Série discrète unitaire, caractères, fusion de Connes et sous-facteurs pour l'algèbre Neveu-Schwarz.

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1 Introduction en français

1.1 Contexte

Dans les années 90, V. Jones and A. Wassermann ont commencé un programme dont le but est de comprendre la théorie (unitaire) conforme des champs du point de vue des algèbres d'opérateurs (voir [46], [98]). Dans [99], Wassermann définit et calcule la fusion de Connes des représentations d'énergie positive irreductibles du groupe de lacets $LSU(n)$ à niveau fixé ℓ, en utilisant des champs primaires, et avec des conséquences en théorie des sous-facteurs. Dans [87] V. Toledano Laredo prouve les règles de fusion de Connes pour $LSpin(2n)$ en utilisant des méthodes similaires. Maintenant, soit $\mathrm{Diff}(\mathbb{S}^1)$ le groupe de difféomorphisme du cercle, son algèbre de Lie est l'algèbre de Witt $\mathfrak{W}$ engendrée par d_n ($n \in \mathbb{Z}$), avec $[d_m, d_n] = (m-n)d_{m+n}$. Elle admet une unique extension centrale appelée algèbre de Virasoro $\mathfrak{Vir}$. Ses représentations unitaires d'énergie positive et les formules de caractères peuvent être déduites de la construction 'coset' de Goddard-Kent-Olive (GKO), à partir de la théorie de $LSU(2)$ et des formules de Kac-Weyl (voir [100], [35]). Dans [66], T. Loke utilise la construction 'coset' pour calculer la fusion de Connes pour $\mathfrak{Vir}$. Maintenant, l'algèbre de Witt admet deux extensions supersymétriques $\mathfrak{W}_0$ et $\mathfrak{W}_{1/2}$ avec des extensions centrales appelées algèbres Ramond et Neveu-Schwarz, notées $\mathfrak{Vir}_0$ et $\mathfrak{Vir}_{1/2}$. Dans ce travail, on donne une preuve complète de la classification des représentations unitaires d'énergie positive de $\mathfrak{Vir}_{1/2}$, on calcule leur caractères et la fusion de Connes, avec des conséquences en théorie des sous-facteurs. On pourrait faire de même avec l'algèbre Ramond $\mathfrak{Vir}_0$, en utilisant des modules vertex tordus sur l'algèbre d'opérateurs vertex de l'algèbre Neveu-Schwarz $\mathfrak{Vir}_{1/2}$, comme R. W. Verrill [96] et Wassermann [102] l'ont fait pour les groupes de lacets tordus.

1.2 Aperçu

Tout d'abord, on regarde les représentations unitaires projectives d'énergie positive de $\mathfrak{W}_{1/2}$. La projectivité donne des 2-cocycles, donnant à $\mathfrak{W}_{1/2}$, une unique extension centrale $\mathfrak{Vir}_{1/2}$. Ces représentations sont complètement réductibles et les irréductibles sont données par les représentations unitaires de plus haut poids de $\mathfrak{Vir}_{1/2}$: des modules de Verma $V(c,h)$ quotientés par les 'vecteurs nuls', dans le cas 'sans fantôme'.

A partir de l'algèbre des fermions sur $H = \mathcal{F}_{NS}$, on construit le champ fermion $\psi(z)$. La localité et le lemme de Dong permettent, grâce à des OPE (operator product expansion), d'engendrer un ensemble de champs $\mathcal{S}$, tel qu'il existe une bijection $V : H \to \mathcal{S}$, avec $Id = V(\Omega)$ et un champ Virasoro $L = V(\omega)$. Ensuite, on donne les axiomes de superalgèbre d'opérateurs vertex, permettant d'aller jusque là dans un cadre général (H, V, Ω, ω), avec H un espace préhilbertien.

Soit $\mathfrak{g}$ une algèbre de Lie simple de dimension fini, $\widehat{\mathfrak{g}}_+$ l'algèbre $\mathfrak{g}$-boson (une extension centrale de l'algèbre de lacets $L\mathfrak{g}$) et $\widehat{\mathfrak{g}}_-$ l'algèbre $\mathfrak{g}$-fermion. On construit un module de superalgèbre d'opérateur vertex à partir de $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$ sur $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$, tel que $\mathfrak{Vir}_{1/2}$ y agit avec $(c,h) = (\frac{3}{2} \cdot \frac{\ell + 1/3g}{\ell + g} dim(\mathfrak{g}), \frac{c_{V_\lambda}}{2(\ell+g)})$, avec g le nombre de Coxeter dual et c_{V_λ} le nombre de Casimir.

Soit $\mathfrak{g} = \mathfrak{sl}_2$, en utilisant le cadre des fonctions thêta, on obtient la décomposition de $H = \mathcal{F}_{NS}^{\mathfrak{g}} \otimes (L(j,\ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}})$ comme $\widehat{\mathfrak{g}}$-module. Les espaces de multiplicité des composantes irréductibles sont des espaces de superentrelacement $Hom_{\widehat{\mathfrak{g}}}(H_k, H)$; on en déduit leur caractère comme module de $\mathfrak{Vir}_{1/2}$ qui agit avec $L(c_m, h_{pq}^m)$ comme sous-module par construction 'coset'. L'unitarité de la série discrète s'ensuit.

On définit des polynômes irréductibles $\varphi_{pq}(c,h)$ à partir de (c_m, h_{pq}^m). Le déterminant de Kac $det_n(c,h)$ de la forme sesquilinéaire sur $V(c,h)$ à niveau n est facilement interpolé comme un produit de φ_{pq}, en calculant les exemples pour n petit. Pour le prouver, on met en lumière des liens entre des résultats précedents sur les caractères et des vecteurs singuliers s (i.e. $G_{1/2}.s = G_{3/2}.s = 0$), dont l'existence annule det_n.

Un déterminant de Kac négatif montre facilement un 'fantôme' dans la région entre les courbes $h = h_{pq}^c$. Maintenant, on part de la région 'sans fantôme' $h > 0$, $c > 3/2$, vers une courbe d'annulation C d'ordre 1; ainsi, de l'autre côté de C, il y a un 'fantôme'. Par transversalité, il reste sur la prochaine courbe intersectant C; et ainsi de suite sur chaque courbes, à l'exception des 'premières intersections': la série discrète. Le théorème 1.2 s'ensuit.

Finalement, un argument de cohérence entre les caractères des espaces de multiplicité M^m_{pq} et ses irréductibles (dans la série discrète par FQS), montre M^m_{pq} sans autre irréductible que $L(c_m, h^m_{rs})$. Ainsi, $M^m_{pq} = L(c_m, h^m_{p,q})$ et on obtient son caractère comme celui de M^m_{pq}, déjà connu par la construction 'coset'. Le théorème 1.3 s'ensuit.

Maintenant, $\widehat{\mathfrak{g}}$ et $\mathfrak{Vir}_{1/2}$ donnent des superalgèbres locales $\widehat{\mathfrak{g}}(I)$ et $\mathfrak{Vir}_{1/2}(I)$ par couplage avec les fonctions lisses s'annulant en dehors de I (un intervalle propre de $\mathbb{S}^1$). Par des estimées de Sobolev, l'action sur les représentations d'énergie positive est continue. On engendre leur algèbre de von Neumann, contenue dans une algèbre de fermions. Par le dévissage de Takesaki et la construction 'coset', on obtient que ces algèbres sont le facteur hyperfini de type III_1, dont les supercommutants sont engendrés par des chaines de compression de fermions. Il y a également une dualité de Haag-Araki dans le vide, et en dehors, un sous-facteur de Jones-Wassermann, comme défaut de dualité.

Les fermions compressés sont des exemples de champs primaires. On les construit en général, à partir d'applications entrelaçant deux représentations irréductibles, et à coefficients dans un espace de densités. On voit que ces applications sont complètement caractérisées, bornées et classifiées par 'coset', pour deux charges particulières α, β. On obtient également leurs relations de tressage, qui permettent de donner le terme dominant d'une sorte d'OPE pour les champs primaires couplés, ce qui permet d'avoir la densité de von Neumann et l'irréductibilité des sous-facteurs.

Ainsi, on obtient des bimodules irréductibles d'algèbres de von Neumann locales, donnant un cadre pour définir la fusion de Connes. Ses règles sont une conséquence directe de la formule de transport (expliquant l'entrelacement pour des chaînes), qui est prouvée en utilisant les relations de tressage et la densité de von Neumann. Les règles donnent la dimension de l'espace des champs primaires; elles montrent également que les sous-facteurs sont d'indice fini et explicitement donné par le carré de la dimension quantique, un caractère de l'anneau de fusion, donné comme l'unique valeur propre positive d'une matrice de fusion, et un produit de deux dimensions quantiques pour $LSU(2)$ par le théorème de Perron-Frobenius.

1.3 Résultats principaux

Partie I: Série unitaire et caractères pour $\mathfrak{Vir}_{1/2}$

Les représentations irréductible d'énergie positive de l'algèbre Neveu-Schwarz sont notées $L(c,h)$ avec Ω leur vecteur cyclique. Notre propos est de donner une preuve complète de la classification des représentations unitaires, de telle manière qu'on obtienne directement les caractères de la série discrète, sans résolution de Feigin-Fuchs [20]. L'algèbre Neveu-Schwarz est définie par :

$$\begin{cases} [L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3-m)\delta_{m+n} \\ [G_r, L_n] &= (m-\frac{n}{2})G_{r+n} \\ [G_r, G_s]_+ &= 2L_{r+s} + \frac{C}{3}(r^2-\frac{1}{4})\delta_{r+s} \end{cases}$$

avec m, $n \in \mathbb{Z}$, r, $s \in \mathbb{Z}+\frac{1}{2}$, $L_n^\star = L_{-n}$, $G_r^\star = G_{-r}$.
La propriété d'énergie positive signifie que $L(c,h) = H = \bigoplus H_n$, avec $n \in \frac{1}{2}\mathbb{N}$, tel que $L_0\xi = (n+h)\xi$ sur H_n et $H_0 = \mathbb{C}\Omega$ (avec $C\Omega = c\Omega$).

Lemme 1.1. *Si $L(c,h)$ est unitaire, alors $c, h \geq 0$*

Théorème 1.2. *La classification des représentations unitaires $L(c,h)$ est :*

(a) *Série continue: $c \geq 3/2$ et $h \geq 0$.*

(b) *Série discrète: $(c,h) = (c_m, h_{pq}^m)$ avec:*

$$c_m = \frac{3}{2}(1 - \frac{8}{m(m+2)}) \quad et \quad h_{pq}^m = \frac{((m+2)p-mq)^2-4}{8m(m+2)}$$

et les entiers $m \geq 2$, $1 \leq p \leq m-1$, $1 \leq q \leq m+1$ et $p \equiv q[2]$.

Théorème 1.3. *Les caractères de la série discrète sont:*

$$ch(L(c_m, h_{pq}^m))(t) = tr(t^{L_0 - c_m/24}) = \chi_{NS}(t).\Gamma_{pq}^m(t).t^{-c_m/24} \quad avec$$

$$\chi_{NS}(t) = \prod_{n\in\mathbb{N}^\star} \frac{1+t^{n-1/2}}{1-t^n}, \quad \Gamma_{pq}^m(t) = \sum_{n\in\mathbb{Z}} (t^{\gamma_{pq}^m(n)} - t^{\gamma_{-pq}^m(n)}) \quad et$$

$$\gamma_{pq}^m(n) = \frac{[2m(m+2)n-(m+2)p+mq]^2-4}{8m(m+2)}$$

Partie II: Fusion de Connes et sous-facteurs pour $\mathfrak{Vir}_{1/2}$

Soit $p = 2i+1$, $q = 2j+1$ et $m = \ell+2$, on note H_{ij}^{ℓ} la L^2-complétion de $L(c_m, h_{pq}^m)$. On définit la fusion de Connes $\boxtimes$ sur les représentations de charge c_m de la série discrète, comme des bimodules du facteur de type III_1 hyperfini engendré par l'algèbre Neveu-Schwarz locale $\mathfrak{Vir}_{1/2}(I)$ (i.e. couplée avec $C_I^{\infty}(\mathbb{S}^1)$) , avec I un intervalle propre de $\mathbb{S}^1$.

Théorème 1.4 (Fusion de Connes)**.**

$$H_{ij}^{\ell} \boxtimes H_{i'j'}^{\ell} = \bigoplus_{(i'',\, j'') \in \langle i,i' \rangle_{\ell} \times \langle j,j' \rangle_{\ell+2}} H_{i''j''}^{\ell}$$

avec $\langle a,b \rangle_n = \{c = |a-b|, |a-b|+1, ..., a+b \mid a+b+c \leq n\}$.

Soit $\mathcal{M}_{ij}^{\ell}(I)$ l'algèbre de von Neumann engendrée sur H_{ij}^{ℓ}, par les fonctions bornées d'opérateurs auto-adjoints de $\mathfrak{Vir}_{1/2}(I)$.

Théorème 1.5 (Dualité de Haag-Araki dans le vide)**.**

$$\mathcal{M}_{00}^{\ell}(I) = \mathcal{M}_{00}^{\ell}(I^c)^{\natural}$$

avec $X^{\natural}$ *le supercommutant de* X.

Comme défaut de dualité de Haag-Araki hors du vide, on a:

Théorème 1.6 (Sous-facteurs de Jones-Wassermann)**.**

$$\mathcal{M}_{ij}^{\ell}(I) \subset \mathcal{M}_{ij}^{\ell}(I^c)^{\natural}$$

C'est un sous-facteurs de profondeur fini, irréductible, de type III_1 *hyperfini, isomorphe au facteur de type* III_1 *hyperfini* $\mathcal{R}_{\infty}$ *tensorisé avec le sous-facteur de type* II_1*:*

$$(\bigcup \mathbb{C} \otimes End_{\mathfrak{Vir}_{1/2}}(H_{ij}^{\ell})^{\boxtimes n})'' \subset (\bigcup End_{\mathfrak{Vir}_{1/2}}(H_{ij}^{\ell})^{\boxtimes n+1})''$$

d'indice $\frac{sin^2(p\pi/m)}{sin^2(\pi/m)} . \frac{sin^2(q\pi/m+2)}{sin^2(\pi/m+2)}$*, avec* $p = 2i+1$*,* $q = 2j+1$*,* $m = \ell+2$*.*

1.4 L'algèbre Neveu-Schwarz

On commence avec $\mathfrak{W}_{1/2}$, la superalgèbre de Witt du secteur (NS):

$$\begin{cases} [d_m, d_n] &= (m-n)d_{m+n} \quad m,n \in \mathbb{Z} \\ [\gamma_m, d_n] &= (m-\frac{n}{2})\gamma_{m+n} \quad m \in \mathbb{Z}+\frac{1}{2}, n \in \mathbb{Z} \\ [\gamma_m, \gamma_n]_+ &= 2d_{m+n} \quad m,n \in \mathbb{Z}+\frac{1}{2} \end{cases}$$

avec $d_n^\star = d_{-n}$ et $\gamma_m^\star = \gamma_{-m}$; on étudie les représentations qui sont:
(a) Unitaire: $\pi(A)^\star = \pi(A^\star)$
(b) Projective: $A \mapsto \pi(A)$ est linéaire et $[\pi(A), \pi(B)] - \pi([A,B]) \in \mathbb{C}$.
(c) Energie positive : H admet une décomposition orthogonale $H = \bigoplus_{n\in\frac{1}{2}\mathbb{N}} H_n$ telle que $\exists D$ agissant sur H_n comme multiplication par n, $\dim(H_n) < +\infty$, $H_0 \neq \{0\}$. Ici, $\exists h \in \mathbb{C}$ tel que $D = \pi(d_0) - hI$. Maintenant, la projectivité donne des 2-cocycles et on voit que $H_2(\mathfrak{W}_{1/2}, \mathbb{C})$ est de dimension 1, $\mathfrak{W}_{1/2}$ admet une unique extension centrale (à équivalence près):

$$0 \to H_2(\mathfrak{W}_{1/2}, \mathbb{C}) \to \mathfrak{Vir}_{1/2} \to \mathfrak{W}_{1/2} \to 0$$

$\mathfrak{Vir}_{1/2}$ est l'algèbre SuperVirasoro (du secteur NS) ou algèbre Neveu-Schwarz:

$$\begin{cases} [L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3-m)\delta_{m+n} \\ [G_m, L_n] &= (m-\frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ &= 2L_{m+n} + \frac{C}{3}(m^2-\frac{1}{4})\delta_{m+n} \end{cases}$$

avec $L_n^\star = L_{-n}$, $G_m^\star = G_{-m}$ et $C = cI$, $c \in \mathbb{C}$ appelé la **charge centrale**.
Les représentations sont complètement réductibles, les irréductibles sont déterminées par deux nombres c, h, et sont complètement données par les représentations unitaires de plus haut poids de $\mathfrak{Vir}_{1/2}$, décrites comme suit:
Les modules de Verma $H = V(c,h)$ sont librement engendrés par: $0 \neq \Omega \in H$ (vecteur cyclique), $C\Omega = c\Omega$, $L_0\Omega = h\Omega$ et $\mathfrak{Vir}_{1/2}^+\Omega = \{0\}$. Maintenant, $(\Omega, \Omega) = 1$, $\pi(A)^\star = \pi(A^\star)$ et $(u,v) = \overline{(v,u)}$ donne la forme sesquilinéaire $(.,.)$. $V(c,h)$ peut admettre des 'fantômes': $(u,u) < 0$ et des 'vecteurs nuls': $(u,u) = 0$. Dans le cas 'sans fantôme', l'ensemble des 'vecteurs nuls' est $K(c,h)$ le noyau de $(.,.)$, le sous-module propre maximal. Soit $L(c,h) = V(c,h)/K(c,h)$, la représentation unitaire de plus haut poids.
Le théorème 1.2 sera prouvé en classifiant les cas 'sans fantôme'.

1.5 Superalgèbres d'opérateurs vertex

Notre approche des superalgèbres d'opérateurs vertex est librement inspirée des références suivantes: Borcherds [11], Goddard [37], Kac [57].

On démarre en travaillant sur l'algèbre fermion: $[\psi_m, \psi_n]_+ = \delta_{m+n} I, \psi_n^\star = \psi_{-n}$ $(m, n \in \mathbb{Z} + \frac{1}{2})$. Comme pour $\mathfrak{W}_{1/2}$, on construit son module de Verma $H = \mathcal{F}_{NS}$ et la forme sesquilinéaire $(.,.)$, qui est un produit scalaire. H est un espace préhilbertien, l'unique représentation unitaire de plus haut poids de l'algèbre fermion. Soit la série formelle $\psi(z) = \sum_{n\in\mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1}$ appelée champ fermion. On définit inductivement les opérateurs D donnant la structure d'énergie positive ($\Leftrightarrow [D, \psi] = z.\psi' + \frac{1}{2}\psi$) et T donnant une dérivation ($[T, \psi] = \psi'$). On calcule $(\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w}$ $(|z| > |w|)$, ce qui permet de prouver inductivement une relation d'anticommutation brièvement écrite comme: $\psi(z)\psi(w) = -\psi(w)\psi(z)$. On définit cette relation dans un cadre général comme la localité: Soit H un espace préhilbertien, et soit $A \in (EndH)[[z, z^{-1}]]$ une série formelle de la forme $A(z) = \sum_{n\in\mathbb{Z}} A(n) z^{-n-1}$ avec $A(n) \in End(H)$. De tels champs A et B sont locaux si $\exists \varepsilon \in \mathbb{Z}_2$, $\exists N \in \mathbb{N}$ tel que $\forall c, d \in H$, $\exists X(A, B, c, d) \in (z-w)^{-N}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ tel que:

$$X(A,B,c,d)(z,w) = \begin{cases} (A(z)B(w)c, d) & \text{si } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{si } |w| > |z| \end{cases}$$

Maintenant, en utilisant la localité et un argument de contour d'intégration, on peut construire un champ $A_n B$ à partir de A et B, avec $(A_n B)(m) =$

$$\begin{cases} \sum_{p=0}^{n} (-1)^p C_n^p [A(n-p), B(m+p)]_\varepsilon & \text{if } n \geq 0 \\ \\ \sum_{p\in\mathbb{N}} C_{p-n-1}^p (A(n-p)B(m+p) - (-1)^{\varepsilon+n} B(m+n-p)A(p)) & \text{if } n < 0 \end{cases}$$

On obtient l' 'operator product expansion' (OPE) brièvement écrit comme $A(z)B(w) \sim \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}}$; par un autre argument de contour d'intégration:

$$[A(m), B(n)]_\varepsilon = \begin{cases} \sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) & \text{if } m \geq 0 \\ \\ \sum_{p=0}^{N-1} (-1)^p C_{p-m-1}^p (A_p B)(m+n-p) & \text{if } m < 0 \end{cases}$$

Grâce au lemme de Dong, l'opération $(A, B) \mapsto A_n B$ permet d'engendrer de nombreux champs. Pour avoir un bon comportement, on définit un système de générateurs comme:

$\{A_1,..., A_r\} \subset (EndH)[[z, z^{-1}]]$ avec $D, T \in End(H)$, $\Omega \in H$ tel que:
(a) $\forall i, j$ A_i et A_j sont locaux avec $N = N_{ij}$ et $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii}.\varepsilon_{jj}$
(b) $\forall i$ $[T, A_i] = A_i'$
(c) $H = \bigoplus_{n \in \mathbb{N}+\frac{1}{2}}$ pour D, $dim(H_n) < \infty$
(d) $\forall i$ $[D, A_i] = z.A_i' + \alpha_i A_i$ avec $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$
(e) $\Omega \in H_0$, $\|\Omega\| = 1$, et $\forall i$ $\forall m \in \mathbb{N}$, $A_i(m)\Omega = D\Omega = T\Omega = 0$
(f) $\mathcal{A} = \{A_i(m), \forall i \ \forall m \in \mathbb{Z}\}$ agit irréductiblement sur H,et $< \mathcal{A} > .\Omega = H$
Ensuite, on engendre l'espace $\mathcal{S}$, avec $V : H \longrightarrow \mathcal{S}$ une application linéaire de correspondance état-champ. $V(a)(z)$ est noté $V(a, z)$ et $V(a, z)\Omega_{|z=0} = a$. Maintenant, $\{\psi\}$ est un système de générateur, on engendre $\mathcal{S}$ et l'application V avec $\psi(z) = V(\psi_{-\frac{1}{2}}\Omega, z)$; mais $\psi(z)\psi(w) \sim \frac{Id}{z-w} + 2L(w)$, avec $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{2}\psi_{-2}\psi(z) = V(\omega, z)$ avec $\omega = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$. Ainsi, en utilisant l'OPE et le crochet de Lie, on trouve que $D = L_0$, $T = L_{-1}$, $L(z)L(w) \sim \frac{(c/2)Id}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)}$, et $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$ avec $c = 2\|L_{-2}\Omega\|^2 = \frac{1}{2}$, la charge centrale. Comme corollaire, $\mathfrak{Vir}$ agit sur $H = \mathcal{F}_{NS}$, et admet sa représentation unitaire de plus haut poids $L(c, h) = L(\frac{1}{2}, 0)$ comme sous-module minimal contenant Ω. On appelle $\omega \in H$ le vecteur Virasoro, et L le champ Virasoro. On est maintenant en mesure de définir les superalgèbres d'opérateurs vertex en général: Une superalgèbre d'opérateurs vertex est donné par un quadruplet (H, V, Ω, ω) avec:
(a) $H = H_{\bar{0}} \oplus H_{\bar{1}}$ un superespace préhilbertien.
(b) $V : H \to (EndH)[[z, z^{-1}]]$ une application linéaire.
(c) Ω, $\omega \in H$ les vecteurs vide et Virasoro.
Soient $\mathcal{S}_\varepsilon = V(H_\varepsilon)$, $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ et $A(z) = V(a, z) = \sum_{n\in\mathbb{Z}} A(n)z^{-n-1}$, alors (H, V, Ω, ω) satisfait les axiomes suivant:
(1) $\forall n \in \mathbb{N}$, $\forall A \in \mathcal{S}$, $A(n)\Omega = 0$, $V(a, z)\Omega_{|z=0} = a$, et $V(\Omega, z) = Id$
(2) $\mathcal{A} = \{A(n)|A \in \mathcal{S}, n \in \mathbb{Z}\}$ agit irréductiblement sur H, avec $\mathcal{A}.\Omega = H$.
(3) $\forall A \in \mathcal{S}_{\varepsilon_1}$, $\forall B \in \mathcal{S}_{\varepsilon_2}$, A and B sont locaux avec $\varepsilon = \varepsilon_1.\varepsilon_2$, $A_n B \in \mathcal{S}_{\varepsilon_1+\varepsilon_2}$
(4) $V(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $[L_m, L_n] = (m-n)L_{m+n} + \frac{\|2\omega\|^2}{12}m(m^2-1)\delta_{m+n}$
(5) $H = \bigoplus_{n\in\mathbb{N}+\frac{1}{2}} H_n$ for L_0, $dim(H_n) < \infty$ and $H_\varepsilon = \bigoplus_{n\in\mathbb{N}+\frac{\varepsilon}{2}} H_n$
(6) $[L_0, V(a, z)] = z.V'(a, z) + \alpha.V(a, z)$ pour $a \in H_\alpha$
(7) $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$
Comme corollaires, on obtient qu'un système de générateurs, engendrant un champ Virasoro $L \in \mathcal{S}$, avec $D = L_0$ et $T = L_{-1}$, engendre une superalgèbre d'opérateurs vertex; les champs fermion ψ et Virasoro L, en donnent chacun une; on a l'associativité de Borcherds: $V(a, z)V(b, w) = V(V(a, z - w)b, w)$.

1.6 $\mathfrak{g}$-superalgèbre d'opérateur vertex et modules

Soit $\mathfrak{g}$ une algèbre de Lie simple de dimension N, une base (X_a) bien normalisée (voir remarque 5.2) telle que $[X_a, X_b] = i\sum_c \Gamma^c_{ab} X_c$ avec $\Gamma^c_{ab} \in \mathbb{R}$ totalement antisymétrique. Soit son nombre de Coxeter dual $g = \frac{1}{4}\sum_{a,c}(\Gamma^b_{ac})^2$.

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$dim(\mathfrak{g})$	n^2+2n	$2n^2+n$	$2n^2+n$	$2n^2-n$	78	133	248	52	14
g	$n+1$	$2n-1$	$n+1$	$2n-2$	12	18	30	9	4

Par exemple, $\mathfrak{g} = A_1 = \mathfrak{sl}_2$, $dim(\mathfrak{g}) = 3$ et $g = 2$.
Soit $\widehat{\mathfrak{g}}_+$, l'algèbre $\mathfrak{g}$-boson: $[X^a_m, X^b_n] = [X_a, X_b]_{m+n} + m\delta_{ab}\delta_{m+n}.\mathcal{L}$, unique extension centrale (par $\mathcal{L}$) de l'algèbre de lacets $L\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{g})$ (voir [100] p 43). Les représentations de plus haut poids unitaires de $\widehat{\mathfrak{g}}_+$ sont $H = L(V_\lambda, \ell)$, avec $\ell \in \mathbb{N}$ tel que $\mathcal{L}\Omega = \ell\Omega$ (le niveau de H), $H_0 = V_\lambda$ une représentation irréductible de $\mathfrak{g}$ avec $(\lambda, \theta) \leq \ell$ et λ le plus haut poids et θ la plus haute racine (voir [100] p 45). La catégorie $\mathscr{C}_\ell$ des représentations pour ℓ fixé est fini. Par exemple $\mathfrak{g} = \mathfrak{sl}_2$, $H = L(j, \ell)$, avec $V_\lambda = V_j$ de spin $j \leq \frac{\ell}{2}$.

On définit l'algèbre $\mathfrak{g}$-fermion $\widehat{\mathfrak{g}}_-$ et les champs fermions, composés de N fermions; et comme pour $N = 1$, on engendre une superalgèbre d'opérateurs vertex, mais maintenant, elle contient des champs $\mathfrak{g}$-bosons (S^a), dont l'algèbre associée est représentée avec $L(V_0, g)$; et grâce au contexte vertex de $\widehat{\mathfrak{g}}_-$, les champs (S^a) engendrent une superalgèbre d'opérateurs vertex; de la même manière, on est en mesure d'en engendrer une, à partir de $\widehat{\mathfrak{g}}_+$ et $H = L(V_0, \ell)$, $\forall \ell \in \mathbb{N}$. On remarque qu'à cause de l'axiome associé au vecteur vide, la structure vertex impose $V_\lambda = V_0$, la représentation triviale; en général, on a des modules vertex (voir les prochains paragraphes).

Maintenant, soit $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$ l'algèbre $\mathfrak{g}$-supersymétrique, on prouve qu'elle admet $H = L(V_\lambda, \ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$ comme représentations de plus haut poids unitaires. On génère une superalgèbre d'opérateurs vertex, avec un champ Virasoro L, et également un champ SuperVirasoro G, ce qui donne la supersymétrie boson-fermion: soient $B^a = X^a + S^a$ les champ bosons de niveau $d = \ell + g$, alors $B^a(z)G(w) \sim d^{\frac{1}{2}}\frac{\psi^a(w)}{(z-w)^2}$ et $\psi^a(z)G(w) \sim d^{-\frac{1}{2}}\frac{B^a(w)}{(z-w)}$.

Finallement, à partir de $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$, on définit un module vertex (H^λ, V^λ) sur (H^0, V, Ω, ω), et on prouve que $\mathfrak{Vir}_{\frac{1}{2}}$ agit unitairement sur H^λ et admet $L(c, h)$ comme sous-module minimal contenant le vecteur cyclique Ω^λ, avec $c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} dim(\mathfrak{g})$, $h = \frac{c_{V_\lambda}}{2(\ell+g)}$ et c_{V_λ} le nombre de Casimir de V_λ.

1.7 Le cadre de Goddard-Kent-Olive

On prend $\mathfrak{g} = \mathfrak{sl}_2$. Soit H une représentation unitaire projective d'énergie positive de l'algèbre de lacets $L\mathfrak{g}$. Soit $ch(H)(t,z) = tr(t^{L_0 - \frac{C}{24}} z^{X_3})$ le caractère de H. $L\mathfrak{g}$ agit sur $\mathcal{F}_{NS}^{\mathfrak{g}}$, et par l'identité du triple produit de Jacobi $\sum_{k\in\mathbb{Z}} t^{\frac{1}{2}k^2} z^k = \prod_{n\in\mathbb{N}^\star}(1+t^{n-\frac{1}{2}}z)(1+t^{n-\frac{1}{2}}z^{-1})(1-t^n)$, on prouve que $ch(\mathcal{F}_{NS}^{\mathfrak{g}})(t,z) = t^{-1/16}\chi_{NS}(t)\theta(t,z)$ avec $\chi_{NS}(t) = \prod_{k\in\mathbb{N}^\star}(\frac{1+t^{n-\frac{1}{2}}}{1-t^n})$ et $\theta(t,z) = \sum_{k\in\mathbb{Z}} t^{\frac{1}{2}k^2} z^k$. Ensuite, soient $H = L(j,\ell)$, et les fonctions thêta $\theta_{n,m}(t,z) = \sum_{k\in\frac{n}{2m}+\mathbb{Z}} t^{mk^2} z^{mk}$, alors par les formules de Kac-Weyl pour $L\mathfrak{g}$: $ch(L(j,\ell)) = \frac{\theta_{2j+1,\ell+2}-\theta_{-2j-1,\ell+2}}{\theta_{1,2}-\theta_{-1,2}}$ (see [49], [56] or [100] p 62). Maintenant, en adaptant la preuve dans [54] p 122, on obtient la formule produit: $\theta(t,z).\theta_{p,m}(t,z) = \sum_{\substack{0\le q<2(m+2)\\ p\equiv q[2]}}(\sum_{n\in\mathbb{Z}} t^{\alpha_{pq}^m(n)})\theta_{q,m+2}(t,z)$ avec $\alpha_{p,q}^m(n) = \frac{[2m(m+2)n-(m+2)p+mq]^2}{8m(m+2)}$.
Mais $L\mathfrak{g}$ agit sur $L(j,\ell)\otimes\mathcal{F}_{NS}^{\mathfrak{g}}$ à niveau $\ell+2$; on en déduit: $ch(L(j,\ell)\otimes\mathcal{F}_{NS}^{\mathfrak{g}}) = \sum_{\substack{1\le q\le m+1\\ p\equiv q[2]}} F_{pq}^m.ch(L(k,\ell+2))$, $F_{pq}^m(t) = t^{-1/16}\chi_{NS}(t)\sum_{n\in\mathbb{Z}}(t^{\alpha_{p,q}^m(n)} - t^{\alpha_{-p,q}^m(n)})$, $p = 2j+1$, $q = 2k+1$ et $m = \ell+2$; et la décomposition du produit tensoriel: $L(j,\ell)\otimes\mathcal{F}_{NS}^{\mathfrak{g}} = \bigoplus_{\substack{1\le q\le m+1\\ p\equiv q[2]}} M_{pq}^m\otimes L(k,\ell+2)$ avec M_{pq}^m l'espace de multiplicité.
Cadre GKO général: Soit $\mathfrak{h}$ une $\star$-superalgèbre de Lie agissant unitairement sur une somme directe finie $H = \bigoplus M_i\otimes H_i$ avec H_i irréductible et M_i l'espace de mulitiplicité. On voit que M_i est l'espace préhilbertien des super-entrelacements $Hom_{\mathfrak{h}}(H_i,H)$. Maintenant, si $\mathfrak{d}$ est une $\star$-superalgèbre de Lie agissant sur H et H_i comme représentations unitaires, projectives, d'énergie positive, et dont la différence $(\pi(D)-\sum\pi_i(D))$ supercommute avec $\mathfrak{h}$, alors, idem sur M_i, avec pour cocycle, la différence des deux autres. Ensuite, en prenant $\mathfrak{h} = \hat{\mathfrak{g}}$ et $\mathfrak{d} = \mathfrak{W}_{1/2}$, on trouve $c_{M_{pq}^m} = \frac{dim(\mathfrak{g})}{2}(1-\frac{2g^2}{(\ell+g)(\ell+2g)}) = \frac{3}{2}(1-\frac{8}{m(m+2)}) =: c_m$, car $m = \ell+2$, $g = 2$ et $dim(\mathfrak{g}) = 3$. Maintenant, le caractère d'un $\mathfrak{Vir}_{1/2}$-module H est : $ch(H)(t) = tr(t^{L_0-\frac{C}{24}})$, ainsi: $ch(M_{pq}^m)(t) = t^{-\frac{c(m)}{24}}.\chi_{NS}(t).\Gamma_{pq}^m(t)$, $\Gamma_{pq}^m(t) = \sum_{n\in\mathbb{Z}}(t^{\gamma_{pq}^m(n)} - t^{\gamma_{-pq}^m(n)})$, $\chi_{NS}(t) = \prod_{n\in\mathbb{N}^\star}\frac{1+t^{n-1/2}}{1-t^n}$ et $\gamma_{pq}^m(n) = \frac{[2m(m+2)n-(m+2)p+mq]^2-4}{8m(m+2)}$. Alors, $h_{pq}^m = \frac{[(m+2)p-mq]^2-4}{8m(m+2)}$ est la plus petite valeur propre de L_0 sur M_{pq}^m; soit $(p',q') = (m-p,m+2-q)$, alors $ch(M_{pq}^m)\sim t^{-\frac{c_m}{24}}.\chi_{NS}(t).t^{h_{pq}^m}.(1-t^{\frac{pq}{2}}-t^{\frac{p'q'}{2}})$. Ainsi, $ch(M_{pq}^m).t^{\frac{c_m}{24}}\sim t^{h_{pq}^m}$, et le h_{pq}^m-espace propre de L_0 est de dimension 1, donc $L(c_m,h_{pq}^m)$ est un $\mathfrak{Vir}_{1/2}$-sous-module de M_{pq}^m, et $ch(L(c_m,h_{pq}^m))\le ch(M_{pq}^m)$. Finalement, comme M_{pq}^m est unitaire, il en est de même pour $L(c_m,h_{pq}^m)$ dans la série discrète.

1.8 La formule du déterminant de Kac

A partir de (c_m, h^m_{pq}), on définit h^c_{pq}, $\forall c \in \mathbb{C}$. Soit $\varphi_{pp}(c,h) = (h - h^c_{pp})$, $\varphi_{pq}(c,h) = (h - h^c_{pq})(h - h^c_{qp})$ si $p \neq q$; $\varphi_{pq} \in \mathbb{C}[c,h]$ et est irreductible. Soit $V_n(c,h)$ le n-espace propre de $D = L_0 - hI$ et $d(n)$ sa dimension. Soit $M_n(c,h)$ la matrice de $(.,.)$ sur $V_n(c,h)$ et $det_n(c,h) = det(M_n(c,h))$.

Par exemple, $M_0(c,h) = (\Omega,\Omega) = (1)$, $M_{\frac{1}{2}}(c,h) = (G_{-\frac{1}{2}}\Omega, G_{-\frac{1}{2}}\Omega) = (2h)$, $M_1(c,h) = (L_{-1}\Omega, L_{-1}\Omega) = (2h)$, et $M_{\frac{3}{2}}(c,h) =$

$$\begin{pmatrix} (G_{-\frac{1}{2}}L_{-1}\Omega, G_{-\frac{1}{2}}L_{-1}\Omega) & (G_{-\frac{1}{2}}L_{-1}\Omega, G_{-\frac{3}{2}}\Omega) \\ (G_{-\frac{3}{2}}\Omega, G_{-\frac{1}{2}}L_{-1}\Omega) & (G_{-\frac{3}{2}}\Omega, G_{-\frac{3}{2}}\Omega) \end{pmatrix} = \begin{pmatrix} 2h + 4h^2 & 4h \\ 4h & 2h + \frac{2}{3}c \end{pmatrix}$$

Maintenant, $det_{\frac{3}{2}}(c_m,h) = 8h[h^2 - (\frac{3}{2} - \frac{c_m}{3})h + c/6] = 8h(h - h^m_{13})(h - h^m_{31})$, alors, $det_{\frac{3}{2}}(c,h) = 8h(h - h^c_{13})(h - h^c_{31}) = 8\varphi_{11}(c,h).\varphi_{13}(c,h)\ \ \forall c \in \mathbb{C}$. Ainsi, d'autres exemples permettent d'intérpoler la formule du déterminant de Kac:

$$det_n(c,h) = A_n \prod_{\substack{0<pq/2\leq n \\ p\equiv q[2]}} (h - h^c_{pq})^{d(n-pq/2)} = A_n \prod_{\substack{0<pq/2\leq n \\ p\leq q,\ p\equiv q[2]}} \varphi_{pq}^{d(n-pq/2)}(c,h)$$

avec $A_n > 0$ indépendant de c et h. Pour la démontrer, on utilise des vecteurs singuliers $s \in V(c,h)$, i.e. $L_0.s = (h+n)s$ avec $n > 0$ son niveau, et $\mathfrak{Vir}^+_{1/2}.s = 0$. Ceci est équivalent à $G_{1/2}.s = G_{3/2}.s = 0$, ainsi, on trouve facilement les singuliers $(mG_{-3/2} - (m+2)L_{-1}G_{-1/2})\Omega \in V_{3/2}(c_m, h^m_{13})$, $G_{-1/2}\Omega \in V_{1/2}(c, h^c_{11})$, ou $(L^2_{-1} - \frac{4}{3}h^c_{22}L_{-2} - G_{-3/2}G_{-1/2})\Omega \in V_2(c, h^c_{22})$. Maintenant, $ch(V(c,h)) = t^{h-\frac{c}{24}}\chi_{NS}(t)$ et les vecteurs singuliers engendrent $K(c,h)$. Ainsi, $V(c,h)$ a un vecteur singulier de niveau minimal $n \in \frac{1}{2}\mathbb{N}$ ssi

$$ch(L(c,h)) \sim t^{h-\frac{c}{24}}\chi_{NS}(t)(1 - t^n),$$

mais grâce à la construction 'coset':

$$ch(L(c_m, h^m_{pq})) \leq ch(M^m_{pq}) \sim t^{-\frac{c_m}{24}}.\chi_{NS}(t).t^{h^m_{pq}}.(1 - t^{\frac{pq}{2}} - t^{\frac{p'q'}{2}})$$

Donc $V(c_m, h^m_{pq})$ admet un vecteur singulier s de niveau $n' \leq min(pq/2, p'q'/2)$, et pour $n > n'$, det_n s'annule en (c_m, h^m_{pq}) avec m, un entier suffisamment grand. Alors il s'annule sur une infinité de zéros de l'irréductible φ_{pq}, donc φ_{pq} divise det_n. Mais au niveau n, s engendre un sous-espace de dimension $d(n-n')$, ainsi $d_n(c,h) = \prod_{\substack{0<pq/2\leq n \\ p\equiv q[2]}} (h - h^c_{pq})^{d(n-pq/2)}$ divise det_n.

Finalement, un argument de cardinalité montre d_n et det_n avec le même degré en h. Le résultat s'ensuit.

1.9 Le critère d'unitarité de Friedan-Qiu-Shenker

Le critère de FQS a été découvert pour $\mathfrak{Vir}$, par Friedan, Qiu et Shenker [25], mais des mathématiciens estimaient leur preuve trop rapide, et alors, FQS [28] et Langlands [64] publièrent en même temps une preuve complète. Au début de notre travail sur $\mathfrak{Vir}_{1/2}$, on avait décidé d'adapter la preuve de Langlands, mais on a trouvé une erreure dans son papier ([64] lemma 7b p 148: $p = 2$, $q = 1$, $m = 2$, $h^m_{pq} = \frac{5}{8}$, $M = 4$ ou $p = 4$, $q = 1$, $m = 3$, $h^m_{pq} = \frac{7}{2}$, $M = 13$ correspondent au cas (B), mais $(p, q) \neq (1, 1)$ et $m \not> q + p - 1$. En fait, on a besoin de distinguer entre $q \neq 1$ et $q = 1$, pas entre $(p, q) \neq (1, 1)$ et $q = (1, 1)$). Ensuite, on a découvert que Sauvageot avait déjà publié une telle adaptation, mais sans correction ([82] lemma 2 (ii) p 648). On a alors choisi d'adapter la méthode de FQS.

On cherche une condition nécessaire sur (c, h) pour que $V(c, h)$ n'ait pas de 'fantôme'. Tout d'abord, si $V(c, h)$ n'admet pas de 'fantôme' alors $c, h \geq 0$ (facile). Maintenant, le déterminant de Kac ne s'annule pas dans la région $h > 0$, $c > 3/2$, et pour (c, h) large, on prouve que la forme $(.,.)$ est positive. Ainsi, par continuité, si $h \geq 0$ et $c \geq 3/2$, $V(c, h)$ n'admet pas de 'fantôme'. Maintenant, dans la région $0 \leq c < 3/2$, $h \geq 0$, le critère FQS dit que $V(c, h)$ admet des 'fantômes' si (c, h) n'appartient pas à (c_m, h^m_{pq}), avec des entiers $m \geq 2$, $1 \leq p \leq m - 1$, $1 \leq q \leq m + 1$ et $p \equiv q[2]$, i.e., exactement la série discrète donnée par la construction 'coset' ! Pour démontrer ce résultat, on exploite l'ensemble des zéros des déterminants de Kac, constitué par des courbes C_{pq} d'équation $h = h^c_{pq}$ avec $0 \neq p \equiv q[2]$. Tout d'abord, on se limite à C'_{pq}, le sous-ensemble ouvert de C_{pq}, entre $c = 3/2$ et sa première intersection au niveau $pq/2$. Soit $p'q' > pq$, $C_{p'q'}$ est une première intersectrice de C'_{pq} si au niveau $p'q'/2$, elle est la première à intersecter C'_{pq} en partant de $c = 3/2$. On voit que ces premières intersections constituent exactement la série discrète. Maintenant, pour chaque région ouverte entre les courbes C'_{pq}, on peut trouver n avec det_n négatif. Cela signifie que $V(c, h)$ y admet un 'fantôme', on peut donc éliminer ces régions. Donc maintenant, il reste à éliminer les intervalles de C'_{pq} entre les points de la série discrète. On commence depuis la région 'sans-fantôme' $h > 0$, $c > 3/2$ et on se dirige vers un tel intervalle. Sur le chemin, on rencontre une courbe d'annulation (bien choisie) d'ordre 1: donc de l'autre côté il y a un 'fantôme'. On continue le long de cette courbe avec notre 'fantôme', jusqu'à un point d'intersection. Maintenant, puisque les intersections sont transverses, on peut distinguer entre les vecteurs 'nuls' de la première et la deuxième courbe, et ainsi, notre 'fantôme' continue d'exister sur l'autre courbe. En répétant ce principe, on peut aller jusqu'à l'intervalle, sans perdre le 'fantôme'. Le critère FQS et le théorème 1.2 s'ensuivent.

1.10 L'argument de Wassermann

On montre que l'espace de multiplicité par la construction 'coset', est une représentation irréductible de l'algèbre Neveu-Schwarz, ce qui donne directement (comme dans [100] p 72 pour $\mathfrak{Vir}$) les caractères de la série discrète, sans résolution de Feigin-Fuchs [20]: Comme corollaire de la preuve du critère FQS, aux niveaux $\leq M = max(pq/2, p'q'/2)$, il existe seulement deux vecteurs singuliers s et s', aux niveaux $pq/2$ et $p'q'/2$. Ainsi, $ch(L(c_m, h_{pq}^m)) \sim t^{h_{pq}^m - c_m/24}\chi_{NS}(t)(1 - t^{pq/2} - t^{p'q'/2})$, comme pour l'espace de multiplicité M_{pq}^m, et alors, $ch(M_{pq}^m) - ch(L(c_m, h_{pq}^m)) = \chi_{NS}(t).t^{-c_m/24}o(t^{h_{pq}^m + M})$. Maintenant, on sait que $L(c_m, h_{pq}^m)$ est un sous-module de M_{pq}^m; si M_{pq}^m admet un autre sous-module irréductible, par le critère FQS, il est de la forme $L(c_m, h_{rs}^m)$; mais par le lemme: $h_{pq}^m + M > m^2/8$ et $h_{rs}^m \leq \frac{m(m-2)}{8}$, on obtient par cohérence sur les caractère, la contradiction: $\frac{m^2}{8} < M + h_{pq}^m < h_{rs}^m \leq \frac{m(m-2)}{8}$. Ainsi, $M_{pq}^m = L(c(m), h_{p,q}^m)$ et $ch(L(c_m, h_{pq}^m)) = ch(M_{pq}^m)$, mais la construction 'coset' a déjà donné les caractères des espaces de multiplicité. Le théorème 1.3 s'ensuit.

1.11 Algèbres de von Neumann locales

Pour l'algèbre de lacets $L\mathfrak{g}$ et l'algèbre Virasoro $\mathfrak{Vir}$, on peut travailler avec les groupes correspondant: LG et Diff($\mathbb{S}^1$). Pour l'algèbre Neveu-Schwarz, il n'y a pas de groupe correspondant aux supergenerateurs G_r, et ainsi on a besoin de travailler avec des opérateurs non bornés. A partir de l'algèbre $\mathfrak{g}$-supersymetrique $\widehat{\mathfrak{g}}$, on construit une superalgèbre de Lie locale $\widehat{\mathfrak{g}}(I)$ (avec I un intervalle propre de $\mathbb{S}^1$), en couplant avec les fonctions lisses s'annulant en dehors de I. De la même manière, on définit la superalgèbre de Lie locale Neveu-Schwarz $\mathfrak{Vir}_{1/2}(I)$. Grâce aux estimés de Sobolev, ces algèbres locales (contenant des opérateurs non bornés) sont représentées continûment sur la complétion L_0-lisse de leurs représentations d'énergie positive. Maintenant, on définit les algèbres de von Neumann par ces algèbres locales, comme les algèbres de von Neumann engendrées par les fonctions bornées de leurs opérateurs auto-adjoints; ce sont des algèbres de von Neumann $\mathbb{Z}_2$-graduées. Ensuite, $\widehat{\mathfrak{g}}$ agit sur un espace de Fock de fermions réels et complexes, qui se décompose en toutes ses représentations d'énergie positive (avec multiplicités), et par construction 'coset', on peut faire de même avec $\mathfrak{Vir}_{1/2}$. Ainsi, on voit que les précédentes algèbres de von Neumann sont incluses avec espérance conditionnelle dans une grande algèbre de von Neumann $\mathcal{M}(I)$, engendrée par des fermions couplés réels et complexes, qui est (par [99] et une construction doublante) le facteur hyperfini de type III_1. Maintenant, l'action modulaire est ergodique, ainsi, par dévissage de Takesaki $\mathcal{N}(I) = \pi(\mathfrak{Vir}_{1/2}(I))''$ est également le facteur hyperfini de type III_1, et par définition du type III, il en est de même pour chaque sous-représentations, donc en particulier pour $\pi_i(\mathfrak{Vir}_{1/2}(I))''$, avec π_i une représentation d'énergie positive irréductible générique. On en déduit l'équivalence locale, ie, les représentations de la séries discrète sont unitairement équivalentes quand on se restreint à $\mathfrak{Vir}_{1/2}(I)$; on en déduit également la dualité de Haag-Araki:

$$\pi_0(\mathfrak{Vir}_{1/2}(I^c))^\natural = \pi_0(\mathfrak{Vir}_{1/2}(I))''$$

avec $X^\natural$ le supercommutant de X, à partir de la dualité de Haag-Araki connue pour $\mathcal{M}(I)$, car le vecteur vide de H_0 est invariant par l'opérateur modulaire $\triangle$ de $\mathcal{M}(I)$. En dehors du vide, on a un sous-facteur de Jones-Wassermann:

$$\pi_i(\mathfrak{Vir}_{1/2}(I))'' \subset \pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural$$

comme défaut de dualité de Haag-Araki.

1.12 Champs primaires

Soit p_0 la projection sur la représentation vide H_0. Par la relation de Jones $p_0\mathcal{M}(I)p_0 = \mathcal{N}(I)p_0$, l'algèbre $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ est engendré par des produits de fermions réels et complexes compressés: $p_0\psi_1(f_1)p_{i_1}\psi_2(f_2)p_{i_2}...\psi_n(f_n)p_0$, avec p_i la projection sur $H_i \subset H$ et f_s localisé en I. Les $p_i\psi(f)p_j$ sont des opérateurs bornés (super)entrelaçant l'action de $\mathfrak{Vir}_{1/2}(I^c)$ entre les représentations H_i et H_j. On veut interpréter ces compressions comme des champs primaires couplés. On définit un champ primaire par un opérateur linéaire:

$$\phi_{ij}^k : H_j \otimes \mathcal{F}_{\lambda,\mu}^{\sigma} \to H_i$$

qui super-entrelace l'action de $\mathfrak{Vir}_{1/2}$; avec H_i, H_j dans la série discrète de $\mathfrak{Vir}_{1/2}$ (k est appelé la charge de ϕ_{ij}^k), et $\mathcal{F}_{\lambda,\mu}^{\sigma}$ une représentation ordinaire de $\mathfrak{Vir}_{1/2}$ avec la base $(v_i)_{i\in\mathbb{Z}+\frac{\sigma}{2}}$, $(w_j)_{j\in\mathbb{Z}+\frac{1-\sigma}{2}}$, et:

(a) $L_n.v_i = -(i+\mu+\lambda n)v_{i+n}$
(b) $G_s.v_i = w_{i+s}$
(c) $L_n.w_j = -(j+\mu+(\lambda-\frac{1}{2})n)w_{j+n}$
(d) $G_s.w_j = -(j+\mu+(2\lambda-1)s)v_{j+s}$
avec $\lambda = 1-h_k$, $\mu = h_j - h_i$, $\sigma = 0, 1$.
Soit l'espace des densités $\{f(\theta)e^{i\mu\theta}(d\theta)^{\lambda} | f \in C^{\infty}(\mathbb{S}^1)\}$ où un recouvrement fini de Diff($\mathbb{S}^1$) agit par reparamétrisation $\theta \to \rho^{-1}(\theta)$ (si $\mu \in \mathbb{Q}$). Ainsi, son algèbre de Lie agit aussi, donc c'est un $\mathfrak{Vir}$-module annulant le centre. Finalement, une construction équivalente avec des superdensités donne un modèle de $\mathcal{F}_{\lambda,\mu}^{\sigma}$ comme $\mathfrak{Vir}_{1/2}$-module.

Ce champ primaire est équivalent à deux opérateurs vertex généralisés $\phi_{ij}^k(z)$ (appelé la partie ordinaire) et $\theta_{ij}^k(z) = [G_{-1/2}, \phi_{ij}^k(z)]$ (appelé la partie super), et on prouve que pour i, j, k et σ fixés, de tels opérateurs sont complètement caractérisés par quelques conditions de compatibilité, donc l'espace des champs primaires associés est au plus de dimension un. Notons que $\sigma = 0$ donne ϕ_{ij}^k avec des modes entiers et $\sigma = 1$, avec des modes demi-entiers. Pour la charge $\alpha = (1/2, 1/2)$, on construit ces opérateurs de la manière suivante (une adaptation d'une idée de Loke pour $\mathfrak{Vir}$ [66], simplifié par A. Wassermann): on commence avec la construction 'coset' GKO $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes H_i^{\ell} = \bigoplus H_{ii'}^{\ell} \otimes H_{i'}^{\ell+2}$, on prend le champ primaire vertex de $LSU(2)$ de niveau ℓ et spin 1/2: $I \otimes \phi_{ij}^{1/2,\ell}(z, v) : \mathcal{F}_{NS}^{\mathfrak{g}} \otimes H_j^{\ell} \to \mathcal{F}_{NS}^{\mathfrak{g}} \otimes H_i^{\ell}$, avec $v \in V_{1/2}$ (la représentation vectorielle de $SU(2)$). Soit $p_{i'}$ la projection sur le bloc $H_{ii'}^{\ell} \otimes H_{i'}^{\ell+2}$. Par relations de compatibilité et unicité, $p_{i'}(I \otimes \phi_{ij}^{1/2,\ell}(z, v))p_{j'} =$

$C.z^r\phi^{\alpha}_{ii'jj'}(z)\otimes\phi^{\frac{1}{2},\ell+2}_{i'j'}(z,v)$, avec C une constante éventuellement nulle et $r\in\mathbb{Q}$. Maintenant, $I\otimes\phi^{1/2,\ell}_{ij}(z,v)=\sum_{i'j'}p_{i'}(I\otimes\phi^{1/2\ell}_{ij}(z,v))p_{j'}$, donc au moins un terme est non nul. Plus précisement, on prouve par un argument d'irréductibilité que $\forall j'$, $\exists i'$ avec un terme non-nul, et ainsi $\phi^{\alpha}_{ii'jj'}(z)$ est non-nul. Notons que les simples relations de localité entre les fermions non-compressés couplés concentrés sur des intervalles disjoints (ie $\psi(f)\psi(g)=-\psi(g)\psi(f)$), admettent un équivalent un peu plus compliqué aprés la compression: les relations de tressage. Maintenant, en utilisant la même idée que Tsuchiya-Nakanishi [92], on déduit les relations de tressage pour $\mathfrak{Vir}_{1/2}$: sa matrice de tressage est la matrice de tressage pour $LSU(2)$ de niveau ℓ, fois la transposée de l'inverse de la matrice de tressage pour $LSU(2)$ de niveau $\ell+2$ (c'est prouvé par la contribution d'une transformation de jauge inverse de l'équation de Knizhnik-Zamolodchikov pour le tressage de $LSU(2)$). On obtient alors des coefficients non nuls:

$$\phi^{\alpha\ell}_{ii'jj'}(z)\phi^{\alpha\ell}_{jj'kk'}(w)=\sum\mu_{rr'}\phi^{\alpha\ell}_{ii'rr'}(w)\phi^{\alpha\ell}_{rr'kk'}(z)\text{ avec }\mu_{rr'}\neq 0.$$

Maintenant si $\phi^{\alpha}_{ii'jj'}=0$ avec $\phi^{1/2,\ell}_{ij}$et $\phi^{\frac{1}{2},\ell+2}_{i'j'}$ non nuls, alors la relation de tressage de $\phi^{\alpha}_{ii'jj'}$ avec son adjoint est nulle, mais produit quelques termes non-nuls $\phi^{\alpha}_{ii'kk'}$par le précédent argument d'irréductibilité: contradiction. Ainsi, on voit que $\phi^{\alpha}_{ii'jj'}$ est non nul ssi $\phi^{1/2\ell}_{ij}$ et$\phi^{1/2,\ell+2}_{i'j'}$ sont non nuls, ie, $i'=i\pm 1/2$ et $j'=j\pm 1/2$ (à quelques conditions de bord près). Maintenant, pour la charge $\beta=(0,1)$ et le tressage avec α, on fait de même, à partir de champ fermion Neveu-Schwarz $\psi(u,z)\otimes I$ commutant avec $I\otimes\phi^{\frac{1}{2},\ell}_{ij}(v,w)$.

Ensuite, par un argument de convolution, le tressage fonctionne également avec deux champs primaires couplés concentrés sur des intervalles disjoints. On déduit également que les algèbres de von Neumann $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ sont engendrées par des chaînes de champs primaires. Cette nouvelle caractérisation est essentielle pour prouver la densité de von Neumann: si I est un intervalle propre de $\mathbb{S}^1$ et I_1, I_2 sont des intervalles obtenus en enlevant un point de I alors, $\pi_i(\mathfrak{Vir}^{I_1}_{1/2})''\vee\pi_i(\mathfrak{Vir}^{I_2}_{1/2})''=\pi_i(\mathfrak{Vir}_{1/2}(I))''$. Par équivalence locale, on a seulement besoin de le prouver dans le vide; dans lequel l'algèbre locale sur I est engendrée par des chaînes concentrées en I. Par linéarité, le contexte L^2 et une sorte d'OPE, on peut séparer en produit de chaînes sur I_1 et I_2. Ensuite, la densité de von Neumann implique l'irréductibilité du sous-facteur de Jones-Wassermann: $\pi_i(\mathfrak{Vir}_{1/2}(I))^{\natural}\cap\pi_i(\mathfrak{Vir}_{1/2}(I^c))^{\natural}=\mathbb{C}$, ce qui signifie que les représentations H_i sont des $\mathfrak{Vir}_{1/2}(I)\oplus\mathfrak{Vir}_{1/2}(I^c)$-modules irréductibles.

1.13 Fusion de Connes et sous-facteurs

Par ce qui précède, les représentations de la série discrète sont des bimodules irréductibles sur l'algèbre de von Neumann $\mathcal{M} = \pi_0(\mathfrak{Vir}_{1/2}(I))''$. On définit un produit tensoriel relatif appelé fusion de Connes $\boxtimes$, en utilisant des fonctions 4-points: Soit $Hom_{-\mathcal{M}}(H_0, H_i) \otimes Hom_{\mathcal{M}-}(H_0, H_j)$, un $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathbb{Z}_2$-gradué, on définit un produit scalaire par:

$$(x_1 \otimes y_1, x_2 \otimes y_2) = (-1)^{(\partial x_1 + \partial x_2)\partial y_2}(x_2^\star x_1 y_2^\star y_1 \Omega, \Omega)$$

La L^2-complétion est toujours un $\mathcal{M}$-$\mathcal{M}$ bimodule $\mathbb{Z}_2$-gradué, appelé la fusion de Connes entre H_i et H_j et noté $H_i \boxtimes H_j$. La fusion est associative. On obtient l'anneau de fusion pour $\oplus$ et $\boxtimes$. L'outil clé pour calculer la fusion est la formule de transport qui montre explicitement comment les chaînes de la représentation vide, se transforment en chaînes sur d'autre représentations par les relations d'entrelacement. Grâce aux relations de tressage connues pour la charge α, on sait prouver la formule de transport suivante:

$$\pi_j(\bar{a}_{0\alpha}.a_{\alpha 0}) = \sum \lambda_k \bar{a}_{jk}.a_{kj} \quad \text{avec } \lambda_k > 0.$$

avec a_{kj} un champ primaire couplé de charge α (et partie ordinaire, donc paire) de $\mathfrak{Vir}_{1/2}$ entre H_j et H_k concentré en I, $\bar{a}_{jk} = a_{kj}^\star$, et $\pi_j : H_0 \to H_j$ l'équivalence locale. Maintenant, $a_{\alpha 0} \in Hom_{-\mathcal{M}}(H_0, H_\alpha)$, donc:

$$\|a_{\alpha 0} \otimes y\|^2 = (a_{\alpha 0}^\star a_{\alpha 0} y^\star y \Omega, \Omega) = (y^\star \pi_j(a_{\alpha 0}^\star a_{\alpha 0}) y \Omega, \Omega) = \sum \lambda_k \|a_{kj} y \Omega\|^2.$$

Ainsi, en utilisant le fait que $a_{\alpha 0}\mathcal{M}$ est dense dans $Hom_{-\mathcal{M}}(H_0, H_\alpha)$ (par densité de von Neumann), une polarisation et l'irréductibilité des bimodules, on obtient une application unitaire entre $H_\alpha \boxtimes H_j$ et $\bigoplus_{k \in \langle \alpha, j \rangle} H_k$, avec $k \in \langle \alpha, j \rangle$ ssi $\phi_{jk}^\alpha \neq 0$. On obtient les règles de fusion avec α:

$$H_\alpha \boxtimes H_j = \bigoplus\nolimits_{k \in \langle \alpha, j \rangle} H_k.$$

Maintenant, idem avec les relations de tressages entre des champs primaires de charge α et β, on obtient une formule de transport partielle et des règles de fusion partielles avec β:

$$H_\beta \boxtimes H_j \leq \bigoplus\nolimits_{k \in \langle \beta, j \rangle} H_k.$$

Mais, les règles de fusion avec α permettent de calculer un caractère de l'anneau de fusion appelé la dimension quantique (par le théorème de Perron-Frobenius). Une manière simple de calculer les dimensions quantiques est de voir que l'anneau de fusion pour l'algèbre Neveu-Schwarz à charge c_m est le produit tensoriel des anneaux de fusion pour l'algèbre de lacets aux niveaux ℓ et $\ell+2$ (avec $m=\ell+2$), modulo un automorphisme de période deux. Ainsi, les dimensions quantiques pour l'algèbre Neveu-Schwarz sont des produits de deux dimensions quantiques de l'algèbre de lacets (correspondant à la construction 'coset'):

$$d(H_{ij}^{\ell}) = d(H_i^{\ell}).d(H_j^{\ell+2}) = \frac{sin((2i+1)\pi/(\ell+2))}{sin(\pi/(\ell+2))}.\frac{sin((2j+1)\pi/(\ell+4))}{sin(\pi/(\ell+4))}$$

Les dimensions quantiques montrent que les règles partielles avec β sont en fait exactes. Ensuite, on voit que les règles de fusion pour α et β permettent de calculer toutes les autres. Finalement, les sous-facteurs (de type III_1 hyperfini) de Jones-Wassermann sont isomorphes à des sous-facteurs de type II_1 hyperfini, tensorisés par le facteur de type III_1 hyperfini, par H. Wenzl [103] et S. Popa [77]. Ces derniers sous-facteurs sont irréductibles, de profondeur fini et d'indices finis, donnés par le carré des dimensions quantiques.

2 Introduction in english

2.1 Background

In the 90's, V. Jones and A. Wassermann started a program whose goal is to understand the unitary conformal field theory from the point of view of operator algebras (see [46], [98]). In [99], Wassermann defines and computes the Connes fusion of the irreducible positive energy representations of the loop group $LSU(n)$ at fixed level ℓ, using primary fields, and with consequences in the theory of subfactors. In [87] V. Toledano Laredo proves the Connes fusion rules for $LSpin(2n)$ using similar methods. Now, let $\text{Diff}(\mathbb{S}^1)$ be the diffeomorphism group on the circle, its Lie algebra is the Witt algebra $\mathfrak{W}$ generated by d_n ($n \in \mathbb{Z}$), with $[d_m, d_n] = (m-n)d_{m+n}$. It admits a unique central extension called the Virasoro algebra $\mathfrak{Vir}$. Its unitary positive energy representation theory and the character formulas can be deduced by a so-called Goddard-Kent-Olive (GKO) coset construction from the theory of $LSU(2)$ and the Kac-Weyl formulas (see [100], [35], [100]). In [66], T. Loke uses the coset construction to compute the Connes fusion for $\mathfrak{Vir}$. Now, the Witt algebra admits two supersymmetric extensions $\mathfrak{W}_0$ and $\mathfrak{W}_{1/2}$ with central extensions called the Ramond and the Neveu-Schwarz algebras, noted $\mathfrak{Vir}_0$ and $\mathfrak{Vir}_{1/2}$. In this work, we give a complete proof of the classification of the unitary positive energy representations of $\mathfrak{Vir}_{1/2}$, we compute their character and the Connes fusion, with consequences in subfactors theory. Note that we could do the same for the Ramond algebra $\mathfrak{Vir}_0$, using twisted vertex module over the vertex operator algebra of the Neveu-Schwarz algebra $\mathfrak{Vir}_{1/2}$, as R. W. Verrill [96] and Wassermann [102] do for twisted loop groups.

2.2 Overview

First, we look unitary, projective, positive energy representations of $\mathfrak{W}_{1/2}$. The projectivity gives 2-cocycles, so that $\mathfrak{W}_{1/2}$ admits a unique central extension $\mathfrak{Vir}_{1/2}$. Such representations are completely reducible, and the irreducibles are given by the unitary highest weight representations of $\mathfrak{Vir}_{1/2}$: Verma modules $V(c,h)$ quotiented by null vectors, in no-ghost cases.

From the fermion algebra on $H = \mathcal{F}_{NS}$, we build the fermion field $\psi(z)$. Locality and Dong's lemma permit, via OPE, to generate a set of fields $\mathcal{S}$, so that there is a $1-1$ map $V : H \to \mathcal{S}$, with $Id = V(\Omega)$ and a Virasoro field $L = V(\omega)$. Then, we give vertex operator superalgebra's axioms, permitting to come so far, in a general framework (H, V, Ω, ω), with H prehilbert.

Let $\mathfrak{g}$ a simple finite-dimensional Lie algebra, $\widehat{\mathfrak{g}}_+$ the $\mathfrak{g}$-boson algebra (central extension of the loop algebra $L\mathfrak{g}$) and $\widehat{\mathfrak{g}}_-$ the $\mathfrak{g}$-fermion algebra. We build a module vertex operator superalgebra from $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$ on $H = L(V_\lambda, \ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$, so that $\mathfrak{Vir}_{1/2}$ acts on with $(c,h) = (\frac{3}{2}\cdot\frac{\ell+1/3g}{\ell+g}dim(\mathfrak{g}), \frac{c_{V_\lambda}}{2(\ell+g)})$, with g the dual Coxeter number and c_{V_λ} the Casimir number.

Let $\mathfrak{g} = \mathfrak{sl}_2$, using theta functions framework, we obtain the decomposition of $H = \mathcal{F}^{\mathfrak{g}}_{NS} \otimes (L(j,\ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS})$ as $\widehat{\mathfrak{g}}$-module. The multiplicity spaces of irreducible components H_k are superintertwiners space $Hom_{\widehat{\mathfrak{g}}}(H_k, H)$; we deduce their character as module of $\mathfrak{W}_{1/2}$, which acts on with $L(c_m, h^m_{pq})$ as submodule by GKO construction. The unitarity of the discrete series follows.

We define irreducible polynomial $\varphi_{pq}(c,h)$ from (c_m, h^m_{pq}). The Kac determinant $det_n(c,h)$ of the sesquilinear form on $V(c,h)$ at level n is easily interpolate, as a product of φ_{pq}, computing the first examples. To prove it, we enlight links between previous characters results and singular vectors s (i.e. $G_{1/2}.s = G_{3/2}.s = 0$), whose the existence vanishes det_n.

A negative Kac determinant shows easily a ghost on the region between the curves $h = h^c_{pq}$. Now, we go from the no-ghost region $h > 0$, $c > 3/2$ to an order 1 vanishing curve C; then, on the other side, there is a ghost. By transversality, it pass on the curve intersecting C next. And so on each curves, excepting 'first intersections': discrete series. Theorem 2.2 follows.

Finally, a coherence argument between the characters of the multiplicity spaces M^m_{pq} and its irreducibles (on discrete series by FQS), shows M^m_{pq} without others irreducibles that $L(c_m, h^m_{rs})$. So, $M^m_{pq} = L(c_m, h^m_{p,q})$ and we obtain the character of $L(c_m, h^m_{p,q})$ as the character of M^m_{pq}, ever known by GKO construction. Theorem 2.3 follows.

Now, $\widehat{\mathfrak{g}}$ and $\mathfrak{Vir}_{1/2}$ give local superalgebras $\widehat{\mathfrak{g}}(I)$ and $\mathfrak{Vir}_{1/2}(I)$ by smearing with the smooth functions vanishing outside of I a proper interval of $\mathbb{S}^1$. By Sobolev estimates, the action on the positive energy representations is continuous. We generate their von Neumann algebra, included in an algebra of fermions. By Takesaki devissage and coset construction, we obtain that these algebras are the hyperfinite III_1 factor, whose the supercommutants are generated by chains of compressed fermions. Also, there is Haag-Araki duality on the vacuum, and outside, a Jones-Wassermann subfactor as a failure of duality.

The compressed fermions are examples of primary fields. We construct them in general from maps intertwining two irreducible representations, dealing with spaces of densities. We see that these maps are completely characterized, bounded and classified by coset for two particular charges α, β. We obtain also their braiding relations, which allow to give the leading term of a kind of OPE for smeared primary fields, which permit, to have the von Neumann density and the irreducibility of the subfactors.

Then, we obtain irreducible bimodules of local von Neumann algebras, giving the framework to define the Connes fusion. Its rules are a direct consequence of the transport formula (explaining the intertwining for chains), which is proved by the braiding relations and the von Neumann density. The rules give the dimension of the space of primary fields, they show also that the subfactors are finite index, explicitly given by the square of the quantum dimension, a fusion ring character given as unique positive eigenvalue of a fusion matrix, and a product of two quantum dimensions of $LSU(2)$ by Perron-Frobenius theorem.

2.3 Main results

Part I: Unitary series and characters for $\mathfrak{Vir}_{1/2}$

The irreducible positive energy representations of the Neveu-Schwarz algebra $\mathfrak{Vir}_{1/2}$ are denoted $L(c,h)$ with Ω its cyclic vector. Our purpose is to give a complete proof of the classification of unitary representations, in such a way that we obtain directly the characters of the discrete series, without Feigin-Fuchs resolution [20]. The Neveu-Schwarz algebra is defined by:

$$\begin{cases} [L_m, L_n] & = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n} \\ [G_r, L_n] & = (m - \frac{n}{2})G_{r+n} \\ [G_r, G_s]_+ & = 2L_{r+s} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s} \end{cases}$$

with m, $n \in \mathbb{Z}$, r, $s \in \mathbb{Z} + \frac{1}{2}$, $L_n^\star = L_{-n}$, $G_r^\star = G_{-r}$.
Positive energy means that $L(c,h) = H = \bigoplus H_n$, with $n \in \frac{1}{2}\mathbb{N}$, such that $L_0\xi = (n+h)\xi$ on H_n and $H_0 = \mathbb{C}\Omega$ (with $C\Omega = c\Omega$).

Lemma 2.1. *If $L(c,h)$ is unitary, then $c, h \geq 0$*

Theorem 2.2. *The classification of unitary representations $L(c,h)$ is:*

(a) *Continuous series: $c \geq 3/2$ and $h \geq 0$.*

(b) *Discrete series: $(c,h) = (c_m, h_{pq}^m)$ with:*

$$c_m = \frac{3}{2}(1 - \frac{8}{m(m+2)}) \quad and \quad h_{pq}^m = \frac{((m+2)p - mq)^2 - 4}{8m(m+2)}$$

with integers $m \geq 2$, $1 \leq p \leq m-1$, $1 \leq q \leq m+1$ and $p \equiv q[2]$.

Theorem 2.3. *The characters of the discrete series are:*

$$ch(L(c_m, h_{pq}^m))(t) = tr(t^{L_0 - c_m/24}) = \chi_{NS}(t).\Gamma_{pq}^m(t).t^{-c_m/24} \quad with$$

$$\chi_{NS}(t) = \prod_{n \in \mathbb{N}^\star} \frac{1 + t^{n-1/2}}{1 - t^n}, \quad \Gamma_{pq}^m(t) = \sum_{n \in \mathbb{Z}} (t^{\gamma_{pq}^m(n)} - t^{\gamma_{-pq}^m(n)}) \quad and$$

$$\gamma_{pq}^m(n) = \frac{[2m(m+2)n - (m+2)p + mq]^2 - 4}{8m(m+2)}$$

Part II: Connes fusion and subfactors for $\mathfrak{Vir}_{1/2}$

Let $p = 2i+1$, $q = 2j+1$ and $m = \ell+2$, we note H^{ℓ}_{ij} the L^2-completion of $L(c_m, h^m_{pq})$. We define the Connes fusion $\boxtimes$ on the discrete series representations of charge c_m, as bimodules of the hyperfinite III$_1$-factor generated by the local Neveu-Schwarz algebra $\mathfrak{Vir}_{1/2}(I)$, with I a proper interval of $\mathbb{S}^1$.

Theorem 2.4. *(Connes fusion)*

$$H^{\ell}_{ij} \boxtimes H^{\ell}_{i'j'} = \bigoplus_{(i'',\, j'') \in \langle i,i' \rangle_{\ell} \times \langle j,j' \rangle_{\ell+2}} H^{\ell}_{i''j''}$$

with $\langle a, b \rangle_n = \{c = |a-b|, |a-b|+1, ..., a+b \mid a+b+c \leq n\}$.

Let $\mathcal{M}^{\ell}_{ij}(I)$ be the von Neumann algebra generated on H^{ℓ}_{ij}, by the bounded function of the self-adjoint operators of $\mathfrak{Vir}_{1/2}(I)$.

Theorem 2.5. *(Haag-Araki duality on the vacuum)*

$$\mathcal{M}^{\ell}_{00}(I) = \mathcal{M}^{\ell}_{00}(I^c)^{\natural}$$

with $X^{\natural}$ *be the supercommutant of* X*.*

As a failure of Haag-Araki duality out of the vacuum, we have:

Theorem 2.6. *(Jones-Wassermann subfactor)*

$$\mathcal{M}^{\ell}_{ij}(I) \subset \mathcal{M}^{\ell}_{ij}(I^c)^{\natural}$$

It's a finite depth, irreducible, hyperfinite III$_1$-subfactor, isomorphic to the hyperfinite III$_1$-factor $\mathcal{R}_{\infty}$ tensor the II$_1$-subfactor :

$$(\bigcup \mathbb{C} \otimes End_{\mathfrak{Vir}_{1/2}}(H^{\ell}_{ij})^{\boxtimes n})'' \subset (\bigcup End_{\mathfrak{Vir}_{1/2}}(H^{\ell}_{ij})^{\boxtimes n+1})''$$

of index $\frac{sin^2(p\pi/m)}{sin^2(\pi/m)} . \frac{sin^2(q\pi/(m+2))}{sin^2(\pi/(m+2))}$*, with* $p = 2i+1$*,* $q = 2j+1$*,* $m = \ell+2$*.*

2.4 The Neveu-Schwarz algebra

We start with $\mathfrak{W}_{1/2}$, the Witt superalgebra of sector (NS):

$$\begin{cases} [d_m, d_n] &= (m-n)d_{m+n} \quad m, n \in \mathbb{Z} \\ [\gamma_m, d_n] &= (m - \frac{n}{2})\gamma_{m+n} \quad m \in \mathbb{Z} + \frac{1}{2}, n \in \mathbb{Z} \\ [\gamma_m, \gamma_n]_+ &= 2d_{m+n} \quad m, n \in \mathbb{Z} + \frac{1}{2} \end{cases}$$

together with $d_n^\star = d_{-n}$ and $\gamma_m^\star = \gamma_{-m}$; we study representations which are:
(a) Unitary: $\pi(A)^\star = \pi(A^\star)$
(b) Projective: $A \mapsto \pi(A)$ is linear and $[\pi(A), \pi(B)] - \pi([A, B]) \in \mathbb{C}$.
(c) Positive energy : H admits an orthogonal decomposition $H = \bigoplus_{n \in \frac{1}{2}\mathbb{N}} H_n$ such that $\exists D$ acting on H_n as multiplication by n, $H_0 \neq \{0\}$, $\dim(H_n) < +\infty$
Here, $\exists h \in \mathbb{C}$ such that $D = \pi(d_0) - hI$.
Now, the projectivity gives the 2-cocycles and we see that $H_2(\mathfrak{W}_{1/2}, \mathbb{C})$ is 1-dimensional, $\mathfrak{W}_{1/2}$ admits a unique central extension up to equivalence:

$$0 \to H_2(\mathfrak{W}_{1/2}, \mathbb{C}) \to \mathfrak{Vir}_{1/2} \to \mathfrak{W}_{1/2} \to 0$$

$\mathfrak{Vir}_{1/2}$ is the SuperVirasoro (of sector NS) or Neveu-Schwarz algebra:

$$\begin{cases} [L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] &= (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ &= 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

with $L_n^\star = L_{-n}$, $G_m^\star = G_{-m}$ and $C = cI$, $c \in \mathbb{C}$ called the **central charge**. The representations are completely reducible, the irreducibles are determined by the two numbers c, h, and are completely given by unitary highest weight representations of $\mathfrak{Vir}_{1/2}$, described as follows: The Verma modules $H = V(c, h)$ are freely generated by: $0 \neq \Omega \in H$ (cyclic vector), $C\Omega = c\Omega$, $L_0\Omega = h\Omega$ and $\mathfrak{Vir}_{1/2}^+\Omega = \{0\}$. Now, $(\Omega, \Omega) = 1$, $\pi(A)^\star = \pi(A^\star)$ and $(u, v) = \overline{(v, u)}$ give the sesquilinear form $(.,.)$. $V(c, h)$ can admit ghost: $(u, u) < 0$, and null vectors: $(u, u) = 0$. In no ghost case, the set of null vectors is $K(c, h)$ the kernel of $(.,.)$, the maximal proper submodule.
Let $L(c, h) = V(c, h)/K(c, h)$, the unitary highest weight representations.
The theorem 2.2 will be proved classifying no ghost cases.

2.5 Vertex operators superalgebras

Our approch of vertex operators superalgebras is freely inspired by the followings references: Borcherds [11], Goddard [37], Kac [57]. We start by working on the femion algebra: $[\psi_m, \psi_n]_+ = \delta_{m+n} I$ and $\psi_n^\star = \psi_{-n}$ $(m, n \in \mathbb{Z}+\frac{1}{2})$. As for $\mathfrak{W}_{1/2}$, we build its Verma module $H = \mathcal{F}_{NS}$ and the sesquilinear form $(.,.)$, which is a scalar product. H is a prehilbert space, the unique unitary highest weight representation of the fermion algebra. Let the formal power series $\psi(z) = \sum_{n\in\mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1}$ called fermion field. We inductively defined operators D giving positive energy structure ($\Leftrightarrow [D, \psi] = z.\psi' + \frac{1}{2}\psi$) and T giving derivation ($[T, \psi] = \psi'$). We compute $(\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w}$ $(|z| > |w|)$, which permits to prove inductively an anticommutation relation shortly written as: $\psi(z)\psi(w) = -\psi(w)\psi(z)$. We define this relation in a general framework as locality: Let H prehilbert space, and let $A \in (EndH)[[z, z^{-1}]]$ formal power series of the form $A(z) = \sum_{n\in\mathbb{Z}} A(n) z^{-n-1}$ with $A(n) \in End(H)$. Such fields A and B are **local** if $\exists \varepsilon \in \mathbb{Z}_2$, $\exists N \in \mathbb{N}$ such that $\forall c, d \in H$, $\exists X(A, B, c, d) \in (z-w)^{-N}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ such that:

$$X(A,B,c,d)(z,w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases}$$

Now, using locality and a contour integration argument, we can explicitly construct a field $A_n B$ from A and B, with $(A_n B)(m) =$

$$\begin{cases} \sum_{p=0}^{n} (-1)^p C_n^p [A(n-p), B(m+p)]_\varepsilon & \text{if } n \geq 0 \\ \\ \sum_{p\in\mathbb{N}} C_{p-n-1}^p (A(n-p)B(m+p) - (-1)^{\varepsilon+n} B(m+n-p)A(p)) & \text{if } n < 0 \end{cases}$$

We obtain the operator product expansion (OPE) shortly written as $A(z)B(w) \sim \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}}$; and by an other contour integration argument:

$$[A(m), B(n)]_\varepsilon = \begin{cases} \sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) & \text{if } m \geq 0 \\ \\ \sum_{p=0}^{N-1} (-1)^p C_{p-m-1}^p (A_p B)(m+n-p) & \text{if } m < 0 \end{cases}$$

Thanks to Dong's lemma, the operation $(A, B) \mapsto A_n B$ permits to generate many fields. To have a good behaviour, we define a system of generators as:

$\{A_1,..., A_r\} \subset (EndH)[[z, z^{-1}]]$ with $D, T \in End(H)$, $\Omega \in H$ such that:
(a) $\forall i, j$ A_i and A_j are local with $N = N_{ij}$ and $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii}.\varepsilon_{jj}$
(b) $\forall i$ $[T, A_i] = A'_i$
(c) $H = \bigoplus_{n\in\mathbb{N}+\frac{1}{2}}$ for D, $dim(H_n) < \infty$
(d) $\forall i$ $[D, A_i] = z.A'_i + \alpha_i A_i$ with $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$
(e) $\Omega \in H_0$, $\|\Omega\| = 1$, and $\forall i$ $\forall m \in \mathbb{N}$, $A_i(m)\Omega = D\Omega = T\Omega = 0$
(f) $\mathcal{A} = \{A_i(m), \forall i\ \forall m \in \mathbb{Z}\}$ acts irreducibly on H, so that $< \mathcal{A} > .\Omega = H$
Hence, we generate a space $\mathcal{S}$, with $V : H \longrightarrow \mathcal{S}$ a state-field correspondence linear map. $V(a)(z)$ is noted $V(a, z)$ and $V(a, z)\Omega_{|z=0} = a$.
Now, $\{\psi\}$ is a system of generator, we generate $\mathcal{S}$ and the map V with $\psi(z) = V(\psi_{-\frac{1}{2}}\Omega, z)$; but, $\psi(z)\psi(w) \sim \frac{Id}{z-w} + 2L(w)$, with $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{2}\psi_{-2}\psi(z) = V(\omega, z)$ with $\omega = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$. Then, using OPE and Lie bracket, we find that $D = L_0$, $T = L_{-1}$, $L(z)L(w) \sim \frac{(c/2)Id}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)}$, and $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$ with $c = 2\|L_{-2}\Omega\|^2 = \frac{1}{2}$, the central charge. As corollary, $\mathfrak{Vir}$ acts on $H = \mathcal{F}_{NS}$, and admits its unitary highest weight representation $L(c, h) = L(\frac{1}{2}, 0)$ as minimal submodule containing Ω. We call $\omega \in H$ the Virasoro vector, and L the Virasoro field.
We are now able to define vertex operators superalgebras in general.
A vertex operator superalgebra is an (H, V, Ω, ω) with:
(a) $H = H_{\bar{0}} \oplus H_{\bar{1}}$ a prehilbert superspace.
(b) $V : H \to (EndH)[[z, z^{-1}]]$ a linear map.
(c) Ω, $\omega \in H$ the vacuum and Virasoro vectors.
Let $\mathcal{S}_\varepsilon = V(H_\varepsilon)$, $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ and $A(z) = V(a, z) = \sum_{n\in\mathbb{Z}} A(n)z^{-n-1}$, then (H, V, Ω, ω) satisfies the followings axioms:
(1) $\forall n \in \mathbb{N}$, $\forall A \in \mathcal{S}$, $A(n)\Omega = 0$, $V(a, z)\Omega_{|z=0} = a$, and $V(\Omega, z) = Id$
(2) $\mathcal{A} = \{A(n)|A \in \mathcal{S}, n \in \mathbb{Z}\}$ acts irreducibly on H, so that $\mathcal{A}.\Omega = H$.
(3) $\forall A \in \mathcal{S}_{\varepsilon_1}$, $\forall B \in \mathcal{S}_{\varepsilon_2}$, A and B are local with $\varepsilon = \varepsilon_1.\varepsilon_2$, $A_nB \in \mathcal{S}_{\varepsilon_1+\varepsilon_2}$
(4) $V(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $[L_m, L_n] = (m-n)L_{m+n} + \frac{\|2\omega\|^2}{12}m(m^2-1)\delta_{m+n}$
(5) $H = \bigoplus_{n\in\mathbb{N}+\frac{1}{2}} H_n$ for L_0, $dim(H_n) < \infty$ and $H_\varepsilon = \bigoplus_{n\in\mathbb{N}+\frac{\varepsilon}{2}} H_n$
(6) $[L_0, V(a, z)] = z.V'(a, z) + \alpha.V(a, z)$ for $a \in H_\alpha$
(7) $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$
As corollaries, we have that a system of generators, generating a Virasoro field $L \in \mathcal{S}$, with $D = L_0$ and $T = L_{-1}$, generates a vertex operator superalgebra; the fermion field ψ and the Virasoro field L generate one, each; and we prove the Borcherds associativity: $V(a, z)V(b, w) = V(V(a, z - w)b, w)$.

2.6 Vertex $\mathfrak{g}$-superalgebras and modules

Let $\mathfrak{g}$ be a simple Lie algebra of dimension N, a basis (X_a), well normalized (see remark 5.2), such that $[X_a, X_b] = i\sum_c \Gamma_{ab}^c X_c$ with $\Gamma_{ab}^c \in \mathbb{R}$ totally antisymmetric. Let its dual coxeter number $g = \frac{1}{4}\sum_{a,c}(\Gamma_{ac}^b)^2$:

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$dim(\mathfrak{g})$	n^2+2n	$2n^2+n$	$2n^2+n$	$2n^2-n$	78	133	248	52	14
g	$n+1$	$2n-1$	$n+1$	$2n-2$	12	18	30	9	4

For example, $\mathfrak{g} = A_1 = \mathfrak{sl}_2$, $dim(\mathfrak{g}) = 3$ and $g = 2$.
Let $\widehat{\mathfrak{g}}_+$ the $\mathfrak{g}$-boson algebra: $[X_m^a, X_n^b] = [X_a, X_b]_{m+n} + m\delta_{ab}\delta_{m+n}.\mathcal{L}$, unique central extension (by $\mathcal{L}$) of the loop algebra $L\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{g})$ (see [100] p 43). The unitary highest weight representations of $\widehat{\mathfrak{g}}_+$ are $H = L(V_\lambda, \ell)$, with $\ell \in \mathbb{N}$ such that $\mathcal{L}\Omega = \ell\Omega$ (the level of H), $H_0 = V_\lambda$ irreducible representation of $\mathfrak{g}$ such that $(\lambda, \theta) \leq \ell$ with λ the highest weight and θ the highest root (see [100] p 45). The category $\mathscr{C}_\ell$ of representations for fixed ℓ is finite. For example $\mathfrak{g} = \mathfrak{sl}_2$, $H = L(j, \ell)$, with $V_\lambda = V_j$ representations of spin $j \leq \frac{\ell}{2}$.

We define the $\mathfrak{g}$-fermion algebra $\widehat{\mathfrak{g}}_-$ and the fermion fields, composed by N fermions; and as for $N = 1$, we generate a vertex operator superalgebra, but now, it contains $\mathfrak{g}$-boson fields (S^a) whose related algebra is represented with $L(V_0, g)$; and thanks to $\widehat{\mathfrak{g}}_-$vertex background, the fields (S^a) generate a vertex operator superalgebra; by this way, we are able to generate one, from $\widehat{\mathfrak{g}}_+$ and $H = L(V_0, \ell)$, $\forall \ell \in \mathbb{N}$. Remark that because of the vacuum axiom, the vertex structure need to take $V_\lambda = V_0$ trivial representation; in general, we have vertex modules (see further).

Now, let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$ the $\mathfrak{g}$-supersymmetric algebra; we prove it admits $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ as unitary highest weight representations. We generate a vertex operator superalgebra, with a Virasoro field L, and also a SuperVirasoro field G, which gives the supersymmetry boson-fermion: Let $B^a = X^a + S^a$ boson fields of level $d = \ell + g$, then:

$$B^a(z)G(w) \sim d^{\frac{1}{2}} \frac{\psi^a(w)}{(z-w)^2} \quad \text{and} \quad \psi^a(z)G(w) \sim d^{-\frac{1}{2}} \frac{B^a(w)}{(z-w)}.$$

Finally, from $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$, we define the vertex module (H^λ, V^λ) over (H^0, V, Ω, ω), and we prove that $\mathfrak{Vir}_{\frac{1}{2}}$ acts unitarily on H^λ and admits $L(c, h)$ as minimal submodule containing the cyclic vector Ω^λ, with
$c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} dim(\mathfrak{g})$, $h = \frac{c_{V_\lambda}}{2(\ell+g)}$ and c_{V_λ} the Casimir number of V_λ.

2.7 Goddard-Kent-Olive framework

We take $\mathfrak{g} = \mathfrak{sl}_2$. Let H an irreducible unitary, projective, positive energy representation of the loop algebra $L\mathfrak{g}$. We define the character of H as: $ch(H)(t,z) = tr(t^{L_0 - \frac{C}{24}} z^{X_3})$. $L\mathfrak{g}$ acts on $\mathcal{F}^{\mathfrak{g}}_{NS}$, and by Jacobi's triple product identity $\sum_{k\in\mathbb{Z}} t^{\frac{1}{2}k^2} z^k = \prod_{n\in\mathbb{N}^\star}(1+t^{n-\frac{1}{2}}z)(1+t^{n-\frac{1}{2}}z^{-1})(1-t^n)$, we prove that $ch(\mathcal{F}^{\mathfrak{g}}_{NS})(t,z) = t^{-1/16}\chi_{NS}(t)\theta(t,z)$ with $\chi_{NS}(t) = \prod_{k\in\mathbb{N}^\star}(\frac{1+t^{n-\frac{1}{2}}}{1-t^n})$ and $\theta(t,z) = \sum_{k\in\mathbb{Z}} t^{\frac{1}{2}k^2} z^k$. Hence, let $H = L(j,\ell)$, and the theta functions $\theta_{n,m}(t,z) = \sum_{k\in\frac{n}{2m}+\mathbb{Z}} t^{mk^2} z^{mk}$, then applying the Weyl-Kac formula to $L\mathfrak{g}$: $ch(L(j,\ell)) = \frac{\theta_{2j+1,\ell+2} - \theta_{-2j-1,\ell+2}}{\theta_{1,2} - \theta_{-1,2}}$ (see [49], [56] or [100] p 62). Now, adapting the proof in [54] p 122, we obtain the product formula: $\theta(t,z).\theta_{p,m}(t,z) = \sum_{\substack{0\leq q<2(m+2)\\ p\equiv q[2]}}(\sum_{n\in\mathbb{Z}} t^{\alpha^m_{pq}(n)})\theta_{q,m+2}(t,z)$ with $\alpha^m_{p,q}(n) = \frac{[2m(m+2)n-(m+2)p+mq]^2}{8m(m+2)}$.
Now, $L\mathfrak{g}$ acts on $L(j,\ell)\otimes\mathcal{F}^{\mathfrak{g}}_{NS}$ at level $\ell+2$; we deduce: $ch(L(j,\ell)\otimes\mathcal{F}^{\mathfrak{g}}_{NS}) = \sum_{\substack{1\leq q\leq m+1\\ p\equiv q[2]}} F^m_{pq}.ch(L(k,\ell+2))$, $F^m_{pq}(t) = t^{-1/16}\chi_{NS}(t)\sum_{n\in\mathbb{Z}}(t^{\alpha^m_{p,q}(n)} - t^{\alpha^m_{-p,q}(n)})$, $p = 2j+1$, $q = 2k+1$ and $m = \ell+2$; and the tensor product decomposition: $L(j,\ell)\otimes\mathcal{F}^{\mathfrak{g}}_{NS} = \bigoplus_{\substack{1\leq q\leq m+1\\ p\equiv q[2]}} M^m_{pq}\otimes L(k,\ell+2)$ with M^m_{pq} the multiplicity space.
General GKO framework: Let $\mathfrak{h}$ be Lie $\star$-superalgebra acting unitarily on a finite direct sum $H = \bigoplus M_i\otimes H_i$ with H_i irreducible and M_i the multiplicity space. We see that M_i is the inner product space of superintertwiners $Hom_{\mathfrak{h}}(H_i,H)$. Now, if $\mathfrak{d}$ is a Lie $\star$-superalgebra acting on H and H_i as unitary, projective, positive energy representations, whose difference $(\pi(D) - \sum\pi_i(D))$ supercommutes with $\mathfrak{h}$, then, so is on M_i, with cocycle, the difference of the others. Then, taking $\mathfrak{h} = \hat{\mathfrak{g}}$ and $\mathfrak{d} = \mathfrak{W}_{1/2}$, we find $c_{M^m_{pq}} = \frac{dim(\mathfrak{g})}{2}(1 - \frac{2g^2}{(\ell+g)(\ell+2g)}) = \frac{3}{2}(1 - \frac{8}{m(m+2)}) =: c_m$, because $m = \ell+2$, $g = 2$ and $dim(\mathfrak{g}) = 3$. Now, the character of a $\mathfrak{Vir}_{1/2}$-module H is : $ch(H)(t) = tr(t^{L_0 - \frac{C}{24}})$, then: $ch(M^m_{pq})(t) = t^{-\frac{c(m)}{24}}.\chi_{NS}(t).\Gamma^m_{pq}(t)$ with $\Gamma^m_{pq}(t) = \sum_{n\in\mathbb{Z}}(t^{\gamma^m_{pq}(n)} - t^{\gamma^m_{-pq}(n)})$, $\chi_{NS}(t) = \prod_{n\in\mathbb{N}^\star}\frac{1+t^{n-1/2}}{1-t^n}$ and $\gamma^m_{pq}(n) = \frac{[2m(m+2)n-(m+2)p+mq]^2-4}{8m(m+2)}$. Hence, $h = h^m_{pq} = \frac{[(m+2)p-mq]^2-4}{8m(m+2)}$ is the lowest eigenvalue of L_0 on M^m_{pq}; let $(p',q') = (m-p, m+2-q)$, then:

$$ch(M^m_{pq}) \sim t^{-\frac{c_m}{24}}.\chi_{NS}(t).t^{h^m_{pq}}.(1 - t^{\frac{pq}{2}} - t^{\frac{p'q'}{2}}).$$

Hence, $ch(M^m_{pq}).t^{\frac{c_m}{24}} \sim t^{h^m_{pq}}$, and the h^m_{pq}-eigenspace of L_0 is one-dimensional, so $L(c_m, h^m_{pq})$ is a $\mathfrak{Vir}_{1/2}$-submodule of M^m_{pq}, and $ch(L(c_m,h^m_{pq})) \leq ch(M^m_{pq})$. Finally, because M^m_{pq} is unitary, so is for $L(c_m,h^m_{pq})$ on the discrete series.

2.8 Kac determinant formula

From (c_m, h^m_{pq}), we define h^c_{pq}, $\forall c \in \mathbb{C}$. Let $\varphi_{pp}(c,h) = (h - h^c_{pp})$, $\varphi_{pq}(c,h) = (h - h^c_{pq})(h - h^c_{qp})$ if $p \neq q$, then $\varphi_{pq} \in \mathbb{C}[c,h]$ is irreducible.
Let $V_n(c,h)$ the n-eigenspace of $D = L_0 - hI$ and $d(n)$ its dimension.
Let $M_n(c,h)$ the matrix of $(.,.)$ on $V_n(c,h)$ and $det_n(c,h) = det(M_n(c,h))$.
For example, $M_0(c,h) = (\Omega,\Omega) = (1)$, $M_{\frac{1}{2}}(c,h) = (G_{-\frac{1}{2}}\Omega, G_{-\frac{1}{2}}\Omega) = (2h)$, $M_1(c,h) = (L_{-1}\Omega, L_{-1}\Omega) = (2h)$, and $M_{\frac{3}{2}}(c,h) =$

$$\begin{pmatrix} (G_{-\frac{1}{2}}L_{-1}\Omega, G_{-\frac{1}{2}}L_{-1}\Omega) & (G_{-\frac{1}{2}}L_{-1}\Omega, G_{-\frac{3}{2}}\Omega) \\ (G_{-\frac{3}{2}}\Omega, G_{-\frac{1}{2}}L_{-1}\Omega) & (G_{-\frac{3}{2}}\Omega, G_{-\frac{3}{2}}\Omega) \end{pmatrix} = \begin{pmatrix} 2h+4h^2 & 4h \\ 4h & 2h+\frac{2}{3}c \end{pmatrix}$$

Now, $det_{\frac{3}{2}}(c_m,h) = 8h[h^2 - (\frac{3}{2} - \frac{c_m}{3})h + c/6] = 8h(h - h^m_{13})(h - h^m_{31})$, then, $det_{\frac{3}{2}}(c,h) = 8h(h - h^c_{13})(h - h^c_{31}) = 8\varphi_{11}(c,h).\varphi_{13}(c,h) \ \ \forall c \in \mathbb{C}$.
Hence, others examples permits to interpolate the Kac determinant formula:

$$det_n(c,h) = A_n \prod_{\substack{0<pq/2\leq n \\ p\equiv q[2]}} (h - h^c_{pq})^{d(n-pq/2)} = A_n \prod_{\substack{0<pq/2\leq n \\ p\leq q,\ p\equiv q[2]}} \varphi_{pq}^{d(n-pq/2)}(c,h)$$

with $A_n > 0$ independent of c and h.
To prove it, we will use singular vectors $s \in V(c,h)$, i.e. $L_0.s = (h+n)s$ with $n > 0$ its level, and $\mathfrak{Vir}^+_{1/2}.s = 0$. This is equivalent to $G_{1/2}.s = G_{3/2}.s = 0$, and so we easily find $(mG_{-3/2} - (m+2)L_{-1}G_{-1/2})\Omega \in V_{3/2}(c_m, h^m_{13})$, $G_{-1/2}\Omega \in V_{1/2}(c, h^c_{11})$, or $(L^2_{-1} - \frac{4}{3}h^c_{22}L_{-2} - G_{-3/2}G_{-1/2})\Omega \in V_2(c, h^c_{22})$.
Now, $ch(V(c,h)) = t^{h-\frac{c}{24}}\chi_{NS}(t)$ and the singular vectors generate $K(c,h)$. So, $V(c,h)$ admits a singular vector of minimal level $n \in \frac{1}{2}\mathbb{N}$ if and only if

$$ch(L(c,h)) \sim t^{h-\frac{c}{24}}\chi_{NS}(t)(1-t^n).$$

Now, thanks to GKO coset construction:

$$ch(L(c_m, h^m_{pq})) \leq ch(M^m_{pq}) \sim t^{-\frac{c_m}{24}}.\chi_{NS}(t).t^{h^m_{pq}}.(1 - t^{\frac{pq}{2}} - t^{\frac{p'q'}{2}})$$

So $V(c_m, h^m_{pq})$ admits a singular vector s at level $n' \leq min(pq/2, p'q'/2)$ and for $n > n'$, det_n vanishes at (c_m, h^m_{pq}) for m sufficiently large integer. Then it vanishes at infinite many zeros of the irreducible φ_{pq}, which so φ_{pq} divides det_n. But s generates a subspace of dimension $d(n-n')$ at level n, so $d_n(c,h) = \prod_{\substack{0<pq/2\leq n \\ p\equiv q[2]}}(h - h^c_{pq})^{d(n-pq/2)}$ divides det_n. Finally, a cardinality argument shows d_n and det_n, with the same degree in h. The result follows.

2.9 Friedan-Qiu-Shenker unitarity criterion

The FQS criterion was discovered for $\mathfrak{Vir}$ by Friedan, Qiu and Shenker [25], but mathematicians estimated their proof too light, and then, in the same time, FQS [28] and Langlands [64] published a complete proof. At the beginning of our research on $\mathfrak{Vir}_{1/2}$, we decided to adapt the way of Langlands, but we find a mistake in this paper ([64] lemma 7b p 148: $p = 2$, $q = 1$, $m = 2$, $h^m_{pq} = \frac{5}{8}$, $M = 4$ or $p = 4$, $q = 1$, $m = 3$, $h^m_{pq} = \frac{7}{2}$, $M = 13$ yield case (B), but $(p, q) \neq (1, 1)$ and $m \not> q + p - 1$. In fact, we need to distinguish between $q \neq 1$ and $q = 1$, but not between $(p, q) \neq (1, 1)$ and $q = (1, 1)$). Next, we discovered that Sauvageot has ever published such an adaptation, without correction ([82] lemma 2 (ii) p 648). Then, we chose the way of FQS:

We are looking for a necessary condition on (c, h) for $V(c, h)$ has no ghost. First of all, if $V(c, h)$ admits no ghost then $c, h \geq 0$ (easy). Now, Kac determinant doesn't vanish on the region $h > 0$, $c > 3/2$, and for (c, h) large, we prove that the form $(., .)$ is positive. So by continuity, if $h \geq 0$ and $c \geq 3/2$, $V(c, h)$ admits no ghost. Now, on the region $0 \leq c < 3/2$, $h \geq 0$, the FQS criterion says that $V(c, h)$ admits ghosts if (c, h) does not belong to (c_m, h^m_{pq}), with integers $m \geq 2$, $1 \leq p \leq m-1$, $1 \leq q \leq m+1$ and $p \equiv q[2]$, ie, exactly the discrete series given by GKO construction ! To prove this result, we exploit the zero set of Kac determinants, constitutes by curves C_{pq} of equation $h = h^c_{pq}$ with $0 \neq p \equiv q[2]$. First of all, we restrict to C'_{pq}, the open subset of C_{pq}, between $c = 3/2$ and its first intersection at level $pq/2$. Let $p'q' > pq$, $C_{p'q'}$ is a first intersector of C'_{pq} if at level $p'q'/2$, it is the first to intersect C'_{pq} starting from $c = 3/2$. We see that all these first intersections constitutes exactly the discrete series. Now, for each open region between the curves C'_{pq}, we can find n with det_n negative on. This significate that $V(c, h)$ admits ghost on, and so we can eliminate these regions. Hence now, we have to eliminate the intervals on C'_{pq} between the points of the discrete series. We start from the no-ghost region $h > 0$, $c > 3/2$ and we go towards such an interval. On the way, we encounter a (well choosen) curve vanishing to order 1; so on the other side, there is a ghost. We continue along the area of this curve with our ghost, up to an intersection point. Now, because the intersections are transversals, we can distinguish null vectors from the first curve to the second, and so our ghost continues to be a ghost on the other curve. Repeating this principle, we can go to the interval, without losing the ghost. Then, FQS criterion and theorem 2.2 follow.

2.10 Wassermann's argument

We show that the multiplicity space of the coset construction, is an irreducible representation of the Neveu-Schwarz algebra, which (as in [100] p 72 for $\mathfrak{Vir}$) gives directly the characters on the discrete series without the Feigin-Fuchs resolution [20]:

As a corollary of FQS criterion's proof, at levels $\leq M = max(pq/2, p'q'/2)$, there exists only two singular vectors s and s', at levels $pq/2$ and $p'q'/2$. Hence, $ch(L(c_m, h_{pq}^m)) \sim t^{h_{pq}^m - c_m/24}\chi_{NS}(t)(1 - t^{pq/2} - t^{p'q'/2})$, as for the multiplicity space M_{pq}^m, and so $ch(M_{pq}^m) - ch(L(c_m, h_{pq}^m)) = \chi_{NS}(t).t^{-c_m/24}o(t^{h_{pq}^m + M})$. Now, we know that $L(c_m, h_{pq}^m)$ is a submodule of M_{pq}^m; if M_{pq}^m admits an other irreducible submodule, by FQS criterion, it is of the form $L(c_m, h_{rs}^m)$; but through the lemma: $h_{pq}^m + M > m^2/8$ and $h_{rs}^m \leq \frac{m(m-2)}{8}$, we obtain, by coherence on the characters, the contradiction: $\frac{m^2}{8} < M + h_{pq}^m < h_{rs}^m \leq \frac{m(m-2)}{8}$. Then, $M_{pq}^m = L(c(m), h_{p,q}^m)$ and $ch(L(c_m, h_{pq}^m)) = ch(M_{pq}^m)$, but the characters of the multiplicity spaces are ever known by GKO. The theorem 2.3 follows.

2.11 Local von Neumann algebras

For the loop algebra $L\mathfrak{g}$ and the Virasoro algebra $\mathfrak{Vir}$, we can work with the cooresponding groups: LG and Diff($\mathbb{S}^1$). For the Neveu-Schwarz algebra, there is no group corresponding to the supergenerators G_r, and so we need to work with unbounded operators. From the $\mathfrak{g}$-supersymmetric algebra $\widehat{\mathfrak{g}}$, we build a local Lie superalgebra $\widehat{\mathfrak{g}}(I)$, with I a proper interval of $\mathbb{S}^1$, by smearing with the smooth functions vanishing outside of I. In the same way, we define the local Neveu-Schwarz Lie superalgebra $\mathfrak{Vir}_{1/2}(I)$. Thanks to Sobolev estimates, these local algebras (containing unbounded operators) are represented continuously on the L_0-smooth completion of their positive energy representation. Now, we define the von Neumann algebras generated by these local algebras as the von Neumann algebra generated by the bounded functions of our self-adjoint operators; they are $\mathbb{Z}_2$-graded von Neumann algebras. Now, $\widehat{\mathfrak{g}}$ acts on a complex and real fermionic Fock space which decomposes into all its irreducible positive energy representations (with multiplicity spaces), and by coset construction we can do the same with $\mathfrak{Vir}_{1/2}$. Then, we see that the previous von Neumann algebras are included with conditional expectation in a big von Neumann algebra $\mathcal{M}(I)$ generated by smeared real and complex fermions, which is known (by [99] and a doubling construction) to be the hyperfinite III_1 factor; now, the modular action is ergodic, so by Takesaki devissage, $\mathcal{N}(I) = \pi(\mathfrak{Vir}_{1/2}(I))''$ is also the hyperfinite III_1 factor, and by the definition of type III, so is for every subrepresentations, so in particular for $\pi_i(\mathfrak{Vir}_{1/2}(I))''$, with π_i a generic irreducible positive energy representation. We deduce local equivalence, ie, the discrete series representations are unitary equivalent when they are restricted to $\mathfrak{Vir}_{1/2}(I)$; we deduce also Haag-Araki duality:

$$\pi_0(\mathfrak{Vir}_{1/2}(I^c))^\natural = \pi_0(\mathfrak{Vir}_{1/2}(I))''$$

with $X^\natural$ the supercommutant of X, from the known Haag-Araki duality of $\mathcal{M}(I)$, because the vacuum vector of H_0 is invariant by the modular operator $\triangle$ of $\mathcal{M}(I)$. Outside of the vacuum, we have a Jones-Wassermann subfactor:

$$\pi_i(\mathfrak{Vir}_{1/2}(I))'' \subset \pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural$$

as a failure of Haag-Araki duality.

2.12 Primary fields

Let p_0 be the projection on the vacuum representation H_0. The Jones relation $p_0\mathcal{M}(I)p_0 = \mathcal{N}(I)p_0$, implies that $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ is generated by products of compressed real and complex fermions: $p_0\psi_1(f_1)p_{i_1}\psi_2(f_2)p_{i_2}...\psi_n(f_n)p_0$, with p_i the projection on $H_i \subset H$ and f_s localized in I. The $p_i\psi(f)p_j$ are bounded operators intertwining the action of $\mathfrak{Vir}_{1/2}(I^c)$ between the representations H_i and H_j. We want to interpret these compressions as smeared primary fields. We define a primary field as a linear operator:

$$\phi_{ij}^k : H_j \otimes \mathcal{F}_{\lambda,\mu}^{\sigma} \to H_i$$

that superintertwines the action of $\mathfrak{Vir}_{1/2}$; with H_i, $H_j$on the discrete series of $\mathfrak{Vir}_{1/2}$ (k is called the charge of ϕ_{ij}^k), and $\mathcal{F}_{\lambda,\mu}^{\sigma}$ an ordinary representation of $\mathfrak{Vir}_{1/2}$ with base $(v_i)_{i\in\mathbb{Z}+\frac{\sigma}{2}}$, $(w_j)_{j\in\mathbb{Z}+\frac{1-\sigma}{2}}$, and:
(a) $L_n.v_i = -(i+\mu+\lambda n)v_{i+n}$
(b) $G_s.v_i = w_{i+s}$
(c) $L_n.w_j = -(j+\mu+(\lambda-\frac{1}{2})n)w_{j+n}$
(d) $G_s.w_j = -(j+\mu+(2\lambda-1)s)v_{j+s}$
with $\lambda = 1-h_k$, $\mu = h_j - h_i$, $\sigma = 0, 1$.

Let the space of densities $\{f(\theta)e^{i\mu\theta}(d\theta)^{\lambda} | f \in C^{\infty}(\mathbb{S}^1)\}$ where a finite covering of Diff($\mathbb{S}^1$) acts by reparametrisation $\theta \to \rho^{-1}(\theta)$ (if $\mu \in \mathbb{Q}$). Then its Lie algebra acts on too, so that it's a $\mathfrak{Vir}$-module vanishing the center. Finally, an equivalent construction with superdensities gives a model for $\mathcal{F}_{\lambda,\mu}^{\sigma}$ as $\mathfrak{Vir}_{1/2}$-module.

This primary field is equivalent to general vertex operators $\phi_{ij}^k(z)$ (called the ordinary part) and $\theta_{ij}^k(z) = [G_{-1/2}, \phi_{ij}^k(z)]$ (called the super part), and we prove that for i, j, k and σ fixed, such operators are completeley characterized by some compatibility conditions, so the space of primary fields associated is at most one dimensional. Note that $\sigma = 0$ gives ϕ_{ij}^k integer moded and $\sigma = 1$, half-integer moded. For charge $\alpha = (1/2, 1/2)$, we build these operators in the following way (an adaptation of an idea of Loke for $\mathfrak{Vir}$ [66], simplify by A. Wassermann): we start from the GKO coset construction $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes H_i^{\ell} = \bigoplus H_{ii'}^{\ell} \otimes H_{i'}^{\ell+2}$, we take the vertex primary field of $LSU(2)$ of level ℓ and spin 1/2: $I \otimes \phi_{ij}^{1/2,\ell}(z,v) : \mathcal{F}_{NS}^{\mathfrak{g}} \otimes H_j^{\ell} \to \mathcal{F}_{NS}^{\mathfrak{g}} \otimes H_i^{\ell}$, with $v \in V_{1/2}$ (the vector representation of $SU(2)$). Let $p_{i'}$ be the projection on the block $H_{ii'}^{\ell} \otimes H_{i'}^{\ell+2}$. By compatibility relations and unicity,

$p_{i'}(I\otimes\phi_{ij}^{1/2,\ell}(z,v))p_{j'} = C.z^r\phi^{\alpha}_{ii'jj'}(z)\otimes\phi_{i'j'}^{\frac{1}{2},\ell+2}(z,v)$, with C a constant possibly zero and $r\in\mathbb{Q}$. Now, $I\otimes\phi_{ij}^{1/2,\ell}(z,v)=\sum_{i'j'}p_{i'}(I\otimes\phi_{ij}^{1/2\ell}(z,v))p_{j'}$, so at least one is non-zero. More precisely, we prove by an irreducibility argument that $\forall j'$, $\exists i'$ with a non-zero term, and so $\phi^{\alpha}_{ii'jj'}(z)$ non-zero. Note that the simple locality relations between non-compressed smeared fermions concentrated on disjoint intervals (ie $\psi(f)\psi(g)=-\psi(g)\psi(f)$), admit a bit more complicated equivalent after compression: the braiding relations.

Now using the same idea as Tsuchiya-Nakanishi [92], we deduce the braiding relations for $\mathfrak{Vir}_{1/2}$: its braiding matrix is the braiding matrix for $LSU(2)$ at level ℓ, times the transposed of the inverse of the braiding matrix for $LSU(2)$ at level $\ell+2$ (it's proved by the contribution of the inverse of a gauge transformation of the Knizhnik-Zamolodchikov equation for the braiding of $LSU(2)$). Then, we obtain non-zero coefficients:

$$\phi^{\alpha\ell}_{ii'jj'}(z)\phi^{\alpha\ell}_{jj'kk'}(w)=\sum\mu_{rr'}\phi^{\alpha\ell}_{ii'rr'}(w)\phi^{\alpha\ell}_{rr'kk'}(z) \text{ with } \mu_{rr'}\neq 0.$$

Now if $\phi^{\alpha}_{ii'jj'}=0$ and $\phi_{ij}^{1/2,\ell}$ and $\phi_{i'j'}^{\frac{1}{2},\ell+2}$ non-zero, then, the braiding relation of $\phi^{\alpha}_{ii'jj'}$ with its adjoint is zero, but produced some non-zero terms $\phi^{\alpha}_{ii'kk'}$ by the previous irreducibility argument, contradiction. Then, we see that $\phi^{\alpha}_{ii'jj'}$ is non-zero iff $\phi_{ij}^{1/2\ell}$ and $\phi_{i'j'}^{1/2,\ell+2}$ are non-zero, ie, $i'=i\pm 1/2$ and $j'=j\pm 1/2$ (up to some boundary restrictions). Now, for charge $\beta=(0,1)$ and the braiding with α, we do the same, from the Neveu-Schwarz fermion field $\psi(u,z)\otimes I$ commuting with $I\otimes\phi_{ij}^{\frac{1}{2},\ell}(v,w)$.

Next, by a convolution argument, the braiding runs also with two smeared primary fields concentrate on disjoint intervals. We deduce also that the von Neumann algebras $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ are generated by chains of primary fields. This new characterization is essential to prove the so-called von Neumann density: if I is a proper interval of $\mathbb{S}^1$ and I_1, I_2 are the intervals obtained by removing a point of I then, $\pi_i(\mathfrak{Vir}_{1/2}^{I_1})''\vee\pi_i(\mathfrak{Vir}_{1/2}^{I_2})''=\pi_i(\mathfrak{Vir}_{1/2}(I))''$. By local equivalence, we only need to prove it on the vacuum, on which the local algebra on I as generated by chains concentred on I. By linearity, the L^2-context, and a kind of OPE, we can separate into products of chains on I_1 and I_2. Next the von Neumann density implies the irreducibility of the Jones-Wassermann subfactor: $\pi_i(\mathfrak{Vir}_{1/2}(I))^{\natural}\cap\pi_i(\mathfrak{Vir}_{1/2}(I^c))^{\natural}=\mathbb{C}$, which significate that the representations H_i are irreducibles $\mathfrak{Vir}_{1/2}(I)\oplus\mathfrak{Vir}_{1/2}(I^c)$-modules.

2.13 Connes fusion and subfactors

Then, the discrete series representations are irreducibles bimodules over the local von Neumann algebra $\mathcal{M} = \pi_0(\mathfrak{Vir}_{1/2}(I))''$. We define a relative tensor product called Connes fusion $\boxtimes$ using a 4-points functions:
Consider the $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodule $Hom_{-\mathcal{M}}(H_0, H_i) \otimes Hom_{\mathcal{M}-}(H_0, H_j)$, we define a pre-inner product on by:

$$(x_1 \otimes y_1, x_2 \otimes y_2) = (-1)^{(\partial x_1 + \partial x_2)\partial y_2}(x_2^\star x_1 y_2^\star y_1 \Omega, \Omega)$$

The L^2-completion is also a $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodule, called the Connes fusion between H_i and H_j and noted $H_i \boxtimes H_j$. The fusion is associative.

We obtain a fusion ring for $\oplus$ and $\boxtimes$. The key tool to compute this fusion is the transport formula which gives explicitly how the chains on the vacuum representation, transform into chains on any representations through the intertwining relations. Thanks to the braiding relations known at charge α, we are able to prove the transport formula:

$$\pi_j(\bar{a}_{0\alpha}.a_{\alpha 0}) = \sum \lambda_k \bar{a}_{jk}.a_{kj} \quad \text{with } \lambda_k > 0.$$

with a_{kj} a charge α (ordinary part, so even) smeared primary field of $\mathfrak{Vir}_{1/2}$ between H_j and H_k concentrated on I, $\bar{a}_{jk} = a_{kj}^\star$, and $\pi_j : H_0 \to H_j$ the local equivalence. Now, $a_{\alpha 0} \in Hom_{-\mathcal{M}}(H_0, H_\alpha)$, so:

$$\|a_{\alpha 0} \otimes y\|^2 = (a_{\alpha 0}^\star a_{\alpha 0} y^\star y \Omega, \Omega) = (y^\star \pi_j(a_{\alpha 0}^\star a_{\alpha 0}) y \Omega, \Omega) = \sum \lambda_k \|a_{kj} y \Omega\|^2.$$

Then using the fact that $a_{\alpha 0}\mathcal{M}$ is dense in $Hom_{-\mathcal{M}}(H_0, H_\alpha)$ (by von Neumann density), a polarization and the irreduibility of the bimodules, we obtain a unitary map between $H_\alpha \boxtimes H_j$ and $\bigoplus_{k \in \langle \alpha, j \rangle} H_k$, with $k \in \langle \alpha, j \rangle$ iff ϕ_{jk}^α is a non-zero primary field. We obtain the fusion rule with α:

$$H_\alpha \boxtimes H_j = \bigoplus_{k \in \langle \alpha, j \rangle} H_k.$$

Now, idem, with the braiding relations between charge α and β primary fields, we obtain a partial transport formula and partial fusion rules with β:

$$H_\beta \boxtimes H_j \leq \bigoplus_{k \in \langle \beta, j \rangle} H_k.$$

But, the fusion rules with α permit to compute a character of the fusion ring called the quantum dimension (by Perron-Frobenius theorem). An easy way to comptute the quantum dimensions is to see that the fusion ring for the Neveu-Schwarz algebra at charge c_m is the tensor product of the fusion rings for the loop algebra at level ℓ and $\ell+2$ (with $m = \ell+2$), modulo a period two automorphism. Then the quantum dimensions for the Neveu-Schwarz algebra is a product of the two (coset corresponding) quantum dimensions for the loop algebra:

$$d(H_{ij}^{\ell}) = d(H_i^{\ell}).d(H_j^{\ell+2}) = \frac{sin((2i+1)\pi/(\ell+2))}{sin(\pi/(\ell+2))}.\frac{sin((2j+1)\pi/(\ell+4))}{sin(\pi/(\ell+4))}$$

Then the quantum dimensions show that these partial rules with β are the exact ones. Next, we see that the rules for α and β permit to compute all fusion rules. Finally, the Jones-Wassermann III_1-subfactors are isomorphic to II_1-subfactors tensor the hyperfinite III_1-factor, by H. Wenzl [103] and S. Popa [77]. These last subfactors are irreducibles, finite depth and finite index given by the square of the quantum dimensions.

Part I
Unitary series and characters for the Neveu-Schwarz algebra

3 The Neveu-Schwarz algebra

3.1 Witt superalgebras and representations

Definition 3.1. *A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{d} = \mathfrak{d}_{\bar{0}} \oplus \mathfrak{d}_{\bar{1}}$, together with a graded Lie bracket $[.,.] : \mathfrak{d} \times \mathfrak{d} \to \mathfrak{d}$, such that $[.,.]$ is a bilinear map with $[\mathfrak{d}_i, \mathfrak{d}_j] \subseteq \mathfrak{d}_{i+j}$, and for homogeneous elements $X \in \mathfrak{d}_x$, $Y \in \mathfrak{d}_y$, $Z \in \mathfrak{d}_z$:*

- $[X, Y] = -(-1)^{xy}[Y, X]$
- $(-1)^{xz}[X, [Y, Z]] + (-1)^{xy}[Y, [Z, X]] + (-1)^{yz}[Z, [X, Y]] = 0$

Definition 3.2. *The Witt algebra $\mathfrak{W}$ is the Lie $\star$-algebra of vector fields on the circle, generated by $d_n = ie^{i\theta n}\frac{d}{d\theta}$ $(n \in \mathbb{Z})$.*

Remark 3.3. *$\mathfrak{W}$ admits two supersymmetrics extensions, $\mathfrak{W}_0$ the Ramond sector (R) and $\mathfrak{W}_{1/2}$ the Neveu-Schwarz sector (NS) ((see [55], [42] chap 9).*

Here, we trait only the (NS) sector.

Definition 3.4. *Let $\mathfrak{d} = \mathfrak{W}_{1/2}$ the Witt superalgebra with:*

$$\begin{cases} [d_m, d_n] = (m-n)d_{m+n} \\ [\gamma_m, d_n] = (m - \frac{n}{2})\gamma_{m+n} \\ [\gamma_m, \gamma_n]_+ = 2d_{m+n} \end{cases}$$

together with the $\star$-structure, $d_n^\star = d_{-n}$ and $\gamma_m^\star = \gamma_{-m}$, and the super-structure: $\mathfrak{d}_{\bar{0}} = \mathfrak{W} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}d_n$, $\mathfrak{d}_{\bar{1}} = \bigoplus_{m\in\mathbb{Z}+1/2} \mathbb{C}\gamma_m$

Now we investigate representations π of $\mathfrak{W}_{1/2}$, which are :

Definition 3.5. *Let H be a prehilbert space.*

(a) *Unitary: $\pi(A)^\star = \pi(A^\star)$*

(b) *Projective: $A \mapsto \pi(A)$ is linear and $[\pi(A), \pi(B)] - \pi([A, B]) \in \mathbb{C}$.*

(c) *Positive energy : H admits an orthogonal decomposition $H = \bigoplus_{n\in\frac{1}{2}\mathbb{N}} H_n$ such that, $\exists D$ acting on H_n as multiplication by n, $H_0 \neq \{0\}$ and $dim(H_n) < +\infty$. Here, $\exists h \in \mathbb{C}$ such that $D = \pi(d_0) - hI$.*

3.2 Investigation

Definition 3.6. *Let $b : \mathfrak{W}_{1/2} \times \mathfrak{W}_{1/2} \to \mathbb{C}$ be the bilinear map defined by*

$$[\pi(A), \pi(B)] - \pi([A, B]) = b(A, B)I \quad \textit{(b is a 2-cocycle)}$$

Definition 3.7. *Let $f : \mathfrak{W}_{1/2} \to \mathbb{C}$ be a $\star$-linear form.*
$\partial f = (A, B) \mapsto f([A, B])$ is a 2-coboundary.

Remark 3.8. *$A \mapsto \pi(A) + f(A)I$ define also a projective, unitary, positive energy representation, where $b(A, B)$ becomes $b(A, B) - f([A, B])$.*

Proposition 3.9. *(SuperVirasoro extension) $\mathfrak{W}_{1/2}$ has a unique central extension, up to equivalent, i.e. $H_2(\mathfrak{W}_{1/2}, \mathbb{C})$ is 1-dimensional. This extension admits the basis $(L_n)_{n\in\mathbb{Z}}$, $(G_m)_{m\in\mathbb{Z}+\frac{1}{2}}$, C central, with $L_n^\star = L_{-n}$, $G_m^\star = G_{-m}$, $C = cI$, $c \in \mathbb{C}$ called the **central charge**; and relations:*

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] = (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ = 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

Proof.
Let $L_n = \pi(d_n)$ and $G_m = \pi(\gamma_m)$ then:
$[L_m, L_n] = (m-n)L_{m+n} + b(d_m, d_n)I$
$[G_m, L_n] = (m - \frac{n}{2})G_{m+n} + b(\gamma_m, d_n)I$
$[G_m, G_n]_+ = 2L_{m+n} + b(\gamma_m, \gamma_n)I$
In particular:
$[L_0, L_n] = -nL_n + b(d_0, d_n)I$
$[L_0, G_n] = -nG_n + b(d_0, \gamma_n)I$
$[L_1, L_{-1}] = 2L_0 + b(d_1, d_{-1})I$
We choose :
$f(d_n) = -n^{-1}b(d_0, d_n)$
$f(\gamma_m) = -m^{-1}b(d_0, \gamma_m)$
$f(d_0) = \frac{1}{2}b(d_1, d_{-1})$
Then, after adjustment by f:
$[L_0, L_n] = -nL_n$
$[L_0, G_n] = -nG_n$
$[L_1, L_{-1}] = 2L_0$

Now $D = L_0 - hI$ and if $v \in H_k$, $Dv = kv$, then:
$DL_n v = L_n Dv + [D, L_n]v = kL_n v + [L_0, L_n]v = (k-n)L_n v$
So, $L_n : H_k \to H_{k-n}$ ($= \{0\}$ if $n > k$).
Similary, $G_m : H_k \to H_{k-m}$, then:

$$\begin{cases} [L_m, L_n] - (m-n)L_{m+n} : H_{m+n+k} \to H_k \\ [G_m, L_n] - (m - \frac{n}{2})G_{m+n} : H_{m+n+k} \to H_k \\ [G_m, G_n]_+ - 2L_{m+n} : H_{m+n+k} \to H_k \end{cases}$$

But $b(d_m, d_n)I$, $b(\gamma_m, d_n)I$, $b(\gamma_m, \gamma_n)I : H_{m+n+k} \to H_{m+n+k}$, so:

$$\begin{cases} b(d_m, d_n) = A(m)\delta_{m+n} \\ b(\gamma_m, d_n) = B(m)\delta_{m+n} = 0 \text{ because } 0 \notin \mathbb{Z} + 1/2 \ni m+n \\ b(\gamma_m, \gamma_n) = C(m)\delta_{m+n} \end{cases}$$

Now, on $\mathfrak{W} = \mathfrak{d}_{\bar{0}}$, $b(A, B) = -b(B, A)$, so, $A(m) = -A(-m)$ and $A(0) = 0$, and Jacobi identity implies $b([A, B], C) + b([B, C], A) + b([C, A], B) = 0$, then, for d_k, d_n, d_m with $k + n + m = 0$:

$$(n-m)A(k) + (m-k)A(n) + (k-n)A(m) = 0$$

Now, with $k = 1$ and $m = -n-1$, $(n-1)A(n+1) = (n+2)A(n) - (2n+1)A(1)$. Then $A(n)$ is completely determined by the knowledge of $A(1)$ and $A(2)$, and so, the solutions are a 2-dimensional space.
Now, n and n^3 are solutions, so $A(n) = a.n + b.n^3$.
Finally, because $[L_1, L_{-1}] = 2L_0$, $A(1) = 0$ and $a + b = 0$, we obtain:

$$A(n) = b(n^3 - n) = \frac{c}{12}(n^3 - n), \quad c \in \mathbb{C} \text{ the central charge.}$$

Process 3.9.
$[[A, B]_+, C] = [A, [B, C]]_+ + [B, [A, C]]_+$ then:
$[[G_r, G_s]_+, L_n] = [G_r, [G_s, L_n]]_+ + [G_s, [G_r, L_n]]_+$
$= [2L_{r+s}, L_n] = [G_r, (s - \frac{1}{2}n)G_{n+s}]_+ + [G_s, (r - \frac{1}{2}n)G_{n+r}]_+$
$= 2(r + s - n)L_{r+s+n} - \delta_{r+s+n}\frac{c}{6}(n^3 - n)$
$= (s - \frac{1}{2}n)(2L_{r+s+n} + C(r)\delta_{r+s+n}) - (r - \frac{1}{2}n)(2L_{r+s+n} + C(s)\delta_{r+s+n})$
Then taking $r + s + n = 0$, $\frac{c}{6}(n^3 - n) + (s - \frac{1}{2}n)C(r) + (r - \frac{1}{2}n)C(s) = 0$.
Finally, with $n = 2s$ and $r = -3s$, $C(s) = \frac{c}{3}(s^2 - \frac{1}{4})$. □

Definition 3.10. *The central extension of* $\mathfrak{W}_{1/2}$ *is called* $\mathfrak{Vir}_{1/2}$*, the Super-Virasoro algebra (on sector NS), also called Neveu-Schwarz algebra.*

Theorem 3.11. *(Complete reducibility)*

(a) *If* H *is a unitary, projective, positive energy representation of* $\mathfrak{W}_{1/2}$*, then any non-zero vector* v *in the lowest energy subspace* H_0 *generates an irreducible submodule.*

(b) H *is an orthogonal direct sum of irreducibles such representations.*

Proof. (a) Let K be the minimal $\mathfrak{W}_{1/2}$-submodule containing v.
Clearly, since $L_n v = G_m v = 0$ for $m, n > 0$ and $L_0 v = hv$, we see that K is spanned by all products $R.v$ with :

$$R = G_{-j_\beta} \dots G_{-j_1} L_{-i_\alpha} \dots L_{-i_1}, \quad 0 < i_1 \leq \dots \leq i_\alpha, \quad \frac{1}{2} \leq j_1 < \dots < j_\beta$$

But then, $K_0 = \mathbb{C}v$. Let K' be a submodule of K, and let p be the orthogonal projection onto K'. By unitarity, p commutes with the action of $\mathfrak{W}_{1/2}$, and hence with D. Thus p leaves $K_0 = \mathbb{C}v$ invariant, so $pv = 0$ or v.
But $pRv = Rpv$, hence $K' = 0$ or K and K is irreducible.

(b) Take the irreducible module M_1 generated by a vector of lowest energy. Now (changing h into $h' = h+m$ if necessary),we repeat this process for $M_1^\perp$, to get $M_2, M_3, \dots$ The positive energy assumption shows that $H = \oplus M_i$ □

Theorem 3.12. *(Uniqueness) If* H *and* H' *are irreducibles with* $c = c'$ *and* $h = h'$*, then they are unitarily equivalents as* $\mathfrak{W}_{1/2}$*-modules.*

Proof. $H_0 = \mathbb{C}u$ and $H'_0 = \mathbb{C}u'$ with u, u' unitary.
Let $U : H \to H', Au \mapsto Au'$, we want to prove that $U^\star U = UU^\star = Id$.
Let $Au \in H_n$, $Bu \in H_m$:
If $n \neq m$, for example, $n < m$, then $B^\star Au \in H_{n-m} = 0$ and
$(Au, Bu) = (B^\star Au, u) = 0 = (Au', Bu')$.
If $n = m$, then $D = B^\star A$ is a constant energy operator, so in $\mathbb{C}L_0 \oplus \mathbb{C}C$.
Now, $(L_0 u, u) = h = (L_0 u', u')$ iff $h = h'$ and $(Cu, u) = c = (Cu', u')$ iff $c = c'$.
Finally, $(v, w) = (Uv, Uw)\ \forall v, w \in H$ and $(v', w') = (U^\star v', U^\star w')\ \forall v', w' \in H'$ iff $h = h'$ and $c = c'$.
So, $U^\star U = UU^\star = Id$, ie, H and H' are unitarily equivalents. □

Definition 3.13. $\mathfrak{Vir}_{1/2} = \mathfrak{Vir}_{1/2}^{-} \oplus \mathfrak{Vir}_{1/2}^{0} \oplus \mathfrak{Vir}_{1/2}^{+}$ with $\mathfrak{Vir}_{1/2}^{0} = \mathbb{C}\mathrm{L}_0 \oplus \mathbb{C}\mathrm{C}$

$$\mathfrak{Vir}_{1/2}^{+} = \bigoplus_{m,n>0} \mathbb{C}L_m \oplus \mathbb{C}G_n \qquad \mathfrak{Vir}_{1/2}^{-} = \bigoplus_{m,n<0} \mathbb{C}L_m \oplus \mathbb{C}G_n$$

Remark 3.14. *This decomposition pass to the universal envelopping :*

$$\mathcal{U}(\mathfrak{Vir}_{1/2}) = \mathcal{U}(\mathfrak{Vir}_{1/2}^{-}) \cdot \mathcal{U}(\mathfrak{Vir}_{1/2}^{0}) \cdot \mathcal{U}(\mathfrak{Vir}_{1/2}^{+})$$

Remark 3.15. *We see that an irreducible, unitary, projective, positive energy representation of $\mathfrak{W}_{1/2}$ is exactly given by a unitary highest weight representation of $\mathfrak{Vir}_{1/2}$ (see the following section).*

3.3 Unitary highest weight representations

Definition 3.16. *Let the Verma module $H = V(c,h)$ be the $\mathfrak{Vir}_{1/2}$-module freely generated by followings conditions:*

(a) $\Omega \in H$*, called the cyclic vector* ($\Omega \neq 0$)*.*

(b) $L_0\Omega = h\Omega$, $C\Omega = c\Omega$ $\ (h, c \in \mathbb{R})$

(c) $\mathfrak{Vir}_{1/2}^{+}\Omega = \{0\}$

Lemma 3.17. $\mathcal{U}(\mathfrak{Vir}_{1/2}^{-})\Omega = H$ *and a set of generators is given by:*
$G{-}j_\beta \dots G_{-j_1} L_{-i_\alpha} \dots L_{-i_1}\Omega, \quad 0 < i_1 \leq \dots \leq i_\alpha, \quad \frac{1}{2} \leq j_1 < \dots < j_\beta$

Proof. It's clear. □

Lemma 3.18. $V(c,h)$ *admits a canonical sesquilinear form* $(.,.)$*, completely defined by:*

(a) $(\Omega, \Omega) = 1$

(b) $\pi(A)^\star = \pi(A^\star)$

(c) $(u,v) = \overline{(v,u)}$ $\ \forall u, v \in H$ *(in particular* $(u,u) = \overline{(u,u)} \in \mathbb{R}$*).*

Proof. It's clear. □

Definition 3.19. $u \in V(c,h)$ *is a ghost if* $(u,u) < 0$*.*

Lemma 3.20. *If $V(c,h)$ admits no ghost then $c, h \geq 0$*

Proof. Since $L_n L_{-n}\Omega = L_{-n}L_n\Omega + 2nh\Omega + c\frac{n(n^2-1)}{12}\Omega$,
we have $(L_{-n}\Omega, L_{-n}\Omega) = 2nh + \frac{n(n^2-1)}{12}c \geq 0$.
Now, taking n first equal to 1 and then very large, we obtain the lemma. □

Definition 3.21. *Let $K(c,h) = ker(.,.) = \{x \in V(c,h); (x,y) = 0\ \forall y\}$ the maximal proper submodule of $V(c,h)$, and $L(c,h) = V(c,h)/K(c,h)$, irreducible highest weight representation of $\mathfrak{Vir}_{1/2}$, with $(.,.)$ well-defined on.*

Definition 3.22. *$u \in V(c,h)$ is a null vector if $(u,u) = 0$.*

Lemma 3.23. *On no ghost case, the set of null vectors is $K(c,h)$.*

Proof. Let x be a null vector, and $y \in V(c,h)$.
By assumption $\forall \alpha, \beta \in \mathbb{C}$, $(\alpha x + \beta y, \alpha x + \beta y) \geq 0$. We develop it, with $\alpha = (y,y)$ and $\beta = -(x,y)$, we obtain : $|(x,y)|^2(y,y) \leq (x,x)(y,y)^2 = 0$.
So if y is not a null vector then $(x,y) = 0$. Else, $(x,x) = (y,y) = 0$, so taking $\alpha = 1$ and $\beta = -(x,y)$, we obtain $2|(x,y)|^2 \leq 0$ and so $(x,y) = 0$ □

Corollary 3.24. *$L(c,h)$ is a unitary highest weight representation.*

Proof. Without ghost, $(.,.)$ is a scalar product on $L(c,h)$. □

Remark 3.25. *Theorem 2.2 will be proved classifying no ghost cases.*

4 Vertex operators superalgebras

We give a progressive introduction to vertex operators superalgebras structure.
We start with the fermion algebra as example.
We work on to obtain, at the end of the section, natural vertex axioms.

4.1 Investigation on fermion algebra

Definition 4.1. *Let the fermion algebra (of sector NS), generated by* $(\psi_n)_{n\in\mathbb{Z}+\frac{1}{2}}$*, and* I *central, with the relations:*

$$[\psi_m, \psi_n]_+ = \delta_{m+n} I \quad \textit{and} \quad \psi_n^\star = \psi_{-n}$$

Definition 4.2. *(Verma module) Let* $H = \mathcal{F}_{NS}$ *freely generated by:*

(a) $\Omega \in H$ *is called the vacuum vector ,* $\Omega \neq 0$.

(b) $\psi_m \Omega = 0 \ \forall m > 0$

(c) $I\Omega = \Omega$

Lemma 4.3. *A set of generators of* H *is given by:*
$\psi_{-m_1} \dots \psi_{-m_r}\Omega \quad m_1 < \dots < m_r \quad r \in \mathbb{N}, \ m_i \in \mathbb{N} + \frac{1}{2}$

Proof. It's clear. □

Lemma 4.4. H *admits the sesquilinear form* $(.,.)$ *completely defined by :*

(a) $(\Omega, \Omega) = 1$

(b) $(u, v) = \overline{(v, u)} \quad \forall u, v \in H$

(c) $(\psi_n u, v) = (u, \psi_{-n} v) \quad \forall u, v \in H \quad$ *ie* $\pi(\psi_n)^\star = \pi(\psi_n^\star)$

$(.,.)$ *is a scalar product and* H *is a prehilbert space.*

Proof. It's clear. □

Remark 4.5. H *is an irreducible representation of the fermion algebra. It is its unique unitary highest weight representation.*

Remark 4.6. $\psi_n^2 = \frac{1}{2}[\psi_n, \psi_n]_+ = 0$ *if* $n \neq 0$

Definition 4.7. *(Operator D) Let* $D \in End(H)$ *inductively defined by :*

(a) $D\Omega = 0$

(b) $D\psi_{-m}a = \psi_{-m}Da + m\psi_{-m}a \quad \forall m \in \mathbb{N} + \frac{1}{2}$ *and* $\forall a \in H$

Lemma 4.8. *D decomposes H into* $\bigoplus_{n \in \mathbb{N}+\frac{1}{2}} H_n$ *with* $D\xi = n\xi$
$\forall \xi \in H_n$*,* $dim(H_n) < \infty$ *and* $H_n \perp H_m$ *if* $n \neq m$

Proof. Let $a = \psi_{-m_1} \dots \psi_{-m_r}\Omega$ be a generic element of the base of H, then $D.a = (\sum m_i)a$. □

Remark 4.9. $[D, \psi_m] = -m\psi_m$ *and* $\Omega \in H_0$*, so* $\psi_m : H_{m+n} \to H_n$.

Definition 4.10. *(Operator T) Let* $T \in End(H)$ *inductively defined by :*

(a) $T\Omega = 0$

(b) $T\psi_{-m}a = \psi_{-m}Ta + (m - \frac{1}{2})\psi_{-m-1}a \quad \forall m \in \mathbb{N} + \frac{1}{2}$ *and* $\forall a \in H$

Remark 4.11. $[T, \psi_m] = -(m - \frac{1}{2})\psi_{m-1}$.

Definition 4.12. *Let* $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}}.z^{-n-1}$ *the fermion operator.*

Remark 4.13. $\psi \in (EndH)[[z, z^{-1}]]$ *is a formal power series.*

Lemma 4.14. *(Relations with* ψ_n*, D and T)*

(a) $[\psi_{m+\frac{1}{2}}, \psi]_+ = z^m$

(b) $[D, \psi] = z.\psi' + \frac{1}{2}\psi$

(c) $[T, \psi] = \psi'$

Proof. $[\psi_{m+\frac{1}{2}}, \psi(z)]_+ = \sum[\psi_{m+\frac{1}{2}}, \psi_{n+\frac{1}{2}}]_+.z^{-n-1} = z^m$
$[D, \psi(z)] = \sum(-n - \frac{1}{2})\psi_{n+\frac{1}{2}}.z^{-n-1} = z.\psi'(z) + \frac{1}{2}\psi(z)$
$[T, \psi(z)] = \sum(-n)\psi_{n-\frac{1}{2}}.z^{-n-1} = \sum(-n-1)\psi_{n+\frac{1}{2}}.z^{-n-2} = \psi'(z)$ □

Remark 4.15. $(.,.)$ *induces* $(\psi(z_1)...\psi(z_n)c, d) \in \mathbb{C}[[z_1^{\pm 1}, ..., z_n^{\pm 1}]]$*,* $\forall c, d \in H$.

Lemma 4.16. $(\psi(z)\Omega, \Omega) = 0 \quad$ *and* $\quad (\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w} \quad$ *if* $|z| > |w|$.

Proof. $(\psi(z)\Omega,\Omega) = \sum_{n\in\mathbb{Z}}(\psi_{n+\frac{1}{2}}\Omega,\Omega).z^{-n-1} = 0$
$(\psi(z)\psi(w)\Omega,\Omega) = \sum_{m,n\in\mathbb{Z}}(\psi_{m+\frac{1}{2}}\Omega,\psi_{-n-\frac{1}{2}}\Omega).z^{-n-1}w^{-m-1}$
$= \sum_{m,n\in\mathbb{Z}}(\psi_{m-\frac{1}{2}}\Omega,\psi_{-n-\frac{1}{2}}\Omega).z^{-n-1}w^{-m} = \sum_{n\in\mathbb{N}}(\psi_{-n-\frac{1}{2}}\Omega,\psi_{-n-\frac{1}{2}}\Omega).z^{-n-1}w^{n}$
$= z^{-1}\sum_{n\in\mathbb{N}}(\frac{w}{z})^{n} = \frac{1}{z-w}$ if $|z|>|w|$ □

Lemma 4.17. *$\forall c,d\in H$, $(\psi(z)c,d)\in\mathbb{C}[z,z^{-1}]$.*

Proof. $(\psi(z)\psi_{-n-\frac{1}{2}}c,d) = (c,d).z^{-n-1} - (\psi(z)c,\psi_{n+\frac{1}{2}}d)$
$(\psi(z)c,\psi_{-n-\frac{1}{2}}d) = (c,d).z^{n} - (\psi(z)\psi_{n+\frac{1}{2}}c,d)$
Then, the result follows by lemma 4.16 and induction. □

Proposition 4.18. *$\forall c,d\in H$, $\exists X(c,d)\in(z-w)^{-1}\mathbb{C}[z^{\pm1},w^{\pm1}]$ such that:*

$$X(c,d)(z,w) = \begin{cases} (\psi(z)\psi(w)c,d) & \text{if } |z|>|w| \\ -(\psi(w)\psi(z)c,d) & \text{if } |w|>|z| \end{cases}$$

Proof. $(\psi(z)\psi(w)\psi_{-n-\frac{1}{2}}c,d) = (\psi(z)c,d)w^{-n-1} - (\psi(w)c,d)z^{-n-1} + (\psi(z)\psi(w)c,\psi_{n+\frac{1}{2}}d)$
$(\psi(z)\psi(w)c,\psi_{-n-\frac{1}{2}}d) = (\psi(w)c,d)z^{n} - (\psi(z)c,d)w^{n} + (\psi(z)\psi(w)\psi_{n+\frac{1}{2}}c,d)$
Then, the result follows by lemma 4.16, 4.17, symmetry and induction. □

4.2 General framework

Definition 4.19. *Let H prehilbert and $A\in(EndH)[[z,z^{-1}]]$ a formal power series defined as $A(z)=\sum_{n\in\mathbb{Z}}A(n)z^{-n-1}$ with $A(n)\in End(H)$.*

Definition 4.20. *Let $A,B\in(EndH)[[z,z^{-1}]]$*
A and B are ***local*** *if $\exists\varepsilon\in\mathbb{Z}_2$, $\exists N\in\mathbb{N}$ such that $\forall c,d\in H$:*
$\exists X(A,B,c,d)\in(z-w)^{-N}\mathbb{C}[z^{\pm1},w^{\pm1}]$ such that:

$$X(A,B,c,d)(z,w) = \begin{cases} (A(z)B(w)c,d) & \text{if } |z|>|w| \\ (-1)^{\varepsilon}(B(w)A(z)c,d) & \text{if } |w|>|z| \end{cases}$$

Example 4.21. *ψ is local with itself, with $N=1$ and $\varepsilon=\bar{1}$*

Notation 4.22. $[X,Y]_{\varepsilon} = \begin{cases} XY-YX & \text{if } \varepsilon=\bar{0} \\ XY+YX & \text{if } \varepsilon=\bar{1} \end{cases}$

Remark 4.23. *Let $n\in\mathbb{N}$, then, $(z-w)^{n}=\sum_{p=0}^{n}C_{n}^{p}(-1)^{p}w^{p}z^{n-p}$ and,*
$(z-w)^{-n} = \begin{cases} \sum_{p\in\mathbb{N}}C_{p+n-1}^{p}w^{p}z^{-p-n} & \text{if } |z|>|w| \\ (-1)^{n}\sum_{p\in\mathbb{N}}C_{p+n-1}^{p}z^{p}w^{-p-n} & \text{if } |w|>|z| \end{cases}$

Proposition 4.24. *Let A, B local and $c, d \in H$ then:*
$X(A,B,c,d)(z,w) = \sum_{n\in\mathbb{Z}} X_n(A,B,c,d)(w)(z-w)^{-n-1}$,
$X_n(A,B,c,d)(w) = (A_nB(w)c,d)$,
$A_nB(w) = \sum_{m\in\mathbb{Z}}(A_nB)(m)w^{-m-1}$ *and* $(A_nB)(m) =$

$$\begin{cases} \sum_{p=0}^{n}(-1)^p C_n^p [A(n-p), B(m+p)]_\varepsilon & \text{if } n \geq 0 \\ \sum_{p\in\mathbb{N}} C_{p-n-1}^p (A(n-p)B(m+p) - (-1)^{\varepsilon+n}B(m+n-p)A(p)) & \text{if } n < 0 \end{cases}$$

Proof. $X(A,B,c,d) \in \mathbb{C}[z^{\pm1}, w^{\pm1}, (z-w)^{-1}]$, we develop it around $z = w$:
$X(A,B,c,d)(z,w) = \sum_{n\in\mathbb{Z}} X_n(A,B,c,d)(w)(z-w)^{-n-1}$
with $X_n(A,B,c,d)(w) = \frac{1}{2\pi i}\oint_w (z-w)^n X(A,B,c,d)(z,w)dz$.
By contour integration argument ($\oint_w = \int_{|z|=R>|w|} - \int_{|z|=r<|w|}$), we obtain:

$X_n(A,B,c,d)(w) = \frac{1}{2\pi i}(\int_{|z|=R>|w|} - \int_{|z|=r<|w|})(z-w)^n X(A,B,c,d)(z,w)dz$
$= \frac{1}{2\pi i}\int_{|z|=R>|w|}(z-w)^n(A(z)B(w)c,d)dz - \frac{(-1)^\varepsilon}{2\pi i}\int_{|z|=r<|w|}(z-w)^n(B(w)A(z)c,d)dz$
$= \frac{1}{2\pi i}\sum_{q\in\mathbb{Z},p=0}^{n}(\int_{|z|=R>|w|} C_n^p(-1)^p z^{n-p}w^p(A(q)B(w)c,d)z^{-q-1}dz$
$-(-1)^\varepsilon \int_{|z|=r<|w|} C_n^p(-1)^p z^{n-p}w^p(B(w)A(q)c,d)z^{-q-1}dz)$
$= (\sum_{p=0}^{n}(-1)^p w^p C_n^p[A(n-p), B(w)]_\varepsilon c, d)$, with $n \in \mathbb{N}$.

$X_{-n}(A,B,c,d)(w) = \frac{1}{2\pi i}\sum_{q\in\mathbb{Z},p\in\mathbb{N}}(\int_{|z|=R>|w|} C_{p+n-1}^p z^{-n-p}w^p(A(q)B(w)c,d)z^{-q-1}dz$
$-(-1)^\varepsilon\int_{|z|=r<|w|} C_{p+n-1}^p(-1)^n w^{-n-p}z^p(B(w)A(q)c,d)z^{-q-1}dz)$
$= (\sum_{p\in\mathbb{N}} C_{p+n-1}^p(w^pA(-n-p)B(w) - (-1)^{\varepsilon+n}w^{-n-p}B(w)A(p))c,d)$ □

Definition 4.25. *Let the operation $(A,B) \to A_nB$ as for proposition 4.24.*

Formula 4.26. *The formula of $(A_nB)(m)$ on proposition 4.24.*

Corollary 4.27. *(Operator product expansion) Let A, B local, and $c, d \in H$:*

$$(A(z)B(w)c,d) \sim (\sum_{n=0}^{N-1}\frac{(A_nB)(w)}{(z-w)^{n+1}}c,d) \quad \text{near } z = w$$

Proof. $X(A,B,c,d)(z,w) = \sum_{n\in\mathbb{Z}}(A_nB)(w)c,d)(z-w)^{-n-1}$
$\in (z-w)^{-N}\mathbb{C}[z^{\pm1}, w^{\pm1}]$, so $A_nB = 0$ for $-n-1 < -N$ ie $n \geq N$. □

Remark 4.28. *We write OPE as:* $A(z)B(w) \sim \sum_{n=0}^{N-1} \frac{(A_nB)(w)}{(z-w)^{n+1}}$.

Remark 4.29. $z^m = \begin{cases} \sum_{k=0}^{m} C_m^k (z-w)^k w^{m-k} & \text{if } m \geq 0 \\ \sum_{k\in\mathbb{N}} (-1)^k C_{k-m-1}^k (z-w)^k w^{m-k} & \text{if } m < 0 \end{cases}$

Proposition 4.30. *(Lie bracket) Let A, B local, with $\varepsilon \in \mathbb{Z}_2$ then:*

$$[A(m), B(n)]_\varepsilon = \begin{cases} \sum_{p=0}^{N-1} C_m^p (A_pB)(m+n-p) & \text{if } m \geq 0 \\ \\ \sum_{p=0}^{N-1} (-1)^p C_{p-m-1}^p (A_pB)(m+n-p) & \text{if } m < 0 \end{cases}$$

Proof. $\forall c, d \in H$, $([A(m), B(n)]_\varepsilon c, d) =$
$\frac{1}{(2\pi i)^2}(\int\int_{|z|=R>|w|} - \int\int_{|z|=r<|w|}) z^m w^n X(A, B, c, d)(z, w) dz dw$

By contour integration argument $(\int\int_{|z|=R>|w|} - \int\int_{|z|=r<|w|} = \oint_0 \oint_w)$:

$([A(m), B(n)]_\varepsilon c, d) = \frac{1}{2\pi i}\oint_0 w^n \frac{1}{2\pi i}\oint_w z^m (\sum_{p=0}^{N-1} \frac{(A_pB)(w)}{(z-w)^{p+1}} c, d) dz dw$

We suppose $m \geq 0$, then by previous remark, $([A(m), B(n)]_\varepsilon c, d) =$
$= \frac{1}{2\pi i}\oint_0 w^n \frac{1}{2\pi i}\oint_w \sum_{k=0}^{m} C_m^k w^{m-k} (\sum_{p=0}^{N-1} \frac{(A_pB)(w)}{(z-w)^{p+1-k}} c, d) dz dw$
$= \frac{1}{2\pi i}\oint_0 (\sum_{p=0}^{N-1} w^{n+m-p} C_m^p (A_pB)(w) c, d) dw$
$= \frac{1}{2\pi i}\oint_0 (\sum_{r\in\mathbb{Z}p=0}^{N-1} w^{n+m-p-r-1} C_m^p (A_pB)(r) c, d) dw$
$= (\sum_{p=0}^{N-1} C_m^p (A_pB)(m+n-p) c, d)$ (we take $C_m^p = 0$ if $p > m$).

Similary for $m < 0$..., and the result follows. □

Formula 4.31. *The formula of $[A(m), B(n)]_\varepsilon$ on proposition 4.30.*

Definition 4.32. *(Operator D) Let $D \in End(H)$ decomposing H into $\bigoplus_{n\in\mathbb{N}+\frac{1}{2}} H_n$ with $D\xi = n\xi$ $\forall \xi \in H_n$, $dim(H_n) < \infty$ and $H_n \perp H_m$ if $n \neq m$.*

Notation 4.33. *Let $A'(z) = \frac{d}{dz}A(z) = \sum_{n\in\mathbb{Z}} (-n)A(n-1)z^{-n-1}$.*

Definition 4.34. *$A \in (EndH)[[z, z^{-1}]]$ is graded if:*
$\exists \alpha \in \frac{1}{2}\mathbb{N}$ such that $[D, A(z)] = zA'(z) + \alpha A(z)$

Lemma 4.35. *A is graded with $\alpha \iff$*
$A(n) : H_m \to H_{m-n+\alpha-1}$ $\forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}$

Proof. $[D, A(z)] = zA'(z) + \alpha A(z) = \sum_{n\in\mathbb{Z}}(\alpha - 1 - n)A(n)z^{-n-1}$
$\iff [D, A(n)] = (\alpha - 1 - n)A(n) \;\; \forall n \in \mathbb{Z}$
$\iff \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}, \forall \xi \in H_m$
$DA(n)\xi = A(n)D\xi + [D, A(n)]\xi = (m - n + \alpha - 1)A(n)\xi$
$\iff A(n) : H_m \to H_{m-n+\alpha-1} \;\; \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}.$ □

Lemma 4.36. *Let A, B local and graded with α and β then:*
$[D, A_nB(z)] = z(A_nB)'(z) + (\alpha + \beta - n - 1)A_nB(z).$

Proof. $A(n) : H_m \to H_{m-n+\alpha-1}$ and $B(n) : H_m \to H_{m-n+\beta-1}$
Now, by formula 4.26, $A_pB(n) : H_m \to H_{m-n+(\alpha+\beta-p-1)-1}$
The result follows by the previous lemma. □

Lemma 4.37. *Let A, $B \in (EndH)[[z, z^{-1}]]$, graded with α and β, then:*
A and B are local $\iff \exists \varepsilon \in \mathbb{Z}_2$, $\exists N \in \mathbb{N}$ *such that* $\forall c, d \in H$:
$(z - w)^N(A(z)B(w)c, d) = (-1)^\varepsilon(z - w)^N(B(w)A(z)c, d)$ *as formal series.*

Proof. ($\Rightarrow$) True by definition.
($\Leftarrow$) Let $c \in H_p$, $d \in H_q$
$A(n)c \in H_{p-n+\alpha-1} = 0$ for $n > p + \alpha - 1$,
$B(m)c \in H_{p-m+\beta-1} = 0$ for $m > p + \beta - 1$,
$A(n)B(m)c$, $B(m)A(n)c \in H_{p-(m+n)+\alpha+\beta-2}$, $d \in H_q$ and $H_r \perp H_q$ if $q \neq r$.

Let $S = \{(m, n) \in \mathbb{Z}^2; m + n = p - q + \alpha + \beta - 2, m \leq p + \beta - 1\}$
and $S' = \{(m, n) \in \mathbb{Z}^2; m + n = p - q + \alpha + \beta - 2, n \leq p + \alpha - 1\}$

$$(z - w)^N(A(z)B(w)c, d) = \sum_{S,k=0}^{N} C_N^k(A(n)B(m)c, d)z^{-n-1-k}w^{-m-1+N-k}$$

$$(z-w)^N(B(w)A(z)c, d) = (-1)^\varepsilon \sum_{S',k=0}^{N} C_N^k(B(m)A(n)c, d)z^{-n-1-k}w^{-m-1+N-k}$$

But, $S \cap S'$ is a finite subset of $\mathbb{Z}^2$, so the formal series is a polynom: $P(A, B, c, d) \in \mathbb{C}[z^{\pm1}, w^{\pm1}]$; now, using remark 4.23, and the fact that $A(n)c = 0$ for $n > p + \alpha - 1$ and $B(m)c = 0$ for $m > p + \beta - 1$, then:

$$(z, w)^{-N}P(A, B, c, d)(z, w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon(B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases}$$

□

Remark 4.38. *(associativity)* $(A_nB)_mC = A_n(B_mC) = A_nB_mC$

Lemma 4.39. *Let* A_1*,...,* A_R *graded,* A_i *and* A_j *local with* $N = N_{ij} \in \mathbb{N}$*. Then,* $\forall c, d \in H$*:*

$$\prod_{i<j}(z_i - z_j)^{N_{ij}}(A_1(z_1)...A_R(z_R)c, d) \in \mathbb{C}[z_1^{\pm 1}, ..., z_R^{\pm 1}]$$

Proof. It is exactly as the previous lemma:
We can put each $A_i(z_i)$ on the first place by commutations.
We obtain equalities between R series with support $S_i \cup T$, with T the support due to $\prod_{i<j}(z_i - z_j)^{N_{ij}}$ (finite), and as the previous lemma:
$S_i = \{(m_1, ..., m_R) \in \mathbb{Z}^R; m_1 + ... + m_R = K, m_i \leq k_i\}$
So, $\bigcap S_i$ is a finite subset of $\mathbb{Z}^R$, and the result follows. □

Lemma 4.40. *(Dong's lemma) Let* A*,* B*,* C *graded and pairwise local, then* A_nB *and* C *are local.*

Proof. Let $Q(z_1, z_2, z_3) = \prod_{i<j}(z_i - z_j)^{N_{ij}}$, by lemma 4.39, $\forall d, e \in H$:

$$Q.(A(z_1)B(z_2)C(z_3)d, e) = Q.(-1)^{\varepsilon_1+\varepsilon_2}(C(z_3)A(z_1)B(z_2)d, e) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$$

Now, we divide this polynom by Q, we fix z_2 and we develop around $z_1 = z_2$.
Then $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{Z}$ if P_n is the coefficient of $(z_1 - z_2)^{-n-1}$ then $S_n = (z_2 - z_3)^N P_n \in \mathbb{C}[z_2^{\pm 1}, z_3^{\pm 1}]$.
Now, on one hand $S_n = (z_2 - z_3)^N(A_nB(z_2)C(z_3)d, e)$ and on the other hand $S_n = (-1)^{\varepsilon}(z_2 - z_3)^N(C(z_3)A_nB(z_2)d, e)$, with $\varepsilon = \varepsilon_1 + \varepsilon_2$.
Then, the result follows by lemmas 4.36 and 4.37. □

Proof's corollary 4.41. *If in addition,* A *and* C *are local with* $\varepsilon_1 \in \mathbb{Z}_2$ *, and,* B *and* C*, local with* ε_2*, then,* A_nB *and* C *are local with* $\varepsilon = \varepsilon_1 + \varepsilon_2$*.*

Lemma 4.42. *If* A *and* B *are local with* $\varepsilon \in \mathbb{Z}_2$*, so is* A' *and* B

Proof. $(z - w)^N(A(z)B(w)c, d) = (-1)^{\varepsilon}(z - w)^N(B(w)A(z)c, d)$
Then, applying $\frac{d}{dz}$ and the lemma 4.37, the result follows. □

Definition 4.43. *(Operator T) Let* $T \in End(H)$*.*

Lemma 4.44. *Let* A*,* B *local such that* $[T, A] = A'$ *and* $[T, B] = B'$*. Then,* $[T, A_nB] = (A_nB)' = A'_nB + A_nB'$ *and [T,A'] = A"*

Proof. $(z-w)^N([T, A(z)B(w)]c, d) = (z-w)^N((A'(z)B(w)+A(z)B'(w))c, d)$

$= (z-w)^N \sum_{n\in\mathbb{Z}}((A'_nB + A_nB')(w)c, d)(z-w)^{-n-1}$ on one hand
$= (z-w)^N(\frac{d}{dz} + \frac{d}{dw})(\sum_{n\in\mathbb{Z}} A_nB(w)(z-w)^{-n-1}c, d)$ on the other hand

$= (z-w)^N[(\sum_{n\in\mathbb{Z}}(-n-1)A_nB(w)(z-w)^{-n-2}c, d) +$
$(\sum_{n\in\mathbb{Z}}(A_nB)'(w)(z-w)^{-n-1}c, d) + (\sum_{n\in\mathbb{Z}}(n+1)A_nB(w)(z-w)^{-n-2}c, d)]$
$= (z-w)^N \sum_{n\in\mathbb{Z}}((A_nB)'(w)c, d)(z-w)^{-n-1}$

By identification: $[T, A_nB] = (A_nB)' = A'_nB + A_nB'$
Now, $[T, A] = A' \Rightarrow [T, A(n)] = -nA(n-1)$, so $[T, A'] = A''$ □

Lemma 4.45. *Let $\Omega \in H$; A, B local with $A(m)\Omega = B(m)\Omega = 0$ $\forall m \in \mathbb{N}$, then $A'(m)\Omega = A_nB(m)\Omega = 0$ $\forall m \in \mathbb{N}, \forall n \in \mathbb{Z}$.*

Proof. $A'(m) = -mA(m-1)$, so $A'(m)\Omega = 0$ $\forall m \in \mathbb{N}$
On the formula 4.26 , $A(n-p)\Omega = B(m+p)\Omega = A(p)\Omega = 0$ because $n-p, m+p, p \in \mathbb{N}$, then, $A_nB(m)\Omega = 0$ $\forall m \in \mathbb{N}, \forall n \in \mathbb{Z}$. □

4.3 System of generators

Definition 4.46. *Let H prehilbert space; $\{A_1,\dots, A_r\} \subset (EndH)[[z, z^{-1}]]$ is a system of generators if $\exists D, T \in End(H)$, $\Omega \in H$ such that:*

(a) *$\forall i, j$ A_i and A_j are local with $N = N_{ij}$ and $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii}.\varepsilon_{jj}$*

(b) *$\forall i$ $[T, A_i] = A'_i$*

(c) *D decomposes $H = \bigoplus_{n\in\mathbb{N}+\frac{1}{2}} H_n$ with $D\xi = n\xi$ $\forall \xi \in H_n$, $dim(H_n) < \infty$, $H_n \perp H_m$ if $n \neq m$ and $\forall i$ A_i is graded with $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$*

(d) *$\Omega \in H_0$, $\|\Omega\| = 1$, and $\forall i$ $\forall m \in \mathbb{N}$, $A_i(m)\Omega = D\Omega = T\Omega = 0$*

(e) *$\mathcal{A} = \{A_i(m), \forall i$ $\forall m \in \mathbb{Z}\}$ acts irreducibly on H, so that H is the minimal space containing Ω and stable by the action of $\mathcal{A}$*

Definition 4.47. *Let $S \subset (EndH)[[z, z^{-1}]]$, the minimal subset containing Id, $A_1,\dots, A_r$, stable by the operations:*

$$(A, B) \mapsto (A_nB) \ (\forall n \in \mathbb{Z}) \qquad , \qquad A \mapsto A'$$

Let $S_\varepsilon = \{A \in S \mid A$ is local with itself with $\varepsilon \in \mathbb{Z}_2\}$, so that $S = S_{\bar{0}} \amalg S_{\bar{1}}$.
Let $\mathcal{S}_\varepsilon = lin < S_\varepsilon >$ and $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$.

Remark 4.48. *All is well defined by previous lemmas.*

Lemma 4.49. *$\forall A, B \in \mathcal{S}$, they are local, $A_n B \in \mathcal{S}$ and $[T, A] = A' \in \mathcal{S}$*

Proof. By previous lemmas and linearizing Dong's lemma. □

Lemma 4.50. *Let $E \in \mathcal{S}_{\varepsilon_1}$ and $F \in \mathcal{S}_{\varepsilon_2}$ then:*

(a) *$E_n F \in \mathcal{S}_{\varepsilon_1+\varepsilon_2}$*

(b) *E and F are local with $\varepsilon = \varepsilon_1.\varepsilon_2$*

Proof. (a) E and F are local with an $\varepsilon \in \mathbb{Z}_2$.
We use the corollary 4.41 with $A = E$, $B = F$, $C = E$,with $A = E$, $B = F$, $C = F$ and finally with $A = E$, $B = F$, $C = E_n F$. Then we see that $E_n F$ is local with itself with $\varepsilon' = \varepsilon_1 + \varepsilon + \varepsilon_2 + \varepsilon = \varepsilon_1 + \varepsilon_2$, so, $E_n F \in \mathcal{S}_{\varepsilon_1+\varepsilon_2}$
(b) By induction:
Base case: $\forall i, j$ $A_i \in \mathcal{S}_{\varepsilon_{ii}}$, $A_j \in \mathcal{S}_{\varepsilon_{jj}}$ and are local with $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii}.\varepsilon_{jj}$ by definition 4.46.
Inductive step: We suppose the property for $E \in \mathcal{S}_{\varepsilon_1}$, $F \in \mathcal{S}_{\varepsilon_2}$ and $G \in \mathcal{S}_{\varepsilon_3}$.
We prove it for $E_n F$ and G:
E and G are local with $\varepsilon = \varepsilon_1.\varepsilon_3$
F and G are local with $\varepsilon = \varepsilon_2.\varepsilon_3$
Now, $E_n F \in \mathcal{S}_{\varepsilon_1+\varepsilon_2}$, $G \in \mathcal{S}_{\varepsilon_3}$ and by corollary 4.41 with $A = E$, $B = F$, $C = G$, $E_n F$ and G are local with $\varepsilon = \varepsilon_1.\varepsilon_3 + \varepsilon_2.\varepsilon_3 = (\varepsilon_1 + \varepsilon_2).\varepsilon_3$
The following lemma completes the proof. □

Lemma 4.51. *$A \in \mathcal{S}_\varepsilon \Rightarrow A' \in \mathcal{S}_\varepsilon$*

Proof. By lemma 4.42, if A and B are local with $\varepsilon \in \mathbb{Z}_2$, so is A' and B.
The result follows by taking $B = A$ and then $B = A'$. □

Definition 4.52. *(well defined by lemma 4.45)*

$$R: \begin{array}{ccc} \mathcal{S} & \longrightarrow & H \\ A & \longmapsto & a := A(z)\Omega_{|z=0} \end{array} \quad \text{linear.}$$

Examples 4.53.

(a) $R(Id) = \Omega, \ \ R(A) = A(-1)\Omega$

(b) $R(A') = A(-2)\Omega = T.R(A)$

(c) $R(A_n B) = A(n)R(B)$ *(by formula 4.26)*

(d) $R(A_n Id) = A(n)\Omega$

Lemma 4.54. *A is graded with* $\alpha \iff R(A) \in H_\alpha$

Proof. By lemma 4.35 and 4.36, inductions and linear combinations. □

State-Field correspondence:

Lemma 4.55. *(Existence)* $\forall a \in H$, $\exists A \in \mathcal{S}$ *such that* $R(A) = a$.

Proof. $R((A_{i_1})_{m_1}(A_{i_2})_{m_2}...(A_{i_k})_{m_k}Id) = A_{i_1}(m_1)R((A_{i_2})_{m_2}...(A_{i_k})_{m_k}Id)$
$= ... = A_{i_1}(m_1)...A_{i_k}(m_k)\Omega$
Now, the action of the $A_i(m)$ on Ω generates H by definition 4.46. □

Lemma 4.56. *Let* $A \in \mathcal{S}$, *then* $A(z)\Omega = e^{zT}R(A)$.

Proof. Let $F_A(z) = A(z)\Omega = \sum_{n\in\mathbb{N}} A(-n-1)\Omega z^n$,
Then, $\forall b \in H$, $(F_A(z), b) \in \mathbb{C}[z]$
Now, $\frac{d}{dz}(F_A(z), b) = (\frac{d}{dz}F_A(z), b) = (A'(z)\Omega, b)$
$= ([T, A(z)]\Omega, b) = (T.A(z)\Omega, b) = (T.F_A(z)\Omega, b)$
But, $F_A(0) = R(A)$, so we see that: $(F_A(z), b) = (e^{zT}R(A), b)$ $\forall b \in H$
Finally, $F_A(z) = e^{zT}R(A)$ □

Lemma 4.57. *(Unicity)* $R(A) = R(B) \Rightarrow A = B$.

Proof. Let $C = A - B$, then $R(C) = R(A) - R(B) = 0$
and $F_C(z) = e^{zT}R(C) = 0$
Now, $\forall e \in H$, $\exists E \in \mathcal{S}$ such that $R(E) = e$.
Then $\forall f \in H$, $\exists N \in \mathbb{N}$ $\exists \varepsilon \in \mathbb{Z}_2$ such that :
$(z-w)^N(C(z)E(w)\Omega, f) = (-1)^\varepsilon(z-w)^N(E(w)C(z)\Omega, f)$
Now, $(E(w)C(z)\Omega, f) = (E(w)F_C(z), f) = 0 = (C(z)E(w)\Omega, f)$
So, $(C(z)E(w)\Omega, f)_{|w=0} = (C(z)e, f) = 0$ $\forall e, f \in H$
Finally, $C = 0$ and $A = B$ □

Now, we can well defined:

Definition 4.58. *(State-Field correspondence map)*

$$V: \begin{array}{ccc} H & \longrightarrow & \mathcal{S} \\ a & \longmapsto & V(a) \end{array} \quad \textit{linear.}$$

such that : $\left(\begin{array}{ll} \forall a \in H & R(V(a)) = a \\ \forall A \in \mathcal{S} & V(R(A)) = A \end{array} \right.$

Notation 4.59. $V(a)(z)$ *is noted* $V(a,z)$ *and* $A(z) = V(R(A), z)$

Examples 4.60.

(a) $V(0,z) = 0, \;\; V(\Omega, z) = Id$

(b) $V'(a,z) = V(T.a, z)$

(c) $(A_n B)(z) = V(A(n)R(B), z)$

Definition 4.61. *Let* $H_\varepsilon = \bigoplus_{n \in \mathbb{N} + \frac{\varepsilon}{2}} H_n$ *so that* $H = H_{\bar{0}} \oplus H_{\bar{1}}$.

Lemma 4.62. $R(\mathcal{S}_\varepsilon) = H_\varepsilon \quad (\varepsilon \in \mathbb{Z}_2)$

Proof. Base step: by definition 4.46 and lemma 4.54,
$\forall i \; A_i \in \mathcal{S}_{\varepsilon_{ii}}$ and $R(A_i) \in H_{\alpha_i}$ with $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$
Inductive step: by lemma 4.50 □

Corollary 4.63. *(Relation with* T *and* D*) Let* $a \in H_\alpha$*, we have that:*

(a) $[T, V(a,z)] = V'(a,z) = V(T.a, z) \in \mathcal{S}$

(b) $[D, V(a,z)] = z.V'(a,z) + \alpha.V(a,z) \quad$ *(* $\notin \mathcal{S}$ *in general)*

4.4 Application to fermion algebra

$H = \mathcal{F}_{NS}$, $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1}$ with $[\psi_m, \psi_n]_+ = \delta_{m+n} Id$.

Proposition 4.64. $\{\psi\}$ *is a system of generator.*

Proof. ψ is local with itself with $N = 1$ and $\varepsilon = \bar{1} = \bar{1}.\bar{1}$ (see definition 4.46)
We have construct D and T (p 51), $\Omega \in H_0$, $\|\Omega\| = 1$, $D\Omega = T\Omega = 0$.
$[T, \psi(z)] = \psi'(z)$, $[D, \psi(z)] = z.\psi'(z) + \frac{1}{2}\psi(z)$ and $\frac{1}{2} \in \mathbb{N} + \frac{1}{2}$
Finally, $\{\psi_n, n \in \frac{1}{2}\mathbb{N}\}$ acts irreducibly on H □

Corollary 4.65. *$\{\psi\}$ generates an $\mathcal{S}$ with a state-field correspondence with:*

$$R(\psi) = \psi_{-\frac{1}{2}}\Omega \text{ and } \psi(z) = V(\psi_{-\frac{1}{2}}\Omega, z)$$

Lemma 4.66. *(OPE)* $\psi(z)\psi(w) \sim \frac{Id}{z-w}$

Proof. $\psi_n\psi(w) = V(\psi_{n+\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega, w) = 0$ if $n \geq 1$ (here $N = 1$)
Now, for $0 \leq n \leq N - 1$ i.e $n = 0$:
$\psi_{\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega = ([\psi_{\frac{1}{2}}, \psi_{-\frac{1}{2}}]_+ - \psi_{-\frac{1}{2}}\psi_{\frac{1}{2}})\Omega = \Omega$, so $\psi_0\psi(w) = Id$ □

Remark 4.67. *(Next operator) $\psi_{-\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega = 0$, so $\psi_{-1}\psi = 0$; and the next operator of the expansion is $2L(w) := \psi_{-2}\psi(w) = 2\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$*
Now, $R(L) = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$, then $L(w) = V(\frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, w)$.

Remark 4.68. *$L(n) = L_{n-1}$ so, $L_0\Omega = L_{-1}\Omega = 0$ by lemma 4.45.*

Lemma 4.69. *(OPE)* $\psi(z)L(w) \sim \frac{1/2\psi(w)}{(z-w)^2} - \frac{1/2\psi'(w)}{(z-w)}$

Proof. $\psi_n L(w) = \frac{1}{2}V(\psi_{n+\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, w) = 0$ if $n \geq 2$ (here $N = 2$)
Now, $\psi_{\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = -\psi_{-\frac{3}{2}}\Omega = R(\psi')$, $\psi_{\frac{3}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = \psi_{-\frac{1}{2}}\Omega = R(\psi')$ □

Lemma 4.70. *(Lie bracket)* $[L_m, \psi_n] = -(n + \frac{1}{2}m)\psi_{m+n}$

Proof. By lemma 4.50, ψ and L are local with $\varepsilon = \bar{0}$, and by formula 4.31:
$[\psi(m), L(n+1)] = -\frac{1}{2}C^0_m\psi'(m+n+1) + \frac{1}{2}C^1_m\psi(m+n+1-1)$
$= \frac{1}{2}(m+n+1)\psi(m+n) + \frac{1}{2}m\psi(m+n) = (m + \frac{1}{2} + \frac{1}{2}n)\psi(m+n)$
We have computed for $m \geq 0$, we find the same result for $m < 0$.
Now, $\psi(m) = \psi_{m+\frac{1}{2}}$ and $L(n+1) = L_n$, so the result follows. □

Lemma 4.71. *$D = L_0$ and $T = L_{-1}$*

Proof. $[L_0, \psi_n] = -n\psi_n = [D, \psi_n]$, $[L_{-1}, \psi_n] = -(n - \frac{1}{2})\psi_{n-1} = [T, \psi_n]$
So, by irreducibility and Schur's lemma, $L_0 - DandL_{-1} - T \in \mathbb{C}Id$
Now, $L_0\Omega = D\Omega = L_{-1}\Omega = T\Omega = 0$, then, $D = L_0$ and $T = L_{-1}$ □

Corollary 4.72. *$\forall a \in H_s$:*

(a) $[L_{-1}, V(a,z)] = V'(a,z) = V(L_{-1}.a, z) \in \mathcal{S}$

(b) $[L_0, V(a,z)] = z.V'(a,z) + s.V(a,z)$

Remark 4.73. *$\forall A \in \mathcal{S}$, $A' = (L_0 A)$, so, by Dong's lemma, we finally don't need here to $A \mapsto A'$ for the construction of $\mathcal{S}$.*

Lemma 4.74. *(OPE)* $\quad L(z)L(w) \sim \frac{(c/2)Id}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)}$

Proof. $\quad L_n L(w) = V(L(n)L(-1)\Omega, w) = V(L_{n-1}L_{-2}\Omega, w) = 0$ if $n \geq 4$. Then, here, $N = 4$, so, for $0 \leq n \leq N-1$:

(a) $V(L_{-1}L_{-2}\Omega, w) = L'(w)$

(b) $L_0 L_{-2}\Omega = 2L_{-2}\Omega = 2R(L)$ $\quad$ because $L_{-2}\Omega \in H_2$

(c) $L_1 L_{-2}\Omega = \frac{1}{2}L_1 \psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = \frac{1}{2}[L_1, \psi_{-\frac{3}{2}}]\psi_{-\frac{1}{2}}\Omega = \frac{1}{2}\psi^2_{-\frac{1}{2}}\Omega = 0$

(d) $L_2 L_{-2}\Omega \in H_0 = \mathbb{C}\Omega$, so, $L_2 L_{-2}\Omega = K\Omega$ with $K = \|L_{-2}\Omega\|^2$

□

Notation 4.75. $c := 2\|L_{-2}\Omega\|^2$, *the central charge.*
(here c $= \frac{1}{2}(\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, \psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega) = \frac{1}{2}$)

Notation 4.76. *Let* $\delta_k = \begin{cases} 0 & \text{if} \quad \text{k} \neq 0 \\ Id & \text{if} \quad \text{k} = 0 \end{cases}$

Lemma 4.77. *(Lie bracket)* $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$.

Proof. By lemma 4.50, $L \in \mathcal{S}_{\bar{0}}$, and by formula 4.31:
If $m+1 \geq 0$, then: $[L(m+1), L(n+1)] =$
$C^0_{m+1}L'(m+n+2) + 2C^1_{m+1}L(m+n+2-1) + \frac{c}{2}C^3_{m+1}Id(m+n+2-3)$
$= -(m+n+2)L(m+n+2) + 2(m+1)L(m+n+1) + \frac{c}{2}\frac{m(m^2-1)}{6}\delta_{m+n}$
$= (m-n)L(m+n+1) + \frac{c}{12}m(m^2-1)\delta_{m+n}$
We find the same result for $m+1 < 0$ □

Remark 4.78. $L^\star_m = L_{-m}$

Proof. $[\psi_{-n}, L^\star_m] = [L_m, \psi_n]^\star = -(n + \frac{1}{2}m)\psi_{-m-n} = [\psi_{-n}, L_{-m}]$, then the result follows by irreducibility, Schur's lemma and grading. □

Remark 4.79. *The (L_n) generate a Virasoro algebra $\mathfrak{Vir}$.*

Corollary 4.80. $\mathfrak{Vir}$ *acts on $H = \mathcal{F}_{NS}$, and admits $L(c,h) = L(\frac{1}{2}, 0)$ as minimal submodule containing Ω.*

Definition 4.81. *Let call L the Virasoro operator, and $\omega = R(L) = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$, the Virasoro vector.*

4.5 Vertex operator superalgebra

Definition 4.82. *A vertex operator superalgebra is an (H, V, Ω, ω) with:*

(a) $H = H_{\bar{0}} \oplus H_{\bar{1}}$ *a prehilbert superspace.*

(b) $V : H \to (End H)[[z, z^{-1}]]$ *a linear map.*

(c) $\Omega, \omega \in H$ *the vacuum and Virasoro vectors.*

Let $\mathcal{S}_\varepsilon = V(H_\varepsilon)$, $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ and $A(z) = V(a,z) = \sum_{n\in\mathbb{Z}} A(n) z^{-n-1}$, then (H, V, Ω, ω) satisfies the followings axioms:

1. *(vacuum axioms): $\forall A \in \mathcal{S}$ and $\forall n \in \mathbb{N}$, $A(n)\Omega = 0$, $V(a,z)\Omega_{|z=0} = a$ and $V(\Omega, z) = Id$*
2. *(irreducibility axiom): Let $\mathcal{A} = \{A(n) | A \in \mathcal{S}, n \in \mathbb{Z}\}$ then, $\mathcal{A}$ acts irreducibly on H, so that $\mathcal{A}.\Omega = H$*
3. *(locality axiom): $\forall A \in \mathcal{S}_{\varepsilon_1}$, $\forall B \in \mathcal{S}_{\varepsilon_2}$, A and B are local (see definition 4.20 and lemma 4.37), with $\varepsilon = \varepsilon_1.\varepsilon_2$ and $A_n B \in \mathcal{S}_{\varepsilon_1+\varepsilon_2}$*
4. *(Virasoro axiom): $V(\omega, z) = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ Virasoro operator ($L_0\Omega = L_{-1}\Omega = 0$ and $\omega = L_{-2}\Omega$). Let $c = 2\|\omega\|^2$ the central charge: $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$*
5. *(L_0 axioms) L_0 decomposes H into $\bigoplus_{n\in\mathbb{N}+\frac{1}{2}} H_n$ with $dim(H_n) < \infty$, $H_n \perp H_m$ if $n \neq m$, $H_\varepsilon = \bigoplus_{n\in\mathbb{N}+\frac{\varepsilon}{2}} H_n$, $\Omega \in H_0$, $\omega \in H_2$, and $\forall a \in H_\alpha$, $[L_0, V(a,z)] = z.V'(a,z) + \alpha.V(a,z)$*
6. *(L_{-1} axioms): $[L_{-1}, V(a,z)] = V'(a,z) = V(L_{-1}.a, z) \in \mathcal{S}$*

Corollary 4.83. *A system of generators, generating a Virasoro operator $L \in \mathcal{S}$, with $D = L_0$ and $T = L_{-1}$, generates a vertex operator superalgebra.*

Corollary 4.84. *The fermion operator ψ generates a vertex operator superalgebra , with Virasoro vector $\omega = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$.*

Remark 4.85. *The Virasoro operator L alone, generates the minimal vertex operator (super)algebra.*

Remark 4.86. *Let $A(z) = V(a,z)$ and $B(w) = V(b,w)$; the formula 4.26 is general, so similary, by vacuum axioms, $A_n B(w) = V(A(n)b, w)$.*

Proposition 4.87. *(Borcherds associativity) $\exists N \in \mathbb{N}$ such that $\forall c, d \in H$: $(z-w)^N (V(a,z)V(b,w)c, d) = (z-w)^N (V(V(a, z-w)b, w)c, d)$*

Proof. To simplify the proof, we don't write:
"$\exists N \in \mathbb{N}$ such that $\forall c, d \in H$ $(z-w)^N(\ . \ c, d)$", but it is implicit.

$$V(a,z)V(b,w) = A(z)B(w) = \sum A_n B(w)(z-w)^{-n-1}$$
$$= \sum V(A(n)b, w)(z-w)^{-n-1} = V(\sum A(n)b(z-w)^{-n-1}, w)$$
$$= V(\sum A(n)(z-w)^{-n-1}b, w) = V(V(a, z-w)b, w).$$
□

5 Vertex $\mathfrak{g}$-superalgebras and modules

5.1 Preliminaries

5.1.1 Simple Lie algebra $\mathfrak{g}$

Let $\mathfrak{g}$ be a simple Lie algebra of dimension N, a basis (X_a) with $[X_a, X_b] = i\sum_c \Gamma_{ab}^c X_c$ with $\Gamma_{ab}^c \in \mathbb{R}$ totally antisymmetric.

Lemma 5.1. *Let $\mathcal{C} = \sum_b X_b^2$, then $[\mathfrak{g}, \mathcal{C}] = 0$*

Proof. It suffices to prove $[X_a, \mathcal{C}] = 0$ for each X_a.
$[X_a, \mathcal{C}] = \sum_b [X_a, X_b^2] = \sum_b([X_a, X_b]X_b + X_b[X_a, X_b]) = i\sum_{b,c}\Gamma_{ab}^c X_c X_b +$
$i\sum_{b,c}\Gamma_{ab}^c X_b X_c = i\sum_{b,c}(\Gamma_{ab}^c + \Gamma_{ac}^b)X_c X_b = 0$ by antisymmetry. □

Remark 5.2. *$\mathcal{C}$ is a multiple of the **Casimir** of $\mathfrak{g}$. We suppose to have well normalized the basis such that $\mathcal{C}$ is exactly the Casimir.*

Corollary 5.3. *By Schur's lemma, $\mathcal{C}$ acts as multiplicative constant c_V on each irreducible representation V.*

Example 5.4. *$\mathfrak{g}$ is simple, it acts irreducibly on $V = \mathfrak{g}$ with ad.*

Lemma 5.5. $\sum_{a,c}\Gamma_{ac}^b.\Gamma_{ac}^d = \delta_{bd}c_{\mathfrak{g}}$

Proof. $(\sum_a ad_{X_a}^2)(X_b) = c_{\mathfrak{g}}X_b = \sum_a [X_a, [X_a, X_b]]$
$= i^2\sum_{a,c,d}\Gamma_{ab}^c.\Gamma_{ac}^d X_d = \sum_{a,c,d}\Gamma_{ac}^b.\Gamma_{ac}^d X_d$.
Then, $\sum_{a,c}\Gamma_{ac}^b.\Gamma_{ac}^d = \delta_{bd}c_{\mathfrak{g}}$ □

Definition 5.6. *$g = \frac{c_{\mathfrak{g}}}{2}$ is called the dual Coxeter number.*

Example 5.7. *$\mathfrak{g} = A_1 = \mathfrak{sl}_2$, $dim(\mathfrak{g}) = 3$*
$[E, F] = H$, $[H, E] = 2E$, $[H, F] = -2F$, with Casimir $EF + FE + \frac{1}{2}H^2$
We choose the basis: $X_1 = \frac{i\sqrt{2}}{2}(E - F)$, $X_2 = \frac{\sqrt{2}}{2}(E + F)$, $X_3 = \frac{\sqrt{2}}{2}H$,
with relations: $[X_1, X_2] = i\sqrt{2}X_3$, $[X_3, X_1] = i\sqrt{2}X_2$, $[X_2, X_3] = i\sqrt{2}X_1$
$\mathcal{C} = \sum_a X_a^2 = EF + FE + \frac{1}{2}H^2$ and $g = \frac{1}{2}\sum_{a,b}(\Gamma_{ab}^c)^2 = 2$

Table (see [54] p 111)

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$dim(\mathfrak{g})$	n^2+2n	$2n^2+n$	$2n^2+n$	$2n^2-n$	78	133	248	52	14
g	$n+1$	$2n-1$	$n+1$	$2n-2$	12	18	30	9	4

5.1.2 Loop algebra $L\mathfrak{g}$

Definition 5.8. *Let $L\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{g})$ the loop algebra of $\mathfrak{g}$.*
It's an infinite dimensional Lie $\star$-algebra, admitting the $X_n^a = X_a e^{in\theta}$ as basis, with $n \in \mathbb{Z}$ and (X_a) the base of $\mathfrak{g}$; so:

$$[X_m^a, X_n^b] = [X_a, X_b]_{m+n} \quad \text{and} \quad (\mathrm{X_n^a})^\star = \mathrm{X_{-n}^a}$$

Proposition 5.9. *(Boson cocycle) $L\mathfrak{g}$ has a unique central extension, up to equivalent, i.e. $H_2(L\mathfrak{g}, \mathbb{C})$ is 1-dimensional. $H_2(L\mathfrak{g}, \mathbb{C})$ is 1-dimensional. Let $\mathcal{L}$ the central element and $\widehat{\mathfrak{g}}_+ = L\mathfrak{g} \oplus \mathbb{C}\mathcal{L}$ called $\mathfrak{g}$-boson algebra, then:*

$$[X_m^a, X_n^b] = [X_a, X_b]_{m+n} + m\delta_{ab}\delta_{m+n}.\mathcal{L}$$

Proof. See [100] p 43. □

Theorem 5.10. *The unitary highest weight representations of $\widehat{\mathfrak{g}}_+$ are $H = L(V_\lambda, \ell)$ with:*

(a) *$\ell \in \mathbb{N}$ such that $\mathcal{L}\Omega = \ell\Omega$ (the level of H).*

(b) *$H_0 = V_\lambda$ irreducible representation of $\mathfrak{g}$ such that:*
$(\lambda, \theta) \leq \ell$ with λ the highest weight and θ the highest root.

Proof. See [100] p 45. □

Remark 5.11. *Let $\mathscr{C}_\ell$ the category of such representations for ℓ fixed.*
$\mathscr{C}_\ell$ is a finite set and $\mathscr{C}_\ell \subset \mathscr{C}_{\ell+1}$

Remark 5.12. *The irreducible unitary projective positive energy representations of $L\mathfrak{g}$ are given by the unitary highest weight representation of $\widehat{\mathfrak{g}}_+$.*

Example 5.13. *We take $\mathfrak{g} = \mathfrak{sl}_2$, then $H = L(j, \ell)$ with:*

- $\mathcal{L}\Omega = \ell\Omega, \quad \ell \in \mathbb{N}$
- *$H_0 = V_j$ with $j \in \frac{1}{2}\mathbb{N}$ the spin and $j \leq \frac{\ell}{2}$, such that $\mathcal{C}\Omega = c_{V_j}\Omega$ with $\mathcal{C} = \sum_a (X_0^a)^2$ the Casimir and $c_{V_j} = 2j^2 + 2j$*

5.2 $\mathfrak{g}$-vertex operator superalgebras

5.2.1 $\mathfrak{g}$-fermion

Definition 5.14. *Let $\widehat{\mathfrak{g}}_-$ be the $\mathfrak{g}$-fermion algebra, generated by (ψ^a_m) with $a \in \{1, ..., N\}$, $N = dim(\mathfrak{g})$, $m \in \mathbb{Z} + \frac{1}{2}$ and relations:*

$$[\psi^a_m, \psi^b_n]_+ = \delta_{ab}\delta_{m+n} \quad and \quad (\psi^a_m)^\star = \psi^a_{-m}$$

Remark 5.15. *As for the fermion algebra of section 4.1, we generate the Verma module $H = \mathcal{F}^{\mathfrak{g}}_{NS}$, and the sesquilinear form $(.,.)$ which is a scalar product; $\pi(\psi^a_n)^\star = \pi((\psi^a_n)^\star)$, $\mathcal{F}^{\mathfrak{g}}_{NS}$ is a prehilbert space, an irreducible representation of $\widehat{\mathfrak{g}}_-$ and its unique unitary highest weight representation.*

Definition 5.16. *Let $\psi^a(z) = \sum_{n\in\mathbb{Z}} \psi^a_{n+\frac{1}{2}}.z^{-n-1}$ the fermion operators.*

Remark 5.17. $\psi^a(z)\psi^b(w) \sim \frac{\delta_{ab}}{(z-w)}$

Remark 5.18. *As for the single fermion operator ψ, of section 4.4, $\{\psi^a, a \in \{1, ..., N\}\}$ generates a vertex operator superalgebra with:*

$$\omega = \frac{1}{2}\sum_a \psi^a_{-\frac{3}{2}}\psi^a_{-\frac{1}{2}}\Omega \quad \text{and} \quad \text{c} = 2\|\omega\|^2 = \frac{\dim(\mathfrak{g})}{2}$$

Definition 5.19. *Let $S^c(z) = V(s^c, z) = \sum_{n\in\mathbb{Z}} S^c_n z^{-n-1}$ with:*

$$s^c = -\tfrac{i}{2}\textstyle\sum_{a,b} \Gamma^c_{ab}\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\Omega \in H_1 \subset H_{\bar{0}}$$

Lemma 5.20. *(OPE and Lie bracket)*

$$\psi^a(z)S^b(w) \sim \frac{i\sum_c \Gamma^c_{ab}\psi^c(w)}{(z-w)} \quad \text{and} \quad [\psi^{\mathrm{a}}_{\mathrm{m}}, \mathrm{S}^{\mathrm{b}}_{\mathrm{n}}] = \mathrm{i}\sum_{\mathrm{c}} \Gamma^{\mathrm{c}}_{\mathrm{ab}}\psi^{\mathrm{c}}_{\mathrm{m+n}} = [\mathrm{S}^{\mathrm{a}}_{\mathrm{m}}, \psi^{\mathrm{b}}_{\mathrm{n}}]$$

Proof. $\psi^d_{n+\frac{1}{2}}.s^c = 0$ if $n \geq 1$ and $\psi^d_{\frac{1}{2}}.s^c = i\sum_a \Gamma^a_{dc}\psi^a_{-\frac{1}{2}}\Omega$. □

Remark 5.21. $[S^a_m, \psi^a_n] = 0$

Lemma 5.22. $(S^b_m)^\star = S^b_{-m}$

Proof. $[(S^b_n)^\star, \psi^a_{-m}] = [\psi^a_m, S^b_n]^\star = -i\sum_c \Gamma^c_{ab}\psi^c_{-m-n} = [S^b_{-n}, \psi^a_{-m}]$
The result follows by irreducibility, Schur's lemma and grading. □

Remark 5.23. *(Jacobi)* $[X_a,[X_b,X_c]] = [[X_a,X_b],X_c] + [X_b,[X_a,X_c]]$
$\Leftrightarrow \sum_d \Gamma^d_{bc}\Gamma^e_{ad} = \sum_d(\Gamma^d_{ab}\Gamma^e_{dc} + \Gamma^d_{ac}\Gamma^e_{bd}) \Leftrightarrow \sum_e(\Gamma^e_{ab}\Gamma^e_{cd} + \Gamma^e_{da}\Gamma^e_{cd} + \Gamma^e_{db}\Gamma^e_{ac}) = 0$

Notation 5.24. $[S^a,S^b] := i\sum_c \Gamma^c_{ab}S^c$

Lemma 5.25. *(OPE and Lie bracket)*

$$S^a(z)S^b(w) \sim \frac{[S^a,S^b](w)}{(z-w)} + \frac{g.\delta_{ab}}{(z-w)^2}$$

and $[S^a_m,S^b_n] = [S^a,S^b](m+n) + \ell.m\delta_{ab}\delta_{m+n}$ *(with* $\ell = g \in \mathbb{N}$*)*

Proof. $S^d_n s^c = 0$ if $n \geq 2$ and:

(a) $S^d_0 s^c = -\frac{i}{2}\sum_{a,b}\Gamma^c_{ab}S^d_0\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\Omega$
$= -\frac{i}{2}(i\sum_{a,b,e}\Gamma^c_{ab}\Gamma^e_{da}\psi^e_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\Omega + i\sum_{a,b,e}\Gamma^c_{ab}\Gamma^e_{db}\psi^a_{-\frac{1}{2}}\psi^e_{-\frac{1}{2}}\Omega)$
$= -\frac{i}{2}(i\sum_{a,b,e}(\Gamma^c_{eb}\Gamma^a_{de} + \Gamma^c_{ae}\Gamma^b_{de})\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\Omega$
$= i\sum_e \Gamma^e_{dc}\frac{-i}{2}\sum_{a,b}\Gamma^e_{ab}\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\Omega = i\sum_e\Gamma^e_{dc}s^e = [S^d,S^c](-1)$

(b) $S^d_1 s^c = -\frac{i}{2}i\sum_{a,b,e}\Gamma^c_{ab}\Gamma^e_{da}\psi^e_{\frac{1}{2}}\psi^b_{-\frac{1}{2}}\Omega = \frac{1}{2}\sum_{a,b}\Gamma^c_{ab}\Gamma^d_{ab} = g.\delta_{cd}$

□

Corollary 5.26. (S^a_m) *is the basis of a* $\mathfrak{g}$*-boson algebra.*
It admits $L(V_0,g)$ *as minimal submodule of* $\mathcal{F}^{\mathfrak{g}}_{NS}$ *containing* Ω
$($*with* $V_0 = \mathbb{C}$ *the trivial representation of* $\mathfrak{g})$.

Lemma 5.27. $\sum_a (S^a_{-1})^2\Omega = 4g\omega$

Proof. $\sum_e (S^e_{-1})^2\Omega = -\frac{i}{2}\sum_{a,b,e}\Gamma^c_{ab}S^e_{-1}\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\Omega$
$= -\frac{1}{4}\sum_{a,b,c,d,e}\Gamma^e_{ab}\Gamma^e_{cd}\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\psi^c_{-\frac{1}{2}}\psi^d_{-\frac{1}{2}}\Omega - \frac{i}{2}\sum_{a,b,c}\Gamma^e_{ab}[S^e_{-1},\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}]\Omega$
$= -\frac{1}{12}\sum_{a,b,c,d}(\sum_e(\Gamma^e_{ab}\Gamma^e_{cd} + \Gamma^e_{da}\Gamma^e_{cb} + \Gamma^e_{db}\Gamma^e_{ac})\psi^a_{-\frac{1}{2}}\psi^b_{-\frac{1}{2}}\psi^c_{-\frac{1}{2}}\psi^d_{-\frac{1}{2}}\Omega)$
$+ \sum_{a,b,c,e}\Gamma^b_{ea}\Gamma^c_{ea}\psi^c_{-\frac{3}{2}}\psi^b_{-\frac{1}{2}}\Omega \;=\; 4g\omega$ □

Lemma 5.28. *(OPE and Lie bracket)*

$$S^a(z)L(w) \sim \frac{S^a(w)}{(z-w)^2} \quad \text{and} \quad [\mathrm{L_m},\mathrm{S^a_n}] = -\mathrm{nS^a_{m+n}}$$

Proof. $S^a_n.\omega = 0$ for $n \geq 3$ and:

(a) $S^a_0.\omega = \frac{1}{4g}\sum_b S^a_0(S^b_{-1})^2\Omega = \frac{1}{4g}\sum_b([S^a_0, S^b_{-1}]S^b_{-1}\Omega + S^b_{-1}[S^a_0, S^b_{-1}]\Omega)$
$= \frac{i}{4g}\sum_{b,c}(\Gamma^c_{ab} + \Gamma^b_{ac})S^c_{-1}S^b_{-1}\Omega = 0$

(b) $S^a_2.\omega = \frac{1}{4g}\sum_b([S^a_2, S^b_{-1}]S^b_{-1}\Omega + S^b_{-1}[S^a_2, S^b_{-1}]\Omega)$
$= \frac{i}{4g}\sum_{b,c}\Gamma^c_{ab}S^c_1S^b_{-1}\Omega = \frac{i}{4g}\sum_{b,c}\Gamma^c_{ab}\delta_{bc}\ell = 0$

(c) $S^a_1.\omega = \frac{1}{4g}\sum_b([S^a_1, S^b_{-1}]S^b_{-1}\Omega + S^b_{-1}[S^a_1, S^b_{-1}]\Omega)$
$= \frac{i}{4g}(2\ell + i\sum_{b,c}\Gamma^c_{ab}S^c_0S^b_{-1}\Omega) = \frac{2(\ell+g)}{4g}S^a_{-1}\Omega = S^a_{-1}\Omega \quad (\star)$

□

Corollary 5.29. *(S^a) generate a vertex operator (super)algebra with $\omega = \frac{1}{4g}\sum_a(S^a_{-1})^2\Omega$ as Virasoro vector.*

5.2.2 $\mathfrak{g}$-boson

Definition 5.30. *Let $X^a(z) = \sum_{n\in\mathbb{Z}} X^a_n z^{-n-1}$ the boson operators with $[X^a_m, X^b_n] = [X^a, X^b]_{m+n} + m\delta_{ab}\delta_{m+n}.\mathcal{L}$*

Corollary 5.31. *The $\mathfrak{g}$-boson algebra $\widehat{\mathfrak{g}}_+$ generates a vertex operator (super)algebra on $H = L(V_0, g)$, and also on $H = L(V_0, \ell)$ for any $\ell \in \mathbb{N}$, with $\omega = \frac{1}{2(\ell+g)}\sum_a(X^a_{-1})^2\Omega$ as Virasoro vector; and:*

$$X^a(z)X^b(w) \sim \frac{[X^a, X^b](w)}{(z-w)} + \frac{g.\delta_{ab}}{(z-w)^2}$$

$$X^a(z)L(w) \sim \frac{X^a(w)}{(z-w)^2} \quad \text{and} \quad [\mathrm{L_m}, \mathrm{X^a_n}] = -\mathrm{nX^a_{m+n}}$$

Proof. By the previous work on (S^a) and $(\star)$. □

Lemma 5.32. $c = 2\|\omega\|^2 = \frac{\ell dim(\mathfrak{g})}{\ell+g}$

Proof. $4(\ell+g)^2\|\omega\|^2 = \sum_{a,b}((X^a_{-1})^2\Omega, (X^b_{-1})^2\Omega) = \sum_{a,b}(\Omega, (X^a_1)^2(X^b_{-1})^2\Omega)$
$= \sum_{a,b}(\Omega, X^a_1X^b_{-1}[X^a_1, X^b_{-1}]\Omega + X^a_1[X^a_1, X^b_{-1}]X^b_{-1}\Omega)$
$= (\sum_{a,b,c} i\Gamma^c_{ab}(\Omega, X^a_1X^c_0X^b_{-1}\Omega)) + 2\ell\sum_a(\Omega, X^a_1X^a_{-1}\Omega)$
$= (\sum_{a,b,c,d}(-1)\Gamma^c_{ab}\Gamma^d_{cb}(\Omega, X^a_1X^d_{-1}\Omega) + 2\ell^2 dim(\mathfrak{g}))$
$= (2g\ell dim(\mathfrak{g}) + 2\ell^2 dim(\mathfrak{g})) = 2\ell dim(\mathfrak{g})(\ell+g)$ □

Remark 5.33. *By vacuum axiom of vertex operator superalgebra, $X^a_0\Omega = 0$, then, the representation $H_0 = V_\lambda$ of $\mathfrak{g}$ is necessary the trivial one V_0.*
At section 5.3, we see that general $L(V_\lambda, \ell)$ admits the structure of vertex module over $L(V_0, \ell)$.

5.2.3 $\mathfrak{g}$-supersymmetry

By lemma 5.20, the $\mathfrak{g}$-boson algebra $\widehat{\mathfrak{g}}_+$ acts on the $\mathfrak{g}$-fermion algebra $\widehat{\mathfrak{g}}_+$, then, we can build their semi-direct product:

Definition 5.34. *Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$ the $\mathfrak{g}$-supersymmetric algebra.*

Proposition 5.35. *The unitary highest weight representations (irreducible) of $\widehat{\mathfrak{g}}$ are $H = L(V_\lambda, \ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$ (see [52]).*

Proof. Let H be such a representation of $\widehat{\mathfrak{g}}$, then, $\widehat{\mathfrak{g}}_-$ acts on, but it admits a unique irreducible representation: $\mathcal{F}^{\mathfrak{g}}_{NS}$, so $H = M \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$, with M a multiplicity space. Now, $\widehat{\mathfrak{g}}_+$ acts on H and on $\mathcal{F}^{\mathfrak{g}}_{NS}$ (corollary 5.26), and the difference commutes with $\widehat{\mathfrak{g}}_-$; but $\widehat{\mathfrak{g}}_-$ acts irreducibly on $\mathcal{F}^{\mathfrak{g}}_{NS}$, so, the commutant of $\widehat{\mathfrak{g}}_-$ is $End(M) \otimes \mathbb{C}$ by Schur's lemma. So, $\widehat{\mathfrak{g}}_+$ acts on M, and this action is necessarily irreducible. Finally, by unitary highest weight context, $\exists\lambda$ such that $M = L(V_\lambda, \ell)$. □

Remark 5.36. *Using the previous notations, $\widehat{\mathfrak{g}}_+$ acts on $L(V_\lambda, \ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$ as $B^a_n = X^a_n + S^a_n$, bosons of level $d = \ell + g$.*

Corollary 5.37. *From $(\psi^a(z))$ and $(B^a(z))$, we generate $S^a(z)$ and $X^a = B^a - S^a$ a vertex operator superalgebra on $H = L(V_0, \ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$ with the Virasoro vector:*

$$\omega = \frac{1}{2}\sum_a \psi^a_{-\frac{3}{2}}\psi^a_{-\frac{1}{2}}\Omega + \frac{1}{2(\ell+g)}\sum_a (X^a_{-1})^2\Omega \quad \text{and :}$$

$$c = 2\|\omega\|^2 = \frac{dim(\mathfrak{g})}{2} + \frac{\ell dim(\mathfrak{g})}{\ell+g} = \frac{3}{2}\cdot\frac{\ell+\frac{1}{3}g}{\ell+g}dim(\mathfrak{g})$$

Definition 5.38. *(SuperVirasoro operator)*
Let $\tau_1 = \sum_a \psi^a_{-\frac{1}{2}}X^a_{-1}\Omega$, $\tau_2 = \frac{1}{3}\sum_a \psi^a_{-\frac{1}{2}}S^a_{-1}\Omega$ and $\tau = (\ell+g)^{-\frac{1}{2}}(\tau_1+\tau_2)$.
Let $G(z) = V(\tau, z) = \sum_{n\in\mathbb{Z}} G_{n-\frac{1}{2}}z^{-n-1} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} G_n z^{-n-\frac{3}{2}}$

Proposition 5.39. *(Supersymmetry boson-fermion)*

$$B^a(z)G(w) \sim d^{\frac{1}{2}} \frac{\psi^a(w)}{(z-w)^2} \quad and \quad \psi^a(z)G(w) \sim d^{-\frac{1}{2}} \frac{B^a(w)}{(z-w)}$$

$$[G_m, B^a_n] = -nd^{\frac{1}{2}} \psi^a_{m+n} \quad and \quad [G_m, \psi^a_n]_+ = d^{-\frac{1}{2}} B^a_{m+n}$$

Proof. $\psi^a_{n+\frac{1}{2}} \tau_i = 0$ for $n \geq 2$ and:

(a) $\psi^a_{\frac{1}{2}} \tau_1 = X^a_{-1}\Omega$

(b) $\psi^a_{\frac{1}{2}} \tau_2 = \frac{1}{3}(S^a_{-1}\Omega - \sum_b \psi^b_{-\frac{1}{2}} \psi^a_{\frac{1}{2}} S^b_{-1}\Omega) = \frac{1}{3}(S^a_{-1}\Omega - i\sum_{b,c} \Gamma^c_{ab} \psi^b_{-\frac{1}{2}} \psi^c_{-\frac{1}{2}}\Omega)$
$= S^a_{-1}\Omega$

(c) $\psi^a_{\frac{3}{2}} \tau_1 = \psi^a_{\frac{3}{2}} \tau_2 = 0.$

$S^a_n \tau_i, X^a_n \tau_i = 0$ for $n \geq 2$ and:

(a) $S^a_0 \tau_1 = \sum_b S^a_0 \psi^b_{-\frac{1}{2}} X^b_{-1}\Omega = i\sum_{b,c} \Gamma^c_{ab} \psi^c_{-\frac{1}{2}} X^b_{-1}\Omega$

(b) $S^a_0 \tau_2 = \frac{1}{3}\sum_b S^a_0 \psi^b_{-\frac{1}{2}} S^b_{-1}\Omega = \frac{1}{3}(i\sum_{b,c} \Gamma^c_{ab} \psi^c_{-\frac{1}{2}} S^b_{-1}\Omega + \sum_b \psi^b_{-\frac{1}{2}} S^a_0 S^b_{-1}\Omega)$
$= \frac{1}{3}(i\sum_{b,c} \Gamma^c_{ab} \psi^c_{-\frac{1}{2}} S^b_{-1}\Omega + i\sum_{b,c} \Gamma^c_{ab} \psi^b_{-\frac{1}{2}} S^c_{-1}\Omega)$
$= \frac{i}{3}\sum_{b,c} (\Gamma^c_{ab} + \Gamma^b_{ac}) \psi^c_{-\frac{1}{2}} S^b_{-1}\Omega = 0$

(c) $X^a_0 \tau_1 = \sum_b \psi^b_{-\frac{1}{2}} X^a_0 X^b_{-1}\Omega = i\sum_{b,c} \Gamma^c_{ab} \psi^b_{-\frac{1}{2}} X^c_{-1}\Omega = -S^a_0\tau_1$

(d) $X^a_0 \tau_2 = X^a_1 \tau_2 = S^a_1 \tau_1 = 0$

(e) $X^a_1 \tau_1 = \ell \psi^a_{-\frac{1}{2}}\Omega$

(f) $S^a_1 \tau_2 = \frac{1}{3}\sum_b S^a_1 \psi^b_{-\frac{1}{2}} S^b_{-1}\Omega = \frac{1}{3}(i\sum_{b,c} \Gamma^c_{ab} \psi^c_{\frac{1}{2}} S^b_{-1}\Omega + \sum_b \psi^b_{-\frac{1}{2}} S^a_1 S^b_{-1}\Omega)$
$= \frac{1}{3}(\sum_{b,c,d} \Gamma^c_{ab} \Gamma^d_{bc} \psi^d_{-\frac{1}{2}}\Omega + g\psi^a_{-\frac{1}{2}}\Omega) = g\psi^a_{-\frac{1}{2}}\Omega$

□

Remark 5.40. $G^\star_m = G_{-m}$ *(as lemma 5.22)*

Lemma 5.41. *(OPE and Lie bracket)*

$$L(z)G(w) \sim \frac{G'(w)}{(z-w)} + \frac{\frac{3}{2}G(w)}{(z-w)^2} \quad \textit{and} \quad [G_m, L_n] = (m - \frac{1}{2}n)G_{m+n}$$

Proof. $L(n)\tau = L_{n-1}\tau = 0$ for $n \geq 3$ and:

(a) $L_{-1}\tau = R(G')$ (see L_{-1} axioms and definition 4.52)

(b) $L_0\tau = \frac{3}{2}R(G)$ (see L_0 axioms)

(c) $L_1(\tau_1 + \tau_2) = \sum_a L_1\psi^a_{-\frac{1}{2}}(X^a_{-1} + \frac{1}{3}S^a_{-1})\Omega = \sum_a \psi^a_{-\frac{1}{2}}L_1(X^a_{-1} + \frac{1}{3}S^a_{-1})\Omega$
$= \sum_a \psi^a_{-\frac{1}{2}}(X^a_0 + \frac{1}{3}S^a_0)\Omega = 0$

□

Remark 5.42. $[[A,B]_+, C] = [A,[B,C]_+] + [B,[A,C]_+]$
$= [A,[B,C]]_+ + [B,[A,C]]_+$

Lemma 5.43. *(OPE and Lie bracket)*

$$G(z)G(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{(z-w)} \quad \textit{and} \quad [G_m, G_n]_+ = 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n}$$

Proof. By supersymmetry:

(a) $[[G_m, G_n]_+, B^a_r] = -2rB^a_{m+n+r} = [2L_{m+n}, B^a_r]$

(b) $[[G_m, G_n]_+, \psi^a_r] = -2(r + \frac{1}{2}(m+n))\psi^a_{m+n+r} = [2L_{m+n}, \psi^a_r]$

Then, $[[G_m, G_n]_+ - 2L_{m+n}, B^a_r] = [[G_m, G_n]_+ - 2L_{m+n}, \psi^a_r] = 0$.
Now, (B^a_r), (ψ^a_r) act irreducibly on H, so by Schur's lemma:

$$[G_m, G_n]_+ - 2L_{m+n} = k_{m,n}I$$

Now, among the $G_n\tau$, $G_{\frac{3}{2}}\tau$ is the only to give a constant term and:
$G_{\frac{3}{2}}\tau = (\ell + g)^{-1}\sum_a G_{\frac{3}{2}}\psi^a_{-\frac{1}{2}}(X^a_{-1} + \frac{1}{3}S^a_{-1})\Omega$
$= (\ell + g)^{-1}\sum_a (X^a_1 + S^a_1)(X^a_{-1} + \frac{1}{3}S^a_{-1})\Omega$
$= (\ell + g)^{-1}dim(\mathfrak{g})(\ell + \frac{1}{3}g)\Omega = \frac{2}{3}c\Omega.$
Finally, by formulas 4.26 and 4.31, $k_{m,n} = \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n}$. □

Summary 5.44.

$$\begin{cases} L(z)L(w) \sim \frac{(c/2)}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)} \\ L(z)G(w) \sim \frac{G'(w)}{(z-w)} + \frac{\frac{3}{2}G(w)}{(z-w)^2} \\ G(z)G(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{(z-w)} \end{cases}$$

and:

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] = (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ = 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

$$L_n^\star = L_{-n},\ G_m^\star = G_{-m}, \quad \textit{and } c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} dim(\mathfrak{g})$$

the SuperVirasoro algebra of sector (NS), or Neveu-Schwarz algebra $\mathfrak{Vir}_{1/2}$.

Corollary 5.45. $\mathfrak{Vir}_{\frac{1}{2}}$ *acts unitarily on* $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ *and admits* $L(c, 0)$ *as minimal submodule containing* Ω *(see definition 3.21).*

5.3 Vertex modules

Remark 5.46. *If* $\ell = 0$*, then* $\lambda = 0$ *and* $L(V_0, 0) = \mathbb{C}$ *trivial, and what we will show is ever proved by the previous section. So, we suppose* $\ell \in \mathbb{N}^\star$ *fixed.*

5.3.1 Summary

Let $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$, the vacuum representation of the $\mathfrak{g}$-supersymmetric algebra $\widehat{\mathfrak{g}}$, with $\pi : \widehat{\mathfrak{g}} \longrightarrow End(H)$.
We have construct the vertex operator superalgebra (H, Ω, ω, V) with $V : H \longrightarrow (EndH)[[z, z^{-1}]]$ the state-field correspondance map.
$\mathcal{S} = V(H)$ is generated by $(V(\psi^a_{-\frac{1}{2}}\Omega))_a$, $(V(X^b_{-1}\Omega))_b$, and $V(\mathcal{L}\Omega)$, pairwise local, with the operations, $(A, B) \mapsto A_n B$ and linear combinations.
We write $V(\psi^a_{-\frac{1}{2}}\Omega, z) = \sum_{n\in\mathbb{Z}} \pi(\psi^a_{n+\frac{1}{2}})z^{-n-1}$,
$V(X^b_{-1}\Omega, z) = \sum_{n\in\mathbb{Z}} \pi(X^b_n)z^{-n-1}$ and $V(\mathcal{L}\Omega, z) = \pi(\mathcal{L})$ $(= \ell Id_H)$.

5.3.2 Modules

Let $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ a unitary highest weight representation of $\widehat{\mathfrak{g}}$ and $\pi^\lambda : \widehat{\mathfrak{g}} \longrightarrow End(H^\lambda)$

Remark 5.47. *H^λ is itself the minimal subspace containing Ω^λ and stable by the action of $\widehat{\mathfrak{g}}$: Ω^λ is the cyclic vector of H^λ.*
On the vacuum representation, Ω is called the vacuum vector.

Lemma 5.48. $(\sum_{n\in\mathbb{Z}} \pi^\lambda(\psi^a_{n+\frac{1}{2}})z^{-n-1})_a$, $(\sum_{n\in\mathbb{Z}} \pi^\lambda(X^b_n)z^{-n-1})_b$ *and* $\pi^\lambda(\mathcal{L})$ *are pairwise local (definition 4.20).*

Proof. Let A, $B \in \widehat{\mathfrak{g}}[[z, z^{-1}]]$; π and π^λ are faithful representations of $\widehat{\mathfrak{g}}$.
Then, as formal power series, with $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{Z}_2$:
$(z-w)^N \pi(A(z))\pi(B(w))c, d)$
$= (-1)^\varepsilon (z-w)^N (\pi(B(w))\pi(A(z))c, d) \quad \forall c, d \in H$ if and only if
$(z-w)^N (\pi^\lambda(A(z))\pi^\lambda(B(w))e, f)$
$= (-1)^\varepsilon (z-w)^N (\pi^\lambda(B(w))\pi^\lambda(A(z))e, f) \quad \forall e, f \in H^\lambda$

We generate inductively an operator D decomposing H^λ into $\bigoplus H^\lambda_n$ by:
$D\Omega^\lambda = 0$, $D\psi^a_{-m}\xi = \psi^a_{-m}D\xi + m\psi^a_{-m}\xi$, $DX^b_{-n}\xi = X^b_{-n}D\xi + nX^b_{-n}\xi$, $\xi \in H^\lambda$, clearly well defined; but, $\psi^a_m : H^\lambda_p \to H^\lambda_{p-m}$ and $X^b_n : H^\lambda_p \to H^\lambda_{p-n}$, so, by lemmas 4.35, 4.36, 4.37, the result follows. □

Lemma 5.49. $D = L_0 - \frac{c_{V_\lambda}}{2(\ell+g)}$,
with c_{V_λ} the Casimir number of V_λ (see corollary 5.3)

Proof. $[L_0, \psi^a_n] = [D, \psi^a_n]$ and $[L_0, X^a_n] = [D, X^a_n]$, so, by irreducibility and Schur's lemma, $L_0 - D \in \mathbb{C}Id_{H^\lambda}$. Now, $D\Omega^\lambda = 0$ and $L_0\Omega^\lambda = h\Omega^\lambda \neq 0$ in general. Now, writing explicitly L_0 with formula 4.26, we obtain:

$$2(\ell+g)L_0\Omega^\lambda = \sum_a (X^a_0)^2\Omega^\lambda = \mathcal{C}.\Omega^\lambda = c_{V_\lambda}\Omega^\lambda$$ □

Theorem 5.50. $\mathfrak{Vir}_{\frac{1}{2}}$ *acts unitarily on* $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}^{\mathfrak{g}}_{NS}$ *and admits* $L(c,h)$ *as minimal submodule containing* Ω^λ, *with* $c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell+g} dim(\mathfrak{g})$ *and* $h = \frac{c_{V_\lambda}}{2(\ell+g)}$.

Proof. We generate $\mathcal{S}^\lambda$ from generators of previous lemma, with the operations $(A,B) \mapsto A_n B$ (now available) and linear combinations. The formula 4.26 is independant of the choice between the faithful representations π and π^λ. So, we identify $\mathcal{S}$ and $\mathcal{S}^\lambda$, which gives the isomorphism $i : \mathcal{S} \to \mathcal{S}^\lambda$; we compose it with the state-field correspondence map $V : H \to \mathcal{S}$ to give:

$$\begin{array}{cccc} V^\lambda : & H & \longrightarrow & (EndH^\lambda)[[z,z^{-1}]] \\ & a & \longmapsto & i(V(a)) \end{array} \qquad (1)$$

Then, $\sum_{n\in\mathbb{Z}} \pi^\lambda(\psi^a_{n+\frac{1}{2}}) z^{-n-1} = V^\lambda(\psi^a_{-\frac{1}{2}}\Omega, z)$,
$\sum_{n\in\mathbb{Z}} \pi^\lambda(X^b_n) z^{-n-1} = V^\lambda(X^b_{-1}\Omega, z)$ and $\pi^\lambda(\mathcal{L}) = V^\lambda(\mathcal{L}\Omega, z)$

Now, $V(a)_n V(b) = V(V(a,n)b)$ $\forall a,b \in H$, so, by construction:

$$V^\lambda(a)_n V^\lambda(b) = V^\lambda(V(a,n)b) \qquad (2)$$

Then, $V^\lambda(\omega, z) = \sum L_n z^{-n-2}$, $V^\lambda(\tau, z) = \sum G_{m-\frac{1}{2}} z^{-m-1}$, $L^\star_n = L_{-n}$ and $G^\star_m = G_{-m}$, with (L_n), (G_m) verifying superVirasoro relations. (3) $\square$

Remark 5.51. $[L_m, \psi^a_n] = -(n + \frac{1}{2}m)\psi^a_{m+n}$ *and* $[L_m, X^a_n] = -nX^a_{m+n}$, *so:*

$$\left\{ \begin{array}{l} [L_{-1}, V^\lambda(a,z)] = (V^\lambda)'(a,z) \\ [L_0, V^\lambda(a,z)] = z.(V^\lambda)'(a,z) + rV^\lambda(a,z) \quad (a \in H_r) \end{array} \right. \qquad (4)$$

Remark 5.52. $V^\lambda(\Omega, z) = Id_{H^\lambda}$ *because* π *and* π^λ *are at same level* ℓ. (5)

Definition 5.53. *By* (1)...(5), (H^λ, V^λ) *is called a* ***vertex module*** *of* (H, V, Ω, ω).

We now apply the theorem 5.50 to GKO construction with $\mathfrak{g} = \mathfrak{sl}_2$.

6 Goddard-Kent-Olive framework

6.1 Characters of $L\mathfrak{g}$-modules

In this section, we take $\mathfrak{g} = \mathfrak{sl}_2$. Let H a unitary, projective and positive energy representation of the loop algebra $L\mathfrak{g}$ (see section 5.1.2).

Remark 6.1. *Thanks to* $\mathfrak{g} \hookrightarrow L\mathfrak{g}: \ X_a \mapsto X_0^a$, $\mathfrak{g}$ *acts on* H, *and by the previous work, the Virasoro algebra* $\mathfrak{Vir}$ *acts on too:* $[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}m(m^2-1)\delta_{m+n} \qquad (n \in \mathbb{Z},\ C$ *central*).

Definition 6.2. *A character of* H *as* $L\mathfrak{g}$*-module is definied by:*

$$ch(H)(t,z) = tr(t^{L_0 - \frac{C}{24}} z^{X_3})$$

Lemma 6.3. *(Jacobi's triple product identity)*

$$\sum_{k\in\mathbb{Z}} t^{\frac{1}{2}k^2} z^k = \prod_{n\in\mathbb{N}^\star} (1+t^{n-\frac{1}{2}}z)(1+t^{n-\frac{1}{2}}z^{-1})(1-t^n)$$

Proof. See [100] p 62. □

Remark 6.4. *At the section 5.2.1,* $L\mathfrak{g}$ *acts on* $\mathcal{F}_{NS}^{\mathfrak{g}}$, *with* $\pi_{\mathcal{F}_{NS}^{\mathfrak{g}}}(X_3) = S_0^3$.

Proposition 6.5. $\quad ch(\mathcal{F}_{NS}^{\mathfrak{g}})(t,z) = t^{-1/16}\chi_{NS}(t)\theta(t,z) \quad$ *with*

$$\chi_{NS}(t) = \prod_{n\in\mathbb{N}^\star} (\frac{1+t^{n-\frac{1}{2}}}{1-t^n}) \quad \text{and} \quad \theta(t,z) = \sum_{k\in\mathbb{Z}} t^{\frac{1}{2}k^2} z^k$$

Proof. C acts as multiplicative constant $c_{\mathcal{F}_{NS}^{\mathfrak{g}}} = \frac{dim(\mathfrak{g})}{2} = \frac{3}{2}$, so, $-\frac{c}{24} = -1/16$ $[S_m^a, \psi_n^b] = i\sum_c \Gamma_{ab}^c \psi_{m+n}^c$, so, $[S_0^3, \psi_n^3] = 0$, $[S_0^3, \psi_n^1] = i\psi_n^2$, $[S_0^3, \psi_n^2] = -i\psi_n^1$. Let $\varphi_n^3 = \psi_n^3$, $\varphi_n^1 = i\psi_n^1 - \psi_n^2$, $\varphi_n^2 = \psi_n^1 - i\psi_n^2$, then, $[S_0^3, \varphi_n^3] = 0$, $[S_0^3, \varphi_n^1] = \varphi_n^1$ and $[S_0^3, \varphi_n^2] = -\varphi_n^2$. Now, if $M = PDP^{-1}$, then, $tr(M) = tr(D)$ and $tr(z^M) = tr(z^D)$, but, $ad_{S_0^3}$ acts diagonally on $\widehat{\mathfrak{g}}_-$ with basis (φ_n^i), $[L_0, \varphi_m^i] = -m\varphi_m^i$, and $S_0^3\Omega = 0$, so, it suffices to associate: $t^{n-\frac{1}{2}}$ to $\varphi_{-n+\frac{1}{2}}^3$, $t^{n-\frac{1}{2}}z$ to $\varphi_{-n+\frac{1}{2}}^1$, and $t^{n-\frac{1}{2}}z^{-1}$ to $\varphi_{-n+\frac{1}{2}}^2$ to find:

$$ch(\mathcal{F}_{NS}^{\mathfrak{g}})(t,z) = t^{-1/16} \prod_{n\in\mathbb{N}^\star} (1+t^{n-\frac{1}{2}})(1+t^{n-\frac{1}{2}}z)(1+t^{n-\frac{1}{2}}z^{-1})$$

The result follows by the Jacobi's triple product identity. □

Definition 6.6. *Let $m \in \mathbb{N}^\star$, $n \in \mathbb{Z}$, $t, z \in \mathbb{C}$ with $\|t\| < 1$.*
Let the theta functions:

$$\theta_{n,m}(t,z) = \sum_{k \in \frac{n}{2m} + \mathbb{Z}} t^{mk^2} z^{mk}$$

Theorem 6.7. *Let $H = L(j,\ell)$, irreducible representation of $L\mathfrak{sl}_2$, then*

$$ch(L(j,\ell)) = \frac{\theta_{2j+1,\ell+2} - \theta_{-2j-1,\ell+2}}{\theta_{1,2} - \theta_{-1,2}}$$

Proof. An application of the Weyl-Kac character formula to $L\mathfrak{sl}_2$ (see [49], [56] or [100] p 62). □

Proposition 6.8. *(Product formula)*

$$\theta(t,z).\theta_{p,m}(t,z) = \sum_{\substack{0 \le q < 2(m+2) \\ p \equiv q[2]}} \Big(\sum_{n \in \mathbb{Z}} t^{\alpha^m_{pq}(n)}\Big)\theta_{q,m+2}(t,z)$$

$$\textit{with} \qquad \alpha^m_{p,q}(n) = \frac{[2m(m+2)n - (m+2)p + mq]^2}{8m(m+2)}$$

Proof. We adapt the proof in [51] or [54] p 122, to the super case:

$$\theta(t,z).\theta_{p,m}(t,z) = \sum_{k,k'} t^{\frac{1}{2}k^2 + mk'^2} z^{k+mk'}$$

Let $k = i$, $k' = \frac{p}{2m} + i'$ where i, $i' \in \mathbb{Z}$; we define s, s' by:

- $(m+2)s = k - 2k' = i - 2i' - \frac{p}{m}$
- $(m+2)s' = k + mk' = (m+2)(k' + s)$

Now, $p + 2(i - 2i') = 2(m+2)n + q$ with $0 \le q < 2(m+2)$, $p \equiv q[2]$, then:

$$s = n - \frac{(m+2)p - mq}{2m(m+2)} \quad \text{and} \quad s' = n' + \frac{q}{2(m+2)} \quad n, n' \in \mathbb{Z} \quad (\text{with} \quad n' = n + i').$$

This gives a bijection between pairs (k, k') and triples (q, s, s').
Now, $\frac{1}{2}k^2 + mk'^2 = \frac{1}{2}(ms + 2s')^2 + m(s - s')^2 = \frac{1}{2}m(m+2)s^2 + (m+2)s'^2$
and $\frac{1}{2}m(m+2)s^2 = \frac{1}{2}m(m+2)(n - \frac{(m+2)p - mq}{2m(m+2)})^2 = \alpha^m_{p,q}(n)$ □

Remark 6.9. *By section 5.2.3, $L\mathfrak{g}$ acts on $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(j,\ell)$ as unitary, projective, positive energy representation of level $\ell+2$ (see definition 5.36).*

Corollary 6.10. *Let $p = 2j+1$, $q = 2k+1$ and $m = \ell+2$, then:*

$$ch(\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(j,\ell)) = \sum_{\substack{1 \le q \le m+1 \\ p \equiv q[2]}} F_{pq}^{m}.ch(L(k,\ell+2))$$

$$\textit{with} \qquad F_{pq}^{m}(t) = t^{-1/16}\chi_{NS}(t)\sum_{n\in\mathbb{Z}}(t^{\alpha_{p,q}^{m}(n)} - t^{\alpha_{-p,q}^{m}(n)})$$

We apply theorem 6.7, propositions 6.5 and

Proof. $L\mathfrak{g}$ acts on H as $(I \otimes X + X \otimes I)$, then:
$ch(\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(j,\ell)) = ch(\mathcal{F}_{NS}^{\mathfrak{g}}).ch(L(j,\ell))$; now by proposition 6.8:

$$\theta(t,z).(\theta_{p,m}(t,z) - \theta_{-p,m}(t,z)) = \sum_{\substack{0 \le q < 2(m+2) \\ p \equiv q[2]}} (\sum_{n\in\mathbb{Z}} t^{\alpha_{pq}^{m}(n)} - t^{\alpha_{-p,q}^{m}(n)})\theta_{q,m+2}(t,z)$$

But for $m+2 \le q' < 2(m+2)$, $q' = 2(m+2)-q$ with $1 \le q \le m+2$. Now by symmetry, $\theta_{2(m+2)-q,m+2} = \theta_{-q,m+2}$, and $F_{p,2(m+2)-q}^{m} = -F_{pq}^{m}$ because $\alpha_{p,2(m+2)-q}^{m}(n) = \alpha_{-p,q}^{m}(-n-1)$. Finally, $F_{p0}^{m} = F_{p,m+2}^{m} = 0$ because $\alpha_{p,0}^{m}(n) = \alpha_{-p,0}^{m}(-n)$ and $\alpha_{p,m+2}^{m}(n) = \alpha_{-p,m+2}^{m}(-n-1)$; the result follows. ☐

Corollary 6.11. *(Tensor product decomposition)*

$$\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(j,\ell) = \bigoplus_{\substack{1 \le q \le m+1 \\ p \equiv q[2]}} M_{pq}^{m} \otimes L(k,\ell+2)$$

with M_{pq}^{m} the multiplicity space.

Proof. By complete reducibility and remark 6.9, $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(j,\ell)$ is a direct sum of irreducibles of type $L(k,\ell+2)$; the result follows by corollary 6.10. ☐

Corollary 6.12. *As $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{+} \ltimes \hat{\mathfrak{g}}_{-}$ representations, we obtain;*

$$\mathcal{F}_{NS}^{\mathfrak{g}} \otimes (L(j,\ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}) = \bigoplus_{\substack{1 \le q \le m+1 \\ p \equiv q[2]}} M_{pq}^{m} \otimes (L(k,\ell+2) \otimes \mathcal{F}_{NS}^{\mathfrak{g}})$$

Proof. Recall proposition 5.35 and remark 5.36.
Next, the characters of $\hat{\mathfrak{g}}$-modules are defined as for $\hat{\mathfrak{g}}_{+}$-modules. ☐

6.2 Coset construction

6.2.1 General framework

Let $\mathfrak{h}$ be a Lie $\star$-superalgebra acting unitarily on an inner product space H, a direct sum of irreducibles of finitely many isomorphic type H_i:

$$H = \bigoplus_i M_i \otimes H_i \quad \text{with } M_i \text{ the multiplicity space.}$$

Remark 6.13. *$\mathfrak{h}$ acts on H as $\pi(X) = \sum I \otimes \pi_i(X)$.*

Definition 6.14. *Let $K_i = Hom_{\mathfrak{h}}(H_i, H)$,
space of homomorphisms that supercommute with $\mathfrak{h}$ (graded intertwinners).*

Recall 6.15. *$Hom_{\mathfrak{h}}(H_i, H_j) = \delta_{ij}\mathbb{C}$, $End_{\mathfrak{h}}(H) = \bigoplus End(M_i) \otimes \mathbb{C}$.*

Lemma 6.16. *K_i admits a natural inner product.*

Proof. If $S, T \in K_i$, then $T^\star S \in End_{\mathfrak{h}}(H_i) = \mathbb{C}$, and so, $(S,T) = T^\star S$ defines the inner product. □

Lemma 6.17. *$\rho : \bigoplus K_i \otimes H_i \to H$ such that: $\rho(\sum \xi_i \otimes \eta_i) = \sum \xi_i(\eta_i)$,
is a unitary isomorphism of $\mathfrak{h}$-modules.*

Proof. Let $\sum m_i \otimes \eta_i \in H$ and $\xi_i : \eta_i \mapsto m_i \otimes \eta_i$, then $\xi_i \in K_i$, because $\mathfrak{h}$ acts on H as $\sum I \otimes \pi_i$; and so, ρ is surjective.
Now, $(\rho(\sum \xi_i' \otimes \eta_i'), \rho(\sum \xi_j \otimes \eta_j)) = \sum(\xi_i'(\eta_i'), \xi_j(\eta_j)) =$
$\sum(\xi_j^\star \xi_i'(\eta_i'), \eta_j) = \sum(\xi_j^\star, \xi_i')(\eta_i', \eta_j) = (\sum \xi_i' \otimes \eta_i', \sum \xi_j \otimes \eta_j)$ □

Remark 6.18. *An operator A on H which supercommutes with $\mathfrak{h}$, acts by definition, on each K_i by an A_i, and, identifying M_i and K_i, $A = \sum A_i \otimes I$*

Let $\mathfrak{d}$ be a Lie $\star$-superalgebra acting as $\pi(D)$ on H, and as $\pi_i(D)$ on H_i.

Corollary 6.19. *If $\forall D \in \mathfrak{d}$, $\sigma(D) = \pi(D) - \sum I \otimes \pi_i(D)$ supercommutes with $\mathfrak{h}$, then $\mathfrak{d}$ acts on M_i as $\sigma_i(D)$ with $\sigma(D) = \sum \sigma_i(D) \otimes I$.*

Definition 6.20. *Let $B_F(D_1, D_2) := [\pi_F(D_1), \pi_F(D_2)] - \pi_F[D_1, D_2]$.*

Remark 6.21. *If F is unitary, projective and positive energy (see definition 3.5), the cocycle b_F is defined by $B_F(D_1, D_2) = b_F(D_1, D_2) I_F$.*

Proposition 6.22. *If in addition to corollary 6.19, π and π_i are unitary, projective, positive energy representations, then, so is σ_i, and the cocycle of $\mathfrak{d}$ on M_i is the difference of the cocycles on H and on H_i.*

Proof. $\pi = \sum(I \otimes \pi_i + \sigma_i \otimes I)$ and $B_H = \sum(I \otimes B_{H_i} + B_{M_i} \otimes I)$.
$M_i \otimes H_i \subset H$, so, $b_H I = b_{M_i \otimes H_i} I = I \otimes B_{H_i} + B_{M_i} \otimes I$.
Finally, $B_{M_i} \otimes I = b_H I - I \otimes B_{H_i} = (b_H - b_{H_i}) I \otimes I$ □

6.2.2 Application

We apply the previous result to corollary 6.12 with $\mathfrak{h} = \hat{\mathfrak{g}}$ and $\mathfrak{d} = \mathfrak{W}_{1/2}$.

Convention 6.23. *To have a graded Lie bracket coherent with tensor product, we need to introduce the following convention: let A, B be superalgebras, then, the product on $A \otimes B$ is defined as follows:*

$$(a \otimes b).(c \otimes d) = (-1)^{\varepsilon(b)\varepsilon(c)} ac \otimes bd \quad \textit{with } \varepsilon(b) \ , \varepsilon(c) \in \mathbb{Z}_2$$

Lemma 6.24. *Let $\mathfrak{t}$ be a Lie superalgebra, then, by the previous convention:*

$$[X \otimes I + I \otimes X, Y \otimes I + I \otimes Y]_\varepsilon = [X, Y]_\varepsilon \otimes I + I \otimes [X, Y]_\varepsilon$$

Corollary 6.25. *The Witt superalgebra $\mathfrak{W}_{1/2}$ acts on the multiplicity space M_{pq}^m as unitary, projective and positive energy representation, with central charge,*

$$c_{M_{pq}^m} = \frac{dim(\mathfrak{g})}{2}(1 - \frac{2g^2}{(\ell + g)(\ell + 2g)}) = \frac{3}{2}(1 - \frac{8}{m(m+2)})$$

$m = \ell + 2$, $g = 2$ and $dim(\mathfrak{g}) = 3$.

Proof. $\mathfrak{W}_{1/2}$ acts as $\sum I \otimes X$ on $\bigoplus M_{pq}^m \otimes (L(k, \ell+2) \otimes \mathcal{F}_{NS}^{\mathfrak{g}})$, as $X \otimes I + I \otimes X$ on $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes (L(j, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}))$, it's projective thanks to lemma 6.24, unitary, positive energy, and their difference supercommutes with $\hat{\mathfrak{g}}$ by proposition 5.39. Now by proposition 6.22:

$$c_{M_{pq}^m} = c_{\mathcal{F}_{NS}^{\mathfrak{g}} \otimes (L(j,\ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}})} - (c_{L(k,\ell+2) \otimes \mathcal{F}_{NS}^{\mathfrak{g}})}) =$$
$$c_{\mathcal{F}_{NS}^{\mathfrak{g}}} + c_{L(j,\ell)} + c_{\mathcal{F}_{NS}^{\mathfrak{g}}} - (c_{L(k,\ell+2)} + c_{\mathcal{F}_{NS}^{\mathfrak{g}}}) = \tfrac{3}{2} \cdot \tfrac{\ell + \frac{1}{3}g}{\ell + g} dim(\mathfrak{g}) - \tfrac{\ell + g}{\ell + 2g} dim(\mathfrak{g})$$

□

Remark 6.26. *Let $\hat{\mathfrak{g}} \subset \hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ be the diagonal inclusion, then the previous construction is equivalent to the Kac-Todorov one [52]: the coset action of $\mathfrak{Vir}_{1/2}$ is given by $L_n^{\hat{\mathfrak{g}}\oplus\hat{\mathfrak{g}}} - L_n^{\hat{\mathfrak{g}}}$ and $G_r^{\hat{\mathfrak{g}}\oplus\hat{\mathfrak{g}}} - G_r^{\hat{\mathfrak{g}}}$. There exists also an manner to write this action only with ordinary loop algebra, due to Goddard, Kent, Olive [35] (used and discussed in section 10.7).*

6.3 Character of the multiplicity space

Definition 6.27. *$\mathfrak{Vir}_{1/2}$-module's character is $ch(H)(t) = tr(t^{L_0 - \frac{C}{24}})$.*

Corollary 6.28. *(Character of the multiplicity space)*

$$ch(M_{pq}^m)(t) = t^{-\frac{c(m)}{24}}.\chi_{NS}(t).\Gamma_{pq}^m(t) \quad with$$

$$\Gamma_{pq}^m(t) = \sum_{n\in\mathbb{Z}}(t^{\gamma_{pq}^m(n)} - t^{\gamma_{-pq}^m(n)}), \quad \chi_{NS}(t) = \prod_{n\in\mathbb{N}^\star}\frac{1+t^{n-1/2}}{1-t^n} \quad and$$

$$\gamma_{pq}^m(n) = \frac{[2m(m+2)n-(m+2)p+mq]^2-4}{8m(m+2)}$$

Proof. It follows by corollaries 6.10, 6.11, and, $\gamma_{pq}^m(n) = \alpha_{pq}^m(n) - \frac{1}{16} + \frac{c_m}{24}$. □

Lemma 6.29. *The lowest eigenvalue of L_0 on M_{pq}^m is:*

$$h = h_{pq}^m = \frac{[(m+2)p-mq]^2-4}{8m(m+2)}$$

Proof. $\chi_{NS}(t) \sim 1 + t^{\frac{1}{2}}$ and $min\{\gamma_{pq}^m(n), \gamma_{-pq}^m(n), n\in\mathbb{Z}\} = \gamma_{pq}^m(0) = h_{p,q}^m$ □

Lemma 6.30. *Let $(p',q') = (m-p, m+2-q)$, then:*

$$ch(M_{pq}^m) \sim t^{-\frac{c_m}{24}}.\chi_{NS}(t).t^{h_{pq}^m}.(1 - t^{\frac{pq}{2}} - t^{\frac{p'q'}{2}})$$

Proof. $\gamma_{-pq}^m(0) = \gamma_{pq}^m(0) + \frac{pq}{2}$, $\gamma_{-pq}^m(-1) = \gamma_{pq}^m(0) + \frac{p'q'}{2}$; and, $\gamma_{pq}^m(0)$, $\gamma_{-pq}^m(0)$, $\gamma_{-pq}^m(-1)$ are the three lowest numbers of $\{\gamma_{pq}^m(n), \gamma_{-pq}^m(n), n\in\mathbb{Z}\}$. □

Corollary 6.31. *$L(c_m, h_{pq}^m)$ is a $\mathfrak{Vir}_{1/2}$-submodule of M_{pq}^m*

Proof. $ch(M_{pq}^m).t^{\frac{c_m}{24}} \sim t^{h_{pq}^m}$, then, the h_{pq}^m-eigenspace of L_0 is one-dimensional; $L(c(m), h_{pq}^m)$ is the minimal $\mathfrak{Vir}_{1/2}$-submodule of M_{pq}^m containing it. □

Corollary 6.32. $ch(L(c_m, h_{pq}^m)) \leq ch(M_{pq}^m) \sim t^{h_{pq}^m - \frac{c_m}{24}}.\chi_{NS}(t)(1 - t^{\frac{pq}{2}} - t^{\frac{p'q'}{2}})$

Theorem 6.33. *(Unitarity sufficient condition)*
Let integers $m \geq 2$, $1 \leq p \leq m-1$, $1 \leq q \leq m+1$ *and* $p \equiv q[2]$*, then:* $L(c_m, h_{pq}^m)$ *is a unitary highest weight representation of* $\mathfrak{Vir}_{1/2}$

Proof. Recall definitions 3.5 and 3.21.
M_{pq}^m is unitary; so is its $\mathfrak{Vir}_{1/2}$-submodule $L(c_m, h_{pq}^m)$. □

Remark 6.34. *FQS criterion proves this is all its discrete series.*

7 Kac determinant formula

7.1 Preliminaries

Let $c,\ h \in \mathbb{C}$, recall section 3.3 for definitions of Verma module $V(c,h)$, sesquilinear form $(.,.)$ and maximal proper submodule $K(c,h)$.
Let $(c,h) = (c_m, h^m_{pq}) = (\frac{3}{2}(1-\frac{8}{m(m+2)}), \frac{[(m+2)p-mq]^2-4}{8m(m+2)})$.

Lemma 7.1. $h^m_{pq} + h^m_{qp} = \frac{p^2+q^2-2}{16}(1-2c_m/3) + \frac{(p-q)^2}{4}$ *and* $h^m_{pq}.h^m_{qp} = \frac{1}{16^2}[2(p-q)^2-(1-2c_m/3)(pq-p-q-1)].[2(p-q)^2-(1-2c_m/3)(pq+p+q+1)]$

Then, solving the system of the lemma, we can define h^c_{pq}, $\forall c \in \mathbb{C}$.

Definition 7.2. $\varphi_{pp}(c,h) = (h - h^c_{pp})$ *and*
$\varphi_{pq}(c,h) = (h-h^c_{pq})(h-h^c_{qp})$ *if* $p \neq q$

Lemma 7.3. $\varphi_{pq} \in \mathbb{C}[c,h]$ *is irreducible.*

Definition 7.4. *Let* $V_n(c,h)$ *the* n*-eigenspace of* $D = L_0 - hId$ *generated by the vectors* $G_{-j_\beta}\dots G_{-j_1}L_{-i_\alpha}\dots L_{-i_1}\Omega$ *such that* $\sum i_s + \sum j_s = n$, *with* $0 < i_1 \leq \dots \leq i_\alpha$, $\frac{1}{2} \leq j_1 < \dots < j_\beta$; *let* $d(n)$ *its dimension.*

Remark 7.5. $d(n) < \infty$, $d(n) = 0$ *for* $n < 0$.
Clearly $(V_n(c,h), V_{n'}(c,h)) = 0$ *if* $n \neq n'$ *and* $V(c,h) = \bigoplus V_n(c,h)$.

Definition 7.6. *Let* $M_n(c,h)$ *the matrix of* $(.,.)$ *on* $V_n(c,h)$
and $det_n(c,h) = det(M_n(c,h))$

Examples 7.7. $M_0(c,h) = (\Omega,\Omega) = (1)$, $M_{\frac{1}{2}}(c,h) = (G_{-\frac{1}{2}}\Omega, G_{-\frac{1}{2}}\Omega) = (2h)$, $M_1(c,h) = (L_{-1}\Omega, L_{-1}\Omega) = (2h)$, *and,* $M_{\frac{3}{2}}(c,h) =$

$$\begin{pmatrix} (G_{-\frac{1}{2}}L_{-1}\Omega, G_{-\frac{1}{2}}L_{-1}\Omega) & (G_{-\frac{1}{2}}L_{-1}\Omega, G_{-\frac{3}{2}}\Omega) \\ (G_{-\frac{3}{2}}\Omega, G_{-\frac{1}{2}}L_{-1}\Omega) & (G_{-\frac{3}{2}}\Omega, G_{-\frac{3}{2}}\Omega) \end{pmatrix} = \begin{pmatrix} 2h+4h^2 & 4h \\ 4h & 2h+\frac{2}{3}c \end{pmatrix}$$

Remark 7.8. $det_{\frac{3}{2}}(c_m,h) = 8h[h^2 - (\frac{3}{2}-\frac{c_m}{3})h + c/6] = 8h(h-h^m_{13})(h-h^m_{31})$, *then,* $det_{\frac{3}{2}}(c,h) = 8h(h-h^c_{13})(h-h^c_{31}) = 8\varphi_{11}(c,h).\varphi_{13}(c,h)$ $\forall c \in \mathbb{C}$

Theorem 7.9. *(Kac determinant formula)*

$$det_n(c,h) = A_n \prod_{\substack{0<pq/2\leq n \\ p\equiv q[2]}} (h-h^c_{pq})^{d(n-pq/2)} = A_n \prod_{\substack{0<pq/2\leq n \\ p\leq q,\ p\equiv q[2]}} \varphi_{pq}^{d(n-pq/2)}(c,h)$$

with $A_n > 0$ *independent of* c *and* h.

7.2 Singulars vectors and characters

Definition 7.10. *A vector* $s \in V(c,h)$ *is singular if:*

(a) $L_0.s = (h+n)s \quad$ *with* $n > 0$ *(its level)*

(b) $\mathfrak{Vir}^+_{1/2}.s = 0 \qquad$ *(recall definition 3.13)*

Remark 7.11. *Let* $n > 0$*,* $s \in V_n(c,h)$ *is singular iff* $G_{1/2}.s = G_{3/2}.s = 0$

Examples 7.12. $(mG_{-3/2} - (m+2)L_{-1}G_{-1/2})\Omega \in V_{3/2}(c_m, h^m_{13})$,
$G_{-1/2}\Omega \in V_{1/2}(c, h^c_{11})$, $\;(L^2_{-1} - \frac{4}{3}h^c_{22}L_{-2} - G_{-3/2}G_{-1/2})\Omega \in V_2(c, h^c_{22})$

Definition 7.13. $K_n(c,h) = ker(M_n(c,h)) = \{x \in V_n(c,h); (x,y) = 0 \ \forall y\}$

Proposition 7.14. *The singular vectors generate* $K(c,h)$*.*

Proof. They clearly generate a subspace of $K(c,h)$. Now, let $v \in K_n(c,h)$, then $\mathfrak{Vir}^+_{1/2}.v$ is of level $< n$ and $\exists n'$ such that $(\mathfrak{Vir}^+_{1/2})^{n'+1}.v = \{0\}$ and $(\mathfrak{Vir}^+_{1/2})^{n'}.v \neq \{0\}$ and contains a singular vector generating v. □

Definition 7.15. *Let* $V^s(c,h)$ *the minimal* $\mathfrak{Vir}_{1/2}$*-submodule of* $V(c,h)$ *containing* s *and* $V^s_n(c,h) = V^s(c,h) \cap V_n(c,h)$*.*

Lemma 7.16. *Let* s *singular of level* n'*, then* $dim(V^s_n(c,h)) = d(n-n')$*.*

Proof. $D.(A.s) = nA.s \iff D.(A\Omega) = (n-n')A\Omega$ □

Lemma 7.17. $ch(V(c,h)) = t^{h-\frac{c}{24}}\chi_{NS}(t)$

Proof. $ch(V(c,h)) = tr(t^{L_0 - \frac{c}{24}}) = t^{h-\frac{c}{24}} \sum_{m \in \frac{1}{2}\mathbb{N}} d(m)t^m$
$\chi_{NS}(t) = \prod_{n\in\mathbb{N}^\star}(\frac{1+q^{n-\frac{1}{2}}}{1-q^n}) = \prod_{n\in\mathbb{N}^\star}(1+q^{n-\frac{1}{2}})(1+q^n+q^{2n}+...)$
Identifying $q^{n-\frac{1}{2}}$ to $G_{n-\frac{1}{2}}$, q^n to L_n, the coefficient of q^m is exactly $d(m)$. □

Corollary 7.18. $ch(V^s(c,h)) = t^{n+h-\frac{c}{24}}\chi_{NS}(t)$*, with* n *the level of* s*.*

Remark 7.19. $dim(L_n(c,h)) = dim(V_n(c,h)) - dim(K_n(c,h))$*, then,*
$ch(L(c,h)) = ch(V(c,h)) - \sum_s ch(V^s(c,h)) + \sum_{s,s'} ch(V^s \cap V^{s'}) - \dots$

Corollary 7.20. $V(c,h)$ *admits a singular vector* s *of minimal level* n *if and only if* $ch(L(c,h)) \sim t^{h-\frac{c}{24}}\chi_{NS}(t)(1-t^n)$

7.3 Proof of the theorem

Proposition 7.21. *For a fixed c, det_n is polynomial in h of degree*

$$M = \sum_{\substack{0<pq/2\leq n \\ p\equiv q[2]}} d(n - pq/2)$$

Proof. It's clear that only the product of the diagonal entries of $M_n(h, c)$ gives a non-zero contribution to the highest power of h (and that its coefficient is > 0 and independent of c); and that M is the sum of possibles $\sum m_i + \sum n_j$ such that $\sum im_i + \sum jn_j = n$ with $i \in \mathbb{N} + \frac{1}{2}$, $j \in \mathbb{N}$, $m_i \in \{0, 1\}$, $n_j \in \mathbb{N}$.
Let $m_n(p, q)$ be the number of such partitions of n, in which $p/2$ appears exactly q times; then, $M = \sum_{0<pq/2\leq n} q.m_n(p, q)$.
Now, if $p \equiv 0[2]$, the number of such partitions in which $p/2$ appears $\geq q$ times is $d(n - pq/2)$; so, $m_n(p, q) = d(n - pq/2) - d(n - p(q + 1)/2)$.
If $p \equiv 1[2]$, then, $m_n(p, q) = 0$ if $q > 1$ and $m_n(p, 1) = d(n - p/2) - m_{n-p/2}(p, 1)$; so, by induction, $m_n(p, 1) = \sum_q (-1)^{q+1} d(n - pq/2)$, where $d(0) = 1$ and $d(k) = 0$ if $k < 0$. Now:

$$M = \sum_{\substack{0<pq/2\leq n \\ p\equiv 0[2]}} q.m_n(p, q) + \sum_{\substack{0<p/2\leq n \\ p\equiv 1[2]}} m_n(p, 1)$$

$$= \sum_{\substack{0<pq/2\leq n \\ p\equiv 0[2]}} q.(d(n-pq/2)-d(n-p(q+1)/2)) + \sum_{\substack{0<p/2\leq n \\ p\equiv 1[2]}} (\sum_q (-1)^{q+1} d(n-pq/2))$$

$$= \sum_{\substack{0<pq/2\leq n \\ p\equiv 0[2]}} d(n - pq/2) + \sum_{\substack{0<pq/2\leq n \\ p\equiv 1[2]}} (-1)^{q+1} d(n - pq/2)$$

Finally, the (p, q) term with $q \equiv 1[2]$ of the first sum, vanishes with the $(p', q') = (q, p)$ term of the second, so the result follows. □

Lemma 7.22. *If $t \mapsto A(t)$ is a polynomial mapping into $d \times d$ matrices and $dim(ker A(t_0)) = k$, then $(t - t_0)^k$ divides $det(A(t))$.*

Proof. Take a basis v_i such that $A(t_0)v_i = 0$ for $i = 1 \ldots k$.
Thus, $(t-t_0)$ divides $A(t)v_i$ for $i = 1 \ldots k$, and $(t-t_0)^k$ divides $det(A(t))$. □

Lemma 7.23. *Consider $det_n(c,h)$ as polynomial in h for c fixed. If n' is minimal such that $det_{n'}$ vanishes at $h = h_0$, then $(h-h_0)^{d(n-n')}$ divides det_n.*

Proof. Clearly $V(c,h_0)$ admits a singular vector s of level n'.
Now, $V_n^s(c,h_0)$ is $d(n-n')$ dimensional, and is contained in $ker(M_n(c,h_0))$.
So, the result follows by previous lemma. □

Lemma 7.24. *det_n vanishes at h_{pq}^c, for $0 < pq/2 \leq n$, $p \equiv q[2]$.*

Proof. Let $m \geq 2$ integer, $1 \leq p \leq m-1$, $1 \leq q \leq m+1$, $p \equiv q[2]$.
Thanks to GKO construction, we have corollary 6.32:

$$ch(L(c_m, h_{pq}^m)) \leq ch(M_{pq}^m) \sim t^{-\frac{c_m}{24}}.\chi_{NS}(t).t^{h_{pq}^m}.(1 - t^{\frac{pq}{2}} - t^{\frac{p'q'}{2}})$$

So, $V(c_m, h_{pq}^m)$ admits a singular vector at level $\leq min(pq/2, p'q'/2)$ by corollary 7.20, and then, $dim(ker(M_n(c_m, h_{pq}^m))) > 0$ for $n \geq pq/2$. Hence, det_n vanishes at h_{pq}^m for m sufficiently large integer. But then, det_n vanishes at infinite many zeros of the irreducible φ_{pq}, which so, divides det_n. □

Proof of the theorem 7.9 By lemma 7.23 and 7.24, det_n is divisible by $d_n(c,h) = \prod_{\substack{0<pq/2\leq n \\ p\equiv q[2]}} (h - h_{pq}^c)^{d(n-pq/2)}$ since the h_{pq}^c are distincts for generic c.
Now, by proposition 7.21, det_n and d_n have the same degree M, and the coefficient of h^M is > 0 and independent of c, h. So, the result follows. □

8 Friedan-Qiu-Shenker unitarity criterion

8.1 Introduction

Recall section 3.3 for definitions of Verma module $V(c,h)$, sesquilinear form $(.,.)$ and ghost. The goal of this section is to give a proof of the FQS theorem for the Neveu-Schwarz algebra, in a parallel way that [28] give for the Virasoro algebra, expoiting Kac determinant formula:

$$det_n(c,h) = A_n \prod_{\substack{0<pq/2\leq n \\ p\equiv q[2]}} (h - h^c_{pq})^{d(n-pq/2)}$$

with $A_n > 0$ independent of c and h.

Lemma 8.1. *If $V(c,h)$ admits no ghost then $c, h \geq 0$*

Proof. Since $L_n L_{-n}\Omega = L_{-n}L_n\Omega + 2nh\Omega + c\frac{n(n^2-1)}{12}\Omega$,
we have $(L_{-n}\Omega, L_{-n}\Omega) = 2nh + \frac{n(n^2-1)}{12}c \geq 0$.
Now, taking n first equal to 1 and then very large, we obtain the lemma. $\square$

Proposition 8.2. *If $h \geq 0$ and $c \geq 3/2$ then $V(c,h)$ admits no ghost.*

Now, it suffices to classify no ghost cases for $h \geq 0$ and $0 \leq c < 3/2$.

Lemma 8.3. *$m \mapsto c_m$ is an inscreasing bijection from $[2, +\infty[$ to $[0, 3/2[$.*

The FQS theorem gives as necessary condition exactly the same discrete series that GKO construction gives as sufficient condition (theorem 6.33):

Theorem 8.4. *(FQS unitary criterion)*
Let $h \geq 0$ and $0 \leq c < 3/2$; $V(c,h)$ admits ghost if (c,h) does not belong to:

$$c = c_m = \frac{3}{2}(1 - \frac{8}{m(m+2)}), \quad h = h^m_{p,q} = \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)}$$

with integers $m \geq 2$, $1 \leq p \leq m-1$, $1 \leq q \leq m+1$ and $p \equiv q[2]$.

Remark 8.5. *Combining theorem 6.33 and lemma 8.1, we see that $h^m_{pq} \geq 0$*

8.2 Proof of proposition 8.2

Proof. By continuity, it suffices to treat the region $R = \{h > 0\ ,\ c > 3/2\}$. Now, we see that $(c, h^c_{pq}) \notin R$, so by Kac determinant formula (theorem 7.9), $det_n(c,h)$ is nowhere zero on R. So, it suffices to prove that the form is positive for one pair $(c,h) \in R$.

If $\alpha = (a_1, ..., a_{r_1}; b_1, ..., b_{r_2})$, let $n(\alpha) = \sum a_i + \sum b_j$, $r(\alpha) = r_1 + r_2$. Let $u_\alpha = A_\alpha \Omega$, with A_α the product of L_{-a_i} and G_{-b_j} in the following order: if $n \leq m$ then L_{-n} or G_{-n} is before L_{-m} or G_{-m}; example: $G_{-1/2}L^2_{-1}G_{-5/2}\Omega$. (u_α) form a basis of $V(c,h)$.
Now, thanks to this order, we easily prove by induction on $n(\alpha) + n(\beta)$ that:

$$(u_\alpha, u_\beta) = \begin{cases} c_\alpha h^{r(\alpha)}(1 + o(1)) & \text{with } c_\alpha > 0 \quad \text{if } \alpha = \beta \\ o(h^{(r(\alpha)+r(\beta))/2}) & \quad\quad\quad\quad\quad \text{if } \alpha \neq \beta \end{cases}$$

So, $\forall n \in \frac{1}{2}\mathbb{N}$ and $\forall u \in V_n(c,h)$, $u = \sum_{n(\alpha)=n} \lambda_\alpha u_\alpha$ and:

$$(u,u) = \sum_{\alpha,\beta} \lambda_\alpha \bar{\lambda_\beta}(u_\alpha, u_\beta) = \sum_\alpha |\lambda_\alpha|^2 (u_\alpha, u_\alpha) + \frac{1}{2}\sum_{\alpha\neq\beta} Re(\lambda_\alpha \bar{\lambda_\beta})(u_\alpha, u_\beta) > 0$$

for h sufficiently large and independent of u.
Then, the form is positive for h large, and so is $\forall (c,h) \in R$ by continuity. □

8.3 Proof of theorem 8.4

Definition 8.6. *Let C_{pq} be the curve $h = h^c_{pq}$ with $0 \neq p \equiv q[2]$.*

Remark 8.7. *C_{pq} intersects the line $c = 3/2$ at $h = \frac{(p-q)^2}{8} = lim_{m\to\infty}(h^m_{pq})$. For $0 \leq c < \frac{3}{2}$, we see the curve as (c_m, h^m_{pq}) with $m \in [2, +\infty[$.*

Definition 8.8. *Let* $\kappa = \begin{cases} 1 & \text{if} \quad q < p+1 \\ 0 & \text{if} \quad q > p+1 \end{cases}$

Proposition 8.9. *When the curve C_{pq} first appears at level $n = pq/2$, if $q = 1$, it intersects no other vanishing curves, else, its first intersection moving forward $c = 3/2$ is with $C_{q-2+\kappa,p+\kappa}$, at $m = p+q-2+\kappa$.*

Proof. Let $(p',q') \neq (p,q)$ with $p'q' \leq pq$, then the intersection points $C_{pq} \cap C_{p'q'}$ are given by $[(m+2)p - mq]^2 = [(m+2)p' - mq']^2$, with two

solutions m_+ and m_- such that $[(p-q) \pm (p'-q')]m_\pm = 2(\mp p' - p)$.
Now, if $[(p-q) \pm (p'-q')] = 0$ then $0 = -(p+p') \leq -2$ or $(p,q) = (p',q')$, contradiction; hence, $m_\pm = 2\frac{\mp p' - p}{(p-q)\pm(p'-q')}$ and $\frac{1}{m_\pm} = \frac{1}{2}(\frac{q \pm q'}{p \pm p'} - 1)$.
If $q = 1$, we see that $\frac{q\pm q'}{p \pm p'} > 0 \Rightarrow p'q' > pq$, contradiction.
Else, $q \neq 1$; let $(p-q) \pm (p'-q') = -2s$ with $s \in \mathbb{Z}^\star$.
The goal is to find the biggest $m_\pm \in [2, +\infty[$ among the following solutions, parametered by $s \in \mathbb{Z}^\star$, $k \in \mathbb{Z}$, with $p'q' \leq pq$:

- $(p'_+, q'_+) = (q-s+k, p+s+k)$ and $m_+ = \frac{p+q+k-s}{s}$
- $(p'_-, q'_-) = (p+s+k, q-s+k)$ and $m_- = -\frac{k-s}{s}$

But, at fixed s and k, $m_+ - m_- = \frac{p+q+2k}{s}$, and $p+q+2k = p'_+ + p'_- > 0$, so, if $s > 0$, we choose m_+, and if $s < 0$, we choose m_-.

Let $s > 0$, $k \in \mathbb{Z}$ and $(p',q') = (q-s+k, p+s+k)$. $p'q' \leq pq \Rightarrow k < s$. The biggest m is given by $s = 1$ and $k = 0$. Now, $(q-1)(p+1) > pq$ if $q > p+1$, so we take $k = -1$ in this case and so $(p',q') = (q-2+\kappa, p+\kappa)$, at $m = p+q-2+\kappa$.

Let $s < 0$, $k \in \mathbb{Z}$ and $(p',q') = (p+s+k, q-s+k)$. $p'q' \leq pq \Rightarrow k < -s$. Now if $-\frac{k-s}{s} = m > p+q-2$, then $k > -s(p+q-1) \geq -s$, contradiction. □

Definition 8.10. *For $q = 1$, let C'_{p1} be all of C_{p1} for $m \geq 2$, ie, $0 \leq c \leq \frac{3}{2}$, else, define C'_{pq} to be the part of C_{pq} for which $m > p+q-2+\kappa$.*

C'_{pq} is the open subset of C_{pq} between $c = \frac{3}{2}$ and its first intersection at level $pq/2$. The first step of the proof of theorem 8.4 is to eliminate all on $0 \leq c \leq \frac{3}{2}$, except the curves C'_{pq}.

Definition 8.11. *Let $n \in \frac{1}{2}\mathbb{N}$:*

$$S_n = \bigcup_{\substack{0 < pq/2 \leq n \\ p \leq q, \ p \equiv q[2]}} \{(c,h) \mid 0 \leq c < \frac{3}{2} \ , \ h^c_{pq} \leq h \leq h^c_{qp} \ \text{or} \ h \leq h^c_{pp}\}$$

Lemma 8.12. *$lim_{n\to\infty} S_n$ is all $0 \leq c < \frac{3}{2}$ of the plane.*

Proof. $lim_{pq/2\to\infty}(c_{p+q-2}) = 3/2$ and $lim_{c\to 3/2}(h^c_{pq}) = h^{3/2}_{pq} = \frac{(p-q)^2}{8}$. □

Definition 8.13. *Let $p'q' > pq$; $C_{p'q'}$ is a first intersector of C'_{pq}, if at level $p'q'/2$, it's the first starting from $c = 3/2$.*

Proposition 8.14. *The first intersectors on C'_{pq} are $C_{q-1+k,p+1+k}$, $k \geq \kappa$, at $m = p + q + k - 1$.*

Proof. We take the same structure that proof of proposition 8.9.
$(p', q') = (q - 1 + k, p + 1 + k)$ corresponds to $s = 1$ and $k \geq \kappa \Leftrightarrow p'q' > pq$.
Now, let $(u, v) = (q - s' + k', p + s' + k')$ or $(p + s' + k', q - s' + k')$,
if $m' = \frac{p+q+k'-s'}{s'}$ or $-\frac{k'-s'}{s'} \geq m$ and $uv \leq p'q'$, then, $k' = k$ and $s' = 1$.
So, $C_{q-1+k,p+1+k}$ first intersects C'_{pq}. Now, if $m' > m - 1$ and $s' \neq 1$, then, $uv > p'q'$; so, there is no other first intersector. □

Lemma 8.15. *The discrete series of theorem 8.4 consists exactly of these first intersections F_{pqk}, on all the C'_{pq}.*

Proof. $m = p + q + k - 1$ with $k \geq \kappa$, so, the set of such m is $\mathbb{N}_{\geq 2}$.
Now, let $m \geq 2$ fixed, then, $p + q \leq m + 1 - \kappa$
But, $h^m_{pq} = h^m_{m-p,m+2-q}$, so we obtain the discrete series:
Integers $m \geq 2$, $1 \leq p \leq m - 1$, $1 \leq q \leq m + 1$ and $p \equiv q[2]$. □

Remark 8.16. *We can write the series without redondancy as:*
$m \geq 2$, $1 \leq p < q - 1 \leq m$ and $p \equiv q[2]$.

Definition 8.17. *Let $R_{11} = \{0 \leq c < 3/2, h < 0\}$;*
for $p \neq 1$, let $R_{1p} = R_{p1}$ be the open region bounded by C'_{p1}, C'_{1p} and $C'_{p-2,1}$;
for $q \neq 1$, R_{pq}, the open region bounded by C'_{pq}, $C'_{p-1,q-1}$ and $C'_{q-2+\kappa,p+\kappa}$.

Lemma 8.18. *No vanishing curves at level $n = pq/2$ intersect R_{pq}.*

Proof. A vanishing curve which did intersect R_{pq}, would have to intersect its boundary. This does not happen by proposition 8.14. □

Lemma 8.19. $S_n - S_{n-1/2} = \bigcup_{\substack{pq/2=n \\ p\equiv q[2]}} R_{pq} \cup C'_{pq}$

Proof. $S_{1/2} = R_{11} \cup C'_{11}$, $C_{pq} - C'_{pq} \subset S_{n-1/2}$ and lemma 8.18. □

Lemma 8.20. *All S_n is eliminated, except C'_{pq}, $pq/2 \leq n$.*

Proof. By previous lemma, $S_n = \bigcup_{\substack{pq/2\leq n\\ p\equiv q[2]}} R_{pq} \cup C'_{pq}$.
Now, we see that, for $p \neq q$, R_{pq} is between C_{pq} and C_{qp}; R_{pp} is under C_{pp}, and for $p'q' \leq pq$ with $(p',q') \neq (p,q)$, R_{pq} is necessarily over $C_{p'q'}$ and $C_{q'p'}$, or under them. So (recall section 7.1), $\varphi_{pq}(c,h) < 0$ and $\varphi_{p'q'}(c,h) > 0$ on R_{pq}, and $d(0) = 1$; then, $det_{pq/2}(c,h) < 0$ and $V(c,h)$ admits ghosts on R_{pq}. □

Now, given lemma 8.12 and 8.20, we have to eliminate the intervals on C'_{pq}, between the points of the discrete series.

Definition 8.21. *Let I_{pqk} be the open subset of C'_{pq} between $F_{p,q,k-1}$ and $F_{p,q,k}$ for $k > \kappa$; and $I_{pq\kappa}$, beyond $F_{pq\kappa}$.*

Lemma 8.22. $C'_{pq} = \bigcup_{k\geq k_0} I_{pqk} \cup F_{pqk}$.

The goal is to eliminate the open subset I_{pqk}, $k \geq \kappa$.
Recall that when $C_{p'q'} = C_{q-1+k,p+1+k}$ first appears at level $n' = p'q'/2$, there is a ghost on $R_{p'q'}$; we will show that this ghost continue to exist on I_{pqk}.

Proposition 8.23. *At level $n' = p'q'/2$, the first $k-\kappa+1$ successives intersections on $C_{p'q'}$ are with $C'_{p+k-j,q+k-j}$ ($\kappa \leq j \leq k$) at its first intersection $F_{p+k-j,q+k-j,j}$, with $m = p+q+2k-j-1$*

Proof. Let $(p'',q'') = (q'-s+k', p'+s+k')$.
If $p''q'' \leq p'q'$ and, $\frac{p'+q'+k'-s'}{s'}$ or $-\frac{k'+s'}{s'} \geq m = p+q+k-1$, (ie, with $j = k$), then $s' = 1$; now, by proposition 8.9, the first is with $j = \kappa$. □

Lemma 8.24. *Let M_t be an d-dimensional polynomial matrix with $det(M_t)$ vanishing to first order at $t = 0$; then, the null space is 1-dimensional.*

Proof. Let $\alpha_1(t), ..., \alpha_d(t)$ be the eigenvalues of M_t; they are analytic in t. Now, $det(M_t) = \prod \alpha_i(t) = \prod(\alpha_i^0 + \alpha_i^1 t + ...)$, vanishing to first order at $t = 0$, so, there exists a unique i such that $\alpha_i^0 = 0$, and $dimker M_0 = 1$. □

Corollary 8.25. *Let $(c,h) \in C_{pq}$, not on an intersection at level $pq/2$, then, the null space of $V_{pq/2}(c,h)$ is 1-dimensional.*

Lemma 8.26. *Let $(c,h) = F_{pqk}$, then, $det_{(p'q'-pq)/2}(c, h+pq/2) \neq 0$.*

Proof. If this determinant were zero, then $(c, h+pq)$ would be on a vanishing curve C_{uv} of level $\leq \frac{1}{2}(p'q'-pq)$: $h^m_{pq} + pq/2 = h^m_{uv}$ and $uv \leq p'q'-pq$.
Then, we find (u,v) or $(v,u) = (ms'-p, (m+2)s'+q)$, with $s' \in \mathbb{Z}^\star$.
So now, $uv \leq p'q'-pq$ is equivalent to $((1+s')m-p)((1-s')(m+2)-q) \geq 0$, but $1 \leq p < m$ and $1 \leq q < m+2$, so, $s'=0$, contradiction. □

To read the followings proposition and its proof, recall section 7.12.
It's strictly parallel that in [28] for the Virasoro algebra.

Proposition 8.27. *For $j = \kappa, ..., k$ there is an open neighborhood $U_{p'q'j}$ of $F_{p+k-j,q+k-j,j} = F_{q'-1-j,p'+1-j,j}$ and a nowhere zero analytic function $v_j(c,h)$ defined on $U_{p'q'j}$ with values in $V_{n'}(c,h)$, with $n' = p'q'/2$, such that:*

$$v_j(c,h) \in K_n(c,h) \Leftrightarrow (c,h) \in C_{p'q'}$$

Proof. Write $p'' = p+k-j$, $q'' = q+k-j$ and $n'' = p''q''/2 < n'$.
Let $U = U_{p'q'j}$ be a neighborhood of $F_{p+k-j,q+k-j,j}$, small enough that it intersects no vanishing curves but $C_{p'q'}$ and $C_{p''q''}$ at level n'. Choose coordinates (x,y) in U, real analytic in (c,h), such that $C_{p''q''}$ is given by $x=0$ and $C_{p'q'}$ by $y=0$. This is possible because the intersection is transversal. At level n'', $x=0$ is the only vanishing curve in U. $K_{n''}(0,y)$ is one dimensional and form a line bundle over the vanishing curve $x=0$ near $y=0$. Let $v''_j(0,y)$ be a nowhere zero analytic section of this line bundle, and let $v''_j(x,y)$ be an analytic function on U with values in $V_{n''}(x,y)$, which extends this section. Let $V''(x,y) = V^{v''_j}_{n'}(x,y)$ of dimension $d(n'-n'')$. For $y \neq 0$, the order of vanishing of $det_{n'}(x,y)$ at $x=0$ is also $d(n'-n'')$. Therefore, for $y \neq 0$, $V''(0,y) = K_{n'}(0,y)$. Let $V'(x,y)$ such that $V_{n'} = V'' \oplus V'$ and we write:

$$M_{n'}(x,y) = \begin{pmatrix} xQ(x,y) & xR(x,y) \\ xR(x,y)^t & S(x,y) \end{pmatrix}$$

with Q, S symmetric and 3 blocks divisible by x because $V''(0,y) \subset K_{n'}(0,y)$.

The key point now, is that $Q(0,0)$ is non-degenerate.
To see this, first note that $v''_j(0,y)$ is singular, $M_{n'}(0,y)v''_j(0,y) = 0$ and $L_0 v''_j(0,y) = (h + p''q''/2)v''_j(0,y)$; recall that $(0,y) = (c,h) \in C_{p''q''}$.
Now, since all is analytic, $\forall \alpha, \beta \in V''(x,y)$:

$$(\alpha, \beta) = (A.v''_j(x,y), B.v''_j(x,y)) = ([B^\star, A]v'', v'') + (B^\star v'', A^\star v'')$$

$$= ([B^\star, A]\widetilde{\Omega}, \widetilde{\Omega})(v'', v'') + o(x) = cte.x(A.\widetilde{\Omega}, B.\widetilde{\Omega}) + o(x),$$

with $\widetilde{\Omega}$ the cyclic vector of $V(c, h + p''q''/2)$; so:

$$Q(x, y) = M_{(p'q'-p''q'')/2}(c, h + p''q''/2) + x.M'(x, y).$$

Since $(0,0) = F_{p''q''j}$, lemma 8.26 gives $det(Q(0,0)) \neq 0$; so, $Q(x,y)$ is non-degenerate on all U (we can replace U by a small neighborhood of $(0,0)$). Let $W = \begin{pmatrix} 1 & -Q^{-1} \\ 0 & 1 \end{pmatrix}$ and make the change of basis:

$$M_{n'} \mapsto W^t M_{n'} W = \begin{pmatrix} xQ(x,y) & 0 \\ 0 & T(x,y) \end{pmatrix}$$

Let $V'''(x,y)$ be the new complement of $V''(x,y)$, on which $T(x,y)$ defined the inner product. The order of vanishing argument implies that $det(T(x,y))$ is non-zero for $y \neq 0$ and vanishes to first order at $y = 0$. The one dimensional null space of $T(x,0)$ is $K_{n'}(x,0)$ for $x \neq 0$. At $x = y = 0$, the one dimensional null space of $T(0,0)$ and $V''(0,0)$, span the d(n'-n")+1 dimensional $K_{n'}(0,0)$. By the same argument which gave $v''_j(x,y)$, we can choose a nowhere zero analytic function $v_j(x,y)$ on U, with values in $V'''(x,y)$ such that $v_j(x,0)$ is in the null space of $T(x,0)$ and therefore in $K_{n'}(x,0)$. Since $T(x,y)$ is non-degenerate for $y \neq 0$, $v_j(x,0)$ is not in $K_{n'}(x,y)$ if $y \neq 0$ □

Definition 8.28. *Let $J_{p'q'j}$, $\kappa < j \leq k$, be the open interval on $C_{p'q'}$ between $F_{p+k-j,q+k-j,j}$ and $F_{p+k-j-1,q+k-j-1,j}$, and let $J_{p'q'\kappa}$ be the open interval on $C_{p'q'}$ lying between $c = 3/2$ and $F_{p+k-\kappa,q+k-\kappa,\kappa}$.*

Definition 8.29. *Let $W_{p'q'j}$, $\kappa \leq j \leq k$ be a neighborhood of a point of $J_{p'q'j}$, which intersects no other vanishing curves on level n', such that: :*
$J_{p'q'j} \subset U_{p'q'j-1} \cup W_{p'q'j} \cup U_{p'q'j}$ if $j > \kappa$, and $\varnothing \neq U_{p'q'\kappa} \cap W_{p'q'\kappa} \subset R_{p'q'}$

Lemma 8.30. *For each j, $\kappa \leq j \leq k$, there is a nowhere zero analytic function $w_j(c,h)$ on $W_{p'q'j}$ with values in $V_{n'}(c,h)$, such that $w_j(c,h)$ is in $K_{n'}(c,h)$ if and only if (c,h) is on $J_{p'q'j}$, and:*

$$w_j = \begin{cases} f_j v_j & \text{on } W_{p'q'j} \cap U_{p'q'j} \\ g_j v_{j-1} & \text{on } W_{p'q'j} \cap U_{p'q'j-1} \ (j \neq \kappa) \end{cases}$$

where f_j, g_j are nonzero function.

Proof. $K_{n'}(c,h)$ is trivial on $W_{p'q'j}$, except on $J_{p'q'j}$, where $dim(K_{n'}) = 1$. □

Lemma 8.31. *I_{pqk} is eliminated on level $n' = (q-1+k)(p+1+k)/2$.*

Proof. By proposition 8.2, $M_{n'}(c,h)$ is positive on $h \geq 0$, $c \geq 3/2$.
Now, at level n', we can go from this sector to $W_{p'q'\kappa}$ without crossing a vanishing curve, so, $(w_\kappa, w_\kappa) > 0$ before crossing $C_{p'q'}$. But it vanishes to first order on $C_{p'q'}$, so, after crossing it, w_κ becomes a ghost. Now, by lemma 8.30 and induction, so is for v_κ, $w_{\kappa+1}$, $v_{\kappa+1}$, ... up to $v_k(c,h) \in I_{pqk} \cap U_{p'q'k}$.
Finally, $v_k(c,h)$ continues to be a ghost on all I_{pqk}, because I_{pqk} cross no other vanishing curve on level n'. □

Lemmas 8.12, 8.20, 8.22 and 8.31 imply theorem 8.4 and theorem 2.2.

9 Wassermann's argument

We need to recall sections 6.3 and 7.12; by lemma 8.15 the discrete series are the intersections of C'_{pq} and $C_{p'q'}$ at $m = p+q+k-1$, $k \geq \kappa$, with $(p',q') = (q-1+k, p+1+k) = (m-p, m+2-q)$, ie, $h^m_{pq} = h^m_{m-p,m+2-q}$. Let $M = max(pq/2, p'q'/2)$. This section will prove theorem 2.3, thanks to an argument that A. Wassermann uses for the Virasoso case in [100].

Lemma 9.1. *At level $\leq M$, we find only two singular vectors s and s' at level $pq/2$ and $p'q'/2$.*

Proof. We can suppose $p'q' > pq$; by proof of proposition 8.27:

$$K_n(c_m, h^m_{pq}) = \begin{cases} \{0\} & \text{if} \quad n < pq/2 \\ \mathbb{C}s & \text{if} \quad n = pq/2 \\ V^s_n(c_m, h^m_{pq}) & \text{if} \quad pq/2 \leq n < p'q'/2 \\ V^s_n(c_m, h^m_{pq}) \oplus \mathbb{C}s' & \text{if} \quad n = p'q'/2 \end{cases}$$

Then, by proposition 7.14, the result follows. □

Corollary 9.2. $ch(L(c_m, h^m_{pq})) \sim \chi_{NS}(t).t^{h^m_{pq} - c_m/24}(1 - t^{pq/2} - t^{p'q'/2})$

Proof. By section 7.12 and lemma 9.1. □

Lemma 9.3. $h^m_{pq} + M > m^2/8$

Proof. $h^m_{pq} + M = max(\gamma^m_{-p,q}(0), \gamma^m_{-p,q}(-1))$.
$\gamma^m_{-p,q}(0) = \frac{x^2-4}{8m(m+2)}$, $\gamma^m_{-p,q}(-1) = \frac{(x-2m(m+2))^2-4}{8m(m+2)}$, with $x = (m+2)p + mq$.
If $\gamma^m_{-p,q}(0) > m^2/8$, it's ok.
Else, $\frac{x^2-4}{8m(m+2)} \leq m^2/8 \Leftrightarrow x^2 \leq m^4 + 2m^2 + 4 < (m+1)^4$
So, $\gamma^m_{-p,q}(-1) = \frac{[2m(m+2)-x]^2-4}{8m(m+2)} > \frac{[2m(m+2)-(m+1)^2]^2-4}{8m(m+2)} \geq \frac{m^4+2m^3}{8m(m+2)} = m^2/8$. □

Theorem 9.4. *The multiplicity space M^m_{pq} is exactly $L(c_m, h^m_{p,q})$.*

Proof. By corollary 6.31, $L(c_m, h^m_{p,q})$ is a $\mathfrak{Vir}_{1/2}$-submodule of M^m_{pq}; if M^m_{pq} admits another irreducible submodule (of central charge c_m), then, by theorem 8.4, it is on the discrete series, of the form $L(c_m, h^m_{rs})$. Now, by lemma 6.30 and corollary 9.2: $ch(M^m_{pq}) - ch(L(c_m, h^m_{pq})) = \chi_{NS}(t).t^{-c_m/24}o(t^{h^m_{pq}+M})$.
So we need $h^m_{rs} > M + h^m_{pq}$; but, $h^m_{rs} = \frac{[(m+2)r-ms]^2-4}{8m(m+2)} \leq \frac{(m^2-2)^2-4}{8m(m+2)} = \frac{m(m-2)}{8}$.
So, by lemma 9.3, $\frac{m^2}{8} < M + h^m_{pq} < h^m_{rs} \leq \frac{m(m-2)}{8}$, contradiction. □

Theorem 9.5. *The characters of the discrete series are:*

$$ch(L(c_m, h^m_{pq}))(t) = \chi_{NS}(t).\Gamma^m_{pq}(t).t^{-c_m/24} \quad with$$

$$\chi_{NS}(t) = \prod_{n \in \mathbb{N}^\star} \frac{1+t^{n-1/2}}{1-t^n}, \quad \Gamma^m_{pq}(t) = \sum_{n \in \mathbb{Z}} (t^{\gamma^m_{pq}(n)} - t^{\gamma^m_{-pq}(n)}) \quad and$$

$$\gamma^m_{pq}(n) = \frac{[2m(m+2)n - (m+2)p + mq]^2 - 4}{8m(m+2)}$$

Proof. $ch(L(c_m, h^m_{pq})) = ch(M^m_{pq})$, the result follows by corollary 6.28. □

Remark 9.6. *(Tensor product decomposition)*

$$\mathcal{F}^{\mathfrak{g}}_{NS} \otimes L(j, \ell) = \bigoplus_{\substack{1 \le q \le m+1 \\ p \equiv q[2]}} L(c_m, h^m_{pq}) \otimes L(k, \ell + 2)$$

with $p = 2j + 1$, $q = 2k + 1$, $m = \ell + 2$ *and* $\mathfrak{g} = \mathfrak{sl}_2$.

We then recover a result due to Frenkel in [22]:

Corollary 9.7. $\mathcal{F}^{\mathfrak{g}}_{NS} = L(0, 2) \oplus L(1, 2)$ *as* $L\mathfrak{g}$*-module.*

Proof. It suffices to take $j = \ell = 0$, and to see that $c_2 = h^2_{11} = h^2_{13} = 0$. □

Corollary 9.8. *(Duality)Let* H *be an irreducible positive energy represemtation of the loop superalgebra* $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$*, let* A *be the operator algebra generated by the modes of the coset operators* L_n *and* G_r*, let* B *be the operator algebra generated by the modes of the diagonal loop superalgebra* $\widehat{\mathfrak{g}}$*. Then,* A *and* B *are each other algebraic graded commutant (see [100]).*

Definition 9.9. *(Vertex algebra supercommutant or centralizer algebra)*
Let V *be a vertex superalgebra and* W *a vertex sub-superalgebra, then, the vertex algebra supercommutant of* W *is the vertex superalgebra corresponding to the vectors* $v \in V$ *such that the modes of the corresponding field supercommute with the modes of fields for vectors of* W *(see [57]).*

Corollary 9.10. *(Vertex superalgebra duality) In the vertex superalgebra generated by* $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$*, the vertex superalgebras generated by the Neveu-Schwarz coset and the diagonal loop superalgebra, are each others supercommutants.*

Part II

Connes fusion and subfactors for the Neveu-Schwarz algebra

10 Local von Neumann algebras

10.1 Recall on von Neumann algebras

Let H be an Hilbert space and $\mathcal{A}$ a unital $\star$-algebra of bounded operators.

Definition 10.1. *The commutant $\mathcal{A}'$ of $\mathcal{A}$ is the set of $b \in B(H)$ such that, $\forall a \in \mathcal{A}$, then $[a,b] := ab - ba = 0$*

Definition 10.2. *The weak operator topology closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ is the set of $a \in B(H)$ such that $\exists a_n \in \mathcal{A}$ with $(a_n \eta, \xi) \to (a\eta, \xi)$, $\forall \eta, \xi \in H$.*

Recall 10.3. *(Bicommutant theorem) Let $\mathcal{M}$ be a unital $\star$-algebra, then:*

$$\mathcal{M}'' = \mathcal{M} \iff \bar{\mathcal{M}} = \mathcal{M}$$

Definition 10.4. *Such a $\mathcal{M}$ verifying one of these equivalents properties is called a von Neumann algebra.*

Definition 10.5. *A factor is a von Neumann algebra $\mathcal{M}$ with $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}$.*

Recall 10.6. *(Murray and von Neumann theorem) The set of all the factors on H is a standard borelian space X and every von Neumann algebra $\mathcal{M}$ decompose into a direct integral of factors: $\mathcal{M} = \int_X^{\oplus} \mathcal{M}_x d\mu_x$*

Recall 10.7. *(Murray and von Neumann's classification of factors)*
Let $\mathcal{M} \subset B(H)$ be a factor. We shall consider H as a representation of $\mathcal{M}'$. Thus subrepresentations of H correspond to projections in $\mathcal{M}$. If $p, q \in \mathcal{M}$ are projections, then pH and qH are unitarily equivalent as representations of $\mathcal{M}'$ iff there is a partial isometry $u \in \mathcal{M}$ between pH and qH; thus $u^*u = p$ and $uu^* = q$. We can immediately distinguish three mutually exclusive cases.
I. H has an irreducible subrepresentation.
II. H has no irreducible subrepresentation, but has a subrepresentation not equivalent to any proper subrepresentation of itself.
III. H has no irreducible subrepresentation and every subrepresentation is equivalent to some proper subrepresentation of itself.
We shall call $\mathcal{M}$ a factor of type I , II or III acording to the above cases.

Recall 10.8. *The type I and II corresponds to factors admitting non-trivial trace, with only integer values on the projectors for the type I ($M_n(\mathbb{C})$ or $B(H)$), and non-integer values for the type II (factors generated by ICC groups for example). On the type III, the values are only 0 or ∞.*

Recall 10.9. *(Tomita-Takesaki theory) We suppose the existence of a vector Ω (called vacuum vector) such that $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in H (ie Ω is cyclic and separating). Let $S : H \to H$ the closure of the antilinear map: $\star : x\Omega \to x^\star\Omega$. Then, S admits the polar decomposition $S = J\triangle^{\frac{1}{2}}$ with J antilinear unitary, and $\triangle^{\frac{1}{2}}$ positve; so that $J\mathcal{M}J = \mathcal{M}'$, $\triangle^{it}\mathcal{M}\triangle^{-it} = \mathcal{M}$ and $\sigma_t^{\Omega}(x) = \triangle^{it}x\triangle^{-it}$ gives the one parameter modular group action.*

Recall 10.10. *(Radon-Nikodym theorem) Let Ω' be another vacuum vector, then there exists a Radon-Nikodym map $u_t \in \mathcal{U}(\mathcal{M})$, define such that $u_{t+s} = u_t\sigma_t^{\Omega'}(u_s)$ and $\sigma_t^{\Omega'}(x) = u_t\sigma_t^{\Omega}(x)u_t^\star$. Then, modulo $Int(\mathcal{M})$, σ_t^{Ω} is independant of the choice of Ω, ie, there exist an intrinsic $\delta : \mathbb{R} \to Out(\mathcal{M}) = Aut(\mathcal{M})/Int(\mathcal{M})$. On type I or II the modular action is internal, and so δ trivial. It's non-trivial for type III.*

Definition 10.11. *We can then define two invariants of $\mathcal{M}$, $T(\mathcal{M}) = ker(\delta)$ and $\mathcal{S}(\mathcal{M}) = Sp(\delta) = \bigcap Sp(\triangle_\Omega) \setminus \{0\}$ called the Connes spectrum of $\mathcal{M}$.*

Recall 10.12. *(see [17]) Let $\mathcal{M}$ be a type III factor, then $\mathcal{S}(M) = \{1\}$, $\lambda^{\mathbb{Z}}$ or $\mathbb{R}_+^\star$, and then, $\mathcal{M}$ is called a III_0, III_λ or III_1 factor (with $0 < \lambda < 1$).*

Recall 10.13. *Let $\mathcal{M} \neq \mathbb{C}$ be a von Neumann algebra on (H, Ω) then it's a III_1 factor if and only if the modular action (i.e. the action of $\mathbb{R}$ on $\mathcal{M}$ via σ_t^{Ω}) is ergodic (i.e. it fixes only the scalar operators).*

10.2 $\mathbb{Z}_2$-graded von Neumann algebras

Definition 10.14. *A $\mathbb{Z}_2$-graded von Neumann algebra $(\mathcal{M}, \tau)$ is a von Neumann algebra $\mathcal{M}$ given with a period two automorphism $\tau \in Aut(\mathcal{M})$ and $\tau^2 = I$. Now $\forall x \in \mathcal{M}$, $x = x_0 + x_1$ with $x_0 = \frac{1}{2}(x+\tau(x))$ and $x_1 = \frac{1}{2}(x-\tau(x))$ called the even and the odd part of x. Then $\tau(x_0) = x_0$ and $\tau(x_1) = -x_1$. Hence $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$; if $a \in \mathcal{M}_{\varepsilon_1}$ and $b \in \mathcal{M}_{\varepsilon_2}$ then $a.b \in \mathcal{M}_{\varepsilon_1+\varepsilon_2}$.*

Definition 10.15. *A $\mathbb{Z}_2$-graded Hilbert space is an Hilbert space given with a period two unitary operator: $u \in \mathcal{U}(H)$ and $u^2 = I$, so that $H = H_0 \oplus H_1$, with H_0 and H_1 the eigenspaces of u for the eigenvalues 1 and -1. Let p_0 and p_1 the corresponding projection, then $u = p_0 - p_1 = 2p_0 - 1$.*

Remark 10.16. *Let $\mathcal{M}$ be a von Neumann algebra on H with Ω its cyclic, separating vector. Then a period two automorphism τ of $\mathcal{M}$ gives a period* *two unitary operator u of H by $u : x\Omega \to \tau(x)\Omega$. Conversely, a period two unitary operator u of H with $u.\Omega = \Omega$ gives $\tau \in Aut(\mathcal{M})$ by $\tau(x) = uxu$.*

Definition 10.17. *Let $x \in B(H)$, then, $\tau(x) = uxu$ defined a period two automorphism on $B(H)$. Then as for definition 10.14, $x = x_0 + x_1$.*
We see that $x_0 = p_0xp_0 + p_1xp_1$ and $x_1 = p_1xp_0 + p_0xp_1$.

Definition 10.18. *(Supercommutator)*
Let $[x, y]_\tau = [x_0, y_0] + [x_0, y_1] + [x_1, y_0] + [x_1, y_1]_+$

Remark 10.19. *A projection p is even, then, $\forall x \in B(H)$, $[x, p]_\tau = [x, p]$. In particular $[x, I]_\tau = [x, I] = 0$.*

Definition 10.20. *The supercommutant $\mathcal{A}^\natural$ of $\mathcal{A}$ is the set of $b \in B(H)$ such that, $\forall a \in \mathcal{A}$, then $[a, b]_\tau = 0$.*

Definition 10.21. *Let $\kappa = p_0 + ip_1$ the Klein transformation.*

Remark 10.22. *κ is unitary, $\kappa^{-1} = \kappa^\star = p_0 - ip_1$ and $\kappa^2 = u$.*

Remark 10.23. *$ux_0u = x_0$, $ux_1u = -x_1$, $\kappa x_0\kappa^\star = x_0$, $\kappa x_1\kappa^\star = -iux_1$*

Lemma 10.24. *Let $\mathcal{A}$ be a von Neumann algebra $\mathbb{Z}_2$-graded for τ, then:*

$$\mathcal{A}^\natural = \kappa\mathcal{A}'\kappa^\star.$$

Proof. Let $a \in \mathcal{A}$ and $x \in B(H)$ such that $[x, a] = 0$.
By the relations of the remarks 10.23:
If x is even, then $[\kappa x\kappa^\star, a]_\tau = [x, a] = 0$.
If a is even, then $[\kappa x\kappa^\star, a]_\tau = \kappa[x, \kappa^\star a\kappa]\kappa^\star = \kappa[x, a]\kappa^\star = 0$.
Else, $[\kappa x\kappa^\star, a]_\tau = [-i\tau x, a]_+ = -i(uxa + aux) = -iu[x, a] = 0$
Then, $\kappa\mathcal{A}'\kappa^\star \subset \mathcal{A}^\natural$; idem, $\kappa^\star\mathcal{A}^\natural\kappa \subset \mathcal{A}'$; the result follows. □

Corollary 10.25. *$\mathcal{A}^\natural$ is unitary equivalent to $\mathcal{A}'$.*

Proof. κ is a unitary operator. □

Lemma 10.26. *Let $(\mathcal{A}, \tau)$ be a $\mathbb{Z}_2$-graded von Neumann algebra then:*

$$\mathcal{A}^{\natural\natural} = \mathcal{A}.$$

Proof. $\mathcal{A}^{\natural\natural} = \kappa(\kappa\mathcal{A}'\kappa^\star)'\kappa^\star = \kappa\kappa(\mathcal{A}'')\kappa^\star\kappa^\star$, because a von Neumann algebra is generated by its projections, and a projection is even, so commute with κ. Then $\mathcal{A}^{\natural\natural} = u\mathcal{A}u = \tau(\mathcal{A}) = \mathcal{A}$. □

10.3 Global analysis

The generic discrete series representation $L(c_m, h^m_{pq})$ is a prehilbert space of finite level vectors, we note H^m_{pq} its L^2-completion.

Definition 10.27. *Let $s \in \mathbb{R}$, we define the Sobolev norms $\|.\|_{(s)}$ as follows:*

$$\|\xi\|_{(s)} := \|(I + L_0)^s \xi\| \quad \forall \xi \in L(c_m, h^m_{pq})$$

Remark 10.28. $((1 + L_0)^{2s}\xi, \xi) = \|\xi\|^2_{(s)}$

Proposition 10.29. *(Sobolev estimate) $\exists k_n, k_r > 0$ such that $\forall \xi \in L(c_m, h^m_{pq})$:*

(a) $\|L_n\xi\|_{(s)} \leq k_n(1 + |n|)^{|s|+3/2}\|\xi\|_{(s+1)}$

(b) $\|G_r\xi\|_{(s)} \leq k_r(1 + |r|)^{|s|+1/2}\|\xi\|_{(s+1/2)}$

Proof. (a) See Goodman-Wallach [40] (proposition 2.1 p 307).
(b)$2L_0 = G_rG_{-r} + G_{-r}G_r$. Then, $2(L_0\xi, \xi) = (G_r\xi, G_r\xi) + (G_{-r}\xi, G_{-r}\xi)$. So, $\|G_r\xi\|^2 \leq k_1\|L_0^{1/2}\xi\|^2$ for any r. Now, it suffices to show the result for an eigenvector of L_0: $L_0\xi = \mu\xi$. We can take $r \leq \mu$ (otherwise $G_r\xi = 0$).
$\|G_r\xi\|^2_s = \|(1+L_0)^sG_r\xi\|^2 \leq (1+\mu-r)^{2s}\|G_r\xi\|^2 \leq (1+\mu-r)^{2s}k_1\|L_0^{1/2}\xi\|^2 \leq (1+\mu-r)^{2s}k_1\mu\|\xi\|^2 \leq \frac{(1+\mu-r)^{2s}}{(1+\mu)^{2s}}k_1\|\xi\|^2_{s+1/2} \leq (1+|r|)^{2|s|+1}k_1\|\xi\|^2_{s+1/2}$. □

Remark 10.30. *Thanks to $L_n = [G_{n-1/2}, G_{1/2}]_+$, we obtain directly the estimate $\|L_n\xi\|_{(s)} \leq k(1 + |n|)^{|2s|+1}\|\xi\|_{(s+1)}$ without Goodman-Wallach result.*

Definition 10.31. *Let $H^{m,s}_{pq}$ be the $\|.\|_s$-completion of $L(c_m, h^m_{pq})$ and:*

$$\mathcal{H}^m_{pq} = \bigcap_{s>0} H^{m,s}_{pq}$$

with the usual Fréchet topology from the norms $\|.\|_s$

Corollary 10.32. *$L(c_m, h^m_{pq})$ extends to a continuous representation of $\mathfrak{Vir}_{1/2}$ on $\mathcal{H}^m_{pq}$.*

Definition 10.33. *Let $d = -i\frac{d}{d\theta}$ the unbounded operator of $L^2(\mathbb{S}^1)$, let F be the subspace of finite Fourier series as a dense domain of d. Let $s \in \mathbb{R}$ and $\|f\|_{(s)} := \|(I + |\delta|)^s.f\|_1$ a Sobolev norm on F. Let F_s be the completion of F relative to $\|.\|_{(s)}$. Idem for $e^{i\theta/2}F$.*

Definition 10.34. *Let* $L_f = \sum a_n L_n$ *and* $G_h = \sum b_r G_r$ *such that* $f(\theta) = \sum a_n e^{in\theta}$, $h(z) = \sum b_r e^{ir\theta}$ *and* $f \in F$ *and* $h \in e^{i\theta/2}F$.

Notation 10.35. *Let* $(f,h)_{\mathbb{R}} := \frac{1}{2\pi i}\int_0^{2\pi} f(\theta)h(\theta)d\theta$, *with* $f.h \in F$

Lemma 10.36. *(Lie bracket relation)*

$$\begin{cases} [L_f, L_h] & = L_{d(f)h - fd(h)} + \frac{C}{12}((d^3-d)(f),h)_{\mathbb{R}} \\ [G_f, L_h] & = G_{d(f)h - \frac{1}{2}fd(h))} \\ [G_f, G_h]_+ & = 2L_{fh} + \frac{C}{3}((d^2-1)(f),h)_{\mathbb{R}} \end{cases}$$

The $\star$*-structure:* $L_f^\star = L_{\bar{f}}$, $G_h^\star = G_{\bar{h}}$.

Proof. Direct by computation from proposition 3.9. □

Proposition 10.37. *(Sobolev estimate)*
$\exists k > 0$ *such that* $\forall \xi \in H_{pq}^m$ *and* $f \in F$, $h \in e^{i\theta/2}F$*:*

(a) $\|L_f\xi\|_{(s)} \leq k\|f\|_{(|s|+3/2)}\|\xi\|_{(s+1)}$

(b) $\|G_h\xi\|_{(s)} \leq k\|h\|_{(|s|+1/2)}\|\xi\|_{(s+1/2)}$

Proof. It's immediate from proposition 10.29. □

Recall 10.38. $\bigcap_{s>0} F_s = C^\infty(\mathbb{S}^1)$.

Corollary 10.39. *The operators* L_f *and* G_h *act continuously on* $\mathcal{H}_{pq}^m$, *with* $f \in C^\infty(\mathbb{S}^1)$ *and* $h \in e^{i\theta/2}C^\infty(\mathbb{S}^1)$.

Recall 10.40. *Let* T *be an operator on a Hilbert space* H. *A subspace* $D(T)$ *of* H *is called a domain of* T *if* $T.D(T) \subset H$. *Then let* $\Gamma(T) = \{(x, T.x), x \in D(T)\}$ *be the graph of* T. *The operator* T *is closed if its graph* $\Gamma(T)$ *is closed in* $H \times H$. *An operator* $\tilde{T}$ *is an extension of* T *if* $\Gamma(T) \subset \Gamma(\tilde{T})$, *we write* $T \subset \tilde{T}$. *The operator* T *is closable if it admits a closed extension; let* $\bar{T}$ *be the smallest one. Then,* T *is closable iff* $\overline{\Gamma(T)}$ *is the graph of a linear operator (not always true). If* T *is densely defined, then its adjoint* $T^\star$ *is closed because its graph is an orthogonal. From now, every domain is dense in* H. *The operator* T *is symmetric or formally self-adjoint if* $T \subset T^\star$, *essentially self-adjoint if* $\bar{T} = T^\star$, *and self-adjoint if* $T = T^\star$.

Recall 10.41. *(Glimm-Jaffe-Nelson commutator theorem [81] X.5)*
Let D be a diagonalizable, positve, compact resolving operator and X formally self-adjoint, with common dense domain. If $(D+I)^{-1}X$, $X(D+I)^{-1}$ and $(D+I)^{-1/2}[D,X](D+I)^{-1/2}$ are bounded, then X is essentially self-adjoint.

Lemma 10.42. *Let f, $h \in C^\infty(\mathbb{S}^1)$ and real, then, L_f and G_h act on $\mathcal{H}^m_{pq}$ as essentially self-adjoint operators.*

Proof. The function f is real, so $\bar{f} = f$, then, by the $\star$-structure and the unitarity of the action, L_f is formally self-adjoint. Now, L_0 is positive and by Sobolev estimate: $\|(L_0+I)^{-1}L_f\xi\| = \|L_f\xi\|_{(-1)} \leq k\|\xi\|_{(0)} = k\|\xi\|$, so $(L_0+I)^{-1}L_f$ is bounded. Now, $\|L_f\eta\| \leq k\|\eta\|_1 = k\|(L_0+I)\eta\|$, so taking $\xi = (L_0+I)\eta$, we find $\|L_f(L_0+I)^{-1}\xi\| \leq k\|\xi\|$. Finally, $[L_0, L_f] = L_h$ with $h(z) = -zf'(z)$, so combining the two previous tips with $\xi = (L_0+I)^{1/2}\eta$, we find $(L_0+I)^{-1/2}[L_0,L_f](L_0+I)^{-1/2}$ bounded too. We can do the same with G_h because $\|\xi\|_{(s+1/2)} \leq \|\xi\|_{(s+1)}$. Then, the result follows by recall 10.41. □

Remark 10.43. *This result was already known for Diff$(\mathbb{S}^1)$ and hence the L_f. On the other hand $G_f^2 = L_{f^2} + kId$, so the essential self-adjointness follows by Nelson's theorem:*

Recall 10.44. *(Nelson's theorem [72]) Let H be an Hilbert space, A and B be formally self-adjoint operator acting on a dense subspace $D \subset H$, such that $AB\xi = BA\xi$ $\forall \xi \in D$, and $A^2 + B^2$ essentially self-adjoint, then A, B are essentially self-adjoint, and their bounded function commute on H.*
Remark that we have the same result for supercommutation introducing κ.

Recall 10.45. *Let T be a self-adjoint operator with $D(T)$ dense in H. There exist a finite measure space (Y,μ), a unitary operator $U : H \to L^2(Y,\mu)$ and a real function f, finite up to a null set on Y, such that, if M_f is the operator of multiplication by f, with domain $D(M_f)$, then $\nu \in D(T) \iff U\nu \in D(M_f)$, and $\forall g \in D(M_f)$, $UTU^\star g = fg$. Let h be a borelian function bounded on $\mathbb{R}$. The bounded operator $h(T)$ on H is defined by $h(T) = U^\star M_{h(f)}U$.*

10.4 Definition of local von Neumann algebras

Definition 10.46. *(Dixmier) Let H be an Hilbert space. An unbounded self-adjoint operator T is affiliated to a von Neumann algebra $\mathcal{M}$ if it satisfy one of the followings equivalent properties:*

(a) $\mathcal{M}$ contains all the spectral projection of T.

(b) $\mathcal{M}$ contains every bounded functions of T.

(c) $\forall u \in \mathcal{M}'$ unitary, $uD(T) = D(T)$ and $uT\xi = Tu\xi$, $\forall \xi \in D(T)$.

We note $T \,\eta\, \mathcal{M}$.

Remark 10.47. *By lemma 10.26, if $(\mathcal{M}, \tau)$ is a $\mathbb{Z}_2$-graded von Neumann algebra, we can add:*

(c') $\forall u \in \mathcal{M}^\natural$ unitary, $uD(T) = D(T)$, $uT\xi = (-1)^{\partial T \partial u} Tu\xi$, $\forall \xi \in D(T)$.

Definition 10.48. *Let I be a proper interval of $\mathbb{S}^1$.
We define $C_I^\infty(\mathbb{S}^1)$ as the algebra of smooth functions vanishing out of I.*

Definition 10.49. *Let $\mathfrak{Vir}_{1/2}(I)$ be the local Neveu-Schwarz Lie superalgebra, generated by L_f, G_f with $f \in C_I^\infty(\mathbb{S}^1)$, and C central.*

Lemma 10.50. *(Locality) $\mathfrak{Vir}_{1/2}(I)$ and $\mathfrak{Vir}_{1/2}(I^c)$ supercommute.*

Proof. By lemma 10.36, the computation of the brackets involve product of functions in $C_I^\infty(\mathbb{S}^1)$ and $C_{I^c}^\infty(\mathbb{S}^1)$, but $C_I^\infty(\mathbb{S}^1).C_{I^c}^\infty(\mathbb{S}^1) = \{0\}$. □

Definition 10.51. *Let p_0 be the projection on the space generated by the vectors of integer level, $p_1 = 1 - p_0$, $u = p_0 - p_1$ and $\tau(x) = uxu$.*

Definition 10.52. *Let the von Neumann algebra $\mathcal{N}_{pq}^m(I)$ be the minimal von Neumann subalgebra of $B(H_{pq}^m)$ such that the self-adjoint operators of $\mathfrak{Vir}_{1/2}(I)$ (i.e L_f, G_f with $f \in C_I^\infty(\mathbb{S}^1)$ real), are affiliated to it. See definition 10.46 for equivalent definitions. $(\mathcal{N}_{pq}^m(I), \tau)$ is a $\mathbb{Z}_2$-graded von Neumann algebra.*

Corollary 10.53. *(Jones-Wassermann subfactor) $\mathcal{N}_{pq}^m(I) \subset \mathcal{N}_{pq}^m(I^c)^\natural$*

Proof. $\mathfrak{Vir}_{1/2}(I)$ and $\mathfrak{Vir}_{1/2}(I^c)$ supercommute, then, by lemma 10.42 and Nelson's theorem, G_f and G_g supercommute for f and g concentrated on I and I^c. So is for the von Neumann algebra they generate. □

Theorem 10.54. *(Reeh-Schlieder theorem) Let $v \in H_{pq}^m$ be a non-null vector of finite level, then, $\mathcal{N}_{pq}^m(I).v$ is dense in H_{pq}^m (i.e. v is a cyclic vector).*

Proof. It's a general principle of local algebra, see [99] p 502. □

10.5 Real and complex fermions

Recall 10.55. *(The complex Clifford algebra, see [99]) Let H be a complex Hilbert space, the complex Clifford algebra Cliff(H) is the unital $\star$-algebra generated by a complex linear map $f \mapsto a(f)$ $f \in H$ satisfying:*

$$[a(f), a(g)]_+ = 0 \quad \text{and} \quad [a(f), a(g)^\star]_+ = (f, g)$$

The complex Clifford algebra as a natural irreducible representation π on the fermionic Fock space $\mathcal{F}(H) = \Lambda H = \bigoplus_{n=0}^{\infty} \Lambda^n H$ (with $\Lambda^0 H = \mathbb{C}\Omega$ and Ω the vacuum vector), given by $\pi(a(f))\omega = f \wedge \omega$ bounded. Let $c(f) = a(f) + a(f)^\star$ satisfying $[c(f), c(g)]_+ = 2Re(f, g)$ and generating the real Clifford algebra. Warning, c is only $\mathbb{R}$-linear. We have the correspondence $a(f) = \frac{1}{2}(c(f) - ic(if))$. Now if P is a projector on H, we can define a new irreducible representation π_P of the complex Clifford algebra by $\pi_P(a(f)) = \frac{1}{2}(c(f) - ic(\mathcal{I}f))$, where $\mathcal{I}$ is the multiplication by i on PH and by $-i$ on $(I - P)H$, ie, $\mathcal{I} = iP - i(I - P) = i(2P - I)$. We know that π_P and π_Q are unitary equivalent if $P - Q$ is an Hilbert-Schmidt operator. Now, a unitary $u \in U(H)$ is implemented in π_P if $\pi_P(a(u.f)) = U\pi_P(a(f))U^\star$ with U unitary, unique up to a phase. But $\pi_P(a(u.f)) = \pi_Q(a(f))$ with $Q = u^\star P u$. Then, u is implemented in π_P if $[P, u]$ is Hilbert-Schmidt.

Recall 10.56. *More generally, taking a real Hilbert space H, we have the real Clifford algebra: $[c(f), c(g)]_+ = 2(f, g)$, $f, g \in H$. Then, we define a complex structure $\mathcal{I}$ with $\mathcal{I}^2 = -Id$. We obtain the complex Hilbert space $H_{\mathcal{I}}$ and then we can define the complex Clifford algebra by: $A(f) = \frac{1}{2}(c(f) - ic(\mathcal{I}f))$, acting irreducibly on the fermionic Fock space $\mathcal{F}_{\mathcal{I}} = \Lambda H_{\mathcal{I}}$. Now, the quantisation condition is: $u \in O(H)$ is implemented in $\mathcal{F}_{\mathcal{I}}$ if $[u, \mathcal{I}]$ is Hilbert-Schmidt. This quantisation due to Segal can be deduce form the condition on the complex case, using the doubling construction described below.*

Example 10.57. *(The Neveu-Schwarz real fermions)*
Let the real Hilbert space of anti-periodic functions $H_{NS} = \{f : \mathbb{R} \to \mathbb{R} | f(\theta + 2\pi) = -f(\theta)\}$ with basis, $\{cos(r\theta), sin(r\theta) | r \in \mathbb{Z} + 1/2\}$, let the complex structure $\mathcal{I}$ defined by $\mathcal{I}cos(r\theta) = sin(r\theta)$ and $\mathcal{I}sin(r\theta) = -cos(r\theta)$. Then we obtain the operators $c(f)$ acting irreducibly on the fermionic Fock space we call $\mathcal{F}_{NS}$. Then, we define $\psi_n = c(cos(n\theta)) + ic(sin(n\theta))$. Now, $\psi_n^\star = \psi_{-n}$ and $[\psi_m, \psi_n]_+ = \delta_{m+n} Id$. The fermionic Fock space can be identified with the irreducible positive energy representation already studied.

Recall 10.58. *(The doubling construction) This is a precise mathematical version of the following physicists slogan: "a complex fermion is equivalent to two real fermions ". We start with a real Hilbert space H and we take $H \oplus iH$ (as a real Hilbert space, iH is the same as H). Let $v = \xi \oplus i\eta$, we define a real Clifford algebra by $c(v) = c(\xi) + c(i\eta)$, acting irreducibly on $\mathcal{F}(H) \otimes \mathcal{F}(iH)$. Then we define $a(v) = \frac{1}{2}(c(v) - ic(iv))$ satisfying the complex Clifford relation on the complex Hilbert space $H \oplus iH$. The operator $\mathcal{I}$ on H extends naturally into a unitary operator on $H \oplus iH$. Now, because $\mathcal{I}^2 = -Id$, it has the form $\mathcal{I} = i(2P - I)$, with P an orthogonal projection. Then the action of the operator $a(v)$ on $\mathcal{F}(H) \otimes \mathcal{F}(iH)$ can be identified with the representation π_P above, by the unique unitary sending $\Omega \otimes \Omega$ to Ω.*

Example 10.59. *We apply to the previous example: in this case, $H_{NS} \oplus iH_{NS} = \{f : \mathbb{R} \to \mathbb{C} | f(\theta + 2\pi) = -f(\theta)\}$. But then the multiplication with $e^{i\theta/2}$ gives an identification with $L^2(\mathbb{S}^1, \mathbb{C})$. This construction was already use on [87].*

Recall 10.60. *(The local algebra for complex fermions) Let V be a complex finite dimensional complex vector space and $H = L^2(\mathbb{S}^1, V)$, let P be the projection on the Hardy space $H^2(\mathbb{S}^1, V)$ (the space of function without negative Fourier coefficient). Let I be a proper interval of $\mathbb{S}^1$ and $\mathcal{M}(I)$ be the von Neumann algebra generated by $\pi_P(Cliff(L^2(I, V)))$, then:*

(a) (Haag-Araki duality) $\quad \mathcal{M}(I)^\natural = \mathcal{M}(I^c)$

(b) (Covariance) $u_{\varphi^{-1}} : f \mapsto \sqrt{\varphi'}.f \circ \varphi$ defines a unitary action of $\varphi \in$ Diff$(\mathbb{S}^1)$ on H; this action is implemented in π_P.

(c) The modular action on $\mathcal{M}(I)$ is $\sigma_t(x) = \pi_P(\varphi_t) x \pi_P(\varphi_t)^\star$, with $\varphi_t \in$ Diff$(\mathbb{S}^1)$ the Möbius flow fixing the end point of I. For example, if I is the upper half-circle, then $\partial I = \{-1, +1\}$ and $\varphi_t(z) = \frac{ch(t)z + sh(t)}{sh(t)z + ch(t)}$.

(d) The modular action is ergodic (ie it fixes only the scalar operators), so that $\mathcal{M}(I)$ is a III_1 factor (the hyperfinite one).

Remark 10.61. *By the doubling construction, Diff$(\mathbb{S}^1)$ acts on H_{NS} by:*

$$\pi(\varphi)^{-1}.f = |\varphi'|^{1/2} f \circ \varphi$$

and the action is quantised. We verify directly that $H_{\mathbb{C}} := H_{NS} \oplus i H_{NS}$ admits the orthogonal basis $e_r = e^{ir\theta}$ with $r \in \mathbb{Z} + 1/2$, that $\mathcal{I} = (2P - I)i$, with P the Hardy projection (on the positive modes $r \geq 0$). Now, the Lie algebra of Diff($\mathbb{S}^1$) is the Witt algebra. The infinitesimal version of the previous action is $d_n e_r = -(r + n/2)e_{r+n}$: the action of the Witt algebra on the 1/2-density (see below or [54] p 4). This infinitesimal action of the Witt algebra is implemented on the Fock space $\mathcal{F}_{\mathbb{C}} = \mathcal{F}_{NS} \otimes \mathcal{F}_{NS}$ into the Virasoro derivation on the real fermions: $[L_n, \psi_r] = -(r + n/2)\psi_{r+n}$ (consistent with section 4.4). Let $SU(1,1)$ be the group of $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 - |\beta|^2 = 1$. By the Mobius transformation: $g(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$, $SU(1,1)$ is injected in Diff($\mathbb{S}^1$), and its Lie algebra is generated by d_{-1}, d_0, d_1. Now, we can see directly that $SU(1,1)$ is quantised, because it acts unitarily and commutes with P:

$$\pi(g)^{-1} f(z) = \frac{1}{|\bar{\beta} z + \bar{\alpha}|} f(g(z))$$

Using $|\bar{\beta} z + \bar{\alpha}| = (\bar{\beta} z + \bar{\alpha})^{1/2}(\beta \bar{z} + \alpha)^{1/2}$, for $k \geq 0$:

$$\pi(g)^{-1} z^{k+1/2} = \frac{(\alpha z + \beta)^k z^{1/2}}{(\beta z + \alpha)^{k+1}} \in PH_{\mathbb{C}}$$

Now, the quantised action of $SU(1,1)$ fixes the vacuum vector of the fermionic Fock space, because L_{-1}, L_0, L_1 vanish on the vacuum vector.
Note that the Lie algebra of the modular action is generated by $L_1 - L_{-1}$.

Recall 10.62. *(Takesaki devissage [89]) Let $M \subset B(H)$ be a von Neumann algebra, $\Omega \in H$ cyclic for M and M', $\triangle^{it}$, J the corresponding modular operators ($\triangle^{it} M \triangle^{-it} = M$ and $JMJ = M'$). If $N \subset M$ is a von Neumann subalgebra such that $\triangle^{it} N \triangle^{-it} = N$ (conditional expectation), then:*

(a) $\triangle^{it}$ and J restrict to the modular automorphism group $\triangle_1^{it}$ and conjugation operator J_1 of N for Ω on the closure H_1 of $N\Omega$.

(b) $\triangle_1^{it} N \triangle_1^{-it} = N$ and $J_1 N J_1 = N'$ on H_1.

(c) If p is the projection onto H_1, then $pMp = Np$ and $N = \{x \in M \mid xp = px\}$ (the Jones relations [45])

(d) $H_1 = H \iff M = N$

(e) *The modular group fixes the center. In fact* $\triangle^{it} x \triangle^{-it} = x$ *and* $JxJ = x^\star$ *for* $x \in Z(M) = M \cap M'$.

Definition 10.63. *Let* $\mathcal{M}_{NS}(I)$ *be the von Neumann algebra generated by the real Neveu-Schwarz* ψ_f *with* f *localised on* I.

Lemma 10.64. *(Reeh-Schlieder theorem) Let* $v \in \mathcal{F}_{NS}$ *be a non-null vector of finite level, then,* $\mathcal{M}_{NS}(I).v$ *is dense in* $\mathcal{F}_{NS}$ *(i.e.* v *is a cyclic vector).*

Proof. It's a general principle of local algebra, see [99]. □

Recall 10.65. *A von Neumann algebra* $\mathcal{M}$ *is hyperfinite iff it is injective, ie* $\mathcal{M} \subset B(H)$ *with conditional expectation (see [17]).*

Proposition 10.66. *The local algebra* $\mathcal{M}_{NS}(I)$ *satisfy Haag-Araki duality, covariance for Diff*$(\mathbb{S}^1)$*, and the modular action is geometric and ergodic. In particular,* $\mathcal{M}_{NS}(I)$ *is the hyperfinite* III_1 *factor*

Proof. The covariance is shown in remark 10.61. Then, $\mathcal{M}_{NS}(I)$ is stable by the modular action of $\mathcal{M}(I)$. Now, $\pi_P(\mathcal{M}_{NS}(I)) \subset \mathcal{M}(I) \subset B(H_{\mathbb{C}})$ with conditional expectation, so $\pi_P(\mathcal{M}_{NS}(I))$ is hyperfinite. Next by Takesaki devissage the modular action of $\pi_P(\mathcal{M}_{NS}(I))$ is ergodic, so it's the hyperfinite III_1 factor. Now, by definition of the type III, every subrepresentations are equivalents, but one copy of $\mathcal{F}_{NS}$ is a subrepresentation. So $\mathcal{M}_{NS}(I)$ is the hyperfinite III_1 factor. Finally, the Haag-Araki duality for $\mathcal{M}_{NS}(I)$ comes from the Haag-Araki duality for $\mathcal{M}(I)$, the Reeh-Schlieder theorem and the Takesaki devissage. □

10.6 Properties of local algebras deducable by devissage from loop superalgebras

In this section we will deduce a few partial results on the local von Neumann algebra of Neveu-Schwarz, using devissage from the loop superalgebras, but it's not enough. In the next section, we will prove more general definitive result by devissage from real and complex fermions (in particular this will imply all the result proved here).

Remark 10.67. $\mathcal{F}_{NS}^{\mathfrak{g}} = \mathcal{F}_{NS}^{\otimes 3}$.

Lemma 10.68. *Let N_1, N_2 be von Neumann algebra, with modular action $\sigma_t^{\Omega_1}$ and $\sigma_t^{\Omega_2}$, then, the modular action on $N_1 \overline{\otimes} N_2$ is $\sigma_t^{\Omega_1 \otimes \Omega_2} = \sigma_t^{\Omega_1} \otimes \sigma_t^{\Omega_2}$*

Proof. By KMS uniqueness (see [99] p 493). □

Definition 10.69. *Let $L(j,\ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ be the irreducible representation of the $\mathfrak{g}$-supersymmetric algebra $\widehat{\mathfrak{g}}$. Let the local von Neumann algebra $\mathcal{N}_j^{\ell}(I)$ generated by $\pi_j^{\ell}(g) \otimes \pi_{NS}^{\mathfrak{g}}(g)$ and $1 \otimes x$, with $g \in L_I G$ and $x \in \mathcal{M}_{NS}^{\mathfrak{g}}(I)$.*

Proposition 10.70. *$\mathcal{N}_j^{\ell}(I) = \pi_j^{\ell}(L_I G) \otimes \mathcal{M}_{NS}^{\mathfrak{g}}(I)$.*

Proof. $\pi_{NS}^{\mathfrak{g}}(g)$ supercommutes with $\mathcal{M}_{NS}^{\mathfrak{g}}(I^c)$, so by the Haag-Araki duality $\pi_{NS}^{\mathfrak{g}}(g) \in \mathcal{M}_{NS}^{\mathfrak{g}}(I)$. We deduce that $\mathcal{N}_j^{\ell}(I)$ is generated by $\pi_j^{\ell}(g) \otimes 1$ and $1 \otimes x$. The result follows. □

Theorem 10.71. *Combining the work of A. Wassermann [99] on local loop group and the previous work on Neveu-Schwarz fermions, we obtain*

(a) *(Local equivalence) For every representations H_j^{ℓ}, there is a unique $\star$-isomorphism $\pi_j^{\ell} : \mathcal{N}_0^{\ell}(I) \to \mathcal{N}_j^{\ell}(I)$ coming from $\pi_0^{\ell}(B_f^a) \mapsto \pi_j^{\ell}(B_f^a) = U.\pi_0^{\ell}(B_f^a).U^{\star}$ and $\pi_0^{\ell}(\psi_g^a) \mapsto \pi_j^{\ell}(\psi_g^a) = U.\pi_0^{\ell}(\psi_g^a).U^{\star}$, with $U : H_0^{\ell} \to H_j^{\ell}$ unitary.*

(b) *(Covariance) $\varphi \in$Diff$(\mathbb{S}^1)$ acts unitarily on H_j^{ℓ} with $\pi_j^{\ell}(\varphi) B_f^a \pi_j^{\ell}(\varphi)^{\star} = B_{f \circ \varphi^{-1}}^a$ and $\pi_j^{\ell}(\varphi) \psi_g^b \pi_j^{\ell}(\varphi)^{\star} = \psi_{\alpha.g \circ \varphi^{-1}}^b$, with $\alpha = \sqrt{(\varphi^{-1})'}$, a kind of Radon-Nikodym correction (which preserves the group action) to be compatible with the Lie structure, ie be unitary on $L^2(\mathbb{S}^1)_{\mathbb{R}}$.*

(c) *The modular action on $\mathcal{N}_0^{\ell}(I)$ is $\sigma_t(x) = \pi_0^{\ell}(\varphi_t) x \pi_0^{\ell}(\varphi_t)^{\star}$, with $\varphi_t \in$ Diff$(\mathbb{S}^1)$ the Möbius flow fixing the end point of I. For example, if I is the upper half-circle, then $\partial I = \{-1, +1\}$ and $\varphi_t(z) = \frac{ch(t)z + sh(t)}{sh(t)z + ch(t)}$.*

(d) *$\mathcal{N}_j^{\ell}(I)$ is the hyperfinite III_1 factor.*

(e) *$\mathcal{N}_0^{\ell}(I) = \mathcal{N}_0^{\ell}(I^c)^{\natural}$ (Haag-Araki duality)*

(f) *$\mathcal{N}_j^{\ell}(I) \subset \mathcal{N}_j^{\ell}(I^c)^{\natural}$ (Jones-Wassermann subfactor)*

(g) *$\mathcal{N}_j^{\ell}(I)^{\natural} \cap \mathcal{N}_j^{\ell}(I^c)^{\natural} = \mathbb{C}$ (irreducibility of the subfactor)*

Lemma 10.72. *The operators G_f and L_h act continuously on $\mathcal{H}_j^\ell$, the L_0-smooth completion of $L(j,\ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$.*

Proof. $\mathcal{H}_j^\ell$ decompose into some irreducible smooth representations of the discrete series $(\mathcal{H}_{pq}^m)$, the result follows by corollary 10.39 □

Notation 10.73. *Let $p = 2j+1$, $q = 2k+1$ and $m = \ell+2$, then, from now, we can note $\mathcal{H}_{pq}^m$ as $\mathcal{H}_{jk}^\ell$. It will be a more conveniant notation for the fusion rules computations*

Recall 10.74. *(Kac-Todorov coset construction) (see section 6.2 or [52])*

$$\mathcal{H}_0^0 \otimes \mathcal{H}_j^\ell = \bigoplus_{\substack{1\leq q\leq m+1 \\ p\equiv q[2]}} \mathcal{H}_{jk}^\ell \otimes \mathcal{H}_k^{\ell+2}, \text{ and}$$

$$\pi_0^0(G_f)\otimes I + I \otimes \pi_j^\ell(G_f) = \sum [\pi_{jk}^\ell(G_f) \otimes I + I \otimes \pi_k^{\ell+2}(G_f)]$$

Lemma 10.75. *We write some usefull relations on $\mathcal{H}_j^\ell$:*

(a) $[\psi_f^a, \psi_h^b]_+ = \delta_{a,b}(f,h)_{\mathbb{R}}$

(b) $[B_f^a, B_h^b] = [B^a, B^b]_{f.h} + (\ell+2)\delta_{a,b}(d(f),h)_{\mathbb{R}}$

(c) $[G_f, B_h^a] = -(\ell+2)^{1/2}\psi_{f.d(h)}^a$

(d) $[G_f, \psi_h^a]_+ = (\ell+2)^{-1/2}B_{f.h}^a$

Proof. Direct by computation from section 5. □

Let π be a positive energy representation of the loop superalgebra $\widehat{\mathfrak{g}}$. We know, it is always of the form $H \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$, where H is a positive energy representation σ of LG (non necessarily irreducible). The Clifford algebra acts on the second factor and the loop group acts by tensor product. We have already seen that the von Neumann algebra $\pi(\widehat{\mathfrak{g}}_I)''$ is naturally a tensor product of von Neumann algebras (proposition 10.70). On the other hand, we have the operators $\pi(L_f)$, $\pi(G_f)$, given by the Sugawara construction (first sections) and $\pi(\varphi)$ with $\varphi \in$Diff($\mathbb{S}^1$). The L_f gives a projective representation of the Witt algebra, so exponentiate them give the element of Diff($\mathbb{S}^1$). The action of Diff($\mathbb{S}^1$) is also given by a tensor product of representation. The following property will be fundamental.

Theorem 10.76. *$\pi(\varphi) \in \pi(\widehat{\mathfrak{g}}_I)''$ and $\pi(L_f)$, $\pi(G_f)$ are affilated to $\pi(\widehat{\mathfrak{g}}_I)''$, if φ and f are concentrate on I^c.*

Remark 10.77. *We will prove it for Diff($\mathbb{S}^1$), and so for the L_f, in general, but for G_f, only for the vacuum representation (general proof on the next section).*

Proof. For the vacuum representation, it's an immediate consequence of the Haag-Araki duality. Now, we can restrict to π irreducible. For Diff($\mathbb{S}^1$), because we have Haag-Araki duality on $\mathcal{F}^{\mathfrak{g}}_{NS}$, it's sufficient to prove that $\sigma(\mathrm{Diff}_I(\mathbb{S}^1))'' \subset \sigma(L_I G)''$. By local equivalence, there exists a unitary U intertwining σ and the vacuum representation σ_0. By Haag duality and covariance $\sigma_0(\mathrm{Diff}_I(\mathbb{S}^1))'' \subset \sigma_0(L_I G)''$. Then, $U\sigma_0(\varphi)U^\star \subset \sigma(L_I G)'' \subset \sigma(L_{I^c}G)'$. On the other hand, $\sigma(\varphi) \in \sigma(L_{I^c}G)'$. So, $T = \sigma(\varphi^{-1})U\sigma_0(\varphi)U^\star \in \sigma(L_{I^c}G)'$. But, $T \in \sigma(L_I G)'$ by covariance relation. Now, by irreducibility $\sigma(L_I G)' \cap \sigma(L_{I^c}G)' = \mathbb{C}$, so T is a constant. The result follows. □

Theorem 10.78. *Haag-Araki duality holds for the Neveu-Schwarz algebra.*

Proof. Let K_0 be the vacuum representation Π_0 of $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}}$. The operators L_f and G_f of the coset construction act on K_0. We will prove that if f is concentrate on the interval I, G_f is affiliated with $\Pi_0(\widehat{\mathfrak{g}_I} \oplus \widehat{\mathfrak{g}_I})''$. Then, because $[G_{f_1}, G_{f_2}]_+ = L_{f_1 f_2} + constant$, L_f is also affiliated. By Haag-Araki duality, it suffices to prove that the operators G_f supercommutes with the bosons (elememt of the loop algebra) and the fermions, concentrate on I^c. Let $A = G_f$ and let B be either the bosonic operator or the fermionic operator conjugate by the Klein transformation. They are formally self-adjoint for f real. By relation 10.75, they commute formally. By the Sobolev estimates and the Glimm-Jaffe-Nelson theorem, $A^2 + B^2$ is essentially self-adjoint. So Nelson's theorem imply the commutation in term of bounded function.

Now, by the coset construction, and the Reeh-Schlieder theorem, the bounded functions of the G_f and L_f applied on the vacuum vector of K_0 generate the vacuum positive energy representation of the Neveu-Schwarz algebra. The Haag-Araki duality follows by Takesaki devissage. □

Lemma 10.79. *(Covariance) Let $\varphi \in$ Diff($\mathbb{S}^1$), then $\pi^\ell_j(\varphi)\pi^\ell_j(G_f)\pi^\ell_j(\varphi)^\star = \pi^\ell_j(G_{\beta.f\circ\varphi^{-1}})$, with $\beta = 1/\alpha$, and $\alpha = \sqrt{(\varphi^{-1})'}$ and $f \in C^\infty(\mathbb{S}^1)$.*

Proof. $\pi_j^\ell(\varphi)[G_f, B_h^a]\pi_j^\ell(\varphi)^\star = -(\ell+2)^{-1/2}\psi^a_{\alpha.(f\circ\varphi^{-1}).(d(h)\circ\varphi^{-1})} =$
$-(\ell+2)^{-1/2}\psi^a_{\beta.(f\circ\varphi^{-1}).d(h\circ\varphi^{-1})} = [G_{\beta.f\circ\varphi^{-1}}, \pi_j^\ell(\varphi)B_h^a\pi_j^\ell(\varphi)^\star]$
Idem, $\pi_j^\ell(\varphi)[G_f, \psi_h^a]_+\pi_j^\ell(\varphi)^\star = [G_{\beta.f\circ\varphi^{-1}}, \pi_j^\ell(\varphi)\psi_h^a\pi_j^\ell(\varphi)^\star]_+$.
Then, by irreducibility, $\pi_j^\ell(\varphi)G_f\pi_j^\ell(\varphi)^\star - G_{\beta.f\circ\varphi^{-1}}$ is a constant operator; it's also an odd operator, so it's zero. □

Corollary 10.80. *By the coset construction, the covariance relation runs also on the discrete series representations of the Neveu-Schwarz algebra.*

10.7 Local algebras and fermions

In [99], the representation of $LSU(2)$ at level 1 are constructed using two complex fermions. This corresponds to the complex Clifford algebra construction on $\Lambda(L^2(\mathbb{S}^1, \mathbb{C}^2)) = \mathcal{F}_{\mathbb{C}^2}$. The level ℓ representations are obtained taking $\mathcal{F}_{\mathbb{C}^2}^{\otimes\ell}$. Then, the level ℓ representations of the corresponding loop superalgebra are realized on the tensor product of this Fock space and the space $\mathcal{F}_{NS}^{\mathfrak{g}}$, of three fermions. As vertex superalgebra, the vertex superalgebra of the loop superalgebra defines a vertex sub-superalgebra of the vertex superalgebra of $\mathcal{F}_{\mathbb{C}^2}^{\otimes\ell} \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$.

Let $H = \{f : \mathbb{R} \to \mathbb{R} \mid f(x+2\pi) = -f(x)\}$. Let $\mathcal{F}_{NS}^V = \Lambda(H \otimes V)$, then, $\mathcal{F}_{NS}^{V_1\oplus V_2} = \mathcal{F}_{NS}^{V_1} \otimes \mathcal{F}_{NS}^{V_2}$. Now, considering the diagonal inclusion $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$, $H \otimes (\mathfrak{g}\oplus\mathfrak{g}) \ominus H \otimes \mathfrak{g} = H \otimes [(\mathfrak{g}\oplus\mathfrak{g}) \ominus \mathfrak{g}] = H \otimes [(\mathfrak{g}\oplus\mathfrak{g})/\mathfrak{g}]$. Then, we easily seen that in the Kac-Todorov construction described before:

$$\mathcal{F}_{NS}^{\mathfrak{g}} \otimes (\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(i,\ell)) = \bigoplus L(c_m, h_{pq}^m) \otimes (\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(j, \ell+2)),$$

we can simplify by a factor $\mathcal{F}_{NS}^{\mathfrak{g}}$ to obtain the following GKO construction:

$$\mathcal{F}_{NS}^{\mathfrak{g}} \otimes L(i,\ell) = \bigoplus L(c_m, h_{pq}^m) \otimes L(j, \ell+2)),$$

preserving the coset action of the Neveu-Schwarz algebra. It's also true replacing $L(i,\ell)$ by a (non necessarily irreducible) positive energy representation $\mathcal{H}$ of level ℓ. Then the coset action of the Neveu-Schwarz algebra on $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes \mathcal{H}$ is described by (see also [35] p114):

(a) $L_n^{gko} = L_n^{\mathfrak{g}\oplus\mathfrak{g}} - L_n^{\mathfrak{g}}$

(b) $G^{gko}(z) = \sum G_r^{gko} z^{-r-3/2} = \Phi(\tau_{gko}, z)$
with Φ the module-vertex operator on $\mathcal{F}_{NS}^{\mathfrak{g}} \otimes \mathcal{H}$ (see section 5.3), and $\tau_{gko} = (2(\ell+2)(\ell+4))^{-1/2}(\ell\tau_1 - 2\tau_2)$, with τ_1, τ_2 as in definition 5.38.

Note that: $\Phi(\ell\tau_1 - 2\tau_2, z) = [\sum_k(\ell\psi_k(z) \otimes X_k(z) - I \otimes \psi_k(z)S_k(z))]$
$= [\sum_k(\ell\psi_k(z) \otimes X_k(z) - \frac{i}{3}\sum_{ij}\Gamma_{ij}^k I \otimes \psi_i(z)\psi_j(z)\psi_k(z))]$
$= [\ell\sum_k(\psi_k(z) \otimes X_k(z) - 2i\sqrt{2}I \otimes \psi_1(z)\psi_2(z)\psi_3(z)]$.

Now, $\mathcal{F}_{\mathbb{C}^2}^{\otimes\ell}$ is a level ℓ representation of the loop algebra (containing all the irreducibles). We apply the previous GKO construction on $\mathcal{F}_{\mathbb{C}^2}^{\otimes\ell} \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$. Let $\mathcal{N}(I) = \mathcal{M}(I)^{\otimes\ell} \otimes \mathcal{M}_{NS}^{\mathfrak{g}}(I)$ be the local von Neumann algebra generated by the corresponding real and complex fermions. Let π_{gko} be the coset representation of $\mathfrak{Vir}_{1/2}$ on. Now, as previously, $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))$ supercommutes with $\mathcal{N}(I^c)$, then by Haag-Araki duality $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))'' \subset \mathcal{N}(I)$. Now, π_{gko} is a direct sum of all the irreducible positive energy representation π_i (with multiplicities) of the Neveu-Schwarz algebra. As previously (see lemma 10.79), $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))''$ is stable under the modular action of $\mathcal{N}(I)$. So we can apply the Takesaki devissage. We deduce that $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))''$ is the hyperfinite III_1 factor. By the property of the type III, every subrepresentations of π_{gko} are equivalents; in particular all the $\pi_i(\mathfrak{Vir}_{1/2}(I))''$ are the hyperfinite III_1-factor, and are equivalents to $\pi_0(\mathfrak{Vir}_{1/2}(I))''$: it's the local equivalence for the Neveu-Schwarz algebra. Finally, let Ω be the vacuum vector of $\mathcal{F}_{\mathbb{C}^2}^{\otimes\ell} \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$, then clearly $\pi_{gko}(\mathfrak{Vir}_{1/2}(I))''\Omega$ is dense (Reeh-Schlieder theorem) on the vacuum representation of $\mathfrak{Vir}_{1/2}$ tensor its corresponding multiplicity M_0. Let P be the projection on, then P commutes with the modular operators (because the vacuum vector is invariant) and with the Klein operator κ. But by Takeaki devissage $PN(I)P = \pi_{gko}(\mathfrak{Vir}_{1/2}(I))''P = [\pi_0^{\otimes M_0}(\mathfrak{Vir}_{1/2}(I))'']$.
So $\kappa JPN(I)PJ\kappa^\star = P\kappa JN(I)J\kappa^\star P = PN(I)^\natural P = PN(I^c)P$
$= [\pi_0^{\otimes M_0}(\mathfrak{Vir}_{1/2}(I^c))''] = [\pi_0^{\otimes M_0}(\mathfrak{Vir}_{1/2}(I))^\natural]$. The Haag-Araki duality for the Neveu-Schwarz algebra follows.

Corollary 10.81. *(Generalized Haag-Araki duality)*
$\pi_{gko}(\mathfrak{Vir}_{1/2}(I))'' = \pi_{gko}(\mathfrak{Vir}_{1/2})'' \cap \mathcal{N}(I)$

Corollary 10.82. $\pi_0(\mathfrak{Vir}_{1/2}(I))''$ *is generated by chains of compressed fermions concentrate in I.*

Proof. Immediate from Jones relation: $p_0\mathcal{N}(I)p_0 = \pi_{gko}(\mathfrak{Vir}_{1/2}(I))''p_0$. □

Now because π_{gko} contains all the irreducible positive energy representations π_i of charge c_m , we deduce that:

Corollary 10.83. *Let π be the direct sum of all the π_i.*
To simplify we note $\pi = \pi_0 \oplus ... \oplus \pi_n$. Then $\mathcal{A} := \pi(\mathfrak{Vir}_{1/2}(I))''$
$= \{T = \begin{pmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_n \end{pmatrix}$ *| T supercommutes with $\mathcal{B}\}$ with $\mathcal{B} = \{ \begin{pmatrix} S_{11} & \dots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & ... & S_{nn} \end{pmatrix}$*
such that S_{ij} is a chain of compressed fermions $p_i\phi(f)p_j$ concentrate on $I^c\}$

By definition $\mathcal{A}^\natural = \mathcal{B}$. Now, let q_i the projection on π_i, then $q_i \in \mathcal{A}^\natural$, so, $(q_i\mathcal{A})^\natural = p_i\mathcal{B}p_i$. Then $(q_i\mathcal{A})^\natural = \pi_i(\mathfrak{Vir}_{1/2}(I))^\natural = \{S_{ii} \mid ...\}$.

Corollary 10.84. *$\pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural$ is generated by chains of compressed fermions concentrated in I.*

Remark 10.85. *In the next section, we will see by unicity that the compression of complex fermions give a primary fields of charge $\alpha = (1/2, 1/2)$, and the compression of a real fermions give primary fields of charge $\beta = (0, 1)$.*

Remark 10.86. *We will see that the supercommutation relation on the vacuum (Haag-Araki duality) is replaced by braiding relations of primary fields. As consequence, we directly see that $\pi_i(\mathfrak{Vir}_{1/2}(I))^\natural$ and $\pi_i(\mathfrak{Vir}_{1/2}(I^c))^\natural$ do not necessarily supercommute if $i \neq 0$. Then, the formulation of the local von Neumann algebra, generated by chains of primary fields (with braiding), shows explicitly the failure of Haag-Araki duality outside of the vacuum.*

11 Primary fields

11.1 Primary fields for $LSU(2)$

This section is an overview of the primary field theory of $LSU(2)$, for a more detailed exposition see [99] and [91].
Let V be a representation of $G = SU(2)$ or $\mathfrak{g} = \mathfrak{sl}_2$.

Definition 11.1. *Let λ, $\mu \in \mathbb{C}$, we define the ordinary representations of $L\mathfrak{g} \rtimes \mathfrak{Vir}$ as $\mathcal{V}_{\lambda,\mu}$, generated by (v_i), $v \in V$ and $i \in \mathbb{Z}$, and:*

(a) $L_n.v_i = -(i + \mu + n\lambda)v_{i+n}$

(b) $X_m.v_i = (X.v)_{m+i} \quad (X \in \mathfrak{g})$

Definition 11.2. *Let L_i^ℓ and L_j^ℓ be irreducible representation of $L\mathfrak{g}$, of level ℓ and spin i and j. We define a primary field as a linear operator:*

$$\phi : L_j^\ell \otimes \mathcal{V}_{\lambda,\mu} \to L_i^\ell$$

that intertwines the action of $L\mathfrak{g} \rtimes \mathfrak{Vir}$. We call V the charge of ϕ.

Recall 11.3. *Let $h_i^\ell = \frac{i^2+i}{\ell+2}$ the lowest eigenvalue of L_0 on L_i^ℓ (see theorem 5.50). The eigenspace is the $\mathfrak{sl}_2$-module V_i.*

Definition 11.4. *For $w \in \mathcal{V}_{\lambda,\mu}$, let $\phi(w) : L_j^\ell \to L_i^\ell$*

Lemma 11.5. *Let $X \in L\mathfrak{g} \rtimes \mathfrak{Vir}$, then $[X, \phi(w)] = \phi(X.v)$*

Proof. As for the proof of lemma 11.31. □

Lemma 11.6. *ϕ non-null implies that $\mu = h_j^\ell - h_i^\ell$.*

Lemma 11.7. *(Gradation) $\phi(v_n).(L_j^\ell)_{s+h_j^\ell} \subset (L_i^\ell)_{s-n+h_i^\ell}$*

Definition 11.8. *Let $h = 1 - \lambda$ be the conformal dimension of ϕ, and $\triangle = 1 - \lambda + \mu = h + h_j^\ell - h_i^\ell$; we define:*

$$\phi(v, z) = \textstyle\sum_{n\in\mathbb{Z}} \phi(v_n) z^{-n-\triangle} \quad (v \in V).$$

Lemma 11.9. *(Compatibility condition)*

(a) $[L_n, \phi(v,z)] = z^n[z\frac{d}{dz} + (n+1)h]\phi(v,z)$

(b) $[X_m, \phi(v,z)] = z^m\phi(X.v, z)$

Proof. Direct from the definition. □

Lemma 11.10. *If $\widetilde{\phi}(z,v)$ satisfy the compatibility condition, then, it gives a primary fields for $LSU(2)$.*

Proof. It's an easy verification. □

Proposition 11.11. *(Initial term) A primary field $\phi : L_j^\ell \otimes \mathcal{V}_{\lambda,\mu} \to L_i^\ell$ with every parameters fixed, is completly determined by its initial term:*

$$\phi_0 : V_j \otimes V \to V_i$$

Proof. Idem, by intertwining relation; see [99] p 513 for details. □

Proposition 11.12. *(Unicity) If $V = V_k$ is irreducible, the space of such primary field is at most one-dimensional.*

Proof. ϕ_0 is an intertwining operator, ie, $\phi_0 \in Hom_{\mathfrak{g}}(V_j \otimes V_k, V_i)$ the multiplicity space at V_i of $V_j \otimes V_k = V_{|j-k|} \oplus V_{|j-k|+1} \oplus ... \oplus V_{j+k}$ (Clebsch-Gordan), so at most one-dimensional. □

Remark 11.13. *As for $\mathfrak{Vir}_{1/2}$ (see remark 11.39), with $(A_nB)(z)$ formula, we define inductively the $L\mathfrak{g}$-module L_k^ℓ from ϕ.*

Corollary 11.14. $\mu = h_j^\ell - h_i^\ell$ *and* $1 - \lambda = h = h_k^\ell$.

Definition 11.15. *We note ϕ as $\phi_{ij}^{k\ell}$, $\triangle$ as $\triangle_{ij}^{k\ell} = h_j^\ell - h_i^\ell + h_k^\ell$.*

We call ϕ a primary field of spin k; in our work, we just need to consider primary fields of spin 1/2 and 1:

Proposition 11.16. *Up to a multiplication by a rational power of z:*
(a) The compression of complex fermions gives primary fields of spin $1/2$.
(b) The compression of real fermions gives primary fields of spin 1.

Proof. We just check the compatibility condition. The calculation can also be made on the vertex algebra of the fermions. See also [99] p 515. □

Definition 11.17. *We note $\phi_{ij}^{k\ell}$ be the primary field from L_j^ℓ to L_i^ℓ, of spin k. It's defined up to a multiplicative constant and is possibly null.*

Recall 11.18. *(Constructible primary fields of spin $1/2$ or 1, see [99]).*

(a) $\phi_{ij}^{\frac{1}{2}\ell}$ is non-null iff $j = i \pm 1/2$ and $i + j + 1/2 \leq \ell$

(b) $\phi_{ij}^{1\ell}$ is non-null iff $j = i - 1, i,$ or $i + 1$ and $i + j + 1 \leq \ell$

with the restriction that: $0 \leq \ i, j \ \leq \ell/2$

Proposition 11.19. *Every primary fields $\phi_{ij}^{k\ell}(w) : L_j^\ell \to L_i^\ell$ of spin $k = 1/2$ or 1, are constructibles as compressions of complex and real fermions. respectively.*

Proof. For spin 1/2 primary fields see [99] p 515.

Now, for spin 1: note that at level $\ell = 2$, there are only 0, 1/2 and 1 as possible spins. But, the real Neveu-Schwarz fermions $\mathcal{F}_{NS}^{\mathfrak{g}}$ equals to $L_0^2 \oplus L_1^2$, and the real Ramond fermions $\mathcal{F}_R^{\mathfrak{g}}$ equals to $L_{1/2}^2$, as $LSU(2)$ module (see corollary 9.7 and [35] p116). Then, compressions of the fermion field $\psi(z, v)$, with $v \in V_1 = \mathfrak{g}$ on $\mathcal{F}_{NS}^{\mathfrak{g}}$ or $\mathcal{F}_R^{\mathfrak{g}}$ give the spin 1 primary fields at level 2, by unicity and compatibility condition.

Now, $L_j^\ell \otimes L_k^{\ell'} = L_{|j-k|}^{\ell+\ell'} \oplus L_{|j-k|+1}^{\ell+\ell'} \oplus ... \oplus L_{j+k}^{\ell+\ell'}$, so:

(a) $\phi_{i,i-1}^{1\ell+2}(v)$ is the compression of $\phi_{01}^{1,2}(v) \otimes I : L_1^2 \otimes L_{i-1}^\ell \to L_0^2 \otimes L_{i-1}^\ell$.

(b) $\phi_{i,i+1}^{1\ell+2}(v)$ is the compression of $\phi_{10}^{1,2}(v) \otimes I : L_0^2 \otimes L_i^\ell \to L_1^2 \otimes L_i^\ell$.

(c) $\phi_{i,i}^{1\ell+2}(v)$ is the compression of $\phi_{01}^{1,2}(v) \otimes I : L_0^2 \otimes L_i^\ell \to L_1^2 \otimes L_i^\ell$.

The result follows. □

Corollary 11.20. *The primary fields of spin $k = 1/2$ or 1 are bounded and identifying the L^2-completion of $\mathcal{V}_{\lambda,\mu}$ with $L^2(\mathbb{S}^1, V_k)$, we obtain $\phi(f)$ for $f \in L^2(\mathbb{S}^1, V_k)$, with: $\|\phi(f)\| \leq K\|f\|_2$.*

Recall 11.21. *(Braiding relations)*
In [91] and [99], the braiding relations of spin $1/2$ primary fields are given by reduced 4-point functions $f : \mathbb{C} \to W$, with W finite dimensional. We give an overview of this theory:

Let $F_j(z, w) = (\phi_{kj}^{\frac{1}{2}\ell}(u, z).\phi_{ji}^{\frac{1}{2}\ell}(v, w)\Omega_i, \Omega_k)$, then by gradation it equals:

$$\sum_{m\geq 0}(\phi_{kj}^{\frac{1}{2}\ell}(u,m).\phi_{ji}^{\frac{1}{2}\ell}(v,-m)\Omega_i,\Omega_k)z^{-m-\triangle}w^{m-\triangle'} = f_j(\zeta)z^{-\triangle}w^{-\triangle'}$$

with $f_j(\zeta) = \sum_{m\geq 0}(\phi_{kj}^{\frac{1}{2}\ell}(u,m).\phi_{ji}^{\frac{1}{2}\ell}(v,-m)\Omega_i,\Omega_k)\zeta^m$ *and* $\zeta = w/z$. *The function* f_j *are holomorphic for* $|\zeta| < 1$. *Now,* $\phi_{kj}^{\frac{1}{2}\ell}$ *and* $\phi_{ji}^{\frac{1}{2}\ell}$ *are non-zero field, ie* $Hom_{\mathfrak{g}}(V_j \otimes V_{1/2}, V_k)$ *and* $Hom_{\mathfrak{g}}(V_i \otimes V_{1/2}, V_j)$ *are* 1*-dimensional space. Then, the set of possible such* j *generate the space* $W = Hom_{\mathfrak{g}}(V_{1/2} \otimes V_{1/2} \otimes V_i, V_j)$. *Then, we consider the vector* $f_j(\zeta)$ *as a vector in* W. *Let* $\tilde{f}_j = \zeta^{\lambda_j} f_j$, *(with* $\lambda_j = (j^2 + j - i^2 - i - 3/4)/(\ell+2)$*), called the reduced four points functions.* $\tilde{f}_j(z)$ *is defined on* $\{z : |z| < 1, z \notin [0,1[\}$. *It satisfy the Knizhnik-Zamolodchikov ordinary differential equation, equivalent to the hypergeometric equation of Gauss:*

$$\tilde{f}'(z) = A(z)\tilde{f}(z), \text{ with } A(z) = \frac{P}{z} + \frac{Q}{1-z}$$

with P, $Q \in End(W)$. *It's proved in [99] section 19, the existence of a holomorphic gauge transformation* $g : \mathbb{C}\backslash[1,\infty[\to GL(W)$ *with* $g(0) = I$ *such that:* $g^{-1}Ag - g^{-1}g' = P/z$. *The solution of the ODE is then* $\tilde{f}(z) = g(z)z^P T$, *with* T *an eigenvector of* P. *So, up to a power of* z, *the solutions* $\tilde{f}_j(z)$ *are just the columns of* $g(z)$ *(in the spectral base of* P*). Now, let* $r_j(z) = \tilde{f}_j(z^{-1})$ *on* $\{z : |z| > 1, z \notin [1,\infty[\}$, *then* r_j *satisfy clearly the equation:*

$$r'(z) = B(z)r(z), \text{ with } B(z) = \frac{Q-P}{z} + \frac{Q}{1-z}$$

The function r_j *and* $\tilde{f}_j$ *extend to holomorphic functions on* $\mathbb{C}\backslash[0,\infty[$. *It's proved in [99], that the solutions of these two equations are related by a transport matrix* $c = (c_{ij})$ *with* $c_{ij} \neq 0$, *so that:*

$$\tilde{f}_j(z) = \sum c_{jm}\tilde{f}_m(z^{-1})$$

We then obtain, up to an analytic continuation, the following equality:

$$(\phi_{kj}^{\frac{1}{2}\ell}(u,z).\phi_{ji}^{\frac{1}{2}\ell}(v,w)\Omega_i,\Omega_k) = \sum c_{jm}(\phi_{km}^{\frac{1}{2}\ell}(v,w).\phi_{mi}^{\frac{1}{2}\ell}(u,z)\Omega_i,\Omega_k)$$

This relation extends to any finite energy vectors:

$$(\phi_{kj}^{\frac{1}{2}\ell}(u,z).\phi_{ji}^{\frac{1}{2}\ell}(v,w)\eta,\xi) = \sum c_{jm}(\phi_{km}^{\frac{1}{2}\ell}(v,w).\phi_{mi}^{\frac{1}{2}\ell}(u,z)\eta,\xi)$$

This analysis runs idem for braiding relations between spin 1/2 *and spin* 1 *primary fields, then:*

Theorem 11.22. *(Braiding relations)*
Let $(k_1, k_2) = (1/2, 1/2)$*,* $(1, 1/2)$ *or* $(1/2, 1)$*;* $v_1 \in V_{k_1}$ *and* $v_2 \in V_{k_2}$*.*

$$\phi_{ij}^{k_1\ell}(v_1, z)\phi_{jk}^{k_2\ell}(v_2, w) = \sum \mu_r \phi_{ir}^{k_2\ell}(v_2, w)\phi_{rk}^{k_1\ell}(v_1, z) \text{ with } \mu_r \neq 0$$

To simplify, we don't write the dependence of μ_r *on the other coefficients.*

Remark 11.23. *The way to write the braiding relations is a simplification. In fact, the left side is defined for* $|z| < |w|$*, and the right side for* $|z| > |w|$*, but each sides admit the same rational extension out of* $z = w$*. The braiding relations generalise the locality of vertex operator (see definition 4.19).*

Remark 11.24. *To prove that all the coefficients are non-null for* $(k_1, k_2) = (1, 1)$*, we should solve Dotsenko-Fateev equations (see [87]).*

Recall 11.25. *(Localised braiding relation) Let* $f \in L^2(I, V_{k_1})$ *and* $g \in L^2(J, V_{k_2})$*, with* I*,* J *be two disjoint proper intervals of* $\mathbb{S}^1$*. Using an argument of convolution (as [99] p 516), we can write the following localised braiding relations:*

$$\phi_{ij}^{k_1\ell}(f)\phi_{jk}^{k_2\ell}(g) = \sum \mu_r \phi_{ir}^{k_2\ell}(e_\alpha g)\phi_{rk}^{k_1\ell}(e_{-\alpha} f) \text{ with } \mu_r \neq 0$$

with $e_\alpha = e^{i\alpha\theta}$*,* $\alpha = h_i^\ell + h_k^\ell - h_j^\ell - h_r^\ell$ *and* (k_1, k_2) *as previously.*

Recall 11.26. *(Contragrediant braiding) Let the previous ODE:*

$$\tilde{f}'(z) = A(z)\tilde{f}(z), \text{ with } A(z) = \tfrac{P}{z} + \tfrac{Q}{1-z}$$

and the previous gauge relation: $g^{-1}Ag - g^{-1}g' = P/z$*.*
In the same way, we can choose $h(z)$ *with* $h(0) = I$ *and* $hAh^{-1} - h'h^{-1} = -P/z$*. This corresponds to take* $-A(z)^t$ *instead of* $A(z)$*. But then* $(hg)' = [P, hg]/z$*, which admits only the constant solutions, but* $h(0)g(0) = I$*, so* $h(z) = g(z)^{-1}$*. Then, the columns of* $(g(z)^{-1})^t$ *are the fundamental solutions of* $k'(z) = -A(z)^t k(z)$*. The transport matrix of this equation is just the transposed of the inverse of the original one, ie* $(c^{-1})^t$*.*

11.2 Primary fields for $\mathfrak{Vir}_{1/2}$

Definition 11.27. *Let λ, $\mu \in \mathbb{C}$, $\sigma = 0, 1$, we define the ordinary representations of $\mathfrak{Vir}_{1/2}$ as $\mathcal{F}^{\sigma}_{\lambda,\mu}$, with base $(v_i)_{i\in\mathbb{Z}+\frac{\sigma}{2}}$, $(w_j)_{j\in\mathbb{Z}+\frac{1-\sigma}{2}}$, and:*

(a) $L_n.v_i = -(i + \mu + \lambda n)v_{i+n}$

(b) $G_s.v_i = w_{i+s}$

(c) $L_n.w_j = -(j + \mu + (\lambda - \frac{1}{2})n)w_{j+n}$

(d) $G_s.w_j = -(j + \mu + (2\lambda - 1)s)v_{j+s}$

Remark 11.28. *Let the space of densities $\{f(\theta)e^{i\mu\theta}(d\theta)^{\lambda} | f \in C^{\infty}(\mathbb{S}^1)\}$ where a finite covering of Diff$(\mathbb{S}^1)$ acts by reparametrisation $\theta \to \rho^{-1}(\theta)$ (if $\mu \in \mathbb{Q}$). Then its Lie algebra acts on too, so that it's a $\mathfrak{Vir}$-module vanishing the center (see [54]). Finally, an equivalent construction with superdensities gives a model for $\mathcal{F}^{\sigma}_{\lambda,\mu}$ as $\mathfrak{Vir}_{1/2}$-module (see [43]).*

Definition 11.29. *Let L^m_{pq} and $L^m_{p'q'}$ on the unitary discrete series of $\mathfrak{Vir}_{1/2}$. We define a primary field as a linear operator:*

$$\phi : L^m_{p'q'} \otimes \mathcal{F}^{\sigma}_{\lambda,\mu} \to L^m_{pq}$$

that superintertwines the action of $\mathfrak{Vir}_{1/2}$.

Definition 11.30. *For $v \in \mathcal{F}^{\sigma}_{\lambda,\mu}$, let $\phi(v) : L^m_{p'q'} \to L^m_{pq}$*

Lemma 11.31. *Let $X \in \mathfrak{Vir}_{1/2}$, then $[X, \phi(v)]_{\tau} = \phi(X.v)$*

Proof. We can suppose X to be homogeneous for the $\mathbb{Z}_2$-gradation τ. Now, ϕ superintertwines the action of $\mathfrak{Vir}_{1/2}$: $\phi.[X \otimes I + I \otimes X] = (-1)^{\partial X} X.\phi$
Let $\xi \otimes v \in L^m_{p'q'} \otimes \mathcal{F}^{\sigma}_{\lambda,\mu}$, then $\phi.[X \otimes I + I \otimes X](\xi \otimes v) = [\phi(v)X + \phi(Xv)]\xi$ and $X.\phi(\xi \otimes v) = X\phi(v)\xi$, then $[X, \phi(v)]_{\tau} = \phi(X.v)$. □

Lemma 11.32. *ϕ non-null implies that $\mu = h^m_{p'q'} - h^m_{pq}$.*

Lemma 11.33. *(Gradation)*

(a) $\phi(v_n).(L^m_{p'q'})_{s+h^m_{p'q'}} \subset (L^m_{pq})_{s-n+h^m_{pq}}$

(b) $\phi(w_r).(L^m_{p'q'})_{s+h^m_{p'q'}} \subset (L^m_{pq})_{s-r+h^m_{pq}}$

Definition 11.34. *Let $h = 1 - \lambda$ be the conformal dimension of ϕ, and $\triangle = 1 - \lambda + \mu = h + h^m_{p'q'} - h^m_{pq}$; we define:*

$$\phi(z) = \sum_{n \in \mathbb{Z} + \frac{\sigma}{2}} \phi(v_n) z^{-n-\triangle} \text{ and } \quad \theta(z) = \sum_{n \in \mathbb{Z} + \frac{1-\sigma}{2}} \phi(w_n) z^{-n-1/2-\triangle}$$

$\phi(z)$ is called the ordinary part and $\theta(z) = [G_{-1/2}, \phi(z)]$, the super part of the primary field.

Lemma 11.35. *(Covariance relations).*

(a) $[L_n, \phi(z)] = [z^{n+1} \frac{d}{dz} + h(n+1) z^n] \phi(z)$

(b) $[G_{n-1/2}, \phi(z)] = z^n \theta(z)$

(c) $[L_n, \theta(z)] = [z^{n+1} \frac{d}{dz} + (h + 1/2)(n+1) z^n] \theta(z)$

(d) $[G_{n-1/2}, \theta(z)]_+ = [z^n \frac{d}{dz} + 2hn.z^{n-1}] \phi_(z)$

Proof. Direct from the definition. ☐

Lemma 11.36. *(Compatibility condition)*

(a) $[L_n, \phi(z)] = [z^{n+1} \frac{d}{dz} + (n+1) z^n (1 - \lambda)] \phi(z)$

(b) $[G_r, \phi(z)] = z^{r+1/2} [G_{-1/2}, \phi(z)]$

Proof. Immediate. ☐

Lemma 11.37. *If $\widetilde{\phi}(z)$ satisfy the compatibility condition, then, it gives a primary fields for the Neveu-Schwarz algebra, with $\widetilde{\theta}(z) = [G_{-1/2}, \widetilde{\phi}(z)]$ as super part.*

Proof. It's an easy verification. ☐

Proposition 11.38. *(Initial term) The space of primary fields $\phi : L^m_{p'q'} \otimes \mathcal{F}^\sigma_{\lambda,\mu} \to L^m_{pq}$ with every parameter fixed, is at most one-dimensional.*

Proof. Let Ω and Ω' be the cyclic vectors of the positive energy representations and $v \in \mathcal{F}^\sigma_{\lambda,\mu}$. Then, by the intertwining relations, $(\phi(v)\eta, \xi)$ is completly determined by the initial term $(\phi(v)\Omega, \Omega')$. Next $(\phi(v)\Omega, \Omega')$ is nonzero for v in a subspace of $\mathcal{F}^\sigma_{\lambda,\mu}$ of at most dimension one (lemma 11.33). ☐

Remark 11.39. *Using a slightly modified $(A_nB)(z)$ formula (see proposition 4.24), we can inductively generate many fields from a given field ψ.*
For example we find:

$$(L_n\psi)(z) = [\sum C_{n+1}^r(-z)^r L_{n-r}]\psi(z) - \psi(z)[\sum C_{n+1}^r(-z)^{n+1-r}L_{r-1}]$$

We can also write a formula for G_r. Now, we see that:

$$(L_0\phi)(z) = [L_0, \phi(z)] - z[L_{-1}, \phi(z)] = h\phi(z)$$

It's easy to see that using this machinery from $\phi(z)$ we generate the unitary $\mathfrak{Vir}_{1/2}$-module $L(h, c_m)$. Then, by FQS criterion, $h = h^m_{p''q''}$. We note the elements $\Phi(a, z)$ with $a \in L^m_{p''q''}$, $\phi(z) = \Phi(\Omega^m_{p''q''}, z)$ and if $\psi(z) = \Phi(a, z)$ then $(L_n\psi)(z) = \Phi(L_n.a, z)$. We do the same with G_r. We call Φ a general vertex operator, it generalizes the vertex operator of the section 4 , it admits many properties, but we don't need to enter into details.

Corollary 11.40. $\mu = h^m_{p'q'} - h^m_{pq}$ *and* $1 - \lambda = h = h^m_{p''q''}$.

Definition 11.41. *We note ϕ as $\phi^{p''q''m}_{pqp'q'}$, $\triangle$ as $\triangle^{p''q''m}_{pqp'q'} = h^m_{p''q''} - h^m_{p'q'} + h^m_{pq}$.*

Definition 11.42. *With $p'' = 2k + 1$ and $q'' = 2k' + 1$, we call ϕ a primary field of charge (k, k').*

Note that the charge and the spaces between which the field acts fixes λ and μ, but σ can be 0 or 1. Now, $\sigma = 0$ or 1 corresponds to $\phi(z)$ with integers or half-integers modes respectively. On our work, we only need to consider primary fields of charge $\alpha = (1/2, 1/2)$ and $\beta = (0, 1)$:

Proposition 11.43. *Up to a multiplication by a rational power of z:*
(a) The compression of complex fermions gives primary fields of charge α.
(b) The compression of real fermions gives primary fields of charge β.

Proof. We just check the compatibility condition using the explicit formula of GKO for G_r. The calculation can also be made on the vertex algebra of the fermions. □

11.3 Constructible primary fields and braiding for $\mathfrak{Vir}_{1/2}$

Lemma 11.44. *Let $m = \ell + 2$. and* $\begin{cases} p = 2i+1 & p' = 2j+1 & p'' = 2k+1 \\ q = 2i'+1 & q' = 2j'+1 & q'' = 2k'+1 \end{cases}$

(a) $h_i^\ell = h_{pq}^m + h_{i'}^{\ell+2} - \frac{1}{2}(i-i')^2$

(b) $\triangle_{ij}^{k\ell} = \triangle_{pqp'q'}^{p''q''m} + \triangle_{i'j'}^{k'\ell+2} - C_{ii'jj'}^{kk'}$

with $C_{ii'jj'}^{kk'} = \frac{1}{2}[(i-i')^2 - (j-j')^2 + (k-k')^2]$

Proof. $h_{pq}^m = \frac{[(m+2)p-mq]^2-4}{8m(m+2)} = \frac{2p^2(m+2)-2q^2m-4}{8m(m+2)} + \frac{(p-q)^2}{8} = h_i^\ell - h_{i'}^{\ell+2} + \frac{1}{2}(i-i')^2$

Next, (b) is immediate by (a). □

Notation 11.45. *We note $h_{ii'}^\ell$, $L_{ii'}^\ell$, $\phi_{ii'jj'}^{kk'\ell}$ and $\triangle_{ii'jj'}^{kk'\ell}$ instead of h_{pq}^m, L_{pq}^m, $\phi_{pqp'q'}^{p''q''m}$ and $\triangle_{pqp'q'}^{p''q''m}$.*

Definition 11.46. *A non-zero primary field of charge $\alpha = (1/2, 1/2)$ or $\beta = (0, 1)$ is called constructible if it's a compression fermions.*

Theorem 11.47. *(Constructible primary fields)*
(1)$\phi_{ii'jj'}^{\alpha\ell}$ is constructible iff:
$i + i' + 1/2 \le \ell$ and $j + j' + 1/2 \le \ell + 2$
(a) If $\sigma = 0$: $i' = i \pm 1/2$ and $j' = j \pm 1/2$,
(b) If $\sigma = 1$: $i' = i \pm 1/2$ and $j' = j \mp 1/2$.

(2) $\phi_{ii'jj'}^{\beta\ell}$ is constructible iff:
$i + i' \le \ell$ and $j + j' + 1 \le \ell + 2$
(a) If $\sigma = 0$: $i' = i$ and $j' = j \pm 1$.
(b) If $\sigma = 1$: $i' = i$ and $j' = j$

with the restriction that $0 \le i, i' \le \ell/2$ and $0 \le j, j' \le (\ell + 2)/2$.

This section is devoted to prove the theorem.

Remark 11.48. *If we ignore σ, we see that the dimension of the spaces of constructible primary fields are 0, 1 or 2-dimensional, and it's correspond to the fusion rules obtained below.*

Remark 11.49. *The dimension of the space of all the primary fields (non necessarily construcible as above) have been calculated by Iohara and Koga [43], using the action on the singular vectors of $\mathcal{F}^{\sigma}_{\lambda,\mu}$. Their result shows that in the previous cases, every primary fields are constructibles.*

Corollary 11.50. *Let ϕ of charge α or β, then $\phi \neq 0$ iff ϕ constructible.*

Recall 11.51. *(GKO construction) (see section 9)*

$$\mathcal{F}^{\mathfrak{g}}_{NS} \otimes L^{\ell}_{i} = \bigoplus L^{\ell}_{ij} \otimes L^{\ell+2}_{j}$$

and $\mathcal{F}^{\mathfrak{g}}_{NS} = L^2_0 \oplus L^2_1$ as $L\mathfrak{g}$-module.

Corollary 11.52. *(Braiding relations)*
Let $(\gamma_1, \gamma_2) = (\alpha, \alpha)$, (β, α) or (α, β):

$$\phi^{\gamma_1 \ell}_{ii'jj'}(z)\phi^{\gamma_2 \ell}_{jj'kk'}(w) = \sum \mu_{rr'}\phi^{\gamma_2 \ell}_{ii'rr'}(w)\phi^{\gamma_1 \ell}_{rr'kk'}(z) \text{ with } \mu_{rr'} \neq 0.$$

To simplify, we don't write the dependence of $\mu_{rr'}$ on the other coefficients.

proof of theorem 11.47 and corollary 11.52
This proof is an adaptation of the proof of Loke [66] for $\mathfrak{Vir}$.
I thank A. Wassermann to have simplified it.
Let H^{ℓ}_{j}, $H^{\ell}_{jj'}$ be the L^2-completion of L^{ℓ}_{j} and $L^{\ell}_{jj'}$.
Let $\Phi(v, z) = I \otimes \phi^{\frac{1}{2}\ell}_{ij}(v, z) : \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{j} \to \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{i}$. By the coset construction:

$$\mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{j} = \bigoplus H^{\ell}_{jj'} \otimes H^{\ell+2}_{j'} \text{ and } \mathcal{F}^{\mathfrak{g}}_{NS} \otimes H^{\ell}_{i} = \bigoplus H^{\ell}_{ii'} \otimes H^{\ell+2}_{i'}$$

Let $p_{i'}$, $p_{j'}$ be the projection on $H^{\ell}_{ii'} \otimes H^{\ell+2}_{i'}$ and $H^{\ell}_{jj'} \otimes H^{\ell+2}_{j'}$.
Let $\eta \in H^{\ell}_{ii'}$, $\xi \in H^{\ell}_{jj'}$ be non-zero fixed L_0-eigenvectors.
Let $\phi(v, z) : H^{\ell+2}_{j'} \to H^{\ell+2}_{i'}$, defined by: $\forall \eta' \in H^{\ell+2}_{i'}$ and $\forall \xi' \in H^{\ell+2}_{j'}$,

$$(p_{i'}\Phi(v, z)p_{j'}.(\xi \otimes \xi'), \eta \otimes \eta') = (\phi(v, z).\xi', \eta').$$

Now, by compatibility condition for $LSU(2)$:

$$[X(n), \Phi(v, z)] = z^n \Phi(X.v, z) \text{ and } [L_n, \Phi(v, z)] = z^n[z\tfrac{d}{dz} + (n+1)h^{\ell}_{1/2}]\Phi(v, z)$$

Now, $X(n)$ and L_n commute with $p_{i'}$, $p_{j'}$ and z^r with $s \in \mathbb{Q}$, then, by easy manipulation we see that, up to multiply by a rational power of z:

$$[X(n), \phi(v,z)] = z^n(\phi(X.v,z)) \text{ and}$$
$$[L_n, \phi(v,z)] = z^n[z\tfrac{d}{dz} + (n+1)h^{\ell+2}_{1/2}](\phi(v,z))$$

By compatibility and uniqueness theorem, $\exists s \in \mathbb{Q}$ such that $z^s\phi(v,z)$ is the spin 1/2 and level $\ell+2$ primary field $\phi^{\ell+2}_{i'j'}(v,z)$ (up to a multiplicative constant) of $LSU(2)$. The power s can be compute using lemma 11.44. It follows that $p_{i'}\Phi(v,z)p_{j'} = \phi^{\ell+2}_{i'j'}(v,z) \otimes \rho(z)$. Now, $h^{\ell}_{\frac{1}{2},\frac{1}{2}} = h^{\ell}_{\frac{1}{2}} - h^{\ell+2}_{\frac{1}{2}}$, it follows that up to multiply by a rational power of z:

$$[L_n, \rho(z)] = z^n[z\tfrac{d}{dz} + (n+1)h^{\ell}_{\frac{1}{2},\frac{1}{2}}]\rho(z)$$

We verify also, using the explicit formula for G_r that:

$$[G_{-1/2}, \rho(z)] = z^{-r-1/2}[G_r, \rho(z)]$$

Finally, by compatibility condition and uniqueness, $\exists s' \in \mathbb{Q}$ such that $z^{s'}\rho(z)$, is the charge $(1/2,1/2)$ primary field $\phi^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z)$ of $\mathfrak{Vir}_{1/2}$ between $H^{\ell}_{jj'}$ and $H^{\ell}_{ii'}$ (up to a multiplicative constant). Finally by lemma 11.44:

$$p_{i'}[I \otimes \phi^{\frac{1}{2}\ell}_{ij}(v,z)]p_{j'} = C.z^{-C^{kk'}_{ii'jj'}}\phi^{\ell+2}_{i'j'}(v,z) \otimes \phi^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z)$$

the value of σ follows using characterization: integer and half-integer moded.

Now, the constant C is possibly zero. So, we will prove it's non-zero for the annonced constructible fields:

If it exists j' such that, $\Phi(v,z)p_{j'} = 0$ $\forall v$, then, $\forall u \in L^{\ell}_{jj'} \otimes L^{\ell+2}_{j'}$, $\Phi(v,z)u = 0$, but, by commutation relation with $I \otimes \psi(x,r)$ and $X(n) \otimes I$, it follows by irreducibility that $u \neq 0$ is cyclic and $\Phi(v,z)u' = 0$ $\forall u' \in \mathcal{F}^{\mathfrak{g}}_{NS} \otimes L^{\ell}_j$. Then, $\Phi(v,z) = 0$ contradiction. So, $\forall j'$, $\Phi(v,z)p_{j'} \neq 0$, so it exists i' such that $p_{i'}\Phi(v,z)p_{j'} \neq 0$. By the beginning of the proof, a necessary condition for i' is that $\phi^{\frac{1}{2}\ell+2}_{i'j'}$ is a non-zero primary field of $LSU(2)$. We will prove that this condition is also sufficient. For now, we know that for this i', $\phi^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}$ is a non-zero primary field of $\mathfrak{Vir}_{1/2}$.

Now, $\forall i'$ let $\rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}$ a multiple (possibly zero) of $\phi^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}$, such that:

$$\Phi(v,z) = \Phi_{ij}(v,z) = \sum \rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z) \otimes \phi^{\frac{1}{2},\ell+2}_{i'j'}(v,z)$$

Now, $(\Phi_{ij}(u,z)\Phi_{jk}(v,w)\Omega_{jj'kk'} \otimes \Omega_{j'k'}, \Omega_{jj'ii'} \otimes \Omega_{j'i'})$
$= \sum(\rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z)\rho^{\frac{1}{2}\frac{1}{2},\ell}_{jj'kk'}(w)\Omega_{jj'kk'}, \Omega_{jj'ii'}).(\phi^{\frac{1}{2},\ell+2}_{i'j'}(u,z)\phi^{\frac{1}{2},\ell+2}_{j'k'}(v,w)\Omega_{j'k'}, \Omega_{j'i'})$
We can write it as a relation between reduced 4-point function:

$$F_j(\zeta) = \sum f_{j'}(\zeta) h_{jj'}(\zeta)$$

We return in the context of recall 11.21 and 11.26: F_j and $f_{j'}$ are holomorphic function from $\mathbb{C}\backslash[0,\infty[$ to W. Let $v_{j'} \in W$ such that $g(\zeta)v_{j'} = \zeta^{\mu_{j'}} f_{j'}(\zeta)$. We apply the gauge transformation $g(\zeta)^{-1}$ on the previous equality:

$$g(\zeta)^{-1}F_j(\zeta) = \sum \zeta^{-\mu_{j'}} v_{j'} h_{jj'}(\zeta)$$

It follows that $h_{jj'}$ is holomorphic on $\mathbb{C}\backslash[0,\infty[$, we get a formula for it:

$$h_{jj'}(\zeta) = C.\zeta^{\mu_{j'}}(g(\zeta)^{-1}F_j(\zeta), v_{j'})$$

with C a non-zero constant.

This formula gives exactly the duality for braiding discovered by Tsuchiya-Nakanishi [92]. Then by recall 11.26, the braiding matrix for the $\rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z)$ is the product of the braiding matrix for $LSU(2)$ at spin 1/2 and level ℓ, times the transposed of the inverse of the braiding matrix for $LSU(2)$ at spin 1/2 and level $\ell+2$. All the coefficients are non-zero. Now, suppose that $\rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z) = 0$, with $\phi^{\frac{1}{2},\ell}_{ij}$ and $\phi^{\frac{1}{2},\ell+2}_{i'j'}$ constructible then:

$$0 = \rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'jj'}(z)\rho^{\frac{1}{2}\frac{1}{2},\ell}_{jj'ii'}(w) = \sum \lambda_{kk'}\rho^{\frac{1}{2}\frac{1}{2},\ell}_{ii'kk'}(w)\rho^{\frac{1}{2}\frac{1}{2},\ell}_{kk'ii'}(z)$$

with all braiding coefficients non-zero. But as we see previously by irreducibility, the right side admits at least a non-zero term, contradiction.

For the braiding between charge $(0,1)$ and charge $(1/2,1/2)$ primary fields, we do the same starting with the Neveu-Schwarz fermion field $\psi(u,z)\otimes I$ commuting with $I\otimes\phi^{\frac{1}{2}\ell}_{ij}(v,w)$. We find also that every possible braiding coefficients are non-zero. The result follows. **End of the proof.**

Remark 11.53. *As a consequence of remark 11.24, we know that such a braiding exists for $(\gamma_1,\gamma_2) = (\beta,\beta)$, but we don't know if every coefficients $\mu_{rr'}$ are non-null.*

Proposition 11.54. *The primary fields of charge α or β are bounded and identifying the L^2-completion of $\mathcal{F}^{\sigma}_{\lambda,\mu}$ with $L^2(\mathbb{S}^1)e^{\sigma i\theta/2}\oplus L^2(\mathbb{S}^1)e^{(1-\sigma)i\theta/2}$, we obtain $\phi(f)$ for $f\in L^2(\mathbb{S}^1)e^{\sigma i\theta/2}$, $\theta(g)$ for $g\in L^2(\mathbb{S}^1)e^{(1-\sigma)i\theta/2}$ with:*

$$\|\phi(f)\| \leq K\|f\|_2 \quad \text{and} \quad \|\theta(g)\| \leq K'\|g\|_2$$

Proof. The primary fields of charge α or β are constructibles, and the compressions of fermions are bounded operators. □

Corollary 11.55. *(Localised braiding relation) Let* $f \in L^2_I(\mathbb{S}^1)e^{\sigma i\theta/2}$ *and* $g \in L^2_J(\mathbb{S}^1)e^{\sigma i\theta/2}$, *with* I, J *be two disjoint proper intervals of* $\mathbb{S}^1$. *Using an argument of convolution (as [99] p 516), we can write the following localised braiding relations:*

$$\phi^{\gamma_1 \ell}_{ii'jj'}(f)\phi^{\gamma_2 \ell}_{jj'kk'}(g) = \sum \mu_{rr'}\phi^{\gamma_2 \ell}_{ii'rr'}(e_\lambda g)\phi^{\gamma_1 \ell}_{rr'kk'}(\bar{e}_\lambda f) \text{ with } \mu_{rr'} \neq 0.$$

with $e_\lambda = e^{i\lambda\theta}$, $\lambda = h^\ell_{ii'} + h^\ell_{kk'} - h^\ell_{jj'} - h^\ell_{rr'}$ *and* (γ_1, γ_2) *as previously.*

11.4 Application to irreducibility

Definition 11.56. *Let* $\mathcal{M}$, $\mathcal{N} \subset B(H)$ *be von Neumann algebra, then,* $\mathcal{M} \vee \mathcal{N}$ *is the von Neumann algebra generated by* $\mathcal{M}$ *and* $\mathcal{N}$.

Notation 11.57. *We simply note* $\phi^k_{ij}(f)$ *for primary field of charge* k *for* $\mathfrak{Vir}_{1/2}$; *the charge* c_m *is fixed and* $i = 0$ *significate* $i = (0,0)$.

Proposition 11.58. *The chains of constructible primary fields of the form:*

$$\phi^\alpha_{0i_1}(f_1)\phi^\alpha_{i_1i_2}(f_2)...\phi^\alpha_{i_{r-1}i_r}(f_r)\phi^\alpha_{i_r0}(f_{r+1}) \text{ with } \alpha = (\tfrac{1}{2}, \tfrac{1}{2}) \text{ and } f_i \text{ on } I.$$

are bounded operators and generate the von Neumann algebra $\mathcal{N}^\ell_{00}(I)$.

Proof. By corollary 10.84 and proposition 11.43. □

Remark 11.59. *Let* σ_t *be the geometric modular action described on recall 10.60. Let* $\psi^k_{ij}(f)$ *be a bounded primary field of charge* k *concentrated on a proper interval* J. *Let* $\sigma_t(\psi^\alpha_{ij}(f)) := \pi_i(\varphi_t)\psi^\alpha_{ij}(f)\pi_j(\varphi_t)^\star = \psi^\alpha_{ij}(u_t.f)$ *by the covariance relations. Then,* $\sigma_t(\psi^\alpha_{ij}(f))$ *is a primary field concentrated on* $\varphi_t(J) \to \{1\}$ *(when* $t \to \infty$ *).*

Recall 11.60. *(Cancellation theorem) If a unitary representation of a connected semisimple non-compact group with finite center has no fixed vectors, then its matrix coefficients vanish at* ∞. *We can find a proof on Zimmer's book [105]. For example,* $G = SU(1,1) \simeq SL(2,\mathbb{R})$ *(non-compact) is implemented on the irreducible positive energy representations* H *of* $\mathfrak{Vir}_{1/2}$, *which give a unitary representation of a central cyclic extension* $\mathcal{G}$ *of* G, *whose Lie* *algebra is generated by L_{-1}, L_0 and L_1. But if $\xi \in H$, $L_0\xi = 0$ implies immediatly that $H = H_0$ and $\xi = \Omega$ (up to a multiplicative constant). So G admits no fixed vectors outside of the vacuum. But the modular operators U_t go to ∞ when $t \to \infty$. Then, their matrix coefficients vanish at ∞. In our case, we can prove the cancellation theorem directly, because H decomposes into a direct sum of irreducible positive energy representation of $\mathcal{G}$ and each summands is a discrete series representation of $\mathcal{G}$, so can be realized as a subrepresentation of $L^2(\mathcal{G})$, and then has matrix coefficient tending to zero at ∞ (see Pukanszky [79]).*

Proposition 11.61. *(Generically non-zero) Let $a = \phi^{\alpha}_{\alpha 0}(f)$ and $b = \phi^{\alpha}_{0\alpha}(g)$ with f, g on proper intervals. Then, $(ba\Omega, \Omega)$ is non-zero in general.*

Proof. Let $a = \phi^{\alpha}_{\alpha 0}(f) \neq 0$ and R_θ be the quantized rotation action: $R_\theta = e^{iL_0\theta}$ (see remark 10.61). Let $b_\theta = R_\theta^\star a^\star R_\theta$. We suppose that $(b_\theta a\Omega, \Omega) = 0$ for $|\theta - \theta_1| \leq \varepsilon$ with θ_1 fixed and $\varepsilon > 0$. Then $(R_\theta^\star a^\star R_\theta a\Omega, \Omega) = 0$. But $L_0\Omega = 0$ on the vacuum representation. Then, $R_\theta\Omega = \Omega$ and $(R_\theta a\Omega, a\Omega) = 0$. Now, by positive energy of the representation $a\Omega = \sum_{n\in\frac{1}{2}\mathbb{N}} \xi_n$ (coming from the orthogonal decomposition for L_0) and $\|a\Omega\|^2 = \sum \|\xi_n\|^2$. Now, with $z = e^{i\theta/2}$, $(R_\theta a\Omega, a\Omega) = \sum_{n\in\mathbb{N}} z^n \|\xi_{n/2}\|^2 = f(z)$, let $g(z) = f(e^{-i\theta_1/2}z)$. Then, g extends to a continuous function on the closed unit disc, holomorphic in the interior and vanishing on the unit circle near $\{1\}$. By the Schwarz reflection principle and the Cayley transfrom, g must vanishes identically in z. So, $(R_0 a\Omega, a\Omega) = \|a\Omega\|^2 = 0$. Then $a\Omega = 0$, so $a^\star a\Omega = 0$. But Ω is a separating vector on the von Neumann algebra, so $a^\star a = 0$, and $a = 0$, contradiction. □

Proposition 11.62. *(Leading term in OPE of primary fields)*
Let I be a proper interval of $\mathbb{S}^1$, and I_1, I_2 be subintervals be subintervals obtained by removing a point. Let $a_{\nu\mu}$ and $b_{\mu\nu}$ be non-zero primary fields of charge α, localised in I_1 and I_2 respectively, then $\sigma_t(a_{\nu\mu}b_{\mu\nu}) \to^w Id_{H_i}$ (up to a multiplicative constant).

Proof. We adapt to $\mathfrak{Vir}_{1/2}$, a proof of A. Wassermann [] for $LSU(2)$.

Without a loose of generality, we can take $\{1\} \in \bar{I}_1 \cap \bar{I}_2$. Let a and b be generic primary fields of charge α concentrate on I_2 and I_1 respectively.

(1)We first prove that $\sigma_t(a_{0\alpha}b_{\alpha 0}) \to^w C$ non-zero constant: $\|\sigma_t(a_{0\alpha}b_{\alpha 0})\|$ is clearly bounded, then by the weak compacity of the unit ball, it exists a sequence t_n such that $\sigma_{t_n}(a_{0\alpha}b_{\alpha 0}) \to^w T$. By the remark 11.59, $\sigma_{t_n}(b_{0\alpha}a_{\alpha 0})$ is concentrated on J_n with $\bigcap J_n = \{1\}$. We obtain that T supercommutes with $\bigvee \mathcal{N}_{00}^{\ell}(J_n^c)$. By Araki-Haag duality, $(\bigvee \mathcal{N}_{00}^{\ell}(J_n^c))^{\natural} = \bigcap \mathcal{N}_{00}^{\ell}(J_n)) = \mathcal{N}_{00}^{\ell}(\{1\}) = \mathbb{C}$. Then $T \in \mathbb{C}Id$. Now, $(\sigma_{t_n}(a_{0\alpha}b_{\alpha 0})\Omega, \Omega) = (a_{0\alpha}b_{\alpha 0}\Omega, \Omega)$ because $\pi_0(U_t)\Omega = \Omega$ (see remark 10.61). Now $(a_{0\alpha}b_{\alpha 0}\Omega, \Omega) = k$ generically non-zero (proposition 11.61) and $T = kId$. Now, k is independant on the sequence (t_n), so $\sigma_t(a_{0\alpha}b_{\alpha 0}) \to^w k.Id \neq 0$.

(2) We now prove that $\sigma_t(a_{\gamma\alpha}b_{\alpha 0}) \to^w 0$ if $\gamma \neq 0$.
Idem, it exists a sequence t_n such that $X_n = \sigma_{t_n}(a_{\gamma\alpha}b_{\alpha 0}) \to^w T$. Let ξ be a finite energy vector in H_γ, then $(X_n\Omega, \xi) = (\pi_\gamma(U_{t_n})a_{\gamma\alpha}b_{\alpha 0}\Omega, \xi) = ((\pi_\gamma(U_{t_n})\eta, \xi) \to 0$ when $t_n \to \infty$ by the cancellation theorem (recall 11.60)Then, $T\Omega = 0$, so $T^\star T\Omega = 0$. But Ω is a separating vector on the von Neumann algebra, so $T^\star T = 0$ and $T = 0$. Now, the 0 is independent of the choice of the sequence, then: $\sigma_t(a_{\gamma\alpha}b_{\alpha 0}) \to^w 0$.

(3) We prove that if $a_{\nu\mu} \neq 0$, then $\sigma_t(a_{\nu\mu}b_{\mu\nu}) \to^w C'$ non-zero constant:
Idem, it exists a sequence such that $\sigma_{t_n}(a_{\nu\mu}b_{\mu\nu}) \to^w R$. Now, let $y_{\nu 0} = x_{\nu\lambda_1}x_{\lambda_1\lambda_2}...x_{\lambda_r 0}$ be a chain between ν and 0 with the minimal number of primary fields of charge α, concentrate on a proper closed K interval out of $\{1\}$. Then for t sufficiently large, we can apply the braiding formulas on $\sigma_t(a_{\nu\mu}b_{\mu\nu})y_{\nu 0}$. We obtain necessarily $\sigma_t(a_{\nu\mu}b_{\mu\nu})y_{\nu 0} = \sum_{\gamma\neq 0} A_\gamma \sigma_t(a_{\gamma\alpha}b_{\alpha 0}) + \lambda y_{\nu 0}\sigma_t(a_{0\alpha}b_{\alpha 0})$, with $\lambda \neq 0$, A_γ a linear sum of non-minimal chains between ν and γ (note that in general, there are many ways to go between 0 and ν minimally, but by the structure of the braiding rules, only the way chosen for $y_{\nu 0}$ can appear at the end). Now, by (1) and (2), the previous equality (with $t = t_n$) weakly converge to $Ry_{\nu 0} = \lambda y_{\nu 0}C = \lambda C y_{\nu 0}$ with λC a non-zero constant. Now, $R \in \mathcal{N}_\nu^\ell(K^c)$, then $Ry_{\nu 0} = y_{\nu 0}\pi_0(R) = \lambda C y_{\nu 0}$. Now $\sigma_t(y_{\nu 0})$ is also a minimal chain of charge α between ν and 0, concentrate on a proper closed interval out of $\{1\}$, so $\sigma_t(y_{\nu 0})\pi_0(R) = C'\sigma_t(y_{\nu 0})$ with C' a non-zero constant. Then $\sigma_t(y_{\nu 0})^\star\sigma_t(y_{\nu 0})\pi_0(R) = C'\sigma_t(y_{\nu 0})^\star\sigma_t(y_{\nu 0})$. But $\sigma_t(y_{\nu 0})^\star\sigma_t(y_{\nu 0}) = \sigma_t(y_{\nu 0}^\star y_{\nu 0}) \to_w k.Id \neq 0$ as for (1). So $\pi_0(R) = C' = R$. □

Proposition 11.63. *(von Neumann density) Let I be a proper interval of $\mathbb{S}^1$, and I_1, I_2 be subintervals such that $I = I_1 \cup I_2$.*

$$\mathcal{N}_{ij}^\ell(I_1) \vee \mathcal{N}_{ij}^\ell(I_2) = \mathcal{N}_{ij}^\ell(I).$$

Proof. By the local equivalence for $\mathfrak{Vir}_{1/2}$ (see section 10.7), we only need to prove the result on the vacuum. By proposition 11.58 we only need to work with chains. Consider the chain $\phi^{\alpha}_{0i_1}(f_1)\phi^{\alpha}_{i_1i_2}(f_2)...\phi^{\alpha}_{i_{r-1}i_r}(f_r)\phi^{\alpha}_{i_r0}(f_{r+1}) \in \mathcal{N}^{\ell}_{00}(I)$, with $f_k \in L^2_I(\mathbb{S}^1)$. Now, $f_k = f_k^{(1)} + f_k^{(2)}$, with $f_k^{(i)}$ concentrated on I_i. Now, a primary field $\phi^k_{ij}(f)$ is linear in f, so, we can develop the chain into a sum of chains of primary filed localized exclusively on I_1 or I_2. Next, applying the braiding relations, we can obtain a linear combination of chains, on which the primary field localized on I_1 and I_2 are separated; generically of the form:

$$\phi^{\alpha}_{0j_1}(g_1)\phi^{\alpha}_{j_1j_2}(g_2)...\phi^{\alpha}_{j_{s-1}j_s}(g_{s-1})\phi^{\alpha}_{j_sj_{s+1}}(h_{s+1})...\phi^{\alpha}_{j_{r-1}j_r}(h_r)\phi^{\alpha}_{j_r0}(h_{r+1})$$

with g_k and h_k concentrate on I_1 and I_2 respectively. Now, if $j_s = 0$, then, the previous chain is a product $a.b$ with $a \in \mathcal{N}^m_{11}(I_1)$ and $b \in \mathcal{N}^m_{11}(I_2)$.
Else, if $j_s \neq 0$, using the previous proposition step by step, we see that the chain is the weak limit of chains with 0 on the middle, the result follows. □

Lemma 11.64. *(Covering lemma) Let (I_n) be a covering of $\mathbb{S}^1$ by open proper intervals. Then $\mathfrak{Vir}_{1/2}(\mathbb{S}^1)$ is the linear span of the $\mathfrak{Vir}_{1/2}(I_n)$. And so $\bigvee \pi(\mathfrak{Vir}_{1/2}(I_n))'' = \pi(\mathfrak{Vir}_{1/2}(\mathbb{S}^1))'' = B(H)$.*

Proof. With a partition of the unity. □

Theorem 11.65. *Let I be a proper interval of $\mathbb{S}^1$, then, the Jones-Wassermann subfactor $\mathcal{N}^{\ell}_{ij}(I) \subset \mathcal{N}^{\ell}_{ij}(I)^{\natural}$ is irreducible, i.e. $\mathcal{N}^{\ell}_{ij}(I)^{\natural} \cap \mathcal{N}^{\ell}_{ij}(I^c)^{\natural} = \mathbb{C}$.*

Proof. Let I_1, I_2 be two proper subintervals of I obtained by removing a point. Let $J_1 = I$, $J_2 = \overline{I_1 \cup I^c}$ and $J_3 = \overline{I^c \cup I_2}$. Let $\mathcal{M} = \mathcal{N}^{\ell}_{ij}(I) \vee \mathcal{N}^{\ell}_{ij}(I^c)$, then $\mathcal{N}^{\ell}_{ij}(I)$,$\mathcal{N}^{\ell}_{ij}(I^c)$,$\mathcal{N}^{\ell}_{ij}(I_1)$ and $\mathcal{N}^{\ell}_{ij}(I_2) \subset \mathcal{M}$. By von Neumann density, $\mathcal{N}^{\ell}_{ij}(J_2) = \mathcal{N}^{\ell}_{ij}(I_1) \vee \mathcal{N}^{\ell}_{ij}(I^c) \subset \mathcal{M}$, and idem $\mathcal{N}^{\ell}_{ij}(J_3) \subset \mathcal{M}$. Let K_1, K_2, K_3 be open subintervals of J_1, J_2 and J_3 such that $K_1 \cup K_2 \cup K_3 = \mathbb{S}^1$. Now, $N^{\ell}_{ij}(K_1) \vee N^{\ell}_{ij}(K_2) \vee N^{\ell}_{ij}(K_3) \subset \mathcal{M}$, but $N^{\ell}_{ij}(K_1) \vee N^{\ell}_{ij}(K_2) \vee N^{\ell}_{ij}(K_3) = B(H^{\ell}_{ij})$ by covering lemma. So $\mathcal{M} = B(H^{\ell}_{ij})$ and $\mathbb{C} = \mathcal{M}^{\natural} = \mathcal{N}^{\ell}_{ij}(I)^{\natural} \cap \mathcal{N}^{\ell}_{ij}(I^c)^{\natural}$. □

12 Connes fusion and subfactors

12.1 Recall on subfactors

See the book [47] for a complete introduction to subfactors.

Definition 12.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebra, then, an inclusion $\mathcal{N} \subset \mathcal{M}$ is called a subfactor.*

Recall 12.2. *A factor $\mathcal{M}$ of type II admits a canonical trace tr. The image of tr on the subset of projection of $\mathcal{M}$ is $[0,1]$ or $[0,\infty]$.*
Then, $\mathcal{M}$ is said to be a factor of type II_1 or II_∞.

Recall 12.3. *(Basic construction) Let the subfactor $\mathcal{N} \subset \mathcal{M}$, with $\mathcal{M}$ and $\mathcal{N}$ II_1 factors. Let tr be the trace on $\mathcal{M}$, then, it admit the following inner product: $(x,y) := tr(xy^\star)$. Let $H = L^2(\mathcal{M}, tr)$ and $L^2(\mathcal{N}, tr)$ be the L^2-completions of $\mathcal{M}$ and $\mathcal{N}$. Let $e_\mathcal{N}$ be the orthogonal projection of $L^2(\mathcal{M}, tr)$ onto $L^2(\mathcal{N}, tr)$.*
Let $\langle \mathcal{M}, e_\mathcal{N} \rangle = (\mathcal{M} \cup \{e_\mathcal{N}\})'' \subset B(H)$. It admit a trace called $tr_{\langle \mathcal{M}, e_\mathcal{N} \rangle}$.
The tower $\mathcal{N} \subset \mathcal{M} \subset \langle \mathcal{M}, e_\mathcal{N} \rangle$ is called the basic construction.

Recall 12.4. *(Index of subfactors) Let the previous subfactor $\mathcal{N} \subset \mathcal{M}$.*
Then we can define its index $[\mathcal{M} : \mathcal{N}] = (tr_{\langle \mathcal{M}, e_\mathcal{N} \rangle}(e_\mathcal{N}))^{-1} \in [1, \infty]$.
The index admits another definition as the von Neumann dimension (see [47]) of the $\mathcal{N}$-module $H = L^2(\mathcal{M}, tr)$, ie $[\mathcal{M} : \mathcal{N}] = dim_\mathcal{N}(H)$.

Recall 12.5. *(Jones' theorem, see [45]) Every possible index of II_1-subfactors:*

$$\{4cos^2(\tfrac{\pi}{m}) | m = 3, 4, ...\} \cup [4, \infty]$$

In the continuation of the basic construction, we can build a graph from a subfactor, called its principal graph. If the subfactor admits a finite index then the square of the norm of the matrix of its principal graph is exactly the index. Now, this matrix admits only integers values, and a theorem of Kronecker said that the norm of an integer valued matrix is in $\{2cos(\frac{\pi}{m}) | m = 3, 4, ...\} \cup [2, \infty]$. Finally, it's proved that every possible such norms are realized from subfactors.

Definition 12.6. *A subfactor of finite index $\mathcal{M} \subset \mathcal{N}$ is said to be irreducible if either of the following equivalent conditions are satisfied:*

(a) $L^2(\mathcal{M})$ is irreducible as an $\mathcal{N}$-$\mathcal{M}$-bimodule.

(b) The relative commutant $\mathcal{N}' \cap \mathcal{M}$ is $\mathbb{C}$.

12.2 Bimodules and Connes fusion

Definition 12.7. *If $\mathcal{M}$, $\mathcal{N}$ are $\mathbb{Z}_2$-graded von Neumann algebra, a $\mathbb{Z}_2$-graded Hilbert space H is said to be a $\mathcal{M}$-$\mathcal{N}$-bimodule if:*

(a) H is a left $\mathcal{M}$-module.

(b) H is a right $\mathcal{N}$-module.

(c) the action of $\mathcal{M}$ and $\mathcal{N}$ supercommute; i.e.,
$\forall m \in \mathcal{M}$, $n \in \mathcal{N}$, $\xi \in H$, $(m.\xi).n = (-1)^{\partial m \partial n} m.(\xi.n)$.

Definition 12.8. *Let $\Omega \in H_0$ be a vacuum vector, then H_0 is a $\mathcal{M}$-$\mathcal{M}$ bimodule, because by Tomita-Takesaki theory, $J\mathcal{M}J = \mathcal{M}'$, by lemma 10.24, $\mathcal{M}^{\natural} = \kappa\mathcal{M}'\kappa^{\star} \simeq \mathcal{M}' \simeq \mathcal{M}^{opp}$. Now, $y^{\star}x^{\star} = (xy)^{\star}$ and $\mathcal{M}^{opp}$ is the opposite algebra: $a \times b = b.a$. Then $x.(\xi.y) := x(\kappa J y^{\star} J \kappa^{\star})\xi$ gives the bimodule action.*

Definition 12.9. *(Intertwinning operators) Let X, Y be $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodules, $\mathcal{X} = Hom_{-\mathcal{M}}(H_0, X)$ and $\mathcal{Y} = Hom_{\mathcal{M}-}(H_0, Y)$ be the space of bounded operators that superintertwin the left (resp. the right) action of $\mathcal{M}$.*

Lemma 12.10. *Consider the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$, we define a pre-inner product by:*

$$(x_1 \otimes y_1, x_2 \otimes y_2) = (-1)^{(\partial x_1 + \partial x_2)\partial y_2}(x_2^{\star} x_1 y_2^{\star} y_1 \Omega, \Omega)$$

Proof. As for [99] p 525-526. □

Definition 12.11. *The L^2-completion is called the Connes fusion between X and Y, and noted $X \boxtimes Y$, naturally a $\mathbb{Z}_2$-graded $\mathcal{M}$-$\mathcal{M}$ bimodule.*

Lemma 12.12. *There are canonical unitary isomorphism*

$$H_0 \boxtimes X \simeq X \simeq X \boxtimes H_0.$$

Proof. If $Y = H_0$, the unitary $X \boxtimes H_0 \to X$ is given by $x \otimes y \mapsto xy\Omega$, and the unitary $H_0 \boxtimes X \to X$ is given by $y \otimes x \mapsto (-1)^{\partial x \partial y} xy\Omega$. □

Lemma 12.13. *$\mathcal{X}$ can be seen as a dense subspace of X via $x \leftrightarrow x\Omega$.*

Proof. $\mathcal{X} = \mathcal{X}.\pi_0(\mathcal{M}(I^c))$, so by Reeh-Schlieder $\mathcal{X}\Omega$ is dense in $\mathcal{X}H_0$. Now, $\mathcal{X}H_0 = [\pi_X(\mathcal{M}(I^c))\mathcal{X}].[\pi_0(\mathcal{M}(I)).H_0] = \pi_X(\mathcal{M}(I^c).\mathcal{M}(I))\mathcal{X}\mathcal{H}_0$ $= \pi_X(\langle\mathcal{M}(I^c).\mathcal{M}(I)\rangle_{lin})\mathcal{X}\mathcal{H}_0$. But, because $\mathcal{M}(I^c)$ and $\mathcal{M}(I)$ supercommute, the $\star$-algebra generated by $\mathcal{M}(I^c).\mathcal{M}(I)$ is exactly its linear span, then, $\pi_X(\langle\mathcal{M}(I^c).\mathcal{M}(I)\rangle_{lin})$ is weakly dense in $\pi_X(\mathcal{M}(I^c).\mathcal{M}(I))''$. So, by von Neumann density $\mathcal{X}H_0$ is dense in $\bigoplus B(H_i)\mathcal{X}H_0 = X$, with $X = \bigoplus H_i$. □

Lemma 12.14. *(Hilbert space continuity lemma)*
The natural map $\mathcal{X}\otimes\mathcal{Y} \to X \boxtimes Y$ extends canonically to continuous maps $X\otimes\mathcal{Y} \to X\boxtimes Y$ and $\mathcal{X}\otimes Y \to X\boxtimes Y$. In fact $\|x_i\otimes y_i\|^2 \leq \|x_i x_i^\star\| \sum\|y_i\Omega\|^2$ and $\|x_i\otimes y_i\|^2 \leq \|y_i y_i^\star\| \sum\|x_i\Omega\|^2$

Proof. As for [99] p 526. □

Lemma 12.15. $\boxtimes$ *is associative.*

Proof. As for [99] p 527. □

12.3 Connes fusion with H_α on $\mathfrak{Vir}_{1/2}$

Remark 12.16. *Note that the primary fields ϕ we consider are always the ordinary part and so even operators. In fact, we only need to consider even intertwiner operators because each odd intertwiner operator is the product of an even one and an odd operator on the vacuum local von Neumann algebra.*

Definition 12.17. *Let $\langle i,j\rangle := \{k \mid \phi_{ij}^k \neq 0\}$.*

Recall that the primary field of charge $\alpha = (1/2, 1/2)$ are bounded. Let the graph $\mathcal{G}_\alpha$ with vertices $\{i\}$ and an edge between i and j if $j \in \langle\alpha, i\rangle$; then, α is a weak generator in the sense that the graph $\mathcal{G}_\alpha$ is connected. Let I be a non-trivial interval of $\mathbb{S}^1$, and let f and g be L^2-functions localized in I and I^c respectively. Recall that every possible braiding at charge α admits non-null coefficients, ie; $\phi_{ij}^\alpha(z)\phi_{jk}^\alpha(w) = \sum\lambda_l\phi_{il}^\alpha(w)\phi_{lk}^\alpha(z)$ with $\lambda_l \neq 0$ iff $l \in \langle\alpha, i\rangle \cap \langle\alpha, k\rangle$. Then, by the standard convolution argument: $\phi_{ij}^\alpha(f)\phi_{jk}^\alpha(g) = \sum\lambda_l\phi_{il}^\alpha(e_l g)\phi_{lk}^\alpha(\bar{e}_l f)$ with e_l the phase correction. We note $a_{0\alpha} = \phi_{0\alpha}^\alpha(f)$, $b_{\alpha 0} = \phi_{\alpha 0}^\alpha(g)$ called the principal part. We define the non-principal parts a_{ij} and b_{ij} such that they incorporate the phase correction in the braiding relations. Next, if $a_{ij} = \phi_{ij}^\alpha(h)$ then $a_{ij}^\star = \phi_{ji}^\alpha(\bar{h})$, so we note $\bar{a}_{ji} = a_{ij}^\star$:

Corollary 12.18. *(Braiding relations)*

$$b_{ij}a_{jk} = \sum \nu_l a_{il} b_{lk} \quad \textit{with } \nu_l \neq 0 \textit{ iff } l \in \langle \alpha, i \rangle \cap \langle \alpha, k \rangle$$

Corollary 12.19. *(Abelian braiding) If $\#(\langle \alpha, i \rangle \cap \langle \alpha, k \rangle) = 1$ then:*

$$b_{ij}a_{jk} = \nu a_{ij} b_{jk} \quad \textit{with } \nu \neq 0$$

Lemma 12.20. *The set of vectors of the form $\eta = (\eta_i)$ with, $\eta_i = \pi_i(x)b_{ij}\xi$, $i \in \langle \alpha, j \rangle$, $x \in \mathcal{M}(I^c)$ and $\xi \in H_j$, spans a dense subspace of $\bigoplus H_i$.*

Proof. By Reeh-Schlieder, choosing a non-null vector $v_j \in F_j$, $\pi_j(\mathcal{M}(I^c))v_j$ is dense in H_j. Now, by intertwining, $b_{ij}\pi_j(\mathcal{M}(I)) = \pi_i(\mathcal{M}(I))b_{ij}$. Then, if $b_{ij}v_j = 0$, then, b_{ij} vanishes on a dense subspace, and so by continuity, $b_{ij} = 0$, contradiction. So, $b_{ij}v_j \neq 0$. Now, clearly, the set of vector $\rho = (\rho_i)$, with $\rho_i = \pi_i(x)b_{ij}\pi_j(y)v_j$, $x \in \mathcal{M}(I^c)$ and $y \in \mathcal{M}(I)$, is a subset of the set of the lemma. Now, by intertwining $\rho_i = \pi_i(x)\pi_i(y)b_{ij}v_j$. Let $\pi = \bigoplus \pi_i$ and $w = (w_i)$, with $w_i = b_{ij}v_j \neq 0$. Then, the set of ρ is exactly $\pi(\mathcal{M}(I^c).\mathcal{M}(I)).w$. Next, because $\mathcal{M}(I^c)$ and $\mathcal{M}(I)$ commute, the linear span of $\pi(\mathcal{M}(I^c).\mathcal{M}(I))$ is weakly dense in $\pi(\mathcal{M}(I^c).\mathcal{M}(I))'' = \bigoplus B(H_i)$ by von Neumann density. So, the set spans a dense subspace of $(\bigoplus B(H_i))w = \bigoplus H_i$ because $w_i \neq 0$. □

Remark 12.21. $\bar{a}_{ij}.a_{ji} \in Hom_{\mathcal{M}(I^c)}(H_i, H_i) = \pi_i(\mathcal{M}(I^c))'$.
In particular, $\bar{a}_{0\alpha}.a_{\alpha 0} \in \pi_0(\mathcal{M}(I))$ by Haag-Araki duality.

Definition 12.22. *Let $|i|$ be the less number of edges from i to 0 in the connected graph $\mathcal{G}_\alpha$.*

Theorem 12.23. *(Transport formula)*

$$\pi_i(\bar{a}_{0\alpha}.a_{\alpha 0}) = \sum_{j \in \langle \alpha, i \rangle} \lambda_j \bar{a}_{ij}.a_{ji} \quad \textit{with } \lambda_j > 0.$$

Proof. We prove by induction on $|i|$. We suppose that:

$$\pi_i(\bar{a}_{0\alpha}.a_{\alpha 0}) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda_j \bar{a}_{ij}.a_{ji} \quad \text{and} \quad \pi_i(\bar{b}_{0\alpha}.b_{\alpha 0}) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda'_j \bar{b}_{ij}.b_{ji}$$

(1) Polarizing the second identity, we get:

$$\pi_i(\bar{b}_{0\alpha}.b'_{\alpha 0}) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda'_j \bar{b}_{ij}.b'_{ji}$$

Now, with $x \in \mathcal{M}(I^c)$ and $b'_{ij} = \pi_i(x)b_{ij}\pi_j(x)^\star$, we get:

$$\pi_i(\bar{b}_{0\alpha}.\pi_\alpha(x)b_{\alpha 0}\pi_0(x)^\star) = \textstyle\sum_{j \in \langle \alpha, i \rangle} \lambda'_j \bar{b}_{ij}.\pi_j(x).b_{ji}\pi_i(x)^\star$$

Now, $\pi_i(\pi_0(x)^\star) = \pi_i(x)^\star$, so you can simplify by $\pi_i(x)^\star$:

$$\pi_i(\bar{b}_{0\alpha}.\pi_\alpha(x)b_{\alpha 0}) = \sum_{j\in\langle\alpha,i\rangle} \lambda'_j \bar{b}_{ij}.\pi_j(x).b_{ji}$$

(2) Next, by (1) and the braiding relations, $\bar{a}_{ik}\pi_k(\bar{b}_{0\alpha}\pi_\alpha(x)b_{\alpha 0})a_{ki} = \pi_i(\bar{b}_{0\alpha}\pi_\alpha(x)b_{\alpha 0})\bar{a}_{ik}a_{ki} = \sum_j \sum_{l,s} \lambda'_j \nu_l \mu_s \bar{b}_{ij}\bar{a}_{jl}a_{ls}\pi_s(x)b_{si}$.
Let $y = \bar{a}_{ik}\pi_k(\bar{b}_{0\alpha}\pi_\alpha(x^\star x)b_{\alpha 0})a_{ki} = a^\star_{ki}\pi_k(b^\star_{\alpha 0}\pi_\alpha(x^\star x)b_{\alpha 0})a_{ki}$ clearly a positive operator, then, $\forall\xi \in H_i$, $(y\xi,\xi) \geq 0$. Then, with $\eta_s = \pi_s(x)b_{si}\xi$, we obtain:

$$\sum \lambda'_j \nu_l \mu_s (a_{ls}\eta_s, a_{lj}\eta_j) \geq 0$$

(3) We now show that this inequality is linear in η:
Let $\tilde{\eta} = \sum \eta^r$ with $\eta^r = (\eta^r_s)$, $\eta^r_s = \pi_s(x_r)b_{si}\xi_r$, $x_r \in \mathcal{M}(I^c)$ and $\xi_r \in H_i$. Idem, $Y = (y_{rt})$ with $y_{rt} = a^\star_{ik}\pi_k(b^\star_{\alpha 0}\pi_\alpha(x^\star_r x_t)b_{\alpha 0})a_{ik}$, is a positive operator-valued matrix, so that $\sum_{r,t}(y_{rt}\xi_t,\xi_r) \geq 0$, which is exactly the inequality $\sum \lambda'_j \nu_l \mu_s (a_{ls}\tilde{\eta}_s, a_{lj}\tilde{\eta}_j) \geq 0$, and the linearity follows.
(4) Next, by lemma 12.20, the set of such η span a dense subspace of $\bigoplus H_s$, then, by linearity and continuity, the inequality runs $\forall\eta \in \bigoplus H_s$.
In particular, taking all but one η_j equal to zero, we obtain $\forall\eta_j \in H_j$:

$$\lambda'_j \mu_j \sum_l \nu_l \|a_{lj}\eta_j\|^2 \geq 0$$

(5) Now, restarting from $\tilde{Y} = (\pi_k(z_u)^\star Y \pi_k(z_v))$ with $z_u \in \mathcal{M}(I)$, we obtain:

$$\lambda'_j \mu_j \sum_l \nu_l \|\rho_l\|^2 \geq 0 \quad \forall(\rho_l) \in \bigoplus H_l$$

Choosing all but one ρ_l equal to zero, we have $\lambda'_j \nu_l \mu_j > 0$, and so $\nu_l \mu_j > 0$.
(6) Let $Z = (z_{rt})$, with $z_{rt} = b^\star_{ji}\pi_j(a^\star_{\alpha 0}a_{\alpha 0})\pi_j(x^\star_r x_t)b_{ji}$, and $x_r \in \mathcal{M}(I^c)$.
Z is a positive operator-valued matrix, so by the same process, induction and intertwining, we get:

$$\sum \lambda_k \nu_l \mu_s (a_{ls}\eta_s, a_{lj}\eta_j) = (\pi_j(a^\star_{\alpha 0}a_{\alpha 0})\eta_j, \eta_j)$$

Since it's true for all $\eta_s \in \bigoplus H_s$, all the term with $s \neq j$ are null:

$$(\pi_j(a^\star_{\alpha 0}a_{\alpha 0})\eta_j, \eta_j) = \sum \lambda_k \nu_l \mu_j (a_{lj}\eta_j, a_{lj}\eta_j)$$

But, we know that $\nu_l \mu_j > 0$, then, by induction hypothesis;

$$\pi_j(\bar{a}_{0\alpha}a_{\alpha 0}) = \sum \Lambda_l \bar{a}_{jl}a_{lj}, \text{ with } \Lambda_l > 0$$

The result follows because α is a weak generator and $j \in \langle\alpha,i\rangle$. $\square$

Corollary 12.24. *(Connes fusion for charge α)*

$$H_\alpha \boxtimes H_i = \bigoplus_{j\in\langle\alpha,i\rangle} H_j$$

Proof. Let $\mathcal{X}_0 \subset Hom_{\mathcal{M}(I^c)}(H_0, H_\alpha)$, be the linear span of intertwiners $x = \pi_\alpha(h)a_{\alpha 0}$, with $h \in \mathcal{M}(I)$ and $a_{\alpha 0}$ a primary field localised in I. Since $x\Omega = (\pi_\alpha(h)a_{\alpha 0}\pi_0(h)^\star)\pi_0(h)\Omega$ with h unitary, and $\pi_\alpha(h)a_{\alpha 0}\pi_0(h)^\star$ also a primary field, it follows by the Reeh-Schlieder theorem (and by the fact that the unitary operators generate the von Neumann algebra) that $\mathcal{X}_0\Omega$ is dense in $\mathcal{X}_0 H_0$. Now, using the von Neumann density in the same way that for the lemma 12.13, $\mathcal{X}_0\Omega$ is also dense in H_α. Let $x = \sum \pi_\alpha(h^{(r)})a_{\alpha 0} \in \mathcal{X}_0$, $x_{ji} = \sum \pi_j(h^{(r)})a_{ji}^{(r)}$ and $y \in \mathcal{Y} := Hom_{\mathcal{M}(I)}(H_0, H_i)$. By the transport formula: $(x^\star x y^\star y\Omega, \Omega) = (y^\star \pi_i(x^\star x)y\Omega, \Omega) = \sum \lambda_j \|x_{ji}y\Omega\|^2$. Now, polarising this identity, we get an isometry U of the closure of $\mathcal{X}_0 \otimes \mathcal{Y}$ in $H_\alpha \boxtimes H_i$ into $\bigoplus H_j$, sending $x \otimes y$ to $\bigoplus \lambda_j^{1/2} x_{ji} y\Omega$. By the Hilbert space continuity lemma, $\mathcal{X}_0 \otimes \mathcal{Y}$ is dense in $H_\alpha \boxtimes H_i$. Now, each a_{ji} can be non-zero, so by the unicity of the decomposition into irreducible, U is surjective and then a unitary operator. □

Corollary 12.25. *(Commutativity for charge α)*

$$H_\alpha \boxtimes H_i = H_i \boxtimes H_\alpha$$

Proof. We prove in the same way that $H_i \boxtimes H_\alpha = \bigoplus_{j\in\langle\alpha,i\rangle} H_j$. □

12.4 Connes fusion with H_β

Recall that $\beta = (0, 1)$ and $\phi^\alpha_{\alpha,\beta}$ is non-zero.

$$\phi^\alpha_{ij}(z)\phi^\beta_{jk}(w) = \sum \lambda_l \phi^\beta_{il}(w)\phi^\alpha_{lk}(z) \text{ with } \lambda_l \neq 0 \text{ iff } l \in \langle\beta, i\rangle \cap \langle\alpha, k\rangle$$

Remark 12.26. *We proceed as previously: this braiding pass to the local primary field, we make principal and non-principal part incorporating the phase correction. Now, β is not a weak generator, so, to prove a transport formula, we prove by induction on $|i|$ that $a_{i0}c^\star_{\beta 0}c_{\beta 0} = [\sum \lambda_l c^\star_{li} c_{li}]a_{i0}$, with a_{i0} a chain of even primary field of charge α localised on I, (c_{ij}) even primary fields of charge β localised on I^c, and $\lambda_l \geq 0$ iff $l \in \langle\beta, i\rangle$. The proof uses the same arguments with positive operators... then by intertwining we obtain the following partial transport formula, and next, a partial fusion rules:*

Corollary 12.27. *(Transport formula)*

$$\pi_i(\bar{c}_{0\beta}.a_{\alpha 0}) = \sum_{j\in\langle\beta,i\rangle} \lambda_j \bar{c}_{ij}.c_{ji} \quad \text{with } \lambda_j \geq 0.$$

Corollary 12.28. *(partial Connes fusion for β)*

$$H_\beta \boxtimes H_i \leq \bigoplus_{j\in\langle\beta,i\rangle} H_j$$

12.5 The fusion ring

We define the fusion ring $(\mathcal{T}_m, \oplus, \boxtimes)$ generated as the $\mathbb{Z}$-module, by the discrete series of $\mathfrak{Vir}_{1/2}$ at fixed charge c_m, with $m = \ell + 2$

Lemma 12.29. *(closure under fusion)*

(a) Each H_i is contains in some $H_\alpha^{\boxtimes n}$.

(b) The H_i's are closed under Connes fusion.

(c) $H_i \boxtimes H_j = \bigoplus m_{ij}^k H_k$ with $m_{ij}^k \in \mathbb{N}$

Proof. (a) Direct because α is a weak generator.
(b) Since $H_i \subset H_\alpha^{\boxtimes m}$ and $H_j \subset H_\alpha^{\boxtimes n}$ for some m, n, we have $H_i \boxtimes H_j \subset H_\alpha^{\boxtimes m+n}$, which is, by induction, a direct sum of some H_i. Now, by Schur's lemma any subrepresentations of a direct sum of irreducibles, is a direct sum of irreducibles; then, so is for $H_i \boxtimes H_j$.
(c) By induction, $H_\alpha^{\boxtimes m+n}$ admits only finite multiplicities. □

Definition 12.30. *(Quantum dimension) A quantum dimension is an application $d : \mathcal{T}_m \to \mathbb{R} \cup \{\infty\}$, which is additive and multiplicative for $\oplus$ and $\boxtimes$, and positive (possibly infinite) on the base (H_i).*

Recall 12.31. *On a fusion ring, finite as $\mathbb{Z}$-module, the quantum dimension d is finite if $\forall A \in \mathcal{T}_m$, $\exists B \in \mathcal{T}_m$ such that $H_0 \leq A \boxtimes B$. If so, B is unique and called the dual of A, noted $A^\star$.*

Remark 12.32. $H_0 \leq H_\alpha \boxtimes H_\alpha$. *Then, H_α^ℓ is self-dual and $d(H_\alpha)$ finite.*

Corollary 12.33. *The quantum dimension is finite on the fusion ring.*

Proof. Because H_α is a weak generator, $\forall i$, $H_i \leq H_\alpha^{\boxtimes n}$ for some n, then $d(H_i) \leq d(H_\alpha)^n$ finite. □

Recall 12.34. *(Frobenius reciprocity) If $nA \leq B \boxtimes C$ then $nC \leq B^\star \boxtimes A$.*

Recall 12.35. *(Perron-Frobenius theorem) An irreducible matrix with positive entries admits one and only one positive eigenvalues. The corresponding eigenspace is generated by a single vector $v = (v_i)$, with $v_i > 0$.*

Corollary 12.36. *A quantum dimension on $\mathcal{T}_m$ with $d(H_0) = 1$ is uniquely determined, and given by the fusion matrix of $H_\alpha = H_\alpha^\star$.*

Proof. $H_\alpha \boxtimes (\sum d(H_j)H_j) = \sum n_{\alpha j}^k d(H_j)H_k = \sum d(\sum n_{\alpha j}^k H_j)H_k$
$= \sum d(\sum n_{\alpha k}^j H_j)H_k = \sum d(H_\alpha H_k)H_k = d(H_\alpha)(\sum d(H_k)H_k)$.
Note that $n_{\alpha j}^k = n_{\alpha k}^j$ is immediate from Frobenius reciprocity and H_α self-dual. Next, α is a weak generator, so the fusion matrix M_α, is irreducible. The result follows with the Perron-Frobenius theorem, with $v_i = d(H_i)$. □

12.6 The fusion ring and index of subfactor.

Definition 12.37. *Let $\langle a, b\rangle_n = \{c = |a-b|, |a-b|+1, ..., a+b \mid a+b+c \leq n\}$.*

Corollary 12.38. *(Connes fusion rules for α and β)*

$$(a) \quad H_\alpha^\ell \boxtimes H_{i'j'}^\ell = \bigoplus_{(i'',\, j'') \in \langle \frac{1}{2}, i'\rangle_\ell \times \langle \frac{1}{2}, j'\rangle_{\ell+2}} H_{i''j''}^\ell$$

$$(b) \quad H_\beta^\ell \boxtimes H_{i'j'}^\ell \leq \bigoplus_{(i'',\, j'') \in \langle 0, i'\rangle_\ell \times \langle 1, j'\rangle_{\ell+2}} H_{i''j''}^\ell$$

Proof. Immediate from theorem 11.47 and sections 12.3, 12.4. □

Recall 12.39. *(Connes fusion rules for $L\mathfrak{g}$ at level ℓ [99])*

$$H_i^\ell \boxtimes H_j^\ell = \bigoplus_{k \in \langle i,j\rangle_\ell} H_k^\ell$$

Recall 12.40. *(Quantum dimension [99])*

$$d(H_i^\ell) = \frac{sin(p\pi/m)}{sin(\pi/m)}$$

with $m = \ell + 2$ *and* $p = dim(V_i) = 2i + 1$.

Definition 12.41. *Let* $(\mathcal{R}_\ell, \oplus, \boxtimes)$ *be the fusion ring generated as* $\mathbb{Z}$*-module by discrete series of* $LSU(2)$ *at level* ℓ.

Remark 12.42. H^m_{pq} *and* $H^m_{m-p,m+2-q}$ *are the same representation of* $\mathfrak{Vir}_{1/2}$ *because* h^m_{pq} *and* $h^m_{m-p,m+2-q}$.

Definition 12.43. *Let* $\tilde{\mathcal{T}}_m$ *be a formal associative fusion ring, generated by* $(\tilde{H}^m_{pq})$ *(or* $(\tilde{H}^\ell_{ij})$ *with the other notation), with every* $\tilde{H}^m_{pq}$ *distinct (in particular* $\tilde{H}^m_{pq} \neq \tilde{H}^m_{m-p,m+2-q}$*), using the fusion rules of corollary 12.38.*

Proposition 12.44. *The ring* $\tilde{\mathcal{T}}_m$ *is isomorphic to* $\mathcal{R}_\ell \otimes_\mathbb{Z} \mathcal{R}_{\ell+2}$.

Proof. Let the bijection $\varphi : \tilde{\mathcal{T}}_m \to \mathcal{R}_\ell \otimes_\mathbb{Z} \mathcal{R}_{\ell+2}$ with $\varphi(\tilde{H}^\ell_{ij}) = (H^\ell_i, H^\ell_j)$. The fusion matrix of $\tilde{H}^\ell_\alpha$ is clearly equal to the fusion matrix of $(H^\ell_{1/2}, H^{\ell+2}_{1/2})$. Then, by Perron-Frobenius theorem, $\tilde{H}^\ell_{ij}$ and (H^ℓ_i, H^ℓ_j) has the same quantum dimension. Now, $d(\tilde{H}^\ell_\beta).d(\tilde{H}^\ell_{i'j'}) \leq \sum d(\tilde{H}^\ell_{i''j''})$, and $d(H^\ell_0, H^\ell_1).d(H^\ell_{i'}, H^\ell_{j'}) = \sum d(H^\ell_{i''}, H^\ell_{j''})$. So, by positivity, the previous inequality is an equality and:

$$\tilde{H}^\ell_\beta \boxtimes \tilde{H}^\ell_{i'j'} = \bigoplus\nolimits_{(i'',\, j'') \in \langle 0, i' \rangle_\ell \times \langle 1, j' \rangle_{\ell+2}} \tilde{H}^\ell_{i''j''}$$

So, the fusion rules for $\tilde{H}^\ell_\beta$ is also the same that for (H^ℓ_0, H^ℓ_1). Now, by associativity, the fusion rules for $\tilde{H}^\ell_\alpha$ and $\tilde{H}^\ell_\beta$ give all the fusion rules.
The result follows. □

Corollary 12.45. $\mathcal{T}_m$ *is isomorphic to the subring of* $(\mathcal{R}_\ell \otimes_\mathbb{Z} \mathcal{R}_{\ell+2}) \otimes \mathbb{Q}$ *generated by* $\frac{1}{2}[(H^\ell_i, H^\ell_j) + (H^\ell_{\frac{\ell}{2}-i}, H^\ell_{\frac{\ell+2}{2}-j})]$*; or to* $(\mathcal{R}_\ell \otimes_\mathbb{Z} \mathcal{R}_{\ell+2})/((H^\ell_i, H^\ell_j) - (H^\ell_{\frac{\ell}{2}-i}, H^\ell_{\frac{\ell+2}{2}-j}))$*. In particular, the fusion is commutative.*

Proof. Immediate. □

Theorem 12.46. *(Connes fusion for* $\mathfrak{Vir}_{1/2}$*)*

$$H^\ell_{ij} \boxtimes H^\ell_{i'j'} = \bigoplus_{(i'',\, j'') \in \langle i, i' \rangle_\ell \times \langle j, j' \rangle_{\ell+2}} H^\ell_{i''j''}$$

Proof. Immediate. □

Remark 12.47. $H^\ell_{00} \leq (H^\ell_{ij})^{\boxtimes 2}$, *so that* H^ℓ_{ij} *is self-dual.*

Theorem 12.48. *(Quantum dimension for* $\mathfrak{Vir}_{1/2}$*)*

$$d(H^\ell_{ij}) = d(H^\ell_i).d(H^{\ell+2}_j) = \tfrac{sin(p\pi/m)}{sin(\pi/m)}.\tfrac{sin(q\pi/(m+2))}{sin(\pi/(m+2))}$$

with $m = \ell + 2$, $p = 2i + 1$ *and* $q = 2j + 1$.

Proof. Immediate. □

Theorem 12.49. *(Jones-Wassermann subfactor)*

$$\pi^\ell_{ij}(\mathfrak{Vir}_{1/2}(I))'' \subset \pi^\ell_{ij}(\mathfrak{Vir}_{1/2}(I^c))^\natural$$

It's a finite depth, irreducible, hyperfinite III_1*-subfactor, isomorphic to the hyperfinite* III_1*-factor* $\mathcal{R}_\infty$ *tensor the* II_1*-subfactor :*

$$(\bigcup \mathbb{C} \otimes End_{\mathfrak{Vir}_{1/2}}(H^\ell_{ij})^{\boxtimes n})'' \subset (\bigcup End_{\mathfrak{Vir}_{1/2}}(H^\ell_{ij})^{\boxtimes n+1})'' \text{ of index } d(H^\ell_{ij})^2.$$

Proof. It's finite depth because there is only finitely many irreducible positive energy representations of charge c_m. Next, the hyperfinite III$_1$-subfactor and the irreducibility has already been proven before. The higher relative commutants can be calculated using the method of H. Wenzl [103]. The rest follows from the work of S. Popa [77]. □

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Résumé: L'algèbre Neveu-Schwarz $\mathfrak{Vir}_{1/2}$ est une extension supersymétrique et centrale de $\mathfrak{W}$, l'algèbre de Lie des champs de vecteurs polynomiaux sur $\mathbb{S}^1$. Soit $\mathfrak{g}$ une algèbre de Lie simple compacte de dimension N, alors, $\mathfrak{Vir}_{1/2}$ émerge du module vertex de l'algèbre supersymétrique $\widehat{\mathfrak{g}}$. Par la construction GKO avec $\mathfrak{g} = \mathfrak{sl}_2$, chaque espace de multiplicité est une représentation unitaire de la série discrète de $\mathfrak{Vir}_{1/2}$, ça donne leur caractère; qui permettent de prouver la formule du déterminant de Kac; on exploite ses courbes d'annulation pour prouver le critère FQS, qui permet par un argument de Wasserman, de montrer que chaque espace de multiplicité est irréductible, et constituent exactement la série discrète; leur caractère s'ensuit.

Par la suite, les algèbres de von Neumann des algèbres Neveu-Schwarz locales $\mathfrak{Vir}_{1/2}(I)$ sont isomorphes au facteur hyperfini de type III_1, par le dévissage de Takesaki depuis les fermions; et leurs supercommutants sont engendrés par des chaînes de champs primaires: indispensable pour prouver l'irréductibilité des sous-facteurs de Jones-Wassermann. Les relations de tressage (déduites par construction 'coset') et la formule de transport permettent de calculer la fusion de Connes de la série discrète vue comme des bimodules, les formules d'indices s'ensuivent.

Title: Unitary discrete series, characters, Connes fusion and subfactors for the Neveu-Schwarz algebra

Abstract: We give a complete proof of the classification of the unitary positive energy representations of the Neveu-Schwarz algebra, in such a way that we obtain directly the characters of the discrete series. Next, we explicit their Connes fusion rules and prove that the Jones-Wassermann subfactors are irreducible of finite index, we give their formula.

Key-words von Neumann algebra, conformal field theory, primary fields, intertwining operator, subfactors, Connes fusion, unitary representations, local algebra, III_1 factor, bimodules, braiding, supersymmetry, boson, fermion, vertex algebra, Virasoro algebra

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