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Université de Versailles - Saint-Quentin-en-Yvelines

THÈSE

présentée en vue de l'obtention du grade de

Docteur de l'Université de Versailles - Saint-Quentin-en-Yvelines
Mention Mathématiques et Applications

par

Claudio MUÑOZ C.

Dynamique et collision de solitons pour quelques équations dispersives nonlinéaires

Soutenue le 23 Juin 2010 devant la Commission d'examen:

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Résumé

Dans cette thèse, nous étudions quelques propriétés dynamiques des solutions de type soliton de quelques équations dispersives nonlinéaires généralisées.

La première partie de ce travail est consacrée à l'étude de l'existence, de l'unicité et du comportement global de solitons pour des équations de KdV généralisées, à variation lente. On donnera une description détaillée de la dynamique pour tout temps et on montrera la non-existence de solitons purs, ce qui est une très grande différence avec l'équation gKdV standard.

Dans une deuxième partie, on étudiera le cas de l'équation de Schrödinger nonlinéaire. Pour cette équation, nous allons améliorer tous les résultats précédents en donnant une description précise pour tout temps de la dynamique du soliton dans le régime à variation lente. En plus, sous des hypothèses générales, on montrera ce résultat dans le cas 2-D.

Finalement, on considère le problème de collision de deux solitons pour l'équation de KdV généralisée. Complétant les résultats récents de Martel et Merle, concernant le cas quartique, nous montrons que la seule possibilité d'avoir une collision de type élastique est donnée par les cas intégrables.

La preuve de tous ces résultats sont des développements et des améliorations de la théorie de Martel et Merle pour la collision de deux solitons des équations gKdV sous différents régimes asymptotiques.

Mots-clefs : équation de Korteweg-de Vries généralisée, équation de Schrödinger nonlinéaire, dynamique du soliton, potentiels à variation lente, collision de deux solitons, intégrabilité.

SOLITON DYNAMICS AND COLLISION FOR SOME NONLINEAR DISPERSIVE EQUATIONS

Abstract

This work deals with long time dynamics of soliton solutions for generalizations of well-known dispersive equations.

The first part of this work is devoted to the study of existence, uniqueness and global behavior of soliton-like solutions for slowly varying, but still large perturbations of generalized KdV equations. We give an accurate description of the dynamics for all time and prove in addition the nonexistence of pure soliton-like solutions, a big difference with the standard gKdV equations.

Next, the same kind of results are proven in the case of nonlinear Schrödinger equations. We improve all the existing results by constructing a unique global soliton solution in this regime, and studying in detail its behavior. In addition, under some mild assumptions we extend this result to the two-dimensional case and under general incident velocities.

Finally, we consider the scenario of a 2-soliton collision between a small and a very small soliton, for generalized KdV equations. We prove a classification result which completes the Martel-Merle results –concerning the quartic case– asserting that in a very general framework the unique possibilities for having an elastic collision are given by the integrable cases.

The proof of all these results are reminiscent of the very recent Martel-Merle theory of 2-soliton's collision for gKdV equations under different asymptotic regimes.

Keywords : generalized Korteweg- de Vries and nonlinear Schrödinger equations, soliton dynamics, slowly varying potentials, 2-soliton collision, integrability.

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À mes amis chiliens, mexicains et français, merci beaucoup pour m'avoir reçu à Paris, et pour les super soirées partagées avec vous.

Finalmente, esta tesis está dedicada a mi familia, a las personas que quiero, y especialmente a *Martina*.

Ésta es mi última transmisión desde el planeta de los monstruos.
No me sumergiré nunca más en el mar de mierda de la literatura.
En adelante escribiré mis poemas con humildad
y trabajaré para no morirme de hambre y no intentaré publicar.

Roberto Bolaño, *Estrella Distante*.

Table des matières

Part I

Introduction

Summary

1 Préliminaires

Dans cette thèse, nous considérons des propriétés qualitatives et en temps long de quelques **équations dispersives nonlinéaires**, principalement les équations de **Korteweg-de Vries** et **Schrödinger nonlinéaires**.

On parlera des équations dispersives pour désigner des équations sous la forme

$$u_t = Lu + F(u, Du, \ldots),$$

où $u(t, x)$ est une fonction à valeurs réelles ou complexes, et $t \in \mathbb{R}, x \in \mathbb{R}^N$. L'opérateur linéaire L est supposé anti-adjoint, c'est-à-dire

$$\mathcal{F}(Lu)(\xi) = ip(\xi)\hat{u}(\xi), \quad p(\xi) \in \mathbb{R},$$

en termes de transformées de Fourier. Enfin, F est un terme dit nonlinéaire, contenant possiblement des dérivées d'ordre supérieur. De façon plus mathématique, l'équation est dite dispersive si de plus la matrice $D^2p(\xi)$ est non singulière.

L'heuristique nous dit qu'une équation est plus ou moins dispersive en fonction de la taille de p. Par exemple, l'équation linéaire d'Airy $u_t + u_{xxx} = 0$, qui satisfait $p(\xi) = \xi^3$, est plus dispersive que l'équation de Schrödinger linéaire $iu_t + u_{xx} = 0$, qui satisfait $p(\xi) = -\xi^2$.

Quelques autres exemples d'équations dispersives sont les équations du type Korteweg-de Vries, les équations de Schrödinger nonlinéaires, l'équation de Benjamin-Ono, l'équation BBM, KPI, KPII, etc.

Une fois que l'on a des résultats appropriés sur le caractère localement ou globalement bien posé de l'équation, une liste de questions très intéressantes peuvent être envisagées. L'une de ces questions est bien sûr la dynamique en temps long, ou bien une description qualitative d'une classe de solutions, même des solutions très particulières.

Justement, ce travail se focalise sur l'étude d'un type très particulier de solutions d'une classe d'équations nonlinéaires dispersives. Ces solutions sont appelées **solitons**, ou plus généralement **ondes solitaires**. Ce sont des concepts légèrement différents, mais l'esprit est le même, et nous pensons qu'un exemple simple peut l'illustrer.

On considère l'un des plus simples cas d'équations dispersives, l'équation de Korteweg-de Vries généralisée (gKdV). Elle est donnée par[1]

$$u_t + (u_{xx} + u^m)_x = 0, \quad u = u(t, x), \quad (t, x) \in \mathbb{R}^2, \quad m = 2, 3 \text{ ou } 4. \tag{1.1}$$

L'équation de Korteweg-de Vries (KdV), c'est-à-dire le cas $m = 2$ au-dessus, apparaît en physique comme un modèle de propagation des ondes dans des eaux peu profondes, comme décrit par J. S. Russel en 1834 [48]. La formulation exacte de cette équation a été donnée par Korteweg et de Vries (1895) [33]. Finalement, cette équation a été étudiée du point de vue numérique par N. Zabusky et M. Kruskal en 1965 [34].

L'équation précédente a quelques propriétés surprenantes. Tout d'abord, notons que si $u = u(t, x)$ est une solution à cette équation, alors $u(t - t_0, x - x_0)$ et $u_c(t, x) := c^{1/(m-1)}u(ct, \sqrt{c}x)$ sont aussi des solutions, quels que soient $c > 0$ et $t_0, x_0 \in \mathbb{R}$. Elles illustrent l'invariance par **translation** et par **scaling**, respectivement. Les **solitons** sont précisément des solutions **localisées** de (1.1) de la forme

$$u(t, x) := Q_c(x - ct),$$

[1]Le fait de considérer uniquement des puissances $m < 5$ sera plus clair ci-dessous.

avec $Q_c(s) := c^{1/(m-1)}Q(\sqrt{c}s)$. Si cette classe de solutions existe bien, alors elle a un profil invariant et se propage vers la droite au cours du temps. De plus, elle est une solution **globalement bien définie**.

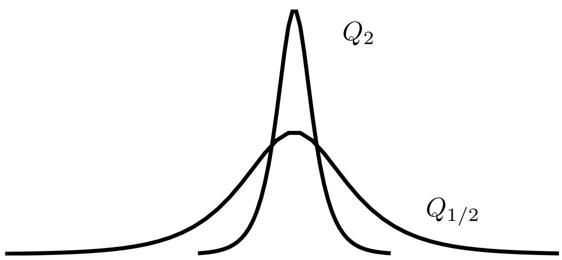
D'après ces considérations, il est clair que la fonction Q doit être une solution de l'équation nonlinéaire de deuxième ordre suivante

$$Q'' - Q + Q^m = 0 \ \text{ dans } \mathbb{R}, \quad Q > 0, \tag{1.2}$$

laquelle a une unique solution dans $H^1(\mathbb{R})$ modulo des translations en espace. Cette solution est donnée par la formule suivante

$$Q(x) = \left[\frac{m+1}{2\cosh^2(\frac{(m-1)}{2}x)}\right]^{\frac{1}{m-1}}.$$

Notons que la solution $Q_c(s)$ nous donne l'heuristique suivante : les solitons petits sont plus plats que les grands solitons, mais en même temps plus lents. Il s'agit de l'équivalence taille-vitesse dans l'équation gKdV. La figure suivante décrit deux solitons avec deux tailles différentes :



De plus, comme Q est une fonction positive, solution d'une équation elliptique nonlinéaire, elle peut être vue comme la solution d'un problème de minimisation dans $H^1(\mathbb{R})$, dont Q représente *l'état fondamental*, avec la quantité d'énergie minimale. Pour montrer l'existence de cette solution de façon générale, il est nécessaire d'utiliser des méthodes du type **concentration-compacité** [6], car l'espace $H^1(\mathbb{R})$ ne contient pas des propriétés de compacité évidentes.

Un deuxième exemple important est donné par l'équation de Schrödinger nonlinéaire (NLS)

$$iu_t + u_{xx} + |u|^{m-1}u = 0, \quad \text{dans } \mathbb{R}_t \times \mathbb{R}_x, \quad m \in [2,5), \tag{1.3}$$

où $u = u(t,x)$ est une fonction à valeurs complexes. Le cas cubique est connu en physique comme un modèle de propagation des ondes en fibre optique dans un milieu nonlinéaire. En deux dimensions, la cas cubique est aussi très important.

Étant données $c_0 > 0$, $v_0, x_0, \phi_0 \in \mathbb{R}$, cette dernière équation a des solutions de la forme

$$Q_{c_0}(x - v_0 t - x_0)e^{\frac{i}{2}v_0 x}e^{i(c_0 - \frac{1}{4}v_0^2)t}e^{i\phi_0},$$

où Q_{c_0} est le soliton déjà mentionné. Cette solution – localisée – est appelée *onde solitaire*, et son module est un profil invariant, traversant l'espace dans la direction du signe de la vitesse v_0. Derrière cette solution, il y a deux symétries additionnelles que satisfont les solutions de (1.3) : **phase et invariance galiléenne**.

Ces deux types de solutions, solitons et ondes solitaires, partagent une propriété très importante : elles sont **pures**, dans le sens où aucun terme supplémentaire n'est nécessaire pour satisfaire l'équation. Ce phénomène est assez étrange dans une solution générale quelconque, en raison du caractère dispersif de l'équation considérée, et il peut être expliqué par un équilibre délicat entre la dispersion donnée par le terme de la dérivée et la nonlinéarité de l'équation.

Au début de ce paragraphe, nous avons mentionné les notions de localement et globalement bien posé (LWP-GWP) comme un élément-clé pour étudier les propriétés qualitatives en temps long pour les solutions des équations dispersives. Voici quelques raisons à cela.

À partir des solitons, on peut considérer l'étude analytique ou numérique de **petites perturbations** de ceux-ci, ou encore plus difficile, le comportement de solutions composées de plusieurs solitons. Si l'on considère deux solitons ou plus, il est naturel de penser aux possibles collisions entre eux, ce qui est une question très difficile à résoudre en raison du caractère nonlinéaire de l'équation. Dans les deux cas, on commence près de solutions du type solitons, et pour décrire les deux problèmes, les propriétés LWP-GWP nous donnent l'existence d'une solution à ce problème pour une période de temps.

Ainsi, dans les prochaines lignes, nous allons donner une compte rendu informel sur la théorie LWP-GWP pour les équations gKdV et NLS.

On suppose $m = 2, 3$ ou 4 dans (1.1). L'existence locale pour gKdV est maintenant un résultat standard. D'après les travaux de Kenig, Ponce et Vega [31], (1.1) est localement bien posée dans $H^1(\mathbb{R})$ et donc de façon globale grâce à la conservation de la masse et de l'énergie

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)\,dx = \frac{1}{2}\int_{\mathbb{R}} u_0^2(x)\,dx = M[u](0), \quad \text{(Masse)}, \tag{1.4}$$

$$\begin{aligned} E[u](t) \quad := \quad & \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} u^{m+1}(t,x)\,dx \\ & = \frac{1}{2}\int_{\mathbb{R}} (u_0)_x^2(x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} u_0^{m+1}(x)\,dx = E[u](0), \ \text{(Énergie)} \end{aligned} \tag{1.5}$$

et l'inégalité de Gagliardo-Nirenberg suivante :

$$\int_{\mathbb{R}} u^{p+1} \leq K(p)\Big(\int_{\mathbb{R}} u^2\Big)^{\frac{p+3}{4}}\Big(\int_{\mathbb{R}} u_x^2\Big)^{\frac{p-1}{4}}. \tag{1.6}$$

Rappelons que, pour $m = 5$, le problème de Cauchy pour l'équation gKdV a des solutions qui explosent en temps fini, voir [**?**, 42, 38] et les références citées à l'intérieur. On pense que pour $m > 5$ la situation est la même. Pour les deux équations, gKdV et NLS, le cas $m = 5$ est noté comme le cas L^2-critique, car tous les solitons ont la même taille :

$$\|Q_c\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})}, \quad \text{pour tout } c > 0.$$

Toutefois, dans le *cas surcritique* ($m > 5$), les solitons de taille petite (par rapport à la norme L^∞) sont plus grands que les solitons grands (par rapport à la norme L^2).

La preuve de Kenig, Ponce et Vega utilise un argument de point fixe dans un sous-espace de $C([0,T], H^1(\mathbb{R}))$ en fonction de la nonlinéarité, pour un petit temps $T > 0$. La preuve d'un effet régularisant de l'opérateur linéaire d'Airy $e^{-t\partial_x^3}$, associé à l'équation linéaire, dans l'esprit de Kato, est l'une des étapes-clés pour boucler la procédure itérative.

Dans le cas NLS pour $2 \leq m < 5$, le problème de Cauchy est *localement bien posé* pour des données $u_0 \in H^1(\mathbb{R})$ (voir Ginibre et Velo [17]), et donc globalement bien posé grâce à la conservation de la masse et de l'énergie

$$M[u](t) := \frac{1}{2} \int_{\mathbb{R}} |u|^2(t,x)\, dx = M[u](0), \quad \text{(Masse)}, \tag{1.7}$$

et

$$E[u](t) := \frac{1}{2} \int_{\mathbb{R}} |u_x|^2(t,x)\, dx - \frac{1}{m+1} \int_{\mathbb{R}} |u|^{m+1}(t,x)\, dx = E[u](0), \text{ (Énergie)}. \tag{1.8}$$

La preuve de Ginibre et Velo est basée sur l'utilisation des estimations de **Strichartz** (dans des espaces $L_t^p L_x^q$ appropriés) et un argument de point fixe dans une petite boule de l'espace $C((-T,T), H^1(\mathbb{R}))$, pour $T > 0$ petit, comme dans le cas gKdV. Nous renvoyons à [8] pour une preuve plus détaillée.

Finalement, rappelons que l'équation NLS a une troisième quantité conservée, appelée *moment* :

$$P[u](t) := \frac{1}{2} \operatorname{Im} \int_{\mathbb{R}} \bar{u} u_x(t,x)\, dx. \tag{1.9}$$

Cette quantité est bien définie pour des solutions dans $H^1(\mathbb{R})$. Comme dans le cas gKdV, pour $m \geq 5$, le problème de Cauchy pour cette équation a des solutions explosives en temps fini, voir [8] et les références à l'intérieur.

L'étude des perturbations des ondes solitaires ou solitons conduit à l'introduction des concepts de *stabilité orbitale et asymptotique*. Par **stabilité orbitale**, nous entendons la propriété suivante. Prenons, par exemple, l'équation gKdV (1.1) et un soliton fixe $Q_c(x - x_0)$ pour $c > 0$, $x_0 \in \mathbb{R}$ données. Supposons qu'il existe une solution $u = u(t)$ de (1.1), globale en temps, et telle que

$$\|u(t_0) - Q_c(\cdot - x_0)\|_{H^1(\mathbb{R})} \leq \alpha,$$

pour une petite constante $\alpha > 0$. On dit que Q_c est stable s'il existe $K, \alpha_0 > 0$ et une fonction régulière $\rho = \rho(t)$ définie pour tout $t \geq 0$ telles que, pour tout $0 < \alpha < \alpha_0$,

$$\|u(t) - Q_c(\cdot - \rho(t))\|_{H^1(\mathbb{R})} + |\rho'(t) - c| \leq K\alpha, \quad |\rho(t_0) - x_0| \leq \alpha. \tag{1.10}$$

Autrement dit, une petite perturbation d'une solution de type soliton reste assez proche d'un soliton avec un paramètre de translation corrigé à chaque temps. La stabilité orbitale des ondes solitaires de (1.1) est par exemple vraie sous l'hypothèse sous-critique $m < 5$.

On peut donner quelques idées de la preuve de (1.10). On va suivre la méthode proposée par Weinstein : s'il existe une constante $\mu > 0$ telle que, pour toute fonction $v \in H^1(\mathbb{R})$ qui satisfait

$$\int_{\mathbb{R}} Q_c' v = 0,$$

on a

$$\mathcal{F}[v](t) := \int_{\mathbb{R}} (v_x^2 + cv^2 - mQ_c^{m-1} v^2) \geq \mu \int_{\mathbb{R}} (v_x^2 + v^2) - |\int_{\mathbb{R}} Q_c z|^2, \tag{1.11}$$

alors (1.10) est satisfait. En effet, par le théorème des fonctions implicites, il existe une fonction $\rho(t)$ et une constante $\bar{c} > 0$ telles que $z(t) := u(t) - Q_{\bar{c}}(\cdot - \rho(t))$ satisfait pour tout temps

$$\int_{\mathbb{R}} Q_{\bar{c}}' z = 0, \quad \int_{\mathbb{R}} Q_{\bar{c}}^2 = \int_{\mathbb{R}} u^2(t_0).$$

D'après la définition de la masse et l'énergie, on obtient

$$E[u](t) + \bar{c}M[u](t) = E[Q_{\bar{c}}](t) + \bar{c}M[Q_{\bar{c}}](t) + \mathcal{F}[z](t) + O(\|z(t)\|^3_{H^1(\mathbb{R})}).$$

Par la conservation de la masse et de l'énergie, et par l'équation que satisfait $Q_{\bar{c}}$, on a

$$\mathcal{F}[z](t) \lesssim \|z(t_0)\|^2_{H^1(\mathbb{R})}.$$

Finalement, d'après (1.11), on a

$$\|z(t)\|^2_{H^1(\mathbb{R})} \lesssim \|z(t_0)\|^2_{H^1(\mathbb{R})} + |\int_{\mathbb{R}} Q_{\bar{c}} z|^2.$$

Enfin, d'après la conservation de la masse, on a

$$\int_{\mathbb{R}} Q_{\bar{c}} z = O(\|z(t)\|^2_{L^2(\mathbb{R})}),$$

et par suite le résultat.

La stabilité orbitale de petites perturbations de solitons pour gKdV a été considérée dans [3, 5, 9, 45]. De façon similaire, dans le cas NLS, on a les travaux de Cazenave et Lions [9], Weinstein [62, 63], Grillakis, Shatah et Strauss [18, 19], Cuccagna [10], et Martel, Merle et Tsai [45]. Voir les références à l'intérieur de ces articles pour une bibliographie plus détaillée.

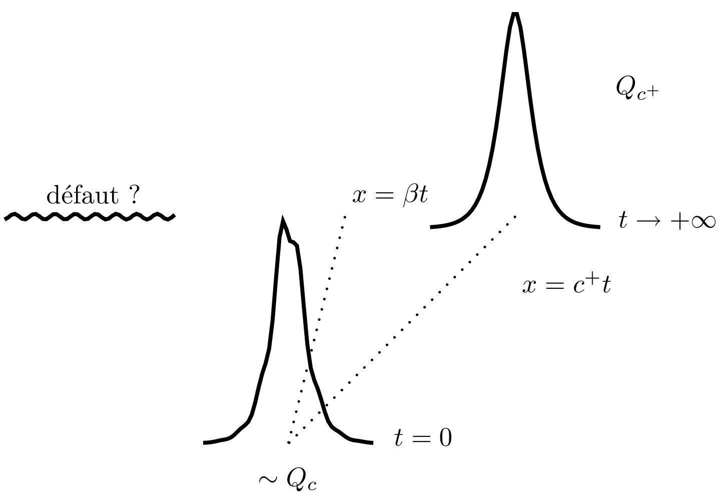
D'autre part, la **stabilité asymptotique** étudie le petit résidu donné par le résultat de stabilité ci-dessus. Il est légitime de se demander si $u(t)$ devrait en fait converger vers un soliton dans un certain sens.

Tout d'abord, quelques remarques. Notons que deux solitons Q_c et $Q_{\tilde{c}}$, avec $c \sim \tilde{c}$ mais différentes, et ayant la même trajectoire, sont toujours à une distance positive dans $H^1(\mathbb{R})$. D'autre part, un résultat de classification standard affirme que si $\|u(t) - Q_c(\cdot - x(t))\|_{H^1(\mathbb{R})} \to 0$ lorsque $t \to +\infty$, pour une certaine fonction $x(t)$, alors $u(t)$ est un *soliton pur*. Ces deux arguments suggèrent la non existence d'une convergence générale dans H^1 vers un soliton pour une petite perturbation d'un soliton.

Afin de résoudre cette problématique, il faut reformuler la propriété de stabilité asymptotique, soit en introduisant des espaces adaptés [56], soit en considérant des normes localisées [41]. Dans cette dernière formulation, on a l'existence d'une constante $\beta > 0$, et $c^+ > 0$ avec $|c^+ - c| \lesssim \alpha$, telles que

$$\|u(t) - Q_{c^+}(\cdot - \rho(t))\|_{H^1(x>\beta t)} \to 0,$$

lorsque $t \to +\infty$. De plus, $\lim_{t\to+\infty} \rho'(t) = c^+$. Autrement dit, on a convergence H^1 fort autour du soliton. Ce résultat peut être résumé par le schéma suivant :



Derrière la preuve de cette propriété se trouve l'identité de **Kato**, que nous allons fréquemment utiliser dans ce travail. Dans sa forme la plus simple, elle indique que, pour $u(t) \in H^1(\mathbb{R})$ solution de (1.1) satisfaisant la propriété de stabilité orbitale (1.10), on a

$$\partial_t \int_{\mathbb{R}} \varphi(x - \beta t) u^2(t, x) \lesssim 0, \tag{1.12}$$

pour $0 < \beta < c$ et pour toute fonction positive, croissante et bornée φ telle que $\varphi' \sim Q$. Formellement, cette dernière inégalité implique que la masse près du soliton est déplacée vers la gauche quand il avance, ce qui donne finalement le résultat. Une autre application intéressante de cette inégalité est *l'effet régularisant* associé aux équations du type gKdV.

Finalement, on remarquera que cette propriété et des résultats de scattering ont été étudiés dans [59, 60, 7, 57, 11, 58].

Revenons à l'équation gKdV considérée en (1.1), dans un cadre encore plus général. On suppose maintenant qu'on a une nonlinéarité quelconque f, ne dépendant que de $u = u(t, x)$, et on considère l'équation

$$u_t + (u_{xx} + f(u))_x = 0, \quad \text{dans } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.13}$$

Rappelons que $f(s) = s^2$ correspond au cas de KdV. D'autres cas physiquement importants sont le cas cubique $f(s) = s^3$, et la nonlinéarité *quadratique-cubique*, c'est-à-dire $f(s) = s^2 - \mu s^3$, $\mu \in \mathbb{R}_+$. Dans le premier cas, l'équation (1.13) est souvent désignée comme l'équation de KdV modifiée (mKdV), et dans le second, elle est connue comme l'équation de *Gardner*. Ces trois équations sont des modèles **complètement intégrables**. L'équation de Schrödinger cubique est aussi intégrable [64]. Finalement, la complète intégrabilité implique l'existence d'un nombre infini de quantités conservées pour l'équation associée.

Dans la forme générale (1.13), gKdV a seulement deux quantités $H^1(\mathbb{R})$ conservées : la masse et l'énergie. Cette dernière est maintenant donnée par la formule

$$E[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(t, x)\, dx - \int_{\mathbb{R}} F(u(t, x))\, dx,$$

où on a noté

$$F(s) := \int_0^s f(\sigma)\, d\sigma. \tag{1.14}$$

Dans ce travail, on ne va considérer que des nonlinéarités $f \in C^3(\mathbb{R})$ de la forme

$$f \in C^{m+2}(\mathbb{R}), \quad f(u) := u^m + f_1(u), \quad m = 2, 3, 4, \quad \text{avec} \quad \lim_{s \to 0} \frac{|f_1(s)|}{|s|^m} = 0. \tag{1.15}$$

Le signe positif devant f permet l'existence [4] de solitons pour (1.13) de la forme

$$u(t, x) := Q_c(x - x_0 - ct),$$

avec $c > 0$ petit et $x_0 \in \mathbb{R}$. La fonction Q_c est maintenant solution de l'équation elliptique

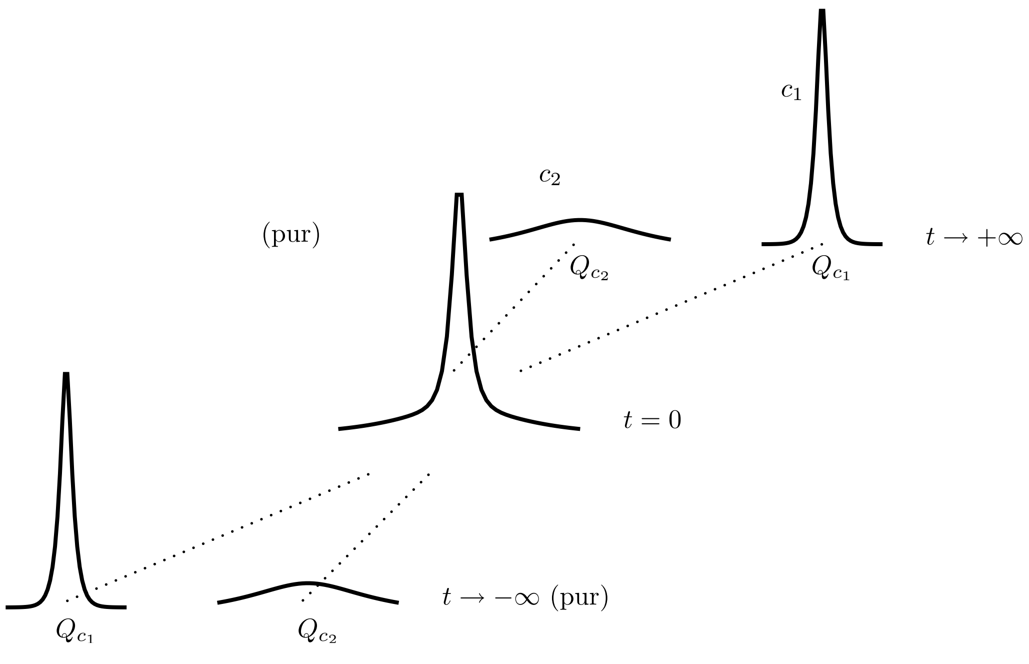
$$Q_c'' + f(Q_c) = cQ_c, \quad Q_c \in H^1(\mathbb{R}). \tag{1.16}$$

Pour tout $c > 0$, s'il existe une solution $Q_c > 0$ de (1.16) alors elle peut être choisie *paire* sur $\mathbb{R}$ et *exponentiellement décroissante* sur $\mathbb{R}^+$ (de même si $Q_c < 0$).

L'un des problèmes les plus intéressants du point de vue physique est la collision de solitons. Le problème dynamique de la collision de solitons est un problème classique dans la propagation des ondes nonlinéaires (voir [39] pour plus de références), bien que mathématiquement loin d'être compris. À présent, nous réduisons notre recherche au problème des collisions de 2-soliton.

Par collision de 2-soliton, nous entendons le problème suivant : étant donnés deux solitons, solutions de (1.13), bien séparés à l'instant initiale et ayant des vitesses différentes, nous attendons à ce qu'ils croisent à un temps fini. La solution obtenue après la collision est précisément l'objet de l'étude. En particulier, on s'intéresse à tout changement de taille, de position ou de forme, ou même à la destruction des solitons, après un certain temps ensemble.

Les premiers travaux dans cette direction ont été les résultats numériques de Fermi, Pasta et Ulam [**?**], et Zabusky et Kruskal [34], qui ont montré le caractère *élastique* de la collision entre deux solitons. Par *élastique*, on veut dire que les solitons restent inchangés et qu'il n'y a pas de défaut de masse positive pour des temps longs. La seule conséquence de la collision sont des shifts sur chaque soliton, en fonction de leurs tailles. Ensuite, le travail de Lax [35] et la méthode de *scattering inverse* (voir [1] et [48] pour plus de détails) ont fourni des formules explicites pour des N-solitons (Hirota [24]) ne présentant pas de termes résiduels avant et après la collision. En d'autres termes, la collision est *élastique*, et le N-soliton est pur, comme le montre le schéma ci-dessous pour le cas $n = 2$:



Ces propriétés sont aussi valables pour l'équation *cubique* mKdV (voir [1], p. 390), et pour l'équation de *Gardner* (voir [16, 61] et les références citées). En particulier, la collision de deux solitons est élastique. On doit remarquer que, pour l'équation de Gardner

$$u_t + (u_{xx} + u^2 - \mu u^3)_x = 0, \tag{1.17}$$

étant donné $\mu \in \mathbb{R}$, il existe une solution de type soliton pour tout $c > 0$ dans le cas $\mu < 0$, et si $c < \frac{2}{9\mu}$ dans le cas $\mu > 0$. Ces solutions sont explicites, et sont données par $u(t, x) = Q_{\mu,c}(x - ct)$, où $Q_{\mu,c}$ est la fonction [61]

$$Q_{\mu,c}(x) := \frac{3c}{1 + \rho \cosh(\sqrt{c}x)} \; ; \quad \rho := (1 - \frac{9}{2}\mu c)^{1/2}. \tag{1.18}$$

En particulier, il n'existe pas de soliton dans le cas $\mu > 0$ et $c > 0$ assez grand.

Nous rappelons que ces techniques sont connues pour être trop rigides pour être appliquées à des modèles plus généraux, et n'ont pas d'équivalent dans le cas de l'équation gKdV (1.13) avec une nonlinéarité générale. Le but est de confirmer que la collision de deux solitons n'est pas élastique en général, à l'exception des équations KdV, mKdV et des équations de Gardner.

Notons que même *l'existence* d'une solution 2-soliton pour des équations non intégrables était une question ouverte. Dans [37, 36], les auteurs ont construit une solution du type 2-soliton, pure à l'infini, ne dépendant que de quelques paramètres donnés, dans les cas gKdV sous-critique et critique, et pour NLS. Dans le cas gKdV, cette solution est aussi *unique*, en raison d'une propriété de monotonie similaire à (1.12). La preuve de ces résultats sera adaptée plusieurs fois dans ce travail pour obtenir l'existence et l'unicité de certaines solutions de type soliton.

Nous finissons ce paragraphe avec quelques remarques sur la littérature déjà existante. La propriété d'intégrabilité a été étudiée dans de nombreuses équations différentielles, comme l'équation NLS cubique, KPI, Benjamin-Ono, etc., voir par exemple [1]. En particulier, lorsque cette propriété est perdue, il n'existe que très peu de résultats. Nous mentionnons les récents travaux de Perelman [57], Holmer, Marzuola et Zworski [26, 27, 28] et Abou Salem, Fröhlich et Sigal [2] sur le problème de la collision de deux solitons pour l'équation de Schrödinger nonlinéaire sous l'action d'un potentiel ou bien avec des vitesses assez grandes.

2 Résultats principaux

Nous avons examiné en détail le comportement des solitons pour les équations gKdV ou NLS, et la collision de deux solitons dans les cas intégrables. Dans cette thèse, nous avons considéré quelques **généralisations non triviales de ces équations** et nous avons donné une description précise de certains phénomènes intéressants, via des calculs explicites, dans des régimes asymptotiques particuliers. Les résultats au cœur de ce travail peuvent être divisés en trois parties, que nous décrivons maintenant.

La première partie traite de la dynamique d'une solution de type soliton pour l'équation gKdV à coefficients variables et de variation lente. Cette solution peut être considérée comme un exemple d'interaction **soliton-potentiel**. Premièrement, nous étudions l'existence et l'unicité d'une telle solution, puis nous décrivons complètement la dynamique à l'intérieur de la région d'interaction. Il apparaît que la dynamique induit de très intéressants effets de dispersion, qui ne sont pas présents dans le régime de coefficients constants. Ensuite, nous décrivons le comportement asymptotique de cette solution pour des temps longs et, enfin, nous montrons que cette solution possède un petit défaut, différent de zéro à l'infini.

La deuxième partie de ce travail porte sur des questions similaires pour une équation généralisée de type NLS. Comme ci-dessus, nous décrivons avec beaucoup de détails la dynamique d'une solution de type soliton pour une large classe de potentiels. Nous en démontrons l'existence et l'unicité et en établissons le comportement global pour tout temps, ce qui est une amélioration considérable de tous les résultats précédents existant dans la littérature.

Enfin, dans la troisième partie, nous traitons le cas de la collision de deux solitons pour une équation de KdV généralisée, avec une nonlinéarité générale. Plus précisément, nous considérons la collision d'un soliton petit contre un très petit. Nous montrons que pour tous les cas non intégrables la collision n'est plus élastique, et aucune solution pure 2-soliton n'existe dans ce régime. Notre preuve est basée sur des résultats récents de Martel-Merle concernant le cas gKdV quartique. Nous généralisons ces résultats dans un cadre général complet.

Dans ces cas, une propriété-clé est perdue : les solutions ne sont jamais complètement pures, et des termes dispersifs non nuls sont toujours présents au long de la dynamique.

Les trois parties susmentionnées font partie des articles suivants :

1. Muñoz C., *On the soliton dynamics under slowly varying medium for generalized Korteweg-de Vries equations*, arXiv :0912.4725. To appear, Analysis and PDE.
2. Muñoz C., *On the soliton dynamics under slowly varying medium for Nonlinear Schrödinger equations*, arXiv :1002.1295.
3. Muñoz C., *On the inelastic 2-soliton collision for gKdV equations with general nonlinearity*, arXiv :0903.1240, accepté dans IMRN.

Des résultats complémentaires seront bientôt disponibles dans [52, 53, 54].

2.1 Première partie : dynamique d'un soliton pour le cas gKdV

La première partie de notre travail a été consacrée à l'étude de la dynamique des solitons pour des perturbations des équations gKdV sous-critiques. En effet, nous étudions la dynamique des solitons pour l'équation de KdV généralisée suivante :

$$\begin{cases} u_t + (u_{xx} - \lambda u + a(\varepsilon x)u^m)_x = 0 \quad \text{dans } \mathbb{R}_t \times \mathbb{R}_x, \\ m = 2, 3 \text{ ou } 4\,; \quad 0 < \varepsilon \leq \varepsilon_0, \quad 0 \leq \lambda < 1, \end{cases} \tag{2.1}$$

avec a une fonction régulière (le potentiel), satisfaisant des hypothèses raisonnables, comme

$$a'(r) > 0, \quad \lim_{r\to-\infty} a(r) = 1, \quad \text{et} \quad \lim_{r\to+\infty} a(r) = 2.$$

Nous supposons aussi $\varepsilon_0 > 0$ assez petit.

Le problème que nous considérons dispose d'une littérature physique étendue, qui commence par les travaux de Kaup et Newell [30] et Karpman et Maslov [29]. La motivation physique était l'étude de perturbations en temps de modèles intégrables.

Les auteurs ci-dessus ont effectué une analyse perturbative de la théorie de scattering inverse pour décrire la dynamique d'un soliton (de l'équation intégrable) dans ce régime variable. Curieusement, l'existence d'une *queue dispersive* sous la forme d'un plateau derrière le soliton a été formellement décrite. Ce phénomène est en fait lié à *l'absence de conservation de l'énergie* dans l'équation.

Ultérieurement, le problème a été considéré dans plusieurs travaux pour différents modèles intégrables, voir par exemple [32, 15, 20, 21]. On peut aussi consulter le manuscrit de Newell [55], pp. 87–97, pour une exposition plus détaillée du problème.

Notons qu'une solution de (2.1) non nulle peut perdre ou gagner de la masse, selon le signe de la fonction u, d'après l'identité

$$\partial_t M[u](t) = -\frac{\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}. \tag{2.2}$$

De plus, l'énergie est donnée par la formule ($\lambda \geq 0$)

$$E_a[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx + \frac{\lambda}{2}\int_{\mathbb{R}} u^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)u^{m+1}(t,x)\,dx, \tag{2.3}$$

et reste conservée pour tout temps.

Le problème de la description *analytique* de la dynamique des solitons pour différents modèles intégrables dans un milieu à variation lente a reçu une attention croissante au cours des dernières années. En ce qui concerne l'équation de KdV, notre conviction est que le premier résultat dans cette direction a été donné par Dejak et Sigal, et récemment amélioré par Holmer, dans [13, 25]. Ils ont estimé la dynamique en temps long des ondes solitaires (solitons) de perturbations plus lentement variables des équations KdV et mKdV suivantes :

$$u_t + (u_{xx} - b(\varepsilon t, \varepsilon x)u + u^2)_x = 0 \quad \text{dans } \mathbb{R}_t \times \mathbb{R}_x, \tag{2.4}$$

où b est une fonction bornée et régulière. Avec ces hypothèses, les auteurs ont montré que dans le cas de données initiales suffisamment proches d'un soliton, on obtient que pour tout temps $t \lesssim \varepsilon^{-1}$ la solution peut être décomposée comme

$$u(t,x) = Q_{c(t)}(x - \rho(t)) + w(t,x),$$

où $\|w(t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}$ et $\rho(t), c(t)$ satisfont un système dynamique particulier.

Pour obtenir de meilleurs résultats dans cette direction, et passer la barrière ε^{-1} en temps, il a été nécessaire de prendre des potentiels **moins généraux**. Cette hypothèse nous permet de comprendre, pour encore une très grande classe de potentiels, la complète dynamique des solitons généralisés.

Afin de présenter les résultats dans la forme la plus claire possible, nous avons choisi d'énoncer notre résultat concernant le cas cubique. Pour un compte rendu plus détaillé des résultats, y compris les cas quadratique et quartique, voir Partie 2, Théorème 1.1.

Théorème 2.1 (Dynamique d'interaction soliton-potentiel pour des équations gKdV)**.**

Soit $m = 3$, et soit $0 \leq \lambda \leq \frac{1}{3}$ un paramètre fixe. Il existe une petite constante $\varepsilon_0 > 0$ telle que, pour tout $0 < \varepsilon < \varepsilon_0$, on a les propriétés suivantes.

1. Existence d'une solution de type soliton. *Il existe une* unique *solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ de (2.1), globale en temps, telle que*

$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = 0.$$

2. Interaction soliton-potentiel et stabilité. *Il existe $K > 0$, $c^+ \geq 1$ et une fonction C^1 $\rho(t)$ définie pour tout $t \gg \frac{1}{\varepsilon}$, telles que*

$$w^+(t,\cdot) := u(t,\cdot) - \frac{1}{\sqrt{2}} Q_{c^+}(\cdot - \rho(t))$$

satisfait, pour tout $t \gg \frac{1}{\varepsilon}$,

$$\|w^+(t)\|_{H^1(\mathbb{R})} + |\rho'(t) - c^+ + \lambda| \leq K\varepsilon^{1/2}.$$

La preuve de ce résultat est basée sur la construction d'une solution approchée de (2.1) dans la région d'interaction, satisfaisant certaines symétries. Cette solution approximative comporte essentiellement un soliton modulé plus un petit terme de correction. Un des premiers points importants est le fait que la position et le scaling du soliton suivent un système dynamique en la variable lente εt. D'après la nature des potentiels considérés, et sous la condition de petitesse $\lambda \leq \frac{1}{3}$, nous pouvons montrer que le soliton sort par la droite, en temps de l'ordre de ε^{-1}. Cependant, à un moment donné, nous obtenons formellement que la correction a un terme de **masse infinie**, voir aussi [43] pour un problème similaire. Il apparaît que, pour obtenir une solution localisée, nous avons besoin de casser la symétrie de cette solution, ce qui est une différence essentielle par rapport au soliton de gKdV. Ce manque de symétrie conduit à l'erreur $\varepsilon^{1/2}$ indiquée dans le théorème ci-dessus. À ce prix, nous avons une description complète de la dynamique dans la région d'interaction, un résultat tout à fait nouveau.

Une question fondamentale d'après ces résultats est de savoir si le soliton est pur à l'infini. Cette question est équivalente à décider si

$$\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} = 0.$$

Notre dernier résultat montre que ce comportement n'existe pas.

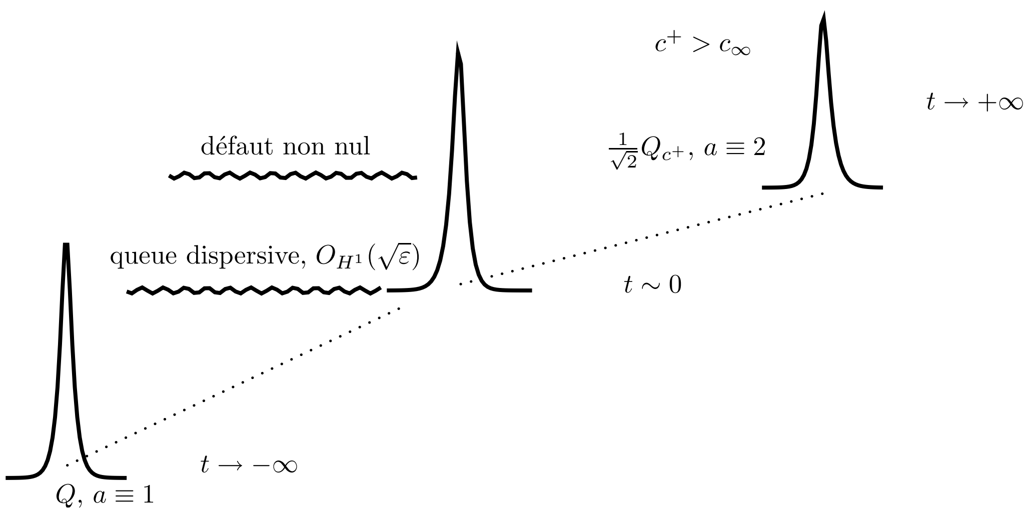
Théorème 2.2 (Non existence d'une solution de type soliton pur)**.**

Sous le contexte du dernier résultat, on suppose $0 < \lambda \leq \frac{1}{3}$*. Il existe* $\varepsilon_0 > 0$ *tel que, pour tout* $0 < \varepsilon < \varepsilon_0$*,*

$$\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} > 0. \tag{2.5}$$

La preuve de ce résultat est basée sur une idée très simple. En effet, si l'on suppose que la solution est un soliton pur en $+\infty$, alors par unicité elle converge à vitesse exponentielle vers une solution de type soliton. Par l'utilisation de cette décroissance en temps, nous pouvons obtenir la décroissance en espace et donc que la solution est effectivement dans L^1, et conserve l'intégrale. Du fait que le scaling varie largement, et en utilisant la précédente loi de conservation, nous obtenons la contradiction souhaitée.

Le deux derniers résultats peuvent être schématisés par la figure suivante :



Quel est le comportement de la solution pour des coefficients $\frac{1}{3} < \lambda < 1$? Nous avons inclus un paragraphe à la fin de la Partie 3 (voir Addendum) qui décrit formellement le comportement de la solution dans ce cas. Pour un compte rendu détaillé de ce problème, voir la dernière section ci-dessous.

2.2 Deuxième partie : le cas Schrödinger

Un problème non trivial est de traiter de questions similaires pour le cas Schrödinger. Tel est l'objectif de la deuxième partie de ce travail : l'étude de la dynamique des solitons pour l'équation NLS dans le cas d'un potentiel à variation lente.

En effet, on considère l'équation de *Schrödinger nonlinéaire généralisée* (NLS)

$$iu_t + u_{xx} + a(\varepsilon x)|u|^{m-1}u = 0, \quad \text{dans } \mathbb{R}_t \times \mathbb{R}_x, \quad m \in [2,5). \tag{2.6}$$

Ici $u = u(t,x)$ est une fonction à valeurs complexes, $\varepsilon > 0$ est un paramètre petit et a satisfait les mêmes hypothèses qu'avant.

La littérature dans ce cas est beaucoup plus développée, commençant à partir de travaux physiques de Kaup et Newell [30] et Grimshaw [21]. D'un point de vue mathématique, le premier résultat dans cette direction a été donné par Bronski et Jerrard [6]. Gustafson et al. [22, 23] et Holmer et al. [26, 27, 28] ont examiné la dynamique d'un soliton pour des potentiels généraux, et pour des périodes $t \sim \frac{1}{\varepsilon}$. Voir aussi [12] pour un résultat similaire dans le cas d'une équation généralisée de Hartree. D'après ces résultats, il semble évident qu'une meilleure compréhension de la dynamique des solitons pour des temps longs dépend fortement du caractère spécifique du potentiel considéré. Comme ci-dessus, l'idée est de considérer des potentiels un peu plus simples, mais encore très généraux afin de comprendre le comportement de la solution.

Maintenant, une solution non nulle de (2.6) peut *gagner du moment*, dans le sens où, au moins de façon formelle, la quantité $P[u](t)$ définie en (1.9) satisfait l'identité suivante :

$$\partial_t P[u](t) = \frac{\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)|u|^{m+1} \geq 0. \tag{2.7}$$

Par conséquent, le moment est toujours une quantité croissante. Ce simple fait a des conséquences importantes dans nos résultats ; en particulier, nous obtiendrons de cette propriété la *stabilité et l'unicité* de notre solution.

D'autre part, la masse $M[u](t)$ définie en (1.7) et la nouvelle énergie

$$E_a[u](t) := \frac{1}{2}\int_{\mathbb{R}} |u_x|^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)|u|^{m+1}(t,x)\,dx$$

sont formellement conservées.

Notre résultat principal est une description complète, pour tout temps, de l'interaction soliton-potentiel pour l'équation NLS (2.6). Comme ci-dessus, afin de simplifier l'exposition, nous allons présenter le cas le plus simple, $m = 3$. Pour en savoir plus, voir le Théorème A 1 dans la Partie 3.

Théorème 2.3 (Dynamique de solutions de type soliton pour des équations NLS généralisées).

On suppose que $a(\cdot)$ satisfait les mêmes hypothèses que dans le cas gKdV. Soit $v_0 > 0$. Il existe une petite constante $\varepsilon_0 > 0$ telle que, pour tout $0 < \varepsilon < \varepsilon_0$, on a les propriétés suivantes.

1. Existence d'un soliton.
   Il existe une unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ de (2.6), globale en temps, telle que

$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - v_0 t)e^{i(\cdot)v_0/2}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0. \tag{2.8}$$

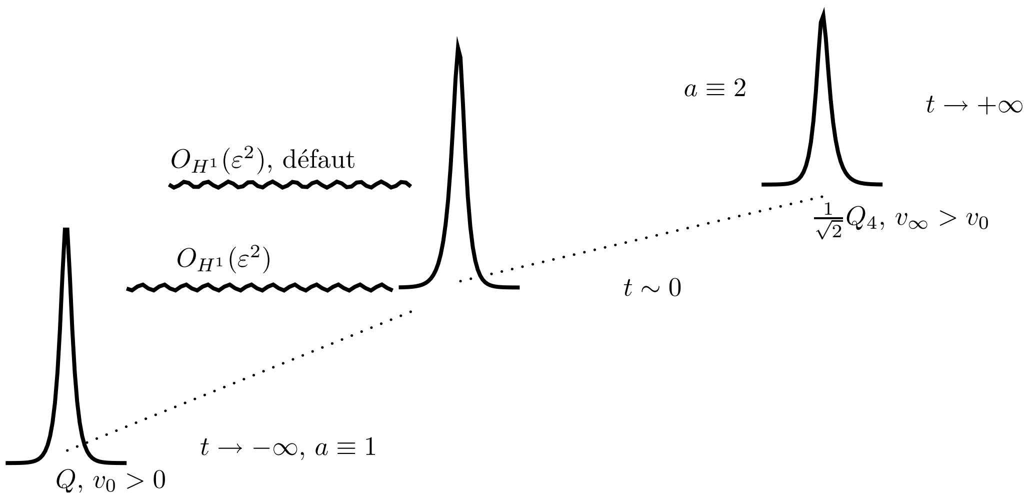
2. Stabilité de l'interaction soliton-potentiel. *Soit $v_\infty := (v_0^2+4)^{\frac{1}{2}}$ $(> v_0)$. Il existe $K > 0$, et des fonctions C^1 $\rho(t), \gamma(t) \in \mathbb{R}$ définies pour tout $t \gg \frac{1}{\varepsilon}$, telles que la fonction*

$$w(t,x) := u(t,x) - \frac{1}{\sqrt{2}}Q_4(x - v_\infty t - \rho(t))e^{\frac{i}{2}xv_\infty}e^{i\gamma(t)}$$

satisfait, pour tout $t \gg \frac{1}{\varepsilon}$,

$$\|w(t)\|_{H^1(\mathbb{R})} + |\rho'(t)| + |\gamma'(t) - 4 + \frac{1}{4}v_\infty^2| \leq K\varepsilon^2. \quad (2.9)$$

Les deux derniers résultats peuvent être illustrés par le schéma suivant :



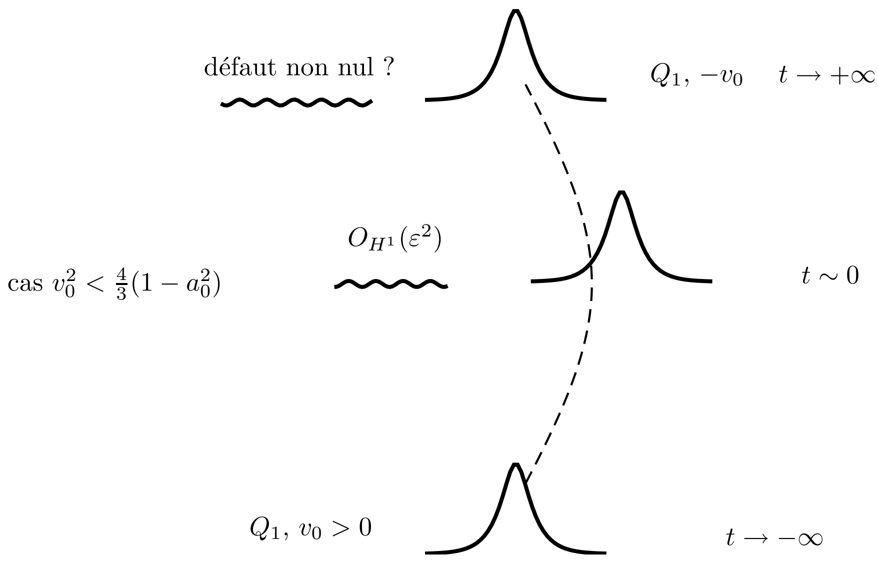
On peut comparer ce résultat avec les résultats pour le cas gKdV, où une borne de l'ordre de $\varepsilon^{1/2}$ a été trouvée. Notre résultat présent est meilleur en raison de l'absence d'une **queue dispersive** derrière le soliton, précisément d'ordre $\varepsilon^{1/2}$ dans $H^1(\mathbb{R})$, et présente dans le cas gKdV. Rappelons enfin que ces éléments dispersifs ne sont pas présents dans le cas d'un soliton pur de NLS ou gKdV.

De plus, ce résultat est aussi vrai dans le cas **deux dimensionnel**, si l'on considère le potentiel ne dépendant que d'une seule variable, et pour une vitesse d'arrivée quelconque (voir Théorème 1.24 de la Partie 3). La restriction à la dimension deux est une conséquence de la perte de régularité pour les puissances dans les dimensions supérieures.

Ces résultats peuvent être aussi généralisés au cas de potentiels décroissants pour des temps pas trop longs, mais encore beaucoup plus grands que ε^{-1}. Le point important ici est que la construction d'une solution de type soliton ne dépend pas du signe de a, mais de la platitude de $a(\cdot)$ à l'infini. La dynamique dans la région d'interaction peut être décrite de la même manière que dans le cas croissant, avec une différence essentielle : pour de petites vitesses initiales, la solution est réfléchie. Pour de grandes vitesses initiales, le soliton sort de la région d'interaction toujours par la droite, mais la stabilité dans ce cas n'est pas connue, et donc le comportement de cette solution pour des temps longs est encore inconnu. Toutefois, si la vitesse est assez petite (mais indépendante de ε), nous pouvons décrire la solution pour tout temps. En effet, supposons que le potentiel $a(\cdot)$ décroît strictement d'un état initial $a \equiv 1$ à un état final $a_0 \in (0,1)$. Alors, si $v_0^2 < \frac{4}{3}(1 - a_0^2)$, il existe une solution de type soliton satisfaisant (2.8) et

$$\sup_{t \gg \frac{1}{\varepsilon}} \|u(t) - Q(\cdot + v_0 t - \rho(t))e^{-i(\cdot)v_0/2}e^{i\gamma(t)}\|_{H^1(\mathbb{R})} \leq K\varepsilon^2,$$

avec $\rho'(t)$ petit (voir **Addendum 1** dans la Partie 3 pour plus de détails). Autrement dit, le comportement dans ce régime est très similaire à la dynamique exprimée dans la figure suivante :



Voir [54] pour plus d'informations. Comme expliqué plus haut, le comportement de cette solution dans le régime $v_0^2 > \frac{4}{3}(1-a_0^2)$ et pour des temps assez grands est une question ouverte. Dans le cas limite $v_0^2 = \frac{4}{3}(1-a_0^2)$ les choses sont apparemment plus compliquées.

2.3 Troisième partie : dynamique de la collision pour deux solitons

La troisième partie de ce travail est consacrée à l'étude de la collision de deux solitons pour les équations gKdV avec une nonlinéarité générale.

Nous rappelons que la théorie de scattering inverse est inutile dans le cas d'une équation non intégrable. Ici, notre but est de confirmer que sous des hypothèses raisonnables sur la nonlinéarité, la collision de deux solitons n'est pas élastique en général, sauf par les cas KdV, mKdV et les équations de Gardner.

On considère ce problème pour (1.13) avec une nonlinéarité $f(u)$, et deux solitons Q_{c_1}, Q_{c_2}, $0 < c_2 < c_1 < c_*(f)$, et on suppose c_2 plus petit que c_1.

Sous ces hypothèses, Martel et Merle [39] ont considéré la collision pour (1.13) dans le cas de nonlinéarité quartique, $f(s) = s^4$. Ils ont montré que la collision est presque élastique, mais inélastique.

La question qui suit dans ce cas est de généraliser ces résultats à (1.13) sous l'hypothèse (1.15). Dans ce cas, Martel et Merle [40] ont montré que la collision est toujours stable, en donnant des bornes supérieures sur la taille du défaut figurant après la collision. Dans [40], la question de savoir si la collision est élastique ou pas dans ce cas général – et donc l'inexistence de purs 2-solitons – a été laissée ouverte, voir [40], Remarque 1.

On a été capable de donner une réponse satisfaisante à cette question à partir de l'amélioration des techniques développées dans [39, 40] et de nouveaux calculs.

Théorème 2.4 (Non existence de solution 2-soliton pure, cas général)**.**

Soit f comme dans (1.15), avec $m = 2$ ou 3, et

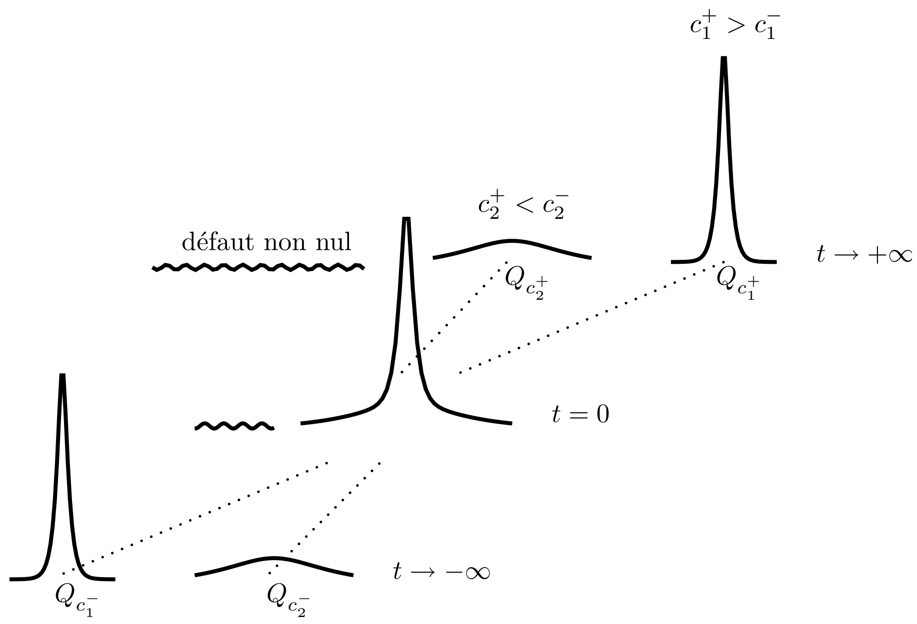
$$f \in C^{p+1}(\mathbb{R}), \quad f^{(p)}(0) \neq 0 \quad \textit{pour un} \quad p \geq 4. \tag{2.10}$$

Pour $0 < c_2 \ll c_1 \ll 1$, l'équation (1.13) n'a pas de solution 2-soliton pure de tailles c_1 et c_2.

La condition $f^{(p)}(0) \neq 0$ pour un $p \geq 4$ n'inclut pas les cas intégrables $f(s) = s^m$, $m = 2$ ou 3 et la nonlinéarité de Gardner $f(s) = s^2 - \mu s^3$.

Notre preuve suit l'approche décrite par Martel, Merle et Mizumachi [39, 40, 44]. L'hypothèse c_2 petite permet de linéariser autour de c_2, et donc de réduire la non existence d'une solution 2-soliton pure au calcul d'un *coefficient ne dépendant que de* c_1. Ce coefficient est une partie d'une solution approchée de (1.13) avec un ordre de précision élevé, et il est évidemment nul dans le cas intégrable. Pour des fonctions f et des coefficients $c_1 > 0$ généraux, calculer ce coefficient est une question ouverte. D'après cette étude, nous calculons les asymptotiques de ce coefficient quand c_1 est petit. C'est le seul endroit où c_1 petit est nécessaire.

La dynamique de cette solution peut être décrite à l'aide du schéma suivant (avec l'autorisation de Y. Martel).



Pour finir, quelques mots sur la littérature du problème. La collision du 2-soliton a été considérée pour le cas NLS par Perelman [57], Holmer, Marzuola et Zworski [26, 27, 28] et Abou Salem, Fröhlich et Sigal [2].

3 Perspectives

Dans cette section finale, nous parlons de quelques problèmes ouverts qui n'ont pas été abordés dans ce travail.

3.1 Borne inférieure dans la taille du défaut à l'infini pour les équations gKdV et NLS

Une des questions ouvertes de ce travail est la preuve d'une borne inférieure sur le défaut à l'infini pour une solution de type soliton. La solution à ce problème est probablement positive, au moins dans le cas $\lambda > 0$, après quelques calculs formels. Nous supposons que pour le cas gKdV, on a

$$\liminf_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} \geq \frac{\varepsilon}{K}, \quad \text{pour } m = 2, 4,$$

et de façon surprenante à l'ordre ε^2 pour le cas cubique (voir [53] pour plus de détails).

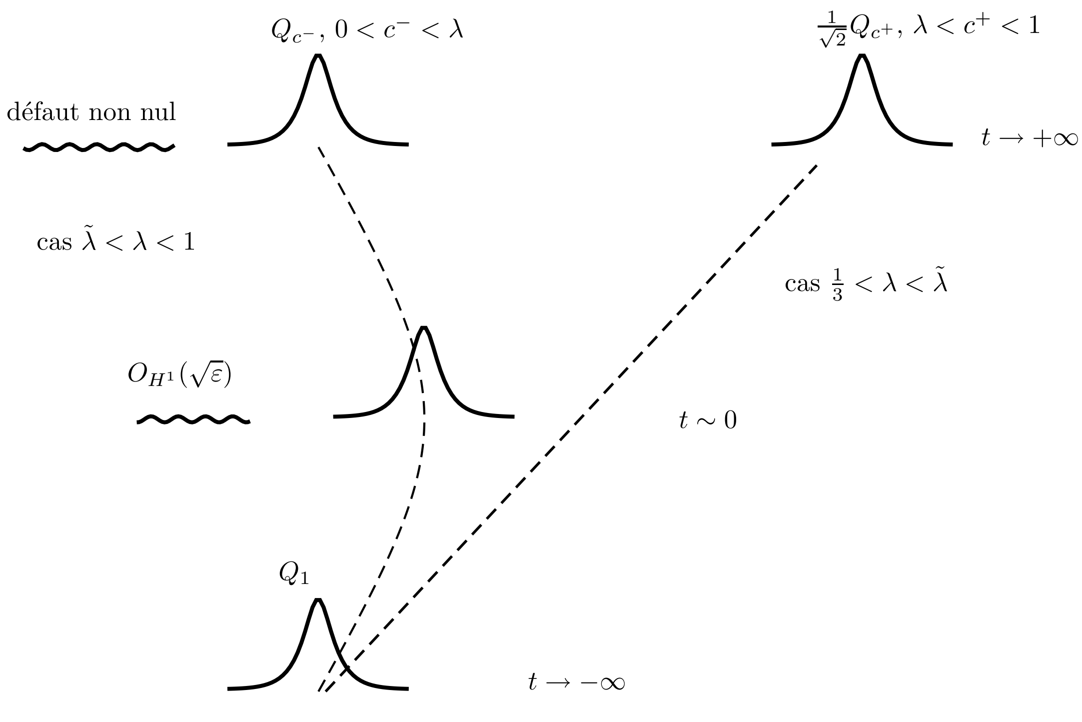
Dans le cas Schrödinger, nous conjecturons l'existence d'une constante $K > 0$ telle que, pour tout $\varepsilon > 0$ assez petit,

$$\liminf_{t\to+\infty} \|w(t)\|_{H^1(\mathbb{R})} \geq \frac{1}{K}\varepsilon^2.$$

Autrement dit, la borne (2.9) est optimale. Ces estimations sont dans un certain sens équivalentes aux résultats de non existence de solutions 2-soliton pour les équations gKdV, comme cela a été montré dans [39, 44, 49, 43].

3.2 Dynamique du soliton pour gKdV, cas $\frac{1}{3} < \lambda < 1$

Un autre problème intéressant est de comprendre le comportement de la solution décrite dans le Théorème 2.1 pour un coefficient $\frac{1}{3} < \lambda < 1$. Ici, la principale différence avec le cas précédent est que formellement le scaling **décroît** avec le temps. Par conséquence, une première tâche non triviale est de montrer l'existence d'un état final du scaling, positif et loin de zéro, de manière uniforme en ε. Toutefois, si l'échelle est petite par rapport à λ, la solution peut être réfléchie vers la gauche par le potentiel. En effet, nous pensons qu'il existe une constante explicite $\tilde{\lambda} \in (\frac{1}{3}, 1)$ telle que, pour tout $\varepsilon > 0$ suffisamment petit et pour tout $\frac{1}{3} < \lambda < \tilde{\lambda}$, le soliton sort toujours de la région d'interaction par le côté droit, et si $\tilde{\lambda} < \lambda < 1$, alors la solution est réfléchie par le potentiel. Cette hypothèse est formellement soutenue par nos calculs dans la partie 2 (cf. Addendum). La figure suivante décrit le comportement possible dans le cas $\frac{1}{3} < \lambda < 1$.



Voir [52] pour plus de détails.

Enfin, quelques résultats annoncés avant dans cette introduction seront bientôt disponibles sur la page web `http ://www.math.uvsq.fr/~munoz`.

1 Preliminaries

In this Ph.D. thesis we deal with long time and qualitative properties of some **nonlinear dispersive equations**, mainly **Korteweg-de Vries** and **semilinear Schrödinger equations**.

By dispersive equations we mean equations of the form

$$u_t = Lu + F(u, Du, \ldots),$$

where $u(t,x)$ is real or complex valued, and $t \in \mathbb{R}, x \in \mathbb{R}^N$. The linear operator L is assumed anti-adjoint, that is

$$\mathcal{F}(Lu)(\xi) = ip(\xi)\hat{u}(\xi), \quad p(\xi) \in \mathbb{R},$$

in terms of Fourier transform. Finally, F is a nonlinear term, possibly containing high order derivatives. Mathematically speaking, the equation is dispersive if in addition $D^2_\xi p(\xi)$ is nonsingular.

Heuristically, an equation is more or less dispersive depending on the size and boundedness of p. For example, the linear Airy equation $u_t + u_{xxx} = 0$, which satisfies $p(\xi) = \xi^3$, is more dispersive than the linear Schrödinger equation $iu_t + u_{xx} = 0$, which satisfies $p(\xi) = -\xi^2$.

Examples of (nonlinear) dispersive equations are the Korteweg-de Vries equation, the nonlinear Schrödinger equation, the Benjamin-Ono equation, the BBM equation, KPI, KPII, etc.

Once suitable local and/or global well-posedness results are valid for a given evolution equation, a list of very interesting questions can be considered. One of these questions is the long-time dynamics, or qualitative description of a class of solutions, even some very particular solutions.

Precisely, this work is focused on a special type of solutions for a wide class of hamiltonian and nonlinear dispersive equations. These solutions are called *solitons*, or more generally *solitary waves*. These concepts slightly vary from one equation to another, but the spirit is the same, and we think that a simple example can illustrate it.

So, let us consider one of the simplest cases of dispersive equations, the generalized Korteweg-de Vries equation (gKdV). It reads[2]

$$u_t + (u_{xx} + u^m)_x = 0, \quad u = u(t,x), \quad (t,x) \in \mathbb{R}^2, \quad m = 2, 3 \text{ or } 4. \tag{1.1}$$

The Korteweg-de Vries equation (KdV), that is, the case $m = 2$ above, arises in Physics as a model of propagation of dispersive long waves, as was pointed out by J. S. Russel in 1834 [48]. The exact formulation of the KdV equation comes from Korteweg and de Vries (1895) [33]. This equation was studied in a numerical work by N. Zabusky and M. Kruskal in 1965 [34].

In addition, the above equation enjoys some surprising properties. First of all, note that if $u = u(t,x)$ is solution of the above equation, then also are $u(t - t_0, x - x_0)$ and $u_c(t,x) := c^{1/(m-1)}u(ct, \sqrt{c}x)$, for any $c > 0$ and $t_0, x_0 \in \mathbb{R}$. These are examples of translation and scaling invariance, respectively. **Solitons** are precisely **localized** solutions of (1.1) of the form

$$u(t,x) := Q_c(x - ct),$$

with $Q_c(s) := c^{1/(m-1)}Q(\sqrt{c}s)$. If this class of solutions exists, then it has an invariant profile and moves to the right as time increases. Moreover, they are **globally defined** solutions.

[2]The fact that we only consider the cases satisfying $m < 5$ will be clear below.

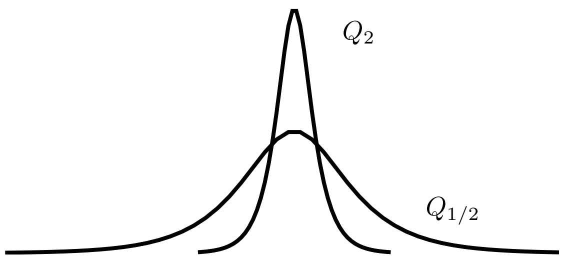
From the above considerations, it is clear that Q must satisfy the second order nonlinear differential equation

$$Q'' - Q + Q^m = 0, \tag{1.2}$$

which has a unique positive $H^1(\mathbb{R})$ solution modulo translations. This solution is given by

$$Q(x) = \Big[\frac{m+1}{2\cosh^2(\frac{(m-1)}{2}x)}\Big]^{\frac{1}{m-1}}.$$

Note that the scaled function $Q_c(s)$ gives us the following heuristic: small solitons are wider than large solitons, but at the same time slower. This is the equivalence scaling-velocity present in the gKdV equation. The following figure describes two different solitons:



In addition, since Q is a positive function, solution of a nonlinear elliptic equation, it can be related to a minimization problem on $H^1(\mathbb{R})$, for which Q represents the *ground state*, the minimum level of energy. In order to obtain the existence of such a solution in a general framework, one needs to apply **concentration-compactness** methods [6], due to the lack of compactness on the entire space.

Another important example is given by the Nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + |u|^{m-1}u = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad m \in [2, 5). \tag{1.3}$$

Here $u = u(t, x)$ is a complex-valued function. The cubic nonlinear Schrödinger equation (namely the case $m = 3$) arises in Physics as a model of wave propagation in fiber optics in a nonlinear medium, and also describes the evolution of the envelope of modulated wave groups in water waves. In two dimensions, the cubic NLS also possesses an important physical meaning.

Given $c_0 > 0$, $v_0, x_0, \phi_0 \in \mathbb{R}$, this equation has solutions of the form

$$Q_c(x - v_0 t - x_0)e^{\frac{i}{2}v_0 x}e^{i(c-\frac{1}{4}v_0^2)t}e^{i\phi_0},$$

with Q_c the aforementioned soliton solution. This localized solution is called *solitary wave*, and its module has an invariant profile. Behind this solution, there are two more additional symmetries satisfied by solutions of (1.3): **phase and Galilean invariances**.

These two types of solutions, solitons and solitary waves, share an important property: they are **pure**, in the sense that no additional terms are required to satisfy the equation. This phenomenon is rather strange in a completely general solution, considering the dispersive character of the considered equation, and it can be explained via a delicate balance between the dispersive member given by the derivative term and the nonlinearity included in the equation.

At the beginning of this paragraph we have mentioned the concepts of local and global well-posedness as a key element to investigate qualitative or long-time properties for solutions of dispersive equations. Let us see some reasons of this.

Once one has soliton solutions, one may consider the analytical or numerical study of **small perturbations**, or even harder, the behavior of several soliton solutions. If one considers two or more solitons, it is natural to think about possible **collisions** among themselves, a difficult question to answer due to the nonlinear character of the equation. In both cases one begins near solitons solutions, and in order to describe both problems, local and/or global well-posedness (LWP-GWP) ensure the existence of such a solution for a period of time.

So, in the next lines we would like to give an informal account of the –by now– well-known LWP and GWP theory for gKdV and NLS.

Let us suppose $m = 2, 3$ or 4 in (1.1). Local well-posedness for gKdV is by now a standard issue. From the work of Kenig, Ponce and Vega [31], (1.1) is locally well-posed in $H^1(\mathbb{R})$ and then globally due to the mass and energy conservation laws

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)\,dx = \frac{1}{2}\int_{\mathbb{R}} u_0^2(x)\,dx = M[u](0), \quad \text{(Mass)}, \tag{1.4}$$

$$\begin{aligned} E[u](t) \quad := \quad & \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} u^{m+1}(t,x)\,dx \\ & = \frac{1}{2}\int_{\mathbb{R}} (u_0)_x^2(x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} u_0^{m+1}(x)\,dx = E[u](0), \ \text{(Energy)}, \end{aligned} \tag{1.5}$$

and the Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}} u^{p+1} \leq K(p)\big(\int_{\mathbb{R}} u^2\big)^{\frac{p+3}{4}}\big(\int_{\mathbb{R}} u_x^2\big)^{\frac{p-1}{4}}. \tag{1.6}$$

Let us emphasize that for $m = 5$, the Cauchy problem for the corresponding gKdV equation has finite-time blow-up solutions, see [47, 42, 38] and references therein. It is believed that for $m > 5$ the situation is the same. For both gKdV and NLS equations, the case $m = 5$ is denoted as the L^2-critical case, in the sense that every soliton has the same size:

$$\|Q_c\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})}, \quad \text{for all } c > 0.$$

However, in the *supercritical case* ($m > 5$), small solitons (with respect to the L^∞-norm) are larger than big solitons (with respect to the L^2-norm).

The proof of Kenig, Ponce and Vega uses a fixed point argument in a subspace of $C([0,T], H^1(\mathbb{R}))$ depending on the nonlinearity, for a small but fixed time $T > 0$. The proof of a smoothing effect for the linear Airy operator $e^{-t\partial_x^3}$, in the spirit of Kato, is one of the key steps to close the iterative procedure.

In the NLS case, $2 \leq m < 5$, the Cauchy problem is *locally well-posed* for $u_0 \in H^1(\mathbb{R})$ (see Ginibre and Velo [17]) and then globally well-posed due to the mass and energy conservation laws

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} |u|^2(t,x)\,dx = M[u](0), \quad \text{(Mass)}, \tag{1.7}$$

and

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} |u_x|^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} |u|^{m+1}(t,x)\,dx = E[u](0), \ \text{(Energy)}. \tag{1.8}$$

The proof of Ginibre and Velo is based on a tough use of **Strichartz estimates** (in suitable $L^p_t L^q_x$-spaces) and a fixed point argument in a small ball of $C((-T,T), H^1(\mathbb{R}))$, for $T > 0$ small, as in the gKdV case. We refer to [8] for a detailed proof.

Finally, let us recall that NLS possesses a third conserved quantity, called Momentum

$$P[u](t) := \frac{1}{2}\,\mathrm{Im}\int_{\mathbb{R}} \bar{u}u_x(t,x)\,dx = P[u](0). \tag{1.9}$$

This quantity is well defined for $H^1(\mathbb{R})$ solutions. Similarly to the gKdV case, for $m \geq 5$, the Cauchy problem for the corresponding NLS equation has finite-time blow-up solutions, see [8] and references there in.

The study of perturbations of solitons or solitary waves lead to the introduction of the concepts of *orbital and asymptotic stability*. By **orbital stability** we mean the following property. Consider the gKdV equation (1.1), and a soliton solution $Q_c(x - x_0)$, for c, x_0 given. Let us suppose that there exists a solution $u = u(t)$ of (1.1), global in time and such that

$$\|u(t_0) - Q_c(\cdot - x_0)\|_{H^1(\mathbb{R})} \leq \alpha,$$

for some constant $\alpha > 0$ small. Then we say that Q_c is orbitally stable if there exist $K, \alpha_0 > 0$ and a smooth $\rho = \rho(t)$ defined for all $t \geq 0$ such that for all $0 < \alpha < \alpha_0$,

$$\|u(t) - Q_c(\cdot - \rho(t))\|_{H^1(\mathbb{R})} + |\rho'(t) - c| \leq K\alpha, \quad |\rho(0) - x_0| \leq \alpha. \tag{1.10}$$

In other words, a small perturbation of a soliton solution stays close enough to a soliton with a corrected translation parameter. Orbital stability of solitary waves of (1.3) holds under the subcritical assumption $m < 5$.

Let us give some ideas of one of the known proofs of (1.10). We follow the Weinstein method, stated as follows: if there exists a constant $\mu > 0$ such that for all $v \in H^1(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} Q_c' v = 0,$$

one has

$$\mathcal{F}[v](t) := \int_{\mathbb{R}} (v_x^2 + cv^2 - mQ_c^{m-1}v^2) \geq \mu \int_{\mathbb{R}} (v_x^2 + v^2) - K\Big|\int_{\mathbb{R}} Q_c v\Big|^2, \tag{1.11}$$

then (1.10) holds. Indeed, by the Implicit Function Theorem there exist $\bar{c} > 0$, $\rho(t)$ such that $z(t) := u(t) - Q_{\bar{c}}(\cdot - \rho(t))$ satisfies for all time

$$\int_{\mathbb{R}} Q_{\bar{c}}' z = 0, \quad \int_{\mathbb{R}} u^2(t_0) = \int_{\mathbb{R}} Q_{\bar{c}}^2, \quad |\rho'(t) - \bar{c}| + |\bar{c} - c| \leq K\alpha.$$

Using the definition of mass and energy, and the equation of $Q_{\bar{c}}$, we have

$$E[u](t) + \bar{c}M[u](t) = E[Q_{\bar{c}}](t) + \bar{c}M[Q_{\bar{c}}](t) + \mathcal{F}[z](t) + O(\|z(t)\|^3_{H^1(\mathbb{R})}),$$

From the energy and mass conservation, we finally have

$$\mathcal{F}[z](t) \lesssim \|z(t_0)\|^2_{H^1(\mathbb{R})} + O(\|z(t)\|^3_{H^1(\mathbb{R})}).$$

Finally, using the mass conservation, one has $\int_{\mathbb{R}} Q_{\bar{c}} z = O(\sup_t \|z(t)\|^2_{L^2(\mathbb{R})})$, so from (1.11), we get the final result.

Orbital stability of small perturbations of solitons for gKdV have been considered in particular in [3, 5, 9, 45]. Similarly, orbital stability of ground states for NLS equations has been widely studied during last decades; we mention the works of Cazenave and Lions [9], Weinstein [62, 63], Grillakis, Shatah and Strauss [18, 19], Cuccagna [10], and Martel, Merle and Tsai [45]. See references therein for a more detailed bibliography.

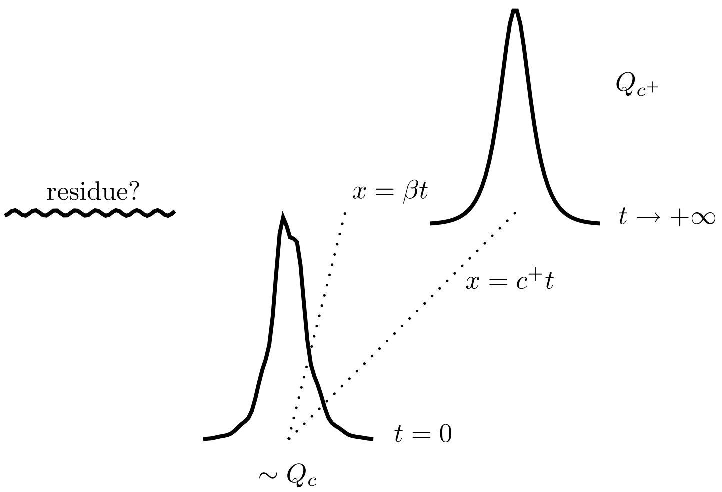
On the other hand, asymptotic stability concerns with the small residue given by the stability result above mentioned. It is legitimate to wonder whether $u(t)$ should actually converge to a soliton in some sense.

First, some remarks. Note that two solitons $Q_c(x-\rho(t))$ and $Q_{\tilde{c}}(x-\rho(t))$, with $c \sim \tilde{c}$ but different are always at a positive distance in $H^1(\mathbb{R})$. In addition, a standard classification result tells you that if $\|u(t) - Q_c(\cdot - x(t))\|_{H^1(\mathbb{R})} \to 0$ as $t \to +\infty$, for some $x(t)$, then $u(t)$ is a *pure soliton solution*. These two arguments suggest the non existence of a completely and general H^1 convergence to a soliton solution of a small perturbation of a soliton.

In order to solve this problem, one needs to reformulate the asymptotic stability property, either by introducing some suitable weighted spaces [56], or by considering only local norms [41]. In this last formulation, one has the existence of $\beta > 0$ depending of α small enough and $c^+ > 0$ with $|c^+ - c| \lesssim \alpha$ such that

$$\|u(t) - Q_{c^+}(\cdot - \rho(t))\|_{H^1(x>\beta t)} \to 0,$$

as $t \to +\infty$. Moreover, $\lim_{t\to+\infty} \rho'(t) = c^+$. In other words, there is strong H^1-convergence near the soliton. This result can be schematized by the following design



Behind the proof of the asymptotic stability is the **Kato identity**, that we will frequently use in this work. In its simplest form, it states that for a $H^1(\mathbb{R})$ solution $u(t)$ of (1.1) satisfying the orbital stability property (1.10), one has

$$\partial_t \int_{\mathbb{R}} \varphi(x-\beta t) u^2(t,x) \lesssim 0, \tag{1.12}$$

for $0 < \beta < c$ and for any positive, increasing and bounded function φ such that $\varphi' \sim Q$. Formally, this last inequality implies that the mass near the soliton is moving to the left as the soliton advances, which finally gives the desired result. We recall that the historical application of this inequality is the so-called **local smoothing effect** associated to gKdV equations.

Finally, let us remark that **asymptotic stability** of solitary waves and related scattering results have been studied in [59, 60, 7, 57, 11, 58].

Let us come back to the gKdV equation considered in (1.1), but now in complete generality. We suppose the nonlinearity being a general function f depending on $u = u(t,x)$, as follows

$$u_t + (u_{xx} + f(u))_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.13}$$

Let us recall that $f(s) = s^2$ corresponds to the well-known KdV equation. Other physically important cases are the cubic one $f(s) = s^3$, and the *quadratic-cubic* nonlinearity, namely $f(s) = s^2 - \mu s^3$, $\mu \in \mathbb{R}$. In the former case, the equation (1.13) is often called the (focusing) *modified* KdV equation (mKdV), and in the latter, it is known as the *Gardner* equation. These three equations are **completely integrable** models. Cubic NLS is also integrable [64]. The complete integrability implies the existence of an infinite number of conserved quantities for the associated equation.

In the general form (1.13), gKdV possesses only two $H^1(\mathbb{R})$ conserved quantities: mass and energy. The energy is now given by

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \int_{\mathbb{R}} F(u(t,x))\,dx,$$

where we have denoted

$$F(s) := \int_0^s f(\sigma)\,d\sigma. \tag{1.14}$$

In this thesis, we will consider nonlinearities $f \in C^3(\mathbb{R})$ of the form

$$f \in C^{m+2}(\mathbb{R}), \quad f(u) := u^m + f_1(u), \quad m = 2,3,4, \quad \text{with} \quad \lim_{s\to 0}\frac{|f_1(s)|}{|s|^m} = 0. \tag{1.15}$$

The positive sign leading in front of f allows the existence [4] of solitons for (1.1) of the form

$$u(t,x) := Q_c(x - x_0 - ct),$$

with $c > 0$ small enough and $x_0 \in \mathbb{R}$, and where the function Q_c satisfies the elliptic equation

$$Q_c'' + f(Q_c) = cQ_c, \quad Q_c \in H^1(\mathbb{R}). \tag{1.16}$$

For all $c > 0$, if a solution $Q_c > 0$ of (1.16) exists, then it can be chosen *even* on $\mathbb{R}$ and *exponentially decreasing* on $\mathbb{R}^+$ (and similarly if $Q_c < 0$).

Finally we consider only *nonlinear stable* solitons in the sense of Weinstein [63], i.e. such that

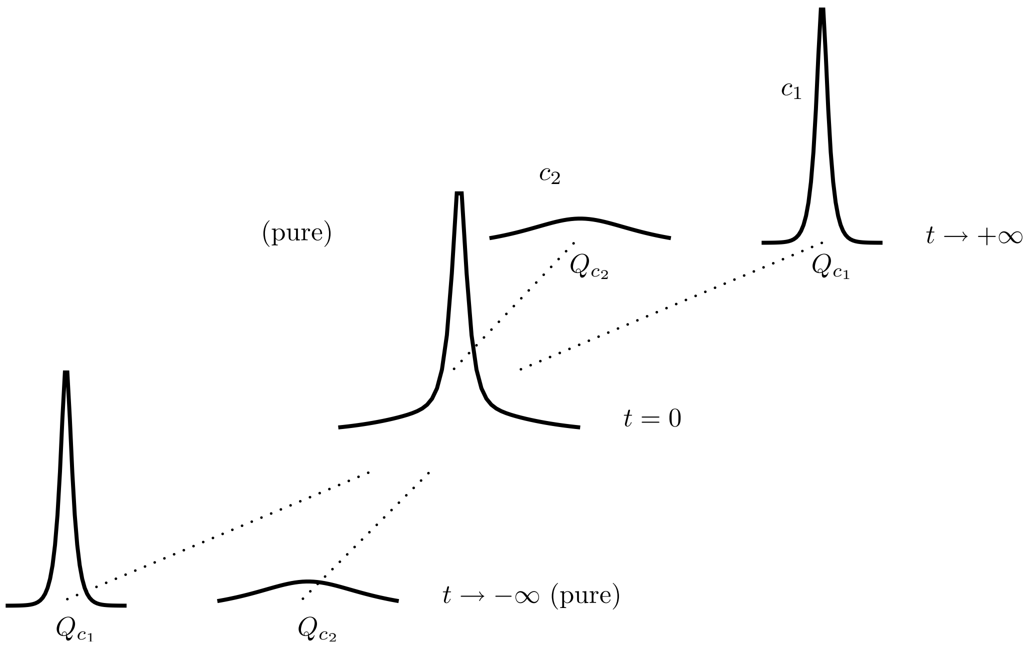
$$\frac{d}{dc'}\int Q_{c'}^2(x)dx\Big|_{c'=c} > 0. \tag{1.17}$$

Note that since $m = 2,3,4$ in (1.10), this condition is automatically satisfied for $c > 0$ small enough (in the pure power case $f(s) = s^m$, this condition is satisfied for any $c > 0$ provided $m < 5$, see [63]).

One of the most interesting problems from the physical point of view is the solitons collision. The dynamical problem of N-soliton collision is a classical problem in nonlinear wave propagation (see [39] for a review and references therein), although mathematically speaking is far from being understood. By now we reduce our research to the problem of 2-soliton collisions.

By 2-soliton collision we mean the following problem: given two solitons, solutions of (1.13), well separated at some early time and having different velocities, we expect that they have to collide at some finite time. The resulting solution after the collision is precisely the object of study. In particular, one considers if any change in size, position, or shape, even destruction of the solitons, after some large time, may be present.

The first works under this direction were the numerical works of Fermi, Pasta and Ulam [14] and Zabusky and Kruskal [34], showing the *elastic* character of the collision between two solitons. By *elastic* we mean that the collision keeps the solitons unchanged and does not produce any residual term of positive mass for large times. The unique consequence of the collision is a **shift translation** on each soliton, depending on their sizes. Next, the work of Lax [35] and the *inverse scattering method* (we refer e.g. to [1] and [48] for a review) provided explicit formulas for N-soliton solutions (Hirota [24]), showing no residual terms before and after the collisions. In other words, the collision is *elastic*, and the N-soliton is pure, as shows the following schema for the case $N = 2$:



These properties are also valid for the *cubic* mKdV, (see [1], p. 390) and for the *Gardner* equation (see [16, 61] and references therein). In particular, complete integrability and elastic collisions are still present. Let us recall that for the Gardner equation

$$u_t + (u_{xx} + u^2 - \mu u^3)_x = 0, \tag{1.18}$$

given $\mu \in \mathbb{R}$, soliton solutions exist for all $c > 0$ in the case $\mu < 0$, and provided $c < \frac{2}{9\mu}$ if $\mu > 0$. These solutions are explicit and given by $u(t,x) = Q_{\mu,c}(x - ct)$, where $Q_{\mu,c}$ is the Schwartz function [61]

$$Q_{\mu,c}(x) := \frac{3c}{1 + \rho\cosh(\sqrt{c}x)}; \quad \rho := (1 - \frac{9}{2}\mu c)^{1/2}. \tag{1.19}$$

In particular, no soliton-solution exists provided $\mu > 0$, and $c > 0$ large enough, where the character of the equation becomes *defocusing*.

We point out that these techniques are known to be too rigid to be applied to more general models, and have no equivalent for the case of the gKdV equation (1.13) with a general non-linearity. The purpose of this thesis is to confirm this belief under reasonable hypotheses on the nonlinearity: the collision of two solitons is not elastic in general, except for KdV, mKdV and the Gardner equations.

Let us emphasize that even the *existence* of a 2-soliton solution for non integrable equations was an open issue. In [37, 36], the authors constructed a 2-soliton solution, pure at infinity, for suitable given parameters, in the case of subcritical and critical gKdV and NLS equations. For the gKdV case, this solution turns out to be *unique*, due to a monotonicity property similar to (1.12). The proof of these results will be adapted throughout this work to obtain existence and uniqueness of suitable soliton solutions.

2 Main results

We have reviewed in some detail the behavior of soliton solutions for gKdV or NLS equations, and the 2-soliton collision in the integrable cases. In this thesis, we have considered **non trivial generalizations of these equations** and we have given a precise description of some interesting phenomena via explicit computations in special asymptotic regimes. In an operative level, the results inside this work can be split in three parts, that we describe now.

The first part deals with the dynamics of a soliton-like solution for a variable coefficients and slowly varying gKdV equation. This soliton solution can be seen as an example of **soliton-potential interaction**. First we study existence and uniqueness of such a solution, and next we completely describe the dynamics inside the interaction region. It turns out that the dynamics induces some very interesting dispersive effects, which are not present in the constant coefficients case. Finally, we describe the asymptotic of this solution for large times, and finally we prove that this solution possesses an additional small defect, nonzero at infinity for all the considered cases.

The second part of this work is concerned with similar questions for a generalized NLS equation. As above, we describe in detail the dynamics of a soliton-like solution for a large class of potentials. We prove existence and uniqueness, and global behavior for all time, improving all the preceding results in the literature.

Finally, in the third part, we treat the case of a two-soliton collision for generalized KdV equations, with general nonlinearity. More precisely, we consider a small soliton colliding against a very small one. We prove that, for all the non integrable cases, the collision is not elastic anymore, and no pure 2-soliton solution exists in this regime. Our proof is based in the recent Martel-Merle results concerning the quartic gKdV case, but now we generalize their results to a complete general nonlinearity.

In these cases a key property is lost: solutions are not pure anymore, and nonzero dispersive terms are always present throughout the dynamics.

The three parts before mentioned represent precisely the following three articles:

1. Muñoz C., *On the soliton dynamics under slowly varying medium for generalized Korteweg-de Vries equations*, arXiv:0912.4725. To appear, Analysis and PDE.

2. Muñoz C., *On the soliton dynamics under slowly varying medium for Nonlinear Schrödinger equations*, arXiv:1002.1295.

3. Muñoz C., *On the inelastic 2-soliton collision for gKdV equations with general nonlinearity*, arXiv:0903.1240. To appear, IMRN.

Further developments, not included in this thesis, will be available in [52, 53, 54].

2.1 First part: one-soliton dynamics in the gKdV case

The first part of our work is devoted to the study of the soliton dynamics for small perturbations of subcritical gKdV equations. Indeed, we study the soliton dynamics for the following *generalized Korteweg-de Vries equation* (gKdV) on the real line

$$\begin{cases} u_t + (u_{xx} - \lambda u + a(\varepsilon x)u^m)_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ m = 2, 3 \text{ and } 4; \quad 0 < \varepsilon \leq \varepsilon_0, \quad 0 \leq \lambda < 1, \end{cases} \tag{2.1}$$

with a a smooth function (= the potential) satisfying suitable hypotheses, the most important fact being that

$$a'(r) > 0, \quad \lim_{r \to -\infty} a(r) = 1 \quad \text{and} \quad \lim_{r \to +\infty} a(r) = 2.$$

We also assume ε_0 small enough. we have chosen $1, 2$ as initial and final states, but our results do not depend on these values and can be replaced by any pair of numbers $0 < a(-\infty) < a(+\infty) < +\infty$.

The problem we consider possesses a long and large physical literature, starting from the works of Kaup and Newell [30] and Karpman and Maslov [29]. The physical motivation was the study of time perturbation of integrable models.

The above authors performed a perturbative analysis of the inverse scattering theory to describe the dynamics of a soliton (for the integrable equation) in this variable regime. Interestingly enough, the existence of a *dispersive shelf-like tail behind the soliton* was formally described. This phenomenon is indeed related to the *lack of energy conservation*.

Subsequently, this problem has been addressed in several other works and for different integrable models, see for example [32, 15, 20, 21]. The reader may consult e.g. the monograph by Newell [55], pp. 87–97, for a more detailed account of the problem.

Note that a nonzero solution of (2.1) *might lose or gain some mass*, depending on the sign of u, in the sense that, at least formally, the mass satisfies the identity

$$\partial_t M[u](t) = -\frac{\varepsilon}{m+1} \int_{\mathbb{R}} a'(\varepsilon x) u^{m+1}. \tag{2.2}$$

On the other hand, the novel energy ($\lambda \geq 0$)

$$E_a[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(t,x)\, dx + \frac{\lambda}{2} \int_{\mathbb{R}} u^2(t,x)\, dx - \frac{1}{m+1} \int_{\mathbb{R}} a_\varepsilon(x) u^{m+1}(t,x)\, dx \tag{2.3}$$

remains formally constant for all time.

The problem of describing analytically the soliton dynamics of different integrable models under a slowly varying medium has received some increasing attention during the last years. Concerning the gKdV equation, our belief is that the first result in this direction was given by Dejak and Sigal, and recently by Holmer in [13, 25]. They considered the long time dynamics of solitary waves (solitons) over a slowly varying perturbation of the gKdV equation

$$u_t + (u_{xx} - b(\varepsilon t, \varepsilon x)u + u^2)_x = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x, \tag{2.4}$$

and where b is a bounded and smooth function. With these hypotheses, the authors show that if the initial condition u_0 is close to a soliton solution, then for any time $t \lesssim \varepsilon^{-1}$, the solution can be decomposed as

$$u(t,x) = Q_{c(t)}(x - \rho(t)) + w(t,x),$$

where $\|w(t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}$ and $\rho(t), c(t)$ satisfy a related dynamical system.

In order to obtain improved results in this direction, and to overpass the barrier ε^{-1} in time, it is necessary to pick **less general potentials**. This assumption allows to understand, for a still large class of potentials, the complete dynamics of the corresponding soliton-like solution.

In order to present the results in the clearest possible form, we have chosen to state the result concerning the cubic case. For a detailed account of results, including the quadratic and quartic cases, see Part 1, Theorem 1.1.

Theorem 2.1 (Dynamics of interaction of solitons for mKdV equations under variable medium)**.**

Let $m = 3$, *and let* $0 \leq \lambda \leq \lambda_0 := \frac{1}{3}$ *be a fixed number. There exists a small constant* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the following holds.*

1. Existence of a soliton-like solution. *There exists a* unique *solution* $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *of (2.1), global in time, such that*

$$\lim_{t \to -\infty} \|u(t) - Q(\cdot - (1 - \lambda)t)\|_{H^1(\mathbb{R})} = 0.$$

2. Interaction soliton-potential and stability. *There exist* $K > 0$, $c^+ \geq 1$ *and a* C^1*-function* $\rho(t)$ *defined for all* $t \gg \varepsilon^{-1}$ *such that*

$$w^+(t, \cdot) := u(t, \cdot) - \frac{1}{\sqrt{2}} Q_{c^+}(\cdot - \rho(t))$$

satisfies, for any $t \gg \varepsilon^{-1}$,

$$\|w^+(t)\|_{H^1(\mathbb{R})} + |\rho'(t) - c^+ + \lambda| \leq K\varepsilon^{1/2}.$$

The proof of this result is based on the construction of an approximate solution of (2.1) in the interaction region, satisfying certain symmetries. This approximate solution is basically composed of a modulated soliton plus a small correction term. One of the first important points is the fact that the soliton's position and scaling follow a dynamical system in the slow variable εt. From the nature of the potential considered, and under the smallness condition $\lambda \leq \frac{1}{3}$, we can show that the soliton exits by the right hand side, at time of order ε^{-1}. However, at some point we formally obtain that the correction term possesses an **infinite mass**, see also [43] for a similar problem. It turns out that, to obtain a localized solution, we need to break the symmetry of this solution, a key difference with respect to the soliton solution of gKdV. This lack of symmetry leads to the error $\varepsilon^{1/2}$ in the theorem above stated. At this price, we completely describe the interaction region, a completely new result.

A fundamental question arises from the above results, namely is the final solution an exactly pure soliton for the gKdV equation (2.1) with $a_\varepsilon \equiv 2$? This question is equivalent to decide whether

$$\limsup_{t \to +\infty} \|w^+(t)\|_{H^1(\mathbb{R})} = 0.$$

Our last result shows that indeed this behavior cannot happen.

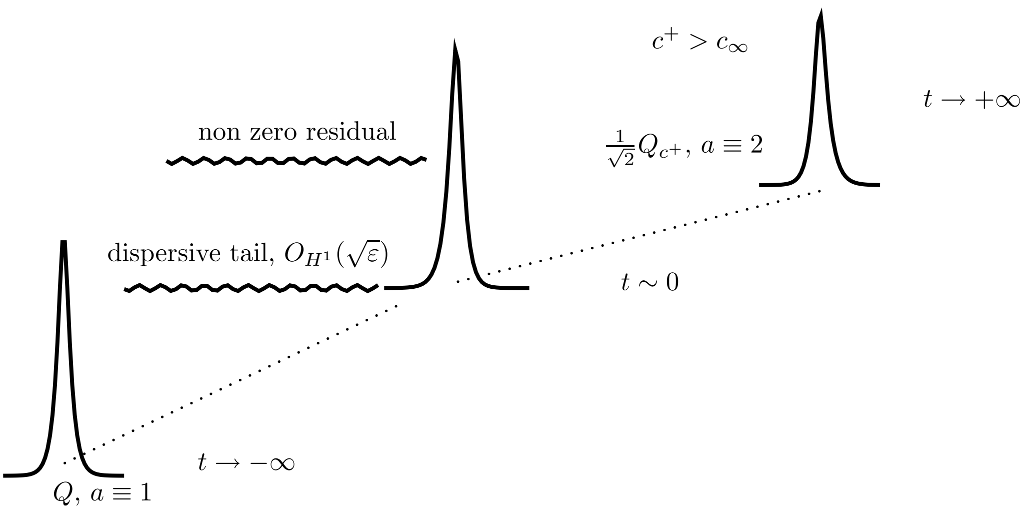
Theorem 2.2 (Non-existence of pure soliton-like solution for generalized gKdV equations)**.**

Under the context of the preceding result, suppose $0 < \lambda \leq \frac{1}{3}$. *There exists* $\varepsilon_0 > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$,

$$\limsup_{t \to +\infty} \|w^+(t)\|_{H^1(\mathbb{R})} > 0. \tag{2.5}$$

The proof of this result follows a very simple idea. Indeed, if we suppose that the soliton solution is pure at $+\infty$, then by uniqueness it converges at exponential rate to a soliton solution. Using this decay in time we can obtain space decay and therefore the solution is actually in L^1, and conserves the L^1-integral. From the fact that the scaling varies in a nontrivial way, and using the integral conservation law, we obtain the desired contradiction.

The last two results can be schematized in the following figure:



What is the behavior of the solution for a coefficient $\frac{1}{3} < \lambda < 1$? We have included a paragraph at the end of Part 2 (see Addendum) where we formally describe the behavior of the soliton solution for this case. For a detailed account of this problem, see the last section below.

2.2 Second part: the Schrödinger case

A basic but non trivial question is to deal with similar questions for the Schrödinger case. This is the objective of the second part of this work: the study of the soliton dynamics for the NLS equation in the case of a slowly varying medium.

Indeed, we considered the following subcritical *generalized nonlinear Schrödinger equation* (NLS)

$$iu_t + u_{xx} + a(\varepsilon x)|u|^{m-1}u = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad m \in [2, 5). \tag{2.6}$$

Here $u = u(t, x)$ is a complex-valued function, $\varepsilon > 0$ a small parameter and a satisfies the same hypotheses as above.

The literature in this case is by far larger, starting from the physical works of Kaup and Newell [30] and Grimshaw [21]. Mathematically speaking, the first result in this direction was given by Bronski and Jerrard [6]. Gustafson et al. [22, 23] and Holmer et al. [26, 27, 28] have considered the dynamics of a soliton under general potentials, for short times, namely $t \sim \frac{1}{\varepsilon}$. See also [12] for a similar result in the case of a generalized Hartree equation. From these results, it seems clear that a deeper understanding of the soliton dynamics for large times strongly depends on the specific character of the considered potential. As above, the idea is to consider simple but still general potentials in order to understand the complete behavior of the soliton solution.

Now, a nonzero solution of (2.6) *might gain momentum*, in the sense that, at least formally, the quantity $P[u](t)$ defined in (1.9) now satisfies the identity

$$\partial_t P[u](t) = \frac{\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)|u|^{m+1} \geq 0. \tag{2.7}$$

Therefore the momentum is always a non decreasing quantity. This simple fact will have important consequences in our results, in particular we will obtain from this property the *stability* and *uniqueness* of our solution. On the other hand, the mass $M[u](t)$ defined in (1.7) and the novel energy

$$E_a[u](t) := \frac{1}{2}\int_{\mathbb{R}} |u_x|^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)|u|^{m+1}(t,x)\,dx$$

remain formally constant for all time.

Our main result is a complete description, for all time, of the interaction soliton-potential for the aNLS equation 2.6. As above, in order to simplify the exposition, we present the simplest case, $m = 3$. For the complete result, see Theorem 1 1 in Part 3.

Theorem 2.3 (Dynamics of a generalized soliton-solution for the cubic NLS equation)**.**

Let $m = 3$ in (2.6). Assume that $a(\cdot)$ satisfies the same hypotheses as above. Let $v_0 > 0$. There exists a small constant $\varepsilon_0(v_0) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.

1. Existence of a soliton-like solution.

   There exists a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (2.6), global in time, such that

$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - v_0 t)e^{i(\cdot)v_0/2}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0. \tag{2.8}$$

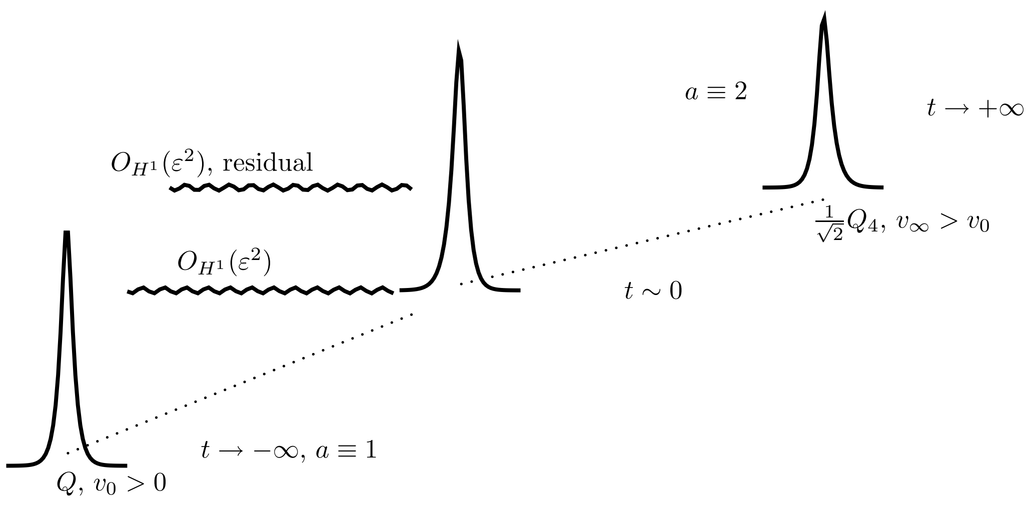
2. Stability of interaction soliton-potential. *Let $v_\infty := (v_0^2+4)^{\frac{1}{2}}$ $(> v_0)$. There exist $K > 0$, and C^1- functions $\rho(t), \gamma(t) \in \mathbb{R}$ defined for all $t \gg \frac{1}{\varepsilon}$ such that the function*

$$w(t,x) := u(t,x) - \frac{1}{\sqrt{2}}Q_4(x - v_\infty t - \rho(t))e^{\frac{i}{2}xv_\infty}e^{i\gamma(t)}$$

   satisfies, for all $t \gg \frac{1}{\varepsilon}$,

$$\|w(t)\|_{H^1(\mathbb{R})} + |\rho'(t)| + |\gamma'(t) - 4 + \frac{1}{4}v_\infty^2| \leq K\varepsilon^2. \tag{2.9}$$

The last two results can be schematized in the following figure:



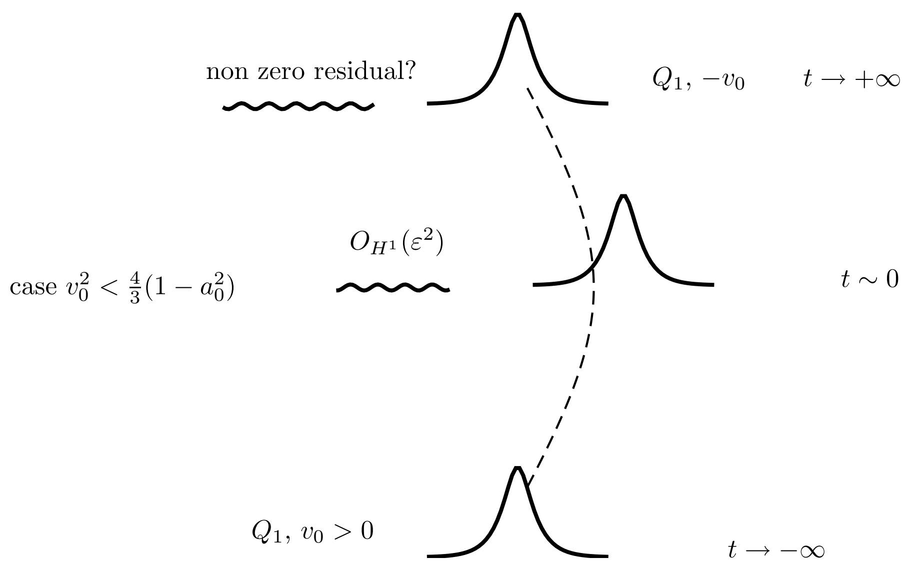
One may compare the above result with the results for the gKdV case, where a bound of order $\varepsilon^{1/2}$ is showed. Our present result is better due to the absence of a **dispersive tail** behind the soliton, precisely of order $\varepsilon^{1/2}$ in $H^1(\mathbb{R})$, and present in the gKdV case. A first mathematical treatment of this phenomenon can be found in [43]. Let us finally recall that such dispersive elements in a soliton solution are not present in the case of a pure NLS or gKdV equation.

In addition, this result is also true for the **two dimensional case**, if we consider a potential depending only on **one variable**, and for any incident velocity (see Theorem B 1 and Corollary C 1). The restriction to the two dimensional case is a consequence of the lack of smoothness for the power nonlinearity in higher dimensions.

These results can also be generalized to **decreasing potentials** for not too large times. The important point here is that the construction of a soliton-like solution does not depend on the sign of a', but on the flatness at infinity. The dynamics on the interaction region can be described in the same way as in the increasing case, with a key difference: for small initial velocities, the soliton solution is **reflected**. For large initial velocities, the soliton exits the interaction region by the right hand side, but stability is not known in this case, so the behavior of this solution for large times is by now unknown. However, if the velocity is small enough (but still independent of ε), we can describe the soliton solution for all time. Indeed, let us suppose that the potential $a(\cdot)$ strictly decreases from an initial state $a \equiv 1$ to a final state $a_0 \in (0,1)$. Then, if $v_0^2 < \frac{4}{3}(1-a_0^2)$, there exists a soliton solution satisfying (2.8) and

$$\sup_{t \gg \frac{1}{\varepsilon}} \|u(t) - Q(\cdot + v_0 t - \rho(t)) e^{-i(\cdot)v_0/2} e^{i\gamma(t)}\|_{H^1(\mathbb{R})} \leq K\varepsilon^2,$$

with $\rho'(t)$ small (see **Addendum 1 in Part 3** for more details.) In other words, the behavior in this regime is very similar to the dynamics expressed by the following figure:



See [54] for more details. As above expressed, the behavior of this solution in the regime $v_0^2 > \frac{4}{3}(1-a_0^2)$ and for large times is an open problem. What happens in the limiting case $v_0^2 = \frac{4}{3}(1-a_0^2)$ seems to be a hard open problem.

2.3 Third part: two-soliton dynamics

The third part of this work concerns the study of the 2-soliton collision for gKdV equations with a general nonlinearity.

We emphasize that inverse scattering theory is useless in the case of a non integrable equation. Here, our purpose is to confirm that under reasonable hypotheses on the nonlinearity, the collision of two solitons is not elastic in general, except for the KdV, mKdV and the Gardner equations.

We deal with this problem for (1.13) with a general nonlinearity $f(u)$ in a particular setting: we consider two positive solitons Q_{c_1}, Q_{c_2}, $0 < c_2 < c_1 < c_*(f)$, and we assume c_2 is smaller than c_1.

Under these assumptions, Martel and Merle [39] considered the collision problem for (1.13) in the quartic case, $f(s) = s^4$. They showed that the collision is almost elastic, but inelastic, by showing the nonexistence of a pure 2-soliton solution.

The next question arising from this result is to generalize these results to (1.13) under assumption (1.15). In this case, Martel and Merle [40] proved that the collision is still stable, giving upper bounds on the residual terms appearing after the collision. In [40], the question of whether the collision is elastic or inelastic in the general case –and thus the nonexistence of pure 2-soliton solutions– was left open, see [40], Remark 1.

By extending some techniques from [39, 40] and developing new computations, we are able to provide a satisfactory answer to this open question.

Theorem 2.4 (Non-existence of pure 2-soliton solution, general case)**.**

Let f be as in (1.15), with $m = 2$ or 3, and

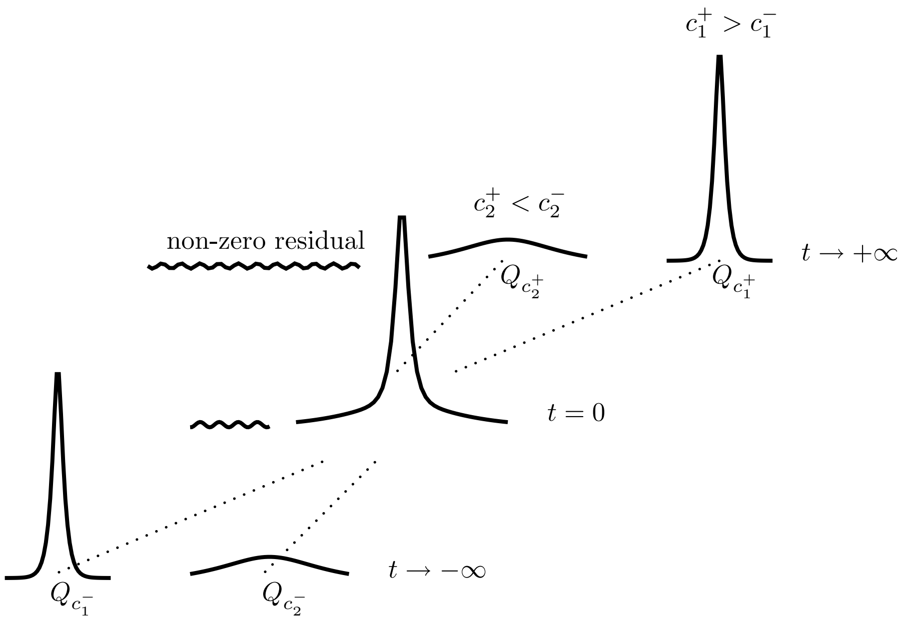
$$f \in C^{p+1}(\mathbb{R}), \quad f^{(p)}(0) \neq 0 \quad \textit{for some} \quad p \geq 4. \tag{2.10}$$

For $0 < c_2 \ll c_1 \ll 1$, equation (1.13) has no pure 2-soliton solution of sizes c_1, c_2.

The nonzero condition $f^{(p)}(0) \neq 0$ for some $p \geq 4$ rules out the integrable cases $f(s) = s^m$, $m = 2$ or 3 and the Gardner nonlinearity $f(s) = s^2 - \mu s^3$.

Our proof follows the approach described by Martel, Merle and Mizumachi [39, 40, 44]. The assumption c_2 small allows to linearize in c_2, and then to reduce the non-existence of a pure 2-soliton solution to the computation of a *coefficient* depending only on c_1. This coefficient is part of an approximate solution of (1.13) with high order of accuracy, and it is evidently zero in the integrable cases. For general f and general $c_1 > 0$, it is an open question to compute this coefficient. According to this, we compute the asymptotic of this coefficient as c_1 is small. This is the only place where c_1 small is needed.

The behavior of this solution can be represented schematically by the following picture (by courtesy of Y. Martel).



Finally, some words about related literature. The 2-soliton collision has been considered for the case of NLS equation very recently. We mention the recent works of Perelman [57], Holmer, Marzuola and Zworski [26, 27, 28] and Abou Salem, Fröhlich and Sigal [2].

3 Perspectives

In this section, we describe some interesting open problems following in this work.

3.1 Lower bound on the defect at infinite for gKdV and NLS equations

One of the questions left open in this work is the obtention of a quantitative lower bound on the defect at infinity for a soliton-like solution. The answer to this question is probably positive, at least in the case $\lambda > 0$, after some formal computations. We believe that in the gKdV case one has

$$\liminf_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} \geq \frac{\varepsilon}{K}, \quad \text{for } m = 2, 4,$$

and surprisingly of order ε^2 in the cubic case (cf. [53] for more details).

In the Schrödinger case, we conjecture the existence of a constant $K > 0$ such that, for all $\varepsilon > 0$ small enough,

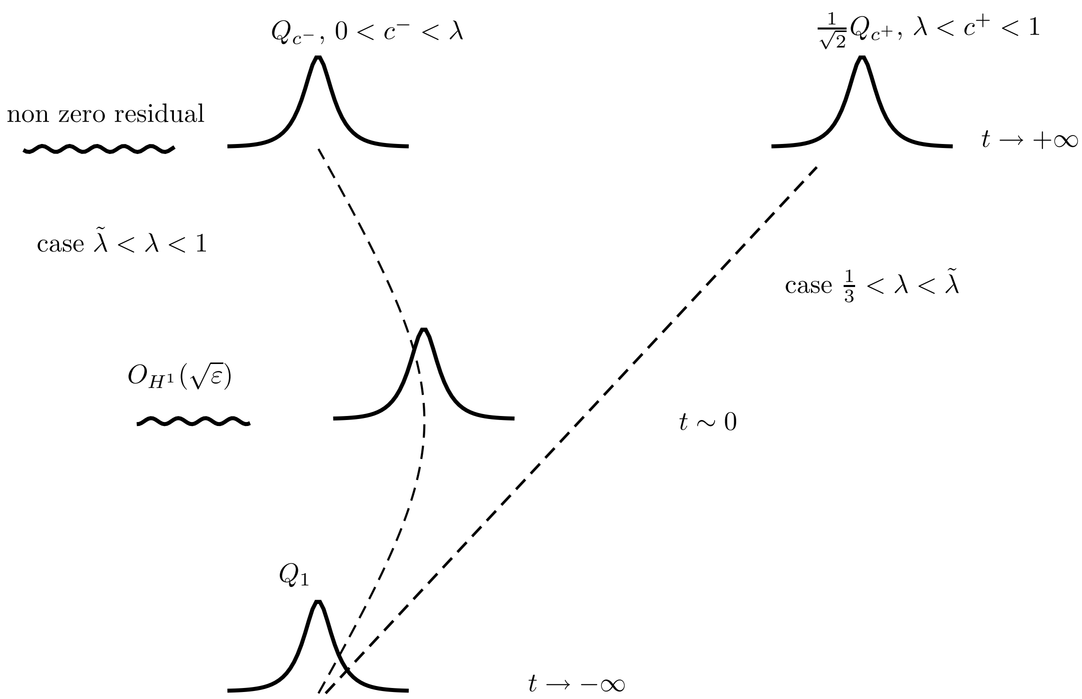
$$\liminf_{t\to+\infty} \|w(t)\|_{H^1(\mathbb{R})} \geq \frac{1}{K}\varepsilon^2.$$

In other words, the behavior in (2.9) is sharp.

These estimates are in a certain sense equivalent to the results of nonexistence of pure 2-soliton solution for gKdV equations and BBM equations outside the integrable cases, as showed in [39, 44, 49, 43].

3.2 Dynamics of soliton solutions for gKdV, case $\frac{1}{3} < \lambda < 1$

Another interesting problem is to understand the behavior of the solution described in Theorem 2.1 for values of the coefficient $\frac{1}{3} < \lambda < 1$. Here the main difference with the preceding case is that formally the scaling **decreases** in time. Therefore a first non trivial task is to show the existence of a final state for the scaling, positive and far from zero, uniformly in ε. However, if the scaling is smaller compared with λ, the soliton solution may be reflected to the left by the potential. Indeed, we believe that there exists a explicit constant $\tilde{\boldsymbol{\lambda}} \in (\frac{1}{3}, 1)$ such that, for any $\varepsilon > 0$ small enough and any $\frac{1}{3} < \lambda < \tilde{\boldsymbol{\lambda}}$, the soliton still exits the interaction region by the right hand side, and if $\tilde{\boldsymbol{\lambda}} < \lambda < 1$ then the soliton solution is reflected by the potential. This claim is formally supported by our computations in Part 2 (cf. Addendum). The following figure describes the conjectured behavior in the case $\frac{1}{3} < \lambda < 1$.



We refer to the work [52], in preparation.

Let us finally remark that some of the results announced in this introduction are available in the web page `http://www.math.uvsq.fr/~munoz`.

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Part II

On the soliton dynamics in a slowly varying medium for generalized Korteweg-de Vries equations

Summary

Abstract

We consider the problem of the soliton propagation, in a slowly varying medium, for a generalized Korteweg - de Vries equations (gKdV). We study the effects of inhomogeneities on the dynamics of a standard soliton. We prove that slowly varying media induce on the soliton solution large dispersive effects at large time. Moreover, unlike gKdV equations, we prove that there is no pure-soliton solution in this regime.
Keywords : generalized Korteweg- de Vries equation, soliton dynamics, slowly varying potentials.

1 Introduction and Main Results

In this work we consider the following *generalized Korteweg-de Vries equation* (gKdV) on the real line

$$u_t + (u_{xx} + f(x,u))_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.1}$$

Here $u = u(t,x)$ is a real-valued function, and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ a nonlinear function. This equation represents a mathematical generalization of the *Korteweg-de Vries* equation (KdV), namely the case $f(x,s) \equiv s^2$,

$$u_t + (u_{xx} + u^2)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x; \tag{1.2}$$

another physically important case is the cubic one, $f(x,s) \equiv s^3$. In this case, the equation (1.1) is often refered as the (focusing) *modified* KdV equation (mKdV). In general, mathematicians denote by *generalized Korteweg-de Vries* (gKdV) the following equation

$$u_t + (u_{xx} + u^m)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x; \quad m \geq 2 \text{ integer}. \tag{1.3}$$

Concerning the KdV equation, it arises in Physics as a model of propagation of dispersive long waves, as was pointed out by J. S. Russel in 1834 [62]. The exact formulation of the KdV equation comes from Korteweg and de Vries (1895) [41]. This equation was re-discovered in a numerical work by N. Zabusky and M. Kruskal in 1965 [42].

After this work, a great amount of literature has emerged, physical, numerical and mathematical, for the study of this equation, see for example [8, 7, 75, 63, 62]. This continuous, focused research on the KdV (and gKdV) equation can be in part explained by some striking algebraic properties. One of the first properties is the existence of localized, exponentially decaying, stable and smooth solutions called *solitons*. Given two real numbers x_0 and $c > 0$, solitons are solutions of (1.3) of the form

$$u(t,x) := Q_c(x - x_0 - ct), \quad Q_c(s) := c^{\frac{1}{m-1}} Q(c^{1/2}s), \tag{1.4}$$

and where Q is a explicit Schwartz function satisfying the second order nonlinear differential equation

$$Q'' - Q + Q^m = 0, \quad Q(x) = \Big[\frac{m+1}{2\cosh^2(\frac{(m-1)}{2}x)}\Big]^{\frac{1}{m-1}}. \tag{1.5}$$

In particular, this solution represents a *solitary wave* defined for all time moving to the right without *any change* in shape, velocity, etc.

In addition, equation (1.3) remains invariant under space and time *translations*. From the Noëther theorem, these symmetries are related to *conserved quantities*, invariant under the gKdV flow, usually called *mass* and *energy*:

$$M[u](t) := \int_{\mathbb{R}} u^2(t,x)\,dx = \int_{\mathbb{R}} u_0^2(x)\,dx = M[u](0), \quad \text{(Mass)}, \tag{1.6}$$

and

$$\begin{aligned} E[u](t) \;&:=\; \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} u^{m+1}(t,x)\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}} (u_0)_x^2(x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} u_0^{m+1}(x)\,dx = E[u](0). \text{ (Energy)} \end{aligned} \tag{1.7}$$

Let us now review some facts about the gKdV equation (1.3), with $m \geq 2$ integer. The Cauchy problem for equation (1.3) (namely, adding the initial condition $u(t=0) = u_0$) is *locally well-posed* for $u_0 \in H^1(\mathbb{R})$ (see Kenig, Ponce and Vega [39]). In the case $m < 5$, any $H^1(\mathbb{R})$ solution is global in time thanks to the conservation of mass and energy (1.6)-(1.7), and the Galiardo-Nirenberg inequality

$$\int_{\mathbb{R}} u^{p+1} \leq K(p)\Big(\int_{\mathbb{R}} u^2\Big)^{\frac{p+3}{4}}\Big(\int_{\mathbb{R}} u_x^2\Big)^{\frac{p-1}{4}}. \tag{1.8}$$

For $m = 5$, solitons are shown to be *unstable* and the Cauchy problem for the corresponding gKdV equation has finite-time blow-up solutions, see [61, 56, 50] and references therein. It is believed that for $m > 5$ the situation is the same. Consequently, in this work, *we will discard high-order nonlinearities*, at leading order.

In addition, there exists another conservation law, formally valid only for $L^1(\mathbb{R})$ solutions:

$$\int_{\mathbb{R}} u(t,x)dx = \text{constant}. \tag{1.9}$$

The problem to consider in this paper possesses a long and extense physical literature. In the next subsection we briefly describe the main results concerning the propagation of solitons in slowly varying medium.

1.1 Statement of the problem, historical review

The dynamical problem of soliton interaction with a slowly varying medium is by now a classical problem in nonlinear wave propagation. By soliton-medium interaction we mean, loosely speaking, the following problem: In (1.1), consider a nonlinear function $f = f(t,x,s)$, slowly varying in space and time, possibly of small amplitude, of the form

$$f(t,x,s) \sim s^m \quad \text{as } x \to \pm\infty, \quad \text{for all time;}$$

(namely (1.1) behaves like a gKdV equation at spatial infinity.) *Consider* a soliton solution of the corresponding variable coefficient equation (1.1) with this nonlinearity, at some early time. Then we expect that this solution does interact with the medium in space and time, here represented by the nonlinearity $f(t,x,s)$. In a slowly varying medium this interaction, small locally in time, may be significantly important on the long time behavior of the solution. The resulting solution after the interaction is precisely the object of study. In particular, one considers if any change in size, position, or shape, even creation or destruction of solitons, after some large time, may be present.

Let us review some relevant works in this direction. After the works of Fermi, Pasta and Ulam [19], Zabusky and Kruskal [42] (see [62] for a review), where complete integrability was established for KdV and other equations, a new branch of research emerged to study the dynamics of solitons solutions of KdV under a slowly varying (*in time*) medium. Kaup and Newell [38] and after Karpman and Maslov [37] considered the study of perturbations of integrable equations, in particular, they considered the perturbed (in time τ) gKdV equation

$$u_\tau + (\beta(\varepsilon\tau)u_{xx} + \alpha(\varepsilon\tau)u^m)_x = 0, \quad m = 2,3; \quad \alpha, \beta > 0. \tag{1.10}$$

This last equation models, for example, the propagation of a wave governed by the KdV equation along a *canal of varying depth*, among many other physical situations, see [37, 4] and references therein.

Note that this equation leaves invariant (1.6) and (1.9), but the corresponding energy for this equation is not conserved anymore. After the transformation $t := \int_0^\tau \beta(\varepsilon s)ds$, $\tilde{u}(t,x) := \big(\frac{\alpha}{\beta}\big)^{\frac{1}{m-1}}(\varepsilon\tau)u(\tau,x)$, the above equation becomes

$$\tilde{u}_t + (\tilde{u}_{xx} + \tilde{u}^m)_x = \varepsilon\gamma(\varepsilon t)\tilde{u}, \quad \text{where} \quad \varepsilon\gamma(\varepsilon t) := \frac{1}{m-1}\partial_t\big[\log(\frac{\alpha}{\beta})(\varepsilon\tau(t))\big]. \tag{1.11}$$

The authors then performed a perturbative analysis of the inverse scattering theory to describe the dynamics of a soliton (for the integrable equation) in this variable regime. Interestingly enough, the existence of a *dispersive shelf-like tail behind the soliton* was formally described. This phenomena is indeed related to the *lack of energy conservation* (1.7) for the equation (1.11).

Subsequently, this problem has been addressed in several other works and for different integrable models, see for example [40, 20, 25, 26]. Moreover, using inverse-scattering techniques, the production of a *second –small– solitary wave* was pointed out in [85], see also [27], but an analytical and satisfactory mathematical proof of this phenomenon is by now out of reach of the current technology. The reader may consult e.g. the monograph by Newell [70], pp. 87–97, for a more detailed account of the problem.

In addition, another important motivation for the study of this problem comes from an interesting point of view, given in Lochak [45], see also [46] for a more detailed description. Based in formal conservation laws, the author points out that, in the case of equation (1.11), well-modulated solitons are good candidates to be *adiabatically stable* objects for this infinite dimensional dynamical system. See [45, 46] for more details.

In this paper we address the problem of soliton dynamics in the case of a slowly varying, inhomogeneous medium, but constant in time. This model, from the mathematical point of view, introduces several difficulties to the study of the dynamical problem, as we will see below; but at the same time reproduces the production of a shelf-like tail behind the soliton, formally seen by physicists. Our main result states that, as a consequence of this tail, there is no pure soliton-solution (unlike gKdV) for this regime. This result illustrates the *lack of pure solutions* of non-trivial perturbations of gKdV equations.

Now let us explain in detail the model we will study along this paper.

1.2 Setting and hypotheses

Let us come back to the general equation (1.1), and consider $\varepsilon > 0$ a small parameter. Following equation (1.10), along this work we will assume that the nonlinearity f is a slowly varying x-dependent function of the power cases, independent of time, plus a (possibly zero) linear term:

$$\begin{cases} f(x,s) := -\lambda s + a_\varepsilon(x) s^m, & \lambda \geq 0, \ m = 2,3 \text{ and } 4. \\ a_\varepsilon(x) := a(\varepsilon x) \in C^3(\mathbb{R}). \end{cases} \tag{1.12}$$

We will suppose the parameter λ fixed, independent of ε. Concerning the function a we will assume that there exist constants $K, \gamma > 0$ such that

$$\begin{cases} 1 < a(r) < 2, \quad a'(r) > 0 \ \text{ for all } r \in \mathbb{R}, \\ 0 < a(r) - 1 \leq K e^{\gamma r}, \ \text{ for all } r \leq 0, \ \text{ and} \\ 0 < 2 - a(r) \leq K e^{-\gamma r} \ \text{ for all } r \geq 0. \end{cases} \tag{1.13}$$

In particular, $\lim_{r\to -\infty} a(r) = 1$ and $\lim_{r\to +\infty} a(r) = 2$. We emphasize that the special choice (1 and 2) of the limits are irrelevant for the results of this paper. The only necessary conditions are that

$$0 < a_{-\infty} := \lim_{r\to -\infty} a(r) < \lim_{r\to +\infty} a(r) =: a_\infty < +\infty.$$

Of course the decay hypothesis on a in (1.13) can be relaxed, and the results of this paper still should hold, with more difficult proofs; but for brevity and clarity of the exposition these issues will not be considered in this work.

Finally, to deal with a special stability property of the mass in Theorems 3.1 and 6.1 (cf. also (6.22)), we will need the following additional (but still general) hypothesis: there exists $K > 0$ such that for $m = 2,3$ and 4,

$$|(a^{1/m})^{(3)}(s)| \leq K (a^{1/m})'(s), \qquad \text{for all } s \in \mathbb{R}. \tag{1.14}$$

This condition is generally satisfied, however a' must not be a compact supported function.

Recapitulating, given $0 \leq \lambda < 1$, we will consider the following *aKdV* equation

$$\begin{cases} u_t + (u_{xx} - \lambda u + a_\varepsilon(x) u^m)_x = 0 \quad \text{ in } \mathbb{R}_t \times \mathbb{R}_x, \\ m = 2,3 \text{ and } 4; \quad 0 < \varepsilon \leq \varepsilon_0; \quad a_\varepsilon \text{ satisfying (1.13)-(1.14)}. \end{cases} \tag{1.15}$$

The main issue that we will study in this paper is the interaction problem between a soliton and a slowly varying medium, here represented by the *potential* a_ε. In other words, we intend to study for (1.15) whether it is possible to generalize the well-known soliton-like solution Q of gKdV. Of course, it is by now well-known that in the case $f(t,x,s) = f(s)$, and under reasonable assumptions (see for example Berestycki and Lions [6]), there exist soliton-like solutions, constructed via *ground states* of the corresponding elliptic equation for a *bound state*. However, in this paper our objective will be the study of soliton solutions under a variable coefficient equation, where no evident ground state is present.

To support our beliefs, note that at least heuristically, (1.15) behaves at infinity as a gKdV equation:

$$\begin{cases} u_t + (u_{xx} - \lambda u + 1u^m)_x = 0 \quad \text{ as } x \to -\infty, \\ u_t + (u_{xx} - \lambda u + 2u^m)_x = 0 \quad \text{ as } x \to +\infty. \end{cases} \tag{1.16}$$

In particular, one should be able of to construct a soliton-like solution $u(t)$ of (1.15) such that

$$u(t) \sim Q(\cdot - (1-\lambda)t), \qquad \text{as } t \to -\infty,$$

in some sense to be defined. Here Q is the soliton of the standard gKdV equation given by (1.5). Indeed, note that $Q(\cdot - (1-\lambda)t)$ is an actual solution for the first equation in (1.16), but on the whole real line, going to the left , to the right or being a steady state depending on $\lambda > 1$, $\lambda < 1$ or $\lambda = 1$ respectively.

On the other hand, after passing the interaction region, by stability properties, this solution *should behave* like (for small ε)

$$\frac{1}{2^{\frac{1}{m-1}}} Q_{c_\infty}(x - (c_\infty - \lambda)t) + \text{ lower order terms in } \varepsilon, \quad \text{as } t \to +\infty, \tag{1.17}$$

where c_∞ is a unknown, positive number, a kind of limiting scaling parameter. In fact, note that if $v = v(t)$ is a solution of (1.3) then $u(t) := 2^{-\frac{1}{m-1}} v(t)$ is a solution of

$$u_t + (u_{xx} - \lambda u + 2u^m)_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.18}$$

In conclusion, this heuristic suggests that even if the potential varies slowly, the soliton will experiment non trivial transformations on its scaling and shape, of the same order that of the amplitude variation of the potential a.

Before stating our results, some important facts are in order. First, unfortunately equation (1.15) is not anymore invariant under scaling and spatial translations. Moreover, a nonzero solution of (1.15) *might lose or gain some mass*, depending on the sign of u, in the sense that, at least formally, the quantity

$$M[u](t) = \frac{1}{2} \int_{\mathbb{R}} u^2(t, x)\, dx \tag{1.19}$$

satisfies the identity

$$\partial_t M[u](t) = -\frac{\varepsilon}{m+1} \int_{\mathbb{R}} a'(\varepsilon x) u^{m+1}. \tag{1.20}$$

Another key observation is the following: in the cubic case $m = 3$, with our choice of a_ε, the mass is always non increasing. This simple fact will have important consequences in our results, at the point of saying that the cubic case corresponds to a well-behaved problem, a sort of good generalization of the pure power case.

On the other hand, the novel energy ($\lambda \geq 0$)

$$E_a[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(t, x)\, dx + \frac{\lambda}{2} \int_{\mathbb{R}} u^2(t, x)\, dx - \frac{1}{m+1} \int_{\mathbb{R}} a_\varepsilon(x) u^{m+1}(t, x)\, dx \tag{1.21}$$

remains formally constant for all time. Moreover, a simple energy balance at $\pm\infty$ allows to determine heuristically the limiting scaling in (1.17), for example in the case $\lambda = 0$, if we suppose that the *lower order terms* are of zero mass at infinity. Indeed, from (1.17) we have

$$E_{a\equiv 1}[u](-\infty) = E[Q] \sim 2^{-\frac{2}{m-1}} c_\infty^{\frac{2}{m-1}+\frac{1}{2}} E[Q] = E_{a\equiv 2}[u](+\infty), \quad E[Q] \neq 0,$$

(cf. Appendix G.1). This implies that $c_\infty \sim 2^{\frac{4}{m+3}} > 1$. These formal arguments suggest the following definition.

Definition 1.1 (Pure generalized soliton-solution for aKdV)**.**

Let $0 \leq \lambda < 1$ be a fixed number. We will say that (1.15) admits a *pure* generalized soliton-like solution (of scaling equals 1) if there exist a C^1 real valued function $\rho = \rho(t)$ defined for all large times and a global in time $H^1(\mathbb{R})$ solution $u(t)$ of (1.15) such that

$$\lim_{t \to -\infty} \|u(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = \lim_{t \to +\infty} \left\|u(t) - 2^{-\frac{1}{m-1}} Q_{c_\infty}(\cdot - \rho(t))\right\|_{H^1(\mathbb{R})} = 0,$$

with $\lim_{t\to+\infty} \rho(t) = +\infty$, and where $c_\infty = c_\infty(\lambda)$ is the scaling suggested by the energy conservation law (1.21).

Remark 1.1. Note that the existence of a translation parameter $\rho(t)$ is a necessary condition: it is even present in the orbital stability of small perturbations of solitons for gKdV, see e.g. [5, 9, 13]. Note that we have not included the case $\rho(t) \to -\infty$ as $t \to +\infty$ (= a reflected soliton), but we hope to consider this case elsewhere.

1.3 Previous analytic results on the soliton dynamics under slowly varying medium

The problem of describing analytically the soliton dynamics of different integrable models under a slowly varying medium has received some increasing attention during the last years. Concerning the KdV equation, our belief is that the first result in this direction was given by Dejak, Jonsson and Sigal in [17, 18]. They considered the long time dynamics of solitary waves (solitons) over slowly varying perturbations of KdV and mKdV equations

$$u_t + (u_{xx} - b(t,x)u + u^m)_x = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x, \quad m = 2, 3, \tag{1.22}$$

and where b is assumed having small size and small variation, in the sense that for ε small,

$$|\partial_t^n \partial_x^p b| \leq \varepsilon^{n+p+1}, \quad \text{for } 0 \leq n + p \leq 2.$$

(Actually their conclusions hold in more generality, but for our purposes we state the closest version to our approach, see [17] for the detailed version.) With these hypotheses the authors show that if $m = 2$ and the initial condition u_0 satisfies the *orbital stability* condition

$$\inf_{\substack{0<c_0<c<c_1 \\ a\in\mathbb{R}}} \|u_0 - Q_c(\cdot - a)\|_{H^1(\mathbb{R})} \leq \varepsilon^{2s}, \quad s < \frac{1}{2}, \; c_0, c_1 \text{ given},$$

then for any for time $t \leq K\varepsilon^{-s}$ the solution can be decomposed as

$$u(t,x) = Q_{c(t)}(x - \rho(t)) + w(t,x),$$

where $\|w(t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^s$ and $\rho(t), c(t)$ satisfies the following differential system

$$\rho'(t) = c(t) - b(t, a(t)) + O(\varepsilon^{2s}), \quad c'(t) = O(\varepsilon^{2s});$$

during the above considered interval of time. In the cubic case ($m = 3$) their results are slightly better, see [17].

Note that our model can be written as a generalized, time independent Dejak-Jonsson-Sigal equation of the type (1.22), after writting $v(t,x) := \tilde{a}(\varepsilon x)u(t,x)$, with $\tilde{a}(\varepsilon x) := a^{\frac{1}{m-1}}(\varepsilon x)$. From these considerations we expect to recover and to improve the results that they have obtained.

Recently Holmer [32] have announced some improvements on the Dejak-Jonsson-Sigal results, without assuming b small. In this paper, in order to achieve a deep understanding of the phenomenon we have preferred to avoid the inclusion of a time depending potential, and to treat the infinite time prescribed and pure data, instead of the standard Cauchy problem. This election will be positively reflexed in the main Theorem, where we will describe with accuracy the dynamical problem, including its asymptotics as $t \to +\infty$.

The interaction soliton-potential can be also considered in the case of the nonlinear Schrödinger equation

$$iu_t + u_{xx} - V(\varepsilon x)u + |u|^2 u = 0, \quad \text{on} \quad \mathbb{R}_t \times \mathbb{R}_x. \tag{1.23}$$

We mention the recent works of Holmer, Marzuola and Zworski [33, 34, 35] and Gustafson et al. [28, 29], where similar results to the above one were obtained. Finally we point out the very recent work of Perelman [73], concerning the critical quintic NLS equation.

1.4 Main Results

Let

$$T_\varepsilon := \frac{1}{1-\lambda}\varepsilon^{-1-\frac{1}{100}} > 0. \tag{1.24}$$

This parameter can be understood as the *interaction time* between the soliton and the potential. In other words, at time $t = -T_\varepsilon$ the soliton should remain almost unperturbed, and at time $t = T_\varepsilon$ the soliton should have completely crossed the influence region of the potential. Note that the asymptotic $\lambda \sim 1$ is a degenerate case and it will be discarded for this work.

In this paper we will prove (cf. Theorems 1.1, 1.2 and 1.3) that for a suitable general case a pure soliton-like solution as in Definition 1.1 does not exist, in the sense that the lower order terms appearing after the interaction have always positive mass. This phenomenon will be a consequence of the dispersion produced during the crossing of the soliton with the main core of the potential a_ε.

In what follows, we assume the validity of above hypotheses, namely (1.12), (1.13) and (1.14). As has been previously claimed, our first result describes in accuracy the dynamics of the *pure* soliton-like solution for aKdV (1.15).

Theorem 1.1 (Dynamics of interaction of solitons for gKdV equations under variable medium)**.**

Let $m = 2, 3$ and 4, and let $0 \le \lambda \le \lambda_0 := \frac{5-m}{m+3}$ be a fixed number. There exists a small constant $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.

1. Existence of a soliton-like solution. *There exists a solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (1.15), global in time, such that*
$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = 0, \tag{1.25}$$
with conserved energy $E_a[u](t) = (\lambda - \lambda_0)M[Q] \le 0$. This solution is unique in the cases $m = 3$; and $m = 2, 4$ provided $\lambda > 0$.

2. Interaction soliton-potential. *There exist $K > 0$ and numbers $c_\infty(\lambda) \ge 1$, $\rho_\varepsilon, \tilde{T}_\varepsilon \in \mathbb{R}$ such that the solution $u(t)$ above constructed satisfies*
$$\left\|u(\tilde{T}_\varepsilon) - 2^{-1/(m-1)}Q_{c_\infty}(x - \rho_\varepsilon)\right\|_{H^1(\mathbb{R})} \le K\varepsilon^{1/2}. \tag{1.26}$$
Moreover,
$$c_\infty(\lambda = 0) = 2^{\frac{4}{m+3}}, \quad \textit{and} \quad c_\infty(\lambda = \lambda_0) = 1. \tag{1.27}$$
Finally we have the bounds
$$|T_\varepsilon - \tilde{T}_\varepsilon| \le \frac{T_\varepsilon}{100}; \quad (1-\lambda)T_\varepsilon \le \rho_\varepsilon \le (2c_\infty(\lambda) - \lambda - 1)T_\varepsilon. \tag{1.28}$$

Remark 1.2. Let us say some words about the special parameter λ_0 from above. First, note that $\lambda_0 = \lambda_0(m)$ is always less than 1 for $m = 2, 3$ and 4; with $\lambda_0(m = 5) = 0$ (= the L^2-critical case). In addition, note that for $\lambda = \lambda_0$ we have $E_a[u](t) = (\lambda - \lambda_0)M[Q] = 0$; and if $\lambda < \lambda_0$ one has $E_a[u](t) < 0$, for all $t \in \mathbb{R}$. For more details about the consequences of this property, and a detailed study of $c_\infty(\lambda)$ see Lemma 4.4.

Remark 1.3. The proof of this result is based on the construction of an approximate solution of (1.15) in the interaction region, satisfying certain symmetries. However, at some point we formally obtain an **infinite mass term**, see also [57] for a similar problem. It turns out that to obtain a localized solution we need to break the symmetry of this solution (see Proposition

4.6 for the details), a key difference with respect to the soliton solution of gKdV and the recent result of Holmer et al. [36], up to ε^{2^-} for the cubic case. This lack of symmetry leads to the error $\varepsilon^{1/2}$ in the theorem above stated. At this price we overtake the interaction region, a completely new result.

The next step is the understanding of the long time behavior of our generalized soliton solution.

Theorem 1.2 (Long time behavior)**.**

Under the assumptions of Theorem 1.1, suppose now in addition that $0 < \lambda \leq \lambda_0$ *for the cases* $m = 2, 4$*, and* $0 \leq \lambda \leq \lambda_0$ *if* $m = 3$*. Let* $0 < \beta < \frac{1}{2}(c_\infty(\lambda) - \lambda)$*. There exists a constant* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$ *the following hold.*

There exist $K, c^+ > 0$ *and a* C^1*-function* $\rho_2(t)$ *defined in* $[T_\varepsilon, +\infty)$ *such that*

$$w^+(t,\cdot) := u(t,\cdot) - 2^{-1/(m-1)} Q_{c^+}(\cdot - \rho_2(t))$$

satisfies

1. Stability. *For any* $t \geq T_\varepsilon$*,*

$$\|w^+(t)\|_{H^1(\mathbb{R})} + |c^+ - c_\infty(\lambda)| + |\rho_2'(t) - (c_\infty(\lambda) - \lambda)| \leq K\varepsilon^{1/2}. \tag{1.29}$$

2. Asymptotic stability.

$$\lim_{t \to +\infty} \|w^+(t)\|_{H^1(x > \beta t)} = 0. \tag{1.30}$$

3. Bounds on the parameters. *Define* $\theta := \frac{1}{m-1} - \frac{1}{4} > 0$*. The limit*

$$\lim_{t \to +\infty} E_a[w^+](t) =: E^+ \tag{1.31}$$

exists and satisfies the identity

$$E^+ = \frac{(c^+)^{2\theta}}{2^{2/(m-1)}}(\lambda_0 c^+ - \lambda) M[Q] + (\lambda - \lambda_0) M[Q], \tag{1.32}$$

and for all $m = 2, 3, 4$ *and* $0 < \lambda \leq \lambda_0$ *there exists* $K(\lambda) > 0$ *such that*

$$\frac{1}{K} \limsup_{t \to +\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq E^+ \leq K\varepsilon. \tag{1.33}$$

Furthermore, in the case $m = 3$*,* $\lambda = 0$*, we have* $\frac{3}{2}E^+ = \big(\frac{c^+}{c_\infty}\big)^{3/2} - 1$*, and for all* $\lambda > 0$*,*

$$\frac{1}{K} \limsup_{t \to +\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq \Big(\frac{c^+}{c_\infty}\Big)^{2\theta} - 1 \leq K\varepsilon. \tag{1.34}$$

Remark 1.4. Stability and asymptotic stability of solitary waves for generalized KdV equations have been widely studied since the '80s. The main ideas of our proof are classical in the literature. For more details, see e.g. [5, 13, 9, 59, 71].

Remark 1.5. The sign of $a'(\cdot)$ is a sufficient condition to ensure stability; however, it can be relaxed by assuming for example the weaker condition $a'(s) > 0$ for all $s > s_0$. In this paper we will not pursue on these assumptions. It is not known whether under more general potentials stability still holds true, see also below.

Remark 1.6 (Decreasing potential). Pick now a potential $a(\cdot)$ satisfying $a'(s) < 0$ and

$$1 = \lim_{s\to-\infty} a(s) > a(s) > \lim_{t\to+\infty} a(s) = \frac{1}{2}.$$

Let us explain the main changes in the above theorems. First of all, Theorem 1.1 part (1) holds true, however we do not know whether the solution constructed is unique. On the other hand, part (2) holds true with the coefficient $2^{\frac{1}{m-1}}$ in front of Q_{c_∞}, $\frac{\lambda}{\lambda_0} < c_\infty(\lambda) < 1$, and $c_\infty(\lambda = 0) = 2^{-p}$ (see Lemma 4.4 to see this). (1.28) holds true with the obvious changes. Finally, Long time stability (=Theorem 1.2) for this case is an open question.

A fundamental question arises from the above results, namely is the final solution an exactly pure soliton for the aKdV equation with $a_\varepsilon \equiv 2$? (cf. Definition 1.1.) This question is equivalent to decide whether

$$\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} = 0.$$

Our last result shows that indeed this behavior cannot happen.

Theorem 1.3 (Non-existence of pure soliton-like solution for aKdV).

Under the context of Theorems 1.1 and 1.2, suppose $m = 2, 3, 4$ *with* $0 < \lambda \leq \lambda_0$. *There exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$,

$$\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} > 0. \tag{1.35}$$

Remark 1.7. We have been unable to solve several questions related to these results. In addition to the classical problem of the extension of these results to more general *potentials* $a(\cdot)$, we have the following questions in mind:

1. A first basic question is to decide if every solution of (1.15) with $H^1(\mathbb{R})$ data is globally bounded in time. In Proposition 2.2 we prove that every solution is globally well defined for all positive times, and uniformly bounded if $\lambda > 0$ or $m = 3$. However, for the cases $m = 2, 4$ and $\lambda = 0$ we only have been able to find an exponential upper bound on the H^1-norm of the solution. Is every solution described in Theorem 1.1 globally bounded?

2. In the cases $m = 2, 4$ and $\lambda = 0$, is the solution constructed in Theorem 1.1 unique? Is it stable for large times? (cf. Theorem 6.1).

3. What is the behavior of the solution for a coefficient $\lambda_0 < \lambda < 1$? We believe in this situation the soliton still survives, but becomes reflected to the left by the potential.

4. It is possible to obtain in Theorem 1.3 a quantitative lower bound on the defect at infinity?

5. Is there scattering modulo the soliton solution, at infinity?

Remark 1.8. The case of the Schrödinger equation considered in (1.23) will be treated in another publication (see [65].)

Remark 1.9 (Case of a time dependent potential). As expected, our results are also valid, with easier proofs, for the following time dependent gKdV equation:

$$u_t + (u_{xx} - \lambda u + a(\varepsilon t)u^m)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.36}$$

Here a satisfies (1.13)-(1.14) now in the time variable. Note that this equation is invariant under scaling and space translations. In addition, the L^1 integral and the mass $M[u]$ remain constants and the energy

$$\tilde{E}[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2 + \frac{\lambda}{2}\int_{\mathbb{R}} u^2 - \frac{a(\varepsilon t)}{m+1}\int_{\mathbb{R}} u^{m+1}$$

satisfies

$$\partial_t \tilde{E}[u](t) = -\frac{\varepsilon a'(\varepsilon t)}{m+1}\int_{\mathbb{R}} u^{m+1}.$$

Furthermore, Theorems 1.1 and 1.2 still hold with $c_\infty(\lambda = 0) = 2^{4/(5-m)}$ (because of the mass conservation), for **any** $\lambda \geq 0$, $m = 2, 3$ and 4 (follow Lemma 4.4 to see this). We left the details to the reader.

Before starting the computations, let us explain how the proof of the main results work.

1.5 Sketch of proof

Our arguments are an adaptation of a series of works by Martel, Merle and Mizumachi [49, 53, 58, 52, 57], and some new computations. The idea is as follows: we separate the analysis among three different time intervals: $t \ll -\varepsilon^{-1}$, $|t| \leq \varepsilon$ and $\varepsilon^{-1} \ll t$. On each interval the solution possesses a specific behavior which we briefly describe:

1. ($t \ll -\varepsilon^{-1}$). In this interval of time we prove that $u(t)$ remains very close to a soliton-solution with no change in the scaling and shift parameters (cf. Theorem 3.1). This result is possible for negative very large times, where the soliton is still far from the interacting region $|t| \leq \varepsilon^{-1}$.

2. ($|t| \leq \varepsilon^{-1}$). Here the soliton-potential interaction leads the dynamics of $u(t)$. The novelty here is the construction of an *approximate solution* of (1.15) with high order of accuracy such that (a) at time $t \sim -\varepsilon^{-1}$ this solution is close to the soliton solution and therefore to $u(t)$; (b) it describes the soliton-potential interaction inside this interval, in particular we show the existence of a remarkable dispersive tail behind the soliton; and (d) it is close to $u(t)$ in the whole interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$, uniformly on time, modulo a modulation on a translation parameter (cf. Theorem 4.1).

3. ($t \gg \varepsilon^{-1}$) Here some *stability* properties (see Theorem 6.1) will be used to establish the convergence of the solution $u(t)$ to a soliton-like solution with modified parameters.

Additionally, by using a contradiction argument, it will be possible to show that the residue of the interaction at time $t \sim \varepsilon^{-1}$ is still present at infinity. This gives the conclusion of the main Theorems 1.1, 1.3. Indeed, recall the L^1 conserved quantity from (1.9). This expression is in general useless when the equation is considered on the whole real line $\mathbb{R}$, however it has some striking applications in the blow-up theory (see [61]). In our case, it will be useful to discard the existence of a pure soliton-like solution.

Remark 1.10 (General nonlinearities). We believe that the main results of this paper are also valid for general, subcritical nonlinearities, with stable solitons. In this case the scaling property of the soliton is no longer valid, so in order to construct an approximate solution one should modify the main argument of the proof.

Finally, some words about the organization of this paper, according to the sketch above mentioned. First in Section 2 we introduce some basic tools to study the interaction and asymptotic problems. Next, Section 3 is devoted to the construction of the soliton like solution for negative large time. Sections 4.1 and 5 deal with the proof of Theorem 1.1. In Section 6 we proof the asymptotic behavior as $t \to +\infty$, namely Theorem 1.2. Finally we prove Theorem 1.3 (Section 7).

2 Preliminaries

In this section we will state several basic but important properties needed in the course of this paper.

2.1 Notation

Along this paper, both $C, K, \gamma > 0$ will denote fixed constants, independent of ε, and possibly changing from one line to the other.

Finally, in order to treat the case $\lambda > 0$ we need to extend the energy (1.7) by adding a mass term. Let us define

$$E_1[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t) + \frac{\lambda}{2}\int_{\mathbb{R}} u^2(t) - \frac{1}{m+1}\int_{\mathbb{R}} u^{m+1}(t), \tag{2.1}$$

namely $E_1[u] = E_{a\equiv 1}[u]$.

2.2 Cauchy Problem

First we develop a suitable local well-posedness theory for the Cauchy problem associated to (1.15).

Let $u_0 \in H^s(\mathbb{R})$, $s \geq 1$, $\lambda \geq 0$. We consider the following initial value problem

$$\begin{cases} u_t + (u_{xx} - \lambda u + a_\varepsilon(x)u^m)_x = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x \\ u(t=0) = u_0, \end{cases} \tag{2.2}$$

where $m = 2, 3$ or 4. The equivalent problem for the generalized KdV equations (1.3) has been extensively studied, but for our purposes, in order to deal with (2.2), we will follow closely the contraction method developed in [39]. We have the following result.

Proposition 2.1 (Local well-posedness in $H^s(\mathbb{R})$, see also [39]**).**

Suppose $u_0 \in H^s(\mathbb{R})$, $s \geq 1$. Then there exist a maximal interval of existence I (with $0 \in I$), and a unique (in a certain sense) solution $u \in C(I, H^s(\mathbb{R}))$ of (2.2). Moreover, the following properties hold:

1. Blow-up alternative. *If* $\sup I < +\infty$, *then*

$$\lim_{t\uparrow \sup I} \|u(t)\|_{H^s(\mathbb{R})} = +\infty. \tag{2.3}$$

The same conclusion holds in the case $\inf I > -\infty$.

2. Energy conservation. *For any $t \in I$ the energy $E_a[u](t)$ from (1.21) remains constant.*

3. Mass variation. *For all $t \in I$ the mass $M[u](t)$ defined in (1.19) satisfies (1.20).*

4. *Suppose $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$. Then (1.9) is well defined and remains constant for all $t \in I$.*

Proof. The proof is standard, and it is based in straightforward application of the Picard iteration procedure and the tools developed in [39]. We skip the details. □

Once a local-in-time existence theory is established, the next step is to ask for the possibility of a *global well-posedness theorem*. In many cases the proof reduces to the *use of conservation laws* to obtain some bounds on the norm of the solution for every time. In the case of gKdV equations ($m \leq 4$) this was proved in [39] by using the mass and energy conservation; however, in our case relation (1.20) is not enough to control the L^2 norm of the solution. As we had stated in the Introduction, the global existence for cubic case $m = 3$ follows from the mass decreasing property. However, to deal with the remaining cases, we will modify our arguments by introducing a perturbed mass, almost decreasing in time, in order to prove global existence. Indeed, define for each $t \in I$, $m = 2, 3$ and 4,

$$\hat{M}[u](t) := \frac{1}{2}\int_{\mathbb{R}} a_\varepsilon^{1/m}(x) u^2(t,x)dx. \tag{2.4}$$

It is clear that $\hat{M}[u](t)$ is a well defined quantity, for any time $t \in I$ and u solution of (2.2) in $H^1(\mathbb{R})$. Note also that for all $t \in I$ we have the equivalence relation $M[u](t) \leq \hat{M}[u](t) \leq 2^{1/m} M[u](t)$.

This modified mass enjoys of a striking property, as is showed in the following

Proposition 2.2 (Global existence in $H^1(\mathbb{R})$).

Consider $u(t)$ the solution of the Cauchy problem (2.2) with $u(0) = u_0 \in H^1(\mathbb{R})$ and maximal interval of existence I. Then $u(t)$ is continuously well defined in $H^1(\mathbb{R})$ for any $t \geq 0$. More precisely, the following properties hold.

1. Cubic case. *Suppose $m = 3$, $\lambda \geq 0$. Then $I = (\tilde{t}_0, +\infty)$ for some $-\infty \leq \tilde{t}_0 < 0$ and there exists $K = K(\|u_0\|_{H^1(\mathbb{R})}) > 0$ such that*

$$\sup_{t\geq 0} \|u(t)\|_{H^1(\mathbb{R})} \leq K. \tag{2.5}$$

2. Almost monotony of the modified mass $\hat{M}$ and global existence. *For any $m = 2, 3$ and 4, and for all $t \in I$ we have*

$$\partial_t \hat{M}[u](t) = -\frac{3}{2}\varepsilon \int_{\mathbb{R}} (a^{1/m})'(\varepsilon x) u_x^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}} [\lambda (a^{1/m})' - \varepsilon^2 (a^{1/m})^{(3)}](\varepsilon x) u^2. \tag{2.6}$$

In particular,

(a) *I is of the same form as above;*

(b) *If $\lambda > 0$ there exists $\varepsilon_0 > 0$ small such that (2.5) holds;*

(c) *if $\lambda = 0$ and $m = 2, 4$, then we have for all $t \geq 0$ the exponential bound*

$$\|u(t)\|_{H^1(\mathbb{R})} \leq K e^{K\varepsilon^3 t}, \tag{2.7}$$

for some $K = K(\|u_0\|_{H^1(\mathbb{R})})$.

Proof of Proposition 2.2. First we consider the cubic case, namely $m = 3$. From (1.20) we have for any $t \in I$, $t \geq 0$
$$M[u](t) \leq M[u](0).$$
This bound implies the global existence for positive times. Indeed, the above bound rules out the L^2 blow-up in (positive) *finite and infinite time* scenario, namely (2.3). In order to control the $H^1(\mathbb{R})$ norm, we use the energy conservation, the Galiardo-Nirenberg inequality (1.8) and the preceding bound on the mass. Indeed, for any $t \in I$, $t \geq 0$, and redefining the constant K if necessary, we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} u_x^2 &= E_a[u](0) - \frac{1}{2}\lambda\int_{\mathbb{R}} u^2 + \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon u^{m+1} \\ &\leq E_a[u](0) + \lambda M[u](0) + K\|u(t)\|_{L^2(\mathbb{R})}^{(m+3)/2}\|u_x(t)\|_{L^2(\mathbb{R})}^{(m-1)/2}.\end{aligned}$$

Noticing that $\frac{1}{4}(m-1) < 1$ for $m = 2, 3$ and 4, we have that

$$\int_{\mathbb{R}} u_x^2 \leq K(\lambda, \|u_0\|_{H^1(\mathbb{R})});$$

for a large constant K. This bound implies the $H^1(\mathbb{R})$ global existence for all positive times and the uniform bound in time (2.5). The bound (2.5) is direct.

In order to prove (2.6), we proceed by formally taking the time derivative. Every step can be rigorously justified by introducing mollifiers. From the equation (1.15) we have

$$\begin{aligned}\partial_t \hat{M}[u](t) &= \int_{\mathbb{R}} a_\varepsilon^{1/m} u u_t = \int_{\mathbb{R}} (a_\varepsilon^{1/m} u)_x (u_{xx} - \lambda u + a_\varepsilon u^m) \\ &= \varepsilon\int_{\mathbb{R}} ((a^{1/m})'(\varepsilon x) u u_{xx} - \frac{1}{2}(a^{1/m})'(\varepsilon x) u_x^2) - \frac{\lambda}{2}\varepsilon\int_{\mathbb{R}} (a^{1/m})'(\varepsilon x) u^2 \\ &\quad + \varepsilon\int_{\mathbb{R}} a_\varepsilon (a^{1/m})'(\varepsilon x) u^{m+1} - \frac{\varepsilon}{m+1}\int_{\mathbb{R}} (a^{1/m+1})'(\varepsilon x) u^{m+1} \\ &= -\frac{1}{2}\varepsilon\int_{\mathbb{R}} [\lambda(a^{1/m})'(\varepsilon x) - \varepsilon^2 (a^{1/m})^{(3)}(\varepsilon x)] u^2 - \frac{3}{2}\varepsilon\int_{\mathbb{R}} (a^{1/m})'(\varepsilon x) u_x^2.\end{aligned}$$

This proves (2.6). Now, in order to establish global $H^1(\mathbb{R})$ existence for positive times, we first control the L^2 norm using $\hat{M}[u](t)$. Let us consider the case $\lambda > 0$. In this case, taking ε_0 small enough, and thanks to (1.14), we have

$$\partial_t \hat{M}[u](t) \leq 0,$$

and thus $\hat{M}[u](t) \leq \hat{M}[u](0)$ for all $t \in I$, $t \geq 0$. The rest of the proof is identical to the cubic case.

Now we consider the last case, namely $m = 2, 4$ and $\lambda = 0$. Here the above argument is not valid anymore and then we have only the existence of $K > 0$ independent of ε such that

$$\partial_t \hat{M}[u](t) \leq K\varepsilon^3 \hat{M}[u](t).$$

This implies that for any $t \in I$, $t \geq 0$,

$$M[u](t) \leq \hat{M}[u](t) \leq K\hat{M}[u](0)e^{K\varepsilon^3 t}.$$

This bound rules out the L^2 blow-up in *finite time* scenario for positive times. To control the $H^1(\mathbb{R})$ norm, we use the same argument from the preceding case. Indeed, for any $t \in I$, redefining the constant K if necessary, we have

$$\int_{\mathbb{R}} u_x^2 \leq K e^{K\varepsilon^3 t},$$

for some large constant K. This bound implies the $H^1(\mathbb{R})$ global existence for positive times. The proof is now complete. □

Remark 2.1 (Mass monotony). A conclusion of the above Proposition is the following. Consider $u(t) \in H^1(\mathbb{R})$ a solution of (1.15). Define de following modified mass

$$\tilde{M}[u](t) := \begin{cases} M[u](t), & \text{if } m = 3, \\ \hat{M}[u](t), & \text{if } m = 2, 4 \text{ and } \lambda > 0. \end{cases} \tag{2.8}$$

Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and for all $t \in \mathbb{R}$, $t \geq t_0$, one has $\tilde{M}[u](t) - \tilde{M}[u](t_0) \leq 0$.

We will also need some properties of the corresponding linearized operator of (1.15). All the results here presented are by now well-known, see for example [53].

2.3 Spectral properties of the linear gKdV operator

In this paragraph we consider some important properties concerning the linearized KdV operator associated to (1.15). Fix $c > 0$, $m = 2, 3$ or 4, and let

$$\mathcal{L}w(y) := -w_{yy} + cw - mQ_c^{m-1}(y)w, \quad \text{where} \quad Q_c(y) := c^{\frac{1}{m-1}} Q(\sqrt{c}y). \tag{2.9}$$

Here $w = w(y)$. We also denote $\mathcal{L}_0 := \mathcal{L}_{c=1}$.

Lemma 2.3 (Spectral properties of $\mathcal{L}$, see [54]).

The operator $\mathcal{L}$ defined (on $L^2(\mathbb{R})$) by (2.9) *has domain $H^2(\mathbb{R})$, it is self-adjoint and satisfies the following properties:*

1. First eigenvalue. *There exist a unique $\lambda_m > 0$ such that $\mathcal{L}Q_c^{\frac{m+1}{2}} = -\lambda_m Q_c^{\frac{m+1}{2}}$.*
2. *The kernel of $\mathcal{L}$ is spawned by Q_c'. Moreover,*

$$\Lambda Q_c := \partial_{c'} Q_{c'}\big|_{c'=c} = \frac{1}{c}\Big[\frac{1}{m-1}Q_c + \frac{1}{2}xQ_c'\Big], \tag{2.10}$$

   satisfies $\mathcal{L}(\Lambda Q_c) = -Q_c$. Finally, the continuous spectrum of $\mathcal{L}$ is given by $\sigma_{cont}(\mathcal{L}) = [c, +\infty)$.
3. Inverse. *For all $h = h(y) \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} hQ_c' = 0$, there exists a unique $\hat{h} \in H^2(\mathbb{R})$ such that $\int_{\mathbb{R}} \hat{h}Q_c' = 0$ and $\mathcal{L}\hat{h} = h$. Moreover, if h is even (resp. odd), then $\hat{h}$ is even (resp. odd).*
4. Regularity in the Schwartz space $\mathcal{S}$. *For $h \in H^2(\mathbb{R})$, $\mathcal{L}h \in \mathcal{S}$ implies $h \in \mathcal{S}$.*
5. Coercivity.

   (a) There exists $K, \sigma_c > 0$ such that for all $w \in H^1(\mathbb{R})$

$$\mathcal{B}[w, w] := \int_{\mathbb{R}} (w_x^2 + cw^2 - mQ_c^{m-1}w^2) \geq \sigma_c \int_{\mathbb{R}} w^2 - K\Big|\int_{\mathbb{R}} wQ_c\Big|^2 - K\Big|\int_{\mathbb{R}} wQ_c'\Big|^2.$$

   In particular, if $\int_{\mathbb{R}} wQ_c = \int_{\mathbb{R}} wQ_c' = 0$, then the functional $\mathcal{B}[w, w]$ is positive definite in $H^1(\mathbb{R})$.

   (b) Now suppose that $\int_{\mathbb{R}} wQ_c = \int_{\mathbb{R}} wxQ_c = 0$. Then the same conclusion as above holds.

Now we introduce some notation, taken from [53]. We denote by $\mathcal{Y}$ the set of C^∞ functions f such that for all $j \in \mathbb{N}$ there exist $K_j, r_j > 0$ such that for all $x \in \mathbb{R}$ we have

$$|f^{(j)}(x)| \leq K_j(1+|x|)^{r_j} e^{-|x|}.$$

Now we recall the following function to describe the effect of *dispersion* on the solution. Let $c > 0$ and

$$\varphi(x) := -\frac{Q'(x)}{Q(x)}, \qquad \varphi_c(x) := -\frac{Q_c'}{Q_c} = \sqrt{c}\varphi(\sqrt{c}x). \tag{2.11}$$

Note that φ is an odd function. Moreover, we have

Claim 1 (See [54]). The function φ above defined satisfies:

1. $\lim_{x\to-\infty}\varphi(x) = -1$; $\lim_{x\to+\infty}\varphi(x) = 1$.
2. For all $x \in \mathbb{R}$, we have $|\varphi'(x)| + |\varphi''(x)| + |\varphi^{(3)}(x)| \leq Ce^{-|x|}$.
3. Both $\varphi', (1-\varphi^2) \in \mathcal{Y}$.

Remark 2.2. The function φ has been already used to describe the main order effect of the collision of two solitons for the quartic KdV equation (see [53]). In that case, φ represented the nonlinear effect on the shift of solitons due to the collision. In this paper, φ will describe the dispersive tail behind the soliton product of the interaction with the potential a_ε. For more details, see Lemma 4.3.

We finish this section with a preliminary Claim taken from [53].

Claim 2 (Non trivial kernel, see [53]). There exists a unique even solution of the problem

$$\mathcal{L}_0 V_0 = mQ^{m-1}, \quad V_0 \in \mathcal{Y}.$$

Moreover, this solution is given by the formula (cf. Lemma 2.3 for the definitions)

$$V_0(y) = \begin{cases} -\frac{1}{2}\Lambda Q(y), & \text{for } m=2, \\ -Q^2(y), & \text{for } m=3, \text{ and} \\ \frac{1}{3}[Q'(y)\int_0^y Q^2 - 2Q^3(y)], & \text{for } m=4. \end{cases}$$

Finally, this solution satisfies $(\mathcal{L}_0(1+V_0))' = (1)' = 0$.

3 Construction of a soliton-like solution

3.1 Statement of the result

Our first effort concerns with the proof of existence of a *pure* soliton-like solution for (1.15) for $t \to -\infty$. Indeed, we prove that, at exponential order in time, there exists a solution $u(t)$ of the form

$$u(t) \sim_{H^1(\mathbb{R})} Q(\cdot - (1-\lambda)t), \quad t \to -\infty,$$

and where Q is a soliton for the gKdV equation.

Theorem 3.1 (Existence and uniqueness of a pure soliton-like solution)**.**

Suppose $0 \leq \lambda < 1$ *fixed. There exists* $\varepsilon_0 > 0$ *small enough such that the following holds for any* $0 < \varepsilon < \varepsilon_0$.

1. Existence. *There exists a solution* $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *of (1.15) such that*

$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = 0, \tag{3.1}$$

and energy $E_a[u](t) = (\lambda - \lambda_0)M[Q]$. *Moreover, there exist constants* $K, \gamma > 0$ *such that for all time* $t \leq -\frac{1}{2}T_\varepsilon$ *and* $s \geq 1$,

$$\|u(t) - Q(\cdot - (1-\lambda)t)\|_{H^s(\mathbb{R})} \leq K\varepsilon^{-1}e^{\varepsilon\gamma t}. \tag{3.2}$$

In particular,

$$\|u(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{-1}e^{-\gamma\varepsilon^{-\frac{1}{100}}} \leq K\varepsilon^{10}, \tag{3.3}$$

provided $0 < \varepsilon < \varepsilon_0$ *small enough.*

2. Uniqueness. *In addition, this solution is unique in the following cases:* (i) $m = 3$*; and* (ii) $m = 2, 4$ *and* $0 < \lambda < 1$.

Remark 3.1. This result follows basically from the fact that inside the region $x \leq -\frac{1}{2}T_\varepsilon$ the potential a_ε is constant ($\equiv 1$) at exponential order (see (1.13)). In other words, the equation (1.15) behaves asymptotically as a gKdV equation, for which soliton solutions exist globally.

Remark 3.2. Note that the energy identity in (1) above follows directly from (3.1), Appendix G.1 and the energy conservation law from Proposition 2.1.

Remark 3.3. The uniqueness of $u(t)$ in the general case is an interesting open question.

The proof of this Theorem is standard and follows closely [49], where the existence of a unique N-soliton solution for gKdV equations was established. Although there exist possible different proofs of this result, the method employed in [49] has the advantage of giving an explicit uniform bound in time (cf. (3.2)). This bound is indeed consequence of compactness properties. For the sake of completeness, we sketch the proof in Appendix B.

Remark 3.4. An easy consequence of the above result is the following. Consider $u(t)$ the solution constructed in Theorem 3.1. Then from the negativity of the energy E_a and the Galiardo-Nirenberg inequality (1.8) there exists a constant $K > 0$ such that for all time $t \in \mathbb{R}$,

$$\frac{1}{K}\|u(t)\|_{H^1(\mathbb{R})} \leq \|u(t)\|_{L^2(\mathbb{R})} \leq K\|u(t)\|_{H^1(\mathbb{R})}. \tag{3.4}$$

Moreover, if $m = 3$ or $m = 2, 4$ and $\lambda > 0$, then we have

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1(\mathbb{R})} \leq K\|u(-\frac{1}{2}T_\varepsilon)\|_{H^1(\mathbb{R})}. \tag{3.5}$$

This last estimate shows that, in order to understand the limiting behavior at large times of $u(t)$, we may consider only the L^2-norm.

4 Description of interaction soliton-potential

Once we have proven the existence (and uniqueness) of a pure soliton-like solution for early times, the next step consists on the study of the interaction soliton-potential. In this sense, note that the region $[-T_\varepsilon, T_\varepsilon]$ can be understood as this nonlinear interaction regime, because of $a_\varepsilon(-T_\varepsilon) \sim 1$ and $a_\varepsilon(T_\varepsilon) \sim 2$ (cf. (1.12)-(1.13)).

The next result shows explicitly that perturbations induced by the potential a_ε are significative, of order one, mainly focused in the scaling and shift parameters. Moreover, the soliton exits the interaction region as a first order solution of the aKdV equation (1.15) with $a_\varepsilon \equiv 2$, and a small error, dispersive term, of order $\varepsilon^{1/2}$ in $H^1(\mathbb{R})$.

Before state the main result of this section, let us recall λ_0 the parameter introduced in Theorem 1.1. These coefficients have a crucial role to distinguish different asymptotic behaviors.

Theorem 4.1 (Dynamics of the soliton in the interaction region)**.**

Suppose $0 \leq \lambda \leq \lambda_0$. *There exist constants* $\varepsilon_0 > 0$, *and* $c_\infty(\lambda) > 1$ *such that the following holds for any* $0 < \varepsilon < \varepsilon_0$. *Let* $u = u(t)$ *be a globally defined* H^1 *solution of (1.15) such that*

$$\|u(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}. \tag{4.1}$$

Then there exist $K_0 = K_0(K) > 0$ *and* $\rho(T_\varepsilon), \rho_1(T_\varepsilon) \in \mathbb{R}$ *such that*

$$\|u(T_\varepsilon + \rho_1(T_\varepsilon)) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \leq K_0 \varepsilon^{1/2}. \tag{4.2}$$

In addition, $c_\infty(\lambda = 0) = 2^p$, $p = \frac{4}{m+3}$, *and* $c_\infty(\lambda = \lambda_0) = 1$. *Finally, we have the bounds*

$$|\rho_1(T_\varepsilon)| \leq \frac{T_\varepsilon}{100}, \qquad (1-\lambda)T_\varepsilon \leq \rho(T_\varepsilon) \leq (2c_\infty(\lambda) - \lambda - 1)T_\varepsilon, \tag{4.3}$$

valid for ε_0 *sufficiently small.*

Remark 4.1. The above theorem is a stability result ensuring that, under the hypotheses of Theorem 1.1, the soliton survives the interaction, with the scaling predicted by the conservation of energy.

Remark 4.2. Even if from Theorem 3.1 we have an exponential decay on the error term at time $t = -T_\varepsilon$ (cf. (3.3) and (4.1)), we are unable to get a better estimate on the solution at time $t = T_\varepsilon$. This problem is due to the emergency of some order $\varepsilon^{1/2}$ dispersive terms, hard to describe using soliton based functions. This new phenomenon has high similarity with a recent description obtained by Martel and Merle for the collision of two solitons of similar sizes for the BBM and KdV equations, see [57].

Remark 4.3. We do not know whether the above result is still valid in the range $\lambda > \lambda_0$. Formal computations suggest that in this regime the soliton might be *reflected* after the interaction. We hope to consider this regime in a forthcoming publication.

The proof of this Theorem requires several steps, in particular this Section and Section 5 deal with the proof of this result. As we have mentioned in the introduction of this paper, we will construct an approximate solution of (1.15). In the next section we prove that the actual solution describing the interaction of the soliton and the potential a_ε is sufficiently close to our approximate solution.

4.1 Construction of an approximate solution describing the interaction

Let us remark that, after the time $-T_\varepsilon$, the interaction begins to be nontrivial and must be considered in our computations. The objective of the following sections is to construct an approximate solution of (1.15), which describes the first order interaction between the soliton and the potential on the interval of time $[-T_\varepsilon, T_\varepsilon]$. The final conclusion of this construction is presented in Proposition 4.6 below.

Our first step towards the proof of Proposition 4.6 is the introduction of a suitable notation.

4.2 Decomposition of the approximate solution

We look for $\tilde{u}(t,x)$, the approximate solution for (1.1), carring out a specific structure. In particular, we construct $\tilde{u}$ as a suitable modulation of the soliton $Q(x-(1-\lambda)t)$, solution of the KdV equation

$$u_t + (u_{xx} - \lambda u + u^m)_x = 0. \tag{4.4}$$

Let $c = c(\varepsilon t) \geq 1$ be a bounded function to be chosen later and

$$y := x - \rho(t) \quad \text{and} \quad R(t,x) := \frac{Q_{c(\varepsilon t)}(y)}{\tilde{a}(\varepsilon\rho(t))}, \tag{4.5}$$

where

$$\tilde{a}(s) := a^{\frac{1}{m-1}}(s), \quad \rho(t) := -(1-\lambda)T_\varepsilon + \int_{-T_\varepsilon}^{t} (c(\varepsilon s) - \lambda) ds. \tag{4.6}$$

The parameter $\tilde{a}$ intends to describe the shape variation of the soliton along the interaction.

The form of $\tilde{u}(t,x)$ will be the sum of the soliton plus a correction term:

$$\tilde{u}(t,x) := R(t,x) + w(t,x), \tag{4.7}$$

$$w(t,x) := \varepsilon A_c(\varepsilon t; y), \tag{4.8}$$

where $A_c := A_{c(\varepsilon t)}(\varepsilon t; y) = c^{\frac{1}{m-1}} A(\varepsilon t; \sqrt{c} y)$ and A is a unknown function to be determined.

We want to measure the size of the error produced by inserting $\tilde{u}$ as defined in (4.8) in the equation (1.1). For this, let

$$S[\tilde{u}](t,x) := \tilde{u}_t + (\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon \tilde{u}^m)_x. \tag{4.9}$$

Finally, let us recall the definition of the linear operator $\mathcal{L}$ given in (2.9). Our first result is the following

Proposition 4.2 (First decomposition of $S[\tilde{u}]$)**.**

For every $t \in [-T_\varepsilon, T_\varepsilon]$, the following nonlinear decomposition of the error term $S[\tilde{u}]$ holds:

$$S[\tilde{u}](t,x) = \varepsilon[F_1 - (\mathcal{L}A_c)_y](\varepsilon t; y) + \varepsilon^2[(A_c)_t + c'(\varepsilon t)\Lambda A_c](\varepsilon t; y) + \varepsilon^2 \mathcal{E}(t,x),$$

where $\Lambda A_c(y) := \frac{1}{c}(\frac{1}{m-1} A_c(y) + \frac{1}{2} y (A_c)_y(y))$ (cf. Lemma 2.3) and

$$F_1(\varepsilon t; y) := \frac{c'(\varepsilon t)}{\tilde{a}(\varepsilon\rho(t))} \Lambda Q_c(y) + \frac{a'(\varepsilon\rho(t))}{\tilde{a}^m(\varepsilon\rho(t))}\big[-\frac{1}{m-1}(c(\varepsilon t) - \lambda) Q_c(y) + (y Q_c^m(y))_y \big], \tag{4.10}$$

and $\mathcal{E}(t,x)$ is a bounded function in $[-T_\varepsilon, T_\varepsilon] \times \mathbb{R}$.

Proof. We prove this result in Appendix C. □

Note that if we want to improve the approximation $\tilde{u}$, the unknown function A_c must be chosen such that

$$(\Omega) \qquad (\mathcal{L}A_c)_y(\varepsilon t; y) = F_1(\varepsilon t; y), \quad \text{for all } y \in \mathbb{R}.$$

Then the error term will be reduced to the second order quantity $S[\tilde{u}] = \varepsilon^2[(A_c)_t + c'(\varepsilon t)\Lambda A_c](\varepsilon t; y) + \varepsilon^2 \mathcal{E}(t,x)$. We prove such a solvability result in a new section, of independent interest.

4.3 Resolution of (Ω)

When solving problem (Ω), we will see below that it is not always possible to find a solution of finite mass. In fact, we will look for solutions such that time and space variables are separated:

$$A_c(t,y) = b(\varepsilon t)\varphi_c(y) + d(\varepsilon t) + h(\varepsilon t)\hat{A}_c(y); \tag{4.11}$$

where $b(s)$, $d(s)$ and $h(s)$ are exponentially decreasing in s, φ_c is the bounded function defined in (2.11) and $\hat{A}_c \in \mathcal{Y}$ (recall that $\lim_{\pm\infty}\varphi_c = \pm\sqrt{c}$.)

This choice gives us a crucial property. Recall that $c \geq 1$. We say that A_c satisfies the **(IP)** property (**IP** = important property) if and only if

$$(\mathbf{IP})\begin{cases} \text{Any spatial derivative of } A_c(\varepsilon t,\cdot) \text{ is a localized } \mathcal{Y}\text{-function,} \\ \text{and there exists } K,\gamma>0 \text{ such that } \|A_c(\varepsilon t,\cdot)\|_{L^\infty(\mathbb{R})} \leq Ke^{-\gamma\varepsilon|t|} \text{ for all } t\in\mathbb{R}. \end{cases}$$

Note that a solution of the form (4.11) satisfies the **(IP)** property.

4.3.1 Resolution of a time independent model problem

In this subsection we address the following existence problem. Let us recall from (2.9), $\mathcal{L}_0 := -\partial_{yy}^2 + 1 - mQ^{m-1}(y)$.

Given a bounded and even function $F = F(y)$, we look for a bounded solution $A = A(y)$ of the following model problem

$$(\mathcal{L}_0 A)' = F. \tag{4.12}$$

satisfying A bounded. The following result deals with the solvability theory for problem (4.12), in the same spirit that Proposition 2.1 in [53] and Proposition 3.2 in [64].

Lemma 4.3 (Existence theory for (4.12))**.**

Suppose $F \in \mathcal{Y}$ even and satisfying the orthogonality condition

$$\int_{\mathbb{R}} FQ = 0. \tag{4.13}$$

Let $\beta = \frac{1}{2}\int_{\mathbb{R}} F$. For any $\delta \in \mathbb{R}$, problem (4.12) has a bounded solution A of the form

$$A(y) = \beta\varphi(y) + \delta + A_1(y), \quad \text{with } A_1(y) \in \mathcal{Y}. \tag{4.14}$$

Finally, this solution is unique in $L^2(\mathbb{R})$ modulo the addition of a constant times Q'.

Proof. Let us write $A := \beta\varphi + \delta(1+V_0) + A_1$, where $\beta, \delta \in \mathbb{R}$ and $A_1 \in \mathcal{Y}$ are to be determined. Inserting this decomposition in (4.12), we have $(\mathcal{L}_0 A_1)' = F - \beta(\mathcal{L}_0\varphi)'$, namely

$$\mathcal{L}_0 A_1 = H - \beta\mathcal{L}_0\varphi + \gamma, \quad H(y) := \int_{-\infty}^{y} F(s)ds, \tag{4.15}$$

and where $\gamma := \mathcal{L}_0 A_1(0) - \int_{-\infty}^0 H(s)ds$. Without loss of generality we can suppose the constant term $\gamma = -\beta$, because from Claim 2 $\mathcal{L}_0(1+V_0) = 1$, thus any constant term can be associated to the free parameter δ.

Now, from Lemma 2.3 the problem (4.12) is solvable if and only if

$$\int_{\mathbb{R}} (H - \beta(\mathcal{L}_0\varphi + 1))Q' = \int_{\mathbb{R}} HQ' = \int_{\mathbb{R}} FQ = 0,$$

namely (4.13) (recall that $\mathcal{L}_0 Q' = 0$.) Thus there exists a solution A_1 of (4.15) satisfying $\int_{\mathbb{R}} A_1 Q' = 0$. Moreover, since

$$\lim_{y\to-\infty}(H - \beta(\mathcal{L}_0\varphi + 1))(y) = 0, \qquad \lim_{y\to+\infty}(H - \beta(\mathcal{L}_0\varphi + 1))(y) = \int_{\mathbb{R}} F - 2\beta,$$

we get $A_1 \in \mathcal{Y}$ provided $\beta = \frac{1}{2}\int_{\mathbb{R}} F$, by Lemma 2.3. This finishes the proof. □

4.3.2 Existence of dynamical parameters

Our first result concerns to the existence of a dynamical system involving the evolution of first order scaling and translation parameters on the main interaction region. This system is related to the orthogonality condition $\int_{\mathbb{R}} F_1 Q_c = 0$, see proof of Lemma 4.5.

Lemma 4.4 (Existence of dynamical parameters)**.**

Suppose $m = 2, 3$ or 4. Let $\lambda_0, p, a(\cdot)$ be as in Theorem 4.1 and (1.13). There exists a unique solution (ρ, c), with c bounded positive, monotone, defined for all $t \geq -T_\varepsilon$, with the same regularity than $a(\varepsilon\cdot)$, of the following system

$$\begin{cases} c'(\varepsilon t) = p\, c(\varepsilon t)\big[c(\varepsilon t) - \frac{\lambda}{\lambda_0}\big]\frac{a'}{a}(\varepsilon\rho(t)), \qquad c(-\varepsilon T_\varepsilon) = 1, \\ \rho'(t) = c(\varepsilon t) - \lambda, \qquad \rho(-T_\varepsilon) = -(1-\lambda)T_\varepsilon. \end{cases} \tag{4.16}$$

In addition,

1. *If $\lambda = \lambda_0$, one has $c \equiv 1$.*
2. *If $0 \leq \lambda < \lambda_0$ then for all $t \geq -T_\varepsilon$ one has $c(\varepsilon t) > 1$ and $\lim_{t\to+\infty} c(\varepsilon t) = c_\infty + O(\varepsilon^{10})$, where $c_\infty = c_\infty(\lambda) > 1$ is the unique solution of the following algebraic equation*

$$c_\infty^{\lambda_0}(c_\infty - \frac{\lambda}{\lambda_0})^{1-\lambda_0} = 2^p(1 - \frac{\lambda}{\lambda_0})^{1-\lambda_0}, \quad c_\infty > 1. \tag{4.17}$$

Moreover, $\lambda \in [0, \lambda_0] \mapsto c_\infty(\lambda) \geq 1$ is a smooth decreasing application, and $c_\infty(\lambda = 0) = 2^p$.

Remark 4.4 (Case $\lambda = 0$). In this situation, there exists a simple implicit expression for $c(\varepsilon t)$:

$$\rho'(t) = c(\varepsilon t) = \frac{a^p(\varepsilon\rho(t))}{a^p(-\varepsilon T_\varepsilon)}.$$

Using the strict monotony of a, from this identity we can find explicitly $c(\varepsilon t)$.

Remark 4.5. Note that the critical value λ_0 can be seen as the exact value of λ such that the solution $u(t)$ constructed in Theorem 3.1 has zero energy. Indeed, note that from Theorem 3.1 we have $E_a[u] = (\lambda - \lambda_0)M[Q]$. This implies that $E_a[u] = 0$ ($> 0, < 0$ resp.) if $\lambda = \lambda_0$ ($\lambda > \lambda_0, \lambda < \lambda_0$ resp.). Because of this phenomenon the study of the soliton dynamics for $\lambda > \lambda_0$ is an open question.

Proof of Lemma 4.4. The local existence of a solution (c, ρ) of (4.16) is a direct consequence of the Cauchy-Lipschitz-Picard theorem.

Now we use (4.16) to prove a priori estimates on the solution c. Note that

$$\frac{(c(\varepsilon t) - \lambda)}{c(\varepsilon t)(c(\varepsilon t) - \frac{\lambda}{\lambda_0})}c'(\varepsilon t) = p(c(\varepsilon t) - \lambda)\frac{a'}{a}(\varepsilon\rho) = p\rho'(t)\frac{a'}{a}(\varepsilon\rho).$$

In particular,

$$(1-\lambda_0)\partial_t \log(c(\varepsilon t) - \frac{\lambda}{\lambda_0}) + \lambda_0 \partial_t \log c(\varepsilon t) = p\partial_t \log a(\varepsilon\rho).$$

By integration on $[-T_\varepsilon, t]$, using $c(-\varepsilon T_\varepsilon) = 1$, we obtain

$$c^{\lambda_0}(\varepsilon t)(c(\varepsilon t) - \frac{\lambda}{\lambda_0})^{1-\lambda_0} = (1-\frac{\lambda}{\lambda_0})^{1-\lambda_0}\frac{a^p(\varepsilon\rho(t))}{a^p(-\varepsilon(1-\lambda)T_\varepsilon)}. \tag{4.18}$$

Since $1 \leq a \leq 2$, c is bounded and ρ is bounded on compact sets and consequently we obtain the global existence. One proves in particular $c' > 0$ and

$$c^{\lambda_0}(\varepsilon t) < a^p(\varepsilon\rho), \quad \text{ and thus } \quad 1 \leq c(\varepsilon t) \leq 2^{\frac{4}{5-m}}. \tag{4.19}$$

Moreover, $\lim_{t\to+\infty} c(\varepsilon t)$ exists and satisfies $\lim_{t\to+\infty} c(\varepsilon t) = c_\infty + O(\varepsilon^{10})$, where c_∞ is a solution of (4.17), after passing to the limit in (4.18). In order to prove the uniqueness of the solution of (4.17), consider for $\mu \geq 1$ the smooth function

$$g(\mu;\lambda) := \mu^{\lambda_0}(\mu - \frac{\lambda}{\lambda_0})^{1-\lambda_0} - 2^p(1-\frac{\lambda}{\lambda_0})^{1-\lambda_0}.$$

Note that in the case $\lambda < \lambda_0$ we have $g(1;\lambda) < 0$ and

$$\partial_\mu g(\mu;\lambda) = \mu^{\lambda_0-1}(\mu - \frac{\lambda}{\lambda_0})^{-\lambda_0}(\mu-\lambda) \geq (1-\frac{\lambda}{\lambda_0})^{-\lambda_0} > 0.$$

This implies that there exists a unique $c_\infty(\lambda) > 1$ such that $g(c_\infty(\lambda);\lambda) = 0$. This proves the uniqueness. The smoothness of the application $\lambda \in [0,\lambda_0] \mapsto c_\infty(\lambda)$ is an easy consequence of the Implicit Function Theorem.

Finally we prove that $\lambda \mapsto c_\infty(\lambda)$ is a decreasing map. To do this, we take derivative in (4.17). We obtain

$$\begin{aligned}\frac{c_\infty(\lambda)^{\lambda_0-1}(c_\infty(\lambda)-\lambda)}{(c_\infty(\lambda)-\frac{\lambda}{\lambda_0})^{\lambda_0}}c'_\infty(\lambda) &= (\frac{1}{\lambda_0}-1)\Big[\frac{c_\infty^{\lambda_0}(\lambda)}{(c_\infty(\lambda)-\frac{\lambda}{\lambda_0})^{\lambda_0}} - \frac{2^p}{(1-\frac{\lambda}{\lambda_0})^{\lambda_0}}\Big] \\ &\leq (\frac{1}{\lambda_0}-1)(1-\frac{\lambda}{\lambda_0})^{-\lambda_0}(1-2^p) < 0.\end{aligned}$$

□

4.3.3 Conclusion of resolution of (Ω)

Lemma 4.5 (Resolution of (Ω))**.**

Suppose $0 \leq \lambda \leq \lambda_0$ *and* $c(\varepsilon t)$ *given by (4.16). There exists a solution* $A_c = A_c(\varepsilon t; y)$ *of*

$$(\mathcal{L}A_c)'(\varepsilon t; y) = F_1(\varepsilon t; y), \tag{4.20}$$

satisfying **(IP)** *and such that*

1. *For every* $t \in [-T_\varepsilon, T_\varepsilon]$,

$$\begin{cases} A_c(\varepsilon t;\cdot) \in L^\infty(\mathbb{R}), \quad A_c(\varepsilon t; y) = b(\varepsilon t)(\varphi_c(y) - c^{1/2}) + h(\varepsilon t)\hat{A}_c(y), \\ \hat{A}_c \in \mathcal{Y}, \quad |b(\varepsilon t)| + |h(\varepsilon t)| \leq K e^{-\gamma\varepsilon|t|}. \end{cases}$$

2. $\lim_{y\to+\infty} A_c(y) = 0.$

Remark 4.6. The function A_c models, at first order in ε, the shelf-like tail behind the soliton, a dispersive effect of the interaction soliton-potential.

Proof. We prove this lemma in three steps.

Step 1. *Reduction to a time independent problem*. We suppose c given as in Lemma 4.4. Note that F_1 in (4.10) can be written as follows

$$F_1(\varepsilon t; y) = \frac{a'}{\tilde{a}^m}\big[pc(c-\frac{\lambda}{\lambda_0})\Lambda Q_c - \frac{1}{m-1}(c-\lambda)Q_c + (yQ_c^m)'\big](y).$$

Consider now the functions

$$\tilde{F}_1(y) := p\Lambda Q - \frac{1}{m-1}Q + (yQ^m)'; \quad \hat{F}_1(y) := \frac{1}{m-1}Q - \frac{p}{\lambda_0}\Lambda Q = \frac{1}{m-1}Q - \frac{4}{5-m}\Lambda Q.$$

We claim that if $c(\varepsilon t)$ satisfies (4.16) then every term in F_1 has the correct scaling, as shows the following result.

Claim 3. Suppose $\tilde{A}(y), \hat{A}(y)$ solve the stationary problems

$$(\mathcal{L}_0\tilde{A})' = \tilde{F}_1, \quad (\mathcal{L}_0\hat{A})' = \hat{F}_1. \tag{4.21}$$

Then for all $t \in \mathbb{R}$,

$$A_c(\varepsilon t; y) := \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}c^{\frac{1}{m-1}}(\varepsilon t)\Big[\tilde{A} + \lambda c^{-1}(\varepsilon t)\hat{A}\Big](c^{1/2}(\varepsilon t)y)$$

is a solution of (4.20).

Proof. Note that

$$\begin{aligned}
(\mathcal{L}A_c)' &= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}c^{\frac{1}{m-1}+1}\big[-\tilde{A}'' + \tilde{A} - mQ^{m-1}\tilde{A}\big]'(c^{1/2}y) \\
&\qquad + \lambda\frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}c^{\frac{1}{m-1}}\big[-\hat{A}'' + \hat{A} - mQ^{m-1}\hat{A}\big]'(c^{1/2}y) \\
&= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}c^{\frac{1}{m-1}+1}\tilde{F}_1(c^{1/2}y) + \lambda\frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}c^{\frac{1}{m-1}}\hat{F}_1(c^{1/2}y) \\
&= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}[pc^2\Lambda Q_c - \frac{1}{m-1}cQ_c + (yQ_c^m)'] + \lambda\frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}[\frac{1}{m-1}Q_c - \frac{p}{\lambda_0}c\Lambda Q_c] \\
&= F_1(\varepsilon t; y).
\end{aligned}$$

This finishes the proof. □

The above Claim reduces to time independent problems.

Step 2. *Resolution of (4.21)*.

Claim 4. There exists $\tilde{A}, \hat{A}$ solutions of (4.21) satisfying (4.14).

Proof. According to Lemma 4.3 it suffices to verify the orthogonality conditions

$$\int_{\mathbb{R}}\tilde{F}_1 Q = \int_{\mathbb{R}}\hat{F}_1 Q = 0.$$

Indeed, using Lemma G.1 in Appendix G

$$\begin{aligned}\int_{\mathbb{R}} \tilde{F}_1 Q &= p\int_{\mathbb{R}} \Lambda Q Q - \frac{1}{m-1}\int_{\mathbb{R}} Q^2 + \int_{\mathbb{R}} Q(yQ^m)_y \\ &= p\int_{\mathbb{R}} \Lambda Q Q - \frac{1}{m-1}\int_{\mathbb{R}} Q^2 + \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1} \\ &= \frac{(5-m)}{4(m-1)}[p - \frac{4}{m+3}]\int_{\mathbb{R}} Q^2 = 0.\end{aligned}$$

Similarly

$$\begin{aligned}\int_{\mathbb{R}} \hat{F}_1 Q &= -\frac{4}{5-m}\int_{\mathbb{R}} \Lambda Q Q + \frac{1}{m-1}\int_{\mathbb{R}} Q^2 \\ &= -\frac{4}{5-m}\times\frac{5-m}{4(m-1)}\int_{\mathbb{R}} Q^2 + \frac{1}{m-1}\int_{\mathbb{R}} Q^2 = 0.\end{aligned}$$

Thus, by invoking Lemma 4.3 there exist solutions $\tilde{A}$, $\hat{A}$ of (4.21) of the form

$$\begin{cases}\tilde{A}(y) = \tilde{\beta}\varphi(y) + \tilde{\delta} + \tilde{A}_1(y), & \tilde{A}_1 \in \mathcal{Y}, \\ \hat{A}(y) = \hat{\beta}\varphi(y) + \hat{\delta} + \hat{A}_1(y), & \hat{A}_1 \in \mathcal{Y},\end{cases}$$

and where $\tilde{\beta}, \hat{\beta}, \tilde{\delta}, \hat{\delta} \in \mathbb{R}$. Moreover, $\tilde{\beta}, \hat{\beta}$ are given by the formulae

$$\begin{aligned}\tilde{\beta} &:= \frac{1}{2}\int_{\mathbb{R}} \tilde{F}_1 = \frac{1}{2}\int_{\mathbb{R}} (p\Lambda Q - \frac{1}{m-1}Q) = \frac{1}{2}[p(\frac{1}{m-1} - \frac{1}{2}) - \frac{1}{m-1}]\int_{\mathbb{R}} Q \\ &= -\frac{3}{2(m+3)}\int_{\mathbb{R}} Q < 0,\end{aligned}$$

for each $m = 2, 3$ and 4. On the other hand

$$\begin{aligned}\hat{\beta} &:= \frac{1}{2}\int_{\mathbb{R}} \hat{F}_1 = \frac{1}{2}\int_{R} (\frac{1}{m-1}Q - \frac{4}{5-m}\Lambda Q) \\ &= \frac{1}{2}[\frac{1}{m-1} - \frac{4}{5-m}\times\frac{3-m}{2(m-1)}]\int_{\mathbb{R}} Q = \frac{1}{2(5-m)}\int_{\mathbb{R}} Q > 0,\end{aligned}$$

for each $m = 2, 3$ and 4. □

Final step. Finally, to get $\lim_{y\to+\infty}\tilde{A}(y) = \lim_{y\to+\infty}\hat{A}(y) = 0$ we choose $\tilde{\delta} = -\tilde{\beta}$ and $\hat{\delta} = -\hat{\beta}$. This proves the last part of the lemma. With this choice we have

$$\tilde{A}(y) = \tilde{\beta}(\varphi(y) - 1) + \tilde{A}_1(y), \quad \hat{A}(y) = \hat{\beta}(\varphi(y) - 1) + \hat{A}_1(y), \quad \tilde{A}_1, \hat{A}_1 \in \mathcal{Y}.$$

Using Claim 3, an actual solution $A_c(\varepsilon t; y)$ of (4.20) is obtained by considering

$$\begin{aligned}A_c(\varepsilon t; y) &:= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)} c^{\frac{1}{m-1}}(\varepsilon t)\big[\tilde{A} + \lambda c^{-1}(\varepsilon t)\hat{A}\big](c^{1/2}y) \\ &=: b(\varepsilon t)(\varphi_c(y) - c^{1/2}) + h(\varepsilon t)\hat{A}_c(y), \quad \hat{A}_c \in \mathcal{Y},.\end{aligned}$$

where

$$b(\varepsilon t) := \frac{a'(\varepsilon\rho)c^{\frac{1}{m-1}-\frac{1}{2}}}{\tilde{a}^m(\varepsilon\rho)}(\tilde{\beta} + \lambda c^{-1}(\varepsilon t)\hat{\beta}), \quad h(\varepsilon t) := \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}.$$

This finishes the proof of Lemma 4.5. □

Remark 4.7. We emphasize that **in any case** $A_c \in L^2(\mathbb{R})$, even if it is exponentially decreasing in time. This non summable solution must be modified in order to obtain a finite mass solution.

Before continuing with the construction of the approximate solution, we need some crucial estimates on the parameter $c(\varepsilon t)$.

Remark 4.8 (Bounds for $c(\varepsilon t)$). From the bound on $c(\varepsilon t)$ in (4.18) we conclude that for all $t \in [-T_\varepsilon, T_\varepsilon]$

$$1 \le c(\varepsilon t) \le 2^{\frac{4}{5-m}}.$$

4.4 Correction to the solution of Problem (Ω)

Consider the cutoff function $\eta \in C^\infty(\mathbb{R})$ satisfying the following properties:

$$\begin{cases} 0 \le \eta(s) \le 1, \quad 0 \le \eta'(s) \le 1, \;\; \text{for any } s \in \mathbb{R}; \\ \eta(s) \equiv 0 \;\; \text{for } s \le -1, \quad \eta(s) \equiv 1 \;\; \text{for } s \ge 1. \end{cases} \tag{4.22}$$

Define

$$\eta_\varepsilon(y) := \eta(\varepsilon y + 2), \tag{4.23}$$

and for $A_c = A_c(\varepsilon t; y)$ *solution* of (4.20) constructed in Lemma 4.5, denote

$$A_\#(\varepsilon t; y) := \eta_\varepsilon(y) A_c(\varepsilon t; y). \tag{4.24}$$

Now redefine

$$\tilde{u} := R + w = R + \varepsilon A_\#. \tag{4.25}$$

where R is the modulated soliton from (4.5).

The following Proposition, which deals with the error associated to this cut-off function and the new approximate solution $\tilde{u}$, is the principal result of this section.

Proposition 4.6 (Construction of an approximate solution for (1.15))**.**

There exist constants $\varepsilon_0, K > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.

1. *Consider the localized function $A_\#$ defined in (4.23)-(4.24). Then we have*

   (a) New behavior. *For all $t \in [-T_\varepsilon, T_\varepsilon]$,*

   $$\begin{cases} A_\#(\varepsilon t, y) = 0 \quad \text{for all } y \le -\frac{3}{\varepsilon}, \\ A_\#(\varepsilon t, y) = A_c(\varepsilon t, y) \quad \text{for all } y \ge -\frac{1}{\varepsilon}. \end{cases} \tag{4.26}$$

   (b) Integrable solution. *For all $t \in [-T_\varepsilon, T_\varepsilon]$, $A_\#(\varepsilon t, \cdot) \in H^1(\mathbb{R})$ with*

   $$\|\varepsilon A_\#(\varepsilon t, \cdot)\|_{H^1(\mathbb{R})} \le K \varepsilon^{\frac{1}{2}} e^{-\gamma\varepsilon|t|}. \tag{4.27}$$

2. *The error associated to the new function $\tilde{u}$ satisfies*

   $$\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} \le K \varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}, \tag{4.28}$$

   and the following integral estimate holds

   $$\int_{\mathbb{R}} \|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} dt \le K \varepsilon^{1/2}.$$

Proof. The proof of (4.26) is direct from the definition. To prove (4.27) it is enough to recall that
$$\|\eta_c'\|_{L^2(\mathbb{R})} \leq K\varepsilon^{-1/2}.$$
For the proof of (4.28), see Appendix D. □

4.5 Recomposition of the solution

In this subsection we present some important estimates concerning our approximate solution. More precisely, we will show that $\tilde{u}$ at time $\pm T_\varepsilon$ behaves as a modulated soliton with the scaling given by the formal computations at infinity. We start out with some model H^1-estimates.

Lemma 4.7 (First estimates on $\tilde{u}$)**.**

1. *Decay away from zero. Suppose $f = f(y) \in \mathcal{Y}$. Then there exist $K, \gamma > 0$ constants such that for all $t \in [-T_\varepsilon, T_\varepsilon]$*
$$\|a'(\varepsilon x) f(y)\|_{H^1(\mathbb{R})} \leq K e^{-\gamma\varepsilon|t|}. \tag{4.29}$$

2. *Almost soliton solution. The following estimates hold for all $t \in [-T_\varepsilon, T_\varepsilon]$.*
$$\|\tilde{u}_t + (c-\lambda)\tilde{u}_x\|_{H^1(\mathbb{R})} \leq K\varepsilon e^{-\gamma\varepsilon|t|}, \quad \|\tilde{u}_t + (c-\lambda)\tilde{u}_x\|_{L^\infty(\mathbb{R})} \leq K\varepsilon e^{-\gamma\varepsilon|t|}, \tag{4.30}$$
$$\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon \tilde{u}^m = (c-\lambda)\tilde{u} + O_{L^2(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}), \tag{4.31}$$
and
$$\|(\tilde{u}_{xx} - c\tilde{u} + a_\varepsilon\tilde{u}^m)_x\|_{H^1(\mathbb{R})} \leq K\varepsilon e^{-\gamma\varepsilon|t|} + K\varepsilon^2. \tag{4.32}$$

Proof. The proof of (4.29) is a direct consequence of (1.13) and the fact that $\rho'(t) = c(\varepsilon t) - \lambda \geq 1-\lambda$, for all $t \in \mathbb{R}$.

Now let us prove (4.30). From (4.25) we obtain
$$\begin{aligned}\tilde{u}_t + (c-\lambda)\tilde{u}_x &= \varepsilon\frac{c'}{\tilde{a}}\Lambda Q_c - \varepsilon\frac{\tilde{a}'}{\tilde{a}^2}(c-\lambda)Q_c + \varepsilon[(A_\#)_t + c(A_\#)_x] \\ &= \varepsilon[(A_\#)_t + c(A_\#)_x] + O_{H^1(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}).\end{aligned}$$
Now, from (D.3) in Appendix D, we know that
$$\varepsilon[(A_\#)_t + c(A_\#)_x] = \varepsilon^2(c-\lambda)\eta_c' A_c + O_{H^1(\mathbb{R})}(\varepsilon^{\frac{3}{2}}e^{-\gamma\varepsilon|t|}) = O_{H^1(\mathbb{R})}(\varepsilon^{\frac{3}{2}}e^{-\gamma\varepsilon|t|}).$$
This estimate completes the proof of the H^1-estimate. The L^∞-estimate follows directly from the continuous Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(R)$.

Concerning (4.31), note that from (4.27)
$$\begin{aligned}\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon\tilde{u}^m &= (c-\lambda)\tilde{u} + \varepsilon[(A_\#)_{xx} + m a_\varepsilon R^{m-1}A_\#] \\ &\qquad + O_{L^2(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}) + O(\varepsilon^2|A_\#|^2) \\ &= (c-\lambda)\tilde{u} + O_{L^2(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}).\end{aligned}$$

Finally we deal with (4.32). Note that $[\tilde{u}_{xx} - c\tilde{u} + a_\varepsilon\tilde{u}^m]_x = S[\tilde{u}] - ((c-\lambda)\tilde{u}_x + \tilde{u}_t)$; the conclusion follows directly from (4.28) and (4.30). □

The next result describes the behavior of the almost solution $\tilde{u}$ at the endpoints $t = -T_\varepsilon, T_\varepsilon$.

Proposition 4.8 (Behavior at $t = \pm T_\varepsilon$).

There exist constants $K, \varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the approximate solution $\tilde{u}$ constructed in Proposition 4.6 satisfies

1. *Closeness to Q at time $t = -T_\varepsilon$.*
$$\|\tilde{u}(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10}. \tag{4.33}$$
2. *Closeness to $2^{-1/(m-1)}Q_{c_\infty}$ at time $t = T_\varepsilon$. Let $c_\infty(\lambda) > 1$ be as defined in Lemma 4.4. Then*
$$\|\tilde{u}(T_\varepsilon) - 2^{-1/(m-1)}Q_{c_\infty}(\cdot - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10}. \tag{4.34}$$

Proof. By definition,
$$\tilde{u}(-T_\varepsilon) - Q(\cdot - \rho(-T_\varepsilon)) = R(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon) + w(-T_\varepsilon).$$
From Lemma 4.6 we have
$$\|w(\pm T_\varepsilon)\|_{H^1(\mathbb{R})} = \|\varepsilon A_\#(\pm T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}e^{-\gamma\varepsilon^{-\frac{1}{100}}} \leq K\varepsilon^{10},$$
for ε small enough. On the other hand, from $\rho(-T_\varepsilon) = -(1-\lambda)T_\varepsilon$ and using the monotony of a, we have
$$1 \leq c(-\varepsilon T_\varepsilon) \leq a^{\frac{4}{5-m}}(\varepsilon\rho(-T_\varepsilon)) \leq 1 + \varepsilon^{10}.$$
In conclusion we have
$$\|R(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10},$$
as desired. Estimate (4.34) is totally analogous, and we skip the details. □

In concluding this section, we have constructed and approximate solution $\tilde{u}$ describing, at least formally, the interaction soliton-potential. In the next section we will show that the *solution u* constructed in Theorem 3.1 actually behaves like $\tilde{u}$ inside the interaction box $[-T_\varepsilon, T_\varepsilon]$.

5 First stability results

In this section our objective is to prove that the approximate solution $\tilde{u}$ describes the actual dynamics of interaction in the interval $[-T_\varepsilon, T_\varepsilon]$. The next proposition is the principal result of this section.

Proposition 5.1 (Exact solution close to the approximate solution $\tilde{u}$).

Let $\kappa > \frac{1}{100}$. There exists $\varepsilon_0 > 0$ such that the following holds for any $0 < \varepsilon < \varepsilon_0$. Suppose that
$$\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} \leq K\varepsilon^{1+\kappa}e^{-\gamma\varepsilon|t|}, \quad \int_{\mathbb{R}} \|S[\tilde{u}](t)\|_{H^2(\mathbb{R})}\, dt \leq K\varepsilon^{\kappa}, \tag{5.1}$$
and
$$\|u(-T_\varepsilon) - \tilde{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{\kappa}, \tag{5.2}$$
with $u = u(t)$ a $H^1(\mathbb{R})$ solution of (1.15) in a vicinity of $t = -T_\varepsilon$. Then $u(t)$ is defined for any $t \in [-T_\varepsilon, T_\varepsilon]$ and there exist $K_0 = K_0(\kappa, K)$ and a C^1-function $\rho_1 : [-T_\varepsilon, T_\varepsilon] \to \mathbb{R}$ such that, for all $t \in [-T_\varepsilon, T_\varepsilon]$,
$$\|u(t + \rho_1(t)) - \tilde{u}(t)\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{\kappa}, \quad |\rho_1'(t)| \leq K_0\varepsilon^{\kappa}. \tag{5.3}$$

Before the proof, we clarify some important details about the statement of the proposition.

Remark 5.1. Note that u has to be modulated in order to get the correct result. However, in this case we have not modulated on the scaling and spatial translation parameters because (1.15) is not invariant under these transformations. Nevertheless, we still have another degeneracy, due to time translations, which fortunately allows to control the dynamics of the solution u for every $t \in [-T_\varepsilon, T_\varepsilon]$. In this sense, the *new time* $s(t) := t + \rho_1(t)$ can be interpreted as a *retarded (or advanced)* time of the actual solution with respect to the approximate solution. Moreover, note that for ε small enough,

$$s'(t) = 1 + \rho'(t) > \frac{99}{100} > 0,$$

for all $t \in [-T_\varepsilon, T_\varepsilon]$. This means that we can inverse $s(t)$ on $s([-T_\varepsilon, T_\varepsilon]) \subseteq \frac{99}{100}[-T_\varepsilon, T_\varepsilon]$.

From the proof we do not know the sign of $\rho_1'(t)$, so in particular we do not know if the solution u is retarded or in advance with respect to the approximate solution $\tilde{u}$.

Proof of Proposition 5.1. Let $K^* > 1$ be a constant to be fixed later. Let us recall that from Proposition 2.2 we have that $u(t)$ is globally well-defined in $H^1(\mathbb{R})$. Since $\|u(-T_\varepsilon) - \tilde{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^\kappa$, by continuity in time in $H^1(\mathbb{R})$, there exists $-T_\varepsilon < T^* \leq T_\varepsilon$ with

$$\begin{aligned} T^* \quad := \quad & \sup\big\{T \in [-T_\varepsilon, T_\varepsilon], \text{ such that for all } t \in [-T_\varepsilon, T], \text{ there exists } r(t) \in \mathbb{R} \text{ with} \\ & \qquad \|u(t + r(t)) - \tilde{u}(t)\|_{H^1(\mathbb{R})} \leq K^*\varepsilon^\kappa\big\}. \end{aligned}$$

The objective is to prove that $T^* = T_\varepsilon$ for K^* large enough. To achieve this, we argue by contradiction, assuming that $T^* < T_\varepsilon$ and reaching a contradiction with the definition of T^* by proving some independent estimates for $\|u(t+r(t)) - \tilde{u}(t)\|_{H^1(\mathbb{R})}$ on $[-T_\varepsilon, T^*]$, for a special modulation parameter $r(t)$.

5.1 Modulation

By using the Implicit function theorem we will construct a modulation parameter and to estimate its variation in time:

Lemma 5.2 (Modulation in time). *Assume $0 < \varepsilon < \varepsilon_0(K^*)$ small enough. There exists a unique C^1 function $\rho_1(t)$ such that, for all $t \in [-T_\varepsilon, T^*]$,*

$$z(t) = u(t + \rho_1(t)) - \tilde{u}(t) \quad \textit{satisfies} \quad \int_{\mathbb{R}} z(t,x) Q_c'(y)dx = 0. \tag{5.4}$$

Moreover, we have, for all $t \in [-T_\varepsilon, T^*]$,

$$|\rho_1(-T_\varepsilon)| + \|z(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^\kappa, \ \|z(t)\|_{H^1(\mathbb{R})} \leq 2K^*\varepsilon^\kappa. \tag{5.5}$$

In addition, $z(t)$ satisfies the following equation

$$z_t + (1 + \rho_1')\big\{z_{xx} - \lambda z + a_\varepsilon[(\tilde{u} + z)^m - \tilde{u}^m]\big\}_x - \rho_1'\tilde{u}_t + (1 + \rho_1')S[\tilde{u}] = 0. \tag{5.6}$$

Finally, there exist $K, \gamma > 0$ independent of K^* such that for every $t \in [-T_\varepsilon, T^*]$

$$|\rho_1'(t)| \leq \frac{K}{c(\varepsilon t) - \lambda}\Big[\|z\|_{L^2(\mathbb{R})} + \varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{L^2(\mathbb{R})} + \|z(t)\|_{L^2(\mathbb{R})}^2 + \|S[\tilde{u}]\|_{L^2(\mathbb{R})}\Big]. \tag{5.7}$$

Proof. The proof of (5.4)-(5.5) is by now well-know and it is a consequence of the Implicit Function Theorem. See e.g. [53] for a detailed proof. On the other hand, the proof of (5.6) follows after a simple calculation using (1.15).

Finally, we prove (5.7). From (5.4)-(5.6) we take time derivative and replace z_t to obtain

$$\begin{aligned} 0 &= (1+\rho_1')\int_{\mathbb{R}}\{z_{xx}-cz+a_\varepsilon[(\tilde{u}+z)^m-\tilde{u}^m]\}Q_c'' \\ &\quad +\rho_1'\int_{\mathbb{R}}(\tilde{u}_t-(c-\lambda)z_x)Q_c'-(1+\rho_1')\int_{\mathbb{R}}S[\tilde{u}]Q_c'+\varepsilon c'(\varepsilon t)\int_{\mathbb{R}}z\Lambda Q_c'. \end{aligned}$$

First, note that

$$\rho_1'\int_{\mathbb{R}}(\tilde{u}_t-(c-\lambda)z_x)Q_c'=-\frac{\rho_1'}{\tilde{a}}\Big[(c-\lambda)\int_{\mathbb{R}}Q_c'^2+O(\varepsilon+\|z(t)\|_{L^2(\mathbb{R})})\Big].$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}}\{z_{xx}-cz+a_\varepsilon[(\tilde{u}+z)^m-\tilde{u}^m]\}Q_c'' &= -\int_{\mathbb{R}}z\mathcal{L}Q_c''+O(\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{L^2(\mathbb{R})}) \\ &\quad +O(\|z(t)\|_{L^2(\mathbb{R})}^2). \end{aligned}$$

Collecting these estimates, and using the fact that $\|z(t)\|_{H^1(\mathbb{R})}$ is small, we get desired result. □

5.2 Control on the Q_c direction

We recall from (1.7) that the energy of the function $u(t+\rho_1(t))$ is conserved, moreover, $E_a[u(t+\rho_1(t))]=E_a[u](t)$ for any $t\in[-T_\varepsilon,T^*]$. In what follows, we will made use of this identity to estimate z against the degenerate direction Q_c. First we prove that the approximate solution $\tilde{u}$ has almost conserved energy.

Lemma 5.3 (Almost conservation of energy)**.**

Consider $\tilde{u}$ the approximate solution constructed in Proposition 4.6. Then

$$\partial_t E_a[\tilde{u}](t)=-\int_{\mathbb{R}}(\tilde{u}_{xx}-\lambda\tilde{u}+a_\varepsilon\tilde{u}^m)S[\tilde{u}]. \tag{5.8}$$

In particular, there exists $K>0$ independent of K^* such that

$$|E_a[\tilde{u}](t)-E_a[\tilde{u}](-T_\varepsilon)|\leq K\varepsilon^\kappa. \tag{5.9}$$

Proof. We start by showing (5.8). From (4.9) we have

$$\begin{aligned} \int_{\mathbb{R}}S[\tilde{u}](\tilde{u}_{xx}-\lambda\tilde{u}+a_\varepsilon\tilde{u}^m) &= \int_{\mathbb{R}}\tilde{u}_t(\tilde{u}_{xx}-\lambda\tilde{u}+a_\varepsilon\tilde{u}^m) \\ &= -\partial_t\frac{1}{2}\int_{\mathbb{R}}\tilde{u}_x^2-\partial_t\frac{\lambda}{2}\int_{\mathbb{R}}\tilde{u}^2+\frac{1}{m+1}\partial_t\int_{\mathbb{R}}a_\varepsilon\tilde{u}^{m+1} \\ &= -\partial_t E_a[\tilde{u}](t), \end{aligned}$$

as desired.

Now we consider (5.9). From Cauchy-Schwarz inequality, we have

$$|\partial_t E_a[\tilde{u}](t)|\leq K\|S[\tilde{u}](t)\|_{L^2(\mathbb{R})},$$

for some constant $K>0$. After integration and considering (5.1), we get the result. □

Lemma 5.4 (Control in the Q_c direction)**.**

There exists $K,\gamma>0$, independent of K^* such that for $0<\varepsilon<\varepsilon_0$ small enough,

$$|\int_{\mathbb{R}} Q_c(y)z| \leq \frac{K}{c(\varepsilon t)-\lambda}\Big[\varepsilon^{\kappa}+\varepsilon^{1/2}e^{-\varepsilon\gamma|t|}\|z(t)\|_{L^2(\mathbb{R})}+\|z(t)\|^2_{H^1(\mathbb{R})}\Big].$$

Proof. Consider the conserved energy $E_a[u(t+\rho_1)]$; we expand this term and make use of the identity $u(t+\rho_1)=\tilde{u}(t)+z(t)$ to obtain

$$\begin{aligned} E_a[\tilde{u}+z](t) \ &= \ E_a[\tilde{u}](t)-\int_{\mathbb{R}} z(\tilde{u}_{xx}-\lambda\tilde{u}+a_\varepsilon\tilde{u}^m)+\frac{1}{2}\int_{\mathbb{R}} z_x^2+\frac{\lambda}{2}\int_{\mathbb{R}} z^2 \\ &\quad -\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon[(\tilde{u}+z)^{m+1}-\tilde{u}^{m+1}-(m+1)\tilde{u}^m z]. \end{aligned}$$

First, note that

$$\begin{aligned} \int_{\mathbb{R}} z(\tilde{u}_{xx}-\lambda\tilde{u}+a_\varepsilon\tilde{u}^m)(t) \ &= \ \int_{\mathbb{R}} z(\tilde{u}_{xx}-\lambda\tilde{u}+a_\varepsilon\tilde{u}^m)(-T_\varepsilon)+\Big\{E_a[\tilde{u}](t)-E_a[\tilde{u}](-T_\varepsilon)\Big\} \\ &\quad +O(\|z(t)\|^2_{H^1(\mathbb{R})}). \end{aligned}$$

We use now (4.31):

$$\int_{\mathbb{R}} z(\tilde{u}_{xx}-\lambda u+a_\varepsilon\tilde{u}^m)=(c-\lambda)\int_{\mathbb{R}}\tilde{u}z+O(\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{L^2(\mathbb{R})})$$

The conclusion follows from the above identity and (5.9). □

5.3 Energy functional for z

Consider the functional

$$\mathcal{F}(t):=\frac{1}{2}\int_{\mathbb{R}}(z_x^2+c(\varepsilon t)z^2)-\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon[(\tilde{u}+z)^{m+1}-\tilde{u}^{m+1}-(m+1)\tilde{u}^m z]. \tag{5.10}$$

Lemma 5.5 (Modified coercivity for $\mathcal{F}$, second version)**.**

There exist $K,\nu_0>0$, independent of K^* and ε such that for every $t\in[-T_\varepsilon,T_\varepsilon]$

$$\mathcal{F}(t)\geq\nu_0\|z(t)\|^2_{H^1(\mathbb{R})}-|\int_{\mathbb{R}} Q_c(y)z|^2-K(\varepsilon e^{-\gamma\varepsilon|t|}+\varepsilon^2)\|z(t)\|^2_{L^2(\mathbb{R})}-K\|z(t)\|^3_{L^2(\mathbb{R})}.$$

Proof. We write

$$\mathcal{F}(t) \ = \ \frac{1}{2}\int_{\mathbb{R}}(z_x^2+cz^2-ma_\varepsilon\tilde{u}^{m-1}z^2) \tag{5.11}$$

$$-\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon[(\tilde{u}+z)^{m+1}-\tilde{u}^{m+1}-(m+1)\tilde{u}^m z-\frac{1}{2}m(m+1)\tilde{u}^{m-1}z^2]. \tag{5.12}$$

In the case $m=2$ the term (5.12) above is identically zero, and for $m=3,4$ we have $|(5.12)|\leq K\|z(t)\|^3_{L^2(\mathbb{R})}$.

On the other hand, the first term above looks as follows

$$(5.11)=\frac{1}{2}\int_{\mathbb{R}}(z_x^2+c(\varepsilon t)z^2-mQ_c^{m-1}z^2)-\varepsilon\frac{ma'(\varepsilon\rho)}{2a(\varepsilon\rho)}\int_{\mathbb{R}} yQ_c^{m-1}z^2+O(\varepsilon^2\|z(t)\|^2_{L^2(\mathbb{R})}).$$

It is clear that

$$|\varepsilon \frac{ma'(\varepsilon\rho)}{2a(\varepsilon\rho)} \int_{\mathbb{R}} yQ_c^{m-1}z^2| \leq K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})}.$$

Finally, from Lemma 2.3, we have the existence of constants $K, \nu_0 > 0$ such that for all $t \in [-T_\varepsilon, T^*]$

$$\frac{1}{2}\int_{\mathbb{R}}(z_x^2 + c(\varepsilon t)z^2 - mQ_c^{m-1}z^2) \geq \nu_0\|z(t)\|^2_{H^1(\mathbb{R})} - K\Big|\int_{\mathbb{R}} Q_c z\Big|^2.$$

□

Now we use a coercivity argument to obtain independent estimates for $\mathcal{F}(T^*)$.

Lemma 5.6 (Estimates on $\mathcal{F}(T^*)$)**.**

The following properties hold for any $t \in [-T_\varepsilon, T^*]$.

1. *First time derivative.*

$$\begin{aligned}\mathcal{F}'(t) &= -\int_{\mathbb{R}} z_t\big\{z_{xx} - cz + a_\varepsilon[(\tilde u + z)^m - \tilde u^m]\big\} + \frac{1}{2}\varepsilon c'(\varepsilon t)\int_{\mathbb{R}} z^2 \\ &\quad -\int_{\mathbb{R}} a_\varepsilon \tilde u_t[(\tilde u + z)^m - \tilde u^m - m\tilde u^{m-1}z]. \qquad (5.13)\end{aligned}$$

2. *Integration in time. There exist constants $K, \gamma > 0$ such that*

$$\begin{aligned}\mathcal{F}(t) - \mathcal{F}(-T_\varepsilon) &\leq K(K^*)^4\varepsilon^{4\kappa - \frac{1}{100}} + K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa} \\ &\quad + K\int_{-T_\varepsilon}^{t} \varepsilon e^{-\varepsilon\gamma|t|}\|z(t)\|^2_{H^1(\mathbb{R})}dt.\end{aligned}$$

Proof. First of all, (5.13) is a simple computation. Let us consider (5.14). Replacing (5.6) in (5.13) we get

$$\begin{aligned}\mathcal{F}'(t) &= (c(\varepsilon t) - \lambda)(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon[(\tilde u + z)^m - \tilde u^m]z_x \qquad (5.14)\\ &\quad -\rho_1'\int_{\mathbb{R}} \tilde u_t\big\{z_{xx} - cz + a_\varepsilon[(\tilde u+z)^m - \tilde u^m]\big\} \qquad (5.15)\\ &\quad +(1+\rho_1')\int_{\mathbb{R}} S[\tilde u]\big\{z_{xx} - cz + a_\varepsilon[(\tilde u + z)^m - \tilde u^m]\big\} \qquad (5.16)\\ &\quad +\frac{1}{2}\varepsilon c'(\varepsilon t)\int_{\mathbb{R}} z^2 - \int_{\mathbb{R}} a_\varepsilon\tilde u_t[(\tilde u + z)^m - \tilde u^m - m\tilde u^{m-1}z]. \qquad (5.17)\end{aligned}$$

Now we consider separate cases. First let us suppose $m = 2$. After some simplifications, we get

$$\begin{aligned}(5.14) &= (c-\lambda)(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon[2\tilde u z + z^2]z_x \\ &= -(c-\lambda)(1+\rho_1')\int_{\mathbb{R}}[a_\varepsilon \tilde u_x z^2 + \varepsilon a'(\varepsilon x)\tilde u z^2 + \frac{1}{3}\varepsilon a'(\varepsilon x)z^3].\end{aligned}$$

From this

$$|(5.14) + (c-\lambda)(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon\tilde u_x z^2| \leq K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})} + K\varepsilon\|z(t)\|^3_{H^1(\mathbb{R})}.$$

On the other hand,

$$
\begin{aligned}
(5.15) \quad = \quad & -\rho_1' \int_{\mathbb{R}} (\tilde{u}_t + (c-\lambda)\tilde{u}_x)\big\{z_{xx} - cz + a_\varepsilon[2\tilde{u}z + z^2]\big\} + (c-\lambda)\rho_1' \int_{\mathbb{R}} a_\varepsilon \tilde{u}_x z^2 \\
& + (c-\lambda)\rho_1' \int_{\mathbb{R}} z[\tilde{u}_{xx} - c\tilde{u} + a_\varepsilon \tilde{u}^2]_x - (c-\lambda)\rho_1' \varepsilon \int_{\mathbb{R}} a'(\varepsilon x)\tilde{u}^2 z.
\end{aligned}
$$

In particular, using estimates (4.29), (4.32) and (4.30) we obtain

$$
|(5.15) - (c-\lambda)\rho_1' \int_{\mathbb{R}} a_\varepsilon \tilde{u}_x z^2| \leq K\varepsilon|\rho_1'|e^{-\gamma\varepsilon|t|}\|z(t)\|_{H^1(\mathbb{R})}
$$

We also have

$$
(5.16) = (1+\rho_1') \int_{\mathbb{R}} z[S[\tilde{u}]_{xx} - cS[\tilde{u}] + 2a_\varepsilon \tilde{u} S[\tilde{u}] + a_\varepsilon z S[\tilde{u}]],
$$

thus using (5.7)

$$
|(5.16)| \leq K\|z(t)\|_{L^2(\mathbb{R})}\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})}.
$$

Finally,

$$
(5.17) \quad = \quad \frac{1}{2}\varepsilon c'(\varepsilon t) \int_{\mathbb{R}} z^2 - \int_{\mathbb{R}} a_\varepsilon(\tilde{u}_t + (c-\lambda)\tilde{u}_x)z^2 + (c-\lambda) \int_{\mathbb{R}} a_\varepsilon \tilde{u}_x z^2.
$$

We get then from (4.30)

$$
|(5.17) - (c-\lambda) \int_{\mathbb{R}} a_\varepsilon \tilde{u}_x z^2| \leq K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})}.
$$

Collecting the above estimates and (5.7), and after an integration, we finally get

$$
|\mathcal{F}(t) - \mathcal{F}(-T_\varepsilon)| \leq K(K^*)^3\varepsilon^{3\kappa - \frac{1}{100}} + KK^*\varepsilon^{2\kappa} + K\int_{-T_c}^{t} \varepsilon e^{-\gamma\varepsilon|s|}\|z(s)\|^2_{L^2(\mathbb{R})}ds.
$$

The cases $m = 3, 4$ are similar, but more involved. From (5.14)-(5.17), and after some integration by parts, the result is the following:

$$\begin{aligned}
\mathcal{F}'(t) = {} & (c-\lambda)(1+\rho_1') \times \\
& \Big[\int_{\mathbb{R}} a_\varepsilon \big\{ (\tilde u + z)^m - \tilde u^m - m\tilde u^{m-1} z - \frac{m}{2}(m-1)\tilde u^{m-2} z^2 \big\} z_x \qquad (5.18)\\
& -\frac{m}{2}\varepsilon \int_{\mathbb{R}} a'(\varepsilon x)\tilde u^{m-1} z^2 - \frac{\varepsilon}{6} m(m-1)\int_{\mathbb{R}} a'(\varepsilon x)\tilde u^{m-2} z^3 \\
& -\frac{m}{2}\int_{\mathbb{R}} a_\varepsilon (\tilde u^{m-1})_x z^2 - \frac{m}{6}(m-1)\int_{\mathbb{R}} a_\varepsilon (\tilde u^{m-2})_x z^3 \Big] \qquad (5.19)\\
& -\rho_1' \int_{\mathbb{R}} (\tilde u_t + (c-\lambda)\tilde u_x)\big\{ z_{xx} - cz + a_\varepsilon[(\tilde u + z)^m - \tilde u^m] \big\} \\
& +(c-\lambda)\rho_1' \Big[\int_{\mathbb{R}} z(\tilde u_{xx} - c\tilde u + a_\varepsilon \tilde u^m)_x - \varepsilon \int_{\mathbb{R}} a'(\varepsilon x)\tilde u^m z \Big] \\
& +(c-\lambda)(1+\rho_1') \int_{\mathbb{R}} \tilde u_x a_\varepsilon \big\{ (\tilde u + z)^m - \tilde u^m - m\tilde u^{m-1} z \\
& \qquad -\frac{m}{2}(m-1)\tilde u^{m-2} z^2 - \frac{m}{6}(m-1)(m-2)\tilde u^{m-3} z^3 \big\} \qquad (5.20)\\
& +\frac{m}{2}(c-\lambda)\rho_1' \Big[\int_{\mathbb{R}} a_\varepsilon (\tilde u^{m-1})_x z^2 + \frac{1}{3}(m-1)\int_{\mathbb{R}} a_\varepsilon (\tilde u^{m-2})_x z^3 \Big] \qquad (5.21)\\
& +(1+\rho_1') \int_{\mathbb{R}} z \big\{ S[\tilde u]_{xx} - cS[\tilde u] + m a_\varepsilon \tilde u^{m-1} S[\tilde u] \big\} \\
& +(1+\rho_1') \int_{\mathbb{R}} a_\varepsilon \big\{ (\tilde u + z)^m - \tilde u^m - m\tilde u^{m-1} z \big\} S[\tilde u] \\
& +\frac{\varepsilon}{2} c' \int_{\mathbb{R}} z^2 - \int_{\mathbb{R}} a_\varepsilon (\tilde u_t + (c-\lambda)\tilde u_x)[(\tilde u + z)^m - \tilde u^m - m\tilde u^{m-1} z] \\
& +\frac{m}{2}(c-\lambda) \Big[\int_{\mathbb{R}} a_\varepsilon (\tilde u^{m-1})_x z^2 + \frac{1}{3}(m-1)\int_{\mathbb{R}} a_\varepsilon (\tilde u^{m-2})_x z^3 \Big]. \qquad (5.22)
\end{aligned}$$

Note that (5.19), (5.21) and (5.22) disappear. With (5.18) and (5.20), we need a little more care. Indeed, for $m = 3$,

$$|(5.18) + (5.20)| = |\frac{1}{4}\varepsilon(c-\lambda)(1+\rho_1') \int_{\mathbb{R}} a'(\varepsilon x) z^4| \leq \varepsilon \|z(t)\|^4_{L^2(\mathbb{R})};$$

In the case $m = 4$,

$$\begin{aligned}
(5.18) + (5.20) &= (c-\lambda)(1+\rho_1') \int_{\mathbb{R}} a_\varepsilon [z_x(4\tilde u z^3 + z^4) + \tilde u_x z^4] \\
&= -\varepsilon(c-\lambda)(1+\rho_1') \int_{\mathbb{R}} a'(\varepsilon x)(\tilde u z^4 + z^5).
\end{aligned}$$

Consequently we have

$$|(5.18) + (5.20)| \leq K\varepsilon e^{-\gamma\varepsilon|t|} \|z(t)\|^4_{L^2(\mathbb{R})} + K\varepsilon \|z(t)\|^5_{L^2(\mathbb{R})}.$$

Finally, using (4.29), (4.32), (4.30) we obtain

$$\begin{aligned}
\mathcal{F}'(t) \leq {} & K\varepsilon \|z(t)\|^4_{H^1(\mathbb{R})} + K\varepsilon e^{-\gamma\varepsilon|t|} \|z(t)\|^2_{L^2(\mathbb{R})} + K\varepsilon \|z(t)\|^3_{H^1(\mathbb{R})} \\
& + K|\rho_1'(t)|\varepsilon e^{-\gamma\varepsilon|t|} \|z(t)\|_{H^1(\mathbb{R})} + K\|S[\tilde u](t)\|_{H^2(\mathbb{R})} \|z(t)\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Integrating and using (5.7), we obtain

$$\begin{aligned}
\mathcal{F}(t) - \mathcal{F}(-T_\varepsilon) \leq {} & K(K^*)^4 \varepsilon^{4\kappa - \frac{1}{100}} + K(K^*)^3 \varepsilon^{3\kappa - \frac{1}{100}} + KK^* \varepsilon^{2\kappa} \\
& + K\int_{-T_\varepsilon}^{t} \varepsilon e^{-\gamma\varepsilon|s|} \|z(s)\|^2_{H^1(\mathbb{R})} ds.
\end{aligned}$$

This finishes the proof. □

We are finally in position to show that $T^* < T_\varepsilon$ leads to a contradiction.

5.4 End of proof of Proposition 5.1

From Lemma 5.2, $\mathcal{F}(-T_\varepsilon) \leq Kc^{2\kappa}$, and from Lemmas 5.5, 5.4 and (5.14) we get

$$\begin{aligned}
\|z(t)\|^2_{L^2(\mathbb{R})} \quad &\leq \quad K|\int_{\mathbb{R}} zQ_c(y)|^2 + K\varepsilon^{2\kappa} + K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} + K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} \\
&\quad + KK^*\varepsilon^{2\kappa} + K\int_{-T_\varepsilon}^{t} \varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})}dt \\
&\leq K|\varepsilon^{\kappa} + K^*\varepsilon^{\frac{1}{2}+\kappa}e^{-\gamma\varepsilon|t|} + (K^*)^2\varepsilon^{2\kappa} + \|S[\tilde{u}]\|_{L^2(\mathbb{R})}|^2 + K\varepsilon^{2\kappa} \\
&\quad + K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} + K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa} \\
&\quad + K\int_{-T_\varepsilon}^{t} \varepsilon e^{-\gamma\varepsilon|s|}\|z(s)\|^2_{L^2(\mathbb{R})}ds \\
&\leq K\varepsilon^{2\kappa} + K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} + K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa} \\
&\quad + K\int_{-T_\varepsilon}^{t} \varepsilon e^{-\gamma\varepsilon|s|}\|z(s)\|^2_{L^2(\mathbb{R})}ds.
\end{aligned}$$

Using Gronwall's inequality (see e.g. [81]) we conclude that for some large constant $K > 0$, but independent of K^* and ε,

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq K\varepsilon^{2\kappa} + K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} + K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa}.$$

From this estimate and taking ε small, and K^* large enough, we obtain that for all $t \in [-T_\varepsilon, T^*]$,

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq \frac{1}{2}(K^*)^2\varepsilon^{2\kappa}.$$

This estimate contradicts the definition of T^*, and concludes the proof of Proposition 5.1. □

5.5 Proof of Theorem 4.1

Now we prove the main result of this section, which describes the core of interaction soliton-potential.

Proof of Theorem 4.1. Consider $u(t)$ the solution constructed in Theorem 3.1. We first compare $u(t)$ with the approximate solution $\tilde{u}(t)$ constructed in Proposition 4.6 at time $t = -T_\varepsilon$.

Behavior at $t = -T_\varepsilon$. From (3.3), Proposition 4.8 and more specifically (4.33) we have that

$$\|u(-T_\varepsilon) - \tilde{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10}.$$

Behavior at $t = T_\varepsilon$. Thanks to the above estimate and (4.28) we can invoke Proposition 5.1 with $\kappa := \frac{1}{2}$ to obtain the existence of $K_0, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\|u(T_\varepsilon + \rho_1(T_\varepsilon)) - \tilde{u}(T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{1/2}, \quad |\rho_1(T_\varepsilon)| \leq K_0\varepsilon^{-\frac{1}{2}-\frac{1}{100}} \leq \frac{T_\varepsilon}{100}.$$

Therefore from (4.34) and triangular inequality,

$$\|u(T_\varepsilon + \rho_1(T_\varepsilon)) - 2^{-1/(m-1)}Q_{c_\infty}(\cdot - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{1/2}.$$

(cf. also (4.5).) Finally note that $(1-\lambda)T_\varepsilon \leq \rho(T_\varepsilon) \leq (2c_\infty(\lambda) - \lambda - 1)T_\varepsilon$. This finishes the proof. □

Next step will be the study of long time properties, on the interval $[T_\varepsilon, +\infty)$.

6 Asymptotic for large times

6.1 Statement of the results

The purpose of this Section is to prove the asymptotic behavior of the solution $u(t)$ as described in Theorem 1.2. Recall the parameters λ_0 and $c_\infty(\lambda)$ from Theorems 1.1 and 4.1.

Theorem 6.1 (Stability and Asymptotic stability in H^1).

Suppose $m = 2, 4$ with $0 < \lambda \leq \lambda_0$; or $m = 3$ with $0 \leq \lambda \leq \lambda_0$. Let $0 < \beta < \frac{1}{2}(c_\infty(\lambda) - \lambda)$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ the following hold. Suppose that for some time $t_1 \geq \frac{1}{2}T_\varepsilon$ and $t_1 \leq X_0 \leq 2t_1$

$$\left\| u(t_1) - 2^{-1/(m-1)} Q_{c_\infty}(x - X_0) \right\|_{H^1(\mathbb{R})} \leq \varepsilon^{1/2}. \tag{6.1}$$

where $u(t)$ is a H^1-solution of (1.15). Then $u(t)$ is defined for every $t \geq t_1$ and there exists $K, c^+ > 0$ and a C^1-function $\rho_2(t)$ defined in $[t_1, +\infty)$ such that

1. Stability.

$$\sup_{t \geq t_1} \left\| u(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho_2(t)) \right\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}, \tag{6.2}$$

where

$$|\rho_2(t_1) - X_0| \leq K\varepsilon^{1/2}, \quad \text{and for all } t \geq t_1, \quad |\rho_2'(t) - c_\infty(\lambda) + \lambda| \leq K\varepsilon^{1/2}.$$

2. Asymptotic stability. *One has*

$$\lim_{t \to +\infty} \left\| u(t) - 2^{-1/(m-1)} Q_{c^+}(\cdot - \rho_2(t)) \right\|_{H^1(x > \beta t)} = 0. \tag{6.3}$$

In addition,

$$\lim_{t \to +\infty} \rho_2'(t) = c^+ - \lambda, \qquad |c^+ - c_\infty| \leq K\varepsilon^{1/2}. \tag{6.4}$$

Remark 6.1. We do not know if stability results are valid in the cases $m = 2, 4$ and $\lambda = 0$. In particular, note that the stability property as stated above is false if we have

$$\limsup_{t \to +\infty} \|u(t)\|_{L^2(\mathbb{R})} = +\infty.$$

Remark 6.2. Let us recall that for any $0 < \lambda < \lambda_0$ the asymptotic stability property (6.3) holds for any $\beta > -\lambda$, provided ε_0 small enough, however we will not pursue on this improvement.[3]

We shall split the proof in two different parts, according with the proof of stability (cf. (6.2)) and asymptotic stability (cf. (6.3)).

The proof of the stability result is standard and similar to Proposition 5.1, see also [5, 59]. For this reason, our proof will be in some sense very sketchy. We invite to the reader to consult the references above mentioned for the original proof. Concerning the asymptotic stability result, the proof will follow closely the papers [55, 52].

[3] In [66] we made use of this property.

Let us recall that for large times ($t \geq T_\varepsilon$) the soliton-like solution is expected to be far away from the region where a_ε varies. In particular, from (1.13), the stability and asymptotic stability properties will follow from the fact that in this region (1.13) behaves like the gKdV equation

$$u_t + (u_{xx} - \lambda u + 2u^m)_x = 0, \quad \text{in } \{t \geq T_\varepsilon\} \times \mathbb{R}_x.$$

Of course, this formal argument must be stated in a rigorous way.

6.2 Stability

Proof of Theorem 6.1, stability part. Let us prove (6.2). Let us assume that for some $K > 0$ fixed,

$$\|u(t_1) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - X_0)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}. \tag{6.5}$$

From the local and global Cauchy theory exposed in Proposition 2.1 and Theorems 3.1 and 4.1, we know that the solution u is well defined for all $t \geq t_1$.

In order to simplify the calculations, note that from (1.18) the function $v := 2^{1/(m-1)}u$ solves

$$v_t + (v_{xx} - \lambda v + \frac{a_\varepsilon}{2} v^m)_x = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x,$$

and (6.5) now becomes

$$\|v(t_1) - Q_{c_\infty}(\cdot - X_0)\|_{H^1(\mathbb{R})} \leq \tilde{K}\varepsilon^{1/2}. \tag{6.6}$$

With a slight abuse of notation we will rename $v := u$ and $\tilde{K} := K$, and we will assume the validity of (6.6) for u. The parameters X_0 and c_∞ remains unchanged.

Let $D_0 > 2K$ be a large number to be chosen later, and set

$$\begin{aligned} T^* \;:=\; \sup\Big\{ & t \geq t_1 \mid \forall\, t' \in [t_1, t),\ \exists\, \tilde{\rho}_2(t') \in \mathbb{R} \text{ smooth s.t. } |\tilde{\rho}_2'(t') - c_\infty + \lambda| \leq \frac{1}{100}, \\ & |\tilde{\rho}_2(t_1) - X_0| \leq \frac{1}{100}, \text{ and } \|u(t') - Q_{c_\infty}(\cdot - \tilde{\rho}_2(t'))\|_{H^1(\mathbb{R})} \leq D_0\varepsilon^{1/2} \Big\} \end{aligned} \tag{6.7}$$

Observe that $T^* > t_1$ is well-defined since $D_0 > 2K$, (6.5) and the continuity of $t \mapsto u(t)$ in $H^1(\mathbb{R})$. The objective is to prove $T^* = +\infty$, and thus (6.2). Therefore, for the sake of contradiction, in what follows **we shall suppose** $T^* < +\infty$.

The first step to reach a contradiction is now to decompose the solution on $[t_1, T^*]$ using modulation theory around the soliton. In particular, we will find a special $\rho_2(t)$ satisfying the hypotheses in (6.7) but with

$$\sup_{t \in [t_1, T^*]} \|u(t) - Q_{c_\infty}(\cdot - \rho_2(t))\|_{H^1(\mathbb{R})} \leq \frac{1}{2} D_0 \varepsilon^{1/2}, \tag{6.8}$$

a contradiction with the definition of T^*.

Lemma 6.2 (Modulated decomposition)**.**

For $\varepsilon > 0$ small enough, independent of T^*, there exist C^1 functions ρ_2, c_2, defined on $[t_1, T^*]$, with $c_2(t) > 0$ and such that the function $z(t)$ given by

$$z(t,x) := u(t,x) - R(t,x), \tag{6.9}$$

where $R(t,x) := Q_{c_2(t)}(x - \rho_2(t))$, satisfies for all $t \in [t_1, T^*]$,

$$\int_{\mathbb{R}} R(t,x) z(t,x)dx = \int_{\mathbb{R}} (x - \rho_2(t)) R(t,x) z(t,x) dx = 0, \ (\textit{Orthogonality}), \tag{6.10}$$

$$\|z(t)\|_{H^1(\mathbb{R})} + |c_2(t) - c_\infty| \leq K D_0 \varepsilon^{1/2}, \ \textit{and} \tag{6.11}$$

$$\|z(t_1)\|_{H^1(\mathbb{R})} + |\rho_2(t_1) - X_0| + |c_2(t_1) - c_\infty| \leq K\varepsilon^{1/2}, \tag{6.12}$$

where K is not depending on D_0. In addition, $z(t)$ now satisfies the following modified gKdV equation

$$
\begin{aligned}
z_t + \{z_{xx} - \lambda z + \frac{a_\varepsilon}{2}[(R+z)^m - R^m] + (\frac{a_\varepsilon(x)}{2} - 1)Q_{c_2}^m\}_x \\
+ c_2'(t)\Lambda Q_{c_2} + (c_2 - \lambda - \rho_2')(t)Q_{c_2}' = 0.
\end{aligned} \quad (6.13)
$$

Furthermore, for some constant $\gamma > 0$ independent of ε, we have the improved estimates:

$$
\begin{aligned}
|\rho_2'(t) + \lambda - c_2(t)| \quad \leq \quad & K(m-3)\Big[\int_{\mathbb{R}} e^{-\gamma|x-\rho_2(t)|}z^2(t,x)dx\Big]^{\frac{1}{2}} \\
& + K\int_{\mathbb{R}} e^{-\gamma|x-\rho_2(t)|}z^2(t,x)dx + Ke^{-\gamma\varepsilon t};
\end{aligned} \quad (6.14)
$$

and

$$
\frac{|c_2'(t)|}{c_2(t)} \leq K\int_{\mathbb{R}} e^{-\gamma|x-\rho_2(t)|}z^2(t,x)dx + Ke^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})} + K\varepsilon e^{-\varepsilon\gamma t}. \quad (6.15)
$$

Remark 6.3. Note that from (6.11) and taking ε small enough we have an improved the bound on $\rho_2(t)$. Indeed, for all $t \in [t_1, T^*]$,

$$
|\rho_2'(t) - c_\infty + \lambda| + |\rho_2(t_1) - X_0| \leq 2D_0\varepsilon^{1/2}.
$$

Thus, in order to reach a contradiction, we only need to show (6.8).

Proof of Lemma 6.2. As in Lemma B.4 and 5.2, the proof of (6.9)-(6.12) are based in a Implicit Function Theorem application, and is very similar to the proof of Lemma A.1 in appendix A of [52].

On the other hand, equation (6.13) is a simple computation, completely similar to (B.11) and (5.6).

Now we claim that from the definition of T^* we can obtain an extra estimate on the parameter $\rho_2(t)$. We claim that for any $t \geq t_1$,

$$
\rho_2(t) \geq \frac{1}{10}(c_\infty(\lambda) - \lambda)t_1. \quad (6.16)
$$

Indeed, from (6.7) and after integration between t_1 and $t \in [t_1, T^*]$ we have the bound

$$
|\rho_2(t) - \rho_2(t_1) - (c_\infty - \lambda)(t - t_1)| \leq \frac{1}{100}(t - t_1), \quad |\rho_2(t_1) - X_0| \leq \frac{1}{100}.
$$

Thus we have

$$
|\rho_2(t) - (c_\infty - \lambda)t| \leq \frac{1}{100}(t - t_1 + 1) + |(c_\infty - \lambda)t_1 - X_0|.
$$

In particular, for any $t \in [t_1, T^*]$ (recall that $\rho_2(t_1) \sim X_0 > 0$)

$$
\rho_2(t) \geq (c_\infty - \lambda)t - \frac{1}{100}(t - t_1 + 1) \geq \frac{1}{10}c_\infty t.
$$

This inequality implies that the soliton position is far away from the potential interaction region.

Now we prove the estimates in (6.14) and (6.15). For this, first denote $y := x - \rho_2(t)$. Taking time derivative in the first orthogonality condition in (6.10) and using the equation (6.13) we obtain

$$
\begin{aligned}
0 \quad = \quad & -c_2'(t)\int_{\mathbb{R}} \Lambda Q_{c_2}(Q_{c_2} - z) + (c_2 - \lambda - \rho_2')(t)\int_{\mathbb{R}} Q_{c_2}'z - \frac{1}{2}\int_{\mathbb{R}} Q_{c_2}^m[(a_\varepsilon - 2)z]_x \\
& - \frac{\varepsilon}{2(m+1)}\int_{\mathbb{R}} a'(\varepsilon x)Q_{c_2}^{m+1}(y) + \frac{1}{2}\int_{\mathbb{R}} Q_{c_2}'a_\varepsilon[(R+z)^m - R^m - mR^{m-1}z]..
\end{aligned}
$$

First of all, note that by scaling arguments

$$\int_{\mathbb{R}} \Lambda Q_{c_2} Q_{c_2} = \theta c_2^{2\theta-1}(t) \int_{\mathbb{R}} Q^2. \tag{6.17}$$

Secondly, by redefining γ if necessary,

$$|\varepsilon \int_{\mathbb{R}} a'(\varepsilon x) Q_{c_2}^{m+1}(y)| \leq K\varepsilon e^{-\gamma\varepsilon c_2(t)\rho_2(t)} \leq K\varepsilon e^{-\gamma\varepsilon t}.$$

Similarly, from (6.16) and following (B.13) we have

$$|\int_{\mathbb{R}} Q_{c_2}^m [(a_\varepsilon - 2)z]_x| \leq K\|z(t)\|_{H^1(\mathbb{R})} e^{-\gamma\varepsilon t}.$$

Finally, note that for $\gamma > 0$ independent of ε,

$$|\int_{\mathbb{R}} Q'_{c_2} a_\varepsilon [(R+z)^m - R^m - mR^{m-1}z]| \leq K \int_{\mathbb{R}} e^{-\gamma|y|} z^2.$$

Collecting the above estimates, we have

$$\begin{aligned} \frac{|c_2'(t)|}{c_2(t)} \leq\ & K \int_{\mathbb{R}} e^{-\gamma|y|} z^2 + K|c_2(t) - \lambda - \rho_2'(t)| \Big[\int_{\mathbb{R}} e^{-\gamma|y|} z^2 \Big]^{\frac{1}{2}} \\ & + K e^{-\gamma\varepsilon t} \|z(t)\|_{H^1(\mathbb{R})} + K\varepsilon e^{-\gamma\varepsilon t}. \end{aligned} \tag{6.18}$$

On the other hand, by using the second orthogonality condition in (6.10), we have

$$\begin{aligned} 0 =\ & (c_2 - \lambda - \rho_2')(t) \int_{\mathbb{R}} z(yR)_x + c_2'(t) \int_{\mathbb{R}} y\Lambda Q_{c_2} z + \frac{1}{2}(c_2 - \lambda - \rho_2')(t) \int_{\mathbb{R}} Q_{c_2}^2 \\ & + \int_{\mathbb{R}} (yR)_x \Big\{ \frac{1}{2} a_\varepsilon [(R+z)^m - R^m - mR^{m-1}z] + (\frac{a_\varepsilon(x)}{2} - 1) Q_{c_2}^m \Big\} \\ & + \int_{\mathbb{R}} (yR)_x (z_{xx} - c_2 z + mR^{m-1}z) + \frac{m}{2} \int_{\mathbb{R}} (yR)_x (a_\varepsilon - 2) R^{m-1} z. \end{aligned}$$

Note that by integration by parts,

$$\int_{\mathbb{R}} (yR)_x (z_{xx} - c_2 z + mR^{m-1}z) = \int_{\mathbb{R}} z(2R + (m-3)R^m) = (m-3) \int_{\mathbb{R}} zR^m.$$

Using the same arguments as in the precedent computations, we have

$$\begin{aligned} |(c_2 - \lambda - \rho_2')(t)| \leq\ & K(m-3)(1 + \frac{|c_2'(t)|}{c_2(t)}) \Big[\int_{\mathbb{R}} z^2 e^{-\gamma|y|} \Big]^{\frac{1}{2}} \\ & + K \int_{\mathbb{R}} z^2 e^{-\gamma|y|} + |\int_{\mathbb{R}} Q_{c_2}^m(y)(a_\varepsilon - 2)|. \end{aligned}$$

From (6.16) and following (B.13) we have

$$|\int_{\mathbb{R}} Q_{c_2}^m(y)(a_\varepsilon - 2)| \leq K e^{-\gamma\varepsilon t}.$$

Putting together (6.18) and the last estimates, we finally obtain the bounds in (6.11), and further we obtain (6.14) and (6.15), as desired. □

6.2.1 Almost conserved quantities and monotonicity

We continue with a complete analogous proof to Proposition B.1 from Section 3. Recall from (2.8) the definition of the modified mass $\tilde{M}$.

Lemma 6.3 (Almost conservation of modified mass and energy)**.**

Consider $\tilde{M} = \tilde{M}[R]$ *and* $E_a = E_a[R]$ *the modified mass and energy of the soliton* R *(cf. (6.9)). Then for all* $t \in [t_1, T^*]$ *we have*

$$\tilde{M}[R](t) = \frac{1}{2}c_2^{2\theta}(t)\int_{\mathbb{R}} Q^2 + O(e^{-\varepsilon\gamma t}); \tag{6.19}$$

$$E_a[R](t) = \frac{1}{2}c_2^{2\theta}(t)(\lambda - \lambda_0 c_2(t))\int_{\mathbb{R}} Q^2 + O(e^{-\varepsilon\gamma t}). \tag{6.20}$$

Furthermore, we have the bound

$$\begin{aligned}|E_a[R](t_1) - E_a[R](t) + (c_2(t_1) - \lambda)(\tilde{M}[R](t_1) - \tilde{M}[R](t))|\\ \leq K|\Big[\frac{c_2(t)}{c_2(t_1)}\Big]^{2\theta} - 1|^2 + Ke^{-\varepsilon\gamma t_1}.\end{aligned} \tag{6.21}$$

Proof. We start by showing the first identity, namely (6.19). We consider the case $m = 2, 4$, the case $m = 3$ being easier. First of all, note that from (2.8),

$$\tilde{M}[R](t) = \hat{M}[R](t) = \frac{1}{2}\int_{\mathbb{R}}\big(\frac{a_\varepsilon}{2}\big)^{1/m}R^2 = \frac{1}{2}c_2^{2\theta}(t)\int_{\mathbb{R}} Q^2 + \frac{1}{2}\int_{\mathbb{R}}\big[\big(\frac{a_\varepsilon(x)}{2}\big)^{1/m} - 1\big]R^2.$$

From (6.16)-(6.17) and following the calculations in (B.13),

$$|\int_{\mathbb{R}}(a_\varepsilon^{1/m}(x) - 2^{1/m})R^2| \leq Ke^{-\gamma\varepsilon t},$$

for some constants $K, \gamma > 0$. Now we consider (6.20). Here we have

$$\begin{aligned} E_a[R](t) &= \frac{1}{2}\int_{\mathbb{R}} R_x^2 + \frac{\lambda}{2}\int_{\mathbb{R}} R^2 - \frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon R^{m+1}\\ &= c_2^{2\theta}(t)\Big[c_2(t)\big(\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1}\big) + \frac{\lambda}{2}\int_{\mathbb{R}} Q^2\Big] + \frac{1}{m+1}\int_{\mathbb{R}}(1 - \frac{a_\varepsilon}{2})R^{m+1}. \end{aligned}$$

Similarly to a recent computation, we have

$$|\int_{\mathbb{R}}(2 - a_\varepsilon(x))R^{m+1}| \leq Ke^{-\gamma\varepsilon t},$$

for some constants $K, \gamma > 0$. On the other hand, from Appendix G we have that $\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1} = -\frac{\lambda_0}{2}\int_{\mathbb{R}} Q^2$, $\lambda_0 = \frac{5-m}{m+3}$, and thus

$$E_a[R](t) = \frac{1}{2}c_2^{2\theta}(t)(\lambda - \lambda_0 c_2(t))\int_{\mathbb{R}} Q^2 + O(e^{-\gamma\varepsilon t}).$$

Adding both identities we have

$$E_a[R](t) + (c_2(t_1) - \lambda)\hat{M}[R](t) = c_2^{2\theta}(t)(c_2(t_1) - \lambda_0 c_2(t))M[Q] + O(e^{-\varepsilon\gamma t}).$$

In particular,

$$\begin{aligned}&E_a[R](t_1) - E_a[R](t) + (c_2(t_1) - \lambda)(\hat{M}[R](t_1) - \hat{M}[R](t)) =\\ &= \lambda_0 M[Q]\Big[c_2^{2\theta+1}(t) - c_2^{2\theta+1}(t_1) - \frac{c_2(t_1)}{\lambda_0}[c_2^{2\theta}(t) - c_2^{2\theta}(t_1)]\Big] + O(e^{-\varepsilon\gamma t_1}).\end{aligned}$$

To obtain the last estimate (6.21) we perform a Taylor development up to the second order (around $y = y_0$) of the function $g(y) := y^{\frac{2\theta+1}{2\theta}}$; and where $y := c_2^{2\theta}(t)$ and $y_0 := c_2^{2\theta}(t_1)$. Note that $\frac{2\theta+1}{2\theta} = \frac{1}{\lambda_0}$ and $y_0^{1/2\theta} = c_2(t_1)$. The conclusion follows at once. □

In order to establish some stability properties for the function $u(t)$ we recall the mass $\tilde{M}[u]$ introduced in (2.8). We have that for $m = 3$ and $0 \leq \lambda \leq \lambda_0$; and for $m = 2, 4$ and $0 < \lambda \leq \lambda_0$,

$$\tilde{M}[u](t) - \tilde{M}[u](t_1) \leq 0. \tag{6.22}$$

for any $t \in [t_1, T^*]$. This result is a consequence of Remark 2.1.

Now our objective is to estimate the quadratic term involved in (6.21). Following [59], we should use a "mass conservation" identity. However, since the mass is not conserved, estimate (6.22) is not enough to obtain a satisfactory estimate. In order to avoid this problem, we shall introduce a virial-type identity.

6.2.2 Virial estimate

First, we define some auxiliary functions. Let $\phi \in C(\mathbb{R})$ be an *even* function satisfying the following properties

$$\begin{cases} \phi' \leq 0 \ \text{ on } [0, +\infty); \quad \phi(x) = 1 \ \text{ on } [0, 1], \\ \phi(x) = e^{-x} \ \text{ on } [2, +\infty) \quad \text{and} \quad e^{-x} \leq \phi(x) \leq 3e^{-x} \ \text{ on } [0, +\infty). \end{cases} \tag{6.23}$$

Now, set $\psi(x) := \int_0^x \phi$. It is clear that ψ an odd function. Moreover, for $|x| \geq 2$,

$$\psi(+\infty) - \psi(|x|) = e^{-|x|}. \tag{6.24}$$

Finally, for $A > 0$, denote

$$\psi_A(x) := A(\psi(+\infty) + \psi(\frac{x}{A})) > 0; \quad e^{-|x|/A} \leq \psi_A'(x) \leq 3e^{-|x|/A}. \tag{6.25}$$

Note that $\lim_{x \to -\infty} \psi(x) = 0$. We are now in condition of state the following

Lemma 6.4 (Virial-type estimate)**.**

There exist $K, A_0, \delta_0 > 0$ such that for all $t \in [t_1, T^*]$ and for some $\gamma = \gamma(c_\infty, A_0) > 0$,

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} z^2(t,x) \psi_{A_0}(x - \rho_2(t)) \leq \\ &\leq -\delta_0 \int_{\mathbb{R}} (z_x^2 + z^2)(t,x) e^{-\frac{1}{A_0}|x - \rho_2(t)|} + KA_0 \|z(t)\|_{H^1(\mathbb{R})} e^{-\gamma\varepsilon t}. \end{aligned} \tag{6.26}$$

Proof. See Appendix E. □

From Lemma 6.4 we can improve the estimate (6.21) to obtain

Corollary 6.5 (Quadratic control on the variation of $c_2(t)$)**.**

$$\begin{aligned} &|E_a[R](t_1) - E_a[R](t) + (c_2(t_1) - \lambda)(\tilde{M}[R](t_1) - \tilde{M}[R](t))| \\ &\quad \leq K\|z(t)\|_{H^1(\mathbb{R})}^4 + K\|z(t_1)\|_{H^1(\mathbb{R})}^4 + Ke^{-\varepsilon\gamma t_1}. \end{aligned} \tag{6.27}$$

Proof. From (6.15) and taking A_0 large enough (but fixed and independent of ε) in Lemma 6.4, we have after an integration of (6.26) that

$$|c_2(t) - c_2(t_1)| \leq KA_0\|z(t)\|_{L^2(\mathbb{R})}^2 + KA_0\|z(t_1)\|_{L^2(\mathbb{R})}^2 + KA_0 D_0 \varepsilon^{-1/2} e^{-\gamma\varepsilon t_1}.$$

Plugin this estimate in (6.21) and taking γ even smaller, we get the conclusion. □

6.2.3 Energy estimates

Let us now introduce the second order functional

$$\mathcal{F}_2(t) \;:=\; \frac{1}{2}\int_{\mathbb{R}} \Big\{ z_x^2 + [\lambda + (c_2(t_1)-\lambda)(\frac{a_\varepsilon}{2})^{1/m}]z^2 \Big\} \\ -\frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon[(R+z)^{m+1} - R^{m+1} - (m+1)R^m z].$$

This functional, related to the Weinstein functional, have the following properties.

Lemma 6.6 (Energy expansion)**.**

Consider $E_a[u]$ *and* $\tilde{M}[u]$ *the energy and mass defined in (1.21)-(2.8). Then we have for all* $t \in [t_1, T^*]$,

$$E_a[u](t) + (c_2(t_1)-\lambda)\tilde{M}[u](t) \;=\; E_a[R] + (c_2(t_1)-\lambda)\tilde{M}[R] + \mathcal{F}_2(t) \\ + O(e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).$$

Proof. Using the orthogonality condition (6.10), we have

$$E_a[u](t) \;=\; E_a[R] - \int_{\mathbb{R}} z(a_\varepsilon - 2)R^m + \frac{1}{2}\int_{\mathbb{R}} z_x^2 + \frac{\lambda}{2}\int_{\mathbb{R}} z^2 \\ -\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon[(R+z)^{m+1} - R^{m+1} - (m+1)R^m z].$$

Moreover, following (B.13), we easily get

$$|\int_{\mathbb{R}} z(a_\varepsilon - 2)R^m| \le Ke^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}.$$

Similarly,

$$\hat{M}[u](t) = \hat{M}[R] + \hat{M}[z] + \int_{\mathbb{R}}((\frac{a_\varepsilon}{2})^{1/m} - 1)Rz = \hat{M}[R] + \hat{M}[z] + O(e^{-\varepsilon\gamma t}\|z(t)\|_{H^1(\mathbb{R})}).$$

Collecting the above estimates, we have

$$E_a[u](t) + (c_2(t_1)-\lambda)\tilde{M}[u](t) = \\ E_a[R] + (c_2(t_1)-\lambda)\tilde{M}[R] + \frac{1}{2}\int_{\mathbb{R}} \{ z_x^2 + [(c_2(t_1)-\lambda)(\frac{a_\varepsilon}{2})^{1/m} + \lambda]z^2 \} \\ -\frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon[(R+z)^{m+1} - R^{m+1} - (m+1)R^m z] + O(e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).$$

This concludes the proof. □

Lemma 6.7 (Modified coercivity for $\mathcal{F}_2$)**.**

There exists $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the following hold. There exist* $K, \tilde{\lambda}_0 > 0$, *independent of* K^* *such that for every* $t \in [t_1, T^*]$

$$\mathcal{F}_2(t) \ge \tilde{\lambda}_0\|z(t)\|^2_{H^1(\mathbb{R})} - K\varepsilon e^{-\gamma\varepsilon t}\|z(t)\|^2_{L^2(\mathbb{R})} + O(\|z(t)\|^3_{L^2(\mathbb{R})}). \tag{6.28}$$

Proof. First of all, note that

$$\mathcal{F}_2(t) \;=\; \frac{1}{2}\int_{\mathbb{R}} \{ z_x^2 + [(c_2(t_1)-\lambda)(\frac{a_\varepsilon}{2})^{1/m} + \lambda]z^2 \} \\ -\frac{m}{2}\int_{\mathbb{R}} Q_{c_2}^{m-1}z^2 + O(\|z(t)\|^3_{H^1(\mathbb{R})}) + O(e^{-\gamma\varepsilon t}\|z(t)\|^2_{H^1(\mathbb{R})}).$$

Now take $R_0 > 0$ independent of ε, to be fixed later. Consider the function $\phi_{R_0}(t,x) := \phi((x-\rho_2(t))/R_0)$, where ϕ is defined in (6.23). We split the analysis according to the decomposition $1 = \phi_{R_0} + (1-\phi_{R_0})$. Inside the region $|x-\rho_2(t)| \leq R_0$, we have

$$2 - a_\varepsilon(x) \leq Ke^{-\gamma\varepsilon|x|} \leq Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}.$$

This last estimate is a consequence of (1.13). Outside this region, we have $\phi_{R_0} \geq e^{-R_0}$. We have then

$$\int_{\mathbb{R}} \phi_{R_0}[(c_2(t_1)-\lambda)(\frac{a_\varepsilon}{2})^{1/m} + \lambda]z^2 \geq [c_2(t_1) - Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}]\int_{\mathbb{R}} \phi_{R_0} z^2;$$

for some fixed constants $K, \gamma > 0$.

On the other hand, $|(1-\phi_{R_0})Q_{c_2}| \leq Ke^{-\gamma R_0}$, and thus

$$\begin{aligned}\int_{\mathbb{R}}(1-\phi_{R_0})[(c_2(t_1)-\lambda)(\frac{a_\varepsilon}{2})^{1/m} + \lambda]z^2 - \frac{m}{2}\int_{\mathbb{R}}(1-\phi_{R_0})Q_{c_2}^{m-1}z^2 \\ \geq [(c_2(t_1)-\lambda)(\frac{1}{2})^{1/m} + \lambda - Ke^{-\gamma R_0}]\int_{\mathbb{R}}(1-\phi_{R_0})z^2,\end{aligned} \tag{6.29}$$

for some fixed $K, \gamma > 0$. Taking $R_0 = R_0(m,\lambda)$ large enough, we have

$$(6.29) \geq \frac{1}{2^{1/m}}c_2(t_1)\int_{\mathbb{R}}(1-\phi_{R_0})z^2.$$

Therefore,

$$\begin{aligned}\mathcal{F}_2(t) \quad \geq \quad & \frac{1}{2}\int_{\mathbb{R}}\phi_{R_0}\{z_x^2 + c_2(t_1)z^2 - mQ_{c_2}^{m-1}z^2\} + \frac{1}{2}\int_{\mathbb{R}}(1-\phi_{R_0})\{z_x^2 + \frac{1}{2^{1/m}}c_2(t_1)z^2\} \\ & -Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}\int_{\mathbb{R}}\phi_{R_0}z^2 + O(\|z(t)\|_{H^1(\mathbb{R})}^3) + O(e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}^2).\end{aligned}$$

Taking R_0 even large if necessary (but independent of ε), and using a localization argument as in [56], we obtain that there exists $\tilde{\lambda}_0 > 0$ such that

$$\begin{aligned}\mathcal{F}_2(t) \quad \geq \quad & \tilde{\lambda}_0\int_{\mathbb{R}}(z_x^2 + z^2) - Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}\int_{\mathbb{R}}\phi_{R_0}z^2 + O(\|z(t)\|_{H^1(\mathbb{R})}^3) \\ & + O(e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}^2).\end{aligned}$$

Finally, taking ε_0 smaller if necessary, we have

$$\mathcal{F}_2(t) \geq \tilde{\lambda}_0\int_{\mathbb{R}}(z_x^2 + z^2) + O(\|z(t)\|_{H^1(\mathbb{R})}^3) + O(e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}^2),$$

for a new constant $\tilde{\lambda}_0 > 0$. □

6.2.4 Conclusion of the proof

Now we prove that our assumption $T^* < +\infty$ leads inevitably to a contradiction. Indeed, from Lemmas 6.6 and 6.7, we have for all $t \in [t_1, T^*]$ and for some constant $K > 0$,

$$\begin{aligned}\|z(t)\|_{H^1(\mathbb{R})}^2 \quad \leq \quad & K\mathcal{F}_2(t_1) + E_a[u](t) - E_a[u](t_1) + (c_2(t_1)-\lambda)[\tilde{M}[u](t) - \tilde{M}[u](t_1)] \\ & + E_a[R](t_1) - E_a[R](t) + (c_2(t_1)-\lambda)[\tilde{M}[R](t_1) - \tilde{M}[R](t)] \\ & + K\varepsilon\sup_{t\in[t_1,T^*]} e^{-\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})} + K\sup_{t\in[t_1,T^*]}\|z(t)\|_{L^2(\mathbb{R})}^3.\end{aligned}$$

From Lemmas 6.2 and 6.3, Corollary 6.5 and the energy conservation we have

$$\begin{aligned}\|z(t)\|^2_{H^1(\mathbb{R})} \quad \leq \quad & K\varepsilon + (c_2(t_1) - \lambda)[\tilde{M}[u](t) - \tilde{M}[u](t_1)] \\ & +K \sup_{t\in[t_1,T^*]} \|z(t)\|^4_{H^1(\mathbb{R})} + Ke^{-\varepsilon\gamma t_1}(1 + D_0\varepsilon^{1/2}) + KD_0^3\varepsilon^{3/2}.\end{aligned}$$

Finally, from (6.22) we have $\tilde{M}[u](t) - \tilde{M}[u](t_1) \leq 0$. Collecting the preceding estimates we have for $\varepsilon > 0$ small and $D_0 = D_0(K)$ large enough

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq \frac{1}{4}D_0^2\varepsilon,$$

which contradicts the definition of T^*. The conclusion is that

$$\sup_{t\geq t_1} \left\|u(t) - 2^{-1/(m-1)}Q_{c_2(t)}(\cdot - \rho_2(t))\right\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}.$$

Using (6.11), we finally get (6.2). This finishes the proof. □

6.3 Asymptotic stability

Now we prove (6.3) in Theorem 6.1.

Proof of Theorem 6.1, Asymptotic stability part. We continue with the notation introduced in the proof of the stability property (6.2). We have to show the existence of $K, c^+ > 0$ such that

$$\lim_{t\to+\infty} \|u(t) - Q_{c^+}(\cdot - \rho_2(t))\|_{H^1(x>\frac{1}{10}c_\infty t)} = 0; \quad |c_\infty - c^+| \leq K\varepsilon^{1/2}.$$

From the stability result above stated it is easy to check that the decomposition proved in Lemma 6.2 and all its conclusions hold **for any time** $t \geq t_1$.

6.3.1 Monotonicity for mass and energy

The next step in the proof is to prove some *monotonicity formulae* for local mass and energy.

Let $K_0 > 0$ and

$$\phi(x) := \frac{2}{\pi}\arctan(e^{x/K_0}). \tag{6.30}$$

It is clear that $\lim_{x\to+\infty}\phi(x) = 1$ and $\lim_{x\to-\infty}\phi(x) = 0$. In addition, $\phi(-x) = 1 - \phi(x)$, for all $x \in \mathbb{R}$, and

$$0 < \phi'(x) = \frac{2}{\pi K_0}\frac{e^{x/K}}{1 + e^{2x/K_0}}; \qquad \phi^{(3)}(x) \leq \frac{1}{K_0^2}\phi'(x).$$

Moreover, we have $1 - \phi(x) \leq Ke^{-x/K_0}$ as $x \to +\infty$, and $\phi(x) \leq Ke^{x/K_0}$ as $x \to -\infty$.

Let $\sigma, x_0 > 0$. We define, for $t, t_0 \geq t_1$, and $\tilde{y}(x_0) := x - (\rho_2(t_0) + \sigma(t - t_0) + x_0)$,

$$I_{x_0,t_0}(t) := \int_{\mathbb{R}} u^2(t,x)\phi(\tilde{y}(x_0))dx, \qquad \tilde{I}_{x_0,t_0}(t) := \int_{\mathbb{R}} u^2(t,x)\phi(\tilde{y}(-x_0))dx, \tag{6.31}$$

and

$$J_{x_0,t_0} := \int_{\mathbb{R}}[u_x^2 + u^2 - \frac{2a_\varepsilon}{m+1}u^{m+1}](t,x)\phi(\tilde{y}(x_0))dx.$$

Lemma 6.8 (Monotonicity formulae)**.**

Suppose $0 < \sigma < \frac{1}{2}(c_\infty(\lambda) - \lambda)$ *and* $K_0 > \sqrt{\frac{2}{\sigma}}$. *There exists* $K, \varepsilon_0 > 0$ *small enough such that for all* $0 < \varepsilon < \varepsilon_0$ *and for all* $t, t_0 \geq t_1$ *with* $t_0 \geq t$ *we have*

$$I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \leq K\big[e^{-x_0/K_0} + \varepsilon^{-1}e^{-\gamma\varepsilon T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0}\big]. \tag{6.32}$$

On the other hand, if $t \geq t_0$ *and* $\rho_2(t_0) \geq t_1 + x_0$,

$$\tilde{I}_{x_0,t_0}(t) - \tilde{I}_{x_0,t_0}(t_0) \leq K\big[e^{-x_0/K_0} + \varepsilon^{-1}e^{-\varepsilon\gamma\rho_2(t_0)}e^{\varepsilon\gamma x_0/K_0}\big], \tag{6.33}$$

and finally if $t_0 \geq t$,

$$J_{x_0,t_0}(t_0) - J_{x_0,t_0}(t) \leq K\big[e^{-x_0/K_0} + \varepsilon^{-1}e^{-\gamma\varepsilon T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0}\big]. \tag{6.34}$$

Proof. For the sake of brevity we prove this Lemma in Appendix F. □

6.3.2 Conclusion of the proof

Now we finally sketch the proof of the asymptotic stability theorem, namely (6.3). Consider $0 < \varepsilon < \varepsilon_0$ and $u(t)$ satisfying (6.1). From Lemma 6.2, we can decompose $u(t)$ for all $t \geq t_1$ such that $u(t,x) = 2^{-1/(m-1)}Q_{c_2(t)}(x - \rho_2(t)) + z(t,x)$, where z satisfies (6.10), (6.11), (6.12), (6.14) and (6.15). We claim that there exists $K = K(D_0) > 0$ such that

$$\int_{t_1}^{+\infty}\int_{\mathbb{R}}(z_x^2 + z)(t,x)e^{-\frac{1}{A_0}|x-\rho_2(t)|} \leq K(D_0)\varepsilon. \tag{6.35}$$

This last estimate is a simple consequence of Lemma 6.4 and an integration in time.

Now we claim that

$$c^+ := \lim_{t\to+\infty} c_2(t) < +\infty, \quad \text{and} \quad |c^+ - c_\infty| \leq K\varepsilon^{1/2}. \tag{6.36}$$

In fact, note that from (6.35) there exists a sequence $t_n \uparrow +\infty$, $t_n \in [n, n+1)$ such that

$$\lim_{n\to+\infty}\int_{\mathbb{R}}(z_x^2 + z)(t_n,x)e^{-\frac{1}{A_0}|x-\rho_2(t_n)|} = 0. \tag{6.37}$$

From this and (6.14)-(6.15), and taking $A_0 > 0$ large such that $\frac{1}{A_0} < \gamma$, we get

$$|c_2'(t)| \leq K\int_{\mathbb{R}} z^2(t,x)e^{-\frac{1}{A_0}|x-\rho_2(t)|} + Ke^{-\gamma\varepsilon t}.$$

This inequality combined with (6.35) and (6.12) allow us to conclude (6.36). Note that this proves the first part of (6.4).

The next step is to prove that

$$\limsup_{t\to+\infty}\int_{\mathbb{R}}(z_x^2 + z^2)(t, x + \rho_2(t))\phi(x - x_0) \leq Ke^{-x_0/2K_0} + K\varepsilon^{-1}e^{-\varepsilon\gamma T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0}.$$

First of all, note that the above assertion follows directly from the decay properties of R and the estimate

$$\limsup_{t\to+\infty}\int_{\mathbb{R}}(u_x^2 + u^2)(t, x + \rho_2(t))\phi(x - x_0) \leq Ke^{-x_0/2K_0} + K\varepsilon^{-1}e^{-\varepsilon\gamma T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0}. \tag{6.38}$$

So we are now reduced to prove this last estimate. We start from (6.34): we have for $t_0 \geq t_1$,

$$J_{x_0,t_0}(t_0) \leq J_{x_0,t_0}(t_1) + Ke^{-x_0/K_0} + K\varepsilon^{-1}e^{-\varepsilon\gamma T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0}.$$

From the equivalence between the energy and H^1-norm (we are in a subcritical case), we have

$$\begin{aligned}\int_{\mathbb{R}}(u_x^2+u^2)(t_0,x+\rho_2(t_0))\phi(x-x_0) &\leq K\int_{\mathbb{R}}(u_x^2+u^2)(t_1,x+\rho_2(t_1))\phi(x-y_0)\\ &\quad +Ke^{-x_0/2K}+K\varepsilon^{-1}e^{-\varepsilon\gamma T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0},\end{aligned}$$

where $y_0 := \rho_2(t_0) - \rho_2(t_1) + \sigma(t_1 - t_0) + x_0$. Now we send $t_0 \to +\infty$ noticing that $y_0 \to +\infty$. This gives (6.38), as desired.

The next step in the proof is to prove that

$$\lim_{n\to+\infty}\int_{\mathbb{R}}(z_x^2+z^2)(t_n,x)\phi(x-\rho_2(t_n)+x_0)dx = 0. \tag{6.39}$$

where $(t_n)_{n\in\mathbb{N}}$ is the sequence from (6.39). Indeed, note that for any $x_1 > 0$,

$$\begin{aligned}\int_{\mathbb{R}}(z_x^2+z^2)(t_n,x+\rho_2(t_n))\phi(x+x_0) &\leq K(e^{\frac{x_0}{A_0}}+e^{\frac{x_1}{A_0}})\int_{\mathbb{R}}(z_x^2+z^2)(t_n,x+\rho_2(t_n))e^{-\frac{|x|}{A_0}}\\ &\quad +K\int_{\mathbb{R}}(z_x^2+z^2)(t_n,x+\rho_2(t_n))\phi(x-x_1).\end{aligned}$$

Thus, using (6.39) we are able to take in the above inequality the limit $n \to +\infty$ with x_0, x_1 fixed. Next, we send $x_1 \to +\infty$ to obtain the conclusion.

We finally prove that the above result holds for any sequence $t_n \to +\infty$. Let $\beta < c_\infty(\lambda) - \lambda$ to be fixed. We want to prove that for ε small enough,

$$\lim_{t\to+\infty}\int_{\mathbb{R}}(z_x^2+z^2)(t,x)\phi(x-\beta t)dx = 0.$$

First, we claim that for any $t_2, t_3 > t_1$ with $t_2 < t_3$ and $\rho_2(t_2) > x_0 + t_1$, we have

$$\int_{\mathbb{R}}u^2(t_3,x)\phi(x-y_3)dx \leq \int_{\mathbb{R}}u^2(t_2,x)\phi(x-y_2)dx + Ke^{-x_0/K_0} + K\varepsilon^{-1}e^{-\gamma\varepsilon\rho_2(t_2)}e^{\gamma\varepsilon x_0/K_0}, \tag{6.40}$$

where $y_3 := \rho_2(t_2) + \frac{1}{2}\beta(t_3 - t_2) - x_0$ and $y_2 := \rho_2(t_2) - x_0$. In fact, the left hand side of the above inequality corresponds to $\tilde{I}_{x_0,t_2}(t_3)$ and the right one is $\tilde{I}_{x_0,t_2}(t_2)$, with $\sigma := \frac{1}{2}\beta$ (cf. (6.31) for the definitions). Thus the above inequality is a consequence of Lemma 6.8, more specifically of (6.33).

Now the rest of the proof is similar to [55]. Since $\int_{\mathbb{R}} z(t, x + \rho_2(t))R(x) = 0$, we have

$$\Big|\int_{\mathbb{R}}z(t,x+\rho_2(t))R(x)\phi(x+x_0)\Big| \leq K\varepsilon^{1/2}e^{-x_0/2K_0}$$

Second, we use the decomposition $u(t,x) = 2^{-1/(m-1)}Q_{c_2(t)}(x - \rho_2(t)) + z(t,x)$ in (6.40) to get

$$\begin{aligned}\int_{\mathbb{R}}z^2(t_3,x)\phi(x-y_3)dx &\leq \int_{\mathbb{R}}z^2(t_2,x)\phi(x-y_2)dx + Ke^{-x_0/2K_0}\\ &\quad +K\varepsilon^{-1}e^{-\gamma\varepsilon\rho_2(t_2)}e^{\gamma\varepsilon x_0/K_0} + K|c_2(t_2)-c_2(t_3)|.\end{aligned} \tag{6.41}$$

Third, consider $t > t_1$ large, and define $t' \in (t_1, t)$ such that $\beta t := \rho_2(t') + \frac{\beta}{2}(t - t') - x_0$. Note that $t' \to +\infty$ as $t \to +\infty$. Since $t_n \in [n, n+1)$ there exists $n = n(t)$ such that $0 < t - t_n \leq 2$, and then

$$\beta t := \rho_2(t_n) + \frac{\beta}{2}(t - t_n) - \tilde{x}_0, \qquad \text{with } |\tilde{x}_0 - x_0| \leq 10.$$

Now we apply (6.41) between $t_3 = t$ and $t_2 = t_n$. We get

$$\begin{aligned}\int_{\mathbb{R}} z^2(t,x)\phi(x - \beta t)dx \quad &\leq \quad \int_{\mathbb{R}} z^2(t_n, x)\phi(x - \rho_2(t_n) + \tilde{x}_0)dx + Ke^{-x_0/2K_0} \\ &\quad + K\varepsilon^{-1}e^{-\gamma\varepsilon\rho_2(t_n)}e^{\gamma\varepsilon x_0/K_0} + K|c_2(t) - c_2(t_n)|.\end{aligned}$$

Since $n(t) \to +\infty$ as $t \to +\infty$, by (6.39) and (6.36) we obtain

$$\limsup_{t \to +\infty} \int_{\mathbb{R}} z^2(t,x)\phi(x - \beta t)dx \leq Ke^{-x_0/2K_0},$$

and since x_0 is arbitrary (because of $\lim_{t\to+\infty} \rho_2(t_n) = +\infty$), we get the desired result. The same result is still valid for z_x. We have

$$\limsup_{t \to +\infty} \int_{\mathbb{R}} z_x^2(t,x)\phi(x - \beta t)dx \leq Ke^{-x_0/2K_0}.$$

Finally, let $w^+(t,x) := u(t,x) - 2^{-1/(m-1)}Q_{c^+}(x - \rho_2(t)) = z(t,x) + 2^{-1/(m-1)}[Q_{c_2(t)}(x - \rho_2(t)) - Q_{c^+}(x - \rho_2(t))]$. From (6.36) and the above result we finally obtain (6.3). □

7 Proof of the Main Theorems

In this section we prove the Main Theorems of this work, namely Theorems 1.1, 1.2 and 1.3.

For the proof of Theorem 1.1, we essentially combine Theorems 3.1, and 4.1 to produce the global solution $u(t)$ with the required properties. This method had also been employed in [53, 58, 64]. The proof of Theorem 1.2 is a consequence of Theorem 6.1. Finally, Theorem 1.3 will require several additional arguments, in particular the fundamental Lemma 7.5.

7.1 Proof of Theorem 1.1

From Theorem 3.1 there exists a solution u of (1.15) satisfying $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ and (3.1). This solution also satisfies, from (3.3),

$$\|u(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10},$$

for ε small enough. In addition, u is unique in the cases $\lambda > 0$ and $m = 2, 4$; and $m = 3$, $\lambda \geq 0$. This proves the part (1) in Theorem 1.1.

Next, we invoke Theorem 4.1 to obtain part (2) in Theorem 1.1. In particular, we have (4.2) and (4.3). We define $\tilde{T}_\varepsilon := T_\varepsilon + \rho_1(T_\varepsilon)$, and $\rho_\varepsilon := \rho(T_\varepsilon)$. Then (1.27) - (1.28) are straightforward.

7.2 Proof of Theorem 1.2

The proof of Theorem 1.2 is a consequence of Theorem 6.1. Indeed, suppose $m = 2, 3, 4$ with $\lambda > 0$ for $m = 2, 4$. Define $t_1 := T_\varepsilon + \rho_1(T_\varepsilon)$ and $X_0 := \rho(T_\varepsilon)$. Then, from the above estimates and Theorem 6.1 we have *stability* and *asymptotic stability* at infinity. In other words, there exists a constant $c^+ > 0$ and a C^1 function $\rho_2(t) \in \mathbb{R}$ such that

$$w^+(t) := u(t) - 2^{-1/(m-1)} Q_{c^+}(\cdot - \rho_2(t))$$

satisfies (6.2) and (6.3). This proves (1.29) and (1.30).

We finally prove (1.32) and (1.33). From the energy conservation, we have for all $t \geq t_1$,

$$E_a[u](-\infty) = E_a\big[2^{-1/(m-1)} Q_{c^+}(\cdot - \rho_2(t)) + w^+(t)\big]$$

In particular, from (6.3) and Appendix G.1 we have as $t \to +\infty$

$$(\lambda - \lambda_0) M[Q] = \frac{(c^+)^{2\theta}}{2^{2/(m-1)}} (\lambda - \lambda_0 c^+) M[Q] + E^+. \tag{7.1}$$

From this identity $E^+ := \lim_{t\to+\infty} E_a[w^+](t)$ is well defined. This proves (1.32). To deal with (1.33), note that from the stability result (6.2) and the Morrey embedding we have that for any $\lambda > 0$

$$\begin{aligned} E[w^+](t) &= \frac{1}{2}\int_{\mathbb{R}} (w^+_x)^2(t) + \frac{\lambda}{2}\int_{\mathbb{R}} (w^+)^2(t) - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon (w^+)^{m+1}(t) \\ &\geq \frac{1}{2}\int_{\mathbb{R}} (w^+_x)^2(t) + \frac{\lambda}{2}\int_{\mathbb{R}} (w^+)^2(t) - K\varepsilon^{(m-1)/2}\int_{\mathbb{R}} a_\varepsilon (w^+)^2(t) \\ &\geq \mu \|w^+(t)\|^2_{H^1(\mathbb{R})} \end{aligned}$$

for some $\mu = \mu(\lambda) > 0$. Passing to the limit we obtain (1.33).

Now we prove the bound (1.34). First, the treat the cubic case with $\lambda = 0$. Here, from (7.1) we have

$$E^+ = \lambda_0 \Big(\frac{(c^+)^{3/2}}{2^{2/(m-1)}} - 1\Big) M[Q].$$

Since in this case we have $2^{2/(m-1)} = 2 = c_\infty^{3/2}$, $M[Q] = 2$ and $\lambda_0 = 1/3$, we obtain $\frac{3}{2}E^+ = \big(\frac{c^+}{c_\infty}\big)^{3/2} - 1$.

Now we deal with the case $\lambda > 0$. First of all, note that after an algebraic manipulation the equation for c_∞ in (4.17) can be written in the following form:

$$\frac{c_\infty^{2\theta}}{2^{2/(m-1)}} (\lambda_0 c_\infty - \lambda) M[Q] = (\lambda_0 - \lambda) M[Q].$$

On the other hand, note that from (7.1) and (1.33) we have

$$\mu \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq \frac{(c^+)^{2\theta}}{2^{2/(m-1)}} (\lambda_0 c^+ - \lambda) M[Q] - (\lambda_0 - \lambda) M[Q].$$

Putting together both estimates, we get

$$\tilde\mu \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq (c^+)^{2\theta+1} - c_\infty^{2\theta+1} - \frac{\lambda}{\lambda_0}((c^+)^{2\theta} - c_\infty^{2\theta}),$$

for some $\tilde\mu > 0$. Using a similar argument as in Lemma 6.3 we have

$$\tilde\mu \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq \frac{1}{\lambda_0}(c_\infty - \lambda)((c^+)^{2\theta} - c_\infty^{2\theta}) + O(|(c^+)^{2\theta} - c_\infty^{2\theta}|^2).$$

From this inequality and the bound $|c^+ - c_\infty| \leq K\varepsilon$ we get

$$\big(\frac{c^+}{c_\infty}\big)^{2\theta} - 1 \geq \tilde{\mu} \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})},$$

as desired.

7.3 Proof of Theorem 1.3

In this section we prove that there is no pure soliton at infinity. To obtain this result we will use a contradiction argument, together with a monotonicity formula which provides polynomial decay of the solution and L^1-integrability, something contradictory with the change of scaling of the soliton.

Proof of Theorem 1.3. By contradiction, we suppose that (1.35) is false. In particular,

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} = 0.$$

First of all, note that from the above limit and the sub-criticality we have $E^+ = 0$. Therefore, by using (7.1), and after some basic algebraic manipulations we have that c^+ must satisfy the following algebraic equation (compare with (4.17)):

$$(c^+)^{\lambda_0}(c^+ - \frac{\lambda}{\lambda_0})^{1-\lambda_0} = 2^p(1 - \frac{\lambda}{\lambda_0})^{1-\lambda_0}.$$

This relation and the uniqueness of c_∞ gives

$$c^+ = c_\infty(\lambda). \tag{7.2}$$

In other words, the soliton-solution is *pure* (cf. Definition 1.1).

Let us consider now the decomposition result for $u(t)$ from Lemma 6.2. We claim that $z(t)$ also vanishes at infinity. Indeed, from Lemma 6.2, the fact that for $t \geq t_1$

$$u(t) = R(t) + z(t) = w^+(t) + 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho_2(t)),$$

and estimates (6.11)-(6.36), we have

$$\lim_{t\to+\infty} \|z(t)\|_{H^1(\mathbb{R})} = 0, \tag{7.3}$$

and

$$u(t, \cdot + \rho_2(t)) \to 2^{-1/(m-1)} Q_{c_\infty} \text{ in } H^1(\mathbb{R}) \text{ as } t \to +\infty, \qquad \lim_{t\to+\infty} \rho_2'(t) - (c_\infty(\lambda) - \lambda) = 0.$$

In order to prove the following results, we need a simple but important result.

Lemma 7.1 (Monotonicity of mass backwards in time)**.**

Suppose $u(t)$ solution of (1.15) constructed in Theorem 3.1, satisfying (6.2) and (6.3). Define

$$\mathcal{M}[u](t) := \int_{\mathbb{R}} \frac{u^2(t,x)}{a_\varepsilon(x)} dx. \tag{7.4}$$

Then, under the additional hypothesis $\lambda > 0$ for $m = 2, 3, 4$, we have that for all $t, t' \geq t_1$, with $t' \geq t$,

$$\mathcal{M}[u](t) - \mathcal{M}[u](t') \leq K e^{-\varepsilon\gamma t}. \tag{7.5}$$

Proof. First of all, a simple computation tell us that the time derivative of $\mathcal{M}[u](t)$ is given by

$$\partial_t \int_{\mathbb{R}} \frac{u^2}{a_\varepsilon} = 2\varepsilon \int_{\mathbb{R}} u_x^2 \frac{a_\varepsilon'}{a_\varepsilon^2} + \varepsilon \int_{\mathbb{R}} u^2 \big[\lambda \frac{a_\varepsilon'}{a_\varepsilon^2} - \varepsilon^2 (\frac{a_\varepsilon'}{a_\varepsilon^2})''\big] - 2\varepsilon \int_{\mathbb{R}} \frac{a_\varepsilon'}{a_\varepsilon} u^{m+1}.$$

Replacing the decomposition $u = R + z$ given by Lemma 6.2, assumption (1.14), and using similar estimates to (B.13), and the smallness of $\|z(t)\|_{H^1(\mathbb{R})}$, we get

$$\partial_t \mathcal{M}[u](t) \geq -K\varepsilon e^{-\varepsilon\gamma t},$$

for some $K,\ \gamma > 0$. The final conclusion is direct after integration. □

Remark 7.1. Note that estimate (7.5) in Lemma 7.4 is valid under the additional assumption $0 < \lambda \leq \lambda_0$. This extra hypothesis unfortunately does not hold for the case $m = 3$, $\lambda = 0$.

The last result allows to prove a new version of Theorem 3.1, for positive times.

Proposition 7.2 (Backward uniqueness)**.**

Suppose $m = 2, 3, 4$. Let $\beta \in \mathbb{R}$ and $0 < \lambda \leq \lambda_0$. There exist constants $K, \gamma, \varepsilon_0 > 0$ and a unique solution $v = v_\beta \in C([\frac{1}{2}T_\varepsilon, +\infty), H^1(\mathbb{R}))$ of (1.15) such that

$$\lim_{t\to+\infty} \big\|v(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - (c_\infty(\lambda) - \lambda)t - \beta)\big\|_{H^1(\mathbb{R})} = 0. \tag{7.6}$$

Furthermore, for all $t \geq \frac{1}{2}T_\varepsilon$ and $s \geq 1$ the function $v(t)$ satisfies

$$\big\|v(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - (c_\infty(\lambda) - \lambda)t - \beta)\big\|_{H^s(\mathbb{R})} \leq K\varepsilon^{-1} e^{-\varepsilon\gamma t}. \tag{7.7}$$

Finally, suppose that there exists $\tilde v(t) \in H^1(\mathbb{R})$ solution of (1.15) such that

$$\lim_{t\to+\infty} \|\tilde v(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho_2(t))\|_{H^1(\mathbb{R})} = 0. \tag{7.8}$$

Then $\tilde v \equiv v_\beta$ for some $\beta \in \mathbb{R}$.

Proof. Given $\beta \in \mathbb{R}$, the proof of existence and uniqueness of the solution v_β satisfying (7.6) and (7.7) is identical the proof of Theorem 3.1 in Section 3 and Appendix B. Indeed, first we construct a sequence of functions v_n as in (B.1) for times $t \sim T_n$. Next, we prove a decomposition lemma as in Lemma B.4. This decomposition allows to prove a version of (7.5) for $\mathcal{M}[v_n](t)$. The main difference is given in estimates (B.14)-(B.15), where now we introduce the modified mass $\mathcal{M}[v_n](t)$ defined in (7.4). The energy functional in B.18 is now given by $E_a[v_n](t) + (c_\infty(\lambda) - \lambda)\mathcal{M}[v_n](t)$. The rest of the proof, including the uniqueness, adapts *mutatis mutandis*.

Now consider $\tilde v$ a solution of (1.15) satisfying (7.8). Using monotonicity arguments, similar to the proof of Lemma B.5, we have the existence of $\beta \in \mathbb{R}$ such that

$$\big\|\tilde v(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - (c_\infty(\lambda) - \lambda)t - \beta)\big\|_{H^1(\mathbb{R})} \leq K\varepsilon^{-1} e^{-\varepsilon\gamma t},$$

for some $K, \gamma > 0$. This implies that there exists $\beta \in \mathbb{R}$ such that $\tilde v$ satisfies (7.6). The conclusion follows from the uniqueness of $v(t)$. □

As a consequence of this result together with (7.3), the solution $u(t)$ constructed in Theorem 3.1 satisfies the following exponential decay at infinity: there exist $K, \gamma > 0$ and $\beta \in \mathbb{R}$ such that, for all $t \geq t_1$, if $\tilde\rho_2(t) := (c_\infty(\lambda) - \lambda)t + \beta$, then

$$\tilde z(t) := u(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \tilde\rho_2(t)), \quad \text{satisfies} \quad \|\tilde z(t)\|_{H^2(\mathbb{R})} \leq K\varepsilon^{-1} e^{-\varepsilon\gamma t}. \tag{7.9}$$

Now we prove that this strong H^1-convergence gives rise to strange localization properties.

Lemma 7.3 (L^2-exponential decay on the left the soliton solution)**.**

There exist $K, \tilde{x}_0 > 0$ large enough such that for all $t \geq T_0$ and for all $x_0 \geq \tilde{x}_0$

$$\|u(t, \cdot + \tilde{\rho}_2(t))\|^2_{L^2(x \leq -x_0)} \leq K e^{-x_0/K}. \tag{7.10}$$

Proof. Suppose $x_0 > 0$, $t, t_0 \geq t_1$ and $\sigma > 0$ from (6.3). Consider the modified mass

$$\tilde{I}_{t_0,x_0}(t) := \frac{1}{2}\int_{\mathbb{R}} \frac{u^2(t,x)}{a_\varepsilon(x)}(1-\phi(y))dx,$$

with $y := x - (\tilde{\rho}_2(t_0) + \sigma(t-t_0) - x_0)$ and ϕ defined in (6.30). For this quantity we claim that for $x_0 > \tilde{x}_0$ and for all $t \geq t_0$,

$$\tilde{I}_{t_0,x_0}(t_0) - \tilde{I}_{t_0,x_0}(t) \leq K e^{-x_0/K}(1 + e^{-\frac{1}{2}\sigma(t-t_0)/K}). \tag{7.11}$$

Let us assume this result for a moment. After sending $t \to +\infty$ and using (6.3), we have $\lim_{t\to+\infty} \tilde{I}_{t_0,x_0}(t) = 0$ and thus

$$\tilde{I}_{t_0,x_0}(t_0) \leq K e^{-x_0/K}.$$

From this last estimate (7.10) is a direct consequence of the fact that $t_0 \geq t_1$ is arbitrary.

Finally, let us prove (7.11). A direct calculation tell us that

$$\begin{aligned}
\frac{1}{2}\partial_t \int_{\mathbb{R}} \frac{(1-\phi(y))}{a_\varepsilon} u^2 \quad = \quad & \frac{3}{2}\int_{\mathbb{R}} \frac{\phi'}{a_\varepsilon} u_x^2 + \frac{3}{2}\varepsilon \int_{\mathbb{R}} \frac{a_\varepsilon'}{a_\varepsilon^2}(1-\phi)u_x^2 - \frac{m}{m+1}\int_{\mathbb{R}} \phi' u^{m+1} \\
& + \frac{1}{2}\int_{\mathbb{R}} u^2 \big[(\sigma+\lambda)\frac{\phi'}{a_\varepsilon} - \frac{\phi^{(3)}}{a_\varepsilon} + 3\varepsilon\phi''\frac{a_\varepsilon'}{a_\varepsilon^2} + 3\varepsilon^2\phi'(\frac{a_\varepsilon'}{a_\varepsilon^2})'\big] \\
& + \frac{\varepsilon}{2}\int_{\mathbb{R}} u^2\big[\lambda\frac{a_\varepsilon'}{a_\varepsilon^2} - \varepsilon^2(\frac{a_\varepsilon'}{a_\varepsilon^2})''\big](1-\phi) - \varepsilon\int_{\mathbb{R}} \frac{a_\varepsilon'}{a_\varepsilon} u^{m+1}(1-\phi).
\end{aligned}$$

Using the decomposition (7.9), we have

$$|\int_{\mathbb{R}} \phi' u^{m+1}| \leq K\varepsilon^{(m-1)/2}\int_{\mathbb{R}} \phi' \tilde{z}^2 + K e^{-\frac{1}{2}\sigma(t-t_0)} e^{-x_0/K},$$

and

$$|\int_{\mathbb{R}} \frac{a_\varepsilon'}{a_\varepsilon} u^{m+1}(1-\phi)| \leq K e^{-\frac{1}{2}\sigma(t-t_0)} e^{-x_0/K} + K\varepsilon^{(m-1)/2}\int_{\mathbb{R}} \frac{a_\varepsilon'}{a_\varepsilon}\tilde{z}^2(1-\phi).$$

After these two estimates, it is easy to conclude that

$$\frac{1}{2}\partial_t \int_{\mathbb{R}} \frac{(1-\phi(y))}{a_\varepsilon} u^2 \geq -K e^{-\frac{1}{2}\sigma(t-t_0)} e^{-x_0/K}.$$

The conclusion follows after integration in time. □

The proof of decay on the right hand side of the soliton requires more care, and is valid under the assumption $\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} = 0$ and $\lambda > 0$. We do not expect to have exponential decay in a general situation, but for our purposes we only need a polynomial decay. The following result is due to Y. Martel.

Lemma 7.4 (L^2-polynomial decay on the right the soliton solution)**.**

There exist $K, \tilde{x}_0 > 0$ large enough but independent of ε, such that for all $t \geq T_0$ and for all $x_0 \geq \tilde{x}_0$

$$\int_{\mathbb{R}} (x-x_0)_+^2 \tilde{z}^2(t, x + \tilde{\rho}_2(t))dx \leq K,$$

where $x_+ := \max\{x, 0\}$.

Proof. Take $x_0 > 0$, $t_0, t \geq t_1$ and define

$$\hat{I}_{t_0,x_0}(t) := \int_{\mathbb{R}} \tilde{z}^2(t,x)\phi(\tilde{y})dx; \quad \tilde{y} := x - (\tilde{\rho}_2(t_0) + \tilde{\sigma}(t-t_0) + x_0),$$

and

$$\hat{J}_{t_0,x_0}(t) := \int_{\mathbb{R}} \tilde{z}_x^2(t,x)\phi(\tilde{y})dx.$$

Here ϕ is the cut-off function defined in (6.30), and $\tilde{\sigma}$ is a fixed constant satisfying $\tilde{\sigma} > 2(c_\infty(\lambda) - \lambda)$. First of all we claim that there exists $K > 0$ such that (for simplicity we omit the dependence if no confusion is present)

$$|\partial_t \hat{I}_{t_0,x_0}(t)| \leq K \int_{\mathbb{R}} (\tilde{z}_x^2 + \tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dx + K\|\tilde{z}(t)\|_{H^1(\mathbb{R})} e^{-\varepsilon(t-t_0)/K} e^{-\varepsilon x_0/K}, \tag{7.12}$$

and

$$|\partial_t \hat{J}_{t_0,x_0}(t)| \leq K \int_{\mathbb{R}} (\tilde{z}_{xx}^2 + \tilde{z}_x^2 + \tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dx + K\|\tilde{z}(t)\|_{H^2(\mathbb{R})} e^{-\varepsilon(t-t_0)/K} e^{-\varepsilon x_0/K}. \tag{7.13}$$

Indeed, these estimates are proved in the same way as in Lemma 6.4 and Appendix E. For the sake of brevity we skip the details.

From Proposition 7.2 and the exponential decay of z we have that both right-hand sides in (7.12)-(7.13) are integrable between t_0 and $+\infty$. We get

$$\hat{I}_{t_0,x_0}(t_0) \leq K \int_{t_0}^{+\infty}\int_{\mathbb{R}} (\tilde{z}_x^2 + \tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dxdt + K\varepsilon^{-1} \sup_{t\geq t_0} \|\tilde{z}(t)\|_{H^1(\mathbb{R})} e^{-\varepsilon x_0/K}. \tag{7.14}$$

In the same line, we have

$$\hat{J}_{t_0,x_0}(t_0) \leq K \int_{t_0}^{+\infty}\int_{\mathbb{R}} (\tilde{z}_{xx}^2 + \tilde{z}_x^2 + \tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dxdt + K\varepsilon^{-1} \sup_{t\geq t_0} \|\tilde{z}(t)\|_{H^2(\mathbb{R})} e^{-\varepsilon x_0/K}. \tag{7.15}$$

Note that both quantities above are **integrable** with respect to x_0.

Let us denote $\xi_0(\tilde{y}) := \phi(\tilde{y})$, and $\xi_j(\tilde{y}) := \int_{-\infty}^{\tilde{y}} \xi_{j-1}(s)ds$, for $j = 1, 2$. Recall that ξ_j are positive and increasing functions on $\mathbb{R}$, with $\xi_j(\tilde{y}) \to 0$ as $\tilde{y} \to -\infty$, and $\xi_j(\tilde{y}) - \tilde{y}^j \to 0$ as $\tilde{y} \to +\infty$. Integrating between x_0 and $+\infty$ in (7.14), and using Fubini's theorem we obtain

$$\int_{\mathbb{R}} \xi_1(\tilde{y}(t_0))\tilde{z}^2(t_0) \leq K \int_{t_0}^{+\infty}\int_{\mathbb{R}} (\tilde{z}_x^2 + \tilde{z}^2)[\xi_0 + \varepsilon a'(\varepsilon x)\xi_1] + K\varepsilon^{-2} \sup_{t\geq t_0} \|\tilde{z}(t)\|_{H^1(\mathbb{R})} e^{-\varepsilon x_0/K}, \tag{7.16}$$

and similarly, from (7.15)

$$\int_{\mathbb{R}} \xi_1(\tilde{y}(t_0))\tilde{z}_x^2(t_0,x)dx \leq K\varepsilon^{-3} e^{-2\varepsilon\gamma t_0} + K\varepsilon^{-2} \sup_{t\geq t_0} \|\tilde{z}(t)\|_{H^2(\mathbb{R})} e^{-\varepsilon x_0/K}. \tag{7.17}$$

In conclusion, thanks to the exponential decay of $\tilde{z}$ and (7.16)-(7.17), we have

$$\int_{t_0}^{+\infty}\int_{\mathbb{R}} \xi_1(x - \tilde{\rho}_2(t) - x_0)(\tilde{z}_x^2 + \tilde{z}^2)(t,x)dxdt < +\infty.$$

Furthermore, note that for all $t \geq t_0$ one has $\tilde{\rho}_2(t) \leq \tilde{\rho}_2(t_0) + \sigma(t - t_0)$. Thus we have

$$\int_{t_0}^{+\infty}\int_{\mathbb{R}} \xi_1(\tilde{y}(t))(\tilde{z}_x^2 + \tilde{z}^2)(t,x)dxdt < +\infty. \tag{7.18}$$

In addition, an easier calculation gives us

$$\int_{t_0}^{+\infty}\int_{\mathbb{R}} a'(\varepsilon x)\xi_2(\tilde{y}(t))(\tilde{z}_x^2+\tilde{z}^2)(t,x)dxdt < +\infty. \tag{7.19}$$

From (7.18) and (7.19), we can perform a second integration with respect to x_0 in (7.16) to obtain

$$\int_{\mathbb{R}} \xi_2(\tilde{y}(t_0))\tilde{z}^2(t_0,x)dx \leq K(\varepsilon),$$

uniformly for x_0 large. Since t_0 is arbitrary, this last estimate gives the conclusion. □

Lemma 7.5 (L^1-integrability and smallness).

Under the assumption (7.3) the following holds. There exists $K, T_0 > 0$ large enough such that for all $t \geq T_0$ one has $u(t, \cdot + \tilde{\rho}_2(t)) \in L^1(\mathbb{R})$. Moreover,

$$|\int_{\mathbb{R}} z(t)| \leq \frac{1}{100}. \tag{7.20}$$

Finally, from the L^1 conservation law (1.9), we have $u(t) \in L^1(\mathbb{R})$ for all $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}} u(t) = \int_{\mathbb{R}} Q. \tag{7.21}$$

Proof. Let $x_0 \geq \tilde{x}_0$ to be fixed below. First of all, note that if $|x| \geq x_0$ we have $2^{-1/(m-1)}Q_{c_\infty}(x) \leq Ke^{-\sqrt{c_\infty}|x|}$. In particular, since $\tilde{z}(t,x+\tilde{\rho}_2(t)) = u(t,x+\tilde{\rho}_2(t)) - 2^{-1/(m-1)}Q_{c_\infty}(x)$, by using Lemma 7.3 and the stability bound (6.2), in addition to a Galiardo-Nirenberg type inequality, we get

$$\begin{aligned}|\tilde{z}(t,x+\tilde{\rho}_2(t))| &\leq K\|\tilde{z}(t,\cdot+\tilde{\rho}_2(t))\|_{L^2(y\geq x)}^{\frac12}\|\tilde{z}_y(t,\cdot+\tilde{\rho}_2(t))\|_{L^2(\mathbb{R})}^{\frac12}\\ &\leq K\varepsilon^{1/4}e^{x/K},\end{aligned}$$

for all $x \leq -x_0$.

On the other hand, inside the interval $[-x_0, x_0]$ one has

$$\int_{[-x_0,x_0]} \tilde{z}(t,x+\tilde{\rho}_2(t)) \leq Kx_0^{1/2}\|\tilde{z}(t,x+\tilde{\rho}_2(t))\|_{L^2(\mathbb{R})}^{1/2} \leq Kx_0^{1/2}\varepsilon^{1/4}.$$

The case $x \geq x_0$ requires more care. From Lemma 7.4 and the Cauchy-Schwarz inequality, we have (for clarity we drop the dependence on $x+\tilde{\rho}_2(t)$)

$$|\int_{x\geq x_0} z(t)| \leq \frac{K}{(x_0-\tilde{x}_0)^{1/2-}}\big[\int_{x\geq x_0}(1+(x-\tilde{x}_0)^2)z^2(t)\big]^{1/2} \leq \frac{K}{x_0^{1/2-}},$$

for x_0 large enough, independent of ε. From the above estimates we finally obtain the smallness condition (7.20).

The final assertion, namely $u(t) \in L^1(\mathbb{R})$ for all $t \in \mathbb{R}$, is a consequence of Proposition 2.1. It is clear that from this last fact (1.9) remains constant for all time and (7.21) holds. The proof is now complete. □

7.3.1 Conclusion of the proof

From the above lemma we can use (7.21) to get the desired contradiction. Indeed, from (7.2) and Appendix G.1 we have

$$\lim_{t\to+\infty}\int_{\mathbb{R}} z(t) = [1-\frac{(c^+)^{\theta-\frac14}}{2^{1/(m-1)}}]\int_{\mathbb{R}} Q = (1-\kappa_m)\int_{\mathbb{R}} Q \neq 0; \quad \kappa_m := \frac{c_\infty^{\frac{3-m}{2(m-1)}}}{2^{1/(m-1)}},$$

a contradiction with (7.20). Indeed, for the cases $m = 3, 4$ we easily have $1-\kappa_m > \frac{1}{10}$. In the case $m = 2$ we have $\kappa_2 = \frac12 c_\infty^{1/2}$; but from (4.19) we know that $c_\infty \leq 2^{\frac43}$. Thus we have $1-\kappa_m > \frac{1}{10}$ for every m. In concluding,

$$|\lim_{t\to+\infty}\int_{\mathbb{R}} z(t)| \geq \frac{1}{10}\int_{\mathbb{R}} Q,$$

a contradiction with (7.20). This finishes the proof of (1.35). □

Appendices

A Proof of Proposition 2.1

In this section we deal with the proof of the local well-posedness result from Proposition 2.1. As above mentioned, the proof is basically an adaptation of the original proof from [39]. In particular, we will follow the standard notation used in that paper.

First of all, note that for any $\lambda \geq 0$ we have that $u = u(t,x)$ is a solution of (1.15) if and only if $\hat{u}(t,x) := u(t, x-\lambda t)$ is a corresponding solution of the following time-dependent coefficient gKdV equation

$$\hat{u}_t + (\hat{u}_{xx} + a_\varepsilon(x-\lambda t)\hat{u}^m)_x = 0, \qquad \text{on } \mathbb{R}_t \times \mathbb{R}_x.$$

In what follows, in order to profit of the Kenig-Ponce-Vega machinery, the local existence theorem will be proved for this last equation. Consequently, to avoid some useless notation, we will drop the hat on $\hat{u}$ and we will assume that a_ε means $a_\varepsilon(x-\lambda t)$. In addition, and without loss of generality, we may suppose $t_0 = 0$.

Let $W(t)$ be the unitary group generated by the linear Airy operator, namely $W(t) := e^{-t\partial_x^3}$. By using the Duhamel formula,

$$u(t) = \Phi(u)(t) := W(t)u_0 - \int_0^t W(t-s)[a_\varepsilon u^m]_x \, ds, \tag{A.1}$$

we will construct a unique solution of the fixed point problem $u = \Phi(u)$ in the ball (centered at 0 and of radius $a > 0$) $B_{X_T^s}(a)$, where, for $T > 0$ to be chosen,

$$X_T^s(a) := \{w \in C([-T,T], H^s(\mathbb{R})), \quad \Lambda_\rho^T(v) < a\}.$$

Here $\Lambda_\rho^T(\cdot)$, $\rho > \frac34$ is the specific Kenig-Ponce-Vega norm for the quadratic case, but with the corresponding adaptation to the $H^s(\mathbb{R})$-case, $s \geq 1$ (see (A.3) for the details).

Denote $v(t,x) := m^{\frac{1}{m}} a_\varepsilon^{\frac{1}{m}}(x-\lambda t)u(t,x)$, $m=2,3$ and 4, such that $(a_\varepsilon u^m)_x = v^{m-1}v_x$. From (4.11) in [39] we have that

$$\begin{aligned}\Lambda_\rho^T(\Phi(u)) &\leq K\|u_0\|_{H_x^s(\mathbb{R})} + K\int_{-T}^{T} \|v^{m-1}v_x\|_{H_x^s(\mathbb{R})}\\ &\leq K\|u_0\|_{H_x^s(\mathbb{R})} + K\|v\|^{m-2}_{L_T^\infty H_x^s(\mathbb{R})}\int_{-T}^{T}\|vv_x\|_{H_x^s(\mathbb{R})}\\ &\leq K\|u_0\|_{H_x^s(\mathbb{R})} + KT^{1/2}\|v\|^{m-2}_{L_T^\infty H_x^s(\mathbb{R})}\|vv_x\|_{L_T^2 H_x^s(\mathbb{R})}.\end{aligned}$$

The second inequality is an easy consequence of the following estimate. Suppose $f\in H^s(\mathbb{R})$, $s>\frac{1}{2}$. Then

$$\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})}\|f\|^{\frac{1}{2s}}_{\dot{H}^s(\mathbb{R})} \leq C\|f\|_{H^s(\mathbb{R})}. \tag{A.2}$$

On the other hand, note that it a standard calculation shows that

$$\|v(t)\|_{H_x^s(\mathbb{R})} \leq K(1+\varepsilon^s)\|u(t)\|_{H_x^s(\mathbb{R})},$$

so that the final conclusion comes from the following fact.

Claim 5. There exists $K>0$ such that for all u with $\Lambda_\rho^T(u)<+\infty$, we have

$$\|vv_x\|_{L_T^2 H_x^s(\mathbb{R})} \leq K(1+T)^\rho(\Lambda_\rho^T(u))^2.$$

Proof. From (4.11) in [39] we have

$$\Lambda_\rho^T(v) := \max\left\{\|v\|_{L_T^\infty H_x^s}, \|v_x\|_{L_T^4 L_x^\infty}, \|D_x^s v_x\|_{L_x^\infty L_T^2}, (1+T)^{-\rho}\|v\|_{L_x^2 L_T^\infty}\right\}. \tag{A.3}$$

Now, note that $vv_x \sim \varepsilon a_\varepsilon' u^2 + uu_x$, so that by using estimate (A.2) and the estimates in pp. 581 and 582 from [39], we get

$$\|vv_x\|_{L_T^2 H_x^s(\mathbb{R})} \leq K(1+T)^\rho(\Lambda_\rho^T(u))^2.$$

We are done. □

It is clear that from this Lemma and following the arguments from [39], the rest of the proof of the local well-posedness result is straightforward.

Now we prove the properties (1), (2) and (3). First, the blow-up alternative (2.3) is a standard consequence of the definition of maximal interval of existence. Next, the proof of the energy conservation and mass variation in (2), (3) is also a well-known fact. Indeed, after having $H^3(\mathbb{R})$ local well-posedness, a density argument allows to pass to the limit to obtain the result. See for example Appendix A in [18] for a complete proof.

Finally, the L^1-conservation law is a consequence of a density argument and a formal computation with $H^s(\mathbb{R})$ solutions, for s large. This finishes the proof.

B Proof of Theorem 3.1

In this section we sketch the proof of Theorem 3.1, for the complete proof, see [49].

Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ an increasing sequence with $T_n \geq \frac{1}{2}T_\varepsilon$ for all n and $\lim_{n\to+\infty} T_n = +\infty$. For notational simplicity we denote by $\tilde{T}_n$ the sequence $(1-\lambda)T_n$. Consider $u_n(t)$ the solution of the following Cauchy problem

$$\begin{cases} (u_n)_t + ((u_n)_{xx} - \lambda u_n + a_\varepsilon u_n^m)_x = 0, & \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ u_n(-T_n) = Q(\cdot - \tilde{T}_n). \end{cases} \tag{B.1}$$

In other words, u_n is a solution of (aKdV) that at time $t = -T_n$ corresponds to the soliton $Q(\cdot - \tilde{T}_n)$. It is clear that $Q(\cdot - \tilde{T}_n) \in H^s(\mathbb{R})$ for every $s \geq 0$; moreover, there exists a uniform constant $C = C(s) > 0$ such that

$$\|Q(\cdot - \tilde{T}_n)\|_{H^s(\mathbb{R})} \leq C.$$

According to Proposition 2.1 and Proposition 2.2, we have that u_n is locally well-defined in time, and global for positive times in $H^1(\mathbb{R})$. Let I_n be its maximal interval of existence.

Following [49], the next step is to establish uniform estimates starting from a fixed time $t = -\frac{1}{2}T_\varepsilon < 0$ large enough such that the soliton is sufficiently away from the region where the influence of the potential a_ε is present. This is the purpose of the following

Proposition B.1 (Uniform estimates in H^s for large times, see also [49])**.**

There exist constants $K, \gamma > 0$ and $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon < \varepsilon_0$ and for all $n \in \mathbb{N}$ we have

$$[-T_n, -\frac{1}{2}T_\varepsilon] \subseteq I_n, \quad (\textit{namely } u_n \in C([-T_n, -\frac{1}{2}T_\varepsilon], H^s(\mathbb{R}))),$$

and for all $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$,

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^s(\mathbb{R})} \leq K\varepsilon^{-1}e^{\gamma\varepsilon t}. \tag{B.2}$$

In particular, there exists a constant $C_s > 0$ such that for all $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$

$$\|u_n(t)\|_{H^s(\mathbb{R})} \leq C_s. \tag{B.3}$$

Using Proposition B.1 we will obtain the existence of a *critical element* $u_{0,*} \in H^s(\mathbb{R})$, with several good compact properties, non dispersive and uniformly close to the desired soliton.

Indeed, consider the sequence $(u_n(-\frac{1}{2}T_\varepsilon))_{n\in\mathbb{N}} \subseteq H^s(\mathbb{R})$. We claim the following result.

Lemma B.2 (Compactness property)**.**

Given any number $\delta > 0$, there exist $\varepsilon_0 > 0$ and a constant $K_0 > 0$ large enough such that for all $0 < \varepsilon < \varepsilon_0$ and for all $n \in \mathbb{N}$,

$$\int_{|x|>K_0} u_n^2(-\frac{1}{2}T_\varepsilon) < \delta. \tag{B.4}$$

Proof. The proof is by now a standard result. See [49] for the details. □

Let us come back to the proof of Theorem 3.1. From (B.3) we have that

$$\|u_n(-T_\varepsilon/2)\|_{H^1(\mathbb{R})} \leq C_0,$$

independent of n. Thus, up to a subsequence we may suppose $u_n(-\frac{1}{2}T_\varepsilon) \rightharpoonup u_{*,0}$ in the $H^1(\mathbb{R})$ weak sense, and $u_n(-\frac{1}{2}T_\varepsilon) \to u_{*,0}$ in $L^2_{loc}(\mathbb{R})$, as $n \to +\infty$. In addition, from (B.4) we have the strong convergence in $L^2(\mathbb{R})$. Moreover, from interpolation and the bound (B.3) we have the strong convergence in $H^s(\mathbb{R})$ for any $s \geq 1$.

Let $u_* = u_*(t)$ be the solution of (1.1) with initial data $u_*(-\frac{1}{2}T_\varepsilon) = u_{*,0}$. From Proposition 2.1 we have $u_* \in C(I, H^s(\mathbb{R}))$, where $-\frac{1}{2}T_\varepsilon \in I$, the corresponding maximal interval of existence. Thus, using the continuous dependence of u_n and u_*, we obtain $u_n(t) \to u_*(t)$ in $H^s(\mathbb{R})$ for every $t \leq -\frac{1}{2}T_\varepsilon \subseteq I$. Passing to the limit in (B.2) we obtain for all $t \leq -\frac{1}{2}T_\varepsilon$,

$$\|u_*(t) - Q(\cdot - (1-\lambda)t)\|_{H^s(\mathbb{R})} \leq K\varepsilon^{-1}e^{\varepsilon\gamma t},$$

as desired. This finish the proof of the existence part of Theorem 3.1.

B.1 Uniform H^1 estimates. Proof of Proposition B.1

In this paragraph we explain the main steps of the proof of Proposition B.1 in the H^1 case; for the general case the reader may consult [49].

The first step in the proof is the following bootstrap property:

Proposition B.3 (Uniform estimates with and without decay assumption)**.**

Let $m = 2, 3$ or 4, and $0 \leq \lambda \leq \lambda_0 < 1$. There exist constants $K, \gamma, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following is true.

1. *Suppose $m = 3$ or $m = 2, 4$ with $\lambda > 0$. Then there exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, if for some $-T_{n,*} \in [-T_n, -\frac{1}{2}T_\varepsilon]$ and for all $t \in [-T_n, -T_{n,*}]$ we have*

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \leq 2\alpha, \tag{B.5}$$

   then, for all $t \in [-T_n, -T_{n,*}]$

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{-1}e^{\varepsilon\gamma t}. \tag{B.6}$$

2. *Suppose now $m = 2, 4$ and $\lambda = 0$. Then the same conclusion (B.6) holds if for some $-T_{n,*} \in [-T_n, -\frac{1}{2}T_\varepsilon]$ and for all $t \in [-T_n, -T_{n,*}]$ one has*

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \leq 2K\varepsilon^{-1}e^{\varepsilon\gamma t}, \tag{B.7}$$

Proof of Proposition B.1, assuming the validity of Proposition B.3. We prove the first case, the second one being similar. Firstly note that from (B.1) we have

$$\|u_n(-T_n) - Q(-(1-\lambda)T_n)\|_{H^1(\mathbb{R})} = 0,$$

so there exists $t_0 = t_0(n, \alpha) > 0$ such that (B.5) holds true for all $t \in [-T_n, -T_n + t_0]$. Now let us consider (we adopt the convention $T_{*,n} > 0$)

$$-\tilde{T}_{*,n} := \sup\{t \in [-T_n, -\frac{1}{2}T_\varepsilon] \mid \text{ for all } t' \in [-T_n, t],\ \|u_n(t') - Q(\cdot - (1-\lambda)t')\|_{H^1(\mathbb{R})} \leq 2\alpha\}.$$

Assume, by contradiction, that $-\tilde{T}_{*,n} < -\frac{1}{2}T_\varepsilon$. From Proposition B.3, we have

$$\|u_n(t') - Q(\cdot - (1-\lambda)t')\|_{H^1(\mathbb{R})} \leq K\varepsilon^{-1}e^{\gamma\varepsilon t} \leq \alpha,$$

for ε small enough (recall that $t \leq -\frac{1}{2}T_\varepsilon = -\frac{1}{2(1-\lambda)}\varepsilon^{-1-\frac{1}{100}}$), a contradiction with the definition of $\tilde{T}_{*,n}$. □

Now we are reduced to prove Proposition B.3.

Proof of Proposition B.3. The first step in the proof is to decompose the solution preserving a standard orthogonality condition. To obtain this fact, and without loss of generality, by taking $T_{n,*}$ even large we may suppose that for all $t \in [-T_n, -T_{n,*}]$

$$\|u_n(t) - Q(\cdot - (1-\lambda)t - r_n(t))\|_{H^1(\mathbb{R})} \leq 2\alpha, \tag{B.8}$$

for all smooth $r_n = r_n(t)$ satisfying $r_n(-T_n) = 0$ and $|r_n'(t)| \leq \frac{1}{t^2}$. A posteriori we will prove that this condition can be improved and extended to any time $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$.

For notational simplicity, in what follows we will drop the index n on $-T_{*,n}$ and u_n, if no confusion is present.

Lemma B.4 (Modulation)**.**

There exist $K, \gamma, \varepsilon_0 > 0$ and a unique C^1 function $\rho_0 : [-T_n, -T_*] \to \mathbb{R}$ such that for all $0 < \varepsilon < \varepsilon_0$ the function z defined by

$$z(t,x) := u(t,x) - R(t,x); \quad R(t,x) := Q(x - (1-\lambda)t - \rho_0(t)) \tag{B.9}$$

satisfies for all $t \in [-T_n, -T_*]$,

$$\int_{\mathbb{R}} z(t,x) R_x(t,x)dx = 0, \quad \|z(t)\|_{H^1(\mathbb{R})} \leq K\alpha, \quad \rho_0(-T_n) = 0. \tag{B.10}$$

Moreover, z satisfies the following modified gKdV equation,

$$z_t + \big\{z_{xx} - \lambda z + a_\varepsilon[(R+z)^m - R^m] + (1-a_\varepsilon)R^m\big\}_x - \rho_0'(t)R_x = 0, \tag{B.11}$$

and

$$|\rho_0'(t)| \leq K\big[e^{\varepsilon\gamma t} + \|z(t)\|_{H^1(\mathbb{R})} + \|z(t)\|^2_{L^2(\mathbb{R})}\big]. \tag{B.12}$$

Proof of Lemma B.4. The proof of (B.10) is a standard consequence of the Implicit Function Theorem, the definition of T_* $(= T_{*,n})$, and the definition of $u_n(-T_n)$ given in (B.1), see for example [49] for a detailed proof. Similarly, the proof of (B.11) follows after a simple computation.

Now we deal with (B.12). Taking time derivative in (B.9) and using (B.11), we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}} z_t R_x - (1 - \lambda + \rho_0') \int_{\mathbb{R}} z R_{xx} \\ &= \int_{\mathbb{R}} \big\{z_{xx} - z + a_\varepsilon[(R+z)^m - R^m] + (1-a_\varepsilon)R^m\big\} R_{xx} + \rho_0' \int_{\mathbb{R}} R_x(R_x + z_x). \end{aligned}$$

First of all, note that

$$\int_{\mathbb{R}} R_x(R_x + z_x) = \int_{\mathbb{R}} Q'^2 + O(\|z(t)\|_{L^2(\mathbb{R})}).$$

On the other hand, from (1.13), (B.10), the uniform bound on $\rho_0'(t)$ in the definition of T_* and the exponential decay of R, we have

$$\Big|\int_{\mathbb{R}} (1-a_\varepsilon)R^m R_{xx}\Big| \leq Ke^{\varepsilon\gamma t}. \tag{B.13}$$

Indeed, first note that from (B.8), by integrating between $-T_n$ and t and using (B.10) we get

$$\rho_0(t) \leq -\frac{1}{T_n} - \frac{1}{t} \leq \frac{2}{T_\varepsilon} \leq K\varepsilon^{1+\frac{1}{100}}.$$

Thus $t+\rho_0(t) \leq t + K\varepsilon^{1+\frac{1}{100}} \leq \frac{9}{10}t$. Therefore, by possibly redefining γ, we have from (1.13),

$$\begin{aligned}|\int_{\mathbb{R}}(1-a_\varepsilon)R^m R_{xx}| &\leq K\int_{-\infty}^{0} e^{\gamma\varepsilon x}e^{-(m+1)|x-(t+\rho_0(t))|}dx \\ &\quad +Ke^{(m+1)(t+\rho_0(t))}\int_0^\infty e^{-(m+1)x}dx \\ &\leq K\exp\big[\gamma\varepsilon(t+\rho_0(t))\big] + K\exp\big[\gamma(m+1)(t+\rho_0(t))\big] \leq Ke^{\gamma\varepsilon t}.\end{aligned}$$

Finally,

$$\int_{\mathbb{R}} R_{xx}\{z_{xx} - z + a_\varepsilon[(R+z)^m - R^m]\} = O(\|z(t)\|_{L^2(\mathbb{R})} + \|z(t)\|^2_{L^2(\mathbb{R})}).$$

Collecting the above estimates we obtain (B.12). □

B.1.1 Almost conservation of mass and energy

Now let us recall that from remark 2.1 the modified mass defined in (2.8) satisfies

$$\tilde{M}[u](t) \leq \tilde{M}[u](-T_n). \tag{B.14}$$

for all $-T_n \leq t \leq -\frac{1}{2}T_\varepsilon$. Moreover, in the case $m = 2, 4$ and $\lambda = 0$, since (1.20) and (B.7) hold, there exist $K, \gamma > 0$ such that

$$M[u](t) \leq M[u](-T_n) + K\varepsilon e^{\gamma\varepsilon t}, \tag{B.15}$$

for ε small enough. By extending the definition of $\tilde{M}[u]$ to the latter case, we have almost conservation of mass, with exponential loss for all cases.

Similarly, note that in the region considered the soliton $R(t)$ is an almost solution of (1.15), in particular it must conserve mass $\tilde{M}$ (2.8) and the energy E_a (1.21), at least for large negative time. Indeed, arguing as in Lemma 6.3 (but with easier proof), one has

$$E_a[R](-T_n) - E_a[R](t) + (1-\lambda)\big[\tilde{M}[R](-T_n) - \tilde{M}[R](t)\big] \leq Ke^{\gamma\varepsilon t}. \tag{B.16}$$

for some constant $K > 0$ and all time $t \in [-T_n, T_*]$

The next step is the use the energy conservation law to provide a control of the $R(t)$ direction (note that $R(t)$ is a essential direction to control in order to obtain some coercivity properties, see Lemma 2.3). Following e.g. Lemma 5.4, one has

$$|\int_{\mathbb{R}} Rz(t)| \leq \frac{K}{1-\lambda}\Big[e^{\gamma\varepsilon t} + \|z(t)\|^2_{L^2(\mathbb{R})} + e^{\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}\Big]. \tag{B.17}$$

for some constants $K, \gamma > 0$, independent of ε.

Now, consider $E_a[u]$ and $\tilde{M}[u]$ the energy and mass defined in (1.21)-(2.8). Then one has

$$E_a[u](t) + (1-\lambda)\tilde{M}[u](t) = E_a[R](t) + (1-\lambda)\tilde{M}[R](t) - \int_{\mathbb{R}} z(a_\varepsilon - 1)R^m + \mathcal{F}_0(t), \tag{B.18}$$

where $\mathcal{F}_0$ is the quadratic functional

$$\mathcal{F}_0(t) := \frac{1}{2}\int_{\mathbb{R}}(z_x^2 + \lambda z^2) + (1-\lambda)\tilde{M}[z] - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon[(R+z)^{m+1} - R^{m+1} - (m+1)R^m z].$$

In addition, for any $t \in [-T_n, -T_*]$,

$$|\int_{\mathbb{R}} z(a_\varepsilon - 1)R^m| \leq Ke^{\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}. \tag{B.19}$$

The proof of this identity is essentially an expansion of the energy-mass functional using the relation $u(t) = R(t) + z(t)$. The proof of (B.19) is similar to (B.13).

On the other hand, the functional $\mathcal{F}_0(t)$ above mentioned enjoys the following coercivity property: there exist $K, \lambda_0 > 0$ independent of ε such that for every $t \in [-T_n, -T_*]$

$$\mathcal{F}_0(t) \geq \lambda_0\|z(t)\|^2_{H^1(\mathbb{R})} - |\int_{\mathbb{R}} R(t)z(t)|^2 - Ke^{\gamma\varepsilon t}\|z(t)\|^2_{L^2(\mathbb{R})} - K\|z(t)\|^3_{L^2(\mathbb{R})}. \tag{B.20}$$

This bound is simply a consequence of the inequality $\lambda + (1-\lambda)a_\varepsilon^{1/m}(x) \geq 1$, (B.10) and Lemma 2.3.

B.1.2 End of proof of Proposition B.3

Now by using (B.18), (B.20), and the estimates (B.14)-(B.15) and (B.17) we finally get (B.6). Indeed, note that

$$E_a[u](t) - E_a[u](-T_n) + (1-\lambda)[\tilde{M}[u](t) - \tilde{M}[u](-T_n)] \leq Ke^{\varepsilon\gamma t}.$$

On the other hand, from (B.18) and (B.10),

$$\begin{aligned} E_a[u](t) - E_a[u](-T_n) + (1-\lambda)[\tilde{M}[u](t) - \tilde{M}[u](-T_n)] \\ \geq \mathcal{F}_0(t) - Ke^{\gamma\varepsilon t} - Ke^{\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}. \end{aligned}$$

Finally, from (B.20) and B.17 we get

$$\|z(t)\|_{H^1(\mathbb{R})} \leq Ke^{\gamma\varepsilon t}.$$

Plugging this estimate in (B.12), we obtain that $|\rho_0'(t)| \leq Ke^{\gamma\varepsilon t}$, and thus after integration we get the final uniform estimate (B.6) for the H^1-case. Note that we have also improved the estimate on $\rho_0'(t)$ assumed in (B.8). This finishes the proof. □

B.2 Proof of Uniqueness

First of all let us recall that the solution u above constructed is in $C(\mathbb{R}, H^s(\mathbb{R}))$ for any $s \geq 1$, and satisfies the exponential decay (3.2). Moreover, every solution converging to a soliton satisfies this property.

Proposition B.5 (Exponential decay, see also [49]).

Let $m = 3$, or $m = 2, 4$ with $0 < \lambda \leq \lambda_0$. Let $v = v(t)$ a $C(\mathbb{R}, H^1(\mathbb{R}))$ solution of (1.1) satisfying

$$\lim_{t\to-\infty} \|v(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = 0.$$

Then there exist $K, \gamma, \varepsilon_0 > 0$ such that for every $t \leq -T_\varepsilon$ we have

$$\|v(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{-1}e^{\gamma\varepsilon t}.$$

Proof. Fix $\alpha > 0$ small. Let $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ small enough such that for all $\varepsilon \le \varepsilon_0$ and $t \le -T_\varepsilon$

$$\|v(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \le \alpha.$$

Possibly choosing ε_0 even smaller, we can apply the arguments of Proposition B.5 to the function $v(t)$ on the interval $(-\infty, -\frac{1}{2}T_\varepsilon]$ to obtain the desired result. □

Remark B.1. For the proof of the above result a key ingredient is the monotony of mass from remark 2.1; this property apparently does not hold in the cases $\lambda = 0$, $m = 2, 4$.

Now we are ready to prove the uniqueness part.

Sketch of proof of uniqueness. Let $w(t) := v(t) - u(t)$. Then $w(t) \in H^1(\mathbb{R})$ and satisfies the equation

$$\begin{cases} w_t + (w_{xx} - \lambda w + a_\varepsilon[(u+w)^m - u^m])_x = 0, & \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ \|w(t)\|_{H^1(\mathbb{R})} \le K\varepsilon^{-1}e^{\gamma\varepsilon t} & \text{for all } t \le -\frac{1}{2}T_\varepsilon. \end{cases} \tag{B.21}$$

The idea is to prove that $w(t) \equiv 0$ for all $t \in \mathbb{R}$. For this purpose, one defines the second order functional

$$\mathcal{F}_0(t) := \frac{1}{2}\int_{\mathbb{R}} w_x^2 + \frac{1}{2}\int_{\mathbb{R}} w^2 - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)[(u+w)^{m+1} - u^{m+1} - (m+1)u^m w].$$

It is easy to verify that

1. Lower bound. There exists $K > 0$ such that for all $t \le -\frac{1}{2}T_\varepsilon$,

$$\mathcal{F}_0(t) \ge \frac{1}{2}\int_{\mathbb{R}} (w_x^2 + w^2 - mQ^{m-1}w^2)(t) - K\varepsilon^{-1}e^{\gamma\varepsilon t}\sup_{t' \le t}\|w(t')\|^2_{H^1(\mathbb{R})}.$$

2. Upper bound. There exists $K, \gamma > 0$ such that

$$\mathcal{F}_0(t) \le K\varepsilon^{-2}e^{\gamma\varepsilon t}\sup_{t' \le t}\|w(t')\|^2_{H^1(\mathbb{R})}.$$

These estimates are proved similarly to the proof of Lemma 5.6. However, this functional is not coercive; so in order to obtain a satisfactory lower bound, one has to modify the function w in $(-\infty, -\frac{1}{2}T_\varepsilon]$ as follows. Let

$$\tilde{w}(t) := w(t) + b(t)Q'(\cdot - t), \quad b(t) := \frac{\int_{\mathbb{R}} w(t)Q'(\cdot - t)}{\int_{\mathbb{R}} Q'^2},$$

This modified function satisfies

1. Orthogonality to the Q' direction: $\displaystyle\int_{\mathbb{R}} \tilde{w}(t)Q'(\cdot - t) = 0.$

2. Equivalence. There exists $C_1, C_2 > 0$ independent of ε such that

$$C_1\|w(t)\|_{H^1(\mathbb{R})} \le \|\tilde{w}(t)\|_{H^1(\mathbb{R})} + |b(t)| \le C_2\|w(t)\|_{H^1(\mathbb{R})}.$$

   Moreover,

$$\frac{1}{2}\int_{\mathbb{R}} (w_x^2 + w^2 - mQ^{m-1}w^2)(t) = \frac{1}{2}\int_{\mathbb{R}} (\tilde{w}_x^2 + \tilde{w}^2 - mQ^{m-1}\tilde{w}^2)(t) + O(e^{-\varepsilon\gamma|t|}).$$

3. Control on the Q direction:

$$|\int_{\mathbb{R}} \tilde{w}(t)Q(\cdot - t)| \leq K\varepsilon^{-1}e^{\varepsilon\gamma t}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})}.$$

   This property is proved similarly to the proof of (6.15): We use the fact that variation in time of the above quantity is of quadratic order on $\tilde{w}$.

4. Coercivity. There exists $\lambda > 0$ independent of t such that

$$\frac{1}{2}\int_{\mathbb{R}}(\tilde{w}_x^2 + \tilde{w}^2 - mQ^{m-1}\tilde{w}^2)(t) \geq \lambda\|\tilde{w}(t)\|^2_{H^1(\mathbb{R})} - K|\int_{\mathbb{R}}\tilde{w}(t)Q(\cdot - t)|^2.$$

5. Sharp control. From the equivalence w-$\tilde{w}$ and the coercivity property we obtain

$$\|\tilde{w}(t)\|_{H^1(\mathbb{R})} + \varepsilon|b(t)| \leq K\varepsilon^{-2}e^{\varepsilon\gamma t/2}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})}. \tag{B.22}$$

   Note that the bound on $b(t)$ is proved similarly to (6.14).

The proof of these affirmations follows closely the argument of Proposition 6 in [49], with easier proofs. Finally, from (B.22) we have for ε small enough and $t \leq -\frac{1}{2}T_\varepsilon$,

$$\|w(t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{-2}e^{\varepsilon\gamma t}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})} < \frac{1}{2}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})}.$$

This inequality implies $w \equiv 0$, and in conclusion the uniqueness. □

C Proof of Proposition 4.2

The proof is similar to Proposition 2.2 in [54] and Appendix in [53].

Proof. First of all, we recall the error term $S[\tilde{u}]$ introduced in (4.9), subsection 4.2.

We easily verify that

$$S[\tilde{u}] = \mathbf{I} + \mathbf{II} + \mathbf{III}, \tag{C.1}$$

where (we omit the dependence on t, x)

$$\mathbf{I} := S[R], \quad \mathbf{II} = \mathbf{II}(w) := w_t + (w_{xx} - \lambda w + m\, a_\varepsilon R^{m-1}w)_x, \tag{C.2}$$

and

$$\mathbf{III} := \left\{a_\varepsilon[(R+w)^m - R^m - mR^{m-1}w]\right\}_x. \tag{C.3}$$

In the next lemmas, we expand the terms in (C.1).

Lemma C.1. *Suppose $m = 2, 3$ or 4. We have*

$$\mathbf{I} = \varepsilon F_1(\varepsilon t; y) + \frac{\varepsilon^2 a''}{2\tilde{a}^m}(y^2 Q_c^m)_y + \varepsilon^3 f_I(\varepsilon t)F_c^I(y), \tag{C.4}$$

where

$$F_1(\varepsilon t; y) := \frac{c'}{\tilde{a}}\Lambda Q_c - \frac{\tilde{a}'}{\tilde{a}^2}(c-\lambda)Q_c + \frac{a'}{\tilde{a}^m}(yQ_c^m)_y \in \mathcal{Y},$$

and $|f_I(\varepsilon t)| \leq K$, $F_c^I \in \mathcal{Y}$. Finally, for every $t \in [-T_\varepsilon, T_\varepsilon]$

$$\|\varepsilon^3 f_I(\varepsilon t)F_c^I(y)\|_{H^2(\mathbb{R})} \leq K\varepsilon^3.$$

Proof of Lemma C.1. Recall that $\tilde{a} := a^{\frac{1}{m-1}}$ and

$$R(t,x) = \frac{Q_{c(\varepsilon t)}(y)}{\tilde{a}(\varepsilon\rho(t))}, \quad y = x-\rho(t), \quad \partial_t\rho(t) = c(\varepsilon t)-\lambda.$$

Thus we have

$$\begin{aligned}\mathbf{I} &= R_t + (R_{xx} - \lambda R + a_\varepsilon R^m)_x \\ &= \frac{\varepsilon c'}{\tilde{a}}\Lambda Q_c - \frac{(c-\lambda)}{\tilde{a}}Q_c' - \varepsilon\frac{\tilde{a}'(c-\lambda)}{\tilde{a}^2}Q_c + \frac{1}{\tilde{a}}Q_c^{(3)} - \frac{\lambda}{\tilde{a}}Q_c' + \frac{1}{\tilde{a}^m}(a(\varepsilon x)Q_c^m)_x.\end{aligned}$$

Note that via a Taylor expansion,

$$(a(\varepsilon x)Q_c^m)_x = a(\varepsilon\rho)(Q_c^m)_x + \varepsilon a'(\varepsilon\rho)(yQ_c^m)_x + \frac{1}{2}\varepsilon^2 a''(\varepsilon\rho)(y^2Q_c^m)_x + O_{H^2(\mathbb{R})}(\varepsilon^3).$$

Therefore,

$$\begin{aligned}\mathbf{I} &= \frac{\varepsilon c'}{\tilde{a}}\Lambda Q_c - \frac{(c-\lambda)}{\tilde{a}}Q_c' - \frac{\varepsilon}{m-1}\frac{a'(c-\lambda)}{\tilde{a}^m}Q_c + \frac{1}{\tilde{a}}Q_c^{(3)} - \frac{\lambda}{\tilde{a}}Q_c' + \frac{1}{\tilde{a}}(Q_c^m)' + \frac{\varepsilon a'}{\tilde{a}^m}(yQ_c^m)_x \\ &\quad + \frac{\varepsilon^2 a''}{2\tilde{a}^m}(y^2Q_c^m)_x + \varepsilon^3 f_I(\varepsilon t)F_c^I(y) \\ &= \frac{1}{\tilde{a}}(Q_c'' - cQ_c + Q_c^m)' + \frac{\varepsilon c'}{\tilde{a}}\Lambda Q_c - \varepsilon\frac{\tilde{a}'}{\tilde{a}^2}(c-\lambda)Q_c + \frac{\varepsilon a'}{\tilde{a}^m}(yQ_c^m)_y \\ &\quad + \frac{\varepsilon^2 a''}{2\tilde{a}^m}(y^2Q_c^m)_y + \varepsilon^3 f_I(\varepsilon t)F_c^I(y) \\ &= \varepsilon\Big[\frac{c'}{\tilde{a}}\Lambda Q_c - \frac{\tilde{a}'}{\tilde{a}^2}(c-\lambda)Q_c + \frac{a'}{\tilde{a}^m}(yQ_c^m)_y\Big] + \frac{\varepsilon^2 a''}{2\tilde{a}^m}(y^2Q_c^m)_y + \varepsilon^3 f_I(\varepsilon t)F_c^I(y).\end{aligned}$$

Moreover $|f_I(\varepsilon t)| \le K$, $F_c^I(y) \in \mathcal{Y}$ and

$$\|\varepsilon^3 f_I(\varepsilon t)F_c^I(y)\|_{H^2(\mathbb{R})} \le K\varepsilon^3.$$

This finishes the proof. □

Lemma C.2 (Decomposition of **II**). *We have*

$$\begin{aligned}\mathbf{II} &= -\varepsilon(\mathcal{L}A_c)_y(\varepsilon t;y) + \varepsilon^2[(A_c)_t + c'(\varepsilon t)\Lambda A_c](\varepsilon t;y) \\ &\quad + m\varepsilon^2\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}(yQ_c^{m-1}(y)A_c(\varepsilon t;y))_y + \varepsilon^3 F_c^{\mathbf{II}}(\varepsilon t;y).\end{aligned}$$

with $F_c^{\mathbf{II}}(\varepsilon t;\cdot) \in \mathcal{Y}$, uniformly in time. In addition, suppose **(IP)** *holds for A_c. Then*

$$\|\varepsilon^3 F_c^{\mathbf{II}}(\varepsilon t;y)\|_{H^2(\mathbb{R})} \le K\varepsilon^3 e^{-\gamma\varepsilon|t|}.$$

Proof. We compute:

$$\begin{aligned}\mathbf{II} &= \varepsilon(A_c(\varepsilon t;y))_t + \varepsilon\big[(A_c)_{yy}(\varepsilon t;y) - \lambda A_c(\varepsilon t;y) + \frac{a_\varepsilon}{a(\varepsilon\rho)}mQ_c^{m-1}(y)A_c(\varepsilon t;y)\big]_x \\ &= -\varepsilon(\mathcal{L}A_c)_y(\varepsilon t;y) + \varepsilon^2(A_c)_t(\varepsilon t;y) + \varepsilon^2 c'(\varepsilon t)\Lambda A_c(\varepsilon t,y) \\ &\quad + m\varepsilon^2\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}(yQ_c^{m-1}(y)A_c(\varepsilon t;y))_y + \varepsilon^3 F_c^{\mathbf{II}}(\varepsilon t;y),\end{aligned}$$

where $F_c^{\mathbf{II}}(\varepsilon t;y) = O(y^2Q_c^{m-1}(y)A_c(\varepsilon t;y))_y) \in \mathcal{Y}$ and thus, thanks to the **(IP)** property,

$$\|\varepsilon^3 F_c^{\mathbf{II}}(\varepsilon t;y)\|_{H^2(\mathbb{R})} \le K\varepsilon^3 e^{-\gamma\varepsilon|t|}.$$

This concludes the proof. □

Lemma C.3 (Decomposition of **III**). *Suppose* **(IP)** *holds for A_c. Then we have*

$$\mathbf{III} = \varepsilon^3 a'(\varepsilon x)[\varepsilon^{m-2} A_c^m(\varepsilon t; y) + \tilde{F}_c^{\mathbf{III}}(\varepsilon t; y)] + \varepsilon^2 a_\varepsilon G_c^{\mathbf{III}}(\varepsilon t; y),$$

with $\tilde{F}_c^{\mathbf{III}}(\varepsilon t;\cdot), G_c^{\mathbf{III}}(\varepsilon t;\cdot) \in \mathcal{Y}$, uniformly for every $t \in [-T_\varepsilon, T_\varepsilon]$. Moreover, we have the estimate

$$\|\mathbf{III}\|_{H^2(\mathbb{R})} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}, \tag{C.5}$$

for every $t \in [-T_\varepsilon, T_\varepsilon]$.

Proof. Define $\tilde{\mathbf{III}} := a_\varepsilon[(R+w)^m - R^m - mR^{m-1}w]$. We consider separate cases. First, note that for $m = 2$, $\tilde{\mathbf{III}} = a_\varepsilon w^2 = \varepsilon^2 a_\varepsilon A_c^2$; thus taking derivative

$$\mathbf{III} = \varepsilon^3 a'(\varepsilon x) A_c^2 + \varepsilon^2 a_\varepsilon (A_c^2)'.$$

Note that $(A_c^2)' \in \mathcal{Y}$ because **(IP)** property holds for A_c.

Suppose now $m = 3$. We have $\tilde{\mathbf{III}} = \varepsilon^2 a_\varepsilon[3Q_c A_c^2 + \varepsilon A_c^3]$. From this we get

$$\mathbf{III} = \varepsilon^3 a'(\varepsilon x)[3Q_c A_c^2 + \varepsilon A_c^3] + \varepsilon^2 a_\varepsilon[3(Q_c A_c^2)' + \varepsilon (A_c^3)'].$$

Finally, for the case $m = 4$

$$\begin{aligned}\mathbf{III} &= \left\{a_\varepsilon \varepsilon^2[6Q_c^2 A_c^2 + 4\varepsilon Q_c A_c^3 + \varepsilon^2 A_c^4]\right\}_x \\ &= \varepsilon^3 a'(\varepsilon x)[6Q_c^2 A_c^2 + 4\varepsilon^2 Q_c A_c^3 + \varepsilon^2 A_c^4] + \varepsilon^2 a_\varepsilon[6(Q_c^2 A_c^2)' + 4\varepsilon(Q_c A_c^3)' + \varepsilon^2 (A_c^4)'].\end{aligned}$$

Under the **(IP)** property, for each $m = 2, 3$ and 4, we can estimate **III** as follows

$$\|\mathbf{III}\|_{H^2(\mathbb{R})} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}.$$

□

Now we collect the estimates from Lemmas C.1, C.2 and C.3. We finally get

$$\begin{aligned}S[\tilde{u}] &= \mathbf{I} + \mathbf{II} + \mathbf{III} \\ &= \varepsilon[F_1 - (\mathcal{L}A_c)_y](\varepsilon t; y) + \varepsilon^2[(A_c)_t + c'(\varepsilon t)\Lambda A_c](\varepsilon t; y) + O(\varepsilon^2 e^{-\gamma\varepsilon|t|}),\end{aligned}$$

provided **(IP)** holds for A_c. □

D End of Proof of Lemma 4.6

In this section we will show that for all $t \in [-T_\varepsilon, T_\varepsilon]$ (cf. (4.28))

$$\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} \leq K\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}, \tag{D.1}$$

where $\tilde{u}$ is the modified approximate solution defined in (4.25).

Proof of (D.1). Similarly to the proof of Proposition 4.2 in Appendix C, we claim that we can decompose

$$S[\tilde{u}] = \mathbf{I} + \tilde{\mathbf{II}} + \tilde{\mathbf{III}},$$

(cf. the definitions in (C.1)-(C.3)).

First of all, note that the conclusions of Lemma C.1 in Appendix C remains unchanged. In particular, (C.4) holds without any variation.

Concerning the term $\tilde{\mathbf{III}}$, we have the following

Claim 6 (Decomposition of $\tilde{\mathbf{III}}$ revisited). We have

$$\tilde{\mathbf{III}} = \varepsilon^3 a'(\varepsilon x)[\varepsilon^{m-2}\eta_c^m A_c^m(\varepsilon t;y) + \tilde{F}_c^{\mathbf{III}}(\varepsilon t;y)] + \varepsilon^2 a_\varepsilon[G_c^{\mathbf{III}}(\varepsilon t;y) + \varepsilon^{m-1}(\eta_c^m)' A_c^m],$$

with $\tilde{F}_c^{\mathbf{III}}(\varepsilon t;\cdot), G_c^{\mathbf{III}}(\varepsilon t;\cdot) \in \mathcal{Y}$, uniformly for every $t \in [-T_\varepsilon, T_\varepsilon]$. Moreover, we have the estimate

$$\|\tilde{\mathbf{III}}\|_{H^2(\mathbb{R})} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}, \tag{D.2}$$

for every $t \in [-T_\varepsilon, T_\varepsilon]$.

Proof. The proof is identical to Lemma C.3, being the unique new element in the proof the emergency of the term

$$\varepsilon^{m+1} a_\varepsilon (\eta_c^m)' A_c^m, \quad \text{with} \quad \|\varepsilon^{m+1} a_\varepsilon(\eta_c^m)' A_c^m\|_{H^2(\mathbb{R})} \leq K\varepsilon^{m+\frac{1}{2}} e^{-\gamma\varepsilon|t|}.$$

The other terms and their respective estimates remain unchanged. This finishes the proof. □

Finally we consider the term $\tilde{\mathbf{II}}$.

Claim 7 (Decomposition of $\tilde{\mathbf{II}}$ revisited). We have

$$\tilde{\mathbf{II}} = -\varepsilon\eta_c(y)(\mathcal{L}A_c)_y(\varepsilon t;y) + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}).$$

Proof. We follow on the lines of the proof of Lemma C.2: First we have

$$\begin{aligned}(\varepsilon A_\#(\varepsilon t;y))_t &= -(c-\lambda)\varepsilon^2\eta_\varepsilon' A_c(\varepsilon t;y) - (c-\lambda)\varepsilon\eta_\varepsilon(A_c)_y(\varepsilon t;y) \\ &\quad + \varepsilon^2\eta_\varepsilon(A_c)_t(\varepsilon t;y) + \varepsilon^2 c'(\varepsilon t)\eta_\varepsilon \Lambda A_c(\varepsilon t;y).\end{aligned}$$

We use now Lemma 4.5 and (4.27) to estimate this last term. We get

$$(\varepsilon A_\#(\varepsilon t;y))_t = -(c-\lambda)\varepsilon\eta_\varepsilon(y)(A_c)_y(\varepsilon t;y) + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}). \tag{D.3}$$

On the other hand,

$$\begin{aligned}&\varepsilon((A_\#)_{xx} - \lambda A_\# + \frac{a_\varepsilon}{a(\varepsilon\rho)} m Q_c^{m-1}(y) A_\#)_x \\ &= \varepsilon\big\{\eta_\varepsilon[(A_c)_{yy} - \lambda A_c + \frac{a_\varepsilon}{a(\varepsilon\rho)} m Q_c^{m-1}(y)A_c] + 2\varepsilon\eta_\varepsilon'(A_c)_y + \varepsilon^2\eta_\varepsilon'' A_c\big\}_x \\ &= \varepsilon\eta_\varepsilon\big[(A_c)_{yy} - \lambda A_c + \frac{a_\varepsilon}{a(\varepsilon\rho)} m Q_c^{m-1}(y)A_c\big]_x \\ &\quad + \varepsilon^2\big[3\eta_\varepsilon'(A_c)_{yy} - \lambda\eta_\varepsilon' A_c + a_\varepsilon m\eta_\varepsilon' Q_c^{m-1}A_c + 3\varepsilon\eta_\varepsilon''(A_c)_y + \varepsilon^2\eta_\varepsilon^{(3)}A_c\big] \\ &= \varepsilon\eta_\varepsilon\big[(A_c)_{yy} - \lambda A_c + mQ_c^{m-1}(y)A_c\big]_y + \varepsilon^2\eta_\varepsilon m\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}(yQ_c^{m-1}A_c)_y \\ &\quad + \varepsilon^2\big[3\eta_\varepsilon'(A_c)_{yy} - \lambda\eta_\varepsilon' A_c + a_\varepsilon m\eta_\varepsilon' Q_c^{m-1}A_c + 3\varepsilon\eta_\varepsilon''(A_c)_y + \varepsilon^2\eta_\varepsilon^{(3)}A_c\big] \\ &\quad + O(\varepsilon^3\eta_\varepsilon(y^2Q_c^{m-1}A_c)_y).\end{aligned}$$

We use now Lemma 4.5 and the **(IP)** property to estimate as follows

$$m\varepsilon^2|\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}|\|\eta_\varepsilon(yQ_c^{m-1}A_c)_y\|_{H^2(\mathbb{R})} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|},$$

$$\|O(\varepsilon^3\eta_\varepsilon(y^2Q_c^{m-1}A_c)_y)\|_{H^2(\mathbb{R})} \leq K\varepsilon^3, \quad \varepsilon^4\|\eta_c^{(3)}A_c\|_{H^2(\mathbb{R})} \leq \varepsilon^{\frac{7}{2}} e^{-\gamma\varepsilon|t|},$$

$$\|\varepsilon^2\lambda\eta_\varepsilon' A_c\|_{H^2(\mathbb{R})} \leq K\lambda\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|},$$

and

$$\varepsilon^2\|3\eta_\varepsilon'(A_c)_{yy} + a_\varepsilon m\eta_\varepsilon' Q_c^{m-1}A_c + 3\varepsilon\eta_\varepsilon''(A_c)_y\|_{H^2(\mathbb{R})} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}.$$

Therefore

$$\begin{aligned}\varepsilon[(A_\#)_{xx} - \lambda A_\# + \frac{a_\varepsilon}{a(\varepsilon\rho)} mQ_c^{m-1}(y)A_\#]_x = \\ \varepsilon\eta_\varepsilon\big[(A_c)_{yy} - \lambda A_c + mQ_c^{m-1}(y)A_c\big]_y + O_{H^2(\mathbb{R})}(\varepsilon^2 e^{-\gamma\varepsilon|t|} + \varepsilon^3).\end{aligned} \quad \text{(D.4)}$$

The conclusion follows from (D.3) and (D.4). □

We return to the global estimate on $S[\tilde{u}]$. From (C.4), Claims 6 and 7 and Lemma 4.5 we get

$$\begin{aligned} S[\tilde{u}] &= \varepsilon[F_1(\varepsilon t, y) - \eta_c(y)(\mathcal{L}A_c)_y)(\varepsilon t, y)] + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}) \\ &= \varepsilon(1-\eta_c(y))F_1(\varepsilon t; y) + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}). \end{aligned}$$

The final conclusion of this appendix is a straightforward consequence of the following fact: For every $t \in [-T_\varepsilon, T_\varepsilon]$

$$\|\varepsilon(1-\eta_c(y))F_1(\varepsilon t; y)\|_{H^2(\mathbb{R})} \leq K\varepsilon e^{-\frac{1}{\varepsilon} - \gamma\varepsilon|t|} \ll K\varepsilon^{10}.$$

for ε small enough. Indeed, note that $\operatorname{supp}(1-\eta_c(\cdot)) \subseteq (-\infty, -\frac{1}{\varepsilon}]$. From (C.4),

$$|F_1(\varepsilon t; y)| \leq K e^{-\gamma|y| - \gamma\varepsilon|t|}.$$

From this estimate the desired estimate follows directly. □

E Proof of Lemma 6.4

E.1 Proof of Lemma 6.4

Our proof of the Virial inequality (6.26) follows closely to the proof of Lemma 2 in [55].

Proof. Take $t \in [t_1, T^*]$, and denote $y := x - \rho_2(t)$. Replacing the value of z_t given by (6.13), we have

$$\begin{aligned} \partial_t \int_{\mathbb{R}} z^2\psi_{A_0}(y) &= 2\int_{\mathbb{R}} zz_t\psi_{A_0}(y) - \rho_2'(t)\int_{\mathbb{R}} z^2\psi_{A_0}'(y) \\ &= 2\int_{\mathbb{R}} (z\psi_{A_0}(y))_x(z_{xx} - \lambda z + mQ_{c_2}^{m-1}(y)z) && \text{(E.1)} \\ &\quad -(c_2(t)-\lambda)\int_{\mathbb{R}} z^2\psi_{A_0}'(y) - 2(c_2(t)-\lambda-\rho_2')(t)\int_{\mathbb{R}} zQ_{c_2}'\psi_{A_0}(y) && \text{(E.2)} \\ &\quad +2\int_{\mathbb{R}} (z\psi_{A_0}(y))_x[(R+z)^m - R^m - mR^{m-1}z] && \text{(E.3)} \\ &\quad -2c_2'(t)\int_{\mathbb{R}} z\Lambda Q_{c_2}\psi_{A_0}(y) + (c_2-\lambda-\rho_2')(t)\int_{\mathbb{R}} z^2\psi_{A_0}'(y) && \text{(E.4)} \\ &\quad +\int_{\mathbb{R}} (z\psi_{A_0}(y))_x(a_\varepsilon - 2)(R+z)^m. && \text{(E.5)} \end{aligned}$$

Now, following [55] and by using (6.14) and (6.15) it is easy to check that for A_0 large enough, and some constants δ_0, ε_0 small

$$|(E.3) + (E.4)| \leq \frac{\delta_0}{100} \int_{\mathbb{R}} (z_x^2 + z^2)(t) e^{-\frac{1}{A_0}|y|}.$$

On the other hand, the terms (E.1) and (E.2) goes similarly to the terms B_1 and B_2 in Appendix B of [55]. We get

$$(E.1) + (E.2) \leq -\frac{\delta_0}{10} \int_{\mathbb{R}} (z_x^2 + z^2)(t) e^{-\frac{1}{A_0}|y|}.$$

Finally, the term (E.5) can be estimated as follows. First, from (6.11) and (6.12) we have for $t \geq t_1$

$$c_2(t) = c_\infty + O(\varepsilon^{1/2}), \quad \rho_2(t) = (c_\infty - \lambda)t + O(\varepsilon^{1/2}(t - t_1)),$$

and then

$$\frac{9}{10} c_\infty \leq c_2(t) \leq \frac{11}{10} c_\infty; \quad \rho_2(t) \geq \frac{9}{10}(c_\infty - \lambda)t. \tag{E.6}$$

On the other hand, we can write (E.5) in the following way

$$\begin{aligned} (E.5) &= \int_{\mathbb{R}} (z\psi_{A_0})_x (a_\varepsilon - 2)[(R+z)^m - z^m] + \int_{\mathbb{R}} (z\psi_{A_0})_x (a_\varepsilon - 2) z^m \\ &= \int_{\mathbb{R}} (\psi_{A_0})_x (a_\varepsilon - 2)[(R+z)^m - z^m] z + \int_{\mathbb{R}} \psi_{A_0} (a_\varepsilon - 2)[(R+z)^m - z^m] z_x \\ &\quad + \frac{m}{m+1} \int_{\mathbb{R}} (\psi_{A_0})_x (a_\varepsilon - 2) z^{m+1} - \frac{\varepsilon}{m+1} \int_{\mathbb{R}} \psi_{A_0} a'(\varepsilon x) z^{m+1}. \end{aligned}$$

Then, from (1.13), (6.25) and by using that $t \geq t_1 \geq \frac{1}{2} T_\varepsilon$, we get for some constant $\gamma = \gamma(A_0, c_\infty, \lambda) > 0$ independent of ε and D_0, (cf. (B.13) for a similar computation)

$$\begin{aligned} \left|\int_{\mathbb{R}} (\psi_{A_0})_x (a_\varepsilon - 2)[(R+z)^m - z^m] z\right| &\leq K A_0 e^{-\varepsilon \rho_2(t)/A_0} \|z(t)\|_{H^1(\mathbb{R})} \\ &\leq K A_0 e^{-\gamma \varepsilon t} \|z(t)\|_{H^1(\mathbb{R})}. \end{aligned}$$

Similarly

$$\left|\int_{\mathbb{R}} \psi_{A_0} (a_\varepsilon - 2)[(R+z)^m - z^m] z_x\right| \leq K A_0 e^{-\gamma \varepsilon t} \|z(t)\|_{H^1(\mathbb{R})};$$

and

$$\left|\int_{\mathbb{R}} (\psi_{A_0})_x (a_\varepsilon - 2) z^{m+1}\right| \leq K A_0 e^{-\gamma \varepsilon t} \|z(t)\|_{H^1(\mathbb{R})}^{m+1}.$$

Finally, from (6.24) and (E.6),

$$\left|\varepsilon \int_{\mathbb{R}} \psi_{A_0}(y) a'(\varepsilon x) z^{m+1}\right| \leq K A_0 e^{-\gamma \varepsilon t} \|z(t)\|_{H^1(\mathbb{R})}^{m+1}.$$

In conclusion, $(E.5) = O(A_0 e^{-\gamma \varepsilon t} \|z(t)\|_{H^1(\mathbb{R})})$, for ε small enough.

From (E.6) we obtain the second term in (6.26). Collecting the above estimates we conclude the proof. □

F Proof of Lemma 6.8

The proof of this result is very similar to Lemma 3 in [55].

Proof. First of all, recall that $\phi = \phi(\tilde{y}(x_0))$, with $\tilde{y}(x_0) = x - (\rho_2(t_0) + \sigma(t - t_0) + x_0)$. Then we have

$$\partial_t \int_{\mathbb{R}} u^2\phi = -\int_{\mathbb{R}} \big(3u_x^2 + (\sigma+\lambda)u^2 - \frac{2ma_\varepsilon}{m+1}u^{m+1}\big)\phi' + \int_{\mathbb{R}} u^2\phi^{(3)} - \frac{2\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}\phi,$$

and

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \big(u_x^2 - \frac{2a_\varepsilon(x)}{m+1}u^{m+1}\big)\phi \quad &= \quad \int_{\mathbb{R}} \big(-(u_{xx} + a_\varepsilon u^m)^2 - 2u_{xx}^2 + 2ma_\varepsilon u_x^2 u^{m-1}\big)\phi' \\ &\quad + \int_{\mathbb{R}} u_x^2\phi^{(3)} - \sigma\int_{\mathbb{R}} \big(u_x^2 - \frac{2a_\varepsilon}{m+1}u^{m+1}\big)\phi' \\ &\quad - \frac{\varepsilon}{m+1}\int_{\mathbb{R}} a_\varepsilon' u^{m+1}\phi'' - \frac{\varepsilon^2}{m+1}\int_{\mathbb{R}} a_\varepsilon'' u^{m+1}\phi'. \qquad \text{(F.1)}\end{aligned}$$

(see for example Appendix C in [55]). The conclusion follows from the arguments in [55], after we estimate the unique new different term. In particular, we have

$$-\int_{\mathbb{R}} \big(3u_x^2 + (\sigma+\lambda)u^2 - \frac{2ma_\varepsilon(x)}{m+1}u^{m+1}\big)\phi' + \int_{\mathbb{R}} u^2\phi^{(3)} \leq Ke^{-(t_0-t)/2K_0}e^{-x_0/K_0}. \qquad \text{(F.2)}$$

Indeed, using that $1/K_0^2 \leq \sigma/2$, we have (we discard the term with λ)

$$-\int_{\mathbb{R}} \big(3u_x^2 + \sigma u^2 - \frac{2ma_\varepsilon(x)}{m+1}u^{m+1}\big)\phi' + \int_{\mathbb{R}} u^2\psi^{(3)} \leq -\int_{\mathbb{R}} \big(3u_x^2 + \frac{\sigma}{2}u^2 - \frac{2ma_\varepsilon(x)}{m+1}u^{m+1}\big)\phi'.$$

Now we estimate the nonlinear term. Let $R_0 > 0$ to be chosen later. Consider the region $t \geq t_1$, $|x - \rho_2(t)| \geq R_0$. In this region we have from the stability and the Morrey's embbeding

$$|u(t,x)| \leq \|u(t) - R(t)\|_{L^\infty(\mathbb{R})} + R(t,x) \leq K\varepsilon^{1/2} + Ke^{-\gamma R_0},$$

with $\gamma > 0$ a fixed constant. Taking $0 < \varepsilon \leq \varepsilon_0$ sufficiently small and R_0 large enough, we have $|ma_\varepsilon(x)u^{m-1}| \leq \sigma/4$, in the considered region. Now we deal with the complementary region, $|x - \rho_2(t)| \leq R_0$. From (6.11) and the hypothesis $\sigma < \frac{1}{2}(1-\lambda_0)$ we have

$$|\tilde{y}(x_0)| \geq |\rho_2(t_0) - \rho_2(t) - \sigma(t_0 - t) + x_0| - |x - \rho_2(t)| \geq \frac{1}{2}\sigma(t_0 - t) + x_0 - R_0. \qquad \text{(F.3)}$$

Thus we have $|\phi'(\tilde{y})| \leq Ke^{-\gamma(t_0-t)/K_0}e^{-x_0/K_0}$. Collecting the above estimates we obtain (F.2). Now we claim that

$$\Big|\frac{2\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}\phi\Big| \leq Ke^{-\varepsilon\gamma T_\varepsilon}e^{-\varepsilon\gamma(t_0-t)/K_0}e^{-\gamma\varepsilon x_0/K_0}. \qquad \text{(F.4)}$$

Indeed, denote $\tilde{x}(t) := \rho_2(t_0) + \sigma(t - t_0) + x_0$. Then from $\sigma < \frac{1}{2}(1-\lambda_0)$ and (6.11) we have

$$\begin{aligned}\tilde{x}(t) \quad &= \quad \rho_2(t_0) - \rho_2(t) - \sigma(t_0 - t) + (x_0 + \rho_2(t)) \\ &\geq \quad \frac{1}{2}\sigma(t_0 - t) + \rho_2(t_0) + x_0 \geq \frac{1}{2}\sigma(t_0 - t) + \frac{1}{2}T_\varepsilon + x_0,\end{aligned}$$

and thus for ε small,

$$\begin{aligned}\Big|\frac{2\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}\phi\Big| \quad &\leq \quad K\varepsilon\int_{-\infty}^{\tilde{x}} e^{-\varepsilon\gamma|x|}e^{(x-\tilde{x})/K_0}dx + K\varepsilon\int_{\tilde{x}}^{\infty} e^{-\varepsilon\gamma x} \\ &\leq \quad K\varepsilon e^{-\tilde{x}/K_0} + Ke^{-\varepsilon\gamma\tilde{x}} \\ &\leq \quad Ke^{-\gamma\varepsilon T_\varepsilon}e^{-\gamma\varepsilon(t_0-t)/K_0}e^{-\gamma\varepsilon x_0/K_0}.\end{aligned}$$

This last estimate proves (F.4). Integrating between t and t_0 we get (6.32).

On the other hand, by following the same kind of calculations (see [55]), we have

$$\begin{aligned}\partial_t \int_{\mathbb{R}} (u_x^2 + u^2 - \frac{2a_\varepsilon(x)}{m+1}u^{m+1})\phi \quad &\leq \quad Ke^{-\gamma(t_0-t)/K_0}e^{-x_0/K_0} \\ &\quad +Ke^{-\gamma\varepsilon T_\varepsilon}e^{-\gamma\varepsilon(t_0-t)/K_0}e^{-\gamma\varepsilon x_0/K_0}.\end{aligned}$$

In consequence, after integration we get (6.34).

Now we prove (6.33). The procedure is analogous to (6.32); the main differences are in (F.3) and (F.4). For the first case we have that $\tilde{y}(-x_0) = x - (\rho_2(t_0) + \sigma(t-t_0) - x_0)$ satisfies

$$|\tilde{y}| \geq |\rho_2(t) - \rho_2(t_0) - \sigma(t-t_0) + x_0| - |x - \rho_2(t)| \geq \frac{1}{2}\sigma(t-t_0) + x_0 - R.$$

From the hypothesis we have that $\hat{x}(t) := \rho_2(t_0) + \sigma(t-t_0) - x_0 > t_1 \geq \frac{1}{2}T_\varepsilon$. Therefore (F.4) can be bounded as follows

$$\begin{aligned}|\frac{2\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}\phi| \quad &\leq \quad K\varepsilon\int_{-\infty}^{\hat{x}} e^{-\varepsilon\gamma|x|}e^{(x-\hat{x})/K_0}dx + K\varepsilon\int_{\hat{x}}^{\infty} e^{-\varepsilon\gamma x} \\ &\leq \quad K\varepsilon e^{-\hat{x}/K_0} + Ke^{-\varepsilon\gamma\hat{x}} \\ &\leq \quad Ke^{-\gamma\varepsilon\rho_2(t_0)}e^{-\gamma\varepsilon(t-t_0)/K_0}e^{\gamma\varepsilon x_0/K_0}.\end{aligned}$$

Collecting the above estimates and integrating between t_0 and t, we obtain the conclusion. □

G Some identities related to the soliton Q

This section has been taken from Appendix C in [53].

Lemma G.1 (Identities for the soliton Q). *Suppose $m > 1$ and denote by $Q_c := c^{\frac{1}{m-1}}Q(\sqrt{c}x)$ the scaled soliton. Then*

1. Energy.

$$E_1[Q] = \frac{1}{2}(\lambda - \lambda_0)\int_{\mathbb{R}} Q^2 = (\lambda-\lambda_0)M[Q], \qquad \textit{with } \lambda_0 = \frac{5-m}{m+3}.$$

2. Integrals. *Recall $\theta = \frac{1}{m-1} - \frac{1}{4}$. Then*

$$\int_{\mathbb{R}} Q_c = c^{\theta-\frac{1}{4}}\int_{\mathbb{R}} Q, \quad \int_{\mathbb{R}} Q_c^2 = c^{2\theta}\int_{\mathbb{R}} Q^2, \quad E_1[Q_c] = c^{2\theta+1}E_1[Q].$$

and finally

$$\int_{\mathbb{R}} Q_c^{m+1} = \frac{2(m+1)c^{2\theta+1}}{m+3}\int_{\mathbb{R}} Q^2, \qquad \int_{\mathbb{R}} \Lambda Q_c Q_c = \theta c^{2\theta-1}\int_{\mathbb{R}} Q^2.$$

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8 Addendum. Formal dynamics in the case $\lambda_0 < \lambda < 1$

An important question left open in the preceding pages was the behavior of the soliton solution $u(t)$ in the case of **positive energy**, namely $\lambda_0 < \lambda < 1$. The analysis in this case requires more attention due to the fact that the scaling of the soliton solution decreases as long as the interaction region is being crossed. In this occasion our main objective is to describe in some detail this case. Indeed, in the next paragraph we **formally** state the following surprising result: given a fixed λ close to 1, for any small $\varepsilon > 0$ the soliton **may be reflected** by the potential $a(\varepsilon\cdot)$. This result is basically a consequence of the fact that given $0 < \lambda < 1$ and $c > 0$ fixed, with $c < \lambda$, the small soliton $Q_c(\cdot - (c-\lambda)t)$, solution of

$$u_t + (u_{xx} - \lambda u + u^m)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x,$$

moves towards the **left direction**.

This section is devoted to the study of the approximate dynamical system involving the evolution of first order *scaling* and *translation* parameters $(C(t), P(t))$ on the main interaction region. This system shares many properties with the nonlinear dynamical system considered in [65] for $0 \leq \lambda \leq \lambda_0$ (see Lemma 4.4above), however the large time behavior in the case $\lambda_0 < \lambda < 1$ may differ radically from the previous case.

We start with some basic properties. First of all, let us define, for $m = 2, 3$ and 4, and $\lambda_0 < \lambda < 1$,

$$\mu(\lambda) := (1 - \frac{\lambda_0}{\lambda})^{\frac{1-\lambda_0}{\lambda_0}} > 0, \tag{8.1}$$

and recall that

$$\lambda_0 = \frac{5-m}{m+3}, \quad p = \frac{4}{m+3}.$$

Lemma 8.1 (Existence and basic properties of dynamical parameters, case $\lambda_0 < \lambda < 1$).

Suppose $m = 2, 3$ or 4. Let λ_0, p be as above and $a(\cdot)$ as in (1.13). Then the following holds.

1. Existence. *There exists a unique solution $(P(t), C(t))$, with C bounded positive, monotone decreasing, defined for all $t \geq -T_\varepsilon$, with the same regularity than $a(\varepsilon\cdot)$, of the following system*

$$\begin{cases} C'(t) = -p\varepsilon C(t)\big[\frac{\lambda}{\lambda_0} - C(t)\big]\frac{a'}{a}(\varepsilon P(t)), & C(-T_\varepsilon) = 1, \\ P'(t) = C(t) - \lambda, \quad P(-T_\varepsilon) = -(1-\lambda)T_\varepsilon. \end{cases} \tag{8.2}$$

   In addition for all $t \geq -T_\varepsilon$ one has $0 < C(t) \leq 1$ and

$$C^{\lambda_0}(t)(\frac{\lambda}{\lambda_0} - C(t))^{1-\lambda_0} = (\frac{\lambda}{\lambda_0} - 1)^{1-\lambda_0}\frac{a^p(\varepsilon P(t))}{a^p(-\varepsilon^{-1/100})}. \tag{8.3}$$

   Moreover, $\lim_{t\to+\infty} C(t)$ exists and satisfies $\lim_{t\to+\infty} C(t) > \mu(\lambda) > 0$, for all $\lambda_0 < \lambda < 1$.

2. Asymptotic behavior. *Let $\lambda_0 < \tilde{\lambda} < 1$ be the unique number satisfying*

$$\tilde{\lambda}(\frac{1-\lambda_0}{\tilde{\lambda} - \lambda_0})^{1-\lambda_0} = 2^p. \tag{8.4}$$

   Then one has

   (a) *For all $\lambda_0 < \lambda \leq \tilde{\lambda}$, one has $\lim_{t\to+\infty} C(t) > \lambda$ and $\lim_{t\to+\infty} P(t) = +\infty$.*

   (b) *For all $\tilde{\lambda} < \lambda < 1$, there exists a unique $t_0 \in (-T_\varepsilon, +\infty)$ such that $C(t_0) = \lambda$, with $\lim_{t\to+\infty} C(t) < \lambda$. Moreover, $\lim_{t\to+\infty} P(t) = -\infty$. Finally, one has the bounds $-T_\varepsilon < t_0 \leq K(\lambda)T_\varepsilon$.*

Proof of Lemma 8.1.

1. The local existence of a solution (C, P) of (8.2) is a direct consequence of the Cauchy-Lipschitz-Picard theorem. In addition, $C \equiv 0, \frac{\lambda}{\lambda_0}$ are constant solutions. Since $C(-T_\varepsilon) = 1$ and $\lambda > \lambda_0$, we have C globally defined, strictly decreasing and satisfying $0 < C(t) < \frac{\lambda}{\lambda_0}$ for all $t \geq -T_\varepsilon$.

2. Now we use (8.2) to obtain some a priori estimates on the solution C. Note that

$$\frac{(C(t)-\lambda)}{C(t)(\frac{\lambda}{\lambda_0} - C(t))} C'(t) = -\varepsilon p (C(t)-\lambda)\frac{a'}{a}(\varepsilon P(t)) = -\varepsilon p P'(t)\frac{a'}{a}(\varepsilon P(t)).$$

In particular,

$$(1-\lambda_0)\partial_t \log(\frac{\lambda}{\lambda_0} - C(t)) + \lambda_0 \partial_t \log C(t) = p\partial_t \log a(\varepsilon P(t)).$$

By integration on $[-T_\varepsilon, t]$, and by using $C(-T_\varepsilon) = 1$, we obtain (8.3).

Since $1 \leq a \leq 2$ and c is bounded we have P bounded on compact sets and consequently we obtain global existence. Using $C > 0$ and (8.3) one proves for ε small

$$C^{\lambda_0}(t) \geq \frac{99}{100}(1 - \frac{\lambda_0}{\lambda})^{1-\lambda_0} \implies C(t) > \mu(\lambda).$$

Moreover, $\lim_{t\to+\infty} C(t)$ **exists and is always far from zero**, as long as $\lambda_0 < \lambda < 1$.

3. Now, given $\lambda_0 < \lambda < 1$, we study the existence of a point $t_0 > -T_\varepsilon$ such that $C(t_0) = \lambda$. A priori, replacing this condition in (8.3), we have

$$\lambda(\frac{1}{\lambda_0} - 1)^{1-\lambda_0} = (\frac{\lambda}{\lambda_0} - 1)^{1-\lambda_0} \frac{a^p(\varepsilon P(t_0))}{a^p(-\varepsilon^{-1/100})}. \tag{8.5}$$

By choosing $\lambda := \lambda_0(1+\delta)$, with $\delta > 0$ any small enough number, we obtain a contradiction in the above identity. In conclusion, such a t_0 does not exist if $\lambda = \lambda_0(1+\delta)$, with $\delta > 0$ small. Moreover, let $\tilde{\lambda} \in (\lambda_0, 1)$ be the unique solution of (8.4). Since the function

$$\lambda \in (\lambda_0, 1) \mapsto f(\lambda) := \lambda(\frac{1-\lambda_0}{\lambda - \lambda_0})^{1-\lambda_0} \in (0, +\infty)$$

is strictly decreasing[4], we have $f(\lambda) \geq 2^p$ provided $\lambda_0 < \lambda \leq \tilde{\lambda}$. Therefore, from (8.5) we have

$$2^p \leq f(\lambda) = \frac{a^p(\varepsilon P(t_0))}{a^p(-\varepsilon^{-1/100})} < 2^p.$$

In conclusion, since $f(\lambda)$ is independent of ε, there is no $t_0 \in \mathbb{R}$ such that $C(t_0) = \lambda$. Thus, by continuity we have $C(t) > \lambda$ for all $t \geq -T_\varepsilon$ and $\lim_{+\infty} C(\cdot) \geq \lambda$. Moreover, if $\lim_{+\infty} C(\cdot) = \lambda$, we have from (8.3) after passing to the limit

$$f(\lambda) \leq \limsup_{t\to+\infty} \frac{a^p(\varepsilon P(t))}{a^p(-\varepsilon^{-1/100})} < 2^p, \quad \lambda \leq \tilde{\lambda},$$

[4]More precisely, one has

$$f'(\lambda) = -\frac{(1-\lambda)(1-\lambda_0)^{1-\lambda_0}}{(\lambda-\lambda_0)^{2-\lambda_0}}.$$

a contradiction. Thus, $\lim_{+\infty} C(\cdot) > \lambda$. Moreover, from the equation for P in (8.2) one has for all $t \geq 0$

$$P(t) = P(-T_\varepsilon) + \int_{-T_\varepsilon}^{0} (C(s) - \lambda)ds + \int_{0}^{t} (C(s) - \lambda)ds \geq P(-T_\varepsilon) + (C(0) - \lambda)t;$$

therefore $\lim_{t\to+\infty} P(t) = +\infty$.

4. Now, let us prove that for all $\lambda \in (\tilde{\lambda}, 1)$ there exists $t_0 \in \mathbb{R}$ such that $C(t_0) = \lambda$ (and therefore $\lim_{+\infty} C(\cdot) < \lambda$.) By contradiction, let us suppose $C(t) > \lambda$ for all $t \geq -T_\varepsilon$, with $\tilde{c}_\infty := \lim_{+\infty} C(\cdot) \geq \lambda$.

First let us suppose $\tilde{c}_\infty > \lambda$. Thus we have $\lim_{+\infty} P(\cdot) = +\infty$ and from (8.3) we have

$$\tilde{c}_\infty^{\lambda_0}(\frac{\lambda}{\lambda_0} - \tilde{c}_\infty)^{1-\lambda_0} = (\frac{\lambda}{\lambda_0} - 1)^{1-\lambda_0} \frac{2^p}{a^p(-\varepsilon^{-1/100})}. \tag{8.6}$$

Since $\tilde{c}_\infty > \lambda$ one has

$$\tilde{c}_\infty^{\lambda_0}(\frac{\lambda - \lambda_0 \tilde{c}_\infty}{\lambda - \lambda_0})^{1-\lambda_0} \leq \max_{r\in(0,1)} r^{\lambda_0}(\frac{\lambda - \lambda_0 r}{\lambda - \lambda_0})^{1-\lambda_0} = f(\lambda) < 2^p,$$

a contradiction with (8.6) for small ε.

Now we suppose $\tilde{c}_\infty = \lambda$. Here we have two possibilities: either $P_\infty := \lim_{t\to+\infty} P(t) = +\infty$, or $P_\infty < +\infty$. For the first case, by following the preceding analysis, we have

$$\tilde{c}_\infty^{\lambda_0}(\frac{\lambda - \lambda_0 \tilde{c}_\infty}{\lambda - \lambda_0})^{1-\lambda_0} = f(\lambda) < 2^p,$$

a contradiction with (8.6), for small ε. Otherwise, computing $C''(t)$ in (8.2) we have

$$C''(t) = -p\varepsilon^2 C(t)(\frac{\lambda}{\lambda_0} - C(t))\big[(C - \lambda)\frac{a''}{a}(\varepsilon P(t)) + \frac{a'^2}{a^2}(\varepsilon P(t))(C(t)\lambda_0 - \lambda(\frac{p}{\lambda_0} - 1))\big];$$

and thus

$$\lim_{t\to+\infty} C''(t) = -p\varepsilon^2\lambda^3(\frac{1}{\lambda_0} - 1)\frac{a'^2}{a^2}(P_\infty)(\lambda_0 - \frac{p}{\lambda_0} + 1) \neq 0,$$

for all $m = 2, 3$ and 4. This last result contradicts the fact that $\lim_{+\infty} C(\cdot)$ exists.

In conclusion, we have that there exists at least one $t_0 > -T_\varepsilon$ such that $C(t_0) = \lambda$. From $C' < 0$ we have that such a t_0 is unique.

5. We finally prove some properties of $P(t)$ in the case $\tilde{\lambda} < \lambda < 1$. From (8.3), one has

$$f(\lambda) = \frac{a^p(\varepsilon P(t_0))}{a^p(-\varepsilon^{-1/100})}.$$

Since $f(\lambda) \in (1, 2^p)$ for fixed $\lambda \in (\tilde{\lambda}, 1)$, and it is independent of ε, one has, for small ε,

$$|\varepsilon P(t_0)| \leq K(\lambda);$$

(the constant K becomes singular as λ approaches $\tilde{\lambda}$ or 1.) Therefore, from (8.2) one has

$$C'(t_0) = -\varepsilon p\lambda^2(\frac{1}{\lambda_0} - 1)\frac{a'(\varepsilon P(t_0))}{a(\varepsilon P(t_0))} \leq -\kappa(\lambda)\varepsilon, \qquad \kappa(\lambda) > 0;$$

and thus, for $\alpha > 0$ small enough (but independent of ε), since $C''(t) = O_{L^\infty}(\varepsilon^2)$,

$$C(t_0 - \frac{\alpha}{\varepsilon}) \geq \lambda + \kappa(\lambda)\alpha + O(\alpha^2) \geq \lambda + \frac{9}{10}\kappa(\lambda)\alpha. \tag{8.7}$$

We use this identity to obtain

$$\begin{aligned} P(t_0) &= -(1-\lambda)T_\varepsilon + \int_{-T_\varepsilon}^{t_0-\frac{\alpha}{\varepsilon}} (C(s)-\lambda)ds + \int_{t_0-\frac{\alpha}{\varepsilon}}^{t_0} (C(s)-\lambda)ds \\ &\geq -(1-\lambda)T_\varepsilon + \frac{9}{10}\kappa(\lambda)\alpha(t_0 - \frac{\alpha}{\varepsilon} + T_\varepsilon) - \frac{K\alpha}{\varepsilon}, \end{aligned}$$

and therefore $t_0 \leq K(\lambda)T_\varepsilon$.

Finally, note that $P(t)$ is strictly decreasing for all $t > t_0$. Therefore, for all $t \geq t_0 + 1$ one has $C(t_0+1) < \lambda$ and

$$P(t) = P(t_0) + \int_{t_0}^{t_0+1} (C(s)-\lambda)ds + \int_{t_0+1}^{t} (C(s)-\lambda)ds \leq P(t_0) + (C(t_0+1)-\lambda)(t-t_0-1);$$

thus $\lim_{t\to+\infty} P(t) = -\infty$. The proof is complete.

□

The last properties obtained in the above Lemma lead to the following definition.

Definition 8.1 (Exit time).

Suppose $\lambda_0 < \lambda \leq \tilde{\lambda}$. Let us define $\tilde{T}_\varepsilon \geq -T_\varepsilon$ such that $P(\tilde{T}_\varepsilon) = (1-\lambda)T_\varepsilon$. Otherwise, if $\tilde{\lambda} < \lambda < 1$, let us consider $\tilde{T}_\varepsilon > t_0$ such that $P(\tilde{T}_\varepsilon) = -(1-\lambda)T_\varepsilon$.

The next result states that in the interval $\lambda_0 < \lambda < \tilde{\lambda}$ the soliton leaves the potential zone by the right hand side, with a well determined scaling $0 < c_\infty(\lambda) < 1$. Moreover, the exit time is bounded by $K(\lambda)T_\varepsilon$, with K becoming unbounded as λ approaches $\tilde{\lambda}$.

Lemma 8.2 (Asymptotic behavior, case $\lambda_0 < \lambda < \tilde{\lambda}$).

Suppose now $\lambda_0 < \lambda < \tilde{\lambda}$. There exists a unique solution $c_\infty(\lambda)$ of the following algebraic equation

$$c_\infty^{\lambda_0}(\frac{\lambda - \lambda_0 c_\infty}{\lambda - \lambda_0})^{1-\lambda_0} = 2^p, \quad \lambda < c_\infty < 1. \tag{8.8}$$

In addition, $\lambda \mapsto c_\infty(\lambda)$ is a strictly decreasing map with $c_\infty(\lambda_0) = 1$ and $c_\infty(\lambda) > c_\infty(\tilde{\lambda}) = \tilde{\lambda}$. Furthermore, one has $C(\tilde{T}_\varepsilon) = c_\infty(\lambda)$, and $\tilde{T}_\varepsilon \leq K(\lambda)T_\varepsilon$, with $K(\lambda) \sim (c_\infty(\lambda) - \lambda)^{-1}$.

Remark 8.1. Note that the condition $c_\infty > \lambda$ is essential, because there exists another minimal branch of solutions $c_\infty^*(\lambda) < \lambda$ increasing in λ with $c_\infty^*(\lambda_0) = 0$ and $c_\infty^*(\tilde{\lambda}) = \tilde{\lambda}$.

Proof. The proof of existence and uniqueness of a solution $c_\infty(\lambda)$ of (8.8) is similar to Lemma 4.4 in [65]. We skip the details.

Let $\tilde{c}_\infty(\lambda, \varepsilon) := \lim_{+\infty} C$. From (8.3) and $\lim_{+\infty} P = +\infty$ one has

$$\tilde{c}_\infty^{\lambda_0}(\frac{\lambda - \lambda_0 \tilde{c}_\infty}{\lambda - \lambda_0})^{1-\lambda_0} = \frac{2^p}{a^p(-\varepsilon^{-1/100})}, \quad \lambda < \tilde{c}_\infty < 1. \tag{8.9}$$

Now let us define for $r \in (0,1)$

$$g(r) := r^{\lambda_0}(\frac{\lambda - \lambda_0 r}{\lambda - \lambda_0})^{1-\lambda_0}.$$

Note that $g(r)$ is strictly decreasing in the interval $(\lambda, 1)$. In addition, from (8.8) and (8.9) we have $c_\infty < \tilde c_\infty$. Moreover, from the behavior of a in (1.13) we have $\tilde c_\infty = c_\infty + O(\varepsilon^{10})$, for all ε small. This implies that

$$\tilde c_\infty(\lambda,\varepsilon) - \lambda > c_\infty(\lambda) - \lambda > 0,$$

uniformly for all ε small enough. On the other hand, at time $t = \tilde T_\varepsilon$ one has

$$C(\tilde T_\varepsilon)^{\lambda_0}(\frac{\lambda - \lambda_0 C(\tilde T_\varepsilon)}{\lambda-\lambda_0})^{1-\lambda_0} = \frac{a^p(\varepsilon^{-1/100})}{a^p(-\varepsilon^{-1/100})}, \qquad 0 < C(\tilde T_\varepsilon) < \lambda,$$

therefore $C(\tilde T_\varepsilon) = c_\infty(\lambda) + O(\varepsilon^{10})$. Moreover,

$$(1-\lambda)T_\varepsilon = P(\tilde T_\varepsilon) = P(-T_\varepsilon) + \int_{-T_\varepsilon}^{\tilde T_\varepsilon}(C(s)-\lambda)ds \geq -(1-\lambda)T_\varepsilon + (c_\infty - \lambda)(\tilde T_\varepsilon + T_\varepsilon).$$

From this inequality we obtain, for all $\lambda_0 < \lambda < \tilde\lambda$, the upper bound $\tilde T_\varepsilon \leq K(\lambda)T_\varepsilon$, with $K(\lambda) \sim (c_\infty(\lambda)-\lambda)^{-1}$. Note that $K(\lambda)$ becomes singular as $\lambda \uparrow \tilde\lambda$. □

Now we consider the case $\tilde\lambda < \lambda < 1$. Here we obtain the following striking result: the soliton is formally reflected by the potential. The final scaling is given by a modified parameter $0 < c_\infty(\lambda) < 1$, away from zero provided $\lambda \in (\tilde\lambda, 1)$.

Lemma 8.3 (Asymptotic behavior, case $\tilde\lambda < \lambda < 1$)**.**

Suppose $\tilde\lambda < \lambda < 1$. There exists a unique solution $c_\infty(\lambda)$ of the following algebraic equation

$$c_\infty^{\lambda_0}(\frac{\lambda-\lambda_0 c_\infty}{\lambda-\lambda_0})^{1-\lambda_0} = 1, \quad 0 < c_\infty < \lambda. \tag{8.10}$$

In addition, the map $\lambda \mapsto c_\infty(\lambda)$ is strictly increasing with $c_\infty(\lambda) \geq c_\infty(\tilde\lambda) > \mu(\tilde\lambda)$, and $\lim_{\lambda\uparrow 1} c_\infty(\lambda) = 1$. Finally, one has $C(\tilde T_\varepsilon) = c_\infty(\lambda)$, and $\tilde T_\varepsilon \leq K(\lambda)T_\varepsilon$.

Proof. The proof of existence and uniqueness of a solution $c_\infty(\lambda)$ of (8.10) is similar to Lemma 4.4 in [65]. We skip the details.

Let $\tilde c_\infty(\lambda,\varepsilon) := \lim_{+\infty} C$. From (8.3) and $\lim_{+\infty} P = -\infty$ one has

$$\tilde c_\infty^{\lambda_0}(\frac{\lambda-\lambda_0\tilde c_\infty}{\lambda-\lambda_0})^{1-\lambda_0} = \frac{1}{a^p(-\varepsilon^{-1/100})}.$$

with $0 < \tilde c_\infty < \lambda$. From the behavior of a in (1.13) we have $\tilde c_\infty = c_\infty(\lambda) + O(\varepsilon^{10})$, for all ε small. This implies that

$$\lambda - \tilde c_\infty(\lambda,\varepsilon) \geq \frac{99}{100}(\lambda - c_\infty(\lambda)) > 0,$$

uniformly for all ε small enough. On the other hand, at time $t = \tilde T_\varepsilon$ one has

$$C(\tilde T_\varepsilon)^{\lambda_0}(\frac{\lambda-\lambda_0 C(\tilde T_\varepsilon)}{\lambda-\lambda_0})^{1-\lambda_0} = \frac{a^p(-\varepsilon(1-\lambda)T_\varepsilon)}{a^p(-\varepsilon^{-1/100})} = 1, \qquad 0 < C(\tilde T_\varepsilon) < \lambda,$$

therefore by uniqueness $C(\tilde T_\varepsilon) = c_\infty(\lambda)$.

Finally, we prove the upper bound on $\tilde T_\varepsilon$. We have

$$P(-T_\varepsilon) = -(1-\lambda)T_\varepsilon = -(1-\lambda)T_\varepsilon + \int_{-T_\varepsilon}^{\tilde T_\varepsilon}(C(s)-\lambda)ds.$$

From here we have for $\beta > 0$

$$\begin{aligned} 0 &= \int_{-T_\varepsilon}^{t_0-\frac{\beta}{\varepsilon}} (C(s)-\lambda)ds + \int_{t_0-\frac{\beta}{\varepsilon}}^{t_0+\frac{\beta}{\varepsilon}} (C(s)-\lambda)ds - \int_{t_0+\frac{\beta}{\varepsilon}}^{\tilde{T}_\varepsilon} (\lambda - C(s))ds \\ &\leq (1-\lambda)(t_0 + \frac{\beta}{\varepsilon} + T_\varepsilon) + \frac{K\beta}{\varepsilon} - \int_{t_0+\frac{\beta}{\varepsilon}}^{\tilde{T}_\varepsilon} (\lambda - C(s))ds. \end{aligned}$$

Similarly to estimate (8.7), one has for $\beta > 0$ small, but independent of ε,

$$C(t_0 + \frac{\beta}{\varepsilon}) \leq \lambda - \nu(\lambda)\beta + O(\beta^2), \quad \nu(\lambda) > 0.$$

Inserting this estimate above, and using the estimate on t_0, one has

$$\tilde{T}_\varepsilon \leq K(\lambda) T_\varepsilon,$$

as desired. □

Part III

On the soliton dynamics under slowly varying medium of nonlinear Schrödinger equations

Summary

Abstract

We consider the problem of the soliton propagation, in a slowly varying medium, for a generalized, variable-coefficients nonlinear Schrödinger equation. We prove global existence and uniqueness of soliton-like solutions for a large class of slowly varying media. Moreover, we describe for all time the behavior of this new generalized soliton solution.

Keywords : Nonlinear Schrödinger equations, soliton dynamics, slowly varying potentials.

1 Introduction and Main Results

In this work we continue our study of soliton-propagation under an inhomogeneous medium, started in [65]. Now we consider the following *generalized nonlinear Schrödinger equation* (NLS)

$$iu_t + u_{xx} + f(x, |u|^2)u = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.1}$$

Here $u = u(t, x)$ is a complex-valued function, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a nonlinear function. This equation is a generalization of the –one dimensional– semilinear *nonlinear Schrödinger equation* (NLS)

$$iu_t + u_{xx} + |u|^{m-1}u = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x; \quad m > 1. \tag{1.2}$$

Concerning the *cubic nonlinear Schrödinger* equation (namely the case $m = 3$), it arises in Physics as a model of wave propagation in fiber optics in a nonlinear medium, and also describes the evolution of the envelope of modulated wave groups in water waves. In two dimensions, the cubic NLS also possesses an important physical meaning.

The Cauchy problem for equation (1.2) (namely, adding the initial condition $u(t = 0) = u_0$) is *locally well-posed* for $u_0 \in H^1(\mathbb{R})$ (see Ginibre and Velo [22]). In addition, solutions of (1.2) are invariant under translations in space, time and phase. From the Noëther theorem, these symmetries are related to *conserved quantities*, invariant under the NLS flow, usually called *mass*, *energy* and *momentum*:

$$M[u](t) := \int_{\mathbb{R}} |u|^2(t, x)\, dx = \int_{\mathbb{R}} |u_0|^2(x)\, dx = M[u](0), \quad \text{(Mass)}, \tag{1.3}$$

$$\begin{aligned} E[u](t) \; &:= \; \frac{1}{2}\int_{\mathbb{R}} |u_x|^2(t, x)\, dx - \frac{1}{m+1}\int_{\mathbb{R}} |u|^{m+1}(t, x)\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}} |(u_0)_x|^2(x)\, dx - \frac{1}{m+1}\int_{\mathbb{R}} |u_0|^{m+1}(x)\, dx = E[u](0), \; \text{(Energy)} \end{aligned} \tag{1.4}$$

and

$$P[u](t) := \frac{1}{2}\operatorname{Im}\int_{\mathbb{R}} \bar{u}u_x(t, x)\, dx = \frac{1}{2}\operatorname{Im}\int_{\mathbb{R}} \bar{u}_0(u_0)_x(x)\, dx = P[u](0), \; \text{(Momentum)}. \tag{1.5}$$

In the case $1 < m < 5$, any $H^1(\mathbb{R})$ solution is global in time thanks to the conservation of mass and energy (1.3)-(1.4), and the Galiardo-Nirenberg inequality

$$\int_{\mathbb{R}} u^{p+1} \leq K(p)\big(\int_{\mathbb{R}} u^2\big)^{\frac{p+3}{4}}\big(\int_{\mathbb{R}} u_x^2\big)^{\frac{p-1}{4}}. \tag{1.6}$$

One of the main properties of NLS equations is the existence of localized, exponentially decaying, stable and smooth solutions called *solitons*, or *traveling waves*. Given four real numbers x_0, v_0, γ_0 and $c_0 > 0$, traveling waves are solutions of (1.2) of the form

$$u(t,x) := Q_{c_0}(x - x_0 - v_0 t)e^{i(c_0 - \frac{1}{4}v_0^2)t}e^{i\gamma_0}e^{\frac{i}{2}v_0 x}, \tag{1.7}$$

with $Q_c(s) := c^{\frac{1}{m-1}}Q(c^{1/2}s)$, where Q is the explicit Schwartz function satisfying the second order nonlinear differential equation

$$Q'' - Q + Q^m = 0, \quad Q > 0, \quad Q(x) = \Big[\frac{m+1}{2\cosh^2(\frac{(m-1)}{2}x)}\Big]^{\frac{1}{m-1}} \sim e^{-|x|}. \tag{1.8}$$

In particular, for $v_0 > 0$, this solution represents a *solitary wave*, with *invariant profile*, defined for all time moving to the right with constant velocity.

For $m \geq 5$, solitons are shown to be *orbitally unstable* and the Cauchy problem for the corresponding NLS equation has finite-time blow-up solutions, see [12] and references there in. In this work, in order to guarantee the stability of soliton solutions, *we will discard high-order nonlinearities*. In other words, we will only consider the case $1 < m < 5$.

The study of perturbations of solitary waves lead to the introduction of the concepts of *orbital and asymptotic stability*. Orbital stability of ground states for NLS equations has been widely studied during last decades; we mention the works of Cazenave and Lions [13], Weinstein [83, 84], Grillakis, Shatah and Strauss [23, 24], Cuccagna [14], and Martel, Merle and Tsai [60]. See references therein for a more detailed bibliography. On the other hand, asymptotic stability of solitary waves and related scattering results have been studied in [78, 80, 11, 72, 15, 74].

The problem we consider in this paper possesses a large physical and mathematical literature. In the next subsection we briefly describe the main results concerning the propagation of solitons in slowly varying medium.

1.1 Statement of the problem, historical review

The dynamical problem of soliton interaction with a slowly varying medium is by now a classical problem in nonlinear wave propagation, representing a simple model of several physical applications. By soliton-medium interaction we mean, loosely speaking, the following problem: In (1.1), consider a nonlinear function $f = f(t,x,s)$, slowly varying in space and time, possibly of small amplitude, of the form

$$f(t,x,s^2) \sim |s|^{m-1} \quad \text{as } x \to \pm\infty, \quad \text{for all time;}$$

(namely (1.1) behaves like a NLS equation at spatial infinity.) *Consider* a solitary wave solution (note that this assertion must be actually proved) of the corresponding variable coefficient equation (1.1) with this nonlinearity, at some early time. Then we expect that this solution does interact with the medium, here represented by the nonlinearity $f(t,x,s)$. In a slowly varying medium this interaction, small locally in time, may be significantly important for the long time behavior of the solution. The emerging solution after the interaction is precisely the object of study. In particular, one considers if any change in size, position, or shape, even creation or destruction of solitons, if any, after some large time, may be present.

Let us review some relevant works in this direction. Kaup and Newell [38] studied, via inverse scattering methods, slowly varying perturbations of integrable equations. In particular, they considered the following perturbed NLS equation

$$iu_t + u_{xx} + |u|^2u = a(\varepsilon x)u_x. \tag{1.9}$$

Here the additional term $a(\varepsilon x)u_x$ is intended to describe e.g. depth variations on a surface gravity wave packet. The authors studied the case where $a(\varepsilon x) := \varepsilon x^2$ and showed that, for a small ε, the soliton shape remains unchanged, but both velocity and position parameters evolve following a trapped trajectory of an harmonic oscillator at leading order.

Subsequently, this problem has been addressed in several other works and for different integrable models. In [26], the author considered the time dependent NLS equation

$$iu_t + u_{xx} + a(\varepsilon t)|u|^2u = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x.$$

(See [26] for the physical description associated to this model.) Using a perturbative analysis the author found an approximate solution up to second order in ε. This approximate solution is less dispersive than the corresponding solution for the generalized KdV equation studied in [38], in the sense that it does not present a tail behind the soliton solution as in the gKdV case (see also [25, 65] for more details).

In this paper we address the problem of soliton dynamics in the case of a slowly varying, inhomogeneous medium, but constant in time.

1.2 Setting and hypotheses

Let us come back to the general equation (1.1), and consider $\varepsilon > 0$ a small parameter. Along this work we will assume that the nonlinearity f is sufficiently smooth and slowly varying x-dependent function of the power cases, independent of time:

$$\begin{cases} f(x, s^2) := a_\varepsilon(x)|s|^{m-1}, \quad 2 \le m < 5, \\ a_\varepsilon(x) := a(\varepsilon x); \qquad a \in C^3(\mathbb{R}) \text{ if } m < 3, \ a \in C^4(\mathbb{R}) \text{ if } m \ge 3. \end{cases} \tag{1.10}$$

Concerning the function a we will assume that there exist constants $K, \mu > 0$ such that

$$\begin{cases} 1 < a(r) < 2, \ a'(r) > 0, \ |a^{(k)}(r)| \le Ke^{-\mu|r|} \ \text{ for all } r \in \mathbb{R}, \ k = 1, 2, 3, (4); \\ 0 < a(r) - 1 \le Ke^{\mu r}, \ \text{ for all } r \le 0, \ \text{ and} \\ 0 < 2 - a(r) \le Ke^{-\mu r} \ \text{ for all } r \ge 0. \end{cases} \tag{1.11}$$

In particular, $\lim_{r\to-\infty} a(r) = 1$ and $\lim_{r\to+\infty} a(r) = 2$. We emphasize that the special choice (1 and 2) of the limits is irrelevant for the results of this paper. The only necessary conditions are that

$$0 < a_{-\infty} = \lim_{r\to-\infty} a(r) < \lim_{r\to+\infty} a(r) =: a_\infty < +\infty.$$

Of course the decay hypothesis on a in (1.11) can be relaxed, and the results of this paper still should hold, with more difficult proofs, for asymptotically flat potentials; but for brevity and clarity of the exposition these issues will not be considered in this work.

Recapitulating, we will consider the following 1D *aNLS* equation

$$\begin{cases} iu_t + u_{xx} + a_\varepsilon(x)|u|^{m-1}u = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ 2 \le m < 5; \quad 0 < \varepsilon \le \varepsilon_0; \quad a_\varepsilon \text{ satisfying (1.11)}. \end{cases} \tag{1.12}$$

The main issue that we will study in this paper is the interaction problem between a soliton and a slowly varying medium, here represented by the *potential* a_ε. In other words, we intend to study for (1.12) whether it is possible to generalize the well-known soliton-like solution Q of NLS. Of course, it is by now well-known that in the case $f(t,x,s^2)=f(s^2)$, and under reasonable assumptions (see for example Berestycki and Lions [6]), there exist soliton-like solutions, but our objective here will be the study of soliton solutions under a variable coefficient equation.

To support our beliefs, note that at least heuristically, (1.12) behaves at infinity as similar NLS equations:

$$\begin{cases} iu_t + u_{xx} + |u|^{m-1}u = 0 \quad \text{as } x \to -\infty, \\ iu_t + u_{xx} + 2|u|^{m-1}u = 0 \quad \text{as } x \to +\infty. \end{cases} \tag{1.13}$$

In particular, given $v_0 > 0$, one should be able to construct a soliton-like solution $u(t)$ of (1.12) such that

$$u(t,x) \sim Q(x - v_0 t) e^{\frac{i}{2} v_0 x} e^{i(1-\frac{1}{4}v_0^2)t}, \quad \text{as } t \to -\infty,$$

in some sense to be defined. Here Q is the standard soliton solution introduced in (1.8).

On the other hand, after passing the interaction region, by stability of the soliton, this solution *should behave* like

$$\sim 2^{-\frac{1}{m-1}} Q_{c_\infty}(x - v_\infty t - \rho_\infty(t)) e^{\frac{i}{2}v_\infty x} e^{i\gamma_\infty(t)} + \text{ lower order terms in } \varepsilon, \text{ as } t \to +\infty, \tag{1.14}$$

for ε small enough. Here $c_\infty > 0, v_\infty$ are unknown parameters, and $\rho_\infty(t), \gamma_\infty(t)$ are *small* perturbations. In fact, note that if $v = v(t)$ is a solution of (1.2) then $u(t) := 2^{-1/(m-1)}v(t)$ is a solution of

$$iu_t + u_{xx} + 2|u|^{m-1}u = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.15}$$

In conclusion, this heuristic suggests that even if the potential varies slowly, the soliton should experiment *non trivial* transformations on its shape, scaling and phase, of the same order that of the amplitude of the potential a.

Before stating our main results, some important facts are in order. First, unfortunately equation (1.12) is in general not anymore invariant under scaling and spatial translations. Moreover, a nonzero solution of (1.12) *might gain momentum*, in the sense that, at least formally, the quantity $P[u](t)$ defined in (1.5) now satisfies the identity

$$\partial_t P[u](t) = \frac{\varepsilon}{m+1} \int_{\mathbb{R}} a'(\varepsilon x)|u|^{m+1} \geq 0. \tag{1.16}$$

Therefore the momentum is always a non decreasing quantity. This simple fact will have important consequences in our results, in particular we will obtain from this property the *stability* and *uniqueness* of our solution. The hypothesis $a'(\cdot) > 0$ is crucial in our arguments, although we think it can be relaxed by considering for example a potential satisfying $a'(r) > 0$ for all $|r| > r_0$. We will not pursue on these issues.

On the other hand, the mass $M[u](t)$ defined in (1.3) and the novel energy

$$E_a[u](t) := \frac{1}{2}\int_{\mathbb{R}} |u_x|^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)|u|^{m+1}(t,x)\,dx \tag{1.17}$$

remain formally constant for all time. Moreover, a simple balance of mass and energy at $\pm\infty$ allows to determine heuristically the limiting scaling and velocity parameters in (1.14), if we suppose that the *lower order terms* in (1.14) are of zero mass at infinity. Indeed, we have (cf. Appendix K)

$$M[Q] \sim \frac{c_\infty^{\frac{2}{m-1}-\frac{1}{2}}}{2^{\frac{2}{m-1}}} M[Q], \tag{1.18}$$

and

$$E[Q] + \frac{1}{4}v_0^2 M[Q] \sim \frac{c_\infty^{\frac{2}{m-1}+\frac{1}{2}}}{2^{\frac{2}{m-1}}} E[Q] + \frac{1}{4}v_\infty^2 \frac{c_\infty^{\frac{2}{m-1}-\frac{1}{2}}}{2^{\frac{2}{m-1}}} M[Q], \quad E[Q] \neq 0, \tag{1.19}$$

This implies that $c_\infty \sim 2^{\frac{4}{5-m}} > 1$ and $v_\infty \sim (v_0^2 + 4\frac{(5-m)}{m+3}(c_\infty - 1))^{1/2}$.

These formal arguments suggest the following definition.

Definition 1.1 (Pure generalized soliton-solution for aNLS)**.**

Let $v_0 > 0$ be a fixed number. We will say that (1.12) admits a *pure* generalized soliton-like solution (of scaling equals 1 and velocity equals v_0), if there exist C^1 real valued functions $\rho = \rho(t), \gamma = \gamma(t)$ defined for all large times and a global in time $H^1(\mathbb{R})$ solution $u(t)$ of (1.12) such that

$$\lim_{t\to -\infty} \|u(t) - Q(\cdot - v_0 t) e^{\frac{i}{2}(\cdot)v_0} e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0,$$
$$\lim_{t\to +\infty} \|u(t) - 2^{-\frac{1}{m-1}} Q_{c_\infty}(\cdot - v_\infty t - \rho(t)) e^{\frac{i}{2}(\cdot)v_\infty} e^{i\gamma(t)}\|_{H^1(\mathbb{R})} = 0,$$

with $|\rho'(t)| \ll v_0$ for all large times, and where $c_\infty, v_\infty > 0$ are the scaling and velocity predicted by the mass and energy conservation law, as in (1.18)-(1.19).

As we will see below, a standard method allows us to construct a generalized soliton solution as $t \to -\infty$, as required in the above definition; however, it is expected that any reasonable soliton-like solution would not be able to satisfy the second assertion, because of some very small dispersive effects produced by the potential a_ε.

1.3 Previous analytic results on the soliton dynamics under slowly varying medium

The problem of describing analytically the soliton dynamics of different integrable models under a slowly varying medium has received some increasing attention during the last years. In the framework of NLS equations with non constant potential, the first result in this direction was given by Bronski and Jerrard [10]. In this paper it is proved that in the semiclassical limit, the soliton's mass center obeys the Newton's second law with external force given by the potential's gradient. Gustafson et al. [28, 29] and Holmer et al. [33, 34, 35] have considered the dynamics of a soliton under general potentials, for short times, namely $t \sim \frac{1}{\varepsilon}$. See also [16] for a similar result in the case of a generalized Hartree equation. From these results it seems clear that a deeper understanding of the soliton dynamics for very large times strongly depends on the specific character of the considered potential, as we will see below.

A related problem is the study of the interaction soliton-medium for a generalized Korteweg- de Vries equation, following the physical literature [42, 38, 37, 70]. Dejak, Jonsson and Sigal in [17, 18] considered the long time dynamics of solitary waves (solitons) over slowly varying perturbations of KdV and mKdV equations. Recently Holmer et al. have improved these results, up to quadratic order in ε. Finally, we recall that in the case of the generalized Korteweg- de Vries equation

$$u_t + (u_{xx} - \lambda u + a_\varepsilon(x)u^m)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad \lambda \geq 0,$$

we have described in [65] the dynamics of a generalized soliton solution. We proved, among other things, that no pure soliton solution is present for any small $\varepsilon > 0$ and $\lambda > 0$. In this paper, our main objective is to extend some of these results to (1.12).

1.4 Main Results

Let

$$T_\varepsilon := \frac{1}{v_0}\varepsilon^{-1-\frac{1}{100}} > 0, \tag{1.20}$$

and

$$p_m := \begin{cases} 1, & \text{if } m \in [2,3), \\ 2, & \text{if } m \in [3,5). \end{cases} \tag{1.21}$$

The first parameter can be understood as the *interaction time* between the soliton and the potential. In other words, at time $t = -T_\varepsilon$ the soliton should remain almost unperturbed, and at time $t = T_\varepsilon$ the soliton should have completely crossed the influence region of the potential. Note that the asymptotic $v_0 \sim 0$ depending on ε is a degenerate case and it will be discarded for this work.

Second, the parameter p_m measures the degree of accuracy of the main result, based in a Taylor expansion of the nonlinearity involved. In other words, the smoother the nonlinearity, the more accurate the main result.

In what follows, we assume the validity of above hypotheses, namely (1.10) and (1.11). Our first result is a complete description, for all times, of the interaction soliton-potential for the aNLS equation (1.12).

Theorem A 1 (Dynamics of a generalized soliton-solution for aNLS equation)**.**

Assume that $a(\cdot)$ satisfies (1.11). Let $2 \le m < 5$, $v_0 > 0$, $\lambda_0 := \frac{5-m}{m+3}$ and p_m be as in (1.21). There exists a small constant $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.

1. Existence of a soliton-like solution.

   There exists a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (1.12), global in time, such that

$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - v_0 t)e^{i(\cdot)v_0/2}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0, \tag{1.22}$$

   with conserved mass $M[u](t) = M[Q]$ and energy $E_a[u](t) = (\frac{1}{4}v_0^2 - \lambda_0)M[Q]$.

2. Stability of interaction soliton-potential. *Let*

$$\lambda_\infty := 2^{-\frac{1}{m-1}}, \quad c_\infty := 2^{\frac{4}{5-m}}\ (>1), \quad v_\infty := (v_0^2 + 4\lambda_0(c_\infty - 1))^{\frac{1}{2}}\ (> v_0). \tag{1.23}$$

   There exist $K > 0$, and C^1- functions $\rho(t), \gamma(t) \in \mathbb{R}$ defined for all $t \ge \frac{1}{2}T_\varepsilon$ such that the function

$$w(t,x) := u(t,x) - \lambda_\infty Q_{c_\infty}(x - v_\infty t - \rho(t))e^{\frac{i}{2}xv_\infty}e^{i\gamma(t)},$$

   satisfies for all $t \ge \frac{1}{2}T_\varepsilon$,

$$\|w(t)\|_{H^1(\mathbb{R})} + |\rho'(t)| + |\gamma'(t) - c_\infty + \frac{1}{4}v_\infty^2| \le K\varepsilon^{p_m}. \tag{1.24}$$

Remark 1.1. One may compare the above result with Theorems 1.1 and 1.2 in [65], where a bound of order $\varepsilon^{1/2}$ was showed (note that $p_m \ge 1$). Our present result is better due to the absence of a *dispersive tail* behind the soliton, precisely of order $\varepsilon^{1/2}$ in $H^1(\mathbb{R})$, and present in the gKdV case. A first mathematical treatment this phenomenon can be found in [57]. Let us finally recall that such dispersive elements in a soliton solution are not present in the case of a pure NLS or gKdV equation.

Remark 1.2. We do not discard the existence of very small solitons after the interaction, with size of order at most ε^{p_m} in $H^1(\mathbb{R})$. This question is also related to the question of scattering modulo-solitons.

One may wonder whether Theorem A is available for other potentials. A first answer in that direction, is the following remark.

Remark 1.3 (Decreasing potential). Pick now a potential $a(\cdot)$ and an initial velocity $v_0 > 0$ satisfying e.g. $a'(s) < 0$,

$$1 = \lim_{s\to-\infty} a(s) > a(s) > \lim_{t\to+\infty} a(s) = \frac{1}{2},$$

and $v_0^2 > 4\lambda_0(1 - 2^{-\frac{4}{5-m}})$. Then there exists a solution $u_\#(t)$ satisfying (1.22), and (1.24) for times $t \sim T_\varepsilon$ (and a little bit more), with the following minor modifications:

$$\lambda_\infty := 2^{\frac{1}{m-1}}(> 1), \quad c_\infty := 2^{-4/(5-m)}(< 1), \quad \text{and} \quad v_\infty := (v_0^2 + 4\lambda_0(c_\infty - 1))^{1/2}.$$

The *uniqueness* and *stability* for large times of this solution is not known, mainly due to the fact that the momentum law has now the opposite sign:

$$\partial_t P[u](t) = \frac{\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)|u|^{m+1} \leq 0.$$

What happens in the regime $v_0^2 \leq 4\lambda_0(1 - c_\infty)$ is also an interesting open question. Formal computations suggest a possible reflection of the soliton in the case of small initial velocity. We hope to consider some of these situations in a forthcoming publication.

We believe that the analysis in the interaction region can be carried out in a general situation, under asymptotically flat potentials. However, stability and uniqueness properties are probably highly dependent on the nonlinearity considered.

Remark 1.4 (Non existence of pure soliton-like solution). An important problem arises from the above results. Is the solution $u(t)$ constructed in Theorem A above an exactly pure solitary wave for the aNLS equation? (cf. Definition 1.1.) This question is equivalent to decide whether

$$\lim_{t\to+\infty} \|w(t)\|_{H^1(\mathbb{R})} = 0.$$

We have been unable to solve this problem, due to the lack of *backwards stability* for large times[5]. In other words, if we suppose $\|w(T)\|_{H^1(\mathbb{R})} \leq \alpha$ for small α and very large time $T \gg T_\varepsilon$, we do not know if a suitable modulation of $w(t)$ is still small (of order α) at time T_ε. This result is equivalent to obtain stability for a decreasing potential $a(\varepsilon x)$, also an open question (see Remark 1.3).

Let us recall that Theorem 1.3 in [65] puts in evidence the following conjecture: the presence of a non constant potential induces on any generalized solitary wave nontrivial dispersive effects, contrary to the standard NLS and gKdV equations. We believe that the same phenomenon is present in the Schrödinger case.

Remark 1.5 (Time depending potentials). As expected, our results are also valid, with easier proofs, for the following time dependent gKdV equation:

$$iu_t + u_{xx} + a(\varepsilon t)|u|^{m-1}u = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.25}$$

[5]Note that in [65] one has backward stability for all $\lambda > 0$.

Here a satisfies (1.11) now in the time variable. Note that this equation is invariant under scaling and space translations. In addition, the mass $M[u]$ and momentum $P[u]$ remain constants and the energy

$$\tilde{E}[u](t) := \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 - \frac{a(\varepsilon t)}{m+1}\int_{\mathbb{R}} |u|^{m+1}$$

satisfies

$$\partial_t \tilde{E}[u](t) = -\frac{\varepsilon a'(\varepsilon t)}{m+1}\int_{\mathbb{R}} |u|^{m+1}.$$

Furthermore, Theorem A still holds with $\lambda_\infty = 2^{-1/(m-1)}$, and $c_\infty = 2^{4/(5-m)}$. We left the details to the reader.

1.5 The two dimensional case

A natural question arising from the above results is their extension to higher dimensions. Very few results are valable on this topic, apart from the aforementioned works [29, 28].

Here we shall consider the two dimensional case with a potential $a(\cdot)$ depending only on one spatial variable. Indeed, let $x = (x_1, x_2) \in \mathbb{R}^2$. For $\varepsilon > 0$ small, consider the following aNLS equation

$$iu_t + \Delta u + a(\varepsilon x_1)|u|^{m-1}u = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}^2_x, \quad 2 \leq m < 3. \tag{1.26}$$

We assume $a = a(r)$ satisfying (1.11). The exponent m is chosen to ensure a subcritical regime in L^2 and global wellposedness with L^2 and H^1 data (cf. [22]). The mass $M(t)$, energy $E_a(t)$ and –vectorial– momentum $P(t)$ in (1.3)-(1.5) are defined in the usual way. From the above assumptions we have mass and energy formally conserved, and

$$\partial_t P[u](t) = \frac{\varepsilon e_1}{m+1}\int_{\mathbb{R}} a'(\varepsilon x_1)|u|^{m+1}(t,x)dx \geq 0. \tag{1.27}$$

Here e_1 is the first unitary vector in $\mathbb{R}^2$.

Concerning *solitons* solutions, given $x_0, \tilde{v}_0 \in \mathbb{R}^2$, $\gamma_0 \in \mathbb{R}$ and $c_0 > 0$, there exists a solution of the two-dimensional version of (1.2) of the form

$$u(t,x) := Q_{c_0}(x - x_0 - \tilde{v}_0 t)e^{i(c_0 - \frac{1}{4}|\tilde{v}_0|^2)t}e^{i\gamma_0}e^{\frac{i}{2}\tilde{v}_0\cdot x}, \tag{1.28}$$

with $Q_c(s) := c^{\frac{1}{m-1}}Q(c^{1/2}s)$. Here Q is the unique (modulo translations) Schwartz function satisfying the second order nonlinear elliptic equation

$$\Delta Q - Q + Q^m = 0, \quad Q > 0, \quad |Q(x)| \leq Ke^{-|x|}. \tag{1.29}$$

For this case, we have the following positive result.

Theorem B 1 (Dynamics of a two-dimensional generalized soliton-solution)**.**

Assume the preceding hypotheses. Let $2 \leq m < 3$, and $v_0 > 0$. There exists a small constant $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.

1. Existence of a soliton-like solution.

   There exists a unique *solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^2))$ of (1.26), global in time, such that*

$$\lim_{t\to -\infty} \|u(t) - Q(\cdot - v_0 e_1 t)e^{i(\cdot)v_0 e_1/2}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R}^2)} = 0.$$

2. Stability of interaction soliton-potential.

Let $\lambda_\infty = 2^{-\frac{1}{m-1}}$, $c_\infty := 2^{2/(3-m)}$, *and*

$$v_\infty = v_\infty(v_0) := (v_0^2 + \alpha_0(c_\infty - 1))^{\frac{1}{2}}, \quad \text{with } \alpha_0 := \frac{4(3-m)}{m+1} \times \frac{\int Q^{m+1}}{\int Q^2}. \tag{1.30}$$

There exist $K > 0$ *and* C^1*-functions* $\gamma(t) \in \mathbb{R}$, $\rho(t) \in \mathbb{R}^2$ *defined for all* $t \geq \frac{1}{2}T_\varepsilon$ *such that the function*

$$w(t,x) := u(t,x) - \lambda_\infty Q_{c_\infty}(x - v_\infty e_1 t - \rho(t)) e^{\frac{i}{2}x\cdot v_\infty e_1} e^{i\gamma(t)}$$

satisfies for all $t \geq T_\varepsilon$,

$$\|w(t)\|_{H^1(\mathbb{R}^2)} + |\rho'(t)| + |\gamma'(t) - c_\infty + \frac{1}{4}v_\infty^2| \leq K\varepsilon. \tag{1.31}$$

Remark 1.6. The proof of this theorem is close the proof of Theorem A. Note that uniqueness and stability follow from the fact that for any constant $v > 0$,

$$\partial_t\{ve_1 \cdot P[u](t)\} \geq 0. \tag{1.32}$$

In section 4 we sketch the main lines of the proof.

Remark 1.7. The restriction to the two dimensional case is a consequence of the lack of smoothness for the power nonlinearity in higher dimensions. We believe that the above results remain valid for a sufficiently smooth nonlinearity of the form $f(x,|u|^2)u$ (e.g. $f(x,s^2) := a_\varepsilon(x)(s^2 + a_0 s^4)$, with a_0 small enough in the one dimensional case.)

Last, thanks to the invariance of (1.26) with respect to Galilean boosts on the x_2 direction we obtain the following striking result.

Corolary C 1 (Description of the soliton dynamics for a *general* incident velocity)**.**

Let $\tilde v = (\tilde v_1, \tilde v_2) \in \mathbb{R}^2$ *be an initial velocity such that* $\tilde v \cdot e_1 > 0$. *Then Theorem B holds with the obvious modifications, and with* ε_0 *independent of* $\tilde v_2$. *Moreover, the final velocity is given by* $\tilde v_\infty := (v_\infty(\tilde v_1), \tilde v_2)$.

Remark 1.8. Note that in this situation one has the following *refraction law* among the two velocities and the angles of incidence ($\theta_{-\infty}$) and refraction ($\theta_{+\infty}$):

$$|\tilde v| \sin\theta_{-\infty} = |\tilde v_\infty| \sin\theta_{+\infty}.$$

Proof of Corolary C. Note that $\tilde v_1 > 0$. Since any solution of (1.26) is invariant under the Galilean transformation

$$\mathcal{G}[u](t,x) = \mathcal{G}[u](t,x_1,x_2) = u(t,x_1,x_2 - \tilde v_2 t) e^{\frac{i}{2}x_2\tilde v_2} e^{-\frac{i}{4}\tilde v_2^2 t},$$

we may suppose without loss of generality that $\tilde v = v_0 e_1$, for $v_0 = |\tilde v| > 0$. We apply Theorem B with this new data. The conclusion follows at once. □

Remark 1.9. The proof of non existence of pure soliton-like solutions for this case remains an open problem.

Before starting the computations, let us explain the main ideas behind the proof of Theorems A and B.

1.6 Main ideas of the proof

Similarly to [65], the proof of our results are mainly based on the construction of a new approximate solution of (1.12) in the interaction region, see e.g. [49, 53, 58, 57, 64] for similar computations. The construction requires several new computations, up to second order in ε in the best cases, in order to describe with enough accuracy the behavior of the soliton solution.

The idea is as follows: one separates the analysis among three different time intervals: $t \ll -\varepsilon^{-1}$, $|t| \leq \frac{K}{\varepsilon}$ and $\varepsilon^{-1} \ll t$. On each interval the solution possesses a specific behavior, as is now described.

Indeed, in the first interval of time we prove that $u(t)$ remains very close to a soliton-solution with no change in the scaling, velocity, phase and shift parameters. This result is possible for negative very large times, where the soliton is still far from the interacting region $|t| \leq \varepsilon^{-1}$, and the potential is essentially $a \equiv 1$. The idea is to use a compactness property of the soliton solution to get exponential decay in time of the convergence at infinity in (1.22).

For the second regime, namely $|t| \leq \varepsilon^{-1}$, the soliton-potential interaction leads the dynamics of $u(t)$. The novelty here is the construction of an *approximate solution* of (1.12) with high order of accuracy such that (a) at time $t \sim -\varepsilon^{-1}$ this solution is close to a modulated soliton solution and therefore to $u(t)$; (b) it describes the soliton-potential interaction inside this interval; and (c) it is close to $u(t)$ in the whole interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$, uniformly on time, modulo a modulation on some degenerate directions.

Finally, for times $t \gg \varepsilon^{-1}$, some well known stability properties allow to establish the stability of the solution $u(t)$ as a soliton-like solution, and therefore the proof of Theorem A. These arguments are easy to extrapolate to higher dimensions, giving the proof of Theorem B.

Notation. Along this paper, both $C, K, \mu > 0$ will denote fixed constants, independent of ε, and possibly changing from one line to the other.

Finally, some words about the organization of this paper. First in Section 2 we sketch the proof of Theorem A. Section 3 is devoted to the proof of the main ingredients of Theorem A. In Section 4 we prove Theorem B. Finally Appendices H and I are devoted to the construction of the soliton-like solution for negative large times and to prove the asymptotic behavior as $t \to +\infty$.

2 Proof of Theorem A

The proof is very similar to the proof of Theorem 1.2 in [65], and it is based in three independent results: Propositions 2.1, 2.2 and 2.3. Assuming these three results, the proof of Theorem A is straightforward. For the proof of each Proposition, we did as follows. In Section 3 we prove Proposition 2.2, and in Appendices H and I we prove Propositions 2.1 and 2.3.

Step 1. Construction of a soliton-like solution at infinity. First we prove the existence and uniqueness of a *pure* soliton-like solution for (1.12) for $t \to -\infty$. See e.g. [65], Theorem 1.1 for a related result.

Proposition 2.1 (Existence and uniqueness of a pure soliton-like solution).

There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (1.12) such that

$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0, \tag{2.1}$$

with mass $M[u](t) = M[Q]$ and energy $E_a[u](t) = (\frac{1}{4}v_0^2 - \lambda_0)M[Q]$. Moreover, there exist constants $K, \mu > 0$ such that for all $t \leq -\frac{1}{2}T_\varepsilon$,

$$\|u(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq Ke^{\varepsilon\mu t}. \tag{2.2}$$

In particular,

$$\|u(-T_\varepsilon) - Q(\cdot + v_0 T_\varepsilon)e^{\frac{i}{2}(\cdot)v_0}e^{-i(1-\frac{1}{4}v_0^2)T_\varepsilon}\|_{H^1(\mathbb{R})} \leq Ke^{-\mu\varepsilon^{-\frac{1}{100}}} \leq K\varepsilon^{10}, \tag{2.3}$$

provided $0 < \varepsilon < \varepsilon_0$ small enough.

Proof. See Appendix H. □

Note that the mass and energy identities above follow directly from (2.1), Appendix K and the energy conservation law from Proposition 3.1. Hereafter, we consider *the* solution $u(t)$ given by the above Proposition.

Step 2. Interaction soliton-potential. The next step in the proof consists on the study of the region of time $[-T_\varepsilon, T_\varepsilon]$, which is the zone where the interaction soliton-potential governs the dynamics.

Recall the definition of λ_∞, c_∞ and v_∞ in (1.23), and p_m in (1.21).

Proposition 2.2 (Dynamics of the soliton in the interaction region)**.**

Suppose $v_0 > 0$. There exist a constant $\varepsilon_0 > 0$ such that the following holds for any $0 < \varepsilon < \varepsilon_0$. Let $u = u(t)$ be a globally defined $H^1(\mathbb{R})$ solution of (1.12) such that

$$\|u(-T_\varepsilon) - Q(\cdot + v_0 T_\varepsilon)e^{\frac{1}{2}i(\cdot)v_0}e^{-i(1-\frac{1}{4}v_0^2)T_\varepsilon}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}. \tag{2.4}$$

Then there exist $K_0 = K_0(K) > 0$, and $\rho_\varepsilon, \gamma_\varepsilon \in \mathbb{R}$ such that

$$\|u(T_\varepsilon) - \lambda_\infty Q_{c_\infty}(\cdot - \rho_\varepsilon)e^{\frac{i}{2}(\cdot)v_\infty}e^{i\gamma_\varepsilon}\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{p_m}, \tag{2.5}$$

and

$$\frac{99}{100}v_0 T_\varepsilon \leq \rho_\varepsilon \leq \frac{101}{100}(2v_\infty - v_0)T_\varepsilon. \tag{2.6}$$

Proof. See Section 3 for a proof of this Proposition and some additional but not less important properties related to this result. □

We apply the above Proposition as follows. From (2.3), one has directly (2.4). Then the solution $u(t)$ satisfies (2.5) and (2.6). We are done.

Last step. Long time behavior. The final step of the proof is the use of the following result.

Proposition 2.3 (Stability in $H^1(\mathbb{R})$)**.**

Suppose $2 \leq m < 5$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ the following hold. Suppose that for some time $t_1 \geq \frac{1}{2}T_\varepsilon$, $v_0 t_1 \leq X_0$ and $\gamma_0 \in \mathbb{R}$ and $K > 0$,

$$\|u(t_1) - \lambda_\infty Q_{c_\infty}(\cdot - X_0)e^{\frac{i}{2}xv_\infty}e^{i\gamma_0}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}. \tag{2.7}$$

where $u(t)$ is a global H^1-solution of (1.12).

Then there exist $K_0 > 0$ and C^1-functions $\rho_2(t), \gamma_2(t) \in \mathbb{R}$ defined in $[t_1, +\infty)$ such that

$$w(t) := u(t) - \lambda_\infty Q_{c_\infty}(\cdot - v_\infty t - \rho_2(t))e^{\frac{i}{2}(\cdot)v_\infty}e^{i\gamma_2(t)},$$

satisfies for all $t \geq t_1$,

$$\|w(t)\|_{H^1(\mathbb{R})} + |\rho_2'(t)| + |\gamma_2'(t) - c_\infty + \frac{1}{4}v_\infty^2| \leq K_0\varepsilon^{p_m}, \tag{2.8}$$

where, for some $K > 0$,

$$|\rho_2(t_1) + v_\infty t_1 - X_0| + |\gamma_2(t_1) - \gamma_0| \leq K\varepsilon^{p_m}.$$

Proof. For the proof, see Appendix I. □

End of proof of Theorem A.

We conclude in the following form: define $t_1 := T_\varepsilon$, $X_0 := \rho_\varepsilon$ and $\gamma_0 := \gamma_\varepsilon$. From (2.5)-(2.6) we have (2.7) and therefore (2.8). By renaming $\rho(t) := \rho_2(t)$, $\gamma(t) := \gamma_2(t)$, we have that from (2.8) and (1.23) we obtain (1.24). The proof is now complete, provided Propositions 2.1, 2.2 and 2.3 are valid.

3 Proof of Proposition 2.2

The proof of Proposition 2.2 is divided in four steps. In the first part, we introduce some basic notation. Next, in Step 2 we construct an approximate solution $\tilde{u}$ solving (1.12) up to second order in ε in the best cases. Then in Step 3 we prove that $\tilde{u}$ is close to an actual solution up to order ε^{p_m} in the whole interval $[-T_\varepsilon, T_\varepsilon]$. Finally, in Step 4 we conclude.

Step 1. Preliminaries.

3.1 Cauchy Problem

First we recall the local well-posedness theory for the Cauchy problem associated to (1.12).

Let $u_0 \in H^1(\mathbb{R})$. We consider the following initial value problem

$$\begin{cases} iu_t + u_{xx} + a_\varepsilon(x)|u|^{m-1}u = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad 2 \leq m < 5, \\ u(t=0) = u_0. \end{cases} \tag{3.1}$$

Following [12], thanks to the subcritical character of the nonlinearity and the bounds on the potential, we have the following result.

Lemma 3.1 (Local and global well-posedness in $H^1(\mathbb{R})$, see [12]).

Suppose $u_0 \in H^1(\mathbb{R})$. Then there exist a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (3.1). Moreover, for any $t \in \mathbb{R}$ the mass $M[u](t)$ and the energy $E_a[u](t)$ from (1.17) remain constant, and the momentum $P[u](t)$ defined in (1.5) obeys (1.16). The same result is valid for $L^2(\mathbb{R})$ data.

Proof. The proof is standard, and it is based in a Picard iteration procedure. For the proof see Example 3.2.11, Theorem 4.3.1, Corollary 4.3.3 and Corollary 6.1.2 in [12]. □

We will also need some properties of the corresponding linearized operator of (1.12). For the proofs, see e.g. [53].

3.2 Spectral properties of the linear NLS operator

Fix $c > 0$, $m = 2, 3$ or 4, and let

$$\mathcal{L}_+ w(y) := -w_{yy} + cw - mQ_c^{m-1}(y)w, \quad \text{and} \quad \mathcal{L}_- w(y) := -w_{yy} + cw - Q_c^{m-1}(y)w; \tag{3.2}$$

where $w = w(y)$. Then one has

Lemma 3.2 (Spectral properties of $\mathcal{L}_\pm$, see [54]).

The operators $\mathcal{L}_\pm$ defined (on $L^2(\mathbb{R})$) by (2.9) *have as domain of definition the space $H^2(\mathbb{R})$. In addition, they are self-adjoint and satisfy the following properties:*

1. First eigenvalue. *There exist a unique $\lambda_m > 0$ such that $\mathcal{L}_+ Q_c^{\frac{m+1}{2}} = -\lambda_m Q_c^{\frac{m+1}{2}}$.*

2. *The kernel of $\mathcal{L}_+$ and $\mathcal{L}_-$ is spawned by Q_c' and Q_c respectively. Moreover,*
$$\Lambda Q_c := \partial_{c'} Q_{c'}\big|_{c'=c} = \frac{1}{c}\Big[\frac{1}{m-1} Q_c + \frac{1}{2} x Q_c'\Big], \tag{3.3}$$
satisfies $\mathcal{L}_+(\Lambda Q_c) = -Q_c$. Finally, the continuous spectrum of $\mathcal{L}_\pm$ is given by $\sigma_{cont}(\mathcal{L}_\pm) = [c, +\infty)$.

3. Inverse. *For all $h = h(y) \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} hQ' = 0$ (resp. $\int_{\mathbb{R}} hQ = 0$), there exists a unique $h_+ \in H^2(\mathbb{R})$ (resp. $h_- \in H^2(\mathbb{R})$) such that $\int_{\mathbb{R}} h_+ Q' = 0$ (resp. $\int_{\mathbb{R}} h_- Q = 0$) and $\mathcal{L}_+ h_+ = h$ (resp $\mathcal{L}_- h_- = h$). Moreover, if h is even (resp. odd), then $h_\pm$ is even (resp. odd).*

4. Regularity in the Schwartz space $\mathcal{S}$. *For $h \in H^2(\mathbb{R})$, $\mathcal{L}_\pm h \in \mathcal{S}$ implies $h \in \mathcal{S}$.*

5. Coercivity. *There exists $\nu_0 > 0$ such that the following is satisfied.*

   (a) For $w = w(y) \in H^1(\mathbb{R})$, define
   $$\mathcal{B}[w,w] := \frac{1}{2}\int_{\mathbb{R}} (|w_y|^2 + |w|^2 - Q_c^{m-1}|w|^2 - (m-1)Q_c^{m-1}(\operatorname{Re} w)^2)$$
   Suppose that $\operatorname{Im}\int_{\mathbb{R}} \bar{w} Q_c = \operatorname{Re}\int_{\mathbb{R}} \bar{w} Q_c' = 0$. Then one has
   $$\mathcal{B}[w,w] \geq \nu_0 \int_{\mathbb{R}} |w|^2 - K|\operatorname{Re}\int_{\mathbb{R}} \bar{w} Q_c|^2.$$
   for some $K > 0$.

   (b) Suppose now that for $v \neq 0$, and $\theta \in \mathbb{R}$ one has
   $$\operatorname{Re}\int_{\mathbb{R}} \bar{w} Q_c' e^{iyv/2} e^{i\theta} = \operatorname{Im}\int_{\mathbb{R}} \bar{w} Q_c e^{iyv/2} e^{i\theta} = 0.$$
   Then
   $$\tilde{\mathcal{B}}[w,w] \geq \nu_0 \int_{\mathbb{R}} |w|^2 - K|\operatorname{Re}\int_{\mathbb{R}} \bar{w} Q_c e^{iyv/2} e^{i\theta}|^2,$$
   where $\tilde{\mathcal{B}}[w,w] := \mathcal{B}[we^{iyv/2}e^{i\theta}, we^{iyv/2}e^{i\theta}]$.

We finish this paragraph with a last definition. We denote by $\mathcal{Y}$ the set of C^∞ functions f such that for all $j \in \mathbb{N}$ there exist $K_j, r_j > 0$ such that for all $x \in \mathbb{R}$ we have

$$|f^{(j)}(x)| \leq K_j(1 + |x|)^{r_j} e^{-\frac{1}{2}|x|}. \tag{3.4}$$

Recall that Q_c is a function in $\mathcal{Y}$, for $c \geq \frac{1}{4}$.

Step 2. Construction of the approximate solution.

We look for $\tilde{u}(t,x)$, an approximate solution for (1.1), carrying out a specific structure. We want $\tilde{u}$ as a suitable modulation of a solitary wave, solution of the NLS equation

$$iu_t + u_{xx} + |u|^{m-1}u = 0, \tag{3.5}$$

plus some extra terms, of small order in ε. Indeed, for $t \in [-T_\varepsilon, T_\varepsilon]$, let

$$\rho(t), \gamma(t) \in \mathbb{R}, \quad c(t), v(t) > 0,$$

to be fixed later. Consider

$$y := x - \rho(t), \quad \text{and} \quad \tilde{R}(t,x) := \frac{Q_{c(t)}(y)}{\tilde{a}(\varepsilon\rho(t))} e^{i\Theta(t,x)}, \tag{3.6}$$

where

$$\tilde{a} := a^{\frac{1}{m-1}}, \quad \Theta(t,x) := \int_0^t c(s)ds + \frac{1}{2}v(t)x - \frac{1}{4}\int_0^t v^2(s)ds + \gamma(t). \tag{3.7}$$

In addition, we will search for *bounded* parameters (c, v, ρ, γ) satisfying the following constraints:

$$\frac{1}{2} \leq c(t) \leq 2^5, \quad \frac{1}{2}v_0 \leq v(t) \leq v_0 + 2^5, \quad |\rho'(t) - v(t)| \leq \frac{v_0}{100}, \quad \gamma(t) \in \mathbb{R}. \tag{3.8}$$

By now we only need these hypotheses. Later we will construct a quadruplet (c, v, γ, ρ) with better estimates, see (3.63)-(3.64).

On the other hand, the form of the ansatz $\tilde{u}(t,x)$ is the sum of the soliton plus a small correction term:

$$\tilde{u}(t,x) := \tilde{R}(t,x) + w(t,x), \tag{3.9}$$

where the correction term depends on the nonlinearity we consider:

$$w(t,x) := \begin{cases} \varepsilon(A_{1,c}(t,y) + iB_{1,c}(t,y))e^{i\Theta}, & \text{in the case } 2 \leq m < 3, \\ \displaystyle\sum_{k=1,2} \varepsilon^k(A_{k,c}(t,y) + iB_{k,c}(t,y))e^{i\Theta}, & \text{for the case } 3 \leq m < 5, \end{cases} \tag{3.10}$$

where $A_{k,c}(t,y) := A_k(t, \sqrt{c(t)}y)$, and A_k, B_k are unknown real valued functions to be determined.

Let us be more precise. Given $k = 1$ (for $m < 3$), or $k = 1$ or 2 for $m \geq 3$, we will search for functions $(A_{k,c}(t,y), B_{k,c}(t,y))$ such that for all $t \in [-T_\varepsilon, T_\varepsilon]$ and for some fixed constants $K, \mu > 0$,

$$\|A_{k,c}(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|B_{k,c}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq Ke^{-\mu\varepsilon|\rho(t)|}, \qquad A_{k,c}(t,\cdot), B_{k,c}(t,\cdot) \in \mathcal{Y}. \tag{3.11}$$

We want to measure the size of the error produced by inserting $\tilde{u}$ as defined in (3.10) in the equation (1.1). For this purpose, let

$$S[\tilde{u}](t,x) := i\tilde{u}_t + \tilde{u}_{xx} + a_\varepsilon(x)|\tilde{u}|^{m-1}\tilde{u}. \tag{3.12}$$

The next result gives the error associated to such an approximated solution.

Proposition 3.3 (Decomposition of $S[\tilde u]$).

Let $\Lambda A_c := \partial_c A_c$. For every $t \in [-T_\varepsilon, T_\varepsilon]$, one has the following nonlinear decomposition of the error term $S[\tilde u]$.

1. Case $2 \le m < 3$.

$$\begin{aligned} S[\tilde u](t,x) &:= \mathcal{F}_0(t,y)e^{i\Theta} + \tilde S[\tilde u](t,x) \\ &:= \Big[\mathcal{F}_0(t,y) + \varepsilon \mathcal{F}_1(t,y) + \varepsilon^2 \mathcal{F}_2(t,y) + \varepsilon^3 f(t)\mathcal{F}_c(y)\Big]e^{i\Theta}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \mathcal{F}_0(t,y) &:= -\frac{1}{2}(v'(t) - \varepsilon f_1(t))y\tilde u + i(c'(t) - \varepsilon f_2(t))\partial_c \tilde u \\ &\quad -(\gamma'(t) + \frac{1}{2}v'(t)\rho(t))\tilde u + i(\rho'(t) - v(t))\partial_\rho \tilde u, \end{aligned} \tag{3.14}$$

with $f_1(t) = f_1(c(t), \rho(t))$ and $f_2(t) = f_2(c(t), v(t), \rho(t))$ given by

$$f_1(t) := \frac{8a'(\varepsilon\rho(t))c(t)}{(m+3)a(\varepsilon\rho(t))}, \qquad f_2(t) := \frac{4a'(\varepsilon\rho(t))c(t)v(t)}{(5-m)a(\varepsilon\rho(t))}; \tag{3.15}$$

$$\mathcal{F}_1(t,y) := F_1(t,y) + iG_1(t,y) - \big[\mathcal{L}_+(A_{1,c}) + i\mathcal{L}_-(B_{1,c})\big], \tag{3.16}$$

and

$$F_1(t,y) := \frac{a'(\varepsilon\rho(t))}{\tilde a^m(\varepsilon\rho(t))} yQ_c(y)\big[Q_c^{m-1}(y) - \frac{4c(t)}{m+3}\big], \tag{3.17}$$

$$G_1(t,y) := \frac{a'(\varepsilon\rho(t))v(t)}{\tilde a^m(\varepsilon\rho(t))}\big[\frac{4c(t)}{5-m}\Lambda Q_c(y) - \frac{1}{m-1}Q_c(y)\big]. \tag{3.18}$$

Furthermore, suppose that $(A_{1,c}, B_{1,c})$ satisfy (3.11). Then

$$\|\varepsilon^2(\mathcal{F}_2(t,\cdot) + \varepsilon f(t)\mathcal{F}_c)e^{i\Theta}\|_{H^1(\mathbb{R})} \le K\varepsilon^2(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon), \tag{3.19}$$

uniformly in time.

2. Case $3 \le m < 5$. *Define $\partial_\rho \tilde u := \partial_\rho \tilde R - w_y$. Here one has the improved decomposition*

$$\begin{aligned} S[\tilde u](t,x) &:= \mathcal{F}_0(t,y)e^{i\Theta} + \tilde S[\tilde u](t,x) \\ &:= \big[\mathcal{F}_0(t,y) + \varepsilon \mathcal{F}_1(t,y) + \varepsilon^2 \mathcal{F}_2(t,y) + \varepsilon^3 \mathcal{F}_3(t,y) + \varepsilon^4 f(t)\mathcal{F}_c(y)\big]e^{i\Theta}, \end{aligned} \tag{3.20}$$

with $\mathcal{F}_0$ given now by

$$\begin{aligned} \mathcal{F}_0(t,y) &:= -\frac{1}{2}(v'(t) - \varepsilon f_1(t))y\tilde u + i(c'(t) - \varepsilon f_2(t))\partial_c \tilde u \\ &\quad -(\gamma'(t) + \frac{1}{2}v'(t)\rho(t) - \varepsilon^2 f_3(t))\tilde u + i(\rho'(t) - v(t) - \varepsilon^2 f_4(t))\partial_\rho \tilde u, \end{aligned} \tag{3.21}$$

with f_1, f_2 as in (3.15), and for $\alpha_{(\cdot)}, \beta_{(\cdot)} \in \mathbb{R}$,

$$f_3(t) = f_3(c(t), v(t), \rho(t)) := (\alpha_I + \alpha_{II}\frac{v^2(t)}{c(t)})\frac{a''}{a}(\varepsilon\rho(t)) + (\alpha_{III} + \alpha_{IV}\frac{v^2(t)}{c(t)})\frac{a'^2}{a^2}(\varepsilon\rho(t)), \tag{3.22}$$

and

$$f_4(t) = f_4(c(t), v(t), \rho(t)) := \big\{\beta_I \frac{a''}{a}(\varepsilon\rho(t)) + \beta_{II}\frac{a'^2}{a^2}(\varepsilon\rho(t))\big\}\frac{v(t)}{c(t)}. \tag{3.23}$$

In addition,

$$\mathcal{F}_k(t,y) := F_k(t,y) + iG_k(t,y) - \big[\mathcal{L}_+(A_{k,c}) + i\mathcal{L}_-(B_{k,c})\big], \quad k = 1,2; \tag{3.24}$$

with F_1, G_1 *given by (3.17)-(3.18), and*

$$\begin{aligned} F_2 \;:=\; & \frac{a''}{2\tilde a^m} y^2 Q_c^m + m\frac{a'}{a} Q_c^{m-1} y A_{1,c} - \frac{1}{2} f_1 y A_{1,c} - \frac{1}{\varepsilon}(B_{1,c})_t - f_2 \Lambda B_{1,c} \\ & + \frac{1}{2}(m-1)\tilde a Q_c^{m-2}(mA_{1,c}^2 + B_{1,c}^2) - \frac{f_3(t)}{\tilde a} Q_c, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} G_2 \;:=\; & \frac{1}{\varepsilon}(A_{1,c})_t + f_2 \Lambda A_{1,c} + \frac{a'}{a} Q_c^{m-1} y B_{1,c} - \frac{1}{2} f_1 y B_{1,c} \\ & + (m-1)\tilde a Q_c^{m-2} A_{1,c} B_{1,c} - \frac{f_4(t)}{\tilde a} Q_c'; \end{aligned} \tag{3.26}$$

Moreover, suppose that $(A_{k,c}, B_{k,c})$ *satisfy (3.11) for* $k = 1$ *and* 2*. Then*

$$\|\varepsilon^3(\mathcal{F}_3(t,\cdot) + \varepsilon f(t)\mathcal{F}_c)e^{i\Theta}\|_{H^1(\mathbb{R})} \leq K\varepsilon^3(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon), \tag{3.27}$$

uniformly in time.

Proof. See Appendix J. □

Remark 3.1 (Important notation). In what follows, and in order to simplify the notation, we will assume that decomposition (3.20) is valid for all $2 \leq m < 5$, with the obvious modifications; in particular, we assume $f_3 \equiv f_4 \equiv 0$ in (3.22)-(3.23), for the case $2 \leq m < 3$. This simplification will be useful in Proposition 3.10 below, where a stability results is proved.

From (3.13)-(3.20) we see that in order to improve the accuracy of $\tilde u$ as a solution of (1.12), we have to eliminate some terms $\mathcal{F}_k$. The next subsection is devoted to the proof of the following assertion: we can choose dynamical parameters (c, v, ρ, γ) in the interval $[-T_\varepsilon, T_\varepsilon]$ in such a way that $\mathcal{F}_0(t,\cdot) \sim 0$.

3.3 Existence for a simplified dynamical system

Our first result concerns the existence of solutions of the differential system involving the evolution of velocity, scaling and phase parameters. This system is given by imposing the condition $\mathcal{F}_0 \equiv 0$.

We we are able to prove existence and long time behavior for an approximated differential system given by $\mathcal{F}_0 \equiv 0$. Indeed,

Lemma 3.4 (Existence of approximated dynamical parameters, case $2 \leq m < 5$)**.**

Let $v_0 > 0, \lambda_0, a(s)$ *be as in Theorem A and (1.11). There exists a unique solution* (C, V, P, G) *defined for all* $t \geq -T_\varepsilon$ *with the same regularity than* $a(\varepsilon\cdot)$*, of the following nonlinear system of differential equations (cf. (3.14))*

$$\begin{cases} V'(t) = \varepsilon f_1(C(t), U(t)), & V(-T_\varepsilon) = v_0, \\ C'(t) = \varepsilon f_2(C(t), V(t), U(t)), & C(-T_\varepsilon) = 1, \\ U'(t) = V(t), & U(-T_\varepsilon) = -v_0 T_\varepsilon, \\ H'(t) = -\frac{1}{2}V'(t)U(t), & H(-T_\varepsilon) = 0. \end{cases} \tag{3.28}$$

In addition, for all $t \in [-T_\varepsilon, T_\varepsilon]$*,*

1. *$C(t)$ is strictly increasing with $1 \leq C(t) \leq C(T_\varepsilon)$ and $C(T_\varepsilon) = c_\infty + O(\varepsilon^{10}) = 2^{\frac{4}{5-m}} + O(\varepsilon^{10})$.*

2. *$V(t)$ is strictly increasing with $v_0 \leq V(t) \leq V(T_\varepsilon)$, and where*

$$V(T_\varepsilon) = v_\infty + O(\varepsilon^{10}) = (v_0^2 + 4\lambda_0(c_\infty - 1))^{1/2} + O(\varepsilon^{10}),$$

$$v_0 T_\varepsilon \leq U(T_\varepsilon) \leq (2v_\infty - v_0)T_\varepsilon.$$

Remark 3.2. Note that $(C(t), V(t), U(t), H(t))$ satisfy (3.8) and therefore is a admissible set of parameters for $\tilde{u}$.

Proof of Lemma 3.4. The existence of a local solution of (3.28) is consequence of the Cauchy-Lipschitz-Picard theorem.

Now, in order to prove global existence of such a solution, we derive some a priori estimates. Note that from the first equation in (3.28) we have C strictly increasing in time with $C(t) \geq 1$, $t \in [-T_\varepsilon, T_\varepsilon]$. Moreover, after integration, we have

$$C(t) = \frac{a^{4/(5-m)}(\varepsilon U(t))}{a^{4(5-m)}(-\varepsilon v_0 T_\varepsilon)} = a^{4/(5-m)}(\varepsilon U(t))(1 + O(\varepsilon^{10})). \tag{3.29}$$

Since $1 \leq a \leq 2$, one has that C is bounded and globally well defined with

$$1 \leq C(t) < c_\infty = 2^{\frac{4}{5-m}}, \quad t \geq -T_\varepsilon.$$

Moreover, from the hypothesis on a (cf. (1.11)), it is easy to see that $C(T_\varepsilon) = c_\infty + O(\varepsilon^{10})$.

On the other hand, from the second equation in (3.28), we have V strictly increasing in time. Replacing (3.29), and after multiplication by $V(t)$, one has

$$V(t)V'(t) = \frac{8}{m+3} a^{\frac{m-1}{5-m}}(\varepsilon U(t)) a'(\varepsilon U(t)) V(t) a^{-\frac{4}{5-m}}(-\varepsilon v_0 T_\varepsilon)).$$

After integration in $[-T_\varepsilon, t)$ we obtain $V^2(t) = v_0^2 + 4\lambda_0[C(t) - 1]$. This relation implies the global existence of V and the uniform bound

$$v_0 \leq V(t) < v_\infty := (v_0^2 + 4\lambda_0(c_\infty - 1))^{1/2}; \quad t \geq -T_\varepsilon.$$

In addition, one has $V(T_\varepsilon) = v_\infty + O(\varepsilon^{10})$.

□

Remark 3.3. Note that the parabola $C(t) = (1 - \frac{v_0^2}{4\lambda_0}) + \frac{V^2(t)}{4\lambda_0}$ and the bound (3.29) allow to describe in great detail the dynamics of the soliton solution. In particular, in the case of a decreasing potential $a(\varepsilon x)$ as in Remark 1.3, with initial velocity $v_0 > 0$ small enough (depending on $\min_{\mathbb{R}} a$), one can predict the reflection of the soliton.

In order to construct a reasonable approximate solution describing the interaction we need to improve the error term $S[\tilde{u}]$ from Proposition 3.3 to the second order in ε. This is the objective of the next subsection.

3.4 Resolution of the first order system

In this paragraph we eliminate the term $\mathcal{F}_1$ in (3.13)-(3.20). According to Proposition 3.3, this can be done for any $2 \leq m < 5$. We are then reduced to find $(A_{1,c}(t,y), B_{1,c}(t,y))$ satisfying, for all (t,y),

$$(\Omega_1) \begin{cases} \mathcal{L}_+ A_{1,c}(t,y) = F_1(t,y), \\ \mathcal{L}_- B_{1,c}(t,y) = G_1(t,y). \end{cases}$$

When solving problem (Ω_1), a key property will be the separability between the variables t and y on the source terms F, G. This is a surprising property, not necessarily true for more complicated nonlinearities others than pure powers. Let us recall that this property is also present in the case of generalized KdV equations, see [65].

3.4.1 Resolution of the linear problem (Ω_1)

Recall that from Proposition 3.3 and (3.24) the system (Ω_1) is more explicitly given by

$$(\Omega_1) \begin{cases} \mathcal{L}_+ A_{1,c}(t,y) = \dfrac{a'}{\tilde{a}^m}(\varepsilon\rho(t)) y Q_c(y)(Q_c^{m-1}(y) - \dfrac{4c(t)}{m+3}), \\ \mathcal{L}_- B_{1,c}(t,y) = \dfrac{1}{5-m}\dfrac{a'}{\tilde{a}^m}(\varepsilon\rho(t)) v(t)(Q_c(y) + 2yQ_c'(y)). \end{cases} \quad (3.30)$$

This system is solvable, as shows the following

Lemma 3.5 (Resolution of (Ω_1)).

Suppose $(c(t), v(t), \rho(t), \gamma(t))$ satisfy (3.8) for $t \in [-T_\varepsilon, T_\varepsilon]$. Then both right hand sides in (3.30) are in $\mathcal{Y}$, and there exists a unique solution $(A_{1,c}(t,y), B_{1,c}(t,y))$ of (Ω_1) satisfying (3.11), given by

$$A_{1,c}(t,y) = \frac{a'(\varepsilon\rho(t))}{(m+3)\tilde{a}^m(\varepsilon\rho(t))c(t)}\{c(t)y(yQ_c'(y) - Q_c(y)) + \xi Q_c'(y)\},$$
$$B_{1,c}(t,y) = -\frac{a'(\varepsilon\rho(t))v(t)}{2(5-m)\tilde{a}^m(\varepsilon\rho(t))c(t)}(c(t)y^2 + \chi)Q_c(y). \quad (3.31)$$

for ξ, χ given by[6]

$$\xi := -\frac{\int_{\mathbb{R}}(\frac{1}{2}Q^2 + y^2Q'^2)}{\int_{\mathbb{R}} Q'^2} = -\frac{m+7}{2(m-1)} + \chi, \quad \chi := -\frac{\int_{\mathbb{R}} y^2Q^2}{\int_{\mathbb{R}} Q^2}.$$

Moreover, $A_{1,c}(t,\cdot)$ is odd and $B_{1,c}(t,\cdot)$ is even, and satisfy

$$\int_{\mathbb{R}} A_{1,c}(t,y)Q_c'(y)dy = \int_{\mathbb{R}} A_{1,c}(t,y)Q_c(y)dy = 0,$$
$$\int_{\mathbb{R}} B_{1,c}(t,y)Q_c'(y)dy = \int_{\mathbb{R}} B_{1,c}(t,y)Q_c(y)dy = 0. \quad (3.32)$$

Proof. From (3.8) we have $F_1, G_1 \in \mathcal{Y}$. Using Lemma 3.2, we have the existence of the required solution provided the following two orthogonality conditions

$$\int_{\mathbb{R}} F_1(t,y)Q_c'(y)dy = \int_{\mathbb{R}} G_1(t,y)Q_c(y)dy = 0,$$

[6] See Appendix K for more details on the computations.

are valid for all $t \in [-T_\varepsilon, T_\varepsilon]$. This is an easy computation. Indeed, up to a function of time, we have (cf. Appendix K)

$$\int_{\mathbb{R}} F_1 Q_c' = \int_{\mathbb{R}} y Q_c' Q_c (Q_c^{m-1} - \frac{4c}{m+3}) = c^{2\theta+1}\big[-\frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1} + \frac{2}{m+3}\int_{\mathbb{R}} Q^2\big] = 0.$$

On the other hand,

$$\int_{\mathbb{R}} G_1 Q_c = \int_{\mathbb{R}} Q_c (\frac{4c}{5-m}\Lambda Q_c - \frac{1}{m-1} Q_c) = \big[\frac{4\theta}{5-m} - \frac{1}{m-1}\big] c^{2\theta}\int_{\mathbb{R}} Q^2 = 0.$$

The fact that $A_{1,c}, B_{1,c}$ in (3.31) solve (Ω_1) is a simple verification. This finishes the proof. □

Remark 3.4. Note that (3.31) can be written as follows (we skip the dependence on t of v and c, and the dependence on $\varepsilon\rho(t)$ of the function a)

$$A_{1,c}(t,y) = \frac{a'}{\tilde{a}^m} c^{\frac{1}{m-1}-\frac{1}{2}} A_1(\sqrt{c}y), \quad B_{1,c}(t,y) = \frac{a' v}{\tilde{a}^m} c^{\frac{1}{m-1}-1} B_1(\sqrt{c}y), \tag{3.33}$$

for some $A_1, B_1 \in \mathcal{Y}$ not depending on c. More precisely,

$$A_1(y) := \frac{1}{m+3}(y(yQ' - Q) + \xi Q'), \quad B_1(y) := -\frac{1}{2(5-m)}(y^2 + \chi)Q. \tag{3.34}$$

3.5 Improvement of the approximate solution

In this paragraph we consider the case $m \geq 3$. Our objective is to profit of the smoothness of the nonlinearity in this case (see Proposition 3.3) to go beyond on the computations and solve one more linear system –denoted by (Ω_2)–, and equivalent to solve $\mathcal{F}_2 \equiv 0$ in (3.20). As a consequence, the error term $S[\tilde{u}]$ in (3.20) will become or order $\sim \varepsilon^3$ (see (3.27) and Lemma 3.7 below.)

3.5.1 Improved description of F_2 and G_2

Before solving the second order system $\mathcal{F}_2 \equiv 0$, we need to simplify some useless terms appearing in the description of F_2 and G_2 given in (3.25)-(3.26). Indeed, note that terms like $(A_{1,c})_t$ or $(B_{1,c})_t$ can be expressed by using the system of equations given by $\mathcal{F}_0 \equiv 0$, as we state now.

Claim 8 (Simplified description of F_2 and G_2).

We have

$$F_2(t,y) := \tilde{F}_2(t,y) + O_{H^1(\mathbb{R})}(\varepsilon|\rho' - v - \varepsilon^2 f_4| e^{-\varepsilon\mu|\rho(t)|}) \tag{3.35}$$

and

$$G_2(t,y) = \tilde{G}_2(t,y) + O_{H^1(\mathbb{R})}(\varepsilon|\rho' - v - \varepsilon^2 f_4| e^{-\varepsilon\mu|\rho(t)|}) \tag{3.36}$$

where $\tilde{F}_2(t,\cdot)$ is even and $\tilde{G}_2(t,\cdot)$ is odd. More precisely, they have the form

$$\tilde{F}_2(t,y) = \frac{a''}{\tilde{a}^m}(F_{2,c}^I(y) + \frac{v^2}{c}F_{2,c}^{II}(y)) + \frac{a'^2}{\tilde{a}^{2m-1}}(F_{2,c}^{III}(y) + \frac{v^2}{c}F_{2,c}^{IV}(y)) - \frac{f_3}{\tilde{a}}Q_c(y), \tag{3.37}$$

$$\tilde{G}_2(t,y) := \frac{a'' v}{\tilde{a}^m} G_{2,c}^I(y) + \frac{a'^2 v}{\tilde{a}^{2m-1}} G_{2,c}^{II}(y) - \frac{f_4}{\tilde{a}} Q_c'(y); \tag{3.38}$$

with $F_{2,c}^{(\cdot)}(y) = c^{\frac{1}{m-1}} F_2^{(\cdot)}(\sqrt{c}y)$, and $G_{2,c}^{(\cdot)}(y) = c^{\frac{1}{m-1}-\frac{1}{2}} G_2^{(\cdot)}(\sqrt{c}y)$. Finally, $F_2^{(\cdot)}$ and $G_2^{(\cdot)}$ are explicitly given below.

Remark 3.5. Note that the small order terms in $|\rho' - v - \varepsilon^2 f_4|$ above can be added to $\mathcal{F}_0$ in Proposition 3.3. In what follows, we adopt this convention.

Proof. First, in order to simplify the notation, let $\rho_1'(t) := \rho'(t) - v(t) - \varepsilon^2 f_4(t)$. Note that from (3.33) we have

$$\begin{aligned}(A_{1,c})_t &= \frac{\varepsilon\rho'}{\tilde a^m}\big[a'' - \frac{ma'^2}{(m-1)a}\big]c^{\frac{1}{m-1}-\frac12}A_1(\sqrt{c}y)\\ &= \frac{\varepsilon v}{\tilde a^m}\big[a'' - \frac{ma'^2}{(m-1)a}\big]c^{\frac{1}{m-1}-\frac12}A_1(\sqrt{c}y) + O_{H^1(\mathbb{R})}(\varepsilon|\rho_1'(t)|e^{-\varepsilon\mu|\rho(t)|}),\end{aligned}$$

and similarly, using (3.15),

$$\begin{aligned}(B_{1,c})_t &= \frac{\varepsilon}{\tilde a^m}\big[a''v^2 + a'f_1 - \frac{m}{m-1}\frac{a'^2v^2}{a}\big]c^{\frac{1}{m-1}-1}B_1(\sqrt{c}y) + O(\varepsilon|\rho_1'(t)|e^{-\varepsilon\mu|\rho(t)|})\\ &= \frac{\varepsilon}{\tilde a^m}\big[a''v^2 + \frac{8a'^2c}{(m+3)a} - \frac{m}{m-1}\frac{a'^2v^2}{a}\big]c^{\frac{1}{m-1}-1}B_1(\sqrt{c}y) + O(\varepsilon|\rho_1'(t)|e^{-\varepsilon\mu|\rho(t)|}).\end{aligned}$$

In addition, we replace (3.31) in $\tilde F_2$, $\tilde G_2$. We obtain

$$\begin{aligned}\tilde F_2(t,y) &= \frac{a''}{\tilde a^m}(\varepsilon\rho(t))\big[F_{2,c}^{I}(y) + \frac{v^2(t)}{c(t)}F_{2,c}^{II}(y)\big]\\ &\quad + \frac{a'^2}{\tilde a^{2m-1}}(\varepsilon\rho(t))\big[F_{2,c}^{III}(y) + \frac{v^2(t)}{c(t)}F_{2,c}^{IV}(y)\big] - \frac{f_3(t)}{\tilde a(\varepsilon\rho(t))}Q_c(y),\end{aligned}$$

with $F_{2,c}^{(\cdot)}(y) = c^{\frac{1}{m-1}}F_2^{(\cdot)}(\sqrt{c}y)$, and $F_2^I(y) := \frac12 y^2Q^m(y)$, $F_2^{II}(y) := -B_1(y)$,

$$F_2^{III}(y) := (mQ^{m-1}(y) - \frac{4}{m+3})yA_1(y) + \frac{m}{2}(m-1)Q^{m-2}(y)A_1^2(y) - \frac{8}{(m+3)}B_1(y);$$

$$F_2^{IV}(y) := \frac12(m-1)Q^{m-2}(y)B_1^2(y) - \frac{2}{5-m}yB_1'(y) - \frac{m-8}{5-m}B_1(y).$$

Note that each term above is even and thus orthogonal to Q'.

On the other hand,

$$\tilde G_2(y) := v(t)\Big[\frac{a''}{\tilde a^m}(\varepsilon\rho(t))G_{2,c}^I(y) + \frac{a'^2}{\tilde a^{2m-1}}(\varepsilon\rho(t))G_{2,c}^{II}(y)\Big] - \frac{f_4(t)}{\tilde a(\varepsilon\rho(t))}Q_c'(y);$$

with $G_{2,c}^{(\cdot)}(y) = c^{\frac{1}{m-1}-\frac12}G_2^{(\cdot)}(\sqrt{c}y)$ and $G_2^I(y) := A_1(y)$,

$$G_2^{II}(y) := \frac{m-6}{5-m}A_1(y) + (Q^{m-1}(y) - \frac{4}{m+3})yB_1(y) + \frac{2}{5-m}yA_1'(y) + (m-1)Q^{m-2}(y)A_1(y)B_1(y).$$

The proof is complete. □

3.5.2 Resolution of a modified linear problem (Ω_2)

From Proposition 3.3, more precisely (3.24), and the above Claim, we want to solve the modified system $(\tilde\Omega_2)$ given by

$$(\tilde\Omega_2)\begin{cases}\mathcal{L}_+A_{2,c}(t,y) = \tilde F_2(t,y),\\ \mathcal{L}_-B_{2,c}(t,y) = \tilde G_2(t,y),\end{cases} \tag{3.39}$$

where $\tilde{F}_2$ and $\tilde{G}_2$ are given in (3.35)-(3.36). The particular choice of $f_3(t)$ and $f_4(t)$ done in (3.22)-(3.23) will allow us to find a unique solution of this linear system satisfying suitable orthogonality conditions. Recall the terms $F_2^{(\cdot)}$ and $G_2^{(\cdot)}$ introduced in the above Claim, and $\theta = \frac{1}{m-1} - \frac{1}{4}$.

Lemma 3.6 (Resolution of $(\tilde{\Omega}_2)$).

Suppose $m \geq 3$ and $f_3(t)$, $f_4(t)$ given by (3.22)-(3.23), with

$$\alpha_{(\cdot)} := \frac{1}{2\theta M[Q]} \int_{\mathbb{R}} \Lambda Q F_2^{(\cdot)}, \qquad \beta_{(\cdot)} := -\frac{1}{M[Q]} \int_{\mathbb{R}} yQG_2^{(\cdot)}. \tag{3.40}$$

There exists a unique solution $(A_{c,2}(t,y), B_{c,2}(t,y))$ of $(\tilde{\Omega}_2)$ satisfying (3.11). In addition, $A_{2,c}$ is even and $B_{2,c}$ is odd and satisfy the following decomposition:

$$A_{2,c}(t,y) = \frac{a''}{\tilde{a}^m}(A^I_{2,c}(y) + \frac{v^2}{c}A^{II}_{2,c}(y)) + \frac{a'^2}{\tilde{a}^{2m-1}}(A^{III}_{2,c}(y) + \frac{v^2}{c}A^{IV}_{2,c}(y)) + \frac{f_3}{\tilde{a}}\Lambda Q_c, \tag{3.41}$$

with $A^{(\cdot)}_{2,c}(y) = c^{\frac{1}{m-1}-1}A^{(\cdot)}_2(\sqrt{c}y)$, $A^{(\cdot)}_2$ even, and

$$B_{2,c}(t,y) = \frac{a''v}{\tilde{a}^m}B^I_{2,c}(y) + \frac{a'^2 v}{\tilde{a}^{2m-1}}B^{II}_{2,c}(y) + \frac{f_4}{2\tilde{a}}yQ_c, \tag{3.42}$$

with $B^{(\cdot)}_{2,c}(y) = c^{\frac{1}{m-1}-\frac{3}{2}}B^{(\cdot)}_2(\sqrt{c}y)$ and $B^{(\cdot)}_2$ odd. Moreover, both $A_{2,c}$ and $B_{2,c}$ satisfy

$$\int_{\mathbb{R}} A_{2,c}(t,y)Q'_c(y)dy = \int_{\mathbb{R}} A_{2,c}(t,y)Q_c(y)dy = 0,$$
$$\int_{\mathbb{R}} B_{2,c}(t,y)Q'_c(y)dy = \int_{\mathbb{R}} B_{2,c}(t,y)Q_c(y)dy = 0. \tag{3.43}$$

Remark 3.6. Note that thanks to the introduction of $f_3(t)$ and $f_4(t)$ in (3.22)-(3.23), and from (3.10), (3.32) and (3.43) one has

$$\int_{\mathbb{R}} w(t,x)Q_c(y)e^{-i\Theta} = \int_{\mathbb{R}} w(t,x)Q'_c(y)e^{-i\Theta} = 0. \tag{3.44}$$

Let us remark that $f_3(t)$ and $f_4(t)$ formally represent the *lack of symmetry* of the soliton solution, with respect to the pure soliton solution considered in Definition 1.1. In this case, and similarly to [57], the main order in the defect of the soliton solution is present on the trajectory and phase, rather than in the scaling, as in [53, 58, 64].

Remark 3.7. The exact value of $\alpha_{(\cdot)}$ and $\beta_{(\cdot)}$ can be computed explicitly but their are not necessary for our purposes. Nevertheless, it is simple to see that from Claim 8, Lemma 3.2, (3.34) and Appendix K one has

$$\alpha_I = \frac{1}{2(m+1)M[Q]} \int_{\mathbb{R}} y^2 Q^{m+1} > 0; \quad \alpha_{II} = -\frac{(m-1)}{2M[Q]} \int_{\mathbb{R}} y^2 Q^2 < 0;$$

and

$$\beta_I = \frac{1}{(m+3)M[Q]}\Big[\frac{5}{2}\int_{\mathbb{R}} y^2Q^2 + \frac{\xi}{2}\int_{\mathbb{R}} Q^2\Big] = \frac{-1}{m+3}\Big(\frac{7+m}{2(m-1)} + 4\chi\Big) > 0,$$

for $m \in [3,5)$.

Proof. Note that $\tilde{F}_2(t,\cdot)$ is even and $\tilde{G}_2(t,\cdot)$ is an odd function, and both functions are in $\mathcal{Y}$, uniformly in time. Therefore $\tilde{F}_2$ is orthogonal to Q'_c, and $\tilde{G}_2$ is orthogonal to Q_c. From Lemma 3.2 part (3), we obtain the conclusion.

Moreover, note that $\mathcal{L}_+\Lambda Q_c = -Q_c$, $\mathcal{L}_-(yQ_c) = -2Q_c'$. Thanks to the choice of f_3 and f_4 one has (see Appendix K for more details on the computations)

$$\begin{aligned}
\int_{\mathbb{R}} A_{2,c}Q_c &= -\int_{\mathbb{R}} \mathcal{L}_+ A_{2,c}\Lambda Q_c = -\int_{\mathbb{R}} \tilde{F}_2\Lambda Q_c \\
&= -\frac{a''}{\tilde{a}^m}\int_{\mathbb{R}} \Lambda Q_c(F^I_{2,c} + \frac{v^2}{c}F^{II}_{2,c}) - \frac{a'^2}{\tilde{a}^{2m-1}}\int_{\mathbb{R}} \Lambda Q_c(F^{III}_{2,c} + \frac{v^2}{c}F^{IV}_{2,c}) + \frac{f_3}{\tilde{a}}\int_{\mathbb{R}} \Lambda Q_c Q_c \\
&= -\frac{c^{2\theta-1}}{\tilde{a}}\Big[\frac{a''}{a}(\alpha_I + \alpha_{II}\frac{v^2}{c}) + \frac{a'^2}{a^2}(\alpha_{III} + \alpha_{IV}\frac{v^2}{c}) - f_3\Big] \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\mathbb{R}} B_{2,c}Q_c' &= -\frac{1}{2}\int_{\mathbb{R}} \mathcal{L}_- B_{2,c}yQ_c = -\frac{1}{2}\int_{\mathbb{R}} \tilde{G}_2 yQ_c \\
&= -\frac{a''v}{2\tilde{a}^m}\int_{\mathbb{R}} yQ_cG^I_{2,c} - \frac{a'^2v}{2\tilde{a}^{2m-1}}\int_{\mathbb{R}} yQ_cG^{II}_{2,c} + \frac{f_4}{2\tilde{a}}\int_{\mathbb{R}} yQ_cQ_c' \\
&= \frac{c^{2\theta}}{2\tilde{a}}\Big[(\beta_I\frac{a''}{a} + \beta_{II}\frac{a'^2}{a^2})\frac{v}{c} - f_4\Big] \\
&= 0.
\end{aligned}$$

The proof is complete. □

From Proposition 3.3 and the singular behavior of the nonlinearity $|z|^{m-1}z$ around $z=0$ for $2 \leq m < 4$, $m \neq 3$, we cannot perform a new expansion to improve our estimates. We stop here the search of an approximate solution for the case $3 \leq m < 5$.

3.6 Error estimates

As a consequence of Proposition 3.3 and Lemma 3.4 and Lemma 3.5, we have the following estimates on the error associated to the approximate solution $\tilde{u}$. Recall the definition of $\tilde{S}[\tilde{u}]$ and p_m given in (3.13)-(3.20) and (1.21) respectively.

Lemma 3.7 (Estimation of the error $\tilde{S}[\tilde{u}]$)**.**

There exist constants $\varepsilon_0, K > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds. The error associated to the function $\tilde{u}$ satisfies

$$\|\tilde{S}[\tilde{u}](t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m+1}(\varepsilon + e^{-\varepsilon\mu|\rho(t)|}), \tag{3.45}$$

and the following integral estimate holds

$$\int_{\mathbb{R}} \|\tilde{S}[\tilde{u}](t)\|_{H^1(\mathbb{R})}dt \leq K\varepsilon^{p_m}. \tag{3.46}$$

Proof. First we prove the case $2 \leq m < 3$. Here $p_m = 1$. From Proposition 3.3 and Lemma 3.5 we have

$$\tilde{S}[\tilde{u}] = \varepsilon^2[\mathcal{F}_2(t,y) + \varepsilon f(t)\mathcal{F}_c(y)]e^{i\Theta},$$

From estimate (3.19) we have (3.45) in this case.

Let us consider the case $m \geq 3$, with $p_m = 2$. Here we invoke Proposition 3.3 and Lemmas 3.5 and 3.6 to get

$$\tilde{S}[\tilde{u}] = \varepsilon^4[\mathcal{F}_4(t,y) + \varepsilon f(t)\mathcal{F}_c(y)]e^{i\Theta},$$

From (3.27), the rest of the proof and (3.46) are direct since from (3.8) one has $\rho'(t) \geq \frac{1}{2}v_0 > 0$. □

3.7 Recomposition of the solution

In this subsection we will show that $\tilde{u}$ at time $t = -T_\varepsilon$ behaves as a modulated soliton. We begin with some H^1-estimates.

Lemma 3.8 (First estimates on $\tilde{u}$).

Suppose $0 < \varepsilon < \varepsilon_0$ small enough, and (c, v, ρ, γ) satisfying (3.8). Then the following auxiliary estimates hold.

1. Decay away from zero. *Suppose $f_c = f_c(y) \in \mathcal{Y}$. Then there exist $K, \mu > 0$ constants such that for all $t \in [-T_\varepsilon, T_\varepsilon]$*
$$\|a'(\varepsilon x) f_c(y)\|_{H^1(\mathbb{R})} \leq K e^{-\mu\varepsilon|\rho(t)|}. \tag{3.47}$$

2. Almost soliton-solution. *The following estimates hold for all $t \in [-T_\varepsilon, T_\varepsilon]$.*
$$\tilde{u}_{xx} - (c(t) + \frac{1}{4}v^2(t))\tilde{u} + a_\varepsilon|\tilde{u}|^{m-1}\tilde{u} - iv(t)\tilde{u}_x = O_{H^1(\mathbb{R})}(\varepsilon e^{-\mu\varepsilon|\rho(t)|}), \tag{3.48}$$
and
$$i\tilde{u}_t + iv\tilde{u}_x + (c(t) + \frac{1}{4}v^2(t))\tilde{u} = O_{H^1(\mathbb{R})}(\varepsilon e^{-\mu\varepsilon|\rho(t)|}). \tag{3.49}$$

Proof. (3.47) is a classical result, see [65] Lemma 4.7 for a complete proof. On the other hand, to prove (3.48), note that after some simplifications, and by using (3.9)-(3.11) with $\|w(t)\|_{H^1(\mathbb{R})} \leq K\varepsilon e^{-\varepsilon\mu|\rho(t)|}$, we have

$$\begin{aligned}
&\tilde{u}_{xx} - (c + \frac{1}{4}v^2)\tilde{u} + a_\varepsilon|\tilde{u}|^{m-1}\tilde{u} - iv\tilde{u}_x \\
&= \frac{1}{\tilde{a}}(Q_c'' - cQ_c + Q_c^m)e^{i\Theta} + (\frac{a_\varepsilon(x)}{a_\varepsilon(\rho)} - 1)\frac{Q_c^m}{\tilde{a}}e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon e^{-\mu\varepsilon|\rho(t)|}) \\
&= O_{H^1(\mathbb{R})}(\varepsilon e^{-\mu\varepsilon|\rho(t)|}).
\end{aligned}$$

Let us prove (3.49). From the definition of $S[\tilde{u}]$ and estimate (3.45),

$$\begin{aligned}
i\tilde{u}_t + iv\tilde{u}_x + (c + \frac{1}{4}v^2)\tilde{u} &= S[\tilde{u}] - \{\tilde{u}_{xx} - (c + \frac{1}{4}v^2)\tilde{u} + a_\varepsilon|\tilde{u}|^{m-1}\tilde{u} - iv\tilde{u}_x\} \\
&= O_{H^1(\mathbb{R})}(\varepsilon e^{-\mu\varepsilon|\rho(t)|}).
\end{aligned}$$

The proof is complete. □

The next result describes the behavior of the almost solution $\tilde{u}$ at the endpoint $t = -T_\varepsilon$.

Lemma 3.9 (Behavior at $t = -T_\varepsilon$).

There exist constants $K, \varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the following holds. Let $\hat{u}(t) := \tilde{u}(t; C(t), V(t), U(t), H(t))$ be the approximate solution constructed in Section 3 Step 2 with modulation parameters (C, V, U, H) given by Lemma 3.4. Then one has

$$\|\hat{u}(-T_\varepsilon) - Q(\cdot + v_0T_\varepsilon)e^{\frac{i}{2}(\cdot)v_0}e^{i\gamma_{-1}}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10}, \tag{3.50}$$

with
$$\gamma_{-1} := -\int_{-T_\varepsilon}^0 C(s)ds + \frac{1}{4}\int_{-T_\varepsilon}^0 V^2(s)ds. \tag{3.51}$$

Proof. By definition,

$$\hat{u}(-T_\varepsilon) - Q(\cdot + v_0T_\varepsilon)e^{\frac{i}{2}(\cdot)v_0}e^{i\gamma_{-1}} = \tilde{R}(-T_\varepsilon) - Q(\cdot + v_0T_\varepsilon)e^{\frac{i}{2}(\cdot)v_0}e^{i\gamma_{-1}} + w(-T_\varepsilon).$$

From (3.10), (3.11) and Lemma 3.4 we have

$$\|w(\pm T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon e^{-\mu\varepsilon^{-\frac{1}{100}}} \leq K\varepsilon^{10},$$

for ε small enough. Since from (3.62) $U(-T_\varepsilon) = v_0T_\varepsilon$, $V(-T_\varepsilon) = v_0$ and $C(-T_\varepsilon) = 1$, we have

$$\|\tilde{R}(-T_\varepsilon) - Q(\cdot + v_0T_\varepsilon)e^{\frac{i}{2}(\cdot)v_0}e^{i\gamma_{-1}}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10},$$

as desired. □

Resuming, we have constructed an approximate solution $\tilde{u}$ formally describing the interaction soliton-potential. In the next subsection we will show that a suitable modification of the the *solution* u constructed in Theorem 2.1 actually behaves like $\tilde{u}$ inside the interaction region $[-T_\varepsilon, T_\varepsilon]$.

Step 3. Stability results.

In this paragraph our objective is to prove that the approximate solution $\tilde{u}(t)$ describes the dynamics of the soliton in the interaction interval $[-T_\varepsilon, T_\varepsilon]$. We will prove the following result, cf. Propositions 5.1 in [65] for a similar result for a gKdV equation.

Proposition 3.10 (Exact solution close to the approximate solution $\tilde{u}$)**.**

Let $2 \leq m < 5$, p_m defined in (1.21). There exists $\varepsilon_0 > 0$ such that the following holds for any $0 < \varepsilon < \varepsilon_0$. Suppose that for $\hat{u}(-T_\varepsilon)$ as defined in Lemma 3.9 one has

$$\|u(-T_\varepsilon) - \hat{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}, \tag{3.52}$$

with $u = u(t)$ the $H^1(\mathbb{R})$ solution of (1.12) constructed in Proposition 2.1. Then there exist $K_0 = K_0(m, K) > 0$ and C^1-functions $c, v, \rho, \gamma : [-T_\varepsilon, T_\varepsilon] \to \mathbb{R}$ such that, for all $t \in [-T_\varepsilon, T_\varepsilon]$,

$$\|u(t) - \tilde{u}(t; c(t), v(t), \rho(t), \gamma(t))\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{p_m}, \tag{3.53}$$

and

$$|\rho'(t) - v(t) - \varepsilon^2 f_4(t)| + |\gamma'(t) - \frac{1}{2}v'(t)\rho(t) - \varepsilon^2 f_3(t)| \leq K_0\varepsilon^{p_m}, \tag{3.54}$$

$$|v'(t) - \varepsilon f_1(t)| + |c'(t) - \varepsilon f_2(t)| \leq K_0(\varepsilon^{2p_m} + \varepsilon^{p_m+1}). \tag{3.55}$$

Proof of Proposition 3.10. Let $K^* > 1$ be a constant to be fixed later. Since $\|u(-T_\varepsilon) - \hat{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}$, by continuity in time in $H^1(\mathbb{R})$, there exists $-T_\varepsilon < T^* \leq T_\varepsilon$ with

$$\begin{aligned} T^* \quad := \quad & \sup\big\{T \in [-T_\varepsilon, T_\varepsilon], \text{ such that for all } t \in [-T_\varepsilon, T], \text{ there exists } \rho(t), \gamma(t) \in \mathbb{R}, \\ & \text{with } \|u(t) - \tilde{u}(t; C(t), V(t), \rho(t), \gamma(t))\|_{H^1(\mathbb{R})} \leq K^*\varepsilon^{p_m}\big\}. \end{aligned}$$

The objective is to prove that $T^* = T_\varepsilon$ for K^* large enough. To achieve this, we argue by contradiction, assuming that $T^* < T_\varepsilon$ and reaching a contradiction with the definition of T^* by proving some independent estimates for $\|u(t) - \tilde{u}(t; C(t), V(t), \rho(t), \gamma(t))\|_{H^1(\mathbb{R})}$ on $[-T_\varepsilon, T^*]$, for a special modulation parameters $\rho(t), \gamma(t)$.

3.7.1 Modulation

By using the Implicit function theorem we will construct some modulation parameters and estimate their variation in time:

Lemma 3.11 (Modulation in time)**.**

Assume $0 < \varepsilon < \varepsilon_0(K^*)$ *small enough. There exist* unique C^1 *functions* $c(t), v(t), \rho(t), \gamma(t)$ *such that, for all* $t \in [-T_\varepsilon, T^*]$*, the function*

$$z(t) := u(t) - \tilde{u}(t; c(t), v(t), \rho(t), \gamma(t)), \tag{3.56}$$

satisfies

$$\int_{\mathbb{R}} \bar{z}(t,x) Q_c(y) e^{i\Theta} dx = \int_{\mathbb{R}} \bar{z}(t,x) Q_c'(y) e^{i\Theta} dx = 0, \tag{3.57}$$

and

$$\begin{aligned}|\rho(-T_\varepsilon) - U(-T_\varepsilon)| + |\gamma(-T_\varepsilon) - H(-T_\varepsilon)| \\ +|c(-T_\varepsilon) - C(-T_\varepsilon)| + |v(-T_\varepsilon) - V(-T_\varepsilon)| + \|z(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}.\end{aligned} \tag{3.58}$$

Moreover, we have, for all $t \in [-T_\varepsilon, T^*]$*,*

$$\|z(t)\|_{H^1(\mathbb{R})} + |c(t) - C(t)| + |v(t) - V(t)| \leq KK^*\varepsilon^{p_m}. \tag{3.59}$$

In addition, $z(t)$ *satisfies the following equation*

$$\begin{aligned}&iz_t + z_{xx} + a_\varepsilon[|\tilde{u} + z|^{m-1}(\tilde{u} + z) - |\tilde{u}|^{m-1}\tilde{u}] + \tilde{S}[\tilde{u}] + i(c' - \varepsilon f_2)\partial_c \tilde{u} \\ &-\frac{1}{2}(v' - \varepsilon f_1) y\tilde{u} + i(\rho' - v - \varepsilon^2 f_4)\partial_\rho \tilde{u} - (\gamma' + \frac{1}{2}v'\rho - \varepsilon^2 f_3)\tilde{u} = 0.\end{aligned} \tag{3.60}$$

Finally, there exist $K, \mu > 0$ *independent of* K^* *such that for every* $t \in [-T_\varepsilon, T^*]$

$$\begin{aligned}&|\rho'(t) - v(t) - \varepsilon^2 f_4(t)| + |\gamma'(t) - \frac{1}{2}v'(t)\rho(t) - \varepsilon^2 f_3(t)| \leq \\ &\leq K\Big[\|z(t)\|_{L^2(\mathbb{R})} + \varepsilon e^{-\mu\varepsilon|\rho(t)|}\|z(t)\|_{L^2(\mathbb{R})} + \|z(t)\|_{L^2(\mathbb{R})}^2 + \|\tilde{S}[\tilde{u}](t)\|_{L^2(\mathbb{R})}\Big],\end{aligned} \tag{3.61}$$

and

$$|v'(t) - \varepsilon f_1(t)| + |c'(t) - \varepsilon f_2(t)| \leq K\Big[\varepsilon e^{-\mu\varepsilon|\rho(t)|}\|z(t)\|_{L^2(\mathbb{R})} + \|z(t)\|_{L^2(\mathbb{R})}^2 + \|\tilde{S}[\tilde{u}](t)\|_{L^2(\mathbb{R})}\Big]. \tag{3.62}$$

Proof. The proof of (3.56)-(3.59) is by now well-know and it is a consequence of an Implicit Function Theorem application. See e.g. [53] for a detailed proof. On the other hand, the proof of (3.60) follows after a simple calculation using (1.12).

The proof of (3.61) and (3.62) follow from (3.57)-(3.60) after taking time derivative and replacing z_t. We skip the details. □

3.7.2 Improvement of (3.8)

In this paragraph we prove that the parameters $(c(t), v(t), \rho(t), \gamma(t))$ constructed in Lemma 3.11 satisfy the assumptions required in (3.8). Recall that under these hypotheses, all the results of Section 3, Step 2 are valid. In particular, one has (3.45).

First of all, from (3.58) we have

$$\frac{9}{10} \leq c(t) \leq \frac{11}{10}c_\infty < 2^5, \qquad \frac{9}{10}v_0 \leq v(t) < v_0 + 2^5. \tag{3.63}$$

On the other hand, from (3.61) we have for ε small,

$$|\rho'(t) - v(t)| \leq K^*\varepsilon^{p_m} \leq \frac{v_0}{100}. \tag{3.64}$$

We are done.

3.7.3 Energy functional for z

Consider the $H^1(\mathbb{R})$ functional

$$\begin{aligned}\mathcal{F}(t) \quad := \quad & \frac{1}{2}\int_{\mathbb{R}} |z_x|^2 + \frac{1}{2}(c + \frac{1}{4}v^2)\int_{\mathbb{R}} |z|^2 - \frac{1}{2}v \operatorname{Im}\int_{\mathbb{R}} \bar{z}z_x \\ & - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)[|\tilde{u}+z|^{m+1} - |\tilde{u}|^{m+1} - (m+1)|\tilde{u}|^{m-1}\operatorname{Re}\{\tilde{u}\bar{z}\}]. \end{aligned} \tag{3.65}$$

Lemma 3.12 (Modified coercivity for $\mathcal{F}$)**.**

There exist $K, \nu_0 > 0$, independent of K^* and ε such that for every $t \in [-T_\varepsilon, T_\varepsilon]$

$$\mathcal{F}(t) \geq \nu_0\|z(t)\|^2_{H^1(\mathbb{R})} - K\varepsilon(e^{-\mu\varepsilon|\rho(t)|} + 1)\|z(t)\|^2_{L^2(\mathbb{R})} - K\|z(t)\|^3_{L^2(\mathbb{R})}.$$

In particular, for ε small enough, one has

$$\mathcal{F}(t) \geq \frac{9}{10}\nu_0\|z(t)\|^2_{H^1(\mathbb{R})}.$$

Proof. The proof is similar to the proof of Lemma 5.5 in [65]. First of all it is easy to see that

$$\begin{aligned}\mathcal{F}(t) \quad = \quad & \frac{1}{2}\int_{\mathbb{R}} |z_x|^2 + \frac{1}{2}(c + \frac{1}{4}v^2)\int_{\mathbb{R}} |z|^2 - \frac{1}{2}v \operatorname{Im}\int_{\mathbb{R}} z_x\bar{z} \\ & - \int_{\mathbb{R}} \frac{a(\varepsilon x)}{a(\varepsilon\rho)}Q_c^{m-1}(y)[|z|^2 + (m-1)[\operatorname{Re}(e^{i\Theta}\bar{z})]^2] + O(\varepsilon\|z(t)\|^2_{H^1(\mathbb{R})} + \|z(t)\|^3_{H^1(\mathbb{R})})\end{aligned}$$

On the other hand, it is clear that $|\varepsilon\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}\int_{\mathbb{R}} yQ_c^{m-1}|z|^2| \leq K\varepsilon e^{-\mu\varepsilon|\rho(t)|}\|z(t)\|^2_{L^2(\mathbb{R})}$. Thus we have

$$\begin{aligned}\mathcal{F}(t) \quad = \quad & \frac{1}{2}\int_{\mathbb{R}} |z_x|^2 + \frac{1}{2}(c + \frac{1}{4}v^2)\int_{\mathbb{R}} |z|^2 - \frac{1}{2}v \operatorname{Im}\int_{\mathbb{R}} z_x\bar{z} \\ & - \int_{\mathbb{R}} Q_c^{m-1}(y)[|z|^2 + (m-1)[\operatorname{Re}(e^{i\Theta}\bar{z})]^2] \\ & + O(\varepsilon(1 + e^{-\varepsilon\mu|\rho(t)|})\|z(t)\|^2_{H^1(\mathbb{R})} + \|z(t)\|^3_{H^1(\mathbb{R})}).\end{aligned} \tag{3.66}$$

Finally, from Lemma 3.2 and (3.57), we have the existence of constants $K, \nu_0 > 0$ such that for all $t \in [-T_\varepsilon, T^*]$

$$(3.66) \geq \nu_0\|z(t)\|^2_{H^1(\mathbb{R})} - K\varepsilon(1 + e^{-\varepsilon\mu|\rho(t)|})\|z(t)\|^2_{H^1(\mathbb{R})} - K\|z(t)\|^3_{H^1(\mathbb{R})}.$$

The proof is now complete. □

Now we use a coercivity argument, similar to Lemma 5.6 in [65] to obtain independent estimates for $\mathcal{F}(T^*)$.

Lemma 3.13 (Estimates on $\mathcal{F}(T^*)$).

The following properties hold for any $t \in [-T_\varepsilon, T^*]$.

1. *First time derivative.*

$$\begin{aligned}\mathcal{F}'(t) &= \\ &= \operatorname{Im} \int_{\mathbb{R}} \overline{iz_t}\{z_{xx} - (c + \frac{1}{4}v^2)z + a_\varepsilon[|\tilde{u} + z|^{m-1}(\tilde{u} + z) - |\tilde{u}|^{m-1}\tilde{u}] - ivz_x\} \quad (3.67)\\ &\quad + \operatorname{Im} \int_{\mathbb{R}} a_\varepsilon \overline{i\tilde{u}_t}[|\tilde{u} + z|^{m-1}(\tilde{u} + z) - |\tilde{u}|^{m-1}\tilde{u} - \frac{1}{2}(m+1)|\tilde{u}|^{m-1}z - \frac{1}{2}(m-1)|\tilde{u}|^{m-3}\tilde{u}^2\bar{z}]\\ &\quad +(c' + \frac{1}{4}v'v)\int_{\mathbb{R}} |z|^2 - \frac{1}{2}v' \operatorname{Im}\int_{\mathbb{R}} \bar{z}z_x. \quad (3.68)\end{aligned}$$

2. *Integration in time. There exist constants $K, \mu > 0$ such that*

$$\mathcal{F}(t) - \mathcal{F}(-T_\varepsilon) \leq K(K^*)^4\varepsilon^{4p_m - 1 - \frac{1}{100}} + KK^*\varepsilon^{2p_m} + K\int_{-T_\varepsilon}^t \varepsilon e^{-\varepsilon\mu|\rho(s)|}\|z(s)\|^2_{H^1(\mathbb{R})}ds. \quad (3.69)$$

Proof. First of all, (3.67) follows after derivation in time. Let us consider (3.69). In order to simplify the computations, let $v_1' := \frac{1}{2}(v' - \varepsilon f_1)$, $c_1' := c' - \varepsilon f_2$, $\rho_1' := \rho' - v - \varepsilon^2 f_4$ and $\gamma_1' := \gamma' + \frac{1}{2}v'\rho - \varepsilon^2 f_3$. In addition, consider

$$L[z] := z_{xx} - (c + \frac{1}{4}v^2)z + \frac{1}{2}a_\varepsilon[(m+1)|\tilde{u}|^{m-1}z + (m-1)|\tilde{u}|^{m-3}\tilde{u}^2\bar{z}] - ivz_x,$$

and

$$N[z] := a_\varepsilon(x)[|\tilde{u} + z|^{m-1}(\tilde{u} + z) - |\tilde{u}|^{m-1}\tilde{u} - \frac{1}{2}(m+1)|\tilde{u}|^{m-1}z - \frac{1}{2}(m-1)|\tilde{u}|^{m-3}\tilde{u}^2\bar{z}].$$

Replacing (3.60) in (3.67) we get

$$\begin{aligned}\mathcal{F}'(t) &= \\ &= \operatorname{Im}\int_{\mathbb{R}} \{(c + \frac{1}{4}v^2)z + ivz_x\}\{\frac{1}{2}a_\varepsilon[(m+1)|\tilde{u}|^{m-1}\bar{z} + (m-1)|\tilde{u}|^{m-3}\overline{\tilde{u}}^2 z] + \overline{N[z]}\} \quad (3.70)\\ &\quad - \operatorname{Im}\int_{\mathbb{R}} \overline{\tilde{S}[\tilde{u}]}L[z] - \operatorname{Im}\int_{\mathbb{R}} \overline{L[z]}[\gamma_1'\tilde{u} + v_1'y\tilde{u} - c_1'i\partial_c\tilde{u} - \rho_1'i\partial_\rho\tilde{u}]\\ &\quad - \operatorname{Im}\int_{\mathbb{R}} \overline{N[z]}[i\tilde{u}_t - \tilde{S}[\tilde{u}] + \gamma_1'\tilde{u} + v_1'y\tilde{u} - c_1'i\partial_c\tilde{u} - \rho_1'i\partial_\rho\tilde{u}] \quad (3.71)\\ &\quad +(c' + \frac{1}{4}v'v)\int_{\mathbb{R}}|z|^2 - \frac{1}{2}v'\operatorname{Im}\int_{\mathbb{R}}\bar{z}z_x. \quad (3.72)\end{aligned}$$

From (3.62),

$$|(3.72)| \leq K\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|^2_{H^1(\mathbb{R})} + K\|z(t)\|^4_{H^1(\mathbb{R})}.$$

On the other hand, note that $L[z] = L_Q[z] + O(\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|_{H^1(\mathbb{R})})$, with

$$L_Q[z] := z_{xx} - (c + \frac{1}{4}v^2)z + \frac{1}{2}Q_c^{m-1}(y)[(m+1)z + (m-1)e^{2i\Theta}\bar{z}] - ivz_x.$$

Therefore a simple computation using (3.57) and (3.62)-(3.61) gives us

$$|\operatorname{Im}\int_{\mathbb{R}} \overline{L[z]}[\gamma_1'\tilde{u} + v_1'y\tilde{u} - c_1'i\partial_c\tilde{u} - \rho_1'i\partial_\rho\tilde{u}]| \leq K\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|^2_{H^1(\mathbb{R})}.$$

We also have

$$|\text{Im}\int_{\mathbb{R}}\overline{\tilde{S}[\tilde{u}]}L[z]|\leq K\|z(t)\|_{L^2(\mathbb{R})}\|\tilde{S}[\tilde{u}](t)\|_{H^1(\mathbb{R})}. \tag{3.73}$$

Next, note that from (3.12), Proposition 3.3 and (3.48) one has

$$i\tilde{u}_t-\tilde{S}[\tilde{u}]+\gamma_1'\tilde{u}+v_1'y\tilde{u}-c_1'i\partial_c\tilde{u}-\rho_1'i\partial_\rho\tilde{u}=-(c+\frac{1}{4}v^2)\tilde{u}-iv\tilde{u}_x+O_{H^1(\mathbb{R})}(\varepsilon e^{-\varepsilon\mu|\rho(t)|}),$$

therefore we obtain

$$\begin{aligned}&(3.70)+(3.71)=\\&=\text{Im}\int_{\mathbb{R}}\{(c+\frac{1}{4}v^2)z+ivz_x\}\{\frac{1}{2}a_\varepsilon[(m+1)|\tilde{u}|^{m-1}\bar{z}+(m-1)|\tilde{u}|^{m-3}\bar{\tilde{u}}^2z]+\overline{N[z]}\}\\&+\text{Im}\int_{\mathbb{R}}\{(c+\frac{1}{4}v^2)\tilde{u}+iv\tilde{u}_x\}\overline{N[z]}+O(\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|^2_{H^1(\mathbb{R})}).\end{aligned}$$

Now we claim that

$$\text{Im}\int_{\mathbb{R}}\frac{1}{2}a_\varepsilon z[(m+1)|\tilde{u}|^{m-1}\bar{z}+(m-1)|\tilde{u}|^{m-3}\bar{\tilde{u}}^2z]+\text{Im}\int_{\mathbb{R}}\tilde{u}\overline{N[z]}=\text{Im}\int_{\mathbb{R}}\bar{z}N[z], \tag{3.74}$$

and

$$\begin{aligned}&\text{Im}\int_{\mathbb{R}}\frac{1}{2}a_\varepsilon iz_x[(m+1)|\tilde{u}|^{m-1}\bar{z}+(m-1)|\tilde{u}|^{m-3}\bar{\tilde{u}}^2z]+\text{Im}\int_{\mathbb{R}}i\tilde{u}_x\overline{N[z]}=\\&=-\text{Im}\int_{\mathbb{R}}i\bar{z}_xN[z]+O(\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|^2_{H^1(\mathbb{R})}).\end{aligned} \tag{3.75}$$

Assuming these two identities, we have finally

$$(3.70)+(3.71)=O(\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|^2_{H^1(\mathbb{R})}).$$

Therefore

$$\begin{aligned}|\mathcal{F}'(t)|&\leq K\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|^2_{L^2(\mathbb{R})}+K\|\tilde{S}[\tilde{u}](t)\|_{H^1(\mathbb{R})}\|z(t)\|_{L^2(\mathbb{R})}+K\|z(t)\|^4_{L^2(\mathbb{R})}\\&\leq K\varepsilon e^{-\varepsilon\mu|\rho(t)|}\|z(t)\|^2_{L^2(\mathbb{R})}+KK^*e^{-\mu\varepsilon|\rho(t)|}\varepsilon^{1+2p_m}+K(K^*)^4\varepsilon^{4p_m}.\end{aligned}$$

After integration between $-T_\varepsilon$ and t we obtain (3.69).

Let us prove (3.74) and (3.75). First of all, note that (3.74) is consequence of the identity

$$\text{Im}\{\frac{1}{2}a_\varepsilon z[(m+1)|\tilde{u}|^{m-1}\bar{z}+(m-1)|\tilde{u}|^{m-3}\bar{\tilde{u}}^2z]-\bar{\tilde{u}}N[z]\}=\text{Im}\{\bar{z}N[z]\}.$$

On the other hand, (3.75) is an easy consequence of the following identity

$$\begin{aligned}&\text{Re}\Big\{\frac{1}{2}a_\varepsilon z_x[(m+1)|\tilde{u}|^{m-1}\bar{z}+(m-1)|\tilde{u}|^{m-3}\bar{\tilde{u}}^2z]+\bar{\tilde{u}}_xN[z]\Big\}=\\&=-\text{Re}\{z_x\overline{N[z]}\}+\frac{a_\varepsilon}{m+1}\partial_x\Big\{|\tilde{u}+z|^{m+1}-|\tilde{u}|^{m+1}-(m+1)|\tilde{u}|^{m-1}\text{Re}\{\tilde{u}\bar{z}\}\Big\};\end{aligned}$$

and integration by parts and (3.47). □

End of proof of Proposition 3.10. Using Gronwall's inequality (see e.g. [65] for more details) in (3.69), the fact that $\rho'(t)\geq\frac{1}{2}v_0>0$, estimate (3.59), and Lemma 3.12 we conclude that for some large constant $K>0$, but independent of K^* and ε,

$$\|z(t)\|^2_{H^1(\mathbb{R})}\leq K\varepsilon^{2p_m}+K(K^*)^4\varepsilon^{4p_m-1-\frac{1}{100}}+KK^*\varepsilon^{2p_m}.$$

From this estimate and taking ε small, and K^* large enough, we obtain that for all $t \in [-T_\varepsilon, T^*]$,

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq \frac{1}{2}(K^*)^2\varepsilon^{2p_m}.$$

Next, from the mass conservation law and (3.58) one has

$$|c(t) - C(t)| \leq K\varepsilon^{p_m} + K(K^*)^2\varepsilon^{2p_m} \leq K\varepsilon^{p_m},$$

and from the energy conservation law and once again (3.58)

$$|v(t) - V(t)| \leq K\varepsilon^{p_m} + K(K^*)^2\varepsilon^{2p_m} \leq K\varepsilon^{p_m}.$$

Therefore, for K^* large enough and all $\varepsilon > 0$ small,

$$\|u(T^*) - \tilde{u}(C(T^*), V(T^*), \rho(T^*), \gamma(T^*))\|_{H^1(\mathbb{R})} \leq \frac{2}{3}K^*\varepsilon^{p_m}.$$

This estimate contradicts the definition of T^*, and therefore $T^* = T_\varepsilon$. In addition, from (3.59) we obtain (3.53). Finally (3.54) and (3.55) are consequence of (3.61)-(3.62). The proof of Proposition 3.10 is now complete.

Final Step. Conclusion and Proof of Proposition 2.2. Now we prove the main result of this section, which describes the core of interaction soliton-potential.

Proof of Proposition 2.2. Consider $u(t)$ a solution of (1.12) satisfying (3.52). We first compare $u(t)$ with the approximate solution $\hat{u}(t)$ from Lemma 3.9, at time $t = -T_\varepsilon$.

3.7.4 Behavior at $t = -T_\varepsilon$

We claim that a suitable modification of u matches with our approximate solution $\hat{u}(t)$. Indeed, for γ_{-1} introduced in (3.51), let

$$v(t,x) := u(t,x)e^{i\tilde{\gamma}}, \qquad \tilde{\gamma} := (1 - \frac{1}{4}v_0^2)T_\varepsilon + \gamma_{-1},$$

which still satisfies (1.12). From (3.52) and (3.50) we have that

$$\|v(-T_\varepsilon) - \hat{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{10}.$$

3.7.5 Behavior at $t = T_\varepsilon$

Thanks to the above estimate we can invoke Proposition 3.10 to obtain the existence of $K_0, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\|v(T_\varepsilon) - \tilde{u}(T_\varepsilon, c(T_\varepsilon), v(T_\varepsilon), \rho(T_\varepsilon), \gamma(T_\varepsilon))\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{p_m},$$

with $|c(T_\varepsilon) - C(T_\varepsilon)| + |v(T_\varepsilon) - V(T_\varepsilon)| \leq K_0\varepsilon^{p_m}$. On the other hand, note that from Lemma 3.4, (3.6), (3.9), (3.10), (3.11) and the last estimates

$$\|\tilde{u}(T_\varepsilon, c(T_\varepsilon), v(T_\varepsilon), \rho(T_\varepsilon), \gamma(T_\varepsilon)) - \lambda_\infty Q_{c_\infty}(\cdot - \rho(T_\varepsilon))e^{\frac{i}{2}(\cdot)v_\infty}e^{i\bar{\gamma}(T_\varepsilon)}\|_{H^1(\mathbb{R})} \leq KK_0\varepsilon^{p_m},$$

where

$$|\rho(T_\varepsilon) - U(T_\varepsilon)| \leq \frac{T_\varepsilon}{100}, \quad \bar{\gamma}(T_\varepsilon) := \int_0^{T_\varepsilon} c(s)ds - \frac{1}{4}\int_0^{T_\varepsilon} v^2(s)ds + \gamma(T_\varepsilon).$$

The proof of this last estimate is similar to the proof of Lemma 3.9. Therefore

$$\|v(T_\varepsilon) - \lambda_\infty Q_{c_\infty}(\cdot - \rho(T_\varepsilon))e^{\frac{i}{2}(\cdot)v_\infty}e^{i\bar\gamma(T_\varepsilon)}\|_{H^1(\mathbb{R})} \leq KK_0\varepsilon^{p_m},$$

Returning to the original function u, we obtain that

$$\|u(T_\varepsilon) - \lambda_\infty Q_{c_\infty}(\cdot - \rho(T_\varepsilon))e^{\frac{i}{2}(\cdot)v_\infty}e^{i(\bar\gamma(T_\varepsilon)-\tilde\gamma)}\|_{H^1(\mathbb{R})} \leq KK_0\varepsilon^{p_m}.$$

Finally, note that $\frac{99}{100}v_0T_\varepsilon \leq \rho(T_\varepsilon) \leq \frac{101}{100}(2v_\infty - v_0)T_\varepsilon$. By defining $\rho_\varepsilon := \rho(T_\varepsilon)$, and $\gamma_\varepsilon := \bar\gamma(T_\varepsilon) - \tilde\gamma$, we obtain (2.5)-(2.6). This finishes the proof. □

4 The two dimensional case

In this section we sketch the proof of Theorem A for dimension 2, namely Theorem B. More precisely, our objective is to adapt the proof of Propositions 2.1, 2.2 and 2.3 to the two dimensional case. Recall that $2 \leq m < 3$.

Step 1. Proposition 2.1 revisited. The proof of this result is identical to the one dimensional case (see Appendix H), with the novelty that $\rho_0(t)$ is now a $\mathbb{R}^2$-valued vector. The uniqueness follows essentially from (1.32). No additional modifications are required.

Step 2. Proposition 2.2 revisited. Here we need to introduce several modifications on the computations.

First of all, the Cauchy problem (3.1) in the higher dimensional case is globally well posed for $1 < m < 3$ for L^2 and H^1 data, see [12, 22]. The conservations laws (1.3), (1.17) and identity (1.27) hold without modifications.

On the other hand, (3.2) now reads

$$\mathcal{L}_+w(y) := -\Delta_y w + cw - mQ_c^{m-1}(y)w, \quad \text{and} \quad \mathcal{L}_-w(y) := -\Delta_y w + cw - Q_c^{m-1}(y)w; \quad (4.1)$$

where $w = w(y)$. Lemma 3.2 is also valid in higher dimensions. In particular, one has the following. Assume that $v \in \mathbb{R}^2$, $v \neq 0$, $\theta \in \mathbb{R}$, and for $k = 1, 2$, one has

$$\operatorname{Re}\int_{\mathbb{R}^2} \bar w\partial_{y_k}Q_c e^{iy\cdot v/2}e^{i\theta} = \operatorname{Im}\int_{\mathbb{R}^2} \bar wQ_c e^{iy\cdot v/2}e^{i\theta} = \operatorname{Re}\int_{\mathbb{R}^2} \bar wQ_c e^{iy\cdot v/2}e^{i\theta} = 0.$$

Then

$$\tilde{\mathcal{B}}[w,w] \geq \sigma_c \int_{\mathbb{R}^2} |w|^2,$$

where $\tilde{\mathcal{B}}[w,w]$ is the standard 2-dimensional generalization of the functional $\tilde{\mathcal{B}}$ defined in Lemma 3.2. Finally, the space $\mathcal{Y}$ in (3.4) is easily generalizable to higher dimensions.

Let us consider now the approximate solution $\tilde u$. From the fact that the potential a depends only on x_1, the relevant dynamical system depends only on this variable. Indeed, for t in $[-T_\varepsilon, T_\varepsilon]$, let

$$c(t), \gamma(t) \in \mathbb{R}, \qquad v(t) = (v_1(t), v_2(t)) \in \mathbb{R}^2, \ \rho(t) = (\rho_1(t), \rho_2(t)) \in \mathbb{R}^2,$$

to be fixed later. Consider $y := (y_1, y_2)$, where

$$y := x - \rho(t), \quad \text{and} \quad \tilde R(t,x) := \frac{Q_{c(t)}(y)}{\tilde a(\varepsilon\rho_1(t))}e^{i\Theta(t,x)}, \quad (4.2)$$

where, as in the one-dimensional case

$$\tilde{a} := a^{\frac{1}{m-1}}, \quad \Theta(t,x) := \int_0^t c(s)ds + \frac{1}{2}v(t)\cdot x - \frac{1}{4}\int_0^t |v|^2(s)ds + \gamma(t). \tag{4.3}$$

In addition, we will search for *bounded* parameters (c, v, γ) satisfying the same constraints (3.8), with the obvious modifications.

By now we only need these hypotheses. As in Lemma 3.4 and Proposition 3.10, we will construct a quadruplet (c, v, ρ, γ) with better estimates.

On the other hand, the form of the ansatz $\tilde{u}(t,x)$ is given by (3.9), with

$$w(t,x) := \varepsilon(A_{1,c}(t,y) + iB_{1,c}(t,y))e^{i\Theta}, \tag{4.4}$$

with $A_{1,c}, B_{1,c}$ satisfying condition (3.11) in $\mathbb{R}^2$. Proposition 3.3 now reads

$$S[\tilde{u}](t,x) = \Big[\mathcal{F}_0(t,y) + \varepsilon\mathcal{F}_1(t,y) + \varepsilon^2\mathcal{F}_2(t,y) + \varepsilon^3 f(t)\mathcal{F}_c(y)\Big]e^{i\Theta(t,x)}, \tag{4.5}$$

where $\mathcal{F}_0$ is given now by

$$\begin{aligned}\mathcal{F}_0(t,y) \; &:= \; -\frac{1}{2}(v'(t) - \varepsilon f_1(t))\cdot y\tilde{u} + i(c'(t) - \varepsilon f_2(t))\partial_c\tilde{u} \\ &\qquad -(\gamma'(t) + \frac{1}{2}v'(t)\cdot\rho(t))\tilde{u} + i(\rho'(t) - v(t))\cdot\partial_\rho\tilde{u},\end{aligned} \tag{4.6}$$

$$f_1(t) := (\frac{4\kappa a'(\varepsilon\rho_1(t))c(t)}{(m+1)a(\varepsilon\rho_1(t))}, 0), \quad f_2(t) := \frac{2a'(\varepsilon\rho_1(t))c(t)v_1(t)}{(3-m)a(\varepsilon\rho_1(t))};$$

$$\mathcal{F}_1(t,y) := F_1(t,y) + iG_1(t,y) - \big[\mathcal{L}_+(A_{1,c}) + i\mathcal{L}_-(B_{1,c})\big], \tag{4.7}$$

with

$$F_1(t,y) := \frac{a'(\varepsilon\rho_1(t))}{\tilde{a}^m(\varepsilon\rho_1(t))}y_1 Q_c(y)\big[Q_c^{m-1}(y) - \frac{2\kappa c(t)}{m+1}\big],$$

$$G_1(t,y) := \frac{a'(\varepsilon\rho_1(t))v_1(t)}{\tilde{a}^m(\varepsilon\rho_1(t))}\big[\frac{2c(t)}{3-m}\Lambda Q_c(y) - \frac{1}{m-1}Q_c(y)\big],$$

and $\kappa := \frac{\int Q^{m+1}}{\int Q^2}$. Furthermore

$$\|\varepsilon^2\mathcal{F}_2(t,\cdot)\|_{H^1(\mathbb{R}^2)} \le K\varepsilon^2 e^{-\varepsilon\mu|\rho(t)|}; \quad \|\varepsilon^3 f(t)\mathcal{F}_c\|_{H^1(\mathbb{R}^2)} \le K\varepsilon^3, \tag{4.8}$$

uniformly in time, provided $(A_{1,c}, B_{1,c})$ satisfy (3.11).

Now, let us describe the main differences on the dynamical system concerning the essentially important variables for the dynamics: $c(t), v_1(t), \rho(t)$ and $\gamma(t)$. The result is the following.

Lemma 4.1 (Existence of dynamical parameters)**.**

Suppose $2 \le m < 3$. *Let* $v_0 > 0, \lambda_0, a(s)$ *be as in Theorem C and (1.11). There exists a unique solution* (c, v, ρ, γ) *defined for all* $t \ge -T_\varepsilon$ *with the same regularity than* $a(\varepsilon\cdot)$, *of the following nonlinear system of differential equations*

$$\begin{cases} C'(t) = \dfrac{2\varepsilon a'(\varepsilon U_1(t))}{(3-m)a(\varepsilon U_1(t))}C(t)V_1(t), & C(-T_\varepsilon) = 1, \\ V_1'(t) = \dfrac{4\varepsilon\kappa}{m+1}\dfrac{a'(\varepsilon U_1(t))}{a(\varepsilon U_1(t))}C(t), & V_1(-T_\varepsilon) = v_0, \\ U_1'(t) = V_1(t), & U_1(-T_\varepsilon) = -v_0 T_\varepsilon, \\ H'(t) = -\dfrac{1}{2}V_1'(t)U_1(t), & H(-T_\varepsilon) = 0. \end{cases} \tag{4.9}$$

In addition,

1. *$C(t)$ is strictly increasing with $1 \leq C(t) \leq C(T_\varepsilon)$,, with*

$$C(T_\varepsilon) = c_\infty + O(\varepsilon^{10}) = 2^{\frac{2}{3-m}} + O(\varepsilon^{10}).$$

2. *$V(t)$ is strictly increasing with $v_0 \leq V(t) \leq V(T_\varepsilon)$, with*

$$V(T_\varepsilon) = v_\infty + O(\varepsilon^{10}) = (v_0^2 + 4\alpha_0(c_\infty - 1))^{1/2} + O(\varepsilon^{10}),$$

with α_0 given in (1.30).

On the other hand, the first linear system (Ω_1) is easily solvable, because

$$\int_{\mathbb{R}^2} F_1 \partial_{y_i} Q_c(y) = \int_{\mathbb{R}^2} G_1 Q_c(y) = 0.$$

Moreover, the solution $(A_{1,c}, B_{1,c})$ satisfies (3.11). In addition, Lemma 3.7 now reads

$$\|\tilde{S}[\tilde{u}](t)\|_{H^1(\mathbb{R}^2)} \leq K\varepsilon^2(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon).$$

Similarly Lemma 3.9 holds with no major modifications.

Let us sketch the proof of Proposition 3.10 in the higher dimensional case. As in Lemma 3.11 we consider

$$z(t) := u(t) - \tilde{u}(t, c(t), v(t), \rho(t), \gamma(t)),$$

satisfying for $k = 1, 2$, and for all $t \in [-T_\varepsilon, T_\varepsilon]$,

$$\int_{\mathbb{R}^2} \bar{z}(t,x) Q_c(y) = \int_{\mathbb{R}^2} \bar{z}(t,x) \partial_{y_k} Q_c(y) = 0,$$

and the equation

$$\begin{aligned} iz_t + \Delta z + a_\varepsilon(x_1)[|\tilde{u} + z|^{m-1}(\tilde{u} + z) - |\tilde{u}|^{m-1}\tilde{u}] + \tilde{S}[\tilde{u}] \\ -\frac{1}{2}(v' - \varepsilon f_1) \cdot y\tilde{u} + i(c' - \varepsilon f_2)\partial_c \tilde{u} + i(\rho' - v) \cdot \partial_\rho \tilde{u} + (\gamma' + \frac{1}{2} v' \cdot \rho)\tilde{u} = 0, \end{aligned}$$

in addition to (3.61)-(3.62).

Finally, the functional $\mathcal{F}$ in (3.65) remains the same, up to the obvious modifications: we replace z_x by ∇z and v by its vectorial version. Following these steps, we finally conclude (3.53) and therefore the two dimensional version of Proposition 2.2.

Step 3. Proposition 2.3 revisited. The proof of this result is identical to the one dimensional case. No additional modifications to the standard ones are required.

From this analysis we conclude the proof of Theorem B.

Appendices

H Proof of Proposition 2.1

In this section we sketch the proof of Proposition 2.1. For a similar proof, see e.g. [65].

Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ an increasing sequence with $T_n \geq \frac{1}{2}T_\varepsilon$ for all n and $\lim_{n\to+\infty} T_n = +\infty$. Consider $u_n(t)$ the solution of the following Cauchy problem

$$\begin{cases} i(u_n)_t + (u_n)_{xx} + a_\varepsilon(x)|u_n|^{m-1}u_n = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ u_n(-T_n) = Q(\cdot + v_0T_n)e^{\frac{i}{2}(\cdot)v_0}e^{-i(1-\frac{1}{4}v_0^2)T_n}. \end{cases} \tag{H.1}$$

In other words, u_n is a solution of aNLS that at time $t = -T_n$ corresponds exactly to a solitary wave. It is clear that this function is in $H^1(\mathbb{R})$; moreover, there exists a uniform constant $C = C(v_0) > 0$ such that

$$\|Q(\cdot - v_0t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq C.$$

According to Lemma 3.1, we have that u_n is globally well-defined in $H^1(\mathbb{R})$.

The next step is to establish uniform estimates starting from a fixed time $t = -\frac{1}{2}T_\varepsilon < 0$ large enough such that the soliton is sufficiently away from the region where the influence of the potential a_ε is present. This is the purpose of the following

Proposition H.1 (Uniform estimates in H^1 for large times, see also [47])**.**

There exist constants $K, \mu > 0$ and $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon < \varepsilon_0$ and for all $n \in \mathbb{N}$ we have and for all $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$,

$$\|u_n(t) - Q(\cdot - v_0t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq Ke^{\mu\varepsilon t}. \tag{H.2}$$

In particular, there exists a constant $C > 0$ such that for all $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$,

$$\|u_n(t)\|_{H^1(\mathbb{R})} \leq C. \tag{H.3}$$

Using Proposition H.1 we will obtain the existence of a *critical element* $u_{0,*} \in H^1(\mathbb{R})$, with several interesting properties. Indeed, let us consider the sequence $(u_n(-\frac{1}{2}T_\varepsilon))_{n\in\mathbb{N}} \subseteq H^1(\mathbb{R})$. We claim the following result.

Lemma H.2 (Compactness property)**.**

Given any number $\delta > 0$, there exist $\varepsilon_0 > 0$ and a constant $K_0 > 0$ large enough such that for all $0 < \varepsilon < \varepsilon_0$ and for all $n \in \mathbb{N}$,

$$\int_{|x|>K_0} |u_n|^2(-\frac{1}{2}T_\varepsilon) < \delta. \tag{H.4}$$

Proof. The proof is by now a standard result. See [49] for the details. □

Let us come back to the proof of Theorem 2.1. From (H.3) we have that

$$\|u_n(-\frac{1}{2}T_\varepsilon)\|_{H^1(\mathbb{R})} \leq C,$$

independent of n. Thus, up to a subsequence we may suppose $u_n(-\frac{1}{2}T_\varepsilon) \rightharpoonup u_{*,0}$ in the $H^1(\mathbb{R})$ weak sense, and $u_n(-\frac{1}{2}T_\varepsilon) \to u_{*,0}$ in $L^2_{loc}(\mathbb{R})$, as $n \to +\infty$. In addition, from (H.4) we have the strong convergence in $L^2(\mathbb{R})$.

Let $u_* = u_*(t)$ be the solution of (1.1) with initial data $u_*(-\frac{1}{2}T_\varepsilon) = u_{*,0}$. From Proposition 3.1 we also have $u_* \in C(\mathbb{R}, L^2(\mathbb{R}))$ (that is, L^2 local well-posedness plus conservation of mass). Thus, using the continuous dependence of u_n and u_*, and the bound (H.3), we obtain $u_n(t) \to u_*(t)$ in $H^1(\mathbb{R})$ for every $t \leq -\frac{1}{2}T_\varepsilon$. Passing to the limit in (H.2) we obtain for all $t \leq -\frac{1}{2}T_\varepsilon$,

$$\|u_*(t) - Q(\cdot - v_0t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq Ke^{\varepsilon\mu t},$$

as desired. This finish the proof of the existence part of Theorem 2.1.

H.1 Uniform H^1 estimates. Proof of Proposition H.1

In this paragraph we explain the main steps of the proof of Proposition H.1 in the H^1 case; for the general case the reader may consult [49].

The first step in the proof is the following bootstrap property:

Proposition H.3 (Bootstrap)**.**

There exist constants $K, \mu, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following is true. There exists $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, if for some $-T_{n,*} \in [-T_n, -\frac{1}{2}T_\varepsilon]$ and for all $t \in [-T_n, -T_{n,*}]$ one has

$$\|u_n(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq 2\alpha, \tag{H.5}$$

then, for all $t \in [-T_n, -T_{n,*}]$

$$\|u_n(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq Ke^{\varepsilon\mu t}. \tag{H.6}$$

Proof of Proposition H.1, assuming the validity of Proposition H.3. Let $0 < \alpha < \alpha_0$. Note that from (H.1) there exists $t_0 = t_0(n, \alpha) > 0$ such that (H.5) holds true for all $t \in [-T_n, -T_n + t_0]$. Now let us consider (we adopt the convention $T_{*,n} > 0$)

$$-\tilde{T}_{*,n} := \sup\{t \in [-T_n, -\frac{1}{2}T_\varepsilon] \mid \text{ for all } t' \in [-T_n, t],$$
$$\|u_n(t') - Q(\cdot - v_0 t')e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t'}\|_{H^1(\mathbb{R})} \leq 2\alpha\}.$$

Assume, by contradiction, that $-\tilde{T}_{*,n} < -\frac{1}{2}T_\varepsilon$. From Proposition H.3, we have

$$\|u_n(t') - Q(\cdot - v_0 t')e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t'}\|_{H^1(\mathbb{R})} \leq Ke^{\mu\varepsilon t} \leq \alpha,$$

for ε small enough (recall that $t \leq -\frac{1}{2}T_\varepsilon = -\frac{1}{2v_0}\varepsilon^{-1-\frac{1}{100}}$), a contradiction with the definition of $\tilde{T}_{*,n}$. □

Now we are reduced to prove Proposition H.3.

Proof of Proposition H.3. The first step in the proof is to decompose the solution preserving a standard orthogonality condition. To obtain this, without loss of generality, by taking $T_{n,*}$ larger we may suppose that for all $t \in [-T_n, -T_{n,*}]$

$$\|u_n(t) - Q(\cdot - v_0 t - r_n(t))e^{it}e^{\frac{i}{2}v_0(\cdot)}e^{-\frac{1}{4}iv_0^2 t}e^{ig_n(t)}\|_{H^1(\mathbb{R})} \leq 2\alpha, \tag{H.7}$$

for all smooth r_n, g_n satisfying $r_n(-T_n) = g_n(-T_n) = 0$ and $|r_n'(t)| \leq \frac{1}{t^2}$. A posteriori we will prove that this condition can be improved and extended to any time $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$.

For notational simplicity, in what follows we will drop the index n on $-T_{*,n}$ and u_n, if no confusion is present.

Lemma H.4 (Modulation)**.**

There exist $K, \mu, \varepsilon_0 > 0$ and unique C^1 functions $\rho_0, \gamma_0 : [-T_n, -T_*] \to \mathbb{R}$ such that for all $0 < \varepsilon < \varepsilon_0$ the function z defined by

$$z(t, x) := u(t, x) - \tilde{R}_{v_0}(t, x); \qquad \tilde{R}_{v_0}(t, x) := Q(y)e^{i\theta}, \tag{H.8}$$

with

$$y := x - v_0 t - \rho_0(t), \qquad \theta := t + \frac{1}{2}v_0 x - \frac{1}{4}v_0^2 t + \gamma_0(t), \tag{H.9}$$

satisfies for all $t \in [-T_n, -T_*]$,

$$\mathrm{Re}\int_{\mathbb{R}} \bar{z}(t,x)Q'(y)e^{i\theta}dx = \mathrm{Im}\int_{\mathbb{R}} \bar{z}(t,x)Q(y)e^{i\theta}dx = 0, \tag{H.10}$$

$$\|z(t)\|_{H^1(\mathbb{R})} \leq K\alpha, \quad \rho_0(-T_n) = \gamma_0(-T_n) = 0. \tag{H.11}$$

In addition, z *satisfies the following modified Schrödinger equation,*

$$\begin{aligned} &iz_t + z_{xx} + a_\varepsilon(x)|\tilde{R}_{v_0} + z|^{m-1}(\tilde{R}_{v_0} + z) - a_\varepsilon(x)|\tilde{R}_{v_0}|^{m-1}\tilde{R}_{v_0} \\ &\qquad -i\rho_0'(t)Q'(y)e^{i\theta} - \gamma_0'(t)\tilde{R}_{v_0} + (a_\varepsilon(x)-1)|\tilde{R}_{v_0}|^{m-1}\tilde{R}_{v_0} = 0, \end{aligned} \tag{H.12}$$

and

$$|\rho_0'(t)| + |\gamma_0'(t)| \leq K\big[e^{\varepsilon\mu t} + \|z(t)\|_{H^1(\mathbb{R})} + \|z(t)\|_{L^2(\mathbb{R})}^2\big]. \tag{H.13}$$

Proof of Lemma H.4. The proof of (H.10) is a standard consequence of the Implicit Function Theorem, the definition of T_* $(= T_{*,n})$, and the definition of $u_n(-T_n)$ given in (H.1), see for example [49] for a detailed proof. Similarly, the proof of (H.12) follows after a simple computation.

Now we deal with (H.13). Taking time derivative to the first identity in (H.10) and using (H.12), we get

$$\begin{aligned} 0 &= -\mathrm{Im}\int_{\mathbb{R}} \overline{iz_t}Q'(y)e^{i\theta} + \mathrm{Re}\int_{\mathbb{R}} \bar{z}(Q'(y)e^{i\theta})_t \\ &= \mathrm{Im}\int_{\mathbb{R}} \big\{\bar{z}_{xx} + a_\varepsilon(x)|\tilde{R}_{v_0}+z|^{m-1}\overline{(\tilde{R}_{v_0}+z)} - a_\varepsilon(x)|\tilde{R}_{v_0}|^{m-1}\overline{\tilde{R}_{v_0}}\big\}Q'(y)e^{i\theta} \\ &\quad + \rho_0'(t)\int_{\mathbb{R}} Q'^2 + \mathrm{Im}\int_{\mathbb{R}}(a_\varepsilon(x)-1)|\tilde{R}_{v_0}|^{m-1}\overline{\tilde{R}_{v_0}}Q'(y)e^{i\theta} \\ &\quad + \mathrm{Re}\int_{\mathbb{R}} \bar{z}\big\{-(v_0+\rho_0'(t))Q''(y) + i(1-\frac{1}{4}v_0^2 + \gamma_0'(t))Q'(y)\big\}e^{i\theta} \end{aligned}$$

First of all, note that

$$\begin{aligned} &\mathrm{Im}\int_{\mathbb{R}} \big\{z_{xx} + a_\varepsilon(x)|\tilde{R}_{v_0}+z|^{m-1}(\tilde{R}_{v_0}+z) - a_\varepsilon(x)|\tilde{R}_{v_0}|^{m-1}\tilde{R}_{v_0}\big\}Q'(y)e^{-i\theta} = \\ &\qquad = O(\|z(t)\|_{L^2(\mathbb{R})} + \|z(t)\|_{L^2(\mathbb{R})}^2). \end{aligned}$$

On the other hand, from (1.11), (H.10), the uniform bound on $\rho_0'(t)$ in the definition of T_* and the exponential decay of R, we have

$$|\mathrm{Im}\int_{\mathbb{R}}(a_\varepsilon(x)-1)|\tilde{R}_{v_0}|^{m-1}\overline{\tilde{R}_{v_0}}Q'(y)e^{i\theta}| \leq Ke^{\varepsilon\mu t}. \tag{H.14}$$

Indeed, first note that from (H.7), by integrating between $-T_n$ and t and using (H.10) we get

$$\rho_0(t) \leq -\frac{1}{T_n} - \frac{1}{t} \leq \frac{2}{T_\varepsilon} \leq K\varepsilon^{1+\frac{1}{100}}.$$

Thus $v_0 t + \rho_0(t) \leq v_0 t + K\varepsilon^{1+\frac{1}{100}} \leq \frac{9}{10}v_0 t$. Therefore, by possibly redefining $\mu > 0$, we have from (1.11),

$$\begin{aligned} |\int_{\mathbb{R}}(a_\varepsilon(x)-1)|\tilde{R}_{v_0}|^{m-1}\overline{\tilde{R}_{v_0}}Q'(y)e^{i\theta}| &\leq K\int_{-\infty}^0 e^{\mu\varepsilon x}e^{-|x-v_0t-\rho_0(t)|}dx \\ &\qquad + Ke^{v_0t+\rho_0(t)}\int_0^\infty e^{-x}dx \\ &\leq K\exp\big[\mu\varepsilon(v_0t+\rho_0(t))\big] + K\exp\big[\mu(v_0t+\rho_0(t))\big] \\ &\leq Ke^{\mu\varepsilon t}. \end{aligned}$$

Finally,

$$|\mathrm{Re}\int_{\mathbb{R}} \bar{z}\{ -(v_0+\rho_0'(t))Q''(y) + i(1-\frac{1}{4}v_0^2+\gamma_0'(t))Q'(y)\}e^{i\theta}| \leq K\|z(t)\|_{L^2(\mathbb{R})}(1+|\rho_0'(t)|+|\gamma_0'(t)|).$$

We arrive, for α small enough, to the following estimate

$$|\rho_0'(t)| \leq K(e^{\varepsilon\mu t} + \|z(t)\|_{L^2(\mathbb{R})}(1+|\gamma_0'(t)|) + \|z(t)\|^2_{L^2(\mathbb{R})}). \tag{H.15}$$

Now we consider the second identity in (H.10). Proceeding in a similar way as above, we obtain

$$|\gamma_0'(t)| \leq K(e^{\varepsilon\mu t} + \|z(t)\|_{L^2(\mathbb{R})}(1+|\rho_0'(t)|) + \|z(t)\|^2_{L^2(\mathbb{R})}). \tag{H.16}$$

Collecting estimates (H.15)-(H.16) we obtain (H.13). □

H.1.1 Almost conservation of mass, energy and momentum

Now let us recall that for all $-T_n \leq t \leq -\frac{1}{2}T_\varepsilon$ we have $M[u](t)$ and $E_a[u](t)$ conserved. In addition, from (1.5) we have

$$\partial_t P[u](t) = \frac{\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)|u|^{m+1} \geq 0.$$

Therefore

$$E_a[u](t) - E_a[u](-T_n) + (1+\frac{1}{4}v_0^2)[M[u](t) - M[u](-T_n)] - v_0[P[u](t) - P[u](-T_n)] \leq 0. \tag{H.17}$$

Similarly, note that in the considered region the solitary wave $\tilde{R}_{v_0}(t)$ is an almost solution of (1.12), in particular it must almost conserve the mass M (1.3) and the energy E_a (1.17), at least for large negative time. Indeed, arguing as in Lemma I.2 (but with easier proof), one has

$$E_a[\tilde{R}_{v_0}](-T_n) - E_a[\tilde{R}_{v_0}](t) + (1+\frac{1}{2}v_0^2)\big[M[\tilde{R}_{v_0}](-T_n) - M[\tilde{R}_{v_0}](t)\big] - v_0\big[P[\tilde{R}_{v_0}](-T_n) - P[\tilde{R}_{v_0}](t)\big] \leq Ke^{\mu\varepsilon t}, \tag{H.18}$$

for some constant $K>0$ and all time $t \in [-T_n, T_*]$.

The next step is the use the mass conservation law to provide a control of the $\tilde{R}_{v_0}(t)$ direction. Indeed, one has

$$|\mathrm{Re}\int_{\mathbb{R}} \tilde{R}_{v_0}\bar{z}(t)| \leq K\|z(-T_n)\|^2_{L^2(\mathbb{R})} + K\|z(t)\|^2_{L^2(\mathbb{R})} \leq K\sup_{t\in[-T_n,T_*]}\|z(t)\|^2_{L^2(\mathbb{R})}. \tag{H.19}$$

for a constant $K>0$, independent of ε. On the other hand, note that

$$\begin{aligned} E_a[u](t) + (1+\frac{1}{4}v_0^2)M[u](t) - v_0P[u](t) \ &= \ E_a[\tilde{R}_{v_0}](t) + (1+\frac{1}{4}v_0^2)M[\tilde{R}_{v_0}](t) - v_0P[\tilde{R}_{v_0}] \\ &\quad - \mathrm{Re}\int_{\mathbb{R}}(a_\varepsilon - 1)|\tilde{R}_{v_0}|^{m-1}\tilde{R}_{v_0}\bar{z} + \mathcal{F}_0(t), \end{aligned} \tag{H.20}$$

where $\mathcal{F}_0$ is the following quadratic functional

$$\begin{aligned} \mathcal{F}_0(t) \ &:= \ \frac{1}{2}\int_{\mathbb{R}}\big[|z_x|^2 + (1+\frac{1}{4}v_0^2)|z|^2\big] - \frac{v_0}{2}\,\mathrm{Im}\int_{\mathbb{R}}\bar{z}z_x \\ &\quad -\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)[|\tilde{R}_{v_0}+z|^{m+1} - |\tilde{R}_{v_0}|^{m+1} - (m+1)|\tilde{R}_{v_0}|^{m-1}\,\mathrm{Re}(\tilde{R}_{v_0}\bar{z})]. \end{aligned}$$

In addition, for any $t \in [-T_n, -T_*]$,

$$|\mathrm{Re} \int_{\mathbb{R}} (a_\varepsilon - 1)|\tilde{R}_{v_0}|^{m-1}\tilde{R}_{v_0}\bar{z}| \leq Ke^{\mu\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}. \tag{H.21}$$

The proof of (H.20) is essentially an expansion of the energy-mass functional using the relation $u(t) = \tilde{R}_{v_0}(t) + z(t)$. The proof of (H.21) is similar to (H.14).

On the other hand, the functional $\mathcal{F}_0(t)$ above mentioned enjoys the following coercivity property: there exist $K, \lambda_0 > 0$ independent of ε such that for every $t \in [-T_n, -T_*]$

$$\mathcal{F}_0(t) \geq \lambda_0\|z(t)\|^2_{H^1(\mathbb{R})} - \Big|\mathrm{Re} \int_{\mathbb{R}} \tilde{R}_{v_0}(t)\bar{z}(t)\Big|^2 - Ke^{\mu\varepsilon t}\|z(t)\|^2_{L^2(\mathbb{R})} - K\|z(t)\|^3_{L^2(\mathbb{R})}. \tag{H.22}$$

This bound is a consequence of (H.10) and Lemma 3.2.

H.1.2 End of proof of Proposition H.3

Now by using (H.20), (H.22), and the estimates (H.17)-(H.18) and (H.19) we finally get (H.6). Indeed, note that

$$\begin{aligned} Ke^{\mu\varepsilon t} \;&\geq\; E_a[\tilde{R}_{v_0}](-T_n) - E_a[\tilde{R}_{v_0}](t) + (1 + \frac{1}{2}v_0^2)\big[M[\tilde{R}_{v_0}](-T_n) - M[\tilde{R}_{v_0}](t)\big] \\ &\qquad -v_0\big[P[\tilde{R}_{v_0}](-T_n) - P[\tilde{R}_{v_0}](t)\big] \\ &\geq\; \mathcal{F}_0(t) - Ke^{\mu\varepsilon t} - Ke^{\mu\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})} - K\|z(t)\|^4_{L^2(\mathbb{R})}. \end{aligned}$$

Finally, from (H.22) and H.19 we conclude that for some $K, \mu > 0$,

$$\|z(t)\|_{H^1(\mathbb{R})} \leq Ke^{\mu\varepsilon t}.$$

Plugging this estimate in (H.13), we obtain that $|\rho_0'(t)| \leq Ke^{\mu\varepsilon t}$, and thus after integration and by taking $\mu > 0$ smaller if necessary, we get the final uniform estimate (H.6) for the H^1-case. Note that we have also improved the estimate on $\rho_0'(t)$ assumed in (H.7). This finishes the proof. □

H.2 Proof of Uniqueness

First of all let us recall that the solution u above constructed is in $C(\mathbb{R}, H^1(\mathbb{R}))$ and satisfies the exponential decay (2.2). Moreover, *every* solution converging to a soliton satisfies this property. This property is crucial to obtain the uniqueness.

Proposition H.5 (Exponential decay, see also [47, 65])**.**

There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds. Let v be a $C(\mathbb{R}, H^1(\mathbb{R}))$ solution of (1.1) satisfying

$$\lim_{t\to-\infty} \|v(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0.$$

Then there exist $K, \mu > 0$ such that for every $t \leq -T_\varepsilon$ one has

$$\|v(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq Ke^{\mu\varepsilon t}.$$

Proof. Fix $\alpha > 0$ small. Let $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ small enough such that for all $0 < \varepsilon \leq \varepsilon_0$ and $t \leq -T_\varepsilon$,

$$\|v(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq \alpha.$$

Possibly choosing ε_0 even smaller, we can apply the arguments of Proposition H.3 to the function $v(t)$ on the interval $(-\infty, -\frac{1}{2}T_\varepsilon]$ to obtain the desired result. Recall that a key fact to obtain this result is that

$$\partial_t P[v](t) \geq 0,$$

which is not valid in the case of a pure soliton solution going to $x \sim +\infty$ as $t \to +\infty$. □

Now we are ready to prove the uniqueness part.

Sketch of proof of uniqueness. Let $w(t) := v(t) - u(t)$. Then $w(t) \in H^1(\mathbb{R})$ and satisfies the equation

$$\begin{cases} iw_t + w_{xx} + a_\varepsilon(x)|u+w|^{m-1}(u+w) - a_\varepsilon(x)|u|^{m-1}u = 0, & \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ \|w(t)\|_{H^1(\mathbb{R})} \leq Ke^{\mu\varepsilon t} \quad \text{for all } t \leq -\frac{1}{2}T_\varepsilon. & \text{(cf. Proposition H.5).} \end{cases} \tag{H.23}$$

The idea is to prove that $w(t) \equiv 0$ for all $t \in \mathbb{R}$. For this purpose, one defines the second order functional

$$\begin{aligned} \mathcal{F}_0[w](t) \;&:=\; \frac{1}{2}\int_{\mathbb{R}} |w_x|^2 + \frac{1}{2}(1+\frac{1}{4}v_0^2)\int_{\mathbb{R}} |w|^2 - \frac{1}{2}v_0 \operatorname{Im}\int_{\mathbb{R}} w_x\bar{w} \\ &\quad -\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)[|u+w|^{m+1} - |u|^{m+1} - (m+1)|u|^{m-1}\operatorname{Re}(u\bar{w})]. \end{aligned}$$

It is easy to verify that

1. Asymptotic at $-\infty$.
$$\lim_{t\to-\infty} \mathcal{F}_0[w](t) = 0. \tag{H.24}$$

2. Lower bound. There exists $K > 0$ such that for all $t \leq -\frac{1}{2}T_\varepsilon$,
$$\mathcal{F}_0[w](t) \geq \tilde{\mathcal{F}}_0[w](t) - Ke^{\mu\varepsilon t}\sup_{t'\leq t}\|w(t')\|^2_{H^1(\mathbb{R})},$$
where
$$\begin{aligned} \tilde{\mathcal{F}}_0[w](t) \;&:=\; \frac{1}{2}\int_{\mathbb{R}} |w_x|^2 + \frac{1}{2}\int_{\mathbb{R}}(1+\frac{1}{4}v_0^2)|w|^2 - \frac{1}{2}v_0 \operatorname{Im}\int_{\mathbb{R}} w_x\bar{w} \\ &\quad -\int_{\mathbb{R}} a_\varepsilon(x)[(m-1)|u|^{m-3}(\operatorname{Re}(u\bar{w}))^2 + |u|^{m-1}|w|^2]. \end{aligned}$$

3. First derivative.
$$\begin{aligned} \mathcal{F}_0'[w](t) &= \operatorname{Im}\int_{\mathbb{R}} \overline{iw_t}\{w_{xx} - (1+\frac{1}{4}v_0^2)w + |u+w|^{m-1}(u+w) - |u|^{m-1}u - iv_0 w_x\} \\ &+ \operatorname{Im}\int_{\mathbb{R}} a_\varepsilon(x)\overline{iu_t}\{|u+w|^{m-1}(u+w) - |u|^{m-1}u - \frac{1}{2}(m+1)|u|^{m-1}w - \frac{1}{2}(m-1)|u|^{m-3}u^2\bar{w}\}. \end{aligned}$$

4. Upper bound. There exists $K, \mu > 0$ such that
$$\mathcal{F}_0[w](t) \leq Ke^{\mu\varepsilon t}\sup_{t'\leq t}\|w(t')\|^2_{H^1(\mathbb{R})}.$$

These estimates are proved similarly to the proof of Lemma 3.13, see also [47] for a similar proof. However, the functional $\mathcal{F}_0(t)$ is not necessarily coercive; so in order to obtain a satisfactory lower bound on $\mathcal{F}_0$, one has to modify the function w in $(-\infty, -\frac{1}{2}T_\varepsilon]$ as follows. Let

$$\tilde{w}(t,x) := w(t,x) + b_1(t)Q(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x} + b_2(t)Q'(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x},$$

with

$$\begin{aligned} b_1(t) &:= -\frac{1}{2M[Q]}\operatorname{Im}\int_{\mathbb{R}} \bar{w}(t,x)Q(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x}dx; \\ b_2(t) &:= -\frac{1}{2M[Q']}\operatorname{Re}\int_{\mathbb{R}} \bar{w}(t,x)Q'(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x}dx. \end{aligned}$$

This new function satisfies

1. Orthogonality on the Q and Q' directions:

$$\operatorname{Im}\int_{\mathbb{R}} \overline{\tilde{w}}(t)Q(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x} = \operatorname{Re}\int_{\mathbb{R}} \overline{\tilde{w}}(t)Q'(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x} = 0.$$

2. Equivalence. There exists $C_1, C_2 > 0$ independent of ε such that

$$C_1\|w(t)\|_{H^1(\mathbb{R})} \leq \|\tilde{w}(t)\|_{H^1(\mathbb{R})} + |b_1(t)| + |b_2(t)| \leq C_2\|w(t)\|_{H^1(\mathbb{R})}.$$

   Moreover,

$$\tilde{\mathcal{F}}_0[\tilde{w}](t) = \tilde{\mathcal{F}}_0[w](t) + O(e^{\varepsilon\mu t}\|w(t)\|^2_{H^1(\mathbb{R})}).$$

3. Control on the Q direction: for some $K, \mu > 0$,

$$|\operatorname{Re}\int_{\mathbb{R}} \overline{\tilde{w}}(t,x)Q(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x}| \leq Ke^{\varepsilon\mu t}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})}.$$

   This property is proved similarly to the proof of (I.12): We use the fact that variation in time of the above quantity is of quadratic order in $\tilde{w}$.

4. Coercivity. There exists $\lambda > 0$ independent of t such that

$$\tilde{\mathcal{F}}_0[\tilde{w}](t) \geq \lambda\|\tilde{w}(t)\|^2_{H^1(\mathbb{R})} - K|\operatorname{Re}\int_{\mathbb{R}} \overline{\tilde{w}}(t,x)Q(x-v_0t)e^{i(1-\frac{1}{4}v_0^2)t}e^{\frac{i}{2}v_0x}|^2.$$

5. Sharp control. From the equivalence w-$\tilde{w}$ and the coercivity property we obtain, for some $K, \mu > 0$,

$$\|\tilde{w}(t)\|_{H^1(\mathbb{R})} \leq Ke^{\varepsilon\mu t/2}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})}, \tag{H.25}$$

   and therefore

$$|b_1(t)| + |b_2(t)| \leq Ke^{\varepsilon\mu t/2}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})}. \tag{H.26}$$

   Note that the bounds on $b_1(t)$ and $b_2(t)$ are proved similarly to (I.11).

The proof of all these affirmations follows the argument of Proposition 6 in [47], with easier proofs. Finally, from (H.25)-(H.26) we have for ε small enough and $t \leq -\frac{1}{2}T_\varepsilon$,

$$\|w(t)\|_{H^1(\mathbb{R})} \leq Ke^{\varepsilon\mu t}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})} < \frac{1}{2}\sup_{t'\leq t}\|w(t')\|_{H^1(\mathbb{R})}.$$

This inequality implies $w \equiv 0$, which gives the uniqueness. □

I Proof of Proposition 2.3

The proof of the stability result (2.8) is based in a standard Weinstein argument. Let us assume that for some $K > 0$ fixed,

$$\|u(t_1) - \lambda_\infty Q_{c_\infty}(\cdot - X_0)e^{\frac{i}{2}v_\infty x}e^{i\gamma_0}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}, \tag{I.1}$$

with $\lambda_\infty, v_\infty, c_\infty$ defined in Theorem A, p_m defined in (1.21), and $\gamma_0 \in \mathbb{R}$. From the local and global Cauchy theory (cf. Lemma 3.1), we know that u is well defined for all $t \geq t_1$.

Step 0. Preliminaries. In order to simplify the calculations, note that from (1.15) the function $v(t,x) := \lambda_\infty^{-1}u(t,x)$ solves

$$iv_t + v_{xx} + \frac{a_\varepsilon}{2}|v|^{m-1}v = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x.$$

The energy is now given by

$$\tilde{E}_a[v] := \frac{1}{2}\int_{\mathbb{R}} |v_x|^2 - \frac{1}{m+1}\int_{\mathbb{R}} \frac{a_\varepsilon}{2}|v|^{m+1}; \tag{I.2}$$

the mass (1.3) and momentum (1.5) remain unchanged. In addition (I.1) now becomes

$$\|v(t_1) - Q_{c_\infty}(\cdot - X_0)e^{\frac{i}{2}xv_\infty}e^{i\gamma_0}\|_{H^1(\mathbb{R})} \leq \tilde{K}\varepsilon^{p_m}. \tag{I.3}$$

With a slight abuse of notation we will **rename** $v := u$, $\tilde{K} := K$, and we will assume the validity of (I.3) for u. In addition, and if no confusion is present, we will drop the tilde in (I.2). The parameters X_0 and c_∞ remain unchanged.

Let $D_0 > 2K$ be a large number to be chosen later, and set

$$\begin{aligned} T^* := \sup\Big\{ & t \geq t_1 \mid \text{ for all } t' \in [t_1, t), \text{ there exist } r_2(t'), g_2(t') \in \mathbb{R} \text{ smooth} \\ & \text{such that } |r_2'(t')| + |r_2(t_1) + v_\infty t_1 - X_0| \leq \frac{v_\infty}{100}, \text{ and} \\ & \|u(t') - Q_{c_\infty}(\cdot - v_\infty t - r_2(t'))\exp\big\{\frac{i}{2}xv_\infty - \frac{i}{4}v_\infty^2 t + ig_2(t)\big\}\|_{H^1(\mathbb{R})} \leq D_0\varepsilon^{p_m}\Big\}. \end{aligned} \tag{I.4}$$

Observe that $T^* > t_1$ is well-defined since $D_0 > 2K$, (I.1) and the continuity of $t \mapsto u(t)$ in $H^1(\mathbb{R})$. The objective is to prove that $T^* = +\infty$, and thus (2.8). Therefore, for the sake of contradiction, in what follows **we shall suppose** $T^* < +\infty$.

The first step to reach a contradiction is now to decompose the solution on $[t_1, T^*]$ using modulation theory around the soliton. In particular, we will find some special $\rho_2(t), \gamma_2(t)$ satisfying the hypothesis in (I.4) but with

$$\sup_{t\in[t_1,T^*]} \big\|u(t) - Q_{c_\infty}(\cdot - v_\infty t - \rho_2(t))\exp\big\{\frac{i}{2}xv_\infty - \frac{i}{4}v_\infty^2 t + i\gamma_2(t)\big\}\big\|_{H^1(\mathbb{R})} \leq \frac{1}{2}D_0\varepsilon^{p_m}, \tag{I.5}$$

a contradiction with the definition of T^*.

Step 1. Modulation on the degenerate directions. We will prove the following

Lemma I.1 (Modulated decomposition)**.**

For $\varepsilon > 0$ small enough, independent of T^*, there exist C^1 functions $\rho_2, c_2, \tilde{\gamma}_2$, defined on $[t_1, T^*]$, with $c_2(t) > 0$ and such that the function $z(t)$ given by

$$z(t,x) := u(t,x) - \tilde{R}(t,x), \tag{I.6}$$

where $\tilde{R}(t,x) := Q_{c_2(t)}(y)e^{i\Gamma}$, *with*

$$y := x - v_\infty t - \rho_2(t) \ \ \textit{and} \ \ \Gamma := \frac{1}{2}xv_\infty + \int_{t_1}^{t} c_2(s)ds - \frac{1}{4}v_\infty^2 t + \tilde{\gamma}_2(t),$$

satisfies for all $t \in [t_1, T^*]$,

$$\operatorname{Re}\int_{\mathbb{R}} \tilde{R}(t)\bar{z}(t) = \operatorname{Im}\int_{\mathbb{R}} \tilde{R}(t)\bar{z}(t) = \operatorname{Re}\int_{\mathbb{R}} Q'_{c_2(t)}(y)e^{i\Gamma}\bar{z}(t) = 0, \tag{I.7}$$

$$\|z(t)\|_{H^1(\mathbb{R})} + |c_2(t) - c_\infty| \leq KD_0\varepsilon^{p_m}, \ \ \textit{and} \tag{I.8}$$

$$\|z(t_1)\|_{H^1(\mathbb{R})} + |\rho_2(t_1) + v_\infty t_1 - X_0| + |c_2(t_1) - c_\infty| + |\tilde{\gamma}_2(t_1) - \frac{1}{4}v_\infty t_1 - \gamma_0| \leq K\varepsilon^{p_m}, \tag{I.9}$$

where K *is not depending on* D_0. *In addition,* $z(t)$ *now satisfies the following modified NLS equation*

$$\begin{aligned} &iz_t + z_{xx} + \frac{1}{2}a_\varepsilon(x)\big[|\tilde{R} + z|^{m-1}(\tilde{R} + z) - |\tilde{R}|^{m-1}\tilde{R}\big] \\ &\quad + ic_2'(t)\Lambda Q_{c_2}e^{i\Gamma} - \tilde{\gamma}_2'(t)Q_{c_2}e^{i\Gamma} - i\rho_2'(t)Q'_{c_2}e^{i\Gamma} + (\frac{1}{2}a_\varepsilon(x) - 1)Q^m_{c_2}e^{i\Gamma} = 0. \end{aligned} \tag{I.10}$$

Furthermore, for some constant $\mu > 0$ *independent of* ε, *we have the following estimates:*

$$|\rho_2'(t)| \leq K\Big[\int_{\mathbb{R}} e^{-\mu|y|}|z|^2(t,x)dx\Big]^{\frac{1}{2}} + K\int_{\mathbb{R}} e^{-\mu|y|}|z|^2(t,x)dx + Ke^{-\mu\varepsilon t}; \tag{I.11}$$

$$\frac{|c_2'(t)|}{c_2(t)} \leq K\int_{\mathbb{R}} e^{-\mu|y|}|z|^2(t,x)dx + Ke^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}, \tag{I.12}$$

and finally

$$|\tilde{\gamma}_2'(t)| \leq K\Big[\int_{\mathbb{R}} e^{-\mu|y|}|z|^2(t,x)dx\Big]^{\frac{1}{2}} + K\int_{\mathbb{R}} e^{-\mu|y|}|z|^2(t,x)dx + Ke^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})} + Ke^{-\varepsilon\mu t}. \tag{I.13}$$

Remark I.1. Note that from (I.8) and taking ε small enough we have an improved the bound on $\rho_2(t)$. Indeed, for all $t \in [t_1, T^*]$,

$$|\rho_2'(t)| + |\rho_2(t_1) + v_\infty t_1 - X_0| \leq 2D_0\varepsilon^{p_m}.$$

Thus, in order to reach a contradiction, we only need to show (I.5). Observe that these inequalities impliy that the soliton position is far away from the interaction region.

Proof of Lemma I.1. As in Lemma H.4 and 3.11, the proof of (I.6)-(I.9) are based in a Implicit Function Theorem application.

On the other hand, equation (I.10) is a simple computation, completely similar to (H.12) and (3.60). Finally, estimates (I.11)-(I.13) are similar to the proof of (H.13). We skip the details. □

Step 2. Almost conserved quantities and monotonicity.

Lemma I.2 (Almost conservation of modified mass, energy and momentum)**.**

Consider $M = M[\tilde{R}]$, $E_a = E_a[\tilde{R}]$ *and* $P[\tilde{R}]$ *the mass, energy and momentum of the* soliton $\tilde{R}$ *(cf. (I.6)). Then for all* $t \in [t_1, T^*]$ *we have*

$$M[\tilde{R}](t) = c_2^{2\theta}(t)M[Q]; \tag{I.14}$$

$$E_a[\tilde{R}](t) = c_2^{2\theta}(t)(\frac{1}{4}v_\infty^2 - \lambda_0 c_2(t))M[Q] + O(e^{-\varepsilon\mu t}); \tag{I.15}$$

$$P[\tilde{R}](t) = \frac{1}{2}v_\infty c_2^{2\theta}(t)M[Q]. \tag{I.16}$$

Furthermore, we have the bound

$$\big|E_a[\tilde R](t_1) - E_a[\tilde R](t) + (c_2(t_1) + \frac{1}{4}v_\infty^2)(M[\tilde R](t_1) - M[\tilde R](t)) - v_\infty(P[\tilde R](t_1) - P[\tilde R](t))\big|$$
$$\leq K|\Big[\frac{c_2(t)}{c_2(t_1)}\Big]^{2\theta} - 1|^2 + Ke^{-\varepsilon\mu t_1}. \quad \text{(I.17)}$$

Proof. The first and third identities, namely (I.14) and (I.16), are direct computations. We consider (I.15). Here we have

$$\begin{aligned} E_a[\tilde R](t) &= \frac{1}{2}\int_{\mathbb{R}} |\tilde R_x|^2 - \frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon(x)|\tilde R|^{m+1} \\ &= c_2^{2\theta}(t)\Big[c_2(t)(\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1}) + \frac{1}{8}v_\infty^2\int_{\mathbb{R}} Q^2\Big] \\ &\qquad + \frac{1}{m+1}\int_{\mathbb{R}} (1 - \frac{a_\varepsilon}{2})|\tilde R|^{m+1}. \end{aligned}$$

Similarly to (H.21), we have

$$|\int_{\mathbb{R}} (1 - \frac{1}{2}a_\varepsilon)|\tilde R|^{m+1}| \leq Ke^{-\mu\varepsilon t},$$

for some constants $K, \mu > 0$. On the other hand, from Appendix K we have that

$$\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1} = -\frac{\lambda_0}{2}\int_{\mathbb{R}} Q^2, \quad \lambda_0 = \frac{5-m}{m+3},$$

and thus

$$E_a[\tilde R](t) = c_2^{2\theta}(t)(\frac{1}{4}v_\infty^2 - \lambda_0 c_2(t))M[Q] + O(e^{-\mu\varepsilon t}).$$

Summing up (I.14), (I.15) and (I.16), we obtain

$$E_a[\tilde R](t) + (c_2(t_1) + \frac{1}{4}v_\infty^2)M[u](t) - v_\infty P[\tilde R](t) = c_2^{2\theta}(t)(c_2(t_1) - \lambda_0 c_2(t))M[Q] + O(e^{-\varepsilon\mu t}).$$

In particular,

$$E_a[\tilde R](t_1) - E_a[\tilde R](t) + (c_2(t_1) + \frac{1}{4}v_\infty^2)(M[\tilde R](t_1) - M[\tilde R](t)) - v_\infty[P[\tilde R](t_1) - P[\tilde R](t)] =$$
$$= \lambda_0 M[Q]\Big[c_2^{2\theta+1}(t) - c_2^{2\theta+1}(t_1) - \frac{c_2(t_1)}{\lambda_0}[c_2^{2\theta}(t) - c_2^{2\theta}(t_1)]\Big] + O(e^{-\varepsilon\mu t_1}).$$

To obtain the last estimate (I.17) we perform a Taylor development up to the second order (around $y = y_0$) of the function $g(y) := y^{\frac{2\theta+1}{2\theta}}$; and where $y := c_2^{2\theta}(t)$ and $y_0 := c_2^{2\theta}(t_1)$. Note that $\frac{2\theta+1}{2\theta} = \frac{1}{\lambda_0}$ and $y_0^{1/2\theta} = c_2(t_1)$. The conclusion follows at once. □

Now our objective is to estimate the quadratic term involved in (I.17). Following [59], we use the mass conservation law identity. From (I.6) -(I.7) we have

$$c_2^{2\theta}(t)M[Q] + \frac{1}{2}\int_{\mathbb{R}} |z(t)|^2 = c_2^{2\theta}(t_1)M[Q] + \frac{1}{2}\int_{\mathbb{R}} |z(t_1)|^2. \quad \text{(I.18)}$$

From here we obtain

$$(I.17) \leq K\|z(t)\|_{L^2(\mathbb{R})}^4 + \|z(t_1)\|_{L^2(\mathbb{R})}^4 + Ke^{-\varepsilon\mu t}, \quad \text{(I.19)}$$

for some $K, \mu > 0$, independent of D_0 and ε.

Step 3. Energy estimates. Let us now introduce the second order functional

$$\begin{aligned}\mathcal{F}_2(t) &:= \frac{1}{2}\int_{\mathbb{R}}\Big\{|z_x|^2+(c_2(t_1)+\frac{1}{4}v_\infty^2)|z|^2\Big\}-\frac{1}{2}v_\infty \operatorname{Im}\int_{\mathbb{R}} z_x\bar{z}\\ &\quad -\frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon(x)[|\tilde{R}+z|^{m+1}-|\tilde{R}|^{m+1}-(m+1)|\tilde{R}|^{m-1}\operatorname{Re}(\tilde{R}\bar{z})].\end{aligned}$$

This functional have the following properties.

Lemma I.3 (Energy expansion)**.**

Consider $M[u]$, $E_a[u]$ and $P[u]$ the mass, energy and momentum defined in (1.3), (I.2) and (1.5). Then we have for all $t\in[t_1,T^*]$,

$$\begin{aligned}&E_a[u](t)+(c_2(t_1)+\frac{1}{4}v_\infty^2)M[u](t)-v_\infty P[u](t)=\\ &\qquad E_a[\tilde{R}](t)+(c_2(t_1)+\frac{1}{4}v_\infty^2)M[\tilde{R}](t)-v_\infty P[\tilde{R}](t)+\mathcal{F}_2(t)+O(e^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).\end{aligned}$$

Proof. Using the orthogonality condition (I.7), we have

$$\begin{aligned}E_a[u](t) &= E_a[\tilde{R}]+\operatorname{Re}\int_{\mathbb{R}}\bar{z}[-\tilde{R}_{xx}-|\tilde{R}|^{m-1}\tilde{R}]+\frac{1}{2}\int_{\mathbb{R}}|z_x|^2+\operatorname{Re}\int_{\mathbb{R}}(1-\frac{a_\varepsilon}{2})|\tilde{R}|^{m-1}\tilde{R}\bar{z}\\ &\quad -\frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon(x)[|\tilde{R}+z|^{m+1}-|\tilde{R}|^{m+1}-(m+1)|\tilde{R}|^{m-1}\operatorname{Re}(\tilde{R}\bar{z})].\end{aligned}$$

Moreover, following (H.14), we easily get

$$|\operatorname{Re}\int_{\mathbb{R}}\bar{z}(1-\frac{1}{2}a_\varepsilon)|\tilde{R}|^{m-1}\tilde{R}|\leq Ke^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}.$$

Similarly, by using (I.7),

$$M[u](t)=M[\tilde{R}]+\frac{1}{2}\int_{\mathbb{R}}|z|^2,$$

and

$$P[u](t)=P[\tilde{R}](t)+\operatorname{Im}\int_{\mathbb{R}}\tilde{R}_x\bar{z}+\frac{1}{2}\operatorname{Im}\int_{\mathbb{R}} z_x\bar{z}.$$

Collecting the above estimates, we have

$$\begin{aligned}&E_a[u](t)+(c_2(t_1)+\frac{1}{4}v_\infty^2)M[u](t)-v_\infty P[u](t)=\\ &\quad E_a[\tilde{R}](t)+(c_2(t_1)+\frac{1}{4}v_\infty^2)M[\tilde{R}](t)-v_\infty P[\tilde{R}](t)+\mathcal{F}_2(t)+O(e^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).\end{aligned}$$

Here we have used (I.7), the equation satisfied by Q_{c_2} and the identity

$$\operatorname{Re}\int_{\mathbb{R}}\bar{z}[-\tilde{R}_{xx}-|\tilde{R}|^{m-1}\tilde{R}+iv_\infty\tilde{R}_x]=0.$$

This concludes the proof. □

Lemma I.4 (Modified coercivity for $\mathcal{F}_2$)**.**

There exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$ the following hold. There exist $K,\nu,\mu>0$, independent of K^* such that for every $t\in[t_1,T^*]$

$$\mathcal{F}_2(t)\geq\nu\|z(t)\|_{H^1(\mathbb{R})}^2-Ke^{-\mu\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}^2+O(\|z(t)\|_{L^2(\mathbb{R})}^3).\qquad\text{(I.20)}$$

Proof. First of all, note that

$$\begin{aligned}\mathcal{F}_2(t) &= \frac{1}{2}\int_{\mathbb{R}}\{z_x^2+(c_2(t_1)+\frac{1}{4}v_\infty^2)z^2\}-\frac{1}{2}v_\infty \operatorname{Im}\int_{\mathbb{R}}\bar{z}z_x \\ &\quad -\int_{\mathbb{R}}[|\tilde{R}|^{m-1}|z|^2+(m-1)|\tilde{R}|^{m-3}[\operatorname{Re}(\tilde{R}\bar{z})]^2] \\ &\quad -\frac{1}{2}\int_{\mathbb{R}}(a_\varepsilon(x)-2)[|\tilde{R}|^{m-1}|z|^2+(m-1)|\tilde{R}|^{m-3}[\operatorname{Re}(\tilde{R}\bar{z})]^2]+O(\|z(t)\|_{H^1(\mathbb{R})}^3)\end{aligned}$$

Since $(a_\varepsilon(x)-2)$ is exponentially decreasing along the region where the soliton $\tilde{R}$ is supported, we have

$$|\int_{\mathbb{R}}(a_\varepsilon(x)-2)[|\tilde{R}|^{m-1}|z|^2+(m-1)|\tilde{R}|^{m-3}[\operatorname{Re}(\tilde{R}\bar{z})]^2]|\leq Ke^{-\varepsilon\mu t}\|z(t)\|_{L^2(\mathbb{R})}.$$

(cf. (H.14 for a similar computation.) From Lemma 3.2 and (I.7) we have for $t\geq t_1$,

$$\mathcal{F}_2(t)\geq \nu\|z(t)\|_{H^1(\mathbb{R})}^2-Ke^{-\varepsilon\mu t}\|z(t)\|_{L^2(\mathbb{R})}^2-K\|z(t)\|_{H^1(\mathbb{R})}^3,$$

as desired. □

End of the proof. Now we prove that our assumption $T^*<+\infty$ leads inevitably to a contradiction. Indeed, from Lemmas I.3 and I.4, the mass and energy conservation, and the positivity of (1.16), we have for all $t\in[t_1,T^*]$ and for some constant $K>0$,

$$\begin{aligned}\|z(t)\|_{H^1(\mathbb{R})}^2\leq\ & K\mathcal{F}(t_1)+Ke^{-\mu\varepsilon t_1}\sup_{t\in[t_1,T^*]}\|z(t)\|_{L^2(\mathbb{R})}+K\sup_{t\in[t_1,T^*]}\|z(t)\|_{L^2(\mathbb{R})}^3 \\ &+\big|E_a[\tilde{R}](t_1)-E_a[\tilde{R}](t)+(c_2(t_1)+\frac{1}{4}v_\infty^2)(M[\tilde{R}](t_1)-M[\tilde{R}](t))-v_\infty(P[\tilde{R}](t_1)-P[\tilde{R}](t))\big|.\end{aligned}$$

From Lemmas I.1 and I.19 we have

$$\|z(t)\|_{H^1(\mathbb{R})}^2\leq K\varepsilon^{2p_m}+K\sup_{t\in[t_1,T^*]}\|z(t)\|_{H^1(\mathbb{R})}^4+Ke^{-\varepsilon\mu t_1}D_0\varepsilon^{p_m}.$$

Collecting the preceding estimates we have for $\varepsilon>0$ small and $D_0=D_0(K)$ large enough

$$\|z(t)\|_{H^1(\mathbb{R})}^2\leq\frac{1}{4}D_0^2\varepsilon^{2p_m}.$$

This estimate together with (I.18) and (I.9) gives us $|c_2(t)-c_\infty|\leq K\varepsilon^{p_m}$, independent of D_0, which contradicts the definition of T^*. The conclusion is that

$$\sup_{t\geq t_1}\big\|u(t)-Q_{c_\infty}(\cdot-v_\infty t-\rho_2(t))\exp\{\frac{i}{2}xv_\infty-\frac{i}{4}v_\infty^2t+i\gamma_2(t)\}\big\|_{H^1(\mathbb{R})}\leq K\varepsilon^{p_m}.$$

This finishes the proof of (2.8).

J Proof of Proposition 3.3

In this section we prove the decomposition result for the error $S[\tilde{u}]$ associated to the approximate solution $\tilde{u}$. First of all, it is easy to verify that

$$S[\tilde{u}]=S[\tilde{R}]+\mathcal{L}[w]+\tilde{N}[w],$$

where

$$\mathcal{L}[w] := iw_t + w_{xx} + \frac{a(\varepsilon x)}{2a(\varepsilon\rho)} Q_c^{m-1}(y)[(m+1)w + e^{2i\Theta}(m-1)\bar{w}],$$

and

$$\tilde{N}[w] := a(\varepsilon x)\big\{|\tilde{R}+w|^{m-1}(\tilde{R}+w) - |\tilde{R}|^{m-1}\tilde{R} - \frac{Q_c^{m-1}(y)}{2a(\varepsilon\rho)}[(m+1)w + e^{2i\Theta}(m-1)\overline{w}]\big\}.$$

In the next Claim we expand the first term, $S[\tilde{R}]$.

Claim 9 (Decomposition of $S[\tilde{R}]$).

1. Suppose $2 \leq m < 3$. Then one has

$$S[\tilde{R}] = \big[F_0^R(t,y) + \varepsilon F_1^R(t,y) + \varepsilon^2 F_2^R(t,y) + \varepsilon^3 f^R(t) F_c^R(y)\big]e^{i\Theta}, \tag{J.1}$$

where

$$\begin{aligned} F_0^R(t,y) \;&:=\; -\frac{1}{2}(v'(t) - \varepsilon f_1(t))\frac{yQ_c(y)}{\tilde{a}(\varepsilon\rho(t))} + i(c'(t) - \varepsilon f_2(t))\frac{\Lambda Q_c(y)}{\tilde{a}(\varepsilon\rho(t))} \\ &\quad -(\gamma'(t) + \frac{1}{2}v'(t)\rho(t))\frac{Q_c(y)}{\tilde{a}(\varepsilon\rho(t))} \\ &\quad -i(\rho'(t) - v(t))[\frac{Q_c'(y)}{\tilde{a}(\varepsilon\rho(t))} - \frac{\varepsilon\tilde{a}'(\varepsilon\rho(t))}{\tilde{a}^2(\varepsilon\rho(t))}Q_c(y)] \in \mathcal{Y}, \end{aligned} \tag{J.2}$$

f_1, f_2 are given by (3.15),

$$\begin{aligned} F_1^R(t,y) \;&:=\; \frac{a'(\varepsilon\rho(t))}{\tilde{a}^m(\varepsilon\rho(t))} yQ_c(y)\big[Q_c^{m-1}(y) - \frac{4c(t)}{m+3}\big] \\ &\quad + \frac{ia'(\varepsilon\rho(t))v(t)}{\tilde{a}^m(\varepsilon\rho(t))}\big[\frac{4c(t)}{5-m}\Lambda Q_c(y) - \frac{1}{m-1}Q_c(y)\big], \end{aligned} \tag{J.3}$$

and $|f^R(t)| \leq K$, $F_c^R \in \mathcal{Y}$. Finally, for every $t \in [-T_\varepsilon, T_\varepsilon]$

$$\|\varepsilon^2 F_2^R(t,y) + \varepsilon^3 f^R(t) F_c^R(y)\|_{H^1(\mathbb{R})} \leq K\varepsilon^2(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon).$$

2. Now suppose $3 \leq m < 5$. Then one has

$$S[\tilde{R}] = \big[F_0^R(t,y) + \varepsilon F_1^R(t,y) + \varepsilon^2 F_2^R(t,y) + \varepsilon^3 F_3^R(t,y) + \varepsilon^4 f(t) F_c^R(y)\big]e^{i\Theta}, \tag{J.4}$$

where F_0^R, is given now by the expression

$$\begin{aligned} F_0^R(t,y) \;&:=\; -\frac{1}{2}(v'(t) - \varepsilon f_1(t))\frac{yQ_c(y)}{\tilde{a}(\varepsilon\rho(t))} + i(c'(t) - \varepsilon f_2(t))\frac{\Lambda Q_c(y)}{\tilde{a}(\varepsilon\rho(t))} \\ &\quad -(\gamma'(t) + \frac{1}{2}v'(t)\rho(t) - \varepsilon^2 f_3(t))\frac{Q_c(y)}{\tilde{a}(\varepsilon\rho(t))} \\ &\quad -i(\rho'(t) - v(t) - \varepsilon^2 f_4(t))[\frac{Q_c'(y)}{\tilde{a}(\varepsilon\rho(t))} - \frac{\varepsilon\tilde{a}'(\varepsilon\rho(t))}{\tilde{a}^2(\varepsilon\rho(t))}Q_c(y)] \in \mathcal{Y}, \end{aligned} \tag{J.5}$$

F_1^R is given by (J.3),

$$F_2^R(t,y) := \frac{a''(\varepsilon\rho(t))}{2\tilde{a}^m(\varepsilon\rho(t))} y^2 Q_c^m(y) - \frac{f_3(t)}{\tilde{a}(\varepsilon\rho(t))}Q_c(y) - i\frac{f_4(t)}{\tilde{a}(\varepsilon\rho(t))}Q_c'(y), \tag{J.6}$$

and $|f^R(t)| \leq K$, $F_c^R \in \mathcal{Y}$. Moreover, for every $t \in [-T_\varepsilon, T_\varepsilon]$

$$\|\varepsilon^3 F_3^R(t,y) + \varepsilon^4 f(t) F_c^R(y)\|_{H^1(\mathbb{R})} \leq K\varepsilon^3(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon).$$

Proof of Claim 9. We prove the worst case, namely $3 \leq m < 5$. The remaining case is easier to handle and we skip the details.

Recall the definitions of $\tilde{R}$, y and Θ from (3.6)-(3.7). We have

$$\begin{aligned} S[\tilde{R}] &= i\tilde{R}_t + \tilde{R}_{xx} + a_\varepsilon(x)|\tilde{R}|^{m-1}\tilde{R} \\ &= -\big[\frac{1}{2}xv' + c + \gamma' - \frac{1}{4}v^2\big]\frac{1}{\tilde{a}}Q_c(y)e^{i\Theta} - \frac{i\rho'}{\tilde{a}}Q_c'(y)e^{i\Theta} + \frac{ic'}{\tilde{a}}\Lambda Q_c e^{i\Theta} \\ &\quad + \frac{1}{\tilde{a}}\big[Q_c'' + ivQ_c' - \frac{1}{4}v^2 Q_c + \frac{a(\varepsilon x)}{a(\varepsilon\rho)}Q_c^m\big]e^{i\Theta} - \frac{i\varepsilon a'\rho'}{(m-1)\tilde{a}^m}Q_c e^{i\Theta}. \end{aligned} \tag{J.7}$$

Now we perform a Taylor expansion of the term $a(\varepsilon x)$ based at $x = \rho(t)$, as in [65]. From (J.7),

$$\begin{aligned} S[\tilde{R}] &= \frac{1}{\tilde{a}}\big[\varepsilon\frac{a'}{a}Q_c^{m-1} - \frac{1}{2}v'\big]yQ_c e^{i\Theta} + \frac{i}{\tilde{a}}\big[c'\Lambda Q_c - \frac{\varepsilon a' v}{(m-1)a}Q_c\big]e^{i\Theta} - \frac{1}{\tilde{a}}(\gamma' + \frac{1}{2}v'\rho)Q_c e^{i\Theta} \\ &\quad - \frac{i}{\tilde{a}}(\rho' - v)\big[Q_c'(y) + \frac{\varepsilon a'}{(m-1)a}Q_c\big]e^{i\Theta} + \frac{\varepsilon^2 a''}{2\tilde{a}^m}y^2 Q_c^m e^{i\Theta} \\ &\quad + \frac{\varepsilon^3 a^{(3)}}{6\tilde{a}^m}y^3 Q_c^m e^{i\Theta} + \varepsilon^4 f(t)F_c(y)e^{i\Theta} \\ &=: \big[F_0^R(t,y) + \varepsilon F_1^R(t,y) + \varepsilon^2 F_2^R(t,y) + \varepsilon^3 F_3^R(t,y) + \varepsilon^4 f^R(t)F_c^R(y)\big]e^{i\Theta}. \end{aligned}$$

Additionally, we have $|f(t)| \leq K$ and $F_c^R(y) \in \mathcal{Y}$. In conclusion,

$$\|\varepsilon^3 F_4(t,y) + \varepsilon^4 f^R(t)F_c^R(y)\|_{H^1(\mathbb{R})} \leq K\varepsilon^4(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon).$$

This finishes the proof. □

Next, we consider the linear term. As above, we need to consider three different cases. Recall that $\Lambda A_c(t,y) = \partial_c A_c(t,y)$.

Claim 10 (Decomposition of $\mathcal{L}[w]$).

1. Suppose $2 \leq m < 3$. Then

$$\begin{aligned} \mathcal{L}[w] &= -\varepsilon\big[\mathcal{L}_+(A_{1,c}) + i\mathcal{L}_-(B_{1,c})\big]e^{i\Theta} - (\gamma' + \frac{1}{2}v'\rho)w - \frac{1}{2}(v' - \varepsilon f_1)yw \\ &\quad - i(\rho' - v)w_y + i\varepsilon(c' - \varepsilon f_2)\partial_c w + \varepsilon^2 f^L(t)F_c^L(y)e^{i\Theta}. \end{aligned} \tag{J.8}$$

   Furthermore, suppose that $(A_{1,c}, B_{1,c})$ satisfy (3.11). Then there exist $K, \mu > 0$ such that

$$\|\varepsilon^2 f^L(t)F_c^L e^{i\Theta}\|_{H^1(\mathbb{R})} \leq K\varepsilon^2(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon). \tag{J.9}$$

2. Consider now the case $3 \leq m < 5$. Here one has

$$\begin{aligned} \mathcal{L}[w] &= -\sum_{k=1}^{2}\varepsilon^k\big[\mathcal{L}_+(A_{k,c}) + i\mathcal{L}_-(B_{k,c})\big]e^{i\Theta} - \frac{1}{2}(v' - \varepsilon f_1)yw + i(c' - \varepsilon f_2)\partial_c w \\ &\quad - (\gamma' + \frac{1}{2}v'\rho - \varepsilon^2 f_3)w - i(\rho' - v - \varepsilon^2 f_4)w_y \\ &\quad + \varepsilon^2[F_2^L(t,y) + iG_2^L(t,y)]e^{i\Theta} + \varepsilon^3 f^L(t)F_c^L(y)e^{i\Theta}. \end{aligned} \tag{J.10}$$

   Here

$$F_2^L(t,y) := m\frac{a'}{a}Q_c^{m-1}yA_{1,c} - \frac{1}{2}f_1 yA_{1,c} - \frac{1}{\varepsilon}(B_{1,c})_t - f_2\Lambda B_{1,c}, \tag{J.11}$$

and

$$G_2^L(t,y) := \frac{1}{\varepsilon}(A_{1,c})_t + f_2\Lambda A_{1,c} + \frac{a'}{a}Q_c^{m-1}yB_{1,c} - \frac{1}{2}f_1yB_{1,c}. \tag{J.12}$$

In addition, suppose that $(A_{k,c}(t,y), B_{k,c}(t,y))$, satisfy (3.11) $k = 1,2$. Then there exist $K, \mu > 0$ such that

$$\|\varepsilon^3 f^L(t)F_c^L e^{i\Theta}\|_{H^1(\mathbb{R})} \leq K\varepsilon^3(e^{-\varepsilon\mu|\rho(t)|} + \varepsilon). \tag{J.13}$$

Proof. From the linear character of w we are reduced to handle only two different kind of terms: $\mathcal{L}[A_c(t,y)e^{i\Theta}]$ and $\mathcal{L}[iB_c(t,y)e^{i\Theta}]$. In addition, we expand in several order of ε to consider the case $m \in [3,5)$. Otherwise, the computations are simpler and one does not need an accurate expression for these terms. We left the details to the reader.

First we compute $\mathcal{L}[A_c(t,y)e^{i\Theta}]$, for a given smooth real valued function A. We have (the subscript $()_t$ means derivative on the first variable)

$$\begin{aligned}
\mathcal{L}[A_c(t,y)e^{i\Theta}] &= i(A_c)_te^{i\Theta} + ic'\Lambda A_ce^{i\Theta} - (\frac{1}{2}xv' + c - \frac{1}{4}v^2 + \gamma')A_ce^{i\Theta} - i\rho'(A_c)_xe^{i\Theta} \\
&\quad + \big[(A_c)_{xx} + iv(A_c)_x - \frac{1}{4}v^2A_c\big]e^{i\Theta} + \frac{ma(\varepsilon x)}{a(\varepsilon\rho)}Q_c^{m-1}A_ce^{i\Theta} \\
&= -\mathcal{L}_+(A_c)e^{i\Theta} + (\varepsilon\frac{ma'}{a}Q_c^{m-1} - \frac{1}{2}v')yA_ce^{i\Theta} - (\gamma' + \frac{1}{2}v'\rho)A_ce^{i\Theta} \\
&\quad - i(\rho' - v)(A_c)_ye^{i\Theta} + i((A_c)_t + c'\Lambda A_c)e^{i\Theta} + \frac{m\varepsilon^2a''}{2a}y^2Q_c^{m-1}A_ce^{i\Theta} \\
&\quad + \frac{m\varepsilon^3a^{(3)}}{6a}y^3Q_c^{m-1}A_ce^{i\Theta} + \varepsilon^4f(t)y^4Q_c^{m-1}A_ce^{i\Theta} \\
&= -\mathcal{L}_+(A_c)e^{i\Theta} - \frac{1}{2}(v' - \varepsilon f_1)yA_ce^{i\Theta} - (\gamma' + \frac{1}{2}v'\rho - \varepsilon^2f_3)A_ce^{i\Theta} \\
&\quad - i(\rho' - v - \varepsilon^2f_4)(A_c)_ye^{i\Theta} + i(c' - \varepsilon f_2)\Lambda A_ce^{i\Theta} \\
&\quad + \frac{\varepsilon a'}{a}mQ_c^{m-1}yA_ce^{i\Theta} - \frac{\varepsilon}{2}f_1yA_ce^{i\Theta} + i\big[(A_c)_t + \varepsilon f_2\Lambda A_c\big]e^{i\Theta} \\
&\quad + \frac{m\varepsilon^2a''}{2a}y^2Q_c^{m-1}A_ce^{i\Theta} - \varepsilon^2f_3A_ce^{i\Theta} - i\varepsilon^2f_4(A_c)_ye^{i\Theta} \\
&\quad + \varepsilon^3\frac{ma^{(3)}}{6a}y^3Q_c^{m-1}A_ce^{i\Theta}e^{i\Theta} + \varepsilon^4f(t)F_c^{\mathbf{II}}(y)e^{i\Theta},
\end{aligned}$$

where $F_c^{\mathbf{II}}(y) \in \mathcal{Y}$ and $f(t)$ is exponentially decaying in time. Therefore,

$$\|\varepsilon^4f^{\mathbf{II}}(t)F_c^{\mathbf{II}}(y)\|_{H^1(\mathbb{R})} \leq K\varepsilon^4e^{-\mu\varepsilon|\rho(t)|}.$$

With a similar computation,

$$\begin{aligned}
\mathcal{L}[iB_c(t,y)e^{i\Theta}] &= -i\mathcal{L}_-(B_c)e^{i\Theta} - \frac{i}{2}(v' - \varepsilon f_1)yB_ce^{i\Theta} - i(\gamma' + \frac{1}{2}v'\rho - \varepsilon^2f_3)B_ce^{i\Theta} \\
&\quad + (\rho' - v - \varepsilon^2f_4)(B_c)_ye^{i\Theta} - (c' - \varepsilon f_2)\Lambda B_ce^{i\Theta} \\
&\quad + \frac{i\varepsilon a'}{a}Q_c^{m-1}yB_ce^{i\Theta} - \frac{i}{2}\varepsilon f_1yB_ce^{i\Theta} - \big[(B_c)_t + \varepsilon f_2\Lambda B_c\big]e^{i\Theta} \\
&\quad + i\frac{\varepsilon^2a''}{2a}y^2Q_c^{m-1}B_ce^{i\Theta} + i\varepsilon^3\frac{a^{(3)}}{6a}y^3Q_c^{m-1}B_ce^{i\Theta} \\
&\quad + \varepsilon^2f_4(t)(B_c)_ye^{i\Theta} - i\varepsilon^2f_3B_ce^{i\Theta} + i\varepsilon^4g^{\mathbf{II}}(t)G_c^{II}(y)e^{i\Theta},
\end{aligned}$$

with $\|\varepsilon^4g^{\mathbf{II}}(t)G_c^{II}(y)e^{i\Theta}\|_{H^1(\mathbb{R})} \leq K\varepsilon^4e^{-\mu\varepsilon|\rho(t)|}$. Collecting the above calculations, we finally obtain (J.10). Estimate (J.13) can be directly verified.

□

For the final term $\tilde{N}[w]$ we have the following

Claim 11 (Decomposition of $\tilde{N}[w]$).

1. Suppose that $2 \leq m < 3$ and (3.11) holds for $(A_{1,c}, B_{1,c})$. Then there exists $K, \mu > 0$ such that
$$\|\tilde{N}[w]\|_{H^1(\mathbb{R})} \leq K\varepsilon^2 e^{-\mu\varepsilon|\rho(t)|},$$
uniformly for every $t \in [-T_\varepsilon, T_\varepsilon]$.

2. Suppose now $3 \leq m < 5$, and that (3.11) holds for each $(A_{k,c}, B_{k,c})$, $k = 1, 2$. Then one has
$$\tilde{N}[w] = \varepsilon^2(N^{2,1}(t,y) + iN^{2,2}(t,y))e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^3 e^{-\varepsilon\mu|\rho(t)|}),$$
with
$$N^{2,1} := \frac{1}{2}(m-1)\tilde{a}(\varepsilon\rho)Q_c^{m-2}(mA_{1,c}^2 + B_{1,c}^2), \quad N^{2,2} := (m-1)\tilde{a}(\varepsilon\rho)Q_c^{m-2}A_{1,c}B_{1,c}. \tag{J.14}$$

Proof. First we prove the case $2 \leq m < 3$. Recall that $w = \varepsilon[A_{1,c}(t,y) + iB_{1,c}(t,y)]e^{i\Theta}$, with the functions $A_c(t,\cdot), B_c(t,\cdot) \in \mathcal{Y}$, real valued. Here we have
$$\tilde{N}[w] = O(Q_c^{m-2}|w|^2 + |w|^3) = O_{H^1(\mathbb{R})}(\varepsilon^2 e^{-\varepsilon\mu|\rho(t)|}),$$
uniformly in time.

Finally, let us consider the case $3 \leq m < 5$. From (3.9) we have $w(t,x) = \sum_{k=1}^2 \varepsilon^k(A_{k,c}(t,y) + iB_{k,c}(t,y))e^{i\Theta}$. In order to simplify the computations, we assume $(A_{k,c}, B_{k,c})_{k=1,2}$ satisfy (3.11) on the interval $[-T_\varepsilon, T_\varepsilon]$ (which is indeed the case). We have
$$\begin{aligned}\tilde{N}[w] &= \frac{(m-1)a(\varepsilon x)}{2a^{\frac{m-2}{m-1}}(\varepsilon\rho)}Q_c^{m-2}(y)\{e^{i\Theta}|w|^2 + 2\operatorname{Re}(e^{i\Theta}\bar{w})w + (m-3)e^{i\Theta}(\operatorname{Re}(e^{i\Theta}\bar{w}))^2\} \\ &\quad + O_{H^1(\mathbb{R})}(\varepsilon^3 e^{-\varepsilon\mu|\rho(t)|}).\end{aligned} \tag{J.15}$$

Now we replace w in the above expression and we arrange the terms obtained according to the power of ε and between real and imaginary parts. We perform this computation in several steps. First, note that
$$a(\varepsilon x) = a(\varepsilon\rho) + O(\varepsilon y).$$
On the other hand,
$$|w|^2 = \varepsilon^2\{A_{1,c}^2 + B_{1,c}^2\} + O_{H^1(\mathbb{R})}(\varepsilon^3 e^{-\varepsilon\mu|\rho(t)|}).$$
Similarly $\operatorname{Re}(e^{i\Theta}\bar{w}) = \varepsilon A_{1,c} + \varepsilon^2 A_{2,c}$. Therefore
$$\operatorname{Re}(e^{i\Theta}\bar{w})w = \varepsilon^2(A_{1,c}^2 + iA_{1,c}B_{1,c})e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^3 e^{-\varepsilon\mu|\rho(t)|}),$$
and
$$e^{i\Theta}(\operatorname{Re}(e^{i\Theta}\bar{w}))^2 = \varepsilon^2 A_{1,c}^2 e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^3 e^{-\varepsilon\mu|\rho(t)|}).$$
Collecting these expansions and replacing in (J.15) we obtain
$$\tilde{N}[w] = \frac{1}{2}\varepsilon^2(m-1)\tilde{a}(\varepsilon\rho)Q_c^{m-2}\{mA_{1,c}^2 + B_{1,c}^2 + 2iA_{1,c}B_{1,c}\}e^{i\Theta} + O_{H^1(\mathbb{R})}(\varepsilon^3 e^{-\varepsilon\mu|\rho(t)|}).$$
We are done. □

Collecting the estimates from Claims 9, 10 and 11, we obtain Proposition 3.3. The proof is now complete.

K Some identities related to the soliton Q

This section has been taken in part from Appendix C in [53].

Lemma K.1 (Identities for the soliton Q).

Suppose $m > 1$ and denote by $Q_c := c^{\frac{1}{m-1}} Q(\sqrt{c}x)$ the scaled soliton, with Q solution of $-Q'' + Q - Q^m = 0$ in $\mathbb{R}$. Then

1. Energy. *Let $E_1[u] := E_{a\equiv 1}[u]$. Then*

$$E_1[Q] = -\frac{1}{2}\lambda_0 \int_{\mathbb{R}} Q^2 = -\lambda_0 M[Q], \qquad \textit{with} \quad \lambda_0 = \frac{5-m}{m+3}.$$

2. Integrals. *Recall $\theta = \frac{1}{m-1} - \frac{1}{4}$. Then*

$$\int_{\mathbb{R}} Q_c = c^{\theta - \frac{1}{4}} \int_{\mathbb{R}} Q, \quad \int_{\mathbb{R}} Q_c^2 = c^{2\theta} \int_{\mathbb{R}} Q^2, \quad E_1[Q_c] = c^{2\theta+1} E_1[Q].$$

and finally

$$\int_{\mathbb{R}} Q_c^{m+1} = \frac{2(m+1)c^{2\theta+1}}{m+3} \int_{\mathbb{R}} Q^2, \quad \int_{\mathbb{R}} \Lambda Q_c = (\theta - \frac{1}{4}) c^{\theta - \frac{5}{4}} \int_{\mathbb{R}} Q, \quad \int_{\mathbb{R}} \Lambda Q_c Q_c = \theta c^{2\theta-1} \int_{\mathbb{R}} Q^2.$$

3. Integrals with powers.

$$\int_{\mathbb{R}} Q'^2 = \frac{m-1}{m+3} \int_{\mathbb{R}} Q^2, \quad \int_{\mathbb{R}} y^2 Q^{m+1} = \frac{m+1}{m+3}[2\int_{\mathbb{R}} y^2 Q^2 - \int_{\mathbb{R}} Q^2],$$

and

$$\int_{\mathbb{R}} y^4 Q^{m+1} = \frac{m+1}{m+3}[2\int_{\mathbb{R}} y^4 Q^2 - 6\int_{\mathbb{R}} y^2 Q^2], \quad \int_{\mathbb{R}} y^2 Q'^2 = \frac{2}{m+3}\int_{\mathbb{R}} Q^2 + \frac{m-1}{m+3}\int_{\mathbb{R}} y^2 Q^2.$$

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5 Addendum: decreasing potential and reflection

In this section we address the case of the soliton dynamics for the equation NLS (1.12) in the case of a slowly varying, **strictly decreasing** potential, constant in time. Indeed, we assume that the function a is smooth enough and there exist constants $K, \mu > 0$ and $0 < a_0 < 1$ such that

$$\begin{cases} a_0 < a(r) < 1, \ a'(r) < 0, \ |a^{(k)}(r)| \leq K e^{-\mu|r|} \ \text{for all } r \in \mathbb{R}, \ k = 1,2,3,(4); \\ 0 < a(r) - a_0 \leq K e^{-\mu r}, \ \text{ for all } r \geq 0, \ \text{ and} \\ 0 < 1 - a(r) \leq K e^{\mu r} \ \text{ for all } r \leq 0. \end{cases} \tag{5.1}$$

In particular, $\lim_{r\to-\infty} a(r) = 1$ and $\lim_{r\to+\infty} a(r) = a_0$.

Note that $P[u](t)$ defined in (1.5) now satisfies (1.16) with the opposite sign, therefore the momentum is always a non increasing quantity. In this section we will sketch the proof of the following

Theorem 5.1 (Dynamics of a reflected soliton-solution for aNLS equation)**.**

Assume that $a(\cdot)$ *now satisfies (5.1). Let* $2 \leq m < 5$, $v_0 > 0$, $\lambda_0 := \frac{5-m}{m+3}$ *and* p_m *be as in (1.21). Suppose in addition that*

$$v_0^2 < 4\lambda_0(1 - a_0^{\frac{4}{5-m}}). \tag{5.2}$$

There exists a small constant $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the following holds.*

1. Existence of a soliton-like solution.

   There exists a solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *of (1.12), global in time, such that*

   $$\lim_{t \to -\infty} \|u(t) - Q(\cdot - v_0 t)e^{i(\cdot)v_0/2}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0, \tag{5.3}$$

   with conserved mass $M[u](t) = M[Q]$ *and energy* $E_a[u](t) = (\frac{1}{4}v_0^2 - \lambda_0)M[Q] < 0$.

2. Reflection and stability of soliton-solution.

   There exist $K = K(v_0) > 0$ *and* C^1*- functions* $\rho(t), \gamma(t) \in \mathbb{R}$ *defined for all* $t \geq KT_\varepsilon$ *such that the rest function*

   $$w(t,x) := u(t,x) - Q(x + v_0 t - \rho(t))e^{-\frac{i}{2}xv_0}e^{i\gamma(t)},$$

   satisfies for all $t \geq KT_\varepsilon$,

   $$\|w(t)\|_{H^1(\mathbb{R})} + |\rho'(t)| + |\gamma'(t) - 1 + \frac{1}{4}v_0^2| \leq K\varepsilon^{p_m}. \tag{5.4}$$

Remark 5.1. Let us clarify this last result. Under small but still fixed velocities, the soliton solution is **reflected** by the potential, and modulo a small defect of order $O_{H^1(\mathbb{R})}(\varepsilon^{p_m})$, it has the **same scaling and opposite velocity** to the initially provided.

Let us remark that the extension of this result to the two dimensional case is direct, after the preceding results (cf. Theorem 1). In addition, the constant $K(v_0)$ becomes unbounded as v_0 approaches the equality in (5.2).

The proof of this result is just an extension of the previous sections. First of all, it is not difficult to prove the following result.

Proposition 5.2 (Existence of a pure soliton-like solution)**.**

There exists $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$*, there exists a solution* $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *of (1.12) such that*

$$\lim_{t \to -\infty} \|u(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} = 0, \tag{5.5}$$

with mass $M[u](t) = M[Q]$ *and energy* $E_a[u](t) = (\frac{1}{4}v_0^2 - \lambda_0)M[Q] < 0$*. Moreover, there exist constants* $K, \mu > 0$ *such that for all* $t \leq -\frac{1}{2}T_\varepsilon$,

$$\|u(t) - Q(\cdot - v_0 t)e^{\frac{i}{2}(\cdot)v_0}e^{i(1-\frac{1}{4}v_0^2)t}\|_{H^1(\mathbb{R})} \leq Ke^{\varepsilon\mu t}. \tag{5.6}$$

In particular,

$$\|u(-T_\varepsilon) - Q(\cdot + v_0 T_\varepsilon)e^{\frac{i}{2}(\cdot)v_0}e^{-i(1-\frac{1}{4}v_0^2)T_\varepsilon}\|_{H^1(\mathbb{R})} \leq Ke^{-\mu\varepsilon^{-\frac{1}{100}}} \leq K\varepsilon^{10}, \tag{5.7}$$

provided $0 < \varepsilon < \varepsilon_0$ *small enough.*

Let us remark that the **uniqueness** of this solution is an open issue.

The next step is the study of the interaction soliton-potential. This is the part of the proof where we need some completely new computations. Our objective is to prove the following result.

Proposition 5.3 (Dynamics of the soliton in the interaction region)**.**

Suppose $v_0 > 0$ satisfying (5.2). There exist a constant $\varepsilon_0 > 0$ such that the following holds for any $0 < \varepsilon < \varepsilon_0$. Let $u = u(t)$ be a globally defined $H^1(\mathbb{R})$ solution of (1.12) such that

$$\|u(-T_\varepsilon) - Q(\cdot + v_0 T_\varepsilon) e^{\frac{1}{2}i(\cdot)v_0} e^{-i(1-\frac{1}{4}v_0^2)T_\varepsilon}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}.$$

Then there exist $K_0 = K_0(K) > 0$, and $\rho_\varepsilon, \gamma_\varepsilon \in \mathbb{R}$ such that

$$\|u(T_\varepsilon) - Q(\cdot - \rho_\varepsilon) e^{-\frac{i}{2}(\cdot)v_0} e^{i\gamma_\varepsilon}\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{p_m},$$

and $\rho_\varepsilon \sim -v_0 T_\varepsilon$.

The proof is based in a deep study of the formal dynamical system governing the dynamics of the soliton, see Lemma 5.4. From this result, the conclusion of the above result is direct, by following the lines of the proof of Proposition 2.2.

Remark 5.2. From the proof of this result it will be clear that in the case $v_0^2 > 4\lambda_0(1 - a_0^{\frac{4}{5-m}})$ the soliton exits by the right hand side of the potential. However, no stability result for large time is known in the regime "decreasing potential", so we do not know the asymptotic behavior of this solution. The proof of this result will require some new ideas.

5.1 Study of a dynamical system, revisited

Similarly to [66], the dynamics of a soliton soliton is mainly described by its velocity $V(t)$, position $U(t)$, scaling $C(t)$ and phase $H(t)$. The dynamical system governing these variables is completely analogous to that of [66], Lemma 3.4, with the key difference on the sign of the derivative of the potential a. Indeed, let

$$f_1(C, U) := \frac{8a'(\varepsilon U)C}{(m+3)a(\varepsilon U)}, \qquad f_2(C, V, U) := \frac{4CVa'(\varepsilon U)}{(5-m)a(\varepsilon U)}. \tag{5.8}$$

Our first result is as follows:

Lemma 5.4 (Existence of approximated dynamical parameters, case $2 \leq m < 5$)**.**

Let $v_0 > 0, \lambda_0, a(s)$ be as in Theorem 5.1 and (5.1). There exists a unique solution (C, V, P, G) defined for all $t \geq -T_\varepsilon$ with the same regularity than $a(\varepsilon\cdot)$, of the following nonlinear system of differential equations

$$\begin{cases} V'(t) = \varepsilon f_1(C(t), U(t)), & V(-T_\varepsilon) = v_0, \\ C'(t) = \varepsilon f_2(C(t), V(t), U(t)), & C(-T_\varepsilon) = 1, \\ U'(t) = V(t), & U(-T_\varepsilon) = -v_0 T_\varepsilon, \\ H'(t) = -\dfrac{1}{2}V'(t)U(t), & H(-T_\varepsilon) = 0. \end{cases} \tag{5.9}$$

In addition, for all $t \geq -T_\varepsilon$, $C(t), V(t)$ are strictly decreasing with

$$C(t) = \frac{a^{4/(5-m)}(\varepsilon U(t))}{a^{4(5-m)}(-\varepsilon v_0 T_\varepsilon)} = \frac{a^{4/(5-m)}(\varepsilon U(t))}{a^{4(5-m)}(-\varepsilon^{-1/100})}, \tag{5.10}$$

and satisfy the parabola

$$C(t) = c_0 + \frac{V^2(t)}{4\lambda_0}, \quad c_0 := 1 - \frac{v_0^2}{4\lambda_0} < 1. \tag{5.11}$$

Proof. The existence of a local solution of (5.9) is consequence of the Cauchy-Lipschitz-Picard theorem.

Now, in order to prove global existence of such a solution, we derive some a priori estimates. Note that from the first equation in (4.16) we have C *strictly decreasing* in time with $C(t) \leq 1$, $t \geq -T_\varepsilon$. Moreover, after integration, we have (5.10). Since $\frac{1}{2} < a < 1$, one has that C is bounded and globally well defined with

$$\frac{a_0^{4/(5-m)}}{a^{4(5-m)}(-\varepsilon^{-1/100})} \leq C(t) < 1, \quad t \geq -T_\varepsilon. \tag{5.12}$$

On the other hand, from the second equation in (5.9), we have V *strictly decreasing* in time. Replacing (5.10), and after multiplication by $V(t)$, one has

$$V(t)V'(t) = \frac{8}{m+3} a^{\frac{m-1}{5-m}}(\varepsilon U(t))a'(\varepsilon U(t))V(t)a^{-\frac{4}{5-m}}(-\varepsilon^{-1/100}).$$

After integration in $[-T_\varepsilon, t)$ we obtain (5.11). This last relation and the fact that $C(t) \leq 1$ implies the global existence of V and the uniform bound

$$|V(t)| \leq v_0, \quad t \geq -T_\varepsilon.$$

The proof is complete. □

Now we describe the behavior of (C, V, U) for large times. Interestingly enough, here the long time behavior may be different depending on the initial velocity v_0. Recall that $c_0 = 1 - \frac{v_0^2}{4\lambda_0}$.

Lemma 5.5 (Long time behavior, refracting case)**.**

Suppose $c_0 \leq a_0^{\frac{4}{5-m}}$. *Then* $\lim_{t\to+\infty}(C(t), V(t), U(t)) = (c_\infty, v_\infty, +\infty)$, *with*

$$c_\infty = a_0^{\frac{4}{5-m}}(1 + O(\varepsilon^{10})), \quad \text{and} \quad v_\infty = [4\lambda_0(c_\infty - c_0)]^{1/2} > 0.$$

In addition, there exists $-T_\varepsilon < \tilde{T}_\varepsilon < K(v_0)T_\varepsilon$ *such that* $U(\tilde{T}_\varepsilon) = -U(-T_\varepsilon)$.

Proof. We prove that for ε small $\lim_{t\to+\infty} C(t) = a_0^{\frac{4}{5-m}}(1 + O(\varepsilon^{10}))$. Indeed, note that from (5.10) and (5.11) we have

$$0 \leq a_0^{\frac{4}{5-m}} - c_0 < \frac{V^2(t)}{4\lambda_0};$$

thus $V(t) > 0$ for all $t \geq -T_\varepsilon$. Moreover, if $\lim_{+\infty} V(t) = 0$, we have $\lim_{+\infty} C(t) = c_0 \leq a_0^{\frac{4}{5-m}}$ and thus from (5.10)

$$\lim_{t\to+\infty} C(t) \leq a_0^{\frac{4}{5-m}},$$

a contradiction with (5.12) (after taking $\limsup$). In concluding we have $\lim_{+\infty} V(t) > 0$. Therefore we have $\lim_{+\infty} U(t) = +\infty$. Passing to the limit in (5.10), one obtains

$$\lim_{t\to+\infty} C(t) = \frac{a_0^{4/(5-m)}}{a^{4/(5-m)}(-\varepsilon^{-1/100})} > a_0^{4/(5-m)}.$$

From (5.11) we obtains

$$V(+\infty) = [4\lambda_0(\lim_{t\to+\infty} C(t) - c_0)]^{1/2} > 0.$$

Next, we prove that $U(+\infty) = +\infty$. Indeed,

$$U(t) = U(-T_\varepsilon) + \int_{-T_\varepsilon}^{t} v(t)dt \geq U(-T_\varepsilon) + V(+\infty)(t + T_\varepsilon),$$

and thus $U(+\infty) = +\infty$.

Now, let us define the exit time $\tilde{T}_\varepsilon > -T_\varepsilon$ such that $U(\tilde{T}_\varepsilon) = -U(-T_\varepsilon)$. We have

$$-U(-T_\varepsilon) = U(-T_\varepsilon) + \int_{-T_\varepsilon}^{\tilde{T}_\varepsilon} v(t)dt \geq U(-T_\varepsilon) + V(+\infty)(\tilde{T}_\varepsilon + T_\varepsilon),$$

then, if $c_0 < a_0^{\frac{4}{5-m}}$ we have $\tilde{T}_\varepsilon \leq KT_\varepsilon$. □

Lemma 5.6 (Long time behavior, reflecting case)**.**

Suppose now $c_0 > a_0^{4/(5-m)}$. Then there exists a unique $t_0 > -T_\varepsilon$, satisfying $V(t_0) = 0$ and $t_0 \leq K(v_0)T_\varepsilon$ for some constant $K(v_0) > 0$ but independent of ε. Moreover, one has

$$\lim_{t\to+\infty} (C(t), V(t), U(t)) = (1, v_0, -\infty).$$

In addition, there exist $\tilde{T}_\varepsilon > -T_\varepsilon$ and $K(v_0) > 0$ such that $U(\tilde{T}_\varepsilon) = U(-T_\varepsilon)$, with $\tilde{T}_\varepsilon \leq K(v_0)T_\varepsilon$.

Proof. First we prove the existence of $t_0 > -T_\varepsilon$ such that $V(t_0) = 0$. Note that its existence implies its uniqueness. By contradiction, let us suppose that $V(t) > 0$ for all $t > -T_\varepsilon$. Then $U(t)$ is strictly increasing. Here we have two cases. First, suppose $\lim_{+\infty} V(t) > 0$. Then we have $U(+\infty) = +\infty$ and therefore $C(t)$ is strictly decreasing with

$$\lim_{t\to+\infty} C(t) = \frac{a_0^{4/5-m}}{a^{4/(5-m)}(-\varepsilon^{-1/100})}.$$

Passing to the limit in (5.11), one has

$$\frac{a_0^{4/5-m}}{a^{4/(5-m)}(-\varepsilon^{-1/100})} = c_0 > a_0^{\frac{4}{5-m}},$$

a contradiction for ε small enough.

Now suppose $\lim_{t\to+\infty} V(t) = 0$. Here we have two cases: either $\lim_{t\to+\infty} U(t) = +\infty$, or $U(-T_\varepsilon) \leq \lim_{t\to+\infty} U(t) =: U_\infty < +\infty$. For the first case, similarly to the recent analysis, one has

$$\lim_{t\to+\infty} C(t) =: C_\infty = \frac{a_0^{4/5-m}}{a^{4/(5-m)}(-\varepsilon^{-1/100})} \geq c_0 > a_0^{\frac{4}{5-m}}.$$

This is a contradiction for ε small enough.

In the second case, one has

$$C_\infty = \frac{a^{4/(5-m)}(\varepsilon U_\infty)}{a^{4(5-m)}(-\varepsilon^{-1/100})} > 0,$$

and
$$\lim_{t\to+\infty} V'(t) = \varepsilon f_1(C_\infty, U_\infty) < 0,$$
a contradiction with $\lim_{t\to+\infty} V(t) = 0$ (since $\lim_{t\to+\infty} V'(t) = \lim_{t\to+\infty} \frac{V(t)}{t} = 0$.)

Therefore, there exists $t_0 \in \mathbb{R}$ such that $V(t_0) = 0$, with $V(t) < 0$ for $t > t_0$. In addition, $C(t_0) = c_0$ and $U'(t_0) = 0$. From (5.10) one has $|\varepsilon U(t_0)| \le K(v_0)$.

Moreover, $\lim_{t\to+\infty} U(t) = -\infty$. Indeed, note that for $\nu > 0$ small, since $V''(t) = O(\varepsilon^2)$,
$$V(t_0 \pm \frac{\nu}{\varepsilon}) = \pm V'(t_0)\frac{\nu}{\varepsilon} + O(\nu^2) = \pm\nu f_1(c_0, U(t_0)) + O(\nu^2) = \pm\kappa_0\nu + O(\nu^2).$$
with $\kappa_0 := f_1(c_0, U(t_0)) = \frac{8c_0}{(m+3)}\frac{a'(\varepsilon U(t_0))}{a(\varepsilon U(t_0))} < 0$. Then for $t > t_0 + \frac{\nu}{\varepsilon}$,
$$\begin{aligned} U(t) &= U(-T_\varepsilon) + \int_{-T_\varepsilon}^{t_0-\frac{\nu}{\varepsilon}} v(t)dt + \int_{t_0-\frac{\nu}{\varepsilon}}^{t_0+\frac{\nu}{\varepsilon}} v(t)dt + \int_{t_0+\frac{\nu}{\varepsilon}}^{t} v(t)dt \\ &\le U(-T_\varepsilon) + v_0(T_\varepsilon + t_0 - \frac{\nu}{\varepsilon}) + v_0\frac{\nu}{\varepsilon} + \nu\kappa_0(t - t_0 - \frac{\nu}{\varepsilon}). \end{aligned}$$
and thus $\lim_{t\to+\infty} U(t) = -\infty$. In addition, $\lim_{t\to+\infty} C(t) = a^{-\frac{4}{5-m}}(-\varepsilon^{-1/100}) = 1 + O(\varepsilon^{10})$ and $\lim_{t\to+\infty} V(t) = -v_0 + O(\varepsilon^{10})$.

Similarly,
$$\begin{aligned} U(t_0) &= U(-T_\varepsilon) + \int_{-T_\varepsilon}^{t_0-\frac{\nu}{\varepsilon}} v(t)dt + \int_{t_0-\frac{\nu}{\varepsilon}}^{t_0} v(t)dt \\ &\ge U(-T_\varepsilon) - \nu\kappa_0(t_0 - \frac{\nu}{\varepsilon} + T_\varepsilon) - K\frac{\nu}{\varepsilon}. \end{aligned}$$
This inequality implies that $t_0 \le KT_\varepsilon$, with $K = K(v_0) > 0$ independent of ε. Note that K becomes singular as v_0 approaches $c(v_0) = a_0^{4/(5-m)}$.

Let us define $\tilde{T}_\varepsilon > -T_\varepsilon$ such that $U(\tilde{T}_\varepsilon) = U(-T_\varepsilon)$. Then we have $C(\tilde{T}_\varepsilon) = 1$ and $V(\tilde{T}_\varepsilon) = -v_0$. We finally have
$$\begin{aligned} 0 &= \int_{-T_\varepsilon}^{\tilde{T}_\varepsilon} v(t)dt = \int_{-T_\varepsilon}^{t_0-\frac{\nu}{\varepsilon}} v(t)dt + \int_{t_0-\frac{\nu}{\varepsilon}}^{t_0+\frac{\nu}{\varepsilon}} v(t)dt - \int_{t_0+\frac{\nu}{\varepsilon}}^{\tilde{T}_\varepsilon} |v(t)|dt \\ &\le v_0(t_0 - \frac{\nu}{\varepsilon} + T_\varepsilon) + v_0\frac{\nu}{\varepsilon} + v(t_0 + \frac{\nu}{\varepsilon})(\tilde{T}_\varepsilon - t_0 + \frac{\nu}{\varepsilon}). \end{aligned}$$
In conclusion $\tilde{T}_\varepsilon \le KT_\varepsilon$. □

Proof of Proposition 5.3. The proof of this result is straightforward, just follow the lines of the proof of Proposition 2.2. The main modification is in the a priori assumptions (3.8). Now we assume that
$$|c(t) - C(t)| + |v(t) - V(t)| + |\rho'(t) - U'(t)| \le \varepsilon^{1/100}.$$
It is clear that these estimates are improved in the new version of Lemma 3.11. The last issue is the integrability of the term $\varepsilon e^{-\varepsilon\gamma|\rho(t)|}$, since $\rho'(t) \sim v(t)$ changes of sign during the dynamics. However, from the last assumptions, and a correct splitting of the interval of integration as in the proof of Lemma 5.6, it is easy to see that
$$\int_{\mathbb{R}} \varepsilon e^{-\varepsilon\gamma|\rho(t)|}dt \le K,$$
with K independent of ε. We left the details to the reader. □

5.2 Stability of the reflected soliton solution

The final step towards the proof of Theorem 5.1 is a stability result for a reflected soliton. Let us recall that this result is consequence of the good sign of the following derivative:

$$\partial_t\{(-v_0)P[u](t)\} \geq 0,$$

which does not hold for the normal case of final velocity $v_\infty > 0$.

Proposition 5.7 (Stability in $H^1(\mathbb{R})$, reflected case)**.**

Suppose $2 \leq m < 5$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ the following hold. Suppose that for some time $t_1 \geq \frac{1}{2}T_\varepsilon$, $X_0 \leq -v_0 t_1$ and $\gamma_0 \in \mathbb{R}$ and $K > 0$,

$$\|u(t_1) - Q(\cdot - X_0)e^{-\frac{i}{2}xv_0}e^{i\gamma_0}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}. \tag{5.13}$$

where $u(t)$ is a global H^1-solution of (1.12).

Then there exist $K_0 > 0$ and C^1-functions $\rho(t), \gamma(t) \in \mathbb{R}$ defined in $[t_1, +\infty)$ such that

$$w(t) := u(t) - Q(\cdot + v_0 t - \rho_2(t))e^{-\frac{i}{2}(\cdot)v_0}e^{i\gamma(t)},$$

satisfies for all $t \geq t_1$,

$$\|w(t)\|_{H^1(\mathbb{R})} + |\rho_2'(t)| + |\gamma_2'(t) - 1 + \frac{1}{4}v_0^2| \leq K_0\varepsilon^{p_m}, \tag{5.14}$$

where, for some $K > 0$,

$$|\rho_2(t_1) - v_0 t_1 - X_0| + |\gamma_2(t_1) - \gamma_0| \leq K\varepsilon^{p_m}.$$

End of proof of Theorem 5.1. Define $\rho(t) := \rho_2(t)$ and $\gamma(t) := \gamma_2(t)$. The conclusion follows at once. □

Proof of Proposition 5.7. The proof of this result is based in a standard Weinstein argument. Let us assume that for some $K > 0$ fixed,

$$\|u(t_1) - Q(\cdot - X_0)e^{-\frac{i}{2}v_0(\cdot)}e^{i\gamma_0}\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}, \tag{5.15}$$

with p_m defined in (1.21), and $\gamma_0 \in \mathbb{R}$.

Step 0. Preliminaries.

Let $D_0 > 2K$ be a large number to be chosen later, and set

$$\begin{aligned} T^* := \ \sup\Big\{ & t \geq t_1 \mid \text{ for all } t' \in [t_1, t), \text{ there exist } r_2(t'), g_2(t') \in \mathbb{R} \text{ smooth} \\ & \text{such that } |r_2'(t')| + |r_2(t_1) + v_0 t_1 - X_0| \leq \frac{v_0}{100}, \text{ and} \\ & \|u(t') - Q(\cdot + v_0 t - r_2(t'))\exp\{-\frac{i}{2}xv_0 - \frac{i}{4}v_0^2 t + ig_2(t)\}\|_{H^1(\mathbb{R})} \leq D_0\varepsilon^{p_m}\Big\} \end{aligned} \tag{5.16}$$

Observe that $T^* > t_1$ is well-defined since $D_0 > 2K$, (5.15) and the continuity of $t \mapsto u(t)$ in $H^1(\mathbb{R})$. The objective is to prove that $T^* = +\infty$, and thus (5.14). Therefore, for the sake of contradiction, in what follows **we shall suppose** $T^* < +\infty$.

The first step to reach a contradiction is now to decompose the solution on $[t_1, T^*]$ using modulation theory around the soliton. In particular, we will find some special $\rho_2(t), \gamma_2(t)$ satisfying the hypothesis in (5.16) but with

$$\sup_{t\in[t_1,T^*]} \big\|u(t) - Q_{c_0}(\cdot + v_0 t - \rho_2(t)) \exp\big\{ - \frac{i}{2}x v_0 - \frac{i}{4}v_0^2 t + i\gamma_2(t)\big\}\big\|_{H^1(\mathbb{R})} \leq \frac{1}{2} D_0 \varepsilon^{p_m}, \quad (5.17)$$

a contradiction with the definition of T^*.

Step 1. Modulation on the degenerate directions. The following result is similar to Lemma I.1.

Lemma 5.8 (Modulated decomposition)**.**

For $\varepsilon > 0$ small enough, independent of T^*, there exist C^1 functions $\rho_2, c_2, \tilde\gamma_2$, defined on $[t_1, T^*]$, with $c_2(t) > 0$ and such that the function $z(t)$ given by

$$z(t,x) := u(t,x) - \tilde R(t,x), \quad (5.18)$$

where $\tilde R(t,x) := Q_{c_2(t)}(y) e^{i\Gamma}$, with

$$y := x + v_0 t - \rho_2(t) \ \text{ and } \ \Gamma := -\frac{1}{2} x v_0 + \int_{t_1}^{t} c_2(s) ds - \frac{1}{4} v_0^2 t + \tilde\gamma_2(t),$$

satisfies for all $t \in [t_1, T^*]$,

$$\operatorname{Re} \int_{\mathbb{R}} \tilde R(t) \bar z(t) = \operatorname{Im} \int_{\mathbb{R}} \tilde R(t) \bar z(t) = \operatorname{Re} \int_{\mathbb{R}} Q'_{c_2(t)}(y) e^{i\Gamma} \bar z(t) = 0, \quad (5.19)$$

$$\|z(t)\|_{H^1(\mathbb{R})} + |c_2(t) - 1| \leq K D_0 \varepsilon^{p_m}, \ \text{ and} \quad (5.20)$$

$$\|z(t_1)\|_{H^1(\mathbb{R})} + |\rho_2(t_1) - v_0 t_1 - X_0| + |c_2(t_1) - 1| + |\tilde\gamma_2(t_1) + \frac{1}{4} v_0 t_1 - \gamma_0| \leq K \varepsilon^{p_m}, \quad (5.21)$$

where K is not depending on D_0. In addition, $z(t)$ now satisfies the following modified NLS equation

$$\begin{aligned} &i z_t + z_{xx} + a_\varepsilon(x)\big[|\tilde R + z|^{m-1}(\tilde R + z) - |\tilde R|^{m-1}\tilde R\big] \\ &\quad + i c_2'(t) \Lambda Q_{c_2} e^{i\Gamma} - \tilde\gamma_2'(t) Q_{c_2} e^{i\Gamma} - i\rho_2'(t) Q'_{c_2} e^{i\Gamma} + (a_\varepsilon(x) - 1) Q_{c_2}^m e^{i\Gamma} = 0. \quad (5.22)\end{aligned}$$

Furthermore, for some constant $\mu > 0$ independent of ε, we have the following estimates:

$$|\rho_2'(t)| \leq K\Big[\int_{\mathbb{R}} e^{-\mu|y|} |z|^2(t,x) dx\Big]^{\frac{1}{2}} + K \int_{\mathbb{R}} e^{-\mu|y|} |z|^2(t,x) dx + K e^{-\mu\varepsilon t}; \quad (5.23)$$

$$\frac{|c_2'(t)|}{c_2(t)} \leq K \int_{\mathbb{R}} e^{-\mu|y|} |z|^2(t,x) dx + K e^{-\mu\varepsilon t} \|z(t)\|_{H^1(\mathbb{R})}, \quad (5.24)$$

and finally

$$|\tilde\gamma_2'(t)| \leq K\Big[\int_{\mathbb{R}} e^{-\mu|y|} |z|^2(t,x) dx\Big]^{\frac{1}{2}} + K \int_{\mathbb{R}} e^{-\mu|y|} |z|^2(t,x) dx + K e^{-\mu\varepsilon t} \|z(t)\|_{H^1(\mathbb{R})} + K e^{-\varepsilon\mu t}. \quad (5.25)$$

Remark 5.3. Note that from (5.20) and taking ε small enough we have an improved the bound on $\rho_2(t)$. Indeed, for all $t \in [t_1, T^*]$,

$$|\rho_2'(t)| + |\rho_2(t_1) + v_0 t_1 - X_0| \leq 2 D_0 \varepsilon^{p_m}.$$

Thus, in order to reach a contradiction, we only need to show (5.17). Observe that these inequalities imply that the soliton position is far away from the interaction region.

Step 2. Almost conserved quantities and monotonicity.

Lemma 5.9 (Almost conservation of modified mass, energy and momentum)**.**

Consider $M = M[\tilde{R}]$, $E_a = E_a[\tilde{R}]$ and $P[\tilde{R}]$ the mass, energy and momentum of the soliton $\tilde{R}$ *(cf. (5.18)). Then for all $t \in [t_1, T^*]$ we have*

$$M[\tilde{R}](t) = c_2^{2\theta}(t) M[Q]; \tag{5.26}$$

$$E_a[\tilde{R}](t) = c_2^{2\theta}(t)(\frac{1}{4}v_0^2 - \lambda_0 c_2(t))M[Q] + O(e^{-\varepsilon\mu t}); \tag{5.27}$$

$$P[\tilde{R}](t) = -\frac{1}{2}v_0 c_2^{2\theta}(t) M[Q]. \tag{5.28}$$

Furthermore, we have the bound

$$\begin{aligned}\big|E_a[\tilde{R}](t_1) - E_a[\tilde{R}](t) + (c_2(t_1) + \frac{1}{4}v_0^2)(M[\tilde{R}](t_1) - M[\tilde{R}](t)) + v_0(P[\tilde{R}](t_1) - P[\tilde{R}](t))\big| \\ \leq K|\Big[\frac{c_2(t)}{c_2(t_1)}\Big]^{2\theta} - 1|^2 + K e^{-\varepsilon\mu t_1}. \end{aligned} \tag{5.29}$$

Proof. The first and third identities, namely (5.26) and (5.28), are just direct computations. We consider (5.27). Here we have

$$\begin{aligned} E_a[\tilde{R}](t) &= \frac{1}{2}\int_{\mathbb{R}} |\tilde{R}_x|^2 - \frac{1}{(m+1)}\int_{\mathbb{R}} a_\varepsilon(x)|\tilde{R}|^{m+1} \\ &= c_2^{2\theta}(t)\Big[c_2(t)(\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1}) + \frac{1}{8}v_0^2\int_{\mathbb{R}} Q^2\Big] \\ &\qquad + \frac{1}{m+1}\int_{\mathbb{R}}(1-a_\varepsilon)|\tilde{R}|^{m+1}. \end{aligned}$$

Similarly to (H.21), we have

$$|\int_{\mathbb{R}}(1-a_\varepsilon)|\tilde{R}|^{m+1}| \leq K e^{-\mu\varepsilon t},$$

for some constants $K, \mu > 0$. On the other hand, from Appendix K we have that

$$\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1} = -\frac{\lambda_0}{2}\int_{\mathbb{R}} Q^2, \quad \lambda_0 = \frac{5-m}{m+3},$$

and thus

$$E_a[\tilde{R}](t) = c_2^{2\theta}(t)(\frac{1}{4}v_0^2 - \lambda_0 c_2(t))M[Q] + O(e^{-\mu\varepsilon t}).$$

Summing up (5.26), (5.27) and (5.28), we obtain

$$E_a[\tilde{R}](t) + (c_2(t_1) + \frac{1}{4}v_0^2)M[u](t) + v_0 P[\tilde{R}](t) = c_2^{2\theta}(t)(c_2(t_1) - \lambda_0 c_2(t))M[Q] + O(e^{-\varepsilon\mu t}).$$

In particular,

$$\begin{aligned} &E_a[\tilde{R}](t_1) - E_a[\tilde{R}](t) + (c_2(t_1) + \frac{1}{4}v_0^2)(M[\tilde{R}](t_1) - M[\tilde{R}](t)) + v_0[P[\tilde{R}](t_1) - P[\tilde{R}](t)] = \\ &= \lambda_0 M[Q]\Big[c_2^{2\theta+1}(t) - c_2^{2\theta+1}(t_1) - \frac{c_2(t_1)}{\lambda_0}[c_2^{2\theta}(t) - c_2^{2\theta}(t_1)]\Big] + O(e^{-\varepsilon\mu t_1}). \end{aligned}$$

To obtain the last estimate (5.29) we perform a Taylor expansion up to the second order (around $y = y_0$) of the function $g(y) := y^{\frac{2\theta+1}{2\theta}}$; and where $y := c_2^{2\theta}(t)$ and $y_0 := c_2^{2\theta}(t_1)$. Note that $\frac{2\theta+1}{2\theta} = \frac{1}{\lambda_0}$ and $y_0^{1/2\theta} = c_2(t_1)$. The conclusion follows at once. □

Now our objective is to estimate the quadratic term involved in (5.29). Following [59], we use the mass conservation law identity. From (6.9) -(6.10) we have

$$c_2^{2\theta}(t)M[Q]+\frac{1}{2}\int_{\mathbb{R}}|z(t)|^2=c_2^{2\theta}(t_1)M[Q]+\frac{1}{2}\int_{\mathbb{R}}|z(t_1)|^2. \tag{5.30}$$

From here we obtain

$$(6.21)\leq K\|z(t)\|_{L^2(\mathbb{R})}^4+\|z(t_1)\|_{L^2(\mathbb{R})}^4+Ke^{-\varepsilon\mu t}, \tag{5.31}$$

for some $K,\mu>0$, independent of D_0 and ε.

Step 3. Energy estimates. Let us now introduce the second order functional

$$\begin{aligned}\mathcal{F}_2(t) \quad := \quad & \frac{1}{2}\int_{\mathbb{R}}\Big\{|z_x|^2+(c_2(t_1)+\frac{1}{4}v_0^2)|z|^2\Big\}+\frac{1}{2}v_0\operatorname{Im}\int_{\mathbb{R}}z_x\bar{z}\\ & -\frac{1}{(m+1)}\int_{\mathbb{R}}a_\varepsilon(x)[|\tilde{R}+z|^{m+1}-|\tilde{R}|^{m+1}-(m+1)|\tilde{R}|^{m-1}\operatorname{Re}(\tilde{R}\bar{z})].\end{aligned}$$

This functional have the following properties.

Lemma 5.10 (Energy expansion)**.**

Consider $M[u]$, $E_a[u]$ and $P[u]$ the mass, energy and momentum defined in (1.3), (I.2) and (1.5). Then we have for all $t\in[t_1,T^*]$,

$$\begin{aligned}&E_a[u](t)+(c_2(t_1)+\frac{1}{4}v_0^2)M[u](t)+v_0P[u](t)=\\ &\quad E_a[\tilde{R}](t)+(c_2(t_1)+\frac{1}{4}v_0^2)M[\tilde{R}](t)+v_0P[\tilde{R}](t)+\mathcal{F}_2(t)+O(e^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).\end{aligned}$$

Proof. Using the orthogonality condition (6.10), we have

$$\begin{aligned}E_a[u](t) \quad = \quad & E_a[\tilde{R}]+\operatorname{Re}\int_{\mathbb{R}}\bar{z}[-\tilde{R}_{xx}-|\tilde{R}|^{m-1}\tilde{R}]+\frac{1}{2}\int_{\mathbb{R}}|z_x|^2+\operatorname{Re}\int_{\mathbb{R}}(1-a_\varepsilon)|\tilde{R}|^{m-1}\tilde{R}\bar{z}\\ & -\frac{1}{(m+1)}\int_{\mathbb{R}}a_\varepsilon(x)[|\tilde{R}+z|^{m+1}-|\tilde{R}|^{m+1}-(m+1)|\tilde{R}|^{m-1}\operatorname{Re}(\tilde{R}\bar{z})].\end{aligned}$$

Moreover, following (B.13), we easily get

$$|\operatorname{Re}\int_{\mathbb{R}}\bar{z}(1-a_\varepsilon)|\tilde{R}|^{m-1}\tilde{R}|\leq Ke^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}.$$

Similarly, by using (6.10),

$$M[u](t)=M[\tilde{R}]+\frac{1}{2}\int_{\mathbb{R}}|z|^2,$$

and

$$P[u](t)=P[\tilde{R}](t)+\operatorname{Im}\int_{\mathbb{R}}\tilde{R}_x\bar{z}+\frac{1}{2}\operatorname{Im}\int_{\mathbb{R}}z_x\bar{z}.$$

Collecting the above estimates, we have

$$\begin{aligned}&E_a[u](t)+(c_2(t_1)+\frac{1}{4}v_0^2)M[u](t)+v_0P[u](t)=\\ &E_a[\tilde{R}](t)+(c_2(t_1)+\frac{1}{4}v_0^2)M[\tilde{R}](t)+v_0P[\tilde{R}](t)+\mathcal{F}_2(t)+O(e^{-\mu\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).\end{aligned}$$

Here we have used (6.10), the equation satisfied by Q_{c_2} and the identity

$$\operatorname{Re}\int_{\mathbb{R}}\bar{z}[-\tilde{R}_{xx}-|\tilde{R}|^{m-1}\tilde{R}-iv_0\tilde{R}_x]=0.$$

This concludes the proof. □

Lemma 5.11 (Modified coercivity for $\mathcal{F}_2$)**.**

There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following hold. There exist $K, \nu, \mu > 0$, independent of K^* such that for every $t \in [t_1, T^*]$

$$\mathcal{F}_2(t) \geq \nu \|z(t)\|^2_{H^1(\mathbb{R})} - Ke^{-\mu\varepsilon t}\|z(t)\|^2_{L^2(\mathbb{R})} + O(\|z(t)\|^3_{L^2(\mathbb{R})}). \tag{5.32}$$

Proof. First of all, note that

$$\begin{aligned}\mathcal{F}_2(t) \quad = \quad & \frac{1}{2}\int_{\mathbb{R}}\{z_x^2 + (c_2(t_1) + \frac{1}{4}v_0^2)z^2\} + \frac{1}{2}v_0 \operatorname{Im}\int_{\mathbb{R}} \bar{z}z_x \\ & - \int_{\mathbb{R}}[|\tilde{R}|^{m-1}|z|^2 + (m-1)|\tilde{R}|^{m-3}[\operatorname{Re}(\tilde{R}\bar{z})]^2] \\ & - \int_{\mathbb{R}}(a_\varepsilon(x) - 1)[|\tilde{R}|^{m-1}|z|^2 + (m-1)|\tilde{R}|^{m-3}[\operatorname{Re}(\tilde{R}\bar{z})]^2] + O(\|z(t)\|^3_{H^1(\mathbb{R})})\end{aligned}$$

Since $(1-a_\varepsilon(x))$ is exponentially decreasing along the region where the soliton $\tilde{R}$ is supported, we have

$$|\int_{\mathbb{R}}(1 - a_\varepsilon(x))[|\tilde{R}|^{m-1}|z|^2 + (m-1)|\tilde{R}|^{m-3}[\operatorname{Re}(\tilde{R}\bar{z})]^2]| \leq Ke^{-\varepsilon\mu t}\|z(t)\|_{L^2(\mathbb{R})}.$$

(cf. (B.13 for a similar computation.) From Lemma 2.3 and (6.10) we have for $t \geq t_1$,

$$\mathcal{F}_2(t) \geq \nu\|z(t)\|^2_{H^1(\mathbb{R})} - Ke^{-\varepsilon\mu t}\|z(t)\|^2_{L^2(\mathbb{R})} - K\|z(t)\|^3_{H^1(\mathbb{R})},$$

as desired. □

End of the proof. Now we prove that our assumption $T^* < +\infty$ leads inevitably to a contradiction. Indeed, from Lemmas 5.10 and 5.11, the mass and energy conservation, and the negativity of (1.16), we have for all $t \in [t_1, T^*]$ and for some constant $K > 0$,

$$\begin{aligned}\|z(t)\|^2_{H^1(\mathbb{R})} \leq \quad & K\mathcal{F}(t_1) + Ke^{-\mu\varepsilon t_1}\sup_{t\in[t_1,T^*]}\|z(t)\|_{L^2(\mathbb{R})} + K\sup_{t\in[t_1,T^*]}\|z(t)\|^3_{L^2(\mathbb{R})} \\ + \big| & E_a[\tilde{R}](t_1) - E_a[\tilde{R}](t) + (c_2(t_1) + \frac{1}{4}v_0^2)(M[\tilde{R}](t_1) - M[\tilde{R}](t)) + v_0(P[\tilde{R}](t_1) - P[\tilde{R}](t))\big|.\end{aligned}$$

From Lemmas 6.2 and 5.31 we have

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq K\varepsilon^{2p_m} + K\sup_{t\in[t_1,T^*]}\|z(t)\|^4_{H^1(\mathbb{R})} + Ke^{-\varepsilon\mu t_1}D_0\varepsilon^{p_m}.$$

Collecting the preceding estimates we have for $\varepsilon > 0$ small and $D_0 = D_0(K)$ large enough

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq \frac{1}{4}D_0^2\varepsilon^{2p_m}.$$

This estimate together with (5.30) and (6.12) gives $|c_2(t) - 1| \leq K\varepsilon^{p_m}$, independent of D_0, which contradicts the definition of T^*. The conclusion is that

$$\sup_{t\geq t_1}\big\|u(t) - Q(\cdot + v_0 t - \rho_2(t))\exp\big\{-\frac{i}{2}xv_0 - \frac{i}{4}v_0^2 t + i\gamma_2(t)\big\}\big\|_{H^1(\mathbb{R})} \leq K\varepsilon^{p_m}.$$

This finishes the proof.

□

Part IV

On the inelastic two-soliton collision for generalized Korteweg-de Vries equations

Summary

Abstract

We study the problem of 2-soliton collision for the generalized Korteweg-de Vries equations, completing some recent works of Y. Martel and F. Merle [53, 54]. We classify the nonlinearities for which collisions are elastic or inelastic. Our main result states that in the case of small solitons, with one soliton smaller than the other one, the unique nonlinearities allowing a perfectly elastic collision are precisely the integrable cases, namely the quadratic (KdV), cubic (mKdV) and Gardner nonlinearities.[7]

Keywords : generalized Korteweg- de Vries equations, 2-soliton collision, integrability.

1 Introduction and Main Results

In this work we consider the *generalized Korteweg-de Vries equation* (gKdV) on the real line

$$u_t + (u_{xx} + f(u))_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1.1}$$

Here $u = u(t,x)$ is a real-valued function, and $f : \mathbb{R} \to \mathbb{R}$ a nonlinear function, often refered as the *nonlinearity* of (1.1). This equation represents a mathematical generalization of the *Korteweg-de Vries* equation (KdV), namely the case $f(s) = s^2$,

$$u_t + (u_{xx} + u^2)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x; \tag{1.2}$$

other physically important cases are the cubic one $f(s) = s^3$, and the *quadratic-cubic* nonlinearity, namely $f(s) = s^2 - \mu s^3$, $\mu \in \mathbb{R}$. In the former case, the equation (1.1) is often refered as the (focusing) *modified* KdV equation (mKdV), and in the latter, it is known as the *Gardner* equation.

Concerning the KdV equation, it arises in Physics as a model of propagation of dispersive long waves, as was pointed out by J. S. Russel in 1834 [62]. The exact formulation of the KdV equation comes from Korteweg and de Vries (1895) [41]. This equation was re-discovered in a numerical work by N. Zabusky and M. Kruskal in 1965 [42].

After this work, a great amount of literature has emerged, physical, numerical and mathematical, for the study of this equation, see for example [8, 7, 75, 44, 30, 63, 62]. Although under different points of view, among the main topics treated are the following: existence of explicit solutions and their stability, local and global well posedness, long time behavior properties and, of course, related generalized models, *hierarchies* and their properties.

This continuous, focused research on the KdV equation can be in part explained by some striking algebraic properties. One of the first properties is the existence of localized, rapidly decaying, stable and smooth solutions called *solitons*. Given three real numbers t_0, x_0 and $c > 0$, solitons are solutions of (1.2) of the form

$$u(t,x) := Q_c(x - x_0 - c(t - t_0)), \quad Q_c(s) := cQ(c^{1/2}s), \tag{1.3}$$

and where Q satisfies the second order nonlinear differential equation

$$Q'' - Q + Q^2 = 0, \quad Q(x) = \frac{3}{2\cosh^2(\frac{1}{2}x)}.$$

[7]To appear in IMRN.

The 3-parameter family of solitons (1.3) contains three important symmetries of the equation, namely *scaling* and *translation* in space and time invariances. From the Noëther theorem, these two last symmetries are related to *conserved quantities*, invariant under the KdV flow, usually called *Mass* and *Energy*, represented below in (1.7)-(1.8) (in a general form). Moreover, due to the mass and energy conservation, the Sobolev space $H^1(\mathbb{R})$ appears as an ideal space to study long time properties of KdV.

Even more striking is the fact that KdV, mKdV and the Gardner equation, being infinite dimensional dynamical systems, possess an *infinity number of conserved quantities*, a consequence of the so-called *complete integrability* property. This one is closely related to the existence of a *Lax pair* for these equations (see Lax, [43]). Another important property is the following well known fact: given any Schwartz initial data, the corresponding solution to the Cauchy problem for (1.2) exists globally in time and decouples, as $t \to +\infty$, into a *radiation* part going leftward plus a nonlinear *multisoliton* component going to the right, see [76].

The dynamical problem of 2-soliton collision is a classical problem in nonlinear wave propagation (see [53] for a review and references therein). By 2-soliton collision we mean the following problem: given two solitons, solutions of (1.1), largely separated at some early time and having different velocities, we expect that they have to collide at some finite time. The resulting solution after the collision is precisely the object of study. In particular, one considers if any change in size, position, or shape, even destruction of the solitons, after some large time, may be present.

Let us review some relevant works in this direction. First, the works of Fermi, Pasta and Ulam [19] and Zabusky and Kruskal [42] exhibited numerical results showing a remarkable phenomena related to solitons collision. More precisely, they put in evidence the *elastic* character of the collision between two solitons. By *elastic* we mean that collision keeps the solitons unchanged and does not produce any residual term of positive mass for large times. The unique consequence of the collision is a shift translation on each soliton, depending on their sizes. Next, the work of Lax [43] developed a mathematical framework to study these problems. After this, the *inverse scattering method* (we refer e.g. to [1] and [62] for a review) provided explicit formulas for N-soliton solutions (Hirota [31]). Indeed, let $c_1 > c_2 > 0$ and $\delta_1, \delta_2 \in \mathbb{R}$ be arbitrary given numbers. There exists an explicit solution $U = U_{c_1,c_2}(t,x)$ of (1.2) which satisfies

$$\Big\|U(t,\cdot)-\sum_{j=1}^{2} Q_{c_j}(\cdot-c_jt-\delta_j)\Big\|_{H^1(\mathbb{R})} \underset{t\to-\infty}{\longrightarrow} 0, \quad \Big\|U(t,\cdot)-\sum_{j=1}^{2} Q_{c_j}(\cdot-c_jt-\delta_j')\Big\|_{H^1(\mathbb{R})} \underset{t\to+\infty}{\longrightarrow} 0, \tag{1.4}$$

for some δ_j' such that the shifts $\Delta_j = \delta_j' - \delta_j$ depend only on c_1, c_2. This solution, called 2-soliton, represents the pure collision of two solitons, with no residual terms before and after the collision. In other words, the collision is *elastic*.

These properties are also valid for the *cubic* mKdV, (see [1], p. 390) and for the *Gardner* equation (see [21, 82] and references there in). In particular, complete integrability and elastic collisions are still present. Let us recall that for the Gardner equation

$$u_t + (u_{xx} + u^2 - \mu u^3)_x = 0, \tag{1.5}$$

given $\mu \in \mathbb{R}$, soliton solutions exist for all $c > 0$ in the case $\mu < 0$, and provided $c < \frac{2}{9\mu}$ if $\mu > 0$. These solutions are explicit and given by $u(t,x) = Q_{\mu,c}(x-ct)$, where $Q_{\mu,c}$ is the Schwartz function [82]

$$Q_{\mu,c}(x) := \frac{3c}{1+\rho\cosh(\sqrt{c}x)}; \quad \rho := (1-\frac{9}{2}\mu c)^{1/2}. \tag{1.6}$$

In particular, no soliton-solution exists provided $\mu > 0$, and $c > 0$ large enough, where the character of the equation becomes *defocusing*.

We point out that these techniques are known to be too rigid to be applied to more general models, and have no equivalent for the case of the gKdV equation (1.1) with a general nonlinearity. The first purpose of this paper is to confirm this belief under reasonable hypothesis on the nonlinearity: the collision of two solitons is not elastic in general, except by KdV, mKdV and the Gardner equations. Before establishing our main result we explain the framework where the problem must be posed.

The complete integrability property has been studied in many other differential equations, as NLS, KPI, Benjamin-Ono, etc.; see for example [1]. In particular, when complete integrability is lost, very little is known. We mention the recent works of Perelman [72], Holmer, Marzuola and Zworski [33, 34, 35] and Abou Salem, Fröhlich and Sigal [3] on the problem of 2-soliton collision for the nonlinear Schrödinger equation (NLS) under the action of a potential and considering higher velocities.

1.1 Setting and hypothesis

Let us come back to the general equation (1.1). Assume that the nonlinearity $f \in C^3(\mathbb{R})$. The Cauchy problem for equation (1.1) (namely, adding the initial condition $u(t=0) = u_0$) is *locally well-posed* for $u_0 \in H^1(\mathbb{R})$ (see Kenig, Ponce and Vega [39]).

For $H^1(\mathbb{R})$ solutions, in the general case, unlike the integrable cases, only the following two quantities are conserved by the flow:

$$M(t) := \int_{\mathbb{R}} u^2(t,x)\,dx = \int_{\mathbb{R}} u_0^2(x)\,dx = M(0), \quad \text{(Mass)}, \tag{1.7}$$

and

$$\begin{aligned} E(t) \;&:=\; \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \int_{\mathbb{R}} F(u(t,x))\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}} (u_0)_x^2(x)\,dx - \int_{\mathbb{R}} F(u_0(x))\,dx = E(0), \quad \text{(Energy)} \end{aligned} \tag{1.8}$$

where we have denoted

$$F(s) := \int_0^s f(\sigma)\,d\sigma. \tag{1.9}$$

In the case of a pure power $f(s) = s^m$, $m < 5$, any $H^1(\mathbb{R})$ solution is global in time thanks to the conservation of energy (1.8). For $m = 5$, solitons are shown to be *unstable* and the Cauchy problem for the corresponding gKdV equation has finite-time blow-up solutions, and see [50] and references there in. It is believed that for $m > 5$ the situation is the same. The origin *grosso modo* of this instability comes from the lack of control for the injection $H^1(\mathbb{R}) \to L^p(\mathbb{R})$ for $p \geq 5$. Indeed, from the Galiardo-Nirenberg inequality

$$\int_{\mathbb{R}} |v|^{p+1} \leq C(p)\Big(\int_{\mathbb{R}} v^2\Big)^{\frac{p+3}{4}}\Big(\int_{\mathbb{R}} v_x^2\Big)^{\frac{p-1}{4}},$$

valid for any $v \in H^1(\mathbb{R})$, one can see that the energy (1.8) cannot be controlled by the usual H^1-norm. Consequently, in this work, we will discard high-order nonlinearities at leading order. Indeed, we will consider nonlinearities f of the form

$$f \in C^{m+2}(\mathbb{R}), \quad f(u) := u^m + f_1(u), \quad m = 2,3,4, \quad \text{with} \quad \lim_{s\to 0}\frac{|f_1(s)|}{|s|^m} = 0. \tag{1.10}$$

Moreover, using stability properties of the solitons, we will have only *global in time* solutions, namely $u(t) \in H^1(\mathbb{R})$ for all time $t \in \mathbb{R}$.

The positive sign leading in front of f (see (1.10)) allows the existence of solitons for (1.21) of the form

$$u(t,x) := Q_c(x - x_0 - ct),$$

with $c > 0$ small enough and $x_0 \in \mathbb{R}$, where the function Q_c satisfies the elliptic equation

$$Q_c'' + f(Q_c) = cQ_c, \quad Q_c \in H^1(\mathbb{R}). \tag{1.11}$$

From Berestycki and Lions [6] and (1.1), it follows that there exists $c_*(f) > 0$ (possibly $+\infty$) defined by

$$c_*(f) := \sup\{c > 0 \text{ such that for all } c' \in (0,c), \text{ exists } Q_{c'} \text{ positive solution of (1.11)}\}.$$

For all $c > 0$, if a solution $Q_c > 0$ of (1.11) exists then it can be chosen *even* on $\mathbb{R}$ and *exponentially decreasing* on $\mathbb{R}^+$ (and similarly if $Q_c < 0$). Moreover, in [51], the authors have showed that $0 < c < c_*(f)$ is a sufficient condition for *asymptotic stability* in the energy space H^1 around the soliton Q_c, see also Proposition 3.3 for details.

Finally, in this paper, we consider only *nonlinear stable* solitons in the sense of Weinstein [84], i.e. such that

$$\frac{d}{dc'} \int Q_{c'}^2(x)dx\Big|_{c'=c} > 0. \tag{1.12}$$

Note that since $m = 2,3,4$ in (1.10), this condition is automatically satisfied for $c > 0$ small enough (in the pure power case $f(s) = s^m$, this condition is satisfied for any $c > 0$ provided $m < 5$, see [84]).

1.2 Previous analytic results on 2-soliton collision in non-integrable cases

As pointed out in [53], the problem of describing the collision of two traveling waves or solitons is a general problem for nonlinear PDEs, which is almost completely open, except in the integrable cases described above. On the other hand, these problems have been studied since the 60's from both experimental and numerical points of view.

We deal with these questions for (1.1) with a general nonlinearity $f(u)$ in a particular setting: we consider two positive solitons Q_{c_1}, Q_{c_2}, $0 < c_2 < c_1 < c_*(f)$, and we assume c_2 small compared with c_1.

Under these assumptions, Martel and Merle [53] considered the collision problem for (2.1) in the quartic case, $f(s) = s^4$, with one soliton small with respect to the other. They showed that the collision is almost elastic, but inelastic, by showing the nonexistence of pure 2-soliton solution.

Theorem 1.1 (Non-existence of a pure 2-soliton solution, quartic case [53]). *Let $f(s) := s^4$ and $0 < c := \frac{c_2}{c_1} < 1$. There exists a constant $c_0 > 0$ such that if $c < c_0$ then the following holds. Let $u(t) \in H^1(\mathbb{R})$ be the unique solution of (1.1) such that*

$$\lim_{t\to-\infty} \|u(t) - Q_{c_1}(\cdot - c_1 t) - Q_{c_2}(\cdot - c_2 t)\|_{H^1(\mathbb{R})} = 0. \tag{1.13}$$

Then there exist x_1^+, x_2^+, $c_1^+ > c_2^+ > 0$ and constants $T_0, K > 0$ large enough such that

$$w^+(t,\cdot) := u(t,\cdot) - Q_{c_1^+}(\cdot - x_1^+ - c_1^+ t) - Q_{c_2^+}(\cdot - x_2^+ - c_2^+ t)$$

satisfies

1. *Support on the left of solitons.*
$$\lim_{t\to+\infty}\|w^+(t)\|_{H^1(x>\frac{1}{10}c_2t)}=0.$$
2. *Parameters perturbation. The limit scaling parameters* c_1^+ *and* c_2^+ *satisfy*
$$\frac{1}{K}c^{\frac{17}{6}}\le\frac{c_1^+}{c_1}-1\le Kc^{\frac{11}{6}},\quad \text{and}\quad \frac{1}{K}c^{\frac{8}{3}}\le 1-\frac{c_2^+}{c}\le Kc^{\frac{1}{3}}.$$
In particular, $c_1^+>c_1$ *and* $c_2^+<c_2$.
3. *Non zero residual term. For every* $t\ge T_0$*, the adapted* H^1*-norm of* $w^+(t)$ *satisfies*
$$\frac{1}{K}c_1^{\frac{7}{12}}c^{\frac{17}{12}}\le\|w_x^+(t)\|_{L^2(\mathbb{R})}+\sqrt{c_1c}\|w^+(t)\|_{L^2(\mathbb{R})}\le Kc_1^{\frac{7}{12}}c^{\frac{11}{12}}.$$

Remark 1.1. The existence and uniqueness of the solution of (1.1) satisfying (1.13) was proved in [49].

Remark 1.2. Note that $\|Q_{c_2}\|_{H^1(\mathbb{R})}\sim c^{\frac{1}{12}}\gg Kc^{\frac{5}{12}}\ge\|w^+(t)\|_{L^2(\mathbb{R})}$ for c small. In other words the defect w^+ is really small compared with Q_{c_2}.

The next question arising from this result is to generalize these results to (1.1) under assumption (1.10). In this case, Martel and Merle [54] proved that the collision is still stable, giving upper bounds on the residual terms appearing after the collision. In particular, their result extends the positive part of Theorem 1.1.

Theorem 1.2 (Behavior after collision of a pure 2-soliton solution, [54]**).**
Let f *satisfying* (1.10). *Let* $0<c_2<c_1<c_*(f)$ *be such that the positive solution* Q_{c_1} *of* (1.11) *satisfies* (1.12). *Then there exists* $c_0=c_0(c_1)\in(0,c_1)$ *such that if* $c_2<c_0(c_1)$ *then the following holds. Let* $u(t)$ *be the solution of* (1.21) *satisfying*
$$\lim_{t\to-\infty}\|u(t)-Q_{c_1}(\cdot-c_1t)-Q_{c_2}(\cdot-c_2t)\|_{H^1(\mathbb{R})}=0. \tag{1.14}$$
Then, there exist $\rho_1(t),\rho_2(t)$, $c_1^+>c_2^+>0$ *and* $K>0$ *such that*
$$w^+(t,x):=u(t,x)-Q_{c_1^+}(x-\rho_1(t))-Q_{c_2^+}(x-\rho_2(t))$$
satisfies $\sup_{t\in\mathbb{R}}\|w^+(t)\|_{H^1(\mathbb{R})}\le Kc_2^{\frac{1}{m-1}}$ *and for* $q=q_m:=\frac{2}{m-1}+\frac{1}{4}$,
$$\lim_{t\to+\infty}\|w^+(t)\|_{H^1(x>\frac{1}{10}c_2t)}=0,\quad \limsup_{t\to+\infty}\|w^+(t)\|_{H^1(\mathbb{R})}\le Kc_2^{q-\frac{1}{2}-\frac{1}{100}}, \tag{1.15}$$
$$\lim_{t\to+\infty}|\rho_1'(t)-c_1^+|+|\rho_2'(t)-c_2^+|=0. \tag{1.16}$$
Moreover, $\lim_{t\to+\infty}E(w^+(t))=:E^+$ *and* $\lim_{t\to+\infty}\int_{\mathbb{R}}(w^+)^2(t)=:M^+$ *exist and the following bounds hold*
$$\frac{1}{2}\limsup_{t\to+\infty}\int_{\mathbb{R}}((w_x^+)^2+c_2(w^+)^2)(t)\le 2E^++c_2M^+\le\liminf_{t\to+\infty}\int_{\mathbb{R}}((w_x^+)^2+2c_2(w^+)^2)(t). \tag{1.17}$$
Finally, the limit parameters c_1^+ *and* c_2^+ *satisfy the following bounds*
$$\frac{1}{K}(2E^++c_2M^+)\le\frac{c_1^+}{c_1}-1\le K(2E^++c_2M^+), \tag{1.18}$$
and
$$\frac{1}{K}c_2^{q-\frac{3}{4}}(2E^++c_1M^+)\le 1-\frac{c_2^+}{c_2}\le Kc_2^{q-\frac{3}{4}}(2E^++c_1M^+). \tag{1.19}$$

Remark 1.3. In Theorem 1.2, if $c_1^+ = c_1$ and $c_2^+ = c_2$ (or equivalently $E^+ = M^+ = 0$), then the solution $u(t)$ is a *pure 2-soliton solution and the collision is elastic*.

In [54], the question of whether the collision is elastic or inelastic in the general case –and thus the nonexistence of pure 2-soliton solutions– was left open, see [54], Remark 1. More precisely, the authors conjectured a *classification result* concerning the nonlinearities $f(s)$ allowing small stable solitons. This affirmation asserts that under reasonable stability properties, *the unique nonlinearities for which any 2-soliton collision is pure are the integrable cases, $f(s) = s^2$, $f(s) = s^3$ and a linear combination of both nonlinearities.* Theorem 1.1 from [53] was the first step in this direction. By extending some techniques from [53], [54] and developping new computations, we are able to provide a satisfactory answer to this open question.

1.3 Main results

Consider the framework introduced in Theorem 1.2. In addition to this result, we have the following

Theorem 1.3 (Non-existence of pure 2-soliton solution, general case)**.** *Let f be as in (1.10), with $m = 2$ or 3, and*

$$f \in C^{p+1}(\mathbb{R}), \quad f^{(p)}(0) \neq 0 \quad \text{for some} \quad p \geq 4. \tag{1.20}$$

For $0 < c_2 \ll c_1 \ll 1$ equation (1.1) has no pure 2-soliton solution of sizes c_1, c_2. In particular Theorem 1.2 holds with $c_1^+ > c_1$ and $c_2^+ < c_2$.

Remark 1.4. The nonzero condition $f^{(p)}(0) \neq 0$ for some $p \geq 4$ rules out the integrable cases $f(s) = s^m$, $m = 2$ or 3 and the Gardner nonlinearity $f(s) = s^2 - \mu s^3$.

Remark 1.5. We do not treat the degenerate cases $f(s) = s^m + f_1(s)$, for $f_1(s) \neq 0$ but $f_1^{(p)}(0) = 0$ for all $p \geq 4$. These cases seem to be not physically relevant.

Remark 1.6. The result of Theorem 1.3 in the quartic case $m = 4$ follows directly from the proof of Theorem 1.1 in [53] together with the techniques used in the present paper. This remark and Theorem 1.3 allow to classify the nonlinearities for which 2-soliton collision is elastic. In particular, with the restriction mentioned in Remark 1.5, we obtain that *pure 2-soliton solutions are present for any pair of solitons with different velocities if and only if f corresponds to the integrable cases, $f(s) = s^2, s^3$ and linear combinations*. We recall that for $m \geq 5$ in (1.10), solitons have been shown to be unstable (see [9]). It is believed that collision may produce blow-up solutions in finite time.

Theorem 1.3 is a consequence of the following reduction of the problem. Let p be the smallest integer greater or equal than 4 satisfying (1.20). Let $c_1 > 0$ small. Consider the transformation

$$\tilde{u}(t,x) := c_1^{-\frac{1}{m-1}} u(c_1^{-\frac{3}{2}} t, c_1^{-\frac{1}{2}} x),$$

which maps $Q_{c_1}(x - c_1 t)$ to $Q(x-t)$ and $Q_{c_2}(x - c_2 t)$ to $Q_c(x - ct)$, with $c := \frac{c_2}{c_1}$. If $u = u(t,x)$ is solution of (1.1) then $\tilde{u}$ is solution of the equation

$$\begin{cases} \tilde{u}_t + (\tilde{u}_{xx} + \tilde{f}(\tilde{u}))_x = 0, \\ \text{with} \quad \tilde{f}(\tilde{u}) := \tilde{u}^m + \tilde{f}_1(\tilde{u}), \quad \tilde{f}_1(\tilde{u}) := c_1^{-\frac{m}{m-1}} f_1(c_1^{\frac{1}{m-1}} \tilde{u}). \end{cases}$$

Note that $\tilde{f}_1$ satisfies (1.10). Then, for the case $m = 3$, $\tilde{f}_1$ can be expanded as

$$\begin{aligned} \tilde{f}_1(\tilde{u}) \quad &= \quad c_1^{-\frac{3}{2}} \Big[\frac{1}{p!} f_1^{(p)}(0) (c_1^{\frac{1}{2}} \tilde{u})^p + O((c_1^{\frac{1}{2}} \tilde{u})^{p+1}) \Big] \\ &=: \quad \varepsilon \tilde{u}^p + |\varepsilon|^{1 + \frac{1}{p-3}} \hat{f}_1(\tilde{u}), \end{aligned}$$

where $\varepsilon := \frac{1}{p!} f_1^{(p)}(0) c_1^{\frac{p-3}{2}}$ is small and $\hat{f}_1$ satisfies the decay relation $\lim_{s\to 0} |s|^{-p} \hat{f}_1(s) = 0$.

For the quadratic case, we need more care *because of the Gardner nonlinearity*. We have

$$\begin{aligned}\tilde{f}_1(\tilde{u}) &= c_1^{-2}\Big[\frac{1}{6} f_1^{(3)}(0)(c_1\tilde{u})^3 + \frac{1}{p!} f_1^{(p)}(0)(c_1\tilde{u})^p + O((c_1\tilde{u})^{p+1})\Big] \\ &= \frac{1}{6} f_1^{(3)}(0) c_1 \tilde{u}^3 + \frac{1}{p!} f_1^{(p)}(0) c_1^{p-2} \tilde{u}^p + O(c_1^{p-1}\tilde{u}^{p+1}) \\ &=: \mu(\varepsilon)\tilde{u}^3 + \varepsilon \tilde{u}^p + |\varepsilon|^{1+\frac{1}{p-2}} \hat{f}_1(\tilde{u}),\end{aligned}$$

where $\varepsilon := \frac{1}{p!} f_1^{(p)}(0) c_1^{p-2} \neq 0$ by hypothesis, and $\mu(\varepsilon) := \frac{1}{6} f_1^{(3)}(0) c_1 = \hat{\mu}\varepsilon^{\frac{1}{p-2}}$, $\hat{\mu} \in \mathbb{R}$. Here both ε and μ are small (depending on c_1) and $\hat{f}_1$ satisfies the decay relation $\lim_{s\to 0} |s|^{-p} \hat{f}_1(s) = 0$. Note that in this framework, the quadratic case can be seen as a particular case of the Gardner nonlinearity, for which $\hat{\mu} = 0$.

Finally, we drop the tilde on $\tilde{u}$ and $\tilde{f}$ and the hat on $\hat{f}_1$. We are now reduced to the ε-*dependent* equation

$$u_t + (u_{xx} + f(u))_x = 0, \tag{1.21}$$

where $\mu(\varepsilon) = \hat{\mu}\varepsilon^{\frac{1}{p-2}}$, $\hat{\mu} \in \mathbb{R}$,

$$f = f_\varepsilon \in C^{p+1}(\mathbb{R}),\ f(u) = \begin{cases} u^2 + \mu(\varepsilon)u^3 + \varepsilon u^p + \varepsilon^{1+\frac{1}{p-2}} f_1(u), & m = 2, \\ u^3 + \varepsilon u^p + \varepsilon^{1+\frac{1}{p-3}} f_1(u), & m = 3, \end{cases} \qquad \lim_{s\to 0} \frac{f_1(s)}{|s|^p} = 0, \tag{1.22}$$

and ε is small, and $p \geq 4$. For notational commodity we will skip the ε-dependence on the functions considered along this work, except in some computations performed in Appendix M. Lastly, note that for ε small Q and Q_c satisfy (1.12), see also Remark 3.2.

In this framework, we now claim the main result of this paper:

Theorem 1.4 (Non-existence of pure 2-soliton solution, general case).
Suppose $m = 2, 3$ and f satisfying (1.22) for $p \geq 4$. There exists a constant $\varepsilon_0 > 0$ such that if

$$0 < |\varepsilon| < \varepsilon_0, \quad \textit{and} \quad 0 < c \leq |\varepsilon|^{m-1+\frac{1}{25}}, \tag{1.23}$$

then the following holds. Let $u(t)$ be solution of (1.21) satisfying

$$\lim_{t\to -\infty} \|u(t) - Q(\cdot - t) - Q_c(\cdot - ct)\|_{H^1(\mathbb{R})} = 0, \tag{1.24}$$

there exist $K, T_0 > 0$ such that

1. *Non zero residual term. There exist $\rho_1(t)$, $\rho_2(t)$, $c_1^+ > c_2^+ > 0$ such that*

$$w^+(t) := u(t,x) - Q_{c_1^+}(x - \rho_1(t)) - Q_{c_2^+}(x - \rho_2(t))$$

satisfies, for every $t \geq T_0$,

$$\lim_{t\to +\infty} \|w^+(t)\|_{H^1(x > \frac{1}{10}ct)} = 0, \tag{1.25}$$

and for $q = \frac{2}{m-1} + \frac{1}{4}$,

$$\frac{1}{K}|\varepsilon| c^{q+\frac{1}{2}} \leq \|w_x^+(t)\|_{L^2(\mathbb{R})} + \sqrt{c}\|w^+(t)\|_{L^2(\mathbb{R})} \leq K|\varepsilon| c^q. \tag{1.26}$$

2. *The asymptotic scaling parameters c_1^+ and c_2^+ satisfy*

$$\frac{1}{K}\varepsilon^2 c^{2q+1} \leq c_1^+ - 1 \leq K\varepsilon^2 c^{2q}, \text{ and}$$
$$\frac{1}{K}\varepsilon^2 c^{3q+\frac{1}{4}} \leq 1 - \frac{c_2^+}{c} \leq K\varepsilon^2 c^{3q-\frac{7}{4}}. \tag{1.27}$$

Before sketching the proof of this Theorem, some remarks are in order.

Remark 1.7 (Comments on the assumptions). In the present paper, as in [53, 54], we study the collision of two solitons Q_{c_1} and Q_{c_2}. The assumption c_2 small allows to linearize in c_2, and then to reduce the non existence of a pure 2-soliton solution to the computation of a *coefficient* depending only on c_1. For general f, as in (1.10), and general $c_1 > 0$, it is an open question to compute this coefficient, see Remark 2.7 (for $p = 4$, a special algebraic structure allowed to compute this coefficient, see [53]).

According to this, we compute the asymptotics of this coefficient as c_1 is small (or equivalent, ε is small), see Appendix M. This is the only place where ε small is needed. This asymptotic allows us to conclude under the additional restriction $0 < c < |\varepsilon|^{m-1+\frac{1}{25}}$ (see Proposition 2.11 and estimate (4.6)). The exponent $\frac{1}{25}$ has no special meaning, and can be taken as small as we want, as long as c is taken even smaller.

Two open questions then arise:

1. Can we relax in (1.23) the second condition on c?
2. For general f, do there exist special values of c_1 for which the coefficient is zero? The residue from the collision would then be of smaller order in c_2.

Remark 1.8. For $m = 2$, the smoothness condition (1.22) allows nonlinearities of type $f(s) = s^2 + \nu\varepsilon^{p-2}s^p$, with $p \geq 4$ (possibly non integer). For $m = 3$, the same conclusion follows for nonlinearities of the type $f(s) = s^3 + \varepsilon s^p$, $p = 4$ and $p \geq 5$ (possibly non integer). See also Appendix L and final remarks in Appendix M.

Remark 1.9. Although this theorem asserts that collision is indeed inelastic, near-elastic, the appearance of smaller solitons on the left of the solitons is not discarded by our proof, and (1.25). However, we believe that, at least under the condition of Theorem 1.4, there are no such small solitons.

1.4 Sketch of the proof

Our proof will follow closely the approach described by Martel, Merle and Mizumachi [53, 54, 58]. The argument is as follows: we consider the solution (unique, see [49]) $u(t)$ of (1.21) satisfying (1.14) at time $t \sim -\infty$. Then, we separate the analysis among three different time intervals: $t \ll -c^{-\frac{1}{2}}$, $|t| \leq c^{-\frac{1}{2}}$ and $c^{-\frac{1}{2}} \ll t$. On each interval the solution possesses a specific behavior which we briefly describe:

1. ($t \ll -c^{-\frac{1}{2}}$). In this interval of time we prove that $u(t)$ remains close to a 2-soliton solution with no changes on scaling and shift parameters. This result is possible for negative long enough times, such that both solitons are still far from each other, and is a consequence of [49].
2. ($|t| \leq c^{-\frac{1}{2}}$). This is the interval where solitons collision leads the dynamic of $u(t)$. The novelty in the method is the construction of an *approximate solution* of (1.21) with high

order of accuracy such that (a) at time $t \sim -c^{-\frac{1}{2}}$ this solution is close to a 2-soliton solution and therefore to $u(t)$, (b) it describes the 2-soliton collision in this interval, (c) at time $t \sim c^{-\frac{1}{2}}$, when solitons are sufficiently separated, it possesses an extra, nonzero, residual term product of the collision, and characterized by a number $d(\varepsilon) \neq 0$ (cf. (2.29)-(2.30)), and (d) it is possible to extend the solution $u(t)$ to the whole interval $[-c^{\frac{1}{2}}, c^{\frac{1}{2}}]$ being still close to our approximate solution, uniformly on time, modulo modulation on a translation parameter. This property confirms that our Ansatz is indeed the correct approximate solution describing the collision.

3. $(t \gg c^{-\frac{1}{2}})$ Here some *stability* properties (see Proposition 3.3) will be used to establish the convergence of the solution $u(t)$ to a 2-soliton solution with modified parameters. Moreover, by using a *monotony* argument, it will be possible to show that the residue appearing after the collision at time $t \sim c^{-\frac{1}{2}}$ is still present at infinity. This gives the conclusion of the Theorem.

The plan of this paper is as follows. In Section 2 we construct the aforementioned approximate solution and compute the error term produced in terms of a set of linear problems. Then we solve such linear systems and finally we give the first basic estimates concerning this solution. We finally prove that it is indeed close to a 2-soliton solution. In section 3 we construct an actual solution u close to the approximate solution for small times, and state some stability results to study the long time behavior of the solution u. Finally, in section 4, we prove Theorem using above results.

2 Construction of an approximate 2-soliton solution

The objective of this section is to construct an approximate solution of the gKdV equation (1.21), which will precisely describe the collision of two solitons. Hereafter, we assume the hypothesis of Theorem 1.4. We suppose both solitons are positive (the negative case, for $m = 3$, can be treated in the same way).

Secondly, note that Q and Q_c have *velocity* (and size) 1 and c respectively; so that working with $u(t, x+t)$ instead of $u(t, x)$ we can assume that the great soliton Q is fixed at $x = 0$ and the small soliton has velocity $c - 1 < 0$. Of course, $v(t, x) := u(t, x+t)$ satisfies now the *translated* equation

$$v_t + (v_{xx} - v + f(v))_x = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x. \tag{2.1}$$

Finally, denote

$$T_c := c^{-\frac{1}{2} - \frac{1}{100}} > 0. \tag{2.2}$$

This quantity can be understood as the time of interaction between the two solitons. The exponent $\frac{1}{100}$ can be replaced by any small positive number without relevant modifications.

The following result deals with the problem of describing the collision in the interval of time $[-T_c, T_c]$:

Proposition 2.1 (Construction of an approximate solution of the gKdV equation). *Let $m = 2, 3$ and f as in (1.22). There exist constants $c_0 = c_0(f) > 0$ and $K_0 = K_0(f)$ such that for all $0 < c < c_0$ there exists a function $\tilde{u} = \tilde{u}_{1,c}(t, x)$ such that the following hold:*

1. *Approximate solution on $[-T_c, T_c]$. For all $t \in [-T_c, T_c]$,*

$$\|\tilde{u}_t + (\tilde{u}_{xx} - \tilde{u} + f(\tilde{u}))_x\|_{H^2(\mathbb{R})} \leq K_0 c^{\frac{3}{m-1}+\frac{3}{4}}.$$

2. *Closeness to the sum of two solitons: For all time $t \in [-T_c, T_c]$, the function $\tilde{u}$ belongs to $H^1(\mathbb{R})$ and satisfies*

$$\|\tilde{u}(t) - Q(x-\alpha) - Q_c(x+(1-c)t)\|_{H^1(\mathbb{R})} \leq K_0 c^{\frac{1}{m-1}},$$

where $\alpha = \alpha(t,x)$ is a smooth bounded function, to be defined below, see (2.4).

Remark 2.1. The proof of this proposition requires several steps, starting in Subsection 2.1 to finally ending in Subsection 2.3, Proposition 2.10. However, the proof is intuitively clear to describe: our approximate solution will consists of a *linear combination* of a *nonlinear basis* well behaved under the gKdV flow, together a variable decomposition resembling the classical separation of variables from second order linear PDEs. This description was first introduced by Martel and Merle [53], [54].

First of all we explain how the approximate solution is composed. We follow [53].

2.1 Decomposition of the approximate solution

We look for $\tilde{u}(t,x)$, the approximate solution for (2.1), carring out a specific structure. We first introduce a set of indices, depending on the cases we deal with. Let

$$\Sigma_2 := \{(k,l) = (1,0),\ (1,1),\ (2,0),\ (2,1),\ (1,2),\ (3,0)\},$$

for the quadratic case ($m = 2$), and

$$\Sigma_3 := \{(k,l) = (1,0),\ (1,1),\ (2,0),\ (2,1),\ (3,0),\ (4,0)\},$$

for the cubic one ($m = 3$).

We recall now an order relation for indices $(k,l), (k',l') \in \Sigma_m$ introduced in [53]. We say that

$$(k',l') < (k,l) \quad \text{if and only if} \quad \begin{cases} k' < k \text{ and } l' \leq l, \text{ or} \\ k' \leq k \text{ and } l' < l. \end{cases} \tag{2.3}$$

We set two variables denoting the position of each soliton. For the small soliton, let

$$y_c := x + (1-c)t \quad \text{and} \quad R_c(t,x) := Q_c(y_c),$$

and for the great soliton,

$$y := x - \alpha(y_c) \quad \text{and} \quad R(t,x) := Q(y),$$

where for $(a_{k,l})_{(k,l)\in\Sigma_m}$,

$$\alpha(s) := \int_0^s \beta(s')ds', \quad \beta(s) := \sum_{(k,l)\in\Sigma_m} a_{k,l}\, c^l Q_c^k(s). \tag{2.4}$$

The correction term α is intended to describe the shift on the position of the great soliton. Note that α might be nonzero even in the integrable case, see (1.4). Moreover, in the quartic case $m = 4$, $\varepsilon = 0$, one has $|\alpha| \to +\infty$ as $c \to 0$, see [51]. Along this work α will be a bounded function, uniformly on c.

The form of $\tilde{u}(t,x)$ is, as it should be expected, the sum of the two soliton plus a correction term:

$$\tilde{u}(t,x) := Q(y) + Q_c(y_c) + W(t,x), \tag{2.5}$$

$$W(t,x) := \sum_{(k,l)\in\Sigma_m} c^l \left(Q_c^k(y_c) A_{k,l}(y) + (Q_c^k)'(y_c) B_{k,l}(y) \right), \tag{2.6}$$

where $a_{k,l}$, $A_{k,l}$, $B_{k,l}$ are unknowns to be determined.

The motivation in [53] for choosing W of the form (2.6) is precisely the closeness of the family of functions

$$\left\{ c^l Q_c^k,\ c^l (Q_c^k)',\ k \geq 1,\ l \geq 0 \right\} \tag{2.7}$$

under multiplication and differentiation, due to the specific form of the equation of Q_c (see Lemma 2.1 in [53]). In the case of equation (2.1), for a general nonlinearity this structure is preserved up to a lower order term (see Lemma L.2).

We want to measure the size of the error produced by inserting $\tilde{u}$ as defined in (2.5)-(2.6) in the equation (2.1). For this, let

$$S[\tilde{u}](t,x) := \tilde{u}_t + (\tilde{u}_{xx} - \tilde{u} + f(\tilde{u}))_x. \tag{2.8}$$

Our first result in the above direction is the following

Proposition 2.2 (Decomposition of $S(\tilde{u})$). *Let*

$$\mathcal{L}w := -w_{yy} + w - f'(Q)w. \tag{2.9}$$

Then,

$$\begin{aligned} S[\tilde{u}](t,x) = &\sum_{(k,l)\in\Sigma_m} c^l Q_c^k(y_c) \Big[a_{k,l}(-3Q + 2f(Q))'(y) - (\mathcal{L}A_{k,l})'(y) + F_{k,l}(y) \Big] \\ &+ \sum_{(k,l)\in\Sigma_m} c^l (Q_c^k)'(y_c) \Big[a_{k,l}(-3Q'')(y) + \left(3A''_{k,l} + f'(Q)A_{k,l} \right)(y) - (\mathcal{L}B_{k,l})'(y) + G_{k,l}(y) \Big] \\ &+ \mathcal{E}(t,x) \end{aligned}$$

where $F_{k,l}$, $G_{k,l}$ and $\mathcal{E}$ satisfy, for any $(k,l) \in \Sigma_m$,

(i) *Dependence property of $F_{k,l}$ and $G_{k,l}$: The expressions of $F_{k,l}$ and $G_{k,l}$ depend only on $(a_{k',l'})$, $(A_{k',l'})$, $(B_{k',l'})$ for $(k',l') < (k,l)$.*

(ii) *Parity property of $F_{k,l}$ and $G_{k,l}$: Assume that for any (k',l') such that $(k',l') < (k,l)$ $A_{k',l'}$ is even and $B_{k',l'}$ is odd, then $F_{k,l}$ is odd and $G_{k,l}$ is even.*

  Moreover, $F_{1,0} = (f'(Q))'$ and $G_{1,0} = f'(Q)$, and higher order terms are given in Appendix L.

(iii) *Estimate on $\mathcal{E}$: Assume both $(A_{k,l})$ and $(B_{k,l})$ bounded, and $(A'_{k,l})$, $(B'_{k,l}) \in \mathcal{Y}$ for $(k,l) \in \Sigma_m$. Then there exists $\kappa > 0$ such that for all $j = 0,1,2$, and for every $(t,x) \in [-T_c, T_c] \times \mathbb{R}$,*

$$|\partial_x^j \mathcal{E}(t,x)| \leq \kappa c^{m-1} Q_c(y_c).$$

Remark 2.2. Note that $(\mathcal{L}w)_y$, as defined in (2.9), represents the linear operator associated to the gKdV equation (2.1). Thus, the expression for $S[\tilde{u}]$ above stated can be seen as a generalization of the linearized gKdV equation, with the addition of some correction terms.

Proof. We postpone the proof of the Proposition 2.2, merely calculative, to Appendix L. We note that this Proposition has been already stated in [54], but here we will need an improved version, describing explicitly every term $F_{k,l}, G_{k,l}$ up to a fixed high order. For the details, see Appendix L. □

Note that if we want to improve the approximation $\tilde{u}$, the unknown functions $A_{k,l}$ and $B_{k,l}$ for a fixed (k,l) must be chosen satisfying a sort of modified linear gKdV system where the source terms are composed of preceding, well-known, $A_{k',l'}$ and $B_{k',l'}$ functions. Indeed, if we choose (formally) $A_{k,l}$ and $B_{k,l}$ such that for any $(k,l) \in \Sigma_m$

$$(\Omega_{k,l}) \qquad \begin{cases} (\mathcal{L}A_{k,l})' + a_{k,l}(3Q - 2f(Q))' = F_{k,l}, \\ (\mathcal{L}B_{k,l})' + 3a_{k,l}Q'' - 3A''_{k,l} - f'(Q)A_{k,l} = G_{k,l}, \end{cases}$$

then the error term will be reduced to the quantity

$$S[\tilde{u}] = \mathcal{E}(t,x).$$

Of course the solvability theory for the linear systems $(\Omega_{k,l})$ and the measure of this error term must be stated in a rigorous form. This will be established in the following section.

2.2 Resolution of linear systems $(\Omega_{k,l})$

First, we recall some preliminary notation and results from [53]. We denote by $\mathcal{Y}$ the set of C^∞ functions f such that

$$\forall j \in \mathbb{N},\ \exists K_j,\ r_j > 0,\ \forall x \in \mathbb{R}, \quad |f^{(j)}(x)| \leq K_j(1+|x|)^{r_j} e^{-|x|}. \tag{2.10}$$

We recall some well-known results concerning a *resonance* function and the operator $\mathcal{L}$.

Claim 12 ([54]). The function $\varphi(x) = -\frac{Q'(x)}{Q(x)}$ is odd and satisfies:

(i) $\lim_{x\to-\infty} \varphi(x) = -1$; $\lim_{x\to+\infty} \varphi(x) = 1$;

(ii) $\forall x \in \mathbb{R}$, $|\varphi'(x)| + |\varphi''(x)| + |\varphi^{(3)}(x)| \leq Ce^{-|x|}$.

(iii) $\varphi' \in \mathcal{Y}$, $(1-\varphi^2) \in \mathcal{Y}$.

Lemma 2.3 (Properties of $\mathcal{L}$, see [54]). *The operator $\mathcal{L}$ defined in $L^2(\mathbb{R})$ by* (2.9) *has domain $H^2(\mathbb{R})$, is self-adjoint and satisfies the following properties:*

(i) *There exist a unique $\lambda_0 > 0$, $\chi_0 \in H^1(\mathbb{R})$, $\chi_0 > 0$ such that $\mathcal{L}\chi_0 = -\lambda_0\chi_0$.*

(ii) *The kernel of $\mathcal{L}$ is $\{\lambda Q', \lambda \in \mathbb{R}\}$. Let $\Lambda Q := \frac{d}{dc}Q_{c|c=1}$, then $\mathcal{L}(\Lambda Q) = -Q$.*

(iii) *(Inverse) For all $h \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} hQ' = 0$, there exists a unique $\widetilde{h} \in H^2(\mathbb{R})$ such that $\int_{\mathbb{R}} \widetilde{h}Q' = 0$ and $\mathcal{L}\widetilde{h} = h$; moreover, if h is even (resp. odd), then $\widetilde{h}$ is even (resp. odd).*

(iv) *For $h \in H^2(\mathbb{R})$, $\mathcal{L}h \in \mathcal{Y}$ implies $h \in \mathcal{Y}$.*

(v) *(Coercivity) If $\frac{d}{d\tilde{c}}\int_{\mathbb{R}} Q^2_{\tilde{c}|\tilde{c}=c} > 0$ then there exists $\lambda_c > 0$ such that if*

$$\int_{\mathbb{R}} wQ_c = \int_{\mathbb{R}} wQ'_c = 0 \quad \textit{then} \quad \int_{\mathbb{R}} (w_x^2 + cw^2 - f'(Q_c)w^2) \geq \lambda_c \int_{\mathbb{R}} w^2.$$

(vi) *There exist unique even solutions P and $\bar{P}$ of the ordinary differential equations*

$$\mathcal{L}P = 3Q'' + f'(Q)Q, \quad P \in \mathcal{Y}, \tag{2.11}$$

$$\mathcal{L}\bar{P} = f'(Q), \quad \bar{P} \in \mathcal{Y}. \tag{2.12}$$

Moreover, $P := -(xQ' + \Lambda Q + Q)$.

Remark 2.3. Item (vi) from above Lemma is new; the proof follows directly from (ii), (iii) and (iv). On the other hand, for general nonlinearities $\bar{P}$ is not explicit.

2.2.1 Existence theory for a model problem

We recall that linear systems $(\Omega_{k,l})$ are very similar and then proving existence reduces to prove the result for a model problem. This idea comes from [53], but we will need a simplified version, from [58].

Proposition 2.4 (Existence for a model problem, see [54])**.** *Let $F \in \mathcal{Y}$, odd, and $G \in \mathcal{Y}$, even. Let $\gamma, \kappa \in \mathbb{R}$. Then, there exist $a, b \in \mathbb{R}$, $\widetilde{A} \in \mathcal{Y}$ even, and $\widetilde{B} \in \mathcal{Y}$ odd, such that*

$$A = \widetilde{A} + \gamma, \quad \textit{and} \quad B = \widetilde{B} + b\varphi + \kappa Q'$$

satisfy

$$(\Omega) \quad \begin{cases} (\mathcal{L}A)' + a(3Q - 2f(Q))' = F, \\ (\mathcal{L}B)' + 3aQ'' - 3A'' - f'(Q)A = G \end{cases}$$

Moreover,

$$a = \frac{-1}{\int_{\mathbb{R}} \Lambda Q Q} \Big\{ \gamma \int_{\mathbb{R}} P + \int_{\mathbb{R}} GQ - \int_{\mathbb{R}} F \int_0^x P \Big\} \tag{2.13}$$

and

$$b = \frac{1}{2} \Big[\gamma \int_{\mathbb{R}} \bar{P} + a \int_{\mathbb{R}} \Lambda Q - \int_{\mathbb{R}} F \int_0^x \bar{P} + \int_{\mathbb{R}} G \Big]. \tag{2.14}$$

Proof. We give a sketch of the proof for the sake of completeness. The original result comes from [54], and here it is even simpler since we deal only with $F, G \in \mathcal{Y}$.

Set $A := \widetilde{A} + \gamma$, $B := \widetilde{B} + b\varphi$, where γ is given, while b is a parameter to be found. Since $(\mathcal{L}1)' = (1 - f(Q))' = -(f(Q))'$, we obtain the following system for $\widetilde{A}$, $\widetilde{B}$:

$$\begin{cases} (\mathcal{L}\widetilde{A})' + a(3Q - 2f(Q))' = F + \gamma(f(Q))', \\ (\mathcal{L}\widetilde{B})' + 3aQ'' - 3\widetilde{A}'' - f'(Q)\widetilde{A} = G + \gamma f'(Q) - b(\mathcal{L}\varphi)'. \end{cases}$$

Note that $F \in \mathcal{Y}$ is odd, therefore $\mathcal{H}(x) = \int_{-\infty}^x F(z)dz + \gamma f(Q)$ belong to $\mathcal{Y}$ and is even. By integration of the first line, we are reduced to solve

$$\begin{cases} \mathcal{L}\widetilde{A} + a(3Q - 2f(Q)) = \mathcal{H}, \\ (\mathcal{L}\widetilde{B})' + 3aQ'' - 3\widetilde{A}'' - f'(Q)\widetilde{A} = G + \gamma f'(Q) - b(\mathcal{L}\varphi)'. \end{cases}$$

Since $\int_{\mathbb{R}} \mathcal{H}Q' = 0$ (by parity) and $\mathcal{H} \in \mathcal{Y}$, by Lemma 2.3, there exists $\overline{H} \in \mathcal{Y}$, even, such that $\mathcal{L}\overline{H} = \mathcal{H}$.

Define $\hat{P}$ to be the unique even solution of

$$\mathcal{L}\hat{P} = 3Q - 2f(Q), \quad \hat{P} \in \mathcal{Y}.$$

Indeed, $\hat{P}$ has an explicit formula

$$\hat{P} = -(xQ' + \Lambda Q), \quad \text{with } \mathcal{L}(\Lambda Q) = -Q. \tag{2.15}$$

It follows that $\widetilde{A} := -a\hat{P} + \overline{H}$ is even, belongs to $\mathcal{Y}$ and solves the first line of the previous system. Note that at this stage, the parameters a and b are still free.

Now, we only need to find $\widetilde{B} \in \mathcal{Y}$, odd, such that $(\mathcal{L}\widetilde{B})' = -aZ_0 + D - b(\mathcal{L}\varphi)'$, where

$$D := 3\overline{H}'' + f'(Q)\overline{H} + G + \gamma f'(Q) \in \mathcal{Y}, \text{ even, } Z_0 := 3Q'' + 3\hat{P}'' + f'(Q)\hat{P} \in \mathcal{Y}, \text{ even.}$$

Let

$$E := \int_0^x (D - aZ_0)(z)dz - b\mathcal{L}\varphi.$$

This function *a priori* is in $L^\infty(\mathbb{R})$, independent of a, b.

Claim 13. There exist numbers a and b such that $E \in \mathcal{Y}$ and $\displaystyle\int_{\mathbb{R}} EQ' = 0$.

Assuming Claim 13, we fix a, b so that $E \in \mathcal{Y}$ and $\int_{\mathbb{R}} EQ' = 0$. It follows from Lemma 2.3 that there exists $\widetilde{B} \in \mathcal{Y}$, odd, such that $\mathcal{L}\widetilde{B} = E$. The final solution is then given by $A := \widetilde{A} + \gamma$ and $B := \widetilde{B} + b\varphi + \kappa Q'$, where κ is a free parameter, because $\mathcal{L}Q' = 0$ (see Lemma 2.3, (ii)).

Proof of Claim 13. First, we check a sort of non-degeneracy condition, namely that $\int_{\mathbb{R}} Z_0 Q \neq 0$. Indeed, by (2.11)

$$\int_{\mathbb{R}} Z_0 Q = -3\int_{\mathbb{R}} Q'^2 + \int_{\mathbb{R}} \mathcal{L}P\hat{P} = -3\int_{\mathbb{R}} Q'^2 + \int_{\mathbb{R}} P(3Q - 2f(Q)).$$

We recall now the following auxiliary result.

Claim 14 ([54], Claim 2.2). We have

$$3\int_{\mathbb{R}} Q'^2 - \int_{\mathbb{R}} (3Q - 2f(Q))P = \int_{\mathbb{R}} \Lambda Q Q \neq 0.$$

Remark 2.4. Indeed,

$$\int_{\mathbb{R}} \Lambda Q Q = \frac{1}{2}\partial_c \int_{\mathbb{R}} Q_c^2 \Big|_{c=1} > 0,$$

thanks to (1.12) provided ε small enough (independent of c).

Let us continue with the proof of Claim 13. By the preceding result, it suffices to choose $a := \dfrac{\int_{\mathbb{R}} DQ}{\int_{\mathbb{R}} Z_0 Q}$, and $b := \displaystyle\int_0^{+\infty} (D - aZ_0)(z)dz$ (note that $\lim_{\pm\infty} \mathcal{L}\varphi = \lim_{\pm\infty} \varphi = \pm 1$). This finishes the proof of Claim 13. □

We return to the proof of Proposition 2.4. Now we find the constants a and b in terms of known quantities in (Ω). First, we multiply the equation of B by Q and use $\mathcal{L}Q' = 0$. We get

$$\begin{aligned} -3a\int_{\mathbb{R}} Q'^2 &= \int_{\mathbb{R}} (3Q'' + f'(Q)Q)A + \int_{\mathbb{R}} GQ \\ &= \int_{\mathbb{R}} (\mathcal{L}A)P + \int_{\mathbb{R}} GQ. \end{aligned}$$

Second, we multiply the equation of A by $\int_0^x P(s)\,ds$. We obtain

$$\begin{aligned}\int_{\mathbb{R}} (\mathcal{L}A)' \int_0^x P &= -\int_{\mathbb{R}} (\mathcal{L}A)P + \gamma \int_{\mathbb{R}} P \\ &= a \int_{\mathbb{R}} (3Q - 2f(Q))P + \int_{\mathbb{R}} F \int_0^x P.\end{aligned}$$

Thus, combining the two identities, we get:

$$-a\left\{3\int_{\mathbb{R}} Q'^2 - \int_{\mathbb{R}} (3Q - 2f(Q))P\right\} = \gamma \int_{\mathbb{R}} P + \int_{\mathbb{R}} GQ - \int_{\mathbb{R}} F \int_0^x P.$$

and the expression for a follows from Claim 14.

To find out b, we integrate the equation for B in (Ω) over $\mathbb{R}$ to obtain

$$2b = \int_{\mathbb{R}} f'(Q)A + \int_{\mathbb{R}} G. \tag{2.16}$$

Now we consider $\bar{P}$ the function defined in (2.12). We multiply the equation for A by $\int_0^x \bar{P}(s)\,ds$ and then we integrate. We get

$$\int_{\mathbb{R}} f'(Q)A = \gamma \int_{\mathbb{R}} \bar{P} - a \int_{\mathbb{R}} \bar{P}(3Q - 2f(Q)) - \int_{\mathbb{R}} F \int_0^x \bar{P}.$$

Now, note that

$$\int_{\mathbb{R}} \bar{P}(3Q - 2f(Q)) = \int_{\mathbb{R}} \mathcal{L}\hat{P}\bar{P} = \int_{\mathbb{R}} \hat{P} f'(Q) = \int_{\mathbb{R}} \hat{P}(1 - \mathcal{L}1) = \int_{\mathbb{R}} \hat{P} - \int_{\mathbb{R}} (3Q - 2f(Q)).$$

From (2.15) we replace the explicit value of $\hat{P}$ and we use the equation satisfied by Q, namely $Q'' - Q + f(Q) = 0$, to obtain

$$\int_{\mathbb{R}} \bar{P}(3Q - 2f(Q)) = -\int_{\mathbb{R}} \Lambda Q.$$

With a previously known we replace this quantity in (2.16) to obtain (2.14). This finishes the proof.

□

We have now a good solvability theory for the linear systems $(\Omega_{k,l})$, that avoids the emergency of linearly growing solutions at this order. As an example, the general theory constructed in [53] for the quartic KdV equation deals with possibly growing solutions, see [53] Proposition 2.3.

Here, for each system $(\Omega_{k,l})$, $(k,l) \in \Sigma_m$, we will look for solutions such that

$$A_{k,l} = \widetilde{A}_{k,l} + \gamma_{k,l}, \quad B_{k,l} = \widetilde{B}_{k,l} + b_{k,l}\varphi + \kappa_{k,l}Q', \quad a_{k,l}, b_{k,l}, \kappa_{k,l} \in \mathbb{R}; \tag{2.17}$$

where $\widetilde{A}_{k,l} \in \mathcal{Y}$ is even and $\widetilde{B}_{k,l} \in \mathcal{Y}$ is odd. (see Proposition 2.10 for a justification of this choice).

This election will have several good properties, but we will emphasize a crucial one. Let $(k,l) \in \Sigma_m$ fixed. We say that (k,l) satisfies the **(IP)** property (**IP** = important property) if and only if

$$(\mathbf{IP})\begin{cases}\text{Any derivative of } A_{k,l} \text{ or } B_{k,l} \text{ is a localized } \mathcal{Y}\text{-function.}\\ \text{Moreover, for } (k,l) = (1,0) \text{ we have } A_{1,0} \in \mathcal{Y}.\end{cases}$$

This property, although depending on the specific pair (k, l), will be useful to quickly discard localized terms composing $F_{k,l}, G_{k,l}$, and seeing essentially the bounded but non localized terms. Indeed, note that thanks to Claim 12 any solution as in (2.17) satisfies this property. For the details, see Appendix L.

We start by solving the first system.

2.2.2 Resolution of the system $(\Omega_{1,0})$

From Proposition 2.2 (ii) the system $(\Omega_{1,0})$ is given by

$$(\mathcal{L}A_{1,0})' = -a_{1,0}(3Q - 2f(Q))' + (f'(Q))', \tag{2.18}$$
$$(\mathcal{L}B_{1,0})' = 3A''_{1,0} + f'(Q)A_{1,0} - 3a_{1,0}Q'' + f'(Q) \tag{2.19}$$

This first system is easily solvable, as shows the following

Lemma 2.5 (Resolution of $(\Omega_{1,0})$). *There exists a solution $(A_{1,0}, B_{1,0}, a_{1,0})$ of (2.18)-(2.19) of the form (2.17) and such that $A_{1,0} \in \mathcal{Y}$ is even (and $\gamma_{1,0} = 0$), $B_{1,0}$ is odd and $a_{1,0}, b_{1,0}$ are given by the formulae*

$$a_{1,0} = \frac{\int_{\mathbb{R}} \Lambda Q}{\int_{\mathbb{R}} \Lambda Q Q}, \quad b_{1,0} = \frac{1}{2} a_{1,0} \int_{\mathbb{R}} \Lambda Q + \frac{1}{2} \int_{\mathbb{R}} \bar{P}. \tag{2.20}$$

Moreover, $A_{1,0}$ is given by

$$A_{1,0} = \bar{P} - a_{1,0}\hat{P}. \tag{2.21}$$

(cf. (2.18), (2.15) and (2.12)). Finally, we choose $B_{1,0}$ such that $\displaystyle\int_{\mathbb{R}} Q' B_{1,0} = 0$.

Remark 2.5. Note that from the value of $\hat{P} = -(xQ' + \Lambda Q)$ and (2.21) we get

$$b_{1,0} = \frac{1}{2}\Big[a_{1,0} \int_{\mathbb{R}} Q + \int_{\mathbb{R}} A_{1,0}\Big]. \tag{2.22}$$

Proof. Note that both $(f'(Q))'$ and $f'(Q)$ are odd and even $\mathcal{Y}$-functions respectively, so thanks to Proposition 2.4, a solution with the desired properties does exist. We will chose $\gamma_{1,0} := 0$. The value of $a_{1,0}$ and $b_{1,0}$ comes from (2.13)-(2.14), after some simple computations. These computations have been carried out in [54], but by completeness we rewrite them. Indeed, note that we only need to verify that

$$\int_{\mathbb{R}} f'(Q)(Q + P) = -\int_{\mathbb{R}} \Lambda Q.$$

In fact, from (2.11), the explicit value of P and Claim 2.3 (ii), we have

$$\int_{\mathbb{R}} f'(Q)(Q+P) = -\int_{\mathbb{R}} f'(Q)(xQ' + \Lambda Q) = \int_{\mathbb{R}} f(Q) - \int_{\mathbb{R}} (1 - \mathcal{L}1)\Lambda Q = \int_{\mathbb{R}} (f(Q) - Q) - \int_{\mathbb{R}} \Lambda Q,$$

but $f(Q) - Q = -Q''$, so we are done.

On the other hand, note that $\mathcal{L}(1 + \bar{P}) = 1$, thus

$$\int_{\mathbb{R}} f'(Q)(\bar{P} + 1) = \int_{\mathbb{R}} \mathcal{L}\bar{P}(1 + \bar{P}) = \int_{\mathbb{R}} \bar{P}1.$$

This give finally the expected value of $b_{1,0}$.

Finally, the constant $\kappa_{1,0}$ in the expression of $B_{1,0}$ is a free parameter that we will fix such that $\int_{\mathbb{R}} B_{1,0}Q' = 0$ for convenience in some future computations (see Proposition 2.4 and (M.43) in Appendix M). We have

$$0 = \int_{\mathbb{R}} Q'B_{1,0} = \int_{\mathbb{R}} \widetilde{B}_{1,0}Q' + b_{1,0}\int_{\mathbb{R}} \varphi Q' + \kappa_{1,0}\int_{\mathbb{R}} Q'^2.$$

where we can obtain $\kappa_{1,0}$. □

2.2.3 Resolution of the system $(\Omega_{2,0})$

From Proposition L.1 (iii) in Appendix L, the system $(\Omega_{2,0})$ is given by

$$(\mathcal{L}A_{2,0})' = a_{2,0}(3Q - 2f(Q))' + F_{2,0}, \tag{2.23}$$
$$(\mathcal{L}B_{2,0})' = 3A''_{2,0} + f'(Q)A_{2,0} - 3a_{2,0}Q'' + G_{2,0} \tag{2.24}$$

where the source terms are given by

1. Case $m = 2$,

$$\begin{aligned} F_{2,0} \quad = \quad & -(3A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0}) + \frac{1}{2}(f''(Q)(2A_{1,0} + A^2_{1,0}))' \\ & -a_{1,0}(3A''_{1,0} - Q + f'(Q)(1 + A_{1,0}))' + 3a^2_{1,0}Q^{(3)} + \frac{1}{2}(f''(Q) - 2)', \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} G_{2,0} \quad = \quad & \frac{1}{2}(f''(Q) - 2) - (A_{1,0} + 3B'_{1,0}) + \frac{1}{2}f''(Q)(2A_{1,0} + A^2_{1,0}) + \frac{3}{2}a^2_{1,0}Q'' \\ & -\frac{1}{2}a_{1,0}(9A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0})' + \frac{1}{2}(f''(Q)(B_{1,0} + A_{1,0}B_{1,0}))'. \end{aligned} \tag{2.26}$$

2. Case $m = 3$,

$$F_{2,0} = (\frac{1}{2}f''(Q)(1 + A_{1,0})^2)' + 3a^2_{1,0}Q^{(3)} - a_{1,0}(f'(Q) + 3A''_{1,0} + f'(Q)A_{1,0})' \tag{2.27}$$

and

$$\begin{aligned} G_{2,0} \quad = \quad & \frac{1}{2}f''(Q)(1 + A_{1,0})^2 + \frac{3}{2}a^2_{1,0}Q'' - \frac{1}{2}a_{1,0}(9A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0})' \\ & +\frac{1}{2}(f''(Q)(1 + A_{1,0})B_{1,0})'. \end{aligned} \tag{2.28}$$

Proposition 2.6 (Resolution of $(\Omega_{2,0})$)**.** *Let f be as in (1.22). There exists a constant $\varepsilon_0 > 0$ not depending on c such that the following holds.*

1. *(Case $m = 2$) There exists a solution $(A_{2,0}, B_{2,0}, a_{2,0})$ of $(\Omega_{2,0})$ satisfying (2.17) and such that*

$$\lim_{+\infty} A_{2,0} = -\frac{1}{2}b^2_{1,0} = \gamma_{2,0}, \quad A_{2,0} - \gamma_{2,0} \in \mathcal{Y},$$
$$\lim_{+\infty} B_{2,0} = b_{2,0}, \quad B_{2,0} - b_{2,0}\varphi \in \mathcal{Y},$$

but for all $|\varepsilon| \in (0, \varepsilon_0)$

$$d(\varepsilon) := b_{2,0}(f) + \frac{1}{6}b^3_{1,0}(f) = c_{2,p}\varepsilon + o(\varepsilon), \quad \textit{with} \quad c_{2,p} \neq 0 \ \textit{for all} \ p \geq 4. \tag{2.29}$$

2. *(Case $m = 3$) There exists a solution $(A_{2,0}, B_{2,0}, a_{2,0})$ of $(\Omega_{2,0})$ such that $A_{2,0} \in \mathcal{Y}$ is even, $B_{2,0}$ is bounded, odd and*

$$\lim_{+\infty} B_{2,0} = b_{2,0}, \quad B_{2,0} - b_{2,0}\varphi \in \mathcal{Y},$$

but for all $|\varepsilon| \in (0, \varepsilon_0)$,

$$d(\varepsilon) := b_{2,0}(f) = c_{3,p}\varepsilon + o(\varepsilon), \quad \textit{and} \quad c_{3,p} \neq 0 \ \textit{for all} \ p \geq 4. \tag{2.30}$$

Moreover, in both cases the solution found satisfies **(IP)**.

Remark 2.6. Note that in the case $m = 2$, one has $c_{2,p} = 0$ for $p = 3$ (see (2.33)). This cancelation is consequence of the complete integrability of the Gardner equation.

Proof. Note that in both cases, $m = 2$ and $m = 3$ the source terms $F_{2,0}, G_{2,0}$ belongs to $\mathcal{Y}$, with the former being an odd function and the last one being even. Thus the existence of solutions to (2.23)-(2.24) with the desired properties follows directly from Proposition 2.4 above.

In particular we will choose $\gamma_{2,0} := -\frac{1}{2}b_{1,0}^2$ for the quadratic case and $\gamma_{2,0} := 0$ in the cubic one.

Let us now check that, being fixed $\gamma_{1,0}, a_{1,0}, b_{1,0}$ and $\gamma_{2,0}$, the value of $a_{2,0}$ and $b_{2,0}$ is uniquely determined. Indeed, from (2.13)-(2.14)

$$a_{2,0} = -\frac{1}{\int_{\mathbb{R}} \Lambda Q Q}\Big[\gamma_{2,0}\int_{\mathbb{R}} P + \int_{\mathbb{R}} G_{2,0}Q - \int_{\mathbb{R}} F_{2,0}\int_0^x P\Big], \tag{2.31}$$

and

$$b_{2,0} = \frac{1}{2}\Big[\gamma_{2,0}\int_{\mathbb{R}} \bar{P} + a_{2,0}\int_{\mathbb{R}} \Lambda Q - \int_{\mathbb{R}} F_{2,0}\int_0^x \bar{P} + \int_{\mathbb{R}} G_{2,0}\Big]. \tag{2.32}$$

We claim (2.29) and (2.30) with

$$c_{2,p} := -\Big[\frac{(p-3)(2p-1)(24-23p+3p^2+2p^3)}{36(p^2-1)(p-2)}\Big]\int_{\mathbb{R}}\Big[\frac{3}{2\cosh^2(\frac{1}{2}x)}\Big]^p, \tag{2.33}$$

and

$$c_{3,p} := -\Big[\frac{(p-1)(p-3)(p^2-3p+8)}{8(p-2)(p+1)}\Big]\int_{\mathbb{R}}\Big[\frac{\sqrt{2}}{\cosh x}\Big]^p. \tag{2.34}$$

The end of the proof of (2.29)-(2.30), and (2.33)-(2.34) is a lengthy but straightforward computation. For the sake of continuity we postpose the proof to Appendix M. □

Remark 2.7. An explicit expression for $d(\varepsilon)$ for any nonlinearity has escaped to us (see Claim 22), and we only have in our hands an asymptotic expression for small values of ε. We believe, however, that it may exist a –necessarily– large ε_0 for which $d(\varepsilon_0) = 0$, and even more, a pure 2-soliton solution may exist at any order.

Remark 2.8. The expressions (2.29)-(2.30) above say roughly speaking that the second order linear system $(\Omega_{2,0})$ has a solution that does not obey (at third order derivatives) the Taylor expansion of a small soliton shifted. Indeed,

$$Q_c(y_c + b_{1,0}\varphi) \sim Q_c(y_c) + b_{1,0}\varphi Q_c'(y_c) + \frac{1}{2}b_{1,0}^2 Q_c''(y_c) + \frac{1}{6}b_{1,0}^3 Q_c^{(3)}(y_c).$$

Note that (cf. (L.4) and (L.5))

$$Q_c''(y_c) \sim cQ_c(y_c) - Q_c^m(y_c), \quad Q_c^{(3)}(y_c) \sim cQ_c'(y_c) - (Q_c^m)'(y_c),$$

and thus for a perfect collision we should have $b_{2,0} = -\frac{1}{6}b_{1,0}^3$ for $m = 2$ and $b_{2,0} = 0$ for $m = 3$, as in the integrable cases. This formal discussion will be justified in the proof of Proposition 2.10.

2.2.4 Resolution of system $(\Omega_{1,1})$, cases $m = 2, 3$

Now we consider the first mixed system, $(\Omega_{1,1})$. Note that this system has a different order depending on the power of leading nonlinearity: for $m = 2$, cQ_c is of quadratic order in Q_c, meanwhile, in the cubic one, cQ_c is a term of cubic order.

From Proposition L.1 the system $(\Omega_{1,1})$ is given by

$$(\mathcal{L}A_{1,1})' = a_{1,1}(3Q - 2f(Q))' + (3A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0}), \tag{2.35}$$
$$(\mathcal{L}B_{1,1})' = 3A''_{1,1} + f'(Q)A_{1,1} - 3a_{1,1}Q'' + 3B'_{1,0}. \tag{2.36}$$

For this system, we recall its source terms

$$F_{1,1} := 3A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0}, \quad G_{1,1} := 3B'_{1,0}. \tag{2.37}$$

Note that as $(k, l) = (1, 0)$ satisfies the **(IP)** property, we have both $F_{1,1}, G_{1,1} \in \mathcal{Y}$.

Lemma 2.7 (Resolution of $(\Omega_{1,1})$, $m = 2, 3$)**.** *There exists a solution $(A_{1,1}, B_{1,1}, a_{1,1})$ of $(\Omega_{1,1})$ such that $A_{1,1}$ is even, $B_{1,1}$ is odd and*

$$\lim_{+\infty} A_{1,1} = \gamma_{1,1} := \frac{1}{2}b_{1,0}^2, \quad A_{1,1} - \gamma_{1,1} \in \mathcal{Y},$$
$$\lim_{+\infty} B_{1,1} = b_{1,1}, \quad B_{1,1} - b_{1,1}\varphi \in \mathcal{Y}.$$

Besides, this solution implies that **(IP)** *holds for $(k, l) = (1, 1)$.*

Proof. From Proposition 2.2, it is clear that $F_{1,1}$ and $G_{1,1}$ given in (2.37) satisfy the assumptions of Proposition 2.4. The choice of $\gamma_{1,1}$ will be justified in Proposition 2.10. In the rest of this paper, we will not need the expression of $b_{1,1}$ (note that it would be possible to compute it as in the proof of Proposition 2.6). □

2.2.5 Resolution of high order systems, quadratic case

From now on, we consider the triplet

$$(A_{k,l}, B_{k,l}, a_{k,l})$$

defined for all $(k, l) \in \Sigma_m$, $1 \le k + l \le 2$ in Lemma 2.5, Proposition 2.6 and Lemma 2.7. We now solve the systems $(\Omega_{k,l})$ for $k + l = 3$. Denote $\delta_{33} := 1$ and $\delta_{p3} := 0$ for $p \ge 4$.

Lemma 2.8 (Resolution of $(\Omega_{k,l})$ for $k + l = 3$ and $m = 2$)**.** *For all $(k, l) \in \Sigma_2$ such that $k + l = 3$, $F_{k,l}$ is odd and $G_{k,l}$ even; both are in the class $\mathcal{Y}$, and there exists a solution $(A_{k,l}, B_{k,l}, a_{k,l})$ of $(\Omega_{k,l})$ such that $A_{k,l}$ is even, $B_{k,l}$ is odd and*

$$\lim_{+\infty} A_{k,l} = \gamma_{k,l}, \quad A_{k,l} - \gamma_{k,l} \in \mathcal{Y},$$
$$\lim_{+\infty} B_{k,l} = b_{k,l}, \quad B_{k,l} - b_{k,l}\varphi \in \mathcal{Y}.$$

Moreover, we will choose the particular values

$$\gamma_{3,0} := \frac{5}{36}b_{1,0}^4 + \frac{10}{3}d(\varepsilon)b_{1,0} + \frac{1}{2}\mu(\varepsilon)b_{1,0}^2, \quad \gamma_{2,1} := \frac{1}{24}b_{1,0}^4 - b_{1,0}b_{1,1} - 4d(\varepsilon)b_{1,0},$$
$$\gamma_{1,2} := -\frac{3}{24}b_{1,0}^4 + b_{1,0}b_{1,1},$$

where $d(\varepsilon)$ satisfies (2.29)-(2.30).

Proof. The proof of this result is easy after the validity of the following claim:

For all $(k,l) \in \Sigma_2$ such that $k+l=3$, we have $F_{k,l} \in \mathcal{Y}$ is odd, $G_{k,l} \in \mathcal{Y}$ is even. (2.38)

Assuming (2.38), Lemma 2.8 is a direct consequence of Proposition 2.4.

Let us prove (2.38). From the Appendix L and Proposition L.1 several (bounded but) nonlocalized terms appear in the expression of $F_{k,l}$ and $G_{k,l}$ for $k+l=3$, but **all these terms eventually cancel**.

Indeed, thanks to the **(IP)** property, terms containing **derivatives** of $B_{1,0}$, $A_{1,1}$ and $A_{2,0}$ are in $\mathcal{Y}$ as well as terms of the kind $f'(Q)B_{1,0}$ and so on. Thus, we focus on the terms containing only $B_{1,0}$, $A_{1,1}$ and $A_{2,0}$ without derivatives nor multiplication by functions of Q. Note also that $A_{1,0} \in \mathcal{Y}$, so we also discard it. For simplicity of notation, we will skip the variables y_c and y.

Now, we recollect all the non-localized terms (due to $B_{1,0}$, $A_{1,1}$ and $A_{2,0}$) in $S[\tilde{u}]$ of order $c^l Q_c^k$ or $c^l (Q_c^k)'$ with $k+l=3$. We have only three cases: the pairs $(3,0)$, $(2,1)$ and $(1,2)$. From Proposition L.1 we obtain

1. (Case $(3,0)$). Here
$$F_{3,0} = \widetilde{F}_{3,0}, \quad G_{3,0} = \widetilde{G}_{3,0} - \frac{2}{3}(B_{1,0}^2 + 2A_{2,0}), \text{ with } \widetilde{F}_{3,0}, \widetilde{G}_{3,0} \in \mathcal{Y};$$
2. (Case $(2,1)$). Here
$$F_{2,1} = \widetilde{F}_{2,1}, \quad G_{2,1} = \widetilde{G}_{2,1} + (B_{1,0}^2 + A_{1,1} + 3A_{2,0}), \text{ with } \widetilde{F}_{2,1}, \widetilde{G}_{2,1} \in \mathcal{Y};$$
3. (Case $(1,2)$). Here $F_{1,2}, G_{1,2} \in \mathcal{Y}$.

Using the following relations among the limits of $A_{2,0}$, $A_{1,1}$ and $B_{1,0}^2$ at $\pm\infty$ (see Proposition 2.6 and Lemma 2.7):

$$\lim_{\pm\infty} A_{2,0} = -\frac{1}{2}\lim_{\pm\infty} B_{1,0}^2, \quad \lim_{\pm\infty} A_{1,1} = -\lim_{\pm\infty} A_{2,0},$$

we observe that the source functions in $(\Omega_{k,l})$ are in fact all localized. This proves (2.38). □

2.2.6 Resolution of high order systems, cubic case

Finally we claim the existence of bounded solutions for the *third* and *fourth* order systems in the cubic case. The proof of these results is identical to the previous Lemma.

Lemma 2.9 (Resolution of $(\Omega_{3,0})$, $(\Omega_{4,0})$ and $(\Omega_{2,1})$ for $m=3$)**.** *For all $(k,l) \in \Sigma_3$ with $k \geq 2$ there exists a solution $(A_{k,l}, B_{k,l}, a_{k,l})$ of $(\Omega_{k,l})$ such that $A_{k,l}$ is even, $B_{k,l}$ is odd and*

$$\lim_{+\infty} A_{k,l} = \gamma_{k,l}, \quad A_{k,l} - \gamma_{k,l} \in \mathcal{Y},$$
$$\lim_{+\infty} B_{k,l} = b_{k,l}, \quad B_{k,l} - b_{k,l}\varphi \in \mathcal{Y}.$$

In particular, we choose

$$\gamma_{3,0} := -\frac{1}{2}b_{1,0}^2, \ \gamma_{2,1} := -4b_{1,0}d(\varepsilon), \ \gamma_{4,0} := 3d(\varepsilon)b_{1,0} + \frac{1}{2}\varepsilon b_{1,0}^2\delta_{p4}. \tag{2.39}$$

In this case $d(\varepsilon) := b_{2,0}(\varepsilon)$ (cf. (2.30)).

Proof. We note that, thanks to the **(IP)** property and Proposition L.1, the only a priori non localized source term is

$$G_{2,1} = 3A_{2,0} + \tilde{G}_{2,0}, \text{ with } \tilde{G}_{2,0} \in \mathcal{Y}.$$

Then the conclusion of the Lemma follows from the fact that, from Proposition 2.6, in the cubic case, we have a priori chosen $A_{2,0} \in \mathcal{Y}$. □

For further purposes, we recall the important quantities (see (2.2) and (2.29)-(2.30))

$$T_c = c^{-\frac{1}{2}-\frac{1}{100}}, \quad d(\varepsilon) = b_{2,0}(\varepsilon) + \frac{1}{6} b_{1,0}^3(\varepsilon)\delta_{m2}, \tag{2.40}$$

with $\delta_{m2} = 0$ for $m = 3$, and $\delta_{22} = 1$.

2.3 Recomposition of the approximate solution. Proof of Proposition 2.1

Having solved several linear systems we now are able to prove Proposition 2.1. Indeed, we have now the enough knowlegde about the notation, so we can go further and claim the following improved result on $\tilde{u}$.

Proposition 2.10 (Construction of a symmetric approximate solution of gKdV, improved version)**.** *The solution $\tilde{u}$ above constructed satisfies, for any $0 < c < c_0$, the following properties:*

1. *For all (t, x) $\tilde{u}(t, x) = \tilde{u}(-t, -x)$.*
2. *For every time $t \in [-T_c, T_c]$,*
$$\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} \leq K c^{\frac{3}{m-1}+\frac{3}{4}}. \tag{2.41}$$
3. *Closeness to the sum of two soliton solution: For all time $t \in [-T_c, T_c]$, the function $\tilde{u}$ is in $H^1(\mathbb{R})$ and satisfies the estimate*
$$\|\tilde{u}(t) - Q(y) - Q_c(y_c)\|_{H^1(\mathbb{R})} \leq K_0 c^{\frac{1}{m-1}}. \tag{2.42}$$
4. *Closeness to a shifted two soliton solution plus a strange term: Denote*
$$\Delta_1 := \sum_{(k,l)\in\Sigma_m} a_{k,l}\, c^l \int_{\mathbb{R}} Q_c^k, \quad \tilde{b}_{1,1} := b_{1,1} - \frac{1}{6} b_{1,0}^3, \quad \Delta_2 := 2(b_{1,0} + c\tilde{b}_{1,1}\delta_{m2}). \tag{2.43}$$
Then $\tilde{u}$ satisfies at time $\pm T_c$
$$\begin{aligned} &\|\tilde{u}(\pm T_c) - Q(\cdot \mp \frac{1}{2}\Delta_1) - Q_c(\cdot \pm (1-c)T_c \mp \frac{1}{2}\Delta_2) \\ &\quad \pm d(\varepsilon)(Q_c^2)'(\cdot \pm (1-c)T_c \mp \frac{1}{2}\Delta_2)\}\|_{H^1(\mathbb{R})} \leq K c^{\frac{3}{m-1}+\frac{1}{4}}, \end{aligned} \tag{2.44}$$
provided for each $(k, l) \in \Sigma_m$, the constants $\gamma_{k,l}$ must be chosen as in Lemma 2.5, Proposition 2.6, Lemmas 2.7, 2.8 and 2.9. Recall that $d(\varepsilon)$ satisfies (2.29)-(2.30).

Remark 2.9. The quantity $\bar{b}_{1,1}$ in (2.43) represents the difference between the expected value of $b_{1,1}$ given by the integrable case and the actual one; namely, for $\varepsilon = 0$ we have $\bar{b}_{1,1} = 0$.

Proof. Let us start by proving (2.41). This follows from from Proposition L.1, and the choice of $a_{k,l}$, $A_{k,l}$, $B_{k,l}$ for $(k,l) \in \Sigma_m$, solving each linear system $(\Omega_{k,l})$, so that

$$S[\tilde{u}] = \mathcal{E}(t,x).$$

Now we deal with (2.42). This is an easy consequence of the fact that $y = x - \alpha$, $y_c = x + (1-c)t$, and

$$\tilde{u}(t) - Q(y) - Q_c(y_c) = W(t,x), \quad \|W(t)\|_{H^1(\mathbb{R})} \leq K_0 c^{\frac{1}{m-1}}.$$

Proof of (2.44). We begin with some preliminary estimates.

Claim 15.

$$\|\alpha\|_{L^\infty} \leq K c^{\frac{1}{m-1}-\frac{1}{2}}, \quad \|\alpha'\|_{L^\infty} \leq K c^{\frac{1}{m-1}}. \tag{2.45}$$

Suppose $f = f(y) \in \mathcal{Y}$. Then for all $t \in [-T_c, T_c]$,

$$\|f(y)Q_c^k(y_v)\|_{L^2(\mathbb{R})} + \frac{1}{\sqrt{c}}\|f(y)(Q_c^k)'(y_v)\|_{L^2(\mathbb{R})} \leq K c^{\frac{k}{m-1}} e^{-(1-c)\sqrt{c}|t|}, \tag{2.46}$$

and for $g = g(y) \in L^\infty(\mathbb{R})$,

$$\|g(y)Q_c^k(y_v)\|_{L^2(\mathbb{R})} + \frac{1}{\sqrt{c}}\|g(y)(Q_c^k)'(y_v)\|_{L^2(\mathbb{R})} \leq K c^{\frac{k}{m-1}-\frac{1}{4}}. \tag{2.47}$$

In particular, if $t = T_c$ and $f \in \mathcal{Y}$, we have, for $c > 0$ small,

$$\|f(y)Q_c(y_c)\|_{H^1(\mathbb{R})} \leq K c^{10}, \tag{2.48}$$

$$\|Q(y) - Q(x - \tfrac{1}{2}\Delta_1)\|_{H^1(\mathbb{R})} \leq K c^{10}. \tag{2.49}$$

Proof. The proof of these estimates are similar to Claim C.1 in the Appendix C of [58]. See also Claim 2.6 in [53]. In particular for the proof we use Lemma L.2 from Appendix L. We skip the details. □

We continue the proof of (2.44).

For the sake of brevity, we will prove only the case $m = 3$. The case $m = 2$ is identical to Lemma 2.6 in [58].

Note that from Claim 15,

$$\|Q_c(y_c - b_{1,0}) - Q_c + b_{1,0}Q_c' - \frac{1}{2}b_{1,0}^2 Q_c''\|_{H^1(\mathbb{R})} \leq K c^{\frac{7}{4}}, \tag{2.50}$$

and

$$\|(Q_c^2)'(y_c - b_{1,0}) - (Q_c^2)' + b_{1,0}(Q_c^2)''\|_{H^1(\mathbb{R})} \leq K c^{\frac{9}{4}}, \tag{2.51}$$

(here we have used the fact $\|Q_c^{(3)}\|_{H^1(\mathbb{R})} \leq K c^{\frac{7}{4}}$ and $\|(Q_c^2)^{(3)}\|_{H^1(\mathbb{R})} \leq K c^{\frac{9}{4}}$). From the identities

$$Q_c'' = cQ_c - Q_c^3 - \varepsilon Q_c^p + O(Q_c^{p+1}), \ (Q_c^2)'' = 4cQ_c^2 - 3Q_c^4 + O(Q_c^5),$$

we obtain

$$\begin{aligned} \Big\| Q_c(y_c - b_{1,0}) &- d(\varepsilon)(Q_c^2)'(y_c - b_{1,0}) \\ &- [Q_c - b_{1,0}Q_c' + \frac{1}{2}b_{1,0}^2 cQ_c - \frac{1}{2}b_{1,0}^2 Q_c^3 - \frac{1}{2}\varepsilon b_{1,0}^2 \delta_{p4} Q_c^4] \\ &+ d(\varepsilon)[(Q_c^2)' - 4b_{1,0}cQ_c^2 + 3b_{1,0}Q_c^4]\Big\|_{H^1(\mathbb{R})} \leq K c^{\frac{7}{4}}. \end{aligned} \tag{2.52}$$

On the other hand, using the fact that $\lim_{+\infty} A_{k,l} = \gamma_{k,l}$, $\lim_{+\infty} B_{k,l} = b_{k,l}$, and Claim 15 we get

$$\begin{aligned}&\big\|\tilde{u}(T_c) - Q - Q_c - b_{1,0}Q_c' - \gamma_{2,0}Q_c^2 - b_{2,0}(Q_c^2)' - \gamma_{1,1}cQ_c \\ &\quad -\gamma_{2,1}cQ_c^2 - \gamma_{3,0}Q_c^3 - \gamma_{4,0}Q_c^4\big\|_{H^1(\mathbb{R})} \leq Kc^{7/4}.\end{aligned}$$

Combining this estimate and (2.52), we find

$$\begin{aligned}&\big\|\tilde{u}(T_c) - \{Q(y) + Q_c(y_c - b_{1,0}) - d(\varepsilon)(Q_c^2)'(y_c - b_{1,0})\} \\ &+ (\gamma_{1,1} - \frac{1}{2}b_{1,0}^2)cQ_c + \gamma_{2,0}Q_c^2 + (b_{2,0} - d(\varepsilon))(Q_c^2)' + (\gamma_{2,1} + 4d(\varepsilon)b_{1,0})cQ_c^2 \\ &+ (\gamma_{3,0} + \frac{1}{2}b_{1,0}^2)Q_c^3 + (\gamma_{4,0} - 3d(\varepsilon)b_{1,0} - \frac{1}{2}\varepsilon b_{1,0}^2\delta_{p4})Q_c^4\big\|_{H^1(\mathbb{R})} \leq Kc^{\frac{7}{4}}.\end{aligned}$$

It follows that with the choice

$$\gamma_{1,1} = \frac{1}{2}b_{1,0}^2, \quad \gamma_{2,0} = 0, \quad \gamma_{3,0} = -\frac{1}{2}b_{1,0}^2, \quad b_{2,0} = d(\varepsilon),$$

$$\gamma_{2,1} = -4d(\varepsilon)b_{1,0} \quad \text{and} \quad \gamma_{4,0} = 3d(\varepsilon)b_{1,0} + \frac{1}{2}\varepsilon b_{1,0}^2\delta_{p4}.$$

we obtain

$$\|\tilde{u}(T_c) - Q(y) - Q_c(y_c - b_{1,0}) + d(\varepsilon)(Q_c^2)'(y_c - b_{1,0})\|_{H^1(\mathbb{R})} \leq Kc^{\frac{7}{4}}. \tag{2.53}$$

The case $t = -T_c$ is similar and we left the proof to the reader.

Together with (2.49), we complete the proof of (2.44). This justifies in particular the choices of $\gamma_{k,l}$, $(k,l) \in \Sigma_m$ done in preceding Lemmas. □

2.4 Existence of the approximate pure 2-soliton collision solution

The fact that $d(\varepsilon) \neq 0$ (see Proposition 2.6) in Proposition 2.10 means formally that the collision is not elastic and that the residue due to the collision is of order $(Q_c^2)'$. However, the approximate solution $\tilde{u}(t,x)$ given in Lemma 2.10 being symmetric, it contains the residue at both $-T_c$ and T_c (see (2.44)). To match the solution $u(t)$ considered in Theorem 1.4, which is pure at $-\infty$, we need to introduce a modified approximate solution, which, at main order, will contain a residue only at time $t = T_c$. This will be clear after the following

Proposition 2.11. *There exists a function $\hat{u} = \hat{u}(t,x)$, of the form given by (2.6) such that for some constants $K, c_0 > 0$ and $0 < c < c_0$, the following estimates hold:*

1. *$\hat{u}(t,x) \not\equiv \hat{u}(-t,-x)$ for every t, x.*
2. *Almost solution. For any $t \in [-T_c, T_c]$,*

$$\|S[\hat{u}](t)\|_{H^1(\mathbb{R})} \leq Kc^{\frac{2}{m-1}+\frac{3}{4}}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac{1}{4}}], \tag{2.54}$$

(recall that $d(\varepsilon)$ measures the residue after the collision, introduced in (2.29)-(2.30)).

3. *Closeness to a two-soliton solution at time $t = -T_c$. With the defintions of shifts given in (2.43), the modified function $\hat{u}$ is close to a two solitons solution at time $-T_c$:*

$$\|\hat{u}(-T_c) - \{Q(\cdot + \tfrac{1}{2}\Delta_2) + Q_c(\cdot + (1-c)T_c + \tfrac{1}{2}\Delta_2)\}\|_{H^1(\mathbb{R})} \leq Kc^{\frac{2}{m-1}+\frac{1}{4}}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac{1}{2}}]. \tag{2.55}$$

4. *Non-matching with a two-soliton solution at time $t = T_c$:*

$$\begin{aligned}&\|\hat{u}(T_c) - Q(\cdot - \frac{1}{2}\Delta_1) - Q_c(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2) \\ &\quad + 2d(\varepsilon)(Q_c^2)'(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2)\|_{H^1(\mathbb{R})} \leq Kc^{\frac{2}{m-1}+\frac{1}{4}}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac{1}{2}}].\end{aligned} \tag{2.56}$$

where, from (2.29)-(2.30),

$$\forall 0 < |\varepsilon| \leq \varepsilon_0, \quad d(\varepsilon) = c_{m,p}\varepsilon + o(\varepsilon),$$

and

$$|\Delta_1 - a_{1,0}\int_{\mathbb{R}} Q_c| \leq Kc^{\frac{2}{m-1}-\frac{1}{2}}, \quad |\Delta_2 - 2b_{1,0}| \leq Kc. \tag{2.57}$$

5. *Comparison residue versus error terms: The residue in (2.56) satisfies*

$$\begin{aligned}\big\|2d(\varepsilon)(Q_c^2)'(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2)\big\|_{H^1(\mathbb{R})} \quad &\sim \quad |d(\varepsilon)|c^{\frac{2}{m-1}+\frac{1}{4}} \\ &\gg \quad c^{\frac{2}{m-1}+\frac{1}{4}}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac{1}{2}}],\end{aligned}$$

provided $c^{\frac{1}{m-1}} \ll |d(\varepsilon)|$.

Remark 2.10. The approximate solution $\hat{u}$ above mentioned describes the collision of two pure solitons that at time $t \sim T_c$ (after colliding) differ by a term of order $|d(\varepsilon)|c^{\frac{2}{m-1}+\frac{1}{4}}$ of the ingoing solitons before the collision, at time $t \sim -T_c$, provided (1.23) holds.

For even small values of ε such that condition (1.23) does not hold, we need to go further in our approximate solution and solve even more linear systems. We believe that in this case, more involved, the conclusions of this paper are the same.

Let us return to the proof of Proposition 2.11.

Proof. Let $\hat{u} := \tilde{u} + w_\#$, where

$$w_\#(t,x) := -d(\varepsilon)(Q_c^2)'(y_c)(1 + \bar{P}(y)), \tag{2.58}$$

and $\bar{P}$ was defined in (2.12). Now $w_\#$ can be expressed in the form

$$w_\#(t,x) = Q(y) + Q_c(y_c) + \sum_{(k,l)\in\Sigma_m} c^l\{\hat{A}_{k,l}(y)Q_c^k(y_c) + \hat{B}_{k,l}(y)(Q_c^k)'(y_c)\},$$

where $\hat{A}_{k,l} = A_{k,l}$, $\hat{B}_{k,l} = B_{k,l} + w_\#\delta_{(k,l),(2,0)}$. Here $\delta_{(2,0),(2,0)} = 1$ and $\delta_{(k,l),(2,0)} = 0$ otherwise.

Let us prove (2.56). Replacing $\tilde{u} = \hat{u} - w_\#$ in (2.44), we have

$$\begin{aligned}&\|\hat{u}(T_c) - Q(\cdot - \frac{1}{2}\Delta_1) - Q_c(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2) \\ &\quad + d(\varepsilon)(Q_c^2)'(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2) - w_\#(T_c)\|_{H^1(\mathbb{R})} \leq Kc^{\frac{3}{m-1}+\frac{1}{4}}.\end{aligned}$$

Thus, using (2.48) (note that $\bar{P} \in \mathcal{Y}$)

$$\begin{aligned}&\|\hat{u}(T_c) - Q(\cdot - \frac{1}{2}\Delta_1) - Q_c(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2) + 2d(\varepsilon)(Q_c^2)'(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2)\|_{H^1(\mathbb{R})} \\ &\leq Kc^{\frac{3}{m-1}+\frac{1}{4}} + \|d(\varepsilon)(Q_c^2)'(\cdot + (1-c)T_c - \frac{1}{2}\Delta_2) + w_\#(T_c)\|_{H^1(\mathbb{R})} \\ &\leq Kc^{\frac{3}{m-1}+\frac{1}{4}} + K|d(\varepsilon)|\|(Q_c^2)'(\cdot - \frac{1}{2}\Delta_2) - (Q_c^2)'\|_{H^1(\mathbb{R})} \\ &\leq Kc^{\frac{3}{m-1}+\frac{1}{4}} + K|d(\varepsilon)|c^{\frac{2}{m-1}+\frac{3}{4}}.\end{aligned}$$

Similarly, at time $t = -T_c$

$$\|\hat{u}(-T_c) - Q(\cdot + \frac{1}{2}\Delta_1) - Q_c(\cdot - (1-c)T_c + \frac{1}{2}\Delta_2)$$
$$-d(\varepsilon)(Q_c^2)'(\cdot - (1-c)T_c + \frac{1}{2}\Delta_2) - w_\#(-T_c)\|_{H^1(\mathbb{R})} \leq Kc^{\frac{3}{m-1}+\frac{1}{4}},$$

so that

$$\begin{aligned}&\|\hat{u}(-T_c) - Q(\cdot + \frac{1}{2}\Delta_1) - Q_c(\cdot - (1-c)T_c + \frac{1}{2}\Delta_2)\|_{H^1(\mathbb{R})}\\ &\leq Kc^{\frac{3}{m-1}+\frac{1}{4}} + K|d(\varepsilon)|\|(Q_c^2)'(\cdot + \frac{1}{2}\Delta_2) - (Q_c^2)'\|_{H^1(\mathbb{R})}\\ &\leq Kc^{\frac{3}{m-1}+\frac{1}{4}} + K|d(\varepsilon)|c^{\frac{2}{m-1}+\frac{3}{4}}.\end{aligned}$$

Note that (2.57) is clearly a consequence of (2.43).

Finally, we prove (2.54). Note that (cf. Appendix B for the definitions)

$$\begin{aligned}S[\hat{u}] &= S[\tilde{u} + w_\#]\\ &= S[\tilde{u}] + \mathbf{III}(w_\#) + [f(\tilde{u} + w_\#) - f(\tilde{u}) - f'(Q)w_\#]_x\end{aligned}$$

The following estimates allow to conclude (2.54). We claim

Claim 16. With the choice of $w_\#$ given in (2.58),

$$\|\mathbf{III}(w_\#)\|_{H^1(\mathbb{R})} \leq K|d(\varepsilon)|c^{1+\frac{2}{m-1}}. \tag{2.59}$$

and

$$\|[f(\tilde{u} + w_\#) - f(\tilde{u}) - f'(Q)w_\#]_x\|_{H^1(\mathbb{R})} \leq K|d(\varepsilon)|c^{\frac{1}{2}+\frac{3}{m-1}} \tag{2.60}$$

Proof. The proof is similar to the proof of Proposition L.1 above. We only sketch the main ideas.

Let us prove (2.59). First, note that for $\bar{P}$ defined in (2.12)

$$(\mathcal{L}(1+\bar{P}))' = (1 - f'(Q) + f'(Q))' = 0. \tag{2.61}$$

This property will be useful in what follows. From the calculations performed in (L.14), (2.61) and the fact that $(1+\bar{P})' \in \mathcal{Y}$, we note that (cf. (L.7) and (L.13) for the definition of $\mathbf{III}(\cdot)$ and Σ'_m respectively)

$$\begin{aligned}\mathbf{III}((1+\bar{P})(Q_c^2)') &= -(\mathcal{L}(1+\bar{P}))'(Q_c^2)' - c(1+\bar{P})(Q_c^2)'' + (1+\bar{P})(Q_c^2)^{(3)}\\ &\quad + \sum_{(k,l)\in\Sigma'_m} c^l[\tilde{F}_{k,l}Q_c^k + \tilde{G}_{k,l}(Q_c^k)'] + O(cQ_c^3 + Q_c^5 + c^2Q_c)\\ &= \sum_{(k,l)\in\Sigma'_m} c^l[\tilde{F}_{k,l}Q_c^k + \tilde{G}_{k,l}(Q_c^k)'] + O(c(Q_c^2)' + cQ_c^3 + Q_c^5 + c^2Q_c),\end{aligned}$$

where both $\tilde{F}_{k,l}$ and $\tilde{G}_{k,l}$ are in $\mathcal{Y}$. Moreover, $\tilde{F}_{3,0} = 0$. From here, the definition of $w_\#$ in (2.58) and Claim 15, we obtain

$$\|\mathbf{III}(w_\#)\|_{H^2(\mathbb{R})} \leq K|d(\varepsilon)|c^{\frac{2}{m-1}+1}.$$

Now, we deal with (2.60). We note that

$$\begin{aligned}(2.60) &= [f(\tilde{u}+w_{\#})-f(\tilde{u})-f'(\tilde{u})w_{\#}]_x+[(f'(\tilde{u})-f'(Q))w_{\#}]_x\\ &\quad [\frac{1}{2}f''(\tilde{u})w_{\#}^2+O(w_{\#}^3)]_x+[f''(Q)(\tilde{u}-Q)w_{\#}+\frac{1}{2}f^{(3)}(Q)(\tilde{u}-Q)^2w_{\#}+O((\tilde{u}-Q)^3w_{\#})]_x.\end{aligned}$$

From here, using the expresion for $w_{\#}$ and Claim 15, we obtain

$$\|(2.60)\|_{H^2}\le K|d(\varepsilon)|[|d(\varepsilon)|c^{\frac{3}{2}+\frac{4}{m-1}}+c^{\frac{1}{2}+\frac{4}{m-1}}+c^{\frac{1}{2}+\frac{3}{m-1}}]\le K|d(\varepsilon)|c^{\frac{1}{2}+\frac{3}{m-1}}.$$

This finishes the proof.

□

This Claim allows us to finish the proof of the Proposition.

□

3 Preliminary results for stability of the 2-soliton structure

In this section several stability results will allow to study the long time behavior of the 2-soliton soliton solution. First of all, we recall a general result proved in [54] concerning the existence and properties of an actual function $u=u(t,x)$, solution of (2.1) in the interval $[-T_c,T_c]$ and close enough to our approximate solution $\hat{u}$. This will be done in the next subsection.

Next, we study the stability of a solution $u(t)$ of (1.21) for long time, namely $t\ge T_c$. These results have been proved in great generality by Martel and Merle in [51], [54], and [52]. In particular, we will use the *stability* and *asymptotic stability* of the two solitons (Proposition 3.3) to show the persistence of the 2-soliton structure for long time.

Finally, a key result is the *decomposition result* from Lemma 3.4, which will be essential to show the persistence of the residual term (cf. (2.56)) at infinity.

3.1 Dynamic stability in the interaction region

For any $c>0$ sufficiently small, we will consider the function $\hat{u}(t)$ of the form

$$\hat{u}(t,x)=Q(y)+Q_c(y_c)+\sum_{(k,l)\in\Sigma_m}c^l\left\{Q_c^k(y_c)\hat{A}_{k,l}(y)+(Q_c^k)'(y_c)\hat{B}_{k,l}(y)\right\}$$

defined in Proposition 2.11 (the notation was introduced in (2.5) and (2.6)). Recall the error term

$$S[\hat{u}](t)=\hat{u}_t+(\hat{u}_{xx}-u+f(\hat{u}))_x.$$

Proposition 3.1 (Exact solution close to the approximate solution $\hat{u}$, [54]). *Let $\theta>\frac{1}{m-1}$ and ε small enough such that (1.12) holds for Q. There exists $c_0>0$ such that the following holds for any $0<c<c_0$. Suppose that for all $t\in[-T_c,T_c]$*

$$\|S[\hat{u}](t)\|_{H^2(\mathbb{R})}\le K\frac{c^\theta}{T_c},\tag{3.1}$$

and for some $T_0 \in [-T_c, T_c]$,

$$\|u(T_0) - \hat{u}(T_0)\|_{H^1(\mathbb{R})} \leq Kc^\theta, \tag{3.2}$$

where $u(t)$ *is an* H^1 *solution of (2.1). Then, there exist* $K_0 = K_0(\theta, K, f)$ *and a* C^1 *function* $\rho : [-T_c, T_c] \to \mathbb{R}$ *such that, for all* $t \in [-T_c, T_c]$,

$$\|u(t) - \hat{u}(t, \cdot - \rho(t))\|_{H^1(\mathbb{R})} \leq K_0 c^\theta, \quad |\rho'(t) - 1| \leq K_0 c^\theta. \tag{3.3}$$

Remark 3.1. The proof of the above Proposition is nontrivial and requires some refined techniques such as modulation theory, coercivity properties and the introduction of a modified energy functional adapted to a two soliton collision. It is necessary to emphasize that one of the key elements in the proof is the smallness of the error term $S[\hat{u}]$ along the collision. For the sake of completeness, we will draw the main lines of the argument, see [54] for the actual complete proof.

Proof. It suffices to show the result on the interval $[T_0, T_c]$. By using the transformation $x \to -x$, $t \to -t$, the proof is the same on $[-T_c, T_0]$.

Let $K^* > 1$ be a constant to be fixed later. Since $\|u(T_0) - \hat{u}(T_0)\|_{H^1(\mathbb{R})} \leq c^\theta$, by continuity in time in $H^1(\mathbb{R})$, there exists $T_0 < T^* \leq T_c$ such that

$$\begin{aligned} T^* \quad = \quad & \sup \big\{ T \in [T_0, T_c] \text{ such that for all } t \in [T_0, T], \text{ there exists } r(t) \in \mathbb{R} \text{ with} \\ & \|u(t) - \hat{u}(t, \cdot - r(t))\|_{H^1(\mathbb{R})} \leq K^* c^\theta \big\}. \end{aligned}$$

The objective is to prove that $T^* = T_c$ for K^* large. For this, we argue by contradiction, assuming that $T^* < T_c$ and reaching a contradiction with the definition of T^* by proving some independent estimates on $\|u(t) - \hat{u}(t, \cdot - r)\|_{H^1(\mathbb{R})}$ on $[T_0, T^*]$.

An argument using the Implicit function theorem allows to construct a modulation parameter and to estimate its variation in time:

Claim 17. Assume that $0 < c < c(K^*)$ small enough. There exists a unique C^1 function $\rho(t)$ such that, for all $t \in [T_0, T^*]$,

$$z(t, x) = u(t, x + \rho(t)) - \hat{u}(t, x) \quad \text{satisfies} \quad \int_{\mathbb{R}} z(t, x) Q'(y) dx = 0.$$

Moreover, we have, for all $t \in [T_0, T^*]$,

$$|\rho(T_0)| + \|z(T_0)\|_{H^1(\mathbb{R})} \leq Kc^\theta, \ \|z(t)\|_{H^1(\mathbb{R})} \leq 2K^* c^\theta,$$

$$z_t + (z_{xx} - z + f(z + \hat{u}) - f(\hat{u}))_x = -S[\hat{u}](t) + (\rho'(t) - 1)(\hat{u} + z)_x.$$

$$|\rho'(t) - 1| \leq K\|z(t)\|_{H^1(\mathbb{R})} + K\|S[\hat{u}](t)\|_{H^1(\mathbb{R})},$$

The purpose of the modulation theory is to establish a lower bound in the following energy functional for $z(t)$:

$$\mathcal{F}(t) := \frac{1}{2} \int_{\mathbb{R}} \left((\partial_x z)^2 + (1 + \alpha'(y_c)) z^2 \right) - \int_{\mathbb{R}} (F(\hat{u} + z) - F(\hat{u}) - f(\hat{u}) z).$$

Indeed, this functional enjoys two useful properties: it has a very small time variation and it is coercive up to the direction Q:

Lemma 3.2 (Coercivity of $\mathcal{F}$)**.** *Assume that* $0 < c < c(K^*)$ *small enough. There exists* $K > 0$ *(independent of* K^* *and* c*) such that*

1. *Coercivity of $\mathcal{F}$ under orthogonality conditions:*

$$\forall t \in [T_0, T^*], \quad \|z(t)\|_{H^1(\mathbb{R})}^2 \leq K\mathcal{F}(t) + K\left|\int_{\mathbb{R}} z(t)Q(y)\right|^2.$$

2. *Control of the direction Q:*

$$\forall t \in [T_0, T^*], \quad \left|\int_{\mathbb{R}} z(t)Q(y)\right| \leq Kc^{\theta} + Kc^{\frac{1}{p-1}-\frac{1}{4}}\|z(t)\|_{L^2} + K\|z(t)\|_{L^2}^2.$$

3. *Control of the variation of the energy functional:*

$$\mathcal{F}(T^*) - \mathcal{F}(T_0) \leq Kc^{2\theta}\left((K^*)^2(1+K^*)c^{\frac{1}{2(m-1)}-\frac{1}{8}} + K^*\right).$$

These estimates allow us, after fixing K^* large enough and possibly taking c even smaller, to show that actually

$$\|z(T^*)\|_{H^1(\mathbb{R})}^2 \leq \frac{1}{2}(K^*)^2c^{2\theta}.$$

contradicting the definition of T^*, thus proving that $T^* = T_c$. □

Once the existence of an actual solution (close to our approximate solution $\hat{u}$ in the interval $[-T_c, T_c]$) is established, one would like to investigate the behavior in long time of this solution. We treat this problem in the next subsection.

3.2 Stability and asymptotic stability for large time

Here we consider the stability of the 2-soliton structure after the collision, and for a long time. Let T_c be defined in (2.2). We start with an important

Remark 3.2. Since (1.12) holds for $f(s) = s^m$, $m = 2, 3$, it is clear by a perturbation argument that (1.12) holds also for f as in (1.22) for all $0 < c < 1$, provided $0 < |\varepsilon| < \varepsilon_0$ is small enough.

Proposition 3.3 (Stability of two decoupled solitons, [51], [52]). *Let ε small enough such that (1.12) holds for Q. Then there exist constants $c_0, K > 0$, such that for any $0 < c < c_0$ and for any $\omega > 0$, the following holds. Let $u(t)$ be an H^1 solution of (2.1) such that for some time $t_1 \in \mathbb{R}$ and $\frac{1}{2}T_c \leq X_0 \leq \frac{3}{2}T_c$,*

$$\|u(t_1) - Q - Q_c(\cdot + X_0)\|_{H^1(\mathbb{R})} \leq c^{\frac{1}{4}+\frac{1}{m-1}+\omega}. \tag{3.4}$$

Then there exist C^1-functions $\rho_1(t)$, $\rho_2(t)$ defined on $[t_1, +\infty)$ such that

1. *Stability:*

$$\sup_{t\geq t_1}\|u(t) - Q(\cdot - \rho_1(t)) - Q_c(\cdot - \rho_2(t))\|_{H^1(\mathbb{R})} \leq Kc^{-\frac{1}{4}+\frac{1}{m-1}+\omega}, \tag{3.5}$$

and for all $t \geq t_1$,

$$\frac{1}{2} \leq \rho_1'(t) - \rho_2'(t) \leq \frac{3}{2}, \quad |\rho_1(t_1)| \leq Kc^{\frac{1}{4}+\frac{1}{m-1}+\omega}, \quad |\rho_2(t_1) + X_0| \leq Kc^{\omega}. \tag{3.6}$$

2. *Asymptotic stability: There exist $c_1^+, c_2^+ > 0$ such that on the right hand side limit*

$$\lim_{t\to+\infty}\|u(t) - Q_{c_1^+}(x - \rho_1(t)) - Q_{c_2^+}(x - \rho_2(t)))\|_{H^1(x>\frac{1}{10}ct)} = 0, \tag{3.7}$$

with

$$\left|c_1^+ - 1\right| \leq Kc^{\frac{1}{4}+\frac{1}{m-1}+\omega}, \quad \left|\frac{c_2^+}{c} - 1\right| \leq Kc^{\omega}. \tag{3.8}$$

3.3 A decomposition result

Recall a more precise decomposition of $u(t)$ used in the proof of Proposition 3.3 in [51], [52].

Lemma 3.4 (Decomposition of the solution, [52])**.** *Suppose (1.12) holds for Q. Let $u = u(t)$ be a solution of the gKdV equation (1.21) such that the estimate (3.4) holds. Then there exist C^1-functions $\rho_1(t)$, $\rho_2(t)$, $c_1(t)$, $c_2(t)$, defined on $[t_1, +\infty)$, such that the function*

$$\eta(t,x) := u(t,x) - R_1(t,x) - R_2(t,x),$$

where, for $j = 1, 2$, $R_j(t,x) := Q_{c_j(t)}(x - \rho_j(t))$, satisfies for all $t \geq t_1$,

$$\int_{\mathbb{R}} R_j(t)\eta(t) = \int_{\mathbb{R}} (x - \rho_j(t))R_j(t)\eta(t) = 0, \quad j = 1, 2, \tag{3.9}$$

$$\|\eta(t)\|_{H^1(\mathbb{R})} + |c_1(t) - 1| + c^{\frac{1}{m-1} - \frac{1}{4}} \left| \frac{c_2(t)}{c} - 1 \right| \leq K c^{\omega + \frac{1}{m-1} - \frac{1}{4}}, \tag{3.10}$$

$$\text{and for all } t \geq t_1 \quad |\rho_2'(t)| + |\rho_1'(t) - 1| \leq \frac{1}{10}, \quad \rho_1(t) - \rho_2(t) \geq \frac{1}{2}t + \frac{1}{4}T_c. \tag{3.11}$$

Moreover, we have the convergence $\lim_{t \to +\infty} \bar{c}_j(t) = c_j^+$ for $j = 1, 2$.

At this moment we have all the necessary information about the 2-soliton solution of (2.1). Indeed, recall from the sketch of proof (Subsection 1.4) that the asymptotic in long time will be treated using the tools from this section, more precisely using Proposition 3.3 and Lemma 3.4. On the other hand the collision region will be described by Proposition 3.1. This is the purpose of the next section.

4 Proof of the Theorem 1.4

Now we are in a position to prove the main Theorem of this work.

Proof of Theorem 1.4. Let $1 = c_1 < c_*(f)$ such that (1.12) holds and $0 < c < c_0(\varepsilon)$ small enough (depending on ε). Let $u(t)$ be the *unique* solution of (1.21) such that (see Theorem 1 and Remark 2 in [49])

$$\lim_{t \to -\infty} \|u(t) - Q(x - t) - Q_c(x - ct)\|_{H^1(\mathbb{R})} = 0.$$

1. Behavior at $-T_c$. We claim that for all $t < -\frac{1}{32}T_c$,

$$\|u(t) - Q(\cdot - t) - Q_c(\cdot - ct)\|_{H^1(\mathbb{R})} \leq K e^{\frac{1}{4}\sqrt{c}(1-c)t}. \tag{4.1}$$

This is a consequence of the proof of existence of $u(t)$ in [49]. See Proposition 5.1 in [53] for a proof in the power case.

Now, using (4.1), we will match the function u with the collision solution $\hat{u}$ constructed in Proposition 2.11. For this, we will translate u in time and space, as follows.

Let Δ_1, Δ_2 be defined in Proposition 2.10 and

$$T_c^- := T_c + \frac{1}{2}\frac{\Delta_1 - \Delta_2}{1 - c}, \quad a := \frac{1}{2}\Delta_1 - T_c^-.$$

Since from (2.57)

$$|\Delta_1| \leq K c^{\frac{1}{m-1}-\frac{1}{2}}, \quad \text{and} \quad |\Delta_2| \leq K,$$

we have $-T_c^- \leq -\frac{1}{32}T_c$, and thus, from (4.1) for c small enough, and after a translation by a, we get

$$\|u(-T_c^-, \cdot + a) - Q(\cdot + \frac{\Delta_1}{2}) - Q_c(\cdot - (1-c)T_c + \frac{\Delta_2}{2})\|_{H^1(\mathbb{R})} \leq K e^{-\frac{1}{4}\sqrt{c}(1-c)T_c^-} \leq K c^{10}.$$

By translation invariance, we may assume $T_c^- = T_c$ and $a = 0$, such that

$$\|u(-T_c) - Q(\cdot + \frac{\Delta_1}{2}) - Q_c(\cdot - (1-c)T_c + \frac{\Delta_2}{2})\|_{H^1(\mathbb{R})} \leq K c^{10}. \tag{4.2}$$

2. Behavior at $+T_c$. Now, possibly taking a smaller c, consider $\hat{u} = \hat{u}_{1,c}$ constructed in Proposition 2.11. By (2.55) and (4.2), we have

$$\|u(-T_c) - \hat{u}(-T_c)\|_{H^1(\mathbb{R})} \leq K c^{\frac{2}{m-1}+\frac{1}{4}}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac{1}{2}}].$$

Applying Proposition 3.1 with

$$T_0 = -T_c, \quad c^{\theta} := c^{\frac{2}{m-1}+\frac{1}{4}-\frac{1}{100}}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac{1}{4}}],$$

it follows that there exists a function $\rho(t)$ such that for all $t \in [-T_c, T_c]$,

$$\|u(t) - \hat{u}(t, \cdot - \rho(t))\|_{H^1(\mathbb{R})} \leq K c^{\theta}.$$

In particular, for $r := \rho(T_c)$, we have

$$\|u(T_c) - \hat{u}(T_c, \cdot - r)\|_{H^1(\mathbb{R})} \leq K c^{\theta}.$$

Using (2.56) and triangular inequality , we obtain

$$\|u(T_c) - Q(\cdot - r_1) - Q_c(\cdot - r_2) - 2d(\varepsilon)(Q_c^2)'(\cdot - r_2)\|_{H^1(\mathbb{R})} \leq K c^{\theta}. \tag{4.3}$$

Here,

$$r_1 := \frac{1}{2}\Delta_1 + r, \quad r_2 := (c-1)T_c + \frac{1}{2}\Delta_2 + r,$$

so that $r_1 - r_2 = (1-c)T_c + \frac{1}{2}(\Delta_1 - \Delta_2)$ satisfies

$$\frac{1}{2}(1-c)T_c \leq r_1 - r_2 \leq 32(1-c)T_c. \tag{4.4}$$

Moreover, note that

$$\|(Q_c^2)'\|_{H^1(\mathbb{R})} \leq K c^{\frac{2}{m-1}+\frac{1}{4}},$$

so that

$$\|u(T_c) - Q(\cdot - r_1) - Q_c(\cdot - r_2)\|_{H^1(\mathbb{R})} \leq K c^{\frac{2}{m-1}+\frac{1}{4}}[|d(\varepsilon)| + c^{\frac{1}{m-1}}] \leq K|d(\varepsilon)|c^{\frac{2}{m-1}+\frac{1}{4}}, \tag{4.5}$$

provided

$$|d(\varepsilon)| \geq \kappa_0 c^{\frac{1}{m-1}-\frac{1}{100}}, \tag{4.6}$$

for some $\kappa_0 > 0$ large enough but fixed. We have thus arrived to time $t = +T_c$ with a stability property of the 2-soliton structure, namely (4.5).

3. Behavior as $t \to +\infty$. From (4.5), it follows that we can apply Proposition 3.3 to $u(t, \cdot + r_1)$ for $t \geq T_c$ (that is, $t_1 := T_c$), with $X_0 := r_1 - r_2$, and

$$c^\omega := |d(\varepsilon)|c^{\frac{1}{m-1}}.$$

It follows that there exist $\rho_1(t)$, $\rho_2(t)$, $c_1^+ > 0$, $c_2^+ > 0$ so that

$$w^+(t,x) := u(t,x) - Q_{c_1^+}(x - r_1 - \rho_1(t)) - Q_{c_2^+}(x - r_1 - \rho_2(t)) \tag{4.7}$$

satisfies

$$\sup_{t \geq T_c} \|w^+(t)\|_{H^1(\mathbb{R})} \leq K|d(\varepsilon)|c^{\frac{2}{m-1}-\frac{1}{4}}, \quad \lim_{t\to+\infty} \|w^+(t)\|_{H^1(x > \frac{c}{10}t + r_1)} = 0, \tag{4.8}$$

and

$$|c_1^+ - 1| \leq K|d(\varepsilon)|c^{\frac{2}{m-1}+\frac{1}{4}}, \quad |c_2^+ - c| \leq K|d(\varepsilon)|c^{1+\frac{1}{m-1}}. \tag{4.9}$$

In particular, the behavior of the 2-soliton structure remains stable at infinity, modulo the emergency of a possible H^1-nonzero residual term. This proves the upper bound in (1.26). At this stage, we do not know if this residual term (that is, w^+) can be bounded by below uniformly in time. This is the purpose of the following key step.

4. Lower bound on $w^+(t)$ for $t > T_c$ large. Consider the decomposition of $u(\cdot, \cdot + r_1)$ defined in Lemma 3.4, i.e. the center of mass $\bar\rho_1(t)$, $\bar\rho_2(t)$, the scaling parameters $\bar c_1(t)$, $\bar c_2(t)$ such that, for $t > T_c$,

$$\eta(t,x) := u(t,x) - Q_{\bar c_1(t)}(x - r_1 - \bar\rho_1(t)) - Q_{\bar c_2(t)}(x - r_1 - \bar\rho_2(t)) \tag{4.10}$$

satisfies

$$\begin{gathered} \sup_{t \geq T_c} \|\eta(t)\|_{H^1(\mathbb{R})} \leq K|d(\varepsilon)|c^{\frac{2}{m-1}-\frac{1}{4}}, \quad |\bar c_1(T_c) - 1| \leq K|d(\varepsilon)|c^{\frac{2}{m-1}-\frac{1}{4}}, \\ \bar\rho_1(t) - \bar\rho_2(t) \geq \frac{1}{2}t + \frac{1}{4}T_c, \quad |\bar c_2(t) - c| \leq K|d(\varepsilon)|c^{\frac{1}{m-1}+\frac{3}{2}}, \end{gathered} \tag{4.11}$$

and

$$|\bar\rho_2(T_c) + r_1 - r_2| \leq K|d(\varepsilon)|c^{\frac{1}{m-1}}. \tag{4.12}$$

Moreover, we have for $j = 1, 2$

$$\lim_{t\to+\infty} \bar c_j(t) = c_j^+. \tag{4.13}$$

First, as a consequence of (4.3), we claim the following lower bound at $t = T_c$: for $K_0 > 0$, independent of $c > 0$,

$$\int_{x < \bar\rho_2(T_c) + r_1 + \frac{1}{4}T_c} \eta^2(T_c, x)dx \geq K_0|d(\varepsilon)|^2 c^{\frac{4}{m-1}+\frac{1}{2}}. \tag{4.14}$$

Proof of (4.14). The proof will proceed by a contradiction argument. Indeed, suppose that for any $\alpha > 0$ there exists $c > 0$ small enough such that (4.14) does not hold properly, namely

$$\|\eta(T_c)\|_{L^2(x < \bar\rho_2(T_c) + r_1 + \frac{1}{4}T_c)} \leq \alpha|d(\varepsilon)|c^{\frac{2}{m-1}+\frac{1}{4}}. \tag{4.15}$$

Replacing

$$u(T_c, x) = Q_{\bar c_1(T)}(x - r_1 - \bar\rho_1(T_c)) + Q_{\bar c_2(T_c)}(x - r_1 - \bar\rho_2(T_c)) + \eta(T_c, x)$$

in (4.3), we find

$$\big\|[Q_{\bar c_1(T_c)}(\cdot - r_1 - \bar\rho_1(T_c)) - Q(\cdot - r_1)] + [Q_{\bar c_2(T_c)}(\cdot - r_1 - \bar\rho_2(T_c)) - Q_c(\cdot - r_2)] + \eta(T_c) + 2d(\varepsilon)(Q_c^2)'(\cdot - r_2)\big\|_{H^1(\mathbb{R})} \leq Kc^{\frac{2}{m-1}+\frac14}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac12}].$$

By the decay properties of Q, (4.11) at time $t = T_c$ and $r_1 - r_2 \geq \frac12(1-c)T_c$ (see (4.4)), we obtain

$$\|[Q_{\bar c_2(T_c)}(\cdot - r_1 - \bar\rho_2(T_c)) - Q_c(\cdot - r_2)] + \eta(T_c) + 2d(\varepsilon)(Q_c^2)'(\cdot - r_2)\|_{L^2(x<\bar\rho_2(T_c)+r_1+\frac14 T_c)} \leq Kc^{\frac{2}{m-1}+\frac14}[c^{\frac{1}{m-1}} + |d(\varepsilon)|c^{\frac12}].$$

Then, using (4.15) and (4.6),

$$\|[Q_{\bar c_2(T_c)}(\cdot - r_1 - \bar\rho_2(T_c)) - Q_c(. - r_2)] + 2d(\varepsilon)(Q_c^2)'(\cdot - r_2)\|_{L^2(x<\bar\rho_2(T_c)+r_1+\frac14 T_c)} \leq c^{\frac{2}{m-1}+\frac14}\big[Kc^{\frac12} + 2\alpha + \frac{K}{\kappa_0}c^{\frac{1}{100}}\big]|d(\varepsilon)|.$$

By scaling and translation, and decay of Q, we obtain

$$\|\bar Q - Q + 2d(\varepsilon)c^{\frac12+\frac{1}{m-1}}(Q^2)'\|_{L^2(\mathbb{R})} \leq c^{\frac{1}{m-1}+\frac12}\big[Kc^{\frac12} + 2\alpha + \frac{K}{\kappa_0}c^{\frac{1}{100}}\big]|d(\varepsilon)| + \|\bar Q - Q + 2d(\varepsilon)c^{\frac12+\frac{1}{m-1}}(Q^2)'\|_{L^2(x>\beta)},$$

where $\bar Q(x) = \lambda Q(\mu x - \xi)$, and

$$\lambda := \Big[\frac{\bar c_2(T_c)}{c}\Big]^{\frac{1}{m-1}}, \quad \mu := \sqrt{\frac{\bar c_2(T_c)}{c}},$$

(do not be confused with μ of Theorem 1.4), and

$$\xi = \sqrt{\bar c_2(T_c)}(\bar\rho_2(T_c) + r_1 - r_2), \quad \beta := \sqrt{c}(\frac14 T_c + \bar\rho_2(T_c) + r_1 - r_2).$$

Note that from (4.11) and (4.12),

$$\beta \geq \frac18\sqrt{c}T_c \geq \frac18 c^{-\frac{1}{100}}, \quad \|\bar Q - Q + 2d(\varepsilon)c^{\frac12+\frac{1}{m-1}}(Q^2)'\|_{L^2(x>\beta)} \leq Kc^{10}.$$

Moreover, note that $\bar Q(x) = Q_\mu(x - \frac{\xi}{\mu})$, and that by (4.11), we have

$$|\mu - 1| \leq K|d(\varepsilon)|c^{\frac12+\frac{1}{m-1}}, \quad |\xi| \leq K|d(\varepsilon)|c^{\frac12+\frac{1}{m-1}}.$$

Expanding $\bar Q$ in $\mu - 1$, and ξ/μ, and using parity properties, we find

$$\|\xi Q' + 2d(\varepsilon)c^{\frac12+\frac{1}{m-1}}(Q^2)'\|_{L^2(\mathbb{R})} \leq \big[Kc^{\frac12} + 3\alpha + \frac{K}{\kappa_0}c^{\frac{1}{100}}\big]|d(\varepsilon)|c^{\frac12+\frac{1}{m-1}},$$

so that for some constant $\bar\xi \in \mathbb{R}$,

$$\|\bar\xi Q' + 2d(\varepsilon)(Q^2)'\|_{L^2(\mathbb{R})} \leq \big[Kc^{\frac12} + 4\alpha + \frac{K}{\kappa_0}c^{\frac{1}{100}}\big]|d(\varepsilon)|.$$

Note that exists $\kappa_1 > 0$, independent of ε and c, such that

$$\inf_{\bar\xi\in\mathbb{R}} \|\bar\xi Q' + 2d(\varepsilon)(Q^2)'\|_{L^2(\mathbb{R})} \geq \kappa_1|d(\varepsilon)|,$$

since $Q' \neq \gamma(Q^2)'$ for all $\gamma \in \mathbb{R}$. By choosing κ_0 large enough in (4.6), depending only on κ_1, and c small enough, we find a contradiction for α small. This contradiction proves (4.14).

Now, we finish the proof of the lower bound by proving the following

Lemma 4.1. *There exists* $K_0 > 0$ *such that*

$$\liminf_{t\to+\infty} \|w^+(t)\|_{H^1_c(\mathbb{R})} \geq K_0|d(\varepsilon)|c^{\frac{3}{4}+\frac{2}{m-1}}. \tag{4.16}$$

Note that (4.16) combined with (2.29)-(2.30) prove the lower bound in (1.26). Thus, we are now reduced to prove (4.16).

Proof. We argue by contradiction. Assume that for any $\alpha > 0$, there exist arbitrarily large T_0 and c arbitrarily close to 0 such that

$$\|w^+(T_0)\|_{H^1_c(\mathbb{R})} \leq \alpha|d(\varepsilon)|c^{\frac{3}{4}+\frac{2}{m-1}}. \tag{4.17}$$

By (4.13), we can choose $T_0 > T_c$ large enough so that

$$\|\eta(T_0)\|_{H^1_c(x<m(T_0)+\frac{T_0}{4})} \leq 2\alpha|d(\varepsilon)|c^{\frac{3}{4}+\frac{2}{m-1}}. \tag{4.18}$$

Here $m(t) := r_1 + \frac{1}{2}(\bar\rho_1(t) + \bar\rho_2(t))$ is the *middle point* between the two solitons at time t.

We need to estimate some local in space conservation laws. For this reason we introduce a sort of cutoff function *supported* on the small soliton. Let

$$\begin{aligned}
&\psi(x) = \frac{2}{\pi}\arctan(\exp(x/\kappa)), \quad \text{so that} \quad \lim_{-\infty}\psi = 0,\ \lim_{\infty}\psi = 1, \text{ and for all } x\in\mathbb{R},\\
&\psi(-x) = 1-\psi(x), \quad \psi'(x) = \frac{1}{\pi\kappa\cosh(x/\kappa)}, \quad |\psi'''(x)| \leq \frac{1}{\kappa^2}|\psi'(x)|.
\end{aligned} \tag{4.19}$$

Let

$$a := \frac{E(Q_{\bar c(T_0)}) - E(Q_{\bar c(T_c)})}{M(Q_{\bar c(T_c)}) - M(Q_{\bar c(T_0)})}.$$

We set

$$\begin{aligned}
\mathcal{G}(t) &:= \frac{1}{2}a\int_{\mathbb{R}} u^2(t,x)(1-\psi(x-m(t)))dx + \frac{1}{2}\int_{\mathbb{R}}(u_x^2 - 2F(u))(t,x)(1-\psi(x-m(t)))dx\\
&= aM(u(t)) + E(u(t)) - (a\mathcal{M}_1(t) + \mathcal{E}_1(t)),
\end{aligned}$$

where

$$\mathcal{M}_1(t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)\psi(x-m(t))dx, \quad \mathcal{E}_1(t) := \frac{1}{2}\int_{\mathbb{R}}(u_x^2 - 2F(u))(t,x)\psi(x-m(t))dx.$$

We claim the following results on $m(t)$, a and $\mathcal{G}(t)$.

Claim 18. The following estimates hold

$$\frac{1}{2} \leq m'(t) \leq \frac{3}{2}. \tag{4.20}$$

and for a positive constant k_m,

$$a = k_m c + o(c). \tag{4.21}$$

(Here $o(c)$ means $|c^{-1}o(c)| \to 0$ as $c \to 0$.)

Proof. To prove (4.20), it is enough to consider Lemma 3.4 on the interval $[T_c, T_0]$ to have

$$m'(t) \geq 1 - \frac{1}{10} \geq \frac{1}{2}; \quad m'(t) \leq 1 + \frac{1}{10} \leq \frac{3}{2}.$$

Let us now treat (4.21). It is easy to show that

$$M(Q_c) = c^{\frac{2}{m-1}-\frac{1}{2}} \int_{\mathbb{R}} Q^2 + o(c^{\frac{2}{m-1}-\frac{1}{2}}).$$

On the other hand,

$$E(Q_c) = \frac{1}{2} c^{\frac{2}{m-1}+\frac{1}{2}} \Big[\int_{\mathbb{R}} Q'^2 - \frac{2}{m+1} \int_{\mathbb{R}} Q^{m+1} \Big] + o(c^{\frac{m+1}{m-1}+\frac{1}{2}}).$$

Thus, from (4.11) and the fact that $E(Q) < 0$, a Taylor expansion and L'Hopital rule gives

$$a = -\frac{\partial_c E(Q_c)}{\partial_c M(Q_c)}\Big|_{c=\bar{c}(T_c)} + O(|\bar{c}(T_0) - \bar{c}(T_c)|) = -\frac{E(Q)}{M(Q)} \bar{c}(T_c) + o(c) = k_m c + o(c). \qquad (4.22)$$

where $k_m := -\frac{E(Q)}{M(Q)} > 0$ is a constant depending on m. □

Lemma 4.2. *For $0 < c < c_0$ small enough,*

$$\mathcal{G}(T_c) - \mathcal{G}(T_0) \leq K c^{10}.$$

Proof. We will need the following
Claim 19. Define $h := u_{xx} + f(u)$, such that $u_t = -h_x$. Then

$$\mathcal{M}_1'(t) = -\frac{3}{2} a \int_{\mathbb{R}} u_x^2 \psi' + \frac{a}{2} \int_{\mathbb{R}} u^2 (\psi''' - m'\psi') + a \int_{\mathbb{R}} (uf(u) - F(u))\psi',$$

and

$$\begin{aligned} \mathcal{E}_1'(t) &= -\frac{3}{2} \int_{\mathbb{R}} h^2 \psi' - \frac{1}{2} \int_{\mathbb{R}} u_x^2 (m'\psi' - \psi''') + \int_{\mathbb{R}} F(u)(m'\psi' + \psi''') \\ &\quad + \int_{\mathbb{R}} f^2(u)\psi' - \int_{\mathbb{R}} u_x^2 f'(u)\psi'. \end{aligned}$$

Proof. A direct computation, see for example [58]. □

Now, we follow the proof contained in Appendix D, [58]. From above Claim, we have

$$\begin{aligned} \mathcal{G}'(t) &= \frac{3}{2} \int_{\mathbb{R}} h^2 \psi' + \frac{1}{2} \int_{\mathbb{R}} u_x^2 (m'\psi' - \psi''' + 3a\psi') + \frac{a}{2} \int_{\mathbb{R}} u^2 (m'\psi' - \psi''') \\ &\quad -a \int_{\mathbb{R}} (uf(u) - F(u))\psi' - \int_{\mathbb{R}} F(u)(m'\psi' + \psi''') - \int_{\mathbb{R}} f^2(u)\psi' + \int_{\mathbb{R}} u_x^2 f'(u)\psi'. \end{aligned}$$

From Claim 18 we choose $\kappa > 0$ large enough such that $m'\psi' - \psi''' \geq \frac{1}{4}\psi'$. From here,

$$\frac{3}{2} \int_{\mathbb{R}} h^2 \psi' + \frac{1}{2} \int_{\mathbb{R}} u_x^2 (m'\psi' - \psi''' + 3a\psi') + \frac{a}{2} \int_{\mathbb{R}} u^2 (m'\psi' - \psi''') \geq \frac{c}{4} \int_{\mathbb{R}} (u_x^2 + u^2)\psi'$$

Let us consider now the nonlinear terms in the second row of $\mathcal{G}'(t)$. For this, let

$$I := [r_1 + \bar{\rho}_2(t) + \frac{1}{8} T_c, r_1 + \bar{\rho}_1(t) - \frac{1}{8} T_c]$$

an interval between the two solitons. We have two cases: $x \in I$ and $x \notin I$.

In the first case, from (4.10) we have for all $t \geq T_c$

$$|u(t)| \leq |Q_{\bar{c}_1}|(t) + |Q_{\bar{c}_2}|(t) + |\eta|(t) \leq K|d(\varepsilon)| c^{\frac{2}{m-1}-\frac{1}{4}}.$$

Thus,

$$\begin{aligned}
&|-a\int_I (uf(u)-F(u))\psi' - \int_I F(u)(m'\psi'+\psi''') - \int_I f^2(u)\psi' + \int_I u_x^2 f'(u)\psi'| \\
&\leq K\Big[\|u(t)\|_{L^\infty(I)}^{m-1} + \|u(t)\|_{L^\infty(I)}^{2(m-1)}\Big]\int_{\mathbb{R}}(u^2+u_x^2)\psi' \\
&\leq K|d(\varepsilon)|^{m-1}c^{2-\frac{1}{4}(m-1)}\Big[|d(\varepsilon)|^{m-1}c^{2-\frac{1}{4}(m-1)}+1\Big]\int_{\mathbb{R}}(u^2+u_x^2)\psi' \\
&\leq Kc^{\frac{3}{2}}\int_{\mathbb{R}}(u^2+u_x^2)\psi'.
\end{aligned}$$

In the second case, we have $|x-m(t)| \geq \frac{t}{4}$ and thus $\psi'(x-m(t)) \leq Ke^{-\gamma t}$, with $\gamma > 0$ a fixed constant. From here,

$$|-a\int_{x\notin I}(uf(u)-F(u))\psi' - \int_{x\notin I}F(u)(m'\psi'+\psi''') - \int_{x\notin I}f^2(u)\psi' + \int_{x\notin I}u_x^2 f'(u)\psi'| \leq Ke^{-\gamma t}.$$

In conclusion, putting together above estimates, we get for all $t \in [T_c, T_0]$,

$$\mathcal{G}'(t) \geq -Ke^{-\gamma t},$$

and after integration we obtain the desired result. The proof is now complete. □

Now, define

$$\mathcal{H}(t) := \int_{\mathbb{R}} \big[a\eta^2 + \eta_x^2 - f'(R_2)\eta^2\big](1-\psi).$$

We have the

Lemma 4.3. *For $0 < c < c_0$ small enough,*

1. *Small variation:*

$$\begin{aligned}
\mathcal{G}(T_c) - \mathcal{G}(T_0) \;&=\; \frac{1}{2}(\mathcal{H}(T_c)-\mathcal{H}(T_0)) + O(\alpha^2|d(\varepsilon)|^3 c^{\frac{6}{m-1}+\frac{1}{4}}c^{\frac{m-2}{m-1}}) \\
&\quad + O(|d(\varepsilon)|c^{\frac{2}{m-1}-\frac{1}{4}}c^{\frac{m-2}{m-1}}\int_{\mathbb{R}}\eta^2(T_c)(1-\psi)) + O(c^{10}), \quad (4.23)
\end{aligned}$$

2. *Coercivity:*

$$\mathcal{H}(t) \geq \sigma_0 \int_{\mathbb{R}} \big[c\eta^2 + \eta_x^2\big](t,x)(1-\psi)dx. \quad (4.24)$$

for some $\sigma_0 > 0$ independent of c.

Proof. Let us first prove (4.23). We replace $u = R_1 + R_2 + \eta$ in the definition of $\mathcal{G}$. We obtain

$$\begin{aligned}
\mathcal{M}(t) \;&=\; \frac{1}{2}\int_{\mathbb{R}}(R_1+R_2+\eta)^2(1-\psi) \\
&=\; \frac{1}{2}\int_{\mathbb{R}}R_2^2(1-\psi) + \int_{\mathbb{R}}\eta R_2(1-\psi) + \frac{1}{2}\int_{\mathbb{R}}\eta^2(1-\psi) + O(c^{10}).
\end{aligned}$$

Here we have used the estimate for $t \geq T_c$

$$|\int_{\mathbb{R}} R_1(t)(1-\psi)| \leq Ke^{-\frac{1}{2}t} \leq Kc^{10},$$

among other similar estimates.

In the same way,

$$\frac{1}{2}\int_{\mathbb{R}} u_x^2(1-\psi) = \frac{1}{2}\int_{\mathbb{R}}(R_2)_x^2(1-\psi) + \int_{\mathbb{R}}\eta_x(R_2)_x(1-\psi) + \frac{1}{2}\int_{\mathbb{R}}\eta_x^2(1-\psi) + O(c^{10}).$$

Finally, using the character exponentially decreasing of R_1 where $1-\psi$ is away from zero,

$$\begin{aligned}
\int_{\mathbb{R}} F(u)(1-\psi) &= \int_{\mathbb{R}} F(R_1+R_2+\eta)(1-\psi) \\
&= \int_{\mathbb{R}}[F(R_1+R_2+\eta)-F(R_2+\eta)](1-\psi) \\
&\quad + \int_{\mathbb{R}}[F(R_2+\eta)-F(R_2)-f(R_2)\eta-\frac{1}{2}f'(R_2)\eta^2](1-\psi) \\
&\quad + \int_{\mathbb{R}}[F(R_2)+f(R_2)\eta+\frac{1}{2}f'(R_2)\eta^2](1-\psi) \\
&= \int_{\mathbb{R}}[f(R_2+\eta)R_1+O(R_1^2)](1-\psi) + O(\|R_2\|_{L^\infty(\mathbb{R})}^{m-2}\int_{\mathbb{R}}|\eta(t)|^3(1-\psi)) \\
&\quad + \int_{\mathbb{R}}[F(R_2)+f(R_2)\eta+\frac{1}{2}f'(R_2)\eta^2](1-\psi) \\
&= \int_{\mathbb{R}}[F(R_2)+f(R_2)\eta+\frac{1}{2}f'(R_2)\eta^2](1-\psi) \\
&\quad +O\big[\|R_2\|_{L^\infty(\mathbb{R})}^{m-2}\|\eta(t)\|_{H^1}\int_{\mathbb{R}}\eta^2(t)(1-\psi)\big] + O(c^{10}).
\end{aligned}$$

From this,

$$\mathcal{G}(t) = \mathcal{G}[R_2](t) + \mathcal{H}(t) + O\big[\|R_2\|_{L^\infty(\mathbb{R})}^{m-2}\|\eta(t)\|_{H^1}\int_{\mathbb{R}}\eta^2(t)(1-\psi)\big] + O(c^{10}).$$

Putting together these estimates, using the value of a, evaluating at times $t=T_c$ and $t=T_0$ and using (4.18) and (4.11), we obtain the desired result.

The proof of (4.24) is standard, see e.g. [52] Appendix B.3.

□

Combining Lemmas 4.2 and 4.3, we find

$$\begin{aligned}
\int_{\mathbb{R}}\left[c\eta^2+\eta_x^2\right](T_c)(1-\psi) &\leq K\mathcal{H}(T_c) \\
&\leq K\mathcal{H}(T_0) + K|d(\varepsilon)|c^{\frac{1}{m-1}+\frac{3}{4}}\int_{\mathbb{R}}\eta^2(T_c)(1-\psi) \\
&\qquad + K\alpha^2|d(\varepsilon)|^3c^{\frac{5}{m-1}+\frac{5}{4}} + O(c^{10}) + K(\mathcal{G}(T_c)-\mathcal{G}(T_0)) \\
&\leq K\alpha^2|d(\varepsilon)|^2c^{\frac{4}{m-1}+\frac{3}{2}} + K\alpha^2|d(\varepsilon)|^3c^{\frac{5}{m-1}+\frac{5}{4}} + Kc^{10}. \qquad (4.25)
\end{aligned}$$

The last inequality is consequence of

$$|d(\varepsilon)|c^{\frac{1}{m-1}+\frac{3}{4}} \ll \frac{c}{K}, \quad m=2,3 \text{ and } 4;$$

therefore the term

$$K|d(\varepsilon)|c^{\frac{1}{m-1}+\frac{3}{4}}\int_{\mathbb{R}}\eta^2(T_c)(1-\psi)$$

can be sent to the left hand side of (4.25). Using (4.14) we finally get

$$|d(\varepsilon)|^2 c^{\frac{4}{m-1}+\frac{3}{2}} \leq K\alpha^2 |d(\varepsilon)|^2 c^{\frac{4}{m-1}+\frac{3}{2}} + K\alpha^2 |d(\varepsilon)|^3 c^{\frac{5}{m-1}+\frac{5}{4}}.$$

But this estimate is a contradiction for $\alpha > 0$ small enough and $0 < c < c_0$ small enough (it is enough to put ε even smaller). The proof of Claim 4.1 is now complete. □

5. Lower bounds on the parameters. We finally prove (1.27). This result is a consequence of Theorem 1.2, (1.17), (1.18) and (1.19), see also [54] for the proof. Indeed, from (1.17) and (1.26), we have

$$\frac{1}{K}|d(\varepsilon)|^2 c^{\frac{3}{2}+\frac{4}{m-1}} \leq 2E^+ + cM^+ \leq \frac{1}{K}|d(\varepsilon)|^2 c^{\frac{1}{2}+\frac{4}{m-1}}.$$

The final conclusion follows from (1.18), (1.19) and (2.29)-(2.30).

This finishes the proof of the Theorem 1.4. □

Appendices

L Proof of Proposition 2.2

The proof is similar to Proposition 2.2 in [54] and Appendix in [53]. The main difference consists in the fact that we need to know explicitly all linear systems up to order $m+1$ to show the nonexistence of growing solutions. We will discard several trivial terms by using the property **(IP)**. For this purpose it is better to state an improved version of Proposition 2.2. Before that we introduce a useful notation.

Definition L.1. Consider $f, g : \mathbb{R} \to \mathbb{R}$ given functions. We say that $f = g \mod \mathcal{Y}$ if there exists $h \in \mathcal{Y}$ such that $f = g + h$.

In our case, this definition will be useful to discard localized functions in the source terms. Indeed,

Proposition L.1 (Decomposition of $S(\tilde{u})$, improved version)**.** *Assume that f is of class C^{m+2}. Let*

$$\mathcal{L}w = -w_{yy} + w - f'(Q)w. \tag{L.1}$$

Then,

$$\begin{aligned} S[\tilde{u}](t,x) = &\sum_{(k,l)\in\Sigma_m} c^l Q_c^k(y_c)\Big[a_{k,l}(-3Q+2f(Q))'(y) - (\mathcal{L}A_{k,l})'(y) + F_{k,l}(y)\Big] \\ &+ \sum_{(k,l)\in\Sigma_m} c^l (Q_c^k)'(y_c)\Big[a_{k,l}(-3Q'')(y) + \left(3A''_{k,l} + f'(Q)A_{k,l}\right)(y) - (\mathcal{L}B_{k,l})'(y) + G_{k,l}(y)\Big] \\ &+ \mathcal{E}(t,x) \end{aligned} \tag{L.2}$$

where $F_{k,l}$, $G_{k,l}$ and $\mathcal{E}$ satisfy, for any $(k,l) \in \Sigma_m$,

(i) *Dependence property of $F_{k,l}$ and $G_{k,l}$: The expressions of $F_{k,l}$ and $G_{k,l}$ depend only on $(a_{k',l'})$, $(A_{k',l'})$, $(B_{k',l'})$ for $(k',l') < (k,l)$.*

(ii) *Parity property of $F_{k,l}$ and $G_{k,l}$: Assume that for any (k',l') such that $(k',l') < (k,l)$ $A_{k',l'}$ is even and $B_{k',l'}$ is odd, then $F_{k,l}$ is odd and $G_{k,l}$ is even.*

(iii) *Explicit source terms: We have $F_{1,0} = (f'(Q))'$ and $G_{1,0} = f'(Q)$,*

$$\begin{aligned} F_{2,0} &= -(3A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0}) + \frac{1}{2}(f''(Q)(2A_{1,0} + A_{1,0}^2))' \\ &\quad -a_{1,0}(3A''_{1,0} - Q + f'(Q)(1 + A_{1,0}))' + 3a_{1,0}^2 Q^{(3)} + \frac{1}{2}(f''(Q) - 2)', \end{aligned}$$

and

$$\begin{aligned} G_{2,0} &= \frac{1}{2}(f''(Q) - 2) - (A_{1,0} + 3B'_{1,0}) + \frac{1}{2}f''(Q)(2A_{1,0} + A_{1,0}^2) + \frac{3}{2}a_{1,0}^2 Q'' \\ &\quad -\frac{1}{2}a_{1,0}(9A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0})' + \frac{1}{2}(f''(Q)(B_{1,0} + A_{1,0}B_{1,0}))'. \end{aligned}$$

for the case $m = 2$, and

$$F_{2,0} = (\frac{1}{2}f''(Q)(1 + A_{1,0})^2)' + 3a_{1,0}^2 Q^{(3)} - a_{1,0}(f'(Q) + 3A''_{1,0} + f'(Q)A_{1,0})'$$

and

$$\begin{aligned} G_{2,0} &= \frac{1}{2}f''(Q)(1 + A_{1,0})^2 + \frac{3}{2}a_{1,0}^2 Q'' - \frac{1}{2}a_{1,0}(9A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0})' \\ &\quad +\frac{1}{2}(f''(Q)(1 + A_{1,0})B_{1,0})'. \end{aligned}$$

in the case $m = 3$. If property **(IP)** *holds for $(k,l) = (1,0)$, then each term above is in $\mathcal{Y}$.*

(iv) *Explicit high order source terms modulo $\mathcal{Y}$: Suppose property* **(IP)** *holds for $(k,l) \in \Sigma_m$ with $k + l \leq 2$. Then, for the quadratic case,*

$$F_{1,2}, G_{1,2}, F_{2,1} \text{ and } F_{3,0} \in \mathcal{Y}; \quad G_{3,0} = -\frac{2}{3}(B_{1,0}^2 + 2A_{2,0}) \mod \mathcal{Y},$$

and

$$G_{2,1} = B_{1,0}^2 + A_{1,1} + 3A_{2,0} \mod \mathcal{Y}.$$

For the cubic case,

$$F_{3,0}, G_{3,0}, F_{2,1}, F_{4,0} \text{ and } G_{4,0} \in \mathcal{Y}, \quad G_{2,1} = 3A_{2,0} \mod \mathcal{Y}.$$

(v) *Improved estimate on $\mathcal{E}$: Suppose in addition that property* **(IP)** *holds for any $(k,l) \in \Sigma_m$, then for all $j = 0, 1, 2$*

$$\|\partial_x^j \mathcal{E}(t,x)\|_{H^1(\mathbb{R})} \leq K c^{\frac{3}{4} + \frac{3}{m-1}}.$$

Proof. Expansion (L.2), and items (i) and (ii) were proven in [54], so in what follows we deal with (iii)-(v). For this it is necesary to improve the computation done in [54].

We start with an important lemma concerning the algebra of Q_c.

Lemma L.2 (Properties of Q_c, see Lemma 2.1 in [54]). *Suppose $0 < c \leq 1$, $0 < \varepsilon \leq \varepsilon_0$ small, $k \in \{1, \dots, k_0\}$, and $m = 2, 3$. Then*

1. *There exists a positive constant $K = K(\varepsilon) > 0$ such that*

$$\frac{1}{K} c^{\frac{1}{m-1}} e^{-\sqrt{c}|x|} \leq Q_c(x) \leq K c^{\frac{1}{m-1}} e^{-\sqrt{c}|x|}, \quad |Q'_c(x)| \leq K c^{\frac{1}{m-1} + \frac{1}{2}} e^{-\sqrt{c}|x|}. \qquad \text{(L.3)}$$

2. *For F defined in (1.9) and any $k \geq 1$,*

$$Q_c'' = cQ_c - f(Q_c), \quad Q_c'^2 = cQ_c^2 - 2F(Q_c). \tag{L.4}$$

$$(Q_c^k)'' = ck^2 Q_c^k - 2k(k-1)Q_c^{k-2}F(Q_c) - kf(Q_c)Q_c^{k-1}. \tag{L.5}$$

We recall the notation introduced in Subsection (2.1):

$$S[\tilde{u}] = \tilde{u}_t + (\tilde{u}_{xx} - \tilde{u} + f(\tilde{u}))_x.$$

We easily verify that

$$S[\tilde{u}] = \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}, \tag{L.6}$$

where (we omit the dependence on t, x)

$$\mathbf{I} := S[R], \quad \mathbf{II} := (f(R + R_c) - f(R) - f(R_c))_x,$$

and

$$\begin{aligned} &\mathcal{L} = -\partial_x^2 + 1 - f'(Q), \\ &\mathbf{III} = \mathbf{III}(W) := W_t - (\mathcal{L}W)_x, \\ &\mathbf{IV} := \{f(R + R_c + W) - f(R + R_c) - f(R)W\}_x. \end{aligned} \tag{L.7}$$

Since $Q_c(y_c)$ is a solution to (1.11), we have $S(Q_c) = 0$.

Claim 20. Let $A = A(y)$ and $q = q(y_c)$ be C^3-functions with y, y_c defined in Section 2.1. Then

$$\begin{aligned} \mathbf{III}(Aq) \quad = \quad & -q(\mathcal{L}A)' + q'(3A'' + f'(Q)A) \\ & + q(-3\beta A^{(3)} - \beta A' - 3\beta_x A'' - A'\beta_{xx} + \beta A' - \beta(f'(Q)A)') \\ & + q(3\beta^2 A^{(3)} + 3\beta\beta_x A'' - \beta^3 A^{(3)} + c\beta A') \\ & + q'(-cA - 6A''\beta - 3A'\beta_x + 3A''\beta^2) \\ & + q''(3A' - 3A'\beta) + Aq^{(3)}. \end{aligned}$$

Proof. Direct differentiation, see [54], Proposition 2.2. □

Claim 21. Recall from (2.4),

$$\beta = \sum_{(k,l) \in \Sigma_m} a_{k,l} c^l Q_c^k(y_c) \tag{L.8}$$

Then, for some fixed numbers $\hat{a}_{k,l}^1, \hat{a}_{k,l}^2, \bar{a}_{k,l}, \tilde{a}_{k,l}$ with $(k,l) \in \Sigma_m$, depending only on $a_{k',l'}$ with $(k', l') \leq (k, l)$, we have

$$\begin{cases} \beta_x = \sum_{(k,l)\in\Sigma_m} a_{k,l} c^l (Q_c^k)'(y_c), \\ \beta_{xx} = \sum_{\substack{(k,l)\in\Sigma_m \\ l\geq 1}} \hat{a}_{k,l}^1 c^l Q_c^k(y_c) + \sum_{\substack{(k,l)\in\Sigma_m \\ k\geq m}} \hat{a}_{k,l}^2 c^l Q_c^k(y_c) + O(Q_c^5 + cQ_c^3), \\ \beta^2 = \sum_{\substack{(k,l)\in\Sigma_m \\ k\geq 2}} \bar{a}_{k,l} c^l Q_c^k(y_c) + O(Q_c^5 + cQ_c^3) \\ (\beta^2)_x = \sum_{\substack{(k,l)\in\Sigma_m \\ k\geq 2}} \bar{a}_{k,l} c^l (Q_c^k)'(y_c) + O(Q_c^5 + cQ_c^3), \text{ and} \\ \beta^3 = \sum_{\substack{(k,l)\in\Sigma_m \\ k\geq 3}} \tilde{a}_{k,l} c^l Q_c^k(y_c) + O(Q_c^5 + cQ_c^3). \end{cases}$$

Proof. The proof follows by elementary calculations from (2.4). □

In the next lemmas, we expand the terms in (L.6).

Lemma L.3.

$$\begin{aligned}\mathbf{I} = &\sum_{(k,l)\in\Sigma_m} c^l \left[Q_c^k(y_c)a_{k,l}(2f(Q)-3Q)'(y) + (Q_c^k)'(y_c)(-3a_{k,l}Q''(y))\right]\\ &+ \sum_{(k,l)\in\Sigma_m} c^l \left(Q_c^k(y_c)F^I_{k,l}(y) + (Q_c^k)'(y_c)G^I_{k,l}(y)\right) + c^3 O(Q_c(y_c)),\end{aligned} \tag{L.9}$$

where

$$F^I_{1,0} = G^I_{1,0} = F^I_{1,1} = G^I_{1,1} = 0,$$
$$F^I_{2,0} = 3a_{1,0}^2 Q^{(3)} + a_{1,0}Q'\delta_{m2}, \quad G^I_{2,0} = \frac{3}{2}a_{1,0}^2 Q'',$$

and for all $(k,l)\in\Sigma_m$, $F^I_{k,l}\in\mathcal{Y}$ is odd, $G^I_{k,l}\in\mathcal{Y}$ is even and depend only on $a_{k',l'}$ for $(k',l')<(k,l)$.

Proof of Lemma L.3. We have (here $'$ denotes derivative with respect to y)

$$\begin{aligned}\mathbf{I} &= R_t + (R_{xx} - R + f(R))_x\\ &= -(1-c)\beta Q' + (f(Q))'(1-\beta) - Q'(1-\beta) + (Q''(1-\beta)^2 - Q'\beta_x)_x\\ &= (Q'' - Q + f(Q))' + Q^{(3)}(-3\beta + 3\beta^2 - \beta^3) - 3Q''(\beta_x - \beta\beta_x) - \beta_{xx}Q' - \beta(f(Q))' + c\beta Q'\\ &= -[3\beta Q^{(3)} + 3Q''\beta_x + \beta(f(Q))'] + 3\beta^2 Q^{(3)} + 3\beta\beta_x Q'' - \beta_{xx}Q' + c\beta Q' - \beta^3 Q^{(3)}.\end{aligned}$$

Hence using Claim 21, we obtain

$$\begin{aligned}\mathbf{I} = &\, a_{1,0}(2f(Q)-3Q)'Q_c(y_c) + a_{1,0}(-3Q'')Q_c'(y_c)\\ &+ \big(a_{2,0}(2f(Q)-3Q)' + 3a_{1,0}^2Q^{(3)} + a_{1,0}Q'\delta_{m2}\big)Q_c^2(y_c)\\ &+ \big(a_{2,0}(-3Q'') + \tfrac{3}{2}a_{1,0}^2Q''\big)(Q_c^2)'(y_c)\\ &+ \sum_{k+l=3,4} c^l\big(a_{k,l}(2f(Q)-3Q)'(y)Q_c^k(y_c) + a_{k,l}(-3Q'')(y)(Q_c^k)'(y_c)\big)\\ &+ \sum_{k+l=3,4} c^l\big(F^I_{k,l}Q_c^k(y_c) + G^I_{k,l}(Q_c^k)'(y_c)\big) + c^3 O(Q_c),\end{aligned}$$

where for all $k+l=3$, $F^I_{k,l}\in\mathcal{Y}$ and $G^I_{k,l}\in\mathcal{Y}$, as claimed in the statement of the Lemma. □

Lemma L.4.

$$\mathbf{II} = \sum_{(k,l)\in\Sigma_m} c^l \left(Q_c^k(y_c)F^{II}_{k,l}(y) + (Q_c^k)'(y_c)G^{II}_{k,l}(y)\right) + O(Q_c^{m+2}),$$

where for all $(k,l)\in\Sigma_m$ and for all $p\geq m+1$, $F^{II}_{k,l}, G^{II}_{k,l}\in\mathcal{Y}$ and are odd and even respectively. Moreover, for $m=2$,

$$F^{II}_{1,0} = (f'(Q))', \quad G^{II}_{1,0} = f'(Q), \quad F^{II}_{1,1} = G^{II}_{1,1} = 0,$$
$$F^{II}_{2,0} = (\frac{1}{2}f''(Q) - a_{1,0}f'(Q))', \quad G^{II}_{2,0} = \frac{1}{2}f''(Q) - 1.$$

Finally, if $m=3$,

$$F^{II}_{1,0} = (f'(Q))', \quad G^{II}_{1,0} = f'(Q), \quad F^{II}_{1,1} = G^{II}_{1,1} = 0,$$
$$F^{II}_{2,0} = (\frac{1}{2}f''(Q) - a_{1,0}f'(Q))', \quad G^{II}_{2,0} = \frac{1}{2}f''(Q).$$

Proof. First define $\tilde{\mathbf{II}} := f(R+R_c) - f(R) - f(R_c)$. Note that

$$\tilde{\mathbf{II}} = f'(R)R_c + \frac{1}{2}f''(R)R_c^2 + \frac{1}{6}f^{(3)}(R)R_c^3 - f(R_c) + \frac{1}{24}f^{(4)}(R)R_c^4 + O(R_c^5),$$

Thus taking derivative

$$\begin{aligned}\mathbf{II} &= (f'(Q))'(1-\beta)Q_c + f'(Q)Q_c' + \frac{1}{2}(f''(Q))'(1-\beta)Q_c^2 + \frac{1}{2}f''(Q)(Q_c^2)' + \frac{1}{6}(f^{(3)}(Q))'(1-\beta)Q_c^3 \\ &\quad + \frac{1}{6}f^{(3)}(Q)(Q_c^3)' + \frac{1}{24}(f^{(4)}(Q))'Q_c^4 + \frac{1}{24}f^{(4)}(Q)(Q_c^4)' - (f(Q_c))' + O(Q_c^5). \qquad \text{(L.10)}\end{aligned}$$

Here we have to identify two different results, depending on the value of m. For $m = 2$, namely, the quadratic case, we will need only up to third order terms. After replacing the value of β given by (L.8), we will obtain (recall that $p \geq 3$)

$$\begin{aligned}\mathbf{II} &= (f'(Q))'Q_c + f'(Q)Q_c' + (\frac{1}{2}f''(Q) - a_{1,0}f'(Q))'Q_c^2 + (\frac{1}{2}f''(Q) - 1)(Q_c^2)' - a_{1,1}(f'(Q))'cQ_c^2 \\ &\quad + (\frac{1}{6}f^{(3)}(Q) - \frac{1}{2}a_{1,0}f''(Q) - a_{2,0}f'(Q))'Q_c^3 + \frac{1}{6}f^{(3)}(Q)(Q_c^3)' - \varepsilon(Q_c^p)' - (f_1(Q_c))' + O(Q_c^4).\end{aligned}$$

It is easy to check that every term depending on y up to order Q_c^3, $(Q_c^3)'$ is indeed in the class $\mathcal{Y}$. Even in the worst case, $p = 3$, we will have the cancelation

$$\frac{1}{6}f^{(3)}(Q)(Q_c^3)' - \varepsilon(Q_c^p)' = \frac{1}{6}f_1^{(3)}(Q)(Q_c^3)',$$

with $\frac{1}{6}f_1^{(3)}(Q) \in \mathcal{Y}$.

Let us consider now the cubic case, $m = 3$. The procedure is completely similar, although we must keep the fourth order terms. We start by replacing β in (L.10) and collecting similar terms

$$\begin{aligned}\mathbf{II} &= (f'(Q))'(1-\beta)Q_c + f'(Q)Q_c' + \frac{1}{2}(f''(Q))'(1-\beta)Q_c^2 + \frac{1}{2}f''(Q)(Q_c^2)' + \frac{1}{6}(f^{(3)}(Q))'(1-\beta)Q_c^3 \\ &\quad + \frac{1}{6}f^{(3)}(Q)(Q_c^3)' + \frac{1}{24}(f^{(4)}(Q))'Q_c^4 + \frac{1}{24}f^{(4)}(Q)(Q_c^4)' - (f(Q_c))' + O(Q_c^5). \\ &= (f'(Q))'Q_c + f'(Q)Q_c' + (\frac{1}{2}f''(Q) - a_{1,0}f'(Q))'Q_c^2 + \frac{1}{2}f''(Q)(Q_c^2)' \\ &\quad (\frac{1}{6}f^{(3)}(Q) - \frac{1}{2}a_{1,0}f''(Q) - a_{2,0}f'(Q))'Q_c^3 + (\frac{1}{6}f^{(3)}(Q) - 1)(Q_c^3)' - a_{1,1}(f'(Q))'cQ_c^2 \\ &\quad + (\frac{1}{24}f^{(4)}(Q) - \frac{1}{6}a_{1,0}f^{(3)}(Q) - \frac{1}{2}a_{2,0}f''(Q) - a_{3,0}f'(Q))'Q_c^4 + \frac{1}{24}f^{(4)}(Q)(Q_c^4)' \\ &\quad - (\varepsilon Q_c^p + f_1(Q_c))' + O(cQ_c^3 + cQ_c^4 + Q_c^5).\end{aligned}$$

It is straightforward to check that every function depending on y is indeed in $\mathcal{Y}$. The only complicated terms are (note that $p \geq 4$)

$$\frac{1}{6}f^{(3)}(Q) - 1 = \frac{1}{6}p(p-1)(p-2)\varepsilon Q^{p-3} + \frac{1}{6}f_1^{(3)}(Q) \in \mathcal{Y},$$

which is in front of $(Q_c^3)'$; and for $p = 4$, facing $(Q_c^4)'$ we have

$$\frac{1}{24}f^{(4)}(Q) - \varepsilon = \varepsilon + \frac{1}{24}f_1^{(4)}(Q) - \varepsilon = \frac{1}{24}f_1^{(4)}(Q) \in \mathcal{Y}.$$

□

Lemma L.5.

$$\mathbf{III} = \sum_{(k,l)\in\Sigma_m} c^l \left(Q_c^k(y_c)(-\mathcal{L}A_{k,l})'(y) + (Q_c^k)'(y_c)((-\mathcal{L}B_{k,l})' + 3A''_{k,l} + f'(Q)A_{k,l})(y) \right)$$
$$+ \sum_{(k,l)\in\Sigma_m} c^l \left(Q_c^k(y_c)F_{k,l}^{III}(y) + (Q_c^k)'(y_c)G_{k,l}^{III}(y) \right) + O(Q_c^{m+2}),$$

where

$$F_{1,0}^{II} = 0, \quad G_{1,0}^{II} = 0, \quad F_{1,1}^{II} = 3A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0}, \quad G_{1,1}^{II} = 3B'_{1,0},$$
$$F_{2,0}^{II} = -a_{1,0}(3A''_{1,0} + f'(Q)A_{1,0})' - (3A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0})\delta_{m2}$$
$$G_{2,0}^{II} = -\frac{1}{2}a_{1,0}\left(9A'_{1,0} + 3B''_{1,0} + f'(Q)B_{1,0}\right)' - (A_{1,0} + 3B'_{1,0})\delta_{m2},$$

and for $(k,l) \in \Sigma_m$, $F_{k,l}^{II}$, $G_{k,l}^{II}$ *depend on* $A_{k',l'}$, $B_{k',l'}$ *such that* $(k',l') < (k,l)$. *Moreover, if* $A_{k',l'}$ *are even and* $B_{k',l'}$ *are odd then* $F_{k,l}^{II}$ *are odd and* $G_{k,l}^{II}$ *are even.*

Finally, the following important property holds. Suppose **(IP)** *holds for any* $(k,l) \in \Sigma_m$ *with* $k + l \leq 2$. *Then we have a sharp decomposition for each high order source term:*

1. *For* $m = 2$,
$$F_{3,0}^{III} = 0, \quad G_{3,0}^{III} = -\frac{10}{3}A_{2,0}, \quad F_{2,1}^{III} = 0, \quad G_{2,1}^{III} = -A_{1,1}+3A_{2,0}, \quad F_{1,2}^{III} = G_{1,2}^{III} = 0 \mod \mathcal{Y}. \tag{L.11}$$

2. *For* $m = 3$,
$$F_{3,0}^{III}, G_{3,0}^{III}, F_{2,1}^{III} \in \mathcal{Y}, \quad G_{2,1}^{III} = 3A_{2,0}, \quad F_{4,0}^{III} = 0, \quad G_{4,0}^{III} = -3A_{2,0} \mod \mathcal{Y}. \tag{L.12}$$

Proof. We have, thanks to the linearity of the operator $\mathbf{III}(\cdot)$,

$$\mathbf{III}(W) = \sum_{(k,l)\in\Sigma_m} c^l \left(\mathbf{III}(A_{k,l}(y)Q_c^k(y_c)) + \mathbf{III}(B_{k,l}(y)(Q_c^k)'(y_c)) \right).$$

In what follows, for commodity of notation we omit the variables y, y_c, if there is no related confusion. First, we compute $\mathbf{III}(A_{1,0}(y)Q_c(y_c))$. By Claim 20 and the definition of β, we have

$$\begin{aligned}
\mathbf{III}(A_{1,0}Q_c) &= -Q_c(\mathcal{L}A_{1,0})' + Q_c'(3A''_{1,0} + f'(Q)A_{1,0}) \\
&\quad + Q_c(-3\beta A_{1,0}^{(3)} - \beta A'_{1,0} - 3\beta_x A''_{1,0} - A'_{1,0}\beta_{xx} + \beta A'_{1,0} - \beta(f'(Q)A_{1,0})') \\
&\quad + Q_c(3\beta^2 A_{1,0}^{(3)} + 3\beta\beta_x A''_{1,0} - \beta^3 A_{1,0}^{(3)} + c\beta A'_{1,0}) \\
&\quad + Q_c'(-cA_{1,0} - 6A''_{1,0}\beta - 3A'_{1,0}\beta_x + 3A''_{1,0}\beta^2) \\
&\quad + Q_c''(3A'_{1,0} - 3A'_{1,0}\beta) + A_{1,0}Q_c^{(3)} \\
&= -(\mathcal{L}A_{1,0})'Q_c + (3A''_{1,0} + f'(Q)A_{1,0})Q_c' + 3A''_{1,0}cQ_c \\
&\quad -(3a_{1,0}A''_{1,0} + a_{1,0}f'(Q)A_{1,0} + 3A_{1,0}\delta_{m2})'Q_c^2 - (\frac{9}{2}a_{1,0}A''_{1,0} + A_{1,0}\delta_{m2})(Q_c^2)' \\
&\quad + \sum_{3\leq k+l\leq 4} c^l(F_{k,l}Q_c^k + G_{k,l}(Q_c^k)') + O(cQ_c^3 + Q_c^5 + c^2Q_c).
\end{aligned}$$

Moreover, by hypothesis $A_{1,0} \in \mathcal{Y}$ so we have all the source terms $F_{k,l}, G_{k,l} \in \mathcal{Y}$, as can be verified directly.

Now, we compute $\mathbf{III}(B_{1,0}(y)Q_c'(y_c))$ in a similar way:

$$\begin{aligned}
\mathbf{III}(B_{1,0}Q_c') &= -Q_c'(\mathcal{L}B_{1,0})' + Q_c''(3B_{1,0}'' + f'(Q)B_{1,0}) \\
&\quad +Q_c'(-3\beta B_{1,0}^{(3)} - \beta B_{1,0}' - 3\beta_x B_{1,0}'' - B_{1,0}'\beta_{xx} + \beta B_{1,0}' - \beta(f'(Q)B_{1,0})') \\
&\quad +Q_c'(3\beta^2 B_{1,0}^{(3)} + 3\beta\beta_x B_{1,0}'' - \beta^3 B_{1,0}^{(3)} + c\beta B_{1,0}') \\
&\quad +Q_c''(-cB_{1,0} - 6B_{1,0}''\beta - 3B_{1,0}'\beta_x + 3B_{1,0}''\beta^2) \\
&\quad +Q_c^{(3)}(3B_{1,0}' - 3B_{1,0}'\beta) + B_{1,0}Q_c^{(4)} \\
&= -(\mathcal{L}B_{1,0})'Q_c' - Q_c^2(3B_{1,0}'' + f'(Q)B_{1,0})\delta_{m2} + (3B_{1,0}'' + f'(Q)B_{1,0})cQ_c \\
&\quad -(3a_{1,0}A_{1,0}'' + a_{1,0}f'(Q)A_{1,0} + 3A_{1,0}\delta_{m2})'Q_c^2 - (\frac{9}{2}a_{1,0}A_{1,0}'' + A_{1,0}\delta_{m2})(Q_c^2)' \\
&\quad + \sum_{3\leq k+l\leq 4} c^l(F_{k,l}Q_c^k + G_{k,l}(Q_c^k)') + O(cQ_c^3 + Q_c^5 + c^2Q_c).
\end{aligned}$$

Suppose now that $2 \leq k+l \leq 4$. Here we will use **(IP)** for $k+l \leq 2$ to discard several terms of a tedious but direct computation. Indeed, from Claim 20 we have

$$\begin{aligned}
\mathbf{III}(A_{k,l}Q_c^k) &= -Q_c^k(\mathcal{L}A_{k,l})' + (Q_c^k)'(3A_{k,l}'' + f'(Q)A_{k,l}) \\
&\quad +Q_c^k(-3\beta A_{k,l}^{(3)} - \beta A_{k,l}' - 3\beta_x A_{k,l}'' - A_{k,l}'\beta_{xx} + \beta A_{k,l}' - \beta(f'(Q)A_{k,l})') \\
&\quad +Q_c^k(3\beta^2 A_{k,l}^{(3)} + 3\beta\beta_x A_{k,l}'' - \beta^3 A_{k,l}^{(3)} + c\beta A_{k,l}') \\
&\quad +(Q_c^k)'(-cA_{k,l} - 6A_{k,l}''\beta - 3A_{k,l}'\beta_x + 3A_{k,l}''\beta^2) \\
&\quad +(Q_c^k)''(3A_{k,l}' - 3A_{k,l}'\beta) + A_{k,l}(Q_c^k)^{(3)} \\
&= -Q_c^k(\mathcal{L}A_{k,l})' + (Q_c^k)'(3A_{k,l}'' + f'(Q)A_{k,l}) + A_{k,l}(Q_c^k)^{(3)} - A_{k,l}c(Q_c^k)' \\
&\quad + \sum_{\substack{(k',l')\in\Sigma_m' \\ (k,l)\leq(k',l')}} c^{l'}(F_{k',l'}Q_c^{k'} + G_{k',l'}(Q_c^{k'})') + O(cQ_c^3 + Q_c^5 + c^2Q_c).
\end{aligned}$$

Here $\Sigma_m' \subseteq \Sigma_m$ is the set of indices of *third order* in Σ_m. More specificaly,

$$\Sigma_2' := \{(1,2),(2,1),(3,0)\}, \quad \Sigma_3' := \{(2,1),(3,0),(4,0)\}. \tag{L.13}$$

The terms describing $F_{k',l'}$ and $G_{k',l'}$ with $(k',l') \in \Sigma_m'$ are in $\mathcal{Y}$ provided **(IP)** is satisfied for every $(k,l) \in \Sigma_m \backslash \Sigma_m'$.

Now note that from (L.5)

$$(Q_c^k)^{(3)} = k^2(cQ_c^k)' - \frac{k(2k+m-1)}{m+1}(Q_c^{k+m-1})' - \varepsilon k\frac{k(2k+p-1)}{p+1}(Q_c^{k+p-1})' + O(Q_c^{k+p}).$$

We can finally conclude that

$$\begin{aligned}
\mathbf{III}(A_{k,l}Q_c^k) &= -Q_c^k(\mathcal{L}A_{k,l})' + (Q_c^k)'(3A_{k,l}'' + f'(Q)A_{k,l}) + (k^2-1)A_{k,l}c(Q_c^k)' \\
&\quad -\frac{k(2k+m-1)}{m+1}A_{k,l}(Q_c^{k+m-1})' + \sum_{(k',l')\in\Sigma_m'} c^{l'}(F_{k',l'}Q_c^{k'} + G_{k',l'}(Q_c^{k'})') \\
&\quad +O(cQ_c^3 + Q_c^5 + c^2Q_c),
\end{aligned}$$

where, as described above, the terms $F_{k',l'}$ and $G_{k',l'}$ with $(k',l') \in \Sigma_m'$ are in $\mathcal{Y}$ provided **(IP)** is satisfied for every $(k,l) \in \Sigma_m \backslash \Sigma_m'$.

On the other hand, the terms of the form

$$\mathbf{III}(B_{k,l}(Q_c^k)'), \quad (k,l)\in\Sigma_m, \quad 2\le k+l\le 4, \tag{L.14}$$

can be treated in the same way as above, and we only write the final result (see the computation of $\mathbf{III}(B_{1,0}Q_c')$ for example):

$$\begin{aligned}\mathbf{III}(B_{k,l}(Q_c^k)') &= -(Q_c^k)'(\mathcal{L}B_{k,l})' + B_{k,l}(Q_c^k)^{(4)} - B_{k,l}(cQ_c^k)'' \\ &\quad + \sum_{1\le k'\le 4} c^{l'}(F_{k',l'}Q_c^{k'} + G_{k',l'}(Q_c^{k'})') + O(cQ_c^3 + Q_c^5 + c^2Q_c) \\ &= -(Q_c^k)'(\mathcal{L}B_{k,l})' + \sum_{\substack{(k',l')\in\Sigma_m' \\ (k,l)\le(k',l')}} c^{l'}(F_{k',l'}Q_c^{k'} + G_{k',l'}(Q_c^{k'})') + O(cQ_c^3 + Q_c^5 + c^2Q_c).\end{aligned}$$

To obtain (L.11) and (L.12) we only evaluate the expressions for $\mathbf{III}(A_{k,l}Q_c^k)$ and $\mathbf{III}(B_{k,l}(Q_c^k)')$ for each $(k,l)\in\Sigma_m$ with $2\le k+l$. The final result follows from the sum of each term $\mathbf{III}(A_{k,l}Q_c^k)$, $\mathbf{III}(B_{k,l}(Q_c^k)')$ for $(k,l)\in\Sigma_m$, discarding localized terms. This concludes the proof. □

The final term reads

Lemma L.6.

$$\mathbf{IV} = \sum_{(k,l)\in\Sigma_m} c^l\left(Q_c^k(y_c)F_{k,l}^{IV}(y) + (Q_c^k)'(y_c)G_{k,l}^{IV}(y)\right) + c^3O(Q_c), \tag{L.15}$$

where

$$F_{1,0}^{IV} = G_{1,0}^{IV} = 0, \quad F_{1,1}^{IV} = G_{1,1}^{IV} = 0,$$
$$F_{2,0}^{IV} = \frac{1}{2}(f''(Q)(2A_{1,0}+A_{1,0}^2))', \quad G_{2,0}^{IV} = \frac{1}{2}\big[f''(Q)(2A_{1,0}+A_{1,0}^2) + (f''(Q)(B_{1,0}+A_{1,0}B_{1,0}))'\big],$$

and for $(k,l)\in\Sigma_m'$ (see (L.13)), $F_{k,l}^{IV}$, $G_{k,l}^{IV}$ depend on $A_{k',l'}$, $B_{k',l'}$ for $(k',l')\in\Sigma_m$ with $(k',l')<(k,l)$. Moreover, if $A_{k',l'}$ are even and $B_{k',l'}$ are odd then $F_{k,l}^{IV}$ are odd and $G_{k,l}^{IV}$ are even.

Finally, suppose **(IP)** *holds for $(k,l)\in\Sigma_m$ with $k+l\le 2$. Then the only non localized terms for $(k,l)\in\Sigma_m'$ are given by*

$$F_{2,1}^{IV} = 0, \quad G_{2,1}^{IV} = \frac{1}{2}f''(Q)(2A_{1,1}+B_{1,0}^2) \mod \mathcal{Y}, \tag{L.16}$$

and

$$F_{3,0}^{IV} = 0, \quad G_{3,0}^{IV} = f''(Q)(A_{2,0} - \frac{1}{3}B_{1,0}^2) \mod \mathcal{Y}, \tag{L.17}$$

for the quadratic case, and

$$G_{4,0}^{IV} = \frac{1}{2}f^{(3)}(Q)A_{2,0} \mod \mathcal{Y}. \tag{L.18}$$

in the cubic case.

Proof. As above, first define $\tilde{\mathbf{IV}} := f(R+R_c+W) - f(R+R_c) - f'(R)W$. Note that, using that $R := Q(y)$ and $R_c := Q_c(y_c)$,

$$
\begin{aligned}
\tilde{\mathbf{IV}} &= (f'(Q+Q_c) - f'(Q))W + \frac{1}{2}f''(Q+Q_c)W^2 + \frac{1}{6}f^{(3)}(Q+Q_c)W^3 \\
&\quad + \frac{1}{24}f^{(4)}(Q+Q_c)W^4 + O(W^5) \\
&= [f''(Q)Q_c + \frac{1}{2}f^{(3)}(Q)Q_c^2 + \frac{1}{6}f^{(4)}(Q)Q_c^3 + O(Q_c^4)]W \\
&\quad + \frac{1}{2}[f''(Q) + f^{(3)}(Q)Q_c + \frac{1}{2}f^{(4)}(Q)Q_c^2 + O(Q_c)^3]W^2 \\
&\quad + \frac{1}{6}[f^{(3)}(Q) + f^{(4)}(Q)Q_c + O(Q_c^2)]W^3 + \frac{1}{24}f^{(4)}(Q)W^4 + O(Q_c^5) \\
&= f''(Q)(Q_cW + \frac{1}{2}W^2) + \frac{1}{2}f^{(3)}(Q)(Q_c^2W + W^2Q_c + \frac{1}{3}W^3) \\
&\quad + \frac{1}{2}f^{(4)}(Q)(\frac{1}{3}Q_c^3W + \frac{1}{2}Q_c^2W^2 + \frac{1}{3}Q_cW^3 + \frac{1}{12}W^4) + O(Q_c^5).
\end{aligned}
$$

Now, the final value of $\mathbf{IV}$ depends on the different values of m. We will proceed carefully in both cases.

Case $m = 2$. Here we consider only up to third order, namely

$$
\begin{aligned}
\tilde{\mathbf{IV}} &= \frac{1}{2}f''(Q)(2Q_cW + W^2) + \frac{1}{2}f^{(3)}(Q)(Q_c^2W + W^2Q_c + \frac{1}{3}W^3) + O(Q_c^4) \\
&=: \tilde{\mathbf{IV}}_2 + \tilde{\mathbf{IV}}_3 + O(Q_c^4).
\end{aligned}
$$

First of all let us consider the third order term $\tilde{\mathbf{IV}}_3$. A quickly computation using (2.6) gives us

$$
Q_c^2W = A_{1,0}Q_c^3 + \frac{1}{3}B_{1,0}(Q_c^3)' + O(c^3Q_c),
$$

and

$$
W^2Q_c = A_{1,0}^2Q_c^3 + \frac{2}{3}A_{1,0}B_{1,0}(Q_c^3)' + O(c^3Q_c), \quad W^3 = A_{1,0}^3Q_c^3 + A_{1,0}B_{1,0}(Q_c^3)' + O(c^3Q_c).
$$

If we suppose $A_{1,0} \in \mathcal{Y}$ and $B_{1,0}$ bounded (this is actually the case), we will obtain

$$
\tilde{\mathbf{IV}}_3 = F_{3,0}^{\tilde{IV}_3}Q_c^3 + (G_{3,0}^{\tilde{IV}_3} + \frac{1}{6}f^{(3)}(Q)B_{1,0})(Q_c^3)' + O(c^3Q_c),
$$

where $F_{3,0}^{\tilde{IV}_3}, G_{3,0}^{\tilde{IV}_3} \in \mathcal{Y}$. Moreover, for $p \geq 4$, actually $f^{(3)}(Q) - 6\mu(\varepsilon) \in \mathcal{Y}$, because of

$$
f^{(3)}(Q) = 6\mu(\varepsilon) + p(p-1)(p-2)\varepsilon Q^{p-3} + f_1^{(3)}(Q).
$$

Now, let us compute in detail the term $\tilde{\mathbf{IV}}_2$. These terms above are important because they will give us source terms of second order. Now from the definition of W in (2.6) it is easy to check that, up to third order,

$$
Q_cW = A_{1,0}Q_c^2 + \frac{1}{2}B_{1,0}(Q_c^2)' + A_{1,1}cQ_c^2 + \frac{1}{2}B_{1,1}c(Q_c^2)' + A_{2,0}Q_c^3 + \frac{2}{3}B_{2,0}(Q_c^3)' + O(c^3Q_c),
$$

and

$$
\begin{aligned}
W^2 &= A_{1,0}^2Q_c^2 + A_{1,0}B_{1,0}(Q_c^2)' + (2A_{1,0}A_{1,1} + B_{1,0}^2)cQ_c^2 + (A_{1,0}B_{1,1} + B_{1,0}A_{1,1})c(Q_c^2)' \\
&\quad + (2A_{1,0}A_{2,0} - \frac{2}{3}B_{1,0}^2)Q_c^3 + \frac{2}{3}(A_{1,0}B_{2,0} + B_{1,0}A_{2,0})(Q_c^3)'.
\end{aligned}
$$

From here,

$$\begin{aligned}
\tilde{\mathbf{IV}}_{\mathbf{2}} &= \frac{1}{2}f''(Q)(2A_{1,0}+A_{1,0}^2)Q_c^2+\frac{1}{2}f''(Q)(B_{1,0}+A_{1,0}B_{1,0})(Q_c^2)' \\
&+\frac{1}{2}f''(Q)(F_{2,1}^{\tilde{IV}_2}+2A_{1,1}+B_{1,0}^2)cQ_c^2+\frac{1}{2}f''(Q)(G_{2,1}^{\tilde{IV}_2}+B_{1,1}+A_{1,1}B_{1,0})c(Q_c^2)' \\
&+\frac{1}{2}f''(Q)(F_{3,0}^{\tilde{IV}_2}+2A_{2,0}-\frac{2}{3}B_{1,0}^2)Q_c^3+\frac{1}{2}f''(Q)(G_{3,0}^{\tilde{IV}_2}+\frac{4}{3}B_{2,0}+\frac{2}{3}A_{2,0}B_{1,0})(Q_c^3)' \\
&+O(c^3Q_c),
\end{aligned}$$

where for $k+l=3$ it is satisfied $F_{k,l}^{\tilde{IV}_2}, G_{k,l}^{\tilde{IV}_2}\in\mathcal{Y}$, provided $A_{1,0}\in\mathcal{Y}$ and $B_{1,0}$ is bounded (namely $(k,l)=(1,0)$ satisfies **(IP)**).

Putting all this information together, and after derivation, we obtain (note that $p\geq 3$ and $(f''(Q))'\in\mathcal{Y}$)

$$\begin{aligned}
\mathbf{IV} &= (\tilde{\mathbf{IV}}_2+\tilde{\mathbf{IV}}_3+O(Q_c^4))_x \\
&= \frac{1}{2}(f''(Q))'(1-\beta)(2A_{1,0}+A_{1,0}^2)Q_c^2+\frac{1}{2}f''(Q)(2A_{1,0}+A_{1,0}^2)'(1-\beta)Q_c^2 \\
&+\frac{1}{2}f''(Q)(2A_{1,0}+A_{1,0}^2)(Q_c^2)'+\frac{1}{2}(f''(Q))'(1-\beta)(B_{1,0}+A_{1,0}B_{1,0})(Q_c^2)' \\
&+\frac{1}{2}f''(Q)(1-\beta)(B_{1,0}+A_{1,0}B_{1,0})'(Q_c^2)' \\
&+F_{2,1}^{IV}cQ_c^2+[G_{2,1}^{IV}+\frac{1}{2}f''(Q)(2A_{1,1}+B_{1,0}^2)](cQ_c^2)' \\
&+\big[F_{3,0}^{IV}+f''(Q)(A_{2,0}-\frac{1}{3}B_{1,0}^2)\big](Q_c^3)'+\big[G_{3,0}^{IV}+f''(Q)(\frac{2}{3}B_{2,0}+\frac{1}{3}A_{2,0}B_{1,0})\big](Q_c^3)'' \\
&+O(c^3Q_c), \\
&= \frac{1}{2}\big[f''(Q)(2A_{1,0}+A_{1,0}^2)\big]'Q_c^2 \\
&+\frac{1}{2}\big[f''(Q)(2A_{1,0}+A_{1,0}^2)+(f''(Q)(B_{1,0}+A_{1,0}B_{1,0}))'\big](Q_c^2)' \\
&+F_{2,1}^{IV}cQ_c^2+[G_{2,1}^{IV}+\frac{1}{2}f''(Q)(2A_{1,1}+B_{1,0}^2)](cQ_c^2)'+F_{3,0}^{IV}Q_c^3 \\
&+\big[G_{3,0}^{IV}+f''(Q)(A_{2,0}-\frac{1}{3}B_{1,0}^2)\big](Q_c^3)'+O(c^3Q_c),
\end{aligned}$$

where $F_{2,1}^{IV}, G_{2,1}^{IV}, F_{3,0}^{IV}$ and $G_{3,0}^{IV}$ are $\mathcal{Y}$-functions provided property **(IP)** holds for $k+l\leq 2$. We finally get the Lemma in the quadratic case, the decomposition (L.15), (L.16) and (L.17), with the desired properties.

Case $m=3$. Here we consider up to fourth order in our computations. First of all, we write

$$\begin{aligned}
\tilde{\mathbf{IV}} &= \frac{1}{2}f''(Q)(2Q_cW+W^2)+\frac{1}{2}f^{(3)}(Q)(Q_c^2W+W^2Q_c+\frac{1}{3}W^3) \\
&+\frac{1}{2}f^{(4)}(Q)(\frac{1}{3}Q_c^3W+\frac{1}{2}Q_c^2W^2+\frac{1}{3}Q_cW^3+\frac{1}{12}W^4)+O(Q_c^5) \\
&=: \tilde{\mathbf{IV}}_{\mathbf{2}}+\tilde{\mathbf{IV}}_{\mathbf{3}}+\tilde{\mathbf{IV}}_{\mathbf{4}}+O(Q_c^5).
\end{aligned}$$

From now on, and for the sake of simplicity in our computations, we will consider that property **(IP)** holds for any $(k,l)\in\Sigma_3$ and that $A_{1,0}\in\mathcal{Y}$. We recall that in the cubic case, our correction term is given by

$$\begin{aligned}
W &= A_{1,0}Q_c+B_{1,0}Q_c'+A_{1,1}cQ_c+B_{1,1}cQ_c' \\
&+A_{2,0}Q_c^2+B_{2,0}(Q_c^2)'+A_{3,0}Q_c^3+B_{3,0}(Q_c^3)' \\
&+A_{2,1}cQ_c^2+B_{2,1}c(Q_c^2)'+A_{4,0}Q_c^4+B_{4,0}(Q_c^4)'.
\end{aligned}$$

Let us first consider the term $\mathbf{I\tilde{V}_2}$. Note that

$$\begin{aligned} Q_c W &= A_{1,0}Q_c^2 + \frac{1}{2}B_{1,0}(Q_c^2)' + A_{1,1}cQ_c^2 + \frac{1}{2}B_{1,1}c(Q_c^2)' + A_{2,0}Q_c^3 + \frac{2}{3}B_{2,0}(Q_c^3)' \\ &\quad + A_{3,0}Q_c^4 + \frac{3}{4}B_{3,0}(Q_c^4)' + O(cQ_c^3 + Q_c^5). \end{aligned}$$

Using (L.4) we get

$$\begin{aligned} W^2 &= A_{1,0}^2Q_c^2 + B_{1,0}^2Q_c'^2 + A_{2,0}^4Q_c^4 \\ &\quad + A_{1,0}B_{1,0}(Q_c^2)' + 2A_{1,0}A_{1,1}cQ_c^2 + A_{1,0}B_{1,1}c(Q_c^2)' \\ &\quad + 2A_{1,0}A_{2,0}Q_c^3 + \frac{4}{3}A_{1,0}B_{2,0}(Q_c^3)' + 2A_{1,0}A_{3,0}Q_c^4 + \frac{3}{2}A_{1,0}B_{3,0}(Q_c^4)' \\ &\quad + A_{1,1}B_{1,0}c(Q_c^2)' + \frac{2}{3}A_{2,0}B_{1,0}(Q_c^3)' + \frac{1}{2}A_{3,0}B_{1,0}(Q_c^4)' + O(cQ_c^3 + Q_c^5) \\ &= A_{1,0}^2Q_c^2 + A_{1,0}B_{1,0}(Q_c^2)' + (2A_{1,0}A_{1,1} + B_{1,0}^2)cQ_c^2 \\ &\quad + (A_{1,1}B_{1,0} + A_{1,0}B_{1,1})c(Q_c^2)' + 2A_{1,0}A_{2,0}Q_c^3 + \frac{2}{3}(2A_{1,0}B_{2,0} + A_{2,0}B_{1,0})(Q_c^3)' \\ &\quad + (2A_{1,0}A_{3,0} + A_{2,0}^4 - \frac{1}{2}B_{1,0}^2)Q_c^4 + \frac{1}{2}(3A_{1,0}B_{3,0} + A_{3,0}B_{1,0})(Q_c^4)' + O(cQ_c^3 + Q_c^5). \end{aligned}$$

From here, and using the **(IP)** property, we get (note that $f''(Q) \in \mathcal{Y}$)

$$\begin{aligned} \mathbf{I\tilde{V}_2} &= \frac{1}{2}f''(Q)(2A_{1,0} + A_{1,0}^2)Q_c^2 + \frac{1}{2}f''(Q)(B_{1,0} + A_{1,0}B_{1,0})(Q_c^2)' \\ &\quad + \sum_{(k,l)\in\Sigma_3'} c^l(F_{k,l}^{I\tilde{V}_2}Q_c^k + G_{k,l}^{I\tilde{V}_2}(Q_c^k)') + O(cQ_c^3 + Q_c^5), \end{aligned}$$

where Σ_3' was introduced in (L.13), and $F_{k,l}^{I\tilde{V}_2}, G_{k,l}^{I\tilde{V}_2} \in \mathcal{Y}$.

Now we deal with $\mathbf{I\tilde{V}_3}$. Here we have

$$Q_c^2W = A_{1,0}Q_c^3 + \frac{1}{3}B_{1,0}(Q_c^3)' + A_{2,0}Q_c^4 + \frac{1}{4}B_{2,0}(Q_c^4)' + O(cQ_c^3 + Q_c^5).$$

and

$$\begin{aligned} Q_cW^2 &= A_{1,0}^2Q_c^3 + \frac{2}{3}A_{1,0}B_{1,0}(Q_c^3)' + 2A_{1,0}A_{2,0}Q_c^4 \\ &\quad + \frac{1}{2}(2A_{1,0}B_{2,0} + A_{2,0}B_{1,0})(Q_c^4)' + O(cQ_c^3 + Q_c^5). \end{aligned}$$

Finally

$$\begin{aligned} W^3 &= W^2W \\ &= A_{1,0}^3Q_c^3 + \frac{2}{3}A_{1,0}^2B_{1,0}(Q_c^3)' + 2A_{1,0}^2A_{2,0}Q_c^4 + \frac{1}{2}(2A_{1,0}^2B_{2,0} + A_{1,0}A_{2,0}B_{1,0})(Q_c^4)' \\ &\quad + \frac{1}{3}B_{1,0}A_{1,0}^2(Q_c^3)' + \frac{1}{2}A_{1,0}B_{1,0}A_{2,0}(Q_c^4)' \\ &\quad + A_{2,0}A_{1,0}^2Q_c^4 + \frac{1}{2}A_{2,0}A_{1,0}B_{1,0}(Q_c^4)' + \frac{1}{2}A_{1,0}^2B_{2,0}(Q_c^4)' + O(cQ_c^3 + Q_c^5) \\ &= A_{1,0}^3Q_c^3 + A_{1,0}^2B_{1,0}(Q_c^3)' + 3A_{1,0}^2A_{2,0}Q_c^4 + \frac{3}{2}(A_{1,0}^2B_{2,0} + A_{1,0}A_{2,0}B_{1,0})(Q_c^4)' \\ &\quad + O(cQ_c^3 + Q_c^5). \end{aligned}$$

From here, and using the **(IP)** property, we get

$$\begin{aligned}\tilde{\mathbf{IV}}_{\mathbf{3}} &= \frac{1}{2}f^{(3)}(Q)[\frac{1}{3}B_{1,0}(Q_c^3)' + A_{2,0}Q_c^4 + \frac{1}{4}(2A_{2,0}B_{1,0} + B_{2,0})(Q_c^4)'] \\ &+ \sum_{(k,l)\in\Sigma_3'} c^l(F_{k,l}^{\tilde{IV}_2}Q_c^k + G_{k,l}^{\tilde{IV}_2}(Q_c^k)') + O(cQ_c^3 + Q_c^5),\end{aligned}$$

where Σ_3' was introduced in (L.13), and $F_{k,l}^{\tilde{IV}_3}, G_{k,l}^{\tilde{IV}_3} \in \mathcal{Y}$.

Finally, fourth order terms are easy to compute:

$$Q_c^3W = A_{1,0}Q_c^4 + \frac{1}{4}B_{1,0}(Q_c^4)' + O(Q_c^5 + cQ_c^3),$$

$$Q_c^2W^2 = A_{1,0}^2Q_c^4 + \frac{1}{2}A_{1,0}B_{1,0}(Q_c^4)' + O(Q_c^5 + cQ_c^3),$$

$$Q_cW^3 = A_{1,0}^3Q_c^4 + \frac{3}{4}A_{1,0}^2B_{1,0}(Q_c^4)' + O(Q_c^5 + cQ_c^3),$$

and

$$W^4 = A_{1,0}^4Q_c^4 + A_{1,0}^3B_{1,0}(Q_c^4)' + O(Q_c^5 + cQ_c^3).$$

As we have supposed $A_{1,0} \in \mathcal{Y}$ ($(1,0)$ satisfies **(IP)**), we will obtain

$$\tilde{\mathbf{IV}}_{\mathbf{4}} = F_{4,0}^{\tilde{IV}_4}Q_c^4 + [G_{4,0}^{\tilde{IV}_4} + \frac{1}{24}f^{(4)}(Q)B_{1,0}](Q_c^4)' + O(Q_c^5 + cQ_c^3),$$

where $F_{4,0}^{\tilde{IV}_4}, G_{4,0}^{\tilde{IV}_4} \in \mathcal{Y}$.

We finally collect the expansions of $\tilde{\mathbf{IV}}_{\mathbf{2}}, \tilde{\mathbf{IV}}_{\mathbf{3}}$ and $\tilde{\mathbf{IV}}_{\mathbf{4}}$. We derivate to obtain

$$\begin{aligned}\mathbf{IV} &= (\tilde{\mathbf{IV}}_{\mathbf{2}} + \tilde{\mathbf{IV}}_{\mathbf{3}} + \tilde{\mathbf{IV}}_{\mathbf{4}} + O(Q_c^5))_x \\ &\frac{1}{2}(f''(Q)(2A_{1,0} + A_{1,0}^2))'Q_c^2 + \frac{1}{2}f^{(3)}(Q)A_{2,0}(Q_c^4)' \\ &+\frac{1}{2}\big[f''(Q)(2A_{1,0} + A_{1,0}^2) + (f''(Q)(B_{1,0} + A_{1,0}B_{1,0}))'\big](Q_c^2)' \\ &+ \sum_{(k,l)\in\Sigma_3'} c^l(F_{k,l}^{\tilde{IV}}Q_c^k + G_{k,l}^{\tilde{IV}}(Q_c^k)') + O(cQ_c^3 + Q_c^5),\end{aligned}$$

where, as we have emphasized, $F_{k,l}^{\tilde{IV}}, G_{k,l}^{\tilde{IV}} \in \mathcal{Y}$ provided $A_{k',l'}, B_{k',l'}$ satisfy the **(IP)** property for $(k',l') < (k,l)$, as is the case. Here we have also used that $(f''(Q))', (f^{(4)}(Q))' \in \mathcal{Y}$ for all $p \geq 4$. The set Σ_3' was defined in (L.13).

Let us finally prove (v). From (i), the rest term $\mathcal{E}(t,x)$ is a finite sum of terms of the type $c^lQ_c^k(y_c)f(y)$ or $c^l(Q_c^k)'(y_c)f(y)$, where $(k,l) \notin \Sigma_m$. More specifically, this means $k+l \geq 4$ for $m=2$ and $(k,l) = (1,2), (3,1)$ or higher order terms (excluding $(k,l) = (4,0)$) in the case $m=3$ (see the definition of Σ_m in Section 2.1). Here f is a bounded function such that $f' \in \mathcal{Y}$. Thus, we easily conclude, using Claim 15,

$$\|\mathcal{E}(t)\|_{H^1(\mathbb{R})} \leq Kc^{3(\frac{1}{m-1}+\frac{1}{4})},$$

as desired. This finishes the proof.

□

Putting together Lemmas L.3–L.6, we obtain Proposition L.1, in particular, the explicit expressions of $F_{k,l}$ and $G_{k,l}$ for $1 \leq k+l \leq 2$.

□

M End of proof of Proposition 2.6

Continuing with the proof of Proposition 2.6, we show now the existence of a nonzero residual term appearing after the collision.

M.1 General computations

We proceed to compute the constants $b_{2,0}$ more explicitly. In the course of the proof we will made use several times of the equations satisfied by the functions $A_{k,l}, B_{k,l}, a_{k,l}$ for $(k,l) = (1,0)$ and $(2,0)$, cf. (2.18)-(2.19) for the system $(\Omega_{1,0})$ and (2.23)-(2.24) for the second one.

Claim 22 (Explicit value of $b_{2,0}$). Suppose f as in (1.10). Then the following expressions for the $b_{2,0}$ coefficient hold.

1. Case $m = 2$.

$$\begin{aligned} b_{2,0} &= -\frac{1}{2}b_{1,0}^3 + \frac{1}{4}\int_{\mathbb{R}}(f''(Q)-2)(1+A_{1,0})^3 - 2b_{1,0} + \frac{1}{2}\int_{\mathbb{R}} A_{1,0}(1+A_{1,0}^2) \\ &\quad -\frac{1}{2}a_{1,0}\int_{\mathbb{R}} QA_{1,0} - \frac{3}{4}a_{1,0}^3\int_{\mathbb{R}} Q'^2 + \frac{1}{2}a_{1,0}^2\int_{\mathbb{R}} Q[Q - f'(Q)(1+A_{1,0})] \\ &\quad -\frac{3}{4}a_{1,0}\int_{\mathbb{R}}[f'(Q)(1+A_{1,0}) + 3A''_{1,0}]A_{1,0} + \frac{1}{2}\int_{\mathbb{R}} B_{1,0}[3A'_{1,0} + f'(Q)\int_0^x (A_{1,0} + a_{1,0}Q)] \\ &\quad +3a_{1,0}^2\int_{\mathbb{R}} Q''A_{1,0}. \end{aligned} \tag{M.1}$$

2. Case $m = 3$.

$$\begin{aligned} b_{2,0} &= \frac{1}{4}\int_{\mathbb{R}} f''(Q)(1+A_{1,0})^3 - \frac{3}{4}a_{1,0}\int_{\mathbb{R}} f'(Q)(1+A_{1,0})A_{1,0} + \frac{9}{4}a_{1,0}\int_{\mathbb{R}} A'^2_{1,0} \\ &\quad -\frac{1}{2}a_{1,0}^2\int_{\mathbb{R}} f'(Q)Q(1+A_{1,0}) + 3a_{1,0}^2\int_{\mathbb{R}} A_{1,0}Q'' - \frac{3}{4}a_{1,0}^3\int_{\mathbb{R}} Q'^2. \end{aligned} \tag{M.2}$$

Proof. We treat first the cubic case, being easier. Let us start with (2.31) and (2.32). In this case, we have a priori chosen $A_{2,0} \in \mathcal{Y}$, so that $\gamma_{2,0} = 0$, and then from (2.20)

$$b_{2,0} = \frac{1}{2}\Big[-a_{1,0}\int_{\mathbb{R}} G_{2,0}Q - a_{1,0}\int_{\mathbb{R}} \tilde{F}_{2,0}P + \int_{\mathbb{R}} \tilde{F}_{2,0}\bar{P} + \int_{\mathbb{R}} G_{2,0}\Big], \tag{M.3}$$

where $\tilde{F}'_{2,0} = F_{2,0}$, $\tilde{F}_{2,0} \in \mathcal{Y}$. More precisely,

$$\tilde{F}_{2,0} := \frac{1}{2}f''(Q)(1+A_{1,0})^2 + 3a_{1,0}^2Q'' - a_{1,0}(3A''_{1,0} + f'(Q)(1+A_{1,0})).$$

First, it is easy to see from (2.28) by using the **(IP)** property for $(k,l) = (1,0)$, that

$$\int_{\mathbb{R}} G_{2,0} = \frac{1}{2}\int_{\mathbb{R}} f''(Q)(1+A_{1,0})^2. \tag{M.4}$$

Secondly, from (2.21), (2.11) and (2.15), $\bar{P} - a_{1,0}P = A_{1,0} + a_{1,0}Q$, and thus

$$b_{2,0} = \frac{1}{2}\Big[a_{1,0}\int_{\mathbb{R}}(\tilde{F}_{2,0} - G_{2,0})Q + \int_{\mathbb{R}} \tilde{F}_{2,0}A_{1,0} + \frac{1}{2}\int_{\mathbb{R}} f''(Q)(1+A_{1,0})^2\Big].$$

It is clear that

$$\begin{aligned}\tilde{F}_{2,0} - G_{2,0} &= \frac{3}{2}a_{1,0}^2 Q'' + a_{1,0}[\frac{3}{2}A_{1,0}'' - f'(Q)(1+A_{1,0}) + \frac{3}{2}B_{1,0}^{(3)} + \frac{1}{2}(f'(Q)B_{1,0})'] \\ &\quad -\frac{1}{2}(f''(Q)(1+A_{1,0})B_{1,0})'.\end{aligned}$$

From here, after several integration by parts,

$$\begin{aligned}\int_{\mathbb{R}} (\tilde{F}_{2,0} - G_{2,0})Q &= -\frac{3}{2}a_{1,0}^2 \int_{\mathbb{R}} Q'^2 + a_{1,0}\int_{\mathbb{R}} [\frac{3}{2}A_{1,0}'' - f'(Q)(1+A_{1,0})]Q \\ &\quad -\frac{1}{2}a_{1,0}\int_{\mathbb{R}} B_{1,0}[3Q'' + f(Q)]' + \frac{1}{2}\int_{\mathbb{R}} B_{1,0}(f'(Q))'(1+A_{1,0}) \\ &= -\frac{3}{2}a_{1,0}^2 \int_{\mathbb{R}} Q'^2 + a_{1,0}\int_{\mathbb{R}} [\frac{3}{2}A_{1,0}'' - f'(Q)(1+A_{1,0})]Q \\ &\quad +\frac{1}{2}\int_{\mathbb{R}} B_{1,0}[(f'(Q))'(1+A_{1,0}) - a_{1,0}(3Q - 2f(Q))'].\end{aligned} \tag{M.5}$$

But from (2.18), $(\mathcal{L}(1+A_{1,0}))' = (1 - f'(Q) + \mathcal{L}A_{1,0})' = -a_{1,0}(3Q - 2f(Q))'$. On the other hand, expanding $(\mathcal{L}(1+A_{1,0}))'$, we get

$$(\mathcal{L}(1+A_{1,0}))' = -A_{1,0}^{(3)} + A_{1,0}' - (f'(Q))'(1+A_{1,0}) - f'(Q)A_{1,0}'.$$

From here, the quantity in front of $B_{1,0}$ in (M.5) is nothing but $\mathcal{L}A_{1,0}'$. Coming back to (M.5), and using the equation for $B_{1,0}$ (2.19), we obtain

$$\begin{aligned}\int_{\mathbb{R}} (\tilde{F}_{2,0} - G_{2,0})Q &= -\frac{3}{2}a_{1,0}^2 \int_{\mathbb{R}} Q'^2 + a_{1,0}\int_{\mathbb{R}} [\frac{3}{2}A_{1,0}'' - f'(Q)(1+A_{1,0})]Q - \frac{1}{2}\int_{\mathbb{R}} A_{1,0}(\mathcal{L}B_{1,0})' \\ &= -\frac{3}{2}a_{1,0}^2 \int_{\mathbb{R}} Q'^2 + \frac{3}{2}\int_{\mathbb{R}} A_{1,0}'^2 + 3a_{1,0}\int_{\mathbb{R}} Q''A_{1,0} \\ &\quad -a_{1,0}\int_{\mathbb{R}} f'(Q)Q(1+A_{1,0}) - \frac{1}{2}\int_{\mathbb{R}} f'(Q)(1+A_{1,0})A_{1,0}.\end{aligned} \tag{M.6}$$

Finally, an easy computation shows that

$$\begin{aligned}\int_{\mathbb{R}} \tilde{F}_{2,0}A_{1,0} &= \frac{1}{2}\int_{\mathbb{R}} f''(Q)(1+A_{1,0})^2 A_{1,0} + 3a_{1,0}^2 \int_{\mathbb{R}} A_{1,0}Q'' \\ &\quad -a_{1,0}\int_{\mathbb{R}} (3A_{1,0}'' + f'(Q)(1+A_{1,0}))A_{1,0}\end{aligned} \tag{M.7}$$

Collecting (M.6) and (M.7), we get

$$\begin{aligned}b_{2,0} &= \frac{1}{4}\int_{\mathbb{R}} f''(Q)(1+A_{1,0})^3 - \frac{3}{4}a_{1,0}\int_{\mathbb{R}} f'(Q)(1+A_{1,0})A_{1,0} + \frac{9}{4}a_{1,0}\int_{\mathbb{R}} A_{1,0}'^2 \\ &\quad -\frac{1}{2}a_{1,0}^2\int_{\mathbb{R}} f'(Q)Q(1+A_{1,0}) + 3a_{1,0}^2\int_{\mathbb{R}} A_{1,0}Q'' - \frac{3}{4}a_{1,0}^3\int_{\mathbb{R}} Q'^2.\end{aligned}$$

as desired.

Let us treat now the quadratic case. The procedure is similar, but more involved. Now we assume that $\gamma_{2,0} = -\frac{1}{2}b_{1,0}^2$ as in Proposition 2.6 (i), and consider (2.31)-(2.32). We get

$$b_{2,0} = \frac{1}{2}\Big[-\frac{1}{2}b_{1,0}^2\int_{\mathbb{R}} (\bar{P} - a_{1,0}P) + \int_{\mathbb{R}} F_{2,0}\int_0^x (a_{1,0}P - \bar{P}) + \int_{\mathbb{R}} G_{2,0} - a_{1,0}\int_{\mathbb{R}} G_{2,0}Q\Big].$$

Now several remarks. Note that from (2.21) and the definition of P in (2.11) we have $\bar{P} - a_{1,0}P = A_{1,0} + a_{1,0}Q$, and from (2.22),

$$\int_{\mathbb{R}} (\bar{P} - a_{1,0}P) = 2b_{1,0}.$$

Second, note that from $b_{1,0} = \pm \lim_{\pm\infty} B_{1,0}$ and $\lim_{\pm\infty} f''(Q) = 2$,

$$\int_{\mathbb{R}} G_{2,0} = \frac{1}{2}\int_{\mathbb{R}}(f''(Q) - 2) - \int_{\mathbb{R}} A_{1,0} - 4b_{1,0} + \frac{1}{2}\int_{\mathbb{R}} f''(Q)(2A_{1,0} + A_{1,0}^2).$$

On the other hand, from (2.25), $F_{2,0} = \tilde{F}_{2,0}' - f'(Q)B_{1,0}$, where $\tilde{F}_{2,0} \in \mathcal{Y}$ and is given by

$$\begin{aligned}\tilde{F}_{2,0} \ := \ & -(3A_{1,0} + 3B_{1,0}') + \frac{1}{2}f''(Q)(2A_{1,0} + A_{1,0}^2) \\ & -a_{1,0}(3A_{1,0}'' - Q + f'(Q)(1 + A_{1,0})) + 3a_{1,0}^2 Q'' + \frac{1}{2}(f''(Q) - 2).\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} F_{2,0}\int_0^x (a_{1,0}P - \bar{P}) &= \int_{\mathbb{R}} \tilde{F}_{2,0}(\bar{P} - a_{1,0}P) + \int_{\mathbb{R}} f'(Q)B_{1,0}\int_0^x (\bar{P} - a_{1,0}P) \\ &= \int_{\mathbb{R}} \tilde{F}_{2,0}(A_{1,0} + a_{1,0}Q) + \int_{\mathbb{R}} f'(Q)B_{1,0}\int_0^x (A_{1,0} + a_{1,0}Q).\end{aligned}$$

Repeating the same computation for the cubic case, we obtain

$$\begin{aligned}\int_{\mathbb{R}} Q(\tilde{F}_{2,0} - G_{2,0}) &= -2\int_{\mathbb{R}} QA_{1,0} - \frac{3}{2}a_{1,0}^2\int_{\mathbb{R}} Q'^2 + a_{1,0}\int_{\mathbb{R}} Q(\frac{3}{2}A_{1,0}'' + Q - f'(Q)(1 + A_{1,0})) \\ & \quad -\frac{1}{2}\int_{\mathbb{R}}[f'(Q)(1 + A_{1,0}) + 3A_{1,0}'' - 3a_{1,0}Q'']A_{1,0},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} \tilde{F}_{2,0}A_{1,0} &= -\int_{\mathbb{R}} A_{1,0}(3A_{1,0} + 3B_{1,0}') + \frac{1}{2}\int_{\mathbb{R}} f''(Q)(2A_{1,0} + A_{1,0}^2)A_{1,0} + 3a_{1,0}^2\int_{\mathbb{R}} Q''A_{1,0} \\ & \quad -a_{1,0}\int_{\mathbb{R}}(3A_{1,0}'' - Q + f'(Q)(1 + A_{1,0}))A_{1,0} + \frac{1}{2}\int_{\mathbb{R}}(f''(Q) - 2)A_{1,0}.\end{aligned}$$

Collecting the above identities and after several simplifications we get

$$\begin{aligned}b_{2,0} \ = \ & -\frac{1}{2}b_{1,0}^3 + \frac{1}{4}\int_{\mathbb{R}}(f''(Q) - 2)(1 + A_{1,0})^3 - 2b_{1,0} + \frac{1}{2}\int_{\mathbb{R}} A_{1,0}(1 + A_{1,0}^2) \\ & -\frac{1}{2}a_{1,0}\int_{\mathbb{R}} QA_{1,0} - \frac{3}{4}a_{1,0}^3\int_{\mathbb{R}} Q'^2 + \frac{1}{2}a_{1,0}^2\int_{\mathbb{R}} Q(Q - f'(Q)(1 + A_{1,0})) \\ & -\frac{3}{4}a_{1,0}\int_{\mathbb{R}}[f'(Q)(1 + A_{1,0}) + 3A_{1,0}'']A_{1,0} + \frac{1}{2}\int_{\mathbb{R}} B_{1,0}[3A_{1,0}' + f'(Q)\int_0^x (A_{1,0} + a_{1,0}Q)] \\ & +3a_{1,0}^2\int_{\mathbb{R}} Q''A_{1,0}.\end{aligned}$$

The proof is now complete. □

The objective is now to give the first order terms for the coefficient $b_{2,0}$. For this, we consider separate cases. It turns out that computations in the cubic case are easy to carry out. We first deal with this case.

M.2 Cubic case

The objective of this paragraph is to prove the following

Lemma M.1 (Asymptotic expansions, case $m = 3$)**.** *We have*

$$b_{2,0} = b_{2,0}^1 \varepsilon + o(\varepsilon).$$

where

$$b_{2,0}^1 =: c_{3,p} = -\Big[\frac{(p-1)(p-3)(p^2-3p+8)}{8(p-2)(p+1)}\Big] \int_{\mathbb{R}} (Q^0)^p. \tag{M.8}$$

In particular, for any $p \geq 4$, $b_{2,0}(\varepsilon) \neq 0$ provided $0 < |\varepsilon| \leq \varepsilon_0$ for ε_0 small.

First of all we start with an auxiliary

Claim 23 (Asymptotic expansions, basic functions). Suppose f as in (1.22), $p \geq 4$. The following asymptotic expansions hold.

1. The soliton solution Q can be expanded as

$$Q = Q^0 + \varepsilon Q^1 + o(\varepsilon), \quad o(\varepsilon) \in \mathcal{Y}, \tag{M.9}$$

where Q^0 and Q^1 satisfy the equations

$$-(Q^0)'' + Q^0 - (Q^0)^3 = 0, \quad \mathcal{L}^0 Q^1 := -(Q^1)'' + Q^1 - 3(Q^0)^2 Q^1 = (Q^0)^p. \tag{M.10}$$

Finally,

$$\begin{cases} f(Q) = (Q^0)^3 + \varepsilon(3(Q^0)^2 Q^1 + (Q^0)^p) + o(\varepsilon), \\ f'(Q) = 3(Q^0)^2 + \varepsilon(6Q^0Q^1 + p(Q^0)^{p-1}) + o(\varepsilon), \\ f''(Q) = 6Q^0 + \varepsilon(6Q^1 + p(p-1)(Q^0)^{p-2}) + o(\varepsilon), \end{cases} \tag{M.11}$$

where every term $o(\varepsilon) \in \mathcal{Y}$ uniformly in $\varepsilon < \varepsilon_0$.

2. The operator $\mathcal{L}$ satisfies

$$\mathcal{L} = \mathcal{L}^0 - \varepsilon[6Q^0Q^1 + p(Q^0)^{p-1}] + o(\varepsilon), \quad \mathcal{L}^0 = -\partial_x^2 + 1 - 3(Q^0)^2. \tag{M.12}$$

3. From (2.11), (2.12) and (2.15), the test functions P, $\bar{P}$ and $\hat{P}$ satisfy the following relations

$$\begin{cases} \Lambda Q = \Lambda Q^0 + \varepsilon \Lambda Q^1 + o(\varepsilon) \in \mathcal{Y} \text{ where } \Lambda Q^0 := \frac{1}{2}(x(Q^0)' + Q^0), \quad \mathcal{L}^0 \Lambda Q^0 = -Q^0; \\ \text{and } \mathcal{L}^0 \Lambda Q^1 := (6Q^0 \Lambda Q^0 - 1) Q^1 + p(Q^0)^{p-1} \Lambda Q^0. \end{cases} \tag{M.13}$$

Moreover, the following identities hold

$$\begin{aligned} \int_{\mathbb{R}} \Lambda Q^0 &= 0, \\ \int_{\mathbb{R}} \Lambda Q^1 &= \int_{\mathbb{R}} [-1 + (Q^0)^2 + 6Q^0 \Lambda Q^0 - 6(Q^0)^3 \Lambda Q^0] Q^1 \\ &\quad + p \int_{\mathbb{R}} (Q^0)^{p-1} \Lambda Q^0 (1 - (Q^0)^2). \end{aligned} \tag{M.14}$$

4. *Integrals*. For any $p \geq 1$,

$$\int_{\mathbb{R}} (Q^0)^{p+2} = \frac{2p}{1+p} \int_{\mathbb{R}} (Q^0)^p, \quad \int_{\mathbb{R}} (Q^0)^2 = 4. \tag{M.15}$$

5. Let $D(\varepsilon) = \int_{\mathbb{R}} \Lambda Q Q$. Then
$$D(\varepsilon) = 1 + O(\varepsilon). \tag{M.16}$$

6. *Inverse functions*. The following identities hold
$$\mathcal{L}^0(-\frac{9}{4}x(Q^0)' - \frac{15}{4}Q^0 + \frac{3}{2}(Q^0)^3) = \frac{9}{2}Q^0(1-(Q^0)^2)^2, \tag{M.17}$$
$$\mathcal{L}^0(xQ^0(Q^0)') = -4(Q^0)^2 + 3(Q^0)^4 - 3xQ^0(Q^0)'(1-(Q^0)^2), \tag{M.18}$$
$$\mathcal{L}^0((Q^0)^4) = -15(Q^0)^4 + 7(Q^0)^6. \tag{M.19}$$

Proof. First of all, (M.9)-(M.12) follow by Taylor expansion in ε. Concerning (M.13), it follows from (M.9)-(M.12). Let us see (M.14). From the definition of $\mathcal{L}^0\Lambda Q^1$ and the identity $\mathcal{L}^0(Q^0)^2 = -3(Q^0)^2$, we have
$$\int_{\mathbb{R}} \mathcal{L}^0\Lambda Q^1 = \int_{\mathbb{R}} \Lambda Q^1 + \int_{\mathbb{R}} (Q^0)^2\mathcal{L}^0\Lambda Q^1,$$
thus
$$\int_{\mathbb{R}} \Lambda Q^1 = \int_{\mathbb{R}} [1-(Q^0)^2]\mathcal{L}^0\Lambda Q^1 = \int_{\mathbb{R}} [1-(Q^0)^2][(6Q^0\Lambda Q^0 - 1)Q^1 + p(Q^0)^{p-1}\Lambda Q^0],$$
where we obtain (M.14).

To obtain (M.15) we use integration by parts and the explicit function $Q^0(x) := \frac{\sqrt{2}}{\cosh x}$.

We prove (M.16). It follows from the fact that
$$\int_{\mathbb{R}} Q^0\Lambda Q^0 = \frac{1}{2}\int_{\mathbb{R}} (\frac{1}{2}x((Q^0)^2)' + (Q^0)^2) = \frac{1}{4}\int_{\mathbb{R}} (Q^0)^2 = 1.$$
Finally, (M.17)-(M.19) are obtained by simple differentiation. We left the proof to the reader. □

Claim 24 (Asymptotic expansions, case $m = 3$). The following expansion hold.
$$\begin{cases} a_{1,0} = a^0_{1,0} + \varepsilon a^1_{1,0} + o(\varepsilon), \quad a^0_{1,0} = 0, \\ A_{1,0} = A^0_{1,0} + \varepsilon A^1_{1,0} + o(\varepsilon), \quad o(\varepsilon) \in \mathcal{Y}, \quad A^0_{1,0} = -(Q^0)^2, \\ B_{1,0} = B^0_{1,0} + \varepsilon B^1_{1,0} + o(\varepsilon), \quad B^0_{1,0} = -2\varphi^0 - \frac{3}{4}\sqrt{2}\pi(Q^0)'. \end{cases} \tag{M.20}$$
Here $a^1_{1,0} := \int_{\mathbb{R}} \Lambda Q^1$ and $A^1_{1,0}, B^1_{1,0}$ satisfy the following linear system
$$\begin{cases} (\mathcal{L}^0 A^1_{1,0})' + a^1_{1,0}(3Q^0 - 2(Q_0)^3)' = ((6Q^0Q^1 + p(Q^0)^{p-1})(1 + A^0_{1,0}))', \\ (\mathcal{L}^0 B^1_{1,0})' + 3a^1_{1,0}(Q^0)'' - 3(A^1_{1,0})'' - 3(Q^0)^2A^1_{1,0} = (6Q^0Q^1 + p(Q^0)^{p-1})(1 + A^0_{1,0}). \end{cases} \tag{M.21}$$

Proof. We start with the zeroth order system. From (2.18)-(2.19) and using Claim 23 we get
$$\begin{cases} (\mathcal{L}^0 A^0_{1,0})' + a^0_{1,0}(3Q^0 - 2(Q^0)^3)' = (3(Q^0)^2)'. \\ (\mathcal{L}^0 B^0_{1,0})' + 3a^0_{1,0}(Q^0)'' - 3(A^0_{1,0})'' - 3(Q^0)^2A^0_{1,0} = 3(Q^0)^2. \end{cases}$$
It is easy to verify that $a^0_{1,0} = 0$, $A^0_{1,0} = -(Q^0)^2 \in \mathcal{Y}$ and $B^0_{1,0} = -2\varphi^0 - \frac{3}{4}\sqrt{2}\pi(Q^0)'$ satisfy this system with the required properties. In particular,
$$\int_{\mathbb{R}} B^0_{1,0}(Q^0)' = 0.$$

Concerning the system (M.21), it follows directly from (2.23)-(2.24) and using Claim 23. We will not solve this system explicitly, but we only compute the constant $a^1_{1,0}$.

Indeed, from (2.20) and Claim 23, we have $a_{1,0} = a^0_{1,0} + a^1_{1,0}\varepsilon + o(\varepsilon)$, where

$$a^0_{1,0} := \frac{\int_{\mathbb{R}} \Lambda Q^0}{\int_{\mathbb{R}} \Lambda Q^0 Q^0} = 0, \quad \text{and} \quad a^1_{1,0} := \frac{\int_{\mathbb{R}} \Lambda Q^1}{\int_{\mathbb{R}} \Lambda Q^0 Q^0} = \int_{\mathbb{R}} \Lambda Q^1.$$

This finishes the proof. □

We finally prove Lemma M.1.

Proof of Lemma M.1. From (M.2) and (M.20) we have $b_{2,0} = b^0_{2,0} + \varepsilon b^1_{2,0} + o(\varepsilon)$, where

$$b^0_{2,0} = \frac{1}{4}\int_{\mathbb{R}} 6Q^0(1+A^0_{1,0})^3 = \frac{1}{4}\int_{\mathbb{R}} 6Q^0(1-(Q^0)^2)^3 = 0,$$

and

$$\begin{aligned} b^1_{2,0} &= \frac{1}{4}\int_{\mathbb{R}}(6Q^1 + p(p-1)(Q^0)^{p-2})(1-(Q^0)^2)^3 + \frac{9}{2}\int_{\mathbb{R}} Q^0(1-(Q^0)^2)^2 A^1_{1,0} \\ &\quad + \frac{9}{4}a^1_{1,0}\Big[\int_{\mathbb{R}}(Q^0)^2(1-(Q^0)^2)(Q^0)^2 + \int_{\mathbb{R}} 4(Q^0)^2((Q^0)^2 - \frac{1}{2}(Q^0)^4)\Big]. \end{aligned}$$

From (M.17), the selfadjointness of the operator $\mathcal{L}^0$ and by using (M.21), we get

$$\begin{aligned} \frac{9}{2}\int_{\mathbb{R}} Q^0(1-(Q^0)^2))^2 A^1_{1,0} &= \frac{3}{4}\int_{\mathbb{R}}(-3x(Q^0)' - 5Q^0 + 2(Q^0)^3)\mathcal{L}^0 A^1_{1,0} \\ &= \frac{3}{4}a^1_{1,0}\int_{\mathbb{R}}(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)(3Q^0 - 2(Q^0)^3) \\ &\quad - \frac{3}{4}\int_{\mathbb{R}}(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)(6Q^0Q^1 + p(Q^0)^{p-1})(1-(Q^0)^2). \end{aligned}$$

Therefore,

$$\begin{aligned} b^1_{2,0} &= \frac{3}{2}\int_{\mathbb{R}} Q^1(1-(Q^0)^2)[1 - 2(Q^0)^2 + (Q^0)^4 - 3Q^0(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)] \\ &\quad + \frac{p}{4}\int_{\mathbb{R}}(Q^0)^{p-2}(1-(Q^0)^2)[(p-1)(1-2(Q^0)^2 + (Q^0)^4) - 3Q^0(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)] \\ &\quad + \frac{3}{4}\int_{\mathbb{R}}\Lambda Q^1\Big[\int_{\mathbb{R}}(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)(3Q^0 - 2(Q^0)^3) + 3\int_{\mathbb{R}}(Q^0)^4(5 - 3(Q^0)^2)\Big]. \end{aligned}$$

Note that, from (M.15)

$$\int_{\mathbb{R}}(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)(3Q^0 - 2(Q^0)^3) + 3\int_{\mathbb{R}}(Q^0)^4(5-3(Q^0)^2) = 2,$$

thus from (M.14) we get

$$\begin{aligned} b^1_{2,0} &= \frac{3}{2}\int_{\mathbb{R}} Q^1(1-(Q^0)^2)[-14(Q^0)^2 + 7(Q^0)^4 - 6xQ^0(Q^0)'] \\ &\quad + \frac{p}{4}\int_{\mathbb{R}}(Q^0)^{p-2}(1-(Q^0)^2)[(p-1)(1-2(Q^0)^2 + (Q^0)^4) - 3Q^0(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)] \\ &\quad + \frac{3}{2}p\int_{\mathbb{R}}(Q^0)^{p-1}\Lambda Q^0(1-(Q^0)^2) \end{aligned}$$

Finally, from (M.18), (M.19) and the identity $\mathcal{L}^0(Q^0)^2 = -3(Q^0)^2$, we have

$$\mathcal{L}^0[2(Q^0)^2 - (Q^0)^4 + 2xQ^0(Q^0)'] = (1-(Q^0)^2)[-14(Q^0)^2 + 7(Q^0)^4 - 6xQ^0(Q^0)'].$$

Using the selfadjointness of $\mathcal{L}^0$ and the equation for Q^1 in (M.10), and after integrating by parts, we conclude that

$$\begin{aligned} b^1_{2,0} &= \frac{3}{2}\int_{\mathbb{R}}(Q^0)^p(2(Q^0)^2 - (Q^0)^4 + 2xQ^0(Q^0)') + \frac{3}{2}p\int_{\mathbb{R}}(Q^0)^{p-1}\Lambda Q^0(1-(Q^0)^2) \\ &\quad + \frac{p}{4}\int_{\mathbb{R}}(Q^0)^{p-2}(1-(Q^0)^2)[(p-1)(1-2(Q^0)^2+(Q^0)^4) - 3Q^0(3x(Q^0)' + 5Q^0 - 2(Q^0)^3)] \\ &= \frac{p}{4}(p-1)\int_{\mathbb{R}}(Q^0)^{p-2} - \frac{3}{4}(p^2+3p-2)\int_{\mathbb{R}}(Q^0)^p + \frac{3}{4(p+2)}(p^3+7p^2+12p+4)\int_{\mathbb{R}}(Q^0)^{p+2} \\ &\quad - \frac{1}{4}(p^2+5p+6)\int_{\mathbb{R}}(Q^0)^{p+4}. \end{aligned}$$

Finally, from (M.15), and after some simplifications,

$$b^1_{2,0} = -\Big[\frac{(p-1)(p-3)(p^2-3p+8)}{8(p-2)(p+1)}\Big]\int_{\mathbb{R}}(Q^0)^p. \tag{M.22}$$

The proof is now complete. □

Remark M.1. Note that even though the higher regularity needed in our results ($f \in C^5$ for $m = 3$), we are able to take, at least formally, the limit $p \downarrow 3$ in (M.22), recovering the results from the integrable case (that is, $b^1_{2,0} = 0$). This gain of regularity comes from (2.27) and (M.2): for these identities, we only need $f \in C^3(\mathbb{R})$.

M.3 Gardner and quadratic nonlinearities

These two nonlinearities are very similar to handle. Although computations are harder for the Gardner nonlinearity, a simple trick will allow to link both results. As a consequence, we are reduced to consider only the quadratic case.

Finally, recall the soliton $Q_{\tilde\mu,1}$ introduced in (1.6), well defined for $\tilde\mu < \frac{2}{9}$. Given $\tilde\mu, \nu \in \mathbb{R}$, $\tilde\mu < \frac{2}{9}$ and ν small enough, let $d_{\tilde\mu,\nu}$ be the *defect* (possibly zero) associated the the nonlinearity $f_{\tilde\mu,\nu}(s) := s^2 - \tilde\mu s^3 + \nu s^p$, namely

$$d_{\tilde\mu,\nu} := b_{2,0}(f_{\tilde\mu,\nu}) + \frac{1}{6}b^3_{1,0}(f_{\tilde\mu,\nu}). \tag{M.23}$$

We following reduction Lemma is the key ingredient of the proof.

Lemma M.2.

Let $d(\varepsilon)$ be the defect parameter introduced in (2.29) for the nonlinearity $f(s)$ described in (1.22), $m = 2$, and let $d_{\tilde\mu,\nu}$ be the defect introduced in (M.23), for $\tilde\mu, \nu$ small. Then the following properties are satisfied:

1. *For all $\tilde\mu < \frac{2}{9}$, $\nu \in \mathbb{R}$ small, $d_{\tilde\mu,\nu}$ is a smooth function of $\tilde\mu, \nu$ and for all $\tilde\mu < \frac{2}{9}$,*

$$d_{\tilde\mu,0} = 0. \tag{M.24}$$

2. *Given ε small, let $\tilde{\mu} = \mu(\varepsilon)$ and $\nu = \varepsilon$. Then the following expansion holds*

$$d_{\mu(\varepsilon),\varepsilon} = -\varepsilon\Big[\frac{(p-3)(2p-1)(48-46p+6p^2+4p^3)}{72(p^2-1)(p-2)}\Big]\int_{\mathbb{R}}\Big[\frac{3}{2\cosh^2(x/2)}\Big]^p + o(\varepsilon), \quad \text{(M.25)}$$

for all $|\varepsilon| < \varepsilon_0$ and $p \geq 3$.

3. *The following expansion holds*

$$d(\varepsilon) = d_{\mu(\varepsilon),\varepsilon} + o(\varepsilon), \quad \text{as } \varepsilon \to 0. \quad \text{(M.26)}$$

Proof of (M.26). This is an easy consequence of the definition of f in (1.21), and the fact that $f(Q) = f_{\mu(\varepsilon),\varepsilon}(Q) + o(\varepsilon)$, with $o(\varepsilon) \in \mathcal{Y}$. In particular, the soliton Q and each term $a_{1,0}, b_{1,0}, A_{1,0}, B_{1,0}$ and $b_{2,0}$ depends smoothly in ε and can be expanded in a similar way. □

Proof of (M.24). The smoothness is a direct consequence of the formula for $b_{2,0}$ in Claim 22 and $b_{1,0}$ in (2.22). We have to prove that for all $\tilde{\mu} < \frac{2}{9}$,

$$b_{2,0}(f_{\tilde{\mu},0}) + \frac{1}{6}b_{1,0}^3(f_{\tilde{\mu},0}) = 0,$$

In order to prove this identity, we claim the following

Claim 25 (Basic functions). Let $Q^0 := Q_{\tilde{\mu},1}$ be the soliton for the Gardner equation. Then we have

1. The soliton solution Q^0 satisfies

$$-(Q^0)'' + Q^0 - (Q^0)^2 + \tilde{\mu}(Q^0)^3 = 0, \quad \mathcal{L}^0(Q^0)' = 0,$$

where $\mathcal{L}^0 := -\partial_{xx} + 1 - (2Q^0 - 3\tilde{\mu}(Q^0)^2)$.

2. From the definition of Q^0, we have

$$\Lambda Q^0 := \frac{1}{2}(x(Q^0)' + 2Q^0) + \frac{3\tilde{\mu}}{4\rho^2}(3Q^0 - (Q^0)^2), \quad \mathcal{L}^0\Lambda Q^0 = -Q^0. \quad \text{(M.27)}$$

Moreover,

$$\int_{\mathbb{R}}\Lambda Q^0 = \frac{3}{\rho^2}, \quad \int_{\mathbb{R}}\Lambda Q^0 Q^0 = \frac{9}{2\rho^2}.$$

3. *Resonance functions*. Define $\varphi^0 := -\frac{(Q^0)'}{Q^0}$. Then

$$(\varphi^0)' = \frac{1}{3}Q^0 - \frac{\tilde{\mu}}{2}(Q^0)^2, \quad (\varphi^0)^2 = 1 - \frac{2}{3}Q^0 + \frac{\tilde{\mu}}{2}(Q^0)^2. \quad \text{(M.28)}$$

4. *Integrals*. For any $p \geq 1$,

$$\tilde{\mu}\int_{\mathbb{R}}(Q^0)^{p+2} = \frac{2(2p+1)}{3(1+p)}\int_{\mathbb{R}}(Q^0)^{p+1} - \frac{2p}{p+1}\int_{\mathbb{R}}(Q^0)^p. \quad \text{(M.29)}$$

Proof. A direct computation, see e.g. Claim 23 for a similar proof. □

Now we proceed to give the explicit value the constants and functions related to system $(\Omega_{1,0})$, see (2.18)-(2.19).

Claim 26 (Resolution of $(\Omega_{1,0})$ for the Gardner equation). Denote by $(a^0_{1,0}, A^0_{1,0}, B^0_{1,0})$ the solution of the linear system $(\Omega_{1,0})$, for the Gardner nonlinearity. Then we have

$$\begin{cases} a^0_{1,0} = \frac{2}{3}, \\ A^0_{1,0} = -\frac{4}{3}Q^0 + \tilde{\mu}(Q^0)^2, \quad B^0_{1,0} = -2\varphi^0 + \kappa^0_{1,0}(Q^0)', \\ b^0_{1,0} = \lim_{+\infty} B^0_{1,0} = -2, \end{cases}$$

with

$$\kappa^0_{1,0} = \frac{3\tilde{\mu}(\int_{\mathbb{R}}(Q^0)^2 - 3\int_{\mathbb{R}} Q^0)}{(3\tilde{\mu}-1)\int_{\mathbb{R}}(Q^0)^2 + \int_{\mathbb{R}} Q^0} = -\frac{10}{3} + o_{\tilde{\mu}}(1). \tag{M.30}$$

Remark M.2. It is remarkable the similarity among the functions solution of the Gardner system $(\Omega_{1,0})$ and the corresponding ones for the quadratic nonlinearity (let $\tilde{\mu} \to 0$).

Proof. First of all, the explicit value of $(a^0_{1,0}, A^0_{1,0}, B^0_{1,0})$ comes from a straightforward verification. More precisely, this triplet is a solution of the zeroth order system

$$\begin{cases} (\mathcal{L}^0 A^0_{1,0})' + a^0_{1,0}(3Q^0 - 2(Q^0)^2 + 2\tilde{\mu}(Q^0)^3)' = (2Q^0 - 3\tilde{\mu}(Q^0)^2)', \\ (\mathcal{L}^0 B^0_{1,0})' + 3a^0_{1,0}(Q^0)'' - 3(A^0_{1,0})'' - (2Q^0 - 3\tilde{\mu}(Q^0)^2)A^0_{1,0} = 2Q^0 - 3\tilde{\mu}(Q^0)^2, \end{cases}$$

which comes from (2.18)-(2.19). In particular, we choose $\kappa_{1,0}$ such that $\int_{\mathbb{R}} B^0_{1,0}(Q^0)' = 0$. The value of $b^0_{1,0}$ comes from the fact that $b^0_{1,0} = -2\lim_{+\infty}\varphi^0 = -2$. On the other hand, from (2.22), one has

$$b^0_{1,0} = -\frac{1}{3}\int_{\mathbb{R}} Q^0 + \frac{1}{2}\tilde{\mu}\int_{\mathbb{R}}(Q^0)^2 = -2.$$

□

Now we are able to prove (M.24). (Note that this is also a consequence of the integrability of the Gardner equation.) First, we claim that

$$3(A^0_{1,0})' + (2Q^0 - 3\tilde{\mu}(Q^0)^2)\int_0^x (A^0_{1,0} + a^0_{1,0}Q^0) = 3(A^0_{1,0})' - 2(2Q^0 - 3\tilde{\mu}(Q^0)^2)\varphi^0 = 0.$$

This is an easy consequence of (M.28) and the values of of $A^0_{1,0}$ and $a^0_{1,0}$. Consider now the expression for $b_{2,0}$, $m=2$ in Claim 22. Note that the term containing $B^0_{1,0}$ disappears. Replacing the values of $a^0_{1,0}$ and $A^0_{1,0}$, and using the recursive formula (M.29), we have

$$\begin{aligned} d_{\tilde{\mu},0} &= -\frac{1}{3}(b^0_{1,0})^3 - \frac{3}{2}\tilde{\mu}\int_{\mathbb{R}} Q^0(1+A^0_{1,0})^3 - 2b^0_{1,0} + \frac{1}{2}\int_{\mathbb{R}} A^0_{1,0}(1+(A^0_{1,0})^2) - \frac{1}{3}\int_{\mathbb{R}} Q^0 A^0_{1,0} \\ &\quad -\frac{2}{9}\int_{\mathbb{R}}(Q^0)'^2 + \frac{2}{9}\int_{\mathbb{R}}(Q^0)^2[1-(2-3\tilde{\mu}Q^0)(1+A^0_{1,0})] + \frac{4}{3}\int_{\mathbb{R}}(Q^0)''A^0_{1,0} \\ &\quad -\frac{1}{2}\int_{\mathbb{R}}[(2Q^0 - 3\tilde{\mu}(Q^0)^2)(1+A^0_{1,0}) + 3(A^0_{1,0})'']A^0_{1,0} \\ &= -\frac{1}{3}(b^0_{1,0})^3 - \frac{3}{2}\tilde{\mu}\int_{\mathbb{R}} Q^0 + (\frac{11}{2}\tilde{\mu} - \frac{28}{9})\int_{\mathbb{R}}(Q^0)^2 + (-\frac{9}{2}\tilde{\mu}^2 + \frac{2}{3}\tilde{\mu} + \frac{20}{9})\int_{\mathbb{R}}(Q^0)^3 \\ &\quad +\tilde{\mu}(\frac{15}{2}\tilde{\mu} - \frac{13}{3})\int_{\mathbb{R}}(Q^0)^4 - \frac{9}{2}\tilde{\mu}^3\int_{\mathbb{R}}(Q^0)^5 + \frac{7}{2}\tilde{\mu}^3\int_{\mathbb{R}}(Q^0)^6 - \frac{3}{2}\tilde{\mu}^4\int_{\mathbb{R}}(Q^0)^7 \\ &= -\frac{1}{3}(b^0_{1,0})^3 - \frac{4}{9}\int_{\mathbb{R}} Q^0 + \frac{2}{3}\tilde{\mu}\int_{\mathbb{R}}(Q^0)^2 \\ &= \frac{1}{3}b^0_{1,0}(4 - (b^0_{1,0})^2) = 0. \end{aligned}$$

This proves (M.24). □

Proof of (M.25). The proof of (M.25) is a consequence of (M.24) and the simple relationship

$$d_{\tilde{\mu},\nu} = d_{\tilde{\mu},0} + \nu\partial_\nu d_{\tilde{\mu},\nu} + o_\nu(\nu) = \nu(\partial_\nu d_{0,\nu} + o_{\tilde{\mu}}(1)) + o_\nu(\nu),$$

valid for any $\tilde{\mu}, \nu \in \mathbb{R}$ small enough. This result says that, in order to prove (M.25), we only need to compute the defect for first order expansion in ν of the quadratic nonlinearity $f(s) = s^2 + \nu s^p$, $p \geq 3$. Then we use the fact that $\tilde{\mu} = \mu(\varepsilon) \sim \varepsilon^{1/(p-2)}$ and $\nu = \varepsilon$ to conclude. Consequently, in what follows we are reduced to prove that $\partial_\nu d_{0,\nu} \neq 0$ for all ν small enough. □

Claim 27 (Asymptotic expansions, case $m = 2$, basic functions). Suppose now $f(s) = f_{0,\nu}(s) = s^2 + \nu s^p$. Let $Q^0(x) = \frac{3}{2\cosh^2(x/2)}$ be the soliton solution for the quadratic case. Then the following asymptotic expansions hold.

1. The soliton solution Q for the nonlinearity f can be expanded as

$$Q = Q^0 + \nu Q^1 + o(\nu), \quad o(\nu) \in \mathcal{Y}, \tag{M.31}$$

where Q^1 satisfies the equation $\mathcal{L}^0 Q^1 := -(Q^1)'' + Q^1 - 2Q^0Q^1 = (Q^0)^p$. We also have

$$\begin{cases} f(Q) = (Q^0)^2 + \nu((Q^0)^p + 2Q^0Q^1) + o(\nu), \\ f'(Q) = 2Q^0 + \nu(2Q^1 + p(Q^0)^{p-1}) + o(\nu), \\ f''(Q) = 2 + \nu p(p-1)(Q^0)^{p-2} + o(\nu), \end{cases} \tag{M.32}$$

where every term $o(\nu) \in \mathcal{Y}$ uniformly in $\nu < \nu_0$ small.

2. The operator $\mathcal{L}$ satisfies

$$\mathcal{L} = \mathcal{L}^0 - \nu[2Q^1 + p(Q^0)^{p-1}] + o(\nu).$$

3. From the definition of Q, we have

$$\Lambda Q = \Lambda Q^0 + \nu\Lambda Q^1 + o(\nu), \quad \mathcal{L}^0\Lambda Q^1 = -Q^1 + (2Q^1 + p(Q^0)^{p-1})\Lambda Q^0. \tag{M.33}$$

4. Let $D(\nu) := \int_{\mathbb{R}} \Lambda Q Q$. Then

$$D(\nu) = \frac{9}{2} + \nu\Big(\int_{\mathbb{R}} \Lambda Q^1 Q^0 + \int_{\mathbb{R}} \Lambda Q^0 Q^1\Big) + o(\nu). \tag{M.34}$$

5. *Inverse functions*. We have

$$\mathcal{L}^0\big[1 - \frac{4}{3}\Lambda Q^0\big] = 1 - \frac{2}{3}Q^0. \tag{M.35}$$

$$\mathcal{L}^0\big[(1 - Q^0)(1 + \frac{1}{3}x^2Q^0) - Q^0\big] = 1 - \frac{8}{3}\Lambda Q^0 + \frac{8}{3}(\Lambda Q^0)^2. \tag{M.36}$$

$$\mathcal{L}^0\big[-5 + \frac{68}{9}Q^0 - 6\Lambda Q^0\big] = -5 + 16Q^0 - \frac{68}{9}(Q^0)^2. \tag{M.37}$$

$$\mathcal{L}^0\big[2 + \frac{20}{3}\Lambda Q^0 - \frac{170}{27}Q^0\big] = 2 - \frac{32}{3}Q^0 + \frac{170}{27}(Q^0)^2. \tag{M.38}$$

Proof. First of all, (M.31)-(M.33) are a direct consequence of a Taylor expansion of the considered functions. The expression for $(\Lambda Q^0)^2$ comes from a simple computation.

The expansion of $D(\nu)$ in (M.34) follows from a Taylor expansion and the fact that

$$\int_{\mathbb{R}} \Lambda Q^0 Q^0 = \frac{3}{4}\int_{\mathbb{R}} (Q^0)^2 = \frac{9}{2}.$$

Finally we prove (M.36), (M.37) and (M.38). These follow from the identities $\mathcal{L}^0 1 = 1 - 2Q^0$, $\mathcal{L}^0 Q^0 = -(Q^0)^2$, $\mathcal{L}^0 \Lambda Q^0 = -Q^0$, $\mathcal{L}^0(x^2 Q^0) = -2Q^0 - 4x(Q^0)' - x^2(Q^0)^2$, and

$$\mathcal{L}^0(x^2(Q^0)^2) = -2(Q^0)^2 - 8xQ^0(Q^0)' - 3x^2(Q^0)^2 + \frac{4}{3}x^2(Q^0)^3.$$

This finishes the proof. □

Now we proceed to give an asymptotic expansion of the constants and functions related to system $(\Omega_{1,0})$, see (2.18)-(2.19).

Claim 28 (Asymptotic expansions II, case $m = 2$). There exists ν_0 small enough such that for all $|\nu| \leq \nu_0$, the following holds. Let $f(s) = s^2 + \nu s^p$, then the corresponding solution to the system $(\Omega_{1,0})$ for this case can be expanded as follows:

$$a_{1,0} = \frac{2}{3} + \nu a_{1,0}^1 + o(\nu), \quad A_{1,0} = -\frac{4}{3}Q^0 + \nu A_{1,0}^1 + o(\nu) \in \mathcal{Y}, \quad b_{1,0} = -2 + b_{1,0}^1\nu + o(\nu), \quad \text{(M.39)}$$

where $\nu^{-1}o(\nu) \to 0$ as $\nu \to 0$ and $A_{1,0}^1 \in \mathcal{Y}$ is a solution of the following linear equation

$$(\mathcal{L}^0 A_{1,0}^1)' + a_{1,0}^1(3Q^0 - 2(Q^0)^2)' = [p(Q^0)^{p-1} - \frac{4}{3}(p-1)(Q^0)^p]', \quad \text{(M.40)}$$

Finally, the following two expressions are satisfied

$$a_{1,0}^1 = -\frac{1}{9}\Big[\frac{(p-3)(2p-1)}{p+1}\Big]\int_{\mathbb{R}} (Q^0)^p, \quad \text{(M.41)}$$

and

$$b_{1,0}^1 = \frac{1}{2}\int_{\mathbb{R}} A_{1,0}^1 + \frac{1}{2}a_{1,0}^1\int_{\mathbb{R}} Q^0 + \frac{1}{3}\int_{\mathbb{R}} Q^1. \quad \text{(M.42)}$$

Proof. The proof of (M.39) and (M.40) is direct from Claim 27 and (2.18)-(2.19). To prove (M.41), first note that from Claim 27

$$\begin{aligned} a_{1,0}^1 &= \frac{2}{9}\Big[\int_{\mathbb{R}} (1 - \frac{2}{3}Q^0)\Lambda Q^1 - \frac{2}{3}\int_{\mathbb{R}} Q^1 \Lambda Q^0\Big] \\ &= \frac{2}{9}\int_{\mathbb{R}} [1 - \frac{4}{3}\Lambda Q^0][-Q^1 + (2Q^1 + p(Q^0)^{p-1})\Lambda Q^0] - \frac{4}{27}\int_{\mathbb{R}} Q^1 \Lambda Q^0 \\ &= -\frac{2}{9}\int_{\mathbb{R}} Q^1[1 - \frac{8}{3}\Lambda Q^0 + \frac{8}{3}(\Lambda Q^0)^2] + \frac{2p}{9}\int_{\mathbb{R}} (Q^0)^{p-1}\Lambda Q^0[1 - \frac{4}{3}\Lambda Q^0]. \end{aligned}$$

Thus from (M.36) and Claim 27 (i), we get after integration by parts

$$\begin{aligned} a_{1,0}^1 &= \frac{2}{9}\int_{\mathbb{R}} (Q^0)^{p-1}\big[p\Lambda Q^0(1 - \frac{4}{3}\Lambda Q^0) - Q^0(1 - Q^0)(1 + \frac{1}{3}x^2 Q^0) + (Q^0)^2\big] \\ &= \frac{2}{9}\Big[\int_{\mathbb{R}} (Q^0)^{p-1}\big[(p-1)Q^0 + (2 - \frac{4}{3}p)(Q^0)^2 + \frac{p}{2}x(Q^0)' - \frac{4}{3}pxQ^0(Q^0)'\big] \\ &\qquad -\frac{1}{3}\int_{\mathbb{R}} x^2(Q^0)^{p+1}\big[(p+1) - (1 + \frac{2}{3}p)Q^0\big]\Big] \\ &= \frac{2}{9}\Big[(p - \frac{3}{2})\int_{\mathbb{R}} (Q^0)^p + (\frac{10}{3} - \frac{4}{3}p - \frac{4}{3}\frac{1}{p+1})\int_{\mathbb{R}} (Q^0)^{p+1} \\ &\qquad -\frac{1}{3}\int_{\mathbb{R}} x^2(Q^0)^{p+1}\big[(p+1) - (1 + \frac{2}{3}p)Q^0\big]\Big]. \end{aligned}$$

Now, recall that from the equation satisfied by Q^0, $[(Q^0)^{p+1}]'' = (p+1)(Q^0)^{p+1}[(p+1) - (1+\frac{2}{3}p)Q^0]$, so that

$$\int_{\mathbb{R}} x^2(Q^0)^{p+1}[(p+1) - (1+\frac{2}{3}p)Q^0] = \frac{1}{p+1}\int_{\mathbb{R}} x^2[(Q^0)^{p+1}]'' = \frac{2}{p+1}\int_{\mathbb{R}}(Q^0)^{p+1}.$$

In conclusion, from (M.29),

$$a^1_{1,0} = \frac{2}{9}\Big[(p-\frac{3}{2})\int_{\mathbb{R}}(Q^0)^p + (\frac{10}{3} - \frac{4}{3}p - \frac{2}{p+1})\int_{\mathbb{R}}(Q^0)^{p+1}\Big] = -\frac{1}{9}\Big[\frac{(p-3)(2p-1)}{p+1}\Big]\int_{\mathbb{R}}(Q^0)^p,$$

as desired. Finally, from (2.22) we obtain (M.42). □

Now we deal with the second order system $(\Omega_{2,0})$ written in (2.23), (2.24), (2.25) and (2.26).

Claim 29 (Asymptotic expansions III, case $m=2$). The following identity holds

$$\partial_\nu d_{0,\nu}|_{\nu=0} = -\Big[\frac{(p-3)(2p-1)(24-23p+3p^2+2p^3)}{36(p^2-1)(p-2)}\Big]\int_{\mathbb{R}}(Q^0)^p,$$

for all $p \geq 3$.

Proof. The proof of the above result is equivalent to prove that for the nonlinearity $f_{0,\nu}(s) = s^2 + \mu s^p$, $p \geq 3$ and ν small, we have

$$d_{0,\nu} = b_{2,0}(f_{0,\nu}) + \frac{1}{6}b^3_{1,0}(f_{0,\nu}) = -\nu\Big[\frac{(p-3)(2p-1)(24-23p+3p^2+2p^3)}{36(p^2-1)(p-2)}\Big]\int_{\mathbb{R}}(Q^0)^p + o(\nu).$$

First of all, note that we can expand $b_{2,0} = b^0_{2,0} + \nu b^1_{2,0} + o(\nu)$, with $b^0_{2,0} = \frac{4}{3}$ (cf. [53], Lemma 3.1). By considering (M.1) in Claim 22, Claim 27 and expanding at first order in ν, we get

$$\begin{aligned}
b^1_{2,0} \; = \; & -8b^1_{1,0} + \frac{1}{4}p(p-1)\int_{\mathbb{R}}(Q^0)^{p-2}(1+A^0_{1,0})^3 + \frac{1}{2}\int_{\mathbb{R}} A^1_{1,0}(1+3(A^0_{1,0})^2) - \frac{1}{2}a^1_{1,0}\int_{\mathbb{R}} Q^0A^0_{1,0} \\
& -\frac{1}{3}\int_{\mathbb{R}} Q^1A^0_{1,0} - \frac{1}{3}\int_{\mathbb{R}} Q^0A^1_{1,0} - a^1_{1,0}\int_{\mathbb{R}}(Q^0)'^2 - \frac{4}{9}\int_{\mathbb{R}}(Q^0)'(Q^1)' \\
& +\frac{2}{3}a^1_{1,0}\int_{\mathbb{R}}(Q^0)^2(1-2(1+A^0_{1,0})) + \frac{2}{9}\int_{\mathbb{R}} Q^0Q^1(1-2(1+A^0_{1,0})) \\
& +\frac{2}{9}\int_{\mathbb{R}} Q^0[Q^1 - 2Q^0A^1_{1,0} - (2Q^1 + p(Q^0)^{p-1})(1+A^0_{1,0})] \\
& -\frac{3}{4}a^1_{1,0}\int_{\mathbb{R}}[2Q^0(1+A^0_{1,0}) + 3(A^0_{1,0})'']A^0_{1,0} - \frac{1}{2}\int_{\mathbb{R}}[2Q^0(1+A^0_{1,0}) + 3(A^0_{1,0})'']A^1_{1,0} \\
& -\frac{1}{2}\int_{\mathbb{R}}[(2Q^1 + p(Q^0)^{p-1})(1+A^0_{1,0}) + 2Q^0A^1_{1,0} + 3(A^1_{1,0})'']A^0_{1,0} \\
& +\frac{1}{2}\int_{\mathbb{R}} B^0_{1,0}(2Q^1 + p(Q^0)^{p-1})\int_0^x(A^0_{1,0} + \frac{2}{3}Q^0) + \frac{1}{2}\int_{\mathbb{R}} B^0_{1,0}[3(A^1_{1,0})' \\
& +2Q^0\int_0^x(A^1_{1,0} + a^1_{1,0}Q^0 + \frac{2}{3}Q^1)] + 4a^1_{1,0}\int_{\mathbb{R}}(Q^0)''A^0_{1,0} + \frac{4}{3}\int_{\mathbb{R}}(Q^1)''A^0_{1,0} + \frac{4}{3}\int_{\mathbb{R}}(Q^0)''A^1_{1,0}.
\end{aligned}$$

Now we arrange the above expression according to $a^1_{1,0}$, $A^1_{1,0}$, Q^1, $b^1_{1,0}$, $B^1_{1,0}$ and the rest terms. We obtain,

$$\begin{aligned}
b^1_{2,0} \; = \; & -8b^1_{1,0} + \frac{1}{4}p(p-1)\int_{\mathbb{R}}(Q^0)^{p-2}(1+A^0_{1,0})^3 - \frac{p}{3}\int_{\mathbb{R}}(Q^0)^{p-1}(1+A^0_{1,0})(\frac{3}{2}A^0_{1,0} + \frac{2}{3}Q^0) \\
& +p\int_{\mathbb{R}} B^0_{1,0}(Q^0)^{p-2}(Q^0)' + \int_{\mathbb{R}} B^0_{1,0}Q^0\int_0^x A^1_{1,0} + \frac{1}{2}\int_{\mathbb{R}} A^1_{1,0}F_A + \delta a^1_{1,0} + \int_{\mathbb{R}} Q^1F_Q \\
& +\frac{2}{3}\int_{\mathbb{R}} B^0_{1,0}Q^0\int_0^x Q^1,
\end{aligned}$$

where

$$F_A := 1+3(A^0_{1,0})^2-\frac{2}{3}Q^0-\frac{8}{9}(Q^0)^2-(2Q^0+2Q^0A^0_{1,0}+3(A^0_{1,0})'')-(2Q^0A^0_{1,0}+3(A^0_{1,0})'')$$
$$-3(B^0_{1,0})'+\frac{8}{3}(Q^0)'';$$

$$\delta := -\frac{1}{2}\int_{\mathbb{R}}Q^0A^0_{1,0}+\int_{\mathbb{R}}Q^0(Q^0)''-\frac{2}{3}\int_{\mathbb{R}}(Q^0)^2(1+2A^0_{1,0})$$
$$-\frac{3}{4}\int_{\mathbb{R}}(2Q^0+2Q^0A^0_{1,0}+3(A^0_{1,0})'')A^0_{1,0}+4\int_{\mathbb{R}}(Q^0)''A^0_{1,0};$$

and

$$F_Q := -\frac{1}{3}A^0_{1,0}+\frac{4}{9}(Q^0)''-\frac{4}{9}Q^0(1+2A^0_{1,0})-(1+A^0_{1,0})A^0_{1,0}-2\varphi^0B^0_{1,0}+\frac{4}{3}(A^0_{1,0})''.$$

Note that we have used that

$$\int_{\mathbb{R}}B^0_{1,0}(Q^0)'=0,\qquad \int_0^x(A^0_{1,0}+\frac{2}{3}Q^0)=-2\varphi^0. \tag{M.43}$$

Now we use the expressions (M.41) and (M.42) in Claim 28 to have

$$b^1_{2,0}+2b^1_{1,0} = -\frac{12}{5}a^1_{1,0}+\frac{1}{4}p(p-1)\int_{\mathbb{R}}(Q^0)^{p-2}(1-\frac{4}{3}Q^0)^3+\frac{4p}{9}\int_{\mathbb{R}}(Q^0)^p(1-\frac{4}{3}Q^0)$$
$$+p\int_{\mathbb{R}}(Q^0)^{p-1}\big[2-\frac{14}{3}Q^0+\frac{20}{9}(Q^0)^2\big]+\frac{1}{2}\int_{\mathbb{R}}A^1_{1,0}\Big[-5+16Q^0-\frac{68}{9}(Q^0)^2\Big]$$
$$+\int_{\mathbb{R}}Q^1\Big[2-\frac{32}{3}Q^0+\frac{170}{27}(Q^0)^2\Big].$$

Note that we have also made use of (M.29) with $\tilde{\mu}=0$ to obtain

$$-3\int_{\mathbb{R}}Q^0-\frac{1}{2}\int_{\mathbb{R}}Q^0A^0_{1,0}+\int_{\mathbb{R}}Q^0(Q^0)''-\frac{2}{3}\int_{\mathbb{R}}(Q^0)^2(1+2A^0_{1,0})$$
$$-\frac{3}{4}\int_{\mathbb{R}}(2Q^0+2Q^0A^0_{1,0}+3(A^0_{1,0})'')A^0_{1,0}+4\int_{\mathbb{R}}(Q^0)''A^0_{1,0}=-\frac{12}{5}.$$

Using (M.37), (M.38) and (M.40), we have

$$b^1_{2,0}+2b^1_{1,0}=-\frac{12}{5}a^1_{1,0}+\frac{1}{4}p(p-1)\int_{\mathbb{R}}(Q^0)^{p-2}(1-\frac{4}{3}Q^0)^3+\frac{4p}{9}\int_{\mathbb{R}}(Q^0)^p(1-\frac{4}{3}Q^0)$$
$$+\frac{1}{2}\int_{\mathbb{R}}\Big[-a^1_{1,0}(3Q^0-2(Q^0)^2)+p(Q^0)^{p-1}-\frac{4}{3}(p-1)(Q^0)^p\Big]\Big[-5-6\Lambda Q^0+\frac{68}{9}Q^0\Big]$$
$$+\int_{\mathbb{R}}(Q^0)^p\Big[2+\frac{20}{3}\Lambda Q^0-\frac{170}{27}Q^0\Big]+p\int_{\mathbb{R}}(Q^0)^{p-1}\big[2-\frac{14}{3}Q^0+\frac{20}{9}(Q^0)^2\big].$$

A simple computation using (M.33) and (M.29) with $\tilde{\mu}=0$ shows that

$$\int_{\mathbb{R}}(3Q^0-2(Q^0)^2)(-5-6\Lambda Q^0+\frac{68}{9}Q^0)=-\frac{59}{5}.$$

Thus, replacing the value of $a^1_{1,0}$ given by (M.41),

$$b^1_{2,0}+2b^1_{1,0}=-\frac{7}{18}\Big[\frac{(p-3)(2p-1)}{p+1}\Big]\int_{\mathbb{R}}(Q^0)^p+\frac{4p}{9}\int_{\mathbb{R}}(Q^0)^p(1-\frac{4}{3}Q^0)$$
$$+\frac{1}{4}p(p-1)\int_{\mathbb{R}}(Q^0)^{p-2}(1-\frac{4}{3}Q^0)^3+p\int_{\mathbb{R}}(Q^0)^{p-1}\big[2-\frac{14}{3}Q^0+\frac{20}{9}(Q^0)^2\big]$$
$$+\frac{1}{2}\int_{\mathbb{R}}\Big[p(Q^0)^{p-1}-\frac{4}{3}(p-1)(Q^0)^p\Big]\Big[-5-6\Lambda Q^0+\frac{68}{9}Q^0\Big]$$
$$+2\int_{\mathbb{R}}(Q^0)^p+\frac{10}{3}\frac{2p+1}{p+1}\int_{\mathbb{R}}(Q^0)^{p+1}-\frac{170}{27}\int_{\mathbb{R}}(Q^0)^{p+1}.$$

Simplifiying, we get

$$b^1_{2,0}+2b^1_{1,0}=-\frac{7}{18}\frac{(p-3)(2p-1)}{p+1}\int_{\mathbb{R}}(Q^0)^p+\frac{1}{4}p(p-1)\int_{\mathbb{R}}(Q^0)^{p-2}-p(p-\frac{1}{2})\int_{\mathbb{R}}(Q^0)^{p-1}$$
$$+(\frac{1}{6}-\frac{13}{9}p+\frac{4}{3}p^2)\int_{\mathbb{R}}(Q^0)^p+(-\frac{16}{27}+\frac{32}{27}p+\frac{2}{3}\frac{1}{p+1}-\frac{16}{27}p^2)\int_{\mathbb{R}}(Q^0)^{p+1}.$$

Using (M.29) with $\tilde{\mu}=0$ and the fact that $p\geq 3$, we finally obtain

$$\begin{aligned} b^1_{2,0}+2b^1_{1,0} &= \Big[-\frac{7}{18}\frac{(p-3)(2p-1)}{p+1}+\frac{1}{36}p\frac{(2p-1)(2p-3)}{p-2}-\frac{1}{3}p(p-\frac{1}{2})\frac{(2p-1)}{p-1}+ \\ &\quad +(\frac{1}{6}-\frac{13}{9}p+\frac{4}{3}p^2)+\frac{3p}{1+2p}(-\frac{16}{27}+\frac{32}{27}p+\frac{2}{3}\frac{1}{p+1}-\frac{16}{27}p^2)\Big]\int_{\mathbb{R}}(Q^0)^p \\ &= -\frac{(p-3)(2p-1)(24-23p+3p^2+2p^3)}{36(p^2-1)(p-2)}\int_{\mathbb{R}}(Q^0)^p. \end{aligned}$$

Let us define, for p real, $f(p):=24-23p+3p^2+2p^3$. Then we have

$$f(p)\geq 36 \quad \text{for all } p\geq 3. \tag{M.44}$$

It is clear that this last affirmation allows us to conclude the proof. Let us prove (M.44). Note that $f(3)=36$ and $f'(p)$ is given by $f'(p)=6p^2+6p-23>0$ for all $p\geq 3$. This implies (M.44). The proof is complete. □

Remark M.3. First of all, note that in the above expression we recover the integrability condition of the Gardner equation ($p=3$). Furthermore, note that this term is divergent when we formally take the limit $p\downarrow 2$ and the equation approaches the integrable case. This can be explained by the higher regularity needed in our results ($f\in C^4$ for $m=2$), to justify the asymptotics. Indeed, from (2.25) and (M.1) we need at least $f\in C^3(\mathbb{R})$, and $f(s):=s^2+\varepsilon s^p$ is not C^3 at zero as $p\downarrow 2$, $p>2$. In addition, the terms in (2.25), (2.26)

$$\frac{1}{2}(f''(Q)-2)',\ \frac{1}{2}(f''(Q)-2),$$

vanish in the integrable case $m=p=2$. For the computation in the quadratic case, see Proposition 2.1 and Lemma 3.1 in [53].

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SOLITON DYNAMICS AND COLLISION FOR SOME NONLINEAR DISPERSIVE EQUATIONS

Abstract

This work deals with long time dynamics of soliton solutions for generalizations of well-known dispersive equations.

The first part of this work is devoted to the study of existence, uniqueness and global behavior of soliton-like solutions for slowly varying, but still large perturbations of generalized KdV equations. We give an accurate description of the dynamics for all time and prove in addition the nonexistence of pure soliton-like solutions, a big difference with the standard gKdV equations.

Next, the same kind of results are proven in the case of nonlinear Schrödinger equations. We improve all the existing results by constructing a unique global soliton solution in this regime, and studying in detail its behavior. In addition, under some mild assumptions we extend this result to the two-dimensional case and under general incident velocities.

Finally, we consider the scenario of a 2-soliton collision between a small and a very small soliton, for generalized KdV equations. We prove a classification result which completes the Martel-Merle results –concerning the quartic case– asserting that in a very general framework the unique possibilities for having an elastic collision are given by the integrable cases.

The proof of all these results are reminiscent of the very recent Martel-Merle theory of 2-soliton's collision for gKdV equations under different asymptotic regimes.

Keywords : generalized Korteweg-de Vries and nonlinear Schrödinger equations, soliton dynamics, slowly varying potentials, 2-soliton collision, integrability.