Contribution à la résolution numérique des problèmes de Helmholtz

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Prof. Rabia Djellouli
Dr. Henri Calandra

Outline

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- **Motivation and Context**

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- **A new solution methodology for Helmholtz problems**

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  - **A new DG method**

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  - An improved modified DG method
- Summary and perspectives

Motivation and Context

Applications

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- **Radar**

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- **Radar**
- **Sonar**

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- **Geophysical exploration**

Applications

- Radar
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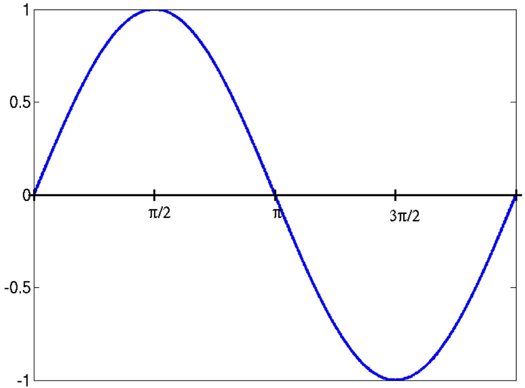
- Medical imaging

Applications

- **Radar**
- **Sonar**
- **Geophysical exploration**
- **Medical imaging**
- **Nondestructive testing**

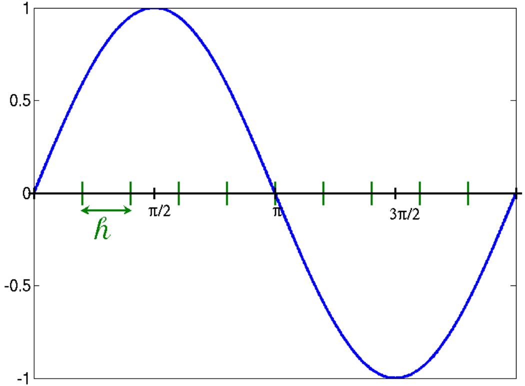
Numerical Difficulties

Numerical Difficulties



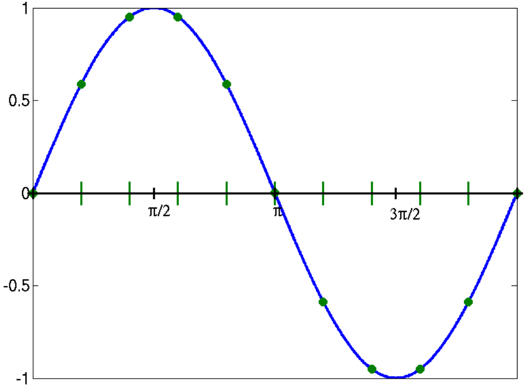
$$ka = 1$$

Numerical Difficulties



$$ka = 1, \quad \frac{h}{a} = \frac{1}{10}$$

Numerical Difficulties



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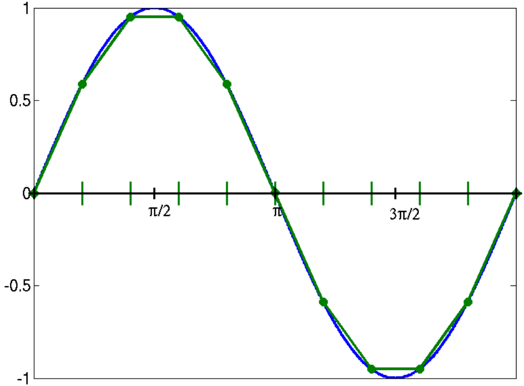

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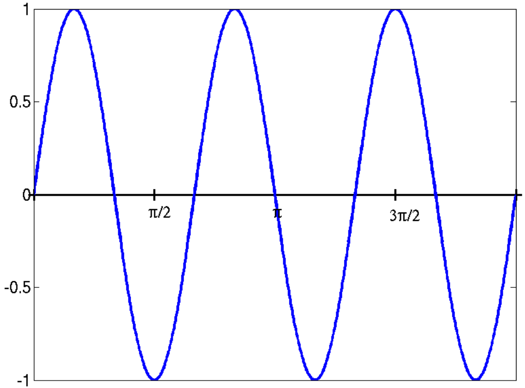
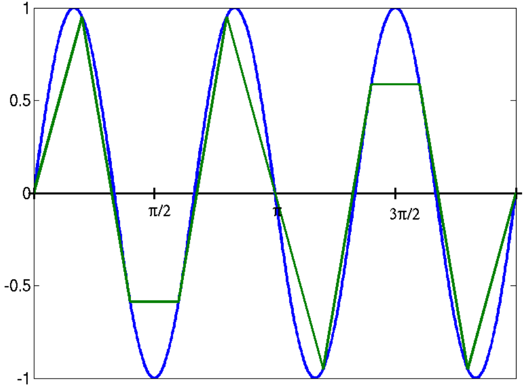


$$ka = 3$$

Numerical Difficulties



$$ka = 3, \quad \frac{h}{a} = \frac{1}{10}$$

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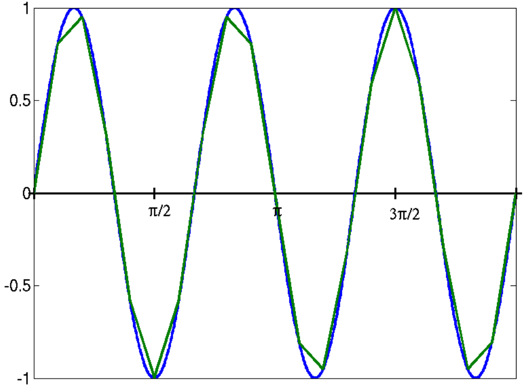
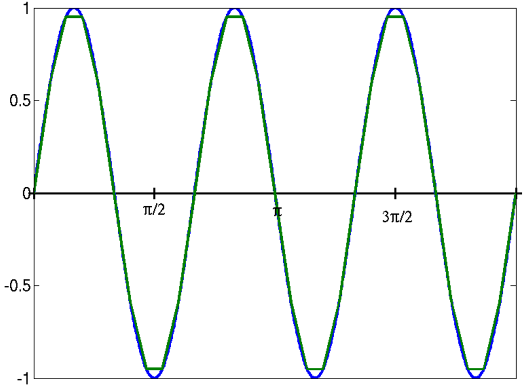


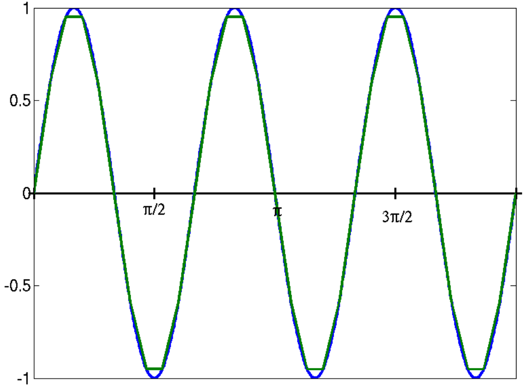
$$ka = 3, \quad \frac{h}{a} = \frac{1}{20}$$

Numerical Difficulties



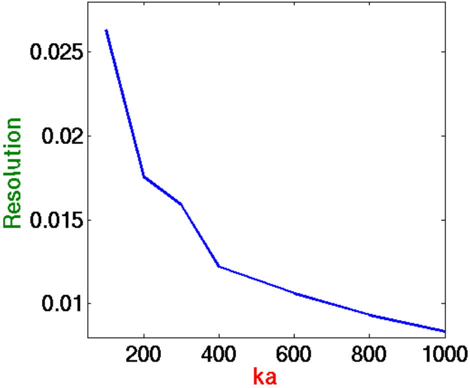
$$ka = 3, \quad \frac{h}{a} = \frac{1}{30}$$

Numerical Difficulties



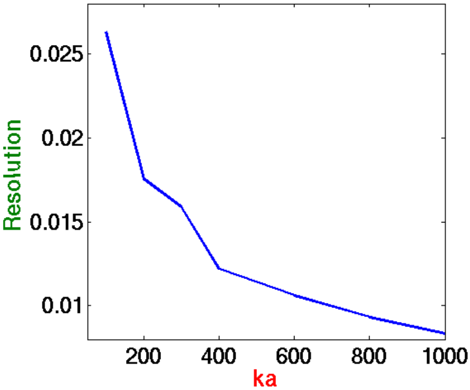
$$ka = 3, \quad \frac{h}{a} = \frac{1}{30} \Longrightarrow kh = \frac{1}{10}$$

Numerical Difficulties



Resolution necessary to achieve 10% on the relative error

Numerical Difficulties



Resolution necessary to achieve 10% on the relative error

$\Longrightarrow kh \neq$ **constant**

Numerical Difficulties

$$\frac{|u - u_h|_1}{|u|_1} \leq C_1 kh + C_2 k^3 h^2; \quad kh < 1$$

(Babuška et al (95, 00))

Prominent Plane Waves Based Approaches

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- **Weak Element Method (Rose (75))**

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- Discontinuous Galerkin Method: DGM (Farhat *et al* (01, 03, 04, 05))

DGM Formulation (Farhat *et al*)

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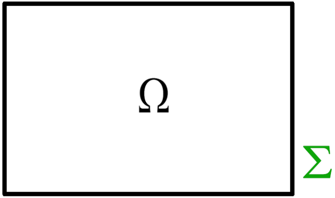
Mathematical model



DGM Formulation (Farhat *et al*)

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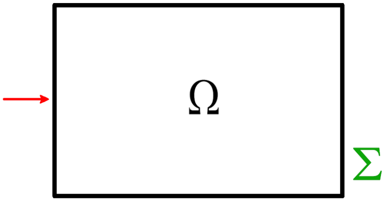

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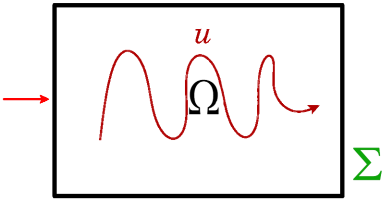

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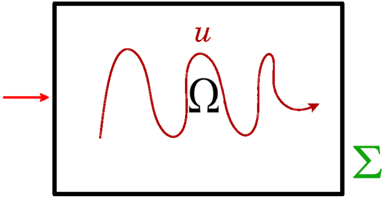

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega$$

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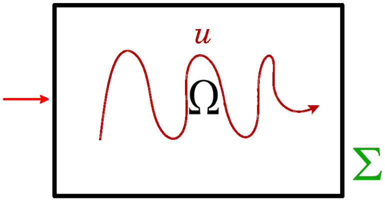

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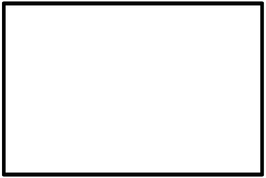
$$\partial_n u = \mathrm{i} k u + g \quad \text{on} \quad \Sigma$$

DGM Formulation (Farhat *et al*)

Approximation



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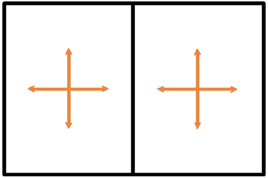

$$\boldsymbol{u} \equiv \sum_{l=1}^{4} \boldsymbol{u}_l^{\boldsymbol{K}} \boldsymbol{\phi}_l^{\boldsymbol{K}} \text{ in } \boldsymbol{K}$$

$$\boldsymbol{\phi}_l^{\boldsymbol{K}} = e^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}\cdot\boldsymbol{d}_l}$$

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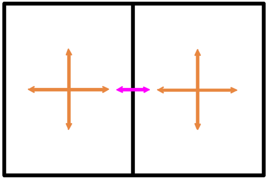


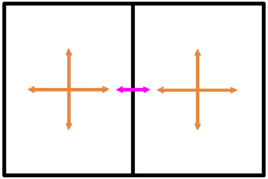
$$u \equiv \sum_{l=1}^{4} u_l^K \phi_l^K \text{ in } K$$

$$\phi_l^K = e^{\mathrm{i} k x \cdot d_l}$$

$$\lambda = \lambda^K \in \mathbb{C} \text{ on } \partial K \cap \partial K'$$

DGM Formulation (Farhat *et al*)

Variational Formulation



$$\left\{ \begin{array}{rcl} a(u, v) + b(v, \lambda) & = & F(v) \\ b(u, \mu) & = & 0 \end{array} \right.$$

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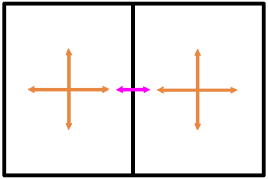
Variational Formulation



$$\begin{cases} a(u,v) + b(v,\lambda) & = & F(v) \\ b(u,\mu) & = & 0 \end{cases}$$

$$a(u,v) = \sum_{K \in \tau_h} \left(\int_K (\nabla u . \nabla \overline{v} - k^2 u \overline{v}) dx - \mathrm{i} k \int_{\partial K \cap \Sigma} u \overline{v} ds \right)$$

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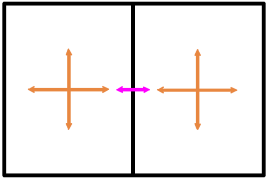
Variational Formulation



$$\begin{cases} a(u,v) + b(v,\lambda) & = & F(v) \\ b(u,\mu) & = & 0 \end{cases}$$

$$b(v,\mu) = \sum_{K\in\tau_h} \int_{\partial K\cap\partial K'} \mu\bar{v}\,ds$$

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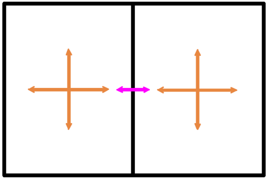
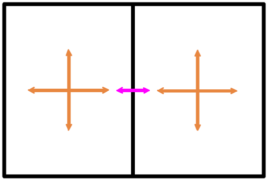


$$\left\{ \begin{array}{rcl} a(u, v) + b(v, \lambda) & = & F(v) \\ b(u, \mu) & = & 0 \end{array} \right.$$

$$F(v) = \sum_{K \in \tau_h} \int_{\partial K \cap \Sigma} g \bar{v} ds$$

DGM Formulation (Farhat *et al*)

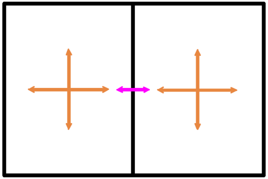
Algebraic Formulation



$$\left| \begin{aligned} \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{\lambda} &= \boldsymbol{f} \\ \boldsymbol{B}^T\boldsymbol{u} &= \boldsymbol{0} \end{aligned} \right.$$

DGM Formulation (Farhat *et al*)

Algebraic Formulation



$$\left| \begin{array}{rcl} A\boldsymbol{u} + B\boldsymbol{\lambda} & = & f \\ B^T\boldsymbol{u} & = & 0 \end{array} \right.$$

$$\Longrightarrow B^T A^{-1} B\boldsymbol{\lambda} = B^T A^{-1} f$$

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DGM Formulation (Farhat *et al*)

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- Global system: **symmetric** and **sparse**

DGM Formulation (Farhat *et al*)

Main Features

- **Plane waves** for **local** approximation of the field
- **Lagrange multipliers** to enforce continuity
- **Analytical** evaluation of the **elementary** matrices
- Global system: **symmetric** and **sparse**
- **Size** of the global system $\equiv \#$ Lagrange multipliers

DGM Formulation (Farhat *et al*)

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DGM **outperforms** high order FE methods:

- **R4-1, R8-2** require 5 to 7 times fewer dof than **Q2**

DGM Formulation (Farhat *et al*)

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- Q16-4 requires 6 times fewer dof than Q4

DGM Formulation (Farhat *et al*)

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DGM outperforms high order FE methods:

- R4-1, R8-2 require 5 to 7 times fewer dof than Q2
- Q16-4 requires 6 times fewer dof than Q4
- Q32-8 requires 25 times fewer dof than Q4

DGM Formulation (Farhat *et al*)

Issues

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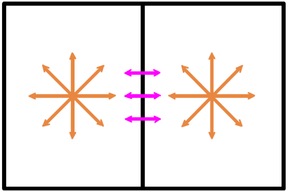
Issues

Inf-Sup condition: Discrete spaces compatibility



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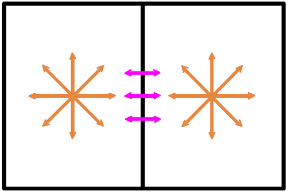
Inf-Sup condition: Discrete spaces compatibility



 # plane waves *vs.* # Lagrange Multipliers

?

DGM Formulation (Farhat *et al*)

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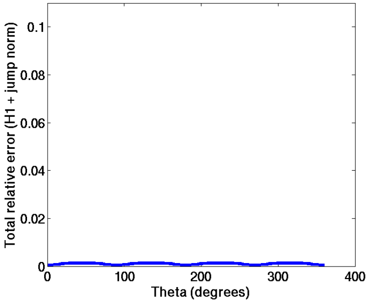
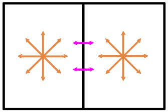


Relative error, ka=10



R8-2 element

DGM Formulation (Farhat *et al*)

Issues

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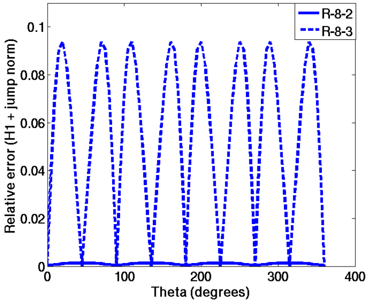
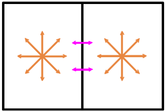
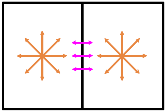


Relative error, ka=10



R8-2 element



R8-3 element

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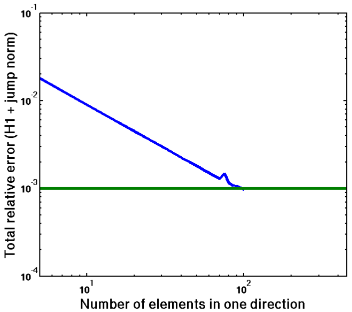
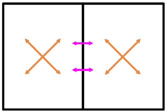


Total relative error, ka=1



R4-2 element

DGM Formulation (Farhat *et al*)

Issues

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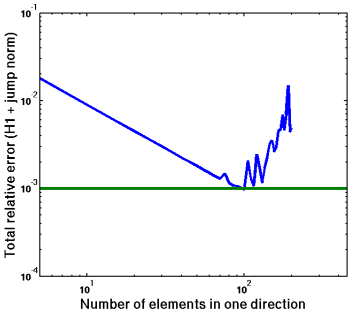
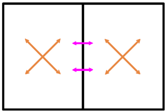


Total relative error, ka=1



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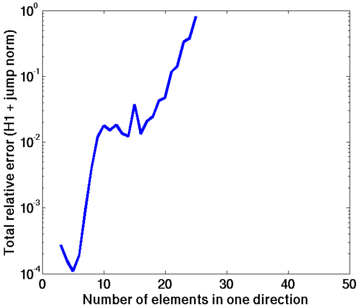
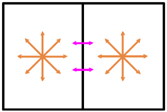


Total relative error, ka=1



R8-2 element

Our objective: build on top of DGM

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- **Preserve the good features of DGM**

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- **Preserve the good features of DGM**
- **Get rid of numerical instabilities**

DGM Formulation: Another point of view

$$\left\{ \begin{array}{rcll} \Delta u + k^2 u & = & 0 & \text{in } \Omega \\ \partial_n u & = & \mathrm{i} k u + g & \text{on } \Sigma \end{array} \right.$$

DGM Formulation: Another point of view

$$\begin{cases} \Delta u + k^2 u &= 0 & \text{in } \Omega \\ \partial_n u &= \mathrm{i}ku + g & \text{on } \Sigma \end{cases}$$

- Split the solution u:

$$u = \varphi + \Phi(\lambda)$$

DGM Formulation: Another point of view

$$\begin{cases} \Delta \varphi^K + k^2 \varphi^K = 0 & \text{in } K \\ \end{cases}$$

$$\begin{cases} \Delta \Phi^K(\lambda) + k^2 \Phi^K(\lambda) = 0 & \text{in } K \\ \end{cases}$$

DGM Formulation: Another point of view

$$\begin{cases} \Delta \varphi^K + k^2 \varphi^K & = & 0 & \text{in } K \\ \partial_n \varphi^K & = & \mathrm{i} k \varphi^K + g & \text{on } \partial K \cap \Sigma \end{cases}$$

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DGM Formulation: Another point of view

$$\begin{cases} \Delta\varphi^K + k^2\varphi^K = 0 & \text{in } K \\ \partial_n\varphi^K = ik\varphi^K + g & \text{on } \partial K \cap \Sigma \\ \partial_n\varphi^K = 0 & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$$\begin{cases} \Delta\Phi^K(\lambda) + k^2\Phi^K(\lambda) = 0 & \text{in } K \\ \partial_n\Phi^K(\lambda) = ik\Phi(\lambda) & \text{on } \partial K \cap \Sigma \\ \partial_n\Phi^K(\lambda) = \lambda & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$\lambda^{K'} = -\lambda^K$ on $\partial K \cap \partial K'$

DGM Formulation: Another point of view

$$\begin{cases} \Delta\varphi^K + k^2\varphi^K = 0 & \text{in } K \\ \partial_n\varphi^K = \mathrm{i}k\varphi^K + g & \text{on } \partial K \cap \Sigma \\ \partial_n\varphi^K = 0 & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$$\begin{cases} \Delta\Phi^K(\lambda) + k^2\Phi^K(\lambda) = 0 & \text{in } K \\ \partial_n\Phi^K(\lambda) = \mathrm{i}k\Phi(\lambda) & \text{on } \partial K \cap \Sigma \\ \partial_n\Phi^K(\lambda) = \lambda & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$\lambda^{K'} = -\lambda^K$ on $\partial K \cap \partial K'$

$[u] = [\varphi + \Phi(\lambda)] = 0$ on each interior edge

DGM Formulation: Another point of view

- **Solve local variational problems in each K:**

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma)\overline{w}\, ds = L(w)$$

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$$\sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)] \overline{\mu} = - \sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\varphi] \overline{\mu}$$

A new solution methodology for Helmholtz problems

A new DGM

A new DGM: first idea

Restore the weak continuity of the field in the least-squares sense:

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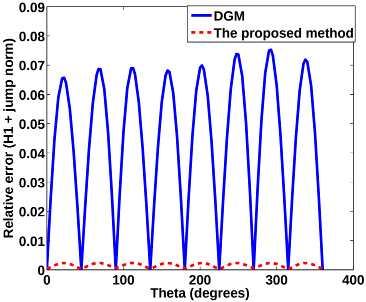
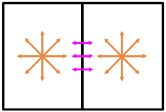
$\Longrightarrow$ **Hermitian and positive semi-definite global matrix**

A new DGM: Illustrative example of the improvement for a fixed resolution

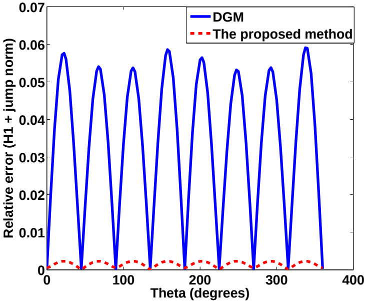


Relative error, ka=10

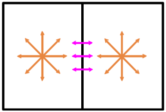


R-8-3 element

A new DGM: Illustrative example of the improvement for a fixed resolution



Relative error, ka=30



R-8-3 element

A new DGM: Persistance of the numerical instabilities

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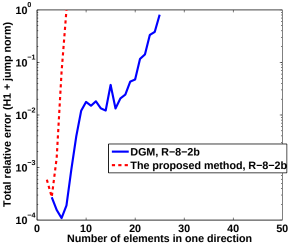
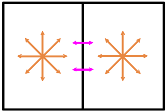


Total relative error, ka=1



R-8-2 element

A new DGM: second idea

Reformulate the local variational problem:

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$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma)(\partial_n \overline{w} + \mathrm{i}\, k \overline{w} \chi_\Sigma) = L(w)$$

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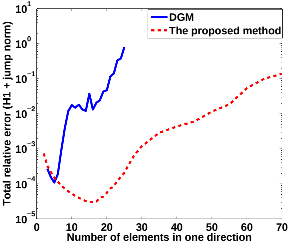
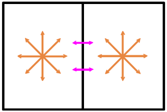


Total relative error, ka=1



R-8-2 element

A new DGM: Strategy resolution

- **Solve local variational problems in each K:**

$$a_K(v, w) = \int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma)(\partial_n \overline{w} + \mathrm{i}\, k \overline{w} \chi_\Sigma)$$

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- Solve **local** variational problems in each K:

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- Solve one **global** variational problem:

$$A(\lambda,\mu) = \sum_{e \subset \mathring{\Omega}} \Big(\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]][[\partial_n \overline{\Phi(\mu)}]]\Big)$$

A new DGM: A priori error estimates

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq \hat{C} \left[\sum_{K \in \tau_h} \frac{1}{k^4 h_K^3} \left(\|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|^2_{0,\partial K \cap \mathring{\Omega}} \right.\right.$$

$$\left.\left. + \|\boldsymbol{\lambda} - \partial_n \boldsymbol{v}_h\|^2_{0,\partial K \cap \mathring{\Omega}} \right)^{1/2} \right]$$

A new DGM: A priori error estimates

$$\|\lambda - \partial_n u_h\|_{0,\partial K \cap \mathring{\Omega}} \leq h_K^{N-\frac{1}{2}} \left(\sum_{l=0}^{N} k^{N+1-l} \left|\Phi(\lambda)\right|_{l,K} + \left|\Phi(\lambda)\right|_{N+1,K} + h_K \left|\Phi(\lambda)\right|_{N+2,K} \right)$$

A new DGM: A priori error estimates. Application

R-m-n element $(m \geq 2N+1)$:

$$\|u - u_h\|_{0,\Omega} \leq \frac{\hat{C}}{k^2} \Big[h^n \, |\Phi(\lambda)|_{n+2,\Omega} + h^{n-1} \, |\Phi(\lambda)|_{n-$$

$$+ h^{N-2} \Big(\sum_{l=0}^{N} k^{N+1-l} \, |\Phi(\lambda)|_{l,\Omega} + |\Phi(\lambda)|_{N+1,\Omega}$$

$$+ h \, |\Phi(\lambda)|_{N+2,\Omega} \Big) \Big]$$

A new DGM: Application in geophysical exploration

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- **Objective**: produce images of the subsurface from tomography measurements

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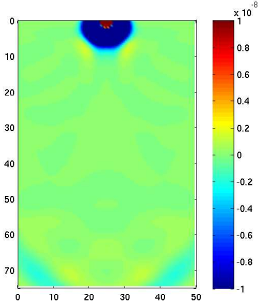
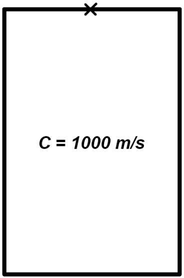
A new DGM: Application in geophysical exploration

- Objective: produce images of the subsurface from tomography measurements
- Wave propagation in time domain: Discrete Fourier Transform
- Solve Helmholtz equation (reduced wave equation): R-4-1 element
- Build the solution in time domain: Inverse Discrete Fourier Transform

A new DGM: Application in geophysical exploration





Homogeneous medium

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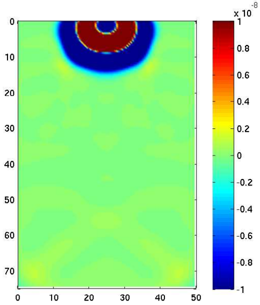

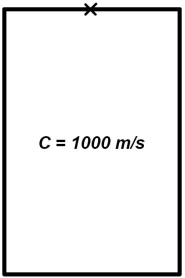

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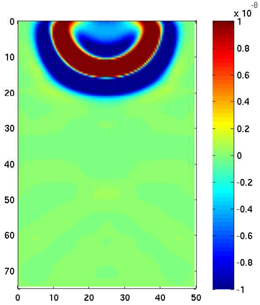

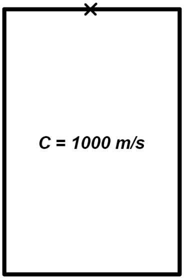

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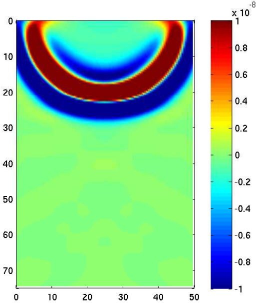

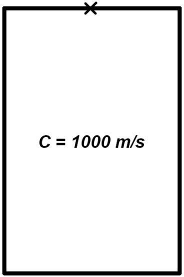

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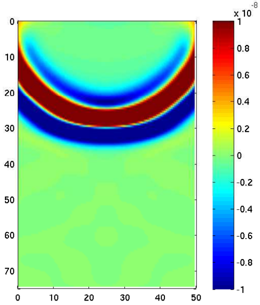

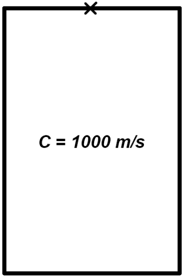

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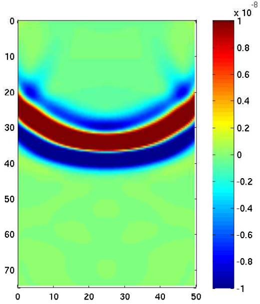

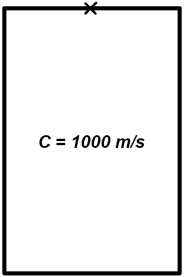

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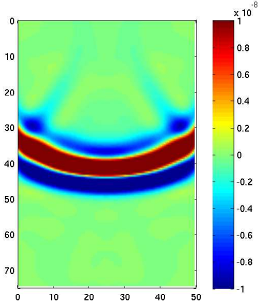

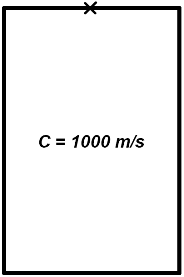

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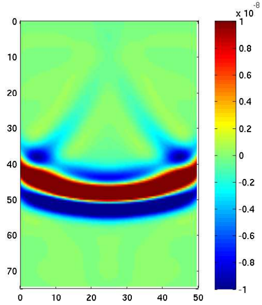

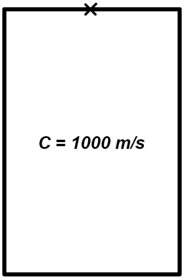

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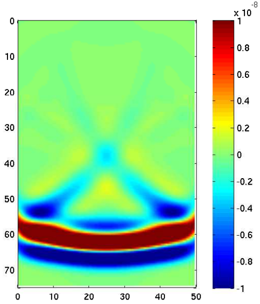

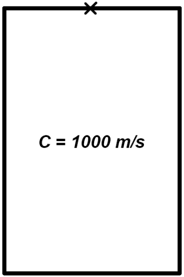

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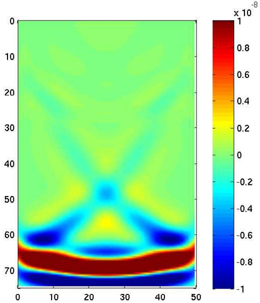

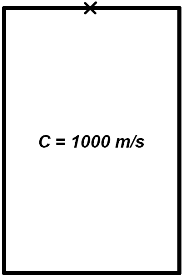

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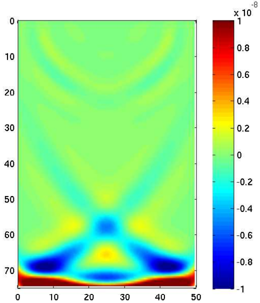

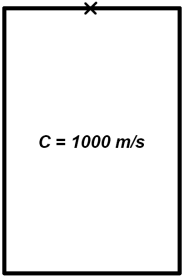

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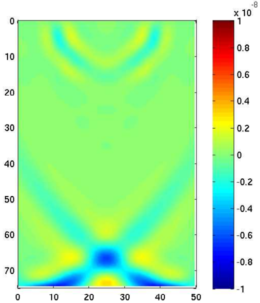

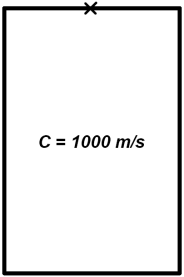

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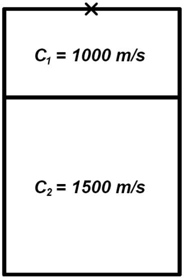

Stratified medium

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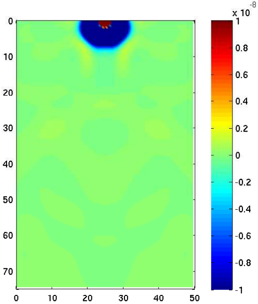

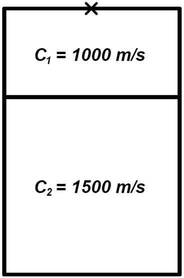

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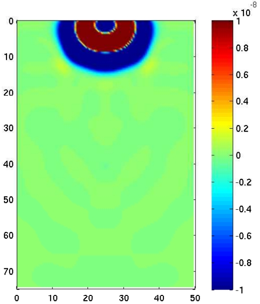

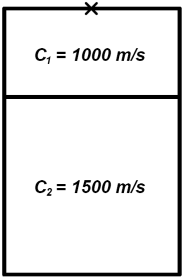

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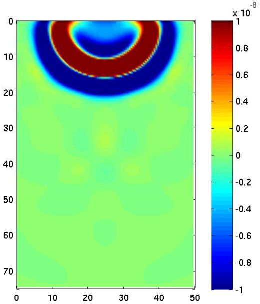

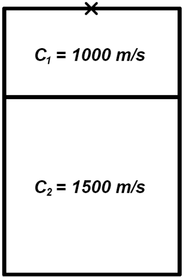

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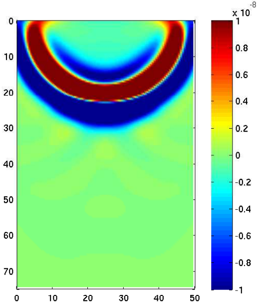

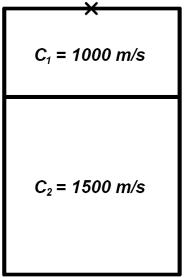

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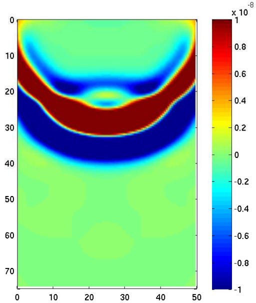

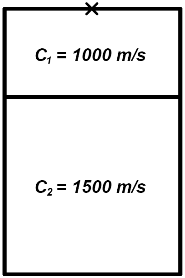

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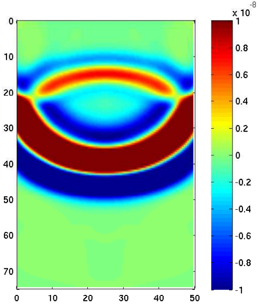

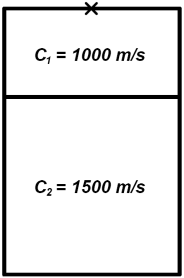

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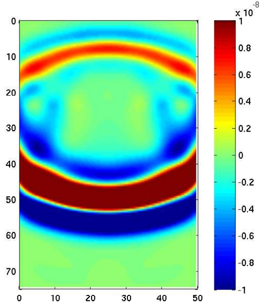

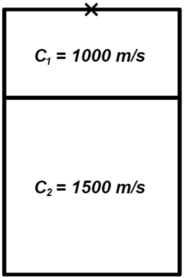

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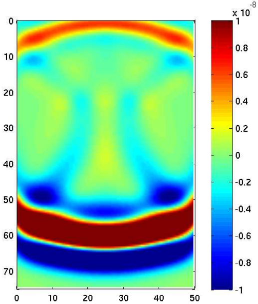

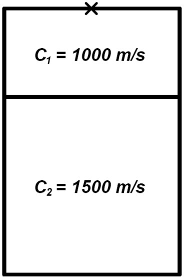

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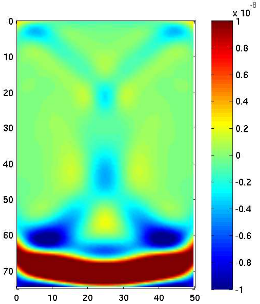

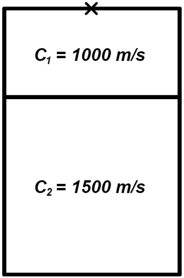

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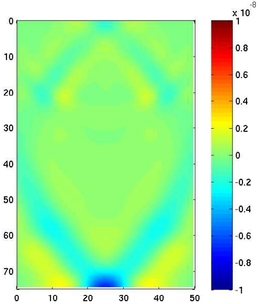

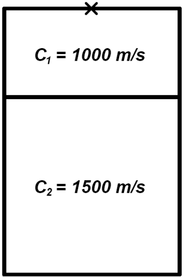

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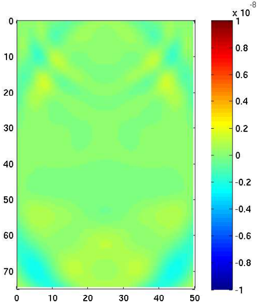
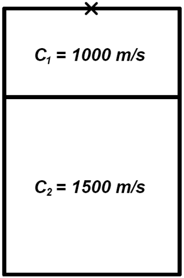
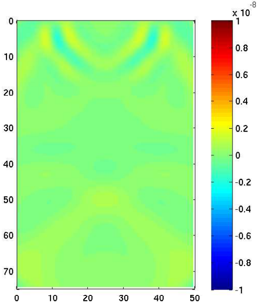
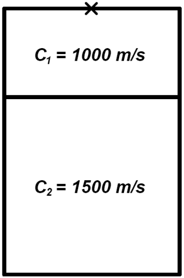




Stratified medium

A new DGM: Application in geophysical exploration





Stratified medium

A new DGM: Application in geophysical exploration

- **Multi-frequency solver**

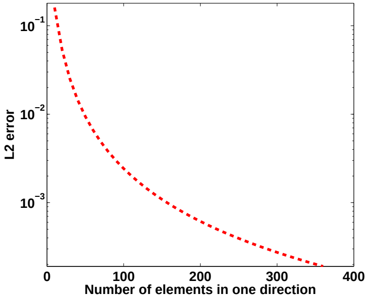
A new DGM: Application in geophysical exploration

- **Multi-frequency solver**
- $\boldsymbol{kh} \in [\frac{1}{50}, 2]$ **corresponds to 3 to 300 elements per wavelength**

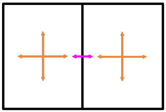
A new DGM: Application in geophysical exploration

- Multi-frequency solver
- $kh \in [\frac{1}{50}, 2]$ corresponds to 3 to 300 elements per wavelength
- R-4-1 element: accurate and stable element

A new DGM: Sensitivity of the error to the mesh refinement



L^2 error, ka=10



R-4-1 element

A new DGM: Summary

DGM:

- Solve **local** variational problems in each K:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma)\overline{w}\, ds = L(w)$$

- Solve **global** variational problem:

$$\sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)]\overline{\mu} = - \sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\varphi]\overline{\mu}$$

A new DGM: Summary

A new DGM:

- NEW local variational problems in each K:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma)(\partial_n \overline{w} + \mathrm{i}\, k \overline{w} \chi_\Sigma) = L(w)$$

- Solve global variational problem:

$$\sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)] \overline{\mu} = - \sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\varphi] \overline{\mu}$$

A new DGM: Summary

A new DGM:

- NEW local variational problems in each K:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma)(\partial_n \overline{w} + \mathrm{i}\, k \overline{w} \chi_\Sigma) = L(w)$$

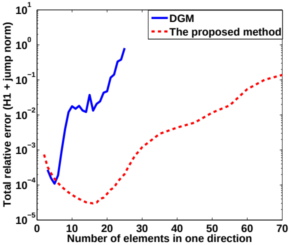
- NEW global variational problem:

$$\sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]][[\partial_n \overline{\Phi(\mu)}]] \right) =$$

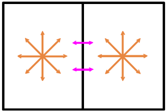
$$- \sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\varphi][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \varphi]][[\partial_n \overline{\Phi(\mu)}]] \right)$$

A new DGM: Summary

- Numerical instabilities: improvement, but not enough



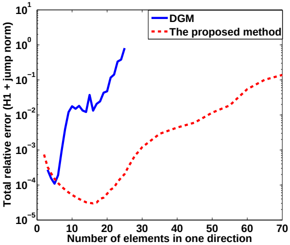
Total relative error, ka=1



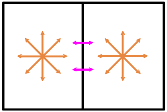
R-8-2 element

A new DGM: Summary

- Numerical instabilities: improvement, but not enough



Total relative error, ka=1



R-8-2 element

- Source of the instabilities: local problems nearly singular

A new solution methodology for Helmholtz problems

A modified DGM (mDGM)

mDGM: main idea

Modify the local problems:

$$\begin{cases} \Delta\varphi^K + k^2\varphi^K & = & 0 & \text{in } K \\ \partial_n\varphi^K - \mathrm{i}k\varphi^K & = & g & \text{on } \partial K \cap \Sigma \\ \partial_n\varphi^K & = & 0 & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$$\begin{cases} \Delta\Phi^K(\lambda) + k^2\Phi^K(\lambda) & = & 0 & \text{in } K \\ \partial_n\Phi^K(\lambda) - \mathrm{i}k\Phi^K(\lambda) & = & 0 & \text{on } \partial K \cap \Sigma \\ \partial_n\Phi^K(\lambda) & = & \lambda & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

mDGM: main idea

Modify the local problems:

$$\begin{cases} \Delta \varphi^K + k^2 \varphi^K = 0 & \text{in } K \\ \partial_n \varphi^K - \mathrm{i} k \varphi^K = g & \text{on } \partial K \cap \Sigma \\ \partial_n \varphi^K - \mathrm{i}\, \alpha \varphi^K = 0 & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$$\begin{cases} \Delta \Phi^K(\lambda) + k^2 \Phi^K(\lambda) = 0 & \text{in } K \\ \partial_n \Phi^K(\lambda) - \mathrm{i} k \Phi^K(\lambda) = 0 & \text{on } \partial K \cap \Sigma \\ \partial_n \Phi^K(\lambda) - \mathrm{i}\, \alpha \Phi^K(\lambda) = \lambda & \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

mDGM: main idea

Modify the local problems:

$$\left\{\begin{array}{rcl} \Delta\varphi^K + k^2\varphi^K & = & 0 \quad \text{in } K \\ \partial_n\varphi^K - \mathrm{i}k\varphi^K & = & g \quad \text{on } \partial K \cap \Sigma \\ \partial_n\varphi^K - \mathrm{i}\,\alpha\varphi^K & = & 0 \quad \text{on } \partial K \cap \mathring{\Omega} \end{array}\right.$$

$$\left\{\begin{array}{rcl} \Delta\Phi^K(\lambda) + k^2\Phi^K(\lambda) & = & 0 \quad \text{in } K \\ \partial_n\Phi^K(\lambda) - \mathrm{i}k\Phi^K(\lambda) & = & 0 \quad \text{on } \partial K \cap \Sigma \\ \partial_n\Phi^K(\lambda) - \mathrm{i}\,\alpha\Phi^K(\lambda) & = & \lambda \quad \text{on } \partial K \cap \mathring{\Omega} \end{array}\right.$$

$\alpha \in \mathbb{R}_+^* \Longrightarrow$ **well posed local problems**

mDGM: main idea

Modify the local problems:

$$\begin{cases} \Delta\varphi^K + k^2\varphi^K &= 0 \quad \text{in } K \\ \partial_n\varphi^K - \mathrm{i}k\varphi^K &= g \quad \text{on } \partial K \cap \Sigma \\ \partial_n\varphi^K - \mathrm{i}\,\alpha\varphi^K &= 0 \quad \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$$\begin{cases} \Delta\Phi^K(\lambda) + k^2\Phi^K(\lambda) &= 0 \quad \text{in } K \\ \partial_n\Phi^K(\lambda) - \mathrm{i}k\Phi^K(\lambda) &= 0 \quad \text{on } \partial K \cap \Sigma \\ \partial_n\Phi^K(\lambda) - \mathrm{i}\,\alpha\Phi^K(\lambda) &= \lambda \quad \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$\alpha \in \mathbb{R}_+^* \implies$ well posed local problems

$[u] = [\varphi + \Phi(\lambda)] = 0$ on each interior edge

mDGM: main idea

Modify the local problems:

$$\begin{cases} \Delta\varphi^K + k^2\varphi^K &= 0 \quad \text{in } K \\ \partial_n\varphi^K - \mathrm{i}k\varphi^K &= g \quad \text{on } \partial K \cap \Sigma \\ \partial_n\varphi^K - \mathrm{i}\,\alpha\varphi^K &= 0 \quad \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$$\begin{cases} \Delta\Phi^K(\lambda) + k^2\Phi^K(\lambda) &= 0 \quad \text{in } K \\ \partial_n\Phi^K(\lambda) - \mathrm{i}k\Phi^K(\lambda) &= 0 \quad \text{on } \partial K \cap \Sigma \\ \partial_n\Phi^K(\lambda) - \mathrm{i}\,\alpha\Phi^K(\lambda) &= \lambda \quad \text{on } \partial K \cap \mathring{\Omega} \end{cases}$$

$\alpha \in \mathbb{R}_+^* \implies$ well posed local problems

$[u] = [\varphi + \Phi(\lambda)] = 0$ on each interior edge

$[[\partial_n u]] = [[\partial_n\varphi + \partial_n\Phi(\lambda)]] = 0$ on each interior edge

mDGM: Strategy resolution

- **Solve local variational problems in each K:**

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v)\overline{w}\, ds = L(w)$$

mDGM: Strategy resolution

- **Solve local variational problems in each K:**

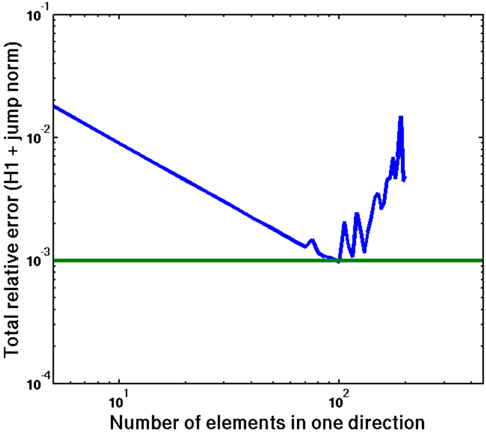
$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v)\overline{w}\, ds = L(w)$$

- **Solve one global variational problem:**

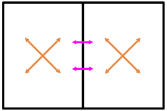
$$\sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]][[\partial_n \overline{\Phi(\mu)}]] \right) =$$

$$-\sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\varphi][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \varphi]][[\partial_n \overline{\Phi(\mu)}]] \right)$$

mDGM: Performance assessment

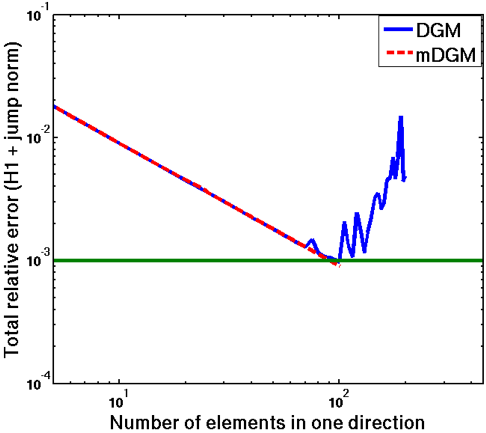


Total relative error, ka=1

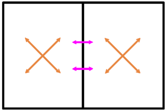


R-4-2 element

mDGM: Performance assessment

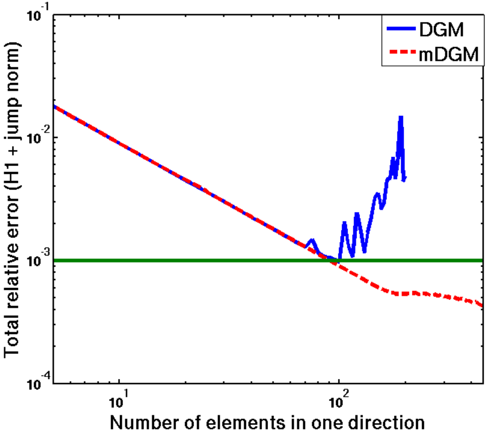


Total relative error, ka=1

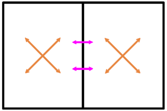


R-4-2 element

mDGM: Performance assessment

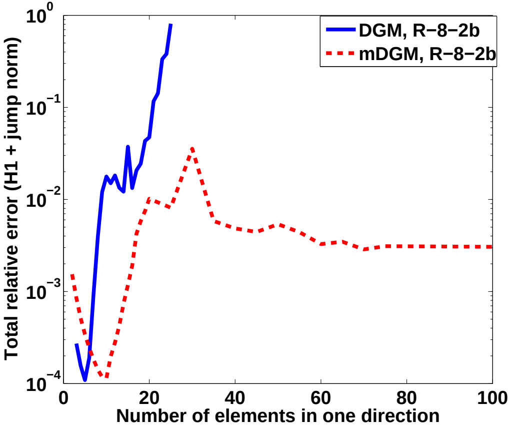


Total relative error, ka=1

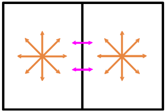


R-4-2 element

mDGM: Performance assessment

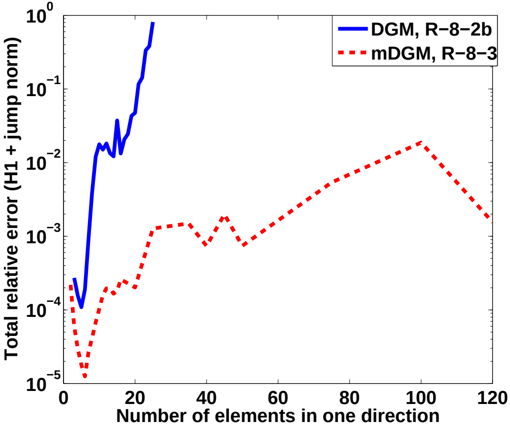


Total relative error, ka=1

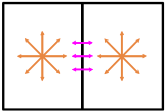


R-8-2 element

mDGM: Performance assessment

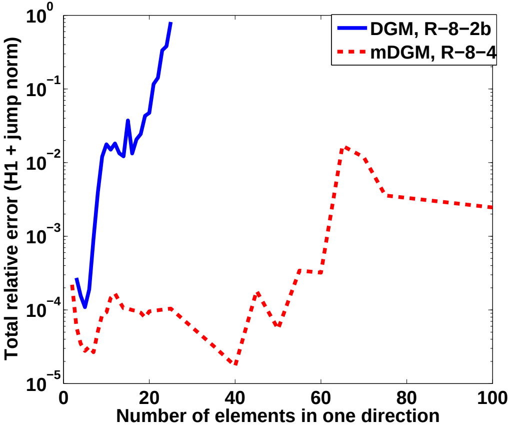


Total relative error, ka=1

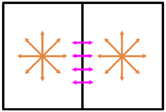


R-8-3 element

mDGM: Performance assessment

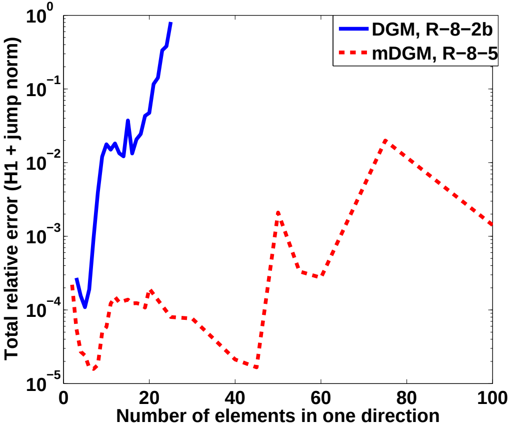


Total relative error, ka=1

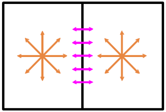


R-8-4 element

mDGM: Performance assessment



Total relative error, ka=1



R-8-5 element

mDGM: Implementation issue

$$\boldsymbol{F}_{jl} = \sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\boldsymbol{\Phi}_h(\boldsymbol{\mu}_l)][\overline{\boldsymbol{\Phi}_h(\boldsymbol{\mu}_j)}] \right.$$

$$\left. + \gamma_e \int_e [[\partial_n \boldsymbol{\Phi}_h(\boldsymbol{\mu}_l)]][[\partial_n \overline{\boldsymbol{\Phi}_h(\boldsymbol{\mu}_j)}]] \right)$$

$\boldsymbol{F}$ is Hermitian

mDGM: Implementation issue

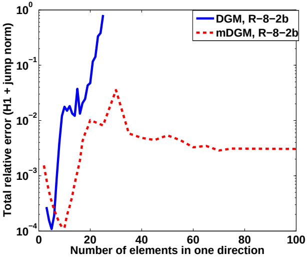
$$F_{jl} = \sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\Phi_h(\mu_l)][\overline{\Phi_h(\mu_j)}] \right.$$

$$\left. + \gamma_e \int_e [[\partial_n \Phi_h(\mu_l)]][[\partial_n \overline{\Phi_h(\mu_j)}]] \right)$$

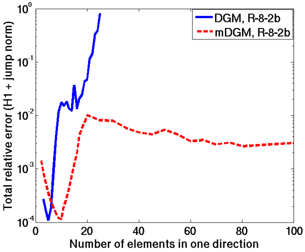
⚠ F is Hermitian, but NOT in practice

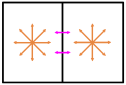
mDGM: Implementation issue

Before



After



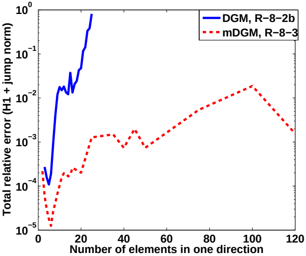


R-8-2 element

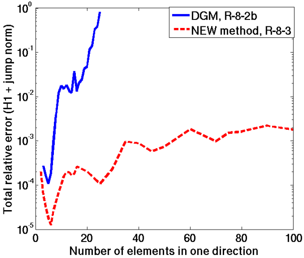
Total relative error, ka=1

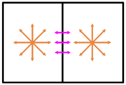
mDGM: Implementation issue

Before



After



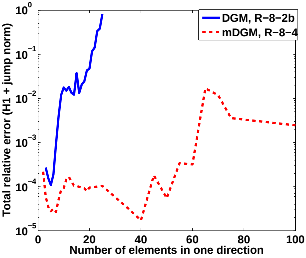


R-8-3 element

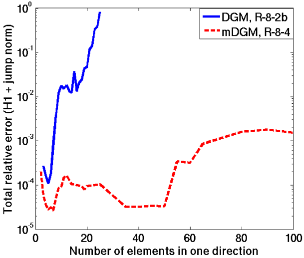
Total relative error, ka=1

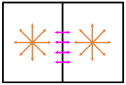
mDGM: Implementation issue

Before



After



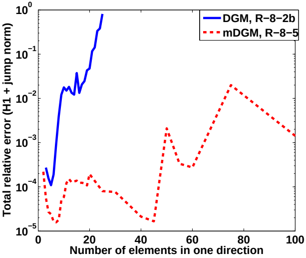


R-8-4
element

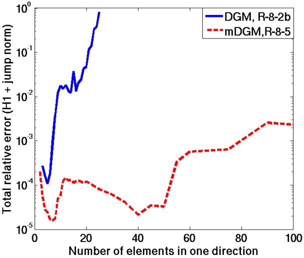
Total relative error, ka=1

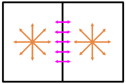
mDGM: Implementation issue

Before



After

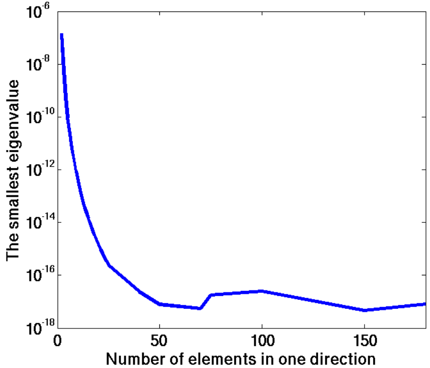




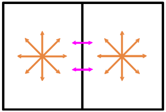
R-8-5
element

Total relative error, ka=1

mDGM: Numerical issue

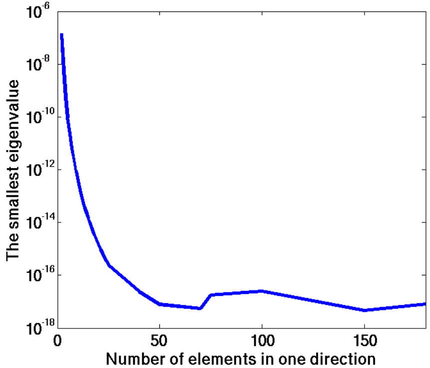


The smallest eigenvalue, ka=1

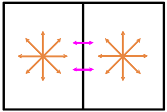


R-8-2 element

mDGM: Numerical issue



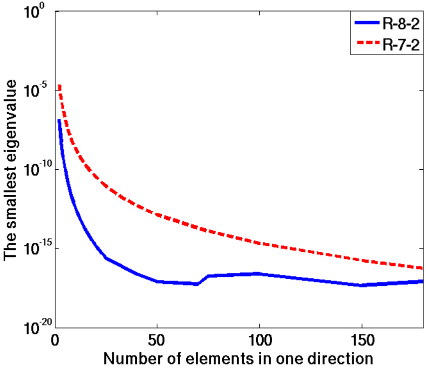
The smallest eigenvalue, ka=1



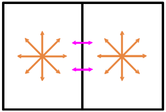
R-8-2 element

Loss of the linear independence

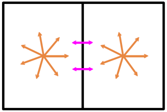
mDGM: Numerical issue



The smallest eigenvalue, ka=1

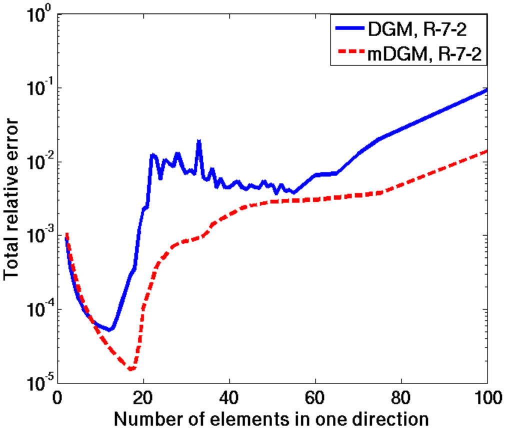


R-8-2 element

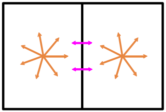


R-7-2 element

mDGM: Numerical issue



Total relative error, ka=1



R-7-2 element

mDGM: Summary

DGM:

- Solve local variational problems in each K:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma) \overline{w}\, ds = L(w)$$

- Solve global variational problem:

$$\sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\Phi(\lambda)] \overline{\mu} = - \sum_{e \subset \mathring{\Omega}} \frac{1}{|e|} \int_e [\varphi] \overline{\mu}$$

mDGM: Summary

A new DGM:

- **NEW local** variational problems in each K:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v \chi_\Sigma)(\partial_n \overline{w} + \mathrm{i}\, k \overline{w} \chi_\Sigma) = L(w)$$

- **NEW global** variational problem:

$$\sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]][[\partial_n \overline{\Phi(\mu)}]] \right) =$$

$$- \sum_{e \subset \Omega} \left(\beta_e \int_e [\varphi][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \varphi]][[\partial_n \overline{\Phi(\mu)}]] \right)$$

mDGM: Summary

mDGM:

- NEW local variational problems in each K:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v)\overline{w}\, ds = L(w)$$

mDGM: Summary

mDGM:

- **NEW local variational problems in each K:**

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v)\overline{w}\, ds = L(w)$$

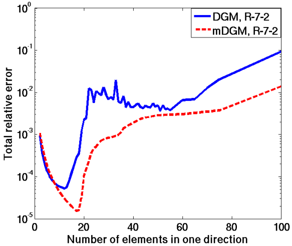
- **Solve one global variational problem:**

$$\sum_{e \subset \mathring{\Omega}} \left(\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]][[\partial_n \overline{\Phi(\mu)}]] \right) =$$

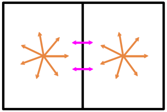
$$- \sum_{e \subset \Omega} \left(\beta_e \int_e [\varphi][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \varphi]][[\partial_n \overline{\Phi(\mu)}]] \right)$$

mDGM: Summary

- Numerical instabilities: improvement, but not enough



Total relative error, ka=1



R-7-2 element

mDGM: Summary

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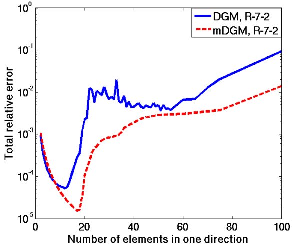
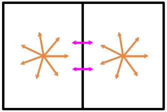


Total relative error, ka=1



R-7-2 element

- Source of the instabilities: loss of the linear independence

A new solution methodology for Helmholtz problems

An improved modified DGM (imDGM)

imDGM: main idea

Reformulate the local variational problems:

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Reformulate the local variational problems:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v)(\partial_n \overline{w} + \mathrm{i}\, k \overline{w})\, ds = L(w)$$

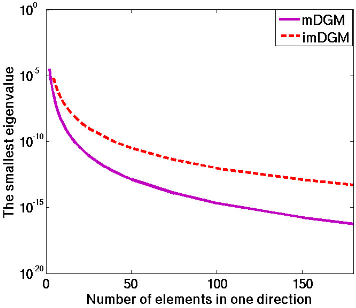
imDGM: main idea

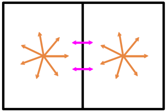
Reformulate the local variational problems:

$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v)(\partial_n \overline{w} + \mathrm{i}\, k \overline{w})\, ds = L(w)$$

$\Longrightarrow$ **Hermitian and positive definite local matrix**

imDGM: Performance assessment

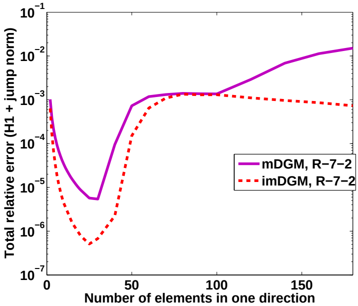




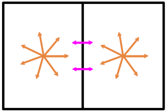
R-7-2 element

The smallest eigenvalue, ka=1

imDGM: Performance assessment



Total relative error, ka=1



R-7-2 element

imDGM: Comparison with DGM and LSM

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- **An improved modified DGM (imDGM)**

$$\sum_{e\subset\mathring{\Omega}}\left(\beta_e\int_e[\Phi(\lambda)][\overline{\Phi(\mu)}]+\gamma_e\int_e[[\partial_n\Phi(\lambda)]][[\partial_n\overline{\Phi(\mu)}]]\right)$$

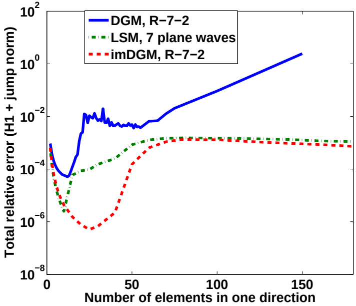
- **Least-Squares Method (LSM) - Monk-Wang (1999)**

$$\sum_{e\subset\mathring{\Omega}}\left(\beta_e\int_e[u][\overline{v}]+\gamma_e\int_e[[\partial_n u]][[\partial_n\overline{v}]]\right)$$

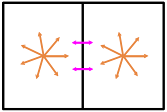
- **Discontinuous Galerkin Method (DGM)**

$$\sum_{e\subset\mathring{\Omega}}\left(\beta_e\int_e[\Phi(\lambda)]\overline{\mu}\right)$$

imDGM: Comparison with DGM and LSM



Total relative error, ka=1



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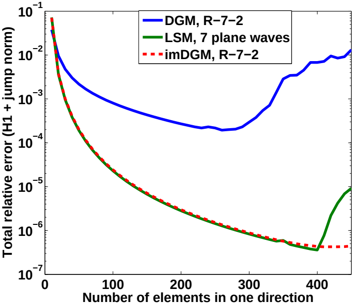
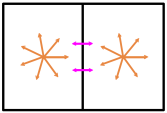


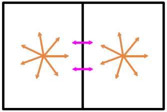


Total relative error, ka=20

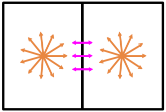
R-7-2 element

imDGM: Performance assessment for a fixed resolution: kh=2

ka	R-7-2	R-11-3
50	28%	0.05%



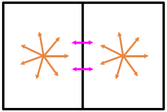
R-7-2 element



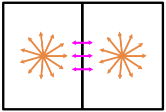
R-11-3 element

imDGM: Performance assessment for a fixed resolution: kh=2

ka	R-7-2	R-11-3
50	28%	0.05%
100	51%	0.07%



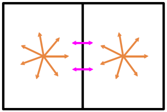
R-7-2 element



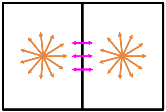
R-11-3 element

imDGM: Performance assessment for a fixed resolution: kh=2

ka	R-7-2	R-11-3
50	28%	0.05%
100	51%	0.07%
200	69%	0.20%



R-7-2 element



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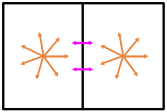
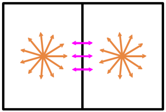


R-7-2 element

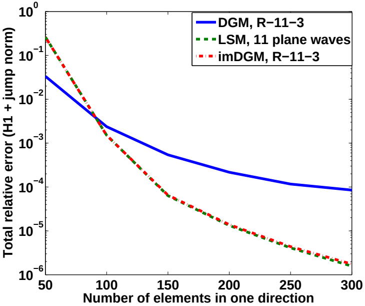


R-11-3 element

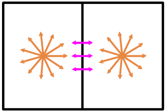
Computational cost increased by 50%

Gain of more than two orders of magnitude

imDGM: Comparison with DGM and LSM for a fixed frequency

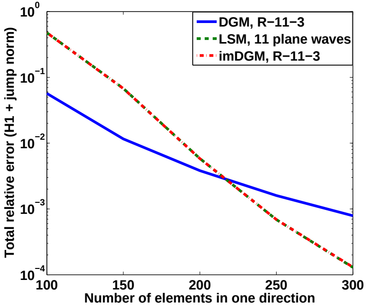


Total relative error, ka=200

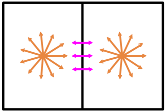


R-11-3 element

imDGM: Comparison with DGM and LSM for a fixed frequency



Total relative error, ka=400



R-11-3 element

Summary and Perspectives

Accomplishment: designed and implemented a new solution methodology for solving Helmholtz problems

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- Small well-posed local problems

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- Small **well-posed** local problems
- Global Hermitian, **positive semi-definite** and sparse system
- Performance: **outperforms** DGM (accuracy and stability)

Short-term goals

- **Assess the performance of higher-order elements**

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- **Extend the mathematical analysis to imDGM**

Long-term goals

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- **Elasto-acoustic scattering problems**

Thank you for your attention!

- **Local space of the primal variable:** $\forall \boldsymbol{K} \in \boldsymbol{\tau_h}$

$$\mathcal{V}(\boldsymbol{K}) = \left\{ \begin{array}{c} \boldsymbol{v^K} \in \boldsymbol{H^1(K)};\ \boldsymbol{\Delta v^K} + \boldsymbol{k^2 v^K} = \boldsymbol{0} \text{ in } \boldsymbol{K}, \\ \\ \boldsymbol{\partial_n v^K} \in \boldsymbol{L^2(\partial K)},\ \boldsymbol{\partial_n v^K} = \mathbf{i}\ \boldsymbol{k v^K} \text{ on } \boldsymbol{\partial K} \cap \boldsymbol{\Sigma} \end{array} \right\}$$

A new DGM: Mathematical framework

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- **Space of the dual variable:**

$$\mathcal{M} = \left\{ \begin{array}{c} \mu \in \prod_{K \in \tau_h} L^2(\partial K),\ \forall K \in \tau_h,\ \mu^K = 0 \text{ on } \partial K \cap \Sigma \\ \forall K, K' \in \tau_h,\ \mu^K + \mu^{K'} = 0 \text{ on } \partial K \cap \partial K' \end{array} \right\}$$

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- $\boldsymbol{\mathcal{V}_h(K) \subset \mathcal{V}(K)}$, $\boldsymbol{\mathcal{M}_h \subset \mathcal{M}}$ - **finite-dimensional spaces**

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$\Longrightarrow$Hermitian and positive semi-definite global matrix

Local formulation

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$$\int_{\partial K} (\partial_n v - \mathrm{i}\, k v) \overline{w}\, ds = L(w)$$

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$$\sum_{e-\text{interior edge}} \left(\beta_e \int_e [\Phi(\lambda)][\overline{\Phi(\mu)}] + \gamma_e \int_e [[\partial_n \Phi(\lambda)]][[\partial_n \overline{\Phi(\mu)}]] \right)$$