Phénomènes d'accrochage et théorie des fluctuations.

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PHÉNOMÈNES D'ACCROCHAGE ET THÉORIE DES FLUCTUATIONS.

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1.1.1 Motivations physiques

Cette thèse est consacrée à l'étude mathématique de phénomènes de localisation et délocalisation pour différents modèles de polymères. Ces dernières années ont vu un essor considérable de la littérature mathématique consacrée à l'étude des polymères, citons par exemple les monographies [Bol02], [Gia07] et [dH09]. Les modèles que nous étudierons au cours de cette thèse visent à décrire des phénomènes physiques variés, comme l'interaction d'un polymère avec une interface entre solvants [DHV92], le phénomène de dénaturation chimique de l'ADN dans l'approximation de Poland-Sheraga [KMP03], l'accrochage et le décrochage d'un polymère et d'un substrat liquide... Un point commun à tous ces modèles est l'existence d'une *transition de phase*, autrement dit l'existence d'une valeur critique de la température T_c telle que le comportement macroscopique du système diffère suivant que sa température T soit supérieure ou inférieure à T_c. Une explication intuitive à l'existence d'une telle transition provient du fait qu'à haute température, l'agitation thermique est forte et que les interactions chimiques ou physiques avec l'environnement peuvent être négligées, et donc qu'il apparaît un phénomène de *délocalisation* caractérisé par le fait que l'influence de l'environnement sur le système devient macroscopiquement négligeable. A basse température par contre, la trajectoire du polymère est fortement conditionnée par l'environnement, on parle alors de phase *localisée*. Ceci justifie de manière informelle l'existence d'une température critique marquant une transition de la phase localisée vers la phase délocalisée.

1.1.2 Formalisme mathématique

Il est courant dans la littérature consacrée à la mécanique statistique d'utiliser le formalisme de Boltzmann-Gibbs, il est utilisé ici pour donner la loi des configurations spatiales de la chaîne de polymères.

On modélise la chaîne de polymères par un chemin de taille finie égale à un entier N dans un ensemble de chemins donné $(S_n)_{n\in[0,N]} \in \Gamma_N$. A chaque trajectoire S on associe une énergie donnée par l'Hamiltonien $H_{N,\beta}(S)$ où $\beta = (k_B T)^{-1}$ et la constante k_B désigne la constante de Boltzmann. Au cours de cette thèse, cette énergie sera toujours donnée par la somme des énergies des sites visités, où la signification du terme "visité" peut varier d'un modèle à l'autre. On dénote par $\mathbf{P}$ la loi de la chaîne *à température infinie*, c'est à dire lorsque le polymère n'interagit pas avec le milieu. La trajectoire de la chaîne de polymères est alors donnée par une mesure de probabilité $\mathbf{P}_{N,\beta}(S)$ sur Γ_N :

$$\frac{d\mathbf{P}_{N,\beta}}{d\mathbf{P}}(S) = \frac{\exp(H_{N,\beta}(S))}{Z_{N,\beta}} \tag{1.1.1}$$

où $Z_{N,\beta}$ vaut

$$Z_{N,\beta} := \mathbf{E}[\exp(H_{N,\beta}(S))]. \tag{1.1.2}$$

Il est clair que ce formalisme peut donner lieu à une très grande variété de modèles. Nous considérons qu'une chaîne de polymère est un objet physique de dimension 1 qui se déploie dans un espace de dimension strictement supérieure et qui ne présente pas d'auto-intersection. Il est donc naturel de prendre pour $\mathbf{P}$ la loi d'une marche aléatoire *auto-évitante* dans $\mathbb{Z}^d$ ou $\mathbb{R}^d$ ($d \geqslant 2$). Il se trouve cependant qu'en pratique, le traitement mathématique de tels objets se révèle extrêmement ardu. Une possibilité naturelle et couramment pratiquée est de se restreindre au cas des *marches aléatoires dirigées*, autrement dit les marches aléatoires dont une des coordonnées est toujours croissante. C'est ce point de vue que nous adopterons dans cette thèse. Notons cependant que même sous cette restriction, on peut encore avoir affaire à bien des modèles présentant des caractéristiques distinctes les uns des autres (à titre d'illustration, on peut se référer par exemple aux travaux de Hryniv et Velenik [HV04] ou encore à ceux de Comets et Yoshida [CY05]).

Nous rencontrerons essentiellement deux modèles différents au cours de ce travail :

- **Le modèle d'accrochage homogène.** Le polymère est modélisé par une marche aléatoire aux plus proches voisins dans $\mathbb{Z}^d$ et reçoit une contribution énergétique lorsque la marche passe par l'origine. En utilisant le formalisme ci-dessus, on pose donc

$$H_{N,\beta}(S) = \beta \sum_{i=1}^{N} \mathbf{1}_{\{S_i=0\}}. \tag{1.1.3}$$

  On peut considérer de manière équivalente que l'on modélise le polymère par le graphe de cette marche (n, S_n), qui est un processus dans Z^{d+1}, et que le polymère reçoit une contribution énergétique lorsque les d dernières coordonnées s'annulent simultanément.

- **Le modèle de mouillage dans une bande.** Dans ce cas, le polymère est modélisé par une marche aléatoire à accroissements continus, autrement dit S_1 possède une densité $h(\cdot)$ continue par rapport à la mesure de Lebesgue. On s'intéressera au cas où les accroissements sont intégrables et de carré sommable (ce qui implique en particulier la convergence de la marche aléatoire renormalisée vers le mouvement brownien). Le terme *mouillage* fait référence au fait que la loi $\mathbf{P}$ est la loi d'une marche aléatoire conditionnée à rester positive sur l'intervalle $[0, N]$. D'autre part, les contributions énergétiques se trouvent dans une bande $\mathbb{R}^+ \times [0, a]$. On pose donc :

$$H_{N,\beta,a}(S) = \beta \sum_{i=1}^{N} \mathbf{1}_{\{S_i \in [0,a]\}} \tag{1.1.4}$$

  et on définit la loi $\mathbf{P}_{N,\beta,a}$ par une dérivée de Radon-Nykodym :

$$\frac{d\mathbf{P}_{N,\beta,a}}{d\mathbf{P}} := \frac{\exp(H_{N,\beta,a}(S))}{Z_{N,\beta,a}} \mathbf{1}_{\{S_1 \geq 0, \ldots, S_N \geq 0\}}. \tag{1.1.5}$$

Pour ces deux modèles, on s'intéressera au comportement asymptotique de la mesure du polymère lorsque N devient grand, et plus précisément on va s'attacher à caractériser l'influence du facteur d'accrochage sur le comportement macroscopique du système en étudiant les limites d'échelle de la chaîne de polymères.

1.2 Modèle de mouillage homogène

1.2.1 Le cadre de la marche (p, q).

On se propose dans cette partie de faire une étude sommaire du modèle de mouillage homogène inspirée des premiers chapitres de [Gia07]. Ce modèle possède la propriété remarquable d'être *exactement résoluble*, au sens où l'on connaît précisément l'ordre de sa transition de phase en fonction d'un paramètre clef du modèle. On rappelle d'abord la définition du modèle en une dimension.

Considérons une suite de variables indépendantes identiquement distribuées (X_i) symétriques de loi $\mathbf{P}$ vérifiant

$$\mathbf{P}[X_1 = 0] = q, \quad \mathbf{P}[X_1 = 1] = \mathbf{P}[X_1 = -1] = p \tag{1.2.1}$$

où p et q sont deux réels positifs non nuls vérifiant $2p + q = 1$. Le processus S défini par $S_n := \sum_{i=1}^n X_i$ est une (p, q) *marche aléatoire*. Pour $n \geqslant 1$, on définit l'événement $\mathcal{C}_N := \{S_1 \geqslant 0, \ldots, S_N \geqslant 0\}$. Pour $\beta \in \mathbb{R}$, on modifie la mesure $\mathbf{P}$ en attribuant un bonus ou un malus d'énergie (selon le signe de β) à la marche S lorsqu'elle passe par l'origine. Ceci nous amène à définir la mesure de polymère suivante :

$$\frac{d\mathbf{P}^c_{N,\beta}}{d\mathbf{P}}(S) := \frac{1}{Z^c_{N,\beta}} \exp\left(\beta \sum_{i=1}^N \delta_i\right) \delta_N \mathbf{1}_{\mathcal{C}_N} \tag{1.2.2}$$

où l'on définit $\delta_i := \mathbf{1}_{S_i=0}$ et où

$$Z^c_{N,\beta} := \mathbf{E}\left[\exp\left(\beta \sum_{i=1}^N \delta_i\right) \delta_N \mathbf{1}_{\mathcal{C}_N}\right]. \tag{1.2.3}$$

La quantité $Z^c_{N,\beta}$ est une constante de normalisation appelée fonction de partition dont le comportement asymptotique se révèle être très riche d'informations sur les caractéristiques macroscopiques du système. Notons que l'exposant c que nous utilisons pour ce modèle fait référence à la contrainte $\{S_N = 0\}$. Il n'est *a priori* pas indispensable d'utiliser cette contrainte, et de fait nous aurons aussi affaire au modèle suivant :

$$\frac{d\mathbf{P}^f_{N,\beta}}{d\mathbf{P}}(S) := \frac{1}{Z^f_{N,\beta}} \exp\left(\beta \sum_{i=1}^N \delta_i\right) \mathbf{1}_{\mathcal{C}_N}. \tag{1.2.4}$$

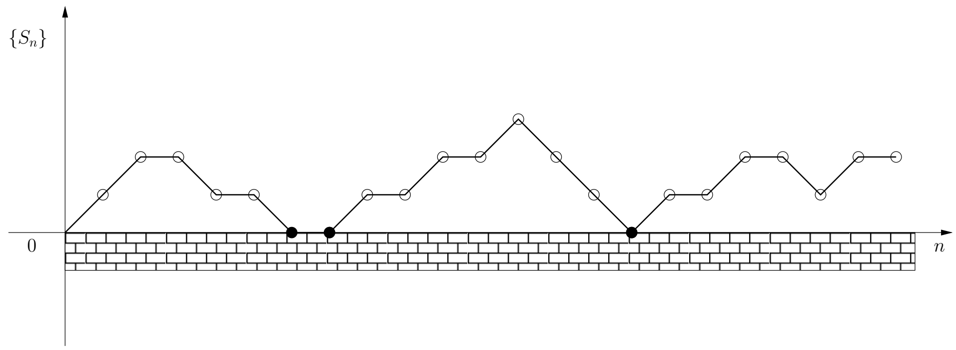

FIGURE 1.1. Une trajectoire de la marche aléatoire (p, q) qu'on interprète comme étant la représentation d'un polymère dirigé. Les 3 contacts qui contribuent à l'énergie totale de la trajectoire sont mis en évidence en noir.

En pratique, le modèle contraint joue un rôle techniquement très important.

Une première question naturelle est de savoir si la marche aléatoire S sous la mesure $\mathbf{P}^f_{N,\beta}$ s'accroche à la ligne d'interaction sous l'influence de la force d'attraction β. Pour ce faire, on s'intéresse à la fraction de contact considérée sous la loi $\mathbf{P}^f_{N,\beta}$, c'est à dire à la quantité $\rho_N(\beta) := \frac{1}{N}\mathbf{E}^f_{N,\beta}\left[\sum_{i=1}^N \delta_i\right]$ et plus particulièrement à son comportement lorsque N devient grand.

Une formalisation utile pour traiter ce problème est la suivante : pour $n \geqslant 1$, posons $K(n) := \mathbf{P}\left[S_1 > 0, \dots, S_{n-1} > 0, S_n = 0\right]$. On considère alors le processus de renouvellement τ défini par $\tau_0 := 0$ et pour $k \geqslant 1$,

$$\begin{aligned} &\mathbf{P}[\tau_k - \tau_{k-1} = n] := K(n) \text{ si } \tau_{k-1} < \infty \\ &\mathbf{P}[\tau_k - \tau_{k-1} = \infty] := 1 - \sum_{n \geqslant 1} K(n) \text{ si } \tau_{k-1} < \infty \\ &\tau_k = \infty \text{ si } \tau_{k-1} = \infty. \end{aligned} \tag{1.2.5}$$

La symétrie et la récurrence de S impliquent que $\sum_{n \geqslant 1} K(n) = \frac{1+q}{2}$, et donc en particulier τ est un renouvellement *transitoire*.

On peut montrer que $K(n) \sim \frac{1}{n^{3/2}}\sqrt{\frac{p}{8\pi}} \sim \frac{c_K}{n^{3/2}}$ lorsque $n \to \infty$ (c'est une conséquence évidente de la Proposition A.10 [Gia07]). On s'autorisera à considérer τ comme un sous-ensemble de $\mathbb{N}$, autrement dit des notations telles que $n \in \tau$ ou $A \cap \tau$ pour $A \subset \mathbb{N}$ sont bien définies.

Notons $F(\beta)$ l'unique solution de l'équation d'inconnue $x(\geqslant 0)$ ci-dessous :

$$\sum_{n \geqslant 1} K(n)e^{-xn} = e^{-\beta} \tag{1.2.6}$$

si cette solution existe (autrement dit si $\beta \geqslant \log(2/(1+q)) =: \beta_c$) et $F(\beta) = 0$ sinon. On a la propriété suivante :

Proposition 1.2.1. *Pour tout réel β, on a la convergence :*

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N} \log(Z^c_{N,\beta}). \tag{1.2.7}$$

Notons que l'on pourrait démontrer l'existence de la limite du membre de droite par suradditivité de manière autonome.

Preuve de la Proposition 1.2.1. Le mécanisme de la preuve consiste à remarquer que $Z^c_{N,\beta}$ est la fonction de Green d'un processus de renouvellement bien choisi. Cette idée sera exploitée dans le chapitre 3 dans le cadre plus général d'un renouvellement Markovien.

Plus concrètement, on a l'égalité :

$$Z^c_{N,\beta} = \sum_{n=1}^{N} \sum_{\substack{l\in\mathbb{N}^n: \\ \sum_{i=1}^n l_i = N}} \prod_{j=1}^{n} e^{\beta} K(l_j). \tag{1.2.8}$$

Considérons d'abord le cas où $\beta > \beta_c$ (et donc $F(\beta) > 0$). On introduit la distribution de probabilité suivante :

$$\widetilde{K}_\beta(n) := K(n) e^{\beta} e^{-F(\beta)n}. \tag{1.2.9}$$

On peut alors écrire :

$$Z^c_{N,\beta} = e^{F(\beta)N} \sum_{n=1}^{N} \sum_{\substack{l\in\mathbb{N}^n: \\ \sum_{i=1}^n l_i = N}} \prod_{j=1}^{n} \widetilde{K}_\beta(l_j) = e^{F(\beta)N} \widetilde{\mathbf{P}}_\beta(N \in \tau) \tag{1.2.10}$$

où $\widetilde{\mathbf{P}}_\beta$ désigne la loi d'un processus de renouvellement dont les probabilités d'interarrivées sont données par $\widetilde{K}_\beta(\cdot)$. Notons que ce nouveau processus de renouvellement est *récurrent positif*, autrement dit qu'il vérifie que $\sum_{i=1}^\infty i\widetilde{K}_\beta(i) < \infty$. Le Théorème du renouvellement (cf [Asm03, Chapitre 1 Théorème 2.2]) pour un énoncé de ce résultat classique) nous indique alors que $\widetilde{\mathbf{P}}_\beta(N \in \tau) \to 1/\sum_{i=1}^\infty i\widetilde{K}_\beta(i)$ lorsque N tend vers l'infini. Par conséquent :

$$Z^c_{N,\beta} \sim \frac{e^{F(\beta)N}}{\sum_{i=1}^\infty i\widetilde{K}_\beta(i)} \tag{1.2.11}$$

et en particulier on retrouve la convergence (1.2.7).

Lorsque $\beta \leqslant \beta_c$, $Z^c_{N,\beta}$ peut être interprétée directement comme étant une fonction de Green : $Z^c_{N,\beta} = \widetilde{\mathbf{P}}_\beta(N \in \tau)$. Pour $\beta = \beta_c$, $\widetilde{\mathbf{P}}_\beta$ est une probabilité, mais pour $\beta < \beta_c$, la densité $\widetilde{K}_\beta(n) := K(n)e^{\beta}$ est une *sous-probabilité* qui peut être interprétée comme étant une probabilité sur $\mathbb{N}\cup\{\infty\}$ où la probabilité d'une interarrivée infinie vaut $1-\exp(\beta-\beta_c)$. Il est alors clair que

$$e^{\beta}K(N) \leqslant Z^c_{N,\beta} \leqslant 1 \tag{1.2.12}$$

ce qui conclut la preuve de la Proposition 1.2.1.

□

Quelques propriétés de l'énergie libre se déduisent immédiatement de cette proposition. En particulier, la fonction $\beta \mapsto \frac{1}{N}\log(Z^c_{N,\beta})$ étant convexe et croissante à N fixé, la Proposition 1.2.1 nous indique que F jouit elle-même de ces propriétés. On peut aisément relier la fraction de contact moyenne à l'énergie libre. Remarquons d'abord que

$$\frac{\partial}{\partial\beta}\frac{1}{N}\log(Z^c_{N,\beta}) = \frac{1}{N}\mathbf{E}^c_{N,\beta}\left[\sum_{i=1}^{N}\delta_i\right]. \tag{1.2.13}$$

Par convexité, on peut passer à la limite dans l'équation ci-dessus, ce qui entraîne que

$$F'(\beta) = \lim_{N\to\infty}\frac{1}{N}\mathbf{E}^c_{N,\beta}\left[\sum_{i=1}^{N}\delta_i\right] \tag{1.2.14}$$

lorsque la limite du membre de droite est bien définie. De la même manière, il est facile de voir que la variance de la variable aléatoire $|\tau\cap[0,N]| =: \mathcal{N}_N(\tau)$ (rappelons que pour une partie A de $\mathbb{N}$, $|A|$ désigne le cardinal de A) vérifie :

$$\frac{\partial^2}{\partial^2\beta}\frac{1}{N}\log(Z^c_{N,\beta}) = \frac{1}{N}\mathbf{E}^c_{N,\beta}\left[\left(\mathcal{N}_N(\tau)-\mathbf{E}^c_{N,\beta}\left[\mathcal{N}_N(\tau)\right]\right)^2\right] \tag{1.2.15}$$

On voit ainsi que la dérivée et la dérivée seconde de l'énergie libre ont une interprétation physique très intuitive, et que l'étude de la régularité de $F(\cdot)$ joue un rôle important dans le compréhension du comportement macroscopique du système.

Dans cet exemple concret, il est possible de calculer la transformée de Laplace de $K(\cdot)$ et ainsi d'en déduire une expression explicite pour $F(\cdot)$ (et donc de ses dérivées). De manière plus intrinsèque, le membre de gauche de (1.2.6) est analytique pour $\beta > \beta_c$ et ceci implique que F l'est aussi sur (β_c,∞), donc sur $\mathbb{R}\setminus\{\beta_c\}$ tout entier (évidemment F ne peut être analytique en β_c). Une question intéressante est alors de déterminer l'ordre de la transition de phase (*ie* le plus petit entier k tel que F soit $k-1$ fois différentiable et ne le soit pas k fois) en β_c. Pour ce faire, nous établissons d'abord l'équivalence suivante :

$$1-\sum_{j\,\geqslant\,1}^{\infty}K(n)e^{-bn} \sim 2c_K\sqrt{\pi b} \tag{1.2.16}$$

valable pour $b \searrow 0$.

En sommant par parties, pour tout b positif, on obtient l'égalité suivante :

$$1 - \sum_{j \geqslant 1}^{\infty} K(n) e^{-bn} = (1 - \exp(-b)) \sum_{n=0}^{\infty} \left(\sum_{j=n+1}^{\infty} K(j) \right) \exp(-bn). \tag{1.2.17}$$

L'équivalence $K(n) \sim c_K n^{-3/2}$ implique alors $\sum_{j=n+1}^{\infty} K(j) \sim 2c_K n^{-1/2}$. On note que dans la somme dans le terme de droite de l'équivalence ci-dessus, on peut négliger un nombre arbitrairement grand de termes (l'erreur que l'on commet ce faisant est un $\mathcal{O}(b)$). D'autre part, pour $\varepsilon > 0$ fixé il existe n_0 assez grand tel que pour tout $n \geqslant n_0$, on ait $c_k - \varepsilon \leqslant K(n) n^{3/2} \leqslant c_k + \varepsilon$. En utilisant les sommes de Riemann, il vient alors :

$$\lim_{b \searrow 0} b^{1/2} \sum_{n \geqslant n_0} n^{-1/2} \exp(-bn) = \int_0^{\infty} \frac{1}{x^{1/2}} \exp(-x) dx = \sqrt{\pi} \tag{1.2.18}$$

et on en déduit immédiatement (1.2.16). Il ne reste plus qu'à rappeler l'équation (1.2.6) pour obtenir l'équivalence suivante, valable pour $\beta \searrow \beta_c$ (et donc $F(\beta) \searrow 0$) :

$$2c_K \sqrt{\pi F(\beta)} \sim (\beta - \beta_c) \tag{1.2.19}$$

dont on déduit immédiatement

$$F(\beta) \sim c(\beta - \beta_c)^2 \tag{1.2.20}$$

où l'on a défini $c := \frac{1}{4c_K^2 \pi}$.

Il résulte de (1.2.20) que F est C^1 en β_c mais n'est pas C^2. En particulier, ceci implique que la fraction de contact $N(\beta) := F'(\beta)$ est continue mais non dérivable au point critique.

1.2.2 Du modèle d'Ising anisotrope au modèle de mouillage dans une bande.

Le modèle de mouillage dont nous venons d'esquisser l'étude permet de modéliser ce qu'il se passe à la frontière entre deux milieux. Un exemple de ce type de situation est la description d'une interface entre un gaz et des goutelettes d'eau reposant sur un mur impénétrable (voir [DGZ05] pour des détails). Une modélisation *a priori* plus répandue pour ce genre de phénomènes est le modèle d'Ising. Il s'avère cependant (et c'est ce que nous illustrons dans ce paragraphe) que le modèle que nous avons développé dans la partie précédente n'est rien d'autre qu'un modèle d'Ising *anisotrope* (pour une présentation exhaustive de ce type de modèles, on peut se

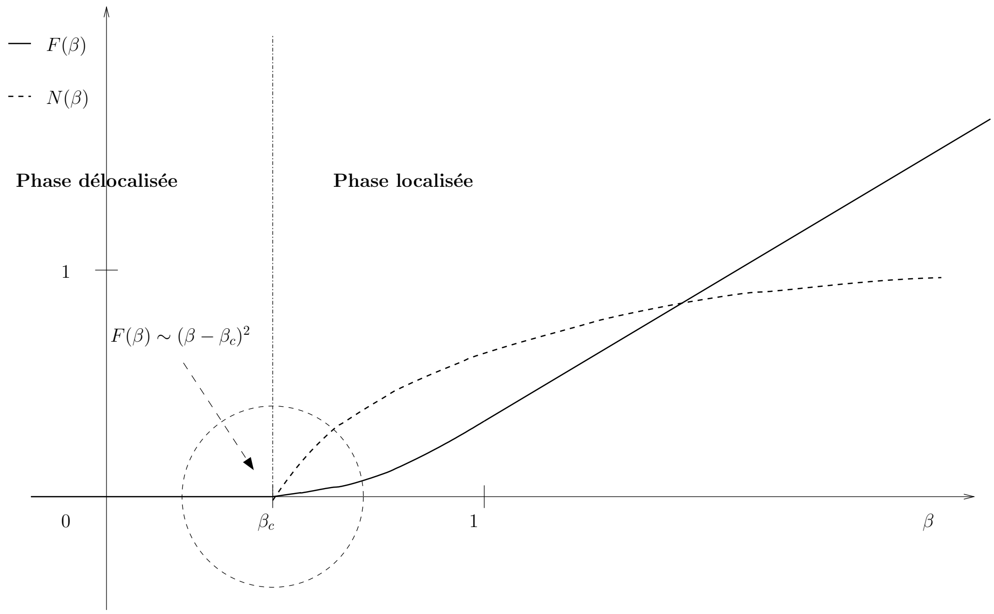

FIGURE 1.2. L'énergie libre et la fraction de contact pour un modèle d'accrochage homogène. $F(\cdot)$ n'est pas analytique en $\beta = \beta_c$. En β_c, $F(\cdot)$ est C^1 mais pas C^2, la transition est donc de second ordre.

référer aux travaux classiques d'Abraham [Abr86], Fisher [Fis84] ou plus récemment de Velenik [Vel06]).

On le définit dans une boîte $\Lambda := \{1, \ldots, L_1\} \times \{1, \ldots, L_2\}$ (où L_1 et L_2 sont deux entiers positifs) en se donnant une loi de probabilité $\mathbf{P}_\Lambda^{J_1,J_2}$ sur les configurations $\sigma \in \{-1, +1\}^{\Lambda \cup \partial\Lambda}$ où J_1, J_2 sont deux paramètres strictement positifs. On fixe les conditions de bord suivantes :

$$\begin{aligned} \sigma_{(x_1,0)} &= +1 \ \text{ pour } \ x_1 \in \{0, \ldots, L_1 + 1\} \ \text{ et} \\ \sigma_{(x_1,x_2)} &= -1 \ \text{ pour } \ (x_1, x_2) \in \partial\Lambda, x_2 \neq 0. \end{aligned} \tag{1.2.21}$$

$\mathbf{P}_\Lambda^{J_1,J_2}$ est alors définie par :

$$\mathbf{P}_\Lambda^{J_1,J_2}(\sigma) := \frac{1}{Z_\Lambda^{J_1,J_2}} \exp(-H_\Lambda^{J_1,J_2}(\sigma)) \tag{1.2.22}$$

où :

$$H_\Lambda^{J_1,J_2}(\sigma) = -\frac{1}{2} \sum_{i=1,2} J_i \sum_{(x,y)\in A_i} \sigma_x \sigma_y \tag{1.2.23}$$

et $A_i := \{(x, y) \in \mathbb{Z}^2, (x, y) \cap \Lambda \neq \emptyset, |x - y| = 1, |x_i - y_i| = 1\}$.

Remarquons que deux spins interagissent si et seulement s'ils sont voisins et qu'ils interagissent différemment selon qu'ils sont disposés de manière verticale ou

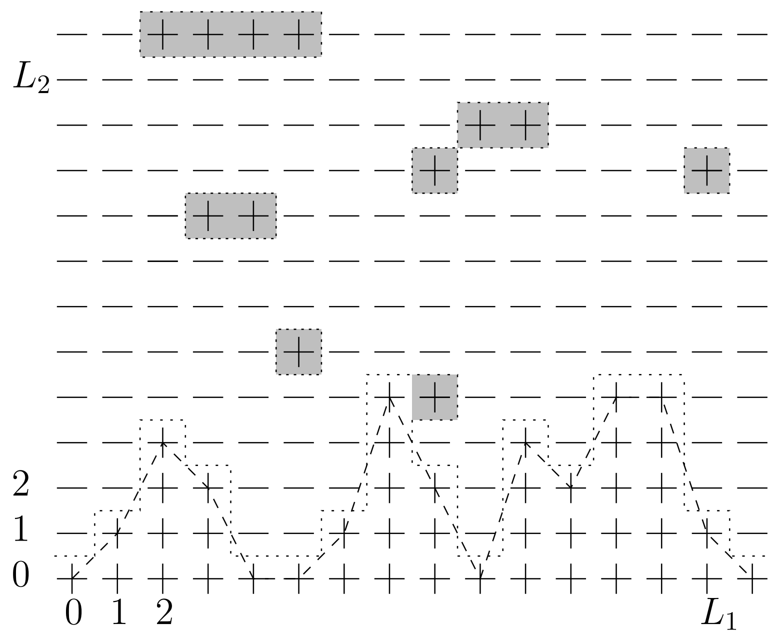

FIGURE 1.3. Une configuration possible. La ligne en pointillés fins représente l'interface. Dans la limite où $J_2 \nearrow \infty$, la configuration ci-dessus devient de probabilité nulle, mais si on flippe les + à l'intérieur des zones ombrées, on obtient une configuration de probabilité positive. Dans la limite $L_2 \to \infty$, la ligne en pointillés gras devient l'excursion d'une marche aléatoire sur l'intervalle $[0, L_1]$.

de manière horizontale l'un par rapport à l'autre. On peut considérer que leur interaction ne contribue de manière effective à l'énergie totale de la configuration que s'ils sont de signes différents. En effet, dans la définition de $H_\Lambda^{J_1,J_2}(\cdot)$, on peut remplacer les termes $\sigma_x\sigma_y$ par $-\frac{1}{2}(\sigma_x - \sigma_y)^2$ sans changer la mesure $\mathbf{P}_\Lambda^{J_1,J_2}(\cdot)$. En particulier, les contributions énergétiques n'ont lieu qu'aux points de contact entre un spin + et un spin − (autrement dit le long de la ligne en pointillés fins de la figure 1.3). Plus précisément, à une constante près, $H_\Lambda^{J_1,J_2}(\cdot)$ est égal à $2J_1$ fois la longueur des segments verticaux de la ligne en pointillés fins de la figure 1.3 additionné à $2J_2$ fois la longueur des segments horizontaux de cette même ligne. On convient d'appeler *interface* la ligne en pointillés fins.

Dans la limite $J_2 \to \infty$, on voit alors que $\mathbf{P}_\Lambda^{J_1,J_2}(\cdot)$ se concentre sur les configurations qui minimisent la longueur horizontale de l'interface (qui vaut donc forcément $L_1 + 1$). En particulier, les zones ombrées de la figure 1.3 disparaissent totalement. Dans cette limite, l'interface est donc réduite à une unique composante connexe. On peut donc considérer l'interface comme le graphe d'une fonction, ou même comme l'image d'une marche aléatoire (la ligne en gras dans la figure 1.3). L'énergie résiduelle est alors simplement donnée par la longueur verticale de l'interface. Au vu des conditions de bord, on voit que cette marche aléatoire est clouée aux extrémités, contrainte à être positive et à ne pas dépasser L_2. On voit aussi que la loi de son accroissement est proportionnelle à $\exp(-2J_1|t|)$. On en déduit que dans la limite $L_2 \nearrow \infty$, l'interface est décrite de manière satisfaisante par l'excursion renormalisée d'une marche aléatoire, autrement dit par le modèle de mouillage sans interaction.

Pour un entier $a \in \mathbb{Z}^+$, on peut alors ajouter des charges magnétiques au Hamiltonien de l'équation (1.2.22), autrement dit on peut considérer la loi $\mathbf{P}_\Lambda^{a,J_1,J_2}(\cdot)$ définie comme ci-dessus où le Hamiltonien vaut :

$$H_\Lambda^{a,J_1,J_2}(\sigma) = -\frac{1}{2}\sum_{i=1,2} J_i \sum_{(x,y)\in A_i} \sigma_x\sigma_y + h\sum_{x\in\mathcal{B}_a} \sigma_x \tag{1.2.24}$$

avec $h \in \mathbb{R}$ et où $\mathcal{B}_a := \{x \in \Lambda, x_2 \leqslant a\}$. Dans le cas où $a = 0$, en considérant la limite des lois $\mathbf{P}_\Lambda^{a,J_1,J_2}$ pour $J_2 \to \infty$ puis $L_2 \nearrow \infty$, des considérations similaires au paragraphe précédent illustrent le fait que l'interface est décrite de manière satisfaisante par un modèle de mouillage homogène similaire à celui que nous avons condidéré dans la partie 1.2.1.

Plus généralement, dans le cas où $a \geqslant 1$, on obtient de la même manière un nouveau modèle de mouillage qui reçoit de l'énergie lorsqu'il se trouve dans la région $\mathcal{B}_a$, le modèle de *mouillage dans la bande* $\mathcal{B}_a$. L'étude de ce modèle dans le cadre plus général où la marche aléatoire sous-jacente n'est plus supposée à pas discret fait l'objet du chapitre 3 de la présente thèse.

1.2.3 Une généralisation naturelle aux processus de renouvellement.

Une caractéristique de l'analyse que nous avons menée au paragraphe 1.2.1 est que seule la loi des retours à l'origine intervient dans l'étude de la régularité de l'énergie libre proche du point critique. Il est facile de voir que la mesure de mouillage homogène transforme la loi de la (p,q) marche aléatoire en changeant la loi des retours en zéro mais ne modifie pas la loi des excursions hors de l'origine lorsque leur taille a été fixée. Pour cette raison, il est naturel de considérer une généralisation de ce modèle. Celle-ci consiste à définir une suite de temps aléatoires $(\tau_n)_{n\geqslant 0}$ par :

1. $\tau_0 := 0$
2. les variables aléatoires $(\tau_n - \tau_{n-1})_{n\geqslant 1}$ sont indépendantes identiquement distribuées à valeurs dans $\mathbb{N}$.
3. il existe $\alpha > 0$ tel que

$$\mathbf{P}\left[\tau_1 = n\right] =: K(n) = \frac{L(n)}{n^{1+\alpha}} \tag{1.2.25}$$

où $L(\cdot)$ désigne une fonction à variation lente, autrement dit une fonction mesurable positive vérifiant que pour tout réel $c > 0$, $\frac{L(cx)}{L(x)} \to 1$ lorsque $x \to \infty$.

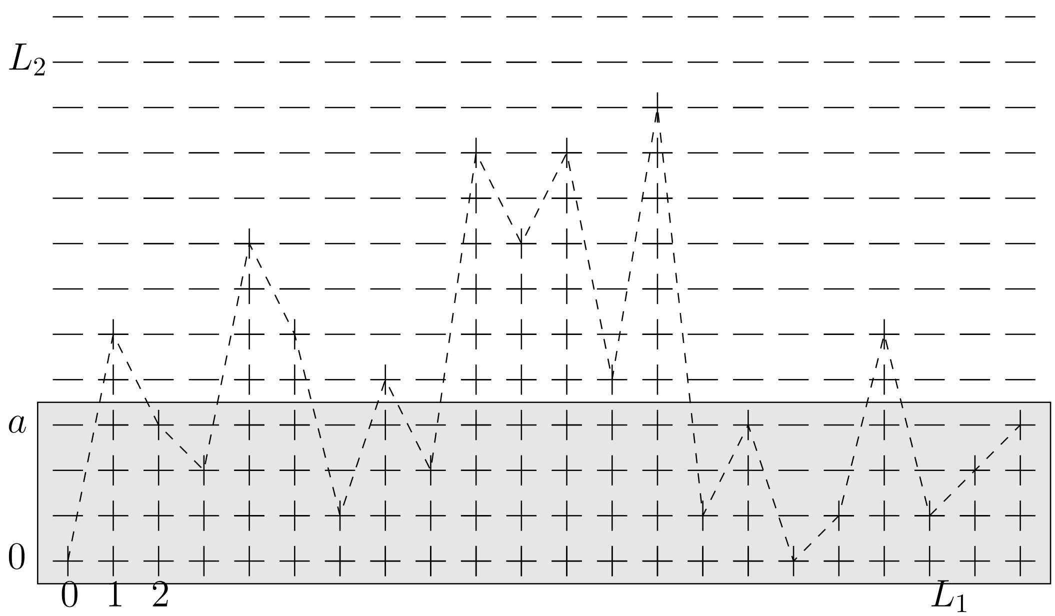

FIGURE 1.4. Une configuration dans laquelle on a déjà considéré la limite $J_2 \to \infty$. La ligne en pointillés représente le graphe d'une marche aléatoire dont la loi du pas est proportionnelle à $\exp(-J_1|t|)$. La zone en gris désigne la bande $\mathcal{B}_a$ dans laquelle le polymère reçoit de l'énergie.

Notons que, comme dans le cas du mouillage, on autorise le renouvellement τ à ne pas être récurrent (c'est aussi le cas par exemple lorsque τ désigne l'ensemble des temps de retour à l'origine de la marche aléatoire symétrique simple dans $\mathbb{Z}^d, d \geqslant 3$, où il est bien connu que $\alpha = d/2-1$), autrement dit il se peut que $\sum_{k=1}^{\infty} K(k) =: \Sigma_K < 1$.

Essentiellement tous les arguments que nous avons utilisés pour traiter le cas de la (p,q) marche aléatoire sont transposables au modèle ci-dessus. Plus précisément, posons $\beta_c := \inf\{\beta \in \mathbb{R}, F(\beta) > 0\}$. On définit

$$\mathcal{L} := \{\beta \in \mathbb{R}, F(\beta) > 0\}; \quad \mathcal{D} := \{\beta \in \mathbb{R}, F(\beta) = 0\}. \tag{1.2.26}$$

La phase $\mathcal{L}$ est appelée phase *phase localisée*, la phase $\mathcal{D}$ la *phase délocalisée*. La croissance et la convexité de $F(\cdot)$ impliquent que $\mathcal{L} = (\beta_c, \infty)$ et que $\mathcal{D} = (-\infty, \beta_c]$.

De plus, l'équation (1.2.6) implique immédiatement le fait que $\beta_c = -\log(\Sigma_K)$, et par conséquent $\beta_c \in [0, \infty)$. De manière similaire à ce que nous avons vu, on peut donner l'ordre de la transition de phase dans ce modèle de manière explicite. Plus précisément, on a le résultat (voir [Gia07, Théorème 2.1]) :

Théorème 1.2.1. *Pour $\beta \searrow \beta_c$, on a l'équivalence :*

$$F(\beta - \beta_c) \sim (\beta - \beta_c)^{\max(1,\alpha^{-1})} \widehat{L}((\beta - \beta_c)^{-1}) \tag{1.2.27}$$

où $\widehat{L}(\cdot)$ est une fonction à variation lente. D'autre part, pour $\beta \to \infty$:

$$F(\beta) = \beta - \log(K(1)) + o(1). \tag{1.2.28}$$

1.2.4 Limites d'échelles du modèle d'accrochage homogène.

On parle de limite d'échelle pour désigner l'objet aléatoire limite vers lequel un processus convenablement renormalisé converge en loi (par exemple, le mouvement Brownien est la limite d'échelle d'une marche aléatoire simple par le Théorème de Donsker). Dans cette section, nous nous intéresserons à la convergence en loi de l'ensemble $\tau_{(N)} := \tau/N \cap [0,1]$. Il s'agit d'un sous ensemble fermé aléatoire de l'intervalle $[0,1]$. On peut définir une topologie sur l'ensemble des fermés de $[0,1]$ (qu'on désigne par $\mathcal{F}$), la topologie de Gromov-Hausdorff (ou de Matheron). Il est bien connu que $\mathcal{F}$ muni de cette topologie est un espace polonais, *ie* compact, métrisable et séparable. On a alors les convergences en loi suivantes ([Gia07, Théorème 2.5]) :

Théorème 1.2.2. *Supposons que le processus τ suive la loi $\mathbf{P}^a_{N,\beta}$ où $a \in \{c, f\}$.*

1. *Si $\beta > \beta_c$, ie $\beta \in \mathcal{L}$, alors la suite $\{\tau_{(N)}\}_N$ converge en loi vers l'intervalle $[0,1]$ tout entier et ce aussi bien pour $a = c$ que pour $a = f$.*
2. *Si $\beta < \beta_c$, ie $\beta \in \overset{\circ}{\mathcal{D}}$, alors la suite $\{\tau_{(N)}\}_N$ converge en loi vers le singleton $\{0\}$ si $a = f$ et vers $\{0,1\}$ si $a = c$.*

Ce résultat justifie de manière satisfaisante la terminologie introduite dans l'équation (1.2.26). Notons aussi qu'il est obtenu en grande partie grâce à des estimées fines sur le comportement asymptotique des fonctions de partition aussi bien dans le cas libre que dans le cas contraint, estimées qui découlent elles mêmes de la représentation (1.2.10).

Pour décrire le comportement du système dans la phase critique, nous introduisons la notion d'*ensemble régénératif d'indice* $\alpha(\geqslant 0)$ que l'on note $\mathcal{A}_\alpha$.

Soit $(\sigma_t)_{t \geqslant 0}$ un processus de Lévy à valeurs dans $\mathbb{R}^+$. Un tel processus est appelé *subordinateur*. La littérature sur les subordinateurs est très vaste, une référence classique est [Ber99]. On peut caractériser entièrement la loi d'un subordinateur à l'aide de sa transformée de Laplace, *ie* la fonction $\Phi(\cdot)$ définie par

$$\mathbf{E}[e^{-\lambda\sigma_t}] =: \exp(-t\Phi(\lambda)). \tag{1.2.29}$$

La fonction $\Phi(\cdot)$ ainsi définie est appelée *exposant de Lévy* du subordinateur. L'ensemble $\overline{\{\sigma_t, t \geqslant 0\}}$ (où bien sûr pour $A \in \mathcal{B}(\mathbb{R})$, $\overline{A}$ désigne le plus petit fermé contenant A) est appelé *ensemble régénératif*. Le cas particulier où $\Phi(\lambda) = \lambda^\alpha$ pour $\alpha \in (0,1)$ correspond au cas où σ est un subordinateur stable d'indice α noté

$(\sigma_t^{(\alpha)})_{t\geqslant 0}$, et l'ensemble $\mathcal{A}_\alpha := \overline{\{\sigma_t^{(\alpha)}, t \geqslant 0\}}$ est alors appelé *ensemble régénératif d'indice* α. Un processus associé à $\sigma^{(\alpha)}$ dont nous allons avoir besoin est le temps local d'indice α, $(L_t^{(\alpha)})_{t\geqslant 0}$ qui est défini comme étant l'inverse généralisé de $\sigma^{(\alpha)}$, autrement dit $L_t^{(\alpha)} := \inf\{s \geqslant 0, \sigma_s^{(\alpha)} \geqslant t\}$.

Notons que $\mathbf{P}[\max \mathcal{A}_\alpha \cap [0,1] = 1] = 0$ et que la variable aléatoire $\max \mathcal{A}_\alpha \cap [0,1]$ suit la loi généralisée de l'arcsinus de paramètre α. Rappelons aussi que $\mathcal{A}_0 = \{0\}$ et que $\mathcal{A}_1 = [0,\infty)$. Pour $\gamma \in (0,1)$, l'ensemble aléatoire $\mathcal{A}_\gamma$ est hautement non trivial et correspond par exemple à l'ensemble des zéros du mouvement Brownien pour $\gamma = 1/2$.

Dans le cas critique, on peut finalement étendre le Théorème 1.2.2 (on donne le résultat dans le cas libre, voir [Gia07, Théorème 2.7]) :

Théorème 1.2.3. *Supposons que* τ *suive la loi* $\mathbf{P}_{N,\beta}^f$.

1. *Si le renouvellement sous-jacent est récurrent (ie si* $\Sigma_K = 1$*), la suite* $\{\tau_{(N)}\}_N$ *converge en loi vers* $\mathcal{A}_{\min(\alpha,1)} \cap [0,1]$ *où* $\mathcal{A}_\gamma$ *désigne l'ensemble régénératif d'indice* $\gamma \in [0,1]$.
2. *Si le renouvellement sous-jacent est transient (ie si* $\Sigma_K < 1$*) et* $\alpha \in (0,1)$*, alors* $\{\tau_{(N)}\}_N$ *converge vers* $\widetilde{\mathcal{A}}_\alpha \subset [0,1]$*, où pour* $\alpha \in (0,1)$*, la loi de* $\widetilde{\mathcal{A}}_\alpha$ *est absolument continue par rapport à celle de* $\mathcal{A}_\alpha \cap [0,1]$ *de dérivée de Radon Nykodyn donnée par* $(\alpha\pi/\sin(\alpha\pi))(1 - \max(\mathcal{A}_\alpha \cap [0,1]))^\alpha$*. D'autre part, si* $\Sigma_K < 1$ *et* $\sum_{n\geqslant 1}^\infty nK(n) < \infty$ *(ce qui est le cas en particulier lorsque* $\alpha > 1$*), alors la suite* $\{\tau_{(N)}\}_N$ *converge en loi vers l'intervalle* $[0,U]$ *où* U *est une variable aléatoire de loi uniforme sur* $[0,1]$.

1.3 La longueur de corrélation du modèle d'accrochage homogène.

On voit ainsi que le caractère totalement résoluble du modèle d'accrochage homogène apporte une très grande quantité d'informations quant au comportement de sa limite d'échelle ; il semble donc pertinent de s'en servir pour caractériser de manière physiquement satisfaisante une notion bien connue des physiciens, la *longueur de corrélation*.

La longueur de corrélation ξ d'un système physique est une quantité qui joue un rôle important dans la littérature consacrée à la physique statistique (citons par exemple l'ouvrage de Cardy, [Car96]). Pour la définir, on introduit d'abord la *fonction de corrélation* $c(\cdot)$ (voir à ce propos [Gia08]) ; on peut montrer ([Gia07, Théorème 2.3]) que pour tout $\beta \in \mathbb{R}$ et pour $a \in \{c,f\}$, la limite de volume infini $\mathbf{P}_{\infty,\beta}^a$ de la suite de lois $\{\mathbf{P}_{N,\beta}^a\}_N$ existe. La loi de $\mathbf{P}_{\infty,\beta}^a$ ne dépend pas de a et est donnée par la loi du processus de renouvellement $\tau(\beta)$ de lois d'interarrivées

$$\mathbf{P}\left[\tau_1(\beta) = k\right] := e^\beta e^{-F(\beta)k} K(k). \tag{1.3.1}$$

Au vu de l'équation (1.2.6), il s'agit d'un renouvellement transient dès lors que $\beta \in \overset{\circ}{\mathcal{D}}$ et récurrent (et même récurrent positif si $\beta \in \overset{\circ}{\mathcal{L}}$) sinon. Pour $\beta \in \mathcal{L}$, on définit alors $c(\cdot)$ par :

$$
\begin{aligned}
&c(m) := \\
&\lim_{n\to\infty} \frac{\mathbf{P}[m \in \tau(\beta), m+n \in \tau(\beta)] - \mathbf{P}[m \in \tau(\beta)]\mathbf{P}[m+n \in \tau(\beta)]}{\sqrt{\mathbf{P}[m \in \tau(\beta)](1-\mathbf{P}[m \in \tau(\beta)])\mathbf{P}[m+n \in \tau(\beta)](1-\mathbf{P}[m+n \in \tau(\beta)])}} \\
&\qquad = \frac{\mathbf{E}[\tau_1(\beta)]}{\mathbf{E}[\tau_1(\beta)]-1}\left(\mathbf{P}[m \in \tau(\beta)] - \frac{1}{\mathbf{E}[\tau_1(\beta)]}\right)
\end{aligned}
\tag{1.3.2}
$$

où l'on s'est servi du Théorème du renouvellement. On peut alors définir la longueur de corrélation comme l'inverse du taux de décroissance de c dans l'asymptotique de Laplace, autrement dit

$$
\begin{aligned}
\widetilde{\xi}(\beta) &:= -1/\limsup_{m\to\infty} m^{-1}\log(|c(m)|) \\
&= -1/\limsup_{m\to\infty} m^{-1}\log\left(\mathbf{P}[m \in \tau(\beta)] - \frac{1}{\mathbf{E}[\tau_1(\beta)]}\right).
\end{aligned}
\tag{1.3.3}
$$

Il a été montré récemment [Gia08] que, pour $\beta \searrow \beta_c$, on a l' équivalence $\widetilde{\xi}(\beta) \sim 1/F(\beta)$, autrement dit dans la phase localisée et proche du point critique, cette longueur se comporte comme l'inverse de l'énergie libre.

Ceci nous amène à définir ξ de manière plus pratique de la manière suivante :

$$
\xi(\beta) = \frac{1}{F(\beta)} \quad \text{si} \quad \beta > \beta_c \quad \text{et} \quad +\infty \quad \text{sinon.} \tag{1.3.4}
$$

Cette définition *a priori* différente de la précédente se justifie par le fait que l'on s'attend à ce que, proche du point critique, quelle que soit la manière dont on la définisse, ξ devienne la seule quantité *pertinente* du système, au sens où lorsque la taille du système devient beaucoup plus grande ou beaucoup plus petite que ξ, il n'apparaît asymptotiquement pas d'effet de volume fini. Dans le cas du modèle d'accrochage homogène, on illustre ce concept de manière plus quantitative.

On supposera dans la suite que $\alpha \in (0,1)$ et pour alléger l'écriture, on suppose que la fonction à variation lente dans la définition (1.2.25) est identiquement constante (disons égale à $C_K > 0$), les techniques que nous utilisons permettent de traiter aussi bien le cas où $L(\cdot)$ n'est pas constante et les limites d'échelle que nous obtenons sont les mêmes dans ce cas là. Le Théorème 1.2.1 nous indique que proche du point critique, $\xi(\cdot)$ se comporte de la manière suivante :

$$
\xi(\beta) \overset{\beta \searrow \beta_c}{\sim} c_\alpha(\beta - \beta_c)^{-1/\alpha}. \tag{1.3.5}
$$

où $c_\alpha > 0$. On s'attend à ce que des effets de volume fini apparaissent seulement lorsque l'on maintient le système à une taille proportionnelle à ξ, autrement dit si l'on contraint la quantité $\xi(\beta)/N =: q$ à rester fixe lorsque N devient grand. Au vu de l'équation (1.3.5), proche du point critique, ceci revient à supposer que la quantité $\frac{(\beta-\beta_c)^{-1/\alpha}}{N}$ reste fixe lorsque N devient grand, et donc *in fine* à ce qu'il existe un réel ε tel que

$$\beta = \beta_c + \varepsilon N^{-\alpha} \tag{1.3.6}$$

On montre que si cette contrainte est satisfaite proche du point critique, les observables du système ont toutes un comportement différent de ce qu'elles ont dans chacune des deux phases.

Il n'est pas difficile de le vérifier pour les quantités physiquement importantes que nous avons déjà mentionnées. Dans le cas de la fraction de contact de volume fini et pour τ supposé récurrent (et donc $\beta_c = 0$), pour $N \to \infty$, on a l' équivalence :

$$\begin{aligned} \rho_N(\beta) &= \mathbf{E}^f_{N,\beta}\left[\frac{\mathcal{N}_N(\tau)}{N}\right] \\ &= \frac{\mathbf{E}\left[\frac{\mathcal{N}_N(\tau)}{N}\exp\left(\frac{\varepsilon\mathcal{N}_N(\tau)}{N^\alpha}\right)\right]}{\mathbf{E}\left[\exp\left(\frac{\varepsilon\mathcal{N}_N(\tau)}{N^\alpha}\right)\right]} \\ &\sim cN^{\alpha-1}\frac{\mathbf{E}\left[L_1^{(\alpha)}\exp\left(c'\varepsilon L_1^{(\alpha)}\right)\right]}{\mathbf{E}\left[\exp\left(c'\varepsilon L_1^{(\alpha)}\right)\right]} =: N^{\alpha-1}\breve{\rho}(q) \end{aligned} \tag{1.3.7}$$

où l' équivalence et l'existence des constantes $c(q), c'(q) > 0$ sont des conséquences des résultats que nous démontrerons au chapitre 2. De manière équivalente, on peut voir qu'il existe une fonction $\widetilde{\rho}(q)$ telle que pour $N \to \infty$, on ait

$$\rho_N(\beta) \sim \xi(\beta)^{\alpha-1}\widetilde{\rho}(q) \tag{1.3.8}$$

autrement dit on peut exprimer la fraction de contact de volume fini comme une fonction de la longueur de corrélation ξ exclusivement.

On montre que cet exemple n'est rien d'autre qu'un cas particulier d'un phénomène beaucoup plus général, à savoir la convergence en loi de la mesure du système.

Pour formaliser cette assertion, on introduit une suite $(b_n)_n$ de réels positifs telle que $\mathcal{N}_N(\tau)/b_N \Rightarrow L_1^{(\alpha)}$ (rappelons que pour une suite de variables aléatoires X_n, $X_n \Rightarrow X$ signifie que X_n converge en loi vers X). Dans le cas où la fonction à variation lente dans (1.2.25) est constante, b_n est simplement la suite $(n^{-\alpha})_{n \geqslant 1}$. On montre le résultat suivant :

Théorème 1.3.1. *Soit ε un réel quelconque.*

1. *Si $\Sigma_K = 1$ (donc $\beta_c = 0$) et si τ suit la loi $\mathbf{P}_{N,\varepsilon/b_N}$, alors :*

$$\tau_{(N)} \Rightarrow \mathcal{B}_{\alpha,\varepsilon} \tag{1.3.9}$$

où $\mathcal{B}_{\alpha,\varepsilon}$ est un élément de $\mathcal{F}$ dont la loi est absolument continue par rapport à la loi de $\mathcal{A}_\alpha$ de densité de Radon-Nykodym valant $\exp(\varepsilon L_1^{(\alpha)})/\mathbf{E}\left[\exp\left(\varepsilon L_1^{(\alpha)}\right)\right]$.

2. *Si $\Sigma_K < 1$ et si τ suit la loi $\mathbf{P}_{N,\beta_c+\varepsilon/b_N}$, alors :*

$$\tau_{(N)} \Rightarrow \widetilde{\mathcal{B}}_{\alpha,\varepsilon} \tag{1.3.10}$$

où $\widetilde{\mathcal{B}}_{\alpha,\varepsilon}$ est un élément de $\mathcal{F}$ dont la loi est absolument continue par rapport à la loi de $\mathcal{A}_\alpha$ de densité de Radon-Nykodym valant $\left(\exp(\varepsilon L_1^{(\alpha)})(1-\max\mathcal{A}_\alpha\cap[0,1])^\alpha\right) / \left(\mathbf{E}\left[\exp(\varepsilon L_1^{(\alpha)})(1-\max\mathcal{A}_\alpha\cap[0,1])^\alpha\right]\right)$.

Ce résultat a fait l'objet d'une publication dans la revue ALEA ([Soh09]).

1.4 Limites d'échelle pour des modèles homogènes.

1.4.1 Le δ modèle de mouillage

Les Théorèmes 1.2.2 et 1.2.3 traitent des limites d'échelle dans le cadre du modèle de renouvellement. Une problématique naturelle est alors de déterminer la limite d'échelle de la (p,q) marche aléatoire S sous les lois $\mathbf{P}^f_{N,\beta}$ et $\mathbf{P}^c_{N,\beta}$. Cette question a été traitée en grande généralité.

Plus concrètement, on peut généraliser les modèles d'accrochage précédents basés sur des marches aléatoires à pas discret au cas où le pas de S n'est plus supposé à valeurs entières. On fait pour cela l'hypothèse que S est à accroissements continus (*ie* de densité $h(\cdot)$ continue) de moyenne nulle et de variance finie ($\mathbf{E}[S_1^2] =: \sigma^2 \in (0,\infty)$). Evidemment, définir précisément l'analogue des lois $\mathbf{P}^c_{N,\beta}$ et $\mathbf{P}^f_{N,\beta}$ dans ce cadre là est moins immédiat. Une approximation naturelle pour ce faire est de se donner un réel $a > 0$ et de définir les lois suivantes :

$$\frac{d\mathbf{P}^f_{N,a,\beta}}{d\mathbf{P}} := \frac{1}{Z^f_{N,a,\beta}} \exp\left(\beta \sum_{k=1}^N \mathbf{1}_{S_k\in[0,a]}\right) \mathbf{1}_{\mathcal{C}_N} \tag{1.4.1}$$

le modèle d'accrochage dans une bande, et sa version contrainte :

$$\frac{d\mathbf{P}^c_{N,a,\beta}}{d\mathbf{P}} := \frac{1}{Z^c_{N,a,\beta}} \exp\left(\beta \sum_{k=1}^N \mathbf{1}_{S_k\in[0,a]}\right) \mathbf{1}_{\mathcal{C}_N}\mathbf{1}_{S_N\in[0,a]} \tag{1.4.2}$$

où l'on rappelle que $\mathcal{C}_N := \{S_1 \geqslant 0,\dots,S_N \geqslant 0\}$ et où $\beta\in\mathbb{R}, N\in\mathbb{N}$.

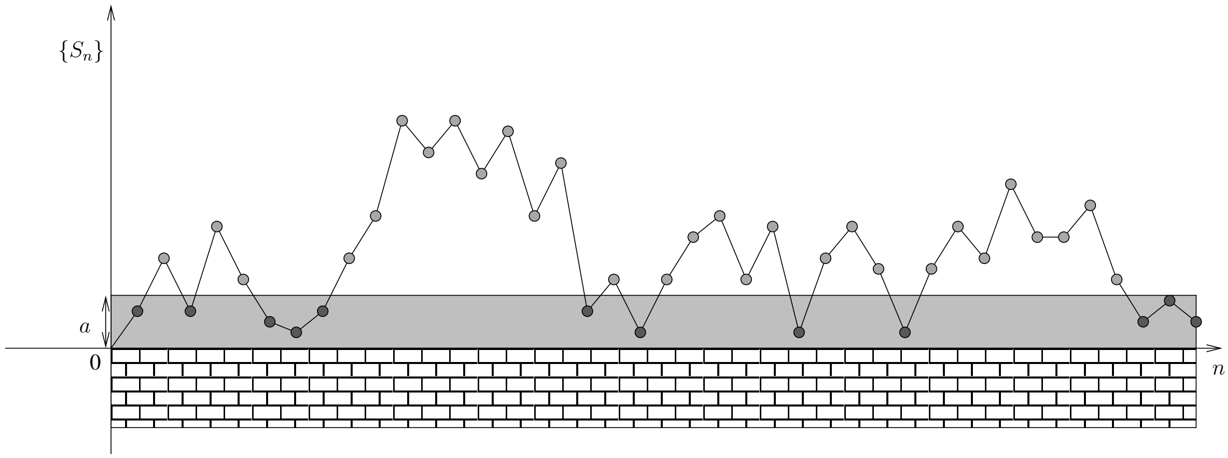

FIGURE 1.5. Une trajectoire de $\mathbf{P}^c_{41,a,\beta}$ avec 12 sites accrochés. Le graphe de (n, S_n) représente l'interface séparant une phase liquide d'une phase gazeuse, la bande $[0, a]$ étant une zone dans laquelle se trouvent des impuretés. L'interface demeure conditionnée à rester au dessus d'un mur qui l'attire ou la repousse lorsqu'elle passe *proche du mur*.

Ce modèle se révèle au premier abord malaisé à traiter, en particulier parce que la suite des temps de retour de S dans la bande $[0, \infty) \times [0, a]$ ne forme pas un processus de renouvellement. C'est pourquoi la littérature dédiée aux limites d'échelle des modèles de mouillage s'est d'abord consacré au modèle de δ *mouillage homogène* dans lequel la structure de renouvellement vue dans le paragraphe précédent est encore présente. Dans le cadre du δ mouillage, on modifie la loi de S en attribuant un bonus d'énergie pour les retours en zéro de la marche, de la forme $\varepsilon\delta_0$, où δ_0 est la mesure de Dirac en zéro (ce Dirac compense le fait que pour notre marche, la probabilité de toucher zéro est nulle). Formellement, pour $u \in \{c, f\}$, la mesure de polymère $\mathbf{P}^u_N$ est la loi de probabilité sur $(\mathbb{R}^+)^N$ définie par :

$$\mathbf{P}^u_{N,\varepsilon}(dx) := \frac{1}{Z^u_{N,\varepsilon}} h^u_N(x_1, \ldots, x_N) \prod_{i=1}^{N} \left(dx_i^+ + \varepsilon\delta_0(dx_i)\right) \tag{1.4.3}$$

où $\varepsilon \in \mathbb{R}$, $\delta_0(\cdot)$ désigne la mesure de Dirac en 0, dx^+ la mesure de Lebesgue sur $\mathbb{R}^+$ et

$$\begin{aligned} h^f_N(x_1, \ldots, x_N) &:= h(x_1)h(x_2 - x_1) \ldots h(x_N - x_{N-1}), \\ h^c_N(x_1, \ldots, x_N) &:= h(x_1)h(x_2 - x_1) \ldots h(-x_{N-1}). \end{aligned} \tag{1.4.4}$$

Evidemment, les fonctions de partition de ces modèles s'écrivent :

$$\begin{aligned} Z^f_{N,\varepsilon} &= \int_{\mathbb{R}^N} h^f_N(x_1,\ldots,x_N)\left(dx_i^+ + \varepsilon\delta_0(dx_i)\right), \\ Z^c_{N,\varepsilon} &= \int_{\mathbb{R}^N} h^c_N(x_1,\ldots,x_N)\left(dx_i^+ + \varepsilon\delta_0(dx_i)\right). \end{aligned} \tag{1.4.5}$$

Les exposants f et c font encore référence respectivement au cas *free* et au cas *constrained*. Pour $u \in \{c, f\}$, les lois $\mathbf{P}^u_{N,\varepsilon}$ ne sont rien d'autre que la limite pour $a \searrow 0$ des lois $\mathbf{P}^u_{N,a,\log(\varepsilon/a)}$.

On peut montrer que les caractéristiques des modèles définis par l'équation (1.4.3) sont très similaires au modèle basé sur la (p, q) marche aléatoire. Par exemple, on peut montrer l'existence d'un réel $\varepsilon_c > 0$ tel qu'il y ait une transition de phase en $\varepsilon = \varepsilon_c$ ([DGZ05]).

Les modèles de δ mouillage ont été très étudiés dans la littérature physique consacrée aux polymères, citons les articles de Fisher [Fis84] et Upton [Upt99] dans lesquels des preuves de la convergence en loi du processus renormalisé dans chacune des phases ont été obtenues. Celles ci sont plus ou moins rigoureuses et essentiellement basées sur des calculs explicites dans des cas particuliers. Dans le cadre de la marche aléatoire (p, q), les limites d'échelle ont été prouvées par Isozaki et Yoshida [IY01] en se servant d'identités combinatoires propres aux marches aléatoires symétriques au plus proche voisin. Deuschel, Giacomin et Zambotti [DGZ05] ont traité ce problème dans un cadre général de manière mathématiquement satisfaisante.

De manière plus quantitative, on définit la fonction $X^N : \mathbb{R}^N \to C([0, 1])$:

$$X^N_t(x) := \frac{x_{[Nt]}}{\sigma\sqrt{N}} + (Nt - [Nt])\frac{x_{[Nt]+1} - x_{[Nt]}}{\sigma\sqrt{N}}, \quad t \in [0,1] \tag{1.4.6}$$

où $[\cdot]$ désigne la partie entière. Pour $u \in \{c, f\}$, on définit la mesure $\mathbf{Q}^u_{\varepsilon,N} := \mathbf{P}^u_{N,\varepsilon} \circ (X^N)^{-1}$. Afin de caractériser la transition de phase à l'aide des limites d'échelle du système, nous aurons besoin de processus bien connus :

- ⋆ le mouvement brownien $(B_t)_{t \geqslant 0}$.
- ⋆ le pont brownien $(\widetilde{B}_t)_{t\in[0,1]}$, le mouvement brownien conditionné à s'annuler en $t = 1$.
- ⋆ le méandre brownien $(m_t)_{t\in[0,1]}$, le mouvement brownien conditionné à être positif sur $[0, 1]$.
- ⋆ l'excursion brownienne normalisée $(e_t)_{t\in[0,1]}$, le pont brownien conditionné à être positif sur $[0, 1]$.

Etant donné que ces processus sont obtenus en conditionnant le mouvement brownien par rapport à des événements de probabilité nulle, les définir rigoureusement n'est pas évident, une référence classique pour ce faire est [RY99].

Le résultat suivant a été montré par Deuschel, Giacomin et Zambotti dans [DGZ05], il donne une caractérisation satisfaisante pour chacune des phases du système :

Théorème 1.4.1. *Soit* $\varepsilon \in \mathbb{R}$.

1. *Si* $\varepsilon \in [0, \varepsilon_c)$,
   - *la suite* $\mathbf{Q}^c_{\varepsilon,N}$ *converge faiblement dans* $C([0,1])$ *vers* e.
   - *la suite* $\mathbf{Q}^f_{\varepsilon,N}$ *converge faiblement dans* $C([0,1])$ *vers* m.
2. *Si* $\varepsilon = \varepsilon_c$,
   - *la suite* $\mathbf{Q}^f_{\varepsilon_c,N}$ *converge faiblement dans* $C([0,1])$ *vers* $|B|$.
   - *la suite* $\mathbf{Q}^c_{\varepsilon_c,N}$ *converge faiblement dans* $C([0,1])$ *vers* $|\widetilde{B}|$.
3. *Si* $\varepsilon > \varepsilon_c$, *alors pour* $a \in \{c, f\}$, *les deux suites* $\mathbf{Q}^a_{\varepsilon,N}$ *convergent en loi vers la loi de la fonction identiquement nulle sur* $[0,1]$.

Ce résultat a à son tour été étendu au cas où S se trouve dans le domaine d'attraction d'une loi normale standard par Caravenna, Giacomin et Zambotti [CGZ06].

La preuve du Théorème 1.4.1 utilise de manière cruciale un découplage remarquable entre l'ensemble des points de contact $\mathcal{I}_N := \{i \in [0,N], S_i = 0\}$ et les excursions de S entre 2 points de contact. Il est facile de voir que condionnellement à $\mathcal{I}_N = \{t_1, \dots, t_k\}$, les excursions $(e_i)_{i \leqslant k} := \{S_{t_i+n}\}_{0 \leqslant n \leqslant t_i - t_{i-1}}$ sont indépendantes sous la loi $\mathbf{P}^f_{N,\varepsilon}$ et ont la même loi que le processus $(S, \mathbf{P})$ conditionné par rapport à l'événement $\{S_{t_i} = 0\}$.

Des résultats analogues au Théorème 1.4.1 ont été obtenus dans des modèles proches du δ mouillage.

Convergences d'échelle en temps continu Considérons le modèle suivant :

$$\frac{d\mathbf{P}_{\beta,T}}{d\mathbf{P}_T} := \frac{\exp\left(\beta \int_0^T v(x(t))dt\right)}{Z_{\beta,T}} \tag{1.4.7}$$

où $\mathbf{P}_T$ est la loi d'un mouvement brownien standard dans $\mathbb{R}^d$, $d \geqslant 1$ considéré jusqu'à l'instant $T > 0$, $\beta \in \mathbb{R}^+$ et $v(\cdot)$ est une fonction $\mathcal{C}^\infty_c(\mathbb{R}^d, \mathbb{R})$. On introduit le processus

$$y^T(t) := x(tT)/\sqrt{T}, \;\; t \in [0,1] \tag{1.4.8}$$

où le processus $x(\cdot)$ suit la loi $\mathbf{P}_{\beta,T}$. Ce modèle présente formellement de grandes similarités avec la loi introduite dans l'équation (1.4.1) (en considérant $v(\cdot) = \mathbf{1}_{[0,a]}(\cdot)$). Cranston, Koralov, Molchanov et Vainberg ont montré récemment le résultat suivant [CKMV09], qui est évidemment très semblable au Théorème 1.4.1 :

Théorème 1.4.2. *Pour $d \geqslant 3$, il existe $\beta_{cr} > 0$ tel que pour $\beta \in [0, \beta_{cr})$, le processus $y^T(\cdot)$ converge en loi dans $C([0,1], \mathbb{R}^d)$ lorsque $T \to \infty$ vers la loi du mouvement brownien standard d-dimensionnel. De la même manière, lorsque $x(\cdot)$ suit la loi $\mathbf{P}_{\beta,T}[\,\cdot\,|x(T) = 0]$, la loi du processus $y^T(\cdot)$ converge vers celle du pont brownien d-dimensionnel.*

Convergences d'échelle multidimensionnelles sous contrainte. Une autre classe de modèles considérés récemment consiste en une généralisation multi dimensionelle des lois de mouillage au cas où le polymère reçoit de l'énergie dans une région de l'espace. Pour $d \geqslant 1$, soit M un sous espace vectoriel de $\mathbb{R}^d$ et soit ν la mesure définie par $\nu(dy) = dy^{(1)}\delta_0(dy^{(2)})$ obtenue en étendant la mesure de surface sur M en utilisant la décomposition $y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^d \cong M \times M^{\perp}$. Pour $a, b \in \mathbb{R}^d_+$, on considère les modèles suivants :

$$\mu_N^{f,\varepsilon,+}(d\phi) := \frac{1}{Z_N^{f,\varepsilon,+}} e^{-H_N(\phi)} \delta_{aN}(\phi_0) \prod_{i=1,\dots,N} \left(\varepsilon\nu(d\phi_i) + d\phi_i^{(+)}\right), \tag{1.4.9}$$

et sa version contrainte

$$\mu_N^{c,\varepsilon,+}(d\phi) := \frac{1}{Z_N^{c,\varepsilon,+}} e^{-H_N(\phi)} \delta_{aN}(\phi_0) \prod_{i=1,\dots,N-1} \left(\varepsilon\nu(d\phi_i) + d\phi_i^{(+)}\right) \delta_{bN}(\phi_N) \tag{1.4.10}$$

où $d\phi_i^{(+)}$ désigne la mesure de Lebesgue sur $\mathbb{R}^d_+$ et

$$H_N(\phi) := \frac{1}{2} \sum_{i=1}^{N-1} |\phi_{i+1} - \phi_i|^2. \tag{1.4.11}$$

Ce modèle est une généralisation multidimensionelle des modèles définis dans l'équation (1.4.3) dans le cas où le processus libre est gaussien et où les extrémités du polymère sont soumis à des contraintes de l'ordre de N. Bolthausen, Funaki et Otobe ont alors prouvé l'existence d'une transition de phase en $\varepsilon_{cr} > 0$ [BFO09] et déterminé son ordre. D'autre part, ils ont en particulier mis en valeur la coexistence de deux limites d'échelle distinctes dans le cas où $\operatorname{codim}(M) = 2$. Ces résultats ont à leur tour été généralisés dans un cadre où le processus libre n'est plus gaussien par Funaki et Otobe [Fun08].

1.4.2 Le modèle de mouillage dans une bande.

Comme dans le paragraphe précédent, on définit les lois $Q^c_{N,a,\beta}$ et $Q^f_{N,a,\beta}$ sur $C([0,1])$ associées à $\mathbf{P}^c_{N,a,\beta}$ et à $\mathbf{P}^f_{N,a,\beta}$.

Dans ce modèle aussi on prouve l'existence d'une transition de phase en $\beta_c^a > 0$. On la caractérise de manière très similaire au cas du δ-modèle de mouillage. Plus précisément, on montre le théorème suivant :

Théorème 1.4.3. *Les deux lois* $\mathbf{P}^f_{N,a,\beta}$ *et* $\mathbf{P}^c_{N,a,\beta}$ *possèdent une transition de phase en* $\beta = \beta_c^a > 0$ *au sens où :*

1. *Dans la phase sous-critique, ie pour* $\beta < \beta_c^a$*,*
   - $(Q^c_{N,a,\beta})_N$ *converge faiblement dans* $C([0,1])$ *vers la loi de* e*.*
   - $(Q^f_{N,a,\beta})_N$ *converge faiblement dans* $C([0,1])$ *vers la loi de* m*.*
2. *Dans la phase surcritique, ie pour* $\beta > \beta_c^a$*, les deux lois* $(Q^c_{N,a,\beta})_N$ *et* $(Q^f_{N,a,\beta})_N$ *convergent dans* $C([0,1])$ *vers la mesure qui se concentre sur la fonction identiquement nulle sur* $[0,1$*.*

Le point manquant de ce résultat est bien sûr la convergence du système au point critique. Notons que la preuve des Théorèmes 1.4.3 et 1.4.1 se base de manière cruciale sur des estimées asymptotiques des fonctions de partition dans chacune des phases du système. Ces estimées se révèlent nettement plus difficiles à obtenir dans le cas critique, il faudrait pour ce faire étendre au cadre du renouvellement markovien un résultat fin de Doney sur les processus de renouvellement récurrents nuls ([Don97]). On peut cependant probablement traiter cette difficulté dans le cas où S est à valeurs discrètes de manière similaire à ce qui a étéfait dans le cas où le désordre est périodique (pour plus de précisions, voir [CGZ07]).

On a déjà remarqué que la différence fondamentale etre les lois définies par (1.4.1), (1.4.2) et les modèles considérés jusqu'alors dans la littérature aussi bien physique que mathématique (et en particulier ceux que nous avons cités au paragraphe précédent) provient du fait que la suite des temps de retour $\tau := (\tau_i)_{i \geqslant 0}$ de S dans la bande $\mathbb{R}^+ \times [0,a]$ (définie par $\tau_0 := 0, \tau_{i+1} := \inf\{k > \tau_i, S_k \in [0,a]\}$) ne forme pas une suite de variables aléatoires indépendantes identiquement distribuées. Cette différence fait qu'il devient délicat d'appliquer les résultats très puissants issus de la littérature consacrée aux processus de renouvellement classiques. Malgré tout, τ conserve certaines propriétés utiles ; en particulier, conditionnellement au processus J défini par $J_i := S_{\tau_i}$, les accroissements de τ sont indépendants mais non identiquement distribués. τ est un *processus de renouvellement markovien* de chaîne modulante J.

Cette considération est l'ingrédient essentiel qui permet d'exprimer l'énergie libre associée à ce système comme la valeur propre de Perron-Frobenius d' un opérateur hilbertien de $L^2([0,a])$, une constatation qui se révèle extrêmement fructueuse. Couplé à la connaissance du comportement asymptotique de l'overshoot (c'est à dire au comportement asymptotique de la loi jointe du premier instant d'entrée dans la bande et du lieu de ce point d'entrée), comportement qui fait lui-même l'objet du chapitre 4 de la présente thèse, il permet d'obtenir des estimées précises sur les fonctions de partition $Z^c_{N,a,\beta}$ et $Z^f_{N,a,\beta}$ dès lors que $\beta \neq \beta_c^a$. Celles-ci impliquent très facilement la convergence du Théorème 1.4.3 dans le régime surcritique. Dans le régime sous-critique, on montre que l'ensemble des points de contact du polymère avec la bande se concentre proche de l'origine dans le cas libre et proche des extrémités du polymère dans le cas contraint. On utilise alors le pendant de la propriété de découplage pour le δ-modèle de mouillage. Plus précisément, conditionnellement à $\tau \cap [0,N] = \{t_1, \ldots, t_k\}$ et au vecteur $(S_{t_1}, \ldots, S_{t_k})$, les excursions $e_i = \{e_i(n)\}_n := \{(S_{t_i+n}\}_{0 \leqslant n \leqslant t_{i+1}-t_i}\}$ sont indépendantes sous la loi $\mathbf{P}^c_{N,a,\varepsilon}$ et suivent la loi $(S, \mathbf{P}_{S_{t_i}})$ conditionnée par rapport à l'événement $\{S_{t_{i+1}-t_i} \in [0,a], S_{t_i+j} > a, j \in \{1, \ldots, t_{i+1}-t_i-1\}\}$.

En associant cette propriété au résultat de convergence vers l'excursion brownienne que l'on montre dans le chapitre 5 de la présente thèse, on peut déduire relativement aisément le Théorème 1.4.3.

1.5 Deux résultats issus de la théorie des fluctuations.

Cette partie est consacrée à l'énoncé de deux résultats issus de la théorie des fluctuations pour les marches aléatoires indispensables à la preuve du Théorème 1.4.3. On a choisi de les présenter de manière séparée afin d'alléger le chapitre 3. Notons que ces deux résultats présentent un intérêt propre.

1.5.1 Comportement asymptotique de l'overshoot

On s'intéresse ici au comportement asymptotique de la loi du point d'entrée dans le demi-plan supérieur pour une marche aléatoire partant d'un réel $-a$ où $a > 0$, *ie* à la quantité $\mathbf{P}_{-a}[S_1 \leqslant 0, \ldots, S_{n-1} \leqslant 0, S_n > 0, S_n \in dx]$ où $x \geqslant 0$. On montre que cette quantité convenablement renormalisée converge *uniformément pour* $x \in \mathbb{R}^+$. Ce résultat permet en particulier d'estimer uniformément le comportement asymptotique du noyau $F_{x,dy}(n)$ défini par :

$$\begin{aligned} &F_{x,dy}(n) := \mathbf{P}_x[S_1 > a, S_2 > a, \ldots, S_{n-1} > a, S_n \in dy]\mathbf{1}_{x,y\in[0,a]} \ \text{ si } \ n \geqslant 2, \\ &F_{x,dy}(1) := h(y-x)\mathbf{1}_{x,y\in[0,a]}dy \end{aligned} \tag{1.5.1}$$

et cette estimation forme la clef de voûte de la preuve du Théorème 1.4.3.

Déterminer le comportement en temps long de $\mathbf{P}_{-a}[S_1 \leqslant 0, \ldots, S_{n-1} \leqslant 0, S_n > 0, S_n \in dx]$ fait appel à la *théorie des fluctuations des marches aléatoires*. Le résultat que nous montrons ici est une extension au cas où le pas de S est continu d'un résultat montré par Alili et Doney [AD99].

Pour un entier n, on désigne par T_n le n-ème temps de renouvellement pour le processus d'échelle, autrement dit $T_0 := 0$ et, pour $n \geqslant 1$, $T_n := \inf\{k \geqslant T_{n-1}, S_k > S_{T_{n-1}}\}$. On définit aussi le processus des *hauteurs d'échelle*, le processus H défini par $H_k := S_{T_k}$. Il est clair que le processus (T, H) est un processus de renouvellement bivarié à valeurs dans $(\mathbb{R}^+)^2$. On introduit alors la fonction de renouvellement $U(\cdot)$

associée à H :

$$U(x) := \sum_{i=0}^{\infty} \mathbf{P}[H_i \leqslant x] = \mathbf{E}[\mathcal{N}_x] \tag{1.5.2}$$

où $\mathcal{N}_x := |\{j \geqslant 0, H_j \leqslant x\}|$. Le résultat suivant est le résultat principal du chapitre 4 :

Théorème 1.5.1. *On a les propositions suivantes :*

(i) *pour tout $a \geqslant 0$, on a l'équivalence :*

$$\mathbf{P}_{-a}[S_1 \leqslant 0, \dots, S_{n-1} \leqslant 0, S_n > 0, S_n \in dx] \sim U(a)\frac{\mathbf{P}[H_1 \geqslant x]}{\sigma\sqrt{2\pi}n^{3/2}}dx \qquad \textit{pour } n \to \infty\,, \tag{1.5.3}$$

et si l'on dénote par $g_{n,a}(x)$ la densité du membre de gauche de la relation ci-dessus, si a est fixé, la convergence de $\sigma\sqrt{2\pi}n^{3/2}g_{n,a}(x)$ vers $U(a)\mathbf{P}[H_1 \geqslant x]$ est uniforme pour $x \in \mathbb{R}$;

(ii) *d'autre part, la suite de fonctions $\sigma\sqrt{2\pi}n^{3/2}g_{n,a}(x)$ est dominée par un multiple de sa limite, autrement dit il existe une constante $\mathcal{C} > 0$ telle que pour tous n, a, x, on ait l'inégalité $n^{3/2}g_{n,a}(x) \leqslant \mathcal{C}U(a)\mathbf{P}[H_1 \geqslant x]$; par conséquent :*

$$\mathbf{P}_{-a}[S_1 \leqslant 0, \dots, S_{n-1} \leqslant 0, S_n > 0] \sim U(a)\frac{\mathbf{E}[H_1]}{\sigma\sqrt{2\pi}n^{3/2}} \qquad \textit{pour } n \to \infty\,.$$

L'autre résultat important dont nous avons besoin pour la preuve du Théorème 1.4.3 concerne la convergence en loi d'une marche aléatoire conditionnée à rester positive.

1.5.2 Un théorème fonctionnel sur des marches aléatoires conditionnées à être positives.

On supposera dans cette partie que S est une marche aléatoire à valeurs entières apériodique qui se trouve dans le domaine d'attraction de la loi normale standard, autrement dit on suppose qu'il existe une suite a_n de réels positifs telle que $S_n/a_n \Rightarrow Z$ où Z est une loi normale standard. C'est alors un résultat classique que la suite de processus $(S_{[nt]}/a_n)_{t\in[0,1]}$ converge en loi vers le mouvement brownien (voir [Bil68]).

Si on dénote par $(S^*, \mathbf{P}_x)$ la marche aléatoire partant de x et conditionnée à être positive sur $\mathbb{N}$ tout entier, il n'a été montré que récemment que si $x/a_n \to 0$ lorsque $n \to \infty$, le processus rééchelonné correspondant converge en loi vers le méandre brownien ([BJD06], [CC08]). Il est alors naturel d'étudier la convergence du même processus conditionné à un retour tardif près de l'origine (*ie* à l'événement $\{S_n = y_n\}$ où $y_n/a_n \to 0$). On montre dans le chapitre 5 que ce processus converge vers l'excursion brownienne e.

Notons qu'étendre les résultats de convergence d'échelle classiques aux marches conditionnées à être positives est une tâche loin d'être immédiate. Parfois, une représentation astucieuse rend les preuves faciles (citons la preuve de la convergence vers le méandre de Bolthausen [Bol76] et son extension naturelle dans le cas où S se trouve dans le domaine d'attraction d'une loi stable [Don85], ou encore la preuve de la convergence vers l'excursion brownienne de Deuschel, Giacomin et Zambotti dans [DGZ05]), mais en général la preuve d'une telle convergence est assez ardue et repose le plus souvent sur la preuve de la convergence des lois fini-dimensionnelles puis de la tension. Citons par exemple les travaux d'Iglehart [Igl74] pour la preuve de la convergence de la marche aléatoire simple conditionnée à être positive vers le méandre ou ceux de Liggett [Lig68] pour une convergence vers le pont brownien. Le cas plus particulier de la convergence en loi vers l'excursion brownienne dans le cas de la marche aléatoire simple symétrique conditionnée à *revenir à l'origine à l'instant* n a été traité par Durrett, Iglehart et Miller [DIM77]. Ce cas a été étendu au cas où S possède un moment d'ordre 2 par Kaigh [Kai76].

Le processus S^* a été introduit par Bertoin et Doney dans un cadre nettement plus général que le notre. Notons $\tau_{(-\infty,0)}$ le temps d'atteinte du demi-plan inférieur pour S, $\tau_{(-\infty,0)} := \inf\{k \geqslant 0, S_k \in (-\infty, 0)\}$. On définit la suite des temps de renouvellement descendants T^- par $T_0^- := 0$ et $T_n^- = \inf\{k > T_{n-1}^-, S_k < S_{T_{n-1}^-}\}$ et la suite des hauteurs d'échelle descendantes H^- définie par $H_k^- := -S_{T_k^-}$. Soit enfin $V(\cdot)$ la fonction de renouvellement associée au processus de renouvellement H^-.

S^* est alors une chaîne de Markov dont les transitions sont données par :

$$\mathbf{P}_x^*[B \cap \{S_n = y\}] := \frac{V(y)}{V(x)} \mathbf{P}_x[B \cap \{S_n = y\} \cap \mathcal{C}_n] \tag{1.5.4}$$

où $\mathcal{C}_n := \{S_1 \geqslant 0, \ldots, S_n \geqslant 0\}$ et $B \in \sigma(S_1, \ldots, S_n)$. Rappelons que la terminologie se justifie par le résultat suivant qui a été montré par Bertoin et Doney [BD94] :

$$\mathbf{P}_x^* = \lim_{n\to\infty} \mathbf{P}_x\left(\cdot|\mathcal{C}_n\right) \tag{1.5.5}$$

où la convergence se fait au sens faible.

Notons $\mathbf{P}_n^{*,x,y}$ la loi de S conditionnellement à l'événement $\{S_0^* = x, S_n^* = y\}$ et

$$\mathbf{Q}_n^{x,y} := \mathbf{P}_n^{*,x,y} \circ (X^n)^{-1} \tag{1.5.6}$$

où l'application $X^n : \mathbb{R}^n \mapsto C([0,1])$ est définie dans l'équation (1.4.6) en y remplaçant les termes $\sigma\sqrt{n}$ par a_n.

Le résultat principal du chapitre 5 est alors le théorème suivant :

Théorème 1.5.2. *Soient x_n et y_n deux suites d'entiers positifs telles que $x_n/a_n \to 0$ et $y_n/a_n \to 0$. Pour $n \to \infty$, on a la convergence en loi :*

$$\mathbf{Q}_n^{x_n,y_n} \Rightarrow e. \tag{1.5.7}$$

Notons que l'on montre le Théorème 1.5.2 dans le cadre des marches aléatoires à pas discret seulement. La preuve de ce résultat est cependant facilement adaptable au cas continu.

Cette preuve repose essentiellement sur le lemme suivant :

Lemme 1.5.3. *Soit $K > 0$. Uniformément sur l'ensemble des suites x telles que $x_n/a_n \to 0$ lorsque $n \to \infty$ et sur l'ensemble des suites y telles que $y_n/a_n \in [0, K]$, on a l'équivalence suivante :*

$$\mathbf{P}_{x_n}\left(\{S_n = y_n\} \cap \mathcal{C}_n\right) \sim \frac{V(x_n)U(y_n)}{n}\mathbf{P}(S_n = y_n) \tag{1.5.8}$$

pour $n \to \infty$.

Ce lemme combiné à plusieurs résultat de Bryn-Jones et Doney [BJD06] permet de montrer la convergence fini-dimensionnelle. La tension est facile à montrer en utilisant des techniques développées au cours de la preuve du Lemme 1.5.3.

Chapitre 2

Finite size scaling

Contents

2.1 Introduction

The model of directed polymers interacting with a one dimensional defect turns out to be quite satisfactory to describe various biological or physical phenomena, such as (1+1) interface wetting [DHV91], the problem of depinning of flux lines from columnar defects in type II superconductors [NV93] and the denaturation transition of DNA in the Poland Sheraga approximation [KMP03]. The first results obtained by the physical community have been summarized in [Fis84], a paper which received a lot of attention from mathematicians. Since then, this model has been under close scrutinity, including in the last few years (the recent mathematical monograph [Gia07], but also [CGZ06], [IY01] or [DGZ05]). Here, we are going to deal with the homogeneous version of the model ; this version is mathematically remarkable in the sense that it is essentially completely solvable. For example, the model can have a transition of any given order [Gia07] depending on the value of α, a parameter in the definition of the system ($1 + \alpha$ is sometimes called loop exponent in the literature).

A central role in statistical mechanics is played by the correlation length, a quantity that we will denote by ξ. Properly defining this quantity in an (infinite volume) model requires some work and in most of the cases, such a concept does not correspond to a unique mathematical object. All the same, it is essential that any "reasonably defined" correlation length behaves in the same way close to criticality. And precisely close to criticality, for a large class of models, ξ becomes the only "relevant" scale in the system (see below for a more precise explanation). This idea is one of the basic concepts of the so called "finite size scaling" theory (see [Car96]). Here we illustrate in very concrete terms this concept via the scaling limit of pinning systems, obtaining a limiting behavior if the parameter is in a small, size dependent window near criticality as the size of the system goes to ∞.

In this first part, we will try to see in which way the above, rather unprecise statements, can be made quantitative. The second part should be seen as a warm up for the two last parts ; actually, in the one dimensional wetting, the existing literature makes the statement quite easy. In the third part, we introduce our general return model, and make the precise statements of our result in terms of scaling limits of the zero level set of our system near criticality.

In this note, α will always be a real number in $(0,1)$. For positive sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$, we will write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$, and for random variables $(X_n)_{n\geqslant 0}$ and X with values in a Polish space (E,d), we will write $X_n \Rightarrow X$ if the sequence $(X_n)_{n\geqslant 0}$ converges in law towards X.

2.1.1 A first model

Let τ be a recurrent renewal process with law $\mathbf{P}$, that is $\tau = \{\tau_n\}_{n\geqslant 0}$ where $\tau_0 = 0$ and $(\tau_{i+1}-\tau_i)_{i\geqslant 0}$ are iid with common law $\mathbf{P}$, where $\mathbf{P}$ is $\mathbf{N}$ valued and verifying, as $k\to\infty$,

$$\mathbf{P}(\tau_{i+1}-\tau_i = k) \sim \frac{C}{k^{1+\alpha}}. \tag{2.1.1}$$

We will actually consider a slightly more general model in section 3, allowing in particular τ to be transient, but for simplicity, we will restrict ourselves temporarily to this setup. The set $\tau\cap[0,N]$ can be considered as a random subset of $\{0,1,\dots,N\}$. We define the law $\mathbf{P}_{N,\beta}$ on the subsets of $\{0,1,\dots,N\}$ by

$$\frac{d\mathbf{P}_{N,\beta}}{d\mathbf{P}}(\tau) = \frac{1}{Z_{N,\beta}}\exp(\beta\mathcal{N}_N(\tau)), \tag{2.1.2}$$

where $\mathcal{N}_N(\tau)$ is the cardinality of the set $\tau\cap[0,N]$. In (2.1.2), $Z_{N,\beta}$ is the partition function of the model and of course

$$Z_{N,\beta} = \mathbf{E}\left[\exp(\beta\mathcal{N}_N(\tau))\right]. \tag{2.1.3}$$

It is not difficult to show that the limit of the quantity $F_N(\beta) := \frac{1}{N}\log Z_{N,\beta}$ exists and is non negative. We denote it by $F(\beta)$. $F(\cdot)$ is a convex, non decreasing and non negative function, and there exists a critical value of β such that $\beta \leqslant \beta_c$ implies $F(\beta) = 0$ and $\beta > \beta_c$ implies the positivity of $F(\beta)$. If $F(\beta) = 0$, the system is said to be *delocalized*, and it is *localized* otherwise. As the underlying renewal is recurrent, we actually have $\beta_c = 0$ (see [Gia07, Chapter 2] and section 3 below).

2.1.2 Finite size scaling

We define the correlation length of our system by

$$\xi(\beta) := \frac{1}{F(\beta)}, \tag{2.1.4}$$

if $\beta > 0$, and $\xi(\beta) = \infty$ otherwise. This definition has first been introduced in the physical literature in [Fis84], and its mathematical relevance with respect to other quantities (in particular with the *correlation function*) is discussed in [Gia08]. Also note that the concept of correlation length has been considered in depth in the inhomogeneous case (that is when the reward-penalty become site-dependent and random), see e.g. [Ton07]. It has been shown in [Gia07] that

$$\xi(\beta) \overset{\beta\searrow\beta_c}{\sim} c_\alpha(\beta-\beta_c)^{-1/\alpha}, \tag{2.1.5}$$

for some (explicit) constant $c_\alpha > 0$.

We expect finite size effects to appear only if the system is of the scale of the correlation length, that is only if $\xi(\beta)/N$ stays fixed (say equal to q). It is clear (considering (2.1.5)) that this is equivalent to keeping $(\beta-\beta_c)^{1/\alpha}N$ fixed (at least close to criticality). In this case, the observables of our system have a different behavior near criticality than their behavior in each of the two phases in the infinite volume limit. In an alternative way, we could say that close to criticality, the only relevant length scale is the correlation length. In order to make this precise, we need two basic objects. The first one is the *stable subordinator* with index α. Recall that a subordinator is an increasing Lévy process ; a subordinator $(\sigma_s^{(\alpha)})_{s\,\geqslant\,0}$ is said to be α *stable* (for $\alpha \in (0,1)$) if, for all $M \in \mathbf{R}^+$, the process $(\sigma_{Ms}^{(\alpha)})_{s\,\geqslant\,0}$ has the same law than the process $(M^{1/\alpha}\sigma_s^{(\alpha)})_{s\,\geqslant\,0}$. The second one is its local time $(L_t^{(\alpha)})_{t\,\geqslant\,0}$, which is simply the generalized inverse of $(\sigma_s^{(\alpha)})_{s\,\geqslant\,0}$, that is

$$L_t^{(\alpha)} := \inf\left\{s \geqslant 0, \sigma_s^{(\alpha)} \geqslant t\right\}. \tag{2.1.6}$$

It will be shown in the last part (Lemma 2.4.1) that there exists $C_K > 0$ such that

$$\left(C_K\frac{\mathcal{N}_N}{N^\alpha}\right)_{N\,\geqslant\,0} \Rightarrow L_1^{(\alpha)}, \tag{2.1.7}$$

and moreover the sequence $\left(C_K \frac{\mathcal{N}_N}{N^\alpha}\right)_{N \geqslant 0}$ is uniformly integrable (Lemma 2.4.2). Using these two facts and (2.1.5), one gets that the *finite size correlation length* $\xi_N(\beta) := 1/F_N(\beta)$ verifies

$$\frac{\xi_N(\beta)}{N} \overset{N\to\infty}{\sim} \left(\log \mathbf{E}\left[\exp\left(q^\alpha c_\alpha^\alpha \frac{\mathcal{N}_N}{N^\alpha}\right)\right]\right)^{-1} \sim f(q), \tag{2.1.8}$$

where $f(q) = \left(\log \mathbf{E}\left[\exp\left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)}\right)\right]\right)^{-1}$.

Let us now consider two very relevant observables (in the usual window) : the (finite volume) *contact fraction* $\rho_N(\beta)$ (that is the expectation of $\mathcal{N}_N/N$ with respect to the finite polymer measure) and the (once again finite volume) *specific heat* $\chi_N(\cdot)$ (that is the second derivative of the finite volume free energy). In our setup, these quantities can be written in an analogous way as above. Specifically, we get

$$\rho_N(\beta) \sim N^{\alpha-1} \frac{1}{C_K} \frac{\mathbf{E}\left[L_1^{(\alpha)} \exp\left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)}\right)\right]}{\mathbf{E}\left[\exp\left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)}\right)\right]} := N^{\alpha-1}\widehat{\rho}(q). \tag{2.1.9}$$

In an equivalent way, introducing $\widetilde{\rho}(x) := x^{1-\alpha}\widehat{\rho}(q)$, we get $\rho_N(\beta) \sim \xi(\beta)^{\alpha-1}\widetilde{\rho}(q)$, that is we can express $\rho_N(\cdot)$ (as $N \to \infty$) as a function of $\xi(\cdot)$. Analogously, we have

$$\frac{\partial^2}{\partial^2 \beta} F_N(\beta) = \chi_N(\beta) \sim N^{2\alpha-1} \frac{1}{C_K^2} \frac{\mathbf{E}\left[(L_1^{(\alpha)} - \mathbf{E}[L_1^{(\alpha)}])^2 \exp\left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)}\right)\right]}{\mathbf{E}\left[\exp\left(\frac{q^\alpha c_\alpha^\alpha}{C_K} L_1^{(\alpha)}\right)\right]}, \tag{2.1.10}$$

so that, once again, we get the following equivalence :

$$\chi_N(\beta) \sim \xi(\beta)^{2\alpha-1}\widehat{\chi}(q). \tag{2.1.11}$$

We show here that these two particular (and physically relevant) examples are actually special cases for a much more general phenomenon, namely the convergence in law of the whole rescaled system as $\xi(\beta)/N$ stays fixed. We are able to compute the scaling limit of the system in terms of a Radon Nykodym derivative with respect to the α regenerative set.

2.2 A look at the wetting model

As a warm-up, we will deal with the case in which the underlying law of our polymer model is the simple symmetric random walk conditioned to stay non negative, that is a sequence $(S_n)_{n \geqslant 0}$ where $S_0 = 0$, the variables $(S_i - S_{i-1})_{i \geqslant 1}$ are i.i.d. and such that $\mathbf{P}[S_i - S_{i-1} = 1] = \mathbf{P}[S_i - S_{i-1} = -1] = \frac{1}{2}$. We introduce the following probability law on $\mathbf{Z}^N$:

$$\frac{d\mathbf{P}_{\beta,N}}{d\mathbf{P}}(x) = \frac{1}{Z_{\beta,N}} \exp(H_{\beta,N}(x)) \mathbf{1}_{x_1 \geqslant 0,\dots,x_N \geqslant 0} \tag{2.2.1}$$

where the Hamiltonian of our system is given by

$$H_{\beta,N}(x) = \beta \sum_{i=1}^{N} \mathbf{1}_{x_i=0} = \beta \mathcal{N}_N. \tag{2.2.2}$$

Here too, it is not hard to prove the existence of $\lim_{N\to\infty} \frac{1}{N} \log(Z_{\beta,N}) = F(\beta)$ (see [Gia07]). In particular, for N even, we have the inequality

$$Z_{\beta,N} = \mathbf{E}\left[\exp(H_{\beta,N}(S)) \mathbf{1}_{S_1 \geqslant 0,\dots,S_N \geqslant 0}\right]$$

$$\geqslant \mathbf{E}\left[\exp(H_{\beta,N}(S)) \mathbf{1}_{S_1>0,\dots,S_N>0}\right] = \mathbf{P}[S_1 > 0, \dots, S_N > 0] \sim \frac{1}{\sqrt{2\pi N}}, \tag{2.2.3}$$

where the last equivalence is well known (cf [Fel71]), which entails that for every $\beta \in \mathbf{R}$, $F(\beta) \geqslant 0$.

In this model, it is possible to show that β_c is actually equal to $\log(2)$ (see [Gia07, Chapter 1] , or [IY01]). We are interested in the scaling limit of the system near criticality. To be more explicit, we define the application : $X^N : \mathbf{R}^N \to C([0,1])$ by :

$$(X_t^N(x))_{t\in[0,1]} = \frac{x_{[Nt]}}{N^{1/2}} + (Nt - [Nt]) \frac{x_{[Nt]+1} - x_{[Nt]}}{N^{1/2}}. \tag{2.2.4}$$

and we introduce the sequence of measures $Q_{\beta,N} := \mathbf{P}_{\beta,N} \circ (X^N)^{-1}, N \geqslant 0$. Of course, in (2.2.4), $[x]$ denotes the integer part of the real number x.

Let $(B_t)_{t \geqslant 0}$ be a standard Brownian motion defined on our probability space $(\Omega, \mathcal{F}, \mathcal{P})$ in the sequel, and $(L_t)_{t \geqslant 0}$ denote its local time at zero. It has been proven in [IY01] that the sequence $(Q_{\beta_c,N})_{N \geqslant 0}$ converges weakly to the law of the process $(|B_t|)_{t \geqslant 0}$. This result is actually very natural since it is quite easy to see that the law of the process $(S_n)_{n \geqslant 0}$ under $\mathbf{P}_{\beta_c,N}$ has the same law as $(|S_n|)_{n \geqslant 0}$ under $\mathbf{P}$.

The following result gives some intuition about the more general theorem we are going to prove in the next section in terms of zero level sets.

Theorem 2.2.1. *Let $\varepsilon \in \mathbf{R}$. The sequence of measures $(Q_{\beta_c+\varepsilon N^{-1/2},N})_{N \geqslant 0}$ converges weakly as $N \to \infty$; the law of the limiting process is absolutely continuous with respect to $(|B_t|)_{t \geqslant 0}$ with Radon-Nykodym density given by $e^{\varepsilon L_1}/\mathbf{E}\left[e^{\varepsilon L_1}\right]$.*

Proof. The following convergence in law is classical :

$$\frac{\mathcal{N}_N}{\sqrt{N}} \Rightarrow L_1, \tag{2.2.5}$$

but a much stronger statement can be proved : given a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$, one can construct a simple random walk $(S_n)_{n \geqslant 0}$ on the same probability space such that, for every $\eta > 0$, one has :

$$\lim_{N \to \infty} N^{-1/4-\eta} |L_N - \mathcal{N}_N| = 0, \tag{2.2.6}$$

almost surely as $N \to \infty$ (see [Rév05, Theorem 10.1]). In particular for $\eta = 1/2$, and by the scaling property of $(L_t)_{t \geqslant 0}$, this implies immediately the convergence of the joint law $(S_{[Nt]}/\sqrt{N}, \mathcal{N}_N/\sqrt{N}) \Rightarrow (B_t, L_1)$.

A proof of the tightness of the sequence $(Q_{\beta_c+\frac{\varepsilon}{\sqrt{N}},N})_{N \geqslant 0}$ can be found in [CGZ]. We are therefore left with showing the convergence in law of the finite dimensional marginals of the process, that is we have to show that, for every $n \in \mathbf{N}$ and for every continuous bounded function $F(\cdot)$ from $[0,1]^n$ to $\mathbf{R}$, the following holds :

$$Q_{\beta_c+\frac{\varepsilon}{\sqrt{N}},N}\left[F\left(\omega_{t_1}, \dots, \omega_{t_n}\right)\right] \underset{N \to \infty}{\to} \frac{\mathbf{E}\left[e^{\varepsilon L_1} F\left(|B_{t_1}|, \dots, |B_{t_n}|\right)\right]}{\mathbf{E}[e^{\varepsilon L_1}]}. \tag{2.2.7}$$

We already know by [IY01] that (2.2.7) holds for $\varepsilon = 0$. The term in the left hand side of (2.2.7) can be written as

$$\mathbf{E}_{\beta_c,N}\left[F\left(\frac{S_{[Nt_1]}}{\sqrt{N}}, \dots, \frac{S_{[Nt_n]}}{\sqrt{N}}\right) e^{\varepsilon \frac{\mathcal{N}_N}{\sqrt{N}}}\right] / \mathbf{E}_{\beta_c,N}\left[e^{\varepsilon \frac{\mathcal{N}_N}{\sqrt{N}}}\right]. \tag{2.2.8}$$

Since by 2.2.5 we have $e^{\frac{\varepsilon \mathcal{N}_N}{\sqrt{N}}} \Rightarrow e^{\varepsilon L_1}$, to prove 2.2.7, we just have to show that the family $\left(e^{\frac{\varepsilon \mathcal{N}_N}{\sqrt{N}}}\right)_{N \geqslant 0}$ is uniformly integrable. This is obvious if $\varepsilon \leqslant 0$, and will be proved in the last section in a more general setup (see Lemma 2.4.2) for $\varepsilon > 0$, although it could be also proven directly using the exact distribution of $\mathcal{N}_N$, which is actually well known in the simple symmetric random walk setup (see for example [Rév05]). This completes the proof. □

2.3 The renewal setup

2.3.1 The model

As we did in the introduction, we denote by τ the points of a renewal process, that is the image set of a sequence of i.i.d. positive random variables $(l_i)_{i \geqslant 0}$ whose law is $\mathbf{P}$. More precisely, $\tau := \{\tau_0, \tau_1, \dots,\}$ where $\tau_k = \sum_{i=1}^k l_i$. In what follows, $\mathbf{P}$ is an integer valued law verifying

$$K(n) := \mathbf{P}(l_1 = n) = \frac{L(n)}{n^{1+\alpha}}, \tag{2.3.1}$$

where $L(.)$ is a slowly varying function. Furthermore, we allow transience for our renewal process, that is we may have $\sum_{n\geqslant 1} K(n) < 1$ (or equivalently $\mathbf{P}(l_1 = \infty) > 0$). For $N \in \mathbf{N}$, we introduce $\overline{K}(N) := \sum_{n>N} K(n)$. We recall that a positive measurable function $L(.)$ is called slowly varying if for all $c > 0$, we have $\lim_{x\to\infty} L(cx)/L(x) = 1$. Such functions are well known, see [BGT89]. One basic example of slowly varying function (apart from the trivial case of constants) at infinity is $x \mapsto \log(1+x), x \in (0,\infty)$; but logarithmic functions are not the only examples, $x \mapsto \exp(a(\log(1+x))^c)$ is also slowly varying for every $a \in \mathbf{R}$ and $c < 1$.

We then define the homogeneous pinning polymer law exactly as in (2.1.2). In all what follows, we will denote by $\tau_{(N)}$ the set $(\tau \cap [0,N])/N \subset [0,1]$. When we use this notation, it will be implicit in the sequel that the set τ has the law $\mathbf{P}_{N,\beta}$, where β will be obvious from the context.

This model has been essentially completely solved, see [Gia07] ; basic facts about it are the existence of a localization delocalization transition for a critical β, whose value is given by $\beta_c = -\log(\Sigma_K)$, where $\Sigma_K := \sum_{n\geqslant 1} K(n)$. In particular, we have the equivalence $\beta_c = 0$ iff the underlying renewal process is recurrent. This transition is actually defined in terms of the free energy, which is defined exactly as in the previous sections. It is actually possible to give a very intuitive description of both phases in terms of the scaling limit of the zero level set. Intuitively, if the system is localized, we expect the set $\tau_{(N)}$ to converge in a suitable way to the full interval $[0,1]$. Similarly, if $\beta < \beta_c$, we expect it to converge to $\{0\}$.

To make such statements quantitative, we introduce $\mathcal{F}$ the set of closed subsets of $[0,1]$. We endow it with the topology of Matheron, which can be described as follows. For $F \in \mathcal{F}$ and $t \in \mathbf{R}^{+,*}$, we set $d_t(F) := \inf(F \cap (t,\infty))$. Notice that $t \mapsto d_t(F)$ is a right-continuous function and that F can actually be identified with $d_{(\cdot)}(F)$ because $F = \{t \in \mathbf{R}^+ : d_{t^-}(F) = t\}$. Then in terms of $d_{(\cdot)}(F)$, the Matheron topology is the standard Skorohod topology on càdlàg functions taking values in $\mathbf{R}^+ \cup \{+\infty\}$. We point out that with this topology, the space $\mathcal{F}$ is metrizable, separable and compact, hence in particular Polish. We denote by $\rho(\cdot)$ the Hausdorff metric given by $\rho(F_1,F_2) := \max_{i\in\{(1,2),(2,1)\}} \sup_{t\in F_{i_1}} \inf_{s\in F_{i_2}} |t-s|$ where $F_1, F_2 \in \mathcal{F}$. It is then a well known result that this metric is equivalent to the metric engendered by the Skorohod distance via the identification through $d_{(\cdot)}$ on $\mathcal{F}$. Using this topology, both of the above statements have actually been proven in [Gia07]. The scaling limit at criticality (that is when $\beta = \beta_c$) is much richer than that. To describe it, we need the notion of α *stable sets*.

2.3.2 α regenerative sets and subordinators.

Recall that a subordinator is a non decreasing Lévy process. A well-known fact about a subordinator $(\sigma_t)_{t\geqslant 0}$ is the existence of its so-called Lévy Khintchine exponent $\Phi(.)$, that is a measurable function verifying $\forall \lambda > 0, \forall t \geqslant 0, \mathbf{E}\left[e^{-\lambda\sigma_t}\right] = e^{-t\Phi(\lambda)}$.

When $\Phi(t) = t^\alpha$ for $\alpha \in (0,1)$, the associated subordinator is referred to as α stable, and we will denote it by $(\sigma_s^{(\alpha)})_{s \geqslant 0}$ in what follows. We already introduced its *local time* in (2.1.6). These objects are very well-known and a classic reference is [Ber99]. We then define the set $\mathcal{A}_\alpha = \overline{\{\sigma_s^{(\alpha)}, s \geqslant 0\}}$ ($\in \mathcal{F}$) the *regenerative* set of index α, where for a subset $A \subset \mathbf{R}$, $\overline{A}$ is the closure of A. An important property that will turn out to be useful is the fact that $\mathbf{P}[1 \in \mathcal{A}_\alpha] = 0$ (see [Kes69]).

2.3.3 Scaling limits at and near criticality.

In [Gia07], it was shown that, if $\beta = \beta_c$, we have the convergence :

$$\tau_{(N)} \Rightarrow \mathcal{A}_\alpha \tag{2.3.2}$$

in the recurrent case (that is $K(\infty) = 0$), and in the transient case

$$\tau_{(N)} \Rightarrow \widetilde{\mathcal{A}}_\alpha, \tag{2.3.3}$$

where $\widetilde{\mathcal{A}}_\alpha$ is a random subset of $[0,1]$ whose law is absolutely continuous with respect to the law of $\mathcal{A}_\alpha$ with Radon-Nykodym density equal to $(\alpha\pi/\sin(\alpha\pi))\,(1-\max(\mathcal{A}_\alpha\cap[0,1]))^\alpha$.

A first step towards this convergence has been made in [IY01] in a discrete random walk set-up ; they actually proved the convergence of the entire process at criticality towards the brownian motion. This work has been strongly generalized in [DGZ05], actually being the most important part of the proof of the convergence of more general pinning models towards the reflected brownian motion at criticality. This was in turn generalized in [CGZ06] using powerful renewal techniques, on which the present work is based.

Let us focus for the moment on the recurrent case, that is we assume for the moment $K(\infty) = 0$. First of all, we recall the following result (see [Fel71, XIII.6 Theorem 2]) : it is possible to choose a positive sequence $(a_n)_{n \geqslant 0}$ such that

$$\frac{n\Gamma(1-\alpha)L(a_n)}{\alpha a_n^\alpha} \to 1, \tag{2.3.4}$$

and furthermore, as soon as a sequence $(a_n)_{n \geqslant 0}$ verifies (2.3.4), we have the convergence :

$$\frac{\sum_{i=1}^n l_i}{a_n} \Rightarrow \sigma_1^{(\alpha)} \tag{2.3.5}$$

where the $(l_i)_{i \geqslant 0}$ are iid with common law $K(.)$ satisfying (2.3.1). We then define the sequence $(b_n)_{n \geqslant 0}$ to be the inverse sequence of the $(a_n)_{n \geqslant 0}$, that is the unique sequence, up to asymptotic equivalence, which verifies

$$a_{[b_n]} \sim b_{[a_n]} \sim n \tag{2.3.6}$$

as $n \to \infty$. Its existence is ensured by [BGT89, Theorem 1.5.12]. It is rather easy to see that $b_n \sim (\Gamma(1-\alpha)\overline{K}(n))^{-1}$ as $n \to \infty$. Let us quickly prove this fact. Using the definition of $(a_n)_{n \geqslant 0}$, we get

$$\frac{b_n \Gamma(1-\alpha) L(a_{[b_n]})}{\alpha a_{[b_n]}^{\alpha}} \sim \frac{b_n \Gamma(1-\alpha) L(n)}{\alpha n^{\alpha}} \sim 1, \tag{2.3.7}$$

and then we note that $\overline{K}(N) \sim L(N)(\alpha N^{\alpha})^{-1}$ (see [Gia07, Page 201],). We also used the fact that $x_n \sim y_n$ yields $L(x_n) \sim L(y_n)$ as $n \to \infty$ for $(x_n)_{n \geqslant 0}$ and $(y_n)_{n \geqslant 0}$ positive sequences. For n large enough, $x_n/y_n \in [1-\eta; 1+\eta]$ for a given $\eta > 0$, and the fact that the convergence $L(cx)/L(x) \to 1$ is uniform in c on every compact subset of $\mathbf{R}^+$ (a basic result on slowly varying functions, see [BGT89]) yields the assumption.

To deal with the transient case, we introduce the recurrent law $\widetilde{\mathbf{P}}$ given by

$$\widetilde{\mathbf{P}}(l_1 = n) := \widetilde{K}(n) = \frac{L(n) e^{\beta_c}}{n^{1+\alpha}} = \frac{L(n)}{n^{1+\alpha} \Sigma_K}. \tag{2.3.8}$$

We have then exactly the same statements as in Eq.(2.3.4) and (2.3.5) making the obvious changes, and we introduce the sequence $(\widetilde{b}_n)_{n \geqslant 0}$, the analogous of $(b_n)_{n \geqslant 0}$ for the new sequence $(\widetilde{a}_n)_{n \geqslant 0}$. We are now ready to state our main result :

Theorem 2.3.1. *Let $\varepsilon \in \mathbf{R}$. The following statements hold :*

1. *if $K(\infty) = 0$, if τ is distributed according to $\mathbf{P}_{\beta_c + \varepsilon/b_N, N}$, then*

$$\tau_{(N)} \Rightarrow \mathcal{B}_{\alpha,\varepsilon}, \tag{2.3.9}$$

where $\mathcal{B}_{\alpha,\varepsilon} (\subset [0,1])$ is the random set whose law is absolutely continuous with respect to the law of $\mathcal{A}_{\alpha}$ with Radon Nykodym density equal to $\exp(\varepsilon L_1^{(\alpha)}) / \mathbf{E}[\exp(\varepsilon L_1^{(\alpha)})]$.

2. *if $K(\infty) > 0$, if τ is distributed according to $\mathbf{P}_{\beta_c + \varepsilon/\widetilde{b}_N, N}$, then*

$$\tau_{(N)} \Rightarrow \widetilde{\mathcal{B}}_{\alpha,\varepsilon}, \tag{2.3.10}$$

where $\widetilde{\mathcal{B}}_{\alpha,\varepsilon} (\subset [0,1])$ has a Radon-Nykodym density given by $\Big(\exp(\varepsilon L_1^{(\alpha)})(1 - \max(\mathcal{A}_{\alpha} \cap [0,1]))^{\alpha}\Big) \Big/ \Big(\mathbf{E}[\exp(\varepsilon L_1^{(\alpha)})(1 - \max(\mathcal{A}_{\alpha} \cap [0,1]))^{\alpha}]\Big)$.

2.4 Proofs

Lemma 2.4.1. *If $K(\infty) = 0$, the following convergence holds :*

$$\frac{|\tau_{(N)}|}{b_N} \Rightarrow L_1^{(\alpha)}. \tag{2.4.1}$$

Proof of Lemma 2.4.1. For $M > 0$, we have the equality :

$$\mathbf{P}\left(\frac{|\tau_{(N)}|}{b_N} \leqslant M\right) = \mathbf{P}\left(\frac{1}{a_{[Mb_N]}}\sum_{i=1}^{[Mb_N]} l_i \geqslant \frac{N}{a_{[Mb_N]}}\right). \tag{2.4.2}$$

The right hand side of (2.4.2) is easily seen to converge towards $\mathbf{P}\left(M^{\frac{1}{\alpha}}\sigma_1^{(\alpha)} \geqslant 1\right)$. Actually, we have $N/a_{[Mb_N]} \to M^{-\frac{1}{\alpha}}$ as $N \to \infty$ (since $Mb_N \sim b_{[M^{\frac{1}{\alpha}}N]}$), and it is easy to see that, if $X_n \Rightarrow X$, if a deterministic sequence $u_n \to u$ and the distribution of X is continuous, then $\mathbf{P}(X_n \geqslant u_n) \to \mathbf{P}(X \geqslant u)$. So we just have to recall that the distribution of $\sigma_1^{(\alpha)}$ is continuous, which is actually a well known fact about stable subordinators (see [Ber99, P.271]).

Because of the scaling property of the stable subordinator, we get

$$\mathbf{P}\left(M^{\frac{1}{\alpha}}\sigma_1^{(\alpha)} \geqslant 1\right) = \mathbf{P}(\sigma_M^{(\alpha)} \geqslant 1)$$

$$= \mathbf{P}(L_1^{(\alpha)} \leqslant M), \tag{2.4.3}$$

which proves our claim. ☐

Lemma 2.4.2. *For $K(\infty) = 0$ and for all $\varepsilon > 0$, the family $\left(\exp(\frac{\varepsilon|\tau_{(N)}|}{b_N})\right)_{N \geqslant 0}$ is uniformly integrable.*

Proof of Lemma 2.4.2. It is enough to show that, for all $a > 0$, we have

$$\sup_{N\in\mathbf{N}} \mathbf{E}\left[e^{a|\tau_{(N)}|/b_N}\right] < \infty. \tag{2.4.4}$$

As

$$\mathbf{E}\left[e^{a|\tau_{(N)}|/b_N}\right] = \int_1^\infty \mathbf{P}\left[|\tau_{(N)}|/b_N > 1/a\log(x)\right] \mathrm{d}x, \tag{2.4.5}$$

to show 2.4.4, we just have to see that for all $C_1 > 0$, there exists $C_2 > 0$ such that for $y \geqslant 1$,

$$\mathbf{P}\left[|\tau_{(N)}|/b_N > y\right] \leqslant C_2\exp(-C_1 y), \tag{2.4.6}$$

as soon as N is large enough. Using the Markov inequality, we see that for every $\lambda > 0$ and $y \geqslant 1$:

$$\mathbf{P}\left(\frac{|\tau_{(N)})|}{b_N} > y\right) = \mathbf{P}\left(\sum_{i=1}^{[yb_N]} l_i < N\right) \leqslant \mathbf{E}\left[e^{-\lambda l_1}\right]^{[yb_N]} e^{\lambda N}. \tag{2.4.7}$$

Noting that τ is recurrent, we have :

$$\log(\mathbf{E}[e^{-\lambda l_1}]) = \log\left(1 - L(1/\lambda)\lambda^{1+\alpha}\sum_{n\geqslant 1}\frac{1-e^{-\lambda n}}{(\lambda n)^{1+\alpha}}\frac{L(\frac{n\lambda}{\lambda})}{L(1/\lambda)}\right). \quad (2.4.8)$$

For any $\eta > 0$ and using the uniform convergence property on compact sets for slowly varying functions, one has the equivalence (for $\lambda \searrow 0$)

$$\lambda\sum_{n\lambda\geqslant\eta}^{1/\eta}\frac{1-e^{-\lambda n}}{(\lambda n)^{1+\alpha}}\frac{L(\frac{n\lambda}{\lambda})}{L(1/\lambda)} \sim \int_{\eta}^{1/\eta}\frac{1-e^{x}}{x^{1+\alpha}}dx, \quad (2.4.9)$$

and it is easily seen that this implies the equivalence (still for $\lambda \searrow 0$) :

$$-\log(\mathbf{E}[e^{-\lambda l_1}]) \sim \Gamma(1-\alpha)/\alpha\lambda^{\alpha}L(1/\lambda). \quad (2.4.10)$$

This entails that there exists $c(\alpha) > 0$ and $\lambda_0(\alpha) > 0$ such that for all $\lambda \in (0, \lambda_0)$, we have $-\log(\mathbf{E}[e^{-\lambda l_1}]) \geqslant c(\alpha)\lambda^{\alpha}L(1/\lambda)$. Note that $(b_N L(N)N^{-\alpha})_{N\geqslant 0}$ is a positive bounded sequence. Then, using 2.4.7, we get, for every N :

$$\mathbf{P}\left(\frac{|\tau_{(N)})|}{b_N} > y\right) \leqslant \inf_{0\leqslant\lambda\leqslant\lambda_0}\exp\left(-c(\alpha)\lambda^{\alpha}L(1/\lambda)y\frac{N^{\alpha}}{L(N)} + \lambda N\right), \quad (2.4.11)$$

possibly modifying $c(\alpha)$ in doing so. Then we consider N_0 such that for $N \geqslant N_0$, $C_1/N < \lambda_0$, and we give us $C_2 \in (C_1; N_0\lambda_0)$. For u in the compact set $[C_1, C_2]$, the convergence of $L(N/u)/L(N)$ towards 1 is uniform, so that, for $\eta > 0$, for N large enough, $L(N/\lambda N)/L(N) \geqslant 1-\eta$ as long as $C_1/N \leqslant \lambda \leqslant C_2/N$. Thus, for N large :

$$\mathbf{P}\left(\frac{|\tau_{(N)})|}{b_N} > y\right) \leqslant \inf_{C_1/N\leqslant\lambda\leqslant C_2/N}\exp\left(-c(\alpha)(1-\eta)(\lambda N)^{\alpha}y + \lambda N\right). \quad (2.4.12)$$

And this last inequality clearly entails 2.4.6, and thus implies the claim of Lemma 2.4.2. □

Lemma 2.4.3. *For $K(\infty) = 0$, we have the convergence :*

$$(|\tau_{(N)}|/b_N, \tau_{(N)}) \Rightarrow (L_1^{(\alpha)}, \mathcal{A}_\alpha). \quad (2.4.13)$$

Proof of Lemma 2.4.3. We introduce the process $(N_{\gamma_N}(t))_{t\geqslant 0}$, a Poisson process of rate γ_N where we define $\gamma_N := \left(\sum_{n\geqslant 1}(1-e^{-n/N})K(n)\right)^{-1}$. We take $(N_{\gamma_N}(t))_{t\geqslant 0}$ independent from τ. It is easy to see that the process $\left(\tau_{N_{\gamma_N}(t)}/N\right)_{t\geqslant 0}$ is a subordinator. In [Gia07], it was shown that the Lévy exponent of $\left(\tau_{N_{\gamma_N}(t)}/N\right)_{t\geqslant 0}$ converges to the one of $\left(\sigma_s^{(\alpha)}\right)_{s\geqslant 0}$, which implies $\overline{\{\tau_{N_{\gamma_N}(s)}/N, s\geqslant 0\}} \Rightarrow \mathcal{A}_\alpha$ as well as the convergence in law of the entire process $\left(\tau_{N_{\gamma_N}(t)}/N\right)_{t\geqslant 0}$ towards $\left(\sigma_s^{(\alpha)}\right)_{s\geqslant 0}$ (see [FFM85]) in the Polish space $\mathcal{D}$. Here, $\mathcal{D}$ denotes the space of càdlàg functions on $[0,1]$ endowed with the standard Skorokhod topology (that is $d(f,g) := \inf_{\lambda\in\Lambda}\{\max(||\lambda - id||, ||f\circ\lambda - g||)\}$ where Λ is the set of non-decreasing homeomorphisms from $[0,1]$ to $[0,1]$) and of course $||f|| = \sup_{t\in[0,1]}|f(t)|$ for $f\in\mathcal{D}$.

We define the function $(F,G) : \mathcal{D}\to\mathbf{R}\times\mathcal{F}$ by

$$(F,G)(f) : \qquad f\mapsto (\sup\{s\geqslant 0 | f(s)\leqslant 1\}, \mathrm{Im}(f)), \tag{2.4.14}$$

where $\mathrm{Im}(f)$ is the image set of f where we endow the space $\mathbf{R}\times\mathcal{F}$ by the topology $\sigma(\mathcal{B}(\mathbf{R})\otimes\mathcal{F})$, and using the distance $\max(|.|,\rho)$ where ρ is the Hausdorff metric. We will show that, for all continuous bounded $\mathcal{G} : \mathbf{R}\times\mathcal{F}\to\mathbf{R}$, we have :

$$\mathbf{E}\left[\mathcal{G}\left(\frac{|\tau_{(N)}|}{b_N}, \tau_{(N)}\right)\right] = \mathbf{E}\left[\mathcal{G}\left(\left(F,G\right)\left(\left(\frac{\tau_{N_{\gamma_N}(s)}}{N}\right)_{s\geqslant 0}\right)\right)\right] + (o_N(1)), \tag{2.4.15}$$

that is that we can consider the joint law in the left hand side of (2.4.13) as a function of $(\tau_{N_{\gamma_N}(s)}/N)_{s\geqslant 0}$ up to negligible corrections.

Assume for the moment that (2.4.15) is true. Taking into account the convergence $\left(\tau_{N_{\gamma_N}(t)}/N\right)_{t\geqslant 0} \Rightarrow \left(\sigma_s^{(\alpha)}\right)_{s\geqslant 0}$ and the fact that $(F,G)\left(\left(\sigma_s^{(\alpha)}\right)_{s\geqslant 0}\right)$ is exactly the limiting law we want, it is enough to show that $(F,G)(\cdot)$ is almost surely continuous at $\left(\sigma_s^{(\alpha)}\right)_{s\geqslant 0}$ to apply the continuous mapping theorem (see [Bil99]) to prove Lemma 2.4.3. So let us show this almost sure continuity first, then we will turn to the proof of (2.4.15).

For $\eta\in(0,1)$, we denote by $\mathcal{B}_\eta$ the subset of $\mathcal{D}$ defined by

$$\mathcal{B}_\eta := \{f\in\mathcal{D} | [1-\eta; 1+\eta]\cap\mathrm{Im(f)} = \emptyset\}. \tag{2.4.16}$$

We already pointed out that $\mathbf{P}(1\in\mathcal{A}_\alpha) = 0$, so that with probability one, $(\sigma_s^{(\alpha)})_{s\geqslant 0}\in\cup_{\eta>0}\mathcal{B}_\eta$. We show that (F,G) restricted to $\cup_{\eta>0}\mathcal{B}_\eta$ is continuous.

Given $f\in\mathcal{B}_\eta$ with $\eta>0$, let us show that F is continuous at f. We introduce $s_f := \sup\{s\geqslant 0|f(s)\leqslant 1\}$. Let $g\in\mathcal{D}$ such that $d(f,g)<\mu/2$ where $\mu\in(0,\eta)$, and in the same way we define $s_g := \sup\{s\geqslant 0|g(s)\leqslant 1\}$. We consider $\lambda\in\Lambda$ such that $||f - g\circ\lambda|| < \mu$ and $||\lambda - id|| < \mu$. Furthermore, for all $\xi>0$, and as $f\in\mathcal{B}_\eta$, we have $f(s_f-\xi) < 1-\eta$ and $f(s_f) > 1+\eta$. Then we get

$$g(\lambda(s_f-\xi)) \leqslant f(s_f-\xi)+\mu \leqslant 1-\eta+\mu < 1, \tag{2.4.17}$$

and similarly

$$g(\lambda(s_f)) \geqslant f(s_f)-\mu > 1+\eta-\mu > 1, \tag{2.4.18}$$

which gives

$$\lambda(s_f - \xi) \leqslant s_g \leqslant \lambda(s_f), \tag{2.4.19}$$

and this entails the desired continuity since ξ is arbitrarily small, λ is continuous and $||\lambda - id|| < \mu$.
Note that, by the virtue of the continuous mapping's theorem, this implies the convergence

$$\sup\left\{s \geqslant 0, \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\} \Rightarrow L_1^{(\alpha)}. \tag{2.4.20}$$

For the second component, it is very easy also ; for g as above, we have :

$$\rho\left(\mathrm{Im(f)}, \mathrm{Im(g)}\right) = \max\left(\sup_{s\in\mathbf{R}} \inf_{t\in\mathbf{R}} |f(t) - g(s)|, \sup_{s\in\mathbf{R}} \inf_{t\in\mathbf{R}} |f(s) - g(t)|\right). \tag{2.4.21}$$

And for λ as above and $s \in \mathbf{R}$, we have $\inf_{t\in\mathbf{R}} |f(s) - g(t)| \leqslant |f(s) - g(\lambda(s))| \leqslant \mu$, which achieves the proof of the continuity of the couple (F, G) on $\cup_{\eta>0}\mathcal{B}_\eta$.

We now go to the proof of (2.4.15). Let $\delta > 0$, there exists a compact set $K_\delta \subset \mathbf{R}^+$ such that for all $N \in \mathbf{N}$, we have :

$$\max\left(\mathbf{P}\left[\frac{|\tau_{(N)}|}{b_N} \in K_\delta^c\right], \mathbf{P}\left[\sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\} \in K_\delta^c\right]\right) < \delta. \tag{2.4.22}$$

This is due to the fact that the sequences $\left(\sup\left\{s \geqslant 0 \middle| \tau_{N_{\gamma_N}(s)}/N \leqslant 1\right\}\right)_{N \geqslant 0}$ and $\left(\sup\left\{\frac{n}{b_N} \middle| \frac{|\tau_n|}{N} \leqslant 1\right\}\right)_{N \geqslant 0}$ are tight because they converge in law (see (2.4.20) and Lemma 2.4.1). Similarly, the convergence of the sequence $(\tau_{(N)})_{N \geqslant 0}$ implies the existence of a compact set $L_\delta \subset \mathcal{F}$ such that for every $N \in \mathbf{N}$,

$$\mathbf{P}\left[\tau_{(N)} \in L_\delta^c\right] < \delta. \tag{2.4.23}$$

For $N \in \mathbf{N}$, we introduce the event

$$\mathcal{H}_{N,\delta} := \left\{\frac{|\tau_{(N)}|}{b_N} \in K_\delta, \sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\} \in K_\delta, \tau_{(N)} \in L_\delta\right\}. \tag{2.4.24}$$

Eqs. (2.4.22) and (2.4.23) say that $\mathbf{P}\left[\mathcal{H}_{N,\delta}^c\right] < \delta$.

Let $M > 0$ be an upper bound for $\left|\mathcal{G}\right|$. For every $N \in \mathbf{N}$ and taking into account the fact that $\gamma_N \sim b_N$ (see [Gia07, Relation A.48]), we have :

$$\begin{aligned}
&\left|\mathbf{E}\left[\mathcal{G}\left(\sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\}, \tau_{(N)}\right) - \mathcal{G}\left(\frac{|\tau_{(N)}|}{\gamma_N}, \tau_{(N)}\right)\right]\right| \\
&\leqslant \mathbf{E}\left[\left|\mathcal{G}\left(\sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\}, \tau_{(N)}\right) - \mathcal{G}\left(\frac{|\tau_{(N)}|}{\gamma_N}, \tau_{(N)}\right)\right|\mathbf{1}_{\mathcal{H}_{N,\delta}}\right] + 2M\delta \\
&\leqslant \mathbf{E}\left[\left|\mathcal{G}\left(\sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\}, \tau_{(N)}\right)\right.\right. \\
&\left.\left. - \mathcal{G}\left(\sup\left\{\frac{N_{\gamma_N}(s)}{\gamma_N} - s + s \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\}, \tau_{(N)}\right)\right|\mathbf{1}_{\mathcal{H}_{N,\delta}}\right] + 2M\delta.
\end{aligned} \tag{2.4.25}$$

For a given $\kappa > 0$, we introduce the event $\mathcal{J}_{\kappa,\delta}$ defined by :

$$\mathcal{J}_{\kappa,\delta} := \left\{\sup_{s \in K_\delta}\left|\frac{N_{\gamma_N}(s)}{\gamma_N} - s\right| \geqslant \kappa\right\}. \tag{2.4.26}$$

Note that the process $\left(\frac{N_{\gamma_N}(s)}{\gamma_N} - s\right)_{s \geqslant 0}$ is a martingale, so that, applying Doob's inequality, we get :

$$\begin{aligned}
\mathbf{P}\left[\mathcal{J}_{\kappa,\delta}\right] &= \mathbf{P}\left[\sup_{s \in K_\delta}\left|\frac{N_{\gamma_N}(s)}{\gamma_N} - s\right|^2 \geqslant \kappa^2\right] \\
&\leqslant \frac{1}{\kappa^2}\sup_{s \in K_\delta}\mathbf{E}\left[\left|\frac{N_{\gamma_N}(s)}{\gamma_N} - s\right|^2\right] = \frac{\max K_\delta}{\gamma_N \kappa^2}.
\end{aligned} \tag{2.4.27}$$

Cutting once again the expectation appearing in Eq.(2.4.25) with respect to the fact that $\mathcal{J}_{\kappa,\delta}$ occurs or not, we get :

$$\begin{aligned}
&\left|\mathbf{E}\left[\mathcal{G}\left(\sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\}, \tau_{(N)}\right) - \mathcal{G}\left(\frac{|\tau_{(N)}|}{\gamma_N}, \tau_{(N)}\right)\right]\right| \\
&\leqslant \mathbf{E}\left[\sup_{|u| \leqslant \kappa}\left\{\left|\mathcal{G}\left(\sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\}, \tau_{(N)}\right)\right.\right.\right. \\
&\left.\left.\left. - \mathcal{G}\left(u + \sup\left\{s \geqslant 0 \middle| \frac{\tau_{N_{\gamma_N}(s)}}{N} \leqslant 1\right\}, \tau_{(N)}\right)\right|\right\}\mathbf{1}_{\mathcal{H}_{N,\delta}}\mathbf{1}_{\mathcal{J}^c_{\kappa,\delta}}\right] \\
&+ 2M\left(\delta + \frac{\max K_\delta}{\gamma_N \kappa^2}\right).
\end{aligned} \tag{2.4.28}$$

In the last inequality, we consider first κ very small, which deals with the expectation term due to the uniform continuity of $\mathcal{G}$ on the compact set $K_\delta \times L_\delta$. Then, we consider δ small enough, and this achieves the proof of (2.4.15), since $\gamma_N \overset{N\to\infty}{\to} \infty$. Thus Lemma 2.4.3 is proved. □

We are now ready to show the main part of Theorem 2.3.1.

Proof of Theorem 2.3.1. We have just shown the following convergence : under the law $K(.)$, assuming $K(\infty) = 0$, for all F continuous bounded function on $\mathcal{F}$ and for all $\varepsilon \in \mathbf{R}$, we have :

$$\mathbf{E}\left[F(\tau_{(N)})e^{\varepsilon|\tau_{(N)}|/b_N}\right] \underset{N\to\infty}{\to} \mathbf{E}\left[F(\mathcal{A}_\alpha)e^{\varepsilon L_1^{(\alpha)}}\right]. \tag{2.4.29}$$

Note that the case where $\varepsilon > 0$ actually uses the fact (see Lemma 2.4.2) that the sequence $(e^{\varepsilon|\tau_{(N)}|/b_N})_{N\geqslant 0}$ is uniformly integrable.

We then have

$$\mathbf{E}\left[F(\tau_{(N)})\right] = \frac{Z_{N,\beta_c}}{Z_{N,\beta_c+\varepsilon/b_N}} \times \frac{1}{Z_{N,\beta_c}}\mathbf{E}\left[F(\tau_{(N)})e^{\varepsilon|\tau_{(N)}|/b_N}\right]. \tag{2.4.30}$$

As in the case $K(\infty) = 0$, $\mathbf{P}_{N,\beta_c}$ is actually $\mathbf{P}$ (because $\beta_c = -\log(\Sigma_K) = 0$) and thanks to the easy remark that $Z_{N,\beta_c+\varepsilon/b_N} = \mathbf{E}\left[e^{\varepsilon|\tau_{(N)}|/b_N}\right] \underset{N\to\infty}{\to} \mathbf{E}\left[e^{\varepsilon L_1^{(\alpha)}}\right]$, we have actually shown the first part of the claim. We also used the fact that $Z_{N,\beta_c} = 1$ for all $N \in \mathbf{N}$ see ([Gia07, Relation (2.17)]).

We now deal finally with the case $K(\infty) > 0$; we can write :

$$\mathbf{E}_{N,\beta_c+\varepsilon/\widetilde{b}_N}\left[F(\tau_{(N)})\right] = \widetilde{\mathbf{E}}\left[F(\tau_{(N)})e^{\varepsilon|\tau_{(N)}|/\widetilde{b}_N}\frac{\overline{K}(N(1-\max(\tau_{(N)}))) + K(\infty)}{Z_{N,\beta_c+\varepsilon/\widetilde{b}_N}\sum_{j>N(1-\max(\tau_{(N)}))}\widetilde{K}(j)}\right]. \tag{2.4.31}$$

The justification for (2.4.31) can be found in [Gia07, Equality 2.48]. Then, on the event $\max\tau_{(N)} < 1-\eta$ for a given $\eta \in (0,1)$, it is known that ([Gia07, Equality 2.50]))

$$\frac{\overline{K}(N(1-\max(\tau_{(N)}))) + K(\infty)}{Z_{N,\beta_c}\sum_{j>N(1-\max(\tau_{(N)}))}\widetilde{K}(j)} \underset{N\to\infty}{\sim} \frac{\alpha\pi}{\sin(\alpha\pi)}(1-\max(\tau_{(N)}))^\alpha, \tag{2.4.32}$$

uniformly in the trajectories of τ. Using once again the convergence of the joint law $(\tau_{(N)}, |\tau_{(N)}|/\widetilde{b}_N)$, where this time τ is distributed according to $\widetilde{\mathbf{P}}$, we get :

$$\mathbf{E}_{N,\beta_c+\varepsilon\widetilde{b}_N}\left[F(\tau_{(N)})\mathbf{1}_{\max\tau_{(N)}<1-\eta}\right] \underset{N\to\infty}{\longrightarrow} \mathbf{E}\left[F(\mathcal{A}_\alpha)e^{\varepsilon L_1^{(\alpha)}}\mathbf{1}_{\max\mathcal{A}_\alpha<1-\eta}\frac{\alpha\pi}{\sin(\alpha\pi)}(1-\max(\mathcal{A}_\alpha))^\alpha\right]. \tag{2.4.33}$$

The fact that $\max(\mathcal{A}_\alpha) < 1$ almost surely allows us to take the above limit for $\eta \searrow 0$. Taking $F \equiv 1$ in (2.4.33), we get

$$\frac{Z_{N,\beta_c+\varepsilon/\widetilde{b}_N}}{Z_{N,\beta_c}} \underset{N\to\infty}{\to} \mathbf{E}\left[e^{\varepsilon L_1^{(\alpha)}} \frac{\alpha\pi}{\sin(\alpha\pi)}(1-\max(\mathcal{A}_\alpha))^\alpha\right]. \tag{2.4.34}$$

This achieves the proof because we can write

$$\begin{aligned}&\mathbf{E}_{N,\beta_c+\varepsilon/\widetilde{b}_N}\left[F(\tau_{(N)})\right] \\ &\quad = \frac{Z_{N,\beta_c}}{Z_{N,\beta_c+\varepsilon/\widetilde{b}_N}}\widetilde{\mathbf{E}}\left[F(\tau_{(N)})e^{\varepsilon|\tau_{(N)}|/\widetilde{b}_N}\frac{\overline{K}(N(1-\max(\tau_{(N)}))) + K(\infty)}{Z_{N,\beta_c}\sum_{j>N(1-\max(\tau_{(N)}))}\widetilde{K}(j)}\right],\end{aligned} \tag{2.4.35}$$

and the term in the right hand side converges towards the desired quantity.

□

Chapitre 3

A homogeneous wetting model

Contents

3.1 Introduction and main results

3.1.1 Definition of the models

We are going to define two strongly related models for a $(1+1)-$dimensional random field. Let $(S_n)_{n \geqslant 0}$ be a random walk such that $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$ where the $X_i's$ are i.i.d. and X_1 has a density $h(\cdot)$ with respect to the Lebesgue measure. We denote by $\mathbf{P}$ the law of S, and by $\mathbf{P}_x$ the law of the same process where $S_0 = x$. Our results will be obtained under the further conditions that $h(\cdot)$ is continuous and bounded on $\mathbb{R}$, that $\mathbf{E}[X] = 0$ and that $\mathbf{E}[X^2] := \sigma^2 \in (0, \infty)$. Note that these conditions will allow us in particular to apply Gnedenko's classical local limit theorem. Note also that the boundedness of $h(\cdot)$ could be substituted without much effort by the assumption that there exists n such that $\sup_y h^{*n}(y) < \infty$ where for two positive measurable functions f, g, $f * g$ denotes the standard convolution operation, and $h^{*n} := \underbrace{h * \ldots * h}_{n \text{ times}}$.

We are now ready to introduce our first model. Let $a > 0$ be fixed in the sequel. We define the probability law (the *wetting model in a stripe*) $\mathbf{P}^f_{N,a,\beta}$ on $\mathbb{R}^N$ by

$$\frac{d\mathbf{P}^f_{N,a,\beta}}{d\mathbf{P}} := \frac{1}{Z^f_{N,a,\beta}} \exp\left(\beta \sum_{k=1}^N \mathbf{1}_{S_k \in [0,a]}\right) \mathbf{1}_{S_k \geqslant 0, k=1\ldots N} \tag{3.1.1}$$

where $N \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $Z^f_{N,a,\beta}$ is the normalization constant usually called the partition function of the system. The second model we define is the *constrained counterpart* of the above, that is

$$\frac{d\mathbf{P}^c_{N,a,\beta}}{d\mathbf{P}} := \frac{1}{Z^c_{N,a,\beta}} \exp\left(\beta \sum_{k=1}^N \mathbf{1}_{S_k \in [0,a]}\right) \mathbf{1}_{S_k \geqslant 0, k=1\ldots N} \mathbf{1}_{S_N \in [0,a]}. \tag{3.1.2}$$

Note in particular that

$$\mathbf{P}^c_{N,a,\beta} = \mathbf{P}^f_{N,a,\beta}\left[\,\cdot\,|S_N \in [0,a]\right], \tag{3.1.3}$$

that $\mathbf{P}^f_{N,a,0}$ is the law of $(S_1, \ldots, S_N)$ under the constraint $\mathcal{C}_N := \{S_1 \geqslant 0, \ldots, S_N \geqslant 0\}$ and that $\mathbf{P}^c_{N,a,0}$ is the law of the same vector under the additional constraint $S_N \in [0,a]$.

Both $\mathbf{P}^c_{N,a,\beta}$ and $\mathbf{P}^f_{N,a,\beta}$ are $(1+1)-$dimensional models for a linear chain of length N which is attracted or repelled to a defect *stripe* of width a. The strength of this interaction with the stripe is tuned by the parameter β. The use of the term *wetting* has become somewhat customary to describe the positivity constraint and refers to the interpretation of the field as an effective model for the interface of separation between a liquid above a wall and a gas, see [DGZ05].

The purpose of this chapter is to investigate the behavior of $\mathbf{P}^c_{N,a,\beta}$ and $\mathbf{P}^f_{N,a,\beta}$ in the large N limit. In particular, we would like to understand when the reward β is strong enough to pin the chain near the defect stripe, a phenomenon that we will call *localization*, and what are the macroscopic effects of the reward on the system. We point out that this kind of questions have been answered in depth in the case of the standard wetting model, that is formally in the $a = 0$ case, and that the problem of extending these results to the case where the defects are in a stripe has been posed in [Gia07, Chapter 3].

Remark 3.1.1. *In the sequel, we will assume without loss of generality that* $\mathbf{P}[S_1 > a] > 0$*. The main purpose of this assumption is to make the exposition lighter, since the assumptions made on* h *imply that there exists a positive integer* k *such that* $\mathbf{P}[S_k > a] > 0$*, and all the results we show by considering events of the kind* $\{S_1 > a, \ldots, S_n > a\}$ *could be substituted without much efforts by events of the kind* $\{S_k > a, \ldots, S_n > a\}$*, which have positive* $\mathbf{P}$ *probability.*

3.1.2 The free energy.

A standard way to define localization for our models is by looking at the Laplace asymptotic behavior of the partition function $Z^f_{N,a,\beta}$ and $Z^c_{N,a,\beta}$ as $N \to \infty$. More precisely, one may define the free energy $F(\beta)$ by

$$F(\beta) := \lim_{N\to\infty} \frac{1}{N} \log\left(Z^f_{N,a,\beta}\right) \tag{3.1.4}$$

where the existence of the limit will follow as a by-product of our approach. Another by-product of our proofs will be that, in the definition (3.1.4) of the free energy, it is actually irrelevant whether one considers the free or the constrained partition function.

The basic observation is that the free energy is non-negative. Indeed, one has the trivial inequality :

$$\begin{aligned} Z^f_{N,a,\beta} &\geqslant \mathbf{E}\left[\exp\left(\beta \sum_{k=1}^{N} \mathbf{1}_{S_k\in[0,a]}\right)\mathbf{1}_{S_k>a,k=1\ldots N}\right] \\ &\geqslant \mathbf{P}\left[S_j > a, j = 1\ldots, N\right]. \end{aligned} \tag{3.1.5}$$

Choose some $M > a$ such that $\mathbf{P}[S_1 \in [a, M]] > 0$. Integrating over S_1, one gets :

$$\mathbf{P}[S_j > a, j = 1\ldots, N] \geqslant \int_{[a,M]} h(t)\mathbf{P}_t\left[S_1 > a, \ldots, S_{N-1} > a\right] dt. \tag{3.1.6}$$

A consequence of fluctuation estimates developed in the chapter 5 of the present thesis (see Lemma 5.6.1) is that for fixed M, the quantity $N^{3/2}\mathbf{P}_t\left[S_1 > a, \dots, S_{N-1} > a\right] \in [c, c']$ for every N and every $t \in [a, M]$ where c, c' are positive constants. Thus :

$$Z^f_{N,a,\beta} \geqslant \frac{c}{N^{3/2}} \int_{[a,M]} h(t)dt. \tag{3.1.7}$$

Therefore $F(\beta) \geqslant 0$ for every β. Since the lower bound has been obtained by ignoring the contribution of the paths that touch the stripe, one is led to the following :

Definition 3.1.2. *For $g \in \{c, f\}$, the model $\{\mathbf{P}^g_{N,a,\beta}\}$ is said to be localized if $F(\beta) > 0$.*

It is easy to show that $F(\cdot)$ is a convex function, in particular it is a continuous function as long as it is finite. Obviously, $F(\cdot)$ is increasing since for fixed N, $Z^f_{N,a,\beta}$ is increasing in β. Therefore, there exists a critical value $\beta^a_c \in \mathbb{R}$ such that the stripe wetting model is localized for $\beta > \beta^a_c$.

It is worth stressing that the free energy has a direct translation in terms of some path properties of the field. For $N \in \mathbb{N}$ and $x \in (\mathbb{R}^+)^N$, we introduce $\mathcal{A} := \mathcal{A}(x)$ a subset of $V_N := \{1, \dots, N\}$ defined by :

$$\mathcal{A}(x) := \{i, x_i \in [0, a]\} \tag{3.1.8}$$

and we denote by l_N the cardinality of $\mathcal{A}$. A standard computation shows that, for $g \in \{c, f\}$, for every $\beta \in \mathbb{R}$ and every $N \in N$, one has

$$\mathbf{E}^g_{N,a,\beta}\left[\frac{l_N}{N}\right] = \frac{\partial}{\partial\beta}\frac{1}{N}\log(Z^g_{N,a,\beta}). \tag{3.1.9}$$

A simple convexity argument shows that, as long as $\beta \neq \beta^a_c$,

$$\frac{\partial}{\partial\beta}\frac{1}{N}\log(Z^g_{N,a,\beta}) \to F'(\beta) \text{ as } N \to \infty \tag{3.1.10}$$

where we used the fact that $F(\cdot)$ is analytic on $\mathbb{R} \setminus \{\beta^a_c\}$ (see the considerations following equation (3.3.6)). Adapting in a straightforward way techniques developed in [CD08] (see in particular their appendix A), one can show the stronger bounds :

- For $\beta > \beta^a_c$, we have $F'(\beta) > 0$; for $g \in \{c, f\}$ and for every $\delta > 0, N \in \mathbb{N}$, the following inequality holds :

$$\mathbf{P}^g_{N,a,\beta}\left[\left|\frac{l_N}{N} - F'(\beta)\right| > \delta\right] \leqslant \exp(-cN) \tag{3.1.11}$$

  where c is a positive constant. This shows that when the model is localized, its typical paths touch the defect stripe a positive fraction of the time.

- On the other hand, for $\beta < \beta_c^a$, we have $F'(\beta) = 0$ and for $g \in \{c, f\}$, for every $\delta > 0, N \in \mathbb{N}$, the following holds :

$$\mathbf{P}^g_{N,a,\beta}\left[\frac{l_N}{N} > \delta\right] \leqslant \exp(-cN) \tag{3.1.12}$$

where c is a positive constant. Thus for $\beta < \beta_c^a$, the typical paths of the model touch the defect stripe only $o(N)$ times ; it is then costumary to say that the model is *delocalized*.

3.1.3 Scaling limits.

We define the map $X^N : \mathbb{R}^N \mapsto C([0,1])$:

$$X_t^N(x) := \frac{x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}} + (Nt - \lfloor Nt \rfloor)\frac{x_{\lfloor Nt \rfloor + 1} - x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}} ; t \in [0,1]. \tag{3.1.13}$$

where $\lfloor Nt \rfloor$ denotes the integer part of Nt. Note that $X_t^N(x)$ is the linear interpolation of the process $\{x_{\lfloor Nt \rfloor}/\sigma N^{1/2}\}_{t \in \mathbb{N}/N \cap [0,1]}$. Then we define the measures

$$Q^c_{N,a,\beta} := \mathbf{P}^c_{N,a,\beta} \circ (X^N)^{-1} \tag{3.1.14}$$

and in an analogous way $Q^f_{N,a,\beta}$. These measures are defined on $C([0,1])$ the space of real continuous functions defined on $[0,1]$. We will frequently use the following standard processes :

- ⋆ the Brownian motion $(B_t)_{t \in [0,1]}$.
- ⋆ the Brownian bridge $(\widetilde{B}_t)_{t \in [0,1]}$.
- ⋆ the Brownian meander $(m_t)_{t \in [0,1]}$ which is the Brownian motion conditioned to stay positive on $[0,1]$.
- ⋆ the normalized Brownian excursion $(e_t)_{t \in [0,1]}$ which is the brownian bridge conditioned to stay positive on $[0,1]$.

Of course defining precisely the last two processes is not an obvious task, see [RY99] for details.

Our main result is the following :

Theorem 3.1.3. *Both the free and the constrained models undergo a wetting transition at $\beta = \beta_c^a$. More precisely :*

1. *In the subcritical regime, that is if $\beta < \beta_c^a$, then*
   - $(Q^c_{N,a,\beta})_N$ *converges weakly in $C([0,1])$ to the law of e.*
   - $(Q^f_{N,a,\beta})_N$ *converges weakly in $C([0,1])$ to the law of m.*

2. *In the supercritical regime, that is if $\beta > \beta_c^a$, then both $(Q^c_{N,a,\beta})_N$ and $(Q^f_{N,a,\beta})_N$ converge in $C([0,1])$ to the measure concentrated on the constant function taking value zero.*

These results characterize the Brownian scaling of the model when $\beta \neq \beta_c^a$. Infinite scaling results like Theorem 3.1.3 have been proved in different contexts involving polymer measures. The first mathematical paper dealing with such an issue is [IY01] where the authors proved an analogous convergence in the homogeneous pinning model for the case where S is a symmetric random walk with increments taking values in $\{-1,0,1\}$. Their results have been strongly generalized in [DGZ05] where the same assumptions are made on S as in this paper, and a further generalization of their results in the case where S is in the domain of attraction of the standard normal law has been obtained in [CGZ06].

Analogous results have also been obtained in [CGZ07] in the case of inhomogeneous, but periodic pinning models , and more recently in [CD09] in the case where the interaction is of *Laplacian* type.

What is left by this analysis is the critical case. We believe this case to be workable in the case where the law $\mathbf{P}$ is lattice, the computations being quite similar to the periodic disorder setup for pinning models (see [CGZ06]). Indeed, in this case one can apply the following deep result by Doney (see [Don97]) :

Theorem 3.1.4. *Let τ be a renewal process such that*

$$\mathbf{P}[\tau_1 = n] \sim \frac{L(n)}{n^{1+\alpha}} \tag{3.1.15}$$

where $\alpha \in (0,1)$ and $L(\cdot)$ is some slowly varying function. Then the following equivalence on the Green function holds :

$$\mathbf{P}[n \in \tau] \sim \frac{\alpha \sin(\pi\alpha)}{\pi L(n) n^{1-\alpha}}. \tag{3.1.16}$$

This fundamental result gives estimates on the partition function at criticality in the standard homogeneous pinning model (see [Gia07, Chapter 2]) and moreover it can be adapted to Markov renewals in the discrete setup. For the time being, such a result is lacking in our continuous setup.

3.1.4 Infinite volume limits results.

Our results go beyond brownian scaling. In fact, in the supercritical regime, on can show that the measure converges with no need of rescaling, thus leading to a strong form of localization. Also in the subcritical regime one can go beyond Theorem 3.1.3.

The following result tells us that in the subcritical phase, the dry region reduces to a finite number of points all being at a finite (or microscopic) distance from zero in the free case, from zero and from N in the constrained case.

Theorem 3.1.5. *For $\beta < \beta_c^a$, the following convergences hold :*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \mathbf{P}^f_{N,\beta,a}\left[\max \mathcal{A} \geqslant L\right] = 0 \tag{3.1.17}$$

and

$$\begin{aligned} &\lim_{L\to\infty} \limsup_{N\to\infty} \mathbf{P}^c_{N,\beta,a}\left[\max(\mathcal{A}\cap[1,N/2]) \geqslant L\right] = 0 \\ &\lim_{L\to\infty} \limsup_{N\to\infty} \mathbf{P}^c_{N,\beta,a}\left[\min(\mathcal{A}\cap[N/2,N]) \leqslant N-L\right] = 0. \end{aligned} \tag{3.1.18}$$

Our main results concern the delocalized phase. For the localized phase, we prove the convergence of the sequence of measures $\mathbf{P}^c_{N,\beta,a}$ and $\mathbf{P}^f_{N,\beta,a}$ in $\mathbb{R}^{\mathbb{N}}$ endowed with the product topology (and suitably redefining the measures $\mathbf{P}^c_{N,\beta,a}$ (respectively $\mathbf{P}^f_{N,\beta,a}$)) as $\mathbf{P}^c_{N,\beta,a}\prod_{j=N+1}^{\infty}\delta_0(dx_j)$ (respectively $\mathbf{P}^f_{N,\beta,a}\prod_{j=N+1}^{\infty}\delta_0(dx_j)$) where $\delta_0(\cdot)$ denotes the Dirac mass on zero.

Proposition 3.1.6. *Let $\beta > \beta_c^a$. Then both $\mathbf{P}^c_{N,\beta,a}$ and $\mathbf{P}^f_{N,\beta,a}$ converge in law towards the law of a finitely recurrent irreducible Markov chain on $\mathbb{R}^+$.*

A more detailed description of the limiting Markov process will be made in the section 3.6 of the present chapter. Note in particular the exponential tails of the limit probability of the law of the returns towards the contact set.

3.1.5 Organization of the chapter

The core of our approach is a precise pathwise description of the law $\mathbf{P}^c_{N,a,\beta}$ based on Markov renewal theory. In analogy to [CGZ06] and [CD08], we would like to stress the importance of Markov renewal theory to derive fundamental results on the large scale behavior of the system. The other basic techniques we use are local limit estimates, an infinite version of the Perron Frobenius Theorem and various limit theorems. Let us describe more in detail the content of this chapter :

- in section 3.2, we recall some fluctuation theory that will be of basic importance and we make a link between the estimates we give there and the tails of the return probability to the stripe for large N.
- in section 3.3 we show that the law $\mathbf{P}^c_{N,a,\beta}$ admits a description in terms of a Markov renewal process. More precisely, we show that the contact points with the stripe under $\mathbf{P}^c_{N,a,\beta}$ are distributed according to the law of a Markov renewal process conditioned to hit the stripe at time N. This representation implies in particular a very useful expression for the partition function $Z^c_{N,a,\beta}$ which will be the key to our main results.

- in sections 3.4 and 3.5, we make use of Markov renewal theorems in the finite mean case and of a uniform equivalence result in the infinite mean case ; these theorems easily imply estimates on $Z^c_{N,a,\beta}$. Deducing the asymptotic behavior of $Z^f_{N,a,\beta}$ in both phases is then a standard procedure.
- in section 3.6, we briefly show in which way the preceding sections easily resolve the infinite volume limit of the system for large N.
- sections 3.7 and 3.8 are devoted to the proof of Theorems 3.1.5 and 3.1.3. The proofs are carried out exploiting the asymptotic estimates obtained in section 3.5 combined with powerful limit theorems.

3.2 Some preliminary facts

3.2.1 Some recurrent notations and terminology

For a_n, b_n two positive sequences, we will write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. More generally, for $a_n(x)$ a positive sequence depending on a parameter $x \in \Delta$ where Δ is a subset of $\mathbb{R}^d, d \geqslant 1$, $\alpha \in \mathbb{R}$ and $b(\cdot)$ a measurable function on Δ, we will often say that the equivalence

$$a_n(x) \sim \frac{b(x)}{n^\alpha} \tag{3.2.1}$$

holds *uniformly for x in Δ* if the following holds :

$$\lim_{n\to\infty} \sup_{x\in\Delta} |n^\alpha a_n(x) - b(x)| = 0. \tag{3.2.2}$$

In this chapter, we will deal with kernels of two kind. Kernels of the first kind are just σ-finite kernels on $\mathbb{R}$, that is functions $A : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \mapsto \mathbb{R}^+$ where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field of $\mathbb{R}$ and such that for each $x \in \mathbb{R}$, $A_{x,\cdot}$ is a σ-finite measure on $\mathbb{R}$ and $A_{\cdot,F}$ is a Borel function for every $F \in \mathcal{B}(\mathbb{R})$. Given two such kernels A and B, their composition is denoted by $(A \circ B)_{x,dy} := \int_{z\in\mathbb{R}} A_{x,dz} B_{z,dy}$ and of course $A^{\circ k}_{x,dy}$ denotes the k-fold composition of A with itself where $A^{\circ 0}_{x,dy} := \delta_x(dy)$. We also use the standard notation

$$(1-A)^{-1}_{x,dy} := \sum_{k=0}^{\infty} A^{\circ k}_{x,dy}, \tag{3.2.3}$$

which may be infinite in general.

The second kind of kernels is obtained by letting a kernel of the first kind depend on a further parameter $n \in \mathbb{Z}^+$, that is we consider objects of the form $A_{x,dy}(n)$ with $x, y \in \mathbb{R}, n \in \mathbb{Z}^+$. Given two such kernels $A_{x,dy}(n), B_{x,dy}(n)$ we define their convolution

$$(A * B)_{x,dy}(n) := \sum_{m=0}^{n} (A(m) \circ B(n-m))_{x,dy} = \sum_{m=0}^{n} \int_{\mathbb{R}} A_{x,dz}(m) B_{z,dy}(n-m). \tag{3.2.4}$$

and the k-fold convolution of the kernel A with itself will be denoted by $A^{*k}_{x,dy}$ where by definition $A^{*0}_{x,dy} := \delta_x(dy)\mathbf{1}_{n=0}$. Finally given two kernels $A_{x,dy}(n)$ and $B_{x,dy}$ and a positive sequence a_n, we will write

$$A_{x,dy}(n) \sim \frac{B_{x,dy}}{a_n} \tag{3.2.5}$$

to mean $A_{x,F}(n) \sim \frac{B_{x,F}}{a_n}$ for every $x \in \mathbb{R}$ and for every bounded set $F \subset \mathbb{R}$.

3.2.2 A Markov renewal setup

Let us introduce the following transition kernel :

$$\begin{aligned} &F_{x,dy}(n) := \mathbf{P}_x[S_1 > a, S_2 > a, \ldots, S_{n-1} > a, S_n \in dy]\mathbf{1}_{x,y\in[0,a]} \text{ if } n \geqslant 2, \\ &F_{x,dy}(1) := h(y-x)\mathbf{1}_{x,y\in[0,a]}dy. \end{aligned} \tag{3.2.6}$$

We denote by $(\tau_n)_{n\,\geqslant\,0}$ the times of return to $[0,a]$ of S, that is $\tau_0 := 0$ and, for $n \geqslant 1$, $\tau_n := \inf\{k > \tau_{n-1} | S_k \in [0,a]\}$. Note that $(\tau_n)_{n\,\geqslant\,0}$ is NOT a true renewal process, and this makes actually the fundamental difference with homogeneous pinning. Introducing the process $(J_n)_{n\,\geqslant\,0}$ where $J_n := S_{\tau_n}$, the process τ is a so called *Markov renewal process* whose modulating chain is the Markov chain J. The topic of Markov renewal theory is quite well known, a classical reference is [Asm03].

We finally denote by l_N the cardinality of $\{k \leqslant N | S_k \in [0,a]\}$. With these notations, we can write the joint law of $(l_N, (\tau_n)_{n\,\leqslant\,l_N}, (J_n)_{n\,\leqslant\,l_N})$ under $\mathbf{P}^c_{N,a,\beta}$ under the following form :

$$\begin{aligned} &\mathbf{P}^c_{N,a,\beta}[l_N = k, \tau_j = t_j, J_j \in dy_i, i = 1, \ldots, k] \\ &\qquad = \frac{e^{\beta k}}{Z^c_{N,a,\beta}} F_{0,dy_1}(t_1) F_{y_1,dy_2}(t_2 - t_1) \ldots F_{y_k,dy_{k-1}}(N - t_{k-1}) \end{aligned} \tag{3.2.7}$$

where $k \in \mathbb{N}, 0 < t_1 < \ldots < t_k = N$ and $(y_i)_{i=1,\ldots,k} \in \mathbb{R}^k$.

It is then obvious that to get asymptotic estimates on the partition functions $Z^c_{N,a,\beta}$ (and thus $Z^f_{N,a,\beta}$), one will need control on the asymptotic behavior of $F_{\cdot,\cdot}(n)$ for large n. This control is obtained via fluctuation theory using the results of chapter 4 of the present thesis.

3.2.3 Sharp fluctuation theory estimates

For n an integer, we denote by T_n the *nth ladder epoch*; that is $T_0 := 0$ and, for $n \geqslant 1$, $T_n := \inf\{k \geqslant T_{n-1}, S_k > S_{T_{n-1}}\}$. We also introduce the so-called *ascending ladder height* $(H_n)_{n\,\geqslant\,0}$, which, for $k \geqslant 1$, is given by $H_k := S_{T_k}$. Note that the process (T, H) is a bivariate renewal process on $(\mathbb{R}^+)^2$. More details about it are given in a discrete setup (but the main features are the same) in the chapter 5 of the present thesis. In a similar way, one may define the strict descending ladder variables process (T^-, H^-) as $(T_0^-, H_0^-) := (0,0)$ and

$$T_n^- := \inf\{k \geqslant T_{n-1}, S_k < S_{T_{n-1}}\} \quad \text{and} \quad H_k^- := -S_{t_k^-}. \tag{3.2.8}$$

It is well known (see [Fel71]) that the following relation holds

$$\sigma^2 = 2\mathbf{E}[H_1]\mathbf{E}[H_1^-]. \tag{3.2.9}$$

In particular, it follows that both $\mathbf{E}[H_1]$ and $\mathbf{E}[H_1^-]$ are finite. It is also a well known fact that the continuity of $h(\cdot)$ implies the continuity of the distribution function of H_1 (see [Fel71]).

The renewal function $U(\cdot)$ associated to the ascending ladder heights process will be of basic importance in the sequel :

$$U(x) := \sum_{k=1}^{\infty} \mathbf{P}[H_k \leqslant x] = \mathbf{E}[\mathcal{N}_x] \tag{3.2.10}$$

where $\mathcal{N}_x$ is the cardinality of $\{k \geqslant 0, H_k \leqslant x\}$. It follows in particular from this definition that $U(\cdot)$ is a subadditive increasing function, and in our context it is also a continuous function. We denote by $V(x)$ the analogous quantity for the process H^-. Finally, for a positive integer n, for $(u,x) \in (\mathbb{R}^+)^2$, we will often use the following transition density :

$$g_{n,u}(x) := \mathbf{P}_{-u}[S_1 \leqslant 0, \dots S_{n-1} \leqslant 0, S_n > 0, S_n \in dx]. \tag{3.2.11}$$

With these notations, we show the following result in the chapter 4 of the present thesis :

Theorem 3.2.1. *Assume that X is given by a bounded density, with zero mean and variance σ^2. Then :*

(i) *for any fixed $u \geqslant 0$ the following convergence holds uniformly for $x \geqslant 0$:*

$$\mathbf{P}_{-u}[S_1 \leqslant 0, \dots S_{n-1} \leqslant 0, S_n > 0, S_n \in dx] \sim U(u)\frac{\mathbf{P}[H_1 \geqslant x]}{\sigma\sqrt{2\pi}n^{3/2}}. \tag{3.2.12}$$

(ii) *moreover, the sequence of functions $\sigma\sqrt{2\pi}n^{3/2}g_{n,u}(x)$ is dominated by a multiple of its limit : $n^{3/2}g_{n,u}(x) \leqslant \mathcal{C}U(a)\mathbf{P}[H_1 \geqslant x]$ for every n, u, x ; consequently, one has also that*

$$\mathbf{P}_{-u}[S_1 \leqslant 0, \dots, S_{n-1} \leqslant 0, S_n > 0] \sim U(u)\frac{\mathbf{E}[H_1]}{\sigma\sqrt{2\pi}n^{3/2}}. \tag{3.2.13}$$

Remark 3.2.2. *Of course, considering the random walk $\widetilde{S}$ whose transition are given by $\mathbf{P}_x[\widetilde{S}_1 \in dy] := h(x-y)$, Theorem 3.2.1 implies immediately that, for fixed $u \geqslant 0$, the following equivalence holds uniformly for $x \geqslant 0$:*

$$\widetilde{g}_{n,u}(x) := \mathbf{P}_u[S_1 \geqslant 0, \ldots, S_{n-1} \geqslant 0, -S_n \in dx] \sim V(u) \frac{\mathbf{P}[H_1^- \geqslant x]}{\sigma\sqrt{2\pi} n^{3/2}} \tag{3.2.14}$$

and obviously a similar statement as (ii) of the above theorem also holds for the quantity $\mathbf{P}_u[S_1 \geqslant 0, \ldots, S_{n-1} \geqslant 0, S_n < 0]$.

3.2.4 Linking Theorem 3.2.1 to the stripe problem

We define the following function :

$$\Phi_a(x,y) := \frac{\mathbf{P}[H_1^- \geqslant a-y]}{\sigma\sqrt{2\pi}} \int_a^{+\infty} h(t-x)V(t-a)dt \;\; \mathbf{1}_{x,y \in [0,a]}. \tag{3.2.15}$$

Note that the integral defining $\Phi_a(x,y)$ is finite for every $(x,y) \in [0,a]^2$. Indeed, making use of the classical renewal theorem and of the finiteness of $\mathbf{E}[H_1^-]$, one has $V(t) \sim \frac{t}{\mathbf{E}[H_1^-]}$ as $t \to \infty$, and then we use that $X_1 \in L^1(\mathbb{R})$. Using the continuity and the boundedness of $h(\cdot)$ (which follows from the continuity of the density function of H_1^-, which is itself a consequence of the identity $\mathbf{P}[H_1^- \in I] = \sum_{k \geqslant 1} \mathbf{P}[T_1^- = k, S_k \in I]$ which is valid for every interval I), it is also quite easy to see that $\Phi_a(\cdot,\cdot)$ is a continuous function on $[0,a]^2$.

The next easy result provides estimates on $F(\cdot,\cdot)(n)$ which will be the basis of our sharp estimates on the partition functions :

Lemma 3.2.3. *The following equivalence holds uniformly on $(x,y) \in [0,a]^2$:*

$$F_{x,dy}(n) \sim \frac{\Phi_a(x,y)}{n^{3/2}} dy. \tag{3.2.16}$$

Moreover, uniformly on $x \in [0,a]$:

$$\mathbf{P}_x[\tau_1 \geqslant n] \sim \frac{\int_{[0,a]} \Phi_a(x,y)dy}{n^{1/2}}. \tag{3.2.17}$$

Proof. For $n \geqslant 2$, integrating over the first step of S which is not in $[0,a]$, one gets :

$$F_{x,dy}(n) = \int_a^{+\infty} h(t-x)\mathbf{P}_t\left[S_1 > a, S_2 > a, \ldots, S_{n-2} > a, S_{n-1} \in dy\right] dt \mathbf{1}_{x,y \in [0,a]}. \tag{3.2.18}$$

Thanks to invariance by translation, by Theorem 3.2.1, for fixed $t > a$, we get that the following equivalence holds uniformly on $y \in [0,a]$:

$$\begin{aligned}\mathbf{P}_t\left[S_1 > a, S_2 > a, \dots, S_{n-2} > a, S_{n-1} \in dy\right] &= \widetilde{g}_{n,t-a}(y-a) \\ &\sim V(t-a)\frac{\mathbf{P}[H_1^- \geqslant a-y]}{\sigma\sqrt{2\pi}n^{3/2}}\end{aligned} \tag{3.2.19}$$

so that for fixed $x \in [0,a]$, one gets :

$$\sup_{y\in[0,a]} |n^{3/2}F_{x,dy}(n) - \Phi_a(x,y)| \leqslant \int_a^{+\infty} dt h(t-x) \sup_{y\in[0,a]} \left| n^{3/2}\widetilde{g}_{n,t-a}(y-a) - V(t-a)\frac{\mathbf{P}[H_1^- \geqslant a-y]}{\sigma\sqrt{2\pi}} \right|. \tag{3.2.20}$$

The point (ii) of Theorem 3.2.1 allows us to use dominated convergence in the above integral and thus for fixed $x \in [0,a]$, the quantity $\sup_{y\in[0,a]} |n^{3/2}F_{x,dy}(n) - \Phi_a(x,y)|$ vanishes as $n \to \infty$.

To get the uniform convergence for $x \in [0,a]$, one just has to find a suitable domination to the integrand to apply the same argument as above. Using once again point (ii) of Theorem 3.2.1, it is easy to see that this reduces to showing that

$$\sup_{x\in[0,a]} \int_a^{\infty} h(t-x)V(t-a)dt < \infty \tag{3.2.21}$$

and using the asymptotic equivalence $V(t) \sim \frac{t}{\mathbf{E}[H_1^-]}$, this will be implied by

$$\sup_{x\in[0,a]} \int_a^{\infty} h(t-x)tdt < \infty \tag{3.2.22}$$

and of course the last term is bounded by

$$a + \sup_{x\in[0,a]} \int_x^{\infty} th(t)dt \leqslant a + \int_0^{\infty} th(t)dt < \infty \tag{3.2.23}$$

thus implying equation (3.2.16).

For equation (3.2.17), we have immediately :

$$\mathbf{P}_x[\tau_1 \geqslant k] = \sum_{j\,\geqslant\,k} j^{-3/2} \int_{[0,a]} j^{3/2}F_{x,dy}(j) \tag{3.2.24}$$

so that using the first point of Lemma 3.2.3 (in particular the uniform convergence part of it) and the standard equivalence $\sum_{j\,\geqslant\,k} j^{-3/2} \sim 2k^{-1/2}$, we are done.

□

3.3 An infinite dimensional problem

3.3.1 Defining the free energy

In this section, we define the free energy in a way that allows us to make use of the Markov renewal structure we already pointed at in section 3.2.2. For $\lambda \geqslant 0$, we introduce the following kernel :

$$B^{\lambda}_{x,dy} := \sum_{n=1}^{\infty} e^{-\lambda n} F_{x,dy}(n) \tag{3.3.1}$$

and the associated the integral operator

$$(B^{\lambda}h)(x) := \int_{[0,a]} B^{\lambda}_{x,dy} h(y). \tag{3.3.2}$$

We then have the

Lemma 3.3.1. *For $\lambda \geqslant 0, B^{\lambda}_{x,dy}$ is a compact operator on the Hilbert space $L^2([0,a])$.*

Proof. Note that for all $n \in \mathbb{N}, F_{x,dy}(n)$ has a density $f_{x,y}(n)$ with respect to the Lebesgue measure restricted to $[0,a]$, and this implies that $B^{\lambda}_{x,dy}$ also has one (we denote it by $b^{\lambda}(x,y)$). To show Lemma 3.3.1, it is sufficient to show that $B^{\lambda}_{x,dy}$ is actually Hilbert-Schmidt, that is that

$$\int b^{\lambda}(x,y)^2 \mathbf{1}_{x,y\in[0,a]} dxdy < \infty. \tag{3.3.3}$$

Note that by the point (ii) from Theorem 3.2.1 and taking into account the fact that $h(\cdot)$ is bounded, there exists $\mathcal{C}' > 0$ such that, for all $n \in \mathbb{N}$ and $x, y \in [0,a]$, $f_{x,y}(n) \leqslant \mathcal{C}' \frac{g(y-a)}{n^{3/2}}$ where $g(\cdot)$ is the density of H_1^-.

It is then straightforward to show (3.3.3). Actually, we have the inequalities :

$$\begin{aligned} &\int b^{\lambda}(x,y)^2 \mathbf{1}_{x,y\in[0,a]} dxdy \\ &\qquad = \int \left(\sum_{n=1}^{\infty} e^{-\lambda n} f_{x,y}(n) \right)^2 \mathbf{1}_{x,y\in[0,a]} dxdy \\ &\qquad \leqslant a \int \sum_{n,m \,\geqslant\, 1} e^{-\lambda(n+m)} \mathcal{C}'^2 g(y-a)^2 n^{-3/2} m^{-3/2} \mathbf{1}_{y\in[0,a]} dy < \infty. \end{aligned} \tag{3.3.4}$$

□

Lemma 3.3.1 enables us to introduce $\delta^a(\lambda)$, the spectral radius of the operator B^λ. One may define $\delta^a(\lambda)$ more explicitly by :

$$\delta^a(\lambda) = \inf\left\{R > 0, \sum_{n=0}^{\infty} \frac{(B^\lambda_{0,[0,a]})^{\circ n}}{R^n} < \infty\right\} \tag{3.3.5}$$

It is easy to check that $\delta^a(\lambda) \in (0,\infty)$ for $\lambda \geqslant 0$; $\delta^a(\lambda)$ is an isolated and simple eigenvalue of $B^\lambda_{x,dy}$ (see Theorem 1 in [Zer87]). The function $\delta^a(\cdot)$ is non-increasing, continuous on $[0,\infty)$ and analytic on $(0,\infty)$ because the operator $B^\lambda_{x,dy}$ has these properties. The analyticity and the fact that $\delta^a(\cdot)$ is not constant (as $\delta^a(\lambda) \to 0$ as $\lambda \to \infty$) force $\delta^a(\cdot)$ to be strictly decreasing.

We denote by $(\delta^a)^{-1}(\cdot)$ its inverse function, defined on $(0, \delta^a(0)]$. We now define β^a_c and $F^a(\beta)$ by :

$$\beta^a_c := -\log(\delta^a(0)), \quad F^a(\beta) := (\delta^a)^{-1}(\exp(\beta)) \text{ if } \beta \geqslant \beta^a_c \text{ and } 0 \text{ otherwise.} \tag{3.3.6}$$

Note that this definition entails in particular the analyticity of $F(\cdot)$ on $\mathbb{R} \setminus \{\beta^a_c\}$.

Of course it is not clear *a priori* that the quantity we define in (3.3.6) actually coincides with the classical definition of the free energy, that is the limit of the quantity $\frac{1}{N}\log\left(Z^f_{N,a,\beta}\right)$. We will show in the next parts that this is the case. Indeed, this definition will entail a representation for the constrained partition function of the system which is explicited in section 3.3.2, and this representation in turn will provide estimates on the partition function of the system in both phases. These estimates will finally validate the coherence of the definition given in equation (3.3.6).

3.3.2 A useful representation for $Z^c_{N,a,\beta}$

The fact that $b^{F^a(\beta)}(x,y) > 0$ for every $(x,y) \in [0,a]$ implies the uniqueness (up to a multiplication by a positive constant) and the positivity almost everywhere of the so-called right and left Perron Frobenius eigenfunctions of $B^{F^a(\beta)}_{x,dy}$; more precisely, Theorem 1 in [Zer87] implies that there exist two functions $v_\beta(\cdot)$ and $w_\beta(\cdot)$ in $L^2([0,a])$ such that $v_\beta(x) > 0$ and $w_\beta(x) > 0$ for almost every $x \in [0,a]$, and moreover :

$$\int_{y\in[0,a]} B^{F^a(\beta)}_{x,dy} v_\beta(y) = \left(\frac{1}{e^\beta} \wedge \frac{1}{e^{\beta^a_c}}\right) v_\beta(x) \tag{3.3.7}$$

$$\int_{x\in[0,a]} w_\beta(x) B^{F^a(\beta)}_{x,dy} dx = \left(\frac{1}{e^\beta} \wedge \frac{1}{e^{\beta^a_c}}\right) w_\beta(y)dy. \tag{3.3.8}$$

Spelling out these equalities, we get that

$$v_\beta(x) = \frac{1}{\frac{1}{e^\beta} \wedge \frac{1}{e^{\beta^a_c}}} \sum_{n\geqslant 0} e^{-F^a(\varepsilon)n} \int_{y\in[0,a]} f_{x,y}(n) v_\beta(y)dy, \tag{3.3.9}$$

which implies in particular the fact that $v_\beta(\cdot)$ is positive on the whole $[0,a]$ and not only almost everywhere. Similarly, the function $w_\beta(\cdot)$ is everywhere positive. These considerations lead us to define the new kernel

$$K^\beta_{x,dy}(n) := e^\beta F_{x,dy}(n) e^{-F^a(\beta)n} \frac{v_\beta(y)}{v_\beta(x)}. \tag{3.3.10}$$

It is then straightforward to check that

$$\begin{aligned} \int_{y\in\mathbb{R}} \sum_{n\in\mathbb{N}} K^\beta_{x,dy}(n) &= \frac{e^\beta}{v_\beta(x)} \int_{y\in\mathbb{R}} \sum_{n\in\mathbb{N}} F_{x,dy}(n) e^{-F^a(\beta)n} v_\beta(y) \mathbf{1}_{y\in[0,a]} \\ &= \frac{e^\beta}{v_\beta(x)} \int_{y\in\mathbb{R}} B^{F^a(\beta)}_{x,dy} v_\beta(y) = 1 \wedge \frac{e^\beta}{e^{\beta^a_c}}. \end{aligned} \tag{3.3.11}$$

With the help of the above considerations, we can define the law $\mathcal{P}_\beta$ under which the joint process $(\tau_k, J_k)_{k\geqslant 0}$ is an inhomogeneous Markov chain (defective if $\beta < \beta^a_c$) on $\mathbb{Z}^+ \times [0,a]$ by :

$$\mathcal{P}_\beta \left[(\tau_{k+1}, J_{k+1}) \in (\{n\}, dy) | (\tau_k, J_k) = (m, x) \right] := K_{x,dy}(n-m). \tag{3.3.12}$$

It is possible to define this object in a more intuitive way ; first, sample the J_i's as a Markov chain following the law

$$\mathcal{P}_\beta(J_{k+1} \in dy | J_k = x) = \sum_{n\geqslant 1} K^\beta_{x,dy}(n). \tag{3.3.13}$$

Then, sample the increments $(\tau_k - \tau_{k-1})_{k\geqslant 1}$ as a sequence of independent but not identically distributed variables according to the law

$$\mathcal{P}_\beta(\tau_k - \tau_{k-1} = n | (J_k)_{k\geqslant 0}) = \frac{k^\beta_{J_{k-1},J_k}(n)}{\sum_{n\geqslant 1} k^\beta_{J_{k-1},J_k}(n)} \tag{3.3.14}$$

where $k^\beta_{x,y}(n)$ is the density of $K^\beta_{x,dy}(n)$ with respect to the Lebesgue measure. Recall that the sequence $(\tau_k)_{k\geqslant 0}$ is what is called a Markov renewal, the process $(J_i)_{i\geqslant 0}$ being its modulating chain. We then have an analogous property as for homogeneous pinning (see [Gia07]), its proof is contained in equality (3.2.7) :

Proposition 3.3.2. *For any $N \in \mathbb{N}$, the vector $(l_N, (\tau_n)_{n\leqslant l_N}, (J_n)_{n\leqslant l_N})$ has the same law under $\mathbf{P}^c_{N,a,\beta}$ as under the conditional law $\mathcal{P}_\beta(\cdot|\mathcal{A}_N)$ where $\mathcal{A}_N := \{\exists k \in \mathbb{N} | \tau_k = N\}$. Equivalently :*

$$\begin{aligned} &\mathbf{P}^c_{N,a,\beta} \left[l_N = k, \tau_j = t_j, J_j \in dy_i, i = 1,\dots,k \right] \\ &\qquad = \mathcal{P}_\beta \left[l_N = k, \tau_j = t_j, J_j \in dy_i, i = 1,\dots,k | \mathcal{A}_N \right]. \end{aligned} \tag{3.3.15}$$

Proposition 3.3.2 shows in particular that, similarly to the classical homogeneous pinning model, the partition $Z^c_{N,a,\beta}$ can be interpreted as the Green function associated to the Markov renewal τ, that is $Z^c_{N,a,\beta} = \mathcal{P}_\beta[N \in \tau]$ and more generally for $x \in [0,a]$, $Z^c_{N,a,\beta}(dx) = \mathcal{P}_\beta[\exists k, \tau_k = N, J_k \in dx]$. Equivalently, we have the equality

$$Z^c_{N,a,\beta} = \exp(F^a(\beta)N) \int_{[0,a]} \frac{v_\beta(0)}{v_\beta(y)} \sum_{k \geqslant 0} (K^\beta)^{*k}_{0,dy}(N) \tag{3.3.16}$$

which is a consequence of the more general equality :

$$Z^c_{N,a,\beta}(dy) = \exp(F^a(\beta)N) \frac{v_\beta(0)}{v_\beta(y)} \sum_{k \geqslant 0} (K^\beta)^{*k}_{0,dy}(N) \tag{3.3.17}$$

which holds for $y \in [0,a]$.

3.4 The localized phase

3.4.1 A key Markov renewal theorem

Let $\beta > \beta^a_c$. In this case, the two functions $w_\beta(\cdot)$ and $v_\beta(\cdot)$ are uniquely defined up to a multiplicative constant, and we use this degree of freedom to fix $\int_{\mathbb{R}} v_\beta(x) w_\beta(x) \mathbf{1}_{x\in[0,a]} = 1$. Thus the measure μ_β defined by

$$\mu_\beta(dx) := v_\beta(x) w_\beta(x) \mathbf{1}_{x\in[0,a]} dx \tag{3.4.1}$$

is a probability measure. It is not difficult to see that, if $\beta > \beta^a_c$, the probability μ_β is invariant for the kernel $\sum_{n\geqslant 1} K^\beta_{x,dy}(n)$, hence for the Markov process (J_n).

To get estimates on the partition function of the Markov renewal process, we need to show an analogous to the classical Markov renewal theorem (which can be found in [Asm03]) in the case where the state space of the J_i's is not countable.

For such a convergence to hold, we need a stronger form of recurrence that in the discrete case. For this we introduce the notion of so called Harris recurrence.

Definition 3.4.1. *A Markov chain (X_n) on a measurable state space $(E, \mathcal{F})$ with positive invariant probability μ is said to be Harris recurrent if the following holds for all $A \in \mathcal{F}$ and all $x \in E$:*

$$\mu(A) > 0 \Longrightarrow \mathbf{P}_x[\tau_A < +\infty] = 1 \tag{3.4.2}$$

where $\tau_A := \inf\{n \geqslant 0, X_n \in A\}$.

We introduce two definitions valid for Markov chains on general state space :

Definition 3.4.2. *Let X_n be a Markov chain on some topological state space $X, \mathcal{B}(X)$.*

- *Given a non negative function Ψ defined on $\mathcal{B}(X)$, we say that X_n is Ψ-irreducible if whenever $\Psi(A) > 0$ (where A is an element of $\mathcal{B}(X)$), there exists some n such that $P^n(x, A) > 0$.*
- *We say that X_n is T-irreducible if there exists a kernel $T_{x,dy}$ such that there exists some $\varepsilon > 0$ verifying*

$$\sum_{n=0}^{\infty} \varepsilon^n P^n(x, A) \geqslant T(x, A) \tag{3.4.3}$$

for all $A \in \mathcal{B}(X)$, where $T(\cdot, A)$ is a lower semi-continuous function for all $A \in \mathcal{B}(X)$.

Of course, in our case X will simply be the space $[0, a] \times \mathbb{N}$.
We then have the fundamental theorem (see [MT09, Theorem 9.0.2]) :

Theorem 3.4.3. *Let X_n be a T-irreducible Markov chain with state space X. X_n is Harris recurrent if and only if $\mathbf{P}_x[X_n \to \infty] = 0$ for each $x \in X$ where the event $\{X_n \to \infty\} := \{$ the trajectory visits each compact set only finitely often $\}$.*

We recall the following very general result which holds for Harris recurrent Markov chains (see [MT09, Theorem 13.3.3]) :

Theorem 3.4.4. *Let X be a Markov chain with transition probability $P(\cdot, \cdot)$ on a measurable state space $(E, \mathcal{F})$ with positive invariant probability μ and denote by $||\cdot||$ the total variation norm. Then the following convergence holds for every initial distribution $\lambda(\cdot)$ on E :*

$$|| \int \lambda(dx) P^n(x, \cdot) - \mu|| \to 0. \tag{3.4.4}$$

We are now ready to state the main result of this part :

Lemma 3.4.5. *In the localized regime, for $x \in [0, a]$ the following convergence holds in total variation norm :*

$$\lim_{N\to\infty} \mathcal{P}_\beta \left[\exists k \in \mathbb{N}, \tau_k = N, J_N \in dx\right] = \frac{\mu_\beta(dx)}{\int_{[0,a]^2} \mu_\beta(du) \sum_{n \geqslant 0} n K^\beta_{u,dy}(n)}. \tag{3.4.5}$$

Proof of Lemma 3.4.5 . We consider the Markov process $(A_k, J'_k)_{k \geqslant 0}$ on $\mathbb{N} \times [0, a]$ whose transitions are given by :

$$P_\beta \left[A_j = k, J'_j \in dy | A_{j-1} = l, J'_{j-1} = x\right] := \delta_{k,l-1} \delta_x(dy) \tag{3.4.6}$$

if $l \geqslant 2$ (where $\delta_x(\cdot)$ is the Dirac measure concentrated on $\{x\}$) and by

$$P_\beta\left[A_j = k, J'_j \in dy | A_{j-1} = 1, J'_{j-1} = x\right] := K^\beta_{x,dy}(k). \tag{3.4.7}$$

Note that this Markov chain is nothing but the well known *forward recurrence chain* associated to the Markov renewal. More specifically, A denotes the time one has to wait at time i until the next renewal happens, the Markov chain J' containing the last location of its modulating chain. Our proof is in a certain sense a generalization of the well-known proof of the classical Renewal Theorem which uses the ergodic properties of the forward recurrence chain (see [Asm03, VII.2]).

We introduce the probability measure on $\mathbb{N} \times [0, a]$ defined by :

$$\Pi^\beta(i, dy) := \frac{1}{\int_{[0,a]^2} \mu_\beta(dx) \sum_{k \geqslant 1} k K^\beta_{x,du}(k)} \int_0^a \mu_\beta(dx) \sum_{j \geqslant i} K^\beta_{x,dy}(j). \tag{3.4.8}$$

Note that $\int_{[0,a]^2} \mu_\beta(dx) \sum_{k \geqslant 1} k K^\beta_{x,du}(k) < \infty$ since $\beta > \beta^a_c$, so that in particular $\Pi^\beta(\cdot,\cdot)$ is non degenerate. It is easy to see that $\Pi^\beta(\cdot,\cdot)$ is the invariant probability of the Markov process $(A_k, J'_k)_{k \geqslant 0}$. Indeed, for all $(i, y) \in \mathbb{N} \times [0, a]$, we have easily :

$$\Pi^\beta P_\beta(i, dy) = \Pi^\beta(i + 1, dy) + \int_0^a \Pi^\beta(1, dx) K^\beta_{x,dy}(i) = \Pi^\beta(i, dy) \tag{3.4.9}$$

where in the second equality we used the fact that $\mu_\beta(\cdot)$ is the invariant probability for the Markov process $(J_k)_{k \geqslant 1}$ (noting that $\Pi^\beta(1, dx)$ is a multiple of $\mu_\beta(dx)$, this is exactly saying that $\int_0^a \mu_\beta(dx) K^\beta_{x,dy}(i) = \mu_\beta(dy)$, which is the second part of equation (3.4.9)).

We note that, making use of the positivity of μ_β on $[0, a] \times \mathbb{N}$, the Markov chain (A, J') satisfies the hypothesis of Theorem 3.4.3 (it is μ_β irreducible and its kernel being positive and continuous it also trivially satisfies the T-condition), so that

$$|| \int \lambda(dx) P^n_\beta(x, \cdot) - \Pi^\beta || \to 0 \tag{3.4.10}$$

as $n \to \infty$, where $|| \cdot ||$ denotes the total variation norm on $\mathbb{N} \times [0, a]$ and $\lambda(\cdot)$ any initial distribution. This implies in particular that, as $j \to \infty$, the following convergence holds in total variation norm :

$$P_\beta\left[A_j = 1, J'_j \in dx\right] \to \frac{\mu_\beta(dx)}{\int_{[0,a]^2} \mu_\beta(du) \sum_{k \geqslant 1} k K^\beta_{u,dy}(k)} =: \frac{\mu_\beta(dx)}{C_\beta} \tag{3.4.11}$$

and since $P_\beta\left[A_j = 1, J'_j \in dx\right] = \mathcal{P}_\beta\left[\exists k \in \mathbb{N}, \tau_k = j, J_k \in dx\right]$, the proof is complete. □

3.4.2 Deducing asymptotic estimates on the partition function

In the localized phase, this result provides an estimate for $Z^c_{N,a,\beta}$.

For $x, y \in [0,a]$, we define :

$$Z^c_{N,a,\beta}(x,dy) := \mathbf{E}_x\left[\exp\left(\beta\sum_{k=1}^N \mathbf{1}_{S_k\in[0,a]}\right)\mathbf{1}_{S_k\geqslant 0, k=1\ldots N}\mathbf{1}_{S_N\in dy}\right], \tag{3.4.12}$$

Theorem 3.4.6. *For $\beta > \beta_c^a$, for every $x \in [0,a], y \in \mathbb{R}^+$, as $N \to \infty$, one has the convergence :*

$$Z^c_{N,a,\beta}(x,dy) \sim \frac{v_\beta(x)v_\beta(y)}{C_\beta}\exp(F^a(\beta)N)dy \tag{3.4.13}$$

where for a fixed $x \in [0,a]$ (and more generally for any initial distribution of x), the convergence of $Z^c_{N,a,\beta}(x,dy)\exp(-F^a(\beta)N)$ towards $\frac{v_\beta(x)v_\beta(y)}{C_\beta}dy$ holds in total variation norm.

These estimates imply in particular that there exist two positive constants $C^a(\beta)$ and $C^a_f(\beta)$ such that, :

1. $Z^c_{N,a,\beta} \sim C^a(\beta)\exp(F^a(\beta)N)$
2. $Z^f_{N,a,\beta} \sim C^a_f(\beta)\exp(F^a(\beta)N)$.

Remark 3.4.7. *Note that an immediate consequence of Theorem 3.4.6 is the convergence of the quantity $1/N\log(Z^f_{N,a,\beta})$ towards $F^a(\beta)$ in the case where this last quantity is positive.*

Proof. Combining identity (3.3.17) and Lemma 3.4.5, we immediately get the following estimate :

$$Z^c_{N,a,\beta}(x,dy) \sim \exp(F^a(\beta)N)\frac{v_\beta(x)}{v_\beta(y)}\frac{\mu_\beta(dy)}{\int_{[0,a]^2}\mu_\beta(dx)\sum_{k\geqslant 1}kK^\beta_{x,dy}(k)}. \tag{3.4.14}$$

The estimate in the free case is easy ; we have the relation :

$$Z^f_{N,a,\beta} = e^{F^a(\beta)N}\sum_{t=0}^N Z_{N-t,a,\beta}(dx)e^{-F^a(\beta)(N-t)}\mathbf{P}_x[S_1 > a, \ldots, S_t > a]e^{-F^a(\beta)t}. \tag{3.4.15}$$

Using the total variation convergence part of Lemma 3.4.5, this entails :

$$Z^f_{N,a,\beta} \sim C_a(\beta)e^{F^a(\beta)N}\sum_{t=0}^\infty e^{-F^a(\beta)t}\int_{[0,a]}\frac{\mu_\beta(dx)}{C_\beta}\mathbf{P}_x[\tau_1 > t+1]. \tag{3.4.16}$$

One just has to note that the integral in the equation above is convergent ; this is the case because on the one hand $\beta > \beta_c^a$ entails $F^a(\beta) > 0$ and on the other hand, making use of Lemma 3.2.3 (in particular the uniform statement part) , the term in the integral is of order $t^{-1/2}$ for large t. We finally get the second statement of Lemma 3.4.6.

□

3.5 The delocalized phase

In the delocalized phase, we can adapt in a quite straightforward way the proof of [CGZ06] using techniques which have been developed in [CD08]. We first recall the following proposition which has been proven in [CD08, Proposition 7.2] :

Proposition 3.5.1. *Assume we are given a kernel $A_{x,dy}(n)$ satisfying the following assumptions :*

1. *the spectral radius of $G_{x,dy} := \sum_{n\in\mathbb{N}} A_{x,dy}(n)$ is strictly smaller than one ;*
2. *uniformly in $x \in [0, a]$:*

$$A_{x,dy}(n) \sim L_{x,dy}/n^{3/2}. \tag{3.5.1}$$

Furthermore, there exists a positive constant $\mathcal{C}$ such that for every $x \in [0, a]$ and every closed set $F \subset [0, a]$, $A_{x,F}(n) \leqslant \mathcal{C}L_{x,F}/n^{3/2}$;

3. *there exists $\gamma > 1$ such that $((1 - \gamma G)^{-1} \circ L \circ (1 - \gamma G)^{-1})_{x,F} < \infty$ for all $x \in [0, a]$ and for all F Borel subset of $[0, a]$.*

Then, as $n \to \infty$, the following relation holds uniformly in $x \in [0, a]$:

$$A^{*k}_{x,F}(n) \sim \frac{((1 - \gamma G)^{-1} \circ L \circ (1 - \gamma G)^{-1})_{x,dy}}{n^{3/2}}. \tag{3.5.2}$$

Actually, the result proven in [CD08] is slightly different, but can be transposed in a straightforward way to our context. Proposition 3.5.1 easily implies the following estimates :

Proposition 3.5.2. *For $\beta < \beta_c^a$,as $N \to \infty$, we have the following :*

1. $Z^c_{N,a,\beta} \sim C'^a(\beta)/N^{3/2}$
2. $Z^f_{N,a,\beta} \sim C'^a_f(\beta)/N^{1/2}$

where $C'^a(\beta)$ and $C'^a_f(\beta)$ are positive functions depending on β.

Note that these estimates coincide (except of course for the precise values of $C'^a(\beta)$ and $C'^a_f(\beta)$) with the ones proven in the homogeneous pinning model case (see [Gia07, Theorem 2.2]). They also complete the proof of the identification of $F^a(\beta)$ as being the free energy (since they tell us that, for $\beta < \beta_c^a$, $1/N \log \left(Z^f_{N,a,\beta}\right)$ converges to zero).

Proof of Proposition 3.5.2. We apply Proposition 3.5.1 considering $A_{x,dy} := e^{\beta} F_{x,dy}$. As $\beta < \beta_c^a$, the spectral radius of $\sum_n A_{x,dy}(n)(= e^{\beta} B^0_{x,dy})$ is equal to $e^{\beta-\beta_c^a} < 1$. The fact that the kernel A satisfies point (2) of Proposition 3.5.1 has been proven in Lemma (3.2.3) with $L_{x,dy} := e^{\beta}\Phi_a(x,y)$. Using equation (3.3.16), it is then clear that we are left with proving that A satisfies the third hypothesis of Proposition 3.5.1 to get the first estimate of Proposition 3.5.2 ; concretely, we are left with showing that, for all closed subset $F \subset [0,a]$,

$$\int_{y,z\in[0,a],t\in F} \left(1-\gamma\sum_{n\in\mathbb{N}} e^{\beta} B^0\right)^{-1}_{x,dy} e^{\beta}\Phi_a(y,z)dz \left(1-\gamma\sum_{n\in\mathbb{N}} e^{\beta}B^0\right)^{-1}_{z,dt} < \infty \quad (3.5.3)$$

for some $\gamma > 1$. We fix some $\gamma \in (1, e^{\beta_c^a - \beta})$; we note that

$$\left(1-\gamma\sum_{n\in\mathbb{N}} e^{\beta}B^0\right)^{-1}_{x,dy} = 1+\gamma e^{\beta}\int_{z\in[0,a]}\sum_{n\in\mathbb{N}} B^0_{x,dz}(\gamma e^{\beta}B^0)^{\circ n}_{z,dy}. \quad (3.5.4)$$

Making use of the explicit form of Φ_a (see equation (3.2.15)), we can split the integral appearing in equation (3.5.3) in two parts, so that we are left with showing that

$$\int_{[0,a]} \left(1+\gamma e^{\beta}\sum_{n\in\mathbb{N}} B^0_{x,dz}\circ(\gamma e^{\beta}B^0)^{\circ n}_{z,dy}\right)\int_a^{+\infty} dt\, h(t-y)V(t-a) < \infty \quad (3.5.5)$$

and

$$\int_{t\in F}\int_{[0,a]} dz \left(1+\gamma e^{\beta}\sum_{n\in\mathbb{N}} B^0_{z,du}\circ(\gamma e^{\beta}B^0)^{\circ n}_{u,dt}\right)\mathbf{P}[H_1^- \geqslant a-z] < \infty. \quad (3.5.6)$$

It is easy to show (3.5.6) ; in fact, making use of Cauchy Schwarz's inequality, we get immediately that the left hand side is smaller than :

$$\int_{t\in F}\left(1+\sum_{n\in\mathbb{N}} ||B^0_{z,du}||\cdot||(\gamma e^{\beta}B^0_{u,dt})^{\circ n}||\right) \quad (3.5.7)$$

where $||\cdot||$ stands for the $L^2([0,a])$ operator norm. Recalling that β_c^a is the spectral radius of the operator B^0, a classical result gives $||(\gamma e^{\beta}B^0_{u,dt})^{\circ n}||^{1/n} \to \gamma e^{\beta-\beta_c^a} < 1$ as $n\to\infty$ (see [Kat76, Chapter III, 6.2]), and using the fact that $||B^0_{z,du}|| < \infty$ (see Lemma (3.3.1)) we get (3.5.6).

To get (3.5.5), using the first point, it is enough to show that the function $y\in[0,a]\mapsto\int_a^{+\infty} dt h(t-y)V(t-a)$ is in $L^2([0,a])$, but we already noted that this function is continuous on $[0,a]$; thus the constrained case is proved.

Conditioning on the last passage in $[0, a]$ of our random walk before N, we finally get :

$$N^{1/2} Z^f_{N,a,\beta} = \int_{[0,a]} \sum_{k=0}^{N} Z^c_{k,a,\beta}(dx) N^{1/2} \mathbf{P}_x[\tau_1 \geqslant N - k + 1] \tag{3.5.8}$$

and using the equivalence part of Proposition 3.5.1 (which gives an estimate on the quantity $Z^c_{k,a,\beta}(dx)$), this implies that $N^{1/2} Z^f_{N,a,\beta} \to C''^a(\beta) \int_{[0,a]^2} \Phi_a(x,y) dx dy =: C''^a_f(\beta)$.

□

3.6 Infinite volume limits

With the above estimates, for $i \in \{c, f\}$, it is quite easy to understand the infinite volume limit of the laws $\mathbf{P}^i_{N,a,\beta}$.

Theorem 3.6.1. *For every $\beta \in \mathbb{R}$ such that $\beta \neq \beta^a_c$ and for $i \in \{c, f\}$, for every $k \in \mathbb{N}$ and for every $(t_1, \ldots, t_k) \in \mathbb{N}^k, t_1 < t_2 < \ldots < t_k, (x_1, \ldots, x_k) \in [0, a]^k$, one has the following convergence :*

$$\mathbf{P}^i_{N,a,\beta}\left[(\tau_1, J_1) \in (\{t_1\}, dx_1), \ldots (\tau_k, J_k) \in (\{t_k\}, dx_k)\right] \\ \to \mathcal{P}_\beta\left[(\tau_1, J_1) \in (\{t_1\}, dx_1), \ldots (\tau_k, J_k) \in (\{t_k\}, dx_k)\right]. \tag{3.6.1}$$

Proof. Using the above equivalences and the representation 3.3.16, as $N \to \infty$, one has both in the localized and in the delocalized regimes :

$$\mathbf{P}^i_{N,a,\beta}\left[(\tau_1, J_1) \in (\{t_1\}, dx_1), \ldots (\tau_k, J_k) \in (\{t_k\}, dx_k)\right] \\ = \frac{\mathcal{P}_\beta\left[(\tau_1, J_1) \in (\{t_1\}, dx_1), \ldots (\tau_k, J_k) \in (\{t_k\}, dx_k)\right] R^i_\beta(N - t_k)}{R^i_\beta(N)} \tag{3.6.2}$$

where $R^i(\cdot)$ is a positive function which converges towards a positive limit if $\beta > \beta^a_c$ and whose limit is a rational function of N if $\beta < \beta^a_c$. Hence the result. □

3.7 The set of contact points in the subcritical regime

The main goal of this part is to prove Lemma 3.1.5, which tells us that in the subcritical regime, the set of contacts points is practically absent. More precisely, we show that both in the constrained case and in the free case, the process visits only finitely many times the interval $[0, a]$ and that these visits are all at a finite distance from the origin for the free case, from the origin and from the fixed endpoint for the constrained case.

Note that the estimates on the asymptotic behavior of the partition functions being the same as in the classical homogeneous wetting case, the proofs in this part are quite similar to the ones in [DGZ05].

3.7.1 The free case

We introduce a new law $p^f_{\beta,N}(\cdot,\cdot)$ on $V_N \times (\mathbb{R}^+)^N$ which is defined as :

$$p^f_{\beta,N}(A, dx) := \frac{1}{Z^f_{N,a,\beta}} e^{\beta|A|} \prod_{j=1}^{|A|} F_{x_{t_{j-1}}, dx_{t_j}}(t_j - t_{j-1}) \mathbf{1}_{x_{t_j} \in [0,a], \forall j \in \{0,\dots,|A|\}}. \tag{3.7.1}$$

where we recall that $V_N := \mathcal{P}(\{1, \dots, N\})$ and where $t_0 := 0$, $x_0 := 0$ and $A := \{t_1 < t_2 < \dots < t_n\}$. We can write :

$$\mathbf{P}^f_{N,a,\beta}(dx) = \sum_{A \subset V_N} \int_{[0,a]^{|A|}} p^f_{\beta,N}(A, dy) \mathbb{P}_{A,y}(dx) \tag{3.7.2}$$

where $\mathbb{P}_{A,y}(\cdot)$ is the law of $(S_1, \dots, S_N)$ conditioned on the event $\mathcal{E}_{N,A,y}$ which is defined by :

$$\mathcal{E}_{N,A,y} := \Big\{ S_i = y_i; i \in A \cup \{0\} \Big\} \cap \Big\{ S_i > a, i \notin A \Big\}. \tag{3.7.3}$$

For $A \in V_N$, we write $\mathcal{L}(A) := \max A$. The following lemma immediately implies the first part of Lemma 3.1.5 :

Lemma 3.7.1. *For $\beta < \beta^a_c$, the following estimates hold :*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^N} p^f_{\beta,N}(\mathcal{L}(A) \geqslant L, x) = 0 \tag{3.7.4}$$

Proof. For $x \in \mathbb{R}^N$, we write :

$$\begin{aligned} p^f_{\beta,N}(\mathcal{L}(A) \geqslant L, x) &= \sum_{k=L}^{N} p^f_{\beta,N}(\mathcal{L}(A) = k, x) \\ &= \frac{\sum_{k=L}^N Z^c_{k,a,\beta}(x) \mathbf{P}_{x_k}[\tau_1 \geqslant N-k]}{\sum_{k=1}^N \int_{[0,a]} Z^c_{k,a,\beta}(du_k) \mathbf{P}_{u_k}[\tau_1 \geqslant N-k]} \end{aligned} \tag{3.7.5}$$

where we defined (with the usual convention $t_{|A|+1} = k$) :

$$Z^c_{k,a,\beta}(x) := \sum_{A \subset V_{k-1}, A=\{t_1<\dots<t_{|A|}\}} e^{\beta(|A|+1)} \prod_{j=1}^{|A|+1} F_{x_{t_{j-1}}, dx_{t_j}}(t_j - t_{j-1}) \mathbf{1}_{x_{t_j} \in [0,a]} \tag{3.7.6}$$

which is simply the partition function $Z^c_{k,a,\beta}$ restricted to the trajectories for which $(J_i)_{i\leqslant |A|}=(x_i)_{i\in A}$. Of course for every $x\in\mathbb{R}^N$, the bound $Z^c_{k,a,\beta}(x)\leqslant Z^c_{k,a,\beta}$ holds, so that applying Lemma 3.2.3 (and in particular the uniform convergence statement), we get that there exists a constant $C_2>0$ such that for every $x\in\mathbb{R}^N$:

$$p^f_{\beta,N}(\mathcal{L}(A)\geqslant L,x)\leqslant C_2\frac{\sum_{k=L}^N Z^c_{k,a,\beta}(N+1-k)^{-1/2}}{\sum_{k=1}^N\int_{[0,a]}Z^c_{k,a,\beta}(dx_k)(N+1-k)^{1/2}}. \tag{3.7.7}$$

Using the estimates of Lemma 3.4.6, we get that the right hand side above is bounded by

$$C_2'\frac{\sum_{k=L}^N k^{-3/2}(N+1-k)^{-1/2}}{\sum_{k=1}^N k^{-3/2}(N+1-k)^{1/2}}. \tag{3.7.8}$$

For every fixed L and large N, we have :

$$\begin{aligned}&\sum_{k=L}^N\frac{1}{k^{3/2}}\frac{1}{(N+1-k)^{1/2}}\\ &\leqslant\frac{2}{\sqrt{N}}\sum_{j=L}^{N/2}\frac{1}{j^{3/2}}+\frac{1}{N}\left(\frac{1}{N}\sum_{j=N/2+1}^{N}\frac{1}{(j/N)^{3/2}}\frac{1}{(1-(j-1)/N)^{1/2}}\right)\\ &:=I_1+I_2.\end{aligned} \tag{3.7.9}$$

Note that $I_1\leqslant(NL)^{-1/2}$ while I_2 is $\frac{1}{N}$ times a Riemann sum approximation of a converging integral. On the other hand, it is easy to see that similar computations yields that the denominator in (3.7.8) is bounded from below by $cN^{-1/2}$. Therefore

$$\limsup_{L\to\infty}\sqrt{L}\limsup_{N\to\infty}\sup_{x\in\mathbb{R}^N}p^f_{\beta,N}(\mathcal{L}(A)\geqslant L,x)<\infty \tag{3.7.10}$$

which proves Lemma 3.7.1, which in turn implies easily Lemma 3.1.5. □

3.7.2 The constrained case

We introduce similar notations to the free case. More precisely, we introduce a law $p^c_{\beta,N}(\cdot,\cdot)$ on $V_{N-1}\times(\mathbb{R}^+)^N$ which is defined as :

$$p^c_{\beta,N}(A,dx):=\frac{1}{Z^c_{N,a,\beta}}e^{\beta(|A|+1)}\prod_{j=1}^{|A|+1}F_{x_{t_{j-1}},dx_{t_j}}(t_j-t_{j-1})\mathbf{1}_{x_{t_j}\in[0,a],\forall j\in\{0,\dots,|A|\}}. \tag{3.7.11}$$

where $t_0:=0$, $t_{|A|+1}:=N$, $x_0:=0$ and $A:=\{t_1<t_2<\ldots<t_{|A|}\}$. We can write :

$$\mathbf{P}^c_{N,a,\beta}(dx)=\sum_{A\subset V_N}\int_{[0,a]^{|A|+1}}p^c_{\beta,N}(A,dy)\mathbb{P}^c_{A,y}(dx) \tag{3.7.12}$$

where for $y \in (\mathbb{R}^+)^N$, $\mathbb{P}_{A,y}(\cdot)$ is the law of $(S_1, \dots, S_N)$ conditioned on the event $\mathcal{E}_{N,A,y}$ which is defined by :

$$\mathcal{E}_{N,A,y} := \Big\{ S_i = y_i ; i \in A \cup \{0\} \cup \{N\} \Big\} \cap \Big\{ S_i > a, i \notin A \cup \{N\} \Big\}. \tag{3.7.13}$$

For $A \subset V_{N-1}$, we recall that $L(A) := \max(A \cap [0, N/2])$ and $R(A) := \min((A \cap [N/2, N]) \cup \{N\})$.

We show the following :

Lemma 3.7.2. *For $\beta < \beta_c^a$, the following estimates hold :*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^N} p^c_{\beta,N}(L(A) \geqslant L, x) = 0 \tag{3.7.14}$$

and

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{x\in\mathbb{R}^N} p^c_{\beta,N}(R(A) \leqslant N - L, x) = 0. \tag{3.7.15}$$

Proof. This proof heavily relies on the estimates on $Z^c_{N,\beta,a}(dy)$ in the $\beta < \beta_c^a$ case from Proposition 3.2.3.

For $x \in \mathbb{R}^N$, we write :

$$\begin{aligned} p^c_{\beta,N}(L(A) \geqslant L, x) &= \sum_{k=L}^{N/2} p^c_{\beta,N}(L(A) = k, x) \\ &\leqslant \frac{1}{Z^c_{N,a,\beta}} \sum_{k=L}^{N/2} Z^c_{k,a,\beta}(dx_k) \sum_{l=N/2+1}^{N-1} F_{x_k,dx_l}(l-k) Z^c_{N-l,a,\beta}(x_l, dx_N) \\ &\leqslant C_1 N^{3/2} \sum_{k=L}^{N/2} k^{-3/2} \sum_{l=N/2+1}^{N-1} (l-k)^{-3/2} ||\Phi_a||_\infty (N-l)^{-3/2} \end{aligned} \tag{3.7.16}$$

where of course $||\Phi_a||_\infty := \sup_{x,y\in[0,a]^2} |\Phi_a(x,y)| < \infty$ because of the continuity of $\Phi_a(\cdot)$. Note that in the last inequality, we used Proposition 3.2.3, and twice Proposition 3.5.2. Thus the right hand side in the expression above is smaller than :

$$C_2 \left[N^{-1/2} \int_{L/N}^{1/2} u^{-3/2} du \right] \left[N^{-1/2} \int_{1/2+1/N}^{1} (s - 1/2)^{-3/2} (1 + 1/N - s)^{-3/2} ds \right] \leqslant C_3 L^{-1/2} \tag{3.7.17}$$

and the first part of Lemma 3.7.2 is proven. For the second part, we simply use the symmetry of S to get :

$$p^c_{\beta,N}(R(A) \leqslant N - L, x) = \sum_{k=N-L}^{N} p^c_{\beta,N}(R(A) = k, x) \tag{3.7.18}$$

$$\leqslant \sup_{x\in\mathbb{R}^N} \sum_{k=L}^{N/2} p^c_{\beta,N}(L(A)=k,x)$$

and we have already treated this quantity above. □

3.8 Scaling limits in the subcritical regime

The goal of this part is to prove Theorem 3.1.3. We fix $\beta < \beta_c^a$. We treat the free case and the constrained case separately, the techniques we use for each case are quite similar but the constrained case is at least notationally a little bit more involved.

The main idea of both proofs is the same. Combining the estimates on the contact set of Lemma 3.1.5 and the very convenient representations of equations (3.7.2) and (3.7.11), one just has to consider the trajectories whose contacts with the stripe are very close to $\{0\}$ (and from the endpoint in the constrained case). After integrating over the first step after the last contact with the stripe and making use of Markov's property, the remaining process is simply the random walk conditioned to stay over the stripe (and to come back in $[0,a]$ close to the endpoint for the constrained case). Finally, in the free case, the convergence towards brownian meander will simply be a consequence of a result from Shimura [Shi83] (see Theorem 3.8.1 below) and of the results we show in the discrete setup in chapter 5 of the present thesis for the constrained case.

We define $\tau_{(-\infty,0)} := \inf\{j \geqslant 0, S_j < 0\}$. Note that of course under $\mathbf{P}$, one has $\tau_{(-\infty,0)} = T_1^-$ but this is not the case under the law $\mathbf{P}_t$ for $t > 0$.

3.8.1 The free case

The main tool in the first part of the proof of Theorem 3.1.3 will be the following result which has been proved in [Shi83, Example 4.1] :

Theorem 3.8.1. *Let x_N be a positive sequence such that $x_N N^{-1/2} \to 0$ as $N \to \infty$. One has the following weak convergence :*

$$\mathbf{P}_{x_N}\Big[\ \cdot\ \Big|\tau_{(-\infty,0)} > N\Big] \circ \Big(X^N\Big)^{-1} \Longrightarrow m(\cdot). \tag{3.8.1}$$

For clarity, we summarize the steps of the first part of the proof of Theorem 3.1.3 in the next key lemma ; then we show that we may apply Lemma 3.8.2 to our setup, and finally we go to its proof.

Lemma 3.8.2. *Let l_N be the random variable $\mathcal{L}(A)$ under $\mathbf{P}^f_{N,a,\beta}$ and let L be a positive integer. Assume the following assumptions hold :*

1. For any $\varepsilon > 0$, one has

$$\lim_{N\to\infty} Q^f_{N,a,\beta}\left[\sup_{t\in[0,\frac{l_N}{N}]} \omega_t \geqslant \varepsilon\right] = 0. \tag{3.8.2}$$

2. For every $A \subset V_N$ such that $\mathcal{L}(A) = L$ and for every $x \in \mathbb{R}^N$, if X follows the law $\mathbb{P}_{A,x}$, one has the convergence in law :

$$\left(\sqrt{1-\frac{L}{N}}X^N_{\frac{L}{N}+t(1-\frac{L}{N})}\right)_{t\in[0,1]} \Rightarrow m \tag{3.8.3}$$

where m denotes the law of the brownian meander.

Then one has the weak convergence

$$Q^f_{N,a,\beta} \Rightarrow m \tag{3.8.4}$$

Proof of Theorem 3.1.3 for the free case

The first point of Lemma 3.8.2 is fulfilled. We write :

$$Q^f_{N,a,\beta}\left[\sup_{t\in[0,\frac{l_N}{N}]} \omega_t \geqslant \varepsilon\right] = \mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geqslant \varepsilon\sigma\sqrt{N}; l_N > L\right] \tag{3.8.5}$$

$$+\mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geqslant \varepsilon\sigma\sqrt{N}; l_N \leqslant L\right]$$

so that choosing L_0 large enough and making use of Lemma 3.1.5, for any fixed $\eta > 0$, we can get the following bound which holds for every N and for every $L \geq L_0$:

$$\left| Q^f_{N,a,\beta}\left[\sup_{t\in[0,\frac{l_N}{N}]} \omega_t \geqslant \varepsilon\right] - \mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geqslant \varepsilon\sigma\sqrt{N}; l_N \leqslant L\right]\right| \leqslant \eta/2. \tag{3.8.6}$$

Then we note that :

$$\mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geqslant \varepsilon\sigma\sqrt{N}; l_N \leqslant L\right] = \frac{\mathbf{E}\left[\mathbf{1}_{\max_{j=1,\dots,l_N} S_j \geqslant \varepsilon\sigma\sqrt{N}} e^{\beta\sum_{i=1}^N \mathbf{1}_{S_i\in[0,a]}}\mathbf{1}_{l_N \leqslant L}\mathbf{1}_{T_1^- > N}\right]}{Z^f_{N,a,\beta}} \tag{3.8.7}$$

so that using the estimates on $Z^f_{N,a,\beta}$ from Proposition 3.5.2, we get that there exists a constant $\mathcal{C} > 0$ (which may vary from line to line) such that :

$$\begin{aligned}\mathbf{P}^f_{N,a,\beta}\left[\max_{j=1,\dots,l_N} S_j \geqslant \varepsilon\sigma\sqrt{N}; l_N \leqslant L\right] &\leqslant \mathcal{C}N^{1/2}e^{\beta L}\mathbf{E}\left[\mathbf{1}_{\max_{j=1,\dots,L} S_j \geqslant \varepsilon\sigma\sqrt{N}}\mathbf{1}_{l_N \leqslant L}\mathbf{1}_{T_1^- > N}\right] \\ &\leqslant \mathcal{C}e^{\beta L}\mathbf{P}\left[\max_{j=1,\dots,L} S_j \geqslant \varepsilon\sigma\sqrt{N} | T_1^- > N\right]\end{aligned} \tag{3.8.8}$$

where in the last equation we made use of the well known fact that the sequence $(N^{1/2}\mathbf{P}\left[S_1 \geqslant 0,\dots,S_N \geqslant 0\right])_{N \geqslant 1}$ converges towards a positive limit as $N \to \infty$ (see [Fel71]).

Then we consider some $\gamma > 0$. As soon as N is large enough, we have of course the bound :

$$\mathbf{P}\left[\max_{j=1,\dots,L} S_j \geqslant \varepsilon\sigma\sqrt{N}\Big| T_1^- > N\right] \leqslant \mathbf{P}\left[\max_{j=1,\dots,\gamma N} S_j \geqslant \varepsilon\sigma\sqrt{N}\Big| T_1^- > N\right]. \tag{3.8.9}$$

We can rewrite the right hand side above as :

$$\mathbf{P}\left[\sup_{t\in[0,\gamma]} S_t^N \geqslant \varepsilon \Big| T_1^- > N\right]. \tag{3.8.10}$$

where of course S^N is the image of $(S_j)_{j \leqslant N}$ under the map X^N. Making use of Theorem 3.8.1, for every bounded continuous function $\Phi(\cdot)$ on $C([0,1])$, one has the convergence

$$\mathbf{E}\Big[\Phi\left((S_t^N)_{t\in[0,1]}\right)\Big| T_1^- > N\Big] \to m(\Phi) \tag{3.8.11}$$

see [Bol76].

Finally, we note that the set of discontinuities of the functional

$$\begin{array}{ccc}\mathcal{C}([0,1],\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \mathbf{1}_{\sup_{[0,\gamma]} f \geqslant \varepsilon}\end{array} \tag{3.8.12}$$

is of null m-measure, so that by the continuous mapping theorem (see [Bil68]) the quantity (3.8.10) converges towards $m(\sup_{t\in[0,\gamma]} \omega_t > \varepsilon)$ which can be made arbitrarily small when γ is chosen accordingly. □

The second point of Lemma 3.8.2 is fulfilled. We first prove the second statement in the case where $A = \emptyset$. Let $\varepsilon > 0$. We consider a Lipschitz bounded functional Φ on $\mathcal{C}([0,1],\mathbb{R})$, that is such that there exist two positive constants c_1 and c_2 verifying that for every $f,g \in \mathcal{C}([0,1],\mathbb{R})$, one has :

$$|\Phi(f)| < c_1 \quad \text{and} \quad |\Phi(f) - \Phi(g)| \leqslant c_2||f-g||_\infty. \tag{3.8.13}$$

Here, $\mathcal{E}_{N,A,x}$ is simply the event $\{S_1 > a, \ldots, S_N > a\}$; conditioning on S_1, for every $M > a$, one gets :

$$\begin{aligned}
&\mathbb{E}_A^f\Big[\Phi\left((X_t^N)_{t\in[0,1]}\right)\Big] \\
&\quad = \int_a^M \mathbf{P}\Big[S_1 \in dt \Big| S_1 > a, \ldots, S_N > a\Big] \\
&\qquad\qquad \times \mathbf{E}\Big[\Phi\left(X^N(t, S_2, \ldots, S_N)\right)\Big| S_1 = t, S_2 > a, \ldots, S_N > a\Big] \\
&\quad + \int_M^{+\infty} \mathbf{P}\Big[S_1 \in dt \Big| S_1 > a, \ldots, S_N > a\Big] \\
&\qquad\qquad \times \mathbf{E}\Big[\Phi\left(X^N(t, S_2, \ldots, S_N)\right)\Big| S_1 = t, S_2 > a, \ldots, S_N > a\Big].
\end{aligned} \tag{3.8.14}$$

Then we note that by the Markov property, for any $t \geqslant a$:

$$\begin{aligned}
&\mathbf{E}\Big[\Phi\left(X^N(t, S_2, \ldots, S_N)\right)\Big| S_1 = t, S_2 > a, \ldots, S_N > a\Big] \\
&\quad = \mathbf{E}_t\Big[\Phi\left(X^N(t, S_1 + a, \ldots, S_{N-1} + a)\right)\Big| \tau_{(-\infty,0)} > N-1\Big].
\end{aligned} \tag{3.8.15}$$

Considering the first part in the right hand side of (3.8.14), it is then easy to see that for any $(x_1, \ldots, x_{N-1}) \in (\mathbb{R}^+)^{N-1}$ and $t \in [a, M]$, one has

$$\begin{aligned}
&\Big|\Phi\left(X^N(t, x_1 + a, \ldots, x_{N-1} + a)\right) - \Phi\left(X^{N-1}(x_1, \ldots, x_{N-1})\right)\Big| \\
&\quad \leqslant \frac{c_2 \sup_{j=1,\ldots,N-1} |x_j - x_{j-1}|}{\sqrt{N}} + c_2 \frac{a+M}{\sqrt{N}}.
\end{aligned} \tag{3.8.16}$$

Theorem 3.8.1 implies that for fixed $t > a, \mathbf{E}_t\Big[\ \cdot\ \Big| \tau_{(-\infty,0)} > N-1\Big] \circ \left(X^{N-1}\right)^{-1}$ converges towards m. In particular, using the tightness criterion of Kolmogorov, this implies the fact that $\frac{\sup_{j=1,\ldots,N-1}|S_j - S_{j-1}|}{\sqrt{N}} + \frac{a+M}{\sqrt{N}} =: \mathcal{Y}_N^a$ converges towards zero in probability when $(S_j)_{j\leqslant N}$ is distributed according to $\mathbf{E}_t\Big[\ \cdot\ \Big| \tau_{(-\infty,0)} > N-1\Big]$.

Thus, one has :

$$\begin{aligned}
&\Big|\mathbf{E}_t\Big[\Phi\left(X^N(t, S_1 + a, \ldots, S_{N-1} + a)\right)\Big| \tau_{(-\infty,0)} > N-1\Big] \\
&\qquad\qquad - \mathbf{E}_t\Big[\Phi\left(X^{N-1}(S_1, \ldots, S_{N-1})\right)\Big| \tau_{(-\infty,0)} > N-1\Big]\Big|
\end{aligned} \tag{3.8.17}$$

$$\leqslant \mathbf{E}_t\Big[\Big|\Phi\left(X^N(t, S_1+a, \dots, S_{N-1}+a)\right) - \Phi\left(X^{N-1}(S_1, \dots, S_{N-1})\right)\Big|\mathbf{1}_{\mathcal{Y}_N^a > \varepsilon}\Big|\tau_{(-\infty,0)} > N-1\Big]$$
$$+ \mathbf{E}_t\Big[\Big|\Phi\left(X^N(t, S_1+a, \dots, S_{N-1}+a)\right) - \Phi\left(X^{N-1}(S_1, \dots, S_{N-1})\right)\Big|\mathbf{1}_{\mathcal{Y}_N^a \leqslant \varepsilon}\Big|\tau_{(-\infty,0)} > N-1\Big]$$
$$\leqslant 2c_1 \mathbf{P}_t\Big[\mathcal{Y}_N^a > \varepsilon\Big|\tau_{(-\infty,0)} > N-1\Big] + c_2\varepsilon$$

where in the last inequality we made use of (3.8.16). We choose N_0 large enough such that the last term above is smaller than say $(1+c_2)\varepsilon$ (which is possible using the convergence in probability we pointed right above).

Now, we have to show that one can choose M large *uniformly in* N so that the second part in the right hand side of (3.8.14) can be made uniformly negligible with respect to the first one as $N \to \infty$. We prove in the appendix that for every $M > 0$, the following convergence holds :

$$\int_M^{+\infty} \mathbf{P}\Big[S_1 \in dt\Big|S_1 > a, \dots, S_N > a\Big] \to \int_M^{+\infty} \frac{\mathbf{P}[S_1 \in dt]V(t-a)}{\int_a^{+\infty} \mathbf{P}[S_1 \in du]V(u-a)}. \quad (3.8.18)$$

Recall the equivalence $V(x) \sim x/\mathbf{E}[H_1^-]$ as $x \to \infty$, so that both integrands in the right hand side of equation (3.8.18) are finite. This implies that for M large enough (and in particular independent from N), the left hand side in the inequality above can be made smaller than ε. Making use of the triangle inequality and considering M large enough, we then have for every N :

$$\begin{aligned}\Big|\mathbb{E}_A^f\Big[\Phi\left((X_t^N)_{t\in[0,1]}\right)\Big] - m(\Phi)\Big| &\leqslant \int_a^M \mathbf{P}[S_1 \in dt | S_1 > a, \dots, S_N > a] \\ &\times \Big|\mathbf{E}\Big[\Phi\left(X^N(t, S_2, \dots, S_N)\right)\Big|S_2 > a, \dots, S_N > a\Big] \\ &\quad - \mathbf{E}_t\Big[\Phi\left(X^{N-1}(S_1, \dots, S_{N-1})\right)\Big|\tau_{(-\infty,0)} > N-1\Big]\Big| \\ &+ \int_a^M \mathbf{P}[S_1 \in dt | S_1 > a, \dots, S_N > a] \\ &\quad \times \Big|\mathbf{E}_t\Big[\Phi\left(X^{N-1}(S_1, \dots, S_{N-1})\right)\Big|\tau_{(-\infty,0)} > N-1\Big] - m(\Phi)\Big| + 2\varepsilon\end{aligned} \quad (3.8.19)$$

The first term in the right hand side of (3.8.19) can be made small using dominated convergence and the considerations following equality (3.8.14), the second one by using Theorem 3.8.1 and dominated convergence, and thus the $A = \emptyset$ case is resolved.

For a generic $A \subset V_N$ and $x \in (\mathbb{R}^+)^N$, we just make use of the Markov property of S and of what we just proved. More precisely, noting that $A \cap [0, L] = A$ and making use of the Markov property, we get :

$$\mathbb{E}_{A,x}\left[\Phi\left(\left(\sqrt{1-\frac{L}{N}}S^N_{\frac{L}{N}+t(1-\frac{L}{N})}\right)_{t\in[0,1]}\right)\right] \\ = \mathbb{E}_{A,x}\Big[\mathbf{E}_{S_L}\Big[\Phi\left((S^{N-L}_t)_{t\in[0,1]}\right)\Big|S_1>a,\dots,S_{N-L}>a\Big]\Big] \quad (3.8.20)$$

We integrate the numerator above on S_L ; it is equal to :

$$\int_0^a \mathbb{P}_{A,x}[S_L\in du]\mathbf{E}_u\Big[\Phi\left((S^{N-L}_t)_{t\in[0,1]}\right)\Big|S_1>a,\dots,S_{N-L}>a\Big] \quad (3.8.21)$$

and we note that for all $x\in[0,a]$, one has the equality :

$$\mathbf{E}_x\Big[\Phi\left(S^{N-L}_t\right)_{t\in[0,1]}\Big)\Big|S_1>a,\dots,S_{N-L}>a\Big] = \mathbb{E}_{\{0\},0}\Big[\Phi\left(S^{N-L}_t\right)_{t\in[0,1]}\Big)\Big] \quad (3.8.22)$$

and the right hand side in the above equality converges towards $m(\Phi)$ as we have already shown.

Thus applying dominated convergence, we get the convergence :

$$\mathbb{E}^f_A\left[\Phi\left(\left(\sqrt{1-\frac{L}{N}}S^N_{\frac{L}{N}+t(1-\frac{L}{N})}\right)_{t\in[0,1]}\right)\right]\to m(\Phi). \quad (3.8.23)$$

as $N\to\infty$, which is the second point of Lemma 3.8.2. □

Proof of Lemma 3.8.2

We consider $\varepsilon,\eta>0$, L_0 large and Φ a continuous function on $C([0,1],\mathbb{R})$. We write :

$$Q^f_{N,a,\beta}\Big[\Phi(\omega)\Big] \\ = \sum_{l=0}^{L_0}\sum_{A\subset V_N;\mathcal{L}(A)=l}\int_{[0,a]^{|A|}} p^f_{\beta,N}(A,dx)\,\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big] + Q^f_{N,a,\beta}\Big[\Phi(\omega)\mathbf{1}_{\mathcal{L}(A)>L_0}\Big]. \quad (3.8.24)$$

Then we note that for each A and x appearing in the right hand side above, the convergence $\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big]\to m(\Phi)$ holds. We note L for the quantity $\mathcal{L}(A)$ and for notational convenience we write $f_N(t):=L/N+t(1-L/N)$ and $g_N(t):=\frac{(t-L/N)}{1-L/N}$ its inverse function (and we set $f_0(t)=g_0(t)=t$).

We first note that for every $n > 0$, for every $t_1 < t_2 < \ldots < t_n \in [0,1]^n$ and for every continuous bounded function $F : [0,1]^n \to \mathbb{R}$, one has the convergence

$$\mathbb{P}_{A,x}\left[F(X^N_{t_1}, X^N_{t_2}, \ldots, X^N_{t_n})\right] \to m\left[F(\omega_{t_1}, \ldots, \omega_{t_n})\right]. \tag{3.8.25}$$

Of course, using the second assumption of Lemma 3.8.2 this is quite obvious. We just note that :

$$\Bigg| \mathbb{P}_{A,x}\Big[F\left(X^N_{t_1}, X^N_{t_2}, \ldots, X^N_{t_n}\right)\Big] - \mathbb{P}_{A,x}\left[F\left(\sqrt{1-\frac{L}{N}}X^N_{f_N(t_1)}, \ldots, \sqrt{1-\frac{L}{N}}X^N_{f_N(t_n)}\right)\right]\Bigg| \to 0 \tag{3.8.26}$$

as $N \to \infty$ by dominated convergence because $F(\cdot)$ is continuous and bounded ; as the convergence of the second term above towards $m(F(\omega_{t_1}, \ldots, \omega_{t_n}))$ is known by hypothesis, the finite dimensional convergence is proven.

We are left with proving the tightness of the sequence X^N under the law $\mathbb{P}_{A,x}$ for $A \subset V_N$ such that $\mathcal{L}(A) \leqslant L_0$. For this, for $\eta > 0$ and for a continuous function f on $[0,1] \to \mathbb{R}^+$ verifying $\sup_{t\in[0,\eta]} f(t) \leqslant \varepsilon$, we introduce its η-cut counterpart $f^{(\eta)}$; namely, $f^{(\eta)}(x) = \frac{xf(\eta)}{\eta}\mathbf{1}_{x\in[0,\eta]} + f(x)\mathbf{1}_{x\,\geqslant\,\eta}$. Clearly, we have $||f^{(\eta)} - f||_\infty \leqslant \varepsilon$.

Note that by the very definition of the brownian meander and using standard properties of the brownian motion, for C large enough, one has $m(\mathcal{B}_C) \geqslant 1-\varepsilon$ where

$$\mathcal{B}_C := \Big\{ f \in C([0,1], \mathbb{R}),\ \sup_{x,y\in[0,1]} |f(x) - f(y)| \leqslant C|x-y|^{2/3} \Big\}. \tag{3.8.27}$$

Therefore for such a C and for N large enough, we have :

$$\mathbb{P}_{A,x}\left[\left(\sqrt{1-\frac{L}{N}}X^N_{f_N(t)}\right)_{t\in[0,1]} \in \mathcal{B}_C\right] \geqslant 1 - 2\eta. \tag{3.8.28}$$

Now we are ready to prove the Kolmogorov criterion for X^N under the law $\mathbb{P}_{A,x}$. We have to show that given $\delta > 0$, there exists N_0 such that :

$$\mathbb{P}_{A,x}\Big[\sup_{s,t,|s-t|\,\leqslant\,\delta} |X^N_s - X^N_t| \geqslant \varepsilon\Big] \leqslant \eta,\ N \geqslant N_0. \tag{3.8.29}$$

Using the first hypothesis of Lemma 3.8.2 , we can restrict ourselves to show (3.8.29) by replacing X^N by $(X^N)^{(L/N)}$ and we write $\widetilde{X}^N$ for this new process. As the modulus of continuity of $\widetilde{X}^N$ is obviously under control on $[0, L/N]$, we just have to show that there exists $\delta > 0$ such that for N large enough, one has :

$$\mathbb{P}_{A,x}\Big[\sup_{s,t>L/N,|s-t|\,\leqslant\,\delta} |\widetilde{X}^N_s - \widetilde{X}^N_t| \geqslant \varepsilon\Big] \leqslant \eta. \tag{3.8.30}$$

Now we write :

$$\left|\widetilde{X}_s^N - \widetilde{X}_t^N\right| \leqslant \left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_{f_N(t)}^N\right| + \left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_s^N\right| + \left|\widetilde{X}_{f_N(t)}^N - \widetilde{X}_t^N\right| \tag{3.8.31}$$

so that, for every $\delta > 0$,

$$\begin{aligned}\mathbb{P}_{A,x}\Big[&\sup_{s,t>L/N,|s-t| \leqslant \delta}\left|\widetilde{X}_s^N - \widetilde{X}_t^N\right| \geqslant \varepsilon\Big]\\ &\leqslant \mathbb{P}_{A,x}\Big[\sup_{|s-t| \leqslant \delta}\left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_{f_N(t)}^N\right| \geqslant \varepsilon/3\Big] + 2\mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]}\left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_s^N\right| \geqslant \varepsilon/3\Big].\end{aligned} \tag{3.8.32}$$

The first term in the right hand side of the above inequality can be made smaller than $\eta/2$ for δ small enough as soon as N is large enough using the second hypothesis of Lemma 3.8.2. For the left hand side, we denote by $g_N(t) = \frac{(t-L/N)}{1-L/N}$, so that

$$\begin{aligned}\mathbb{P}_{A,x}\Big[&\sup_{s\in[0,1]}\left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_s^N\right| \geqslant \varepsilon/3\Big]\\ &= \mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]}\left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_{g_N(f_N(s))}^N\right| \geqslant \varepsilon/3; \left(\widetilde{X}_{f_N(t)}^N\right)_{t\in[0,1]} \in \mathcal{B}_C\Big]\\ &\qquad + \mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]}\left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_{g_N(f_N(s))}^N\right| \geqslant \varepsilon/3; \left(\widetilde{X}_{f_N(t)}^N\right)_{t\in[0,1]} \in \mathcal{B}_C^c\Big].\end{aligned} \tag{3.8.33}$$

The last term of equation (3.8.33) above can be made smaller than $\eta/3$ for N large enough since $\mathcal{B}_C$ is a m continuity set (that is a set whose boundary is of null m measure) and by using the Porte-Manteau theorem, which states that in this case

$$\mathbb{P}_{A,x}\Big[(\widetilde{X}_{f_N(t)}^N)_{t\in[0,1]} \in \mathcal{B}_C\Big] \to m(\mathcal{B}_C) \tag{3.8.34}$$

as $N \to \infty$.

Finally, for $(\widetilde{X}_{f_N(t)}^N)_{t\in[0,1]} \in \mathcal{B}_C$, we have

$$\sup_{s\in[0,1]}\left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_{g_N(f_N(s))}^N\right| \leqslant C \sup_{s\in[0,1]}\left|s - g_N(s)\right|^{2/3} \tag{3.8.35}$$

and $\sup_{s\in[0,1]}\left|s - g_N(s)\right|^{2/3} \leqslant (L/N)^{2/3}$. Thus for N large enough, one has :

$$\mathbb{P}_{A,x}\Big[\sup_{s\in[0,1]}\left|\widetilde{X}_{f_N(s)}^N - \widetilde{X}_{g_N(f_N(s))}^N\right| \geqslant \varepsilon/3; \left(\widetilde{X}_{f_N(t)}^N\right)_{t\in[0,1]} \in \mathcal{B}_C\Big] \leqslant \eta/3 \tag{3.8.36}$$

which proves (3.8.30). Thus we have shown that $\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big] \to m(\Phi)$.

Now we make use of equation (3.8.24) and the triangle equality to get that

$$\begin{aligned}
&\left|Q^f_{N,a,\beta}\Big[\Phi(\omega)\Big] - m(\Phi)\right| \\
&\leqslant \sum_{l=0}^{L_0} \sum_{A\subset V_N;\mathcal{L}(A)=l} \int_{[0,a]^{|A|}} p^f_{\beta,N}(A,x) \left|\mathbb{P}_{A,x}\Big[\Phi\left(X^N\right)\Big) - m(\Phi)\right| \\
&\qquad + m(|\Phi|)Q^f_{N,a,\beta}\Big[\mathbf{1}_{\mathcal{L}(A)>L_0}\Big] + Q^f_{N,a,\beta}\Big[\Big|\Phi(\omega)\Big|\mathbf{1}_{\mathcal{L}(A)>L_0}\Big].
\end{aligned} \tag{3.8.37}$$

As $\Phi(\cdot)$ is bounded, using dominated convergence (recall that $\sum_{l=0}^\infty \sum_{A\subset V_N;\mathcal{L}(A)=l} \int_{[0,a]^{|A|}} p^f_{\beta,N}(A,x) = 1$) and the fact that $\mathbb{P}_{A,x}\Big[\Phi(X^N)\Big] \to m(\Phi)$), we are done by simply considering L_0 large enough and by using Lemma 3.1.5. □

3.8.2 The constrained case

The strategy in this part is essentially the same as in the free case. The analogous for Shimura's theorem is proved in chapter 5 of the present thesis.

Theorem 3.8.3. *Let x_N and y_N two positive sequences such that both $x_N/\sqrt{N}$ and $y_N/\sqrt{N}$ vanish as $N\to\infty$. One has the following weak convergence :*

$$\mathbf{P}_{x_N}\Big[\ \cdot\ \Big| S_N = y_N, \tau_{(-\infty,0)} > N\Big] \circ \left(X^N\right)^{-1} \Rightarrow e(\cdot). \tag{3.8.38}$$

Remark 3.8.4. *The result we show in chapter 5 is actually slightly more general than that, since we show it for S in the domain of attraction of the normal law. On the other hand, we only prove it in the lattice setup (that is in the case where S is integer valued), although the proof should be very similar in our continuous setup.*

Like we did in the free case, we first give a technical lemma which immediately implies the convergence in the constrained case of Theorem 3.1.3 ; its proof is very similar to the free case, so that we choose to skip it.

Lemma 3.8.5. *Let (l_N, r_N) denote the random variables $(L(A), R(A))$ under $\mathbf{P}^c_{N,a,\beta}$. Assume the following holds :*

1. *For any $\varepsilon > 0$, one has*

$$\lim_{N\to\infty} Q^c_{N,a,\beta}\left[\sup_{t\in[0,l_N/N]\cup[r_N/N,1]} w_t \geqslant \varepsilon\right] = 0. \tag{3.8.39}$$

2. *For every $A\subset V_{N-1}$ such that $(L(A),R(A)) = (L,R)$ and for every $x\in\mathbb{R}^N$, if X follows the law $\mathbb{P}^c_{A,x}$, one has the convergence in law :*

$$\left(\sqrt{\frac{R-L}{N}} X^N_{\frac{L}{N}+t(\frac{R-L}{N})}\right)_{t\in[0,1]} \Rightarrow e. \tag{3.8.40}$$

Then one has the second convergence of Theorem 3.1.3.

Proof of Theorem 3.1.3 in the constrained case.

Of course we simply show that the hypothesis of Lemma 3.8.5 are fulfilled.

The first point of Lemma 3.8.5 is fulfilled. Making use of the convergence (3.1.18) of Lemma 3.1.5, for any $\eta > 0$ and for R_0, L_0 large enough, one has :

$$\Bigg| Q^c_{N,a,\beta}\left[\sup_{t\in[0,l_N/N]\cup[r_N/N,1]} w_t \geqslant \varepsilon\right] - \mathbf{P}^c_{N,a,\beta}\left[\max_{j=1,\dots,l_N,r_N,\dots,N} S_j \geqslant \varepsilon\sigma\sqrt{N}; l_N \leqslant L_0, r_N \leqslant R_0\right]\Bigg| \leqslant \eta. \quad (3.8.41)$$

Similarly to the free case, one obtains easily :

$$\begin{aligned}
&\mathbf{P}^c_{N,a,\beta}\left[\max_{j=1,\dots,l_N,r_N,\dots,N} S_j \geqslant \varepsilon\sigma\sqrt{N}; l_N \leqslant L_0, r_N \leqslant R_0\right] \\
&\leqslant e^{\beta(L_0+R_0)} \frac{\mathbf{E}\left[\mathbf{1}_{\max_{j=1,\dots,L_0,R_0,\dots,N} S_j \geqslant \varepsilon\sigma\sqrt{N}} \mathbf{1}_{S_{R_0},S_{L_0}\in[0,a]} \mathbf{1}_{S_N\in[0,a]} \mathbf{1}_{S_j>0, j\leqslant N}\right]}{Z^c_{N,\beta,a}} \\
&\leqslant e^{\beta(L_0+R_0)} \mathbf{P}\left[\max_{j=1,\dots,L_0,R_0,\dots,N} S_j \geqslant \varepsilon\sigma\sqrt{N} \,\Big|\, S_N\in[0,a]; T_1^- > N\right] \\
&\qquad \times \frac{\mathbf{P}\left[S_N\in[0,a]; S_j>0, j\leqslant N\right]}{Z^c_{N,\beta,a}}.
\end{aligned} \quad (3.8.42)$$

Then we have the equivalence :

$$\mathbf{P}[S_N\in[0,a]; S_j>0, j\leqslant N] \sim \frac{\int_0^a U(u)du}{\sqrt{2\pi}\sigma N^{3/2}} \quad (3.8.43)$$

which follows from the more general equivalence

$$\mathbf{P}_y[\tau_{(-\infty,0)} > N-1, S_N\in dx] \sim \frac{U(x)V(y)}{\sqrt{2\pi}\sigma N^{3/2}} \quad (3.8.44)$$

which holds for $y, x \geqslant 0$. We prove this relation in chapter 5, and moreover we note that it holds uniformly for x, y such that $x/a_n \to 0$ and $y/a_n \in [0, K]$. We are left with proving that

$$\lim_{N\to\infty} \mathbf{P}\left[\max_{j=1,\dots,L_0,R_0,\dots,N} S_j \geqslant \varepsilon\sigma\sqrt{N} \,\Big|\, S_N\in[0,a]; T_1^- > N\right] = 0. \quad (3.8.45)$$

This is an easy consequence of Theorem 3.8.3. Actually, like in the free case, this quantity can be rewritten as

$$\mathbf{P}\left[\sup_{s\in[0,L_0/N]\cup[R_0/N,1]} S_t^N \geqslant \varepsilon \middle| S_N \in [0,a]; T_1^- > N\right] \tag{3.8.46}$$

and using the same trick as we did in the free case (and noting that by definition the process $(e_t)_{t\in[0,1]}$ is continuous and $e_1 = 0$), we get that as $N \to \infty$, for every $\lambda, \gamma \in (0,1)$, the term above can be made smaller than a sequence which converges towards $e(\sup_{t\in[0,\lambda]\cup[\gamma,1]} w_t \geqslant \varepsilon)$. Using the fact that here too the set of discontinuities of the map

$$\begin{array}{ccc} \mathcal{C}([0,1],\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & \mathbf{1}_{\sup_{[0,\lambda]\cup[\gamma,1]} f \geqslant \varepsilon} \end{array} \tag{3.8.47}$$

is of null e-measure, choosing λ small enough and γ close enough to 1 we get our first point. □

The second point of Lemma 3.8.5 is fulfilled. Here we make use of Theorem 3.8.3 in a crucial way. We first treat the $A = \emptyset$ case. Once again we consider $\varepsilon > 0$ and Φ a Lipschitz bounded functional on $\mathcal{C}([0,1],\mathbb{R})$ verifying the same properties as in (3.8.13). For a given $M > 0$, we write :

$$\begin{aligned} &\mathbb{E}^c_{A,x}\Big[\Phi\left(X_t^N\right)_{t\in[0,1]}\Big] \\ &\quad= \int_{t,t'\in[a,M]^2}\int_{u\in[0,a]} \mathbf{P}\Big[S_1 = t, S_{N-1} = t', S_N = u\Big| S_1 > a, \dots, S_{N-1} > a, S_N \in [0,a]\Big] \\ &\qquad \times \mathbf{E}\Big[\Phi\left(X^N(t, S_2, \dots, t', u)\right)\Big| S_1 = t, S_2 > a, \dots, S_{N-2} > a, S_{N-1} = t', S_N = u\Big] \\ &\quad+ \int_{t,t'\geqslant M}\int_{u\in[0,a]} \mathbf{P}\Big[S_1 = t, S_{N-1} = t', S_N = u\Big| S_1 > a, \dots, S_{N-1} > a, S_N \in [0,a]\Big] \\ &\qquad \times \mathbf{E}\Big[\Phi\left(X^N(t, S_2, \dots, t', u)\right)\Big| S_1 = t, S_2 > a, \dots, S_{N-2} > a, S_{N-1} = t', S_N = u\Big]. \end{aligned} \tag{3.8.48}$$

Using the Markov property and the homogeneity of S, we note that for every $(t,t',u) \in (a,+\infty)^2 \times [0,a]$, one has :

$$\begin{aligned} &\mathbf{P}[S_1 = t, S_{N-1} = t', S_N = u | S_1 > a, \dots, S_{N-1} > a, S_N \in [0,a]] \\ &\quad= \frac{\mathbf{P}[S_1 = t]\mathbf{P}_{t-a}[\tau_{(-\infty,0)} > N-3, S_{N-2} = t'-a]\mathbf{P}_{t'}[S_1 \in du]}{\mathbf{P}[S_1 > a, \dots, S_{N-1} > a, S_N \in [0,a]]} \\ &\qquad= \frac{h(t)h(t'-u)\mathbf{P}_{t-a}[\tau_{(-\infty,0)} > N-3, S_{N-2} = t'-a]}{\int_{[a,+\infty)^2} h(u)\mathbf{P}_{u-a}[\tau_{(-\infty,0)} > N-3, S_{N-2} = v]\int_{\tau\in[0,a]} h(\tau - v)d\tau}. \end{aligned} \tag{3.8.49}$$

For every $t, t' > a$, we already mentioned in (3.8.44) that the following convergence holds :

$$\mathbf{P}_{t-a}[\tau_{(-\infty,0)} > N-3, S_{N-2} = t'-a] \sim \frac{V(t-a)U(t'-a)}{\sqrt{2\pi}\sigma N^{3/2}} \tag{3.8.50}$$

and moreover for fixed positive K, this convergence holds uniformly for $t, t' \in [a, K\sqrt{N}]^2$. From this, we can deduce that :

$$\begin{aligned}\mathbf{P}\Big[S_1 = t, S_{N-1} = t', S_N = u \Big| S_1 > a, \dots, S_{N-1} > a, S_N \in [0,a]\Big] \\ \to \frac{h(t)h(u-t')U(t-a)V(t'-a)}{\int_{[a,+\infty)^2}\int_0^a h(v)h(u-v')V(v-a)U(v'-a)dvdv'du}.\end{aligned} \tag{3.8.51}$$

This convergence is somewhat lengthy and uses similar arguments as the proof of the convergence in (3.8.18) ; we show it in the appendix 3.8.3. Note that if we are able to show (3.8.51), the $A = \emptyset$ case will be treated, the arguments being just the same as in the free case ; finally, in a similar way as in the free case, the general case $A \neq \emptyset$ will follow (one essentially just has to apply Markov's property twice), and thus we will be done. □

3.8.3 Appendix

We show here the convergences appearing in (3.8.18) and (3.8.51), that is

Lemma 3.8.6. *For $M \geqslant a$, as $N \to \infty$, the following convergences hold :*

$$\begin{aligned}&\int_M^{+\infty} \mathbf{P}[S_1 \in dt | S_1 > a, \dots, S_N > a] \\ &\to \int_M^{+\infty} \frac{\mathbf{P}[S_1 \in dt]V(t-a)}{\int_a^{+\infty}\mathbf{P}[S_1 \in du]V(u-a)}\end{aligned} \tag{3.8.52}$$

and

$$\begin{aligned}&\int_{[M,\infty)^2\times[0,a]} \mathbf{P}[S_1 = t, S_{N-1} = t', S_N = u | S_1 > a, \dots, S_{N-1} > a, S_N \in [0,a]] \\ &\to \frac{\int_{[M,\infty)^2\times[0,a]} h(t)h(u-t')U(t-a)V(t'-a)}{\int_{[a,+\infty)^2}\int_0^a h(v)h(u-v')V(v-a)U(v'-a)dvdv'du}.\end{aligned} \tag{3.8.53}$$

Proof of (3.8.52)

We first prove the convergence appearing in (3.8.52). For $t \geqslant 0$, we define

$$\phi_N(t) := \frac{\mathbf{P}_t[S_1 > 0, \dots, S_N > 0]}{\mathbf{P}[S_1 > 0, \dots, S_N > 0]}. \tag{3.8.54}$$

Note that $\phi_N(\cdot)$ is uniformly bounded by $cN^{1/2}$. We recall that Lemma 5.6.1 in chapter 5 states that the convergence

$$\frac{\mathbf{P}_{x_N}[\tau_{(-\infty,0)} > N]}{\mathbf{P}[T_1^- > N]} \sim V(x_N) \tag{3.8.55}$$

holds uniformly for sequences x_N such that $x_N/N^{1/2} \to 0$ as $N \to \infty$, so that we just have to show that, as $N \to \infty$,

$$\int_a^{+\infty} h(u)\phi_N(u-a)du \to \int_a^{+\infty} h(u)V(u-a)du \tag{3.8.56}$$

where we recall that $h(\cdot)$ is the density of S_1. For any $t > a$, using the Markov property, we have :

$$\begin{aligned}\mathbf{P}\left[S_1 \in dt | S_1 > a, \dots, S_N > a\right] \\ &= \frac{\mathbf{P}[S_1 \in dt]\mathbf{P}_t\left[S_1 > a, \dots, S_{N-1} > a\right]}{\int_a^{+\infty} \mathbf{P}[S_1 \in du]\mathbf{P}_u\left[S_1 > a, \dots, S_{N-1} > a\right]} \\ &= \frac{\mathbf{P}[S_1 \in dt]\mathbf{P}_{t-a}\left[\tau_{(-\infty,0)} > N-1\right]}{\int_a^{+\infty} \mathbf{P}[S_1 \in du]\mathbf{P}_{u-a}\left[\tau_{(-\infty,0)} > N-1\right]} \\ &= \frac{h(t)\phi_{N-1}(t)}{\int_a^{+\infty} h(u)\phi_N(u-a)du}.\end{aligned} \tag{3.8.57}$$

Making use of the triangle inequality, one gets :

$$\begin{aligned}\left|\int_a^{+\infty} h(u)\phi_N(u-a)du - \int_a^{+\infty} h(u)V(u-a)du\right| \\ \leqslant \underbrace{\left|\int_a^{N^{1/4}} \dots\right|}_{(1)} + \underbrace{\left|\int_{N^{1/4}}^{\infty} \dots\right|}_{(2)}\end{aligned} \tag{3.8.58}$$

Let $\varepsilon > 0$. Lemma 5.6.1 in chapter 5 entails that for N large enough :

$$(1) \leqslant \int_a^{N^{1/4}} h(u)\left|\phi_N(u-a) - V(u-a)\right| du \leqslant \varepsilon \int_a^{N^{1/4}} h(u)du \leqslant \varepsilon. \tag{3.8.59}$$

We have easily, still for N large enough :

$$\begin{aligned}(2) &\leqslant \int_{N^{1/4}}^{\infty} h(u)V(u-a)du + \int_{N^{1/4}}^{C\infty} h(u)\phi_N(u-a)du \\ &\leqslant \varepsilon + \int_{N^{1/4}}^{\infty} h(u)u^2\frac{\phi_N(u-a)}{N^{1/2}}du \\ &\leqslant \varepsilon + C\int_{N^{1/4}}^{\infty} u^2 h(u)du \\ &\leqslant 2\varepsilon\end{aligned} \tag{3.8.60}$$

where in the third inequality we used the fact that $\mathbf{E}[S_1^2] < \infty$. This concludes the proof of the convergence (3.8.52). □

Proof of (3.8.53)

We use a bidimensional version of the above arguments, so the proof is somewhat lengthy although technically not very involved. For $t, t' \geqslant a$, we define

$$\psi_N^a(t,t') := \sqrt{2\pi}\sigma N^{3/2}\mathbf{P}_{t-a}\left[\tau_{(-\infty,0)} > N-3, S_{N-2} = t'-a\right]. \tag{3.8.61}$$

It is easy to see that in a similar way as the one dimensional case, we just have to show that, for any $M \geqslant a$, as $N \to \infty$, the following convergence holds :

$$\begin{aligned}&\int_{[M,\infty)^2\times[0,a]} \psi_N^a(t,t')h(t)h(u-t')dtdt'du \\ &\to \int_{[M,\infty)^2\times[0,a]} h(v)h(u-v')V(v-a)U(v'-a)dvdv'du.\end{aligned} \tag{3.8.62}$$

The following result is proved in chapter5, it will be important in what follows. Note that it is of crucial importance in the proof of Theorem 3.8.3.

Lemma 3.8.7. *For $K > 0$, uniformly in $x_n n^{-1/2} \to 0$ as $n \to \infty$ and in y_n such that $y_n n^{-1/2} \in [0, K]$, one has the following equivalence :*

$$\mathbf{P}_{x_n}[\tau_{(-\infty,0)} > n, S_n = y_n] \sim \frac{V(x_n)U(y_n)}{n}\mathbf{P}[S_n = y_n]. \tag{3.8.63}$$

We give us $\kappa, \kappa' > 0$ and we split the integral appearing in the left hand side of (3.8.51) into five parts, namely

$$\begin{aligned}&\int_{t,t'\in[a,+\infty)^2}\int_{u\in[0,a]} h(t)\psi_N^a(t,t')\int_{u\in[0,a]} h(u-t')dtdt'du \\ &= \int_{\mathcal{D}_1^N\times[0,a]} \ldots + \int_{\mathcal{D}_2^N\times[0,a]} \ldots + \int_{\mathcal{D}_3^N\times[0,a]} \ldots + \int_{\mathcal{D}_4^N\times[0,a]} \ldots + \int_{\mathcal{D}_5^N\times[0,a]} \ldots\end{aligned} \tag{3.8.64}$$

where we define (recall the standard notations $a \wedge b := \min(a,b)$ and $a \vee b := \max(a,b)$) :

$$\begin{aligned}
&\mathcal{D}_1^N := \left\{(t,t') \in [a, \kappa N^{1/4}]^2\right\}\\
&\mathcal{D}_2^N := \left\{(t,t') \in \mathbb{R}^2, t\wedge t' \geqslant \kappa N^{1/4}, t \vee t' \geqslant \kappa' N^{1/2}\right\}\\
&\mathcal{D}_3^N := \left\{(t,t') \in \mathbb{R}^2, a \leqslant t\wedge t' \leqslant \kappa N^{1/4}, t \vee t' \geqslant \kappa' N^{1/2}\right\}\\
&\mathcal{D}_4^N := \left(\left[\kappa N^{1/4}, \kappa' N^{1/2}\right] \times \left[a, \kappa N^{1/4}\right]\right) \cup \left(\left[a, \kappa N^{1/4}\right] \times \left[\kappa N^{1/4}, \kappa' N^{1/2}\right]\right)\\
&\mathcal{D}_5^N := [\kappa N^{1/4}, \kappa' N^{1/2}]^2.
\end{aligned} \tag{3.8.65}$$

The domain $\mathcal{D}_1^N$

The uniform convergence of $\psi_N^a(t,t') \to V(t-a)U(t'-a)$ on $\mathcal{D}_1^N$ (see Lemma 3.8.7) is a consequence of the standard local limit theorem and implies immediately that, as $N \to \infty$:

$$\begin{aligned}
&\int_{\mathcal{D}_1^N \times [0,a]} \psi_N^a(t,t') h(t) h(u-t') dt dt' du\\
&\to \int_{\mathcal{D}_1^N \times [0,a]} h(v) h(u-v') V(v-a) U(v'-a) dv dv' du.
\end{aligned} \tag{3.8.66}$$

The domain $\mathcal{D}_2^N$

One has the trivial bounds :

$$\begin{aligned}
&\int_{\mathcal{D}_2^N \times [0,a]} \psi_N^a(t,t') h(t) h(u-t') dt dt' du\\
&\leqslant \sqrt{2\pi}\sigma \int_{\mathcal{D}_2^N \times [0,a]} N^{3/2} h(t) h(u-t') dt dt' du\\
&\leqslant \frac{\sqrt{2\pi}\sigma}{\kappa^2 \kappa'^2} \int_{\mathcal{D}_2^N \times [0,a]} (t\vee t')^2 (t \wedge t')^2 h(t) h(u-t') dt dt' du
\end{aligned} \tag{3.8.67}$$

and as $\mathbf{E}[S_1^2] < \infty$, using Fubini's theorem it is clear that the last term above vanishes as $N \to \infty$.

The domain $\mathcal{D}_3^N$

Note that $\mathcal{D}_3^N$ is the union of $[a, \kappa N^{1/4}] \times [\kappa' N^{1/2}, \infty)$ and of its image by the symmetry with respect to the line $y = x$, so that using symmetry arguments we just have to show that the integral of $h(t)\psi_N^a(t,t') \int_{u\in[0,a]} h(u-t')$ vanishes over

$[a, \kappa N^{1/4}] \times [\kappa' N^{1/2}, \infty)$. Note that for $t, t' \in [a, \kappa N^{1/4}] \times [\kappa' N^{1/2}, \infty) \times [0, a]$, the following bound holds :

$$\psi_N^a(t,t') \leqslant cN \frac{\mathbf{P}_{t-a}\left[\tau_{(-\infty,0)} > N, S_N - a \geqslant \kappa' N^{1/2}\right]}{\mathbf{P}\left[T_1^- > N\right]} \tag{3.8.68}$$

and using the equivalence (3.8.55) (and in particular the uniform part), we get :

$$\begin{aligned} \psi_N^a(t,t') &\leqslant cNV(t)\mathbf{P}_{t-a}[S_N - a \geqslant \kappa' N^{1/2} | \tau_{(-\infty,0)} > N] \\ &\leqslant cNV(t) m(\omega_1 > \kappa') \end{aligned} \tag{3.8.69}$$

where in the last inequality we made use of Theorem 3.8.1. As $V(t) \sim t/\mathbf{E}[H_1^-]$, we get that

$$\begin{aligned} &\int_{[a,\kappa N^{1/4}]\times[\kappa' N^{1/2},\infty)\times[0,a]} V(t)h(t)\psi_N^a(t,t') \int_{u\in[0,a]} h(u-t')dtdt'du \\ &\leqslant c\int_{[a,\kappa N^{1/4}]\times[\kappa' N^{1/2},\infty)\times[0,a]} \frac{t}{\mathbf{E}[H_1^-]} h(t) N m(\omega_1 > \kappa') h(u-t') dtdt'du \\ &\leqslant \frac{c\mathbf{E}[S_1]}{\mathbf{E}[H_1^-]\kappa'^2} m(\omega_1 > \kappa') \int_{[\kappa' N^{1/2},\infty)\times[0,a]} t'^2 h(u-t')dt'du \end{aligned} \tag{3.8.70}$$

and the last term vanishes as $N \to \infty$.

The domain $\mathcal{D}_4^N$

We first consider the rectangle $[a, \kappa N^{1/4}] \times [\kappa N^{1/4}, \kappa' N^{1/2}]$, the other component of $\mathcal{D}_4^N$ will be treated in a similar way via time reversal. We use once again Lemma 3.8.7 and the standard local limit theorem which show that the following convergence holds uniformly for $(t, t') \in [a, \kappa N^{1/4}] \times [\kappa N^{1/4}, \kappa' N^{1/2}]$:

$$\psi_N^a(t,t')h(t)h(u-t') \to h(t)h(u-t')V(t-a)U(t'-a)\phi(t' N^{-1/2}) \tag{3.8.71}$$

where we recall that $\phi(\cdot)$ is the density of the standard normal law ; since this density is bounded over $\mathbb{R}$, one concludes easily that the integral of $\psi_N^a(t,t')h(t)h(u-t')$ over $[a, \kappa N^{1/4}] \times [\kappa N^{1/4}, \kappa' N^{1/2}] \times [0, a]$ vanishes as $N \to \infty$.

The domain $\mathcal{D}_5^N$

Using the same trick as for the domains $\mathcal{D}_2^N$ and $\mathcal{D}_3^N$, it is easy to see that once we are able to prove that

$$\limsup_{N\to\infty} \sup_{(t,t')\in[\kappa N^{1/4},\kappa' N^{1/2}]^2} N^{1/2}\mathbf{P}_t\left[\tau_{(-\infty,0)} > N, S_N = t'\right] < \infty \tag{3.8.72}$$

the integral of ψ_N^a will vanish over the domain $\mathcal{D}_5^N$. Note that the assumptions made on $h(\cdot)$ imply that the convergence

$$N^{1/2}\mathbf{P}\left[\frac{S_N}{\sqrt{N}} = t\right] \to \phi(t) \tag{3.8.73}$$

holds uniformly for $t \in \mathbb{R}$, thus in particular

$$N^{1/2}\mathbf{P}_t\left[\tau_{(-\infty,0)} > N, S_N = t'\right] \leqslant N^{1/2}\mathbf{P}\left[\frac{S_N}{\sqrt{N}} = \frac{t'-t}{\sqrt{N}}\right] < \infty \tag{3.8.74}$$

is valid for $t, t' \in \mathbb{R}$ and thus we are done. □

Chapitre 4

An asymptotic result about the overshoot

Contents

4.1 The setting and the main results

We collect some results about the large-time asymptotics of the density for a random walk with continuous increments conditioned on the first entrance into the positive half line when starting from a negative point. The main result of this part is the key estimate to the results of the chapter 3 of the present thesis.

4.1.1 Definitions and hypothesis

The setting

We are concerned with a continuous random walk, that is a sequence $\{S_n\}_{n \geqslant 0}$ of random variables where $S_0 = 0$ and $S_n - S_{n-1} =: X_n$ are i.i.d., $X_n \sim X$; we assume that the law of X is absolutely continuous with respect to the Lebesgue measure, with density $f : X \sim f(x)dx$. We denote by $\mathbf{P}$ the global law of the random walk, and by $\mathbf{P}_b$ $(b \in \mathbb{R})$ the law of $\{S_n + b\}$ under $\mathbf{P}$.

Our basic hypothesis is that X satisfies the hypothesis of the classical *Local Limit Theorem*, namely :

- X is square integrable, with zero mean : $\mathbf{E}[X] = 0,\ \mathbf{E}[X^2] = \sigma^2$;
- $f(\cdot)$ is a bounded function on $\mathbb{R}$.

Under these assumptions, the Gnedenko's classical local limit theorems holds, namely :

Theorem 4.1.1. *As $n \to \infty$, the following convergence holds uniformly for $x \in \mathbb{R}$:*

$$\sigma\sqrt{n}\ \mathbf{P}[\frac{S_n}{\sigma\sqrt{n}} \in dx] \to \phi(x)dx \tag{4.1.1}$$

where $\phi(\cdot)$ denotes the density of the standard normal law.

We define $f^{*n_o} := \underbrace{f * \ldots * f}_{n_0 \ \text{times}}$ where $*$ denotes the standard convolution.

A simple consequence of Theorem 4.1.1 is that for every $x \in \mathbb{R}$, $\sqrt{n}f_n(x) \to 1/(\sigma\sqrt{2\pi})$ as $n \to \infty$, and the convergence is uniform for $x \in \mathbb{R}$.

Note that the hypothesis on the boundedness of f could be replaced without much efforts by the assumption that there exists $n_o \in \mathbb{N}$ such that the density $f_{n_o} = f^{*n_o}$ of S_{n_o} is a bounded function.

Some ladder theory

We recall some notions from fluctuation theory for random walks ([Fel71], ch. XII). We denote by T_k the k-th ascending *ladder epoch*, defined by $T_0 := 0$ and

$$T_k := \inf\{n > T_{k-1} : S_n > S_{T_{k-1}}\} \qquad k \geqslant 1\,; \tag{4.1.2}$$

then $H_k := S_{T_k}$ is the k-th strict ascending *ladder height*. Since $(T_k, H_k) \sim (T_1, H_1)^{*k}$, the sequence $\{(T_k, H_k), k \geqslant 0\}$ is a two-dimensional renewal process : we will call it the (ascending) *ladder variables process*. Its marginals $\{T_k\}_k$, $\{H_k\}_k$ are of course one-dimensional renewal processes, the ladder epochs and the ladder heights processes.

In a similar way, one can define the descending ladder variables (T_k^-, H_k^-), that is

$$T_k^- := \inf\{n > T_{k-1}^- : S_n < S_{T_{k-1}}\} \qquad k \geqslant 1 \tag{4.1.3}$$

and $H_k^- := -S_{T_k^-}$. In principle all these variables may be defective, but this is not the case in our setup : in fact, since $\mathbf{E}[X] = 0$, the random walk is persistent ([Fel71], ch. VI) and consequently both the ascending and descending ladder processes are proper. It is also a known fact ([Fel71, ch. XVIII.4 p. 611] or [Don80]) that the existence of the variance for X is equivalent to -and hence implies- the existence of the mean for the two *overshoots* H_1 and H_1^-, and the following relation holds :

$$\sigma^2 = 2\, \mathbf{E}[H_1]\, \mathbf{E}[H_1^-]. \tag{4.1.4}$$

We will also need the *renewal function* $U(\cdot)$:

$$U(x) := \sum_{j \geqslant 0} \mathbf{P}[H_j \leqslant x] = \mathbf{E}[\mathcal{N}(x)] \tag{4.1.5}$$

where $\mathcal{N}(x)$ denotes the number of renewals of H (including H_0) in the interval $[0, x]$. It is clear that $U(\cdot)$ is increasing and subadditive, and that $U(0) = 1$.

4.1.2 An estimate about the overshoot

We are now ready to state our main result :

Theorem 4.1.2. *Assume that X is given by a bounded density, with zero mean and variance σ^2. Then :*

(i) *for any fixed $a \geqslant 0$ the following asymptotic relation holds :*

$$\frac{d}{dx}\mathbf{P}_{-a}[S_1 \leqslant 0, \dots, S_{n-1} \leqslant 0, S_n > 0, S_n \in dx] \sim U(a)\frac{\mathbf{P}[H_1 \geqslant x]}{\sigma\sqrt{2\pi}n^{3/2}} \qquad \text{as } n \to \infty\,, \tag{4.1.6}$$

and denoting by $g_{n,a}(x)$ the left hand side in the above relation, we have that the convergence of $\sigma\sqrt{2\pi}n^{3/2}g_{n,a}(x)$ towards $U(a)\mathbf{P}[H_1 \geqslant x]$ is uniform for $x \in \mathbb{R}$ (a is fixed) ;

(ii) *moreover, the sequence of functions $\sigma\sqrt{2\pi}n^{3/2}g_{n,a}(x)$ is dominated by a multiple of its limit : $n^{3/2}g_{n,a}(x) \leqslant \mathcal{C}U(a)\mathbf{P}[H_1 \geqslant x]$ for every n, a, x ; consequently, one has also that*

$$\mathbf{P}_{-a}[S_1 \leqslant 0, \dots, S_{n-1} \leqslant 0, S_n > 0] \sim U(a)\frac{\mathbf{E}[H_1]}{\sigma\sqrt{2\pi}n^{3/2}} \qquad \text{as } n \to \infty\,.$$

4.1.3 Organization of the chapter

This chapter will be organized as follows :

- in section 4.2, we define properly some important tools and discuss some preliminary facts.
- in section 4.3, for fixed k, we show an asymptotic equivalence on the quantity $\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}]$ that will be the cornerstone of our proof.
- in section 4.4, we give asymptotics on the law (H_k, S_n) as $n \to \infty$ and these estimates imply in particular our main result in the $a = 0$ case.
- in section 4.5, we finally deduce our main result as a consequence of the preceding section.
- the appendix are devoted to the proof of two technical lemmas which are both an extension of Iglehart's classical lemma.

4.2 Some preliminary facts

4.2.1 About the densities

A useful notation

In the sequel, we will often use the following formal notation : if Y is a random variable with density and A is an event, the measure $\mathbf{P}[A, Y \in \mathrm{d}x]$ is absolutely continuous, and we denote its density by $\mathbf{P}[A, \text{“}Y = x\text{”}]$: it is simply the density of the possibly defective random variable (of total mass $\mathbf{P}[A]$) that equals Y on A and is undefined otherwise. In the same way, sometimes we write $\mathbf{P}[\text{“}S_n = x\text{”}]$ for $f_n(x)$.

From almost sure equalities to equalities holding everywhere

The only densities we will encounter are that of the random variables $\{H_k\}$ and $\{S_n\}$ (maybe restricted on some events, as in the preceding relations). In general a density is not uniquely defined pointwise, but in our case for S we have a privileged representation, namely the one given by the explicit integral over the trajectories of the random walk˙
One could think that the equations between densities we are going to write should be intended to hold only for almost every x. However, a closer examination of the combinatorial proofs our relations shows that *they hold for every* x, provided that the densities are chosen as above. It is also for this reason that in the following we always implicitly make this choice. Furthermore, in this way all the density asymptotics we are going to prove hold for every x and not only almost everywhere.

4.2.2 The renewal measure

The renewal measure $\mathcal{U}$ associated to the ladder variables process will be of basic importance. This measure is defined for an integer n and for a measurable subset $I \subseteq \mathbb{R}$ by

$$\mathcal{U}(n, I) := \sum_{k=0}^{\infty} \mathbf{P}[T_k = n, H_k \in I] = \mathbf{P}[n \text{ is a ladder epoch, } S_n \in I], \qquad (4.2.1)$$

whose distribution function we denote by $\mathcal{U}(n, x)$. It is a standard result of renewal theory ([Fel71, ch. VI]) that $\mathcal{U}$ is a σ-finite measure on $\mathbb{N} \times \mathbb{R}$ (supported by $\mathbb{N} \times [0, +\infty)$).

Since by definition $(T_0, H_0) = (0, 0)$, $\mathcal{U}(0, \cdot\,)$ is a Dirac measure at the origin. On the other side, H_1 is absolutely continuous with respect to the Lebesgue measure (and consequently so are the random variables $H_k \sim H_1^{*k}$, for all $k \geqslant 1$) : this follows easily from the relation $\mathbf{P}[H_1 \in I] = \sum_{n \geqslant 1} \mathbf{P}[T_1 = n, S_n \in I]$. A look to Equation (4.2.1) then shows that for $n \geqslant 1$ the measure $\mathcal{U}(n, \cdot\,)$ is absolutely continuous too : we denote by $u(n, x)$ the corresponding density, given by

$$u(n, x) := \sum_{k=0}^{\infty} \mathbf{P}[T_k = n, \text{“}H_k = x\text{”}], \qquad (4.2.2)$$

(notice that $u(n, x) = 0$ for $x < 0$).

Note that, summing relation (4.2.1) over n, we obtain the relation :

$$U(x) = \sum_{n \geqslant 0} \mathcal{U}(n, [0, x]) = 1 + \int_{[0,x]} \sum_{n \geqslant 1} u(n, y) dy\,. \qquad (4.2.3)$$

4.2.3 Two combinatorial identities

The Duality Lemma

The power of the ladder theory for the study of random walk fluctuations is linked to some surprising identities, of purely combinatorial nature, enjoyed by the law of a generic random walk ([Fel71, ch. XXII]). The most famous one is the “Duality Lemma”, which reads :

$$\mathbf{P}[n \text{ is a ladder epoch, } S_n \in I] = \mathbf{P}[S_1 > 0, \ldots, S_n > 0,\ S_n \in I],$$

valid for every integer $n \geqslant 1$ and for every measurable $I \in [0, +\infty)$. The left hand side is exactly the renewal measure of the ladder process, so that with our notation we can rewrite this identity as

$$\mathcal{U}(n, I) = \mathbf{P}[S_1 > 0, \ldots, S_n > 0,\ S_n \in I] = \mathbf{P}[T_1^- > n,\ S_n \in I], \qquad (4.2.4)$$

and of course a similar identity holds for the density :

$$u(n,x) = \mathbf{P}[S_1 > 0, \dots, S_n > 0,\ \text{“}S_n = x\text{”}] = \mathbf{P}[T_1^- > n,\ \text{“}S_n = x\text{”}]\,. \tag{4.2.5}$$

A powerful combinatorial identity

It is another identity in the same spirit, which has been first proved in [AD99] (and later by Marchal, [Mar01]), that will play a basic role in our analysis and will be used a lot of times in the sequel. We write it in density form :

$$\mathbf{P}[T_k = n, \text{“}H_k = x\text{”}] = \frac{k}{n}\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \tag{4.2.6}$$

and for clarity we rewrite explicitely the $k = 1$ case :

$$\mathbf{P}[T_1 = n, \text{“}H_1 = x\text{”}] = \frac{1}{n}\mathbf{P}[H_1 \geqslant x, \text{“}S_n = x\text{”}]\,. \tag{4.2.7}$$

Note that a simple proof of this identity can be made with the help of the so-called “Spitzer-Baxter identity” ([Spi60], or [Fel71, ch. XVIII.3]), concerning the joint distribution of the first increasing ladder variables :

$$\left(1 - \mathbf{E}\left[r^{T_1}e^{-\mu H_1}\right]\right)^{-1} = \exp\left(\sum_{m=1}^{\infty}\frac{r^m}{m}\mathbf{E}\left[e^{-\mu S_m}, S_m > 0\right]\right)\,.$$

which is valid for $r, \mu \geqslant 0$. This identity may be itself derived with combinatorial methods, or more easily with Fourier techniques.

We will show that the proof of theorem 4.1.2 will be a consequence of two asymptotics which are interesting in themselves :

1. the one for the $a = 0$ case, that is for $\mathbf{P}[T_1 = n, \text{“}H_1 = x\text{”}]$;
2. the asymptotic for the renewal density $u(n, x)$ of the ladder variables process.

These two problems have been studied and solved in [AD99] in a slightly different context, that is when S is an aperiodic integer valued random walk. Some of their methods can be adapted to the continuum case with little modifications, while some other are peculiar of the lattice case. The statement and the proof of these asymptotics in the continuous setting is the object of the next sections.

4.3 Two preliminary estimates

In the following we will have to calculate the asymptotics for several quantities. However, we will see that, thanks to the fundamental relation (4.2.6), all these asymptotics can be simply expressed in terms of the limiting behavior as $n \to \infty$ of the sequence $\sqrt{n}\,\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}]$. The study of this sequence is the object of this section.

More precisely, we first prove the following domination :

Proposition 4.3.1. *For every $n, k \in \mathbb{N}$ and for every $x \geqslant 0$, the following inequality holds :*

$$\sqrt{n}\,\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \leqslant \mathcal{C}\, k \min\left\{\mathbf{P}[H_{k-2} < x], \mathbf{P}[H_k \geqslant x]\right\}, \tag{4.3.1}$$

where $\mathcal{C}$ is a positive constant.

Then we show the uniform convergence of the same quantity :

Theorem 4.3.2. *For any $k \in \mathbb{N}$ and for every $x \geqslant 0$, as $n \to \infty$,*

$$\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \sim \mathbf{P}[H_{k-1} < x \leqslant H_k] \frac{1}{\sigma\sqrt{2\pi}\sqrt{n}}. \tag{4.3.2}$$

Moreover, for any fixed k, the convergence of $\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}]$ towards $\mathbf{P}[H_{k-1} < x \leqslant H_k]$ is uniform for $x \in \mathbb{R}$.

The rest of this part is devoted to the proofs of the above results; we first show the domination part (that is Proposition 4.3.1), then we prove the uniform convergence result.

4.3.1 Domination

Before going to the proof of Proposition 4.3.1, we need some preliminary results contained in the following two lemmas. The first one is just elementary analysis, the second one is a direct consequence of the identity (4.2.6).

Lemma 4.3.3. *Let $\{d_j\}_{j \geqslant 1}$ be a non negative sequence, and for $0 < \alpha < 1$ define another sequence*

$$a_n := \sum_{j=1}^{n-1} \frac{d_j}{(n-j)^\alpha};$$

then for every n the following relation holds :

$$n^\alpha a_n \leqslant 2\sum_{j=1}^{\infty} d_j + C \sup_{\{j \geqslant n/2\}} (j d_j), \tag{4.3.3}$$

where the constant C does not depend on the sequence $\{d_j\}$.

In particular, if $\sum_{j\geqslant 1} d_j < \infty$ and $\sup_{\{j\geqslant 1\}}(jd_j) < \infty$, Lemma 4.3.3 implies that the sequence $n^\alpha a_n$ is bounded.

Proof. We write :

$$a_n := \sum_{j=1}^{n-1} \frac{d_j}{(n-j)^\alpha} = \sum_{j=1}^{n/2} \frac{d_j}{(n-j)^\alpha} + \sum_{j=n/2}^{n-1} \frac{d_j}{(n-j)^\alpha}\,.$$

The first sum is immediately estimated :

$$\sum_{j=1}^{n/2} \frac{d_j}{(n-j)^\alpha} \leqslant \frac{2}{n^\alpha}\sum_{j=1}^{n/2} d_j \leqslant \frac{2}{n^\alpha}\sum_{j=1}^{\infty} d_j\,. \tag{4.3.4}$$

For the second one, we get

$$\sum_{j=n/2}^{n-1} \frac{d_j}{(n-j)^\alpha} \leqslant \sup_{\{k\geqslant n/2\}}(kd_k) \sum_{j=n/2}^{n-1} \frac{1}{j}\frac{1}{(n-j)^\alpha} \leqslant \sup_{\{k\geqslant n/2\}}(kd_k)\,\frac{2}{n}\sum_{j=n/2}^{n-1}\frac{1}{(n-j)^\alpha}$$
$$\leqslant \frac{2}{n^\alpha}\sup_{\{k\geqslant n/2\}}(kd_k)\left(\frac{1}{n}\sum_{j=n/2}^{n-1}\frac{1}{(1-\frac{j}{n})^\alpha}\right) \leqslant \frac{c}{n^\alpha}\sup_{\{k\geqslant n/2\}}(kd_k)\left(\int_{1/2}^{1}\mathrm{d}y\frac{1}{(1-y)^\alpha}\right), \tag{4.3.5}$$

where we used Riemann sums.

If we denote by $C := c\cdot\int_{1/2}^{1}\mathrm{d}y 1/(1-y)^\alpha$, then relation (4.3.3) easily follows from (4.3.4) and (4.3.5). □

Lemma 4.3.4. *For every $k, n \geqslant 1$ and for every $x \geqslant 0$ the following inequalities hold :*

$$\mathbf{P}[T_k = n, H_k \geqslant x] \leqslant \frac{k}{n}\mathbf{P}[H_k \geqslant x] \tag{4.3.6}$$
$$\mathbf{P}[T_k = n, H_k < x] \leqslant \frac{k}{n}\mathbf{P}[H_{k-1} < x] \tag{4.3.7}$$

Proof. Using relation (4.2.6) one has :

$$\mathbf{P}[T_k = n, H_k \geqslant x] = \int_x^\infty \mathrm{d}y\,\mathbf{P}[T_k = n, \text{“}H_k = y\text{”}]$$
$$= \frac{k}{n}\int_x^\infty \mathrm{d}y\,\mathbf{P}[H_{k-1} < y \leqslant H_k, \text{“}S_n = y\text{”}] \leqslant \frac{k}{n}\int_x^\infty \mathrm{d}y\,\mathbf{P}[H_k \geqslant y, \text{“}S_n = y\text{”}]$$
$$\leqslant \frac{k}{n}\int_x^\infty \mathrm{d}y\,\mathbf{P}[H_k \geqslant x, \text{“}S_n = y\text{”}] = \frac{k}{n}\mathbf{P}[H_k \geqslant x, S_n \geqslant x] \leqslant \frac{k}{n}\mathbf{P}[H_k \geqslant x]\,,$$

where all the inequalities follow by simple inclusion of events. In the same manner,

$$\mathbf{P}[T_k = n, H_k < x] = \int_0^x \mathrm{d}y\, \mathbf{P}[T_k = n, \text{“}H_k = y\text{”}]$$
$$= \frac{k}{n}\int_0^x \mathrm{d}y\, \mathbf{P}[H_{k-1} < y \leqslant H_k, \text{“}S_n = y\text{”}] \leqslant \frac{k}{n}\int_0^x \mathrm{d}y\, \mathbf{P}[H_{k-1} < y, \text{“}S_n = y\text{”}]$$
$$\leqslant \frac{k}{n}\int_0^x \mathrm{d}y\, \mathbf{P}[H_{k-1} < x, \text{“}S_n = y\text{”}] = \frac{k}{n}\mathbf{P}[H_{k-1} < x, S_n \in (0,x)] \leqslant \frac{k}{n}\mathbf{P}[H_{k-1} < x]\,,$$

and the proof is completed. □

Proof of proposition 4.3.1

First note that a direct consequence of the LLT and of the fact that the density of X is bounded is that there exists a positive constant Q such that for every $n \in \mathbb{N}$ and for every $z \in \mathbb{R}$ one has

$$\mathbf{P}[\text{“}S_m = z\text{”}] \leqslant \frac{Q}{\sqrt{n}}\,. \qquad (4.3.8)$$

By conditioning on (T_k, H_k), using the Markov property and the bound (4.3.8), we get

$$\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] = \mathbf{P}[T_k \leqslant n, H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}]$$
$$\leqslant \mathbf{P}[T_k \leqslant n, H_k \geqslant x, \text{“}S_n = x\text{”}] = \sum_{j=1}^{n}\int_x^\infty \mathrm{d}y\, \mathbf{P}[T_k = j, \text{“}H_k = y\text{”}]\mathbf{P}[\text{“}S_{n-j} = x - y\text{”}]$$
$$\leqslant Q\sum_{j=1}^{n}\mathbf{P}[T_k = j, H_k \geqslant x]\frac{1}{\sqrt{n-j}}\,,$$

and since $\sum_{j \geqslant 1}\mathbf{P}[T_k = j, H_k \geqslant x] = \mathbf{P}[H_k \geqslant x]$ we can apply Lemma 4.3.3 so that for every n

$$\sqrt{n}\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \leqslant 2\mathbf{P}[H_k \geqslant x] + C\sup_{\{j \geqslant n/2\}}\left(j\mathbf{P}[T_k = j, H_k \geqslant x]\right)\,.$$

Now we just have to apply relation (4.3.6) to get

$$\sqrt{n}\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \leqslant (2 + Ck)\mathbf{P}[H_k \geqslant x] \leqslant \mathcal{C}\,k\mathbf{P}[H_k \geqslant x]\,, \quad (4.3.9)$$

where $\mathcal{C} := C + 2$.

In a similar way, we have :

$$\begin{aligned}
&\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \leqslant \mathbf{P}[H_{k-1} < x, \text{“}S_n = x\text{”}] \\
&= \sum_{j=1}^{n} \int_0^x \mathrm{d}y\, \mathbf{P}[T_{k-1} = j, \text{“}H_{k-1} = y\text{”}]\mathbf{P}[\text{“}S_{n-j} = x - y\text{”}] \\
&\leqslant Q \sum_{j=1}^{n} \mathbf{P}[T_{k-1} = j, H_{k-1} < x] \frac{1}{\sqrt{n-j}} \,,
\end{aligned}$$

and applying again Lemma 4.3.3 and relation (4.3.7) we obtain that, for every n,

$$\sqrt{n}\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \leqslant 2\mathbf{P}[H_{k-1} < x] + Ck\mathbf{P}[H_{k-2} < x] \leqslant \mathcal{C}\, k\, \mathbf{P}[H_{k-2} < x] \,,$$

so that the proof is concluded. □

4.3.2 Uniform convergence

The idea of the proof is essentially the same as the one of [AD99, Proposition 6], which deals with the discrete case : their proof essentially relies on a suitable application of Iglehart's lemma. In quite the same way, the basis of our proof will be a refinement of the same lemma combined with an estimate on the asymptotic behavior of the joint law (H_k, S_n) where k is fixed and n becomes large.

A refinement of Iglehart's lemma

The statement of Iglehart's lemma is the following : we are given two sequences c_n, d_n of non negative numbers, and we define a new sequence $a_n := \sum_{j=1}^{n-1} c_{n-j} d_j$; if the following hypothesis are satisfied :

- $c_n \sim c/\sqrt{n}$ as $n \to \infty$, with $c > 0$;
- $\sum d_j =: D < \infty$ and $d_n = O(1/n)$ as $n \to \infty$;

then we have the explicit asymptotic behavior : $a_n \sim Dc/\sqrt{n}$ as $n \to \infty$.

In the study of the convergence of $\sqrt{n}\,\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}]$, we will encounter sequences defined like the a_n above, but this time c_n, d_n will depend on some parameters, and we will need to explicitly express the difference $|\sqrt{n}a_n - Dc|$. The following refinement of Iglehart's Lemma goes exactly in this direction.

Lemma 4.3.5 (Refined Iglehart's lemma). *Let $\{c_j(z)\}_{j \geqslant 1}, \{d_j\}_{j \geqslant 1}$ be two sequences of non negative numbers, the first one depending on a parameter z that varies in a set $\mathcal{S}$. Define the new sequence $a_n(z) := \sum_{j=1}^{n-1} c_{n-j}(z) d_j$. If the following conditions hold :*

- *$c_n(z) \sim c(z)/n^\alpha$ as $n \to \infty$, with $c(z) > 0$, $0 < \alpha < 1$, uniformly for $z \in \mathcal{S}$ in the sense that*

$$\lim_{n\to\infty} \left(\sup_{z \in \mathcal{S}} |n^\alpha c_n(z) - c(z)| \right) = 0 \; ; \tag{4.3.10}$$

- *$c_n(z) \leqslant Cc(z)/n^\alpha$ for every $n \geqslant 1$ and $z \in \mathcal{S}$, for some positive constant C ;*

then we have the following explicit estimate (the relation is always valid, with the usual conventions about the use of ∞) : for every $\delta \in (0, 1/2)$, there exists $\bar{n}_\delta$ such that for every $n \geqslant \bar{n}_\delta$, $z \in \mathcal{S}$ and for every non negative sequence $\{d_j\}_{j \geqslant 1}$

$$\left| n^\alpha a_n(z) - c(z) \sum_{j=1}^\infty d_j \right| \leqslant \xi(\delta)\, c(z) \left(\sum_{j=1}^\infty d_j + \sup_{\{m \geqslant n/2\}} (m d_m) \right) + \xi(\delta) \sum_{j=1}^\infty d_j + \frac{4}{\delta^\alpha} c(z) \sum_{j \geqslant n\delta} d_j \,,$$

where the function $\xi(\delta) \to 0$ as $\delta \to 0$. We stress that $\bar{n}_\delta$ and ξ do not depend on the sequence $\{d_j\}$.

If in addition the function $c(z)$ is bounded, $\sup_{z \in \mathcal{S}} c(z) < \infty$, then the preceding relation reduces to :

$$\left| n^\alpha a_n(z) - c(z) \sum_{j=1}^\infty d_j \right| \leqslant \xi(\delta) \left(\sum_{j=1}^\infty d_j + \sup_{\{m \geqslant n/2\}} (m d_m) \right) + \frac{Q}{\delta^\alpha} \sum_{j \geqslant n\delta} d_j \,, \quad (4.3.11)$$

for possibly another $\xi(\delta) \to 0$ as $\delta \to 0$, and Q a constant depending only on the sequence $\{c_j\}$.

The proof of this Lemma 4.3.5 will be given in Appendix 4.7.

Note in particular that the original Iglehart's lemma easily follows from equation (4.3.11). In what follows, we shall use Lemma 4.3.5 when $c_n(z)$ is $\mathbf{P}[\text{“}S_n = z\text{”}]$ and z varies over a compact set K. The hypothesis are easily verified to hold in this case, with $\alpha = 1/2$ and $c(z) = 1/\sigma\sqrt{2\pi}$: in fact, as we recalled in the introduction, a direct consequence of the LLT is that

$$\lim_{n \to \infty} \sup_{z \in K} \left(\sigma \sqrt{2\pi} \sqrt{n} c_n(z) \right) = 1 \,,$$

which is equivalent to (4.3.10). For the second hypothesis, observe that a simple consequence of the LLTis that there exists a constant Q such that $\mathbf{P}[\text{“}S_n = z\text{”}] \leqslant Q/\sqrt{n}$ for every n and every $z \in \mathbb{R}$.

Asymptotic behavior of the joint law (H_k, S_n) as $n \to \infty$.

The other result we will need for the proof of Theorem 4.3.2 is stated in the following :

Proposition 4.3.6. *For any fixed $k \in \mathbb{N}$ and for every $x \geqslant 0$ one has that, as $n \to \infty$,*

$$\mathbf{P}[H_k < x, \text{“}S_n = x\text{”}] \sim \mathbf{P}[H_k < x]\mathbf{P}[\text{“}S_n = x\text{”}] \sim \mathbf{P}[H_k < x]\frac{1}{\sigma\sqrt{2\pi}\sqrt{n}}, \tag{4.3.12}$$

where the convergence of $\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[H_k < x, \text{“}S_n = x\text{”}]$ towards $\mathbf{P}[H_k < x]$ is uniform for x in a compact set.

Equation (4.3.12) is not an unexpected result : considering the Markov property and the LLT, it seems reasonable that for large n the “event” [“$S_n = x$”] becomes nearly independent from the event $[H_k < x]$, and the factorization given in (4.3.12) should consequently follow. More precisely, conditioning on the k-th ladder epoch and height and using the Markov property, one obtains

$$\begin{aligned} \mathbf{P}[H_k < x, \text{“}S_n = x\text{”}] &= \int_0^x \mathrm{d}y \sum_{j=1}^{n-1} \mathbf{P}[T_k = j, \text{“}H_k = y\text{”}, \text{“}S_n = x\text{”}] \\ &= \int_0^x \mathrm{d}y \sum_{j=1}^{n-1} \mathbf{P}[T_k = j, \text{“}H_k = y\text{”}]\mathbf{P}[\text{“}S_{n-j} = x - y\text{”}]\,. \end{aligned} \tag{4.3.13}$$

For each fixed y, the integrand is exactly of the form $\sum_{j=1}^{n-1} d_j c_{n-j}$, and we have already observed that $c_n = \mathbf{P}[\text{“}S_n = x - y\text{”}]$ satisfies the hypothesis of Iglehart's lemma : since $\sum_{j \geqslant 1} d_j = \mathbf{P}[\text{“}H_k = y\text{”}]$, we can thus conclude that the asymptotic behavior of the integrand is given by $\mathbf{P}[\text{“}H_k = y\text{”}]\mathbf{P}[\text{“}S_n = x\text{”}]$. Now, in order to perform the integration over y to obtain (4.3.12), it only remains to find a domination for the sequence of integrands, in order to apply dominated convergence.

Exhibiting such a domination is not hard (indeed, we have already found it in the proof of Proposition 4.3.1). However, in our proof we will proceed in a slightly different way : we will start directly with the finer result provided by Lemma 4.3.5, by which we will be able to prove not only (4.3.12), but also the stronger result by which the convergence is uniform on compact sets ; furthermore, in the proof we will carry out some estimates that will be useful later.

Proof of Proposition 4.3.6. We give a name to the integrand in (4.3.13) :

$$a_n^{[k,y]}(z) := \sum_{j=1}^{n-1} \mathbf{P}[T_k = j, \text{“}H_k = y\text{”}]\mathbf{P}[\text{“}S_{n-j} = z\text{”}] = \sum_{j=1}^{n-1} d_j^{[k,y]} c_{n-j}(z)\,, \tag{4.3.14}$$

where $d_j^{[k,y]} := \mathbf{P}[T_k = j, \text{“}H_k = y\text{”}]$ and $c_n(z) := \mathbf{P}[\text{“}S_{n-j} = z\text{”}]$. Here $k \geqslant 1$ and $y \in \mathbb{R}^+$, while z belongs to the compact interval $[0, x_o]$, where x_o is positive. Observe that $\sum_{j \geqslant 1} d_j^{[k,y]} = \mathbf{P}[\text{“}H_k = y\text{”}]$.

Since it has already been shown that $c_n(z)$ satisfies the hypothesis of Lemma 4.3.5, we can apply relation (4.3.11), specialized to the present setting : given $\delta \in (0,1/2)$, there exists $\bar{n}_\delta$ such that for $n \geqslant \bar{n}_\delta, z \in [0,x_o], y > 0$ and $k \geqslant 1$ one has

$$\left|\sigma\sqrt{2\pi}\sqrt{n}\, a_n^{[k,y]}(z) - \mathbf{P}[\text{“}H_k = y\text{”}]\right| \leqslant \xi(\delta)\left\{\mathbf{P}[\text{“}H_k = y\text{”}] + \sup_{m \geqslant n/2}(md_m^{[k,y]})\right\} + \frac{Q}{\sqrt{\delta}}\mathbf{P}[T_k > n\delta, \text{“}H_k = y\text{”}]\,, \tag{4.3.15}$$

where $\xi(\delta) = o(1)$ as $\delta \to 0$, and Q is a constant. To estimate the second term in brackets, we use equation (4.2.6) combined with Proposition 4.3.1 :

$$\begin{aligned} md_m^{[k,y]} &= m\mathbf{P}[T_k = m, \text{“}H_k = y\text{”}] = k\mathbf{P}[H_{k-1} < y \leqslant H_k, \text{“}S_m = y\text{”}] \\ &\leqslant \frac{\mathcal{C}k^2}{\sqrt{m}} \min\left\{\mathbf{P}[H_{k-2} < y], \mathbf{P}[H_k \geqslant y]\right\} \leqslant \mathcal{C}k^2\mathbf{P}[H_{k-2} < y] \end{aligned} \tag{4.3.16}$$

(we use the convention $\mathbf{P}[H_{k-2} < y] = 1$ for $k = 1$).

Since the right hand side of (4.3.15) does not depend on z, we might set $z = x-y$ in the left hand side; note that for $y \leqslant x$, $a_n^{[k,y]}(x-y) = \mathbf{P}[\text{“}H_k = y\text{”}, \text{“}S_n = x\text{”}]$. From this consideration and from (4.3.16), we can write (4.3.15) in a more convenient way :

$$\begin{aligned} &\left|\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[\text{“}H_k = y\text{”}, \text{“}S_n = x\text{”}] - \mathbf{P}[\text{“}H_k = y\text{”}]\right| \\ &\leqslant \xi(\delta)\left\{\mathbf{P}[\text{“}H_k = y\text{”}] + \mathcal{C}k^2\mathbf{P}[H_{k-2} < y]\right\} + \frac{Q}{\sqrt{\delta}}\mathbf{P}[T_k > n\delta, \text{“}H_k = y\text{”}]\,, \end{aligned} \tag{4.3.17}$$

which is valid for every $n \geqslant \bar{n}_\delta$, $x \in [0,x_o]$, $y \in [0,x]$ and $k \geqslant 1$.By an integration over $y \in [0,x]$, we get an explicit estimate for the tail :

$$\begin{aligned} &\left|\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[H_k < x, \text{“}S_n = x\text{”}] - \mathbf{P}[H_k < x]\right| \\ &\leqslant \xi(\delta)\left\{\mathbf{P}[H_k < x_o] + \mathcal{C}k^2 x_o\mathbf{P}[H_{k-2} < x_o]\right\} + \frac{Q}{\sqrt{\delta}}\mathbf{P}[T_k > n\delta, H_k < x_o]\,, \end{aligned} \tag{4.3.18}$$

which is valid for $\delta \in (0,1/2)$, $n \geqslant \bar{n}_\delta$, $k \geqslant 1$ and $x \in [0,x_o]$.

It remains to show that, for fixed k, this can be made arbitrarily small, uniformly in $x \in [0,x_o]$. Notice that the variable x does not appear in the right hand side, and that the term in brackets is simply a constant $\mathcal{M}$. Furthermore, for fixed δ the last term vanishes as $n \to \infty$, because $\mathbf{P}[T_k > n\delta] \to 0$.

Now, for any $\varepsilon > 0$, we choose a $\delta_\varepsilon \in (0,1/2)$ such that $\xi(\delta_\varepsilon) \cdot \mathcal{M} < \varepsilon/2$; then choose an $\widehat{n}_{\delta_\varepsilon}$ such that for $n \geqslant \widehat{n}_{\delta_\varepsilon}$ one has $\mathbf{P}[T_k > n\delta_\varepsilon, H_k < x_o] < \sqrt{\delta_\varepsilon}\,\varepsilon/(2Q)$.

Then for every $n \geqslant \bar{n}_{\delta_\varepsilon} \vee \widehat{n}_{\delta_\varepsilon}$, we can apply (4.3.18), from which we get that for every $x \in [0, x_o]$

$$\left|\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[H_k < x, \text{“}S_n = x\text{”}] - \mathbf{P}[H_k < x]\right| < \varepsilon\,,$$

so that the proof is concluded.

□

Proof of Theorem 4.3.2

Since obviously

$$\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] = \mathbf{P}[H_{k-1} < x, \text{“}S_n = x\text{”}] - \mathbf{P}[H_k < x, \text{“}S_n = x\text{”}]\,,$$

for any fixed k, Proposition 4.3.6 guarantees the convergence of the sequence of functions $\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}]$ towards $\mathbf{P}[H_{k-1} < x] - \mathbf{P}[H_k < x] = \mathbf{P}[H_{k-1} < x \leqslant H_k]$, uniformly on every compact set. To show that the same convergence holds uniformly on the whole real line, we have to make use of Proposition 4.3.1, which gives an appropriate bound to the sequence :

$$\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] \leqslant \sigma\sqrt{2\pi}\mathcal{C}\,k\mathbf{P}[H_k \geqslant x]\,.$$

For any $\varepsilon > 0$ choose an x_ε such that $\mathbf{P}[H_k \geqslant x_\varepsilon] < \min\{\varepsilon/2,\, \varepsilon/(2\mathcal{C}k\sigma\sqrt{2\pi})\}$: by the triangle inequality and the preceding bound, we have that for every $x > x_\varepsilon$

$$\left|\sigma\sqrt{2\pi}\sqrt{n}\,\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] - \mathbf{P}[H_{k-1} < x \leqslant H_k]\right| < \varepsilon\,;$$

since we have convergence on compact sets, we can now choose an $\widetilde{n}$ such that, for every $n \geqslant \widetilde{n}$ and $x \in [0, x_\varepsilon]$, the left hand side of the preceding relation is smaller than ε, and thus the proof is completed. □

4.4 Asymptotics for the ladder variables and the renewal density

The asymptotics of Theorem 4.1.2 will now be simple consequences of the results of the preceding section. We first give the asymptotic for the ladder variables : the following is a generalization of the continuous analogue of Proposition 6 of [AD99].

Theorem 4.4.1. *Assume that X is given by a bounded density, with zero mean and finite variance σ^2. Then :*

(i) for fixed k and for $x > 0$, the following asymptotic relation holds :

$$\mathbf{P}[T_k = n, \text{“}H_k = x\text{”}] \sim \frac{k}{\sigma\sqrt{2\pi}} \mathbf{P}[H_{k-1} < x \leqslant H_k] \, \frac{1}{n^{3/2}} \qquad \text{as } n \to \infty\,,$$

and the convergence of $\sigma\sqrt{2\pi}n^{3/2}\mathbf{P}[T_k = n, \text{“}H_k = x\text{”}]$ towards $k\mathbf{P}[H_{k-1} < x \leqslant H_k]$ is uniform for $x \in \mathbb{R}$;

(ii) one has the following domination : $\sigma\sqrt{2\pi}n^{3/2}\mathbf{P}[T_k = n, \text{“}H_k = x\text{”}] \leqslant \mathcal{C}k^2\mathbf{P}[H_k \geqslant x]$, for every n, k and x (where $\mathcal{C}$ is a constant). It's then possible to integrate the asymptotic relation in (i), getting

$$\mathbf{P}[T_k = n] \sim \frac{k\mathbf{E}[H_1]}{\sigma\sqrt{2\pi}} \, \frac{1}{n^{3/2}} \qquad \text{as } n \to \infty\,.$$

Proof. We essentially just have to apply identity (4.2.6), so that

$$\sigma\sqrt{2\pi}n^{3/2}\mathbf{P}[T_k = n, \text{“}H_k = x\text{”}] = k \cdot \sigma\sqrt{2\pi}\sqrt{n}\, \mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}]\,;$$

from this relation it is clear that the first part of the statement is an immediate consequence of Theorem 4.3.2. Similarly, the domination part (ii) is a simple consequence of Proposition 4.3.1 ; the asymptotic relation then easily follows from dominated convergence, noting that

$$\begin{aligned} \int_0^\infty \mathrm{d}x\, \mathbf{P}[H_{k-1} < x \leqslant H_k] &= \int_0^\infty \mathrm{d}x\, \left(\mathbf{P}[H_k \geqslant x] - \mathbf{P}[H_{k-1} \geqslant x]\right) \\ &= \mathbf{E}[H_k] - \mathbf{E}[H_{k-1}] = k\mathbf{E}[H_1] - (k-1)\mathbf{E}[H_1] = \mathbf{E}[H_1]\,, \end{aligned}$$

and the proof is completed. □

We explicitly rewrite the result for the $k = 1$ case, which is the most interesting :

$$\mathbf{P}[T_1 = n, \text{“}H_1 = x\text{”}] \sim \frac{1}{\sigma\sqrt{2\pi}} \mathbf{P}[H_1 \geqslant x] \, \frac{1}{n^{3/2}} \qquad \text{as } n \to \infty\,, \tag{4.4.1}$$

uniformly for $x \in \mathbb{R}$ in the sense specified in Theorem 4.4.1, and the rescaled sequence $n^{3/2}\mathbf{P}[T_1 = n, \text{“}H_1 = x\text{”}]$ is dominated by a multiple of its limit $\mathbf{P}[H_1 \geqslant x]$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Finally we show an asymptotic equivalence on $u(\cdot, \cdot)$. Recall that

$$u(n, x) = \sum_{k \geqslant 1} \mathbf{P}[T_k = n, \text{“}H_k = x\text{”}]\,, \tag{4.4.2}$$

which by the Duality Lemma is the same as

$$u(n, x) = \mathbf{P}[S_1 > 0, \ldots, S_n > 0, \ \text{“}S_n = x\text{”}]\,.$$

Note that this quantity has also been studied in [AD99] for the lattice case, but their proof cannot easily be adapted to the continuous setting, since it exploited a property of the increasing ladder heights which is peculiar of the lattice case. The proof given here uses Theorem 4.3.2 and an appropriate bound from renewal theory ; however, the result is weaker than in [AD99] (cf. Theorem 7 and equation (35) therein) since the equivalence in Theorem 4.4.2 holds uniformly only for x in a compact set.

Theorem 4.4.2. *Assume that X is given by a bounded density, with zero mean and finite variance σ^2. Let $u(n,x)$, $n \geqslant 1$ be the density of the renewal measure associated to the ascending ladder process of the random walk generated by X. Then for any fixed $x > 0$ the following asymptotic relation holds :*

$$u(n,x) \sim U(x) \frac{1}{\sigma\sqrt{2\pi} n^{3/2}} \qquad \text{as } n \to \infty\,,$$

and the convergence of $\sigma\sqrt{2\pi}n^{3/2}u(n,x)$ towards $U(x)$ is uniform for x in a compact set.

Proof. Recall that the number of ladder heights in the interval $[0,x]$ is given by the random variable

$$\mathcal{N}(x) = \sum_{k \geqslant 0} \mathbf{1}_{\{H_k \leqslant x\}} = \sum_{k \geqslant 1} k\, \mathbf{1}_{\{H_{k-1} < x \leqslant H_k\}}\,, \tag{4.4.3}$$

and we have the equality

$$U(x) = \mathbf{E}[\mathcal{N}(x)] = \sum_{k \geqslant 1} k\, \mathbf{P}[H_{k-1} < x \leqslant H_k]\,. \tag{4.4.4}$$

Now, directly by the definition (4.4.2) we can write

$$\left(\sigma\sqrt{2\pi}n^{3/2}\right)\, u(n,x) = \sum_{k \geqslant 1} \left(\sigma\sqrt{2\pi}n^{3/2}\right)\, \mathbf{P}[T_k = n, \text{“}H_k = x\text{”}]\,,$$

thus it is clear that the result will follow combining Theorem 4.4.1 and equation (4.4.4).
More precisely, from relation (4.2.6) one deduces :

$$\begin{aligned}
&\left|\sigma\sqrt{2\pi}n^{3/2}u(n,x) - U(x)\right| \\
&\leqslant \sum_{k \geqslant 1} \left|\sigma\sqrt{2\pi}n^{3/2}\mathbf{P}[T_k = n, \text{“}H_k = x\text{”}] - k\,\mathbf{P}[H_{k-1} < x \leqslant H_k]\right| \\
&= \sum_{k \geqslant 1} k \left|\sigma\sqrt{2\pi}\sqrt{n}\mathbf{P}[H_{k-1} < x \leqslant H_k, \text{“}S_n = x\text{”}] - \mathbf{P}[H_{k-1} < x \leqslant H_k]\right|\,,
\end{aligned}$$

and since $[H_{k-1} < x \leqslant H_k] = [H_{k-1} < x] \setminus [H_k < x]$ we get

$$\begin{aligned}
&\left|\sigma\sqrt{2\pi}n^{3/2}u(n,x) - U(x)\right| \\
&\leqslant \sum_{k\,\geqslant\,1} k \left|\sigma\sqrt{2\pi}\sqrt{n}\mathbf{P}[H_{k-1} < x, \text{“}S_n = x\text{”}] - \mathbf{P}[H_{k-1} < x]\right| \\
&\qquad\qquad + \sum_{k\,\geqslant\,1} k \left|\sigma\sqrt{2\pi}\sqrt{n}\mathbf{P}[H_k < x, \text{“}S_n = x\text{”}] - \mathbf{P}[H_k < x]\right| .
\end{aligned}$$

We show how to treat the second term above, the second can obviously be treated similarly ; by relation (4.3.18), for any fixed positive x_o and for every $\delta \in (0, 1/2)$, there exists $\bar{n}_\delta$ such that for every $n \geqslant \bar{n}_\delta$ and $x \in [0, x_o]$, one has :

$$\begin{aligned}
&\sum_{k\,\geqslant\,1} k \left|\sigma\sqrt{2\pi}\sqrt{n}\mathbf{P}[H_k < x, \text{“}S_n = x\text{”}] - \mathbf{P}[H_k < x]\right| \leqslant \qquad (4.4.5) \\
&\leqslant \xi(\delta) \sum_{k\,\geqslant\,1} k \Big\{\mathbf{P}[H_k < x_o] + \mathcal{C}k^2 x_o \mathbf{P}[H_{k-2} < x_o]\Big\} + \frac{Q}{\sqrt{\delta}} \sum_{k\,\geqslant\,1} \mathbf{P}[T_k > n\delta, H_k < x_o] ..
\end{aligned}$$

Since the next lemma from renewal theory guarantees that $\mathbf{P}[H_k < x_o] = \mathbf{P}[\mathcal{N}(x_o) > k]$ has an exponential decay in k, the first series in the right hand side above converges ; from the fact that $\xi = o(1)$ as $\delta \to 0$, we can make the first term in the right hand side as small as we like, and for $\varepsilon > 0$ we can choose a δ_ε to make it $< \varepsilon/2$.

Now notice that for each fixed k the summand in the last series $\mathbf{P}[T_k > n\delta_\varepsilon, H_k < x_o] \leqslant \mathbf{P}[T_k > n\delta_\varepsilon] \to 0$ as $n \to \infty$; moreover, for every n, $\mathbf{P}[T_k > n\delta_\varepsilon, H_k < x_o] \leqslant \mathbf{P}[H_k < x_o]$ which is a summable sequence. By dominated convergence we can make the second term in the right hand side of (4.4.5) above $< \varepsilon/2$ by choosing a large enough n, and the proof is completed. □

We are left with proving the following easy lemma :

Lemma 4.4.3. *Let $\{H_n\}_{n\,\geqslant\,0}$ be a renewal process, that is $H_0 = 0$ and $h_n := H_n - H_{n-1}$ are positive i.i.d. random variables ; let $\mathcal{N}(x)$ be the number of renewal points in the interval $[0, x]$, defined by (4.4.3). Then, for any fixed x, the tail $\mathbb{P}[\mathcal{N}(x) > l]$ has an exponential decay in l. In particular, the random variable $\mathcal{N}(x)$ has finite moments of all orders.*

Proof. Let F be the distribution function of H_1 (by the positivity assumption, $F(0) = 0$). The distribution function of H_n is then $F_n := F^{*n}$. For any non negative z, we have

$$F_2(z) = \int_{\mathbb{R}} F(z-y)F(\,\mathrm{d}y) = \int_0^z F(z-y)F(\,\mathrm{d}y) \leqslant F(z)\int_0^z F(\,\mathrm{d}y) = F(z)^2 ,$$

and by a trivial recursion one deduces that $F_n(z) \leqslant F(z)^n$ for every n and for every non negative z.

If $F(x) < 1$ we then have

$$\mathbb{P}[\mathcal{N}(x) > l] = \mathbb{P}[H_l \leqslant x] = F_l(x) \leqslant (F(x))^l$$

and thus the exponential decay is proved. Otherwise, one can find an integer k_o such that $F_{k_o}(x) < 1$, and in this case we have the exponential decay of the subsequence $\mathbb{P}[\mathcal{N}(x) > k_o l]$, since

$$\mathbb{P}[\mathcal{N}(x) > k_o l] = \mathbb{P}[H_{k_o l} \leqslant x] = F_{k_o l}(x) \leqslant (F_{k_o}(x))^l \,.$$

The exponential decay for $\mathbb{P}[\mathcal{N}(x) > l]$ then follows from the fact that this sequence is decreasing in l. □

4.5 Asymptotics for the first entrance in the positive half-line

We can finally prove our main result.

4.5.1 A last convolution result

As in the case of Proposition 4.3.6, the key idea is quite simple, and so is the proof of the pointwise convergence. The proof of the uniform convergence is more involved, and estimating the tails requires some technical efforts. To deal with them, we will need the following lemma which is much in the spirit of Lemma 4.3.5. Its proof is given in Appendix 4.6.

Lemma 4.5.1. *Let* $\{e_j(z)\}_{j \geqslant 1}, \{d_j\}_{j \geqslant 1}$ *be two non negative sequences, the first one depending on a parameter* z *which belongs to a set* $\mathcal{S}$. *Define a new sequence* $a_n(z) := \sum_{j=1}^{n/2} d_j e_{n-j}(z)$. *If the following hypothesis is satisfied :*

- $e_n(z) \sim e(z)/n^\alpha$ *as* $n \to \infty$, *with* $e(z) > 0,\ \alpha > 1$, *uniformly for* $z \in \mathcal{S}$ *in the sense that*

$$\lim_{n\to\infty} \sup_{z\in\mathcal{S}} |n^\alpha e_n(z) - e(z)| = 0 \,, \tag{4.5.1}$$

then for every $\delta \in (0, 1/2)$ *there exists an* $\bar{n}_\delta$ *such that for every* $n \geqslant \bar{n}_\delta, z \in \mathcal{S}$, *the following holds :*

$$\left| n^\alpha a_n(z) - e(z) \sum_{j=1}^{\infty} d_j \right| \leqslant \mathcal{C}\,\delta\,(1 + e(z)) \sum_{j=1}^{\infty} d_j + \frac{\mathcal{C}}{\delta^\alpha} e(z) \sum_{j \geqslant \delta n} d_j \,, \tag{4.5.2}$$

where $\mathcal{C}$ *is a positive constant.*

Remark. We will use this lemma in the following situation : let e_n, d_n be non negative sequences, and let a_n their convolution, that is

$$a_n := \sum_{j=1}^{n-1} e_{n-j} d_j ;$$

if the following hypothesis are satisfied :
- $e_n \sim e/n^{3/2}$ as $n \to \infty$, with $e > 0$;
- $d_n \sim d/n^{3/2}$ as $n \to \infty$, with $d > 0$;

then, defining $E := \sum_{j=1}^{\infty} e_j$ and $D := \sum_{j=1}^{\infty} d_j$, the sequence a_n has the following asymptotic : $a_n \sim (De_n + Ed_n) = (De + Ed)/n^{3/2}$; moreover, Lemma 4.5.1 enables us to give an estimate on the difference $|n^{3/2}a_n - (De+Ed)|$. This estimate will turn out to be the key in the proof of Theorem 4.1.2.

4.5.2 Proof of Theorem 4.1.2

We can write :

$$\begin{aligned}
&\mathbf{P}_{-a}[S_1 \leqslant 0, \dots, S_{n-1} \leqslant 0, S_n > 0, \text{“}S_n = x\text{”}] \\
&= \mathbf{P}[S_1 \leqslant a, \dots, S_{n-1} \leqslant a, S_n > a, \text{“}S_n = a + x\text{”}] \\
&= \mathbf{P}[n \text{ is a ladder epoch,“}S_n = a + x\text{”, the preceding ladder height is} \leqslant a] \,.
\end{aligned} \tag{4.5.3}$$

Now we condition on the possible ladder epoch and heights, using the Markov property ; recalling the definition (4.2.1) of the renewal measure $\mathcal{U}$, we get

$$\begin{aligned}
&\mathbf{P}_{-a}[S_1 \leqslant 0, \dots, S_{n-1} \leqslant 0, S_n > 0, \text{“}S_n = x\text{”}] = \sum_{i=1}^{n} \mathbf{P}[T_i = n, \text{“}H_i = a + x\text{”}, H_{i-1} \leqslant a] \\
&= \sum_{i=1}^{n} \sum_{m=0}^{n-1} \int_0^a \mathbf{P}[T_{i-1} = m, H_{i-1} \in \mathrm{d}y] \, \mathbf{P}[T_1 = n - m, \text{“}H_1 = a + x - y\text{”}] \\
&= \sum_{m=0}^{n-1} \int_0^a \left(\sum_{i=1}^{\infty} \mathbf{P}[T_{i-1} = m, H_{i-1} \in \mathrm{d}y] \right) \mathbf{P}[T_1 = n - m, \text{“}H_1 = a + x - y\text{”}] \\
&= \sum_{m=0}^{n-1} \int_0^a \mathcal{U}(m, \mathrm{d}y) \mathbf{P}[T_1 = n - m, \text{“}H_1 = a + x - y\text{”}] \,.
\end{aligned} \tag{4.5.4}$$

The domination claimed in part (ii) easily follows from the above relation : using the domination result already proved in the second part of Theorem 4.4.1, that in our case reads as $\mathbf{P}[T_1 = n, \text{“}H_1 = z\text{”}] \leqslant \mathcal{C}\mathbf{P}[H_1 \geqslant z]$ for every $n \in \mathbb{N}$ and $z > 0$, we get

$$\begin{aligned}
&\sum_{m=0}^{n-1}\int_0^a \mathcal{U}(m,\,\mathrm{d}y)\mathbf{P}[T_1=n-m,\,\text{“}H_1=a+x-y\text{”}]\\
&\leqslant \mathcal{C}\int_0^a\sum_{m=0}^{n-1}\mathcal{U}(m,\,\mathrm{d}y)\mathbf{P}[H_1\geqslant a+x-y]\leqslant \mathcal{C}\,\mathbf{P}[H_1\geqslant x]\int_0^a\sum_{m=0}^{n-1}\mathcal{U}(m,\,\mathrm{d}y)\\
&\leqslant \mathcal{C}\,\mathbf{P}[H_1\geqslant x]\int_0^a\sum_{m=0}^{\infty}\mathcal{U}(m,\,\mathrm{d}y) \;=\; \mathcal{C}\,\mathbf{P}[H_1\geqslant x]U(a)\,,
\end{aligned}$$

which is exactly the desired result.

Now we turn to the convergence. Recalling that $\mathcal{U}(0,\,\cdot\,)=\delta_0(\,\cdot\,)$, we use equation (4.5.4) to get

$$\begin{aligned}
&\mathbf{P}_{-a}[S_1\leqslant 0,\dots,S_{n-1}\leqslant 0, S_n>0,\,\text{“}S_n=x\text{”}] \qquad (4.5.5)\\
&=\mathbf{P}[T_1=n,\,\text{“}H_1=a+x\text{”}]+\int_0^a \mathrm{d}y\left(\sum_{m=1}^{n-1}u(m,y)\,\mathbf{P}[T_1=n-m,\,\text{“}H_1=a+x-y\text{”}]\right)\,.
\end{aligned}$$

Now notice that Theorem 4.4.1 and Theorem 4.4.2 give the exact asymptotics of the two sequences appearing in the sum inside the integral above ; in particular, since they are both of order $n^{-3/2}$ as $n\to\infty$, for almost every fixed y, we can make use of the Remark after Lemma 4.5.1 to get the asymptotic behavior of the sum :

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sigma\sqrt{2\pi}n^{3/2}\sum_{m=1}^{n-1}u(m,y)\,\mathbf{P}[T_1=n-m,\,\text{“}H_1=a+x-y\text{”}]\right)\\
&=\left(\sum_{n\,\geqslant\,0}u(n,y)\right)\mathbf{P}[H_1\geqslant a+x-y]+\mathbf{P}[\text{“}H_1=a+x-y\text{”}]U(y)\,.
\end{aligned} \qquad (4.5.6)$$

As usual, it is enough to find a suitable domination to deduce the convergence of the integrals, but using Lemma (4.5.1), we will directly prove the stronger result of the uniform convergence. However, we first want to check that the result we get is really (4.1.6), so assume for the moment that the convergence of the integrals has been proved. From equations (4.5.5) and (4.5.6) and using again Theorem 4.4.1, we want to compute

$$\mathbf{P}[H_1\geqslant a+x]+\int_0^a \mathrm{d}y\left(\left(\sum_{n\,\geqslant\,0}u(n,y)\right)\mathbf{P}[H_1\geqslant a+x-y]+\mathbf{P}[\text{“}H_1=a+x-y\text{”}]U(y)\right)\,. \qquad (4.5.7)$$

We first carry out the integration for the second term in the integral : from relation (4.2.3), we deduce :

$$\begin{aligned}
&\int_0^a \mathrm{d}y\, \mathbf{P}[\text{“}H_1 = a+x-y\text{”}]U(y) \\
&= \int_0^a \mathrm{d}y\, \mathbf{P}[\text{“}H_1 = a+x-y\text{”}] + \int_0^a \mathrm{d}y\, \mathbf{P}[\text{“}H_1 = a+x-y\text{”}] \int_0^y \mathrm{d}z\, \left(\sum_{n\geqslant 0} u(n,z)\right) \\
&= \int_0^a \mathrm{d}y\, \mathbf{P}[\text{“}H_1 = a+x-y\text{”}] + \int_0^a \mathrm{d}z\, \left(\sum_{n\geqslant 0} u(n,z)\right) \int_z^a \mathrm{d}y\, \mathbf{P}[\text{“}H_1 = a+x-y\text{”}] \\
&= U(a)\mathbf{P}[H_1 \geqslant x] - \mathbf{P}[H_1 \geqslant a+x] - \int_0^a \mathrm{d}z\, \left(\sum_{n\geqslant 0} u(n,z)\right) \mathbf{P}[H_1 \geqslant a+x-z]\,,
\end{aligned}$$

and a substitution in equation (4.5.7) shows that

$$\begin{aligned}
&\mathbf{P}[H_1 \geqslant a+x] + \int_0^a \mathrm{d}y\, \left(\left(\sum_{n\geqslant 0} u(n,y)\right)\mathbf{P}[H_1 \geqslant a+x-y] + \mathbf{P}[\text{“}H_1 = a+x-y\text{”}]U(y)\right) \\
&= U(a)\mathbf{P}[H_1 \geqslant x]\,,
\end{aligned} \tag{4.5.8}$$

which is exactly (4.1.6).

We are left with the proof of the uniform convergence. We recall that the uniform convergence of $\sigma\sqrt{2\pi}n^{3/2}\mathbf{P}[T_1 = n, \text{“}H_1 = a+x\text{”}]$ for $x \in \mathbb{R}$ has already been proved in Theorem 4.4.1, so that we can focus on the integral in equation (4.5.4). Note that we just have to show that the uniform convergence holds for x in a compact set $[0, x_o]$: to extend the result to the whole real line, one can proceed as in Theorem 4.3.2, since the sequence of the integrals is dominated by a multiple of $\mathbf{P}[H_1 \geqslant x]$.

We first estimate the tails in the limit (4.5.6), with the help of Lemma 4.5.1 ; for conciseness, we write $a_n := u(n,y)$, $b_n := \mathbf{P}[T_1 = n, \text{“}H_1 = a+x-y\text{”}]$, and accordingly we define a, b such that $a_n \sim a/(\sigma\sqrt{2\pi}n^{3/2})$, $b_n \sim b/(\sigma\sqrt{2\pi}n^{3/2})$. Observe that the following holds :

$$\begin{aligned}
&\left|\sigma\sqrt{2\pi}n^{3/2}\sum_{j=1}^{n-1} a_j b_{n-j} - b\sum_{j=1}^{\infty} a_j - a\sum_{j=1}^{\infty} b_j\right| \\
&\leqslant \left|\sigma\sqrt{2\pi}n^{3/2}\sum_{j=1}^{n/2} a_j b_{n-j} - b\sum_{j=1}^{\infty} a_j\right| + \left|\sigma\sqrt{2\pi}n^{3/2}\sum_{j=1}^{n/2} b_j a_{n-j} - a\sum_{j=1}^{\infty} b_j\right|
\end{aligned} \tag{4.5.9}$$

and to each of these two terms we can apply Lemma 4.5.1. Indeed, hypothesis (4.5.1) is satisfied by both sequences, as it has been proved in the preceding section. We apply 4.5.2 to the first term : given $\delta \in (0,1/2)$, there exists $\bar{n_\delta}$ such that for every $n \geqslant \bar{n}_\delta, y \in [0,a]$ and $x \in [0,x_o]$, one has :

$$\begin{aligned}
&\left|\sigma\sqrt{2\pi}n^{3/2}\sum_{m=1}^{n/2} u(m,y)\,\mathbf{P}[T_1 = n-m,\text{“}H_1 = a+x-y\text{”}] - \left(\sum_{n\,\geqslant\,0} u(n,y)\right)\mathbf{P}[H_1 \geqslant a+x-y]\right|\\
&\leqslant \mathcal{C}\,\delta\,(1+\mathbf{P}[H_1 \geqslant a+x-y])\left(\sum_{n\,\geqslant\,0} u(n,y)\right) + \frac{\mathcal{C}}{\delta^{3/2}}\mathbf{P}[H_1 \geqslant a+x-y]\sum_{j\,\geqslant\,\delta n} u(j,y)\\
&\leqslant 2\mathcal{C}\,\delta\left(\sum_{n\,\geqslant\,0} u(n,y)\right) + \frac{\mathcal{C}}{\delta^{3/2}}\sum_{j\,\geqslant\,\delta n} u(j,y)\,, \qquad (4.5.10)
\end{aligned}$$

and analogously for the second term

$$\begin{aligned}
&\left|\sigma\sqrt{2\pi}n^{3/2}\sum_{m=n/2}^{n-1} u(m,y)\,\mathbf{P}[T_1 = n-m,\text{“}H_1 = a+x-y\text{”}] - \mathbf{P}[\text{“}H_1 = a+x-y\text{”}]U(y)\right|\\
&\leqslant \mathcal{C}\,\delta\,(1+U(y))\mathbf{P}[\text{“}H_1 = a+x-y\text{”}] + \frac{\mathcal{C}}{\delta^{3/2}}U(y)\sum_{j\,\geqslant\,\delta n}\mathbf{P}[T_1 = j,\text{“}H_1 = a+x-y\text{”}]\\
&\leqslant (1+U(a))\mathcal{C}\,\delta\,\mathbf{P}[\text{“}H_1 = a+x-y\text{”}] + \frac{\mathcal{C}U(a)}{\delta^{3/2}}\sum_{j\,\geqslant\,\delta n}\mathbf{P}[T_1 = j,\text{“}H_1 = a+x-y\text{”}]
\end{aligned}$$

Integrating over $y \in [0,a]$, we finally get

$$\begin{aligned}
&\Bigg|\sigma\sqrt{2\pi}n^{3/2}\int_0^a \mathrm{d}y\left(\sum_{m=1}^{n-1} u(m,y)\,\mathbf{P}[T_1 = n-m,\text{“}H_1 = a+x-y\text{”}]\right) -\\
&\qquad - \int_0^a \mathrm{d}y\left(\sum_{n\,\geqslant\,0} u(n,y)\mathbf{P}[H_1 \geqslant a+x-y] + \mathbf{P}[\text{“}H_1 = a+x-y\text{”}]U(y)\right)\Bigg|\\
&\leqslant \int_0^a \mathrm{d}y\Bigg\{2\mathcal{C}\,\delta\left(\sum_{n\,\geqslant\,0} u(n,y)\right) + \frac{\mathcal{C}}{\delta^{3/2}}\sum_{j\,\geqslant\,\delta n} u(j,y) + (1+U(a))\mathcal{C}\,\delta\,\mathbf{P}[\text{“}H_1 = a+x-y\text{”}] +\\
&\qquad + \frac{\mathcal{C}U(a)}{\delta^{3/2}}\sum_{j\,\geqslant\,\delta n}\mathbf{P}[T_1 = j,\text{“}H_1 = a+x-y\text{”}]\Bigg\} \qquad (4.5.11)
\end{aligned}$$

$$
\begin{aligned}
&= \delta \left\{ 2\mathcal{C}(U(a)-1) + \mathcal{C}(1+U(a))\mathbf{P}[H_1 \leqslant x] \right\} \\
&\qquad\qquad + \frac{\mathcal{C}}{\delta^{3/2}} \left\{ \sum_{j \,\geqslant\, \delta n} \mathcal{U}\big(j,(0,a]\big) + U(a) \sum_{j \,\geqslant\, \delta n} \mathbf{P}[T_1 = j, H_1 \leqslant x] \right\} \\
&\leqslant \delta \left\{ 2\mathcal{C}(U(a)-1) + \mathcal{C}(1+U(a))\mathbf{P}[H_1 \leqslant x_o] \right\} \\
&\qquad\qquad + \frac{\mathcal{C}}{\delta^{3/2}} \left\{ \sum_{j \,\geqslant\, \delta n} \mathcal{U}\big(j,(0,a]\big) + U(a) \sum_{j \,\geqslant\, \delta n} \mathbf{P}[T_1 = j, H_1 \leqslant x_o] \right\},
\end{aligned}
$$

and we recall that this holds for every $\delta \in (0,1/2), n \geqslant \bar{n}_\delta$ and $x \in [0, x_o]$.

We consider the last line of the above relation ; choosing δ small enough, for fixed x_0, the first term can be made small (say smaller than $\varepsilon/2$). Observe then that for fixed δ the terms in the second brackets vanish as $n \to \infty$: in fact both series are convergent (their sums are respectively $U((0,a])$ and $\mathbf{P}[H_1 \leqslant x_o]$) and consequently their tails can be made arbitrarily small by choosing n large enough, so that the proof is completed. □

4.6 Proof of Lemma 4.5.1

This section is devoted to the proof of Lemma 4.5.1. Evidently we may assume that $\sum_{j\geqslant 1} d_j < \infty$. The difference in (4.5.2) can be splitted :

$$\left| n^\alpha \sum_{j=1}^{n/2} d_j e_{n-j}(z) - e(z)\sum_{j=1}^{\infty} d_j \right| \leqslant \sum_{j=1}^{n/2} d_j \left| n^\alpha e_{n-j}(z) - e(z)\right| + e(z)\sum_{j=n/2}^{\infty} d_j \,, \quad (4.6.1)$$

We first treat the first term in the right hand side of equation (4.6.1). Using the triangle inequality, we get :

$$\sum_{j=1}^{n/2} d_j |n^\alpha e_{n-j}(z) - e(z)| = \sum_{j=1}^{n/2} d_j \frac{n^\alpha}{(n-j)^\alpha}\left|(n-j)^\alpha e_{n-j}(z) - \frac{(n-j)^\alpha}{n^\alpha} e(z)\right|$$
$$\leqslant \sum_{j=1}^{n/2} d_j \frac{n^\alpha}{(n-j)^\alpha}\left|(n-j)^\alpha e_{n-j}(z) - e(z)\right| \;+\; e(z)\sum_{j=1}^{n/2} d_j \frac{n^\alpha}{(n-j)^\alpha}\left|1 - \frac{n^\alpha}{(n-j)^\alpha}\right| . \quad (4.6.2)$$

For any $\delta \in (0,1/2)$, by hypothesis (4.5.1) we can find $\bar{n}_\delta$ such that $|m^\alpha e_m(z) - e(z)| < \delta$ for every $m \geqslant \bar{n}_\delta/2$ and every $z \in \mathcal{S}$. In particular, if $n \geqslant \bar{n}_\delta$,

$$\sum_{j=1}^{n/2} d_j \frac{n^\alpha}{(n-j)^\alpha}\left|(n-j)^\alpha e_{n-j}(z) - e(z)\right| \leqslant 2^\alpha \delta \sum_{j=1}^{\infty} d_j \,. \quad (4.6.3)$$

On the other hand :

$$e(z)\sum_{j=1}^{n/2} d_j \frac{n^\alpha}{(n-j)^\alpha}\left|1 - \frac{n^\alpha}{(n-j)^\alpha}\right|$$
$$= e(z)\sum_{j=1}^{\delta n} d_j \frac{n^\alpha}{(n-j)^\alpha}\left|1 - \frac{n^\alpha}{(n-j)^\alpha}\right| + e(z)\sum_{j=\delta n}^{n/2} d_j \frac{n^\alpha}{(n-j)^\alpha}\left|1 - \frac{n^\alpha}{(n-j)^\alpha}\right| \quad (4.6.4)$$
$$\leqslant 2^\alpha (C\delta) e(z)\sum_{j=1}^{\infty} d_j \;+\; \frac{2^\alpha}{\delta^\alpha} e(z)\sum_{j\geqslant \delta n} d_j \,,$$

where C is a positive contant (for example such that $1-(1-\delta)^\alpha \leqslant C\delta/2$ for $0<\delta<1/2$).

Observing that the last term in (4.6.1) trivially satisfies

$$e(z)\sum_{j=n/2}^{\infty} d_j \leqslant \frac{2^\alpha}{\delta^\alpha} e(z)\sum_{j\geqslant \delta n} d_j \,,$$

we can finally put everything together, concluding that for any $\delta \in (0,1/2)$ there exists an $\bar{n}_\delta$ such that for every $n \geqslant \bar{n}_\delta$, $z \in \mathcal{S}$ and for every non negative sequence $\{d_j\}$

$$\left| n^\alpha \sum_{j=1}^{n/2} d_j e_{n-j}(z) - e(z) \sum_{j=1}^{\infty} d_j \right| \leqslant \mathcal{C}\, \delta\, (1+e(z)) \sum_{j=1}^{\infty} d_j + \frac{\mathcal{C}}{\delta^\alpha} e(z) \sum_{j \geqslant \delta n} d_j\,,$$

where $\mathcal{C}$ is a positive constant : hence the result. □

4.7 Refined Iglehart's Lemma

Now we give the proof of Lemma 4.3.5, which is quite similar to the proof of Lemma 4.5.1 given in the preceding section. Here too we may assume $\sum_{j \geqslant 1} d_j < \infty$. We split the difference appearing in the left hand side of (4.3.11) into three parts :

$$\begin{aligned} \left| n^\alpha a_n(z) - c(z) \sum_{j=1}^{\infty} d_j \right| &= \left| n^\alpha \sum_{j=1}^{n-1} c_{n-j}(z) d_j - c(z) \sum_{j=1}^{\infty} d_j \right| \\ &\leqslant \sum_{j=1}^{(1-\delta)n} d_j \left| n^\alpha c_{n-j}(z) - c(z) \right| \;+\; n^\alpha \sum_{j=(1-\delta)n}^{n-1} c_{n-j}(z) d_j \;+\; c(z) \sum_{j=(1-\delta)n}^{\infty} d_j\,, \end{aligned} \tag{4.7.1}$$

where δ is a positive number in $(0,1/2)$. We study separately the three terms appearing in the above relation.

4.7.1 First term

Using the triangle inequality, for every n, one gets :

$$\begin{aligned} \sum_{j=1}^{(1-\delta)n} d_j \left| n^\alpha c_{n-j}(z) - c(z) \right| &= \sum_{j=1}^{(1-\delta)n} d_j \frac{n^\alpha}{(n-j)^\alpha} \left| (n-j)^\alpha c_{n-j}(z) - \frac{(n-j)^\alpha}{n^\alpha} c(z) \right| \\ &\leqslant \sum_{j=1}^{(1-\delta)n} d_j \frac{n^\alpha}{(n-j)^\alpha} \left| (n-j)^\alpha c_{n-j}(z) - c(z) \right| + c(z) \sum_{j=1}^{(1-\delta)n} d_j \frac{n^\alpha}{(n-j)^\alpha} \left| 1 - \frac{n^\alpha}{(n-j)^\alpha} \right|. \end{aligned}$$

Making use of hypothesis (4.3.10), we can find an $\bar{n}_\delta$ such that $|m^\alpha e_m(z) - e(z)| < \delta$ for every $m \geqslant \bar{n}_\delta/2$ and every $z \in \mathcal{S}$, so that

$$\sum_{j=1}^{(1-\delta)n} d_j \frac{n^\alpha}{(n-j)^\alpha} \left| (n-j)^\alpha c_{n-j}(z) - c(z) \right| \leqslant 2^\alpha \delta \sum_{j=1}^{\infty} d_j\,. \tag{4.7.2}$$

On the other hand :

$$\begin{aligned}
& c(z) \sum_{j=1}^{(1-\delta)n} d_j \frac{n^\alpha}{(n-j)^\alpha} \left| 1 - \frac{n^\alpha}{(n-j)^\alpha} \right| \\
& = c(z) \sum_{j=1}^{\delta n} d_j \frac{n^\alpha}{(n-j)^\alpha} \left| 1 - \frac{n^\alpha}{(n-j)^\alpha} \right| + c(z) \sum_{j=\delta n}^{(1-\delta)n} d_j \frac{n^\alpha}{(n-j)^\alpha} \left| 1 - \frac{n^\alpha}{(n-j)^\alpha} \right| \\
& \leqslant 2^\alpha \frac{1-(1-\delta)^\alpha}{(1-\delta)^\alpha} c(z) \sum_{j=1}^{\infty} d_j \ + \ \frac{2}{\delta^\alpha} c(z) \sum_{j \,\geqslant\, \delta n} d_j \,,
\end{aligned} \tag{4.7.3}$$

4.7.2 Second term

For the second term in (4.7.1), since by hypothesis $c_n(z) \leqslant Q c(z)/n^\alpha$, we have

$$\begin{aligned}
& n^\alpha \sum_{j=(1-\delta)n}^{n-1} c_{n-j}(z) d_j \leqslant c(z) Q n^\alpha \sum_{j=(1-\delta)n}^{n-1} \frac{1}{(n-j)^\alpha} d_j = c(z) Q \sum_{j=(1-\delta)n}^{n-1} \frac{1}{(1-\frac{j}{n})^\alpha} d_j \\
& \leqslant c(z) Q \left(\sup_{\{k \,\geqslant\, (1-\delta)n\}} (k d_k) \right) \sum_{j=(1-\delta)n}^{n-1} \frac{1}{j} \frac{1}{(1-\frac{j}{n})^\alpha} \\
& \leqslant c(z) \frac{Q}{1-\delta} \left(\sup_{\{k \,\geqslant\, (1-\delta)n\}} (k d_k) \right) \left(\frac{1}{n} \sum_{j=(1-\delta)n}^{n-1} \frac{1}{(1-\frac{j}{n})^\alpha} \right) ,
\end{aligned}$$

and in the last parenthesis one recognizes Riemann sums that converge to the finite integral $\int_{1-\delta}^{1} \mathrm{d}y \, 1/(1-y)^\alpha$. Hence,there exists $\widetilde{Q} > 0$ such that :

$$n^\alpha \sum_{j=(1-\delta)n}^{n-1} c_{n-j}(z) d_j \leqslant c(z) \widetilde{Q} \left(\int_{1-\delta}^{1} \frac{1}{(1-y)^\alpha} \, \mathrm{d}y \right) \sup_{\{k \,\geqslant\, n/2\}} (k d_k) \,. \tag{4.7.4}$$

4.7.3 Third term

As $\delta \in (0, 1/2)$, the following is immediate :

$$c(z) \sum_{j=(1-\delta)n}^{\infty} d_j \leqslant \frac{2}{\delta^\alpha} c(z) \sum_{j=\delta n}^{\infty} d_j \,. \tag{4.7.5}$$

Getting the conclusion is now easy. We define $\xi(\delta)$ as being the maximum between the prefactors in equations (4.7.2), (4.7.3) and (4.7.4), that is

$$\xi(\delta) := \max \left\{ 2^\alpha \delta \,,\ 2^\alpha \frac{1-(1-\delta)^\alpha}{(1-\delta)^\alpha} \,,\ \widetilde{Q} \int_{1-\delta}^{1} \mathrm{d}y \, \frac{1}{(1-y)^\alpha} \right\} ,$$

and note that $\xi(\delta) \to 0$ as $\delta \to 0$. From relations (4.7.1), (4.7.2), (4.7.3), (4.7.4) and (4.7.5) we finally get that for $n \geqslant \bar{n}_\delta$, $k \geqslant 1$, $z \in \mathcal{S}$ and $\{d_j\}_{j \geqslant 1}$,

$$\left| n^\alpha a_n(z) - c(z) \sum_{j=1}^{\infty} d_j \right| \leqslant \xi(\delta)\, c(z) \left(\sum_{j=1}^{\infty} d_j + \sup_{\{k \geqslant n/2\}} (k d_k) \right) + \xi(\delta) \sum_{j=1}^{\infty} d_j + \frac{4}{\delta^\alpha} c(z) \sum_{j \geqslant n\delta} d_j \,,$$

which is exactly the statement of the lemma, so that the proof is completed. □

Chapitre 5

A convergence towards brownian excursion

Contents

5.1 Introduction and the main result

It is a classical result that if a random walk S is in the domain of attraction of the standard normal law with norming sequence a_n, the rescaled process $(S_{\lfloor nt \rfloor}/a_n)_{t \geqslant 0}$

converges in law towards the brownian motion (see [Bil68]). Denoting by $S^*, \mathbf{P}_x$ the random walk starting from x and conditioned to stay always positive (one can make sense of such a definition by means of a so called h-transform), it has recently been shown in [BJD06] and in [CC08] that if x/a_n vanishes as $n \to \infty$, the corresponding rescaled process converges in law towards the brownian meander. A natural question related to these results is whether conditioning on a *late* return near the origin (ie on $\{S^*_n = y\}$ with $y/a_n \to 0$ as $n \to \infty$) implies the convergence of $(S^*, \mathbf{P}_x)$ towards the brownian excursion.

Extending previous results from [BJD06], we show in this chapter that such a convergence holds. Before stating precisely our main results, we recall the essentials of the conditioning to stay positive for an oscillating random walk.

5.1.1 Conditioning a random walk to stay positive

Let $S_n = X_1 + \ldots + X_n$ be an integer valued aperiodic random walk. We write $\mathbf{P}_x$ the law of S started at x and for convenience we put $\mathbf{P} = \mathbf{P}_0$.

Next we introduce the strict descending *ladder process* $(T^-_k, H^-_k)_{k \geqslant 0}$ by setting $(T^-_0, H^-_0) = (0,0)$ and

$$T^-_{k+1} := \min\{j > T^-_k | S_j < S_{T^-_k}\}, \qquad H^-_{k+1} = -S_{T^-_{k+1}}. \tag{5.1.1}$$

Note that under $\mathbf{P}$, (T^-, H^-) is a bivariate renewal process, that is a random walk on $(\mathbb{Z}^+)^2$ with step law supported on the first quadrant. The sequence T^- is the sequence of the so called *(strictly) descending ladder epochs*, the sequence H^- the sequence of *descending ladder heights*.

We denote by $V(\cdot)$ the renewal function associated to H^-, that is the positive function defined by

$$V(x) := \sum_{k \geqslant 0} \mathbf{P}(H^-_k \leqslant x). \tag{5.1.2}$$

Note in particular that $V(y)$ is the expected number of ladder points in the stripe $[0,\infty) \times [0,y]$. It follows that it is a subadditive and increasing function.

The *killed* random walk $\widehat{S}$ is a Markov chain defined in the following way. Let $\tau_{(-\infty,0)}$ denote the first entrance time of S into the negative half plane. Introducing $\{\Delta\}$ a cimetery state, for every n,

$$\widehat{S}_n := S_n \mathbf{1}_{\tau_{(-\infty,0)} > n} + \Delta \mathbf{1}_{\tau_{(-\infty,0)} \leqslant n}. \tag{5.1.3}$$

Then we denote S conditioned to stay non negative by $S^*_n = \sum_{i=1}^n X^*_i$. In our integer valued oscillating case this is a Markov chain on $\mathbb{Z}^+$ whose law is defined for any $n \in \mathbb{N}$ and for any $B \in \sigma(S_1, \ldots, S_n)$ by :

$$\mathbf{P}^*_x[B \cap \{S_n = y\}] := \frac{V(y)}{V(x)} \mathbf{P}_x[B \cap \{S_n = y\} \cap \mathcal{C}_n] = \frac{V(y)}{V(x)} \mathbf{P}_x[B \cap \{\widehat{S}_n = y\}], \tag{5.1.4}$$

where $\mathcal{C}_n = \{S_1 \geqslant 0, \dots, S_n \geqslant 0\}$. The terminology is justified by the following weak convergence result

$$\mathbf{P}_x^* = \lim_{n\to\infty} \mathbf{P}_x\left(\cdot|\mathcal{C}_n\right) \tag{5.1.5}$$

which is proved in [BD94], Theorem 1.

5.1.2 A convergence towards the brownian excursion

From now on, we will always assume that S lies in the domain of attraction of the standard normal law. This means that the sequence (X_k) is iid and that for a suitable norming sequence (a_n) one has the weak convergence

$$S_n/a_n \Rightarrow \phi(x)dx, \qquad \phi(x) := \frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{5.1.6}$$

In particular this is the case when $\mathbf{E}[X_1] = 0$ and $\mathbf{E}[X_1^2] =: \sigma^2 < \infty$ with $a_n = \sigma\sqrt{n}$ by the central limit theorem.

By standard theory of stability, (see [Fel71] IX.8 and XVII.5) for (5.1.6) to hold it is necessary and sufficient that $\mathbf{E}[X_1] = 0$, that the truncated variance $\Phi(t) := \mathbf{E}[X_1^2\mathbf{1}_{|X_1|\,\leqslant\, t}]$ is slowly varying at infinity (that is $\frac{\Phi(ct)}{\Phi(t)} \to 1$ as $t \to \infty$ for any $c > 0$) and that the sequence a_n satisfies $a_n^2 \sim n\Phi(a_n)$ as $n \to \infty$.

We define Ω as being the space $D([0,1],\mathbb{R})$ the set of càdlàg functions on $[0,1]$ endowed with the standard Skorohod topology (see [Bil68]) and for $n \in \mathbb{Z}^+$, we define the application X^n by :

$$X^n : \begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \Omega \\ (u_1,\dots,u_n) & \mapsto & \left(\frac{\sum_{i=1}^{[nt]} u_i}{a_n}\right)_{t\in[0,1]} \end{array}. \tag{5.1.7}$$

For x, y positive integers, we denote by $P_n^{*,x,y}$ the law of S^* conditionally on the event $\{S_0^* = x, S_n^* = y\}$, and we define the probability laws on Ω :

$$Q_n^{x,y} := P_n^{*,x,y} \circ (X^n)^{-1}. \tag{5.1.8}$$

We can now state our main result :

Theorem 5.1.1. *Let x_n and y_n be positive integer valued sequences such that $x_n/a_n \to 0$ and $y_n/a_n \to 0$. Then, as $n \to \infty$, the following convergence holds in Ω :*

$$Q_n^{x_n,y_n} \Rightarrow e \tag{5.1.9}$$

where e denotes the law of the normalized brownian excursion.

The proof of this result will follow the standard procedure of showing finite dimensional convergence and tightness.

5.1.3 Some motivations and a short overview of the literature

The study of invariance principles for random walks is a very classical topic in probability (classical references are [Sko57], [Bil68]). Extending these invariance principles to conditioned random walks is far from being straightforward. Sometimes a clever representation can considerably simplify the proofs (like in [Bol76], [Don85] for the convergence towards the meander) , but generally speaking such an issue demands some technical efforts, see [Igl74] for a convergence towards the meander or [Lig68] for the brownian bridge.

The more particular case of convergence towards the brownian excursion for the conditioned simple random walk conditioned by a late return to zero has first been proved in [DIM77]. Their results have been extended to the case where S has finite variance in [Kai76].

A related result to ours that will turn out to be quite useful in our proofs is the convergence towards the brownian meander of a random walk in the domain of attraction of the normal law starting from x_n where x_n is $o(a_n)$ conditioned on $\mathcal{C}_n$ (see [Shi83, Remark 4]). Combining tightness arguments and local limit estimates, this result has been extended to the case where S is conditioned to stay positive by [BJD06], and their results in turn have been extended by quite different and somewhat lighter techniques in [CC08] to the case where S is in the domain of attraction of a *stable law* with index $\alpha \in (0, 2]$ and with positivity parameter $\rho \in (0, 1)$. Lacking a suitable representation under the form of an h-transform for the brownian excursion, our methods follow the same path as in [BJD06].

Besides the interest they have in their own, invariance principles are important in view of their applications. Let us mention one of them which is actually the main motivation of this chapter. Consider the following homogeneous polymer model (a by now classical reference for polymer models is [Gia07]) : for $N \in \mathbb{N}, y \in \mathbb{R}^+, a > 0$ and $\varepsilon \in \mathbb{R}$, we set

$$\frac{d\mathbf{P}^c_{N,a,\varepsilon}}{d\mathbf{P}} := \frac{1}{Z_{N,a,\varepsilon}} \exp\left(\varepsilon \sum_{i=1}^{N} \mathbf{1}_{S_i \in [0,a]}\right) \mathbf{1}_{S_N \in [0,a]} \tag{5.1.10}$$

where $\mathbf{P}$ is an aperiodic $\mathbb{Z}$ valued random walk in the domain of attraction of the standard normal law. The law $\mathbf{P}^c_{N,a,\varepsilon}$ may be viewed as an effective model for a $(1 + 1)$ dimensional interface above a wall with homogeneous impurities which are concentrated in the stripe $[0, \infty) \times [0, a]$. These impurities are either attracting or repelling the interface (depending on the sign of ε).

One standard goal related to this kind of models is to find the asymptotic behavior of the typical paths in the limit $N \to \infty$ and to study their dependence on ε and a. We study these limits in the chapter 2 of the present thesis.

A common feature shared by this model and the classical homogeneous one is that the measure $\mathbf{P}^c_{N,a,\varepsilon}$ exhibits a remarkable decoupling between the contact level set $\mathcal{I}_N := \{i \leqslant N, S_i \in [0,a]\}$ and the excursions of S between two consecutive contact points (see [DGZ05] for more details in the standard homogeneous pinning case). In fact, conditionally on $I_N = \{t_1, \dots, t_k\}$ and on $(S_{t_1}, \dots, S_{t_k})$, the *bulk* excursions $e_i = \{e_i(n)\}_n := \left\{\{S_{t_i+n}\}_{0 \leqslant n \leqslant t_{i+1}-t_i}\right\}$ are independent under $\mathbf{P}^c_{N,a,\varepsilon}$ and are distributed like the random walk $(S, \mathbf{P}_{S_{t_i}})$ conditioned on the event $\left\{S_{t_{i+1}-t_i} \in [0,a], S_{t_i+j} > a, j \in \{1, \dots, t_{i+1}-t_i-1\}\right\}$. It is therefore clear that to extract scaling limits on $\mathbf{P}^c_{N,a,\varepsilon}$, one has to combine good control over the law of the contact set $\mathcal{I}_N$ and suitable asymptotics properties of the excursions, and for this the utility of Theorem 5.1.1 emerges (see chapter 3 of the present thesis for more details).

5.1.4 Outline of the chapter

The exposition of this chapter will be organized as follows :

- in Section 5.2, we collect some preliminary facts.
- in Section 5.3, we discuss finite dimensional convergence and state our main technical lemma.
- in Section 5.4, we prove Lemma 5.3.1, which implies the finite dimensional convergence in Theorem 5.1.1.
- in Section 5.5, we show the tightness of the sequence of measures $(Q_n^{x_n,y_n})_n$, thus proving Theorem 5.1.1.
- in Section 5.6, we give a uniform equivalence for the tails of the random variable $\tau_{(-\infty,0)}$ under the law $\mathbf{P}_{x_n}$. This estimate is widely used in sections 5.4 and 5.5.

5.2 Some preliminary facts

5.2.1 Regular varying sequences

Throughout this chapter, for positive sequences α_n and β_n, we use the notation $\alpha_n \sim \beta_n$ to indicate that $\alpha_n/\beta_n \to 1$ as $n \to \infty$. Following Doney's terminology, for positive measurable functions g, h on $\mathbb{R}^+$, we will often say that the equivalence

$$g(x_n) \sim h(x_n) \tag{5.2.1}$$

is true *uniformly on the sequences x_n such that $x_n/a_n \to 0$*. By this we mean that, given any positive sequence ε_n such that $\varepsilon_n \to 0$ as $n \to \infty$, the convergence

$$\frac{g(x_n)}{h(x_n)} \to 1 \tag{5.2.2}$$

holds uniformly for every sequence $x_n \in \Delta_{\varepsilon_n}$ where

$$\Delta_{\varepsilon_n} := \{y \in \mathbb{Z}^{\mathbb{N}}, \forall n \geqslant 0, y_n \in [0, \varepsilon_n a_n]\}. \tag{5.2.3}$$

A positive sequence d_n is said to be slowly varying with index $\alpha \in \mathbb{R}$ (which we denote by $d_n \in \mathbb{R}_\alpha$) if $d_n \sim L_n n^\alpha$ where L_n is slowly varying at infinity that is for every positive t, $\lim_{n\to\infty} \frac{L_{[nt]}}{L_n} = 1$. If $d_n \in \mathbb{R}_\alpha$, we can (and will always assume) that $d_n = d(n)$ where $d(\cdot)$ is a continuous strictly monotone function whose inverse will be denoted $d^{-1}(\cdot)$ (see [BGT89, Theorem 1.5.3]). Observe that if $d_n \in \mathcal{R}_\alpha$, $d^{-1}(n) \in \mathcal{R}_{1/\alpha}$ and $1/d_n \in \mathcal{R}_{-\alpha}$.

The following basic uniform convergence property ([BGT89, Theorem 1.2.1]) will be often used in the sequel ; if $d_n \in \mathcal{R}_\alpha$, then for every fixed $\varepsilon > 0$

$$d_{[tn]} = t^\alpha d_n(1 + o(1)) \tag{5.2.4}$$

uniformly for $t \in [\varepsilon, 1/\varepsilon]$.

5.2.2 Fluctuation theory

In a similar way as for the descending ladder process, one can define the weak ascending bivariate renewal process $(T_k^+, H_k^+)_k$ as $T_0^+ := 0$, $T_{k+1} := \min\{j > T_k^+, S_j \geqslant S_{T_k^+}\}$, $H_k^+ := S_{T_k^+}$ and

$$U(x) := \sum_{k \geqslant 0} \mathbf{P}(H_k^+ \leqslant x). \tag{5.2.5}$$

It is known that S_1 is in the domain of attraction (without centering) of a stable law if and only if (T_1^-, H_1^-) lies in a bivariate domain of attraction (see for example [DG93]). We can specialize this fact to our setting. By hypothesis, S_1 lies in the domain of attraction of the standard normal law, so that by standard fluctuation theory, $a_n \in \mathcal{R}_{1/2}$. We then define two sequences

$$\log(\frac{n}{\sqrt{2}}) = \sum_{m=1}^{\infty} \frac{\mathbf{P}[S_m < 0]}{m} e^{-\frac{m}{b_n}}, \qquad c_n := a(b_n). \tag{5.2.6}$$

Then $b_n \in \mathcal{R}_2$, $c_n \in \mathcal{R}_1$ and we have the weak convergence

$$\left(\frac{T_n^-}{b_n}, \frac{H_n^-}{a_n}\right) \Rightarrow Z, \qquad \mathbf{P}[Z \in (dx, dy)] = \frac{e^{-1/2x}}{\sqrt{2\pi} x^{3/2}} \mathbf{1}_{x \geqslant 0} \delta_1(dy), \tag{5.2.7}$$

where $\delta_1(dy)$ denotes the Dirac measure at $y = 1$. Note in particular that, like in the simple random walk case, T_1^- is attracted to Y, the stable law of index 1/2.

$$\frac{T_n^-}{b_n} \Rightarrow Y, \qquad \mathbf{P}[Y \in dx] = \frac{e^{-1/2x}}{\sqrt{2\pi} x^{3/2}} \mathbf{1}_{x \geqslant 0}. \tag{5.2.8}$$

We recall also that b_n is sharply linked to the tails of T_1^- by the relation

$$\mathbf{P}[T_1^- > b_n] \sim \sqrt{\frac{2}{\pi}} \frac{1}{n} \tag{5.2.9}$$

and it is known that this is a necessary and sufficient relation in order for b_n to be such that $T_n^-/b_n \Rightarrow Y$.

Equation (5.2.7) also implies that the process (H^-) follows a generalized law of large numbers, namely $\frac{H_n^-}{c_n} \Rightarrow 1$ (H_1^- is said to be *relatively stable*). Consequently the following equivalence holds (see [BGT89, Theorem 8.8.1])

$$V(x) \sim c^{-1}(x) =: \frac{x}{l^-(x)} \tag{5.2.10}$$

where $l^-(\cdot)$ is slowly varying at infinity. In a similar way, one can prove that the equivalence

$$U(x) \sim \frac{x}{l^+(x)} \tag{5.2.11}$$

is verified for some slowly varying function $l^+(\cdot)$.

5.2.3 The duality lemma and local limit estimates

Let $v(\cdot,\cdot)$ be the renewal mass function of the bivariate renewal process (H^-, T^-), that is

$$v(n,x) := \sum_k \mathbf{P}[T_k^- = n, H_k^- = x] \tag{5.2.12}$$

and $u(\cdot,\cdot)$ its counterpart for the process (H^+, T^+)

$$u(n,x) := \sum_k \mathbf{P}[T_k^+ = n, H_k^+ = x]. \tag{5.2.13}$$

The power of fluctuation theory for the study of random walks is linked to some fundamental identities, the most famous one being the so called "duality lemma" (see [Fel71, Chapter XII]) :

$$\mathbf{P}[\mathcal{C}_n, S_n \in dx] = \mathbf{P}[n \text{ is a ladder epoch}, S_n \in dx] = u(n,x) \tag{5.2.14}$$

where by the event $\{n$ is a ladder epoch$\}$ we mean of course the disjoint union of the events $\cup_k\{T_k^+ = n\}$. The following equivalence about the asymptotics of $u(\cdot,\cdot)$ has been shown independently in [Car05] and in [BJD06]. Note that for the later, it is the chore of the proof of their main result.

Lemma 5.2.1. *Uniformly for $0 \leqslant y_n \leqslant K a_n$, one has the following equivalence :*

$$\mathbf{P}[\widehat{S}_n = y_n] = u(n, y_n) \sim \frac{U(y_n)}{n} \mathbf{P}[S_n = y_n]. \tag{5.2.15}$$

5.3 Finite dimensional convergence in Theorem 5.1.1

5.3.1 The law of the renormalized brownian excursion

For $x, y, t > 0$, we define $q_t(x,y)$ the transition function of the killed Brownian motion, that is

$$\begin{aligned} q_t(x,y) &:= \frac{1}{\sqrt{t}} r(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}) \\ \text{where } r(u,v) &:= \sqrt{\frac{2}{\pi}} \sinh(uv) \exp(-\frac{u^2+v^2}{2}), \end{aligned} \tag{5.3.1}$$

and the following transition function :

$$\begin{aligned} l_t(y) &:= \frac{1}{t} r_0(\frac{y}{\sqrt{t}}) \\ \text{where } r_0(v) &:= \sqrt{\frac{1}{2\pi}} v \exp(-\frac{v^2}{2}). \end{aligned} \tag{5.3.2}$$

It is well known that (see [BS02]) for $k \in \mathbf{N}, 0 < t_1 < \ldots < t_k < 1$ and $f \in \mathcal{C}^b([0,1]^k, \mathbb{R})$, one has :

$$\begin{aligned} &e(f(\omega_{t_1}, \ldots, \omega_{t_1})) \\ &\quad = 2\sqrt{2\pi} \int_{(\mathbb{R}^+)^k} f(x_1, \ldots, x_k) l_{t_1}(x_1) \ldots q_{t_k - t_{k-1}}(x_{k-1}, x_k) l_{1-t_k}(x_k) dx_1 \ldots dx_k. \end{aligned} \tag{5.3.3}$$

To get Theorem 5.1.1, we have to show finite dimensional convergence, that is we show that for every positive integer k, $(t_1, \ldots, t_k) \in (0,1)^k$, $f \in \mathcal{C}^b((\mathbb{R}^+)^k, \mathbb{R})$:

$$\begin{aligned} &\frac{\mathbf{E}_{x_n}[f(\frac{S^*_{\lceil nt_1 \rceil}}{a_n}, \ldots, \frac{S^*_{\lceil nt_k \rceil}}{a_n}) \mathbf{1}_{S^*_n = y_n}]}{\mathbf{P}_{x_n}[S^*_n = y_n]} \\ &\quad \to 2\sqrt{2\pi} \int_{\mathbb{R}^+} f(x_1, \ldots, x_k) l_{t_1}(x_1) q_{t_2 - t_1}(x_1, x_2) \ldots l_{1-t_k}(x_k) dx_1 \ldots dx_k \end{aligned} \tag{5.3.4}$$

as $n \to \infty$.

5.3.2 Getting the convergence (5.3.4)

Our main tool to get this convergence is the following result which we prove in part 5.4 :

Lemma 5.3.1. *For $K > 0$, uniformly in $x_n/a_n \to 0$ as $n \to \infty$ and in y_n such that $y_n/a_n \in [0, K]$, one has the following equivalence :*

$$\mathbf{P}_{x_n}(\widehat{S}_n = y_n) \sim \frac{V(x_n)U(y_n)}{n}\mathbf{P}(S_n = y_n). \tag{5.3.5}$$

The next result is a consequence of the Wiener Hopf factorization, it has been shown in [BJD06] and it will turn out to be useful numerous times in the sequel.

Lemma 5.3.2. *Let $K > 0$. Uniformly in the sequences $(x_n)_{n \geqslant 0}, (y_n)_{n \geqslant 0}$ such that $x_n/a_n \in [0, K], y_n/a_n \in [0, K]$, one has the following equivalence :*

$$\frac{U(x_n)V(y_n)}{n} = 2\frac{x_n}{a_n}\frac{y_n}{a_n} + o(1) \quad as \quad n \to \infty \tag{5.3.6}$$

Lemma 5.3.1 straightforwardly implies the equivalence :

$$\mathbf{P}_{x_n}(S^*_n = y_n) \sim \frac{U(y_n)V(y_n)}{n}\mathbf{P}[S_n = y_n]. \tag{5.3.7}$$

Of course, S^* is not reversible. Nevertheless, using time reversal, combining Lemma 5.3.2 and the equivalence (5.3.7) straightforwardly imply the following :

Lemma 5.3.3. *For $K > 0$, uniformly in $x_n/a_n \in [0, K]$ and in y_n such that $y_n/a_n \to 0$ as $n \to \infty$, one has the following equivalence :*

$$\mathbf{P}_{x_n}(S^*_n = y_n) \sim 2\frac{y_n^2}{a_n^2}\mathbf{P}(S_n = x_n) \sim 2\frac{y_n^2}{a_n^2}\frac{\phi(x_n/a_n)}{a_n}. \tag{5.3.8}$$

We finally recall the following proposition from [BJD06] :

Proposition 5.3.4. *Suppose x_n and y_n are integers such that*

$$x_n/a_n \to u > 0, \qquad y_n/a_n \to v > 0 \tag{5.3.9}$$

as $n \to \infty$. Then one has the convergence :

$$a_n\mathbf{P}[\widehat{S}_n = y_n] \to r(u, v) \tag{5.3.10}$$

It is then easy to check that combining the Lemmas 5.3.1, 5.3.2, 5.3.3 and the Proposition 5.3.4, one gets the convergence in (5.3.4), so that finite dimensional convergence in Theorem 5.1.1 holds.

5.4 Proof of Lemma 5.3.1

5.4.1 The case where y_n/a_n is bounded away from zero

We first assume that there exists $\varepsilon > 0$ such that for every $n, y_n/a_n \geqslant \varepsilon$.

We define $m_n := \inf\{S_j, j \leqslant n\}$ and $\mu_n := \inf\{j \leqslant n, S_j = m\}$ and their all time counterparts $m = \inf\{S_j, j \geqslant 0\}$ and $\mu := \inf\{j \geqslant 0, S_j = m\}$. Let $\eta > 0$ be fixed.

Alili and Doney have used the following equality in [AD01], it is an easy consequence of the duality lemma :

$$\begin{aligned}\mathbf{P}_{x_n}[\widehat{S}_n = y_n] &= \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n < \eta n] + \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n] \\ &= \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n \wedge y_n} \mathbf{P}_{x_n}[S_n = y_n, \mu_n = j, m_n = k] + \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n] \\ &= \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n \wedge y_n} v(j, x_n - k) u(n-j, y_n - k) + \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n].\end{aligned} \tag{5.4.1}$$

We first treat the first term in the right hand side of the above equality. The assumptions on x_n, y_n imply that for large enough n, $x_n \wedge y_n = x_n$. Using Lemma 5.2.1, for large enough n, we get that :

$$\sum_{j=0}^{\eta n} \sum_{k=0}^{x_n \wedge y_n} v(j, x_n - k) u(n-j, y_n - k) \sim \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j, x_n - k) \frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{n-j} \tag{5.4.2}$$

as $n \to \infty$, so that :

$$\begin{aligned}g_n(\eta) \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j,k) &\leqslant \frac{n}{U(y_n)\mathbf{P}[S_n = y_n]} \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j, x_n - k) \frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{n-j} \\ &\leqslant f_n(\eta) \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j,k)\end{aligned} \tag{5.4.3}$$

where we defined

$$f_n(\eta) := \sup_{j \leqslant \eta n, k \in [0, x_n]} \frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{(1-\eta)U(y_n)\mathbf{P}[S_n = y_n]} \tag{5.4.4}$$

and

$$g_n(\eta) := \inf_{j \leqslant \eta n, k \in [0, x_n]} \frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{U(y_n)\mathbf{P}[S_n = y_n]}. \tag{5.4.5}$$

Using the standard local limit theorem and equivalence (5.2.10), one gets easily that $\lim_{\eta\searrow 0}\limsup_{n\to\infty} f_n(\eta) = \lim_{\eta\searrow 0}\liminf_{n\to\infty} g_n(\eta) = 1$. Thus we are left with showing that

$$\sum_{j=0}^{\eta n}\sum_{k=0}^{x_n} v(j,k) \sim V(x_n). \tag{5.4.6}$$

Note that of course

$$\sum_{j=0}^{\infty}\sum_{k=0}^{x_n} v(j,k) = V(x_n), \tag{5.4.7}$$

so that we just have to show that

$$\frac{\sum_{j>\eta n}\sum_{k=0}^{x_n} v(j,k)}{V(x_n)} \to 0 \tag{5.4.8}$$

as $n\to\infty$ uniformly on x_n such that $x_n/a_n \to 0$. For this, we note that Lemma 5.2.1 implies

$$v(n,x) \sim \frac{V(x)\mathbf{P}[S_n = -x]}{n} \tag{5.4.9}$$

as $n\to\infty$ uniformly on $x\in[0, Ka_n]$ where $K>0$, so that

$$\sum_{j>\eta n}\sum_{k=0}^{x_n} v(j,k) \sim \sum_{j>\eta n}\sum_{k=0}^{x_n} \frac{V(k)\mathbf{P}[S_j = -k]}{j}. \tag{5.4.10}$$

Using the fact that $V(\cdot)$ is increasing and the standard local limit theorem (here and later c is a positive constant which may vary from line to line) :

$$\frac{\sum_{j>\eta n}\sum_{k=0}^{x_n} v(j,k)}{V(x_n)} \leqslant c\sum_{j>\eta n}\sum_{k=0}^{x_n}\frac{\phi(k/a_j)}{ja_j} \leqslant c\sum_{j>\eta n}\frac{x_n}{ja_j}. \tag{5.4.11}$$

Finally, as $a_n \in \mathcal{R}_{1/2}$, using property (5.2.4) it is easy to see that

$$\sum_{j>\eta n}\frac{a_n}{ja_j} \sim \int_{\eta}^{\infty} x^{-3/2}dx \tag{5.4.12}$$

and this entails (5.4.6). To conclude the case where y_n/a_n is bounded away from zero, we are left with showing that for any $\eta>0$, one has :

$$\limsup_{n\to\infty}\frac{n\mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n]}{V(x_n)U(y_n)\mathbf{P}[S_n = y_n]} = 0 \tag{5.4.13}$$

as $n \to \infty$. By the standard local limit theorem, there exists $a, b > 0$ such that $a \leqslant a_n \mathbf{P}[S_n = y_n] \leqslant b$. Using Lemma 5.3.2, we get that :

$$\frac{n\mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n]}{V(x_n)U(y_n)\mathbf{P}[S_n = y_n]} = \frac{n\mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n]}{V(y_n)U(y_n)\mathbf{P}[S_n = y_n]} \leqslant \frac{ca_n\mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n]}{\varepsilon^2} \tag{5.4.14}$$

so that we have to show that

$$\limsup_{n\to\infty} a_n \mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n] = 0. \tag{5.4.15}$$

Then we fix $\theta \in (\eta, 1)$ and we have :

$$\begin{aligned} a_n\mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n] &= \underbrace{a_n\mathbf{P}^*_{x_n}[\eta n \leqslant \mu_n \leqslant \theta n]}_{(1)} \\ &\quad + \underbrace{a_n\mathbf{P}^*_{x_n}[\mu_n > \theta n]}_{(2)}. \end{aligned} \tag{5.4.16}$$

Making use of the Markov property, one gets :

$$\begin{aligned} (1) &= a_n \sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n} \mathbf{P}^*_{x_n}\left[\mu_n = j, m_n = k\right]\mathbf{P}^*_k\left[S_{n-j} = y_n, \min_{l \leqslant n-j} S_l \geqslant k\right] \\ &\leqslant a_n \sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n} \mathbf{P}^*_{x_n}[\mu_n = j, m_n = k]\frac{V(y_n)}{V(k)}\mathbf{P}_k\left[\widehat{S}_{n-j} = y_n, \min_{i \leqslant n-j} \widehat{S}_i \geqslant k\right]. \end{aligned} \tag{5.4.17}$$

Noting that one has the equality $\mathbf{P}_k\left[\widehat{S}_{n-j} = y_n, \min_{i \leqslant n-j} \widehat{S}_i \geqslant k\right] = \mathbf{P}\left[\widehat{S}_{n-j} = y_n - k\right]$, we get (note that $V(k) \geqslant 1$ for every k) :

$$(1) \leqslant a_n \sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n} \mathbf{P}^*_{x_n}\left[\mu_n = j, m_n = k\right]V(y_n)\mathbf{P}\left[\widehat{S}_{n-j} = y_n - k\right]. \tag{5.4.18}$$

Making use of Lemma 5.3.2, of Lemma 5.2.1 and of the fact that $x_n/a_n \to 0$ as $n \to \infty$, we get :

$$\begin{aligned} (1) &\leqslant ca_n \sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n} \mathbf{P}^*_{x_n}[\mu_n = j, m_n = k]\frac{U(y_n)V(y_n)}{n-j}\mathbf{P}[S_{n-j} = y_n - k] \\ &\leqslant cK^2 \sum_{j=\eta n}^{\theta n} \mathbf{P}^*_{x_n}[\mu_n = j]\frac{n}{n-j}a_n\mathbf{P}[S_{n-j} = y_n]. \end{aligned} \tag{5.4.19}$$

Making use of the standard local limit theorem, we have easily :

$$(1) \leqslant cK^2(1-\theta)^{-3/2}\mathbf{P}^*_{x_n}\left[\mu_n \geqslant \eta n\right]. \tag{5.4.20}$$

Evidently, for every n, one has $\mu_n \leqslant \mu$, so that

$$\mathbf{P}^*_{x_n}[\mu_n \geqslant \eta n] \leqslant \mathbf{P}^*_{x_n}[\mu \geqslant \eta n], \tag{5.4.21}$$

and it has been shown in [BJD06, Theorem 5.1] that for every $\eta > 0$, uniformly in the sequences x_n such that $x_n/a_n \to 0$ as $n \to \infty$, the quantity $\mathbf{P}^*_{x_n}[\mu \geqslant \eta n]$ vanishes as $n \to \infty$.

For the second term in (5.4.16), we will need the following result which has been proved in [BJD06] :

Proposition 5.4.1. *For any $\kappa > 0$, for $x_n/a_n \to 0$ as $n \to \infty$, one has the following convergence :*

$$\mathbf{P}^*_{x_n}\left[\max_{j \leqslant \mu} S_j \geqslant \kappa a_n\right] \to 0. \tag{5.4.22}$$

We give us $\kappa \in (0,\varepsilon)$ and for $n > 0$, we note $\tau := \inf\{j \geqslant 0, S_j \geqslant \kappa a_n\}$. Then we have :

$$\begin{aligned}(2) = &\underbrace{a_n\mathbf{P}^*_{x_n}[\mu_n \geqslant \eta n, S_n = y_n, \tau \geqslant \theta n]}_{(3)} \\ &+ \underbrace{a_n\mathbf{P}^*_{x_n}[\mu_n \geqslant \eta n, S_n = y_n, \tau < \theta n]}_{(4)}.\end{aligned} \tag{5.4.23}$$

Making use of the Markov property, we have :

$$\begin{aligned}(3) &\leqslant a_n \sum_{j=0}^{\kappa a_n} \mathbf{P}^*_{x_n}\left[\max_{i \leqslant \theta n} S_i \leqslant \kappa a_n, S_{\theta n} = j, S_n = y_n\right] \\ &\leqslant a_n \frac{V(y_n)}{V(x_n)} \sum_{j=0}^{\kappa a_n} \mathbf{P}_{x_n}\left[\max_{i \leqslant \theta n} \widehat{S}_i \leqslant \kappa a_n, \widehat{S}_{\theta n} = j\right] \mathbf{P}_j\left[\widehat{S}_{(1-\theta)n} = y_n\right] \\ &\leqslant a_n \frac{V(y_n)}{V(x_n)} \sum_{j=0}^{\kappa a_n} \mathbf{P}_{x_n}\left[\max_{i \leqslant \theta n} S_i \leqslant \kappa a_n, \tau_{(-\infty,0)} > \theta n, S_{\theta n} = j\right] \mathbf{P}\left[S_{(1-\theta)n} = y_n - j\right],\end{aligned} \tag{5.4.24}$$

where we recall that $\tau_{(-\infty,0)} = \inf\{j \geqslant 1, S_j \in (-\infty,0)\}$. Using the local limit theorem and the fact that $j \in [0,\kappa a_n]$, we get :

$$(3) \leqslant c\frac{V(y_n)\mathbf{P}_{x_n}\left[\tau_{(-\infty,0)} > \theta n\right]}{V(x_n)} \times \mathbf{P}_{x_n}\left[\max_{i\leqslant \theta n} S_i \leqslant \kappa a_n \middle| \tau_{(-\infty,0)} > \theta n\right]\frac{1}{\sqrt{1-\theta}}\phi\left((\varepsilon-\kappa)(1-\theta)^{-1/2}\right). \tag{5.4.25}$$

Using the remark 4 in [Shi83], we note that, as $n \to \infty$,

$$\mathbf{P}_{x_n}\left[\max_{i\leqslant \theta n} S_i \leqslant \kappa a_n \middle| \tau_{(-\infty,0)} > \theta n\right] \to m\left[\sup_{[0,1]} \omega_t \leqslant \frac{\kappa}{\sqrt{\theta}}\right], \tag{5.4.26}$$

where $m(\cdot)$ denotes the measure of the brownian meander.

We prove that the equivalence

$$\mathbf{P}_{x_n}[\tau_{(-\infty,0)} > \theta n] \sim V(x_n)\mathbf{P}[T_1^- > \theta n] \tag{5.4.27}$$

holds uniformly on the sequences x_n such that $x_n/a_n \to 0$ in Lemma 5.6.1, so that finally, using the convergence

$$V(Ka_n)\mathbf{P}[T_1^- > \theta n] \to c\frac{K}{\sqrt{\theta}}, \tag{5.4.28}$$

which one can deduce from part 5.2.2, one gets :

$$\begin{aligned}(3) &\leqslant cV(Ka_n)\mathbf{P}[T_1^- > \theta n]m\left[\sup_{[0,1]} \omega_t \leqslant \frac{\kappa}{\sqrt{\theta}}\right]\frac{1}{\sqrt{1-\theta}}\phi\left((\varepsilon-\kappa)(1-\theta)^{-1/2}\right)\\ &\leqslant cKm\left[\sup_{[0,1]} \omega_t \leqslant \frac{\kappa}{\sqrt{\theta}}\right]\frac{1}{\sqrt{\theta(1-\theta)}}\phi\left((\varepsilon-\kappa)(1-\theta)^{-1/2}\right)\end{aligned} \tag{5.4.29}$$

and for $\theta > 0$ fixed, the quantity in the right hand side above vanishes as $\kappa \searrow 0$.

We are left with the second term in equation (5.4.23). To get this, one notes that looking at the proof of Lemma 5.3.4 in [BJD06], it is not difficult to see that, with $c, c' > 0$ fixed, the convergence in (5.3.10) holds uniformly for (u,v) in the compact set $[c,c'] \times [\varepsilon, K]$. Note in particular the uniformity part in Lemma 5.3.1, the fact that the convergence in the local limit theorem is uniform on the sets $[ca_n, c'a_n]$ and finally the fact that the derivative of the function $(x,u) \mapsto \frac{x}{u^{3/2}}\phi(x/u^2)$ is uniformly bounded for $(x,u) \in [c,c'] \times (0,1)$ (to get the uniform convergence of the Riemann's sums in the proof of Lemma 5.3.4 in [BJD06]).

Making use once again of the Markov property, this implies that :

$$
\begin{aligned}
(4) &\leqslant a_n \sum_{j \leqslant \theta n} \sum_{k \geqslant \kappa a_n} \mathbf{P}^*_{x_n}[\tau = j, S_j = k, \mu > \theta n] \mathbf{P}^*_k [S_{n-j} = y_n] \\
&\leqslant \sum_{j \leqslant \theta n} \sum_{k \geqslant \kappa a_n} \mathbf{P}^*_{x_n}[\tau = j, S_j = k, \mu > \theta n] \frac{V(y_n)}{V(k)} a_n \mathbf{P}_k \left[\widehat{S}_{n-j} = y_n\right].
\end{aligned} \tag{5.4.30}
$$

Note that one can restrict the range of summation of k in the above expression over $[\kappa a_n, K' a_n]$ where $K' > 0$ is large enough and independent of n. Thus, using Proposition 5.3.4 and the fact that $r(\cdot,\cdot)$ is continuous, one obtains :

$$
\begin{aligned}
(4) &\leqslant c \frac{V(K)}{V(\kappa)\sqrt{1-\theta}} \frac{K'}{\varepsilon} \left[\sup_{u \in [\kappa, K'], v \in [\varepsilon, K]} r(u,v) \right] \sum_{j \leqslant \theta n} \sum_{k \geqslant \kappa a_n} \mathbf{P}^*_{x_n}[\tau = j, S_j = k, \mu_n > \theta n] \\
&\leqslant c \frac{V(K)}{V(\kappa)\sqrt{1-\theta}} \frac{K'}{\varepsilon} \left[\sup_{u \in [\kappa, K'], v \in [\varepsilon, K]} r(u,v) \right] \mathbf{P}^*_{x_n}[\max_{j \leqslant \mu_n} S_j \geqslant \kappa a_n]
\end{aligned} \tag{5.4.31}
$$

and as evidently the inclusion of events $\{\max_{j \leqslant \mu_n} S_j \geqslant \kappa a_n\} \subset \{\max_{j \leqslant \mu} S_j \geqslant \kappa a_n\}$ holds, making use of Proposition 5.4.1, the last term in the equation above vanishes as $n \to \infty$ since $x_n / a_n \to 0$.

5.4.2 The case where y_n/a_n vanishes at infinity

This case relies heavily on the previous one. One has the equality :

$$
\begin{aligned}
\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n\right] &= \underbrace{\sum_{z=\varepsilon a_n}^{K a_n} \mathbf{P}_{x_n}\left[\widehat{S}_{n/2} = z\right] \mathbf{P}_z \left[\widehat{S}_{n/2} = y_n\right]}_{(5)} \\
&\quad + \underbrace{\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, S_{n/2} \leqslant \varepsilon a_n, S_{n/2} \geqslant K a_n\right]}_{(6)}
\end{aligned} \tag{5.4.32}
$$

We first show that the term in (5) yields the desired estimate, and then that the term in (6) is negligible with respect to the first one.

For the term in (5) , using time reversal and the case we just treated, one has of course :

$$
\mathbf{P}_z[\widehat{S}_{n/2} = y_n] \sim \frac{U(y_n)V(z)}{n/2} \mathbf{P}[S_{n/2} = z] \tag{5.4.33}
$$

so that, for $n \to \infty$, we have the equivalence :

$$\begin{aligned}\sum_{z=\varepsilon a_n}^{K a_n} \mathbf{P}_{x_n}\left[\widehat{S}_{n/2}=z\right]\mathbf{P}_z\left[\widehat{S}_{n/2}=y_n\right] &\sim \sum_{z=\varepsilon a_n}^{K a_n} \frac{V(x_n)U(z)}{n/2}\frac{U(y_n)V(z)}{n/2}\mathbf{P}[S_{n/2}=z]^2 \\ &\sim \frac{V(x_n)U(y_n)}{n}\sum_{z=\varepsilon a_n}^{K a_n} 8\frac{z^2}{a_n^2}\frac{\phi(z/a_{n/2})^2}{a_{n/2}^2}\end{aligned} \tag{5.4.34}$$

where in the last equivalence we made use of the standard local limit theorem and of Lemma 5.3.3. Thus we are left with showing that

$$\lim_{\varepsilon\searrow 0, K\nearrow\infty}\lim_{n\to\infty} 8\sqrt{2\pi}a_n \sum_{z=\varepsilon a_n}^{K a_n}\frac{z^2}{a_n^2}\frac{\phi(z/a_{n/2})^2}{a_{n/2}^2} = 1. \tag{5.4.35}$$

We use Riemann's sum and the fact that $(a_n) \in \mathcal{R}_{1/2}$ to get :

$$\begin{aligned}8\sum_{z=\varepsilon a_n}^{K a_n}\frac{z^2}{a_n^2}\frac{\phi(z/a_{n/2})^2}{a_{n/2}^2} &\sim 16\sum_{z=\varepsilon a_n}^{K a_n}\frac{z^2}{a_n^2}\frac{\phi(\sqrt{2}z/a_n)^2}{a_n^2} \\ &\sim \frac{16}{\sqrt{2\pi}}\sum_{z=\varepsilon a_n}^{K a_n}\frac{z^2}{a_n^2}\frac{\phi(2z/a_n)}{a_n^2} \\ &\sim \frac{16}{\sqrt{2\pi}a_n}\int_\varepsilon^K u^2\phi(2u)du \\ &\sim \frac{2}{\sqrt{2\pi}a_n}\int_{\varepsilon/2}^{K/2} u^2\phi(u)du.\end{aligned} \tag{5.4.36}$$

and thus (5.4.35) is valid.

We are left with showing that :

$$\begin{aligned}&\limsup_{K\nearrow\infty}\lim_{n\to\infty}\frac{n a_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n=y_n, \widehat{S}_{n/2}\geqslant K a_n\right]=0, \\ &\limsup_{\varepsilon\searrow 0}\lim_{n\to\infty}\frac{n a_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n=y_n, \widehat{S}_{n/2}\leqslant \varepsilon a_n\right]=0.\end{aligned} \tag{5.4.37}$$

We define $\widetilde{S}$ as being the time reversed version of S, that is the random walk whose transitions are given by

$$\mathbf{P}[\widetilde{S}_1 = y] := \mathbf{P}[S_1=-y], \qquad y\in\mathbb{Z}. \tag{5.4.38}$$

Note that

$$\begin{aligned}&\frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \geqslant Ka_n\right]\\&= \frac{na_n}{V(x_n)U(y_n)}\sum_{z \geqslant Ka_n}\mathbf{P}_{x_n}\left[\widehat{S}_{n/2} = z\right]\mathbf{P}_{y_n}\left[\widehat{\widetilde{S}}_{n/2} = z\right].\end{aligned} \tag{5.4.39}$$

We recall that the following equivalences are shown in Lemma 5.6.1 below :

$$\begin{aligned}\mathbf{P}_{x_n}[\tau_{(-\infty,0)} > n/2] &\sim V(x_n)\mathbf{P}[T_1^- > n/2],\\ \mathbf{P}_{y_n}[\tau_{(-\infty,0)} > n/2] &\sim V(y_n)\mathbf{P}[\widetilde{T_1^-} > n/2]\end{aligned} \tag{5.4.40}$$

and that they hold uniformly for x_n, y_n which are $o(a_n)$.

Therefore, one deduces

$$\begin{aligned}&\frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}[\widehat{S}_n = y_n, \widehat{S}_{n/2} \geqslant Ka_n]\\&\quad\sim n\mathbf{P}[T_1^- > n/2]\mathbf{P}[\widetilde{T_1^-} > n/2]\\&\qquad\times\sum_{z \geqslant Ka_n} a_n\mathbf{P}_{x_n}[S_{n/2} = z|\tau_{(-\infty,0)} > n/2]\mathbf{P}_{y_n}[\widetilde{S}_{n/2} = z|\widetilde{\tau}_{(-\infty,0)} > n/2].\end{aligned} \tag{5.4.41}$$

By the local limit theorem for the random walk conditioned to stay positive (see [Car05, Theorem 2]) :

$$\sup_{z\in\mathbb{Z}} a_n\mathbf{P}_{x_n}[S_{n/2} = z|\tau_{(-\infty,0)} > n/2] =: C < \infty. \tag{5.4.42}$$

Recall that T_1^+ and T_1^- are attracted to stable laws of index 1/2, so that by standard Tauberian theorems (see [Fel71, XIII 5.]) :

$$\mathbf{P}[T_1^- > n] \sim \frac{1}{\sqrt{\pi}}\left(1-\mathbf{E}\left[e^{-\frac{1}{n}T_1^-}\right]\right), \mathbf{P}[\widetilde{T}_1^- > n] \sim \frac{1}{\sqrt{\pi}}\left(1-\mathbf{E}\left[e^{-\frac{1}{n}\widetilde{T}_1^-}\right]\right). \tag{5.4.43}$$

On the other hand, by the Wiener-Hopf factorization :

$$\begin{aligned}1-\mathbf{E}[e^{-\lambda T_1^-}] &= \exp\left(-\sum_{n=1}^{\infty}\frac{e^{-\lambda n}}{n}\mathbf{P}[S_n < 0]\right)\\ 1-\mathbf{E}[e^{-\lambda T_1^+}] &= \exp\left(-\sum_{n=1}^{\infty}\frac{e^{-\lambda n}}{n}\mathbf{P}[S_n \geqslant 0]\right)\end{aligned} \tag{5.4.44}$$

hence, for $\lambda \searrow 0$,

$$\left(1-\mathbf{E}\left[e^{-\lambda T_1^-}\right]\right)\left(1-\mathbf{E}\left[e^{-\lambda T_1^+}\right]\right) = \exp\left(-\sum_{n=1}^{\infty}\frac{e^{-\lambda n}}{n}\right) = 1-e^{-\lambda} \sim \lambda \tag{5.4.45}$$

therefore $\lim_{n\to\infty} n\mathbf{P}[T_1^- > n]\mathbf{P}[\widetilde{T}_1^- > n] = \frac{1}{\pi}$. Using finally the convergence towards the brownian meander, we get that

$$\begin{aligned} \frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \geqslant Ka_n\right] &\leqslant \frac{C}{\pi}\mathbf{P}_{y_n}\left[\widetilde{S}_{n/2} \geqslant Ka_n \middle| \widetilde{T}_1^- > n\right] \\ &\leqslant cm\left[\omega_{1/2} > K\right] \end{aligned} \tag{5.4.46}$$

and the last term vanishes as $K \to \infty$. Proceeding in the same way, it is easy to see that

$$\lim_{n\to\infty} \frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \leqslant \varepsilon a_n\right] \leqslant cm\left[\omega_{1/2} \leqslant \varepsilon\right], \tag{5.4.47}$$

and this last quantity also vanishes when $\varepsilon \searrow 0$, and this concludes the proof of Lemma 5.3.1.

5.5 Tightness of the measures $Q_n^{x_n,y_n}$

The proof of tightness is very similar to the one of [BJD06]. We first note that the process S under $\mathbf{P}_{x_n}^*[\cdot|S_n = y_n]$ is still a Markov chain, so that according to [Bil68, Theorem 8.4] , tightness will follow if we can show that for each positive ε and $K \in (0,1)$, there exists $\lambda > 0$ and an integer n_0 such that

$$\mathbf{P}_{x_n}^*\left[\max_{i \leqslant Kn} S_i \geqslant \lambda a_n \middle| S_n = y_n\right] \leqslant \frac{\varepsilon}{\lambda^2} \tag{5.5.1}$$

for all $n \geqslant n_0$.

We proceed quite similarly as in the last part of the proof of Lemma 5.3.1. We write :

$$\begin{aligned} \star &:= \mathbf{P}_{x_n}^*\left[\max_{i \leqslant Kn} S_i \geqslant \lambda a_n \middle| S_n = y_n\right] \\ &= \sum_{j \geqslant 0} \mathbf{P}_{x_n}\left[\max_{i \leqslant Kn} \widehat{S}_i \geqslant \lambda a_n, \widehat{S}_{Kn} = j \middle| \widehat{S}_n = y_n\right] \\ &\sim \frac{na_n}{V(x_n)U(y_n)} \sum_{j \geqslant 0} \mathbf{P}_{x_n}\left[\max_{i \leqslant Kn} \widehat{S}_i \geqslant \lambda a_n, \widehat{S}_{Kn} = j\right] \mathbf{P}_j\left[\widehat{S}_{n(1-K)} = y_n\right] \end{aligned} \tag{5.5.2}$$

Using the same considerations as in the last part of the proof of Lemma 5.3.1 (by simply replacing $n/2$ by Kn or $(1-K)n$) , one gets that there exists a constant $C > 0$ such that :

$$\star \leqslant \frac{C}{\sqrt{1-K}} \sum_{j \geqslant 0} \mathbf{P}_{x_n}\left[\max_{i \leqslant Kn} S_i \geqslant \lambda a_n, S_{Kn} = j \middle| \tau_{(-\infty,0)} > Kn\right] \tag{5.5.3}$$

so that using the weak convergence towards the brownian meander, we get :

$$\star \leqslant \frac{C}{\sqrt{1-K}} m \left[\sup_{t\in[0,1]} \omega_t \geqslant \frac{\lambda}{\sqrt{K}} \right], \tag{5.5.4}$$

which for fixed K vanishes exponentially fast when λ becomes large, and in particular (5.5.1) holds. This concludes the proof of Theorem 5.1.1, and thus we are done.

5.6 Appendix

The following is the main result of this appendix :

Lemma 5.6.1. *Uniformly in x_n such that $x_n a_n^{-1} \to 0$ as $n \to \infty$, one has the following convergence :*

$$\frac{\mathbf{P}_{x_n}[\tau_{(-\infty,0)} > n]}{\mathbf{P}[T_1^- > n]} \sim V(x_n). \tag{5.6.1}$$

Note that it has been proved in [BD94] that

$$\liminf_{n\to\infty} \frac{\mathbf{P}_x[\tau_{(-\infty,0)} > n]}{\mathbf{P}[T_1^- > n]} \geqslant V(x) \tag{5.6.2}$$

in full generality (that is for every oscillating random walk S verifying $\mathbf{P}[S_1 > 0] \in (0,1)$). The convergence (5.6.1) has also been proved in [Kes63] in the lattice case for fixed x.

Proof. For $x > 0$, we denote by $\tau_x = \inf\{k \geqslant 1, S_k < -x\}$. One has the following identity :

$$\begin{aligned}
\mathbf{P}_x[\tau_{(-\infty,0)} > n] &= \mathbf{P}[\tau_x > n] \\
&= \sum_{k=0}^{+\infty} \mathbf{P}[T_k^- \leqslant n < T_{k+1}^-, \tau_x > n] \\
&= \sum_{k=0}^{+\infty} \mathbf{P}[T_k^- \leqslant n < T_{k+1}^-, H_k^- < x] \\
&= \sum_{k=0}^{+\infty} \sum_{l=0}^{n} \mathbf{P}[T_k^- = l, H_k^- < x] \mathbf{P}[T_1^- > n-l]
\end{aligned} \tag{5.6.3}$$

where in the last equality we made use of the Markov property. Thus :

$$\frac{\mathbf{P}_x[\tau_{(-\infty,0)} > n]}{V(x)} = \sum_{l=0}^{n} \frac{\mathbf{P}[\,l \text{ is a descending ladder epoch}, -S_l < x]}{V(x)} \mathbf{P}[T_1^- > n-l]. \tag{5.6.4}$$

We recall a strong version of Iglehart's lemma ([AD99, Lemma 5]) :

Lemma 5.6.2. *Let $c_n, d_n(z)$be two sequences where z belongs to a subset Δ of $\mathbb{R}$. Define e_n on Δ by :*

$$e_n(z) := \sum_{j=0}^{n-1} d_j(z)c_{n-j}. \tag{5.6.5}$$

Assume that there exist $c > 0$ such that the following condition holds uniformly on $z \in \Delta$:

$$\sum_{j=1}^{n} d_j(z) \to d(z) < \infty \text{ and } nd_n(z) \leqslant c \tag{5.6.6}$$

Assume moreover that the sequence c_n is regularly varying with index $-\rho$ where $\rho \in (0, 1)$. Then the equivalence $e_n(z) \sim d(z)c_n$ holds uniformly on $z \in \Delta$.

We already pointed out that :

$$\mathbf{P}[T_1^- > n] \sim \frac{b^{-1}(n)}{\sqrt{2\pi n}} \text{ as } n \to \infty. \tag{5.6.7}$$

Recalling that $b(\cdot) \in \mathcal{R}_2$, one has $b^{-1}(n)/n \in \mathcal{R}_{-1/2}$, which implies that the sequence $\big(\mathbf{P}[T_1^- > n]\big)_n$ verifies the hypothesis of the sequence c of Lemma 5.6.2 with $\rho = 1/2$.

On the other hand, we write

$$1 = \sum_{l \geqslant 0} \frac{\mathbf{P}\,[\,l \text{ is a descending ladder epoch}, -S_l < x]}{V(x)} = \sum_{l \geqslant 0} \frac{\sum_{j\in[0,x]} v(l, j)}{V(x)} \tag{5.6.8}$$

and thus we want to prove that the sequence $d_l(x) = \frac{\sum_{j\in[0,x]} v(l,j)}{V(x)}$ satisfies the second conditions of Lemma 5.6.2 with $\Delta_{(\varepsilon_n)} = \big\{(x_n) \in \mathbb{Z}^{\mathbb{N}}, \forall n, x_n \in [0, \varepsilon_n a_n]\big\}$ where ε_n is a given positive sequence which vanishes at infinity.

We first note that the uniform convergence of the series on $\Delta_{(\varepsilon_n)}$ has already been proved in the first part of the proof of Lemma 5.3.1.

For the second point, we consider a sequence $(x_n)_n \in \Delta_{(\varepsilon_n)}$. For $l > 0$ and making use of Lemma 5.2.1 (note in particular the uniformity part of it) and of the local limit theorem :

$$\begin{aligned} \sum_{j\in[0,x_n]} v(l, j) &\leqslant x_n \sup_{j \,\leqslant\, x_n} v(l, j) \\ &\leqslant c\frac{x_n}{na_n} V(x_n) \\ &\leqslant c\frac{\varepsilon_n}{n} V(x_n) \end{aligned} \tag{5.6.9}$$

and as $\varepsilon_n \to 0$, both conditions of the first part of (5.6.6) are fulfilled by the sequence $\Big(\frac{\sum_{j\in[0,x]} v(l,j)}{V(x)}\Big)_l$.

Thus we get that the following equivalence holds uniformly on $\Delta_{(\varepsilon_n)}$:

$$\frac{\mathbf{P}_{x_n}\left[\tau_{(-\infty,0)} > n\right]}{V(x_n)} \sim \mathbf{P}\left[T_1^- > n\right]. \tag{5.6.10}$$

This entails that the following equivalence holds uniformly on x_n such that $x_n a_n^{-1} \to 0$ as $n \to \infty$:

$$\lim_{n\to\infty} \frac{\mathbf{P}_{x_n}\left[\tau_{(-\infty,0)} > n\right]}{\mathbf{P}[T_1^- > n]} \sim V(x_n) \tag{5.6.11}$$

which is equation (5.6.1). □

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