Observations bruitées d'une diffusion
Estimation, filtrage, applications

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30 septembre 2010

Thèse préparée sous la direction de Valentine Genon-Catalot

Stochastic Differential Equations

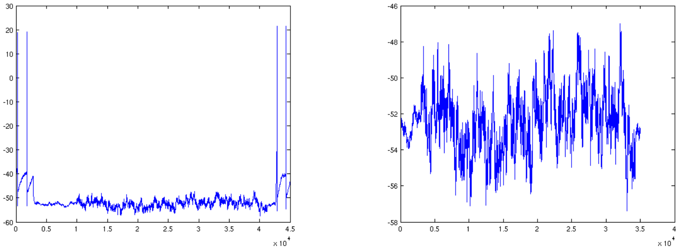


FIGURE: **Diffusion models (neuronal data)**

- Continuous-time stochastic models
- Mean behaviour ruled by an ordinary differential equation
- Aim : estimation of a parameter θ of interest

Model

Stochastic Differential Equation

$$dX_t = b(X_t, \theta)dt + \sigma(X_t, \theta)dB_t, \quad X_0 = \eta \tag{1}$$

N+1 observations at discrete times

$$0 = t_0 < t_1 < \cdots < t_N = T \text{ with } t_{i+1} - t_i = \Delta_N$$

(X_t) ergodic, with stationary probability π_0.

Litterature review

- [Florens-Zmirou, 1989], [Genon-Catalot, 1990], [Yoshida, 1992], [Genon-Catalot and Jacod, 1993], [Kessler, 1997] for the estimation of θ by likelihood-based or contrast-based methods
- [Bibby and Sørensen, 1995], [Kessler and Sørensen, 1999], [Kessler, 2000], [Sørensen, 2009] for the estimating-equations method
- [Ditlevsen and Lansky, 2005], [Höpfner and Brodda, 2006], [Ditlevsen and Ditlevsen, 2007] for examples of applications

Examples of diffusions

- Ornstein-Uhlenbeck (on $\mathbb{R}$)

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t \tag{2}$$

- Cox-Ingersoll-Ross (on $(0, \infty)$)

$$dX_t = \theta(\mu - X_t)dt + \sigma\sqrt{X_t}dB_t \tag{3}$$

- Hyperbolic diffusion (on $\mathbb{R}$)

$$dX_t = \theta X_t dt + \sigma\sqrt{1 + X_t^2}dB_t \tag{4}$$

Hidden diffusions and statistical problems

Observations

$$Y_{t_k} = F(X_{t_k}, \varepsilon_{t_k}) \quad (5)$$

F (known) function, (ε_{t_k}) iid random variables independent of (X_t) : *observation noise*

$\longrightarrow$ to model a measurement error (physical or biological measure), a microstructure noise (finance)...

$\longrightarrow$ Particular case of hidden Markov model, two questions :

1. estimation of $\theta \in \Theta$ when (X_t) is ergodic
2. exact computation and approximation of the filter

$$\pi_{k|k:0}(f) = \mathbb{E}(f(X_{t_k})|Y_{t_0}, \ldots, Y_{t_k})$$

Example of additive noise

$$Y_{t_k} = X_{t_k} + \varepsilon_{t_k} \quad (6)$$

often with Gaussian distribution.

References

$$\begin{cases} dX_t = b(X_t, \theta)dt + \sigma(X_t, \theta)dB_t, \quad X_0 = \eta \\ Y_{t_k} = F(X_{t_k}, \varepsilon_{t_k}) \end{cases} \tag{7}$$

1. Parameter estimation for noisy observations of a diffusion : [Gloter and Jacod, 2001], [Zhang et al., 2005], [Jacod et al., 2009] on $[0, 1]$
2. Optimal filter : *Kalman filter* (specific for Gaussian framework) [Kalman, 1960], [Cappé et al., 2005], *Computable filter* [Chaleyat-Maurel and Genon-Catalot, 2006], [Genon-Catalot and Kessler, 2004]

Survey

1. Parameter estimation for a bidimensional partially observed Ornstein-Uhlenbeck process with biological application (chapters 2 and 3)
    1. Theoretical results [Favetto and Samson, 2010] in Scandinavian journal of statistics
    2. Application to medical data [Favetto et al., 2009], submitted
2. Parameter estimation by contrast minimization for noisy observations of a diffusion process (chapters 4 and 5)
    1. Contrasts, consistency of the estimators and simulations [Favetto, 2010], submitted
    2. Estimating equations for noisy observations of ergodic diffusions, preprint
3. On the asymptotic variance in the Central Limit Theorem for particle filters (chapter 6, [Favetto, 2009] accepted in ESAIM P&S)

Parameter estimation for a bidimensional partially observed Ornstein-Uhlenbeck process

[Favetto and Samson, 2010] for the theoretical results
Medical data : joint work with Adeline Samson (MAP5), Daniel Balvay (HEGP), Isabelle Thomassin (HEGP), Valentine Genon-Catalot (MAP5), Charles-André Cuénod (HEGP) and Yves Rozenholc (MAP5)

Medical problem

- New treatments in anti-cancer therapy : anti-angiogenesis
- Evaluation *in vivo*
- Based on several sequences of medical images
- Comprehension of microvascularization phenomena
- With a pharmacokinetic model for the contrast agent

MRI data

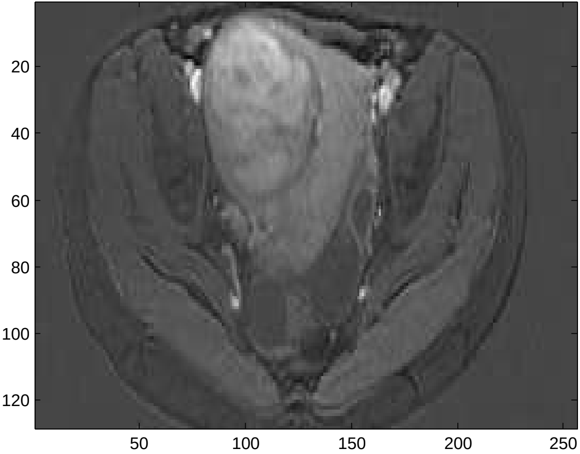


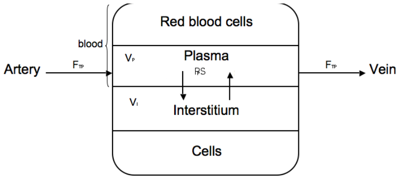
FIGURE: Female pelvis

Biological model

Deterministic pharmacokinetic model

1. $Q_P(t)$ quantity of contrast agent at time t in the plasma
2. $Q_I(t)$ quantity of contrast agent at time t in the interstitium

driven by two coupled ODE.
Only access to the sum $S(t) = Q_P(t) + Q_I(t)$ (quantity of contrast agent in a voxel.) [Brochot et al., 2006], [Fournier et al., 2007], [Brix et al., 2004]



$$
\begin{aligned}
\frac{dQ_P(t)}{dt} &= \frac{F_T}{1-h}\,AIF(t) - \frac{PS}{V_b(1-h)}Q_P(t) \\
&\quad + \frac{PS}{V_e}Q_I(t) - \frac{F_T}{V_b(1-h)}Q_P(t) \\
\frac{dQ_I(t)}{dt} &= \frac{PS}{V_b(1-h)}Q_P(t) - \frac{PS}{V_e}Q_I(t)
\end{aligned}
$$

Stochastic version of the model

Hidden diffusion (bidimensional Ornstein-Uhlenbeck model)

$$\begin{cases} dU(t) = (F_\theta(t) + G_\theta\, U(t))dt + \Sigma_\theta dB(t) \\ U(t_0) = U_0 \end{cases} \quad (8)$$

with $U(t) = \begin{pmatrix} S(t) \\ \star \end{pmatrix}$ and $\Sigma_\theta = \begin{pmatrix} \sigma_1 & \sigma_2 \\ 0 & \sigma_2 \end{pmatrix}$

Discrete-time noisy observations (unidimensional)

$$y_i = J\,U(t_i) + \sigma\varepsilon_i, \quad J = (1\ 0), \quad \varepsilon_i \sim \mathcal{N}(0,1) \quad (9)$$

with $\Delta = t_{i+1} - t_i$.

$\rightarrow$ Aim : estimation of θ (microvascularization parameters and diffusion coefficients).

Stochastic models in biology : [Picchini et al., 2006], [Picchini et al., 2008]

Computation of the Maximum Likelihood Estimator

Log-likelihood

$$\ell_{0:n}(\theta) = \ell(\theta, y_0, \ldots, y_n) = \sum_{i=0}^{n} \log p(y_i | y_{i-1}, \ldots, y_0; \theta).$$

Maximum Likelihood Estimator

Aim : compute $\hat{\theta} = \operatorname{argmax}_{\theta \in \Theta} \ell(\theta, y_0, \ldots, y_n)$.

Kalman algorithm for the log-likelihood

- iterative computation of the log-likelihood using Kalman recursions
- Gradient and Hessian of the log-likelihood also exactly computed by similar recursions
- $\rightarrow \hat{\theta}$ obtained by a gradient method

Advantages of an iterative computation of the likelihood : no need to invert large matrix, faster than direct method

Theoretical properties of $\hat{\theta}$

(in stationary regime)

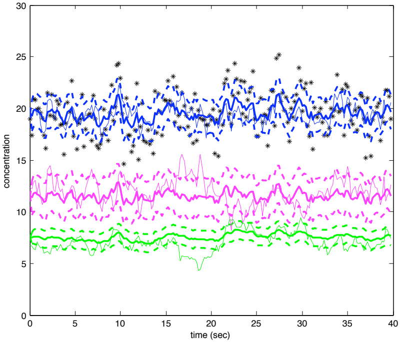
Link with ARMA processes

- (y_i) is an ARMA (2,2) process (can be extended to d-dimensional hidden diffusions)
- Hessian and information matrix $\lim_{n\to\infty}\left(-\frac{1}{n}\frac{\partial^2}{\partial\theta^2}\ell_{0:n}(\theta)\right) = I(\theta)$
- Central Limit Theorem $\sqrt{n}(\hat{\theta}_n - \theta_0) \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0, I^{-1}(\theta_0))$

Identifiability

- difficulty : number of identifiable parameters on the spectral density
- when Δ is small, five parameters (out of six) are identifiable

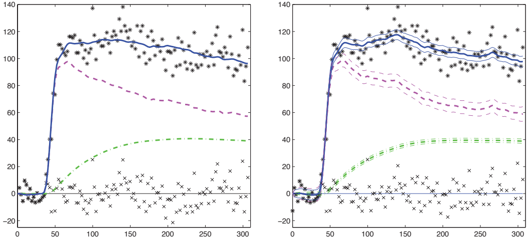
Simulation results



- x-axis : time
- y-axis : quantity of contrast agent
- blue : $S(t)$
- purple : $Q_P(t)$
- green : $Q_I(t)$
- constant AIF

- Comparison with the EM algorithm
- Validation of the method
- Influence of Δ
- Curves obtained with the estimated parameters

Medical data



- left : ODE method
- right : SDE method
- confidence intervals

- comparison with the ODE model (no stochastic part in the hidden process)
- different behaviours for the ODE and the SDE-based methods
  1. ODE similar to SDE
  2. slightly different
  3. important differences

  → the SDE-based method performs at least as well as the ODE-based, and sometimes much better
- stability of the SDE-based method

Conclusion on this part

- strongly based on Gaussian framework
- stability of the SDE method
- importance of numerical results (simulations and medical data)

Parameter estimation by contrast minimization for noisy observations of a diffusion process

Model

$$\begin{array}{lcll} dX_t & = & b(X_t,\kappa)dt + \sigma(X_t,\lambda)dB_t, \quad X_0 = \eta & \text{(hidden)} \\ Y_{i\delta_N} & = & X_{i\delta_N} + \rho_N \varepsilon_{i\delta_N} & \text{(observed)} \end{array}$$

Hidden unidimensional diffusion, with general drift and diffusion coefficient

- Aim : estimate $\theta = (\kappa, \lambda)$
- Discretization step $\delta_N \to 0$, number of observations $N \to \infty$ over $[0, T]$ with $T = N\delta_N \to \infty$ (high frequency data)
- (X_t) ergodic, with stationary distribution $\nu_0(dx) = \nu_0(x)dx$
- ρ_N is known here
- $(\varepsilon_{i\delta_N})$ i.i.d. centered random variables, $\mathbb{E}((\varepsilon_{i\delta_N})^2) = 1$

Assumptions on the noise

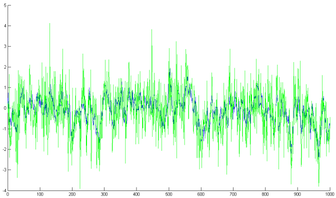


FIGURE: Ornstein-Uhlenbeck process with high-frequency noisy observations. $N = 1000, \delta = 0.1, \rho^2 = 1$

Two possible cases for the observations :

(B1) $\rho_N^2 = \rho^2 > 0$

(B2) $\rho_N^2 \to 0$ when $N \to \infty$

Assumption **(B2)** corresponds to the case $\sqrt{\delta_N}\varepsilon_{i\delta_N} = V_{(i+1)\delta_N} - V_{i\delta_N}$ with (V_t) Brownian motion independent of (B_t).

Minimum contrast estimation for a discretely observed diffusion

The Euler Scheme

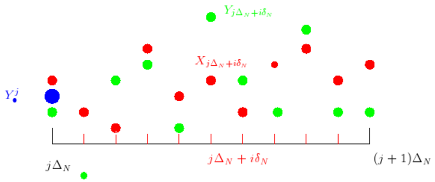
For $\delta_N \to 0$,

$$X_{(j+1)\delta_N} - X_{j\delta_N} \approx \mathcal{N}(b(X_{j\delta_N}, \kappa)\delta_N, \sigma(X_{j\delta_N}, \lambda)^2\delta_N).$$

- [Kessler, 1997] $\hat{\theta} = (\hat{\kappa}, \hat{\lambda})$ consistent and asymptotically Gaussian estimator built as minimum of a contrast based on the loglikelihood of Gaussian observations
- [Gloter, 2000] and [Gloter, 2006] : $\int_{j\delta_N}^{(j+1)\delta_N} X_s ds$ observed $\to$ minimum contrast estimators, for observations on $[0, 1]$ and for $T = N\delta_N \to \infty$

Aim : build a contrast based on noisy data, then obtain an estimator $\hat{\theta} = (\hat{\kappa}, \hat{\lambda})$ and derive consistency and asymptotic normality

Local means of the observations



Local means and noise reduction

p_N, k_N such that $p_N = \delta_N^{-\frac{1}{\alpha}}$ for $1 < \alpha \leq 2$, $N = p_N k_N$. Let $\Delta_N = p_N \delta_N = \delta_N^{1-\frac{1}{\alpha}}$. Hence $N\delta_N = k_N \Delta_N$. Define

$$Y_\bullet^j = \frac{1}{p_N} \sum_{i=0}^{p_N-1} Y_{j\Delta_N + i\delta_N} = X_\bullet^j + \rho_N \varepsilon_\bullet^j$$

Idea : $Y_\bullet^j \approx X_\bullet^j \approx \Delta_N^{-1} \int_{j\Delta_N}^{(j+1)\Delta_N} X_s ds \approx X_{j\Delta_N}$ → Build a contrast function based on $(Y_\bullet^j)$ in the ergodic case.

Minimum contrasts estimation for an hidden diffusion

Contrast for $(Y_\bullet^j)$

$$\mathcal{E}_N(\theta) = \sum_{j=1}^{k_N-2} \left\{ \frac{3}{2} \frac{(Y_\bullet^{j+1} - Y_\bullet^j - \Delta_N b(Y_\bullet^{j-1}, \kappa))^2}{\Delta_N c(Y_\bullet^{j-1}, \lambda)} + \log(c(Y_\bullet^{j-1}, \lambda)) \right\}$$

where $c(., \lambda) = \sigma(., \lambda)^2$.

Modified contrast for $(Y_\bullet^j)$

$$\mathcal{E}_N^{\rho_N}(\theta) = \sum_{j=1}^{k_N-2} \left\{ \frac{3}{2} \frac{(Y_\bullet^{j+1} - Y_\bullet^j - \Delta_N b(Y_\bullet^{j-1}, \kappa))^2}{\Delta_N c_{N,\rho_N}(Y_\bullet^{j-1}, \lambda)} + \log(c_{N,\rho_N}(Y_\bullet^{j-1}, \lambda)) \right\}$$

where $c_{N,\rho_N}(x, \lambda) = \sigma(x, \lambda)^2 + 3\Delta_N^{\frac{2-\alpha}{\alpha-1}} \rho_N^2$, for $1 < \alpha \leq 2$.

Minimum contrast estimators (I)

Let $\hat{\theta}_N = \underset{\theta\in\Theta}{\arginf}\, \mathcal{E}_N(\theta)$ and $\hat{\theta}_N^{\rho_N} = \underset{\theta\in\Theta}{\arginf}\, \mathcal{E}_N^{\rho_N}(\theta)$.

Consistency

1. If **(B1/2)** holds, with $p_N = \delta_N^{-\frac{1}{\alpha}}$ with $\alpha \in (1,2)$, the estimator $\hat{\theta}_N$ is consistent.
2. If **(B1/2)** holds, with $p_N = \delta_N^{-\frac{1}{\alpha}}$, $\alpha \in (1,2]$, the estimator $\hat{\theta}_N^{\rho_N}$ is consistent.

$\rightarrow$ Results based on Taylor expansions for $Y_\bullet^j$, and the asymptotic behaviour of the variation and the quadratic variation of $Y_\bullet^j$
$\rightarrow$ Special case $\alpha = 2$

Minimum contrast estimators (II)

Asymptotic normality

Assume that $N\delta_N^{2-\frac{1}{\alpha}} \to 0$, when $N \to \infty$, $\delta_N \to 0$, $N\delta_N \to \infty$, $k_N = N\delta_N^{\frac{1}{\alpha}} \to \infty$, $\Delta_N = \delta_N^{1-\frac{1}{\alpha}} \to 0$. If $\rho_N = \rho$ **(B1)** and $\alpha \in (1,2)$ or $\rho_N \to 0$ **(B2)** with $\alpha \in (1,2]$,

$$\begin{pmatrix} \sqrt{N\delta_N}(\hat{\kappa}_N^{\rho_N} - \kappa_0) \\ \sqrt{N\delta_N^{\frac{1}{\alpha}}}(\hat{\lambda}_N^{\rho_N} - \lambda_0) \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \mathbf{V}(\theta_0)\right)$$

where

$$\mathbf{V}(\theta_0) = \begin{pmatrix} \left\{\nu_0\left(\frac{(\partial_\kappa b(.,\kappa_0))^2}{c(.,\lambda_0)}\right)\right\}^{-1} & 0 \\ 0 & \frac{9}{4}\left\{\nu_0\left(\frac{(\partial_\lambda c(.,\lambda_0))^2}{c(.,\lambda_0)^2}\right)\right\}^{-1} \end{pmatrix}.$$

In the case $\alpha = 2$ with **(B1)** ($\rho_N = \rho > 0$), the asymptotic variance $\mathbf{V}(\theta_0)$ is increased by the noise variance ρ^2.

$\to$ Estimation rate and asymptotic variance for $\hat{\lambda}_N^{\rho_N}$ ([Gloter, 2006])

Estimation with unknown ρ^2

Quadratic variation of the observations

- $\hat{\rho}_N^2 = \frac{1}{2N} \sum_{i=0}^{N-1} (Y_{(i+1)\delta_N} - Y_{i\delta_N})^2$
- If $N\delta_N^2 \to 0$, then $\sqrt{N}(\hat{\rho}_N^2 - \rho^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 3\rho^4)$

Associated estimator

The estimator $\hat{\theta}_N^{\hat{\rho}_N}$ is consistent.

$\rightarrow$ Essential difference with direct observations, local means to deal with the parameters of the hidden diffusion.

Numerical results

Ornstein-Uhlenbeck model with additive noise

- explicit estimators
- simulations with different δ and N : large number of data needed, with high frequency sampling
- with $\alpha = 2$, $\alpha = 1.5$ and α close to 1 : $\alpha = 1.5$ is a pretty good choice
- no influence of the distribution of $\varepsilon_{i\delta_N}$
- $\hat{\lambda}_N$ is poorly estimated when ρ is very large
- comparison with direct observations

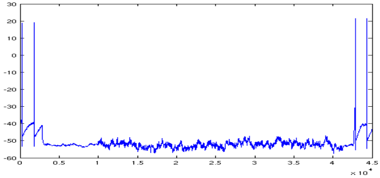
Other models

- Cox-Ingersoll-Ross model with multiplicative noise
- Hyperbolic diffusion with additive noise

Conclusion on this part

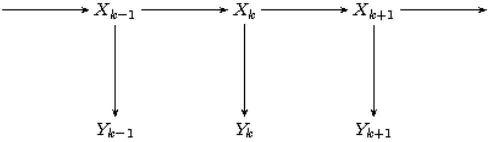
Other results

- Parameter estimation for neuronal data



- Case of an integrated diffusion process observed with small noise
- General framework of estimating functions

On the asymptotic variance in the Central Limit Theorem for particle filters



General hidden Markov model (X_k, Y_k) (not only an hidden diffusion model) :

Quantities of interest

Aim : Compute

$$\pi_{k|k:0}(f) = \mathbb{E}(f(X_k)|Y_k, \ldots, Y_0) \qquad \text{(filter)}$$

$$\eta_{k|k-1:0}(f) = \mathbb{E}(f(X_k)|Y_{k-1}, \ldots, Y_0) \qquad \text{(prediction)}$$

Filtering : problems and goals

- Except for a few models ([Kalman, 1960], [Genon-Catalot and Kessler, 2004], [Chaleyat-Maurel and Genon-Catalot, 2006]), exact algorithmic computation of $\pi_{k|k:0}(f)$ and $\eta_{k|k-1:0}(f)$ untractable
- Numerical approximation needed : particle filter ([Künsch, 2001], [Doucet et al., 2001])
- (Asymptotic) confidence interval for the method ? Accuracy of the method ? ([Del Moral and Jacod, 2001b], [Van Handel, 2009])

Assumptions and notations

On the hidden Markov chain

- Kernel $Q(x, dx')$ with stationary probability $\pi(dx)$
- There exists a probability measure μ and $0 < \epsilon_- \leq \epsilon_+$ such that

$$\forall x \in \mathcal{X}, \forall B \in \mathcal{B}(\mathcal{X}) \quad \epsilon_- \mu(B) \leq Q(x, B) \leq \epsilon_+ \mu(B). \tag{10}$$

On the observations

- $f(y|x)$ conditional density of $\mathcal{L}(Y_k|X_k)$

With $g_k(x) = f(Y_k|x)$ and $L_k(x, dx') = g_k(x)Q(x, dx')$,

$$\mathbb{E}\left(f(X_k)|Y_{k-1}, \ldots Y_0\right) = \eta_{k|k-1:0}(f) = \frac{\eta_0 L_{0,k-1} f}{\eta_0 L_{0,k-1} \mathbf{1}} \tag{11}$$

Particle Monte-Carlo method

- Algorithm which provides recursively empirical measures $\pi^N_{k|k:0}$ and $\eta^N_{k|k-1:0}$ which are good approximations of $\pi_{k|k:0}$ and $\eta_{k|k-1:0}$, for a fixed set of observations $Y_{0:k}$
- Based on the simulation of the evolution of N particles (need to know how to simulate under Q), which give the accuracy of the method

See e.g. [Künsch, 2001], [Doucet et al., 2001], [Del Moral et al., 2001] ...

Central Limit Theorem

[Pitt and Shephard, 1999], [Del Moral and Jacod, 2001a], [Chopin, 2004]

Central Limit Theorem for the prediction

$$\sqrt{N}(\eta^N_{k|k-1:0}(f) - \eta_{k|k-1:0}(f)) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, \Delta_{k|k-1:0}(f))$$

Central Limit Theorem for the filter

$$\sqrt{N}(\pi^N_{k|k:0}(f) - \pi_{k|k:0}(f)) \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, \Gamma_{k|k:0}(f))$$

As $N \to \infty$, $\Gamma_{k|k:0}(f)$ and $\Delta_{k|k-1:0}(f)$ give the accuracy of the approximations (confidence interval) for a fixed sequence of observations. $\Gamma_{k|k:0}(f)$ and $\Delta_{k|k-1:0}(f)$ can be seen as functions of $Y_{0:k}$, *i.e.* random variables depending on the observations.
Question : behaviour when $k \to \infty$? Aim : prove tightness of the sequences, to have uniform (in time) confidence intervals.

Litterature review

Numerous references about particle filter(s). For the asymptotic behaviour :

- [Chopin, 2004] CLT and tightness under stronger assumptions, Bayesian approach of particle algorithms
- [Del Moral and Jacod, 2001b] Tightness in the particular case of Kalman filter, with analytic method
- [Van Handel, 2009] Other criterion of time-stability, under assumptions close to us

The asymptotic variances

Asymptotic variance of the prediction

$$\Delta_{k|k-1:0}(f) = \sum_{i=0}^{k} \eta_{i|i-1:0} \left(\left(\frac{L_{i,k-1}\mathbf{1}(.)}{\eta_{i|i-1:0} L_{i,k-1}\mathbf{1}} \right)^2 \left(\eta_{k|k-1:i} f(.) - \eta_{k|k-1:0} f \right)^2 \right)$$

Asymptotic variance of the filter

$$\Gamma_{k|k:0}(f) = \sum_{i=0}^{k} \frac{\eta_{i|i-1:0} \left(\left(\frac{L_{i,k-1}\mathbf{1}(.)}{\eta_{i|i-1:0} L_{i,k-1}\mathbf{1}} \right)^2 \left(\eta_{k|k-1:i}(g_k f)(.) - \pi_{k|k:0} f \right)^2 \right)}{(\eta_{k|k-1:0}(g_k))^2}$$

Tightness of $(\Delta_{k|k-1:0}(f))_k$ and$(\Gamma_{k|k:0}(f))$

Additional assumption

(B) For some $\delta > 0$

$$\sup_{k\geq 0} \mathbf{E} \left|\log\left(\eta_{k|k-1:0}\left(g_k\right)\right)\right|^{1+\delta} < \infty,$$

where $\mathbf{E}$ denotes the expectation with respect to the distribution of $(Y_k)_{k\geq 0}$.

Remark : Technical assumption, but easy to check on common examples.

Main result :

Assume **(B)**. Then, for any bounded function f, the sequences of variances $(\Delta_{k|k-1:0}(f))$ and $(\Gamma_{k|k:0}(f))$ are tight.

Exponential stability of the prediction

Tool for the proof : forget the initial distribution at exponential rate ([Douc et al., 2009])

Proposition : forgetting the initial distribution

Setting $\rho = 1 - \frac{\epsilon_-^2}{\epsilon_+^2}$, then for all k, ν, ν' and all set $y_{0:k-1}$ of values :

$$\|\eta_{\nu,k}[y_{0:k-1}] - \eta_{\nu',k}[y_{0:k-1}]\|_{TV} \leq \rho^k,$$

where $\|.\|_{TV}$ denotes the total variation distance.

Consequence :

For f bounded,

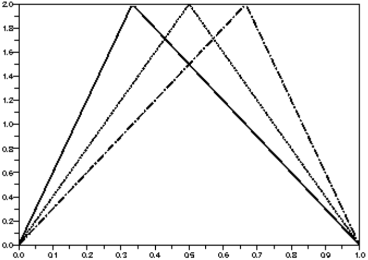
$$\Delta_{k|k-1:0}(f) \leq \|f\|_\infty^2 \frac{\epsilon_+^2}{\epsilon_-^2} \sum_{i=0}^{k} \eta_{i|i-1:0}\left(\left(\frac{g_i}{\eta_{i|i-1:0} g_i}\right)^2\right) \rho^{2(k-i)}.$$

$\rightarrow$ conclusion with the tightness lemma [Del Moral and Jacod, 2001b].

A simple Markov chain

$$X_0 \sim \pi, \qquad X_{k+1} = \mathbf{1}_{X_k < \alpha} U_{k+1} + \mathbf{1}_{X_k \geq \alpha} V_{k+1}$$

(U_k) and (V_k) independent of X_k with distributions $u(x)dx$ and $v(x)dx$.



Then $\mu(dx) = 4(x \wedge 1 - x)dx$, $\epsilon_- = \frac{1}{4}$ and $\epsilon_+ = \frac{3}{2}$.

Simulations and numerical results

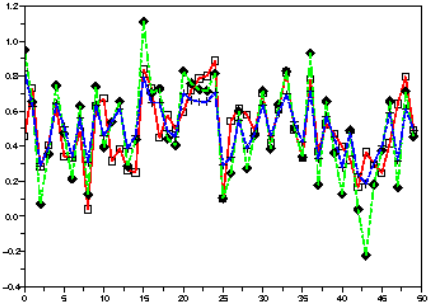


FIGURE: Simulation of the example, with additive noise (observations on $[0, 50]$)

Observations : $Y_i = X_i + \varepsilon_i, \quad \varepsilon_i \sim_{iid} \mathcal{N}(0, 0.2)$.
$N = 500$ particles. In dark blue : result of particle filter. In red : hidden Markov chain. In green : noisy observations.

Concluding remarks and perspectives

- Generalization of the contrast-based method for high-frequency noisy observations of a diffusion to the multidimensional case, with general noise.
- Assumption on the transition kernel of the hidden chain used to prove the tightness could be weakened.
- Other implementations of the contrast-based method on real data.

Concluding remarks and perspectives

- Generalization of the contrast-based method for high-frequency noisy observations of a diffusion to the multidimensional case, with general noise.
- Assumption on the transition kernel of the hidden chain used to prove the tightness could be weakened.
- Other implementations of the contrast-based method on real data.

Thanks !

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