \dots, f_s]) \end{matrix} \\
& \begin{matrix} X(g)(x) \\ (y_1, f_1), \dots, (y_s, f_s) \end{matrix} \arrow[ur, mapsto] &
\end{tikzcd}$$

where $c_i : k_i \to \sum_{j=1}^{s} k_j$ and $c'_i : k_{g(i)} \to \sum_{j=1}^{r} k_{g(j)}$. The arrow $g : r \to s$ induces the arrow $[c_{g(1)}, \dots, c_{g(r)}] : \sum_{j=1}^{r} k_{g(j)} \to \sum_{j=1}^{s} k_j$ that makes the following commute

$$\begin{tikzcd}
& k_{g(i)} \arrow[dl, "c'_i"'] \arrow[dr, "c_{g(i)}"] & \\
\sum_{j=1}^{r} k_{g(j)} \arrow[rr, "{[c_{g(1)}, \dots, c_{g(r)}]}"] \arrow[dr, "{[f_{g(1)}, \dots, f_{g(r)}]}"'] & & \sum_{j=1}^{s} k_j \arrow[dl, "{[f_1, \dots, f_s]}"] \\
& Z(n) &
\end{tikzcd}$$

So by the following universal properties of the coends

$$\begin{tikzcd}[column sep=tiny]
& \begin{matrix} X(r) \times Y(\sum_{j=1}^{r} k_{g(j)})^r \\ \times Z(n)^{k_{g(1)} + \dots + k_{g(r)}} \end{matrix} \arrow[dr] & \\
\begin{matrix} X(r) \times Y(\sum_{j=1}^{r} k_{g(j)})^r \\ \times Z(n)^{k_1 + \dots + k_s} \end{matrix} \arrow[ur, "{\mathrm{id} \times (- \circ [c_{g(1)}, \dots, c_{g(r)}])}"] \arrow[dr, "{\mathrm{id} \times Y([c_{g(1)}, \dots, c_{g(r)}])^r \times \mathrm{id}}"'] & & \begin{matrix} X(r) \times \int^k X(k)^r \\ \times Z(n)^k \end{matrix} \\
& \begin{matrix} X(r) \times Y(\sum_{j=1}^{s} k_j)^r \\ \times Z(n)^{k_1 + \dots + k_s} \end{matrix} \arrow[ur] &
\end{tikzcd} \tag{B.1}$$

and

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c} X(r) \times \int^k X(k)^r \\ \times Z(n)^k \end{array} \arrow[dr] & \\
\begin{array}{c} X(r) \times \int^k X(k)^s \\ \times Z(n)^k \end{array} \arrow[ur, "\mathrm{id}\times(-)\circ g\times \mathrm{id}"] \arrow[dr, "X(g)\times \mathrm{id}"'] & & \begin{array}{c} \int^r X(r) \times \int^k X(k)^r \\ \times Z(n)^k \end{array} \\
 & \begin{array}{c} X(s) \times \int^k X(k)^s \\ \times Z(n)^k \end{array} \arrow[ur] &
\end{tikzcd} \tag{B.2}$$

we obtain

$$(X(g)(x), Y(c_1)(y_1), \ldots, Y(c_s)(y_s), [f_1, \ldots, f_s])$$

by (B.2) is equal to

$$(x, Y(c_{g(1)})(y_{g(1)}), \ldots, Y(c_{g(r)})(y_{g(r)}), [f_1, \ldots, f_s])$$

by (B.1) is equal to

$$(x, Y(c'_1)(y_{g(1)}), \ldots, Y(c'_r)(y_{g(r)}), [f_{g(1)}, \ldots, f_{g(r)}])$$

We check that the two arrows constructed above are inverse to each other. Starting with $(x, y_1, \ldots, y_r, f) \in X(r) \times Y(k)^r \times Z(n)^k$, we have the following composite

$$\begin{tikzcd}
(x, y_1, \ldots, y_r, f) \arrow[d, mapsto] \\
(x, y_1, \ldots, y_r, [f, \ldots, f]) \arrow[d, mapsto] \\
(x, Y(c_1)(y_1), \ldots, Y(c_r)(y_r), [f, \ldots, f])
\end{tikzcd}$$

The two elements $(x, y_1, \ldots, y_r, f)$ and $(x, Y(c_1)(y_1), \ldots, Y(c_r)(y_r), [f, \ldots, f])$ are equal since they come from $(x, Y(c_1)(y_1), \ldots, Y(c_r)(y_r), f) \in X(r) \times Y(r \times k)^r \times Z(n)^k$ with the arrow $[\mathrm{id}_k, \ldots, \mathrm{id}_k] : r \times k \to k$

$$\begin{tikzcd}
 & X(r) \times Y(k)^r \times Z(n)^k \\
X(r) \times Y(r \times k)^r \times Z(n)^k \arrow[ur, "\mathrm{id}\times[\mathrm{id}_k,\ldots,\mathrm{id}_k]^r\times \mathrm{id}"] \arrow[dr, "\mathrm{id}\times(-)\circ[\mathrm{id}_k,\ldots,\mathrm{id}_k]"'] & \\
 & X(r) \times Y(r \times k)^r \times Z(n)^{r \times k}
\end{tikzcd}$$

Starting with $(x, y_1, \ldots, y_r, f) \in X(r) \times Y(p_1) \times \ldots \times Y(p_r) \times Z(n)^{p_1 + \ldots + p_r}$, we obtain the following composite

$$\begin{tikzcd}
(x, y_1, \ldots, y_r, f) \arrow[d, mapsto] \\
(x, Y(c_1)(y_1), \ldots, Y(c_r)(y_r), f) \arrow[d, mapsto] \\
(x, Y(c_1)(y_1), \ldots, Y(c_r)(y_r), [f, \ldots, f])
\end{tikzcd}$$

The two elements $(x, y_1, \ldots, y_r, f)$ and $(x, Y(c_1)(y_1), \ldots, Y(c_r)(y_r), [f, \ldots, f])$ are equal since they come from with the arrows $c_i : p_i \to \sum_{j=1}^{r} p_j$ for $i = 1, \ldots, r$

$$\begin{tikzcd}
 & X(r) \times Y(p_1) \times \ldots \times Y(p_r) \times Z(n)^{p_1+\ldots+p_r} \\
X(r) \times Y(p_1) \times \ldots \times Y(p_r) \times Z(n)^{\sum p_j+\ldots+\sum p_j} \arrow[ur, "\mathrm{id} \times (-)\circ(c_1+\ldots+c_r)"] \arrow[dr, "\mathrm{id} \times Y(c_1)\times\ldots\times Y(c_r)\times \mathrm{id}"'] & \\
 & X(r) \times Y(\sum_{j=1}^{r} p_j)^r \times Z(n)^{\sum p_j+\ldots+\sum p_j}
\end{tikzcd}$$

Next we check naturalities in n, X, Y and Z. Let $h : n \to m$ in $\mathbb{F}$. The following naturality square

$$\begin{tikzcd}
((X \otimes Y) \otimes Z)(n) \arrow[r] \arrow[d] & (X \otimes (Y \otimes Z))(n) \arrow[d] \\
((X \otimes Y) \otimes Z)(m) \arrow[r] & (X \otimes (Y \otimes Z))(m)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
(x, y_1, \ldots, y_r, z) \arrow[r, mapsto] \arrow[d, mapsto] & (x, (y_1, z), \ldots, (y_r, z)) \arrow[d, mapsto] \\
(x, y_1, \ldots, y_r, Z(h) \circ z) \arrow[r, mapsto] & (x, (y_1, Z(h) \circ z), \ldots, (y_r, Z(h) \circ z))
\end{tikzcd}$$

Let $h : X \to W$ in $\mathcal{F}$. The following naturality square

$$\begin{tikzcd}
((X \otimes Y) \otimes Z)(n) \arrow[r] \arrow[d] & (X \otimes (Y \otimes Z))(n) \arrow[d] \\
((W \otimes Y) \otimes Z)(n) \arrow[r] & (W \otimes (Y \otimes Z))(n)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
(x, y_1, \ldots, y_r, z) \arrow[r, mapsto] \arrow[d, mapsto] & (x, (y_1, z), \ldots, (y_r, z)) \arrow[d, mapsto] \\
(h_r(x), y_1, \ldots, y_r, z) \arrow[r, mapsto] & (h_r(x), (y_1, z), \ldots, (y_r, z))
\end{tikzcd}$$

Let $h : Y \to W$ in $\mathcal{F}$. The following naturality square

$$\begin{tikzcd}
((X \otimes Y) \otimes Z)(n) \arrow[r] \arrow[d] & (X \otimes (Y \otimes Z))(n) \arrow[d] \\
((X \otimes W) \otimes Z)(n) \arrow[r] & (X \otimes (W \otimes Z))(n)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
(x, y_1, \ldots, y_r, z) \arrow[r, mapsto] \arrow[d, mapsto] & (x, (y_1, z), \ldots, (y_r, z)) \arrow[d, mapsto] \\
(x, h_k(y_1), \ldots, h_k(y_r), z) \arrow[r, mapsto] & (x, (h_k(y_1), z), \ldots, (h_k(y_r), z))
\end{tikzcd}$$

Let $h : Z \to W$ in $\mathcal{F}$. The following naturality square

$$\begin{tikzcd}
((X \otimes Y) \otimes Z)(n) \arrow[r] \arrow[d] & (X \otimes (Y \otimes Z))(n) \arrow[d] \\
((X \otimes Y) \otimes W)(n) \arrow[r] & (X \otimes (Y \otimes W))(n)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
(x, y_1, \ldots, y_r, z) \arrow[r, mapsto] \arrow[d, mapsto] & (x, (y_1, z), \ldots, (y_r, z)) \arrow[d, mapsto] \\
(x, y_1, \ldots, y_r, h_n(z)) \arrow[r, mapsto] & (x, (y_1, h_n(z)), \ldots, (y_r, h_n(z)))
\end{tikzcd}$$

- We define $\rho_X : \int^k X(k) \times n^k \to X(n)$ by the universal property of the coend. We give a mapping $X(k) \times n^k \to X(n)$ for all $k \in \mathbb{F}$ satisfying the wedge condition. We take it to be the following mapping

$$(x, f) \quad \mapsto \quad X(f)(x)$$

We check the wedge condition. Let $g : k \to \ell$. Then the following diagram commutes

$$\begin{tikzcd}
 & X(k) \times n^k \arrow[dr] & \\
X(k) \times n^\ell \arrow[ur] \arrow[dr] & & X(n) \\
 & X(\ell) \times n^\ell \arrow[ur] &
\end{tikzcd}$$

since we have the following assignation on elements

$$\begin{tikzcd}
 & (x, f \circ g) \arrow[dr, mapsto] & \\
(x, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & X(f \circ g)(x) = X(f) \circ X(g)(x) \\
 & (X(g)(x), f) \arrow[ur, mapsto] &
\end{tikzcd}$$

We define the inverse arrow $X(n) \to \int^k X(k) \times n^k$ by the following mapping composed with the n-th coprojection

$$x \quad \mapsto \quad (x, \mathrm{id}_n)$$

We check that these are inverse to each other. One of the composites yields

$$x \quad \mapsto \quad (x, \mathrm{id}_n) \quad \mapsto \quad X(\mathrm{id}_n)(x) = x$$

which is the identity on x. The other composite yields

$$(x, f) \quad \mapsto \quad X(f)(x) \quad \mapsto \quad (X(f)(x), \mathrm{id}_n)$$

These two elements are identic since they come from $X(k) \times n^n$ with the arrow $f : k \to n$

$$\begin{tikzcd}
 & \begin{matrix} X(k) \times n^k \\ (x, f) \end{matrix} \\
\begin{matrix} X(k) \times n^n \\ (x, \mathrm{id}_n) \end{matrix} \arrow[ur] \arrow[dr] & \\
 & \begin{matrix} X(n) \times n^n \\ (X(f)(x), \mathrm{id}_n) \end{matrix}
\end{tikzcd}$$

We check the naturalities in n and X. Let $h : n \to m$ in $\mathbb{F}$. The naturality square

$$\begin{tikzcd}
(X \otimes U)(n) \arrow[r] \arrow[d] & X(n) \arrow[d] \\
(X \otimes U)(m) \arrow[r] & X(m)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
(x, f) \arrow[r, mapsto] \arrow[d, mapsto] & X(f)(x) \arrow[d, mapsto] \\
(x, h \circ f) \arrow[r, mapsto] & X(h \circ f)(x)
\end{tikzcd}$$

Let $h : X \to Y$ in $\mathcal{F}$. The naturality square

$$\begin{tikzcd}
(X \otimes U)(n) \arrow[r] \arrow[d] & X(n) \arrow[d] \\
(Y \otimes U)(n) \arrow[r] & Y(n)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
(x, f) \arrow[r, mapsto] \arrow[d, mapsto] & X(f)(x) \arrow[d, mapsto] \\
(h_k(x), f) \arrow[r, mapsto] & h_n \circ X(f)(x) = Y(f)(h_k(x))
\end{tikzcd}$$

- We define $\lambda_X : \int^k k \times X(n)^k \to X(n)$ by the universal property of the coend. We give a mapping $k \times X(n)^k \to X(n)$ for all $k \in \mathbb{F}$ satisfying the wedge condition. We take it to be the following mapping

$$(i, f) \quad \mapsto \quad f(i)$$

We check the wedge condition. Let $g : k \to \ell$. The following diagram commutes

$$\begin{tikzcd}
 & k \times X(n)^k \arrow[dr] & \\
k \times X(n)^\ell \arrow[ur] \arrow[dr] & & X(n) \\
 & \ell \times X(n)^\ell \arrow[ur] &
\end{tikzcd}$$

since we have the following assignations on elements

$$\begin{tikzcd}
 & (i, f \circ g) \arrow[dr, mapsto] & \\
(i, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & f \circ g(i) = f(g(i)) \\
 & (g(i), f) \arrow[ur, mapsto] &
\end{tikzcd}$$

We define the inverse arrow $X(n) \to \int^k k \times (X(n)^k$ by the following mapping composed with the corresponding coprojection

$$x \quad \mapsto \quad (1, \langle x \rangle)$$

where $\langle x \rangle : 1 \to X(n), 1 \mapsto x$. We check that these two assignations are inverse to each other. One of the composites yields

$$x \quad \mapsto \quad (1, \langle x \rangle) \quad \mapsto \quad \langle x \rangle(1) = x$$

which is identity on x. The other composite yields

$$(i, f) \quad \mapsto \quad f(i) \quad \mapsto \quad (1, \langle f(i) \rangle)$$

These two are identic since they come from $1 \times X(n)^k$ with the arrow $\langle i \rangle : 1 \to k$

$$\begin{tikzcd}
 & \begin{array}{c} k \times (X(n))^k \\ (i, f) \end{array} \\
\begin{array}{c} 1 \times (X(n))^k \\ (1, f) \end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{c} 1 \times X(n)^1 \\ (1, \langle f(i) \rangle) \end{array}
\end{tikzcd}$$

We check the naturalities in n and X. Let $h : n \to m$ in $\mathbb{F}$. The following naturality square

$$\begin{tikzcd}
(U \otimes X)(n) \arrow[r] \arrow[d] & X(n) \arrow[d] \\
(U \otimes X)(m) \arrow[r] & X(m)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
(i, f) \arrow[r, mapsto] \arrow[d, mapsto] & f(i) \arrow[d, mapsto] \\
(i, X(h) \circ f) \arrow[r, mapsto] & X(h)(f(i))
\end{tikzcd}$$

Let $h : X \to Y$ in $\mathcal{F}$. The following naturality square

$$\begin{tikzcd}
(U \otimes X)(n) \arrow[r] \arrow[d] & X(n) \arrow[d] \\
(U \otimes Y)(n) \arrow[r] & Y(n)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd} (i,f) \arrow[r, mapsto] \arrow[d, mapsto] & f(i) \arrow[d, mapsto] \\ (i, h_n \circ f) \arrow[r, mapsto] & h_n(f(i)) \end{tikzcd}$$

B.2 Proof of lemma 4.1.25

We rewrite the source and target using the coend notation

$$((X \times Y) \otimes Z)(n) = \int^k X(k) \times Y(k) \times Z(n)^k$$

and

$$\begin{aligned} ((X \otimes Z) \times (Y \otimes Z))(n) &= \Big(\int^\ell X(\ell) \times Z(n)^\ell\Big) \times \Big(\int^m Y(m) \times Z(n)^m\Big) \\ &\cong \int^\ell \int^m X(\ell) \times Y(m) \times Z(n)^{\ell+m} \end{aligned}$$

To give an arrow $((X \times Y) \otimes Z)(n) \to (X \otimes Z)(n) \times (Y \otimes Z)(n)$, by universal property of the coend, is equivalent of giving a family of arrows

$$X(k) \times Y(k) \times Z(n)^k \to (X \otimes Z)(n) \times (Y \otimes Z)(n)$$

for all $k \in \mathbb{F}$ that satisfies the wedge condition. We take the following composite

$$\begin{tikzcd} X(k) \times Y(k) \times Z(n)^k \arrow[d] \\ X(k) \times Y(k) \times Z(n)^{k+k} \arrow[d] \\ \int^\ell X(\ell) \times Z(n)^\ell \times \int^m Y(m) \times Z(n)^m \end{tikzcd}$$

We check the wedge condition. Let $g : k \to \ell$. The following diagram commutes

$$\begin{tikzcd} & X(k) \times Y(k) \times Z(n)^k \arrow[dr] & \\ X(k) \times Y(k) \times Z(n)^\ell \arrow[ur] \arrow[dr] & & (X \otimes Z)(n) \times (Y \otimes Z)(n) \\ & X(\ell) \times Y(\ell) \times Z(n)^\ell \arrow[ur] & \end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd} & (x, y, f \circ g) \arrow[dr, mapsto] & \\ (x, y, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} ((x, f \circ g), (y, f \circ g)) \\ = ((X(g)(x), f), (Y(g)(y), f)) \end{array} \\ & (X(g)(x), Y(g)(y), f) \arrow[ur, mapsto] & \end{tikzcd}$$

and by universal properties of the coends $(x, f \circ g) = (X(g)(x), f)$ and $(y, f \circ g) = (Y(g)(y), f)$ since

$$\begin{tikzcd}
 & X(k) \times Z(n)^k \arrow[dr] & \\
X(k) \times Z(n)^\ell \arrow[ur] \arrow[dr] & & (X \otimes Z)(n) \\
 & X(\ell) \times Z(n)^\ell \arrow[ur] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & Y(k) \times Z(n)^k \arrow[dr] & \\
Y(k) \times Z(n)^\ell \arrow[ur] \arrow[dr] & & (Y \otimes Z)(n) \\
 & Y(\ell) \times Z(n)^\ell \arrow[ur] &
\end{tikzcd}$$

commute.

Inversely to give an arrow $(X \otimes Z)(n) \times (Y \otimes Z)(n) \to ((X \times Y) \otimes Z)(n)$ is equivalent, by universal property of the coend, to give a family of arrows

$$X(\ell) \times Y(m) \times Z(n)^{\ell+m} \to ((X \times Y) \otimes Z)(n)$$

for all $\ell, m \in \mathbb{F}$ that satisfies the wedge condition. We take te following composite

$$\begin{tikzcd}
X(\ell) \times Y(m) \times Z(n)^{\ell+m} \arrow[d] \\
X(\ell+m) \times Y(\ell+m) \times Z(n)^{\ell+m} \arrow[d] \\
\int^k X(k) \times Y(k) \times Z(n)^k
\end{tikzcd}$$

We check the wedge condition. Let $g : \ell_1 \to \ell_2$ and $h : m_1 \to m_2$. The following diagram commutes

$$\begin{tikzcd}
 & X(\ell_1) \times Y(m_1) \times Z(n)^{\ell_1+m_1} \arrow[dr] & \\
X(\ell_1) \times Y(m_1) \times Z(n)^{\ell_2+m_2} \arrow[ur] \arrow[dr] & & ((X \times Y) \otimes Z)(n) \\
 & X(\ell_2) \times Y(m_2) \times Z(n)^{\ell_2+m_2} \arrow[ur] &
\end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd}
 & (x, y, f \circ (g+h)) \arrow[dr, mapsto] & \\
(x, y, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{l}(X(i_{\ell_1})(x), Y(i_{m_1})(y), f \circ (g+h)) \\ = (X(i_{\ell_2} \circ g)(x), Y(i_{m_2} \circ h)(y), f)\end{array} \\
 & (X(g)(x), Y(h)(y), f) \arrow[ur, mapsto] &
\end{tikzcd}$$

where $i_{\ell_1} : \ell_1 \to \ell_1 + m_1$, $i_{m_1} : m_1 \to \ell_1 + m_1$, $i_{\ell_2} : \ell_2 \to \ell_2 + m_2$ and $i_{m_2} : m_2 \to \ell_2 + m_2$. By naturality of these inclusions $i_{\ell_2} \circ g = (g+h) \circ i_{\ell_1}$ and $i_{m_2} \circ h = (g+h) \circ i_{m_1}$. Moreover by universal property of the coend $(X(i_{\ell_1})(x), Y(i_{m_1})(y), f \circ (g+h)) = (X((g+h) \circ i_{\ell_1})(x), Y((g+h) \circ i_{m_1})(y), f)$ since they come from $(X(i_{\ell_1})(x), Y(i_{m_1})(y), f)$ with the arrow $g + h$

$$\begin{tikzcd}
 & X(\ell_1+m_1) \times Y(\ell_1+m_1) \times Z(n)^{\ell_1+m_1} \arrow[dr] & \\
X(\ell_1+m_1) \times Y(\ell_1+m_1) \times Z(n)^{\ell_2+m_2} \arrow[ur] \arrow[dr] & & \int^k X(k) \times Y(k) \times Z(n)^k \\
 & X(\ell_2+m_2) \times Y(\ell_2+m_2) \times Z(n)^{\ell_2+m_2} \arrow[ur] &
\end{tikzcd}$$

Now we check that these two arrows are inverse to each other. Starting with $(x,y,z) \in ((X \times Y) \otimes Z)(n)$ we have the following composite

$$(x,y,z) \quad \mapsto \quad (x,y,[z,z]) \quad \mapsto \quad (X(i_\ell)(x), Y(i_m)(y), [z,z])$$

where $i_\ell : \ell \to \ell + m$ and $i_m : \ell + m$. The elements (x,y,z) and $(X(i_\ell)(x), Y(i_m)(y), [z,z])$ come from $X(k+k) \times Y(k+k) \times Z(n)^k$ with the arrow $f = [\mathrm{id}_k, \mathrm{id}_k] : k + k \to k$

$$\begin{tikzcd}
 & X(k) \times Y(k) \times Z(n)^k \\
X(k+k) \times Y(k+k) \times Z(n)^k \arrow[ur, "X(f)\times Y(f)\times \mathrm{id}"] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times(-)\circ f"'] & \\
 & X(k+k) \times Y(k+k) \times Z(n)^{k+k}
\end{tikzcd}$$

on elements

$$\begin{tikzcd}
 & (x,y,z) \\
(X(i_\ell)(x), Y(i_m)(y), z) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & (X(i_\ell)(x), Y(i_m)(y), [z,z])
\end{tikzcd}$$

Starting with $(x,y,z) \in ((X \otimes Z) \times (Y \otimes Z))(n)$ we have the following composite

$$(x,y,z) \quad \mapsto \quad (X(i_\ell)(x), Y(i_m)(y), z) \quad \mapsto \quad (X(i_\ell)(x), Y(i_m)(y), [z,z])$$

The two elements (x,y,z) and $(X(i_\ell)(x), Y(i_m)(y), [z,z])$ come from $X(\ell) \times Y(m) \times Z(n)^{\ell+m+\ell+m}$ with the arrows $i_\ell : \ell \to \ell + m$ and $i_m : m \to \ell + m$

$$\begin{tikzcd}
 & X(\ell) \times Y(m) \times Z(n)^{\ell+m} \\
X(\ell) \times Y(m) \times Z(n)^{\ell+m+\ell+m} \arrow[ur, "\mathrm{id} \times \mathrm{id} \times (-)\circ(i_\ell + i_m)"] \arrow[dr, "X(i_\ell)\times Y(i_m) \times \mathrm{id}"'] & \\
 & X(\ell+m) \times Y(\ell+m) \times Z(n)^{\ell+m+\ell+m}
\end{tikzcd}$$

on elements

$$\begin{tikzcd}
 & (x, y, z) \\
(x, y, [z, z]) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & (X(i_\ell)(x), Y(i_m)(y), [z, z])
\end{tikzcd}$$

Finally we check naturalities in X, Y and Z. Let $f : X \to W$. The naturality square

$$\begin{tikzcd}
(X \times Y) \otimes Z \arrow[r] \arrow[d] & (X \otimes Z) \times (Y \otimes Z) \arrow[d] \\
(W \times Y) \otimes Z \arrow[r] & (W \otimes Z) \times (Y \otimes Z)
\end{tikzcd}$$

commutes because we have on elements

$$\begin{tikzcd}
(x, y, z) \arrow[r, mapsto] \arrow[d, mapsto] & (x, y, [z, z]) \arrow[d, mapsto] \\
(f_k(x), y, z) \arrow[r, mapsto] & (f_k(x), y, [z, z])
\end{tikzcd}$$

Let $f : Y \to W$. The naturality square

$$\begin{tikzcd}
(X \times Y) \otimes Z \arrow[r] \arrow[d] & (X \otimes Z) \times (Y \otimes Z) \arrow[d] \\
(X \times W) \otimes Z \arrow[r] & (X \otimes Z) \times (W \otimes Z)
\end{tikzcd}$$

commutes because we have on elements

$$\begin{tikzcd}
(x, y, z) \arrow[r, mapsto] \arrow[d, mapsto] & (x, y, [z, z]) \arrow[d, mapsto] \\
(x, f_k(y), z) \arrow[r, mapsto] & (x, f_k(y), [z, z])
\end{tikzcd}$$

Let $f : Z \to W$. The naturality square

$$\begin{tikzcd}
(X \times Y) \otimes Z \arrow[r] \arrow[d] & (X \otimes Z) \times (Y \otimes Z) \arrow[d] \\
(X \times Y) \otimes W \arrow[r] & (X \otimes W) \times (Y \otimes W)
\end{tikzcd}$$

commutes because we have on elements

$$\begin{tikzcd}
(x, y, z) \arrow[r, mapsto] \arrow[d, mapsto] & (x, y, [z, z]) \arrow[d, mapsto] \\
(x, y, f_n \circ z) \arrow[r, mapsto] & (x, y, f_n \circ [z, z])
\end{tikzcd}$$

B.3 Proof of lemma 4.1.26

We rewrite the source using the left Kan extension notation

$$(X + Y) \otimes Z = \mathrm{Lan}_U(X + Y) \circ Z$$

Since Lan_U is a left adjoint, it preserves coproducts, so

$$\begin{aligned}
\mathrm{Lan}_U(X + Y) \circ Z &\cong (\mathrm{Lan}_U(X) + \mathrm{Lan}_U(Y)) \circ Z \\
&= \mathrm{Lan}_U(X) \circ Z + \mathrm{Lan}_U(Y) \circ Z \\
&= (X \otimes Z) + (Y \otimes Z)
\end{aligned}$$

Naturalities in X and Y follow from naturalities of the canonical isomorphism $\mathrm{Lan}_U(X+Y) \cong \mathrm{Lan}_U(X)+\mathrm{Lan}_U(Y)$. Naturality in Z is easy to see.

B.4 Proof of lemma 4.2.28

Let A be a set. We rewrite the source and target using the coend notation

$$\ell(X \times Y)(A) = \int^k X(k) \times Y(k) \times A^k$$

and

$$\begin{aligned}
\ell(X)(A) \times \ell(Y)(A) &= \Big(\int^n X(n) \times A^n\Big) \times \Big(\int^m Y(m) \times A^m\Big) \\
&\cong \int^n \int^m X(n) \times Y(m) \times A^{n+m}
\end{aligned}$$

To give an arrow $\ell(X \times Y)(A) \to \ell(X)(A) \times \ell(Y)(A)$, by universal property of the coend, is equivalent of giving a family of arrows

$$X(k) \times Y(k) \times A^k \to \ell(X)(A) \times \ell(Y)(A)$$

for all $k \in \mathbb{F}$ that satisfies the wedge condition. We take the following composite

$$\begin{tikzcd}
X(k) \times Y(k) \times A^k \arrow[d] \\
X(k) \times Y(k) \times A^{k+k} \arrow[d] \\
\int^n X(n) \times A^n \times \int^m Y(m) \times A^m
\end{tikzcd}$$

We check the wedge condition. Let $g : k \to r$. The following diagram commutes

$$\begin{tikzcd}
 & X(k) \times Y(k) \times A^k \arrow[dr] & \\
X(k) \times Y(k) \times A^r \arrow[ur] \arrow[dr] & & \ell(X)(A) \times \ell(Y)(A) \\
 & X(r) \times Y(r) \times A^r \arrow[ur] &
\end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd}
 & (x, y, f \circ g) \arrow[dr, mapsto] & \\
(x, y, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}((x, f \circ g), (y, f \circ g)) \\ = ((X(g)(x), f), (Y(g)(y), f))\end{array} \\
 & (X(g)(x), Y(g)(y), f) \arrow[ur, mapsto] &
\end{tikzcd}$$

and by universal properties of the coends $(x, f \circ g) = (X(g)(x), f)$ and $(y, f \circ g) = (Y(g)(y), f)$ since

$$\begin{tikzcd}
 & X(k) \times A^k \arrow[dr] & \\
X(k) \times A^r \arrow[ur] \arrow[dr] & & \ell(X)(A) \\
 & X(r) \times A^r \arrow[ur] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & Y(k) \times A^k \arrow[dr] & \\
Y(k) \times A^r \arrow[ur] \arrow[dr] & & \ell(Y)(A) \\
 & Y(r) \times A^r \arrow[ur] &
\end{tikzcd}$$

commute.

Inversely to give an arrow $\ell(X)(A) \times \ell(Y)(A) \to \ell(X \times Y)(A)$ is equivalent, by universal property of the coend, to give a family of arrows

$$X(n) \times Y(m) \times A^{n+m} \to \ell(X \times Y)(A)$$

for all $n, m \in \mathbb{F}$ that satisfies the wedge condition. We take te following composite

$$\begin{tikzcd}
X(n) \times Y(m) \times A^{n+m} \arrow[d] \\
X(n+m) \times Y(n+m) \times A^{n+m} \arrow[d] \\
\int^k X(k) \times Y(k) \times A^k
\end{tikzcd}$$

We check the wedge condition. Let $g : n_1 \to n_2$ and $h : m_1 \to m_2$. The following diagram commutes

$$\begin{tikzcd} & X(n_1)\times Y(m_1)\times A^{n_1+m_1} \arrow[dr] & \\ X(n_1)\times Y(m_1)\times A^{n_2+m_2} \arrow[ur] \arrow[dr] & & \ell(X\times Y)(A) \\ & X(n_2)\times Y(m_2)\times A^{n_2+m_2} \arrow[ur] & \end{tikzcd}$$

because we have the following assignations on elements

$$\begin{tikzcd} & (x,y,f\circ(g+h)) \arrow[dr, mapsto] & \\ (x,y,f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{l}(X(i_{n_1})(x), Y(i_{m_1})(y), f\circ(g+h)) \\ = (X(i_{n_2}\circ g)(x), Y(i_{m_2}\circ h)(y), f)\end{array} \\ & (X(g)(x), Y(h)(y), f) \arrow[ur, mapsto] & \end{tikzcd}$$

where $i_{n_1} : n_1 \to n_1+m_1$, $i_{m_1} : m_1 \to n_1+m_1$, $i_{n_2} : n_2 \to n_2+m_2$ and $i_{m_2} : m_2 \to n_2+m_2$. By naturality of these inclusions $i_{n_2}\circ g = (g+h)\circ i_{n_1}$ and $i_{m_2}\circ h = (g+h)\circ i_{m_1}$. Moreover by universal property of the coend $(X(i_{n_1})(x), Y(i_{m_1})(y), f\circ(g+h)) = (X((g+h)\circ i_{n_1})(x), Y((g+h)\circ i_{m_1})(y), f)$ since they come from $(X(i_{n_1})(x), Y(i_{m_1})(y), f)$ with the arrow $g+h$

$$\begin{tikzcd} & \begin{array}{l}X(n_1+m_1)\\ \times Y(n_1+m_1)\\ \times A^{n_1+m_1}\end{array} \arrow[dr] & \\ \begin{array}{l}X(n_1+m_1)\\ \times Y(n_1+m_1)\\ \times A^{n_2+m_2}\end{array} \arrow[ur] \arrow[dr] & & \int^k X(k)\times Y(k)\times A^k \\ & \begin{array}{l}X(n_2+m_2)\\ \times Y(n_2+m_2)\\ \times A^{n_2+m_2}\end{array} \arrow[ur] & \end{tikzcd}$$

Now we check that these two arrows are inverse to each other. Starting with $(x,y,a) \in \ell(X\times Y)(A)$ we have the following composite

$$(x,y,a) \quad\mapsto\quad (x,y,[a,a]) \quad\mapsto\quad (X(i_n)(x), Y(i_m)(y), [a,a])$$

where $i_n : n \to n+m$ and $i_m : n+m$. The elements (x,y,a) and $(X(i_n)(x), Y(i_m)(y), [a,a])$ come from $X(k+k)\times Y(k+k)\times A^k$ with the arrow $f = [\mathrm{id}_k, \mathrm{id}_k] : k+k \to k$

$$\begin{tikzcd} & X(k)\times Y(k)\times A^k \\ X(k+k)\times Y(k+k)\times A^k \arrow[ur, "X(f)\times Y(f)\times \mathrm{id}"] \arrow[dr, "\mathrm{id}\times\mathrm{id}\times(-)\circ f"'] & \\ & X(k+k)\times Y(k+k)\times A^{k+k} \end{tikzcd}$$

on elements

$$\begin{tikzcd}
 & (x,y,a) \\
(X(i_n)(x),Y(i_m)(y),a) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & (X(i_n)(x),Y(i_m)(y),[a,a])
\end{tikzcd}$$

Starting with $(x,y,a) \in \ell(X)(A) \times \ell(Y)(A)$ we have the following composite

$$(x,y,a) \quad \mapsto \quad (X(i_n)(x),Y(i_m)(y),a) \quad \mapsto \quad (X(i_n)(x),Y(i_m)(y),[a,a])$$

The two elements (x,y,a) and $(X(i_n)(x),Y(i_m)(y),[a,a])$ come from $X(n) \times Y(m) \times A^{n+m+n+m}$ with the arrows $i_n : n \to n+m$ and $i_m : m \to n+m$

$$\begin{tikzcd}
 & X(n) \times Y(m) \times A^{n+m} \\
X(n) \times Y(m) \times A^{n+m+n+m} \arrow[ur, "\mathrm{id} \times \mathrm{id} \times (-)\circ(i_n+i_m)"] \arrow[dr, "X(i_n) \times Y(i_m) \times \mathrm{id}"'] & \\
 & X(n+m) \times Y(n+m) \times A^{n+m+n+m}
\end{tikzcd}$$

on elements

$$\begin{tikzcd}
 & (x,y,a) \\
(x,y,[a,a]) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & (X(i_n)(x),Y(i_m)(y),[a,a])
\end{tikzcd}$$

Finally we check naturalities in A and X, Y. Let $f : A \to B$. The naturality square

$$\begin{tikzcd}
\ell(X \times Y)(A) \arrow[r] \arrow[d] & \ell(X)(A) \times \ell(Y)(A) \arrow[d] \\
\ell(X \times Y)(B) \arrow[r] & \ell(X)(B) \times \ell(Y)(B)
\end{tikzcd}$$

commutes because we have on elements

$$\begin{tikzcd}
(x,y,a) \arrow[r, mapsto] \arrow[d, mapsto] & (x,y,[a,a]) \arrow[d, mapsto] \\
(x,y,f \circ a) \arrow[r, mapsto] & (x,y,f \circ [a,a])
\end{tikzcd}$$

Let $f : X \to W$. The naturality square

$$\begin{tikzcd}
\ell(X \times Y)(A) \arrow[r] \arrow[d] & \ell(X)(A) \times \ell(Y)(A) \arrow[d] \\
\ell(W \times Y)(A) \arrow[r] & \ell(W)(A) \times \ell(Y)(A)
\end{tikzcd}$$

commutes because we have on elements

$$\begin{tikzcd}
(x,y,a) \arrow[r, mapsto] \arrow[d, mapsto] & (x,y,[a,a]) \arrow[d, mapsto] \\
(f_k(x),y,a) \arrow[r, mapsto] & (f_k(x),y,[a,a])
\end{tikzcd}$$

Let $f : Y \to W$. The naturality square

$$\begin{tikzcd}
\ell(X\times Y)(A) \arrow[r] \arrow[d] & \ell(X)(A)\times\ell(Y)(A) \arrow[d] \\
\ell(X\times W)(A) \arrow[r] & \ell(X)(A)\times\ell(W)(A)
\end{tikzcd}$$

commutes because we have on elements

$$\begin{tikzcd}
(x,y,a) \arrow[r, mapsto] \arrow[d, mapsto] & (x,y,[a,a]) \arrow[d, mapsto] \\
(x,f_k(y),a) \arrow[r, mapsto] & (x,f_k(y),[a,a])
\end{tikzcd}$$

B.5 Proof of proposition 4.2.30

It is obviously natural in F and G. We verify the commutativity of the monoidal functor axioms. The first one concerning the associativities is as follows.

$$\begin{tikzcd}
\int^p\int^r F(r)\times(G(p))^r\times(H(n))^p \arrow[r] \arrow[d] & \int^r F(r)\times(\int^q G(q)\times(H(n))^q)^r \arrow[d] \\
\int^p F(G(p))\times(H(n))^p \arrow[d] & \int^r F(r)\times(G(H(n)))^r \arrow[d] \\
F(G(H(n))) \arrow[r] & F(G(H(n)))
\end{tikzcd}$$

On elements we find

$$\begin{tikzcd}
(x,g=[g_1,\ldots,g_r]:r\to G(p),h:p\to H(n)) \arrow[r, mapsto] \arrow[d, mapsto] & (x,g_1,\ldots,g_r,h,\ldots,h) \arrow[d, mapsto] \\
(F(g)(x),h) \arrow[d, mapsto] & (x,G(h)(g_1),\ldots,G(h)(g_r)) \arrow[d, mapsto] \\
F(G(h)\circ g)(x) \arrow[r, mapsto] & F(G(h)\circ g)(x)
\end{tikzcd}$$

Now we turn to the right unit axiom. Along the right–hand side of the square we have the following composite

$$\begin{tikzcd}
\int^m (F \circ U)(m) \times (U(n))^m \arrow[d] & (x, f) \arrow[d, mapsto] \\
\begin{array}{c} \int^m (F \circ U)(m) \times (\mathrm{Id} \circ U(n))^m \\ = \int^m F(m) \times (\mathrm{Id} \circ U(n))^m \end{array} \arrow[d] & (x, f) \arrow[d, mapsto] \\
F \circ \mathrm{Id} \circ U(n) \arrow[d] & F(f)(x) \arrow[d, mapsto] \\
F \circ U(n) & F(f)(x)
\end{tikzcd}$$

which yields $\rho_{F,n}(x, f) = F(f)(x)$ as desired. Now we take a look at the left unit axiom. Along the right–hand side of the square we have the following composite

$$\begin{tikzcd}
\begin{array}{c} \int^m U(m) \times (F \circ U(n))^k \\ = \int^m m \times (F(n))^m \end{array} \arrow[d] & (i, f) \arrow[d, mapsto] \\
\int^m \mathrm{Id}(m) \times (F \circ U(n))^m \arrow[d] & (i, f) \arrow[d, mapsto] \\
\mathrm{Id} \circ F \circ U(n) \arrow[d] & f(i) \arrow[d, mapsto] \\
F \circ U(n) & f(i)
\end{tikzcd}$$

which yields $\lambda_{F,n}(i, f) = f(i)$ as desired.

B.6 Proof of proposition 4.2.31

We check the wedge conditions. Let $h_i : m_i \to q_i$ be arrows in $\mathbb{F}$. The following diagram commutes

$$\begin{tikzcd}
& \begin{array}{l} X(n) \times Y(m_1) \times \ldots \times Y(m_n) \\ \times A^{\sum_{i=1}^n m_i} \end{array} \arrow[dr] & \\
\begin{array}{l} X(n) \times Y(m_1) \times \ldots \times Y(m_n) \\ \times A^{\sum_{i=1}^n q_i} \end{array} \arrow[ur] \arrow[dr] & & \ell(X \otimes Y)(A) \\
& \begin{array}{l} X(n) \times Y(q_1) \times \ldots \times Y(q_n) \\ \times A^{\sum_{i=1}^n q_i} \end{array} \arrow[ur] &
\end{tikzcd}$$

because on elements we have the following assignations

$$\begin{tikzcd}
 & x, y_1, \ldots, y_n, a \circ \sum_{i=1}^{n} h_i \arrow[dr, mapsto] & \\
x, y_1, \ldots, y_n, a \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} x, Y(c_1)(y_1), \ldots, \\ Y(c_n)(y_n), a \circ \sum_{i=1}^{n} h_i \\ = x, Y(c_1' \circ h_1)(y_1), \ldots, \\ Y(c_n' \circ h_n)(y_n), a \end{array} \\
 & x, Y(h_1)(y_1), \ldots, Y(h_n)(y_n), a \arrow[ur, mapsto] &
\end{tikzcd}$$

The elements $x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), a \circ \sum_{i=1}^{n} h_i$ and $x, Y(c_1' \circ h_1)(y_1), \ldots, Y(c_n' \circ h_n)(y_n), a$ are equal since they come from $x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), a \in X(n) \times Y(\sum_{i=1}^{n} m_i)^n \times A^{\sum_{i=1}^{n} q_i}$ with the arrow $\sum_{i=1}^{n} h_i$ as illustrated below

$$\begin{tikzcd}
 & \begin{array}{l} X(n) \times \\ \times Y(\sum_{i=1}^{n} m_i)^n \times A^{\sum_{i=1}^{n} m_i} \end{array} \arrow[dr] & \\
\begin{array}{l} X(n) \times \\ \times Y(\sum_{i=1}^{n} m_i)^n \times A^{\sum_{i=1}^{n} q_i} \end{array} \arrow[ur, "\mathrm{id} \times (-) \circ \sum h_i"] \arrow[dr, "\mathrm{id} \times (Y(\sum h_i))^n \times \mathrm{id}"'] & & \begin{array}{l} X(n) \times \\ \times \int^r Y(r)^n \times A^r \end{array} \\
 & \begin{array}{l} X(n) \times \\ \times Y(\sum_{i=1}^{n} q_i)^n \times A^{\sum_{i=1}^{n} q_i} \end{array} \arrow[ur] &
\end{tikzcd}$$

and $\sum_{i=1}^{n} h_i \circ c_i = c_i' \circ h_i$ by naturality of the inclusions.

So by universal property of the coend we have an arrow $X(n) \times \left(\int^m Y(m) \times A^m\right)^n \to \ell(X \otimes Y)(A)$ for all $n \in \mathbb{F}$.

Now let $f : n \to p$ be an arrow. The following diagram commutes

$$\begin{tikzcd}
 & X(n) \times \left(\int^m Y(m) \times A^m\right)^n \arrow[dr] & \\
X(n) \times \left(\int^m Y(m) \times A^m\right)^p \arrow[ur] \arrow[dr] & & \ell(X \otimes Y)(A) \\
 & X(p) \times \left(\int^m Y(m) \times A^m\right)^p \arrow[ur] &
\end{tikzcd}$$

because on elements we find

$$\begin{tikzcd}
 & x, y_{f(1)}, \dots, y_{f(n)}, a \circ h \arrow[dr, mapsto] & \\
x, y_1, \dots, y_p, a \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} x, Y(c_1)(y_{f(1)}), \dots, \\ Y(c_n)(y_{f(n)}), a \circ h \\ = X(f)(x), Y(c'_1)(y_1), \dots, \\ Y(c'_p)(y_p), a \end{array} \\
 & X(f)(x), y_1, \dots, y_p, a \arrow[ur, mapsto] &
\end{tikzcd}$$

and $h : \sum_{i=1}^{n} m_{f(i)} \to \sum_{i=1}^{p} m_i = r \to r'$ is induced by f. By universal property of the outer coend we have

$$\begin{tikzcd}
 & X(n) \times \int^r (Y(r))^n \times A^r \arrow[dr] & \\
X(n) \times \int^r (Y(r))^p \times A^r \arrow[ur, "\mathrm{id} \times ((-)\circ f) \times \mathrm{id}"] \arrow[dr, "X(f)\times \mathrm{id}"'] & & \int^n X(n) \times \int^r (Y(r))^n \times A^r \\
 & X(p) \times \int^r (Y(r))^p \times A^r \arrow[ur] &
\end{tikzcd} \tag{B.3}$$

by universal property of the inner coend we have

$$\begin{tikzcd}
 & Y(r)^n \times A^r \arrow[dr] & \\
Y(r)^n \times A^{r'} \arrow[ur, "\mathrm{id} \times ((-)\circ h)"] \arrow[dr, "X(h)^n \times \mathrm{id}"'] & & \int^r Y(r)^n \times A^r \\
 & Y(r')^n \times A^{r'} \arrow[ur] &
\end{tikzcd} \tag{B.4}$$

So together

$$X(f)(x), Y(c'_1)(y_1), \dots, Y(c'_p)(y_p), a$$

is identic by (B.3) with

$$\begin{aligned} & x, Y(c'_{f(1)})(y_{f(1)}), \dots, Y(c'_{f(n)})(y_{f(n)}), a \\ = {} & x, Y(h \circ c_1)(y_{f(1)}), \dots, Y(h \circ c_n)(y_{f(n)}), a \end{aligned}$$

is identic by (B.4) with

$$x, Y(c_1)(y_{f(1)}), \dots, Y(c_n)(y_{f(n)}), a \circ h$$

Now we check naturality in A, X and Y of the above constructed arrow. Let $f : A \to B$. The naturality square

$$\begin{tikzcd}
\ell X(\ell Y(A)) \arrow[r] \arrow[d] & \ell(X \otimes Y)(A) \arrow[d] \\
\ell X(\ell Y(B)) \arrow[r] & \ell(X \otimes Y)(B)
\end{tikzcd}$$

commutes because on elements we find

$$\begin{tikzcd} x, y_1, \ldots, y_n, a \arrow[r, mapsto] \arrow[d, mapsto] & x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), a \arrow[d, mapsto] \\ x, y_1, \ldots, y_n, f \circ a \arrow[r, mapsto] & x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), f \circ a \end{tikzcd}$$

Let $f : X \to Z$ be an arrow in $\mathcal{F}$. The naturality square

$$\begin{tikzcd} \ell X(\ell Y(A)) \arrow[r] \arrow[d] & \ell(X \otimes Y)(A) \arrow[d] \\ \ell Z(\ell Y(A)) \arrow[r] & \ell(Z \otimes Y)(A) \end{tikzcd}$$

commutes because on elements we find

$$\begin{tikzcd} x, y_1, \ldots, y_n, a \arrow[r, mapsto] \arrow[d, mapsto] & x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), a \arrow[d, mapsto] \\ f_n(x), y_1, \ldots, y_n, a \arrow[r, mapsto] & f_n(x), Y(c_1)(y_1), \ldots, Y(c_n)(y_n), a \end{tikzcd}$$

Let $f : Y \to Z$ be an arrow in $\mathcal{F}$. The naturality square

$$\begin{tikzcd} \ell X(\ell Y(A)) \arrow[r] \arrow[d] & \ell(X \otimes Y)(A) \arrow[d] \\ \ell X(\ell Z(A)) \arrow[r] & \ell(X \otimes Z)(A) \end{tikzcd}$$

commutes because on elements we find

$$\begin{tikzcd} x, y_1, \ldots, y_n, a \arrow[r, mapsto] \arrow[d, mapsto] & x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), a \arrow[d, mapsto] \\ x, f_{m_1}(y_1), \ldots, f_{m_n}(y_n), a \arrow[r, mapsto] & \begin{array}{c} x, f_r(Y(c_1)(y_1)), \ldots, f_r(Y(c_n)(y_n)), a \\ = x, Z(c_1)(f_{m_1}(y_1)), \ldots, Z(c_n)(f_{m_n}(y_n)), a \end{array} \end{tikzcd}$$

Now let us check the left unit axiom. Along the right–hand side we have the following composite

$$\begin{tikzcd} \int^n X(n) \times A^n \arrow[d] & x, a_1, \ldots, a_n \arrow[d, mapsto, "m_i = 1"] \\ \int^n X(n) \times \left(\int^m m \times A^m\right)^n \arrow[d] & x, (1, a_1), \ldots (1, a_n) \arrow[d, mapsto, "r = \sum_{i=1}^n m_i = n"] \\ \int^r \int^n X(n) \times r^n \times A^r \arrow[d] & x, \mathrm{id}_n, a_1, \ldots, a_n \arrow[d, mapsto] \\ \int^r X(r) \times A^r & X(\mathrm{id}_n)(x), a_1, \ldots, a_n \end{tikzcd}$$

which yields identity on $x, a_1, \ldots, a_n$ as desired. As for the right unit axiom we find along the right hand side the following composite

$$
\begin{tikzcd}
\int^m X(m) \times A^m \arrow[d] & x, a_1, \ldots, a_m \arrow[d, mapsto, "n=1"] \\
\int^n n \times \left(\int^m X(m) \times A^m\right)^n \arrow[d] & 1, (x, a_1, \ldots, a_n) \arrow[d, mapsto, "r=\sum_{i=1}^n m_i = m"] \\
\int^r \int^n n \times X(r)^n \times A^r \arrow[d] & 1, x, a_1, \ldots, a_n \arrow[d, mapsto] \\
\int^r X(r) \times A^r & x, a_1, \ldots, a_n
\end{tikzcd}
$$

which yields identity on $x, a_1, \ldots, a_n$ as desired.

The axiom concerning the associativities is as follows

$$
\begin{tikzcd}
((\ell X \circ \ell Y) \circ \ell Z)(A) \arrow[r, equal] \arrow[d] & (\ell X \circ (\ell Y \circ \ell Z))(A) \arrow[d] \\
(\ell(X \otimes Y) \circ \ell Z)(A) \arrow[d] & (\ell X \circ \ell(Y \otimes Z))(A) \arrow[d] \\
\ell((X \otimes Y) \otimes Z)(A) \arrow[r] & \ell(X \otimes (Y \otimes Z))(A)
\end{tikzcd}
$$

On elements along the left hand side we have

$$
\begin{tikzcd}
\begin{array}{l} \int^n \int^{m_1} \int^{k_{1,1}} \ldots \int^{k_{1,m_1}} \ldots \int^{m_n} \int^{k_{n,1}} \ldots \int^{k_{n,m_n}} \\ X(n) \times Y(m_1) \times \ldots \times Y(m_n) \\ \times Z(k_{1,1}) \times \ldots \times Z(k_{1,m_1}) \times A^{k_{1,1}+\ldots+k_{1,m_1}} \\ \times \ldots \\ \times Z(k_{n,1}) \times \ldots \times Z(k_{n,m_n}) \times A^{k_{n,1}+\ldots+k_{n,m_n}} \end{array} \arrow[d] & \begin{array}{c} x, y_1, \ldots, y_m, \\ z_{1,1}, \ldots, z_{1,m_1}, a_1, \\ \ldots, \\ z_{n,1}, \ldots, z_{n,m_n}, a_n \end{array} \arrow[d, mapsto, "r=\sum m_i"] \\
\begin{array}{l} \int^n \int^r \int^{p_1} \ldots \int^{p_r} X(n) \times Y(r)^n \\ \times Z(p_1) \times \ldots \times Z(p_r) \times A^{p_1+\ldots+p_r} \end{array} \arrow[d] & \begin{array}{c} x, \overline{y}_1, \ldots, \overline{y}_m, \\ z_{1,1}, \ldots, z_{1,m_1}, \ldots, \\ z_{n,1}, \ldots, z_{n,m_n}, [a_1, \ldots, a_n] \end{array} \arrow[d, mapsto, "q=\sum p_i = \sum k_{i,j}"] \\
\int^n \int^r \int^q X(n) \times Y(r)^n \times Z(q)^r \times A^q \arrow[d] & \begin{array}{c} x, \overline{y}_1, \ldots, \overline{y}_m, \\ \overline{z}_{1,1}, \ldots, \overline{z}_{n,m_n}, [a_1, \ldots, a_n] \end{array} \arrow[d, mapsto, "\ell_1=\ldots=\ell_n=r=\sum m_i"] \\
\begin{array}{l} \int^n \int^q \int^{\ell_1} \ldots \int^{\ell_n} X(n) \times Y(\ell_1) \times \ldots \times Y(\ell_n) \\ \times Z(q)^{\ell_1+\ldots+\ell_n} \times A^q \end{array} & \begin{array}{c} x, \overline{y}_1, \ldots, \overline{y}_m, \\ \overline{z}_{1,1}, \ldots, \overline{z}_{n,m_n}, \\ \ldots, \\ \overline{z}_{1,1}, \ldots, \overline{z}_{n,m_n}, \\ [a_1, \ldots, a_n] \end{array}
\end{tikzcd}
$$

where $\overline{y}_i \in Y(\sum\limits_{i=1}^{n} m_i)$ and $\overline{z}_{i,j} \in Z(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m_i} k_{i,j})$. Along the right–hand side we have on elements

$$\begin{array}{ccc}
\begin{array}{l}
\int^n \int^{m_1} \int^{k_{1,1}} \dots \int^{k_{1,m_1}} \dots \int^{m_n} \int^{k_{n,1}} \dots \int^{k_{n,m_n}} \\
X(n) \times Y(m_1) \times \dots \times Y(m_n) \\
\times Z(k_{1,1}) \times \dots \times Z(k_{1,m_1}) \times A^{k_{1,1}+\dots+k_{1,m_1}} \\
\times \dots \\
\times Z(k_{n,1}) \times \dots \times Z(k_{n,m_n}) \times A^{k_{n,1}+\dots+k_{n,m_n}}
\end{array}
& &
\begin{array}{c}
x, y_1, \dots, y_m, \\
z_{1,1}, \dots, z_{1,m_1}, a_1, \\
\dots, \\
z_{n,1}, \dots, z_{n,m_n}, a_n
\end{array} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle p_i = \sum_j k_{i,j}} \\
\begin{array}{l}
\int^n \int^{m_1} \dots \int^{m_n} \int^{p_1} \dots \int^{p_n} \\
X(n) \times Y(m_1) \times \dots \times Y(m_n) \\
\times Z(p_1)^{m_1} \times \dots \times Z(p_n)^{m_n} \times A^{p_1+\dots+p_n}
\end{array}
& &
\begin{array}{c}
x, y_1, \dots, y_m, \\
\widehat{z}_{1,1}, \dots, \widehat{z}_{1,m_1}, \\
\dots, \\
\widehat{z}_{n,1}, \dots, \widehat{z}_{n,m_n}, \\
[a_1, \dots, a_n]
\end{array} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle q = \sum p_i} \\
\begin{array}{l}
\int^n \int^q \int^{m_1} \dots \int^{m_n} \\
X(n) \times Y(m_1) \times \dots \times Y(m_n) \\
\times Z(q)^{m_1+\dots+m_n} \times A^q
\end{array}
& &
\begin{array}{c}
x, y_1, \dots, y_m, \\
\overline{z}_{1,1}, \dots, \overline{z}_{n,m_n}, \\
[a_1, \dots, a_n]
\end{array}
\end{array}$$

The two resulting elements are equal since they come from $X(n)\times Y(m_1)\times\dots\times Y(m_n)\times Z(q)^{\sum m_i+\dots+\sum m_i}\times A^q$ with the arrows $c_j : m_j \to \sum m_i$ for $j = 1, \dots, n$

$$\begin{tikzcd}
& \begin{array}{l} X(n) \times Y(m_1) \times \dots \times Y(m_n) \\ \times Z(q)^{m_1+\dots+m_n} \times A^q \end{array} \arrow[dr] & \\
\begin{array}{l} X(n) \times Y(m_1) \times \dots \times Y(m_n) \\ \times Z(q)^{\sum m_i+\dots+\sum m_i} \times A^q \end{array} \arrow[ur, "\mathrm{id} \times ((-)\circ(c_1+\dots+c_n)) \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times Y(c_1)\times\dots\times Y(c_n) \times \mathrm{id}"'] & & \int \dots \\
& \begin{array}{l} X(n) \times Y(\sum m_i) \times \dots \times Y(\sum m_i) \\ \times Z(q)^{\sum m_i+\dots+\sum m_i} \times A^q \end{array} \arrow[ur] &
\end{tikzcd}$$

B.7 Proof of proposition 4.3.34

Let us recall that $\eta_{X,n} : X(n) \to \int^m X(m) \times n^m$ is given by the composite of the following mapping with the n-th coprojection.

$$x \quad \mapsto \quad (x, \mathrm{id}_n)$$

By lemma 4.2.32 the composite $k\ell$ is monidal. The identity functor $\mathrm{Id}_{\mathcal{F}}$ is trivially monoidal. The first monoidal natural transformation axiom is in our case

$$\begin{tikzcd}
& k\ell X \otimes k\ell Y \arrow[d] \\
X \otimes Y \arrow[ur] \arrow[dr] & k(\ell X \circ \ell Y) \arrow[d] \\
& k\ell(X \otimes Y)
\end{tikzcd}$$

The composite along the right–hand side is the following

$$\begin{tikzcd}
\int^m X(m) \times (Y(n))^m \arrow[d] & (x, y_1, \ldots, y_m) \arrow[d, mapsto, "p=m, q_i=n"] \\
\begin{array}{c}\int^m \int^p \int^{q_1} \ldots \int^{q_m} X(p) \times m^p \times \\ \times Y(q_1) \times \ldots \times Y(q_m) \times n^{\sum_{i=1}^m q_i}\end{array} \arrow[d] & (x, \mathrm{id}_m, y_1, \ldots, y_m, [\mathrm{id}_n, \ldots, \mathrm{id}_n]) \arrow[d, mapsto] \\
\begin{array}{c}\int^m \int^{q_1} \ldots \int^{q_m} X(m) \times \\ \times Y(q_1) \times \ldots \times Y(q_m) \times n^{\sum_{i=1}^m q_i}\end{array} \arrow[d] & (X(\mathrm{id}_m)(x), y_1, \ldots, y_m, [\mathrm{id}_n, \ldots, \mathrm{id}_n]) \arrow[d, mapsto, "r=\sum_{i=1}^m q_i"] \\
\int^r \int^m X(m) \times (Y(r))^m \times n^r & \begin{array}{c}(x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), \\ [\mathrm{id}_n, \ldots, \mathrm{id}_n])\end{array}
\end{tikzcd}$$

Along the left–hand side we have

$$\begin{tikzcd}
\int^m X(m) \times (Y(n))^m \arrow[d] & (x, y_1, \ldots, y_m) \arrow[d, mapsto, "r=n"] \\
\int^r \int^m X(m) \times (Y(r))^m \times n^r & (x, y_1, \ldots, y_m, \mathrm{id}_n)
\end{tikzcd}$$

These two elements $(x, y_1, \ldots, y_m, \mathrm{id}_n)$ and $(x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), [\mathrm{id}_n, \ldots, \mathrm{id}_n])$ come from $\int^m X(m) \times (Y(m \times n))^m \times n^n$ with the arrow $[\mathrm{id}_n, \ldots, \mathrm{id}_n] : n + \ldots + n = m \times n \to n$

$$\begin{tikzcd}
& \int^m X(m) \times (Y(n))^m \times n^n \arrow[dr] & \\
\begin{array}{c}\int^m X(m) \times (Y(m \times n))^m \\ \times n^n\end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c}\int^r \int^m X(m) \\ \times (Y(r))^m \times n^r\end{array} \\
& \int^m X(m) \times (Y(m \times n))^m \times n^{m \times n} \arrow[ur] &
\end{tikzcd}$$

and on elements

$$\begin{tikzcd}
& (x, y_1, \ldots, y_m, \mathrm{id}_n) \\
(x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), \mathrm{id}_n) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
& (x, Y(c_1)(y_1), \ldots, Y(c_n)(y_n), [\mathrm{id}_n, \ldots, \mathrm{id}_n])
\end{tikzcd}$$

The triangle

$$\begin{tikzcd}
& U \arrow[dl, "\mathrm{Id}"'] \arrow[dr] & \\
U \arrow[rr, "\eta_U"'] & & k\ell U
\end{tikzcd}$$

becomes the following

$$\begin{tikzcd}
& U \arrow[dl, "\mathrm{Id}_U"'] \arrow[dr, "\psi"] & \\
& & k\,\mathrm{Id} \arrow[d, "k\phi"] \\
U \arrow[rr, "\eta_U"'] & & k\ell U
\end{tikzcd}$$

if we unfold the arrows. Let $n \in \mathbb{F}$ and $i \in n$. On elements we have

$$\begin{tikzcd}
& i \arrow[dl, mapsto, "\mathrm{Id}_U"'] \arrow[dr, mapsto, "\psi"] & \\
& & i \arrow[d, mapsto, "k\phi"] \\
i \arrow[rr, mapsto, "\eta_U"'] & & (1,i) \\
& & = (i, \mathrm{id}_n)
\end{tikzcd}$$

The two elements $(1,i)$ and (i, id_n) are identic since they come from $1 \times n^n$ with the arrow $i : 1 \to n$

$$\begin{tikzcd}
& n \times n^n \arrow[dr] & \\
1 \times n^n \arrow[ur] \arrow[dr] & & \int^m m \times n^m \\
& 1 \times n^1 \arrow[ur] &
\end{tikzcd}$$

on elements

$$\begin{tikzcd}
& (i, \mathrm{id}_n) \\
(1, \mathrm{id}_n) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
& (1, i)
\end{tikzcd}$$

B.8 Proof of proposition 4.3.35

Let us recall that $\varepsilon_{F,A} : \int^n F(n) \times A^n \to F(A)$ is given by the universal property of the coend, and the arrows

$$F(n) \times A^n \to F(A)$$

for all $n \in \mathbb{F}$ are given by the following mapping

$$(x, f) \quad \mapsto \quad F(f)(x)$$

By lemma 4.2.32 the composite ℓk is monidal. The identity functor $\mathrm{Id}_{\mathcal{E}}$ is trivially monoidal.

The first monoidal natural transformation axiom is in our case

$$\begin{tikzcd}
\ell kF \circ \ell kG \arrow[d] \arrow[dr] & \\
\ell(kF \otimes kG) \arrow[d] & F \circ G \\
\ell k(F \circ G) \arrow[ur] &
\end{tikzcd}$$

Along the left hand–side we have the following composite

$$\begin{tikzcd}
\begin{array}{c}\int^n \int^{m_1} \ldots \int^{m_n} F(n)\times \\ \times G(m_1) \times \ldots \times G(m_n) \times A^{\sum_{i=1}^n m_i}\end{array} \arrow[d] & (x, y_1, \ldots, y_n, a) \arrow[d, mapsto, "r=\sum_{i=1}^n m_i"] \\
\int^r \int^n F(n) \times G(r)^n \times A^r \arrow[d] & (x, G(c_1)(y_1), \ldots, G(c_n)(y_n), a) \arrow[d, mapsto, "g=[G(c_i)(y_i)]_{i=1\ldots n}:n\to G(r)"] \\
\int^r FG(r) \times A^r \arrow[d] & (F(g)(x), a) \arrow[d, mapsto] \\
FG(A) & F(G(a) \circ g)(x)
\end{tikzcd}$$

Along the top we find

$$\begin{tikzcd}
\begin{array}{c}\int^n \int^{m_1} \ldots \int^{m_n} F(n)\times \\ \times G(m_1) \times \ldots \times G(m_n) \times A^{\sum_{i=1}^n m_i}\end{array} \arrow[d] & (x, y_1, \ldots, y_n, a_1, \ldots, a_n) \arrow[d, mapsto, "{[G(a_i)(y_i)]_{i=1\ldots n}:n\to GA}"] \\
FG(A) & F([G(a_i)(y_i)]_{i=1\ldots n})(x)
\end{tikzcd}$$

where $a = [a_i]_{i=1\ldots n}$ and $a \circ c_i = a_i$, so $F([G(a_i)(y_i)]_{i=1\ldots n})(x) = F(G(a) \circ g)(x)$. The triangle

$$\begin{tikzcd}
& \mathrm{Id} \arrow[dl] \arrow[dr] & \\
\ell k\,\mathrm{Id} \arrow[rr, "\varepsilon_{\mathrm{Id}}"'] & & \mathrm{Id}
\end{tikzcd}$$

becomes the following

$$\begin{tikzcd}
& & \mathrm{Id} \arrow[dl, "\phi"'] \arrow[ddrr] & & \\
& \ell U \arrow[dl, "\ell\psi"'] & & & \\
\ell k\,\mathrm{Id} \arrow[rrrr, "\varepsilon_{\mathrm{Id}}"'] & & & & \mathrm{Id}
\end{tikzcd}$$

if we unfold the arrows. Let $A \in \mathrm{Set}$ and $a \in A$ we find on elements

$$
\begin{tikzcd}
& & a \arrow[dl, mapsto, "\phi"'] \arrow[ddr, mapsto] & \\
& (1,a) \arrow[dl, mapsto, "\ell\psi"'] & & \\
(1,a) \arrow[rrr, mapsto, "\varepsilon_{\mathbf{Id}}"'] & & & a
\end{tikzcd}
$$

B.9 Proof of proposition 4.7.50

We check the wedge condition. Let $g : m \to p$. The following diagram commutes

$$
\begin{tikzcd}
& X(m+1) \times Y(n)^m \arrow[dr] & \\
X(m+1) \times Y(n)^p \arrow[ur] \arrow[dr] & & \delta(X \otimes Y)(n) \\
& X(p+1) \times Y(n)^p \arrow[ur] &
\end{tikzcd}
$$

because we have the following assignations on elements

$$
\begin{tikzcd}
& (x, f \circ g) \arrow[dr, mapsto] & \\
(x, f) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}(x, [Y(i_n) \circ f \circ g, \overline{y}_n]) \\ = (X(g+1)(x), [Y(i_n) \circ f, \overline{y}_n])\end{array} \\
& (X(g)(x), f) \arrow[ur, mapsto] &
\end{tikzcd}
$$

The elements $(x, [Y(i_n) \circ f \circ g, \overline{y}_n])$ and $(X(g+1)(x), [Y(i_n) \circ f, \overline{y}_n])$ are equal by universal property of the coend since they come from $(x, [Y(i_n) \circ f, \overline{y}_n]) \in X(m+1) \times Y(n+1)^{p+1}$ with the arrow $g+1 : n+1 \to p+1$

$$
\begin{tikzcd}
& X(m+1) \times Y(n+1)^{m+1} \arrow[dr] & \\
X(m+1) \times Y(n+1)^{p+1} \arrow[ur] \arrow[dr] & & \delta(X \otimes Y)(n) \\
& X(p+1) \times Y(n+1)^{p+1} \arrow[ur] &
\end{tikzcd}
$$

Now let us check naturality of $s_{X,Y,n}$ in n, X and Y. Let $f : n \to p$ be an arrow in $\mathbb{F}$. The naturality square

$$
\begin{tikzcd}
(\delta X \otimes Y)(n) \arrow[r] \arrow[d] & \delta(X \otimes Y)(n) \arrow[d] \\
(\delta X \otimes Y)(p) \arrow[r] & \delta(X \otimes Y)(p)
\end{tikzcd}
$$

commutes because we find on elements

$$\begin{tikzcd}
(x,g) \arrow[r, mapsto] \arrow[d, mapsto] & (x,[Y(i)\circ g,\overline{y}_n]) \arrow[d, mapsto] \\
(x,Y(f)\circ g) \arrow[r, mapsto] & \begin{array}{c}(x,Y(f+1)\circ[Y(i)\circ g,\overline{y}_n]) \\ =(x,[Y(i)\circ Y(f)\circ g,\overline{y}_p])\end{array}
\end{tikzcd}$$

and by naturality of $\overline{y}$ we have $Y(f+1)(\overline{y}_n)=\overline{y}_p$ and by naturality of the inclusions we have $f+1\circ i=i\circ f$. Let $f:X\to Z$ an arrow in $\mathcal{F}$. The naturality square

$$\begin{tikzcd}
(\delta X\otimes Y)(n) \arrow[r] \arrow[d] & \delta(X\otimes Y)(n) \arrow[d] \\
(\delta Z\otimes Y)(n) \arrow[r] & \delta(Z\otimes Y)(n)
\end{tikzcd}$$

commutes because on elements we find

$$\begin{tikzcd}
(x,g) \arrow[r, mapsto] \arrow[d, mapsto] & (x,[Y(i)\circ g,\overline{y}_n]) \arrow[d, mapsto] \\
(f_{m+1}(x),g) \arrow[r, mapsto] & (f_{m+1}(x),[Y(i)\circ g,\overline{y}_n])
\end{tikzcd}$$

Let $f:(Y,y)\to(Z,z)$ be an arrow in $U\downarrow\mathcal{F}$. The naturality square

$$\begin{tikzcd}
(\delta X\otimes Y)(n) \arrow[r] \arrow[d] & \delta(X\otimes Y)(n) \arrow[d] \\
(\delta X\otimes Z)(n) \arrow[r] & \delta(X\otimes Z)(n)
\end{tikzcd}$$

commutes because on elements we find

$$\begin{tikzcd}
(x,g) \arrow[r, mapsto] \arrow[d, mapsto] & (x,[Y(i)\circ g,\overline{y}_n]) \arrow[d, mapsto] \\
(x,f_n\circ g) \arrow[r, mapsto] & \begin{array}{c}(x,f_{n+1}\circ[Y(i)\circ g,\overline{y}_n]) \\ =(x,[Z(i)\circ f_n\circ g,\overline{z}_n])\end{array}
\end{tikzcd}$$

by naturality of f we have $f_{n+1}\circ Y(i)=Z(i)\circ f_n$ and $f_{n+1}(\overline{y}_n)=\overline{z}_n$ by definition of an arrow in $U\downarrow\mathcal{F}$.

Now we check the two axioms for $U\downarrow\mathcal{F}$–strengths. The triangle

$$\begin{tikzcd}
\delta X\otimes U \arrow[r] \arrow[d] & \delta(X\otimes U) \arrow[dl] \\
\delta X &
\end{tikzcd}$$

commutes because we find on elements

$$\begin{tikzcd}
(x,f) \arrow[r, mapsto] \arrow[d, mapsto] & (x,[i\circ f,\overline{u}_n])=(x,f+1) \arrow[dl, mapsto] \\
X(f+1)(x) &
\end{tikzcd}$$

The other diagram is the following

$$\begin{tikzcd}
(\delta X \otimes Y) \otimes Z \arrow[r] \arrow[d] & \delta X \otimes (Y \otimes Z) \arrow[dd] \\
\delta(X \otimes Y) \otimes Z \arrow[d] & \\
\delta((X \otimes Y) \otimes Z) \arrow[r] & \delta(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Let $n \in \mathbb{F}$. Along the right–hand side we find on elements

$$\begin{array}{ccc}
\int^k \int^r X(r+1) \times Y(k)^r \times Z(n)^k & & (x, a_1, \dots, a_r, b_1, \dots, b_k) \\
\downarrow & & \downarrow{\scriptstyle p_i = k} \\
\int^r \int^{p_1} \dots \int^{p_r} X(r+1) \times Y(p_1) \times Z(n)^{p_1} \times \dots \times Y(p_r) \times Z(n)^{p_r} & & (x, (a_1, b_1, \dots, b_k), \dots, (a_r, b_1, \dots, b_k)) \\
\downarrow & & \downarrow{\scriptstyle m = r+1} \\
\int^m X(m) \times \left(\int^k Y(k) \times Z(n+1)^k \right)^m & & (x, (a_1, Z(i_n)(b_1), \dots, Z(i_n)(b_k)), \dots, (a_r, Z(i_n)(b_1), \dots, Z(i_n)(b_k)), \overline{yz}_n)
\end{array}$$

where $i_n : n \to n+1$, $y : U \to Y$ and $z : U \to Z$ and so we have $yz : U \to U \otimes U \to Y \otimes Z$. At the component n it assigns to an element $i \in n$, $(y_1(1), z_n(i)) \in \int^k Y(k) \times Z(n)^k$. Along the left–hand side we find on elements

$$\begin{array}{ccc}
\int^k \int^r X(r+1) \times Y(k)^r \times Z(n)^k & & (x, a_1, \dots, a_r, b_1, \dots, b_k) \\
\downarrow & & \downarrow{\scriptstyle m = r+1} \\
\int^k \int^m X(m) \times Y(k+1)^m \times Z(n)^k & & (x, Y(i_k)(a_1), \dots, Y(i_k)(a_r), \overline{y}_k, b_1, \dots, b_k) \\
\downarrow & & \downarrow{\scriptstyle q = k+1} \\
\int^q \int^m X(m) \times Y(q)^m \times Z(n+1)^q & & (x, Y(i_k)(a_1), \dots, Y(i_k)(a_r), \overline{y}_k, Z(i_n)(b_1), \dots, Z(i_n)(b_k), \overline{z}_n) \\
\downarrow & & \downarrow{\scriptstyle p = q = k+1} \\
\int^m X(m) \times \left(\int^p Y(p) \times Z(n+1)^p \right)^m & & (x, (Y(i_k)(a_1), Z(i_n)(b_1), \dots, Z(i_n)(b_k), \overline{z}_n), \dots, (Y(i_k)(a_r), Z(i_n)(b_1), \dots, Z(i_n)(b_k), \overline{z}_n), (\overline{y}_k, Z(i_n)(b_1), \dots, Z(i_n)(b_k), \overline{z}_n))
\end{array}$$

where $i_k : k \to k+1$. The two results along the right and the left–hand side are equal since each pair $(a_j, Z(i_n)(b_1), \dots, Z(i_n)(b_k))$ and $(Y(i_k)(a_j), Z(i_n)(b_1), \dots, Z(i_n)(b_k), \overline{z}_n)$ comes from $(a_j, Z(i_n)(b_1), \dots, Z(i_n)(b_k), \overline{z}_n)$

with the arrow $i_k : k \to k+1$

$$\begin{tikzcd}
& Y(k) \times Z(n)^k \arrow[dr] & \\
Y(k) \times Z(n)^{k+1} \arrow[ur, "\mathrm{id}\times(-\circ i_k)"] \arrow[dr, "Y(i_k)\circ\mathrm{id}"'] & & \int^k Y(k) \times Z(n)^k \\
& Y(k+1) \times Z(n)^{k+1} \arrow[ur] &
\end{tikzcd}$$

the last pair $\overline{yz}_n = yz_{n+1}(1) = (y_1(1), z_{n+1}(1))$ and $(\overline{y}_k, Z(i_n)(b_1), \ldots, Z(i_n)(b_k), \overline{z}_n) =$ $(y_{k+1}(1), Z(i_n)(b_1), \ldots, Z(i_n)(b_k), z_{n+1}(1))$ comes from $(y_1(1), Z(i_n)(b_1), \ldots, Z(i_n)(b_k), z_{n+1}(1))$ with the arrow $i_1 : 1 \to k+1$

$$\begin{tikzcd}
& Y(1) \times Z(n) \arrow[dr] & \\
Y(1) \times Z(n)^{k+1} \arrow[ur, "\mathrm{id}\times(-\circ i_1)"] \arrow[dr, "Y(i_1)\circ\mathrm{id}"'] & & \int^k Y(k) \times Z(n)^k \\
& Y(k+1) \times Z(n)^{k+1} \arrow[ur] &
\end{tikzcd}$$

since by naturality of the inclusions $Y(i_1)(y_1(1)) = y_{k+1}(1)$.

B.10 Proof of proposition 4.7.51

Naturality in A is given since i_l, i_r and g are natural in A. Let $f : F \to H$ be an arrow in $\mathcal{E}$. The naturality square

$$\begin{tikzcd}
F(GA+1) \arrow[r] \arrow[d] & FG(A+1) \arrow[d] \\
H(GA+1) \arrow[r] & HG(A+1)
\end{tikzcd}$$

commutes because f is a natural transformation. Let $f : (G, g) \to (H, h)$ be an arrow in $\mathrm{Id} \downarrow \mathcal{E}$. The naturality square

$$\begin{tikzcd}
F(GA+1) \arrow[r] \arrow[d] & FG(A+1) \arrow[d] \\
F(HA+1) \arrow[r] & FH(A+1)
\end{tikzcd}$$

commutes because the following two squares commute

$$\begin{tikzcd}
GA \arrow[r] \arrow[d] & G(A+1) \arrow[d] \\
HA \arrow[r] & H(A+1)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r] \arrow[d] & A+1 \arrow[r] \arrow[d] & G(A+1) \arrow[d] \\
1 \arrow[r] & A+1 \arrow[r] & H(A+1)
\end{tikzcd}$$

Now we check the two $\mathrm{Id} \downarrow \mathcal{E}$–strength axioms. The trianlge

$$\begin{tikzcd}
F' \circ \mathrm{Id} \arrow[r] \arrow[d] & (F \circ \mathrm{Id})' \arrow[dl] \\
F' &
\end{tikzcd}$$

commutes trivially. The associativity diagram

$$\begin{tikzcd}
F(GHA+1) \arrow[r, equal] \arrow[d] & F(GHA+1) \arrow[dd] \\
FG(HA+1) \arrow[d] & \\
FGH(A+1) \arrow[r, equal] & FGH(A+1)
\end{tikzcd}$$

commutes because along the left we have the composite $FG[Hi_l, h_{A+1} \circ i_r] \circ F[Gi_l', g_{HA+1} \circ i_r']$ and along the right $F[GHi_l, Gh_{A+1} \circ g_{A+1} \circ i_r]$. They are identic since the following diagrams commute

$$\begin{tikzcd}
GHA \arrow[r, "Gi_l'"] \arrow[dr, "GHi_l"'] & G(HA+1) \arrow[d, "{[Gi_l, h_{A+1} \circ i_r]}"] \\
& GH(A+1)
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "i_r"] \arrow[d, "i_r'"'] & A+1 \arrow[r, "g_{A+1}"] \arrow[dl, "h_A+1"'] & G(A+1) \arrow[dl, "G(h_A+1)"'] \arrow[d, "Gh_{A+1}"] \\
HA+1 \arrow[r, "g_{HA+1}"'] & G(HA+1) \arrow[r, "{G[Hi_l, h_{A+1} \circ i_r]}"'] & GH(A+1)
\end{tikzcd}$$

the triangle on the left commutes by naturality of the inclusion, the middle square is a naturality square of g and the right triangle commutes because $Hi_l \circ h_A = h_{A+1} \circ i_l$ and $[h_{A+1} \circ i_l, h_{A+1} \circ i_r] = h_{A+1}$.

B.11 Definition 4.8.58 of α_1^{-1}

We have to check the wedge condition, that is, that the following diagram commutes for all arrow $h : n \to k$

$$\begin{tikzcd}
& X(n) \times (A+1)^n \arrow[r] & X(m+1) \times (A+1)^m \arrow[dr] & \\
X(n) \times (A+1)^k \arrow[ur] \arrow[dr] & & & \begin{array}{l} \int^m X(m+1) \\ \times (A+1)^m \end{array} \\
& X(k) \times (A+1)^k \arrow[r] & X(p+1) \times (A+1)^p \arrow[ur] &
\end{tikzcd}$$

on elements we find

$$\begin{tikzcd}
& \begin{array}{l} x \in X(n) \\ f \circ h : n \to k \to A+1 \end{array} \arrow[r, mapsto] & \begin{array}{l} S(\phi)(x) \in S(m) \\ \overline{f \circ h} : m \to A \end{array} \\
\begin{array}{l} x \in X(n) \\ f : k \to A+1 \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \\
& \begin{array}{l} X(h)(x) \in S(k) \\ f : k \to A+1 \end{array} \arrow[r, mapsto] & \begin{array}{l} X(\psi)S(h)(x) \in S(p+1) \\ \overline{f} : p \to A \end{array}
\end{tikzcd}$$

The two expressions $(X(\phi)(x), \overline{f \circ h})$ and $(X(\psi)S(h)(x), \overline{f})$ come from $X(p+1) \times (A+1)^m$ with the arrow $g : m \to p$ which makes the following square commute

$$\begin{tikzcd} n \arrow[r, "\phi"] \arrow[d, "h"'] & m+1 \arrow[d, "g+1"] \\ k \arrow[r, "\psi"'] & p+1 \end{tikzcd}$$

and this map g factorises $\overline{f \circ h} : m \to A$ by $\overline{f} \circ g : m \to p \to A$.

Now we check naturality in A. Let $g : A \to B$. The naturality square

$$\begin{tikzcd} (\ell X)'(A) \arrow[r] \arrow[d] & \ell(\delta X)(A) \arrow[d] \\ (\ell X)'(B) \arrow[r] & \ell(\delta X)(B) \end{tikzcd}$$

commutes because we find on elements

$$\begin{tikzcd} (x, f) \arrow[r, mapsto] \arrow[d, mapsto] & (X(\phi)(x), \overline{f}) \arrow[d, mapsto] \\ (x, (g+1) \circ f) \arrow[r, mapsto] & (X(\phi)(x), g \circ \overline{f}) \end{tikzcd}$$

since $\{i \in n \text{ such that } (g+1) \circ f(i) \in B\} = \{i \in n \text{ such that } f(i) \in A\}$ and $\overline{(g+1) \circ f} = g \circ \overline{f}$. Let $h : X \to Y$ be an arrow in $\mathcal{F}$. The naturality square

$$\begin{tikzcd} (\ell X)'(A) \arrow[r] \arrow[d] & \ell(\delta X)(A) \arrow[d] \\ (\ell Y)'(A) \arrow[r] & \ell(\delta Y)(A) \end{tikzcd}$$

commutes because we find on elements

$$\begin{tikzcd} (x, f) \arrow[r, mapsto] \arrow[d, mapsto] & (X(\phi)(x), \overline{f}) \arrow[d, mapsto] \\ (h_n(x), f) \arrow[r, mapsto] & (h_{m+1}(X(\phi)(x)), \overline{f}) \end{tikzcd}$$

and by naturality of h we have $h_{m+1}(X(\phi)(x)) = Y(\phi)(h_n(x))$.

Net we check the strength morphism axiom.

$$\begin{tikzcd} (\ell X)' \circ \ell Y \arrow[r, "\alpha_1^{-1} \circ \ell Y"] \arrow[d, "\sigma"'] & \ell\delta X \circ \ell Y \arrow[d, "\phi"] \\ (\ell X \circ \ell Y)' \arrow[d, "\phi'"'] & \ell(\delta X \otimes Y) \arrow[d, "\ell s"] \\ (\ell(X \otimes Y))' \arrow[r, "\alpha_1^{-1}"'] & \ell\delta(X \otimes Y) \end{tikzcd}$$

in proposition 4.9.64 we show that α_1^{-1} has an inverse α_1. So instead of verifying the above diagram, we check the commutativity of the diagram where we inverse the arrows α_1^{-1}. Let A be a set.

$$\begin{tikzcd}
\ell\delta X(\ell Y(A)) \arrow[r, "\alpha_1\circ\ell Y"] \arrow[d, "\phi"'] & (\ell X)(\ell Y(A)+1) \arrow[d, "\sigma"] \\
\ell(\delta X\otimes Y)(A) \arrow[d, "\ell s"'] & \ell X(\ell Y(A+1)) \arrow[d, "\phi'"] \\
\ell\delta(X\otimes Y)(A) \arrow[r, "\alpha_1"'] & \ell(X\otimes Y)(A+1)
\end{tikzcd}$$

Along the right hand–side we have the following assignations on elements

$$\begin{tikzcd}
\int^k X(k+1)\times\left(\int^m Y(m)\times A^m\right)^k \arrow[d] & x, y_1, a_1, \dots, y_k, a_k \arrow[d, mapsto, "\ell=k+1"] \\
\int^\ell X(\ell)\times\left(\int^m Y(m)\times A^m+1\right)^\ell \arrow[d] & x, y_1, a_1, \dots, y_k, a_k, 1 \arrow[d, mapsto, "p_1=m_1,\dots,p_{\ell-1}=m_k"', "p_\ell=1"] \\
\int^\ell X(\ell)\times\left(\int^p Y(p)\times (A+1)^p\right)^\ell \arrow[d] & x, y_1, i_A\circ a_1, \dots, y_k, i_A\circ a_k, y(1), 1 \arrow[d, mapsto, "q=\sum p_i=\sum m_i+1"] \\
\int^q\int^\ell X(\ell)\times Y(q)^\ell\times(A+1)^q & \begin{array}{c} x, Y(c_1)(y_1), \dots, Y(c_k)(y_k), Y(c)(y(1)), \\ {[i_A\circ a_1, \dots, i_A\circ a_k, 1]} \end{array}
\end{tikzcd}$$

where $c_i : m_i \to \sum m_i \to \sum m_i + 1$ and $c : 1 \to \sum m_i + 1$. Along the left hand side we have the following assignations on elements

$$\begin{tikzcd}
\int^k X(k+1)\times\left(\int^m Y(m)\times A^m\right)^k \arrow[d] & x, y_1, a_1, \dots, y_k, a_k \arrow[d, mapsto, "n=\sum m_i"] \\
\int^n\int^k X(k+1)\times Y(n)^k\times A^n \arrow[d] & x, Y(c_1')(y_1), \dots, Y(c_k')(y_k), [a_1, \dots, a_k] \arrow[d, mapsto, "\ell=k+1"] \\
\int^n\int^\ell X(\ell)\times Y(n+1)^\ell\times A^n \arrow[d] & \begin{array}{c} x, Y(i_n\circ c_1')(y_1), \dots, Y(i\circ c_k')(y_k), \overline{y}_n, \\ {[a_1, \dots, a_k]} \end{array} \arrow[d, mapsto, "p=n+1"] \\
\int^p\int^\ell X(\ell)\times Y(p)^\ell\times(A+1)^p & \begin{array}{c} x, Y(i_n\circ c_1')(y_1), \dots, Y(i_n\circ c_k')(y_k), \overline{y}_n, \\ {[i_A\circ a_1, \dots, i_A\circ a_k, 1]} \end{array}
\end{tikzcd}$$

Since $\overline{y}_n = Y(c)(y(1))$, we obtain the same result along the two sides.

B.12 Definition 4.8.59 of β_1

The arrow is obviously natural in n and in F. Now we check the strength morphism axiom

$$\begin{tikzcd}
\delta kF \otimes kG \arrow[r] \arrow[d] & k(F') \otimes kG \arrow[d] \\
\delta(kF \otimes kG) \arrow[d] & k(F' \circ G) \arrow[d] \\
\delta k(F \circ G) \arrow[r] & k((F \circ G)')
\end{tikzcd}$$

which is

$$\begin{tikzcd}
\int^m F(m+1) \times G(n)^m \arrow[r] \arrow[d] & \int^m F(m+1) \times G(n)^m \arrow[d] \\
\int^r F(r) \times G(n+1)^r \arrow[d] & F(G(n)+1) \arrow[d] \\
FG(n+1) \arrow[r] & FG(n+1)
\end{tikzcd}$$

This diagram commutes because on elements we find

$$\begin{tikzcd}
(x,g) \arrow[r, mapsto] \arrow[d, mapsto, "r=m+1"'] & (x,g) \arrow[d, mapsto] \\
(x,[G(i)\circ g, \overline{ke_n}]) \arrow[d, mapsto] & F(g+1)(x) \arrow[d, mapsto] \\
F[G(i)\circ g, \overline{ke_n}](x) \arrow[r, mapsto] & \begin{array}{c} F[G(i), e_{n+1}\circ j]\circ F(g+1)(x) \\ = F[G(i)\circ g, \overline{ke_n}](x) \end{array}
\end{tikzcd}$$

where $e : \mathrm{Id} \to G$, $i : n \to n+1$ and $j : 1 \to n+1$.

B.13 Proof of proposition 4.9.64

Next we check that the following diagram commutes for all arrow $h : m \to k$

$$\begin{tikzcd}
 & X(m+1) \times A^m \arrow[r] & X(m+1) \times (A+1)^{m+1} \arrow[dr] & \\
X(m+1) \times A^k \arrow[ur] \arrow[dr] & & & \int^n X(n) \times (A+1)^n \\
 & X(k+1) \times A^k \arrow[r] & X(k+1) \times (A+1)^{k+1} \arrow[ur] &
\end{tikzcd}$$

on elements we find

$$\begin{tikzcd}
 & \begin{array}{l} x \in X(m+1) \\ f \circ h : m \to k \to A \end{array} \arrow[r, mapsto] & \begin{array}{l} x \in S(m+1) \\ (f \circ h) + 1 : m+1 \to A+1 \end{array} \\
\begin{array}{l} x \in X(m+1) \\ f : k \to A \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \\
 & \begin{array}{l} X(h+1)(x) \in S(k+1) \\ f : k \to A \end{array} \arrow[r, mapsto] & \begin{array}{l} S(h+1)(x) \in S(k+1) \\ f+1 : k+1 \to A+1 \end{array}
\end{tikzcd}$$

The two expressions $(x, (f+1) \circ (h+1))$ and $(X(h+1)(x), f+1)$ come from $X(m+1) \times (A+1)^{k+1}$ with the arrow $h+1 : m+1 \to k+1$.

Now we check the naturality in A. Let $f : A \to B$ be a map. We have to check the commutativity of the following square:

$$\begin{tikzcd}
\int^m X(m+1) \times A^m \arrow[r, "\alpha_{X,A}"] \arrow[d, "\ell\delta X(f)"'] & \int^n S(n) \times (A+1)^n \arrow[d, "(\ell S)'(f)"] \\
\int^m X(m+1) \times B^m \arrow[r, "\alpha_{S,B}"'] & \int^n X(n) \times (B+1)^n
\end{tikzcd}$$

on elements we find

$$\begin{tikzcd}
(x, g : m \to A) \arrow[r, mapsto] \arrow[d, mapsto] & (x, g+1) \arrow[d, mapsto] \\
(x, f \circ g) \arrow[r, mapsto] & \begin{array}{c} (x, (f+1) \circ (g+1)) \\ = (x, (f \circ g) + 1) \end{array}
\end{tikzcd}$$

Next we check the naturality in X. Let $\rho : X_1 \to X_2$ be a morphism. We have to check the commutativity of the following square

$$\begin{tikzcd}
\int^m X_1(m+1) \times A^m \arrow[r, "\alpha_{X_1,A}"] \arrow[d, "\ell\delta\rho_A"'] & \int^n X_2(n) \times (A+1)^n \arrow[d, "(\ell\rho)'_A"] \\
\int^m X_2(m+1) \times A^m \arrow[r, "\alpha_{X_2,A}"'] & \int^n X_2(n) \times (A+1)^n
\end{tikzcd}$$

on elements we find

$$\begin{tikzcd}
(x, g) \arrow[r, mapsto] \arrow[d, mapsto] & (x, g+1) \arrow[d, mapsto] \\
(\rho_{m+1}(x), g) \arrow[r, mapsto] & (\rho_{m+1}(x), g+1)
\end{tikzcd}$$

Next we show that $\alpha_{1,X,A} : \ell(\delta X)(A) \to (\ell X)'(A)$ is the inverse of $\alpha^{-1}_{1,X,1} : (\ell X)'(A) \to \ell(\delta X)(A)$.

$$\begin{tikzcd}
X(m+1) \times A^m \arrow[d] & x, f \arrow[d, mapsto] \\
X(m+1) \times (A+1)^{m+1} \arrow[d] & x, f+1 \arrow[d, mapsto] \\
X(m+1) \times A^m & X(\mathrm{id})(x) = x, \overline{f+1} = f
\end{tikzcd}$$

Since the subset of $A+1$ such that $\{i \in m+1 \text{ s.t. } f+1(i) \in A\}$ is exactly A and therefore the corresponding map $A+1 \to A+1$ is the identity on $A+1$.

$$\begin{tikzcd}
X(n) \times (A+1)^n \arrow[d] & x, f \arrow[d, mapsto] \\
X(m+1) \times A^m \arrow[d] & X(\phi)(x), \overline{f} \arrow[d, mapsto] \\
X(m+1) \times (A+1)^{m+1} & X(\phi)(x), \overline{f}+1
\end{tikzcd}$$

We remark that $(\overline{f}+1) \circ \phi = f$ and therefore the two expressions $(x, (\overline{f}+1) \circ \phi)$ and $(X(\phi)(x), \overline{f}+1)$ come from $S(n) \times (A+1)^{m+1}$ with the arrow $\phi : n \to m+1$.

Appendix C

Proofs of chapter 5

C.1 Monoidal isomorphisms of $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$

C.1.1 Construction α

Let $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$. We rewrite the domain and the codomain of $\alpha_{P,Q,R,u,\Gamma}$ using the coend notation

$$((P_u \bullet Q) \bullet R)(\Gamma) = \int^{\Delta} \int^{\Delta'} P_u(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$$

and

$$\begin{aligned}(P_u \bullet (Q \otimes R))(\Gamma) = \int^{\Delta'} \int^{\Delta_1} \ldots \int^{\Delta_m} & P_u(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ & \times \ldots \\ & \times Q_{u_m}(\Delta_m) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}\end{aligned}$$

where $\Delta' = (u_1, \ldots, u_m)$. To define a map $((P_u \bullet Q) \bullet R)(\Gamma) \to (P_u \bullet (Q \otimes R))(\Gamma)$, it suffices to give a collection of arrows

$$P_u(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \to (P_u \bullet (Q \otimes R))(\Gamma)$$

for all $\Delta, \Delta' \in \mathbb{F}\downarrow\mathcal{T}$ that satisfies the wedge condition. We take the following composite

$$\begin{array}{c}
P_u(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \\
\downarrow \\
\begin{aligned} P_u(\Delta') \times Q_{u_1}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \\ \times \ldots \\ \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{aligned} \\
\downarrow \\
\begin{aligned} P_u(\Delta') \times \int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times \int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{aligned} \\
\downarrow \\
\begin{aligned} \int^{\Delta'} P_u(\Delta') \times \int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times \int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{aligned}
\end{array}$$

We check the wedge condition. Let $f : \Delta_1 \to \Delta_2$ in $\mathbb{F}\downarrow\mathcal{T}$. The diagram

$$\begin{tikzcd}
 & \begin{array}{c} P_u(\Delta') \times Q_{u_1}(\Delta_1) \\ \times \ldots \times Q_{u_m}(\Delta_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} P_u(\Delta') \times Q_{u_1}(\Delta_1) \\ \times \ldots \times Q_{u_m}(\Delta_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & (P_u \bullet (Q \otimes R))(\Gamma) \\
 & \begin{array}{c} P_u(\Delta') \times Q_{u_1}(\Delta_2) \\ \times \ldots \times Q_{u_m}(\Delta_2) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
 & \begin{array}{c} p, q_1, \ldots, \\ q_m, (r_t \circ f_t)_{t \in \mathcal{T}} \end{array} \arrow[dr, mapsto] & \\
\begin{array}{c} p, q_1, \ldots, \\ q_m, (r_t)_{t \in \mathcal{T}} \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, q_1, (r_t \circ f_t)_{t \in \mathcal{T}}, \ldots, \\ q_m, (r_t \circ f_t)_{t \in \mathcal{T}} \\ = p, Q_{u_1}(f)(q_1), (r_t)_{t \in \mathcal{T}}, \ldots, \\ Q_{u_m}(f)(q_m), (r_t)_{t \in \mathcal{T}} \end{array} \\
 & \begin{array}{c} p, Q_{u_1}(f)(q_1), \ldots, \\ Q_{u_m}(f)(q_m), (r_t)_{t \in \mathcal{T}} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

each pair $q_j, (r_t \circ f_t)_{t \in \mathcal{T}} \in Q_{u_j}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}$ and $Q_{u_j}(f)(q_j), (r_t)_{t \in \mathcal{T}} \in Q_{u_j}(\Delta_2) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)}$ is equal since they come from $q_j, (r_t)_{t \in \mathcal{T}} \in Q_{u_j}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)}$ with the arrow f

$$\begin{tikzcd}
 & Q_{u_j}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \arrow[dr] & \\
Q_{u_j}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur, "\mathrm{id} \times (- \circ f_t)_{t \in \mathcal{T}}"] \arrow[dr, "Q_{u_j}(f) \times \mathrm{id}"'] & & \begin{array}{c} \int^{\Delta_j} Q_{u_j}(\Delta_j) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_j^{-1}(t)} \end{array} \\
 & Q_{u_j}(\Delta_2) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

Let $g : \Delta' \to \Delta''$ in $\mathbb{F} \downarrow \mathcal{T}$ where $\Delta' = (u_1, \ldots, u_m)$ and $\Delta'' = (v_1, \ldots, v_k)$. The diagram

$$\begin{tikzcd}
 & \begin{array}{c} P_u(\Delta') \times \int^{\Delta} Q_{u_1}(\Delta) \times \ldots \\ \times Q_{u_m}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} P_u(\Delta') \times \int^{\Delta} Q_{v_1}(\Delta_1) \times \ldots \\ \times Q_{v_k}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & (P_u \bullet (Q \otimes R))(\Gamma) \\
 & \begin{array}{c} P_u(\Delta'') \times \int^{\Delta} Q_{v_1}(\Delta) \times \ldots \\ \times Q_{v_k}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & \begin{array}{l} p, q_{g_{u_1}}, \dots, \\ q_{g_{u_m}}, (r_t)_{t\in\mathcal{T}} \end{array} \arrow[dr, mapsto] & \\
\begin{array}{l} p, q_1, \dots, \\ q_k, (r_t)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, q_{g_{u_1}}, (r_t)_{t\in\mathcal{T}}, \dots, \\ q_{g_{u_m}}, (r_t)_{t\in\mathcal{T}} \\ = P_u(g)(p), q_1, (r_t)_{t\in\mathcal{T}}, \dots, \\ q_k, (r_t)_{t\in\mathcal{T}} \end{array} \\
 & \begin{array}{l} P(g)(p), q_1, \dots, \\ q_k, (r_t)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

The elements $p, q_{g_{u_1}}, (r_t)_{t\in\mathcal{T}}, \dots, q_{g_{u_m}}, (r_t)_{t\in\mathcal{T}} \in P_u(\Delta') \times \big(\int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}\big) \times \dots \times \big(\int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}\big)$ and $P_u(g)(p), q_1, (r_t)_{t\in\mathcal{T}}, \dots, q_k, (r_t)_{t\in\mathcal{T}} \in P_u(\Delta'')$ $\times \big(\int^{\Delta_1} Q_{v_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}\big) \times \dots \times \big(\int^{\Delta_k} Q_{v_k}(\Delta_k) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)}\big)$ are equal since they come from $p, q_1, (r_t)_{t\in\mathcal{T}}, \dots, q_k, (r_t)_{t\in\mathcal{T}} \in P_u(\Delta') \times \big(\int^{\Delta_1} Q_{v_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}\big) \times \dots \times \big(\int^{\Delta_k} Q_{v_k}(\Delta_k) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)}\big)$ with the arrow g

$$\begin{tikzcd}
 & \begin{array}{l} P_u(\Delta') \\ \times \int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \dots \\ \times \int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P_u(\Delta') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1) \\ \quad \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \dots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k) \\ \quad \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & (P_u \bullet (Q \otimes R))(\Gamma) \\
 & \begin{array}{l} P_u(\Delta'') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \dots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

Next we construct an inverse arrow. By universal properties of the coends, it suffices to give a collection of arrows

$$\begin{aligned}P_u(\Delta')\times Q_{u_1}(\Delta_1)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta_1^{-1}(t)}&\\ \times\dots&\\ \times Q_{u_m}(\Delta_m)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta_m^{-1}(t)}&\to((P_u\bullet Q)\bullet R)(\Gamma)\end{aligned}$$

for all Δ', $\Delta_1,\dots,\Delta_m\in\mathbb{F}\downarrow\mathcal{T}$ satisfying the wedge condition. We take the following composite

$$\begin{array}{c}
\begin{array}{l}P_u(\Delta')\times Q_{u_1}(\Delta_1)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta_1^{-1}(t)}\\ \times\dots\\ \times Q_{u_m}(\Delta_m)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta_m^{-1}(t)}\end{array}\\
\downarrow\\
\begin{array}{c}P_u(\Delta')\times Q_{u_1}(\sum_{i=1}^m\Delta_i)\times\dots\times Q_{u_m}(\sum_{i=1}^m\Delta_i)\\ \times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{(\sum_{i=1}^m\Delta_i)^{-1}(t)}\end{array}\\
\downarrow\\
\int^{\Delta}P_u(\Delta')\times Q_{u_1}(\Delta)\times\dots\times Q_{u_m}(\Delta)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta^{-1}(t)}\\
\downarrow\\
\int^{\Delta}\int^{\Delta'}P_u(\Delta')\times Q_{u_1}(\Delta)\times\dots\times Q_{u_m}(\Delta)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta^{-1}(t)}
\end{array}$$

We check the wedge condition. Let $f^i:\Delta_i\to\Xi_i$ in $\mathbb{F}\downarrow\mathcal{T}$ for all $i=1,\dots,m$. The diagram

$$\begin{tikzcd}
& \begin{array}{l}P_u(\Delta')\\ \times Q_{u_1}(\Delta_1)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta_1^{-1}(t)}\\ \times\dots\\ \times Q_{u_m}(\Delta_m)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Delta_m^{-1}(t)}\end{array} \arrow[dr] & \\
\begin{array}{l}P_u(\Delta')\\ \times Q_{u_1}(\Delta_1)\times\\ \quad\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Xi_1^{-1}(t)}\\ \times\dots\\ \times Q_{u_m}(\Delta_m)\\ \quad\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Xi_m^{-1}(t)}\end{array} \arrow[ur] \arrow[dr] & & ((P_u\bullet Q)\bullet R)(\Gamma) \\
& \begin{array}{l}P_u(\Delta')\\ \times Q_{u_1}(\Xi_1)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Xi_1^{-1}(t)}\\ \times\dots\\ \times Q_{u_m}(\Xi_m)\times\prod_{t\in\mathcal{T}}R_t(\Gamma)^{\Xi_m^{-1}(t)}\end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}[column sep=small]
& \begin{array}{c} p, q_1, (r_t^1 \circ f_t^i)_{t\in\mathcal{T}}, \\ \dots, q_m, (r_t^m \circ f_t^i)_{t\in\mathcal{T}} \end{array} \arrow[dr, mapsto] & \\
\begin{array}{c} p, q_1, (r_t^1)_{t\in\mathcal{T}}, \\ \dots, q_m, (r_t^m)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, \overline{q}_1, \dots, \overline{q}_m, \\ [r_t^1 \circ f_t^i, \dots, r_t^m \circ f_t^i]_{t\in\mathcal{T}} \\ = p, \overline{Q_{u_1}(f^1)(q_1)}, \\ \dots, \overline{Q_{u_m}(f^m)(q_m)}, \\ [r_t^1, \dots, r_t^m]_{t\in\mathcal{T}} \end{array} \\
& \begin{array}{c} p, Q_{u_1}(f^1)(q_1), (r_t^1)_{t\in\mathcal{T}}, \\ \dots, Q_{u_m}(f^m)(q_m), (r_t^m)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

By naturality of the inclusions $\Delta_j \to \sum_{i=1}^{m} \Delta_i$ and $\Xi_j \to \sum_{i=1}^{m} \Xi_i$, $\overline{Q_{u_j}(f^j)(q_j)} = \big(\sum_{i=1}^{m} f^i\big)(\overline{q}_j)$ and $[r_t^1 \circ f_t^i, \dots, r_t^m \circ f_t^i]_{t\in\mathcal{T}} = ([r_t^1, \dots, r_t^m] \circ \sum_{i=1}^{m} f_t^i)_{t\in\mathcal{T}}$. The elements $p, \overline{q}_1, \dots, \overline{q}_m, ([r_t^1, \dots, r_t^m] \circ \sum_{i=1}^{m} f_t^i)_{t\in\mathcal{T}}$ and $p, \big(\sum_{i=1}^{m} f^i\big)(\overline{q}_1), \dots, \big(\sum_{i=1}^{m} f^i\big)(\overline{q}_m), [r_t^1, \dots, r_t^m]_{t\in\mathcal{T}}$ are equal since they come from $p, \overline{q}_1, \dots, \overline{q}_m, [r_t^1, \dots, r_t^m]_{t\in\mathcal{T}}$ with the arrow $f := \sum_{i=1}^{m} f^i$

$$\begin{tikzcd}[column sep=small]
& \begin{array}{l} P_u(\Delta') \\ \times Q_{u_1}(\sum_{i=1}^{m} \Delta_i) \times \dots \times Q_{u_m}(\sum_{i=1}^{m} \Delta_i) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P_u(\Delta') \\ \times Q_{u_1}(\sum_{i=1}^{m} \Delta_i) \\ \times \dots \\ \times Q_{u_m}(\sum_{i=1}^{m} \Delta_i) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Xi_i)^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times (-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "\mathrm{id} \times Q_{u_1}(f) \times \dots \times Q_{u_m}(f) \times \mathrm{id}"'] & & ((P_u \bullet Q) \bullet R)(\Gamma) \\
& \begin{array}{l} P_u(\Delta') \\ \times Q_{u_1}(\sum_{i=1}^{m} \Xi_i) \times \dots \times Q_{u_m}(\sum_{i=1}^{m} \Xi_i) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Xi_i)^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

Let $f:\Delta'\to\Delta''$ in $\mathbb{F}\downarrow\mathcal{T}$ where $\Delta'=(u_1,\dots,u_m)$ and $\Delta''=(v_1,\dots,v_k)$. The diagram

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{l} P_u(\Delta') \\ \times \int^{\Delta_1} Q_{u_1}(\Delta_1)\times\prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times\dots \\ \times \int^{\Delta_m} Q_{u_m}(\Delta_m)\times\prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P_u(\Delta') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1) \\ \qquad \times\prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times\dots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k) \\ \qquad \times\prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & ((P_u\bullet Q)\bullet R)(\Gamma) \\
& \begin{array}{l} P_u(\Delta'') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1)\times\prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times\dots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k)\times\prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}
$$

commutes since we have the following assignations on elements

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{l} p, q_{f_{u_1}}, (r_t^{f_{u_1}})_{t\in\mathcal{T}}, \\ \dots, q_{f_{u_m}}, (r_t^{f_{u_m}})_{t\in\mathcal{T}} \end{array} \arrow[dr, mapsto] & \\
\begin{array}{l} p, q_1, (r_t^1)_{t\in\mathcal{T}}, \\ \dots, q_k, (r_t^k)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, \overline{q}_{f_{u_1}}, \dots, \\ \overline{q}_{f_{u_m}}, [r_t^{f_{u_1}}, \dots, r_t^{f_{u_m}}]_{t\in\mathcal{T}} \\ = P_u(g)(p), \overline{q}_1, \dots, \\ \overline{q}_k, [r_t^1, \dots, r_t^k]_{t\in\mathcal{T}} \end{array} \\
& \begin{array}{l} P_u(g)(p), q_1, (r_t^1)_{t\in\mathcal{T}}, \\ \dots, q_k, (r_t^k)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}
$$

The arrow $f:(u_1,\dots,u_m)\to(v_1,\dots,v_k)$ induces an arrow $f':\sum\limits_{i=1}^{m}\Delta_{f_{u_i}}\to\sum\limits_{i=1}^{k}\Delta_i$ such that $Q_{u_j}(f')(\overline{q}_{f_{u_j}})=\overline{q}_{f_{u_j}}$ where we write $\overline{(-)}$ for both the inclusions $\Delta_j\to\sum\limits_{i=1}^{k}\Delta_i$ and $\Delta_{f_{u_j}}\to\sum\limits_{i=1}^{m}\Delta_{f_{u_i}}$ and $[r_t^{f_{u_1}},\dots,r_t^{f_{u_m}}]=$

$[r_t^1, \ldots, r_t^k] \circ f'_t$. So by universal properties of the coends

$$\begin{tikzcd}
& \begin{array}{l} P_u(\Delta') \\ \times \int^{\Gamma'} Q_{u_1}(\Gamma') \times \ldots \times Q_{u_m}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P_u(\Delta') \\ \times \int^{\Gamma'} Q_{v_1}(\Gamma') \\ \times \ldots \\ \times Q_{v_k}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \arrow[ur] \arrow[dr, "P(f) \times \mathrm{id}"'] & & \begin{array}{l} \int^{\Delta'} P_u(\Delta') \\ \times \int^{\Gamma'} Q_{u_1}(\Gamma') \\ \times \ldots \\ \times Q_{u_m}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \\
& \begin{array}{l} P_u(\Delta'') \\ \times \int^{\Gamma'} Q_{v_1}(\Gamma') \times \ldots \times Q_{v_k}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd} \tag{C.1}$$

and

$$\begin{tikzcd}
& \begin{array}{l} P_u(\Delta') \times Q_{u_1}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \\ \times \ldots \times Q_{u_m}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_{f_{u_i}})^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P_u(\Delta') \\ \times Q_{u_1}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \\ \times \ldots \\ \times Q_{u_m}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{k} \Delta_i)^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ f'_t)_{t \in \mathcal{T}}"] \arrow[dr, "\mathrm{id} \times Q_{u_1}(f') \times \ldots \times Q_{u_m}(f') \times \mathrm{id}"'] & & \begin{array}{l} P_u(\Delta') \\ \times \int^{\Gamma'} Q_{u_1}(\Gamma') \\ \times \ldots \\ \times Q_{u_m}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \\
& \begin{array}{l} P_u(\Delta') \times Q_{u_1}(\sum\limits_{i=1}^{k} \Delta_i) \\ \times \ldots \times Q_{u_m}(\sum\limits_{i=1}^{m} \Delta_i) \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd} \tag{C.2}$$

we have

$$P_u(g)(p), \overline{q}_1, \ldots, \overline{q}_k, [r_t^1, \ldots, r_t^k]_{t \in \mathcal{T}}$$

equals to by (C.1)

$$p, \overline{q}_{f_{u_1}}, \ldots, \overline{q}_{f_{u_m}}, [r_t^1, \ldots, r_t^k]_{t \in \mathcal{T}}$$

equals to by (C.2)

$$p, \overline{q}_{f_{u_1}}, \ldots, \overline{q}_{f_{u_m}}, [r_t^{f_{u_1}}, \ldots, r_t^{f_{u_m}}]_{t \in \mathcal{T}}$$

We check that the above defined arrows are inverse to each other. Starting with $p, q_1, \ldots, q_m, r \in P_u(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$, we have the following assignations

$$\begin{tikzcd}
p, q_1, \ldots, q_m, r \arrow[d] \\
p, q_1, r, \ldots, q_m, r \arrow[d] \\
p, \overline{q}_1, \ldots, \overline{q}_m, [r, \ldots, r]
\end{tikzcd}$$

The two elements $p, q_1, \ldots, q_m, r \in P_u(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$ and $p, \overline{q}_1, \ldots, \overline{q}_m, [r, \ldots, r] \in P_u(\Delta') \times Q_{u_1}(m \times \Delta) \times \ldots \times Q_{u_m}(m \times \Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(m \times \Delta)^{-1}(t)}$ are equal since they come from $p, \overline{q}_1, \ldots, \overline{q}_m, r \in P_u(\Delta') \times Q_{u_1}(m \times \Delta) \times \ldots \times Q_{u_m}(m \times \Delta) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$ with the arrow $h : m \times \Delta \to \Delta$

$$\begin{tikzcd}
& \begin{matrix} P_u(\Delta') \times Q_{u_1}(\Delta) \\ \times \ldots \times Q_{u_m}(\Delta) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{matrix} \arrow[dr] & \\
\begin{matrix} P_u(\Delta') \times Q_{u_1}(m \times \Delta) \\ \times \ldots \times Q_{u_m}(m \times \Delta) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{matrix} \arrow[ur, "\mathrm{id} \times Q_{u_1}(h) \times \ldots \times Q_{u_m}(h) \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times (- \circ h_t)_{t \in \mathcal{T}}"'] & & \begin{matrix} \int^{\Delta} P_u(\Delta') \times Q_{u_1}(\Delta) \\ \times \ldots \times Q_{u_m}(\Delta) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{matrix} \\
& \begin{matrix} P_u(\Delta') \times Q_{u_1}(m \times \Delta) \\ \times \ldots \times Q_{u_m}(m \times \Delta) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(m \times \Delta)^{-1}(t)} \end{matrix} \arrow[ur] &
\end{tikzcd}$$

Starting with $p, q_1, r_1, \ldots, q_m, r_m \in P_u(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \times \ldots \times Q_{u_m}(\Delta_m) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}$, we have the following assignations

$$\begin{tikzcd}
p, q_1, r_1, \ldots, q_m, r_m \arrow[d] \\
p, \overline{q}_1, \ldots, \overline{q}_m, [r_1, \ldots, r_m] \arrow[d] \\
p, \overline{q}_1, [r_1, \ldots, r_m], \ldots, \overline{q}_m, [r_1, \ldots, r_m]
\end{tikzcd}$$

The two elements $p, q_1, r_1, \ldots, q_m, r_m \in P_u(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \times \ldots \times Q_{u_m}(\Delta_m) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}$ and $p, \overline{q}_1, [r_1, \ldots, r_m], \ldots, \overline{q}_m, [r_1, \ldots, r_m] \in P_u(\Delta') \times Q_{u_1}(\sum_{i=1}^{m} \Delta_i)$ $\times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \times \ldots \times Q_{u_m}(\sum_{i=1}^{m} \Delta_i) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)}$ are equal since each pair q_j, r_j and $\overline{q}_j, [r_1, \ldots, r_m]$ comes from $q_j, [r_1, \ldots, r_m]$ with the arrow $i_j : \Delta_j \to \sum_{i=1}^{m} \Delta_i$

$$\begin{tikzcd}
& Q_{u_j}(\Delta_j) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_j^{-1}(t)} \arrow[dr] & \\
Q_{u_j}(\Delta_j) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \arrow[ur, "\mathrm{id} \times (- \circ (i_j)_t)_{t \in \mathcal{T}}"] \arrow[dr, "Q_{u_j}(i_j) \times \mathrm{id}"'] & & \int^{\Delta_j} Q_{u_j}(\Delta_j) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_j^{-1}(t)} \\
& Q_{u_j}(\sum_{i=1}^{m} \Delta_i) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

Now we check naturalities in Γ, P, Q and R. Let $f : \Gamma_1 \to \Gamma_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
((P_u \bullet Q) \bullet R)(\Gamma_1) \arrow[r] \arrow[d] & (P_u \bullet (Q \otimes R))(\Gamma_1) \arrow[d] \\
((P_u \bullet Q) \bullet R)(\Gamma_2) \arrow[r] & (P_u \bullet (Q \otimes R))(\Gamma_2)
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
p, q_1, \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t \in \mathcal{T}} \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[d, mapsto] \\
p, q_1, \ldots, q_m, (R_t(f) \circ r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & p, q_1, (R_t(f) \circ r_t)_{t \in \mathcal{T}} \ldots, q_m, (R_t(f) \circ r_t)_{t \in \mathcal{T}}
\end{tikzcd}$$

Let $f : P \to S$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
((P_u \bullet Q) \bullet R)(\Gamma) \arrow[r] \arrow[d] & (P_u \bullet (Q \otimes R))(\Gamma) \arrow[d] \\
((S_u \bullet Q) \bullet R)(\Gamma) \arrow[r] & (S_u \bullet (Q \otimes R))(\Gamma)
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
p, q_1, \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t \in \mathcal{T}} \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[d, mapsto] \\
f_{u,\Delta'}(p), q_1, \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & f_{u,\Delta'}(p), q_1, (r_t)_{t \in \mathcal{T}} \ldots, q_m, (r_t)_{t \in \mathcal{T}}
\end{tikzcd}$$

Let $f : Q \to S$ in $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd}
((P_u \bullet Q)\bullet R)(\Gamma) \arrow[r] \arrow[d] & (P_u \bullet (Q\otimes R))(\Gamma) \arrow[d] \\
((P_u \bullet S)\bullet R)(\Gamma) \arrow[r] & (P_u \bullet (S\otimes R))(\Gamma)
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
p, q_1, \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t\in\mathcal{T}} \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
p, f_{u_1,\Delta}(q_1), \ldots, f_{u_1,\Delta}(q_m), (r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & p, f_{u_1,\Delta}(q_1), (r_t)_{t\in\mathcal{T}} \ldots, f_{u_1,\Delta}(q_m), (r_t)_{t\in\mathcal{T}}
\end{tikzcd}$$

Let $f : R \to S$ in $\mathcal{Y}\downarrow[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd}
((P_u \bullet Q)\bullet R)(\Gamma) \arrow[r] \arrow[d] & (P_u \bullet (Q\otimes R))(\Gamma) \arrow[d] \\
((P_u \bullet Q)\bullet S)(\Gamma) \arrow[r] & (P_u \bullet (Q\otimes S))(\Gamma)
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
p, q_1, \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t\in\mathcal{T}} \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
p, q_1, \ldots, q_m, (f_t\circ r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & p, q_1, (f_t\circ r_t)_{t\in\mathcal{T}} \ldots, q_m, (f_t\circ r_t)_{t\in\mathcal{T}}
\end{tikzcd}$$

C.1.2 Construction λ

The left–hand side is explicitly

$$\begin{aligned}
(\mathcal{Y}\otimes P)_u(\Gamma) &= \int^{\Delta} \mathcal{Y}\langle u\rangle(\Delta) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta^{-1}(t)} \\
&= \int^{\Delta} \Delta^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta^{-1}(t)}
\end{aligned}$$

By universal property of the coend it suffices to give a collection of arrows

$$\Delta^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta^{-1}(t)} \to P_u(\Gamma)$$

for all Δ satisfying the wedge condition. We take the following mapping

$$(x, (h_t)_{t\in\mathcal{T}}) \quad \mapsto \quad h_u(x)$$

and check the wedge condition. Let $f : \Delta_1 \to \Delta_2$, the following diagram commutes

$$\begin{tikzcd}
 & \Delta_1^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta_1^{-1}(t)} \arrow[dr] & \\
\Delta_1^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur, "\mathrm{id}\times(-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "f_u\times\mathrm{id}"'] & & P_u(\Gamma) \\
 & \Delta_2^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

since we have the following assignations on elements

$$\begin{tikzcd}
 & x, (h_t \circ f_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\
x, (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & h_u(f_u(x)) \\
 & f_u(x), (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}$$

We define the arrow in the inverse direction by the composite of the following mapping and the corresponding coprojection in $\langle u\rangle$.

$$x \cong 1 \times x^1 \in \langle u\rangle^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\langle u\rangle^{-1}(t)}$$

These two mappings are inverse to each other. One of the composites is

$$x \quad \mapsto \quad (1, x^1) \quad \mapsto x$$

which is the identity on x. The other composite is

$$(x, (h_t)_{t\in\mathcal{T}}) \quad \mapsto \quad h_u(x) \quad \mapsto (1, h_u(x)^1)$$

This is identity because $(x, (h_u)_{u\in\mathcal{T}})$ and $(1, h_u(x)^1)$ come from
$(1, (h_t)_{t\in\mathcal{T}}) \in \langle u\rangle^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta^{-1}(t)}$ with the arrow $x : \langle u\rangle \to \Delta$

$$\begin{tikzcd}
 & \begin{matrix}\Delta^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta^{-1}(t)} \\ (x, (h_t)_{t\in\mathcal{T}})\end{matrix} \arrow[dr] & \\
\begin{matrix}\langle u\rangle^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\Delta^{-1}(t)} \\ (1, (h_t)_{t\in\mathcal{T}})\end{matrix} \arrow[ur] \arrow[dr] & & (\mathcal{Y} \otimes P)_u(\Gamma) \\
 & \begin{matrix}\langle u\rangle^{-1}(u) \times \prod_{t\in\mathcal{T}} P_t(\Gamma)^{\langle u\rangle^{-1}(t)} \\ (1, h_u(x)^1)\end{matrix} \arrow[ur] &
\end{tikzcd}$$

We check naturalities in Γ and P. Let $f : \Gamma_1 \to \Gamma_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
(\mathcal{Y} \otimes P)_u(\Gamma_1) \arrow[r] \arrow[d] & P_u(\Gamma_1) \arrow[d] \\
(\mathcal{Y} \otimes P)_u(\Gamma_2) \arrow[r] & P_u(\Gamma_2)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
x, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & h_u(x) \arrow[d, mapsto] \\
x, (P_t(f) \circ h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & P_u(f)(h_u(x))
\end{tikzcd}$$

Let $f : P \to Q$ in $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd}
(\mathcal{Y}\otimes P)_u(\Gamma) \arrow[r] \arrow[d] & P_u(\Gamma) \arrow[d] \\
(\mathcal{Y}\otimes Q)_u(\Gamma) \arrow[r] & Q_u(\Gamma)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
x,(h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & h_u(x) \arrow[d, mapsto] \\
x,(f_{t,\Gamma}\circ h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & f_{u,\Delta}(h_u(x))
\end{tikzcd}$$

C.1.3 Construction ρ

The left–hand side is explicitly

$$\begin{aligned}
(Q\otimes\mathcal{Y})_u(\Gamma) &= \int^{\Delta} Q_u(\Delta)\times\prod_{t\in\mathcal{T}}\mathcal{Y}\langle t\rangle(\Gamma)^{\Delta^{-1}(t)}\\
&= \int^{\Delta} Q_u(\Delta)\times\prod_{t\in\mathcal{T}}(\Gamma^{-1}(t))^{\Delta^{-1}(t)}
\end{aligned}$$

By universal property of the coend it suffices to give a collection of arrows

$$Q_u(\Delta)\times\prod_{t\in\mathcal{T}}(\Gamma^{-1}(t))^{\Delta^{-1}(t)} \to Q_u(\Gamma)$$

for all Δ that satisfies the wedge condition. We take the mapping

$$(x,(h_t)_{t\in\mathcal{T}}) \quad\mapsto\quad Q_u(\sum_{t\in\mathcal{T}} h_t)(x)$$

and check the wedge condition. Let $f : \Delta_1 \to \Delta_2$. The following diagram commutes

$$\begin{tikzcd}
& Q_u(\Delta_1)\times\prod_{t\in\mathcal{T}}(\Gamma^{-1}(t))^{\Delta_1^{-1}(t)} \arrow[dr] & \\
Q_u(\Delta_1)\times\prod_{t\in\mathcal{T}}(\Gamma^{-1}(t))^{\Delta_2^{-1}(t)} \arrow[ur, "\mathrm{id}\times(-\circ f_t)"] \arrow[dr, "Q_u(f)\times\mathrm{id}"'] & & Q_u(\Gamma) \\
& Q_u(\Delta_2)\times\prod_{t\in\mathcal{T}}(\Gamma^{-1}(t))^{\Delta_2^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

since we have the following assignations on elements

$$\begin{tikzcd}
& x,(h_t\circ f_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\
x,(h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & Q_u\big(\sum_{t\in\mathcal{T}}(h_t\circ f_t)\big)(x) \\
& Q_u(f)(x),(h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}$$

We define the arrow in the inverse direction by the composite of the following mapping and the corresponding coprojection in Γ.

$$x \quad \mapsto \quad (x, (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}})$$

These two mappings are inverse to each other. One of the composites is

$$x \quad \mapsto \quad (x, (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}}) \quad \mapsto \quad Q_u(\mathrm{id}_\Gamma)(x) = x$$

which is the identity on x. The other composite is

$$(x, (h_t)_{t\in\mathcal{T}}) \quad \mapsto \quad Q_u(\sum_{t\in\mathcal{T}} h_t)(x) \quad \mapsto \quad (Q_u(\sum_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}})$$

This is the identity because $(x, (h_t)_{t\in\mathcal{T}})$ and $(Q_u(\sum\limits_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}})$ come from $(x, (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}}) \in Q_u(\Delta) \times \prod\limits_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Gamma^{-1}(t)}$ with the arrow $\sum\limits_{t\in\mathcal{T}} h_t : \Delta \to \Gamma$

$$\begin{tikzcd}
 & \begin{array}{c} Q_u(\Delta) \times \prod\limits_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Delta^{-1}(t)} \\ (x, (h_t)_{t\in\mathcal{T}}) \end{array} \arrow[dr] & \\
\begin{array}{c} Q_u(\Delta) \times \prod\limits_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Gamma^{-1}(t)} \\ (x, (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}}) \end{array} \arrow[ur] \arrow[dr] & & (Q \otimes \mathcal{Y})_u(\Gamma) \\
 & \begin{array}{c} Q_u(\Gamma) \times \prod\limits_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Gamma^{-1}(t)} \\ (Q_u(\sum_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}}) \end{array} \arrow[ur] &
\end{tikzcd}$$

We check naturalities in Γ and Q. Let $f : \Gamma_1 \to \Gamma_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
(Q \times \mathcal{Y})_u(\Gamma_1) \arrow[r] \arrow[d] & Q_u(\Gamma_1) \arrow[d] \\
(Q \times \mathcal{Y})_u(\Gamma_2) \arrow[r] & Q_u(\Gamma_2)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
x, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & Q_u(\sum\limits_{t\in\mathcal{T}} h_t)(x) \arrow[d, mapsto] \\
x, (f_t \circ h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & Q_u\big(\sum\limits_{t\in\mathcal{T}} (f_t \circ h_t)\big)(x)
\end{tikzcd}$$

Let $f : Q \to R$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd}
(Q \times \mathcal{Y})_u(\Gamma) \arrow[r] \arrow[d] & Q_u(\Gamma) \arrow[d] \\
(R \times \mathcal{Y})_u(\Gamma) \arrow[r] & R_u(\Gamma)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
x, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & Q_u\big(\sum\limits_{t\in\mathcal{T}} h_t\big)(x) \arrow[d, mapsto] \\
f_{u,\Delta}(x), (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & \begin{array}{c} f_{u,\Gamma}\Big(Q_u\big(\sum\limits_{t\in\mathcal{T}} h_t\big)(x)\Big) \\ = R_u\big(\sum\limits_{t\in\mathcal{T}} h_t\big)(f_{u,\Delta}(x)) \end{array}
\end{tikzcd}$$

C.2 Proof of proposition 5.5.89

We check the wedge condition of $a_{P,Q,R}$. Let $f : \Delta_1 \to \Delta_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The diagram

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta') \times Q_{u_1}(\Delta_1) \\ \times \ldots \times Q_{u_m}(\Delta_1) \\ \times \prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} P(\Delta') \times Q_{u_1}(\Delta_1) \\ \times \ldots \times Q_{u_m}(\Delta_1) \\ \times \prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & (P \bullet (Q \otimes R))(\Gamma) \\
 & \begin{array}{c} P(\Delta') \times Q_{u_1}(\Delta_2) \\ \times \ldots \times Q_{u_m}(\Delta_2) \\ \times \prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
 & \begin{array}{c} p, q_1, \ldots, \\ q_m, (r_t \circ f_t)_{t\in\mathcal{T}} \end{array} \arrow[dr, mapsto] & \\
\begin{array}{c} p, q_1, \ldots, \\ q_m, (r_t)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, q_1, (r_t \circ f_t)_{t\in\mathcal{T}}, \ldots, \\ q_m, (r_t \circ f_t)_{t\in\mathcal{T}} \\ = p, Q_{u_1}(f)(q_1), (r_t)_{t\in\mathcal{T}}, \ldots, \\ Q_{u_m}(f)(q_m), (r_t)_{t\in\mathcal{T}} \end{array} \\
 & \begin{array}{c} p, Q_{u_1}(f)(q_1), \ldots, \\ Q_{u_m}(f)(q_m), (r_t)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

each pair $q_j, (r_t \circ f_t)_{t\in\mathcal{T}} \in Q_{u_j}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}$ and $Q_{u_j}(f)(q_j), (r_t)_{t\in\mathcal{T}} \in Q_{u_j}(\Delta_2) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)}$ is equal since they come from $q_j, (r_t)_{t\in\mathcal{T}} \in Q_{u_j}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)}$ with the arrow f

$$
\begin{tikzcd}
& Q_{u_j}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \arrow[dr] & \\
Q_{u_j}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur, "\mathrm{id} \times (-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "Q_{u_j}(f)\times \mathrm{id}"'] & & \int^{\Delta_j} Q_{u_j}(\Delta_j) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_j^{-1}(t)} \\
& Q_{u_j}(\Delta_2) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur] &
\end{tikzcd}
$$

Let $g : \Delta' \to \Delta''$ in $\mathbb{F} \downarrow \mathcal{T}$ where $\Delta' = (u_1, \dots, u_m)$ and $\Delta'' = (v_1, \dots, v_k)$. The diagram

$$
\begin{tikzcd}
& P(\Delta') \times \int^{\Delta} Q_{u_1}(\Delta) \times \dots \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[dr] & \\
P(\Delta') \times \int^{\Delta} Q_{v_1}(\Delta_1) \times \dots \times Q_{v_k}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[ur] \arrow[dr] & & (P \bullet (Q \otimes R))(\Gamma) \\
& P(\Delta'') \times \int^{\Delta} Q_{v_1}(\Delta) \times \dots \times Q_{v_k}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[ur] &
\end{tikzcd}
$$

commutes since we have the following assignations on elements

$$
\begin{tikzcd}
& p, q_{g_{u_1}}, \dots, q_{g_{u_m}}, (r_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\
p, q_1, \dots, q_k, (r_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, q_{g_{u_1}}, (r_t)_{t\in\mathcal{T}}, \dots, \\ q_{g_{u_m}}, (r_t)_{t\in\mathcal{T}} \\ = P(g)(p), q_1, (r_t)_{t\in\mathcal{T}}, \dots, \\ q_k, (r_t)_{t\in\mathcal{T}} \end{array} \\
& P(g)(p), q_1, \dots, q_k, (r_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}
$$

The elements $p, q_{g_{u_1}}, (r_t)_{t\in\mathcal{T}}, \dots, q_{g_{u_m}}, (r_t)_{t\in\mathcal{T}} \in P(\Delta') \times \big(\int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}\big) \times \dots \times \big(\int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}\big)$ and $P(g)(p), q_1, (r_t)_{t\in\mathcal{T}}, \dots, q_k, (r_t)_{t\in\mathcal{T}} \in P(\Delta'') \times \big(\int^{\Delta_1} Q_{v_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}\big) \times \dots \times \big(\int^{\Delta_k} Q_{v_k}(\Delta_k) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)}\big)$ are equal since they come from $p, q_1, (r_t)_{t\in\mathcal{T}}, \dots, q_k, (r_t)_{t\in\mathcal{T}} \in P(\Delta') \times \left(\int^{\Delta_1} Q_{v_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)}\right) \times \dots \times \left(\int^{\Delta_k} Q_{v_k}(\Delta_k) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)}\right)$ with the arrow g

$$\begin{tikzcd}
& \begin{array}{l} P(\Delta') \\ \times \int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \dots \\ \times \int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P(\Delta') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1) \\ \quad \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \dots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k) \\ \quad \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & (P \bullet (Q \otimes R))(\Gamma) \\
& \begin{array}{l} P(\Delta'') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \dots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

Next we construct an inverse arrow. By universal properties of the coends, it suffices to give a collection of arrows

$$\begin{aligned} P(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} & \\ \times \dots & \\ \times Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} & \to ((P \bullet Q) \bullet R)(\Gamma) \end{aligned}$$

for all Δ', $\Delta_1, \ldots, \Delta_m \in \mathbb{F}\downarrow\mathcal{T}$ satisfying the wedge condition. We take the following composite

$$
\begin{tikzcd}
\begin{array}{l} P(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{array} \arrow[d] \\
\begin{array}{c} P(\Delta') \times Q_{u_1}(\sum_{i=1}^{m} \Delta_i) \times \ldots \times Q_{u_m}(\sum_{i=1}^{m} \Delta_i) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \end{array} \arrow[d] \\
\int^{\Delta} P(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] \\
\int^{\Delta} \int^{\Delta'} P(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}
\end{tikzcd}
$$

We check the wedge condition. Let $f^i : \Delta_i \to \Xi_i$ in $\mathbb{F}\downarrow\mathcal{T}$ for all $i = 1, \ldots, m$. The diagram

$$
\begin{tikzcd}
& \begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\Delta_1) \times \\ \quad \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Xi_1^{-1}(t)} \\ \times \ldots \\ \times Q_{u_m}(\Delta_m) \\ \quad \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Xi_m^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & ((P \bullet Q) \bullet R)(\Gamma) \\
& \begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\Xi_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Xi_1^{-1}(t)} \\ \times \ldots \\ \times Q_{u_m}(\Xi_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Xi_m^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}
$$

commutes since we have the following assignations on elements

$$
\begin{tikzcd}
 & \begin{array}{c} p, q_1, (r_t^1 \circ f_t^i)_{t\in\mathcal{T}}, \\ \dots, q_m, (r_t^m \circ f_t^i)_{t\in\mathcal{T}} \end{array} \arrow[dr, mapsto] & \\
\begin{array}{c} p, q_1, (r_t^1)_{t\in\mathcal{T}}, \\ \dots, q_m, (r_t^m)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, \overline{q}_1, \dots, \overline{q}_m, \\ [r_t^1 \circ f_t^i, \dots, r_t^m \circ f_t^i]_{t\in\mathcal{T}} \\ = p, \overline{Q_{u_1}(f^1)(q_1)}, \\ \dots, \overline{Q_{u_m}(f^m)(q_m)}, \\ [r_t^1, \dots, r_t^m]_{t\in\mathcal{T}} \end{array} \\
 & \begin{array}{c} p, Q_{u_1}(f^1)(q_1), (r_t^1)_{t\in\mathcal{T}}, \\ \dots, Q_{u_m}(f^m)(q_m), (r_t^m)_{t\in\mathcal{T}} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}
$$

By naturality of the inclusions $\Delta_j \to \sum_{i=1}^{m} \Delta_i$ and $\Xi_j \to \sum_{i=1}^{m} \Xi_i$, $\overline{Q_{u_j}(f^j)(q_j)} = \big(\sum_{i=1}^{m} f^i\big)(\overline{q}_j)$ and $[r_t^1 \circ f_t^i, \dots, r_t^m \circ f_t^i]_{t\in\mathcal{T}} = ([r_t^1, \dots, r_t^m] \circ \sum_{i=1}^{m} f_t^i)_{t\in\mathcal{T}}$. The elements $p, \overline{q}_1, \dots, \overline{q}_m, ([r_t^1, \dots, r_t^m] \circ \sum_{i=1}^{m} f_t^i)_{t\in\mathcal{T}}$ and $p, \big(\sum_{i=1}^{m} f^i\big)(\overline{q}_1), \dots, \big(\sum_{i=1}^{m} f^i\big)(\overline{q}_m), [r_t^1, \dots, r_t^m]_{t\in\mathcal{T}}$ are equal since they come from $p, \overline{q}_1, \dots, \overline{q}_m, [r_t^1, \dots, r_t^m]_{t\in\mathcal{T}}$ with the arrow $f := \sum_{i=1}^{m} f^i$

$$
\begin{tikzcd}
 & \begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\sum_{i=1}^{m} \Delta_i) \times \ldots \times Q_{u_m}(\sum_{i=1}^{m} \Delta_i) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\sum_{i=1}^{m} \Delta_i) \\ \times \ldots \\ \times Q_{u_m}(\sum_{i=1}^{m} \Delta_i) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Xi_i)^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times (-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "\mathrm{id} \times Q_{u_1}(f) \times \ldots \times Q_{u_m}(f) \times \mathrm{id}"'] & & ((P \bullet Q) \bullet R)(\Gamma) \\
 & \begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\sum_{i=1}^{m} \Xi_i) \times \ldots \times Q_{u_m}(\sum_{i=1}^{m} \Xi_i) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Xi_i)^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}
$$

Let $f : \Delta' \to \Delta''$ in $\mathbb{F} \downarrow \mathcal{T}$ where $\Delta' = (u_1, \ldots, u_m)$ and $\Delta'' = (v_1, \ldots, v_k)$. The diagram

$$\begin{tikzcd}
& \begin{array}{l} P(\Delta') \\ \times \int^{\Delta_1} Q_{u_1}(\Delta_1) \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times \int^{\Delta_m} Q_{u_m}(\Delta_m) \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P(\Delta') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1) \times \\ \qquad \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k) \\ \qquad \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & ((P \bullet Q) \bullet R)(\Gamma) \\
& \begin{array}{l} P(\Delta'') \\ \times \int^{\Delta_1} Q_{v_1}(\Delta_1) \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ \times \ldots \\ \times \int^{\Delta_k} Q_{v_k}(\Delta_k) \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_k^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
& \begin{array}{l} p, q_{f_{u_1}}, (r_t^{f_{u_1}})_{t \in \mathcal{T}}, \\ \ldots, q_{f_{u_m}}, (r_t^{f_{u_m}})_{t \in \mathcal{T}} \end{array} \arrow[dr, mapsto] & \\
\begin{array}{l} p, q_1, (r_t^1)_{t \in \mathcal{T}}, \\ \ldots, q_k, (r_t^k)_{t \in \mathcal{T}} \end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} p, \overline{q}_{f_{u_1}}, \ldots, \\ \overline{q}_{f_{u_m}}, [r_t^{f_{u_1}}, \ldots, r_t^{f_{u_m}}]_{t \in \mathcal{T}} \\ = P(g)(p), \overline{q}_1, \ldots, \\ \overline{q}_k, [r_t^1, \ldots, r_t^k]_{t \in \mathcal{T}} \end{array} \\
& \begin{array}{l} P(g)(p), q_1, (r_t^1)_{t \in \mathcal{T}}, \\ \ldots, q_k, (r_t^k)_{t \in \mathcal{T}} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

The arrow $f : (u_1, \ldots, u_m) \to (v_1, \ldots, v_k)$ induces an arrow $f' : \sum\limits_{i=1}^{m} \Delta_{f_{u_i}} \to \sum\limits_{i=1}^{k} \Delta_i$ such that $Q_{u_j}(f')(\overline{q}_{f_{u_j}}) = \overline{q}_{f_{u_j}}$ where we write $\overline{(-)}$ for both the inclusions $\Delta_j \to \sum\limits_{i=1}^{k} \Delta_i$ and $\Delta_{f_{u_j}} \to \sum\limits_{i=1}^{m} \Delta_{f_{u_i}}$ and $[r_t^{f_{u_1}}, \ldots, r_t^{f_{u_m}}] =$

$[r_t^1, \ldots, r_t^k] \circ f'_t$. So by universal properties of the coends

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{l} P(\Delta') \\ \times \int^{\Gamma'} Q_{u_1}(\Gamma') \times \ldots \times Q_{u_m}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P(\Delta') \\ \times \int^{\Gamma'} Q_{v_1}(\Gamma') \\ \times \ldots \\ \times Q_{v_k}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \arrow[ur] \arrow[dr, "P(f) \times \mathrm{id}"'] & & \begin{array}{l} \int^{\Delta'} P(\Delta') \\ \times \int^{\Gamma'} Q_{u_1}(\Gamma') \\ \times \ldots \\ \times Q_{u_m}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \\
 & \begin{array}{l} P(\Delta'') \\ \times \int^{\Gamma'} Q_{v_1}(\Gamma') \times \ldots \times Q_{v_k}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd} \tag{C.3}$$

and

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \times \ldots \times Q_{u_m}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}})^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \\ \times \ldots \\ \times Q_{u_m}(\sum\limits_{i=1}^{m} \Delta_{f_{u_i}}) \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum\limits_{i=1}^{k} \Delta_i)^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ f'_t)_{t \in \mathcal{T}}"] \arrow[dr, "\mathrm{id} \times Q_{u_1}(f') \times \ldots \times Q_{u_m}(f') \times \mathrm{id}"'] & & \begin{array}{l} P(\Delta') \\ \times \int^{\Gamma'} Q_{u_1}(\Gamma') \\ \times \ldots \\ \times Q_{u_m}(\Gamma') \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\Gamma')^{-1}(t)} \end{array} \\
 & \begin{array}{l} P(\Delta') \\ \times Q_{u_1}(\sum\limits_{i=1}^{k} \Delta_i) \times \ldots \times Q_{u_m}(\sum\limits_{i=1}^{m} \Delta_i) \\ \times \prod\limits_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd} \tag{C.4}$$

we have

$$P(g)(p), \overline{q}_1, \ldots, \overline{q}_k, [r_t^1, \ldots, r_t^k]_{t\in\mathcal{T}}$$

equals to by (C.3)

$$p, \overline{q}_{f_{u_1}}, \ldots, \overline{q}_{f_{u_m}}, [r_t^1, \ldots, r_t^k]_{t\in\mathcal{T}}$$

equals to by (C.4)

$$p, \overline{q}_{f_{u_1}}, \ldots, \overline{q}_{f_{u_m}}, [r_t^{f_{u_1}}, \ldots, r_t^{f_{u_m}}]_{t\in\mathcal{T}}$$

We check that the above defined arrows are inverse to each other. Starting with $p, q_1, \ldots, q_m, r \in P(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$, we have the following assignations

$$\begin{array}{c} p, q_1, \ldots, q_m, r \\ \downarrow \\ p, q_1, r, \ldots, q_m, r \\ \downarrow \\ p, \overline{q}_1, \ldots, \overline{q}_m, [r, \ldots, r] \end{array}$$

The two elements $p, q_1, \ldots, q_m, r \in P(\Delta') \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$ and $p, \overline{q}_1, \ldots, \overline{q}_m, [r, \ldots, r] \in P(\Delta') \times Q_{u_1}(m\times\Delta) \times \ldots \times Q_{u_m}(m\times\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(m\times\Delta)^{-1}(t)}$ are equal since they come from $p, \overline{q}_1, \ldots, \overline{q}_m, r \in P(\Delta') \times Q_{u_1}(m \times \Delta) \times \ldots \times Q_{u_m}(m \times \Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$ with the arrow $h : m \times \Delta \to \Delta$

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta') \times Q_{u_1}(\Delta) \\ \times \ldots \times Q_{u_m}(\Delta) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} P(\Delta') \times Q_{u_1}(m \times \Delta) \\ \times \ldots \times Q_{u_m}(m \times \Delta) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times Q_{u_1}(h) \times \ldots \times Q_{u_m}(h) \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times (-\circ h_t)_{t\in\mathcal{T}}"'] & & \begin{array}{c} \int^{\Delta} P(\Delta') \times Q_{u_1}(\Delta) \\ \times \ldots \times Q_{u_m}(\Delta) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{array} \\
 & \begin{array}{c} P(\Delta') \times Q_{u_1}(m \times \Delta) \\ \times \ldots \times Q_{u_m}(m \times \Delta) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(m\times\Delta)^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

Starting with $p, q_1, r_1, \ldots, q_m, r_m \in P(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \times \ldots \times Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}$, we have the following assignations

$$\begin{array}{c} p, q_1, r_1, \ldots, q_m, r_m \\ \downarrow \\ p, \overline{q}_1, \ldots, \overline{q}_m, [r_1, \ldots, r_m] \\ \downarrow \\ p, \overline{q}_1, [r_1, \ldots, r_m], \ldots, \overline{q}_m, [r_1, \ldots, r_m] \end{array}$$

The two elements $p, q_1, r_1, \ldots, q_m, r_m \in P(\Delta') \times Q_{u_1}(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \times \ldots \times Q_{u_m}(\Delta_m) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_m^{-1}(t)}$ and $p, \overline{q}_1, [r_1, \ldots, r_m], \ldots, \overline{q}_m, [r_1, \ldots, r_m] \in P(\Delta') \times Q_{u_1}(\sum_{i=1}^{m} \Delta_i)$ $\times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \times \ldots \times Q_{u_m}(\sum_{i=1}^{m} \Delta_i) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)}$ are equal since each pair q_j, r_j and $\overline{q}_j, [r_1, \ldots, r_m]$ comes from $q_j, [r_1, \ldots, r_m]$ with the arrow $i_j : \Delta_j \to \sum_{i=1}^{m} \Delta_i$

$$\begin{tikzcd}
 & Q_{u_j}(\Delta_j) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_j^{-1}(t)} \arrow[dr] & \\
Q_{u_j}(\Delta_j) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \arrow[ur, "\mathrm{id} \times (- \circ (i_j)_t)_{t \in \mathcal{T}}"] \arrow[dr, "Q_{u_j}(i_j) \times \mathrm{id}"'] & & \int^{\Delta_j} Q_{u_j}(\Delta_j) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_j^{-1}(t)} \\
 & Q_{u_j}(\sum_{i=1}^{m} \Delta_i) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{(\sum_{i=1}^{m} \Delta_i)^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

Now we check naturalities in Γ, P, Q and R. Let $f : \Gamma_1 \to \Gamma_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
((P \bullet Q) \bullet R)(\Gamma_1) \arrow[r] \arrow[d] & (P \bullet (Q \otimes R))(\Gamma_1) \arrow[d] \\
((P \bullet Q) \bullet R)(\Gamma_2) \arrow[r] & (P \bullet (Q \otimes R))(\Gamma_2)
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
p, q_1, \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t \in \mathcal{T}} \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[d, mapsto] \\
p, q_1, \ldots, q_m, (R_t(f) \circ r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & p, q_1, (R_t(f) \circ r_t)_{t \in \mathcal{T}} \ldots, q_m, (R_t(f) \circ r_t)_{t \in \mathcal{T}}
\end{tikzcd}$$

Let $f : P \to S$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
((P \bullet Q) \bullet R)(\Gamma) \arrow[r] \arrow[d] & (P \bullet (Q \otimes R))(\Gamma) \arrow[d] \\
((S \bullet Q) \bullet R)(\Gamma) \arrow[r] & (S \bullet (Q \otimes R))(\Gamma)
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
p, q_1, \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t \in \mathcal{T}} \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[d, mapsto] \\
f_{\Delta'}(p), q_1, \ldots, q_m, (r_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & f_{\Delta'}(p), q_1, (r_t)_{t \in \mathcal{T}} \ldots, q_m, (r_t)_{t \in \mathcal{T}}
\end{tikzcd}$$

Let $f : Q \to S$ in $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd} ((P \bullet Q) \bullet R)(\Gamma) \arrow[r] \arrow[d] & (P \bullet (Q \otimes R))(\Gamma) \arrow[d] \\ ((P \bullet S) \bullet R)(\Gamma) \arrow[r] & (P \bullet (S \otimes R))(\Gamma) \end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd} p, q_1, \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t\in\mathcal{T}} \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\ p, f_{u_1,\Delta}(q_1), \ldots, f_{u_1,\Delta}(q_m), (r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & p, f_{u_1,\Delta}(q_1), (r_t)_{t\in\mathcal{T}} \ldots, f_{u_1,\Delta}(q_m), (r_t)_{t\in\mathcal{T}} \end{tikzcd}$$

Let $f : R \to S$ in $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd} ((P \bullet Q) \bullet R)(\Gamma) \arrow[r] \arrow[d] & (P \bullet (Q \otimes R))(\Gamma) \arrow[d] \\ ((P \bullet Q) \bullet S)(\Gamma) \arrow[r] & (P \bullet (Q \otimes S))(\Gamma) \end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd} p, q_1, \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, q_1, (r_t)_{t\in\mathcal{T}} \ldots, q_m, (r_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\ p, q_1, \ldots, q_m, (f_t \circ r_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & p, q_1, (f_t \circ r_t)_{t\in\mathcal{T}} \ldots, q_m, (f_t \circ r_t)_{t\in\mathcal{T}} \end{tikzcd}$$

We check the wedge condition of r_P. Let $f : \Delta_1 \to \Delta_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The diagram

$$\begin{tikzcd} & P(\Delta_1) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta_1^{-1}(t)} \arrow[dr] & \\ P(\Delta_1) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta_2^{-1}(t)} \arrow[ur] \arrow[dr] & & P(\Gamma) \\ & P(\Delta_2) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta_2^{-1}(t)} \arrow[ur] & \end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd} & p, (h_t \circ f_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\ p, (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & P\big(\sum_{t\in\mathcal{T}} h_t\big)(P(f)(p)) \\ & P(f)(p), (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] & \end{tikzcd}$$

We define the arrow in the inverse direction by the composite of the following mapping and the corresponding coprojection in Γ.

$$x \quad \mapsto \quad (x, (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}})$$

These two mappings are inverse to each other. One of the composites is

$$x \quad \mapsto \quad (x, (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}}) \quad \mapsto \quad P(\mathrm{id}_\Gamma)(x) = x$$

which is the identity on x. The other composite is

$$(x, (h_t)_{t\in\mathcal{T}}) \quad \mapsto \quad P(\sum_{t\in\mathcal{T}} h_t)(x) \quad \mapsto \quad (P(\sum_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}})$$

This is the identity because $(x, (h_t)_{t\in\mathcal{T}}) \in P(\Delta) \times \prod_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Delta^{-1}(t)}$ and $(P(\sum_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}}) \in P(\Gamma)\times \prod_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Gamma^{-1}(t)}$ come from $(x, (\mathrm{id}_{\Gamma^{-1}(t)})_{t\in\mathcal{T}}) \in P(\Delta)\times \prod_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Gamma^{-1}(t)}$ with the arrow $\sum_{t\in\mathcal{T}} h_t : \Delta \to \Gamma$

$$\begin{tikzcd}
 & P(\Delta) \times \prod_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Delta^{-1}(t)} \arrow[dr] & \\
P(\Delta) \times \prod_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Gamma^{-1}(t)} \arrow[ur, "\mathrm{id} \times (-\circ h_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(\sum_{t\in\mathcal{T}} h_t)\times \mathrm{id}"'] & & (P \bullet \mathcal{Y})(\Gamma) \\
 & P(\Gamma) \times \prod_{t\in\mathcal{T}} (\Gamma^{-1}(t))^{\Gamma^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

We check naturality of $r_{P,\Gamma}$ in Γ and P. Let $f : \Gamma_1 \to \Gamma_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
(P \bullet \mathcal{Y})(\Gamma_1) \arrow[r] \arrow[d] & P(\Gamma_1) \arrow[d] \\
(P \bullet \mathcal{Y})(\Gamma_2) \arrow[r] & P(\Gamma_2)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
p, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & P\big(\sum_{t\in\mathcal{T}} h_t\big)(p) \arrow[d, mapsto] \\
p, (f_t \circ h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & P(f) \circ P\big(\sum_{t\in\mathcal{T}} h_t\big)(p)
\end{tikzcd}$$

Let $f : P \to Q$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
(P \bullet \mathcal{Y})(\Gamma) \arrow[r] \arrow[d] & P(\Gamma) \arrow[d] \\
(Q \bullet \mathcal{Y})(\Gamma) \arrow[r] & Q(\Gamma)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
p, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & P\big(\sum_{t\in\mathcal{T}} h_t\big)(p) \arrow[d, mapsto] \\
f_\Delta(p), (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & f_\Gamma(P\big(\sum_{t\in\mathcal{T}} h_t\big)(p)) = Q\big(\sum_{t\in\mathcal{T}} h_t\big)(f_\Delta(p))
\end{tikzcd}$$

We check the axioms for a right $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$–action. First we check the commutativity of the following diagram

$$
\begin{tikzcd}
(((P \bullet Q) \bullet R) \bullet S)(\Gamma) \arrow[r] \arrow[d] & ((P \bullet Q) \bullet (R \otimes S))(\Gamma) \arrow[dd] \\
((P \bullet (Q \otimes R)) \bullet S)(\Gamma) \arrow[d] & \\
(P \bullet ((Q \otimes R) \otimes S))(\Gamma) \arrow[r] & (P \bullet (Q \otimes (R \otimes S)))(\Gamma)
\end{tikzcd}
$$

Along the left hand side we have the following assignations on elements

$$
\begin{array}{ll}
\begin{array}{l}
\int^{\Delta} \int^{\Delta'} \int^{\Delta''} P(\Delta'') \\
\times Q_{v_1}(\Delta') \times \ldots \times Q_{v_k}(\Delta') \\
\times R_{u_1}(\Delta) \times \ldots \times R_{u_m}(\Delta) \\
\times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Delta^{-1}(t)}
\end{array}
&
\begin{array}{c}
p, \\
q_1, \ldots, q_k, \\
r_1, \ldots, r_m, \\
s
\end{array}
\\
\downarrow & \big\downarrow{\scriptstyle \Delta_1 = \ldots = \Delta_k = \Delta'} \\
\begin{array}{l}
\int^{\Delta} \int^{\Delta''} \int^{\Delta_1} \ldots \int^{\Delta_k} P(\Delta'') \\
\times Q_{v_1}(\Delta_1) \times R_{u_{1,1}}(\Delta) \times \ldots \times R_{u_{1,m_1}}(\Delta) \\
\times \ldots \\
\times Q_{v_k}(\Delta) \times R_{u_{k,1}}(\Delta) \times \ldots \times R_{u_{k,m_k}}(\Delta_k) \\
\times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Delta^{-1}(t)}
\end{array}
&
\begin{array}{c}
p, \\
q_1, r_1, \ldots, r_m, \\
\ldots, \\
q_k, r_1, \ldots, r_m, \\
s
\end{array}
\\
\downarrow & \big\downarrow{\scriptstyle \Gamma_1 = \ldots = \Gamma_k = \Delta} \\
\begin{array}{l}
\int^{\Delta''} \int^{\Gamma_1} \int^{\Delta_1} \ldots \int^{\Gamma_k} \int^{\Delta_k} P(\Delta'') \\
\times Q_{v_1}(\Delta_1) \times R_{u_{1,1}}(\Gamma_1) \times \ldots \times R_{u_{1,m_1}}(\Gamma_1) \times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Gamma_1^{-1}(t)} \\
\times \ldots \\
\times Q_{v_k}(\Delta_k) \times R_{u_{k,1}}(\Gamma_k) \times \ldots \times R_{u_{k,m_k}}(\Gamma_k) \times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Gamma_k^{-1}(t)}
\end{array}
&
\begin{array}{c}
p, \\
q_1, r_1, \ldots, r_m, s \\
\ldots, \\
q_k, r_1, \ldots, r_m, s
\end{array}
\\
\downarrow & \big\downarrow{\scriptstyle \Gamma_{i,j} = \Gamma_j} \\
\begin{array}{l}
\int^{\Delta''} \int^{\Delta_1} \ldots \int^{\Delta_k} P(\Delta'') \\
\times Q_{v_1}(\Delta_1) \\
\qquad \times \int^{\Gamma_{1,1}} R_{u_{1,1}}(\Gamma_{1,1}) \times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Gamma_{1,1}^{-1}(t)} \\
\qquad \times \ldots \\
\qquad \times \int^{\Gamma_{1,m_1}} R_{u_{1,m_1}}(\Gamma_{1,m_1}) \times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Gamma_{1,m_1}^{-1}(t)} \\
\times \ldots \\
\times Q_{v_k}(\Delta_k) \\
\qquad \times \int^{\Gamma_{k,1}} \times R_{u_{k,1}}(\Gamma_{k,1}) \times \times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Gamma_{k,1}^{-1}(t)} \\
\qquad \times \ldots \\
\qquad \times \int^{\Gamma_{k,m_k}} R_{u_{k,m_k}}(\Gamma_{k,m_k}) \times \prod\limits_{t \in \mathcal{T}} S_t(\Gamma)^{\Gamma_{k,m_k}^{-1}(t)}
\end{array}
&
\begin{array}{c}
p, \\
q_1, r_1, s, \ldots, r_m, s \\
\ldots, \\
q_k, r_1, s \ldots, r_m, s
\end{array}
\end{array}
$$

Along the right hand side we have the following assignations

$$
\begin{array}{ll}
\begin{array}{l}
\int^{\Delta}\int^{\Delta'}\int^{\Delta''} P(\Delta'') \\
\times Q_{v_1}(\Delta') \times \ldots \times Q_{v_k}(\Delta') \\
\times R_{u_1}(\Delta) \times \ldots \times R_{u_m}(\Delta) \\
\times \prod_{t\in\mathcal{T}} S_t(\Gamma)^{\Delta^{-1}(t)}
\end{array}
&
\begin{array}{c}
p, \\ q_1,\ldots,q_k, \\ r_1,\ldots,r_m, \\ s
\end{array}
\\
\Big\downarrow & \Big\downarrow{\scriptstyle \Gamma_1=\ldots=\Gamma_m=\Delta}
\\
\begin{array}{l}
\int^{\Delta'}\int^{\Delta''} P(\Delta'') \\
\times Q_{v_1}(\Delta') \times \ldots \times Q_{v_k}(\Delta') \\
\times \int^{\Gamma_1} R_{u_1}(\Gamma_1) \times \prod_{t\in\mathcal{T}} S_t(\Gamma)^{\Gamma_1^{-1}(t)} \\
\times \ldots \\
\times \int^{\Gamma_m} R_{u_m}(\Gamma_m) \times \prod_{t\in\mathcal{T}} S_t(\Gamma)^{\Gamma_m^{-1}(t)}
\end{array}
&
\begin{array}{c}
p, \\ q_1,\ldots,q_k, \\ r_1,s,\ldots,r_m,s
\end{array}
\\
\Big\downarrow & \Big\downarrow{\scriptstyle \Delta_1=\ldots=\Delta_k=\Delta' \quad \Gamma_{i,j}=\Gamma_j}
\\
\begin{array}{l}
\int^{\Delta''}\int^{\Delta_1}\ldots\int^{\Delta_k} P(\Delta'') \\
\times Q_{v_1}(\Delta_1) \\
\qquad \times \int^{\Gamma_{1,1}} R_{u_{1,1}}(\Gamma_{1,1}) \times \prod_{t\in\mathcal{T}} S_t(\Gamma)^{\Gamma_{1,1}^{-1}(t)} \\
\qquad \times \ldots \\
\qquad \times \int^{\Gamma_{1,m_1}} R_{u_{1,m_1}}(\Gamma_{1,m_1}) \times \prod_{t\in\mathcal{T}} S_t(\Gamma)^{\Gamma_{1,m_1}^{-1}(t)} \\
\times \ldots \\
\times Q_{v_k}(\Delta_k) \\
\qquad \times \int^{\Gamma_{k,1}} \times R_{u_{k,1}}(\Gamma_{k,1}) \times \times \prod_{t\in\mathcal{T}} S_t(\Gamma)^{\Gamma_{k,1}^{-1}(t)} \\
\qquad \times \ldots \\
\qquad \times \int^{\Gamma_{k,m_k}} R_{u_{k,m_k}}(\Gamma_{k,m_k}) \times \prod_{t\in\mathcal{T}} S_t(\Gamma)^{\Gamma_{k,m_k}^{-1}(t)}
\end{array}
&
\begin{array}{c}
p, \\ q_1, r_1, s, \ldots, r_m, s \\ \ldots, \\ q_k, r_1, s\ldots, r_m, s
\end{array}
\end{array}
$$

So we obtain the same result. Next we check the commutativity of the two triangles.

$$
\begin{tikzcd}
((P\bullet Q)\bullet\mathcal{Y})(\Gamma) \arrow[r] \arrow[d] & (P\bullet(Q\otimes\mathcal{Y}))(\Gamma) \arrow[dl] \\
(P\bullet Q)(\Gamma) &
\end{tikzcd}
\qquad
\begin{tikzcd}
((P\bullet\mathcal{Y})\bullet Q)(\Gamma) \arrow[r] \arrow[d] & (P\bullet(\mathcal{Y}\otimes Q))(\Gamma) \arrow[dl] \\
(P\bullet Q)(\Gamma) &
\end{tikzcd}
$$

For the left triangle, we have the following assignations along the right–hand side

$$
\begin{array}{cc}
\begin{array}{l}
\int^{\Delta}\int^{\Delta'} P(\Delta') \\
\quad \times Q_{u_1}(\Delta) \times \ldots \times Q_{u_m}(\Delta) \\
\quad \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)}
\end{array}
&
p, q_1, \ldots, q_m, (h_t)_{t\in\mathcal{T}}
\\
\big\downarrow & \big\downarrow{\scriptstyle \Delta_1=\ldots=\Delta_m=\Delta}
\\
\begin{array}{l}
\int^{\Delta'}\int^{\Delta_1}\ldots\int^{\Delta_m} P(\Delta') \\
\quad \times Q_{u_1}(\Delta_1) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta_1^{-1}(t)} \\
\quad \times \ldots \\
\quad \times Q_{u_m}(\Delta_m) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta_m^{-1}(t)}
\end{array}
&
p, q_1, (h_t)_{t\in\mathcal{T}}, \ldots, q_m, (h_t)_{t\in\mathcal{T}}
\\
\big\downarrow & \big\downarrow
\\
\int^{\Delta'} P(\Delta') \times Q_{u_1}(\Gamma) \times \ldots \times Q_{u_m}(\Gamma)
&
\begin{array}{l}
p, Q_{u_1}(\sum_{t\in\mathcal{T}} h_t)(q_1), \\
\ldots, Q_{u_m}(\sum_{t\in\mathcal{T}} h_t)(q_m)
\end{array}
\end{array}
$$

and along the left–hand side

$$
\begin{array}{cc}
\int^{\Delta}(P\bullet Q)(\Delta) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)} & x, (h_t)_{t\in\mathcal{T}} \\
\big\downarrow & \big\downarrow \\
P\bullet Q(\Gamma) & (P\bullet Q)(\sum_{t\in\mathcal{T}} h_t)(x)
\end{array}
$$

and

$$
\begin{aligned}
(P\bullet Q)(\sum_{t\in\mathcal{T}} h_t) : (P\bullet Q)(\Delta) &\to (P\bullet Q)(\Gamma) \\
(p, q_1, \ldots, q_m) &\mapsto (p, Q_{u_1}(\sum_{t\in\mathcal{T}} h_t)(q_1), \ldots, Q_{u_m}(\sum_{t\in\mathcal{T}} h_t)(q_m))
\end{aligned}
$$

For the right triangle we have the following assignations along the right–hand side

$$
\begin{array}{cc}
\begin{array}{l}
\int^{\Delta}\int^{\Delta'} P(\Delta') \\
\quad \times \Delta^{-1}(u_1) \times \ldots \times \Delta^{-1}(u_m) \\
\quad \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta^{-1}(t)}
\end{array}
&
p, x_1, \ldots, x_m, (h_t)_{t\in\mathcal{T}}
\\
\big\downarrow & \big\downarrow{\scriptstyle \Delta_1=\ldots=\Delta_m=\Delta}
\\
\begin{array}{l}
\int^{\Delta'}\int^{\Delta_1}\ldots\int^{\Delta_m} P(\Delta') \\
\quad \times \Delta_1^{-1}(u_1) \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta_1^{-1}(t)} \\
\quad \times \ldots \\
\quad \times \Delta_m^{-1}(u_m) \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta_m^{-1}(t)}
\end{array}
&
p, x_1, (h_t)_{t\in\mathcal{T}}, \ldots, x_m, (h_t)_{t\in\mathcal{T}}
\\
\big\downarrow & \big\downarrow
\\
\int^{\Delta'} P(\Delta') \times Q_{u_1}(\Gamma) \times \ldots \times Q_{u_m}(\Gamma)
&
p, h_{u_1}(x_1), \ldots, h_{u_m}(x_m)
\end{array}
$$

and along the left–hand side

$$\begin{tikzcd} \int^{\Delta}\int^{\Delta'} P(\Delta') \times \prod_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Delta'^{-1}(t)} \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] & p, (f_t)_{t\in\mathcal{T}}, (h_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\ \int^{\Delta} P(\Delta) \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta^{-1}(t)} & P\big(\sum_{t\in\mathcal{T}} f_t\big)(p), (h_t)_{t\in\mathcal{T}} \end{tikzcd}$$

The two elements $p, h_{u_1}(x_1), \ldots, h_{u_m}(x_m) = p, (h_t \circ f_t)_{t\in\mathcal{T}}$ and $P\big(\sum_{t\in\mathcal{T}} f_t\big)(p), (h_t)_{t\in\mathcal{T}}$ are equal since the come from $p, (h_t)_{t\in\mathcal{T}} \in P(\Delta') \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta^{-1}(t)}$ with the arrow $f = \sum_{t\in\mathcal{T}} f_t : \Delta' \to \Delta$

$$\begin{tikzcd} & P(\Delta) \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta^{-1}(t)} \arrow[dr] & \\ P(\Delta') \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta^{-1}(t)} \arrow[ur, "P(f)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times(-\circ f_t)_{t\in\mathcal{T}}"'] & & \int^{\Delta} P(\Delta) \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta^{-1}(t)} \\ & P(\Delta') \times \prod_{t\in\mathcal{T}} Q_t(\Gamma)^{\Delta'^{-1}(t)} \arrow[ur] & \end{tikzcd}$$

C.3 Proof of lemma 5.5.90

We construct two arrows of the bijection, then we show that they are each other's inverses. Let $\Gamma \in \mathbb{F} \downarrow \mathcal{T}$. We have

$$\begin{aligned} ((P \times Q) \bullet R)(\Gamma) &= \int^{\Delta} (P \times Q)(\Delta) \times R^{\times\Delta}(\Gamma) \\ &= \int^{\Delta} P(\Delta) \times Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{aligned}$$

and

$$\begin{aligned} ((P \bullet R) \times (Q \bullet R))(\Gamma) &= (P \bullet R)(\Gamma) \times (Q \bullet R)(\Gamma) \\ &= \Big(\int^{\Delta'} P(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)}\Big) \times \Big(\int^{\Delta''} Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)}\Big) \\ &\cong \int^{\Delta'}\int^{\Delta''} P(\Delta') \times Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta'')^{-1}(t)} \end{aligned}$$

- First we construct the arrow $((P \times Q) \bullet R)(\Gamma) \to ((P \bullet R) \times (Q \bullet R))(\Gamma)$. By universal properties of the coend and the product it suffices to give for all Δ two families of arrows

$$P(\Delta) \times Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \to \int^{\Delta'} P(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)}$$

  and

$$P(\Delta) \times Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \to \int^{\Delta''} Q(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)}$$

that satisfy the wedge condition. We take the following composites

$$
\begin{tikzcd}
P(\Delta)\times Q(\Delta)\times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] \\
P(\Delta)\times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] \\
\int^{\Delta'} P(\Delta')\times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)}
\end{tikzcd}
\qquad
\begin{tikzcd}
P(\Delta)\times Q(\Delta)\times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] \\
Q(\Delta)\times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] \\
\int^{\Delta''} Q(\Delta'')\times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)}
\end{tikzcd}
$$

Let $f:\Delta_1\to\Delta_2$. The following diagram commutes

$$
\begin{tikzcd}
 & \begin{matrix} P(\Delta_1)\times Q(\Delta_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \end{matrix} \arrow[dr] & \\
\begin{matrix} P(\Delta_1)\times Q(\Delta_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{matrix} \arrow[ur, "\mathrm{id}\times(-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(f)\times Q(f)\times\mathrm{id}"'] & & \int^{\Delta'} P(\Delta')\times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)} \\
 & \begin{matrix} P(\Delta_2)\times Q(\Delta_2) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{matrix} \arrow[ur] &
\end{tikzcd}
$$

since we have the following assignations on elements

$$
\begin{tikzcd}
 & p,q,(h_t\circ f_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\
p,q,(h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{matrix} p,(h_t\circ f_t)_{t\in\mathcal{T}} \\ = P(f)(p),(h_t)_{t\in\mathcal{T}} \end{matrix} \\
 & P(f)(p),Q(f)(q),(h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}
$$

The two elements $p,(h_t\circ f_t)_{t\in\mathcal{T}}$ and $P(f)(p),(h_t)_{t\in\mathcal{T}}$ come from $p,(h_t)_{t\in\mathcal{T}}\in P(\Delta_1)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)}$

with the arrow f

$$
\begin{tikzcd}
& {\begin{array}{c} P(\Delta_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \end{array}} \arrow[dr] & \\
{\begin{array}{c} P(\Delta_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array}} \arrow[ur, "\mathrm{id}\times(-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(f)\times\mathrm{id}"'] & & \int^{\Delta'} P(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)} \\
& {\begin{array}{c} P(\Delta_2) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array}} \arrow[ur] &
\end{tikzcd}
$$

To check the wedge condition for the collection of arrows

$$P(\Delta) \times Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \to \int^{\Delta''} Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)}$$

goes the same as above by only replacing P by Q.

- Next we construct the arrow in the inverse direction. By universal properties of the coends, it suffices to give for all Δ' and Δ'' a collection of arrows

$$P(\Delta') \times Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta'')^{-1}(t)} \to \int^{\Delta} P(\Delta) \times Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$$

that satisfies the wedge condition. We take the following composite

$$
\begin{tikzcd}
P(\Delta') \times Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta'')^{-1}(t)} \arrow[d] \\
P(\Delta'+\Delta'') \times Q(\Delta'+\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta'')^{-1}(t)} \arrow[d] \\
\int^{\Delta} P(\Delta) \times Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}
\end{tikzcd}
$$

Let $f' : \Delta'_1 \to \Delta'_2$ and $f'' : \Delta''_1 \to \Delta''_2$. The following diagram commutes

$$
\begin{tikzcd}
& {\begin{array}{c} P(\Delta'_1) \times Q(\Delta''_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'_1+\Delta''_1)^{-1}(t)} \end{array}} \arrow[dr] & \\
{\begin{array}{c} P(\Delta'_1) \times Q(\Delta''_1) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'_2+\Delta''_2)^{-1}(t)} \end{array}} \arrow[ur, "\mathrm{id}\times(-\circ(f'+f'')_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(f')\times Q(f'')\times\mathrm{id}"'] & & {\begin{array}{c} \int^{\Delta} P(\Delta) \times Q(\Delta) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma^{\Delta})^{-1}(t) \end{array}} \\
& {\begin{array}{c} P(\Delta'_2) \times Q(\Delta''_2) \\ \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'_2+\Delta''_2)^{-1}(t)} \end{array}} \arrow[ur] &
\end{tikzcd}
$$

since we have the following assignations on elements

$$
\begin{tikzcd}[column sep=small]
& p,q,(h_t\circ(f'+f'')_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\
p,q,(h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}\overline{p},\overline{q},(h_t\circ(f'+f'')_t)_{t\in\mathcal{T}} \\ =\overline{P(f')(p)},\overline{Q(f'')(q)},(h_t)_{t\in\mathcal{T}}\end{array} \\
& P(f')(p),Q(f'')(q),(h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}
$$

The two elements $\overline{p},\overline{q},(h_t\circ(f'+f'')_t)_{t\in\mathcal{T}}$ and $\overline{P(f')(p)},\overline{Q(f'')(q)},(h_t)_{t\in\mathcal{T}}$ are equal since by naturality of the inclusions $\overline{P(f')(p)} = P(f'+f'')(\overline{p})$ and $\overline{Q(f')(q)} = Q(f'+f'')(\overline{q})$ and they come from $\overline{p},\overline{q},(h_t)_{t\in\mathcal{T}} \in P(\Delta'_1+\Delta'')\times Q(\Delta'+\Delta''_1)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'_2+\Delta''_2)^{-1}(t)}$ with the arrow $f'+f''$

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{c}P(\Delta'_1+\Delta''_1)\times Q(\Delta'_1+\Delta''_1)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'_1+\Delta''_1)^{-1}(t)}\end{array} \arrow[dr] & \\
\begin{array}{c}P(\Delta'_1+\Delta''_1)\times Q(\Delta'_1+\Delta''_1)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'_2+\Delta''_2)^{-1}(t)}\end{array} \arrow[ur, "\mathrm{id}\times(-\circ(f'+f'')_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(f'+f'')\times Q(f'+f'')\times\mathrm{id}"'] & & \begin{array}{c}\int^\Delta P(\Delta)\times Q(\Delta)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma^{\Delta)^{-1}(t)}\end{array} \\
& \begin{array}{c}P(\Delta'_2+\Delta''_2)\times Q(\Delta'_2+\Delta''_2)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'_2+\Delta''_2)^{-1}(t)}\end{array} \arrow[ur] &
\end{tikzcd}
$$

- These arrows are inverse to each other. Starting with $p,q,(h_t)_{t\in\mathcal{T}} \in P(\Delta)\times Q(\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$, we obtain the following composite

$$
p,q,(h_t)_{t\in\mathcal{T}} \quad\mapsto\quad p,(h_t)_{t\in\mathcal{T}},q,(h_t)_{t\in\mathcal{T}} \quad\mapsto\quad \overline{p},\overline{q},[h_t,h_t]_{t\in\mathcal{T}}
$$

The two elements $p,q,(h_t)_{t\in\mathcal{T}}$ and $\overline{p},\overline{q},[h_t,h_t]_{t\in\mathcal{T}}$ are equal since they come from $\overline{p},\overline{q},(h_t)_{t\in\mathcal{T}}\in P(\Delta+\Delta)\times Q(\Delta+\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta)^{-1}(t)}$ with the arrow $f=[\mathrm{id}_\Delta,\mathrm{id}_\Delta]:\Delta+\Delta\to\Delta$

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{c}P(\Delta)\times Q(\Delta)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}\end{array} \arrow[dr] & \\
\begin{array}{c}P(\Delta+\Delta)\times Q(\Delta+\Delta)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}\end{array} \arrow[ur, "P(f)\times Q(f)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times(-\circ f_t)_{t\in\mathcal{T}}"'] & & \begin{array}{c}\int^\Delta P(\Delta)\times Q(\Delta)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma^{\Delta^{-1}(t)}\end{array} \\
& \begin{array}{c}P(\Delta+\Delta)\times Q(\Delta+\Delta)\\ \times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta+\Delta)^{-1}(t)}\end{array} \arrow[ur] &
\end{tikzcd}
$$

- Starting with $p, q, (h_t)_{t\in\mathcal{T}} \in P(\Delta') \times Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta'')^{-1}(t)}$, we obtain the following composite
$$p, q, (h_t)_{t\in\mathcal{T}} \quad\mapsto\quad \overline{p}, \overline{q}, (h_t)_{t\in\mathcal{T}} \quad\mapsto\quad \overline{p}, (h_t)_{t\in\mathcal{T}}, \overline{q}, (h_t)_{t\in\mathcal{T}}$$
The two elements $p, q, (h_t)_{t\in\mathcal{T}}$ and $\overline{p}, (h_t)_{t\in\mathcal{T}}, \overline{q}, (h_t)_{t\in\mathcal{T}}$ are equal since they come from $p, q, [h_t, h_t]_{t\in\mathcal{T}} \in P(\Delta') \times Q(\Delta'') \times \prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta''+\Delta'+\Delta'')^{-1}(t)}$ with the arrows $i' : \Delta' \to \Delta'+\Delta''$, $i'' : \Delta'' \to \Delta'+\Delta''$

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta') \times Q(\Delta'') \\ \times \prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta'')^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} P(\Delta') \times Q(\Delta'') \\ \times \prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta''+\Delta'+\Delta'')^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times (-\circ(i'+i'')_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(i')\times Q(i'')\times \mathrm{id}"'] & & \begin{array}{c} P \bullet R(\Gamma) \\ \times Q \bullet R(\Gamma) \end{array} \\
 & \begin{array}{c} P(\Delta'+\Delta'') \times Q(\Delta'+\Delta'') \\ \times \prod\limits_{t\in\mathcal{T}} R_t(\Gamma)^{(\Delta'+\Delta''+\Delta'+\Delta'')^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

Now we check naturalities in Γ,P,Q and R. Let $f : \Gamma_1 \to \Gamma_2$ be an arrow in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
(P \times Q) \bullet R(\Gamma_1) \arrow[r] \arrow[d] & (P \bullet R)(\Gamma_1) \times (Q \bullet R)(\Gamma_1) \arrow[d] \\
(P \times Q) \bullet R(\Gamma_2) \arrow[r] & (P \bullet R)(\Gamma_2) \times (Q \bullet R)(\Gamma_2)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
p, q, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, (h_t)_{t\in\mathcal{T}}, q, (h_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
p, q, (R_t(f) \circ h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & p, (R_t(f) \circ h_t)_{t\in\mathcal{T}}, q, (R_t(f) \circ h_t)_{t\in\mathcal{T}}
\end{tikzcd}$$

Now let $f : P \to S$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
(P \times Q) \bullet R(\Gamma) \arrow[r] \arrow[d] & (P \bullet R)(\Gamma) \times (Q \bullet R)(\Gamma) \arrow[d] \\
(S \times Q) \bullet R(\Gamma) \arrow[r] & (S \bullet R)(\Gamma) \times (Q \bullet R)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
p, q, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, (h_t)_{t\in\mathcal{T}}, q, (h_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
f_\Delta(p), q, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & f_\Delta(p), (h_t)_{t\in\mathcal{T}}, q, (h_t)_{t\in\mathcal{T}}
\end{tikzcd}$$

Let $f : Q \to S$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
(P \times Q) \bullet R(\Gamma) \arrow[r] \arrow[d] & (P \bullet R)(\Gamma) \times (Q \bullet R)(\Gamma) \arrow[d] \\
(P \times S) \bullet R(\Gamma) \arrow[r] & (P \bullet R)(\Gamma) \times (S \bullet R)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
p, q, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, (h_t)_{t\in\mathcal{T}}, q, (h_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
p, f_\Delta(q), (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & p, (h_t)_{t\in\mathcal{T}}, f_\Delta(q), (h_t)_{t\in\mathcal{T}}
\end{tikzcd}$$

Let $f : R \to S$ in $[\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd}
(P\times Q)\bullet R(\Gamma) \arrow[r] \arrow[d] & (P\bullet R)(\Gamma)\times(Q\bullet R)(\Gamma) \arrow[d] \\
(P\times Q)\bullet S(\Gamma) \arrow[r] & (P\bullet S)(\Gamma)\times(Q\bullet S)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
p, q, (h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & p, (h_t)_{t\in\mathcal{T}}, q, (h_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
p, q, (f_{t,\Gamma}\circ h_t)_{t\in\mathcal{T}} \arrow[r, mapsto] & p, (f_{t,\Gamma}\circ h_t)_{t\in\mathcal{T}}, q, (f_{t,\Gamma}\circ h_t)_{t\in\mathcal{T}}
\end{tikzcd}$$

C.4 Proof of lemma 5.5.92

We construct two arrows of the bijection, then we show that they are each other's inverses. Let $\Gamma \in \mathbb{F}\downarrow\mathcal{T}$. We have

$$\begin{aligned}
((P+Q)\bullet R)(\Gamma) &= \int^{\Delta} (P(\Delta)+Q(\Delta))\times R^{\times\Delta}(\Gamma) \\
&\cong \int^{\Delta} P(\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} + Q(\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}
\end{aligned}$$

and

$$\begin{aligned}
((P\bullet R)+(Q\bullet R))(\Gamma) &= (P\bullet R)(\Gamma)+(Q\bullet R)(\Gamma) \\
&= \int^{\Delta'} P(\Delta')\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)} + \int^{\Delta''} Q(\Delta'')\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)}
\end{aligned}$$

- First we construct the arrow $((P+Q)\bullet R)(\Gamma)\to(P\bullet R)(\Gamma)+(Q\bullet R)(\Gamma)$. By the universal properties of the coend and the coproduct, it suffices to give for all Δ a collection of two arrows

$$P(\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \to (P\bullet R)(\Gamma)+(Q\bullet R)(\Gamma)$$

  and

$$Q(\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \to (P\bullet R)(\Gamma)+(Q\bullet R)(\Gamma)$$

  that satisfies the wedge condition. We take the following the composites

$$\begin{tikzcd}
P(\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] & & Q(\Delta)\times\prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \arrow[d] \\
(P\bullet R)(\Gamma) \arrow[dr] & & (Q\bullet R)(\Gamma) \arrow[dl] \\
& \begin{array}{c}(P\bullet R)(\Gamma)\\ +(Q\bullet R)(\Gamma)\end{array} &
\end{tikzcd}$$

Let $f : \Delta_1 \to \Delta_2$. The following diagram commutes

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_1}^{-1}(t)} \\ +Q(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_1}^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} P(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_2}^{-1}(t)} \\ +Q(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_2}^{-1}(t)} \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} (P \bullet R)(\Gamma) \\ +(Q \bullet R)(\Gamma) \end{array} \\
 & \begin{array}{c} P(\Delta_2) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_2}^{-1}(t)} \\ +Q(\Delta_2) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_2}^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

because each component

$$\begin{tikzcd}
 & P(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_1}^{-1}(t)} \arrow[dr] & \\
P(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_2}^{-1}(t)} \arrow[ur] \arrow[dr] & & (P \bullet R)(\Gamma) \\
 & P(\Delta_2) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{{\Delta_2}^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

and the one with Q instead of P commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & x, (h_t \circ f_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\
x, (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} x, (h_t \circ f_t)_{t\in\mathcal{T}} \\ = P(f)(x), (h_t)_{t\in\mathcal{T}} \end{array} \\
 & P(f)(x), (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}$$

- Next we construct the arrow in the inverse direction. By the universal properties of the coproduct and the coends, it suffices to give a collection of arrows

$$P(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)} \to ((P+Q) \bullet R)(\Gamma)$$

for all Δ' satisfying the wedge condition and a collection of arrows

$$Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)} \to ((P+Q) \bullet R)(\Gamma)$$

for all Δ'' satisfying the wedge condition. We take the following composites

$$\begin{tikzcd} P(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)} \arrow[d] \\ P(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)} + Q(\Delta') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta'^{-1}(t)} \arrow[d] \\ \int^{\Delta} P(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} + Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{tikzcd}$$

and

$$\begin{tikzcd} Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)} \arrow[d] \\ P(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)} + Q(\Delta'') \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta''^{-1}(t)} \arrow[d] \\ \int^{\Delta} P(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} + Q(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)} \end{tikzcd}$$

We check the wedge conditions. Let $f : \Delta_1 \to \Delta_2$. The following diagram commutes

$$\begin{tikzcd} & P(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \arrow[dr] & \\ P(\Delta_1) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur, "\mathrm{id} \times (-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(f)\times \mathrm{id}"'] & & ((P+Q)\bullet R)(\Gamma) \\ & P(\Delta_2) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \arrow[ur] & \end{tikzcd}$$

since we have the following assignations on elements

$$\begin{tikzcd} & p, (h_t \circ f_t)_{t\in\mathcal{T}} \arrow[dr, mapsto] & \\ p, (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}\overline{p, (h_t \circ f_t)_{t\in\mathcal{T}}} \\ = \overline{P(f)(p), (h_t)_{t\in\mathcal{T}}}\end{array} \\ & P(f)(p), (h_t)_{t\in\mathcal{T}} \arrow[ur, mapsto] & \end{tikzcd}$$

the two elements $\overline{p, (h_t \circ f_t)_{t\in\mathcal{T}}}$ and $\overline{P(f)(p), (h_t)_{t\in\mathcal{T}}}$ are equal since they come from $\overline{p, (h_t)_{t\in\mathcal{T}}} \in$

$$P(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} + Q(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)}$$

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \\ +Q(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} P(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \\ +Q(\Delta_1) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ f_t)_{t \in \mathcal{T}} + \mathrm{id} \times (- \circ f_t)_{t \in \mathcal{T}}"] \arrow[dr, "P(f) \times \mathrm{id} + Q(f) \times \mathrm{id}"'] & & ((P+Q) \bullet R)(\Gamma) \\
 & \begin{array}{c} P(\Delta_2) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \\ +Q(\Delta_2) \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
 & \begin{array}{c} Q(\Delta_1) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_1^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{c} Q(\Delta_1) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ f_t)_{t \in \mathcal{T}}"] \arrow[dr, "P(f) \times \mathrm{id}"'] & & ((P+Q) \bullet R)(\Gamma) \\
 & \begin{array}{c} Q(\Delta_2) \\ \times \prod_{t \in \mathcal{T}} R_t(\Gamma)^{\Delta_2^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes by the same reasoning as above.

- Now we check that the above defined arrows are inverse to each other. Starting with $x, (h_t)_{t \in \mathcal{T}} \in ((P+Q) \bullet R)(\Gamma)$, we obtain the following composite

$$x, (h_t)_{t \in \mathcal{T}} \quad \mapsto \quad \overline{x, (h_t)_{t \in \mathcal{T}}} \quad \mapsto \quad x, (h_t)_{t \in \mathcal{T}}$$

Starting with $x, (h_t)_{t \in \mathcal{T}} \in (P \bullet R)(\Gamma) + (Q \bullet R)(\Gamma)$, we obtain the following composite

$$x, (h_t)_{t \in \mathcal{T}} \quad \mapsto \quad \overline{x, (h_t)_{t \in \mathcal{T}}} \quad \mapsto \quad x, (h_t)_{t \in \mathcal{T}}$$

Next we check naturalities in Γ, P, Q and R. Let $f : \Gamma_1 \to \Gamma_2$ in $\mathbb{F} \downarrow \mathcal{T}$. The naturality square

$$\begin{tikzcd}
((P+Q) \bullet R)(\Gamma_1) \arrow[r] \arrow[d] & (P \bullet R)(\Gamma_1) + (Q \bullet R)(\Gamma_1) \arrow[d] \\
((P+Q) \bullet R)(\Gamma_2) \arrow[r] & (P \bullet R)(\Gamma_2) + (Q \bullet R)(\Gamma_2)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
x, (h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & \overline{x, (h_t)_{t \in \mathcal{T}}} \arrow[d, mapsto] \\
x, (R_t(f) \circ h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & \overline{x, (R_t(f) \circ h_t)_{t \in \mathcal{T}}}
\end{tikzcd}$$

Let $f : P \to S$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
((P+Q) \bullet R)(\Gamma) \arrow[r] \arrow[d] & (P \bullet R)(\Gamma) + (Q \bullet R)(\Gamma) \arrow[d] \\
((S+Q) \bullet R)(\Gamma) \arrow[r] & (S \bullet R)(\Gamma) + (Q \bullet R)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements, supposing that $x \in P(\Delta)$

$$\begin{tikzcd}
x, (h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & \overline{x, (h_t)_{t \in \mathcal{T}}} \arrow[d, mapsto] \\
f_\Delta(x), (h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & \overline{f_\Delta(x), (h_t)_{t \in \mathcal{T}}}
\end{tikzcd}$$

Let $f : Q \to S$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
((P+Q) \bullet R)(\Gamma) \arrow[r] \arrow[d] & (P \bullet R)(\Gamma) + (Q \bullet R)(\Gamma) \arrow[d] \\
((P+S) \bullet R)(\Gamma) \arrow[r] & (P \bullet R)(\Gamma) + (S \bullet R)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements, supposing that $x \in Q(\Delta)$

$$\begin{tikzcd}
x, (h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & \overline{x, (h_t)_{t \in \mathcal{T}}} \arrow[d, mapsto] \\
f_\Delta(x), (h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & \overline{f_\Delta(x), (h_t)_{t \in \mathcal{T}}}
\end{tikzcd}$$

Let $f : R \to S$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd}
((P+Q) \bullet R)(\Gamma) \arrow[r] \arrow[d] & (P \bullet R)(\Gamma) + (Q \bullet R)(\Gamma) \arrow[d] \\
((P+Q) \bullet S)(\Gamma) \arrow[r] & (P \bullet S)(\Gamma) + (Q \bullet S)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
x, (h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & \overline{x, (h_t)_{t \in \mathcal{T}}} \arrow[d, mapsto] \\
x, (f_{t,\Gamma} \circ h_t)_{t \in \mathcal{T}} \arrow[r, mapsto] & \overline{x, (f_{t,\Gamma} \circ h_t)_{t \in \mathcal{T}}}
\end{tikzcd}$$

C.5 Proof of proposition 5.7.96

We check the wedge condition. Let $f : \Delta_1 \to \Delta_2$ in $\mathbb{F} \downarrow \mathcal{T}$ where $\Delta_1 = (u_1, \ldots, u_m)$ and $\Delta_2 = (v_1, \ldots, v_k)$. The diagram

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta_1 + \langle u \rangle) \times Q_{u_1}(\Gamma) \\ \times \ldots \times Q_{u_m}(\Gamma) \end{array} \arrow[dr] & \\
\begin{array}{c} P(\Delta_1 + \langle u \rangle) \times Q_{v_1}(\Gamma) \\ \times \ldots \times Q_{v_k}(\Gamma) \end{array} \arrow[ur] \arrow[dr] & & (P \bullet Q)(\Gamma + \langle u \rangle) \\
 & \begin{array}{c} P(\Delta_2 + \langle u \rangle) \times Q_{v_1}(\Gamma) \\ \times \ldots \times Q_{v_k}(\Gamma) \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes because we have on elements the following assignations

$$\begin{tikzcd}
 & b, a_{f_{u_1}}, \ldots, a_{f_{u_m}} \arrow[dr, mapsto] & \\
b, a_1, \ldots, a_k \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} b, \overline{a}_{f_{u_1}}, \ldots, \overline{a}_{f_{u_m}}, \overline{q_u}_{,\Gamma} \\ = P(f + \langle u \rangle)(b), \overline{a}_1, \ldots, \overline{a}_k, \overline{q_u}_{,\Gamma} \end{array} \\
 & P(f + \langle u \rangle)(b), a_1, \ldots, a_k \arrow[ur, mapsto] &
\end{tikzcd}$$

The elements $b, \overline{a}_{f_{u_1}}, \ldots, \overline{a}_{f_{u_m}}, \overline{q_u}_{,\Gamma} \in P(\Delta_1 + \langle u \rangle) \times Q_{u_1}(\Gamma + \langle u \rangle) \times \ldots \times Q_{u_m}(\Gamma + \langle u \rangle) \times Q_u(\Gamma + \langle u \rangle)$ and $P(f + \langle u \rangle)(b), \overline{a}_1, \ldots, \overline{a}_k, \overline{q_u}_{,\Gamma} \in P(\Delta_2 + \langle u \rangle) \times Q_{v_1}(\Gamma + \langle u \rangle) \times \ldots \times Q_{v_k}(\Gamma + \langle u \rangle) \times Q_u(\Gamma + \langle u \rangle)$ are equal since they come from $b, \overline{a}_1, \ldots, \overline{a}_k, \overline{q_u}_{,\Gamma} \in P(\Delta_1 + \langle u \rangle) \times Q_{v_1}(\Gamma + \langle u \rangle) \times \ldots \times Q_{v_k}(\Gamma + \langle u \rangle) \times Q_u(\Gamma + \langle u \rangle)$ with the arrow $f + \langle u \rangle : \Delta_1 + \langle u \rangle \to \Delta_2 + \langle u \rangle$

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta_1 + \langle u \rangle) \times Q_{u_1}(\Gamma + \langle u \rangle) \\ \times \ldots \times Q_{u_m}(\Gamma + \langle u \rangle) \\ \times Q_u(\Gamma + \langle u \rangle) \end{array} \arrow[dr] & \\
\begin{array}{c} P(\Delta_1 + \langle u \rangle) \times Q_{v_1}(\Gamma + \langle u \rangle) \\ \times \ldots \times Q_{v_k}(\Gamma + \langle u \rangle) \\ \times Q_u(\Gamma + \langle u \rangle) \end{array} \arrow[ur] \arrow[dr] & & (P \bullet Q)(\Gamma + \langle u \rangle) \\
 & \begin{array}{c} P(\Delta_2 + \langle u \rangle) \times Q_{v_1}(\Gamma + \langle u \rangle) \\ \times \ldots \times Q_{v_k}(\Gamma + \langle u \rangle) \\ \times Q_u(\Gamma + \langle u \rangle) \end{array} \arrow[ur] &
\end{tikzcd}$$

The constructed arrow $s^{(u)}_{P,Q,\Gamma}$ is natural in Γ. Let $f : \Gamma_1 \to \Gamma_2$. The naturality square

$$\begin{tikzcd}
(P^{\mathcal{Y}\langle u \rangle} \bullet Q)(\Gamma_1) \arrow[r] \arrow[d] & (P \bullet Q)^{\mathcal{Y}\langle u \rangle}(\Gamma_1) \arrow[d] \\
(P^{\mathcal{Y}\langle u \rangle} \bullet Q)(\Gamma_2) \arrow[r] & (P \bullet Q)^{\mathcal{Y}\langle u \rangle}(\Gamma_2)
\end{tikzcd}$$

commutes because we have on elements

$$
\begin{tikzcd}
(p, x_1, \ldots, x_m) \arrow[r, mapsto] \arrow[d, mapsto] & (p, \overline{x}_1, \ldots, \overline{x}_m, \overline{q_{u,\Gamma_1}}) \arrow[d, mapsto] \\
(p, Q_{u_1}(f)(x_1), \ldots, Q_{u_m}(f)(x_m)) \arrow[r, mapsto] & \begin{array}{c} (p, Q_{u_1}(f + \langle u \rangle)(\overline{x}_1), \\ \ldots, \\ Q_{u_m}(f + \langle u \rangle)(\overline{x}_m), \\ Q_u(f + \langle u \rangle)(\overline{q_{u,\Gamma_1}})) \\ = (p, \overline{Q_{u_1}(f)(x_1)}, \\ \overline{\ldots,} \\ \overline{Q_{u_m}(f)(x_m)}, \overline{q_{u,\Gamma_2}}) \end{array}
\end{tikzcd}
$$

By naturality of the inclusions $Q_{u_j}(f + \langle u \rangle)(\overline{x}_j) = \overline{Q_{u_j}(f)(x_j)}$ and $Q_u(f + \langle u \rangle)(\overline{q_{u,\Gamma_1}}) = \overline{q_{u,\Gamma_2}}$.

Let $f : P \to R$ be an arrow in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$
\begin{tikzcd}
(P^{\mathcal{Y}\langle u \rangle} \bullet Q)(\Gamma) \arrow[r] \arrow[d] & (P \bullet Q)^{\mathcal{Y}\langle u \rangle}(\Gamma) \arrow[d] \\
(R^{\mathcal{Y}\langle u \rangle} \bullet Q)(\Gamma) \arrow[r] & (R \bullet Q)^{\mathcal{Y}\langle u \rangle}(\Gamma)
\end{tikzcd}
$$

commutes because we have on elements

$$
\begin{tikzcd}
(p, x_1, \ldots, x_m) \arrow[r, mapsto] \arrow[d, mapsto] & (p, \overline{x}_1, \ldots, \overline{x}_m, \overline{q_{u,\Gamma}}) \arrow[d, mapsto] \\
(f_\Delta(p), x_1, \ldots, x_m) \arrow[r, mapsto] & (f_\Delta(p), \overline{x}_1, \ldots, \overline{x}_m, \overline{q_{u,\Gamma}})
\end{tikzcd}
$$

Let $g : Q \to R$ be an arrow in $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$
\begin{tikzcd}
(P^{\mathcal{Y}\langle u \rangle} \bullet Q)(\Gamma) \arrow[r] \arrow[d] & (P \bullet Q)^{\mathcal{Y}\langle u \rangle}(\Gamma) \arrow[d] \\
(P^{\mathcal{Y}\langle u \rangle} \bullet R)(\Gamma) \arrow[r] & (P \bullet R)^{\mathcal{Y}\langle u \rangle}(\Gamma)
\end{tikzcd}
$$

commutes because we have on elements

$$
\begin{tikzcd}
(p, x_1, \ldots, x_m) \arrow[r, mapsto] \arrow[d, mapsto] & (p, \overline{x}_1, \ldots, \overline{x}_m, \overline{q_{u,\Gamma}}) \arrow[d, mapsto] \\
(p, g_{u_1,\Gamma}(x_1), \ldots, g_{u_m,\Gamma}(x_m)) \arrow[r, mapsto] & \begin{array}{c} (p, g_{u_1,\Gamma+\langle u \rangle}(\overline{x}_1), \ldots, \\ g_{u_m,\Gamma+\langle u \rangle}(\overline{x}_m), g_{u,\Gamma+\langle u \rangle}(\overline{q_{u,\Gamma}})) \\ = (p, \overline{g_{u_1,\Gamma}(x_1)}, \ldots, \overline{g_{u_m,\Gamma}(x_m)}, \overline{r_{u,\Gamma}}) \end{array}
\end{tikzcd}
$$

By naturality of the inclusions $g_{u_j,\Gamma+\langle u \rangle}(\overline{x}_j) = \overline{g_{u_j,\Gamma}(x_j)}$ and since g is a morphism of $\mathcal{Y} \downarrow [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$ we have $g_{u,\Gamma+\langle u \rangle}(\overline{q_{u,\Gamma}}) = \overline{r_{u,\Gamma}}$.

Next we check the commutativity of the following diagrams

$$
\begin{tikzcd}
(P^{\mathcal{Y}\langle u\rangle} \bullet Q) \bullet R \arrow[r, "a"] \arrow[d, "s^{(u)} \bullet R"'] & P^{\mathcal{Y}\langle u\rangle} \bullet (Q \otimes R) \arrow[dd, "s^{(u)}"] \\
(P \bullet Q)^{\mathcal{Y}\langle u\rangle} \bullet R \arrow[d, "s^{(u)}"'] & \\
((P \bullet Q) \bullet R)^{\mathcal{Y}\langle u\rangle} \arrow[r, "a^{\mathcal{Y}\langle u\rangle}"'] & (P \bullet (Q \otimes R))^{\mathcal{Y}\langle u\rangle}
\end{tikzcd}
\qquad
\begin{tikzcd}
P^{\mathcal{Y}\langle u\rangle} \bullet \mathcal{Y} \arrow[r, "s^{(u)}"] \arrow[d, "r"'] & (P \bullet \mathcal{Y})^{\mathcal{Y}\langle u\rangle} \arrow[dl, "r^{\mathcal{Y}\langle u\rangle}"] \\
P^{\mathcal{Y}\langle u\rangle} &
\end{tikzcd}
$$

Along the left–hand side of the rectangular diagram we have the following assignations on elements

$$
\begin{array}{cc}
\begin{array}{l}
\int^{\Delta} \int^{\Delta'} P(\Delta' + \langle u\rangle) \\
\times Q_{v_1}(\Delta) \times \ldots \times Q_{v_k}(\Delta) \\
\times R_{u_1}(\Gamma) \times \ldots \times R_{u_m}(\Gamma)
\end{array}
&
\begin{array}{c}
p, \\
x_1, \ldots, x_k, \\
y_1, \ldots, y_m
\end{array}
\\
\downarrow & \big\downarrow{\scriptstyle \Delta'' = \Delta' + \langle u\rangle} \\
\begin{array}{l}
\int^{\Delta} \int^{\Delta''} P(\Delta'') \\
\times Q_{v_1}(\Delta + \langle u\rangle) \times \ldots \times Q_{v_k}(\Delta + \langle u\rangle) \times Q_u(\Delta + \langle u\rangle) \\
\times R_{u_1}(\Gamma) \times \ldots \times R_{u_m}(\Gamma)
\end{array}
&
\begin{array}{c}
p, \\
\overline{x}_1, \ldots, \overline{x}_k, \overline{q_{u,\Delta}}, \\
y_1, \ldots, y_m
\end{array}
\\
\downarrow & \big\downarrow{\scriptstyle \Gamma' = \Delta + \langle u\rangle} \\
\begin{array}{l}
\int^{\Gamma'} \int^{\Delta''} P(\Delta'') \\
\times Q_{v_1}(\Gamma') \times \ldots Q_{v_k}(\Gamma') \times Q_u(\Gamma') \\
\times R_{u_1}(\Gamma + \langle u\rangle) \times \ldots \times R_{u_m}(\Gamma + \langle u\rangle) \times R_u(\Gamma + \langle u\rangle)
\end{array}
&
\begin{array}{c}
p, \\
\overline{x}_1, \ldots, \overline{x}_k, \overline{q_{u,\Delta}}, \\
\overline{y}_1, \ldots, \overline{y}_m, \overline{r_{u,\Gamma}}
\end{array}
\\
\downarrow & {\scriptstyle \Gamma_1 = \ldots} \big\downarrow {\scriptstyle = \Gamma_{k+1} = \Gamma'} \\
\begin{array}{l}
\int^{\Delta''} \int^{\Gamma_1} \ldots \int^{\Gamma_{k+1}} P(\Delta'') \\
\times Q_{v_1}(\Gamma_1) \times R_{u_1}(\Gamma + \langle u\rangle) \times \ldots \times R_{u_m}(\Gamma + \langle u\rangle) \times R_u(\Gamma + \langle u\rangle) \\
\times \ldots \\
\times Q_{v_k}(\Gamma_k) \times R_{u_1}(\Gamma + \langle u\rangle) \times \ldots \times R_{u_m}(\Gamma + \langle u\rangle) \times R_u(\Gamma + \langle u\rangle) \\
\times Q_u(\Gamma_{k+1}) \times R_{u_1}(\Gamma + \langle u\rangle) \times \ldots \times R_{u_m}(\Gamma + \langle u\rangle) \times R_u(\Gamma + \langle u\rangle)
\end{array}
&
\begin{array}{c}
p, \\
\overline{x}_1, \overline{y}_1, \ldots, \overline{y}_m, \overline{r_{u,\Gamma}}, \\
\ldots, \\
\overline{x}_k, \overline{y}_1, \ldots, \overline{y}_m, \overline{r_{u,\Gamma}}, \\
\overline{q_{u,\Delta}}, \overline{y}_1, \ldots, \overline{y}_m, \overline{r_{u,\Gamma}}
\end{array}
\end{array}
$$

where we write $\Delta = (u_1, \ldots, u_m)$ and $\Delta' = (v_1, \ldots, v_k)$. Along the right–hand side we have the following assignations

$$\begin{array}{ccc}
\begin{array}{l}\int^{\Delta}\int^{\Delta'} P(\Delta'+\langle u\rangle)\\ \times Q_{v_1}(\Delta)\times\ldots\times Q_{v_k}(\Delta)\\ \times R_{u_1}(\Gamma)\times\ldots\times R_{u_m}(\Gamma)\end{array} & & \begin{array}{c}p,\\ x_1,\ldots,x_k,\\ y_1,\ldots,y_m\end{array}\\
\downarrow & & \big\downarrow{\scriptstyle \Delta_1=\ldots=\Delta_k=\Delta}\\
\begin{array}{l}\int^{\Delta'}\int^{\Delta_1}\ldots\int^{\Delta_k} P(\Delta'+\langle u\rangle)\\ \times Q_{v_1}(\Delta)\times R_{u_1}(\Gamma)\times\ldots\times R_{u_m}(\Gamma)\\ \times\ldots\\ \times Q_{v_k}(\Delta)\times R_{u_1}(\Gamma)\times\ldots\times R_{u_m}(\Gamma)\end{array} & & \begin{array}{c}p,\\ x_1,y_1,\ldots,y_m,\\ \ldots,\\ x_k,y_1,\ldots,y_m\end{array}\\
\downarrow & & \big\downarrow{\scriptstyle \Delta''=\Delta'+\langle u\rangle}\\
\begin{array}{l}\int^{\Delta''}\int^{\Delta_1}\ldots\int^{\Delta_k}\int^{\Delta_{k+1}} P(\Delta'')\\ \times Q_{v_1}(\Delta)\times R_{u_1}(\Gamma+\langle u\rangle)\times\ldots\times R_{u_m}(\Gamma+\langle u\rangle)\\ \times\ldots\\ \times Q_{v_k}(\Delta)\times R_{u_1}(\Gamma+\langle u\rangle)\times\ldots\times R_{u_m}(\Gamma+\langle u\rangle)\\ \times Q_{u}(\Delta)\times R_{u_1}(\Gamma+\langle u\rangle)\times\ldots\times R_{u_m}(\Gamma+\langle u\rangle)\end{array} & & \begin{array}{c}p,\\ x_1,\overline{y}_1,\ldots,\overline{y}_m,\\ \ldots,\\ x_k,\overline{y}_1,\ldots,\overline{y}_m,\\ \overline{qr_{u,\Gamma}}\end{array}
\end{array}$$

where $qr : \mathcal{Y} \to \mathcal{Y}\otimes\mathcal{Y} \to Q\otimes R$, by definition

$$\mathcal{Y}_u(\Gamma) \longrightarrow \int^{\Delta} \Delta^{-1}(u) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)} \longrightarrow \int^{\Delta} Q_u(\Delta) \times \prod_{t\in\mathcal{T}} R_t(\Gamma)^{\Delta^{-1}(t)}$$

$$x \xmapsto{\Delta=\langle u\rangle} 1, x \longmapsto q_{u,\langle u\rangle}(1), r_{u,\Gamma}(x)$$

thus $\overline{qr_{u,\Gamma}} : 1 \to (Q_u \bullet R)^{\mathcal{Y}\langle u\rangle}(\Gamma)$, $1 \mapsto qr_{u,\Gamma+\langle u\rangle}(1) = (q_{u,\Delta}(1), r_{u,\Gamma+\langle u\rangle}(1))$.

The two results along the two composites are equal, since each pair $\overline{x}_i, \overline{y}_1, \ldots, \overline{y}_m, \overline{r_{u,\Gamma}}$ and $x_i, \overline{y}_1, \ldots, \overline{y}_m$ comes from $x_i, \overline{y}_1, \ldots, \overline{y}_m, \overline{r_{u,\Gamma}}$ with the arrow $\Delta \to \Delta + \langle u\rangle$

$$\begin{array}{ccccc}
 & & \begin{array}{c}Q_{v_i}(\Delta)\times R_{u_1}(\Gamma+\langle u\rangle)\\ \times\ldots\times R_{u_m}(\Gamma+\langle u\rangle)\end{array} & & \\
 & \nearrow & & \searrow & \\
\begin{array}{c}Q_{v_i}(\Delta)\\ \times R_{u_1}(\Gamma+\langle u\rangle)\\ \times\ldots\\ \times R_{u_m}(\Gamma+\langle u\rangle)\\ \times R_{u}(\Gamma+\langle u\rangle)\end{array} & & & & \begin{array}{c}\int^{\Delta} Q_{v_i}(\Delta)\\ \times R_{u_1}(\Gamma+\langle u\rangle)\\ \times\ldots\\ \times R_{u_m}(\Gamma+\langle u\rangle)\end{array}\\
 & \searrow & & \nearrow & \\
 & & \begin{array}{c}Q_{v_i}(\Delta+\langle u\rangle)\times R_{u_1}(\Gamma+\langle u\rangle)\\ \times\ldots\\ \times R_{u_m}(\Gamma+\langle u\rangle)\times R_{u}(\Gamma+\langle u\rangle)\end{array} & &
\end{array}$$

and the last pair $\overline{q_{u,\Delta}}, \overline{y}_1, \ldots, \overline{y}_m, \overline{r_{u,\Gamma}} = q_{u,\Delta+\langle u\rangle}(1), \overline{y}_1, \ldots, \overline{y}_m, r_{u,\Gamma+\langle u\rangle}(1)$ and $q_{u,\langle u\rangle}(1), r_{u,\Gamma+\langle u\rangle}(1)$

comes from $q_{u,\langle u\rangle}(1), \overline{y}_1, \dots, \overline{y}_m, r_{u,\Gamma+\langle u\rangle}(1)$ with the arrow $i : \langle u\rangle \to \Delta + \langle u\rangle$

$$\begin{tikzcd}
& Q_u(\langle u\rangle) \times R_u(\Gamma + \langle u\rangle) \arrow[dr] & \\
\begin{array}{c} Q_u(\langle u\rangle) \\ \times R_{u_1}(\Gamma + \langle u\rangle) \\ \times \dots \\ \times R_{u_m}(\Gamma + \langle u\rangle) \\ \times R_u(\Gamma + \langle u\rangle) \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} \int^{\Delta} Q_u(\Delta) \\ \times R_{u_1}(\Gamma + \langle u\rangle) \\ \times \dots \\ \times R_{u_m}(\Gamma + \langle u\rangle) \end{array} \\
& \begin{array}{c} Q_u(\Delta + \langle u\rangle) \times R_{u_1}(\Gamma + \langle u\rangle) \\ \times \dots \\ \times R_{u_m}(\Gamma + \langle u\rangle) \times R_u(\Gamma + \langle u\rangle) \end{array} \arrow[ur] &
\end{tikzcd}$$

since $Q_u(i)(q_{u,\langle u\rangle}(1)) = q_{u,\Delta+\langle u\rangle}(1)$.

Now we turn to the triangular axiom. Along the right–hand side we have the following assignations on elements

$$\begin{tikzcd}
\int^{\Delta} P(\Delta + \langle u\rangle) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)} \arrow[d] & x, (h_t)_{t\in\mathcal{T}} \arrow[d, mapsto] \\
\int^{\Delta'} P(\Delta') \times \prod_{t\in\mathcal{T}} (\Gamma + \langle u\rangle)^{-1}(t)^{\Delta'^{-1}(t)} \arrow[d] & x, (h_t + \langle u\rangle^{-1}(t))_{t\in\mathcal{T}} \arrow[d, mapsto] \\
P(\Gamma) & P\big(\sum_{t\in\mathcal{T}} h_t + \langle u\rangle\big)(x)
\end{tikzcd}$$

which is exactly $r_{P\mathcal{Y}\langle u\rangle}(x, (h_t)_{t\in\mathcal{T}})$.

C.6 Proof of proposition 5.8.100

First we check naturality of $t_{X,Y}$ in X. Let $f : X \to Z$ be a morphism of $[\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. It is equivalent to check the naturality of $\widehat{t}_{X,Y}$ in X because of the adjunction $-\otimes Y \dashv Y \multimap -$ and the following square

$$\begin{tikzcd}
TX \otimes Y \arrow[r, "\widehat{t}_{X,Y}\otimes Y"] \arrow[d, "Tf\otimes Y"'] & (Y \multimap T(X\otimes Y))\otimes Y \arrow[r, "\varepsilon_{T(X\otimes Y)}"] \arrow[d, "(Y\multimap T(f\otimes Y))\otimes Y"] & T(X\otimes Y) \arrow[d, "T(f\otimes Y)"] \\
TZ \otimes Y \arrow[r, "\widehat{t}_{Z,Y}\otimes Y"'] & (Y \multimap T(Z\otimes Y))\otimes Y \arrow[r, "\varepsilon_{T(Z\otimes Y)}"'] & T(Z\otimes Y)
\end{tikzcd}$$

which commutes if and only if $\widehat{t}_{X,Y}$ is natural in X. So we have to check the commutativity of the following square

$$\begin{tikzcd}
TX \arrow[r, "\widehat{t}_{X,Y}"] \arrow[d, "Tf"'] & Y \multimap T(X\otimes Y) \arrow[d, "Y\multimap T(f\otimes Y)"] \\
TZ \arrow[r, "\widehat{t}_{Z,Y}"'] & Y \multimap T(Z\otimes Y)
\end{tikzcd}$$

We are going to provide $Y \multimap T(Z\otimes Y)$ with a Σ_X–algebra structure and check that $\widehat{t}_{Z,Y}$ and $Y \multimap T(f\otimes Y)$ are morphisms of Σ_X–algebras. By definition Tf and $\widehat{t}_{X,Y}$ are morphisms of Σ_X–algebras. By initiality of TX we can then conclude that $Y \multimap T(Z\otimes Y) \circ \widehat{t}_{X,Y}$ and $\widehat{t}_{Z,Y} \circ Tf$ are equal.

- $Y \multimap T(Z \otimes Y)$ is a Σ_X–algebra: First we define the arrow $X \to Y \multimap T(Z \otimes Y)$ to be the transpose of
$$X \otimes Y \xrightarrow{f \otimes Y} Z \otimes Y \xrightarrow{\eta_{Z \otimes Y}} T(Z \otimes Y)$$
Then we define the arrow $\beta : \Sigma(Y \multimap T(Z \otimes Y)) \to Y \multimap T(Z \otimes Y)$ to be the transpose of
$$\begin{tikzcd} \Sigma(Y \multimap T(Z \otimes Y)) \otimes Y \arrow[d, "s_{Y \multimap T(Z \otimes Y), Y}"] \\ \Sigma(Y \multimap T(Z \otimes Y) \otimes Y) \arrow[d, "\Sigma(\varepsilon_{T(Z \otimes Y)})"] \\ \Sigma(T(Z \otimes Y)) \arrow[d, "\sigma_{Z \otimes Y}"] \\ T(Z \otimes Y) \end{tikzcd}$$

- $\widehat{t}_{Z,Y}$ is a morphism of Σ_X–algebras: We have to check the commutativity of the following square
$$\begin{tikzcd} X + \Sigma TZ \arrow[r, "\mathrm{id} + \Sigma(\widehat{t}_{Z,Y})"] \arrow[d, "{[\eta_Z \circ f, \sigma_Z]}"'] & X + \Sigma(Y \multimap T(Z \otimes Y)) \arrow[d, "{[\overline{\eta_{Z \otimes Y} \circ (f \otimes Y)}, \beta]}"] \\ TZ \arrow[r, "\widehat{t}_{Z,Y}"'] & Y \multimap T(Z \otimes Y) \end{tikzcd}$$
which commutes if and only if the following two diagrams commute
$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & Y \multimap (X \otimes Y) \arrow[d, "Y \multimap (f \otimes Y)"] \\ Z \arrow[r, "\eta_Z"] \arrow[d, "\eta^T_Z"'] & Y \multimap (Z \otimes Y) \arrow[d, "Y \multimap \eta^T_{Z \otimes Y}"] \\ TZ \arrow[r, "\widehat{t}_{Z,Y}"'] & Y \multimap T(Z \otimes Y) \end{tikzcd}$$
where η denotes the unit of the adjunction $- \otimes Y \dashv Y \multimap -$ and η^T denotes the unit of the monad T. The top square is a naturality square of η and the bottom square commutes because $\widehat{t}_{Z,Y}$ is by definition a morphism of Σ_Z–algebras.
$$\begin{tikzcd} \Sigma TZ \arrow[r, "P(\widehat{t}_{Z,Y})"] \arrow[d, "\sigma_Z"'] & \Sigma(Y \multimap T(Z \otimes Y)) \arrow[d, "\beta"] \\ TZ \arrow[r, "\widehat{t}_{Z,Y}"'] & Y \multimap T(Z \otimes Y) \end{tikzcd}$$
which commutes by definition of $\widehat{t}_{Z,Y}$ being the unique morphism of Σ_Z–algebras $TZ \to Y \multimap T(Z \otimes Y)$.

- $Y \multimap T(f \otimes Y)$ is a morphism of Σ_X–algebras: We have to check the commutativity of the following square

$$\begin{tikzcd}
X+\Sigma(Y\multimap T(X\otimes Y)) \arrow[r, "\mathrm{id}+\Sigma(Y\multimap T(f\otimes Y))"] \arrow[dd, "{[\overline{\eta_{X\otimes Y}},\alpha]}"'] & X+\Sigma(Y\multimap T(Z\otimes Y)) \arrow[d, "f+\mathrm{id}"] \\
 & Z+\Sigma(Y\multimap T(Z\otimes Y)) \arrow[d, "{[\overline{\eta_{Z\otimes Y}\circ(f\otimes Y)},\beta]}"] \\
Y\multimap T(X\otimes Y) \arrow[r, "Y\multimap T(f\otimes Y)"'] & Y\multimap T(Z\otimes Y)
\end{tikzcd}$$

This square commutes if and only if the two following squares commute.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_Y"'] & Z \arrow[d, "\eta_Z"] \\
Y\multimap(X\otimes Y) \arrow[r, "Y\multimap(f\otimes Y)"] \arrow[d, "Y\multimap(\eta^T_{Y\otimes X})"'] & Y\multimap(Z\otimes Y) \arrow[d, "Y\multimap(\eta^T_{Z\otimes Y})"] \\
Y\multimap T(X\otimes Y) \arrow[r, "Y\multimap T(f\otimes Y)"'] & Y\multimap T(Z\otimes Y)
\end{tikzcd}$$

where η denotes the unit of the adjunction $-\otimes Y \dashv Y\multimap -$ and η^T the unit of the monad T. The top square is a naturality square of η and the bottom one a naturality square of η^T.

$$\begin{tikzcd}
\Sigma(Y\multimap T(X\otimes Y)) \arrow[r, "\Sigma(Y\multimap T(f\otimes Y))"] \arrow[d, "\eta_{\Sigma(Y\multimap T(X\otimes Y))}"'] & \Sigma(Y\multimap T(Z\otimes Y)) \arrow[d, "\eta_{\Sigma(Y\multimap T(Z\otimes Y))}"] \\
Y\multimap(\Sigma(Y\multimap T(X\otimes Y))\otimes Y) \arrow[r, "Y\multimap(\Sigma(Y\multimap T(f\otimes Y))\otimes Y)"] \arrow[d, "Y\multimap s_{\Sigma(Y\multimap T(X\otimes Y)),Y}"'] & Y\multimap(\Sigma(Y\multimap T(Z\otimes Y))\otimes Y) \arrow[d, "Y\multimap s_{\Sigma(Y\multimap T(Z\otimes Y)),Y}"] \\
Y\multimap \Sigma(Y\multimap T(X\otimes Y)\otimes Y) \arrow[r, "Y\multimap\Sigma(Y\multimap T(f\otimes Y)\otimes Y)"] \arrow[d, "Y\multimap\Sigma\varepsilon_{T(X\otimes Y)}"'] & Y\multimap \Sigma(Y\multimap T(Z\otimes Y)\otimes Y) \arrow[d, "Y\multimap\Sigma\varepsilon_{T(Z\otimes Y)}"] \\
Y\multimap \Sigma(T(X\otimes Y)) \arrow[r, "Y\multimap\Sigma(T(f\otimes Y))"] \arrow[d, "Y\multimap\sigma_{X\otimes Y}"'] & Y\multimap \Sigma(T(Z\otimes Y)) \arrow[d, "Y\multimap\sigma_{Z\otimes X}"] \\
Y\multimap T(X\otimes Y) \arrow[r, "Y\multimap T(f\otimes Y)"'] & Y\multimap T(Z\otimes Y)
\end{tikzcd}$$

These squares are naturality squares of η, s, ε and of σ (from top to bottom).

We check the compatibility of t with η and μ of the monad T and the compatibility with s the strength of Σ. By definition of $\widehat{t}_{X,Y}$ we have the following commutative square

$$\begin{tikzcd}
X+\Sigma TX \arrow[r, "X+\Sigma\widehat{t}_{X,Y}"] \arrow[d, "{[\eta_X,\sigma_X]}"'] & X+\Sigma(Y\multimap T(X\otimes Y)) \arrow[d, "{[\widehat{\eta}_{X\otimes Y},\widehat{\alpha}]}"] \\
TX \arrow[r, "\widehat{t}_{X,Y}"'] & Y\multimap T(X\otimes Y)
\end{tikzcd}$$

it implies the commutativity of the following two diagrams

$$\begin{tikzcd} X \arrow[d, "\eta_X"'] \arrow[dr, "\overline{\eta}_{X\otimes Y}"] & \\ TX \arrow[r, "\widehat{t}_{X,Y}"'] & Y \multimap T(X\otimes Y) \end{tikzcd} \tag{C.5}$$

$$\begin{tikzcd} \Sigma TX \arrow[r, "\Sigma\widehat{t}_{X,Y}"] \arrow[d, "\sigma_X"'] & \Sigma(Y \multimap T(X\otimes Y)) \arrow[d, "\alpha"] \\ TX \arrow[r, "\widehat{t}_{X,Y}"'] & Y \multimap T(X\otimes Y) \end{tikzcd} \tag{C.6}$$

By transposition (C.5) is equivalent to

$$\begin{tikzcd} X\otimes Y \arrow[d, "\eta_X\otimes Y"'] \arrow[dr, "\eta_{X\otimes Y}"] & \\ TX\otimes Y \arrow[r, "t_{X,Y}"'] & T(X\otimes Y) \end{tikzcd}$$

which proves the compatibility of t with η.

By transposition (C.6) is equivalent to

$$\begin{tikzcd} \Sigma TX\otimes Y \arrow[r, "\Sigma\widehat{t}_{X,Y}\otimes Y"] \arrow[dr, "s_{TX,Y}"'] \arrow[dd, "\sigma_X\otimes Y"'] & \Sigma(Y \multimap T(X\otimes Y))\otimes Y \arrow[r, "s_{Y\multimap T(X\otimes Y),Y}"] \arrow[d, phantom, "I."] & \Sigma((Y \multimap T(X\otimes Y))\otimes Y) \arrow[dd, "\Sigma\varepsilon_{T(X\otimes Y)}"] \\ & \Sigma(TX\otimes Y) \arrow[ur, "\Sigma(\widehat{t}\otimes Y)"'] \arrow[dr, "\Sigma t_{X,Y}"] \arrow[r, phantom, "II."] & {} \\ TX\otimes Y \arrow[r, "t_{X,Y}"'] & T(X\otimes Y) & \Sigma T(X\otimes Y) \arrow[l, "\sigma_{X\otimes Y}"] \end{tikzcd}$$

where the square I. is a naturality square of s and by definition of the transpose of $\widehat{t}$ the triangle II. commutes. So we find the following commutative diagram

$$\begin{tikzcd} \Sigma TX\otimes Y \arrow[r, "s_{TX,Y}"] \arrow[d, "\sigma_X\otimes Y"'] & \Sigma(TX\otimes Y) \arrow[r, "\Sigma t_{X,Y}"] & \Sigma T(X\otimes Y) \arrow[d, "\sigma_{X\otimes Y}"] \\ TX\otimes Y \arrow[rr, "t_{X,Y}"'] & & T(X\otimes Y) \end{tikzcd} \tag{C.7}$$

This proves the compatibility of the two strengths t and s with σ.

In order to check the compatibility of t with μ, we have to check the commutativity of the following diagram for all $X, Y \in [\mathbb{F}, \mathrm{Set}]$

$$\begin{tikzcd} TTX\otimes Y \arrow[r, "t_{TX,Y}"] \arrow[d, "\mu_X\otimes Y"'] & T(TX\otimes Y) \arrow[r, "Tt_{X,Y}"] & TT(X\otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\ TX\otimes Y \arrow[rr, "t_{X,Y}"'] & & T(X\otimes Y) \end{tikzcd}$$

This diagram becomes by transposition

$$\begin{tikzcd}
TTX \arrow[r, "\widehat{t}_{TX,Y}"] \arrow[d, "\mu_X"'] & Y \multimap T(TX \otimes Y) \arrow[r, "Y \multimap Tt_{X,Y}"] & Y \multimap TT(X \otimes Y) \arrow[d, "Y \multimap \mu_{X\otimes Y}"] \\
TX \arrow[rr, "\widehat{t}_{X,Y}"'] & & Y \multimap T(X \otimes Y)
\end{tikzcd}$$

In order to check its commutativity we provide $Y \multimap TT(X \otimes Y)$ and $Y \multimap T(X \otimes Y)$ with a Σ_{TX}–algebra structure and show that $Y \multimap Tt_{X,Y}$, $Y \multimap \mu_{X\otimes Y}$ and $\widehat{t}_{X,Y}$ are morphisms of Σ_{TX}–algebras. Since $\widehat{t}_{TX,Y}$ and μ_X are by definition morphisms of Σ_{TX}–algebras, we can conclude by initiality of TTX that $\widehat{t}_{X,Y} \circ \mu_X$ and $Y \multimap \mu_{X\otimes Y} \circ Y \multimap Tt_{X,Y} \circ \widehat{t}_{TX,Y}$ are equal.

- $Y \multimap TT(X \otimes Y)$ is a Σ_{TX}–algebra: We define the arrow

  $$TX + \Sigma(Y \multimap TT(X \otimes Y)) \to Y \multimap TT(X \otimes Y)$$

  by giving the two arrows $TX \to Y \multimap TT(X \otimes Y)$ and $\Sigma(Y \multimap TT(X \otimes Y)) \to Y \multimap TT(X \otimes Y)$. For the first one we take the transpose of

  $$TX \otimes Y \xrightarrow{t_{X,Y}} T(X \otimes Y) \xrightarrow{\eta^T_{T(X\otimes Y)}} TT(X \otimes Y)$$

  and for the second one we take the transpose of

  $$\begin{tikzcd}
  \Sigma(Y \multimap TT(X \otimes Y)) \otimes Y \arrow[d, "s_{Y \multimap TT(X\otimes Y),Y}"] \\
  \Sigma(Y \multimap TT(X \otimes Y) \otimes Y) \arrow[d, "\Sigma\varepsilon_{TT(X\otimes Y)}"] \\
  \Sigma TT(X \otimes Y) \arrow[d, "\sigma_{T(X\otimes Y)}"] \\
  TT(X \otimes Y)
  \end{tikzcd}$$

- $Y \multimap T(X \otimes Y)$ is a Σ_{TX}–algebra: We define the arrow

  $$TX + \Sigma(Y \multimap T(X \otimes Y)) \to Y \multimap T(X \otimes Y)$$

  by giving the two arrows $TX \to Y \multimap T(X \otimes Y)$ and $\Sigma(Y \multimap T(X \otimes Y)) \to Y \multimap T(X \otimes Y)$. For the first one we take

  $$TX \xrightarrow{\widehat{t}_{X,Y}} Y \multimap T(X \otimes Y)$$

  and for the second one the transpose of

  $$\begin{tikzcd}
  \Sigma(Y \multimap T(X \otimes Y)) \otimes Y \arrow[d, "s_{Y \multimap T(X\otimes Y),Y}"] \\
  \Sigma(Y \multimap T(X \otimes Y) \otimes Y) \arrow[d, "\Sigma\varepsilon_{T(X\otimes Y)}"] \\
  \Sigma T(X \otimes Y) \arrow[d, "\sigma_{X\otimes Y}"] \\
  T(X \otimes Y)
  \end{tikzcd}$$

  (which is α).

- $Y \multimap Tt_{X,Y}$ is a morphism of Σ_{TX}–algebras: We check the commutativity of the Σ_{TX}–algebra morphism axiom separately in the first and the second component.

$$\begin{tikzcd}
TX \arrow[r, "\eta_{TX}"] \arrow[d, "\eta_{TX}"'] & Y \multimap TX \otimes Y \arrow[d, "Y \multimap t_{X,Y}"] \\
Y \multimap TX \otimes Y \arrow[r, "Y \multimap t_{X,Y}"] \arrow[d, "Y \multimap \eta^T_{TX \otimes Y}"'] & Y \multimap T(X \otimes Y) \arrow[d, "Y \multimap \eta^T_{T(X \otimes Y)}"] \\
Y \multimap T(TX \otimes Y) \arrow[r, "Y \multimap Tt_{X,Y}"'] & Y \multimap TT(X \otimes Y)
\end{tikzcd}$$

The top square commutes obviously and the bottom square is a naturality square of η^T.

$$\begin{tikzcd}
\Sigma(Y \multimap T(TX \otimes Y)) \arrow[r, "\Sigma(Y \multimap Tt_{X,Y})"] \arrow[d, "\eta_{\Sigma(Y \multimap T(TX \otimes Y))}"'] & \Sigma(Y \multimap TT(X \otimes Y)) \arrow[d, "\eta_{\Sigma(Y \multimap TT(X \otimes Y))}"] \\
Y \multimap \Sigma(Y \multimap T(TX \otimes Y)) \otimes Y \arrow[r, "Y \multimap \Sigma(Y \multimap Tt_{X,Y}) \otimes Y"] \arrow[d, "Y \multimap p_{Y \multimap T(TX \otimes Y),Y}"'] & Y \multimap \Sigma(Y \multimap TT(X \otimes Y)) \otimes Y \arrow[d, "Y \multimap p_{Y \multimap TT(X \otimes Y),Y}"] \\
Y \multimap \Sigma(Y \multimap T(TX \otimes Y) \otimes Y) \arrow[r, "Y \multimap \Sigma(Y \multimap Tt_{X,Y} \otimes Y)"] \arrow[d, "Y \multimap \Sigma\varepsilon_{T(TX \otimes Y)}"'] & Y \multimap \Sigma(Y \multimap TT(X \otimes Y) \otimes Y) \arrow[d, "Y \multimap \Sigma\varepsilon_{TT(X \otimes Y)}"] \\
Y \multimap \Sigma T(TX \otimes Y) \arrow[r, "Y \multimap \Sigma Tt_{X,Y}"] \arrow[d, "Y \multimap \sigma_{TX \otimes Y}"'] & Y \multimap \Sigma TT(X \otimes Y) \arrow[d, "Y \multimap \sigma_{T(X \otimes Y)}"] \\
Y \multimap T(TX \otimes Y) \arrow[r, "Y \multimap Tt_{X,Y}"'] & Y \multimap TT(X \otimes Y)
\end{tikzcd}$$

These squares are naturality squares of η, s, ε and of σ (from top to bottom).

- $Y \multimap \mu_{X \otimes Y}$ is a morphism of Σ_{TX}–algebras: We check the commutativity of the Σ_{TX}–algebra morphism axiom separately in the first and the second component.

$$\begin{tikzcd}
TX \arrow[d, "\eta_{TX}"'] \arrow[dddr, "\widehat{t}_{X,Y}"] & \\
Y \multimap (TX \otimes Y) \arrow[d, "Y \multimap t_{X,Y}"'] & \\
Y \multimap T(X \otimes Y) \arrow[d, "Y \multimap \eta^T_{T(X \otimes Y)}"'] \arrow[dr, "Y \multimap \mathrm{id}_{T(X \otimes Y)}"] & \\
Y \multimap TT(X \otimes Y) \arrow[r, "Y \multimap \mu^T_{X \otimes Y}"'] & Y \multimap T(X \otimes Y)
\end{tikzcd}$$

The top triangle commutes because t is by definition the transpose of $\widehat{t}$ and the bottom triangle is a monad axiom.

$$\begin{tikzcd}
\Sigma(Y \multimap TT(X\otimes Y)) \arrow[r, "\Sigma(Y\multimap\mu^T_{X\otimes Y})"] \arrow[d, "\eta_{\Sigma(Y\multimap TT(X\otimes Y))}"'] & \Sigma(Y \multimap T(X\otimes Y)) \arrow[d, "\eta_{\Sigma(Y\multimap T(X\otimes Y))}"] \\
Y \multimap \Sigma(Y \multimap TT(X\otimes Y))\otimes Y \arrow[r, "Y\multimap\Sigma(Y\multimap\mu^T_{X\otimes Y})\otimes Y"] \arrow[d, "Y\multimap s_{Y\multimap TT(X\otimes Y),Y}"'] & Y \multimap \Sigma(Y \multimap T(X\otimes Y))\otimes Y \arrow[d, "Y\multimap s_{Y\multimap T(X\otimes Y),Y}"] \\
Y \multimap \Sigma(Y \multimap TT(X\otimes Y)\otimes Y) \arrow[r, "Y\multimap\Sigma(Y\multimap\mu^T_{X\otimes Y}\otimes Y)"] \arrow[d, "Y\multimap\Sigma\varepsilon_{TT(X\otimes Y)}"'] & Y \multimap \Sigma(Y \multimap T(X\otimes Y)\otimes Y) \arrow[d, "Y\multimap\Sigma\varepsilon_{T(X\otimes Y)}"] \\
Y \multimap \Sigma TT(X\otimes Y) \arrow[r, "Y\multimap\Sigma\mu^T_{X\otimes Y}"] \arrow[d, "Y\multimap\sigma_{T(X\otimes Y)}"'] & Y \multimap \Sigma T(X\otimes Y) \arrow[d, "Y\multimap\sigma_{X\otimes Y}"] \\
Y \multimap TT(X\otimes Y) \arrow[r, "Y\multimap\mu^T_{X\otimes Y}"'] & Y \multimap T(X\otimes Y)
\end{tikzcd}$$

The top three squares are naturality squares of η, s and of ε. The bottom square commutes because $\mu^T_{X\otimes Y}$ is by definition the unique morphism of $\Sigma_{T(X\otimes Y)}$–algebras $TT(X\otimes Y) \to T(X\otimes Y)$ such that

$$\begin{tikzcd}
T(X\otimes Y)+\Sigma TT(X\otimes Y) \arrow[r, "\mathrm{id}+\Sigma\mu^T_{X\otimes Y}"] \arrow[d, "{[\eta^T_{T(X\otimes Y)},\sigma_{T(X\otimes Y)}]}"'] & T(X\otimes Y)+\Sigma T(X\otimes Y) \arrow[d, "{[\mathrm{id},\sigma_{X\otimes Y}]}"] \\
TT(X\otimes Y) \arrow[r, "\mu^T_{X\otimes Y}"'] & T(X\otimes Y)
\end{tikzcd}$$

commutes.

- $\widehat{t}_{X,Y}$ is a morphism of Σ_{TX}–algebras: We have to check the commutativity of the following diagram

$$\begin{tikzcd}
TX+\Sigma TX \arrow[r, "\mathrm{id}+\Sigma\widehat{t}_{X,Y}"] \arrow[d, "{[\mathrm{id},\sigma_X]}"'] & TX+\Sigma(Y\multimap T(X\otimes Y)) \arrow[d, "{[\widehat{t}_{X,Y},\overline{\sigma_{X\otimes Y}\circ\Sigma\varepsilon_{T(X\otimes Y)}\circ p}]}"] \\
TX \arrow[r, "\widehat{t}_{X,Y}"'] & Y\multimap T(X\otimes Y)
\end{tikzcd}$$

This square commutes obviously in its first component. The commutativity in its second component is a consequence of the definition of $\widehat{t}_{X,Y}$ being the unique morphism of Σ_X–algebras $TX \to Y \multimap T(X\otimes Y)$.

Appendix D

Proofs of chapter 7

D.1 Definition of $\ell : [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}] \to [\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$

We check that $\ell(X)$ is indeed a functor of $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$. Let $f : \Gamma \to \Delta$ be an arrow in $\mathrm{Set}/\mathcal{T}$. We construct the arrow $\ell(X)(\Gamma) \to \ell(X)(\Delta)$. By universal property of the coend it suffices to give an arrow

$$X(\Gamma') \times \prod_{t \in \mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \to \int^{\Gamma' \in \mathbb{F}\downarrow\mathcal{T}} X(\Gamma') \times \prod_{t \in \mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)}$$

for all $\Gamma' \in \mathbb{F} \downarrow \mathcal{T}$ satisfying the wedge condition. We take the following mapping composed with the corresponding coprojection

$$(x, (h_t)_{t\in\mathcal{T}}) \quad \mapsto \quad (x, (f_t \circ h_t)_{t\in\mathcal{T}})$$

We check the wedge condition. Let $g : \Gamma' \to \Gamma''$ in $\mathbb{F} \downarrow \mathcal{T}$. The following diagram

$$\begin{tikzcd}
 & X(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \arrow[dr] & \\
X(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \arrow[ur, "\mathrm{id} \times (-\circ g_t)_{t\in\mathcal{T}}"] \arrow[dr, "X(g)\times \mathrm{id}"'] & & \int^{\Gamma'} X(\Gamma') \times \prod_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)} \\
 & X(\Gamma'') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & (x, (h_t \circ g_t)_{t\in\mathcal{T}}) \arrow[dr, mapsto] & \\
(x, (h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{l}(x, (f_t \circ h_t \circ g_t)_{t\in\mathcal{T}}) \\ = (X(g)(x), (f_t \circ h_t)_{t\in\mathcal{T}})\end{array} \\
 & (X(g)(x), (h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] &
\end{tikzcd}$$

The two elements $(x,(f_t \circ h_t \circ g_t)_{t\in\mathcal{T}})$ and $(X(g)(x),(f_t \circ h_t)_{t\in\mathcal{T}})$ are equal since they come from $(x,(f_t \circ h_t)_{t\in\mathcal{T}}) \in X(\Gamma') \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma''^{-1}(t)}$ with the arrow g

$$\begin{tikzcd}
& \begin{array}{l} X(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} X(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma''^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id}\times(-\circ g_t)"] \arrow[dr, "X(g)\times\mathrm{id}"'] & & \begin{array}{l} \int^{\Gamma'} X(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)} \end{array} \\
& \begin{array}{l} X(\Gamma'') \\ \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma''^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

Definition on morphisms: We check the wedge condition. Let $g : \Gamma' \to \Gamma''$ in $\mathbb{F} \downarrow \mathcal{T}$. The diagram

$$\begin{tikzcd}
& \begin{array}{l} X(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} X(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id}\times(-\circ g_t)"] \arrow[dr, "X(g)\times\mathrm{id}"'] & & \begin{array}{l} Y(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \end{array} \\
& \begin{array}{l} X(\Gamma'') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
& (x,(h_t \circ g_t)_{t\in\mathcal{T}}) \arrow[dr] & \\
(x,(h_t)_{t\in\mathcal{T}}) \arrow[ur] \arrow[dr] & & \begin{array}{l} (f_{\Gamma'}(x),(h_t \circ g_t)_{t\in\mathcal{T}}) \\ = (f_{\Gamma''}(X(g)(x)),(h_t)_{t\in\mathcal{T}}) \end{array} \\
& (X(g)(x),(h_t)_{t\in\mathcal{T}}) \arrow[ur] &
\end{tikzcd}$$

The two elements $(f_{\Gamma'}(x),(h_t \circ g_t)_{t\in\mathcal{T}})$ and $(f_{\Gamma''}(X(g)(x)),(h_t)_{t\in\mathcal{T}}) = (Y(g)(f_{\Gamma'}(x)))$ are equal since they come from $(f_{\Gamma'}(x), (h_t)_{t\in\mathcal{T}}) \in Y(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)}$ with the arrow g

$$
\begin{tikzcd}
 & Y(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \arrow[dr] & \\
Y(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \arrow[ur, "\mathrm{id}\times(-\circ g_t)"] \arrow[dr, "Y(g)\times\mathrm{id}"'] & & Y(\Gamma') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \\
 & Y(\Gamma'') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \arrow[ur] &
\end{tikzcd}
$$

D.2 Proof of lemma 7.2.127

Let $\Gamma \in \mathrm{Set}/\mathcal{T}$. On the one hand we have

$$(\ell P)(\Gamma) \times (\ell Q)(\Gamma) = \int^{\Delta_1}\int^{\Delta_2} P(\Delta_1) \times Q(\Delta_2) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1+\Delta_2)^{-1}(t)}$$

and on the other hand

$$\ell(P \times Q)(\Gamma) = \int^{\Delta} P(\Delta) \times Q(\Delta) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)}$$

In order to define an arrow $(\ell P)(\Gamma) \times (\ell Q)(\Gamma) \to \ell(P \times Q)(\Gamma)$, by universal property of the coend we give a collection of arrows

$$P(\Delta_1) \times Q(\Delta_2) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1+\Delta_2)^{-1}(t)} \to \ell(P \times Q)(\Gamma)$$

for all $\Delta_1, \Delta_2 \in [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$ that satisfies the wedge condition. We take the following mapping composed with the $\Delta_1 + \Delta_2$-th coprojection.

$$(p, q, (h_t)_{t\in\mathcal{T}}) \quad \mapsto \quad (P(i_1)(p), Q(i_2)(q), (h_t)_{t\in\mathcal{T}})$$

where we write $i_1 : \Delta_1 \to \Delta$ and $i_2 : \Delta_2 \to \Delta$ for the inclusions. We check the wedge condition. Let $f_1 : \Delta_1 \to \Delta_1'$ and $f_2 : \Delta_2 \to \Delta_2'$. The following diagram commutes

$$
\begin{tikzcd}
 & P(\Delta_1) \times Q(\Delta_2) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1+\Delta_2)^{-1}(t)} \arrow[dr] & \\
P(\Delta_1) \times Q(\Delta_2) \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1'+\Delta_2')^{-1}(t)} \arrow[ur, "P(f_1)\times Q(f_2)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times(-\circ(f_1+f_2))_{t\in\mathcal{T}}"'] & & \ell(P \times Q)(\Gamma) \\
 & P(\Delta_1') \times Q(\Delta_2') \times \prod_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1'+\Delta_2')^{-1}(t)} \arrow[ur] &
\end{tikzcd}
$$

since we have the following assignations on elements

$$\begin{tikzcd}[column sep=small]
 & (p,q,(h_t\circ(f_1+f_2)_t)_{t\in\mathcal{T}}) \arrow[dr, mapsto] & \\
(p,q,(h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}(P(i_1)(p),Q(i_2)(q),\\ (h_t\circ(f_1+f_2)_t)_{t\in\mathcal{T}})\\ =(P(i_1'\circ f_1)(p),Q(i_2'\circ f_2)(q),\\ (h_t)_{t\in\mathcal{T}})\end{array} \\
 & (P(f_1)(p),Q(f_2)(q),(h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] &
\end{tikzcd}$$

By naturality of the inclusions we have $i_1'\circ f_1=(f_1+f_2)\circ i_1$ and $i_2'\circ f_2=(f_1+f_2)\circ i_2$. The two elements $(P(i_1)(p),Q(i_2)(q),(h_t\circ(f_1+f_2)_t)_{t\in\mathcal{T}})$ and $(P((f_1+f_2)\circ i_1)(p),Q((f_1+f_2)\circ f_2)(q),(h_t)_{t\in\mathcal{T}})$ are equal since they come from $(P(i_1)(p),Q(i_2)(q),(h_t)_{t\in\mathcal{T}})$ with the arrow f_1+f_2

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c}P(\Delta_1+\Delta_2)\times Q(\Delta_1+\Delta_2)\\ \times\prod\limits_{t\in\mathcal{T}}\Gamma^{-1}(t)^{(\Delta_1+\Delta_2)^{-1}(t)}\end{array} \arrow[dr] & \\
\begin{array}{c}P(\Delta_1+\Delta_2)\times Q(\Delta_1+\Delta_2)\\ \times\prod\limits_{t\in\mathcal{T}}\Gamma^{-1}(t)^{(\Delta_1'+\Delta_2')^{-1}(t)}\end{array} \arrow[ur, "P(f_1+f_2)\times Q(f_1+f_2)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times(-\circ(f_1+f_2)_t)_{t\in\mathcal{T}}"'] & & \ell(P\times Q)(\Gamma) \\
 & \begin{array}{c}P(\Delta_1'+\Delta_2')\times Q(\Delta_1'+\Delta_2')\\ \times\prod\limits_{t\in\mathcal{T}}\Gamma^{-1}(t)^{(\Delta_1'+\Delta_2')^{-1}(t)}\end{array} \arrow[ur] &
\end{tikzcd}$$

In order to define an arrow in the inverse direction, by universal property of the coend we give a collection of arrows

$$P(\Delta)\times Q(\Delta)\times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Delta^{-1}(t)}\to(\ell P)(\Gamma)\times(\ell Q)(\Gamma)$$

for all $\Delta\in\mathbb{F}\downarrow\mathcal{T}$ that satisfies the wedge condition. We take the following mapping composed with the corresponding coprojections.

$$(p,q,(h_t)_{t\in\mathcal{T}})\quad\mapsto\quad(p,q,([h_t,h_t])_{t\in\mathcal{T}})$$

where we write $[h_t,h_t]:(\Delta+\Delta)^{-1}(t)\to\Delta^{-1}(t)$. We check the wedge condition. Let $f:\Delta\to\Delta$ in $\mathbb{F}\downarrow\mathcal{T}$. The following diagram

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c}P(\Delta)\times Q(\Delta)\\ \times\prod\limits_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Delta^{-1}(t)}\end{array} \arrow[dr] & \\
\begin{array}{c}P(\Delta)\times Q(\Delta)\\ \times\prod\limits_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Delta'^{-1}(t)}\end{array} \arrow[ur, "\mathrm{id}\times(-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(f)\times Q(f)\times\mathrm{id}"'] & & (\ell P)(\Gamma)\times(\ell Q)(\Gamma) \\
 & \begin{array}{c}P(\Delta')\times Q(\Delta')\\ \times\prod\limits_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Delta'^{-1}(t)}\end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
 & (p,q,(h_t \circ f_t)_{t\in\mathcal{T}}) \arrow[dr, mapsto] & \\
(p,q,(h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{l}(p,q,([h_t \circ f_t, h_t \circ f_t])_{t\in\mathcal{T}}) \\ = (P(f)(p), Q(f)(q), ([h_t,h_t])_{t\in\mathcal{T}})\end{array} \\
 & (P(f)(p), Q(f)(q), (h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] &
\end{tikzcd}$$

The two elements $(p,(h_t \circ f_t)_{t\in\mathcal{T}})$ and $(P(f)(p),(h_t)_{t\in\mathcal{T}})$ are equal since they come from $(p,(h_t)_{t\in\mathcal{T}}) \in P(\Delta) \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta'^{-1}(t)}$ with the arrow f

$$\begin{tikzcd}
 & \begin{array}{l}P(\Delta) \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta^{-1}(t)}\end{array} \arrow[dr] & \\
\begin{array}{l}P(\Delta) \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta'^{-1}(t)}\end{array} \arrow[ur, "\mathrm{id} \times (-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "P(f)\times \mathrm{id}"'] & & (\ell P)(\Gamma) \\
 & \begin{array}{l}P(\Delta') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Delta'^{-1}(t)}\end{array} \arrow[ur] &
\end{tikzcd}$$

and by an analogous reasoning $(q,(h_t \circ f_t)_{t\in\mathcal{T}})$ and $(Q(f)(p),(h_t)_{t\in\mathcal{T}})$ are equal.

Let us check that these two arrows are inverse to each other. One of the composites yields

$$(p,q,(h_t)_{t\in\mathcal{T}}) \mapsto (P(i_1)(p), Q(i_2)(q), (h_t)_{t\in\mathcal{T}}) \mapsto (P(i_1)(p), Q(i_2)(q), ([h_t,h_t])_{t\in\mathcal{T}})$$

These two elements are identical since they come from $(p,q,([h_t,h_t])_{t\in\mathcal{T}}) \in P(\Delta_1) \times Q(\Delta_2) \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1+\Delta_2+\Delta_1+\Delta_2)^{-1}(t)}$ with the arrows $i_1 : \Delta_1 \to \Delta_1 + \Delta_2$ and $i_2 : \Delta_2 \to \Delta_1 + \Delta_2$

$$\begin{tikzcd}
 & \begin{array}{c}P(\Delta_1) \times Q(\Delta_2)\times \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1+\Delta_2)^{-1}(t)} \\ (p,q,(h_t)_{t\in\mathcal{T}})\end{array} \\
\begin{array}{c}P(\Delta_1) \times Q(\Delta_2)\times \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1+\Delta_2+\Delta_1+\Delta_2)^{-1}(t)} \\ (p,q,([h_t,h_t])_{t\in\mathcal{T}})\end{array} \arrow[ur, "\mathrm{id} \times \prod\limits_{t\in\mathcal{T}} (-\circ(i_1+i_2)_t)"] \arrow[dr, "P(i_1)\times Q(i_2)\times \mathrm{id}"'] & \\
 & \begin{array}{c}P(\Delta_1+\Delta_2) \times Q(\Delta_1+\Delta_2)\times \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{(\Delta_1+\Delta_2+\Delta_1+\Delta_2)^{-1}(t)} \\ (P(i_1)(p), Q(i_2)(q), ([h_t,h_t])_{t\in\mathcal{T}})\end{array}
\end{tikzcd}$$

where $(i_1+i_2)_t$ is the fibre of i_1+i_2 in t and $[h_t,h_t]\circ(i_1+i_2)_t = h_t$.

The other composite yields

$$(p,q,(h_t)_{t\in\mathcal{T}}) \mapsto (p,q,([h_t,h_t])_{t\in\mathcal{T}}) \mapsto (P(i_1)(p),Q(i_2)(q),([h_t,h_t])_{t\in\mathcal{T}})$$

These two elements are identical since they come from $P(\Delta+\Delta)\times Q(\Delta+\Delta)\times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Delta^{-1}(t)}$ with the arrow $[\mathrm{id}_\Delta,\mathrm{id}_\Delta] = p : \Delta+\Delta\to\Delta = 2\times\Delta\to\Delta$

$$\begin{tikzcd}
 & \begin{array}{c} P(\Delta)\times Q(\Delta)\times \\ \times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{(\Delta)^{-1}(t)} \\ (x,y,(h_t)_{t\in\mathcal{T}}) \end{array} \\
\begin{array}{c} P(\Delta+\Delta)\times Q(\Delta+\Delta)\times \\ \times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{\Delta^{-1}(t)} \\ (P(i_1)(x),Q(i_2)(y),(h_t)_{t\in\mathcal{T}}) \end{array} \arrow[ur, "P(p)\times Q(p)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times\prod_{t\in\mathcal{T}}(-\circ p_t)"'] & \\
 & \begin{array}{c} P(\Delta+\Delta)\times Q(\Delta+\Delta)\times \\ \times\prod_{t\in\mathcal{T}}\Gamma^{-1}(t)^{(\Delta+\Delta)^{-1}(t)} \\ (P(i_1)(x),Q(i_2)(y),([h_t,h_t])_{t\in\mathcal{T}}) \end{array}
\end{tikzcd}$$

where p_t is the fibre of p in t and $[\mathrm{id},\mathrm{id}]\circ i_1 = \mathrm{id}$ and $[\mathrm{id},\mathrm{id}]\circ i_2 = \mathrm{id}$.

Now we check naturality in Γ, P and Q. Let $f:\Gamma_1\to\Gamma_2$ in $\mathrm{Set}/\mathcal{T}$. The naturality square

$$\begin{tikzcd}
(\ell P)(\Gamma_1)\times(\ell Q)(\Gamma_1) \arrow[r] \arrow[d] & \ell(P\times Q)(\Gamma_1) \arrow[d] \\
(\ell P)(\Gamma_2)\times(\ell Q)(\Gamma_2) \arrow[r] & \ell(P\times Q)(\Gamma_2)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
(p,q,(h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & (P(i_1)(p),Q(i_2)(q),(h_t)_{t\in\mathcal{T}}) \arrow[d, mapsto] \\
(p,q,(f_t\circ h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] & (P(i_1)(p),Q(i_2)(q),(f_t\circ h_t)_{t\in\mathcal{T}})
\end{tikzcd}$$

Let $f:P\to R$ in $[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
(\ell P)(\Gamma)\times(\ell Q)(\Gamma) \arrow[r] \arrow[d] & \ell(P\times Q)(\Gamma) \arrow[d] \\
(\ell R)(\Gamma)\times(\ell Q)(\Gamma) \arrow[r] & \ell(R\times Q)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
(p,q,(h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & (P(i_1)(p),Q(i_2)(q),(h_t)_{t\in\mathcal{T}}) \arrow[d, mapsto] \\
(f_{\Delta_1}(p),q,(h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] & \begin{array}{c}(f_{\Delta_1+\Delta_2}(P(i_1)(p)),Q(i_2)(q),(h_t)_{t\in\mathcal{T}}) \\ =(R(i_1)(f_{\Delta_1}(p)),Q(i_2)(q),(h_t)_{t\in\mathcal{T}})\end{array}
\end{tikzcd}$$

Let $f : Q \to R$ in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
(\ell P)(\Gamma) \times (\ell Q)(\Gamma) \arrow[r] \arrow[d] & \ell(P \times Q)(\Gamma) \arrow[d] \\
(\ell P)(\Gamma) \times (\ell R)(\Gamma) \arrow[r] & \ell(P \times R)(\Gamma)
\end{tikzcd}$$

commutes because we have the following assignations on elements

$$\begin{tikzcd}
(p, q, (h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & (P(i_1)(p), Q(i_2)(q), (h_t)_{t\in\mathcal{T}}) \arrow[d, mapsto] \\
(p, f_{\Delta_2}(q), (h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] & \begin{array}{c}(P(i_1)(p), f_{\Delta_1+\Delta_2}(Q(i_2)(q)), (h_t)_{t\in\mathcal{T}}) \\ = (P(i_1)(p), R(i_1)(f_{\Delta_2}(q)), (h_t)_{t\in\mathcal{T}})\end{array}
\end{tikzcd}$$

D.3 Unit of $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}] \to [\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$

Let us check naturality in Δ. Let $g : \Delta_1 \to \Delta_2$. The naturality square

$$\begin{tikzcd}
X(\Delta_1) \arrow[r, "\eta_{X,\Delta_1}"] \arrow[d, "X(g)"'] & (k\ell X)(\Delta_1) \arrow[d, "(k\ell X)(g)"] \\
X(\Delta_2) \arrow[r, "\eta_{X,\Delta_2}"'] & (k\ell X)(\Delta_2)
\end{tikzcd}$$

commutes because on elements we find

$$\begin{tikzcd}
x \arrow[r, mapsto, "\eta_{X,\Delta_1}"] \arrow[d, "X(g)"'] & (x, (\mathrm{id}_{\Delta_1^{-1}(t)})_{t\in\mathcal{T}}) \arrow[d, mapsto, "(k\ell X)_t(g)"] \\
X(g)(x) \arrow[r, mapsto, "\eta_{X,\Delta_2}"'] & \begin{array}{c}(x, (g_t)_{t\in\mathcal{T}}) = \\ (X(g)(x), (\mathrm{id}_{\Delta_2^{-1}(t)})_{t\in\mathcal{T}})\end{array}
\end{tikzcd}$$

The elements $(x, (g_t)_{t\in\mathcal{T}})$ and $(X(g)(x), (\mathrm{id}_{\Delta_2^{-1}(t)})_{t\in\mathcal{T}})$ come from $X(\Delta_1) \times \prod_{t\in\mathcal{T}} \Delta_2^{-1}(t)^{\Delta_2^{-1}(t)}$ with the arrow g

$$\begin{tikzcd}
& \begin{array}{c} X(\Delta_2) \times \prod_{t\in\mathcal{T}} \Delta_2^{-1}(t)^{\Delta_2^{-1}(t)} \\ (X(g)(x), (\mathrm{id}_{\Delta_2^{-1}(t)})_{t\in\mathcal{T}}) \end{array} \\
\begin{array}{c} X(\Delta_1) \times \prod_{t\in\mathcal{T}} \Delta_2^{-1}(t)^{\Delta_2^{-1}(t)} \\ (x, (\mathrm{id}_{\Delta_2^{-1}(t)})_{t\in\mathcal{T}}) \end{array} \arrow[ur] \arrow[dr] & \\
& \begin{array}{c} X(\Delta_1) \times \prod_{t\in\mathcal{T}} \Delta_2^{-1}(t)^{\Delta_1^{-1}(t)} \\ (x, (g_t)_{t\in\mathcal{T}}) \end{array}
\end{tikzcd}$$

Now we check naturality in X. Let $f : X \to Y$ be a morphism in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd} X(\Delta) \arrow[r, "\eta_{X,\Delta}"] \arrow[d, "f_\Delta"'] & (k\ell X)(\Delta) \arrow[d, "(k\ell f)_\Delta"] \\ Y(\Delta) \arrow[r, "\eta_{Y,\Delta}"'] & (k\ell Y)(\Delta) \end{tikzcd}$$

commutes because we find on elements

$$\begin{tikzcd} x \arrow[r, mapsto, "\eta_{X,\Delta}"] \arrow[d, mapsto, "f_\Delta"'] & (x, (\mathrm{id}_{\Delta^{-1}(t)})_{t\in\mathcal{T}}) \arrow[d, mapsto, "(k\ell f)_\Delta"] \\ f_\Delta(x) \arrow[r, mapsto, "\eta_{Y,\Delta}"'] & (f_\Delta(x), (\mathrm{id}_{\Delta^{-1}(t)})_{t\in\mathcal{T}}) \end{tikzcd}$$

D.4 Proof of lemma 7.4.131

It remains to check that the two mappings are inverse to each other. One of the composites yields

$$x \quad \mapsto \quad (x, (\mathrm{id}_{\Delta^{-1}(t)})_{t\in\mathcal{T}}) \quad \mapsto \quad X(\mathrm{id}_\Delta)(x) = x$$

which is the identity on x. The other composite is

$$(x, (h_t)_{t\in\mathcal{T}}) \quad \mapsto \quad X(\sum_{t\in\mathcal{T}} h_t)(x) \quad \mapsto \quad (X(\sum_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Delta^{-1}(t)})_{t\in\mathcal{T}})$$

which is the identity because $(x, (h_t)_{t\in\mathcal{T}})$ and $(X(\sum\limits_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Delta^{-1}(t)})_{t\in\mathcal{T}})$ come from $X(\Gamma) \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Delta^{-1}(t)}$ with the arrow $\sum\limits_{t\in\mathcal{T}} h_t : \Gamma' \to \Delta$

$$\begin{tikzcd} & \begin{matrix} X(\Gamma') \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Gamma'^{-1}(t)} \\ (x, (h_t)_{t\in\mathcal{T}}) \end{matrix} \\ \begin{matrix} X(\Gamma') \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Delta^{-1}(t)} \\ (x, (\mathrm{id}_{\Delta^{-1}(t)})_{t\in\mathcal{T}}) \end{matrix} \arrow[ur] \arrow[dr] & \\ & \begin{matrix} X(\Delta) \times \prod\limits_{t\in\mathcal{T}} \Delta^{-1}(t)^{\Delta^{-1}(t)} \\ (X(\sum\limits_{t\in\mathcal{T}} h_t)(x), (\mathrm{id}_{\Delta^{-1}(t)})_{t\in\mathcal{T}}) \end{matrix} \end{tikzcd}$$

D.5 Counit of $\ell \dashv k : [\mathrm{Set}/\mathcal{T}, \mathrm{Set}] \to [\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}]$

We check the wedge condition. Let $g : \Gamma' \to \Gamma''$ in $\mathbb{F}\downarrow\mathcal{T}$. The following diagram

$$\begin{tikzcd}
 & \begin{array}{l} F(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma'^{-1}(t)} \end{array} \arrow[dr] & \\
\begin{array}{l} F(\Gamma') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \end{array} \arrow[ur, "\mathrm{id}\times(-\circ g_t)"] \arrow[dr, "F(g)\times\mathrm{id}"'] & & F(\Gamma) \\
 & \begin{array}{l} F(\Gamma'') \\ \times \prod\limits_{t\in\mathcal{T}} \Gamma^{-1}(t)^{\Gamma''^{-1}(t)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & (x, (h_t \circ g_t)_{t\in\mathcal{T}}) \arrow[dr, mapsto] & \\
(x, (h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & F\big(\sum\limits_{t\in\mathcal{T}} h_t \circ g\big)(x) \\
 & (F(g)(x), (h_t)_{t\in\mathcal{T}}) \arrow[ur, mapsto] &
\end{tikzcd}$$

Let $f : \Gamma_1 \to \Gamma_2$ be an arrow in $\mathrm{Set}/\mathcal{T}$. The following naturality square

$$\begin{tikzcd}
\ell k(F)(\Gamma_1) \arrow[r] \arrow[d] & F(\Gamma_1) \arrow[d] \\
\ell k(F)(\Gamma_2) \arrow[r] & F(\Gamma_2)
\end{tikzcd}$$

commutes since we have on elements

$$\begin{tikzcd}
(x, (h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & (F(\sum\limits_{t\in\mathcal{T}} h_t))(x) \arrow[d, mapsto] \\
(x, (f_t \circ h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] & \begin{array}{c} F(f \circ \sum\limits_{t\in\mathcal{T}} h_t)(x) \\ = (F(\sum\limits_{t\in\mathcal{T}} f_t \circ h_t))(x) \end{array}
\end{tikzcd}$$

Let $f : F \to G$ be an arrow in $[\mathrm{Set}/\mathcal{T}, \mathrm{Set}]$. The naturality square

$$\begin{tikzcd}
\ell k(F)(\Gamma) \arrow[r] \arrow[d] & F(\Gamma) \arrow[d] \\
\ell k(G)(\Gamma) \arrow[r] & G(\Gamma)
\end{tikzcd}$$

commutes since we find on elements

$$\begin{tikzcd}
(x,(h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & (F(\sum_{t\in\mathcal{T}} h_t))(x) \arrow[d, mapsto] \\
(f_{\Gamma'}(x),(h_t)_{t\in\mathcal{T}}) \arrow[r, mapsto] & \begin{array}{c} f_\Gamma F(\sum_{t\in\mathcal{T}} h_t)(x) \\ = (F(\sum_{t\in\mathcal{T}} h_t))(f_{\Gamma'}(x)) \end{array}
\end{tikzcd}$$

D.6 Proof of proposition 7.5.133

We check the wedge condition. Let $f_j : \Delta_j \to \Delta'_j$ in $\mathbb{F}\downarrow\mathcal{T}$ for $j = 1,\ldots,m$. The diagram

$$\begin{tikzcd}
& \begin{array}{c} X_t(\Delta)\times Y_{u_1}(\Delta_1) \\ \times\ldots\times Y_{u_m}(\Delta_m) \\ \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{(\Delta_1+\ldots+\Delta_m)^{-1}(t')} \end{array} \arrow[dr] & \\
\begin{array}{c} X_t(\Delta)\times Y_{u_1}(\Delta_1) \\ \times\ldots\times Y_{u_m}(\Delta_m) \\ \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{(\Delta'_1+\ldots+\Delta'_m)^{-1}(t')} \end{array} \arrow[ur] \arrow[dr] & & \ell(X\otimes Y)(\Gamma)^{-1}(t) \\
& \begin{array}{c} X_t(\Delta)\times Y_{u_1}(\Delta'_1) \\ \times\ldots\times Y_{u_m}(\Delta'_m) \\ \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{(\Delta'_1+\ldots+\Delta'_m)^{-1}(t')} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
& \begin{array}{c} (x,y_1,\ldots,y_m, \\ (h_{t'}\circ(\sum_{j=1}^m f_j)_{t'})_{t'\in\mathcal{T}}) \end{array} \arrow[dr, mapsto] & \\
(x,y_1,\ldots,y_m,(h_{t'})_{t'\in\mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} (x,Y_{u_1}(i_1)(y_1),\ldots,Y_{u_m}(i_m)(y_m), \\ (h_{t'}\circ(\sum_{j=1}^m f_j)_{t'})_{t'\in\mathcal{T}}) \\ = (x,Y_{u_1}(i'_1\circ f_1)(y_1),\ldots, \\ Y_{u_m}(i'_m\circ f_m)(y_m),(h_{t'})_{t'\in\mathcal{T}}) \end{array} \\
& \begin{array}{c} (x,Y_{u_1}(f_1)(y_1),\ldots, \\ Y_{u_m}(f_m)(y_m),(h_{t'})_{t'\in\mathcal{T}}) \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

By naturality of the inclusions we have $Y_{u_j}(i'_j\circ f_j)(y_j) = Y_{u_j}(\sum_{j=1}^m f_j\circ i_j)(y_j)$ for all $j = 1,\ldots,m$. The two elements $(x,Y_{u_1}(i_1)(y_1),\ldots,Y_{u_m}(i_m)(y_m),(h_{t'}\circ(\sum_{j=1}^m f_j)_{t'})_{t'\in\mathcal{T}})$ and $(x,Y_{u_1}(\sum_{j=1}^m f_j\circ i_1)(y_1),$

$\ldots, Y_{u_m}(\sum_{j=1}^{m} f_j \circ i_m)(y_m), (h_{t'})_{t' \in \mathcal{T}})$ are equal since they come from $(x, Y_{u_1}(i_1)(y_1), \ldots, Y_{u_m}(i_m)(y_m),$ $(h_{t'})_{t' \in \mathcal{T}})$ with the arrow $\sum_{j=1}^{m} f_j$

$$
\begin{tikzcd}
& \begin{array}{c} X_t(\Delta) \times Y_{u_1}(\sum_{j=1}^{m} \Delta_j) \\ \times \ldots \times Y_{u_m}(\sum_{j=1}^{m} \Delta_j) \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta_1 + \ldots + \Delta_m)^{-1}(t')} \end{array} \arrow[dr] & \\
\begin{array}{c} X_t(\Delta) \times Y_{u_1}(\sum_{j=1}^{m} \Delta_j) \\ \times \ldots \times Y_{u_m}(\sum_{j=1}^{m} \Delta_j) \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta'_1 + \ldots + \Delta'_m)^{-1}(t')} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ \left(\sum_{j=1}^{m} f_j\right)_{t'})_{t' \in \mathcal{T}}"] \arrow[dr, "\mathrm{id} \times Y_{u_1}(\sum_{j=1}^{m} f_j) \times \ldots \times Y_{u_m}(\sum_{j=1}^{m} f_j) \times \mathrm{id}"'] & & \ell(X \otimes Y)(\Gamma)^{-1}(t) \\
& \begin{array}{c} X_t(\Delta) \times Y_{u_1}(\sum_{j=1}^{m} \Delta'_j) \\ \times \ldots \times Y_{u_m}(\sum_{j=1}^{m} \Delta'_j) \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta'_1 + \ldots + \Delta'_m)^{-1}(t')} \end{array} \arrow[ur] &
\end{tikzcd}
$$

Now let $f : \Delta \to \Delta'$ in $\mathbb{F} \downarrow \mathcal{T}$ where $\Delta = (u_1, \ldots, u_m)$ and $\Delta' = (v_1, \ldots, v_k)$. The diagram

$$
\begin{tikzcd}
& \begin{array}{l} X_t(\Delta) \\ \times \int^{\Delta_1} Y_{u_1}(\Delta_1) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta_1^{-1}(t')} \\ \times \ldots \\ \times \int^{\Delta_m} Y_{u_m}(\Delta_m) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta_m^{-1}(t')} \end{array} \arrow[dr] & \\
\begin{array}{l} X_t(\Delta) \\ \times \int^{\Delta_1} Y_{v_1}(\Delta_1) \\ \qquad \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta_1^{-1}(t')} \\ \times \ldots \\ \times \int^{\Delta_k} Y_{v_k}(\Delta_k) \\ \qquad \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta_k^{-1}(t')} \end{array} \arrow[ur] \arrow[dr] & & \ell(X \otimes Y)(\Gamma)^{-1}(t) \\
& \begin{array}{l} X_t(\Delta') \\ \times \int^{\Delta_1} Y_{v_1}(\Delta_1) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta_1^{-1}(t')} \\ \times \ldots \\ \times \int^{\Delta_k} Y_{v_k}(\Delta_k) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta_k^{-1}(t')} \end{array} \arrow[ur] &
\end{tikzcd}
$$

commutes since we have on elements

$$\begin{tikzcd}
 & \begin{array}{c}(x, y_{f_{u_1}}, (h_{t'}^{f_{u_1}})_{t'\in\mathcal{T}},\\ \dots, y_{f_{u_m}}, (h_{t'}^{f_{u_m}})_{t'\in\mathcal{T}})\end{array} \arrow[dr, mapsto] & \\
\begin{array}{c}(x, y_1, (h_{t'}^{1})_{t'\in\mathcal{T}},\\ \dots, y_k, (h_{t'}^{k})_{t'\in\mathcal{T}})\end{array} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}(x, Y_{f_{u_1}}(i'_1)(y_{f_{u_1}}), \dots,\\ Y_{f_{u_m}}(i'_m)(y_{f_{u_m}}), [h_{t'}^{f_{u_1}}, \dots, h_{t'}^{f_{u_m}}]_{t'\in\mathcal{T}})\\ = (X_t(f)(x), Y_{v_1}(i_1)(y_1), \dots,\\ Y_{v_k}(i_k)(y_k), [h_{t'}^{1}, \dots, h_{t'}^{k}]_{t'\in\mathcal{T}})\end{array} \\
 & \begin{array}{c}(X_t(f)(x), y_1, (h_{t'}^{1})_{t'\in\mathcal{T}},\\ \dots, y_k, (h_{t'}^{k})_{t'\in\mathcal{T}})\end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

where $i_j : \Delta_j \to \sum_{j=1}^{k} \Delta_j$ and $i'_j : \Delta_{f_{u_j}} \to \sum_{j=1}^{m} \Delta_{f_{u_j}}$. The arrow $f : (u_1, \dots, u_m) \to (v_1, \dots, v_k)$ induces an arrow $f' : \sum_{j=1}^{m} \Delta_{f_{u_j}} \to \sum_{j=1}^{k} \Delta_j$ such that $i_{f_{u_j}} = f' \circ i'_j$ for all $j = 1, \dots, k$

$$\begin{tikzcd}
 & \begin{array}{l}X_t(\Delta) \times \int^{\Gamma'} Y_{u_1}(\Gamma')\\ \times \dots \times Y_{u_m}(\Gamma')\\ \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{(\Gamma')^{-1}(t')}\end{array} \arrow[dr] & \\
\begin{array}{l}X_t(\Delta) \times \int^{\Gamma'} Y_{v_1}(\Gamma')\\ \times \dots \times Y_{v_k}(\Gamma')\\ \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{(\Gamma')^{-1}(t')}\end{array} \arrow[ur, "\mathrm{id} \times (-\circ f_t)_{t\in\mathcal{T}}"] \arrow[dr, "X_t(f)\times \mathrm{id}"'] & & \ell(X \otimes Y)(\Gamma)^{-1}(t) \\
 & \begin{array}{l}X_t(\Delta') \times \int^{\Gamma'} Y_{v_1}(\Gamma')\\ \times \dots \times Y_{v_k}(\Gamma')\\ \times \prod_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{(\Gamma')^{-1}(t')}\end{array} \arrow[ur] &
\end{tikzcd} \tag{D.1}$$

and we write short $\Gamma'' := \sum_{j=1}^{m} \Delta_{f_{u_j}}$ and $\Gamma' := \sum_{j=1}^{k} \Delta_j$

$$\begin{tikzcd}[column sep=small]
& \begin{gathered} X_t(\Delta) \times Y_{u_1}(\Gamma'') \\ \times \ldots \times Y_{u_m}(\Gamma'') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Gamma'')^{-1}(t')} \end{gathered} \arrow[dr] & \\
\begin{gathered} X_t(\Delta) \times Y_{u_1}(\Gamma'') \\ \times \ldots \times Y_{u_m}(\Gamma'') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Gamma')^{-1}(t')} \end{gathered} \arrow[ur, "\mathrm{id} \times (- \circ f'_{t'})_{t' \in \mathcal{T}}"] \arrow[dr, "\mathrm{id} \times Y_{u_1}(f') \times \ldots \times Y_{u_m}(f') \times \mathrm{id}"'] & & \begin{gathered} X_t(\Delta) \times \int^{\Gamma'} Y_{u_1}(\Gamma') \\ \times \ldots \times Y_{u_m}(\Gamma') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Gamma')^{-1}(t')} \end{gathered} \\
& \begin{gathered} X_t(\Delta) \times Y_{u_1}(\Gamma') \\ \times \ldots \times Y_{u_m}(\Gamma') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Gamma')^{-1}(t')} \end{gathered} \arrow[ur] &
\end{tikzcd} \tag{D.2}$$

So the two elements $(x, Y_{f_{u_1}}(i'_1)(y_{f_{u_1}}), \ldots, Y_{f_{u_m}}(i'_m)(y_{f_{u_m}}), [h_{t'}^{f_{u_1}}, \ldots, h_{t'}^{f_{u_m}}]_{t' \in \mathcal{T}})$ and $(X_t(f)(x), Y_{v_1}(i_1)(y_1), \ldots, Y_{v_k}(i_k)(y_k), [h_{t'}^1, \ldots, h_{t'}^k]_{t' \in \mathcal{T}})$ are equal since

$$(X_t(f)(x), Y_{v_1}(i_1)(y_1), \ldots, Y_{v_k}(i_k)(y_k), [h_{t'}^1, \ldots, h_{t'}^k]_{t' \in \mathcal{T}})$$

equals to by (D.1)

$$(x, Y_{f_{u_1}}(i_{f_{u_1}})(y_{f_{u_1}}), \ldots, Y_{f_{u_m}}(i_{f_{u_m}})(y_{f_{u_m}}), [h_{t'}^1, \ldots, h_{t'}^k]_{t' \in \mathcal{T}}) =$$
$$(x, Y_{f_{u_1}}(f' \circ i'_1)(y_{f_{u_1}}), \ldots, Y_{f_{u_m}}(f' \circ i'_m)(y_{f_{u_m}}), [h_{t'}^1, \ldots, h_{t'}^k]_{t' \in \mathcal{T}})$$

equals to by (D.2)

$$(x, Y_{f_{u_1}}(i'_1)(y_{f_{u_1}}), \ldots, Y_{f_{u_m}}(i'_m)(y_{f_{u_m}}), [h_{t'}^{f_{u_1}}, \ldots, h_{t'}^{f_{u_m}}]_{t' \in \mathcal{T}})$$

Let $f : \Gamma_1 \to \Gamma_2$. The naturality square

$$\begin{tikzcd}
\ell(X)(\ell(Y)(\Gamma_1))^{-1}(t) \arrow[r] \arrow[d] & \ell(X \otimes Y)(\Gamma_1)^{-1}(t) \arrow[d] \\
\ell(X)(\ell(Y)(\Gamma_2))^{-1}(t) \arrow[r] & \ell(X \otimes Y)(\Gamma_2)^{-1}(t)
\end{tikzcd}$$

commutes because we find on elements

$$\begin{tikzcd}
(x, y_1, \ldots, y_m, (h_{t'})_{t' \in \mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & (x, Y_{u_1}(i_1)(y_1), \ldots, Y_{u_m}(i_m)(y_m), (h_{t'})_{t' \in \mathcal{T}}) \arrow[d, mapsto] \\
(x, y_1, \ldots, y_m, (f_{t'} \circ h_{t'})_{t' \in \mathcal{T}}) \arrow[r, mapsto] & (x, Y_{u_1}(i_1)(y_1), \ldots, Y_{u_m}(i_m)(y_m), (f_{t'} \circ h_{t'})_{t' \in \mathcal{T}})
\end{tikzcd}$$

Now let $f : X \to Z$ be an arrow in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The following naturality square commutes

$$\begin{tikzcd}
\ell(X)(\ell(Y)(\Gamma))^{-1}(t) \arrow[r] \arrow[d] & \ell(X \otimes Y)(\Gamma)^{-1}(t) \arrow[d] \\
\ell(Z)(\ell(Y)(\Gamma))^{-1}(t) \arrow[r] & \ell(Z \otimes Y)(\Gamma)^{-1}(t)
\end{tikzcd}$$

because we have the following assignments on elements

$$\begin{tikzcd}
(x, y_1, \ldots, y_m, h) \arrow[r, mapsto] \arrow[d, mapsto] & (x, Y_{u_1}(i_1)(y_1), \ldots, Y_{u_m}(i_m)(y_m), h) \arrow[d, mapsto] \\
(f_\Delta(x), y_1, \ldots, y_m, h) \arrow[r, mapsto] & (f_\Delta(x), Y_{u_1}(i_1)(y_1), \ldots, Y_{u_m}(i_m)(y_m), h)
\end{tikzcd}$$

Let $g : Y \to Z$ be an arrow in $[\mathbb{F} \downarrow \mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The following naturality square commutes

$$\begin{tikzcd}
\ell(X)(\ell(Y)(\Gamma))^{-1}(t) \arrow[r] \arrow[d] & \ell(X \otimes Y)(\Gamma)^{-1}(t) \arrow[d] \\
\ell(X)(\ell(Z)(\Gamma))^{-1}(t) \arrow[r] & \ell(X \otimes Z)(\Gamma)^{-1}(t)
\end{tikzcd}$$

because we have the following assignments on elements

$$\begin{tikzcd}
(x, y_1, \ldots, y_m, h) \arrow[r, mapsto] \arrow[d, mapsto] & (x, Y_{u_1}(i_1)(y_1), \ldots, Y_{u_m}(i_m)(y_m), h) \arrow[d, mapsto] \\
(x, g_{u_1,\Delta_1}(y_1), \ldots, g_{u_m,\Delta_m}(y_m), h) \arrow[r, mapsto] & \begin{array}{c} (x, g_{u_1,\Gamma'}(Y_{u_1}(i_1)(y_1)), \ldots, g_{u_m,\Gamma'}(Y_{u_m}(i_m)(y_m)), h) \\ (x, Z_{u_1}(i_1)(g_{u_1,\Delta_1}(y_1)), \ldots, Z_{u_m}(i_m)(g_{u_m,\Delta_m}(y_1)), h) \end{array}
\end{tikzcd}$$

and by naturality of g_{u_j} for all $j = 1, \ldots, m$ we have $g_{u_j,\Gamma'} \circ Y_{u_j}(i_j) = Z_{u_j}(i_j) \circ g_{u_j,\Delta_j}$.

Now we construct the arrow $\phi : \mathrm{Id}_{\mathrm{Set}\,/\mathcal{T}} \to \ell(\mathcal{Y})$. The component in $\Gamma \in \mathrm{Set}\,/\mathcal{T}$ at the fibre $t \in \mathcal{T}$ is

$$\phi : \Gamma^{-1}(t) \to \int^{\Gamma'} \Gamma'^{-1}(t) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Gamma'^{-1}(t')}$$

we take the following mapping composed with the $\langle t \rangle$-th coprojection

$$x \quad \mapsto \quad 1 \times x^1 \in \langle t \rangle^{-1}(t) \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\langle t \rangle^{-1}(t')}$$

Let $f : \Gamma_1 \to \Gamma_2$. The naturality square

$$\begin{tikzcd}
\Gamma_1^{-1}(t) \arrow[r] \arrow[d] & \ell(\mathcal{Y})(\Gamma_1)^{-1}(t) \arrow[d] \\
\Gamma_2^{-1}(t) \arrow[r] & \ell(\mathcal{Y})(\Gamma_2)^{-1}(t)
\end{tikzcd}$$

commutes because we find on elements

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & (1, x^1) \arrow[d, mapsto] \\
f_t(x) \arrow[r, mapsto] & (1, f_t(x)^1)
\end{tikzcd}$$

It remains to check the monoidal functor axioms.

1. We check the commutativity of the following diagram for all $\Gamma \in \mathrm{Set}\,/\mathcal{T}$ and $t \in \mathcal{T}$

$$\begin{tikzcd}
\mathrm{Id}_{\mathrm{Set}\,/\mathcal{T}}(\ell X(\Gamma))^{-1}(t) \arrow[r, "\phi \circ \ell X"] \arrow[ddr, "\mathrm{Id}"'] & \ell\mathcal{Y}(\ell X(\Gamma))^{-1}(t) \arrow[d, "\phi_{\mathcal{Y},X}"] \\
& \ell(\mathcal{Y} \otimes X)(\Gamma)^{-1}(t) \arrow[d, "\ell(\lambda_X)"] \\
& \ell X(\Gamma)^{-1}(t)
\end{tikzcd}$$

we find on elements

$$
\begin{array}{cc}
\int^{\Delta'} X_t(\Delta') \times \prod\limits_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')} & (x,(h_{t'})_{t'\in\mathcal{T}}) \\
\Big\downarrow{\scriptstyle \Gamma'=\langle t\rangle, \Delta_1=\Delta'} & \Big\downarrow \\
\begin{array}{l}
\int^{\Gamma'=(u_1,\ldots,u_m)} \Gamma^{-1}(t)\times \\
\quad \times \int^{\Delta_1} X_{u_1}(\Delta_1) \times \prod\limits_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{\Delta_1^{-1}(t')} \times \\
\quad \times \ldots \times \\
\quad \times \int^{\Delta_m} X_{u_m}(\Delta_m) \times \prod\limits_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{\Delta_m^{-1}(t')}
\end{array} & (1,(x,(h_{t'})_{t'\in\mathcal{T}})^1) \\
\Big\downarrow{\scriptstyle \Delta'=\Delta_1, \Gamma'=\langle t\rangle} & \Big\downarrow \\
\begin{array}{l}
\int^{\Delta'} \int^{\Gamma'=(u_1,\ldots,u_m)} \Gamma^{-1}(t) \times X_{u_1}(\Delta') \times X_{u_m}(\Delta') \times \\
\quad \times \prod\limits_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')}
\end{array} & (1,(x,(h_{t'})_{t'\in\mathcal{T}})^1) \\
\Big\downarrow & \Big\downarrow \\
\int^{\Delta'} X_t(\Delta') \times \prod\limits_{t'\in\mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')} & (x,(h_{t'})_{t'\in\mathcal{T}})
\end{array}
$$

which is the identity on $(x,(h_{t'})_{t'\in\mathcal{T}})$.

2. We check the commutativity of the following diagram for all $\Gamma \in \mathrm{Set}/\mathcal{T}$ and $t \in \mathcal{T}$

$$
\begin{tikzcd}
\ell X(\mathrm{Id}_{\mathrm{Set}/\mathcal{T}}(\Gamma))^{-1}(t) \arrow[r, "\ell X(\phi)"] \arrow[ddr, "\rho_{\ell X}"'] & \ell X(\ell\mathcal{Y}(\Gamma))^{-1}(t) \arrow[d, "\phi_{X,\mathcal{Y}}"] \\
& \ell(X\otimes\mathcal{Y})(\Gamma)^{-1}(t) \arrow[d, "\ell(\rho_X)"] \\
& \ell X(\Gamma)^{-1}(t)
\end{tikzcd}
$$

We find on elements

$$
\begin{tikzcd}
\int^{\Delta=(u_1,\ldots,u_m)} X_t(\Delta)\times\Gamma^{-1}(u_1)\times\ldots\times\Gamma^{-1}(u_m) \arrow[d, "\Delta_j=\langle u_j\rangle"] & (x,a_1,\ldots,a_m) \arrow[d, mapsto] \\
\begin{array}{l} \int^{\Delta} X_t(\Delta)\times\int^{\Delta_1}\Delta_1^{-1}(u_1)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_1^{-1}(t')} \\ \qquad \times\ldots \\ \qquad \times\int^{\Delta_m}\Delta_m^{-1}(u_m)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_m^{-1}(t')} \end{array} \arrow[d, "\Gamma'=\sum_{j=1}^{m}\langle u_j\rangle=\Delta"] & \begin{array}{l}(x,1\times a_1^1,\\ \ldots,\\ 1\times a_m^1)\end{array} \arrow[d, mapsto] \\
\begin{array}{l} \int^{\Gamma'}\int^{\Delta} X_t(\Delta)\times\Gamma'^{-1}(u_1)\times\ldots\times\Gamma'^{-1}(u_m) \\ \qquad \times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma'^{-1}(t')} \\ =\int^{\Gamma'}\int^{\Delta} X_t(\Delta)\times\prod\limits_{t'\in\mathcal{T}}\Gamma'^{-1}(t')^{\Delta^{-1}(t')} \\ \qquad \times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma'^{-1}(t')} \end{array} \arrow[d] & \begin{array}{l}(x,i_1(1),\ldots,i_m(1),\\ (h_{u_j}:1\mapsto a_j)_j)\\ =(x,(\mathrm{id}_{\Delta^{-1}(t')})_{t'\in\mathcal{T}},\\ (h_{u_j}:1\mapsto a_j)_j)\end{array} \arrow[d, mapsto] \\
\int^{\Gamma'} X_t(\Gamma')\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma'^{-1}(t')} & \begin{array}{l}(X_t(\sum\limits_{t'\in\mathcal{T}}\mathrm{id}_{\Delta^{-1}(t')})(x)=x,\\ (h_{u_j}:1\mapsto a_j)_j)\end{array}
\end{tikzcd}
$$

which is the identity on $(x,a_1,\ldots,a_m)$.

3. We check the commutativity of the following diagram for all $X,Y,Z\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$, $\Gamma\in\mathrm{Set}/\mathcal{T}$ and $t\in\mathcal{T}$

$$
\begin{tikzcd}
((\ell X\circ\ell Y)\circ\ell Z)(\Gamma)^{-1}(t) \arrow[r, equal] \arrow[d, "\phi_{X,Y}\circ\ell Z"'] & (\ell X\circ(\ell Y\circ\ell Z))(\Gamma)^{-1}(t) \arrow[d, "\ell X\circ\phi_{Y,Z}"] \\
(\ell(X\otimes Y)\circ\ell Z)(\Gamma)^{-1}(t) \arrow[d, "\phi_{X\otimes Y,Z}"'] & (\ell X\circ\ell(Y\otimes Z))(\Gamma)^{-1}(t) \arrow[d, "\phi_{X,Y\otimes Z}"] \\
\ell((X\otimes Y)\otimes Z)(\Gamma)^{-1}(t) \arrow[r, "\ell\alpha_{X,Y,Z}"'] & \ell(X\otimes(Y\otimes Z))(\Gamma)^{-1}(t)
\end{tikzcd}
$$

Along the left–hand side we have the following composite

$$
\begin{array}{l}
\int^{\Delta}\int^{\Delta_1}\int^{\Xi_{1,1}}\dots\int^{\Xi_{1,k_1}}\dots\int^{\Delta_m}\int^{\Xi_{m,1}}\dots\int^{\Xi_{m,k_m}} \\
X_t(\Delta)\times Y_{u_1}(\Delta_1)\times\dots\times Y_{u_m}(\Delta_m) \\
\times Z_{v_{1,1}}(\Xi_{1,1})\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{1,1})^{-1}(t')} \\
\qquad\times\dots \\
\qquad\times Z_{v_{1,k_1}}(\Xi_{1,k_1})\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{1,k_1})^{-1}(t')} \\
\times\dots \\
\times Z_{v_{m,1}}(\Xi_{m,1})\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{m,1})^{-1}(t')} \\
\qquad\times\dots \\
\qquad\times Z_{v_{m,k_m}}(\Xi_{m,k_m})\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{m,k_m})^{-1}(t')} \\
\Big\downarrow \\
\int^{\Gamma'}\int^{\Delta}\int^{\Xi_1}\dots\int^{\Xi_\ell} \\
X_t(\Delta)\times Y_{u_1}(\Gamma')\times\dots\times Y_{u_m}(\Gamma') \\
\times Z_{w_1}(\Xi_1)\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_1)^{-1}(t')} \\
\times\dots \\
\times Z_{w_\ell}(\Xi_\ell)\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_\ell)^{-1}(t')} \\
\Big\downarrow \\
\int^{\Delta'}\int^{\Gamma'}\int^{\Delta}X_t(\Delta) \\
\times Y_{u_1}(\Gamma')\times\dots\times Y_{u_m}(\Gamma') \\
\times Z_{w_1}(\Delta')\times\dots\times Z_{w_\ell}(\Delta') \\
\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Delta')^{-1}(t')} \\
\Big\downarrow \\
\int^{\Delta'}\int^{\Delta}\int^{\Gamma_1}\dots\int^{\Gamma_m}X_t(\Delta) \\
\times Y_{u_1}(\Gamma_1)\times Z_{t_{1,1}}(\Delta')\times\dots\times Z_{t_{1,s_1}}(\Delta') \\
\times\dots \\
\times Y_{u_m}(\Gamma_m)\times Z_{t_{m,1}}(\Delta')\times\dots\times Z_{t_{m,s_m}}(\Delta') \\
\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Delta')^{-1}(t')}
\end{array}
\qquad
\begin{array}{c}
x, y_1,\dots,y_m, \\
z_{1,1}, w_{1,1},\dots,z_{1,k_1}, w_{1,k_1}, \\
\dots, \\
z_{m,1}, w_{m,1},\dots,z_{m,k_m}, w_{m,k_m} \\
\Big\downarrow{\scriptstyle \Gamma'=\Delta_1+\ldots+\Delta_m,\ \ell=k_1+\ldots+k_m} \\
x, \overline{y}_1,\dots,\overline{y}_m, \\
z_{1,1}, w_{1,1},\dots,z_{1,k_1}, w_{1,k_1}, \\
\dots, \\
z_{m,1}, w_{m,1},\dots,z_{m,k_m}, w_{m,k_m} \\
\Big\downarrow{\scriptstyle \Delta'=\sum_{j=1}^{\ell}\Xi_j} \\
x, \overline{y}_1,\dots,\overline{y}_m, \\
\overline{z}_{1,1},\dots,\overline{z}_{1,k_1},\dots,\overline{z}_{m,1},\dots,\overline{z}_{m,k_m}, \\
[w_{1,1},\dots,w_{m,k_m}] \\
\Big\downarrow{\scriptstyle \Gamma_i=\Gamma'} \\
x, \\
\overline{y}_1, \overline{z}_{1,1},\dots,\overline{z}_{m,k_m}, \\
\dots, \\
\overline{y}_m, \overline{z}_{1,1},\dots,\overline{z}_{m,k_m}, \\
[w_{1,1},\dots,w_{m,k_m}]
\end{array}
$$

where we write $\Delta = (u_1,\dots,u_m)$, $\Delta_j = (v_{j,1},\dots,v_{j,k_j})$ for all $j = 1,\dots,m$, $\Gamma' = (w_1,\dots,w_\ell)$ and $\Gamma_i = (t_{i,1},\dots,t_{i,s_i})$.

Along the right–hand side we have the following composite

$$
\begin{array}{l}
\int^{\Delta}\int^{\Delta_1}\int^{\Xi_{1,1}}\dots\int^{\Xi_{1,k_1}}\dots\int^{\Delta_m}\int^{\Xi_{m,1}}\dots\int^{\Xi_{m,k_m}} \\
X_t(\Delta)\times Y_{u_1}(\Delta_1)\times\dots\times Y_{u_m}(\Delta_m) \\
\times Z_{v_{1,1}}(\Xi_{1,1})\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{1,1})^{-1}(t')} \\
\qquad\times\dots \\
\qquad\times Z_{v_{1,k_1}}(\Xi_{1,k_1})\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{1,k_1})^{-1}(t')} \\
\times\dots \\
\times Z_{v_{m,1}}(\Xi_{m,1})\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{m,1})^{-1}(t')} \\
\qquad\times\dots \\
\qquad\times Z_{v_{m,k_m}}(\Xi_{m,k_m})\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Xi_{m,k_m})^{-1}(t')} \\
\Big\downarrow \\
\int^{\Delta}\int^{\Delta_1}\int^{\Gamma_1}\dots\int^{\Delta_m}\int^{\Gamma_m}X_t(\Delta) \\
\times Y_{u_1}(\Delta_1)\times Z_{v_{1,1}}(\Gamma_1)\times Z_{v_{1,k_1}}(\Gamma_1) \\
\qquad\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Gamma_1)^{-1}(t')} \\
\times\dots \\
\times Y_{u_m}(\Delta_m)\times Z_{v_{m,1}}(\Gamma_m)\times\dots\times Z_{v_{m,k_m}}(\Gamma_m) \\
\qquad\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Gamma_m)^{-1}(t')} \\
\Big\downarrow \\
\int^{\Delta'}\int^{\Delta}\int^{\Delta_1}\dots\int^{\Delta_m}X_t(\Delta) \\
\times Y_{u_1}(\Delta_1)\times\dots\times Y_{u_m}(\Delta_m) \\
\times Z_{v_{1,1}}(\Delta')\times\dots\times Z_{v_{1,k_1}}(\Delta')\times\dots\times Z_{v_{m,k_m}}(\Delta') \\
\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Delta')^{-1}(t')}
\end{array}
$$

$$
\begin{array}{c}
x, y_1,\dots,y_m, \\
z_{1,1}, w_{1,1},\dots,z_{1,k_1},w_{1,k_1}, \\
\dots, \\
z_{m,1}, w_{m,1},\dots,z_{m,k_m},w_{m,k_m} \\
\Big\downarrow{\scriptstyle \Gamma_i=\Xi_{i,1}+\dots+\Xi_{i,m_i}} \\
x, \\
y_1,\widehat{z}_{1,1},\dots,\widehat{z}_{1,k_1}, \\
[w_{1,1},\dots,w_{m,1}], \\
\dots, \\
y_m,\widehat{z}_{m,1},\dots,\widehat{z}_{m,k_m}, \\
[w_{m,1},\dots,w_{m,k_m}] \\
\Big\downarrow{\scriptstyle \Delta'=\Gamma_1+\dots+\Gamma_m} \\
x, y_1,\dots,y_m, \\
\overline{z}_{1,1},\dots,\overline{z}_{1,k_1},\dots,\overline{z}_{m,k_m} \\
[w_{1,1},\dots,w_{m,1},\dots,w_{m,k_m}]
\end{array}
$$

The two results are equal since they come from with the arrows $c_i : \Delta_i \to \Delta_1 + \ldots + \Delta_m$

$$\begin{tikzcd}[column sep=tiny]
 & \begin{array}{l} X_t(\Delta) \\ \times Y_{u_1}(\Delta_1) \times Z_{v_{1,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{1,k_1}}(\Delta') \\ \times \ldots \\ \times Y_{u_m}(\Delta_m) \times Z_{v_{m,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{m,k_m}}(\Delta') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta')^{-1}(t')} \end{array} \arrow[dr] & \\
\begin{array}{l} X_t(\Delta) \\ \times Y_{u_1}(\Delta_1) \times Z_{v_{1,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{m,k_m}}(\Delta') \\ \times \ldots \\ \times Y_{u_m}(\Delta_m) \times Z_{v_{1,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{m,k_m}}(\Delta') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta')^{-1}(t')} \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{l} X_t(\Delta) \\ \times \int^{\Delta_1} Y_{u_1}(\Delta_1) \times Z_{v_{1,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{1,k_1}}(\Delta') \\ \times \ldots \\ \times \int^{\Delta_m} Y_{u_m}(\Delta_m) \times Z_{v_{m,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{m,k_m}}(\Delta') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta')^{-1}(t')} \end{array} \\
 & \begin{array}{l} X_t(\Delta) \\ \times Y_{u_1}(\sum_{i=1}^{m} \Delta_i) \times Z_{v_{1,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{m,k_m}}(\Delta') \\ \times \ldots \\ \times Y_{u_m}(\sum_{i=1}^{m} \Delta_i) \times Z_{v_{1,1}}(\Delta') \\ \quad \times \ldots \times Z_{v_{m,k_m}}(\Delta') \\ \times \prod_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{(\Delta')^{-1}(t')} \end{array} \arrow[ur] &
\end{tikzcd}$$

D.7 Proof of proposition 7.5.134

We check the wedge condition. Let $g : \Gamma' \to \Gamma''$ in $\mathbb{F} \downarrow \mathcal{T}$. The diagram

$$\begin{tikzcd}
 & \begin{array}{l} F(\Gamma')^{-1}(t) \\ \times \prod_{u \in \mathcal{T}} G(\Delta)^{-1}(u)^{\Gamma'^{-1}(u)} \end{array} \arrow[dr] & \\
\begin{array}{l} F(\Gamma')^{-1}(t) \\ \times \prod_{u \in \mathcal{T}} G(\Delta)^{-1}(u)^{\Gamma''^{-1}(u)} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ g_u)"] \arrow[dr, "F(g)_t \times \mathrm{id}"'] & & (F \circ G)(\Delta)^{-1}(t) \\
 & \begin{array}{l} F(\Gamma'')^{-1}(t) \\ \times \prod_{u \in \mathcal{T}} G(\Delta)^{-1}(u)^{\Gamma''^{-1}(u)} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & (x,(h_u \circ g_u)_{u\in\mathcal{T}}) \arrow[dr, mapsto] & \\
(x,(h_u)_{u\in\mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & F\big(\sum_{u\in\mathcal{T}} h_u \circ g\big)_t(x) \\
 & (F(g)_t(x),(h_u)_{u\in\mathcal{T}}) \arrow[ur, mapsto] &
\end{tikzcd}$$

Let $f:\Delta_1\to\Delta_2$ be an arrow in $\mathbb{F}\downarrow\mathcal{T}$. The naturality square

$$\begin{tikzcd}
(k(F)\otimes k(G))(\Delta_1)^{-1}(t) \arrow[r] \arrow[d] & (F\circ G)(\Delta_1)^{-1}(t) \arrow[d] \\
(k(F)\otimes k(G))(\Delta_2)^{-1}(t) \arrow[r] & (F\circ G)(\Delta_2)^{-1}(t)
\end{tikzcd}$$

commutes because on elements we find

$$\begin{tikzcd}
(x,(h_u)_{u\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & F\Big(\sum_{u\in\mathcal{T}} h_u\Big)_t(x) \arrow[d, mapsto] \\
(x,(G(f)_u \circ h_u)_{u\in\mathcal{T}}) \arrow[r, mapsto] & F(G(f))_t \circ F\Big(\sum_{u\in\mathcal{T}} h_u\Big)_t(x)
\end{tikzcd}$$

Now let $f:F\to H$ be an arrow in $[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$. The following naturality square commutes

$$\begin{tikzcd}
(kF\otimes kG)(\Delta)^{-1}(t) \arrow[r] \arrow[d] & k(F\circ G)(\Delta)^{-1} \arrow[d] \\
(kH\otimes kG)(\Delta)^{-1}(t) \arrow[r] & k(H\circ G)(\Delta)^{-1}
\end{tikzcd}$$

because we have the following assignments on elements

$$\begin{tikzcd}
(x,(h_u)_{u\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & F\Big(\sum_{u\in\mathcal{T}} h_u\Big)_t(x) \arrow[d, mapsto] \\
((f_{\Gamma'})_t(x),(h_u)_{u\in\mathcal{T}}) \arrow[r, mapsto] & \begin{array}{c}(f_{G\Delta})_t\Big(F\Big(\sum_{u\in\mathcal{T}} h_u\Big)_t(x)\Big) \\ = H\Big(\sum_{u\in\mathcal{T}} h_u\Big)_t(f_{\Gamma'})_t(x)\end{array}
\end{tikzcd}$$

and by naturality of f we have $f_{G\Delta}\circ F(\sum_{u\in\mathcal{T}} h_u) = H(\sum_{u\in\mathcal{T}} h_u)\circ f_{\Gamma'}$. Let $g:G\to K$ be an arrow in $[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$. The following naturality square commutes

$$\begin{tikzcd}
(kF\otimes kG)(\Delta)^{-1}(t) \arrow[r] \arrow[d] & k(F\circ G)(\Delta)^{-1} \arrow[d] \\
(kH\otimes kK)(\Delta)^{-1}(t) \arrow[r] & k(F\circ K)(\Delta)^{-1}
\end{tikzcd}$$

because we have the following assignments on elements

$$\begin{tikzcd}
(x,(h_u)_{u\in\mathcal{T}}) \arrow[r, mapsto] \arrow[d, mapsto] & F\Big(\sum\limits_{u\in\mathcal{T}} h_u\Big)_t(x) \arrow[d, mapsto] \\
(x,((g_\Delta)_u\circ h_u)_{u\in\mathcal{T}}) \arrow[r, mapsto] & \begin{array}{l}(Fg_\Delta)_t\Big(F\Big(\sum\limits_{u\in\mathcal{T}} h_u\Big)_t(x)\Big)\\ = F\Big(\sum\limits_{u\in\mathcal{T}}(g_\Delta)_u\circ h_u\Big)_t(x)\end{array}
\end{tikzcd}$$

The arrow $\psi:\mathcal{Y}\to k(\mathrm{Id})$ is given by the identity on $\mathcal{Y}$.

It remains to check the monoidal functor axioms.

1. We check the commutativity of the following diagram for all $F\in[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$, $\Delta\in\mathbb{F}\downarrow\mathcal{T}$ and $t\in\mathcal{T}$

$$\begin{tikzcd}
(\mathcal{Y}\otimes kF)_t(\Delta) \arrow[r, "\psi\otimes kF"] \arrow[ddr, "\lambda_{kF}"'] & (k\,\mathrm{Id}\otimes kF)_t(\Delta) \arrow[d, "\psi_{\mathrm{Id},F}"] \\
& (k(\mathrm{Id}\otimes F))_t(\Delta) \arrow[d, "k(\lambda_F)"] \\
& (kF)_t(\Delta)
\end{tikzcd}$$

we find on elements

$$\begin{tikzcd}
\int^{\Gamma'}\Gamma'^{-1}(t)\times\prod\limits_{t'\in\mathcal{T}}F\Delta^{-1}(t')^{\Gamma'^{-1}(t')} \arrow[d] & (x,(h_{t'})_{t'\in\mathcal{T}}) \arrow[d, mapsto] \\
\int^{\Gamma'}\Gamma'^{-1}(t)\times\prod\limits_{t'\in\mathcal{T}}F\Delta^{-1}(t')^{\Gamma'^{-1}(t')} \arrow[d] & (x,(h_{t'})_{t'\in\mathcal{T}}) \arrow[d, mapsto] \\
F\Delta^{-1}(t) \arrow[d] & h_t(x) \arrow[d, mapsto] \\
F\Delta^{-1}(t) & h_t(x)
\end{tikzcd}$$

which is λ_{kF}.

2. We check the commutativity of the following diagram for all $F\in[\mathrm{Set}/\mathcal{T},\mathrm{Set}/\mathcal{T}]$, $\Delta\in\mathbb{F}\downarrow\mathcal{T}$ and $t\in\mathcal{T}$

$$\begin{tikzcd}
(kF\otimes\mathcal{Y})_t(\Delta) \arrow[r, "kF\otimes\psi"] \arrow[ddr, "\rho_{kF}"'] & (kF\otimes k\,\mathrm{Id})_t(\Delta) \arrow[d, "\psi_{F,\mathrm{Id}}"] \\
& (k(F\circ\mathrm{Id}))_t(\Delta) \arrow[d, "k\rho_F"] \\
& (kF)_t(\Delta)
\end{tikzcd}$$

we find on elements

$$\begin{array}{ccc}
\int \Gamma' F(\Gamma')^{-1}(t) \times \prod_{t'\in\mathcal{T}} \Delta^{-1}(t')^{\Gamma'^{-1}(t')} & & (x,(h_{t'})_{t'\in\mathcal{T}}) \\
\downarrow & & \downarrow \\
\int \Gamma' F(\Gamma')^{-1}(t) \times \prod_{t'\in\mathcal{T}} \Delta^{-1}(t')^{\Gamma'^{-1}(t')} & & (x,(h_{t'})_{t'\in\mathcal{T}}) \\
\downarrow & & \downarrow \\
F\Delta^{-1}(t) & & F(\sum_{t'\in\mathcal{T}} h_{t'})(x) \\
\downarrow & & \downarrow \\
F\Delta^{-1}(t) & & F(\sum_{t'\in\mathcal{T}} h_{t'})(x)
\end{array}$$

which is ρ_{kF}.

3. We check the commutativity of the following diagram for all $F,G,H \in [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$, $\Delta \in \mathbb{F}\downarrow\mathcal{T}$ and $t\in\mathcal{T}$

$$\begin{tikzcd}
((kF\otimes kG)\otimes kH)_t(\Delta) \arrow[r, "\alpha_{kF,kG,kH}"] \arrow[d, "\psi_{F,G}\otimes kH"'] & (kF\otimes(kG\otimes kH))_t(\Delta) \arrow[d, "kF\otimes\psi_{G,H}"] \\
(k(F\circ G)\otimes kF)_t(\Delta) \arrow[d, "\psi_{F\circ G,H}"'] & (kF\otimes k(G\circ H))_t(\Delta) \arrow[d, "\psi_{F,G\circ H}"] \\
k((F\circ G)\circ H)_t(\Delta) \arrow[r, equal] & k(F\circ(G\circ H))_t(\Delta)
\end{tikzcd}$$

Along the left–hand side we have the following assignations on elements

$$\begin{array}{ccc}
\begin{array}{l}\int^{\Gamma''}\int^{\Gamma'} F\Gamma'^{-1}(t) \\ \times \prod_{u\in\mathcal{T}} G\Gamma''^{-1}(u)^{\Gamma'^{-1}(u)} \\ \times \prod_{u\in\mathcal{T}} H\Delta^{-1}(u)^{\Gamma''^{-1}(u)}\end{array} & & x,(g_u)_{u\in\mathcal{T}},(h_u)_{u\in\mathcal{T}} \\
\downarrow & & \downarrow \\
\begin{array}{l}\int^{\Gamma''} FG(\Gamma'')^{-1}(t) \\ \times \prod_{u\in\mathcal{T}} H\Delta^{-1}(u)^{\Gamma''^{-1}(u)}\end{array} & & F\big(\sum_{u\in\mathcal{T}} g_u\big)_t(x),(h_u)_{u\in\mathcal{T}} \\
\downarrow & & \downarrow \\
FGH\Delta^{-1}(t) & & FG\big(\sum_{u\in\mathcal{T}} h_u\big)_t \circ F\big(\sum_{u\in\mathcal{T}} g_u\big)_t(x)
\end{array}$$

Along the right–hand side we have the following assignations on elements

$$
\begin{array}{ccc}
\begin{array}{l}\int^{\Gamma''}\int^{\Gamma'} F\Gamma'^{-1}(t)\\ \times G\Gamma''^{-1}(u_1)\times\ldots\times G\Gamma''^{-1}(u_m)\\ \times\prod\limits_{u\in\mathcal{T}} H\Delta^{-1}(u)^{\Gamma''^{-1}(u)}\end{array} & & x, y_1, \ldots, y_m, (h_u)_{u\in\mathcal{T}}\\
\downarrow & & \downarrow\\
\begin{array}{l}\int^{\Gamma''}\int^{\Delta_1}\ldots\int^{\Delta_m} F\Gamma'^{-1}(t)\\ \times G\Delta_1^{-1}(u_1)\times\prod\limits_{u\in\mathcal{T}} H\Delta^{-1}(u)^{\Delta_1^{-1}(u)}\\ \times\ldots\\ \times G\Delta_m^{-1}(u_m)\times\prod\limits_{u\in\mathcal{T}} H\Delta^{-1}(u)^{\Delta_m^{-1}(u)}\end{array} & & x, y_1, (h_u)_{u\in\mathcal{T}}, \ldots, y_m, (h_u)_{u\in\mathcal{T}}\\
\downarrow & & \downarrow\\
\begin{array}{l}\int^{\Gamma''} F\Gamma'^{-1}(t)\\ \times GH\Delta^{-1}(u_1)\times\ldots\times GH\Delta^{-1}(u_m)\\ =\int^{\Gamma''} F\Gamma'^{-1}(t)\times\prod\limits_{u\in\mathcal{T}} GH\Delta^{-1}(u)^{\Gamma''^{-1}(u)}\end{array} & & \begin{array}{c}x, G\big(\sum\limits_{u\in\mathcal{T}} h_u\big)_{u_1}(y_1), \ldots, G\big(\sum\limits_{u\in\mathcal{T}} h_u\big)_{u_m}(y_m)\\ = x, G\big(\sum\limits_{u\in\mathcal{T}} h_u\big)\circ\sum\limits_{u\in\mathcal{T}} g_u\end{array}\\
\downarrow & & \downarrow\\
FGH\Delta^{-1}(t) & & F\Big(G\big(\sum\limits_{u\in\mathcal{T}} h_u\big)\circ\sum\limits_{u\in\mathcal{T}} g_u\Big)_t(x)
\end{array}
$$

The two sides yield the same result, so the diagram is commutative.

D.8 Proof of proposition 7.5.135

We check the two monoidal natural transformation axioms.

1. First we check the commutativity of the following diagram for all $\Delta\in\mathbb{F}\downarrow\mathcal{T}$ and $t\in\mathcal{T}$

$$
\begin{tikzcd}
 & \mathcal{Y}_t(\Delta) \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\eta_{\mathcal{Y}}"] & \\
(k\,\mathrm{Id})_t(\Delta) \arrow[rr, "k\phi"'] & & (k\ell\mathcal{Y})_t(\Delta)
\end{tikzcd}
$$

We have explicitly on elements

$$k\phi : x\in\Delta^{-1}(t) \quad\mapsto\quad (1, x^1)\in\langle t\rangle^{-1}(t)\times\prod_{t'\in\mathcal{T}}\Delta^{-1}(t')^{\langle t\rangle^{-1}(t')}$$

and

$$\eta_{\mathcal{Y}} : x\in\Delta^{-1}(t) \quad\mapsto\quad (x, (\mathrm{id}_{\Delta^{-1}(t')})_{t'\in\mathcal{T}})\in\Delta^{-1}(t)\times\prod_{t'\in\mathcal{T}}\Delta^{-1}(t')^{\Delta^{-1}(t')}$$

These two are equal since they come from $\langle t\rangle^{-1}(t)\times\prod_{t'\in\mathcal{T}}\Delta^{-1}(t')^{\Delta^{-1}(t')}$ with the arrow $x:\langle t\rangle\to\Delta$

$$
\begin{tikzcd}
 & \begin{array}{c}\Delta^{-1}(t)\times\prod\limits_{t'\in\mathcal{T}}\Delta^{-1}(t')^{\Delta^{-1}(t')}\\ (x,(\mathrm{id}_{\Delta^{-1}(t')})_{t'\in\mathcal{T}})\end{array} \\
\begin{array}{c}\langle t\rangle^{-1}(t)\times\prod\limits_{t'\in\mathcal{T}}\Delta^{-1}(t')^{\Delta^{-1}(t')}\\ (1,(\mathrm{id}_{\Delta^{-1}(t')})_{t'\in\mathcal{T}})\end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{c}\langle t\rangle^{-1}(t)\times\prod\limits_{t'\in\mathcal{T}}\Delta^{-1}(t')^{\langle t\rangle^{-1}(t')}\\ (1,x^1)\end{array}
\end{tikzcd}
$$

2. Next we check the commutativity of the following diagram for all $X,Y\in[\mathbb{F}\downarrow\mathcal{T},\mathrm{Set}]^{\mathcal{T}}$, $\Delta\in\mathbb{F}\downarrow\mathcal{T}$ and $t\in\mathcal{T}$

$$
\begin{tikzcd}
(X\otimes Y)_t(\Delta) \arrow[r, "\eta_X\otimes\eta_Y"] \arrow[d, "\eta_{X\otimes Y}"'] & (k\ell X\otimes k\ell Y)_t(\Delta) \arrow[d, "\psi_{\ell X,\ell Y}"] \\
k\ell(X\otimes Y)_t(\Delta) & (k(\ell X\circ\ell Y))_t(\Delta) \arrow[l, "k\phi_{X,Y}"]
\end{tikzcd}
$$

On elements along the right–hand side we find

$$
\begin{tikzcd}
\int^{\Delta=(u_1,\dots,u_m)} X_t(\Delta)\times Y_{u_1}(\Gamma)\times\dots\times Y_{u_m}(\Gamma) \arrow[d, "{\Gamma'=\Delta,\Delta_i=\Gamma}"] & (x,a_1,\dots,a_m) \arrow[d, mapsto] \\
\begin{array}{l}\int^{\Delta=(u_1,\dots,u_m)}\int^{\Gamma'} X_t(\Gamma')\times\prod\limits_{t'\in\mathcal{T}}\Delta^{-1}(t')^{\Gamma'^{-1}(t')}\\ \quad\times\int^{\Delta_1} Y_{u_1}(\Delta_1)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_1^{-1}(t')}\\ \quad\times\dots\\ \quad\times\int^{\Delta_m} Y_{u_m}(\Delta_m)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_m^{-1}(t')}\end{array} \arrow[d] & \begin{array}{l}((x,(\mathrm{id}_{\Delta^{-1}(t')})_{t'\in\mathcal{T}}),\\ (a_1,(\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}}),\\ \dots,\\ (a_m,(\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}}))\end{array} \arrow[d, mapsto] \\
\begin{array}{l}\int^{\Delta=(u_1,\dots,u_m)} X_t(\Delta)\\ \quad\times\int^{\Delta_1} Y_{u_1}(\Delta_1)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_1^{-1}(t')}\\ \quad\times\dots\\ \quad\times\int^{\Delta_m} Y_{u_m}(\Delta_m)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_m^{-1}(t')}\end{array} \arrow[d, "\Delta'=\sum\Delta_i"] & \begin{array}{l}(X_t(\mathrm{id}_\Delta)(x),\\ (a_1,(\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}}),\\ \dots,\\ (a_m,(\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}}))\end{array} \arrow[d, mapsto] \\
\begin{array}{l}\int^{\Delta'}\int^{\Delta=(u_1,\dots,u_m)} X_t(\Delta)\\ \qquad\times Y_{u_1}(\Delta')\times\dots\times Y_{u_m}(\Delta')\\ \qquad\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta'^{-1}(t')}\end{array} & \begin{array}{l}(x,\\ Y_{u_1}(i_1)(a_1),\dots,Y_{u_m}(i_m)(a_m),\\ (\sum\limits_{j=1}^{m}\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}})\end{array}
\end{tikzcd}
$$

On elements along the left–hand side we have

$$
\begin{array}{ccc}
\int^{\Delta=(u_1,\dots,u_m)} X_t(\Delta)\times Y_{u_1}(\Gamma)\times\ldots\times Y_{u_m}(\Gamma) & & (x,a_1,\dots,a_m) \\
\Big\downarrow{\scriptstyle \Delta'=\Delta} & & \Big\downarrow \\
\begin{array}{l}\int^{\Delta'}\int^{\Delta=(u_1,\dots,u_m)} X_t(\Delta)\\ \quad\times Y_{u_1}(\Delta')\times\ldots\times Y_{u_m}(\Delta')\\ \quad\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta'^{-1}(t')}\end{array} & & \begin{array}{l}(x,\\ a_1,\dots,a_m,\\ (\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}})\end{array}
\end{array}
$$

This is identity because the two elements come from $(X\otimes Y)_t(m\times\Gamma)\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma^{-1}(t')}$ with the arrow $p:=\langle i_1,\dots,i_m\rangle:\Gamma+\ldots+\Gamma\to\Gamma=m\times\Gamma\to\Gamma$.

$$
\begin{tikzcd}
 & \begin{array}{c}(X\otimes Y)_t(m\times\Gamma)\times\\ \times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(m\times\Gamma)^{-1}(t')}\\ (x,Y_{u_1}(i_1)(a_1),\dots,Y_{u_m}(i_m)(a_m),\\ (\sum_{j=1}^{m}\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}})\end{array} \\
\begin{array}{c}(X\otimes Y)_t(m\times\Gamma)\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma^{-1}(t')}\\ (x,Y_{u_1}(i_1)(a_1),\dots,Y_{u_m}(i_m)(a_m),\\ (\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}})\end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{l}(X\otimes Y)_t(\Gamma)\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Gamma^{-1}(t')}\\ ((X\otimes Y)_t(p)(x,Y_{u_1}(i_1)(a_1),\\ \qquad\qquad\dots,Y_{u_m}(i_m)(a_m)),\\ (\mathrm{id}_{\Gamma^{-1}(t')})_{t'\in\mathcal{T}})\end{array}
\end{tikzcd}
$$

and

$$
\begin{aligned}
&(X\otimes Y)_t(p)(x,Y_{u_1}(i_1)(a_1),\dots,Y_{u_m}(i_m)(a_m))\\
&=(x,Y_{u_1}(p)(Y_{u_1}(i_1)(a_1)),\dots,Y_{u_m}(p)(Y_{u_m}(i_m)(a_m)))\\
&=(x,a_1,\dots,a_m)
\end{aligned}
$$

D.9 Proof of proposition 7.5.136

We check the two monoidal natural transformation axioms.

1. First we check the commutativity of the following diagram for all $\Gamma\in\mathrm{Set}/\mathcal{T}$ and $t\in\mathcal{T}$

$$
\begin{tikzcd}
 & \Gamma^{-1}(t) \arrow[dl, "\phi"'] & \\
(\ell\mathcal{Y})(\Gamma)^{-1}(t) \arrow[rr, "\ell\psi=\mathrm{id}_{\ell\mathcal{Y}}"'] & & (\ell k\,\mathrm{Id})(\Gamma)^{-1}(t) \arrow[ul, "\varepsilon_{\mathcal{Y}}"']
\end{tikzcd}
$$

Explicitly on elements we have

$$
x\in\Gamma^{-1}(t)\quad\mapsto\quad(1,x^1)\in\langle t\rangle^{-1}(t)\times\prod_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\langle t\rangle^{-1}(t')}\quad\mapsto\quad x
$$

2. Next we check the commutativity of the following diagram for all $F, G \in [\mathrm{Set}/\mathcal{T}, \mathrm{Set}/\mathcal{T}]$, $\Gamma \in \mathrm{Set}/\mathcal{T}$ and $t \in \mathcal{T}$

$$\begin{tikzcd}
(\ell kF \circ \ell kG)(\Gamma)^{-1}(t) \arrow[r, "\varepsilon_F \circ \varepsilon_G"] \arrow[d, "\phi_{kF,kG}"'] & (F \circ G)(\Gamma)^{-1}(t) \\
\ell(kF \otimes kG)(\Gamma)^{-1}(t) \arrow[r, "\ell(\psi_{F,G})"'] & (\ell k(F \circ G))(\Gamma)^{-1}(t) \arrow[u, "\varepsilon_{F \circ G}"']
\end{tikzcd}$$

Along the bottom we find on elements

$$\begin{array}{cc}
\begin{array}{l}
\int^{\Gamma'} \int^{\Delta_1} \dots \int^{\Delta_m} F(\Gamma')^{-1}(t) \\
\times G_{u_1}(\Delta_1) \times \prod\limits_{u \in \mathcal{T}} \Gamma^{-1}(u)^{\Delta_1^{-1}(u)} \\
\times \dots \\
\times G_{u_m}(\Delta_m) \times \prod\limits_{u \in \mathcal{T}} \Gamma^{-1}(u)^{\Delta_m^{-1}(u)}
\end{array}
&
\begin{array}{c}
x, \\
y_1, (h_u^1)_{u \in \mathcal{T}}, \\
\dots, \\
y_m, (h_u^m)_{u \in \mathcal{T}}
\end{array}
\\
\Big\downarrow & \Big\downarrow {\scriptstyle \Delta = \Delta_1 + \dots + \Delta_m} \\
\begin{array}{l}
\int^{\Gamma'} \int^{\Delta} F(\Gamma')^{-1}(t) \\
\times G_{u_1}(\Delta) \times \dots \times G_{u_m}(\Delta) \times \prod\limits_{u \in \mathcal{T}} \Gamma^{-1}(u)^{\Delta^{-1}(u)} \\
= \int^{\Gamma'} \int^{\Delta} F(\Gamma')^{-1}(t) \times \prod\limits_{u \in \mathcal{T}} G_u(\Delta)^{-1}(u)^{\Gamma'^{-1}(u)} \\
\times \prod\limits_{u \in \mathcal{T}} \Gamma^{-1}(u)^{\Delta^{-1}(u)}
\end{array}
&
\begin{array}{c}
x, \\
G_{u_1}(i_1)(y_1), \dots, G_{u_m}(i_m)(y_m), \\
([h_u^1, \dots, h_u^m])_{u \in \mathcal{T}} \\
= x, (g'_u)_{u \in \mathcal{T}}, \\
([h_u^1, \dots, h_u^m])_{u \in \mathcal{T}}
\end{array}
\\
\Big\downarrow & \Big\downarrow \\
\int^{\Delta} F(G(\Delta))^{-1}(t) \times \prod\limits_{u \in \mathcal{T}} \Gamma^{-1}(u)^{\Delta^{-1}(u)}
&
\begin{array}{c}
F\big(\sum\limits_{u \in \mathcal{T}} g'_u\big)_t(x), \\
([h_u^1, \dots, h_u^m])_{u \in \mathcal{T}}
\end{array}
\\
\Big\downarrow & \Big\downarrow \\
F(G(H(\Gamma)))^{-1}(t) & FG\big(\sum\limits_{u \in \mathcal{T}} [h_u^1, \dots, h_u^m]\big)_t \circ F\big(\sum\limits_{u \in \mathcal{T}} g'_u\big)_t(x)
\end{array}$$

where $\Gamma' = (u_1, \dots, u_m)$. Along the top we have the assignation

$$x, y_1, (h_u^1)_{u \in \mathcal{T}}, \dots, y_m, (h_u^m)_{u \in \mathcal{T}} \quad \mapsto \quad F\big(\sum_{u \in \mathcal{T}} g_u\big)_t(x)$$

where $(g_u)_{u \in \mathcal{T}} : \prod\limits_{u \in \mathcal{T}} G_u(\Gamma)^{\Gamma'^{-1}(u)}$ with $g_{u_j} : 1 \to G_{u_j}(\Gamma)$, $1 \mapsto G_{u_j}\big(\sum\limits_{u \in \mathcal{T}} h_u^j\big)(y_j)$. These results are equal since

$$\begin{aligned}
G_{u_j}\big(\sum_{u \in \mathcal{T}} [h_u^1, \dots, h_u^m]\big) \circ g'_{u_j} &= G_{u_j}[\big(\sum_{u \in \mathcal{T}} h_u^1\big), \dots, \big(\sum_{u \in \mathcal{T}} h_u^m\big)] \circ g'_{u_j} \\
&= G_{u_j}[\big(\sum_{u \in \mathcal{T}} h_u^1\big), \dots, \big(\sum_{u \in \mathcal{T}} h_u^m\big)] \circ G_{u_j}(i_j) \\
&= G_{u_j}\big(\sum_{u \in \mathcal{T}} h_u^j\big)
\end{aligned}$$

D.10 Definition 7.6.137 of $b^{(u)}$

It is clearly natural in Δ and R. Next we have to check the commutativity of the following diagram

$$\begin{tikzcd}
((kR)^{\mathcal{Y}\langle u\rangle}\otimes(kR))_t(\Delta) \arrow[r, "b_R^{(u)}\otimes kR"] \arrow[d, "s_{kR}^{(u)}"'] & (k(\partial_u R)\otimes(kR))_t(\Delta) \arrow[d, "\psi_{\partial_u R,R}"] \\
((kR)\otimes(kR))_t^{\mathcal{Y}\langle u\rangle}(\Delta) \arrow[d, "(\psi_{R,R})^{\mathcal{Y}\langle u\rangle}"'] & (k(\partial_u R\circ R)_t(\Delta) \arrow[d, "k(\sigma_R^{(u)})"] \\
(k(R\circ R))_t^{\mathcal{Y}\langle u\rangle}(\Delta) \arrow[r, "b_{R\circ R}^{(u)}"'] & (k(\partial_u(R\circ R))_t(\Delta)
\end{tikzcd}$$

Along the left–hand side we have the following composite

$$\begin{tikzcd}
\int^{\Delta'} R(\Delta'+\langle u\rangle)^{-1}(t)\times\prod_{t'\in\mathcal{T}}(R\Delta)^{-1}(t')^{\Delta'^{-1}_{(}t')} \arrow[d] & (x,(h_t)_{t\in\mathcal{T}}) \arrow[d, mapsto, "\Delta''=\Delta'+\langle u\rangle"] \\
\int^{\Delta''}(R\Delta'')^{-1}(t)\times\prod_{t'\in\mathcal{T}}(R(\Delta+\langle u\rangle))^{-1}(t')^{\Delta'^{-1}(t')} \arrow[d] & (x,(\widehat{h}_t)_{t\in\mathcal{T}}) \arrow[d, mapsto] \\
R(R(\Delta+\langle u\rangle))^{-1}(t) & (\sum_{t'\in\mathcal{T}}\widehat{h}_{t'})_t(x)
\end{tikzcd}$$

where $\widehat{h}_{t'}$

$$\widehat{h}_{t'}=\begin{cases} R(i)_{t'}\circ h_{t'} & \text{if } t'\neq u\\ [R(i)_u\circ h_u,\overline{\eta\mathcal{Y}}_{u,\Delta}] & \text{if } t'=u\end{cases}$$

where $i:\Delta\to\Delta+\langle u\rangle$ and $\overline{\eta\mathcal{Y}_u}:1\to R(-+\langle u\rangle)^{-1}(u)$ is the transpose of $\eta\mathcal{Y}_u:\mathcal{Y}\langle u\rangle\to(R\mathcal{Y})_u$.

Along the right–hand side we have the following composite

$$\begin{tikzcd}
\int^{\Delta'} R(\Delta'+\langle u\rangle)^{-1}(t)\times\prod_{t'\in\mathcal{T}}(R\Delta)^{-1}(t')^{\Delta'^{t'}} \arrow[d] & (x,(h_t)_{t\in\mathcal{T}}) \arrow[d, mapsto] \\
R(R\Delta+\langle u\rangle)^{-1}(t) \arrow[d] & R(\sum_{t'\in\mathcal{T}}h_{t'}+\langle u\rangle)|_t(x) \arrow[d, mapsto] \\
R(R(\Delta+\langle u\rangle))^{-1}(t) & \begin{array}{c} R[Ri,\eta_{\Delta+\langle u\rangle}\circ j]_t \\ R(\sum_{t'\in\mathcal{T}}h_{t'}+\langle u\rangle)|_t(x)\end{array}
\end{tikzcd}$$

where $j:\langle u\rangle\to\Delta+\langle u\rangle$. One can easily show that $\overline{\eta\mathcal{Y}}_{u,\Delta}=\eta_{\Delta+\langle u\rangle}\circ j$, so the the two composites yield the same result.

D.11 Definition 7.6.138 of $a^{(u)}$

We check the wedge condition. Let $g : \Delta \to \Xi$. The diagram

$$\begin{tikzcd}
 & P_t(\Delta) \times \prod\limits_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{\Delta^{-1}(t')} \arrow[dr] & \\
\begin{array}{c} P_t(\Delta) \times \\ \prod\limits_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{\Xi^{-1}(t')} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ g_{t'})_{t' \in \mathcal{T}}"] \arrow[dr, "P_t(g) \times \mathrm{id}"'] & & (\ell(P^{\mathcal{Y}\langle u \rangle}))(\Gamma)^{-1}(t) \\
 & P_t(\Xi) \times \prod\limits_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{\Xi^{-1}(t')} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & p, (h_{t'} \circ g_{t'})_{t' \in \mathcal{T}} \arrow[dr, mapsto] & \\
p, (h_{t'})_{t' \in \mathcal{T}} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c} P_t(f^{\Delta})(p), (\widehat{h_{t'} \circ g_{t'}})_{t' \in \mathcal{T}} \\ = P_t(f^{\Xi} \circ g)(p), (\widehat{h}_{t'})_{t' \in \mathcal{T}} \end{array} \\
 & P_t(g)(p), (h_{t'})_{t' \in \mathcal{T}} \arrow[ur, mapsto] &
\end{tikzcd}$$

By naturality of f, there is an arrow $g' : \Delta' \to \Xi'$ such that $f^{\Xi} \circ g = (g' + \langle u \rangle) \circ f^{\Delta}$ and $\widehat{h_{t'} \circ g_{t'}} = \widehat{h}_{t'} \circ g'_{t'}$. The two elements $P_t(f^{\Delta})(p), (\widehat{h_{t'} \circ g_{t'}})_{t' \in \mathcal{T}}$ and $P_t(f^{\Xi} \circ g)(p) = P_t((g' + \langle u \rangle) \circ f^{\Delta})(p), (\widehat{h}_{t'})_{t' \in \mathcal{T}}$ are equal since they come from $P_t(f^{\Delta}(p), (\widehat{h}_{t'})_{t' \in \mathcal{T}}$ with the arrow g'

$$\begin{tikzcd}
 & P_t(\Delta' + \langle u \rangle) \times \prod\limits_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')} \arrow[dr] & \\
\begin{array}{c} P_t(\Delta' + \langle u \rangle) \times \\ \prod\limits_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Xi'^{-1}(t')} \end{array} \arrow[ur, "\mathrm{id} \times (- \circ g'_{t'})_{t' \in \mathcal{T}}"] \arrow[dr, "P_t(g' + \langle u \rangle) \times \mathrm{id}"'] & & \begin{array}{c} \int^{\Delta'} P_t(\Delta' + \langle u \rangle) \times \\ \prod\limits_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Delta'^{-1}(t')} \end{array} \\
 & P_t(\Xi' + \langle u \rangle) \times \prod\limits_{t' \in \mathcal{T}} \Gamma^{-1}(t')^{\Xi'^{-1}(t')} \arrow[ur] &
\end{tikzcd}$$

Next we check naturalities of $a^{(u)}_{P,\Gamma}$ in Γ and P. Let $g : \Gamma_1 \to \Gamma_2$ in $\mathrm{Set}/\mathcal{T}$. The naturality square

$$\begin{tikzcd}
(\ell P)(\Gamma_1 + \langle u \rangle)^{-1}(t) \arrow[r] \arrow[d] & (\ell(P^{\mathcal{Y}\langle u \rangle}))(\Gamma_1)^{-1}(t) \arrow[d] \\
(\ell P)(\Gamma_2 + \langle u \rangle)^{-1}(t) \arrow[r] & (\ell(P^{\mathcal{Y}\langle u \rangle}))(\Gamma_2)^{-1}(t)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
p, (h_{t'})_{t'\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & P_t(f^\Delta)(p), (\widehat{h}_{t'})_{t'\in\mathcal{T}} \arrow[d, mapsto] \\
p, ((g+\langle u\rangle)_{t'} \circ h_{t'})_{t'\in\mathcal{T}} \arrow[r, mapsto] & \begin{array}{c} P_t(f^\Delta)(p), (g \circ \widehat{h}_{t'})_{t'\in\mathcal{T}} \\ = P_t(f^\Delta)(p), (\widehat{(g+\langle u\rangle)_{t'}} \circ h_{t'})_{t'\in\mathcal{T}} \end{array}
\end{tikzcd}$$

Let $g : P \to Q$ in $[\mathbb{F}\downarrow\mathcal{T}, \mathrm{Set}]^{\mathcal{T}}$. The naturality square

$$\begin{tikzcd}
(\ell P)(\Gamma+\langle u\rangle)^{-1}(t) \arrow[r] \arrow[d] & (\ell(P^{\mathcal{Y}\langle u\rangle}))(\Gamma)^{-1}(t) \arrow[d] \\
(\ell Q)(\Gamma+\langle u\rangle)^{-1}(t) \arrow[r] & (\ell(Q^{\mathcal{Y}\langle u\rangle}))(\Gamma)^{-1}(t)
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
p, (h_{t'})_{t'\in\mathcal{T}} \arrow[r, mapsto] \arrow[d, mapsto] & P_t(f^\Delta)(p), (\widehat{h}_{t'})_{t'\in\mathcal{T}} \arrow[d, mapsto] \\
g_{t,\Delta}(p), (h_{t'})_{t'\in\mathcal{T}} \arrow[r, mapsto] & \begin{array}{c} g_{t,\Delta'}(P_t(f^\Delta)(p)) \\ = Q_t(f^\Delta)(g_{t,\Delta}(p)), (\widehat{h}_{t'})_{t'\in\mathcal{T}} \end{array}
\end{tikzcd}$$

Next we have to check the commutativity of the following diagram for all $\Gamma \in \mathrm{Set}/\mathcal{T}$

$$\begin{tikzcd}
(\ell P)(\ell Q(\Gamma)+\langle u\rangle)^{-1}(t) \arrow[r, "a^{(u)}\circ(\ell Q)"] \arrow[d, "s"] & (\ell(P^{\mathcal{Y}\langle u\rangle}))(\ell Q(\Gamma))^{-1}(t) \arrow[d, "\phi"] \\
(\ell P(\ell Q(\Gamma+\langle u\rangle)))^{-1}(t) \arrow[d, "\phi"] & (\ell(P^{\mathcal{Y}\langle u\rangle}\otimes Q))(\Gamma)^{-1}(t) \arrow[d, "\sigma"] \\
(\ell(P\otimes Q))(\Gamma+\langle u\rangle)^{-1}(t) \arrow[r, "a^{(u)}"'] & (\ell((P\otimes Q)^{\mathcal{Y}\langle u\rangle}))(\Gamma)^{-1}(t)
\end{tikzcd}$$

In lemma 7.6.141 we show that $a^{(u)}$ is an isomorphism, its inverse being $d^{(u)}$. We check the above diagram where we inverse the arrows $a^{(u)}$, so explicitly we check the commutativity of the following diagram

$$\begin{tikzcd}
(\ell(P^{\mathcal{Y}\langle u\rangle}))(\ell Q(\Gamma))^{-1}(t) \arrow[r, "d^{(u)}\circ\ell Q"] \arrow[d, "\phi"] & (\ell P)(\ell Q(\Gamma)+\langle u\rangle)^{-1}(t) \arrow[d, "\sigma"] \\
(\ell(P^{\mathcal{Y}\langle u\rangle}\otimes Q))(\Gamma)^{-1}(t) \arrow[d, "s"] & (\ell P(\ell Q(\Gamma+\langle u\rangle)))^{-1}(t) \arrow[d, "\phi"] \\
(\ell((P\otimes Q)^{\mathcal{Y}\langle u\rangle}))(\Gamma)^{-1}(t) \arrow[r, "d"'] & (\ell(P\otimes Q))(\Gamma+\langle u\rangle)^{-1}(t)
\end{tikzcd}$$

Along the left–hand side we have the following composite

$$
\begin{array}{ccc}
\begin{array}{l}
\int^{\Delta}\int^{\Delta_1}\dots\int^{\Delta_m} P_t(\Delta+\langle u\rangle) \\
\times Q_{u_1}(\Delta_1)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_1^{-1}(t')} \\
\times\dots \\
\times Q_{u_m}(\Delta_m)\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_m^{-1}(t')}
\end{array}
& &
\begin{array}{c}
x, \\ y_1, z_1, \\ \dots, \\ y_m, z_m
\end{array}
\\
\Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma'=\Delta_1+\dots+\Delta_m}
\\
\begin{array}{l}
\int^{\Gamma'}\int^{\Delta} P_t(\Delta+\langle u\rangle) \\
\times Q_{u_1}(\Gamma')\times\dots\times Q_{u_m}(\Gamma') \\
\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Gamma')^{-1}(t')}
\end{array}
& &
\begin{array}{c}
x, \\ \overline{y}_1,\dots,\overline{y}_m, \\ [z_1,\dots,z_m]
\end{array}
\\
\Big\downarrow & & \Big\downarrow{\scriptstyle \Delta'=\Delta+\langle s\rangle}
\\
\begin{array}{l}
\int^{\Gamma'}\int^{\Delta'} P_t(\Delta') \\
\times Q_{v_1}(\Gamma'+\langle u\rangle)\times\dots\times Q_{v_k}(\Gamma'+\langle u\rangle) \\
\times\prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{(\Gamma')^{-1}(t')}
\end{array}
& &
\begin{array}{c}
x, \\ i_\Gamma(\overline{y}_1),\dots,i_\Gamma(\overline{y}_m),\overline{q_{u,\Gamma}} \\ [z_1,\dots,z_m]
\end{array}
\\
\Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma''=\Gamma'+\langle s\rangle}
\\
\begin{array}{l}
\int^{\Gamma''}\int^{\Delta'} P_t(\Delta') \\
\times Q_{v_1}(\Gamma'')\times\dots\times Q_{v_k}(\Gamma'') \\
\times\prod\limits_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{(\Gamma'')^{-1}(t')}
\end{array}
& &
\begin{array}{c}
x, \\ i_\Gamma(\overline{y}_1),\dots,i_\Gamma(\overline{y}_m),\overline{q_{u,\Gamma}} \\ i_\Gamma\circ[z_1,\dots,z_m]+\langle u\rangle
\end{array}
\end{array}
$$

where we write $\Delta=(u_1,\dots,u_m)$, $\Delta'=(v_1,\dots,v_k)$, $i_\Gamma:\Gamma\to\Gamma+\langle u\rangle$ and $\overline{q_u}:1\to Q_u^{\mathcal{Y}\langle u\rangle}$ for the transpose of $q_u:\mathcal{Y}\langle u\rangle\to Q_u$.

Along the right–hand side we have the following composite

$$\begin{tikzcd}
\begin{array}{l} \int^{\Delta}\int^{\Delta_1}\dots\int^{\Delta_m} P_t(\Delta+\langle u\rangle) \\ \times Q_{u_1}(\Delta_1)\times \prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_1^{-1}(t')} \\ \times\dots \\ \times Q_{u_m}(\Delta_m)\times \prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_m^{-1}(t')} \end{array} \arrow[d] & \begin{array}{c} x, \\ y_1, z_1, \\ \dots, \\ y_m, z_m \end{array} \arrow[d, mapsto, "\Delta'=\Delta+\langle u\rangle"] \\
\begin{array}{l} \int^{\Delta'} P_t(\Delta') \\ \times\big(\int^{\Delta_1} Q_{v_1}(\Delta')\times \prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_1^{-1}(t')}\big)+\langle u\rangle^{-1}(v_1) \\ \times\dots \\ \times\big(\int^{\Delta_k} Q_{v_k}(\Delta')\times \prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta_k^{-1}(t')}\big)+\langle u\rangle^{-1}(v_1) \end{array} \arrow[d] & \begin{array}{c} x, \\ y_1, z_1, \\ \dots, \\ y_m, z_m, \\ 1 \end{array} \arrow[d, mapsto, "{\Gamma_1=\Delta_1,\dots,\ \Gamma_{k-1}=\Delta_m,\Gamma_k=\langle u\rangle}"] \\
\begin{array}{l} \int^{\Delta'}\int^{\Gamma_1}\dots\int^{\Gamma_k} P_t(\Delta') \\ \times Q_{v_1}(\Gamma_1)\times \prod\limits_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{\Gamma_1^{-1}(t')} \\ \times\dots \\ \times Q_{v_k}(\Gamma_m)\times \prod\limits_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{\Gamma_k^{-1}(t')} \end{array} \arrow[d] & \begin{array}{c} x, \\ y_1, i_\Gamma\circ z_1, \\ \dots, \\ y_m, i_\Gamma\circ z_m, \\ q_u(1), 1 \end{array} \arrow[d, mapsto, "\Gamma''=\Gamma_1+\dots+\Gamma_k"] \\
\begin{array}{l} \int^{\Gamma''}\int^{\Delta'} P_t(\Delta') \\ \times Q_{v_1}(\Gamma'')\times\dots\times Q_{v_k}(\Gamma'') \\ \times \prod\limits_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{(\Gamma'')^{-1}(t')} \end{array} & \begin{array}{c} x, \\ i_\Gamma(\overline{y}_1),\dots,i_\Gamma(\overline{y}_m),\overline{q_{u,\Gamma}} \\ {[i_\Gamma\circ z_1,\dots,i_\Gamma\circ z_m,1]} \end{array}
\end{tikzcd}$$

where we write again $\Delta=(u_1,\dots,u_m)$, $\Delta'=(v_1,\dots,v_k)$, $i_\Gamma:\Gamma\to\Gamma+\langle u\rangle$ and $\overline{q_u}:1\to Q_u^{\mathcal{Y}\langle u\rangle}$ for the transpose of $q_u:\mathcal{Y}\langle u\rangle\to Q_u$. The results along the left–hand side and the right–hand side are equal so the diagram commutes.

D.12 Proof of proposition 7.6.141

We check the wedge condition. Let $f:\Delta'\to\Delta''$ in $\mathbb{F}\downarrow\mathcal{T}$. The diagram

$$\begin{tikzcd}
& \begin{array}{l} P_t(\Delta'+\langle u\rangle) \\ \times \prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta'^{-1}(t')} \end{array} \arrow[dr] & \\
\begin{array}{l} P_t(\Delta'+\langle u\rangle) \\ \times \prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta''^{-1}(t')} \end{array} \arrow[ur, "\mathrm{id}\times(-\circ f_{t'})_{t'\in\mathcal{T}}"] \arrow[dr, "P_t(f+\langle u\rangle)\times\mathrm{id}"'] & & (\ell P)(\Gamma+\langle u\rangle)^{-1}(t) \\
& \begin{array}{l} P_t(\Delta''+\langle u\rangle) \\ \times \prod\limits_{t'\in\mathcal{T}}\Gamma^{-1}(t')^{\Delta''^{-1}(t')} \end{array} \arrow[ur] &
\end{tikzcd}$$

commutes since we have the following assignations on elements

$$\begin{tikzcd}
 & (x,(h_{t'}\circ f_{t'})_{t'\in\mathcal{T}}) \arrow[dr, mapsto] & \\
(x,(h_{t'})_{t'\in\mathcal{T}}) \arrow[ur, mapsto] \arrow[dr, mapsto] & & \begin{array}{c}(x,((h_{t'}\circ f_{t'})+\langle s\rangle_{t'})_{t'\in\mathcal{T}})\\ =(P_t(f+\langle u\rangle)(x),(h_{t'}+\langle u\rangle_{t'})_{t'\in\mathcal{T}})\end{array} \\
 & (P_t(f+\langle u\rangle)(x),(h_{t'})_{t'\in\mathcal{T}}) \arrow[ur, mapsto] &
\end{tikzcd}$$

The two elements $(x,((h_{t'}\circ f_{t'})+\langle u\rangle_{t'})_{t'\in\mathcal{T}})$ and $(P_t(f+\langle u\rangle)(x),(h_{t'}+\langle u\rangle_{t'})_{t'\in\mathcal{T}})$ are equal since they come from $(x,(h_{t'}+\langle u\rangle_{t'})_{t'\in\mathcal{T}})\in P_t(\Delta'+\langle u\rangle)\times\prod_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{(\Delta''+\langle u\rangle)^{-1}(t')}$ with the arrow $f+\langle u\rangle:\Delta'+\langle u\rangle\to\Delta''+\langle u\rangle$

$$\begin{tikzcd}
 & \begin{array}{l}P_t(\Delta'+\langle u\rangle)\times\\ \prod_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{(\Delta'+\langle u\rangle)^{-1}(t')}\end{array} \arrow[dr] & \\
\begin{array}{l}P_t(\Delta'+\langle u\rangle)\times\\ \prod_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{(\Delta''+\langle u\rangle)^{-1}(t')}\end{array} \arrow[ur, "\mathrm{id}\times(-\circ(f+\langle u\rangle)_{t'})_{t'\in\mathcal{T}}"] \arrow[dr, "P_t(f+\langle u\rangle)\times\mathrm{id}"'] & & \begin{array}{l}\int^{\Delta}P_t(\Delta)\times\\ \prod_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{\Delta^{-1}(t')}\end{array} \\
 & \begin{array}{l}P_t(\Delta''+\langle u\rangle)\times\\ \prod_{t'\in\mathcal{T}}(\Gamma+\langle u\rangle)^{-1}(t')^{(\Delta''+\langle u\rangle)^{-1}(t')}\end{array} \arrow[ur] &
\end{tikzcd}$$

Next we have to check that $a_P^{(u)}$ and $d_P^{(u)}$ are inverse to each other. The composite $a_P^{(u)}\circ d_P^{(u)}$ yields on elements

$$\begin{tikzcd}
(x,(h_{t'})_{t'\in\mathcal{T}}) \arrow[d, mapsto] \\
(x,(h_{t'}+\langle u\rangle_{t'})_{t'\in\mathcal{T}}) \arrow[d, mapsto] \\
\begin{array}{c}(P_t(f)(x),(\widehat{h_{t'}+\langle u\rangle}_{t'})_{t'\in\mathcal{T}})\\ =(x,(h_{t'})_{t'\in\mathcal{T}})\end{array}
\end{tikzcd}$$

We remark that $\Delta''^{-1}(u)=\{x\in(\Delta'+\langle u\rangle)^{-1}(u)$ such that $(\widehat{h_u+\langle u\rangle_u})(x)\in\Gamma^{-1}(u)\}=\Delta'^{-1}(u)$, so $f:\Delta'+\langle u\rangle\to\Delta''+\langle u\rangle$ is the identity on $\Delta'+\langle u\rangle$ and $(\widehat{h_{t'}+\langle u\rangle}_{t'})_{t'\in\mathcal{T}}=(h_{t'})_{t'\in\mathcal{T}}$. So this composite yields the identity on $(x,(h_{t'})_{t'\in\mathcal{T}})$.

The composite $d_P^{(u)}\circ a_P^{(u)}$ yields on elements

$$\begin{tikzcd}
(x,(h_{t'})_{t'\in\mathcal{T}}) \arrow[d, mapsto] \\
(P_t(f)(x),((\widehat{h}_{t'})_{t'\in\mathcal{T}}) \arrow[d, mapsto] \\
(P_t(f)(x),((\widehat{h}_{t'}+\langle u\rangle_{t'})_{t'\in\mathcal{T}})
\end{tikzcd}$$

These two elements are identical since they come from $P_t(\Delta) \times \prod_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{(\Delta' + \langle u \rangle)^{-1}(t')}$ with the arrow $f : \Delta \to \Delta' + \langle u \rangle$

$$\begin{tikzcd}
 & \begin{array}{c} P_t(\Delta) \\ \times \prod_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{\Delta^{-1}(t')} \\ (x, (h_{t'})_{t' \in \mathcal{T}}) \end{array} \\
\begin{array}{c} P_t(\Delta) \\ \times \prod_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{(\Delta' + \langle u \rangle)^{-1}(t')} \\ (x, (\widehat{h}_{t'} + \langle u \rangle_{t'})_{t' \in \mathcal{T}}) \end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{c} P_t(\Delta' + \langle u \rangle) \\ \times \prod_{t' \in \mathcal{T}} (\Gamma + \langle u \rangle)^{-1}(t')^{(\Delta' + \langle u \rangle)^{-1}(t')} \\ (P_t(f)(x), (\widehat{h}_{t'} + \langle u \rangle_{t'})_{t' \in \mathcal{T}}) \end{array}
\end{tikzcd}$$

We remark that $(\widehat{h}_{t'} + \langle u \rangle_{t'})_{t'} \circ f_{t'} = h_{t'}$ since in the fibre u we have explicitly

$$x \mapsto \left\{ \begin{array}{c} x \\ 1 \end{array} \right\} \mapsto \left\{ \begin{array}{cl} h_u(x) & \text{if } x \in \Delta'^{-1}(u) \\ 1 = h_u(x) & \text{if } x \notin \Delta'^{-1}(u) \end{array} \right.$$

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Abstract. In order to specify the behavior of programming languages, to investigate their properties and to allow certification of their implementations, one studies formal models of existing programming languages. This study splits into the study of syntax and semantics, where the latter is based on appropriate formal models for the syntax. This PhD thesis is located in the syntactic part and is mainly concerned with two approaches to abstract syntax with variable binding. Both make use of the language of category theory. The first one is in the spirit of the category theoretic approach to algebraic theories. The second one is based on the notion of monads and introduces modules on monads instead of working with functors and their algebras. Furthermore the latter approach is adapted to a larger class of typed syntax with types depending on terms.

Résumé. Afin de spécifier le comportement des langages de programmation, de préciser leurs propriétés et de certifier leurs implémentations, on étudie des modèles formels des langages de programmation. L'étude se divise en l'étude de la syntaxe et en celle de la sémantique. La deuxième est basée sur des modèles formels de la syntaxe. Cette thèse de doctorat se situe dans l'étude de la syntaxe et est consacrée principalement à deux approches à la syntaxe abstraite typée avec liaison de variables. Ces deux approches utilisent le langage de la théorie des catégories. La première approche est dans l'esprit de l'approche catégorique aux théories alébriques. La deuxième est basée sur la notion de monade et introduit la notion d'un module sur une monade qui remplacent les foncteurs et leurs algèbres. En outre la deuxième approche est adaptée pour une classe plus large de syntaxes typées où les types dépendent des termes.