Deformation of Orbits in Minimal Sheets

Jonas Budmiger

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Jonas Budmiger

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Prof. Dr. Hanspeter Kraft und Prof. Dr. Michel Brion

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CHAPTER I

Introduction

In this introductory part we start by fixing some notation and recalling some notions used throughout this Thesis. Then we proceed by giving a rough survey of the contents of this Thesis.

We work over an algebraically closed field k of characteristic zero, and all schemes and rings are supposed to be Noetherian. In this introductory chapter, G is a connected reductive linear algebraic group. In the other chapters, G is a semisimple linear algebraic group. As references for the structure of semisimple groups and their representations, we use [**Hu75**] and [**Bo81**]. We fix once and for all a maximal torus $T \subset G$, a Borel subgroup $B \subset G$ containing T, and we let $U \subset B$ be the corresponding unipotent radical. Further, Λ denotes the weight lattice of (G, B, T), and $\Lambda^+ \subset \Lambda$ denotes the monoid of dominant weights. For a dominant weight $\lambda \in \Lambda^+$ denote by $V(\lambda)$ an irreducible G-module with highest weight λ. Let w_0 be the longest element of the Weyl group, and let $\lambda^* = -w_0\lambda$ for $\lambda \in \Lambda^+$. Then the dual G-module $V(\lambda)^*$ is isomorphic to $V(\lambda^*)$ (see e.g. [**Hu75**], Exercise 21.6).

If V is a rational G-module, and if $\lambda \in \Lambda^+$ is a dominant weight, then $V_{(\lambda)} := \sum\{W \subset V \mid W \cong V(\lambda) \text{ as } G\text{-module}\}$ denotes the *isotypic component of V of type λ*. Then $V_{(\lambda)} \cong V_\lambda^U \otimes V(\lambda)$, where V_λ denotes the weight space of V to the weight λ. If $X = \operatorname{Spec}(R)$ is an affine G-scheme, then X is called *multiplicity-finite* if $\dim_k R_{(\lambda)} < \infty$ for all $\lambda \in \Lambda^+$ (or equivalently, if $\dim_k R^G < \infty$), and X is called *multiplicity-free* if $\dim_k R_{(\lambda)}^U \leq 1$ for all λ. If X is multiplicity-finite, we define the *Hilbert function $h_X \equiv h_R$ of X* (or *of R*) to be the function $h_X \colon \Lambda^+ \to \mathbf{N}$ assigning $\dim_k R_{(\lambda)}^U$ (which is the multiplicity of $V(\lambda)$ in R) to λ (background information can be found e.g. in [**Kr85**]).

The notions of isotypic components and Hilbert functions can be extended to families. These constructions are crucial for this work, and they can be found in [**AB05**]. If S is a scheme, and if $\mathfrak{X}$ is a G-scheme with an affine G-invariant morphism $\pi \colon \mathfrak{X} \to S$ of finite type, then the sheaf of $\mathcal{O}_S$-modules $\mathcal{R} := \pi_*\mathcal{O}_{\mathfrak{X}}$ comes with a G-action, and $\mathcal{R}_{(\lambda)}$ denotes the *isotypic component of $\mathcal{R}$ of type λ*. In this situation $\pi \colon \mathfrak{X} \to S$ is called a *family of affine G-schemes*. If each isotypic component $(\mathcal{R}_{(\lambda)})^U$ is a locally free sheaf of $\mathcal{O}_S$-modules of constant finite rank, then we say that the family is *multiplicity-finite with Hilbert function h*, where $h \colon \Lambda^+ \to \mathbf{N}$ is the function assigning to λ the rank of $(\mathcal{R}_{(\lambda)})^U$ as $\mathcal{O}_S$-module. In this case, the morphism $\pi\colon \mathfrak{X} \to S$ is flat. Let vice versa $\pi\colon \mathfrak{X} \to S$ be a flat family of affine G-schemes over a connected scheme S with the property that $\mathcal{R}^G$ is finitely generated $\mathcal{O}_S$-module, where $\mathcal{R} := \pi_* \mathcal{O}_{\mathfrak{X}}$. Then π is a multiplicity-finite family of affine G-schemes with Hilbert function h, where h is the Hilbert function of the fiber of π over the general point of S. (To see this, one has to verify that each $(\mathcal{R}_{(\lambda)})^U$ is a locally free sheaf of $\mathcal{O}_S$-modules of constant rank $h(\lambda)$: First, $\mathcal{R}^G$ is a finitely generated $\mathcal{O}_S$-module. Moreover, each $(\mathcal{R}_{(\lambda)})^U$ is a finitely generated $\mathcal{R}^G$-module (see [**Kr85**] II.3.2), and hence a finitely generated $\mathcal{O}_S$-module. Localizing in $s \in S$ and taking U-invariants, it follows that $(\mathcal{R}_{(\lambda)})^U_s$ is a finitely generated $\mathcal{O}_{S,s}$-module. Now, $\mathcal{O}_{S,s}$ is Noetherian by assumption, and a finitely generated module over a Noetherian ring is finitely presented ([**Ma86**], Exercise 3.7). Furthermore, $(\mathcal{R}_{(\lambda)})^U_s$ is a flat $\mathcal{O}_{S,s}$-module because $\pi\colon \mathfrak{X} \to S$ is flat. A flat and finitely presented module is projective ([**Ma86**], Corollary to Theorem 7.12). Finally, a finitely generated projective module over a local ring is free ([**Ma86**] Theorem 2.5). This shows that $(\mathcal{R}_{(\lambda)})^U_s$ is a free $\mathcal{O}_{S,s}$-module. Since S is connected, the rank does not depend on $s \in S$.)

In particular, the Hilbert function of a multiplicity-finite affine G-variety X is the same as the Hilbert function of the corresponding family $X \to \operatorname{Spec}(k)$.

If X is an affine G-scheme, and if $h\colon \Lambda^+ \to \mathbf{N}$ is any function, we can consider the (contravariant) functor $\mathcal{H}ilb^G_h(X)$ from the category of schemes to the category of sets assigning to a scheme S the set of all multiplicity-finite families $\mathfrak{X} \xrightarrow{\pi} S$ of affine G-schemes with Hilbert function h, where $\mathfrak{X} \subset S \times X$ is a closed G-stable subscheme, and where $\pi\colon \mathfrak{X} \to S$ is the restriction to $\mathfrak{X}$ of the projection of $S \times X$ onto S. Then Theorem 1.7 in [**AB05**] states that the functor $\mathcal{H}ilb^G_h(X)$ is represented by a quasi-projective scheme $\operatorname{Hilb}^G_h(X)$, called *invariant Hilbert scheme* to the data (G, h, X).

The main object of study of this work are orbits in so-called *minimal sheets* in irreducible representations. The notion of sheets goes back to Dixmier, cf. [**Di75**]: Given a G-module V, the union of all orbits in V of a fixed dimension is a locally closed subset. Its irreducible components are called *sheets of* V. We call a sheet *minimal* if it contains an orbit in V of minimal strictly positive dimension among all orbits in V. In Chapter II, we describe minimal sheets in simple G-modules, and study G-stable deformations of orbits in minimal sheets by means of an invariant Hilbert scheme. This is closely related to the work of Jansou (cf. [**Ja05**]) in the following way: Choose once and for all a non-zero vector $v_\lambda \in V(\lambda)^U$ for each $\lambda \in \Lambda^+$, and let $X_\lambda = \overline{Gv_\lambda} \subset V(\lambda)$ be the closure of the orbit Gv_λ of v_λ in $V(\lambda)$. Since each G-module $V(\lambda)$ contains a unique B-stable line, v_λ is determined uniquely up to scalar multiples, and X_λ is determined independently of the choice of v_λ.

In [**Ja05**], Jansou investigates G-stable deformations of X_λ in $V(\lambda)$. If h_λ denotes the Hilbert function of X_λ, then Jansou proves that the invariant Hilbert scheme $\mathrm{Hilb}^G_{h_\lambda}(V(\lambda))$ is, depending on G and λ, either isomorphic to $\mathbf{A}^0$ or to $\mathbf{A}^1$. Furthermore, he gives a complete list of all pairs (G, λ) such that $\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)) \cong \mathbf{A}^1$ (Théorème 1.1 in [**Ja05**]). In the sequel, we call these weights *Jansou-weights*.

The orbit Gv_λ is of minimal strictly positive dimension among all G-orbits in $V(\lambda)$ (cf. Lemma II.1.4). If λ is a Jansou-weight of G, then there exist other orbits of the same dimension as Gv_λ. Here, we start with a general orbit X of minimal strictly positive dimension in a fixed simple G-module $V(\lambda)$, and we study G-stable deformations of X. In particular, we conjecture that the invariant Hilbert scheme parametrizing the G-stable deformations of X *in the closure of the sheet of* X is either $\mathbf{A}^0$ or $\mathbf{A}^1$. This will stand in contrast to the fact that the invariant Hilbert scheme parametrizing the G-stable deformations of X *in* $V(\lambda)$ can look much more complicated. This is the content of Chapter III, in which we will focus on the group SL_2, and compute some corresponding invariant Hilbert schemes. In particular, we study deformations of orbits of the form $\mathrm{SL}_2 \cdot x^{d/2}y^{d/2}$ in the space $k[x,y]_d = V(d)$ of binary forms of degree d. It turns out that easiest accessible case is when d is a multiple of 4, and even in this case the corresponding invariant Hilbert scheme can become very complicated. This reflects the principle that even in 'simple' cases for invariant Hilbert schemes all possible sort of 'bad' things (different irreducible components, non-reduced points, singularities) occur. (This 'bad' behavior is also encountered in the case of the classical Grothendieck Hilbert scheme parametrizing closed subschemes of projective space with a given Hilbert polynomial — see e.g. [**Mu66**]).

Finally, we turn our attention to not necessarily simple modules. In the multiplicity-free case important work has been done by Bravi and Cupit-Foutou (cf. [**BC08**] and [**Cu08**]). We translate some of their results to the case of not necessarily multiplicity-free modules. A corresponding (but wrongly formulated) result can be found in [**AB05**], so this fourth chapter can be seen as a (minor) erratum to the formulation in [**AB05**]. Chapter IV is independent from the preceding chapters.

In the two following sections, we state two results — one concerning flat quotients of G-schemes and one concerning closed subschemes of invariant Hilbert schemes. These results should not be over-estimated as part of this work: On the one hand, they both fit naturally in the context of the theory, and on the other hand, they have been around as folklore in the community (clearly, the second result and at least up to some minor degree also the first one). However, since both results are quite useful and since they are not stated or proven explicitly in the literature, we prove both of them here.

I.1. Invariant Hilbert schemes and flat quotients

Let $X = \operatorname{Spec}(R)$ be an irreducible affine G-variety. If the quotient $\pi\colon X \to X/\!\!/G$ is flat, then π is a family of affine G-schemes with Hilbert function h'_X, where h'_X is the Hilbert function of the general fiber of π. After identifying X with its graph $\Gamma(X) \subset X/\!\!/G \times X$, one obtains a morphism $X/\!\!/G \to \operatorname{Hilb}^G_{h'_X}(X)$. This morphism is in fact an isomorphism:

Theorem I.1.1. *Suppose that the quotient morphism $\pi\colon X \to X/\!\!/G$ is flat, and let h'_X be the Hilbert function of the general fiber of π. Then the invariant Hilbert scheme $\operatorname{Hilb}^G_{h'_X}(X)$ is isomorphic to $X/\!\!/G$, and the quotient $\pi\colon X \to X/\!\!/G$ is G-isomorphic to the universal family $\operatorname{Univ}^G_{h'_X}(X) \to \operatorname{Hilb}^G_{h'_X}(X)$.*

This result was found independently by M. Brion (cf. Remark 2.7 in [**JR09**]) and by the author. This section is devoted to the proof of Theorem I.1.1.

Lemma I.1.2. *Let A be a (Noetherian) ring, and let $\varphi\colon A \to B$ be a ring homomorphism. If B is locally free of rank 1 as A-module, then φ is an isomorphism.*

Remark I.1.3. It is important that B is an A-algebra and not only an A-module: Consider $T^{-1}\mathbf{Z}[T]$, which is a $\mathbf{Z}[T]$-module via the inclusion $\mathbf{Z}[T] \to T^{-1}\mathbf{Z}[T]$. Clearly, $T^{-1}\mathbf{Z}[T]$ is free of rank 1, but the inclusion $\mathbf{Z}[T] \to T^{-1}\mathbf{Z}[T]$ is no isomorphism.

Proof of Lemma I.1.2. First, suppose that $A = (A, \mathfrak{m})$ is a local ring. Let $\psi\colon B \to A$ be an isomorphism of A-modules. Then $(\psi \circ \varphi\colon A \to A) \in \operatorname{Hom}_A(A, A) \cong A$; hence $\psi\circ\varphi$ is multiplication with some $a_0 \in A$. Since $\psi(\varphi(A)) = a_0 \cdot A$, we see that $\varphi(A) = \psi^{-1}(a_0 \cdot A) = \varphi(a_0) \cdot B$. Now $\varphi(1_A) = 1_B \in \varphi(A)$; hence $1_B \in \varphi(a_0) \cdot B$. This shows that $\varphi(a_0)$ is a unit in B, and $\varphi(A) = B$. Because $\psi \circ \varphi\colon A \to B \xrightarrow{\cong} A$ is surjective, it is multiplication with a unit, and hence injective and finally an isomorphism.

Let now A be arbitrary. Localizing the exact sequence of A-modules

$$0 \to \ker(\varphi) \to A \to B \to B/\varphi(A) \to 0$$

in $\mathfrak{p} \in \operatorname{Spec}(A)$, yields the exact sequence

$$0 \to \ker(\varphi_{\mathfrak{p}}) \to A_{\mathfrak{p}} \xrightarrow{\varphi_{\mathfrak{p}}} B_{\mathfrak{p}} \to (B/\varphi(A))_{\mathfrak{p}} \to 0.$$

From the first part we know that $\varphi_{\mathfrak{p}}$ is an isomorphism, and hence $\ker(\varphi_{\mathfrak{p}}) = 0$ and $(B/\varphi(A))_{\mathfrak{p}} = 0$. Since this holds for all $\mathfrak{p}$, we conclude that $\ker(\varphi) = 0$ and that $\operatorname{coker}(\varphi) = 0$, and the claim follows. $\square$

Using the fact that $h'_X(0) = 1$, we obtain the following corollary:

Corollary I.1.4. *Let S be a scheme, and let $S \times X \supset \mathfrak{X} \to S$ be a closed G-stable subscheme with Hilbert function h'_X. Then the natural morphism $\mathfrak{X}/\!\!/G \to S$ is an isomorphism.*

Lemma I.1.5. *Let $h\colon \Lambda^+ \to \mathbf{N}$ be a function, let S be a scheme and let $\pi\colon \mathfrak{X} \to S$ be a family of affine G-schemes with Hilbert function h. Let $\mathfrak{X}' \subset \mathfrak{X}$ be a closed G-stable subscheme such that also $\pi'\colon \mathfrak{X}' \to S$ is a family of affine G-schemes with Hilbert function h, where π' is the restriction of π to $\mathfrak{X}'$. Then $\mathfrak{X}' = \mathfrak{X}$.*

Proof. Let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ be the ideal sheaf defining $\mathfrak{X}'$ as subscheme of $\mathfrak{X}$. Let $\lambda \in \Lambda^+$, and let $s \in S$. Then we obtain the following short exact sequence of $\mathcal{O}_{S,s}$-modules:

$$0 \to (((\pi_*\mathcal{J})_{(\lambda)})^U)_s \to (((\pi_*\mathcal{O}_{\mathfrak{X}})_{(\lambda)})^U)_s \to (((\pi_*\mathcal{O}_{\mathfrak{X}'})_{(\lambda)})^U)_s \to 0.$$

Both $(((\pi_*\mathcal{O}_{\mathfrak{X}})_{(\lambda)})^U)_s$ and $(((\pi_*\mathcal{O}_{\mathfrak{X}'})_{(\lambda)})^U)_s$ are free $\mathcal{O}_{S,s}$-modules, thus the sequence splits. This implies that

$$(((\pi_*\mathcal{O}_{\mathfrak{X}})_{(\lambda)})^U)_s \cong (((\pi_*\mathcal{J})_{(\lambda)})^U)_s \oplus (((\pi_*\mathcal{O}_{\mathfrak{X}'})_{(\lambda)})^U)_s.$$

This shows that $(((\pi_*\mathcal{J})_{(\lambda)})^U)_s$ is a summand of a free $\mathcal{O}_{S,s}$-module, and hence is projective. Since a projective module over a local ring is free, we conclude that $(((\pi_*\mathcal{J})_{(\lambda)})^U)_s$ is free. But both $(((\pi_*\mathcal{O}_{\mathfrak{X}})_{(\lambda)})^U)_s$ and $(((\pi_*\mathcal{O}_{\mathfrak{X}'})_{(\lambda)})^U)_s$ are free of the same rank (namely $h(\lambda)$). Thus, it follows that $(((\pi_*\mathcal{J})_{(\lambda)})^U)_s$ is free of rank 0. Since this holds for all $\lambda \in \Lambda^+$ and for all $s \in S$, we conclude that $\pi_*\mathcal{J}$ is zero. Hence also $\mathcal{J} = 0$, and $\mathfrak{X}' = \mathfrak{X}$. □

Remark I.1.6. Suppose that $\mathfrak{X} = \mathrm{Spec}(R)$ and $\mathfrak{X}' = \mathrm{Spec}(R')$ are affine G-varieties with G-invariant flat morphisms to a third affine variety $S = \mathrm{Spec}(A)$, making $\mathfrak{X} \to S$ and $\mathfrak{X}' \to S$ into families of affine G-schemes with Hilbert functions h and h', respectively. Assume further that there exists a dominant G-equivariant morphism $\varphi\colon \mathfrak{X} \to \mathfrak{X}'$ of varieties over S. It can happen that the Hilbert functions h and h' coincide, but that φ is no isomorphism (or in other words that $R' \subsetneq R$ is a strict sub-S-module), nevertheless. An example shall be provided in Lemma II.2.7 and in Remark II.2.8. Indeed, if one tries to imitate the proof of Lemma I.1.5 in this situation, one sees the following: Going over to an isotypic component λ and localising in $\mathfrak{a} \in \mathrm{Spec}(A)$, we get a short exact sequence of $A_{\mathfrak{a}}$-modules

$$0 \to (((R')_{(\lambda)})^U)_{\mathfrak{a}} \to (((R)_{(\lambda)})^U)_{\mathfrak{a}} \to (((R/R')_{(\lambda)})^U)_{\mathfrak{a}} \to 0.$$

Even if both $(((R')_{(\lambda)})^U)_{\mathfrak{a}}$ and $(((R)_{(\lambda)})^U)_{\mathfrak{a}}$ are free of equal rank $h(\lambda)$, it does not follow that $(((R/R')_{(\lambda)})^U)_{\mathfrak{a}} = 0$ (as e.g. $0 \to \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \to \mathbf{Z}/(2) \to 0$ is an exact sequence of $\mathbf{Z}$-modules.)

Proof of Theorem I.1.1. In order to show that $\pi\colon X \to X/\!\!/G$ represents the Hilbert functor $\mathcal{H}ilb^G_{h'_X}(X)$, one needs to verify that given a scheme S, each family of affine multiplicity-finite G-schemes $S \times X \supset \mathfrak{X} \xrightarrow{\pi'} S$ with Hilbert function h'_X is isomorphic to the pull-back of $\pi\colon X \to X/\!\!/G$ via a suitable morphism $\varphi\colon S \to X/\!\!/G$. Let $\mathcal{J} \subset \mathcal{O}_{S\times X}$ be the ideal sheaf defining $\mathfrak{X}$ as closed subscheme of $S \times X$. Let $\Phi\colon \mathfrak{X} \to X$ be the composition of the inclusion $\mathfrak{X} \to S \times X$ followed by the projection of $S \times X$ onto X. Now, Φ is G-equivariant, and by functoriality we obtain a morphism $\varphi\colon \mathfrak{X}/\!\!/G \to X/\!\!/G$ such that $\pi \circ \Phi = \varphi \circ \pi'$. If we identify $\mathfrak{X}/\!\!/G$ with S (which is possible according to Corollary I.1.4), we obtain the required morphism $\varphi\colon S \to X/\!\!/G$. By the universal property of a pull-back there is a unique morphism $\psi\colon \mathfrak{X} \to S \times_{X/\!\!/G} X$ of schemes over $X/\!\!/G$ making the diagram

(I.1.1)
$$
\begin{tikzcd}
\mathfrak{X} \arrow[dr, "\psi"] \arrow[drr, bend left, "\Phi"] \arrow[ddr, bend right, "\pi'"'] & & \\
& S \times_{X/\!\!/G} X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& \mathfrak{X}/\!\!/G \cong S \arrow[r, "\varphi"] & X/\!\!/G
\end{tikzcd}
$$

commutative. The pull-back $S\times_{X/\!\!/G}X$ is the closed subscheme of $S\times X$ defined by the ideal sheaf $\mathcal{J}^G \cdot \mathcal{O}_{S\times X} \subset \mathcal{O}_{S\times X}$. Since $\mathcal{J}^G \cdot \mathcal{O}_{S\times X} \subset \mathcal{J}$, we see that $\mathfrak{X}$ is a closed subscheme of $S \times_{X/\!\!/G} X$. It now follows from Lemma I.1.5 that ψ is an isomorphism. $\square$

I.2. Closed subschemes

Let $X = \mathrm{Spec}(R)$ be an affine G-variety, and let $h\colon \Lambda^+ \to \mathbf{N}$ be a function. In this section we prove the following result:

Proposition I.2.1. *If $X' \subset X$ is a closed G-stable subscheme, then $\mathrm{Hilb}^G_h(X')$ is a closed subscheme of $\mathrm{Hilb}^G_h(X)$.*

A particular case of this result is already stated in [**AB05**], Lemma 1.6; and the proof given there can be carried over to our situation. We only state a proof here for the sake of completeness.

Proof. For simplicity, we write Hilb instead of $\mathrm{Hilb}^G_h(X)$ and Univ instead of $\mathrm{Univ}^G_h(X)$. Let $f\colon S \to \mathrm{Hilb}$ be a morphism, let $S\times_{\mathrm{Hilb}}\mathrm{Univ} =: \mathfrak{X} \xrightarrow{p} S$ be the family corresponding to X, and let $\pi\colon \mathrm{Univ} \to \mathrm{Hilb}$ be the projection. Then $\mathfrak{X}$ can be seen as closed subscheme of $S \times X$. We now investigate conditions on f guaranteeing that $\mathfrak{X}$ is already a closed subscheme of $S \times X'$. Let $I \subset R$ be the ideal defining $X' \subset X$.

The natural morphism

$$S \times_{\mathrm{Hilb}} \mathrm{Univ} \to S \times_{\mathrm{Hilb}} (\mathrm{Hilb} \times X) \cong S \times X$$

of schemes over S yields a morphism of sheaves of $\mathcal{O}_S$-modules

(I.2.1)
$$\mathcal{O}_S \otimes R \to p_*\mathcal{O}_{S\times_{\mathrm{Hilb}}\mathrm{Univ}},$$

and $\mathfrak{X}$ is a closed subscheme of $S \times X'$ if and only if $\mathcal{O}_S \otimes I$ is in the kernel of this morphism. Denoting $f' \colon S \times_{\mathrm{Hilb}} \mathrm{Univ} \to \mathrm{Univ}$ the morphism induced by f and using the isomorphism

$$f_* p_* \mathcal{O}_{S \times_{\mathrm{Hilb}} \mathrm{Univ}} = \pi_* f'_* \mathcal{O}_{S \times_{\mathrm{Hilb}} \mathrm{Univ}} \cong f_* \mathcal{O}_S \otimes_{\mathcal{O}_{\mathrm{Hilb}}} \pi_* \mathcal{O}_{\mathrm{Univ}}$$

of sheaves of $\mathcal{O}_{\mathrm{Hilb}}$-modules, we can write (I.2.1) as morphism of sheaves of $\mathcal{O}_{\mathrm{Hilb}}$-modules

$$\varphi \colon f_* \mathcal{O}_S \otimes R \to f_* \mathcal{O}_S \otimes_{\mathcal{O}_{\mathrm{Hilb}}} \pi_* \mathcal{O}_{\mathrm{Univ}}.$$

Now, $\mathfrak{X}$ is a closed subscheme of $S \times X'$ if and only if $f_* \mathcal{O}_S \otimes I \subset \ker(\varphi)$. The image sheaf of $f_* \mathcal{O}_S \otimes I$ under φ equals $f_* \mathcal{O}_S \otimes_{\mathcal{O}_{\mathrm{Hilb}}} \pi_* \mathcal{J}$, where $\mathcal{J} \subset \mathcal{O}_{\mathrm{Univ}}$ is the ideal sheaf defining the scheme-theoretic intersection $\mathrm{Univ} \cap (\mathrm{Hilb} \times X')$ as closed subscheme of Univ, and hence $\mathfrak{X}$ is a closed subscheme of $S \times X'$ if and only if

$$f_* \mathcal{O}_S \otimes_{\mathcal{O}_{\mathrm{Hilb}}} \pi_* \mathcal{J} = 0. \tag{I.2.2}$$

We need to construct a closed subscheme $\mathrm{Hilb}' \subset \mathrm{Hilb}$ with the property that (I.2.2) holds if and only if $f \colon S \to \mathrm{Hilb}$ factors via Hilb'. Going over to isotypic components, we see that (I.2.2) holds if and only if

$$f_* \mathcal{O}_S \otimes_{\mathcal{O}_{\mathrm{Hilb}}} ((\pi_* \mathcal{J})_{(\lambda)})^U = 0 \tag{I.2.3}$$

for all $\lambda \in \Lambda^+$. For $\lambda \in \Lambda^+$ consider the morphism

$$\Phi_{(\lambda)} \colon \mathcal{H}om_{\mathcal{O}_{\mathrm{Hilb}}}(((\pi_* \mathcal{O}_{\mathrm{Univ}})_{(\lambda)})^U, \mathcal{O}_{\mathrm{Hilb}}) \times ((\pi_* \mathcal{J})_{(\lambda)})^U \to \mathcal{O}_{\mathrm{Hilb}}, \quad (\psi, r) \mapsto \psi(r)$$

of sheaves of $\mathcal{O}_{\mathrm{Hilb}}$-modules. We claim that the image sheaf $\mathcal{I}_{(\lambda)}$ of $\Phi_{(\lambda)}$ is a sheaf of ideals in $\mathcal{O}_{\mathrm{Hilb}}$, and that (I.2.3) holds if and only if $\mathcal{I}_{(\lambda)} \subset \ker(f^\#)$ for all λ, or equivalently if and only if $\mathcal{I} := \bigcup_{\lambda \in \Lambda^+} \mathcal{I}_{(\lambda)} \subset \ker(f^\#)$. Then Hilb' is the closed subscheme of Hilb defined by $\mathcal{I}$.

We verify the claim locally and denote $((\pi_* \mathcal{O}_{\mathrm{Univ}})_{(\lambda)})^U$ by M and $((\pi_* \mathcal{J})_{(\lambda)})^U$ by N. First, (I.2.3) holds if and only if

$$(f_* \mathcal{O}_S)_p \otimes_{\mathcal{O}_{\mathrm{Hilb},p}} N_p = 0 \tag{I.2.4}$$

for all $\lambda \in \Lambda^+$ and for all $p \in \mathrm{Hilb}$. For $p \in \mathrm{Hilb}$ the stalk of $(\Phi_{(\lambda)})_p$ is the homomorphism of $\mathcal{O}_{\mathrm{Hilb},p}$-modules

$$(\Phi_{(\lambda)})_p \colon \mathrm{Hom}_{\mathcal{O}_{\mathrm{Hilb},p}}(M_p, \mathcal{O}_{\mathrm{Hilb},p}) \times N_p \to \mathcal{O}_{\mathrm{Hilb},p}, \quad (\psi, r) \mapsto \psi(r).$$

Fix an isomorphism $M_p \cong (\mathcal{O}_{\mathrm{Hilb},p})^{h(\lambda)}$. If we identify $N_p \subset M_p$ with its image in $(\mathcal{O}_{\mathrm{Hilb},p})^{h(\lambda)}$ under the chosen isomorphism, then N_p is generated by elements of the form $(m_{i,1}, \ldots, m_{i,h(\lambda)})_i$ where i runs over some index set and where each $m_{i,j} \in \mathcal{O}_{\mathrm{Hilb},p}$. Then the image $(\mathcal{I}_{(\lambda)})_p$ of $(\Phi_{(\lambda)})_p$ is the ideal of $\mathcal{O}_{\mathrm{Hilb},p}$ generated by all $m_{i,j}$. This shows that $\mathcal{I}_{(\lambda)}$ is a sheaf of ideals in $\mathcal{O}_{\mathrm{Hilb}}$.

Using the above isomorphism $M_p \cong (\mathcal{O}_{\mathrm{Hilb},p})^{h(\lambda)}$, we see that (I.2.4) holds if and only if $m_{i,j} \in \ker(f_p^\#)$ for all i and j. This in turn is equivalent to say that $\sum_{i,j} \mu_{ij} m_{i,j} \in \ker(f_p^\#)$ for all $\mu_{ij} \in \mathcal{O}_{\mathrm{Hilb},p}$. But $\sum_{i,j} \mu_{ij} m_{i,j}$ equals $(\mathcal{I}_{(\lambda)})_p$, and hence (I.2.4) holds if and only if $(\mathcal{I}_{(\lambda)})_p \subset \ker(f_p^\#)$. This shows that (I.2.3) holds if and only if $\mathcal{I}_{(\lambda)} \subset \ker(f^\#)$, and this completes the proof. $\square$

CHAPTER II

Deformations of orbits in minimal sheets

II.1. Minimal sheets

Let G be semisimple, and let V be a finite-dimensional G-module. For $n \in \mathbf{N}$ let $V^{(n)} \subset V$ be the union of all n-dimensional orbits in V. Each $V^{(n)}$ is a locally closed subset of V, and the irreducible components of the $V^{(n)}$ are called *sheets of* V (cf. [**Di75**] or [**BK79**]). We say that a sheet $S \subset V$ is *minimal* if $S \subset V^{(n)}$ for some n such that $V^{(m)}$ is empty for all m with $0 < m < n$.

Recall that we introduced the following notation: We fixed in each simple G-module $V(\lambda)$ a non-zero U-invariant vector v_λ. Moreover, X_λ denotes the closure of the orbit of v_λ in $V(\lambda)$, and $h_\lambda := h_{X_\lambda} : \Lambda^+ \to \mathbf{N}$ is the Hilbert function of X_λ. Then $h_\lambda(\mu) = 1$ if and only if $\mu \in \mathbf{N}\lambda^*$, and $h_\lambda(\mu) = 0$ otherwise. Finally, we call a weight $\lambda \in \Lambda^+$ a Jansou-weight if $\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)) \cong \mathbf{A}^1$. In [**Ja05**] Jansou proves that if λ is no Jansou-weight, then $\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)) \cong \mathbf{A}^0$, and he gives a complete list of all pairs (G, λ) with G semisimple and with λ a Jansou-weight of G. The main result of this section is the following description of minimal sheets in irreducible representations.

Proposition II.1.1. *Let* $0 \neq \lambda \in \Lambda^+$.

a) *The simple G-module $V(\lambda)$ contains a unique minimal sheet S.*
b) *The orbit Gv_λ lies in S.*
c) *The sheet S contains an orbit Gv different from Gv_λ if and only if λ is an integral multiple of a Jansou-weight. In this case, the orbit Gv is closed in $V(\lambda)$.*

This justifies the following definition:

Definition II.1.2. Let $\lambda \in \Lambda^+$. Then $\mathcal{S}(\lambda)$ denotes the closure of the unique minimal sheet in $V(\lambda)$. By abuse of notation, we sometimes call $\mathcal{S}(\lambda)$ also the *minimal sheet of* $V(\lambda)$.

Corollary II.1.3. *Suppose that λ is an integral multiple of a Jansou-weight and that $Gv \subset \mathcal{S}(\lambda)$ is a closed orbit. Then $\mathcal{S}(\lambda)$ consists of $Gv_\lambda \cup \{0\}$ and the orbits $\gamma \cdot Gv$ for $\gamma \in k \setminus \{0\}$. Moreover, $\mathcal{S}(\lambda)$ is a cone in $V(\lambda)$, and its quotient $\mathcal{S}(\lambda)/\!\!/G$ is an irreducible curve.*

The proof of Proposition II.1.1 and of Corollary II.1.3 follow at the end of this section.

In the sequel we make frequently use of the *(asymptotic) cone* $\mathcal{C}X$ *associated to a subscheme* X *of a vector space* V. We now explain how it is defined. If $f = \sum_{n=0}^{N} f_n \in \mathrm{Sym}(V^*)$ with f_n homogeneous of degree n and with $f_N \neq 0$, then $\mathrm{gr}(f) := f_N$. If $X \subset V$ is defined by the ideal $I \subset \mathrm{Sym}(V^*)$, then $\mathcal{C}X \subset V$ is the subscheme defined by the ideal generated by $\mathrm{gr}(I) := \{\mathrm{gr}(f) \mid f \in I\}$. We refer to [**BK79**] or [**Kr85**] for properties of the associated cone. (In contrast to the definitions stated there, the associated cone $\mathcal{C}X$ needs not be reduced here).

Lemma II.1.4. *Let* $0 \neq \lambda \in \Lambda^+$.

a) *If* $0 \neq X \subset V(\lambda)$ *is a reduced closed* G*-stable cone with* $\dim X \leq \dim Gv_\lambda$*, then* $X = X_\lambda$.
b) *A sheet containing* v_λ *is minimal.*
c) *Let* $v \in V(\lambda)$ *be a closed point with* $\dim Gv = \dim Gv_\lambda$*. Then either* $Gv = Gv_\lambda$ *or the orbit* Gv *is closed in* $V(\lambda)$.
d) *Let* $v \in V(\lambda)$ *be a closed point such that* $\dim Gv = \dim Gv_\lambda$ *and such that* Gv *is closed in* V*. Then the cone* $\mathcal{C}(Gv)$ *associated to* Gv *is irreducible, and its underlying variety is* X_λ.
e) *A minimal sheet contains* v_λ.

Proof. a) Let I be the ideal of $X \subset V(\lambda)$. Since X is reduced and $X \neq 0$, it follows that $\mathrm{Sym}^n(V(\lambda)^*)_{(n\lambda)} \not\subset I$ for all $n \geq 0$. But then I is strictly contained in the ideal $I(X_\lambda) = \bigoplus_{n\geq 0} \bigoplus_{\mu<n\lambda^*} \mathrm{Sym}^n(V(\lambda)^*)_{(\mu)}$ of $X_\lambda \subset V(\lambda)$. Hence X_λ is a closed subscheme of X. Since $\dim X \leq \dim Gv_\lambda$, the claim now follows.

b) Let $S \subset V(\lambda)$ be a minimal sheet, and let $v_0 \in S$ be a closed point. Let $X := \overline{Gv_0} \subset V(\lambda)$ be the closure of the orbit of v_0, and let $Y \subset V(\lambda)$ be an irreducible component of maximal dimension of the cone $\mathcal{C}X$ associated to X. Then $\dim X = \dim Y$, and since $Y \neq 0$, part a) implies that $\dim Y \geq \dim X_\lambda$. This proves that $\dim X_\lambda \leq \dim X$, and hence a sheet containing v_λ is minimal.

c) Suppose that Gv is not closed in $V(\lambda)$, and choose $v_0 \in \overline{Gv} \setminus Gv \subset V(\lambda)$. Let $X := \overline{Gv_0} \subset V(\lambda)$ be the closure of the orbit of v_0, and let $Y \subset V(\lambda)$ be an irreducible component of maximal dimension of the cone $\mathcal{C}X$ associated to X. If $Y \neq 0$, then $\dim Y \geq \dim X_\lambda$ according to a). On the other hand, $\dim Y = \dim X < \dim Gv = \dim X_\lambda$. This shows that $Y = 0$, and hence $\mathcal{C}X = 0$. Hence $\dim X = 0$, and because G is connected it follows that v_0 is a fixed point, which in turn implies that $v_0 = 0$. This shows that $0 \in \overline{Gv}$. Since $v_\lambda \in \overline{Gv}$ for all $v \neq 0$ with $0 \in \overline{Gv}$ (cf. [**Kr85**] III.3.6 Bemerkung 2), it follows that $X_\lambda \subset \overline{Gv}$. Since $\dim Gv = \dim X_\lambda$ and since both Gv and X_λ are irreducible, it follows that $Gv = Gv_\lambda$.

d) The variety $\mathcal{C}(Gv)_{\mathrm{red}}$ underlying the cone $\mathcal{C}(Gv)$ is a reduced closed G-stable cone in $V(\lambda)$ with $0 < \dim \mathcal{C}(Gv)_{\mathrm{red}} \leq \dim Gv_\lambda$. It follows with a) that $\mathcal{C}(Gv)_{\mathrm{red}} = X_\lambda$, and hence $\mathcal{C}(Gv)$ is irreducible.

e) Let S be a minimal sheet, and let Gv be an orbit in S. Because of b) and since S is minimal, we see that $\dim Gv = \dim Gv_\lambda$. Hence c) implies that $Gv = Gv_\lambda$ or that Gv is closed in $V(\lambda)$. In the first case, we are done. Otherwise, observe that $\gamma \cdot Gv \subset S$ for all $\gamma \in k \setminus \{0\}$. It follows from [**Kr85**] II.4.2 Satz 2 that $\mathcal{C}(Gv)_{\mathrm{red}} \subset \overline{S}$. Moreover, d) implies that $\mathcal{C}(Gv)_{\mathrm{red}} = X_\lambda$, and hence $Gv_\lambda \subset \overline{S}$. The claim now follows. □

Definition II.1.5. Let $0 \neq \mu \in \Lambda^+$. A reductive subgroup H of G is called *Jansou-subgroup of type* μ if there is an isomorphism of G-modules

$$\mathcal{O}(G/H) = \mathcal{O}(G)^H \cong \bigoplus_{k \geq 0} V(k\mu^*).$$

Remark II.1.6. A reductive subgroup H of G is a Jansou-subgroup if and only if the algebra $\mathcal{O}(G/H)^U$ has (Krull) dimension 1.

Let $\lambda \in \Lambda^+$ be a Jansou-weight. Then there exists a Jansou-subgroup $H \subset G$ of type λ. To see this, observe that there exists a closed subscheme $V(\lambda) \supset X \neq X_\lambda$ with $\mathcal{O}(X) = \bigoplus_{k \geq 0} V(k\lambda^*)$. If $v \in X$ is a closed point, then the stabilizer G_v of v is a Jansou-subgroup of type λ. (Since $X \neq X_\lambda$, it follows from Lemma II.1.4 c) that X is a closed orbit in $V(\lambda)$. Thanks to Matsushima's Theorem (cf. [**Ma's60**], or see [**Lu73**] p. 5), it now follows that G_v is reductive.)

Lemma II.1.7. *Let $0 \neq \mu \in \Lambda^+$, and let X be an irreducible affine G-variety with $\mathcal{O}(X) \cong \bigoplus_{k \geq 0} V(k\mu^*)$ as G-module.*

a) *There exists a closed G-equivariant embedding $X \to V(\mu)$.*
b) *Either $X \cong X_\mu$ as G-variety or there exists a Jansou-subgroup $H \subset G$ of type μ such that $X \cong G/H$ as G-variety.*

Proof. a) Observe that $\dim_k \mathrm{Hom}^G(V(\mu)^*, \mathcal{O}(X)) = 1$ according to Schur's lemma. Any non-zero G-equivariant morphism $V(\mu)^* \to \mathcal{O}(X)$ of G-modules extends to a G-equivariant epimorphism $\mathrm{Sym}(V(\mu)^*) \to \mathcal{O}(X)$ of k-G-algebras, which gives the desired closed G-embedding $X \to V(\mu)$.

b) Consider the cone $\mathcal{C}X$ associated to X. Since $h_{\mathcal{C}X} = h_X = h_{X_\mu}$, it follows either from Lemma II.1.4 or from the discussion in [**Ja05**] that $\mathcal{C}X = X_\mu$. Lemma II.1.4 c) implies that either $X = X_\mu$, or that G acts transitively on X. In the second case, it follows that $\mathrm{Hilb}^G_{h_\mu}(V(\mu))$ contains at least two closed points (one point corresponding to X_μ and one point corresponding to X), and hence that μ is a Jansou-weight. Let $v \in X$ be a closed point with stabilizer $G_v \subset G$. Then $X \cong G/G_v$ as G-scheme, and hence G_v is a Jansou-subgroup of type μ. □

Lemma II.1.8. *Two Jansou-subgroups H and H' of G of the same type μ are conjugate.*

Proof. With Lemma II.1.7 it follows that there are closed G-equivariant embeddings $\varphi\colon G/H \to V(\mu)$ and $\varphi'\colon G/H' \to V(\mu)$. Let $X = \varphi(G/H)$ and $X' = \varphi(G/H')$. The proof of Lemma II.1.7 and the fact that $\mathrm{Hilb}^G_{h_\mu}(V(\mu)) = \mathbf{A}^1$ according to [**Ja05**] imply that $X = \gamma X'$ for some $\gamma \in k \setminus \{0\}$. The stabilizer of $v := \varphi(1H) \in V(\mu)$ equals H, and the stabilizer of $v' := \varphi(1H') \in V(\mu)$ equals H'. Moreover, $v' = tgv$ for some $g \in G$. It now follows that $H' = gHg^{-1}$. □

For a G-module A we let $\Lambda_A := \{\lambda \in \Lambda^+ \mid \mathrm{Hom}^G(A, V(\lambda)) \neq 0\}$. For an affine G-scheme $X = \mathrm{Spec}(A)$, we let $\Lambda_X := \Lambda_A$. The proof of the following result was wrongly stated in a first version of this Thesis. I am most grateful to M. Brion who pointed out the necessary corrections.

Lemma II.1.9. *Let A be a (Noetherian) G-algebra, and let $I \subset A$ be a G-stable nilpotent ideal. Then the following hold:*

a) *If Λ_A is a monoid, then $\Lambda_A \subset \mathbf{Q}\Lambda_{A/I}$.*
b) *If A/I is multiplicity-free, then A is multiplicity-bounded, i.e. there exists $n_0 \in \mathbf{N}$ such that $\dim_k \mathrm{Hom}^G(A, V(\lambda)) \leq n_0$ for all $\lambda \in \Lambda^+$.*

Proof. a) Let R be a G-algebra, let M be a finitely generated R-G-module, and let $\mu \in \Lambda_M$ be a weight with $\mu \notin \mathbf{Q}\Lambda_R$. We claim that $m\mu \notin \Lambda_M$ for all m suitably large. To see this, suppose that M is generated as R-module by a finite-dimensional G-module $N \subset M$. If $\mu_1, \ldots, \mu_n$ are the weights of N, then

$$\Lambda_M \subset \bigcup_{i=1}^{n} (\Lambda_R + \mu_i). \tag{II.1.1}$$

(Because M is generated as R-module by N, there is a surjective homomorphism of G-modules $R \otimes N \to M$. The weights of the U-invariants in $V(\lambda) \otimes V(\mu)$ are of the form $\lambda + \nu$, where ν is a weight of $V(\mu)$ (see e.g. [**FH00**], Ex. 25.33), and (II.1.1) follows.) If the claim is false, there exist $n+1$ different integers $k_1, \ldots, k_{n+1} > 0$ with $k_j\mu \in \Lambda_M$. But then there exist $0 < j_1 < j_2 \leq n+1$ and i such that $k_{j_1}\mu = \lambda_1 + \mu_i$ and $k_{j_2}\mu = \lambda_2 + \mu_i$ for some $\lambda_1, \lambda_2 \in \Lambda_R$. It follows that $0 \neq (k_{j_1} - k_{j_2})\mu = \lambda_2 - \lambda_1 \in \mathbf{Z}\Lambda_R$, and hence that $\mu \in \mathbf{Q}\Lambda_R$. This proves the claim.

Consider $\mathrm{gr}_I(A) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$, which is a finitely generated A/I-algebra. If $I \subset A$ is nilpotent, then $\mathrm{gr}_I(A)$ is finitely generated as A/I-module. Applying the above with $R = A/I$ and with $M = \mathrm{gr}_I(A)$, and observing that $\Lambda_{\mathrm{gr}_I(A)} = \Lambda_A$, it follows that $m\mu \notin \Lambda_A$ for all m suitably large if $\mu \in \Lambda_A \setminus \mathbf{Q}\Lambda_{A/I}$. On the other hand, since Λ_A is a monoid, $m\mu \in \Lambda_A$ for all $m \geq 0$ whenever $\mu \in \Lambda_A$. This implies that $\Lambda_A \subset \mathbf{Q}\Lambda_{A/I}$.

b) Using the above notation, suppose that R is multiplicity-free, and let $\lambda \in \Lambda^+$. Then

$$\begin{aligned}\dim_k \operatorname{Hom}^G(M, V(\lambda)) &\leq \dim_k \operatorname{Hom}^G(R \otimes N, V(\lambda)) \\ &= \dim_k \operatorname{Hom}^G(R, N^* \otimes V(\lambda)).\end{aligned}$$ $\square$

Lemma II.1.10. *Let $0 \neq \lambda \in \Lambda^+$, and let $S \subset V(\lambda)$ be a minimal sheet. Suppose that $v \in S$ is a closed point such that the orbit Gv is closed in $V(\lambda)$. Then there exists an integer $n \in \mathbf{N}$ such that λ/n is a Jansou-weight and such that $Gv \cong G/H$, where $H \subset G$ is a Jansou-subgroup of type λ/n.*

Proof. Let $X := Gv \subset V(\lambda)$ be the (closed) orbit of v, and let $\mathcal{C}X$ be the cone associated to X. It follows from Lemma II.1.4 d) that $\mathcal{C}X$ is irreducible and that the reduced variety underlying $\mathcal{C}X$ equals X_λ. Hence there are the following isomorphisms of G-modules:

$$\mathcal{O}(X) \cong \mathcal{O}(\mathcal{C}X) \cong \mathcal{O}(X_\lambda) \oplus \sqrt{(0)},$$

where $\sqrt{(0)}$ is the nilradical of the $\mathcal{O}(\mathcal{C}X)$. It follows from Lemma II.1.9 a) that $\Lambda_X \subset \mathbf{Q}\Lambda_{X_\lambda} = \mathbf{Q}\lambda^*$. Since X_λ is multiplicity-free, Lemma II.1.9 b) implies that there exists $N \in \mathbf{N}$ such that $N \geq h_X(\mu)(= h_{\mathcal{C}X}(\mu))$ for all $\mu \in \Lambda^+$. If $h_X(\mu) \geq 2$ for some μ, then we find two linearly independent U-invariants $f_1, f_2 \in (\mathcal{O}(X)_{(\mu)})^U$. Since $\mathcal{O}(X)$ is a domain, it follows that $f_1^N, f_1^{N-1}f_2, \ldots, f_2^N \in (\mathcal{O}(X)_{(N\mu)})^U$ are linearly independent. But then $h_X(N\mu) \geq N+1$, a contradiction. This shows that X is multiplicity-free. Moreover, $\Lambda_X \subset \mathbf{Q}\lambda^*$ is a monoid. This shows that there exists $n \in \mathbf{N}$ and $f \in k(X)^U$ of weight λ^*/n such that

$$\mathcal{O}(X)^U = k[f^{n_1}, \ldots, f^{n_k}]$$

for some integers $n_1, \ldots, n_k$ with $\gcd(n_1, \ldots, n_k) = 1$. Since $X = Gv$ is smooth and in particular normal, $\mathcal{O}(X)^U$ is integrally closed in its field of fractions $k(f)$. This shows that $\mathcal{O}(X)^U = k[f]$, or that $\mathcal{O}(X) = \bigoplus_{k \geq 0} V(k\lambda^*/n)$. Lemma II.1.7 b) now implies that $Gv = X \cong G/H$, where H is a Jansou-subgroup of type λ/n. $\square$

Lemma II.1.11. a) *If $H \subset G$ is a Jansou-subgroup, then H^0 is a maximal proper connected reductive subgroup of G.*

b) *If $H, H' \subset G$ are two non-conjugate Jansou-subgroups with $H^0 = H'^0$, then either $H = H^0$ and $H' = N_G(H)$, or vice versa.*

c) *If $H \subset G$ is a Jansou-subgroup that is not self-normalizing (i.e. $H \neq N_G(H)$), then $[N_G(H) : H] = 2$.*

Proof. a) Let $H \subset G$ be a Jansou-subgroup of type μ. Then H^0 is a connected Jansou-subgroup of G. If μ^0 denotes its type, then $\mu \in \mathbf{N}\mu^0$. Let $G \supsetneq H' \supset H^0$ with H' connected and reductive. Then H' is a Jansou-subgroup of G of type $m\mu^0$ for some $m > 1$. It follows that the natural morphism $G/H^0 \to G/H'$ is finite, and hence $\dim G/H' =$

$\dim G/H^0$. Since both H^0 and H' are connected, it follows that $H = H'$.

b) Let $H, H' \subset G$ be two Jansou-subgroups with $H^0 = H'^0$. Then all H, H', and H^0 are Jansou-subgroups of type μ, μ', and μ_0; and both μ and μ' are rational multiples of μ_0. Using the list in Théorème 1.1 in [**Ja05**], we find that two of the three weights coincide. Since $\mu \neq \mu'$, we can assume that $\mu = \mu_0$, and hence that $H = H^0$. Since the normalizer of a Jansou-subgroup is also a Jansou-subgroup, we conclude that $H' = N_G(H)$.

c) Suppose that H is of type μ. Now, $N_G(H)$ is a Jansou-subgroup of type μ/n for some n. Using Jansou's list, we see that $n = 2$. Hence the natural morphism $G/H \to G/N_G(H)$ is finite of degree 2, and the claim follows. □

Remark II.1.12. Let $H \subset G$ be a Jansou-subgroup of type μ, and let $\lambda \in \Lambda^+$ be a dominant weight. Then $\dim V(\lambda)^H = 1$ whenever $\lambda \in \mathbf{N}\mu$, and $\dim V(\lambda)^H = 0$ for all other λ. This follows from Frobenius reciprocity; see [**Jant87**], [**KP96**] 5.5, or also [**Kr85**] III.3.4 Satz.

We conclude this section with the proof of Proposition II.1.1.

Proof of Proposition II.1.1. a) By definition, $V(\lambda)$ contains at least one minimal sheet S. Suppose that $V(\lambda)$ contains a further minimal sheet S'. Since each minimal sheet contains X_λ according to Lemma II.1.4 e), we can assume that both S and S' contain an orbit different from X_λ (if e.g. $S = X_\lambda$, then $S \subset S'$, and S would not be a sheet). Call these orbits Gv and Gv'. Lemma II.1.4 c) implies that Gv and Gv' are closed in $V(\lambda)$. According to Lemma II.1.10 there exist integers $n, n' \in \mathbf{N}$ such that $Gv \cong G/H$ and $Gv' \cong G/H'$ with H and H' Jansou-subgroups of type λ/n and λ/n', respectively. After replacing H by a suitable conjugate subgroup, we can now assume that $H^0 = H'^0$, and hence $H' = H$ or $H' = N_G(H)$ (or vice-versa) according to Lemma II.1.11. Remark II.1.12 implies that $1 = \dim V(\lambda)^H = \dim V(\lambda)^{H'}$, and it follows that the two vector-spaces coincide: $V(\lambda)^{H'} = V(\lambda)^H$. It follows that $S' = G \cdot V(\lambda)^{H'} = G \cdot V(\lambda)^H = S$.

b) It follows from Lemma II.1.4 b) that $Gv_\lambda \subset S$.

c) Suppose that $X := Gv \subset S$ is an orbit different from X_λ. Then Lemma II.1.10 implies that there exists a Jansou-subgroup $H \subset G$ of type λ/n for an integer $n \in \mathbf{N}$ such that $X \cong G/H$. This proves the 'only if' part of c).

For the other direction, suppose that λ/n is a Jansou-weight. Let $w \in V(\lambda/n)$ be a closed point such that $H := G_w$ is a Jansou-subgroup of type λ/n. Then $Gw \cong G/H$, and $\mathcal{O}(G/H) \cong \bigoplus_k V(k\lambda/n)^*$. Using the fact that $h_{G/H}(V(\lambda^*)) = 1$, we obtain a finite morphism $\varphi\colon G/H \to V(\lambda)$. Hence the image Y of G/H in $V(\lambda)$ is a closed multiplicity-free orbit of dimension $\dim G/H$. Because $\dim Gv_{\lambda/n} = \dim Gv_\lambda$, it follows that $\dim Y = \dim G/H = \dim Gv_{\lambda/n} = \dim Gv_\lambda$, which proves with a) and b) that $Y \subset S$. $\square$

Proof of Corollary II.1.3. The statement is now a consequence of the proof of Proposition II.1.1 (and of Schur's lemma). $\square$

II.2. Invariant Hilbert schemes for minimal sheets in simple modules

In this section we use the following notation: A closed G-stable cone $X = \operatorname{Spec}(A)$ in a G-module V is regarded as a $\mathbf{G}_m \times G$-variety, where the $\mathbf{G}_m$-action comes from the scalar multiplication of $\mathbf{G}_m$ on V. For $n \in \mathbf{N}$ and $\lambda \in \Lambda^+$, we denote by $A_{(n,\lambda)}$ the isotypic component of the $\mathbf{G}_m \times G$-algebra A of type (n, λ); which is the same as the intersection of the homogeneous component of A of degree n (with respect to the grading induced by the $\mathbf{G}_m$-action) and the isotypic component $A_{(\lambda)}$ of the G-algebra A.

Recall from Definition II.1.2 that $\mathcal{S}(\lambda)$ denotes the closure of the unique minimal sheet in $V(\lambda)$. In the sequel, $h'_\lambda \colon \Lambda^+ \to \mathbf{N}$ is the Hilbert function of a general orbit of $\mathcal{S}(\lambda)$, whereas $h_\lambda \colon \Lambda^+ \to \mathbf{N}$ still denotes the Hilbert function of $X_\lambda = \overline{Gv_\lambda} \subset V(\lambda)$. In this section we are interested in describing $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))$.

Lemma II.2.1. a) *If λ is a Jansou-weight, then $h'_\lambda = h_\lambda$. Furthermore,* $\dim \mathcal{S}(\lambda)/\!\!/G = 1$, *and* $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \mathbf{A}^1$.

b) *If λ is not an integral multiple of a Jansou-weight, then $h'_\lambda = h_\lambda$. Furthermore, $\mathcal{S}(\lambda) = X_\lambda$, and* $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \mathbf{A}^0$.

The goal of this section is to prove that *under some assumptions* $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \mathbf{A}^1$ when λ is an integral multiple of a Jansou-weight.

Proof of Lemma II.2.1. a) Note that $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))(= \operatorname{Hilb}^G_{h_\lambda}(\mathcal{S}(\lambda)))$ is a closed subscheme of $\operatorname{Hilb}^G_{h_\lambda}(V(\lambda)) = \mathbf{A}^1$ according to Proposition I.2.1. On the other hand, the G-orbits in $V(\lambda)$ with Hilbert function h_λ that are parametrized by $\operatorname{Hilb}^G_{h_\lambda}(V(\lambda))$ are closed subschemes of $\mathcal{S}(\lambda)$. This shows that the universal family $\operatorname{Univ}^G_{h_\lambda}(V(\lambda)) \subset \mathbf{A}^1 \times V(\lambda)$ is in fact a closed subscheme of $\mathbf{A}^1 \times \mathcal{S}(\lambda)$. The claim now follows.

b) Also here, $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \operatorname{Hilb}^G_{h_\lambda}(\mathcal{S}(\lambda))$ is a closed subscheme of $\operatorname{Hilb}^G_{h_\lambda}(V(\lambda)) = \mathbf{A}^0$. Since $X_\lambda \subset \mathcal{S}(\lambda)$ has Hilbert function $h_{X_\lambda} = h_\lambda = h'_\lambda$, the invariant Hilbert scheme $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))$ has a closed point, and hence equals $\mathbf{A}^0$. (Or, one could observe that the quotient morphism $\mathcal{S}(\lambda) \to \mathcal{S}(\lambda)/\!\!/G = \mathbf{A}^0$ is flat, and hence according to Theorem I.1.1 $\operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \mathcal{S}(\lambda)/\!\!/G = \mathbf{A}^0$.) $\square$

However, if λ is an integral multiple of a Jansou-weight but not a Jansou-weight itself, then $h'_\lambda \neq h_\lambda$. (To see this, the discussion of the previous section is useful: If λ is an integral multiple of a Jansou-weight, then the minimal sheet contains a family of orbits different from X_λ. These other orbits are all isomorphic, and their Hilbert function is h'_λ. In particular, $\mathrm{Hilb}^G_{h'_\lambda}(V(\lambda))$ contains more than one closed point. On the other hand, if λ is not a Jansou-weight itself, then $\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)) \cong \mathbf{A}^0$. Thus, h'_λ cannot be equal to h_λ.) In this situation, we only know from Proposition II.1.1 that $\dim \mathcal{S}(\lambda)/\!\!/G = 1$. If $\mathcal{S}(\lambda)/\!\!/G \cong \mathbf{A}^1$, then it follows from Theorem I.1.1 (and from [**Ha77**], Proposition III.9.7) that $\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}_\lambda) \cong \mathcal{S}(\lambda)/\!\!/G \cong \mathbf{A}^1$. However, if $\mathcal{S}(\lambda)/\!\!/G$ is not smooth, then the situation is more delicate. Anyway, we now construct a morphism $\mathbf{A}^1 \to \mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))$.

Since $\mathcal{S}(\lambda) \subset V(\lambda)$ is a cone, the action of $\mathbf{G}_m$ on $V(\lambda)$ by scalar multiplication restricts to a $\mathbf{G}_m$-action on $\mathcal{S}(\lambda)$. The Borel subgroup $\mathbf{G}_m \times B$ of $\mathbf{G}_m \times G$ has a dense orbit in $\mathcal{S}(\lambda)$, hence $R := \mathcal{O}(\mathcal{S}(\lambda))$ is multiplicity-free as k-$(\mathbf{G}_m \times G)$-algebra. Using once more that $\mathcal{S}(\lambda)$ is a cone in $V(\lambda)$, we see that there exist G-invariants $f_1, \ldots, f_s \in R^G$, each f_i homogeneous of degree n_i, such that $R^G = k[f_1, \ldots, f_s]$.

Definition II.2.2. Let

$$\begin{aligned} n_0 :=& \gcd\{n_1, \ldots, n_s\} \\ =& \gcd\{n \in \mathbf{N} \mid \text{ there exists } 0 \neq f \in R^G \text{ of degree } n\}. \end{aligned}$$

Since the normalization of $\mathcal{S}(\lambda)/\!\!/G$ is isomorphic to $\mathbf{A}^1$, there exists an invariant rational function $f \in \mathrm{Quot}(R)$, homogeneous of degree n_0, such that each f_i equals f^{m_i} for some $m_i \in \mathbf{N}$, up to some non-zero scalar factor. (Here, $\mathrm{Quot}(R)$ denotes the quotient field of R.) Then $R^G = k[f^{m_1}, \ldots, f^{m_s}]$.

Let $v \in \mathcal{S}(\lambda)$ be a closed point such that the orbit Gv is closed. Let $H \subset G$ be its stabilizer (which is a Jansou-subgroup). Recall from Lemma II.1.11 that H is either self-normalizing or that $[N_G(H) : H] = 2$.

Lemma II.2.3. *Let n_0 be as above.*

a) *If $H = N_G(H)$, then $n_0 = 1$; and if $H \neq N_G(H)$, then $n_0 = 2$.*
b) *Suppose that the only Jansou-weight in $\mathbf{Q}\lambda$ is $\mu = \lambda/n$. Then $n_0 = 1$.*
c) *Suppose that the Jansou-weights in $\mathbf{Q}\lambda$ are $\mu = \lambda/n$ and 2μ. If n is odd, then $n_0 = 2$. If n is even, then $n_0 = 1$.*

Proof. a) Suppose that $n_0 \neq 1$, and let $\zeta \in k$ be a primitive n_0-th root of 1. Then there exists $g \in G$ such that $gv = \zeta v$. If $h \in H$, then $g^{-1}hgv = g^{-1}h\zeta v = g^{-1}\zeta hv = g^{-1}\zeta v = g^{-1}gv = v$, which shows that $g^{-1}Hg \subset H$. Since $g \notin H$, it follows that $H \neq N_G(H)$. According to Lemma II.1.11, we see that $N_G(H) = H \cup gH$. It is now easy to see that $\zeta^2 = 1$, and hence $n_0 = 2$.

On the other hand, suppose that $n_0 = 1$. If $H \neq N_G(H)$, then there exists $g \in G \setminus H$ with $N_G(H) = H \cup gH$. Then $gv \neq v$, but $g^2v = v$.

Then $v + gv \in V(\lambda)^{N_G(H)} \subset V(\lambda)^H = k \cdot v$. It follows that $gv = -v$. However, if $n_0 = 1$, then v and $-v$ are separated by some invariant function of odd degree, which shows that v and $-v$ cannot lie in the same G-orbit. This contradiction shows that $H = N_G(H)$.

b) Under the present assumptions, neither $\mu/2$ nor 2μ are Jansou-weights. It then follows that each Jansou-subgroup of type μ is self-normalizing. Since H is such a Jansou-subgroup, the claim follows with a).

c) If n is odd, then λ is not an integral multiple of 2μ. Lemma II.1.10 implies that H is a Jansou-subgroup of type μ (and not of type 2μ). But a Jansou-subgroup of type μ is not self-normalizing, and a) implies that $n_0 = 2$.

Let $X = Gv \subset \mathcal{S}(\lambda)$. If n is even, then λ is an integral multiple of 2μ. Let $H' \subset G$ be a Jansou-subgroup of type 2μ. Then $H' = N_G(H')$, and $(H')^0$ is a Jansou-subgroup of G of type μ. As in the proof of Lemma **??**, there is a finite G-morphism $G/H' \to X \cong G/H$. Moreover, $H \cong H'$ or $H \cong (H')^0$. As there is no non-zero G-morphism $G/H' \to G/(H')^0$, we conclude that $H \cong H'$, and hence $H = N_G(H)$. □

Let $J := \{\sum_k r_k t^k \mid \sum_k r_k f^k = 0 \text{ in } \mathrm{Quot}(R)\} \subset R[t]$, and let $\mathcal{S}(\lambda)' := \mathrm{Spec}(R[t]/J) \subset \mathcal{S}(\lambda) \times \mathbf{A}^1$.

Proposition II.2.4. *The morphism $p\colon \mathbf{A}^1 \times \mathcal{S}(\lambda) \supset \mathcal{S}(\lambda)' \to \mathbf{A}^1$ induced by the composition of the natural homomorphisms $k[t] \to R[t] \to R[t]/J$ defines a family in $\mathcal{H}ilb^G_{h'_\lambda}(\mathcal{S}(\lambda))(\mathbf{A}^1)$.*

Proof. The ideal J is the kernel of the homomorphism of k-algebras $R[t] \to \mathrm{Quot}(R)$ mapping t to f. This implies that $R[t]/J$ is a domain, since it can be regarded as a subring of $\mathrm{Quot}(R)$. In particular, the morphism p, mapping an irreducible variety onto $\mathbf{A}^1$, is flat. It now suffices to verify that one fiber of p has h'_λ as Hilbert function. To see this, we compute the fiber of $p\colon \mathcal{S}(\lambda)' \to \mathbf{A}^1$ over the generic point of $\mathbf{A}^1$. This fiber equals $\mathrm{Spec}((R[t]/J) \otimes_{k[t]} k(t)) \cong \mathrm{Spec}(R(f))$, where $f \in \mathrm{Quot}(R)$ is as above. On the other hand, the fiber of $\pi\colon \mathcal{S}(\lambda) \to \mathcal{S}(\lambda)/\!\!/G$ over the generic point of $\mathcal{S}(\lambda)/\!\!/G$ equals $\mathrm{Spec}(R \otimes_{R^G} k(f)) = \mathrm{Spec}(R(f))$. Thus, the two fibers are isomorphic and hence have the same Hilbert function. Since the fiber over the generic point of the quotient morphism π has Hilbert function h'_λ, the claim follows. □

Since R is multiplicity-free as k-$(\mathbf{G}_m \times G)$-algebra, we have

$$R = \bigoplus_{\mu \in \Lambda_R} \bigoplus_{n \geq 0} R_{(n,\mu)}$$

with $R_{(n,\mu)}$ either 0 or G-isomorphic to $V(\mu)$.

Remark II.2.5. The following observation turns out to be useful. If $R_{(n,\mu)} \cong V(\mu)$ and $R_{(n+n_0,\mu)} \cong V(\mu)$, then $R_{(n+n_0,\mu)} = f \cdot R_{(n,\mu)}$: If $0 \neq r \in R^U_{(n,\mu)}$ and if $0 \neq r' \in R^U_{(n+n_0,\mu)}$, it suffices to show that up to a non-zero scalar multiple $r' = f \cdot r$. Because R is a domain, it is enough to show that up to a non-zero scalar multiple $(r')^m = f^m \cdot r^m$ for some $m \geq 1$. For $m \gg 0$, the invariant f^m is a regular function, and thus both $(r')^m$ and $f^m r^m$ are invariant regular functions of the same degree for $m \gg 0$. Because R is multiplicity-free as $\mathbf{G}_m \times G$-algebra, we see that $(r')^m = f^m r^m$ up to some non-zero scalar multiple, and the claim follows.

Remark II.2.6. There is the following description of $\mathcal{S}(\lambda)'$. Consider the variety $(\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda))_{\mathrm{red}}$. It is the pull-back of $\mathcal{S}(\lambda)$ and $\mathbf{A}^1$ over $\mathcal{S}(\lambda)/\!\!/G$ in the category of varieties. It is not hard to see that $(\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda))_{\mathrm{red}}$ is irreducible: First,

$$(\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda))_{\mathrm{red}} = \{(t,v) \in \mathbf{A}^1 \times \mathcal{S}(\lambda) \mid \eta(t) = \pi(v)\} \subset \mathbf{A}^1 \times \mathcal{S}(\lambda),$$

where $\eta\colon \mathbf{A}^1 \to \mathcal{S}(\lambda)/\!\!/G$ is the normalization morphism. From this description one sees that the (reduced) fibers of $p'\colon (\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda))_{\mathrm{red}} \to \mathbf{A}^1$ coincide with the (reduced) fibers of $\pi\colon \mathcal{S}(\lambda) \to \mathcal{S}(\lambda)/\!\!/G$. Thus, $(\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda))_{\mathrm{red}}$ is irreducible, and hence p' is flat. Since both are subschemes of $\mathbf{A}^1 \times \mathcal{S}(\lambda)$, it follows that $(\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda))_{\mathrm{red}}$ isomorphic to $\mathcal{S}(\lambda)'$.

Lemma II.2.7. a) *In general, $\mathcal{S}(\lambda)'$ is not isomorphic to the (scheme-theoretic) pull-back $\mathcal{S}(\lambda) \times_{\mathcal{S}(\lambda)/\!\!/G} \mathbf{A}^1$.*

b) *In general, $\mathcal{S}(\lambda)'$ is not isomorphic to the normalization $\widetilde{\mathcal{S}(\lambda)}$.*

Proof. a) In view of Remark II.2.6, this amounts to show that in general $\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda)$ is not reduced. If $\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda)$ is reduced, the schematic fiber $\mathcal{S}(\lambda)'_0$ of $p\colon \mathcal{S}(\lambda)' \to \mathbf{A}^1$ would coincide with the schematic fiber $\mathcal{S}(\lambda)_0$ of $\pi\colon \mathcal{S}(\lambda) \to \mathcal{S}(\lambda)/\!\!/G$. But this would imply that π is flat. This is not the case in general, and an example is given in Section III.3 (cf. Remark III.3.8).

b) Recall that $\mathcal{S}(\lambda)' = \mathrm{Spec}(R[f])$, where $f \in \mathrm{Quot}(R)$ is as above. On the other hand, $\widetilde{\mathcal{S}(\lambda)} = \mathrm{Spec}(\tilde{R})$, where $\tilde{R} \subset \mathrm{Quot}(R)$ is the integral closure of R in $\mathrm{Quot}(R)$. We now give an example in which $R[f] \subsetneq \tilde{R}$. Let $G = \mathrm{SL}_2$, and let $\lambda = d \geq 8$ be a multiple of 4. Then $R_{(2,4)} \neq 0$, and $R_{(2,8)} \neq 0$, as well as $f^2 \in R_{(2,0)}$, but $R_{(1,4)} = 0$ (as we shall see in detail in Section III.3). Let $0 \neq r \in R^U_{(2,4)}$, and consider r/f, which is a U-invariant rational function of degree 1 and of weight 4. Now $r^2 \in R^U_{(4,8)}$. Choose further $0 \neq r' \in R^U_{(2,8)}$. Because $f^2 \in R_{(2,0)}$ and because R is multiplicity-free as $\mathbf{G}_m \times \mathrm{SL}_2$-algebra, it follows that $r^2 = f^2 r'$ up to some non-zero scalar multiple. But now $(r/f)^2 = r' \in R$, and thus $r/f \in \tilde{R}$. This shows that $\tilde{R}_{(1,4)} \neq 0$. However, clearly $R[f]_{(1,4)} = 0$, because the only SL_2-submodules of $R[f]$ of degree 1 are $k \cdot f \cong V(0)$ and $R_{(1,d)} \cong V(d)$. Thus, $R[f] \subsetneq \tilde{R}$.

$\square$

Remark II.2.8. The inclusion $R[f] \subsetneq \tilde{R}$ corresponds to the $\mathbf{G}_m \times G$-equivariant morphism $\varphi\colon \widetilde{\mathcal{S}(\lambda)} \to (\mathbf{A}^1 \times_{\mathcal{S}(\lambda)/\!\!/G} \mathcal{S}(\lambda))_{\mathrm{red}}$ (that is obtained by the universal property of the pull-back):

$$\begin{tikzcd}
\widetilde{\mathcal{S}(\lambda)} \arrow[dr, "\varphi"] \arrow[drr, bend left] \arrow[ddr, bend right, "\tilde{\pi}"'] & & \\
& (\mathcal{S}(\lambda) \times_{\mathcal{S}(\lambda)/\!\!/G} \mathbf{A}^1)_{\mathrm{red}} \arrow[r] \arrow[d, "\pi'"] & \mathcal{S}(\lambda) \arrow[d, "\pi"] \\
& \mathbf{A}^1 \arrow[r] & \mathcal{S}(\lambda)/\!\!/G.
\end{tikzcd}$$

Both π' and $\tilde{\pi}$ are flat families of G-schemes whose fibers have the same Hilbert function, and both families are isomorphic over $\mathbf{A}^1 \setminus \{0\}$. Nevertheless, in general φ is no isomorphism according to Lemma II.2.7 b). This does not contradict the representability of the functor $\mathcal{H}ilb^G_{h'_\lambda}(\mathcal{S}(\lambda))$ since there is no reason why $\widetilde{\mathcal{S}(\lambda)}$ should be a subscheme of $\mathcal{S}(\lambda) \times \mathbf{A}^1$. (Compare this to Remark I.1.6 in the introductory part.)

From the definition of $\mathcal{S}(\lambda)'$, we see that the schematic fiber $\mathcal{S}(\lambda)'_0$ of $\pi'\colon \mathcal{S}(\lambda)' \to \mathbf{A}^1$ over 0 equals $\mathrm{Spec}(R/I'_0)$, where $I'_0 \subset R$ is the ideal defined by

$$I'_0 = \{r_0 \in R \mid \exists r_1, \ldots, r_l \in R : \sum_{k=0}^{l} r_k f^k = 0 \in \mathrm{Quot}(R)\}.$$

For $\mu \in \Lambda_R$ let $n^\mu := \min\{n \in \mathbf{N}_{>0} \mid R_{(n,\mu)} \neq 0\}$. Then

$$I'_0 = \bigoplus_{\mu \in \Lambda_R} \bigoplus_{n > n^\mu} R_{(n,\mu)}.$$

To see this, let $0 \neq r_0 \in R^U_{(n,\mu)}$ with $n > n_\mu$, and let $0 \neq r \in R^U_{(n^\mu,\mu)}$. Then (after multiplying r_0 with a suitable scalar) $r_0 - r f^{n-n^\mu} = 0 \in \mathrm{Quot}(R)$, and the claim follows.

Proposition II.2.9. *Let n_0 be as in Definition II.2.2, and suppose that $R_{(1+n_0,\lambda^*)} \cong V(\lambda^*)$. Then*

$$\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \mathbf{A}^1,$$

and $p\colon \mathcal{S}(\lambda)' \to \mathbf{A}^1$ from Proposition II.2.4 is the universal family.

Proof. Step 1. Let $\mu \in \Lambda^+$. We claim that $R_{(n,\mu)} \neq 0$ if and only if $n = n^\mu + l n_0$ for some $l \geq 0$ or if $(n,\mu) = (0,0)$. Assume first that $R_{(n,\mu)} \neq 0$, and let $0 \neq r_n \in R^U_{(n,\mu)}$ and $0 \neq r_{n^\mu} \in R^U_{(n^\mu,\mu)}$. Then r_n/r_{n^μ} is a rational invariant function of degree $n - n^\mu$. By definition of n_0, it follows that $n - n^\mu \in n_0\mathbf{N}$.

On the other hand, let $n = n^\mu + l n_0$. Then we need to show that $R_{(n,\mu)} \neq 0$. If $(n,\mu) = (1,\lambda^*)$ or if $(n,\mu) = (1+n_0,\lambda^*)$, then $R_{(n,\mu)} \neq 0$

by assumption. Otherwise, we proceed by induction on l. If $l = 0$, the claim is true by definition of n^μ. So, we assume that $R_{(n^\mu+ln_0,\mu)} \neq 0$. Now, there exists $\mu' \in \Lambda^+$ such that

$$R_{(n^\mu+ln_0,\mu)} \subset R_{(n^\mu+ln_0-1,\mu')} \cdot R_{(1,\lambda^*)}.$$

Multiplying this with f and using Remark II.2.5, we find:

$$\begin{aligned} f \cdot R_{(n^\mu+ln_0,\mu)} &\subset R_{(n^\mu+ln_0-1,\mu')} \cdot f \cdot R_{(1,\lambda^*)} \\ &= R_{(n^\mu+ln_0-1,\mu')} \cdot R_{(1+n_0,\lambda^*)} \subset R. \end{aligned}$$

This shows that $R_{(n^\mu+(l+1)n_0,\mu)} \neq 0$.

Step 2. The preceding step shows that $R_{(n^\mu+ln_0,\mu)}$ is contained in the G-stable ideal generated by $R_{(1+n_0,\lambda^*)}$ for all μ and for all $l > 0$. This shows that the ideal I_0' is the smallest G-stable ideal containing $R_{(1+n_0,\lambda^*)}$ and $R_{(n^0,0)}$.

Step 3. Let $I \subset R$ be a G-stable ideal with $h_{R/I} = h'_\lambda$ and with $\operatorname{Spec}(R/\sqrt{I}) = \operatorname{Spec}(R/I)_{\mathrm{red}} = X_\lambda$. Then $I = I_0'$. To see this, first observe that the condition $\operatorname{Spec}(R/\sqrt{I}) = X_\lambda$ implies that $R_{(1+n_0,\lambda^*)} \subset I$. Consider $R_{(n^0,0)} = kf^{n^0/n_0}$. Suppose that $R_{(n^0,0)} \not\subset I$. Then there exists $0 \neq \gamma \in k$ such that $\gamma - f^{n^0/n_0} \in I$ (because $h_{R/I}(0) = 1$). Hence $f \cdot (\gamma - f^{n^0/n_0}) = f\gamma - f^{n^0/n_0+1} \in I$. According to Step 2, we know that f^{n^0/n_0+1} is in the G-stable ideal generated by $R_{(1+n_0,\lambda^*)}$, and hence $f^{n^0/n_0+1} \in I$. This in turn implies that $f \in I$, which is a contradiction to $f^{n^0/n_0} \notin I$. We conclude that $R_{(n^0,0)} \subset I$. But then $I_0' \subset I$ according to Step 2, and hence $I = I_0'$ because $h_{R/I} = h'_\lambda = h_{R/I_0'}$.

Step 4. We claim that $\dim_k \operatorname{Hom}_R^G(I_0', R/I_0') \leq 1$. Since I_0' is generated as R-G-module by $R_{(1+n_0,\lambda^*)}$ and by $R_{(n^0,0)}$, it follows that

$$\begin{aligned} \dim_k \operatorname{Hom}_R^G(I_0', R/I_0') &\leq \dim_k \operatorname{Hom}_R^G(R_{(1+n_0,\lambda^*)}, R/I_0') \\ &\quad + \dim_k \operatorname{Hom}_R^G(R_{(n^0,0)}, R/I_0') \\ &= 1 + 1 = 2 \end{aligned}$$

according to Schur's Lemma. Let $\varphi \in \operatorname{Hom}_R^G(I_0', R/I_0')$. We now show that the restriction of φ to $R_{(1+n_0,\lambda^*)}$ already determines $\varphi(f^{n^0/n_0})$. Let $r \in R^U_{(1,\lambda^*)}$. Now $\varphi(r \cdot f^{n^0/n_0}) = r\varphi(f^{n^0/n_0}) \in R/I_0'$. Since $r \cdot f^{n^0/n_0} \in R_{(1+n^0,\lambda^*)}$, we see that $r\varphi(f^{n^0/n_0})$ is determined by $\varphi|_{R_{(1+n_0,\lambda^*)}}$. But then also $\varphi(f^{n^0/n_0})$ is determined by $\varphi|_{R_{(1+n_0,\lambda^*)}}$, and the claim follows.

Step 5. Consider the family $p\colon \mathcal{S}(\lambda)' \to \mathbf{A}^1$. Step 3 shows that the induced morphism $\psi\colon \mathbf{A}^1 \to \operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))$ is bijective. Because $\mathcal{S}(\lambda) \to \mathcal{S}(\lambda)/\!\!/G$ is flat in the complement of the zero-fiber, we see that ψ is an isomorphism in the complement of 0. To show that ψ is an isomorphism, it suffices to show that $T_{\mathcal{S}(\lambda)_0'} \operatorname{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))$ is at most one-dimensional. According to [**AB05**], Proposition 1.13, the tangent space is isomorphic to $\operatorname{Hom}_R^G(I_0', R/I_0')$, and now the claim follows with Step 4. $\square$

Let $\lambda \in \Lambda^+$, and let $v \in \mathcal{S}(\lambda)$ be a closed point such that the orbit Gv is closed. Let $H = G_v$ be the stabilizer of v. Then H is a Jansou-subgroup of type λ/n for some n. Let n_0 be as in Lemma II.2.3

Corollary II.2.10. *Suppose that*

(1) $n_0 = 1$, *and that one of the following conditions hold:*
  (a) $R_{(2,\lambda^*)} \neq 0$, *or:*
  (b) $R_{(3,0)} \neq 0$, *or:*
  (c) *if*

$$\cdot_{G/H} \colon \bigoplus_{k\geq 0} V(k\lambda^*/n) \times \bigoplus_{k\geq 0} V(k\lambda^*/n) \to \bigoplus_{k\geq 0} V(k\lambda^*/n)$$

is the multiplication on $\mathcal{O}(G/H)$, *then* $V(\lambda^*) \subset V(\lambda^*) \cdot_{G/H} V(\lambda^*)$;

or that

(2) $n_0 = 2$, *and that one of the following conditions hold:*
  (a) $\mathcal{S}(\lambda)/\!\!/G \cong \mathbf{A}^1$, *or:*
  (b) $R_{(3,\lambda^*)} \neq 0$, *or:*
  (c) *if*

$$\cdot_{G/H} \colon \bigoplus_{k\geq 0} V(k\lambda^*/n) \times \bigoplus_{k\geq 0} V(k\lambda^*/n) \to \bigoplus_{k\geq 0} V(k\lambda^*/n)$$

is the multiplication on $\mathcal{O}(G/H)$, *then* $V(\lambda^*) \subset V(\lambda^*) \cdot_{G/H} V(\lambda^*) \cdot_{G/H} V(\lambda^*)$.

Then

$$\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \mathbf{A}^1,$$

and $p \colon \mathcal{S}(\lambda)' \to \mathbf{A}^1$ *is the universal family.*

Observe that in the case $n_0 = 1$ the quotient $\mathcal{S}(\lambda)/\!\!/G$ cannot be an affine line. Otherwise, $R^G = k[f]$, where f is an invariant function of degree 1 (since $n_0 = 1$). Since R is a quotient of $\mathrm{Sym}(V(\lambda^*))$, it would follow that $\mathrm{Sym}^1(V(\lambda^*))^G \neq 0$, a contradiction.

Proof. (1) (a). This follows from Proposition II.2.9.

(1) (b). Clearly, $R_{(3,0)} \subset \bigoplus_{\mu\in\Lambda_R} R_{(2,\mu)} \cdot \bigoplus_{\mu\in\Lambda_R} R_{(1,\mu)}$. However, $R_{(1,\mu)} = 0$ if $\mu \neq \lambda^*$, hence $R_{(3,0)} \subset \bigoplus_{\mu\in\Lambda_R} R_{(2,\mu)} \cdot R_{(1,\lambda^*)}$. But $R_{(2,\mu)} \cdot R_{(1,\lambda^*)}$ can contain $V(0)$ only if $\mu = \lambda^*$. This shows that $R_{(3,0)} \subset R_{(2,\lambda^*)} \cdot R_{(1,\lambda^*)}$. If $R_{(3,0)} \neq 0$, then also $R_{(2,\lambda^*)} \neq 0$, and the claim follows with (1) (a).

(1) (c). The orbit $Gv \subset \mathcal{S}(\lambda)$ is G-isomorphic to G/H. Let $J \subset R$ be the ideal of Gv, and let $\mu \in \Lambda^+$ be a dominant weight. The isotypic component $(R/J)_{(\mu)}$ is isomorphic to $R_{(\mu)}/J_{(\mu)}$ (see [**Kr85**], II.3.2). If $V(\lambda^*) \subset V(\lambda^*) \cdot_{G/H} V(\lambda^*)$, then $(R/J)_{(\lambda^*)} \subset (R/J)_{(\lambda^*)} \cdot (R/J)_{(\lambda^*)}$. Since $R_{(1,\lambda^*)} \not\subset J$, we find that

$$R_{(1,\lambda^*)} \subset R_{(1,\lambda^*)} \cdot R_{(1,\lambda^*)} + J_{(\lambda^*)}.$$

Using once more that $R_{(1,\lambda^*)} \neq 0$, we find that $(R_{(1,\lambda^*)} \cdot R_{(1,\lambda^*)})_{(\lambda^*)} = R_{(2,\lambda^*)} \neq 0$. The claim now follows with (1) (a).

(2) (a) If $\mathcal{S}(\lambda)/\!\!/G \cong \mathbf{A}^1$, then the quotient morphism $\pi\colon \mathcal{S}(\lambda) \to \mathcal{S}(\lambda)/\!\!/G$ is flat, and the claim follows from Theorem I.1.1.

(2) (b) This follows from Proposition II.2.9.

(2) (c) The proof of (1) (c) can be carried over to this situation using 2 (b). □

We will show in the next chapter that for $G = \mathrm{SL}_2$ either condition (1) (a) or condition (2) (a) holds for every integral multiple of a Jansou-weight.

Example II.2.11. Consider SL_3. The only Jansou-weight of SL_3 is $\lambda := \omega_1 + \omega_3$, and $V(\lambda)$ can be identified with the Lie algebra $\mathfrak{sl}_3$, on which SL_3 acts with the adjoint representation. We claim that

$$\mathrm{Hilb}^{\mathrm{SL}_3}_{h'_{2\lambda}}(\mathcal{S}(2\lambda)) = \mathbf{A}^1.$$

Let $X := \mathrm{SL}_3 \cdot \mathrm{diag}(1, 1, -2) \subset \mathcal{S}(\lambda) \subset \mathfrak{sl}_3$. Then $h_X = h'_\lambda$, and $\mathcal{O}(X) \cong \bigoplus_{k\geq 0} V(k\lambda^*) \cong \bigoplus_{k\geq 0} V(k\lambda)$. In view of Corollary II.2.10 1 (c) the claim is true once we have shown that $V(2\lambda) \subset V(2\lambda) \cdot V(2\lambda)$, where $\cdot$ is the multiplication on $\mathcal{O}(X)$.

Step 1. For a 3×3-matrix matrix $A = (a_{ij}) \in \mathrm{Mat}_3(k)$ the assignment $y_{ij}(A) := a_{ij}$ defines elements in $\mathrm{Mat}_3(k)^*$. Because $((\mathfrak{sl}_3)^*)^U = k \cdot y_{31}$, we see that

$$V(2\lambda) = \mathrm{span}_k\{(gy_{31} \cdot gy_{31}) \mid g \in \mathrm{SL}_3\} \subset \mathrm{Sym}^2((\mathfrak{sl}_3)^*).$$

If we identify $(\mathfrak{sl}_3)^*$ with $\mathfrak{sl}_3$, then $\mathrm{SL}_3 \cdot y_{31}$ consists of all traceless matrices Y of rank 1 with $Y^2 = 0$. Every such matrix is of the form $w_1 \cdot (w_2)^\tau$, where $w_1, w_2 \in k^3$ are two vectors that are orthogonal with respect to the standard inner product on k^3. We now compute some of these matrices. Always assume now that $i \neq j \neq k \neq i$. For $w_1 = e_i + e_j + e_k$ and $w_2 = e_i - e_j$ we see that $y_{ii} + y_{ji} + y_{ki} - y_{ij} - y_{jj} - y_{kj} \in \mathrm{SL}_3 \cdot y_{31}$. Hence $(y_{ii} + y_{ji} + y_{ki} - y_{ij} - y_{jj} - y_{kj})^2 \in V(2\lambda)$. The maximal torus $T \subset \mathrm{SL}_3$ acts with different weights on the summands of this sum. Because the sum of all summands of the same weight also belongs to $V(2\lambda)$, we see that

$$y_{ij}^2 \in V(2\lambda), \tag{II.2.1}$$

$$y_{ki}y_{kj} \in V(2\lambda), \tag{II.2.2}$$

$$y_{ij}(y_{ii} - y_{jj}) \in V(2\lambda), \tag{II.2.3}$$

$$y_{ki}(y_{ii} - y_{jj}) - y_{kj}y_{ji} \in V(2\lambda). \tag{II.2.4}$$

Step 2. From (II.2.3) it follows that

$$y_{ij}^2(y_{ii} - y_{jj})^2 \in V(2\lambda) \cdot V(2\lambda).$$

From (II.2.4) it follows (after permutating the indices cyclically) that

$$y_{ij}^2(y_{jj} - y_{kk})^2 + y_{ik}^2 y_{kj}^2 - 2y_{ij}(y_{jj} - y_{kk})y_{ik}y_{kj} \in V(2\lambda) \cdot V(2\lambda).$$

From (II.2.1) it follows that $y_{ik}^2 y_{kj}^2 \in V(2\lambda)\cdot V(2\lambda)$. From (II.2.2) and (II.2.3) it follows that

$$y_{ij}(y_{jj}-y_{kk})y_{ik}y_{kj} = y_{ij}y_{ik}\cdot y_{kj}(y_{jj}-y_{kk}) \in V(2\lambda)\cdot V(2\lambda).$$

We conclude that

$$y_{ij}^2(y_{jj}-y_{kk})^2 \in V(2\lambda)\cdot V(2\lambda). \tag{II.2.5}$$

Starting with $w_1 = e_i - e_j$ and with $w_2 = e_i + e_j + e_k$, one sees similarly that

$$y_{ij}^2(y_{kk}-y_{ii})^2 \in V(2\lambda)\cdot V(2\lambda). \tag{II.2.6}$$

Now (II.2.5) and (II.2.6) imply that

$$y_{ij}^2 \sum_{k\neq l}(y_{kk}-y_{ll})^2 \in V(2\lambda)\cdot V(2\lambda).$$

But $\sum_{k\neq l}(y_{kk}-y_{ll})^2$ is a symmetric polynomial in the y_{kk}, and hence it is constant on X. Since $\sum_{k\neq l}(y_{kk}-y_{ll})^2(\mathrm{diag}(1,1,-2)) = 36 \neq 0$, we conclude that $y_{ij}^2 \in V(2\lambda)\cdot V(2\lambda)$. In particular, $y_{31}^2 \in V(2\lambda)\cdot V(2\lambda)$, and hence $V(2\lambda) \subset V(2\lambda)\cdot V(2\lambda)$.

Conjecture II.2.12. *Let G be a semisimple group, and let λ be an integral multiple of a Jansou-weight of G. We conjecture that*

$$\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda)) = \mathbf{A}^1.$$

The conjecture is only known to be true if λ is a Jansou-weight (cf. [**Ja05**]) and for $G = \mathrm{SL}_2$, as well as for $G = \mathrm{SL}_3$ with $\lambda = 2(\omega_1+\omega_3)$.

II.3. Deformations and the null-cone

Definition II.3.1. The *schematic null-cone* $\mathcal{N}(\lambda)$ of $V(\lambda)$ is the closed subscheme of $V(\lambda)$ defined by the ideal $(\bigoplus_{n>0}\mathrm{Sym}^n(V(\lambda)^*)^G)$ of all non-constant homogeneous invariant functions on $V(\lambda)$.

Lemma II.3.2. *Let $X \subset V(\lambda)$ be a closed G-stable subscheme with Hilbert function $h_X = h'_\lambda$.*

a) *If $X_{\mathrm{red}} = X_\lambda = \overline{Gv_\lambda}$, then X is a closed subscheme of $\mathcal{N}(\lambda)$.*
b) *If $X_{\mathrm{red}} \neq X_\lambda$, then X is a closed subvariety of $\mathcal{S}(\lambda)$.*

Proof. a) Suppose that $X_{\mathrm{red}} = X_\lambda$, and that $f = \sum_n f_n \in I(X)^G \subset \mathrm{Sym}(V(\lambda^*))^G$ is an invariant function in the ideal $I(X)$ of $X \subset V(\lambda)$, where each f_n is homogeneous of degree n. Then $f \in I(X_{\mathrm{red}})^G = \bigoplus_{n>0}\mathrm{Sym}^n(V(\lambda^*))^G$. Hence $f_0 = 0$. Since $h_X(0) = h'_\lambda(0) = 1$, it follows that $I(X)^G = \bigoplus_{n>0}\mathrm{Sym}^n(V(\lambda^*))^G$. Hence $I(X) \supset I(\mathcal{N}(\lambda))$.

b) If $X_{\mathrm{red}} \neq X_\lambda$, then X_{red} is an orbit in $\mathcal{S}(\lambda)$ (since $\dim X = \dim X_\lambda$ thanks to $h_X = h'_\lambda$). The claim follows from the discussion in Section II.1. □

Remark II.3.3. Let $\mathcal{N}(\lambda)_{\mathrm{red}}$ denote the reduced null-cone, which is the subvariety of $V(\lambda)$ of those closed points $v \in V(\lambda)$ having 0 in the closure of their orbit Gv. It is not known whether one can replace $\mathcal{N}(\lambda)$ in Lemma II.3.2 by $\mathcal{N}(\lambda)_{\mathrm{red}}$.

Suppose that $\lambda \in \Lambda^+$ is an integral multiple of a Jansou-weight. In Section II.2 we have constructed a family $p\colon \mathcal{S}(\lambda)' \to \mathbf{A}^1$ giving an injective morphism $\mathbf{A}^1 \to \mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))_{\mathrm{red}}$. Its image is an irreducible component of $\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))_{\mathrm{red}}$, which we will denote by $\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))^0_{\mathrm{red}}$.

Corollary II.3.4. *Suppose that $\lambda \in \Lambda^+$ is an integral multiple of a Jansou-weight.*

a) *Then*
$$\mathrm{Hilb}^G_{h'_\lambda}(V(\lambda))_{\mathrm{red}} = \mathrm{Hilb}^G_{h'_\lambda}(\mathcal{S}(\lambda))^0_{\mathrm{red}} \cup \mathrm{Hilb}^G_{h'_\lambda}(\mathcal{N}(\lambda))_{\mathrm{red}},$$
and the two latter intersect in exactly one closed point.

b) *The natural morphism $\eta\colon Hilb^G_{h'_\lambda}(\mathcal{N}(\lambda)) \to \mathrm{Spec}(k)$ is proper. In particular, $\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{N}(\lambda))$ is projective.*

Proof. Statement a) follows from Lemma II.3.2. For b) note that the morphism η is the Nakamura-morphism
$$\eta\colon\ \mathrm{Hilb}^G_{h'_\lambda}(\mathcal{N}(\lambda)) \to \mathrm{Hilb}_{h'_\lambda(0)}(\mathcal{N}(\lambda)/\!\!/G) = \mathrm{Hilb}_1(\mathbf{A}^0) = \mathbf{A}^0,$$
which is proper (cf. [**AB05**] p. 92). (Here $\mathrm{Hilb}_n(X)$ is the punctual Hilbert scheme parametrizing the closed subschemes of X of length n.) Moreover, $\mathrm{Hilb}^G_{h'_\lambda}(\mathcal{N}(\lambda))$ is quasi-projective according to the construction of invariant Hilbert schemes, cf. [**HS04**]. The claim now follows. $\square$

CHAPTER III

Examples for SL_2

In this chapter, we focus on the group SL_2 and discuss some examples in detail.

III.1. Statement of the results

The group SL_2 of 2×2-matrices with determinant 1 is semi-simple of type A_1. We fix as maximal torus T the diagonal matrices in SL_2, and as Borel-subgroup B the upper triangular matrices in SL_2. After this choice we can identify the monoid of dominant weights Λ^+ with $\mathbf{N}$ and the weight lattice Λ with $\mathbf{Z}$, and the unipotent radical U equals the subgroup of upper triangular matrices with diagonal coefficients 1.

For each $d \in \mathbf{N}$ the simple SL_2-module $V(d)$ can be realized as the space of binary forms $k[x,y]_d$ of degree d, on which SL_2 acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^i y^{d-i} = (dx - by)^i(-cx + ay)^{d-i},$$

where we accept the ambiguity that d stands for both a matrix entry and the degree of the form. Note that $V(d)^U = k \cdot y^d$.

Recall that $\mathcal{S}(d)$ denotes the closure of the minimal sheet in $V(d)$, and that h'_d is the Hilbert function of the closure of a general orbit in $\mathcal{S}(d)$. The function h'_d is described in Section III.3. In this chapter we are interested in $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))$ and $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))$. Since the Jansou-weights of SL_2 are 2 and 4, the above Hilbert schemes equal $\mathbf{A}^0$ if d is odd, and only the cases with even d are of interest (cf. Lemma II.2.1 b). The starting point is Corollary II.3.4, which states in the case of SL_2 that

$$\text{(III.1.1)} \qquad \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))_{\mathrm{red}} = \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))^0_{\mathrm{red}} \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))_{\mathrm{red}},$$

and that these two subvarieties intersect in exactly one closed point.

In Section III.2 we settle some notation and study some (well-known) Classical Invariant Theory. In Section III.3 we focus on $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))$. In particular, Theorem III.3.2 states that:

Theorem 1. *Let $d \in \mathbf{N}$ be even. Then*

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d)) \cong \mathbf{A}^1.$$

Hence (III.1.1) simplifies to

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))_{\mathrm{red}} \cong \mathbf{A}^1 \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))_{\mathrm{red}}.$$

The group $\mathrm{GL}^G(V(d)) \cong \mathbf{G}_m$ acts in a natural way on $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))$. The description of this action can be found in [**AB05**], in [**Ja05**], or in Section IV.1. In particular, closed $\mathbf{G}_m$-fixed points on $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))$ correspond to homogeneous SL_2-stable subschemes of $V(d)$ with Hilbert function h'_d.

We shall concentrate on the cases where $d = 4, 8, 12$, and 16. Starting with the known case $d = 4$, one has

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_4}(V(4)) \cong \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_4}(\mathcal{S}(4)) \cong \mathbf{A}^1.$$

Carrying on with $d = 8$, one finds (Theorem III.6.5):

Theorem 2.

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_8}(V(8)) \cong \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_8}(\mathcal{S}(8)) \cong \mathbf{A}^1.$$

The closed $\mathbf{G}_m$*-fixed point of* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_8}(V(8))$ *is corresponds to a non-reduced subscheme of* $V(8)$ *with* $\overline{\mathrm{SL}_2 \cdot y^8}$ *as underlying variety.*

The situation is more involved for $d = 12$ (Theorem III.6.6):

Theorem 3.

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))_{\mathrm{red}} \cong \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(\mathcal{S}(12)) \cong \mathbf{A}^1.$$

If p *denotes the unique closed* $\mathbf{G}_m$*-fixed point of* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$ *(which corresponds to a non-reduced subscheme of* $\mathcal{S}(12) \subset V(12)$ *with* $\overline{\mathrm{SL}_2 \cdot y^{12}}$ *as underlying variety), then*

a) *the invariant Hilbert scheme* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$ *is smooth in the complement of* p*, and*
b) *the invariant Hilbert scheme* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$ *is not reduced in* p*.*

Finally, for $d = 16$ one has (Theorem III.6.11):

Theorem 4. a) *There are isomorphisms*

$$\begin{aligned}\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))_{\mathrm{red}} &\cong \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{S}(16)) \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{N}(16)_{\mathrm{red}})_{\mathrm{red}} \\ &\cong \mathbf{A}^1 \cup \mathbf{P}^1 \cup \mathbf{P}^1,\end{aligned}$$

and the three irreducible components intersect in one closed point p*.*

b) *The invariant Hilbert scheme* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))$ *is smooth in the complement of* p*.*
c) *The action of* $\mathbf{G}_m$ *on* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))$ *has three closed fixed points: The point* p*, and on each copy of* $\mathbf{P}^1$ *one further closed fixed point.*

III.2. Classical Invariant Theory

Before studying deformations of SL_2-orbits, we collect some well-known facts on the representation theory of SL_2 and on Classical Invariant Theory. This section does not contain any new results, but is important for the following calculations and is useful to settle some notation. The contents of this section can be found in [**Cl1872**], or in a contemporary language in [**KW99**] or in [**Ol99**]. This section closely follows [**Ol99**].

Invariants and Covariants. A *covariant of $V(d)$ of order k* is an SL_2-equivariant morphism $V(d) \to V(k)$. A covariant of order 0 is an *invariant*. We identify $\mathrm{Sym}(V(d)^*)$ with $k[a_0, a_1, \dots, a_d]$, where

$$a_i(\sum_{j=0}^{d} \binom{d}{j} \lambda_j x^j y^{d-j}) := \lambda_i. \tag{III.2.1}$$

The group SL_2 acts on $\mathrm{Sym}(V(d)^*)$ via the contragredient action:

$$g \cdot a_i(\sum_{j=0}^{d} \binom{d}{j} \lambda_j x^j y^{d-j}) = a_i(g^{-1} \cdot (\sum_{j=0}^{d} \binom{d}{j} \lambda_j x^j y^{d-j})).$$

Consider a covariant $\varphi\colon V(d) \to V(k)$. Then φ can be written as $\varphi = \sum_{i=0}^{k} \varphi_i(a_0, \dots, a_d) x^i y^{k-i}$ with $\varphi_i \in k[a_0, \dots, a_d]$. The SL_2-equivariance implies that

$$\begin{aligned}\sum_i g^{-1}\varphi_i(f) x^i y^{k-i} = \sum_i \varphi_i(gf) x^i y^{k-i} &= \varphi(gf) \\ &= g\varphi(f) = \sum_i \varphi_i(f) g x^i y^{k-i}.\end{aligned}$$

Taking $g = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$, we see that $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \varphi_i = t^{2i-k} \varphi_i$. It is now easy to see that $V := \mathrm{span}_k(\varphi_0, \dots, \varphi_k) \subset \mathrm{Sym}(V(d)^*)$ is a simple SL_2-submodule with $V^U = k \cdot \varphi_k$, and that $\sum_i \varphi_i x^i y^{k-i}$ is an SL_2-invariant expression. Summarizing, to give a covariant $\varphi\colon V(d) \to V(k)$ up to a non-zero scalar multiple is the same as to give a simple submodule $V(k) \subset \mathrm{Sym}(V(d)^*)$, which in turn is the same as to give an SL_2-invariant expression

$$\varphi_0(a_0, \dots, a_d) y^k + \varphi_1(a_0, \dots, a_d) x y^{k-1} + \dots + \varphi_k(a_0, \dots, a_d) x^k, \tag{III.2.2}$$

where each $\varphi_i \in \mathrm{Sym}(V(d)^*) = k[a_0, \dots, a_d]$. The expression (III.2.2) is unique up to a non-zero scalar multiple.

Remark III.2.1. If $\sum_{i=0}^{k} \varphi_i(a_0, \dots, a_d) x^i y^{k-i}$ is a covariant of $V(d)$ of order k, then $\varphi_k(a_0, \dots, a_d)$ is U-invariant.

A covariant $\varphi = \sum_{i=0}^{k} \varphi_i(a_0, \dots, a_d) x^i y^{k-i}$ is called *homogeneous of degree n* if each φ_i is homogeneous of degree n.

Definition III.2.2. Every representation of SL_2 is self-dual. Thus, $\mathrm{Sym}^1(V(d)^*) = V(d)^*$ is a covariant of $V(d)$ of degree 1 and of order d. This covariant can also be written as $\sum_{i=0}^{d} a_i \binom{d}{i} x^i y^{d-i}$, and we denote it by $Q^{(d)}$.

Definition III.2.3. For a homogeneous covariant of $V(d)$ of degree n and of order k we define its *co-order* to be $(nd-k)/2$.

Example III.2.4. [[**Ol99**], p. 26] A cubic binary form possesses an invariant of degree 4 called *discriminant*:

$$\Delta = a_0^2 a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 - 3a_1^2a_2^2 + 4a_1^3a_3,$$

and a covariant of degree 2, of order 2, and of co-order 2 called *Hessian*:

$$H = (a_1a_3 - a_2^2)x^2 + (a_0a_3 - a_1a_2)xy + (a_0a_2 - a_1^2)y^2.$$

Example III.2.5. More generally, we can define the *Hessian* of a binary form of degree d by derivating $Q^{(d)}$ as follows:

$$H = \frac{(d-2)!^2}{d!^2}[Q_{xx}^{(d)}Q_{yy}^{(d)} - (Q_{xy}^{(d)})^2].$$

We shall see in Example III.2.6 that the Hessian is a covariant. Its order equals $2d-4$, and its co-order equals 2. It has the following significance: The Hessian of a binary form vanishes if and only if the form is the power of a linear form $(\alpha x + \beta y)$.

Similarly, for a binary form of degree d there is an invariant Δ called its *discriminant* with the property that it vanishes if and only if the form has a multiple root.

Transvections and transvectants. Given two covariants V_1 and V_2 of $V(d)$ of respective orders k_1 and k_2, we consider the SL_2-submodule $V_1 \cdot V_2 \subset \mathrm{Sym}(V(d)^*)$. Since $V(k_1)\otimes V(k_2) \cong V(k_1+k_2) \oplus V(k_1+k_2-2) \oplus \ldots \oplus V(|k_1-k_2|)$, we see that

$$V_1 \cdot V_2 = W_{k_1+k_2} \oplus W_{k_1+k_2-2} \oplus \ldots \oplus W_{|k_1-k_2|} \subset \mathrm{Sym}(V(d)^*)$$

with W_i either 0 or isomorphic to $V(i)$. Now $W_{k_1+k_2-2r}$ is called the *r-th transvectant of V_1 and V_2*, and the projection of $V_1 \cdot V_2$ onto $W_{k_1+k_2-2r}$ is called the *r-th transvection of V_1 and V_2*. We write $(V_1, V_2)^r$ for $W_{k_1+k_2-2r}$. If both V_1 and V_2 are homogeneous of degree n_1 and n_2, of order k_1 and k_2, and of co-order l_1 and l_2, respectively, then $(V_1, V_2)^r$ is homogeneous of degree n_1+n_2, of order k_1+k_2-2r, and of co-order l_1+l_2+r.

If the covariants V_1 and V_2 are written in the form (III.2.2), then their transvectants can be computed by means of the following formula (cf. [**Ol99**], p. 88):

$$(V_1, V_2)^r = \frac{(d-r)!^2}{d!^2}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{\partial^r V_1}{\partial x^{r-s}\partial y^s}\frac{\partial^r V_2}{\partial x^s \partial y^{r-s}}. \qquad \text{(III.2.3)}$$

The factor $\frac{(d-r)!^2}{d!^2}$ turns out to be useful for computations and appears already in the ancient literature.

Example III.2.6. The Hessian of a binary form now can be expressed as

$$H = \frac{1}{2}(Q^{(d)}, Q^{(d)})^2.$$

For a binary quadratic form ($Q^{(2)} = a_0y^2 + 2a_1xy + a_2x^2$) this yields

$$H = \frac{1}{2}(Q^{(2)}, Q^{(2)})^2 = \frac{1}{4}\left[\frac{\partial^2 Q^{(2)}}{\partial x^2}\frac{\partial^2 Q^{(2)}}{\partial y^2} - \left(\frac{\partial^2 Q^{(2)}}{\partial x \partial y}\right)^2\right] = a_0a_2 - a_1^2.$$

Since a quadratic binary form has a multiple root if and only if it is the power of a linear form, we see that its Hessian H equals its discriminant Δ (up to some non-zero factor).

Example III.2.7. We introduce two fundamental invariants for binary forms, which will be used later: For a form of degree d, we define the *apolar* to be the invariant $i^{(d)} = (Q^{(d)}, Q^{(d)})^d$ of degree 2, and we define $j^{(d)}$ to be the invariant $((Q^{(d)}, Q^{(d)})^{d/2}, Q^{(d)})^d$ of degree 3. We shall see later that $i^{(d)}$ is non-zero if and only if d is even, and that $j^{(d)}$ is non-zero if and only if d is a multiple of 4.

If e.g. $d = 4$, then we find

$$\begin{aligned} i^{(4)} =& 2(3a_2^2 - 4a_1a_3 + a_0a_4), \text{ and} \\ j^{(4)} =& 6(-a_2^3 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 + a_0a_2a_4) = 6\det\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}. \end{aligned}$$

The invariant $j^{(4)}$ of a quartic binary form is called its *Hankelsche Determinante*.

Covariants and the symbolic method. The *symbolic method* provides an elegant and effective tool to compute transvectants. The primary goal is to assign to each covariant a *symbolic expression*. Given a monomial $\prod_j a_{i_j} \in k[a_0, \ldots, a_d]_n$ of degree n, we obtain its symbolic expression by replacing each a_{i_j} by a so-called *symbolic variable* $\xi_j^{i_j}\eta_j^{d-i_j}$. This transition rule is now extended linearly to $k[a_0, \ldots, a_d]_n$. To avoid confusion one usually indexes the ξ_j and η_j with greek letters rather than with integers, so we replace ξ_1 by ξ_α, η_1 by η_α, ξ_2 by ξ_β, etc. To give an example, start with a quadratic binary form and consider $a_0a_2 - a_1^2$. One replaces a_0a_2 by $\eta_\alpha^2\xi_\beta^2$ and a_1^2 by $\xi_\alpha\eta_\alpha\xi_\beta\eta_\beta$. We denote the resulting polynomial by $\tilde{\varphi}$. Given a covariant of $V(d)$ of degree n and of order k, written as

$$\text{(III.2.4)}\quad \varphi_0(a_0, \ldots, a_d)y^k + \varphi_1(a_0, \ldots, a_d)xy^{k-1} + \ldots + \varphi_k(a_0, \ldots, a_d)x^k,$$

first replace each φ_i by the corresponding expression $\tilde{\varphi}_i$ according to the above rules. One obtains an expression

$$\tilde{\varphi}_0 y^k + \tilde{\varphi}_1 x y^{k-1} + \ldots + \tilde{\varphi}_k x^k,$$

which however depends on the arrangement of the a_i in (III.2.4): Consider once more $\Delta = a_0 a_2 - a_1^2$. Its symbolic expression is

$$\eta_\alpha^2 \xi_\beta^2 - \xi_\alpha \eta_\alpha \xi_\beta \eta_\beta. \tag{III.2.5}$$

On the other hand, we can clearly write $\Delta = a_2 a_0 - a_1^2$, which yields the symbolic expression

$$\xi_\alpha^2 \eta_\beta^2 - \xi_\alpha \eta_\alpha \xi_\beta \eta_\beta$$

different from (III.2.5). In order to bypass this problem, one symmetrizes the symbolic expression and considers $\Delta = \frac{1}{2}((a_0 a_2 - a_1^2) + (a_2 a_0 - a_1^2))$ with the symbolic expression

$$\frac{1}{2}(\eta_\alpha^2 \xi_\beta^2 - \xi_\alpha \eta_\alpha \xi_\beta \eta_\beta) + \frac{1}{2}(\xi_\alpha^2 \eta_\beta^2 - \xi_\alpha \eta_\alpha \xi_\beta \eta_\beta) = \frac{1}{2}(\xi_\alpha \eta_\beta - \eta_\alpha \xi_\beta)^2.$$

A symbolic expression is called *symmetric* if it is invariant under any permutation of the indices $\alpha, \beta, \ldots$. One then finds:

Lemma III.2.8 (Theorem 6.5. in [**Ol99**])**.** *Every covariant of $V(d)$ possesses a unique symmetric symbolic expression.*

Note that not every symmetric symbolic expression comes from a covariant. If we are given, however, a symmetric symbolic expression coming from a covariant, we can easily reconstruct the covariant.

Definition III.2.9 (Definitions 6.10 and 6.12 in [**Ol99**])**.** (1) A *bracket factor of the first kind* is a symbolic expression

$$(\alpha\, \boldsymbol{x}) = \xi_\alpha x + \eta_\alpha y.$$

(2) A *bracket factor of the second kind* is a symbolic expression

$$[\alpha\, \beta] = \xi_\alpha \eta_\beta - \eta_\alpha \xi_\beta.$$

(3) A *bracket polynomial* is a symbolic expression that can be written as a polynomial in the bracket factors of the first and second kinds.

Example III.2.10. Consider once more the invariant $\Delta = a_0 a_2 - a_1^2$ of a quadratic binary form. Its symbolic expression $\frac{1}{2}(\xi_\alpha \eta_\beta - \eta_\alpha \xi_\beta)^2$ can now be written as bracket polynomial $\frac{1}{2}[\alpha\, \beta]^2$.

One then has the following fundamental result:

Theorem III.2.11 (First Fundamental Theorem, Theorem 6.14 in [**Ol99**])**.** *Every symmetric symbolic expression coming from a covariant can be written as a bracket polynomial. Vice versa, every bracket polynomial is the symmetric symbolic expression of a covariant.*

Even though the symmetric symbolic expression of a covariant is unique, it is not clear whether the same holds for bracket polynomials. Indeed, there are relations between the bracket factors, like e.g. $[\alpha\,\beta] = -[\beta\,\alpha]$. One has the following relations called *syzygies*:

$$
\begin{aligned}
&[\alpha\,\beta] = -[\beta\,\alpha], \\
&[\alpha\,\beta](\gamma\,\boldsymbol{x}) + [\gamma\,\alpha](\beta\,\boldsymbol{x}) + [\beta\,\gamma](\alpha\,\boldsymbol{x}) = 0, \\
&[\alpha\,\beta][\gamma\,\delta] + [\gamma\,\alpha][\beta\,\delta] + [\beta\,\gamma][\alpha\,\delta] = 0.
\end{aligned}
\tag{III.2.6}
$$

The Second Fundamental Theorem now states that these are all identities:

Theorem III.2.12 (Second Fundamental Theorem, Theorem 6.19 in [**Ol99**]). *Every polynomial identity among the different bracket factors is obtained as a linear combination of the above syzygies.*

Remark III.2.13. Consider a homogeneous covariant of $V(d)$ of degree n, of order k, and of co-order l. Its bracket polynomial is a sum $\sum_i M_i$ of monomials M_i sharing the following properties: Each monomial M_i contains n symbolic letters that all occur exactly d times in M_i. Moreover, each M_i consists of k brackets of the first kind and of l brackets of the second kind.

Transvectants of bracket polynomials. The discussion below is motivated by this example:

Example III.2.14. Given a quadratic binary form, consider both its discriminant $\Delta = a_0a_2 - a_1^2$ and the square of the discriminant $\Delta^2 = a_0^2a_2^2 - 2a_0a_1^2a_2 + a_1^4$. We already found that $\frac{1}{2}[\alpha\,\beta]^2$ is the bracket polynomial of Δ. What about Δ^2? We first calculate the symbolic expression belonging to Δ^2 and find:

$$\eta_\alpha^2\eta_\beta^2\xi_\gamma^2\xi_\delta^2 - 2\eta_\alpha^2\xi_\beta\eta_\beta\xi_\gamma\eta_\gamma\xi_\delta^2 + o\xi_\alpha\eta_\alpha\xi_\beta\eta_\beta\xi_\gamma\eta_\gamma\xi_\delta\eta_\delta.$$

After symmetrizing the expression one finds

$$\frac{1}{4}(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2(\xi_\gamma\eta_\delta - \eta_\gamma\xi_\delta)^2,$$

which can be written as bracket polynomial

$$\frac{1}{4}[\alpha\,\beta]^2[\gamma\,\delta]^2.$$

This is clearly not the same as $\frac{1}{4}[\alpha\,\beta]^4$.

This example shows the need for a adequate definition of the multiplication of two bracket polynomials. Let φ_1 and φ_2 be two covariants of a form of degree d with respective bracket polynomials

$$\prod_{\substack{j,k\\ \text{finite}}}[\alpha_{i_j}\,\alpha_{i_k}]\prod_{l=1}^{L}(\alpha_{i_l}\,\boldsymbol{x}) \quad \text{and} \quad \prod_{\substack{u,v\\ \text{finite}}}[\beta_{i_u}\,\beta_{i_v}]\prod_{w=1}^{W}(\beta i_w\,\boldsymbol{x})$$

in symbolic letters α_i and β_i in *two different alphabets*. Then the bracket polynomial belonging to the covariant $(\varphi_1, \varphi_2)^r$ equals

$$\frac{(d-r)!^2}{d!^2} \sum_{\substack{\sigma \in \mathrm{Perm}\{1,\dots,L\} \\ \tau \in \mathrm{Perm}\{1,\dots,W\}}} \Bigg(\prod_{j,k} [\alpha_{i_j}\, \alpha_{i_k}] \prod_{u,v} [\beta_{i_u}\, \beta_{i_v}] \prod_{l=1}^{r} [\alpha_{i_{\sigma(l)}}\, \beta_{i_{\tau(l)}}] \dots$$

$$\dots \prod_{l=r+1}^{L} (\alpha_{i_{\sigma(l)}}\, \boldsymbol{x}) \prod_{w=r+1}^{W} (\beta_{i_{\tau(w)}}\, \boldsymbol{x}) \Bigg).$$

Example III.2.15. Let $\varphi_1 = \varphi_2 = \frac{1}{2}[\alpha\, \beta]$. Then

$$(\varphi_1, \varphi_2)^0 = \frac{1}{4}[\alpha^1\, \beta^1][\alpha^2\, \beta^2] \quad \text{or} \ = \frac{1}{4}[\alpha\, \beta][\gamma\, \delta],$$

and $(\varphi_1, \varphi_2)^r = 0$ for $r > 0$.

The following example shows that it is much easier to compute transvectants of bracket polynomials than transvectants of the covariants.

Example III.2.16. Consider a binary form of degree 3. Its Hessian has been computed in Example III.2.4 to be

$$H = (a_1a_3 - a_2^2)x^2 + (a_0a_3 - a_1a_2)xy + (a_0a_2 - a_1^2)y^2.$$

Consider $(a_1a_3 - a_2^2)$. We replace a_1 by $\xi_\alpha\eta_\alpha^2$, a_3 by ξ_β^3, and a_2^2 by $\xi_\alpha^2\eta_\alpha\xi_\beta^2\eta_\beta$. After symmetrizing we find its symbolic expression

$$\frac{1}{2}(\xi_\alpha^3\xi_\beta\eta_\beta^2 - 2\xi_\alpha^2\eta_\alpha\xi_\beta^2\eta_\beta + \xi_\alpha\eta_\alpha^2\xi_\beta^3) = \frac{1}{2}\xi_\alpha\xi_\beta(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2.$$

Similarly, we replace $(a_0a_3 - a_1a_2)$ by

$$\frac{1}{2}(\eta_\alpha^3\xi_\beta^3 - \xi_\alpha\eta_\alpha^2\xi_\beta^2\eta_\beta - \xi_\alpha^2\eta_\alpha\xi_\beta\eta_\beta^2 + \xi_\alpha\eta_\beta^3) = \frac{1}{2}(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2(\xi_\alpha\eta_\beta + \eta_\alpha\xi_\beta),$$

and finally $(a_0a_2 - a_1^2)$ by

$$\frac{1}{2}(\eta_\alpha^3\xi_\beta^2\eta_\beta - 2\xi_\alpha\eta_\alpha^2\xi_\beta\eta_\beta^2 + \xi_\alpha^2\eta_\alpha\eta_\beta^3) = \frac{1}{2}\eta_\alpha\eta_\beta(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2.$$

Hence the symmetric symbolic expression corresponding to H is

$$\frac{1}{2}(\xi_\alpha\xi_\beta(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2x^2$$
$$+ (\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2(\xi_\alpha\eta_\beta + \eta_\alpha\xi_\beta)xy + \eta_\alpha\eta_\beta(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2y^2).$$

Bearing in mind that $[\alpha\,\beta] = \xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta$ and that $(\alpha\,\boldsymbol{x}) = \xi_\alpha x + \eta_\alpha y$ we find

$$\begin{aligned}
&\frac{1}{2}(\xi_\alpha\xi_\beta(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2x^2+\\
&\quad(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2(\xi_\alpha\eta_\beta + \eta_\alpha\xi_\beta)xy + \eta_\alpha\eta_\beta(\xi_\alpha\eta_\beta - \eta_\alpha\xi_\beta)^2y^2)\\
=&\frac{1}{2}[\alpha\,\beta]^2(\xi_\alpha\xi_\beta x^2 + (\xi_\alpha\eta_\beta + \eta_\alpha\xi_\beta)xy + \eta_\alpha\eta_\beta y^2)\\
=&\frac{1}{2}[\alpha\,\beta]^2(\alpha\,\boldsymbol{x})(\beta\,\boldsymbol{x})
\end{aligned}$$

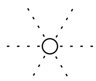
We see that each symbolic letter occurs exactly trice (we started with a binary form of degree 3). There occur two symbolic letters. This corresponds to the fact that the Hessian is a covariant of degree 2. There occur two brackets of the first kind and two brackets of the second kind. This corresponds to the fact that the Hessian is a covariant of order 2 and of co-order 2.

On the other hand, $H = \frac{1}{2}(Q^{(3)}, Q^{(3)})^2$. Computing the transvectant $\frac{1}{2}((\alpha\,\boldsymbol{x})^3, (\alpha\,\boldsymbol{x})^3)^2$ of the bracket polynomials $(\alpha\,\boldsymbol{x})^3$ corresponding to $Q^{(3)}$ yields:

$$\begin{aligned}
\frac{1}{2}((\alpha\,\boldsymbol{x})^3, (\alpha\,\boldsymbol{x})^3)^2 &= \frac{1}{2}\cdot\frac{1}{36}\sum_{\substack{\sigma\in\mathrm{Perm}\{1,2,3\}\\ \tau\in\mathrm{Perm}\{1,2,3\}}}\prod_{l=1}^{2}[\alpha^{(1)}\,\alpha^{(2)}]\prod_{l=3}^{3}(\alpha^{(1)}\,\boldsymbol{x})(\alpha^{(2)}\,\boldsymbol{x})\\
&= \frac{1}{2}\cdot[\alpha^{(1)}\,\alpha^{(2)}]^2(\alpha^{(1)}\,\boldsymbol{x})(\alpha^{(2)}\,\boldsymbol{x}),
\end{aligned}$$

which equals the bracket polynomial of the Hessian (written in $\alpha^{(1)}, \alpha^{(2)}$ instead of α, β).

The algebra of digraphs. We closely follow the treatment in [**OS89**]. Bracket polynomials can be visualized in a very nice manner with directed graphs, or short *digraphs*. An *atom of valence d* is a vertex $\circ$ with d (unlabelled) bond sites, drawn like this:


.

A *d-digraph* (or by abuse of notation simply *digraph*) is a finite set of atoms of valence d, together with a set of directed edges or *arrows* between the bond sites of the atoms. One bond site can be the source or target of at most one arrow. A bond site is called *free* if it is neither the source nor the target of any arrow.

To a bracket monomial P of the form

$$P = \prod_{i,j=1}^{s}[\alpha_i\,\alpha_j]^{\mu_{ij}}\prod_{k=1}^{s}(\alpha_k\,\boldsymbol{x})^{\nu_k}$$

we assign the following d-digraph: For each symbolic letter α_i occurring in P draw one atom v_i of valence d. Draw μ_{ij} arrows from v_i to v_j

(obeying the rule that each bond site can be used only for one arrow). In the end, the atom v_k has ν_k free bond sites. Since the degree d is fixed the number of free bond sites of each atom can always be obtained by subtracting the number of arrows having the atom as source or target from d. This is why the free bond sites are often omitted in the drawings.

Example III.2.17. Consider a cubic binary form. The bracket polynomial corresponding to twice its Hessian is $[\alpha\,\beta]^2(\alpha\,\boldsymbol{x})(\beta\,\boldsymbol{x})$. The 3-digraph associated to this bracket monomial is

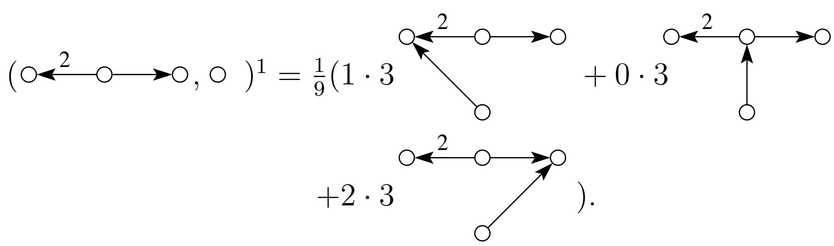
$$\circ \Longrightarrow \circ \;=\; \circ \xrightarrow{2} \circ\,.$$

(If there are $l > 1$ arrows connecting two fixed vertices, we usually only draw one arrow, which we superscribe with l.)

Formal linear combinations of d-digraphs are called d-digraphs, too. The assignment (bracket monomials) $\rightsquigarrow$ (d-digraphs) can now be extended linearly to an assignment (bracket polynomials) $\rightsquigarrow$ ((formal linear combinations of) d-digraphs). It is obvious that one can reconstruct the bracket polynomial from its d-digraph.

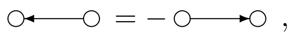
Transvectants can easily be computed with digraphs. Suppose that the r-th transvectant of two d-digraphs D_1 and D_2 is searched. For each atom of each digraph label its ν free bond sites with the integers $1, 2, \dots, \nu$. Now draw r arrows with a free bond site in D_2 as source and a free bond site in D_1 as target and label these arrows with the integers $1, 2, \dots, r$. Generally, there are plenty of possibilities to form a new digraph as described. Now form the sum over all different (partially labelled) digraphs obtained in the above way, and finally forget all the labels. The resulting sum multiplied with $(d-r)!^2/d!^2$ is now the digraph belonging to the transvectant $(D_1, D_2)^r$.

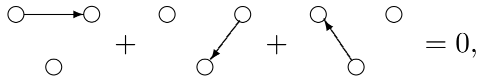
Example III.2.18. We sketch an example of the first transvectant of two forms of degree 3:

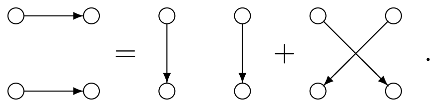


The multplicities have the following meaning: The first summand is multiplied by $1 \cdot 3$ since the upper left atom has one free bond side, whereas the lower atom has three free bond sides. The second summand is multiplied with zero, since the upper middle atom has valence three and therefore cannot be source and target of four atoms. Similarly, the third summand is multiplied with $2 \cdot 3$ since the upper right atom has two free bond sites, whereas the lower atom has three free bond sites.

The syzygies in (III.2.6) can be translated into the following relations:

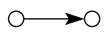
(I) 

(II) 

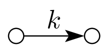
(III) 

This is to be understood as follows. Consider the first relation. If we are given two digraphs D_1 and D_2 that are equal except that in D_1 one particular arrow points in the reverse direction than the corresponding arrow in D_2, then D_1 can be identified with $-D_2$. The other relations are to be applied in the same way.

Remark III.2.19. From relation (I) it follows that

 $= 0,$

and more generally that

 $= 0$

for all odd k. This relation will be used frequently in the sequel.

The results obtained so far can be summarized in the following form: The algebra $\mathcal{D}^d$ of (formal linear combinations) of d-digraphs (endowed with the multiplication given by the zeroth transvection) modulo the ideal generated by the above relations is canonically isomorphic to the algebra of U-invariants of a binary form of degree d, and the U-invariants can in turn be identified with the covariants of the form.

III.3. Deformations of orbits in minimal sheets, Part I

Let $d \in \mathbf{N}$ be a dominant weight of SL_2. Recall that $\mathcal{S}(d)$ denotes the closure of the minimal sheet in $V(d)$, and that $h'_d \colon \mathbf{N} \to \mathbf{N}$ is the Hilbert function of the closure of a general orbit in $\mathcal{S}(d)$. Using Proposition II.2.9, we show in this section that $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d)) = \mathbf{A}^1$ for all even d.

Description of the minimal sheet in the space of binary forms of degree d. We start by collecting a few well-known facts (see e.g. [**Kr85**]). A one-dimensional subgroup of SL_2 is either conjugated to T, to the normalizer $N = N_{\mathrm{SL}_2}(T)$ of T in SL_2, or to U_n for some $n \in \mathbf{N}$, where $U_n = U \rtimes C_n$ is the semidirect product of U with the cyclic group

$C_n = \{\operatorname{diag}(\zeta, \zeta^{-1}) \mid \zeta^n = 1\}$. A two-dimensional subgroup of SL_2 is a Borel-subgroup and hence conjugated to B.

If X is an affine SL_2-variety and if $x \in X$ is B-stable, then x is fixed under the SL_2-action. This implies that an SL_2-variety contains no one-dimensional orbits. The stabilizer of $y^d \in V(d)$ equals U_d, and hence $\dim \mathrm{SL}_2 \cdot y^d = 2$. Therefore, the minimal sheet $S \subset V(d)$ consists of the two-dimensional orbits in $V(d)$. Recall that the Jansou-weights of SL_2 are 2 and 4. Bearing this in mind, the discussion in Section II.2 yields the following picture:

a) Suppose that d is odd. Then $\mathcal{S}(d) = X_d = \overline{\mathrm{SL}_2 \cdot y^d}$, and

$$h'_d = h_{X_d} \colon \mathbf{N} \to \mathbf{N}, \quad k \mapsto \begin{cases} 1 & \text{if } d|k \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d)) = \mathbf{A}^0$.

b) If d is even, then the minimal sheet consists of $\mathrm{SL}_2 \cdot y^d$ and the orbits $\mathrm{SL}_2 \cdot \gamma x^{d/2} y^{d/2}$ for $0 \neq \gamma \in k$. The quotient $\mathcal{S}(d)/\!\!/\mathrm{SL}_2$ is one-dimensional.

i) If $d \equiv 0 \mod 4$, the stabilizer of $x^{d/2}y^{d/2}$ equals $N_{\mathrm{SL}_2}(T)$. In this case

$$h'_d = h_{\mathrm{SL}_2 / N_{\mathrm{SL}_2}(T)} \colon \mathbf{N} \to \mathbf{N}, \quad k \mapsto \begin{cases} 1 & \text{if } 4|k \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

ii) If $d \equiv 2 \mod 4$, the stabilizer of $x^{d/2}y^{d/2}$ equals T. In this case

$$h'_d = h_{\mathrm{SL}_2 / T} \colon \mathbf{N} \to \mathbf{N}, \quad k \mapsto \begin{cases} 1 & \text{if } 2|k \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Hilbert function of the general fiber of $\mathcal{S}(d) \to \mathcal{S}(d)/\!\!/\mathrm{SL}_2$ coincides with the Hilbert function of X_d if and only if d is odd or if $d = 2$ or if $d = 4$, which are the Jansou-weights of SL_2.

Recall that we have introduced the invariants $i^{(d)} = (Q^{(d)}, Q^{(d)})^d$ and $j^{(d)} = ((Q^{(d)}, Q^{(d)})^{d/2}, Q^{(d)})^d$ of degree 2 and 3 in Example III.2.7, where $Q^{(d)}$ is the covariant $\sum_{i=0}^{d} a_i \binom{d}{i} x^i y^{d-i}$.

Proposition III.3.1.

$$\mathcal{S}(d)/\!\!/\mathrm{SL}_2 = \begin{cases} \operatorname{Spec}(k[i^{(d)}]) \cong \mathbf{A}^1 & \text{\textit{if} } d \equiv 2 \mod 4, \\ \operatorname{Spec}(k[i^{(d)}, j^{(d)}]) \cong \operatorname{Spec}(k[t^2, t^3]) & \text{\textit{if} } d \equiv 0 \mod 4. \end{cases}$$

The proof of this proposition will be stated later in this section. The proposition shows that the quotient morphism $\mathcal{S}(d) \to \mathcal{S}(d)/\!\!/\mathrm{SL}_2$ is flat if $d \equiv 2 \mod 4$, and hence Theorem I.1.1 implies that $\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d)) \cong \mathbf{A}^1$ if $d \equiv 2 \mod 4$. Even though the quotient morphism $\mathcal{S}(d) \to \mathcal{S}(d)/\!\!/\mathrm{SL}_2$ is not flat for $d \equiv 0 \mod 4$ (as will be shown in Remark III.3.8), the corresponding invariant Hilbert scheme still equals $\mathbf{A}^1$:

Theorem III.3.2. *If d is even, then* $\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d)) \cong \mathbf{A}^1$.

The proof follows later in this section.

Definition III.3.3. Let X_d^0 be the schematic zero-fiber of the morphism $\mathrm{Univ}_{h'_d}^{\mathrm{SL}_2}(\mathcal{S}(d)) \to \mathrm{Hilb}_{h'_d}^{\mathrm{SL}_2}(\mathcal{S}(d)) \cong \mathbf{A}^1$. Theorem III.3.2 implies that X_d^0 is the unique closed subscheme $X_d^0 \subset \mathcal{S}(d)$ with Hilbert function $h_{X_d^0} = h'_d$ and with $(X_d^0)_{\mathrm{red}} = X_d$. Equivalently, X_d^0 is the unique homogeneous SL_2-stable subscheme of $\mathcal{S}(d)$ with Hilbert function $h_{X_d^0} = h'_d$.

Remark III.3.4. Let $I^0 \subset \mathrm{Sym}(V(d^*))$ be the SL_2-stable ideal defining X_d^0 as closed subscheme of $V(d)$, and let $I'_0 \subset \mathcal{O}(\mathcal{S}(d))$ be the SL_2-stable ideal defining X_d^0 as closed subscheme of $\mathcal{S}(d)$. Then I'_0 coincides with I'_0 from Section II.2, and I^0 is the inverse image of I'_0 under the projection $\mathrm{Sym}(V(d)^*) \to \mathcal{O}(\mathcal{S}(d))$.

Non-vanishing results for covariants. From Remark III.2.19 we know that $(Q^{(d)}, Q^{(d)})^l = 0$ if l is odd. However, we claim that:

Lemma III.3.5. a) *Let d be even, and let $l \in \{0, 2, \ldots, d\}$. Then the covariant $(Q^{(d)}, Q^{(d)})^l$ is not in the ideal $I(\mathcal{S}(d))$ of $\mathcal{S}(d)$.*
b) *If $4|d$, then $j^{(d)} = ((Q^{(d)}, Q^{(d)})^{d/2}), Q^{(d)})^d \notin I(\mathcal{S}(d))$.*

Remark III.3.6. Because $\mathrm{Sym}^n(V(d)^*)$ is multiplicity-free as SL_2-module for $n \leq 2$, it follows from Lemma III.3.5 a) that the ideal $I(\mathcal{S}(d))$ is generated by covariants of degree ≥ 3.

Remark III.3.7. Lemma III.3.5 a) and the proof of Proposition II.2.9 show that $\mathcal{O}(\mathcal{S}(d))_{(n,k)} \neq 0$ for all $n \geq 2$ and for all $k \leq nd$ with $4|k$.

Using the notation of Remark III.3.4, the statement of Lemma III.3.5 a) and the proof of Proposition II.2.9 show in addition that $I'_0 \subset \mathcal{O}(\mathcal{S}(d))$ is the smallest SL_2-stable ideal containing $\mathcal{O}(\mathcal{S}(d))_{(2,d)}$ and $\mathcal{O}(\mathcal{S}(d))_{(2,0)}$.

Proof of Lemma III.3.5. Fix d, and let $Q = Q^{(d)}$.

a) Since $\binom{d}{d/2} x^{d/2} y^{d/2} \in \mathcal{S}(d)$ it suffices to verify that the covariants $(Q, Q)^l$ don't vanish identically on $\binom{d}{d/2} x^{d/2} y^{d/2}$. We compute the transvectant $(Q, Q)^l$:

(III.3.1)

$$(Q,Q)^l = \frac{(d-l)!^2}{d!^2} \sum_{i=0}^{l} (-1)^i \binom{l}{i} \frac{\partial^l Q}{\partial x^{l-i} \partial y^i} \frac{\partial^l Q}{\partial x^i \partial y^{l-i}}$$

$$= \frac{(d-l)!^2}{d!^2} \sum_{i=0}^{l} \sum_{j,j'=0}^{d} (-1)^i \binom{l}{i} \binom{d}{j} \binom{d}{j'} a_j a_{j'} \frac{\partial^l x^j y^{d-j}}{\partial x^{l-i} \partial y^i} \frac{\partial^l x^{j'} y^{d-j'}}{\partial x^i \partial y^{l-i}}.$$

This can be written in the form of (III.2.2) in Section III.2:

$$(Q,Q)^l = \sum_{e=0}^{2d-2l} \varphi_e^l(a_0, \ldots, a_d) x^e y^{2d-2l-e}, \qquad \text{(III.3.2)}$$

where

$$\varphi_e^l(a_0,\dots,a_d) = \sum_{j+j'-l=e} \gamma_{j,j'}^l a_j a_j'$$

for suitable $\gamma_{j,j'}^l \in k$. Now

$$\varphi_{d-l}^l(a_0,\dots,a_d)(\binom{d}{d/2}x^{d/2}y^{d/2}) = \gamma_{d/2,d/2}^l,$$

and this is non-zero if and only if $\gamma_{d/2,d/2}^l \neq 0$. We compute $\gamma_{d/2,d/2}^l$ by means of (III.3.1):

$$\begin{aligned}
\gamma_{d/2,d/2}^l x^{d-l}y^{d-l} &= \frac{(d-l)!^2}{d!^2}\sum_{i=0}^l (-1)^i \binom{l}{i}\binom{d}{d/2}^2 \frac{\partial^l x^{d/2}y^{d/2}}{\partial x^{l-i}\partial y^i}\frac{\partial^l x^{d/2}y^{d/2}}{\partial x^i \partial y^{l-i}} \\
&= \sum_{i=0}^l (-1)^i \binom{l}{i}\binom{d-l}{d/2-i}^2 x^{d-l}y^{d-l} \\
&= \frac{2^d\Gamma(1/2+d/2-l/2)\Gamma(1+d-l/2)}{\Gamma(1+d/2)^2\Gamma(1/2-l/2)\Gamma(1+d/2-l/2)} x^{d-l}y^{d-l},
\end{aligned}$$

where Γ is the Gamma function, and where the last expression was computed with `Mathematica`. We can read off that $\gamma_{d/2,d/2}^l \neq 0$ for all even d and for all even $l \in \{0,2,\dots,d\}$. Thus, $\varphi_{d-l}^l \notin I(\mathcal{S}(d))$, and hence $(Q,Q)^l \not\subset I(\mathcal{S}(d))$.

b) The invariant $j^{(d)} = ((Q,Q)^{d/2},Q)^d$ can be written as

$$j^{(d)} = ((Q,Q)^{d/2},Q)^d = \sum_{\substack{j+k+l=3d/2 \\ j\geq k\geq l}} \delta_{jkl} a_j a_k a_l$$

for suitable $\delta_{jkl} \in k$. In order to show that $j^{(d)} \notin I(\mathcal{S}(d))$, it suffices to show that $j^{(d)}(\binom{d}{d/2}x^{d/2}y^{d/2}) = \delta_{d/2,d/2,d/2} \neq 0$:

$$\begin{aligned}
((Q,Q)^{d/2},Q)^d =& \frac{1}{d!^2}\sum_{i=0}^d (-1)^i\binom{d}{i}\frac{\partial^d (Q,Q)^{d/2}}{\partial x^{d-i}\partial y^i}\frac{\partial^d Q}{\partial x^i \partial y^{d-i}} \\
=& \frac{1}{d!^2}\sum_{i=0}^d (-1)^i\binom{d}{i}\frac{\partial^d}{\partial x^{d-i}\partial y^i}\left[\sum_{e=0}^d \varphi_e^{d/2}x^e y^{d-e}\right]\cdot\dots \\
&\dots\cdot\frac{\partial^d}{\partial x^i\partial y^{d-i}}\left[\sum_{l=0}^d a_l\binom{d}{l}x^l y^{d-l}\right],
\end{aligned}$$

where we have written $(Q,Q)^{d/2}$ in the form of (III.3.2). Recall that the monomial $a_{d/2}^2$ doesn't occur in $\varphi_e^{d/2}$ unless $e = d/2$, and $\varphi_{d/2}^{d/2} =$

$\sum_{f=0}^{d/2} \gamma_{f,d-f}^{d/2} a_f a_{d-f}$. Now a) implies that $\gamma_{d/2,d/2}^{d/2} \neq 0$ if $4|d$. Hence

$$\begin{aligned}
\delta_{d/2,d/2,d/2} &= \frac{1}{d!^2} \sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{\partial^d}{\partial x^{d-i} \partial y^i} \gamma_{d/2,d/2}^{d/2} x^{d/2} y^{d/2} \cdot \ldots \\
&\qquad \ldots \cdot \frac{\partial^d}{\partial x^i \partial y^{d-i}} \left[\binom{d}{d/2} x^{d/2} y^{d/2} \right] \\
&= \frac{1}{d!^2} (-1)^{d/2} \binom{d}{d/2} \gamma_{d/2,d/2}^{d/2} (d/2)!^2 \binom{d}{d/2} (d/2)!^2 \\
&= (-1)^{d/2} \gamma_{d/2,d/2}^{d/2} \neq 0. \qquad \square
\end{aligned}$$

Proof of Proposition III.3.1. Recall from Section II.2 that $\mathcal{S}(d)$ is a multiplicity-free $\mathbf{G}_m \times \mathrm{SL}_2$-variety. Hence in each degree there is up to scalar multiples at most one invariant function on $\mathcal{S}(d)$.

Let first $d \equiv 0 \mod 4$. According to Lemma III.3.5 the invariants $i^{(d)}$ and $j^{(d)}$ do not vanish on $\mathcal{S}(d)$. For this reason, $\mathcal{O}(\mathcal{S}(d))^{\mathrm{SL}_2}$ contains functions of degree 2 and 3, and hence of each degree $n \geq 2$. Since $\mathrm{Sym}^1(V(d)^*) \cong V(d)$, there are no invariant functions of degree 1 on $V(d)$ and hence neither on $\mathcal{S}(d)$. Thus, $\mathcal{O}(\mathcal{S}(d))^{\mathrm{SL}_2} = k[i^{(d)}, j^{(d)}]$.

Let now $d \equiv 2 \mod 4$. According to Lemma III.3.5 the invariant $i^{(d)}$ does not vanish on $\mathcal{S}(d)$, whereas all invariants of odd degree vanish on $\mathcal{S}(d)$ according to Lemma II.2.3 (or as one verifies directly). Therefore, $\mathcal{O}(\mathcal{S}(d))^{\mathrm{SL}_2} = k[i^{(d)}]$, and the claim follows. $\square$

Remark III.3.8. If $d \equiv 0 \mod 4$, then the quotient map $\pi\colon \mathcal{S}(d) \to \mathcal{S}(d)/\!\!/\mathrm{SL}_2 = \mathrm{Spec}(k[i^{(d)}, j^{(d)}])$ is not flat: Consider the schematic zero-fiber $(\mathcal{S}(d))_0$ of π. It equals $\mathrm{Spec}(\mathrm{Sym}(V(d)^*)/(I(\mathcal{S}(d)), i^{(d)}, j^{(d)})$, where $I(\mathcal{S}(d))$ is the ideal of $\mathcal{S}(d) \subset V(d)$. Denote its Hilbert function by h. We claim that

$$\text{(III.3.3)} \qquad (Q^{(d)} \oplus (Q^{(d)}, Q^{(d)})^{d/2})/(I(\mathcal{S}(d)), i^{(d)}, j^{(d)}) \cong V(d) \oplus V(d)$$

as SL_2-module. If this is true, then $h(d) = 2 > 1 = h'_d(d)$. Because h'_d is the Hilbert function of a general fiber of π, it follows that π cannot be flat. We are left to prove (III.3.3). From Remark III.3.6 it follows that $I(\mathcal{S}(d))$ contains only covariants of degree ≥ 3. Hence, the ideal $(I(\mathcal{S}(d)), i^{(d)}, j^{(d)})$ contains no covariants of order < 2, and the only covariant of order 2 in $(I(\mathcal{S}(d)), i^{(d)}, j^{(d)})$ is $k \cdot i^{(d)}$. In particular, neither $Q^{(d)}$ nor $(Q^{(d)}, Q^{(d)})^{d/2}$ is contained in $(I(\mathcal{S}(d)), i^{(d)}, j^{(d)})$. Because $(I(\mathcal{S}(d)), i^{(d)}, j^{(d)})$ is homogeneous, (III.3.3) follows.

Proof of Theorem III.3.2. If $d \equiv 2 \mod 4$, then Proposition III.3.1 implies that $\mathcal{S}(d)/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1$. In this situation, the quotient morphism $\mathcal{S}(d) \to \mathbf{A}^1$ is flat (see [**Ha77**], Proposition III.9.7), and the claim follows from Theorem I.1.1.

If $d \equiv 0 \mod 4$, then the $\mathbf{G}_m \times \mathrm{SL}_2$-isotypic component $\mathcal{O}(\mathcal{S}(d))_{(2,d)}$ of type $(2, d)$ is isomorphic to $V(d)$ according to Lemma III.3.5 a). Now, Proposition II.2.9 applies (with $n_0 = 1$), and the claim follows. $\square$

Let $\mathcal{N}(d) \subset V(d)$ be the (schematic) null-cone as defined in Definition II.3.1. It is well-known that a form $0 \neq v \in V(d)$ is contained in the reduced null-cone $\mathcal{N}(d)_{\mathrm{red}}$ if and only if v contains a linear factor with multiplicity at least $d/2+1$ (see e.g. [**Kr85**], I.5). Using Theorem III.5.10 (whose proof is independent from the following lemma), we can show the following:

Lemma III.3.9. *Let $X_d^0 \subset \mathcal{S}(d)$ be as in Definition III.3.3. Then X_d^0 is a closed subscheme of $\mathcal{N}(d)_{\mathrm{red}}$.*

Proof. Let $I^0 \subset R^d$ be the ideal of X_d^0, and let $J \subset R^d$ be the ideal of $\mathcal{N}(d)_{\mathrm{red}}$. We need to show that $J \subset I^0$. In (III.5.2) in Theorem III.5.10 we shall see that $R^d_{(n,k)} \subset I^0$ if $n \geq 3$ and if $k \leq nd - d$ and that $R^d_{(2,0)} \subset I^0$. Since $\mathcal{N}(d)_{\mathrm{red}}$ is a cone, its ideal J is homogeneous. In order to verify that $J \subset I^0$, it therefore suffices to show that

$$J_{(n,k)} = \{0\} \quad \text{if} \begin{cases} n \geq 3 \text{ and } k > nd - d, \text{ or} \\ n = 2 \text{ and } k > 0. \end{cases} \tag{III.3.4}$$

Recall that in (III.2.1) we identified R^d with $k[a_0, \ldots, a_d]$, where a_i was defined by $a_i(\sum \lambda_j \binom{d}{j} x^j y^{d-j}) = \lambda_i$.

To see (III.3.4), let first $n \geq 3$ and choose k even with $(n-1)d < k \leq nd$. Let $0 \neq f \in (R^d_{(n,k)})^U$. For $i \in \{0, 1, \ldots, d\}^n$ there exist constants $\gamma_i \in k$ such that

$$f = \sum_i \gamma_i a_{i_1} \cdots a_{i_n},$$

and $\sum_s i_s = nd - l$, where $l := (nd - k)/2 < d/2$ is the co-order of f. This is only possible if each $i_s > d/2$. Evaluating f on the nullform $\sum_{j=d/2+1}^d \lambda_j \binom{d}{j} x^j y^{d-j}$ yields $\sum_i \gamma_i \lambda_{i_1} \cdots \lambda_{i_n}$. This is non-zero for suitable values for $\lambda_{d/2+1}, \ldots, \lambda_d$, and hence $f \notin J$.

Let now $n = 2$, let $k > 0$, and let $Q = Q^{(d)}$. If k is not divisible by 4, then $R^d_{(2,k)} = 0$ anyway, and there is nothing to show. We assume now that k is divisible by 4. Let $l := (2d - k)/2$; then $R^d_{(2,k)} = (Q, Q)^l$. Writing $(Q, Q)^l$ in the form of (III.3.1), we see that

(III.3.5)
$$\begin{aligned}(Q,Q)^l &= \frac{(d-l)!^2}{d!^2} \sum_{i=0}^{l} \sum_{j,j'=0}^{d} (-1)^i \binom{l}{i} \binom{d}{j} \binom{d}{j'} a_j a_{j'} \frac{\partial^l x^j y^{d-j}}{\partial x^{l-i} \partial y^i} \frac{\partial^l x^{j'} y^{d-j'}}{\partial x^i \partial y^{l-i}} \\ &= \sum_{s=0}^{2d-2l} \varphi_s^l(a_0, \ldots, a_d) x^s y^{2d-2l-s},\end{aligned}$$

where φ_s^l is of the form

$$\varphi_{2d-2l}^l(a_0,\dots,a_s) = \sum_{j+j'-l=2d-2l} \gamma_{j,j'}^l a_j a_{j'}$$

for constants $\gamma_{j,j'}^l \in k$. Since $\binom{d}{l/2}x^{d-l/2}y^{l/2}$ is a nullform, it now suffices to show that $\varphi_{2d-2l}^l(a_0,\dots,a_s)(\binom{d}{l/2}x^{d-l/2}y^{l/2}) \neq 0$. We see that

$$\varphi_{2d-2l}^l(a_0,\dots,a_s)(\binom{d}{l/2}x^{d-l/2}y^{l/2}) = \gamma_{d-l/2,d-l/2}^l.$$

Using (III.3.5) we can compute $\gamma_{d-l/2,d-l/2}^l$:

$$\begin{aligned}
&\gamma_{d-l/2,d-l/2}^l x^k \\
&\quad = \frac{(d-l)!^2}{d!^2}\sum_{i=0}^{l}(-1)^i\binom{l}{i}\binom{d}{d-l/2}^2 \frac{\partial^l x^{d-l/2}y^{l/2}}{\partial x^{l-i}\partial y^i}\frac{\partial^l x^{d-l/2}y^{l/2}}{\partial x^i\partial y^{l-i}} \\
&\quad = \frac{(d-l)!^2}{d!^2}(-1)^{l/2}\binom{l}{l/2}\binom{d}{d-l/2}^2 \frac{\partial^l x^{d-l/2}y^{l/2}}{\partial x^{l/2}\partial y^{l/2}}\frac{\partial^l x^{d-l/2}y^{l/2}}{\partial x^{l/2}\partial y^{l/2}} \\
&\quad = (-1)^{l/2}\binom{l}{l/2}x^k.
\end{aligned}$$

This shows that $\gamma_{d-l/2} = (-1)^{l/2}\binom{l}{l/2} \neq 0$, and proves (III.3.4). □

III.4. Multiplicities and Stability

For $n \in \mathbf{N}$ and $l \in \mathbf{N}$ with $0 \leq l \leq nd/2$, we define

$$\begin{aligned}
\mathrm{mult}_{n;l}^d :=& \dim_k\{f \in \mathrm{Sym}^n(V(d))^U \mid f \text{ is of co-order } l\} \\
=&\{\dim_k \mathrm{Sym}^n(V(d))_{(nd-2l)}\}/(nd-2l+1),
\end{aligned}$$

which is the *multiplicity* of the isotypic component of type $nd-2l$ in $\mathrm{Sym}^n(V(d))$. Let further $\mathcal{D}_{n;l}^d$ be the space of d-digraphs with n vertices and l arrows modulo relations. In view of Section III.2 the space $\mathcal{D}_{n;l}^d$ is isomorphic to the space of covariants of $V(d)$ of degree n and of co-order l, and hence $\mathrm{mult}_{n;l}^d = \dim_k(\mathcal{D}_{n;l}^d)$.

The computation of these multiplicities goes back to Cayley and Sylvester (cf. [**Cl1872**]); a modern approach can be found in [**Br94**]. Let

$$(n;d,l) := \{(j_0,j_1,\dots,j_d) \in \mathbf{N}^{d+1} \mid \sum_{i=0}^d j_i = n \text{ and } \sum_{i=0}^d i j_i = l\}.$$

The Cayley-Sylvester formula states that

(III.4.1) $$\mathrm{mult}_{n;l}^d = \#(n;d,l) - \#(n;d,l-1).$$

This implies in particular that

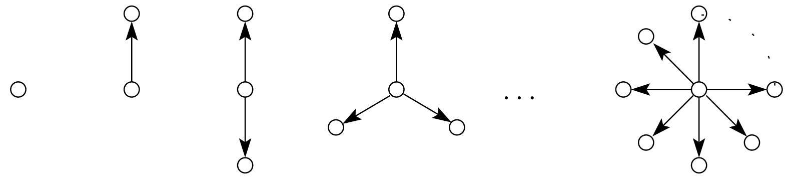
(III.4.2) $$\mathrm{mult}_{n;l}^d = \mathrm{mult}_{n+1;l}^d \quad \text{for all } l \leq n \text{ provided that } d \geq 2.$$

(To see this, suppose first that $l < n$. Then the map $(n; d, l) \to (n+1; d, l)$ mapping $(j_0, \ldots, j_d)$ to $(j_0 + 1, j_1, \ldots, j_d)$ is a bijection. Hence $\#(n; d, l) = \#(n+1; d, l)$. On the other hand, if $l = n$, then one sees similarly that $\#(n; d, l) = \#(n+1; d, l) - 1$. Now the claim follows readily.)

Remark III.4.1. The map $\mathcal{D}^d_{n;l} \to \mathcal{D}^d_{n+1;l}$ assigning $\circ \sqcup D$ to $D \in \mathcal{D}^d_{n;l}$ is an injective homomorphism of k-vector spaces. If further $l \leq n$, then (III.4.2) implies that this map is even an isomorphism.

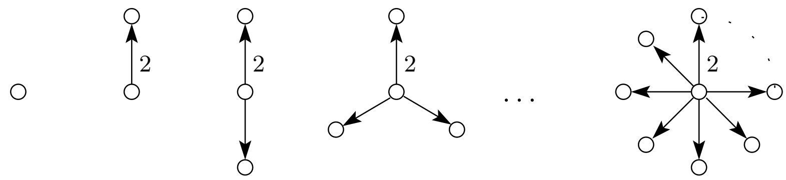
In Proposition III.4.3 bases of the vector spaces $\mathcal{D}^d_{n;l}$ with $n \geq l$ are given. Constructions of such bases might be known, but for lack of references in the literature and for further use we treat the subject in detail.

Consider the following family of d-digraphs indexed by the integers $0 \leq s \leq d$:



$s = 0$ $s = 1$ $s = 2$ $s = 3$ $s = d$

A member of this family with s arrows is denoted $\star^d_s$. Similarly, we introduce a second family:



$s = 0$ $s = 2$ $s = 3$ $s = 4$ $s = d$

A member of this family with s arrows is denoted $\tilde{\star}^d_s$.

Remark III.4.2. Remark III.2.19 states that $\star^d_1 = 0$. This will be used frequently.

Given $\omega \in \mathbf{N}$ and $s = (s_1, \ldots, s_\omega) \in \{0, 2, 3, \ldots, d\}^\omega$, we write

$$\star^d_s := \bigsqcup_{i=1}^{\omega} \star^d_{s_i} \quad \text{and} \quad \tilde{\star}^d_s := \bigsqcup_{i=1}^{\omega} \tilde{\star}^d_{s_i}$$

for the disjoint union of the corresponding digraphs. If $P \in \mathrm{Perm}(\omega)$ is a permutation, then $\star^d_{P(s)} = \star^d_s$ and $\tilde{\star}^d_{P(s)} = \tilde{\star}^d_s$. This motivates the following definition:

$$\Sigma^d_{n;l} := \{s \in \{0, 2, 3, \ldots, d\}^{n-l} \mid |s| = l\}/\sim,$$

where $s \sim t$ if s and t are in the same $\mathrm{Perm}(n-l)$-orbit, and where $|s| = \sum_i s_i$. If now $[s] \in \Sigma^d_{n;l}$, then $\star^d_s \in \mathcal{D}^d_{n;l}$. Similarly, with $|s|_0 := \sum_i \max\{1, s_i\}$, we let

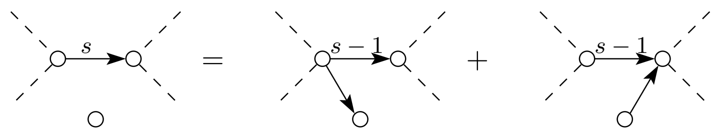
$$\tilde{\Sigma}^d_{n;l} := \bigcup_{\omega \in \mathbf{N}} (\{s \in \{0, 2, 3, \ldots, d\}^\omega \mid |s| = l, |s|_0 = n\}/\sim),$$

where $s \sim t$ if $t = P(s)$ for a permuation P. If $[s] \in \tilde{\Sigma}^d_{n;l}$, then $\tilde{\star}^d_s \in \mathcal{D}^d_{n;l}$. From now on we write $\star^d_{[s]}$ and $\tilde{\star}^d_{[s]}$ instead of $\star^d_s$ and $\tilde{\star}^d_s$ if s is a vector of length > 1.

Proposition III.4.3. *If* $\lfloor \frac{3}{2} l \rfloor \leq n$, *then* $\mathrm{span}_k(\star^d_{[s]} \mid [s] \in \Sigma^d_{n;l}) = \mathcal{D}^d_{n;l}$. *If* $l \leq n$, *then* $\mathrm{span}_k(\tilde{\star}^d_{[s]} \mid [s] \in \tilde{\Sigma}^d_{n;l}) = \mathcal{D}^d_{n;l}$.

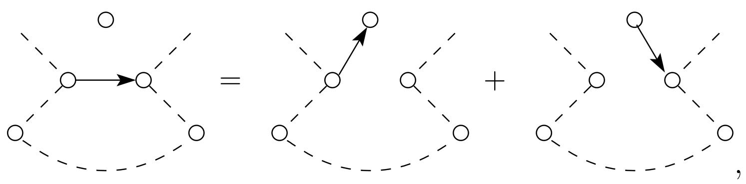
Proof. Step 1. Let $D \in \mathcal{D}^d_{n;l}$. We show that there exists $N \in \mathbf{N}$ such that $D \sqcup \bigsqcup_{i=1}^N \circ \in \mathrm{span}_k(\star^d_{[s]} \mid [s] \in \Sigma^d_{n+N;l})$. We can always assume that there are as many $\circ$ as connected components of D as we wish (if necessary enlarge N).

To start with, we use the relation



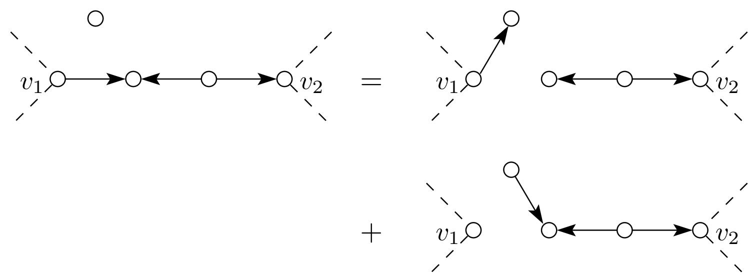
to write D as a sum of digraphs in which two vertices are joint by at most one arrow. One sees inductively that this is always possible — maybe at the cost of augmenting N.

Secondly, the same relation can be used to break up cycles: Applying the relation



one can write D as a sum of digraphs with a fundamental group of smaller rank. This yields a digraph whose underlying graph has trees as connected components.

Finally, we show that trees can be written as sums of stars. Suppose that D' is a tree with n vertices and with l arrows. We proceed by induction on l. If $l \leq 2$, the digraph D' is a star and we are done. Otherwise, if D' is not yet a star, it has at least two vertices v_1 and v_2 being the source or target of at least two arrows. Since D' is connected, there is a path of arrows from v_1 and v_2. Applying the relation



(recall that $\star_1^d = 0$) yields a sum of trees, and each of the summands has less than l arrows. This completes the induction step.

Step 2. If D_1 and D_2 be two digraphs with the property that

$$D_1 \sqcup \bigsqcup_{i=1}^{N} \circ = D_2 \sqcup \bigsqcup_{i=1}^{N} \circ$$

in the algebra $\mathcal{D}^d$ of d-digraphs, then $D_1 = D_2$ in $\mathcal{D}^d$ (cf. Remark III.4.1).

Step 3. If $[s] \in \Sigma_{n;l}^d$ with $n \geq \lfloor \frac{3}{2} l \rfloor$, then $\star_{[s]}^d$ contains at least $n - \lfloor \frac{3}{2} l \rfloor$ times $\circ$ as connected component. (To see this, it suffices to note that the ratio between the number of vertices and the number of arrows in $\star_s^d$ equals $(s+1)/s \leq 3/2$ for all $s \geq 2$.) This, together with Steps 1 and 2 shows that $\mathrm{span}_k(\star_{[s]}^d \mid [s] \in \Sigma_{n;l}^d) = \mathcal{D}_{n;l}^d$ whenever $n \geq \lfloor \frac{3}{2} l \rfloor$.

Step 4. Let $s_0 \in \mathbf{N}$, and consider the relations

$$\begin{aligned} \tilde{\star}_{s_0}^d \sqcup \circ \sqcup \circ &= \star_{s_0}^d \sqcup \circ + \star_{s_0-2}^d \sqcup \star_2^d \quad \text{if } s_0 \geq 2, \text{ and} \\ \tilde{\star}_0^d &= \star_0^d. \end{aligned}$$

This, together with the above steps, shows that $\mathrm{span}_k(\tilde{\star}_{[s]}^d \mid [s] \in \tilde{\Sigma}_{n;l}^d) = \mathcal{D}_{n;l}^d$ whenever n is sufficiently large compared to l. If $[s] \in \tilde{\Sigma}_{n;l}^d$ with $n \geq l$, then $\tilde{\star}_{[s]}^d$ contains at least $n - l$ times $\circ$ as connected component. It now follows that $\mathrm{span}_k(\tilde{\star}_{[s]}^d \mid [s] \in \tilde{\Sigma}_{n;l}^d) = \mathcal{D}_{n;l}^d$ whenever $n \geq l$. $\square$

Remark III.4.4. If $\lfloor \frac{3}{2} l \rfloor \leq n$, then the set $\{\star_{[s]}^d \mid [s] \in \Sigma_{n;l}^d\}$ is a basis of $\mathcal{D}_{n;l}^d$. If $l \leq n$, then the set $\{\tilde{\star}_{[s]}^d \mid [s] \in \tilde{\Sigma}_{n;l}^d\}$ is a basis of $\mathcal{D}_{n;l}^d$. In view of (III.4.1) and Proposition III.4.3 it suffices to verify that $\#\Sigma_{n;l}^d = \#(n;d,l) - \#(n;d,l-1) = \mathrm{mult}_{n;l}^d$ for all (d,n,l) as in the proposition.

Alternatively, one can directly verify that the sets $\{\star_s^d \mid [s] \in \Sigma_{n;l}^d\}$ and $\{\tilde{\star}_{[s]}^d \mid [s] \in \Sigma_{n;l}^d\}$ are linearly independent. This can be done by computing and comparing their U-invariants.

Example III.4.5. For an integer $0 \leq s \leq d$, we calculate the U-invariant belonging to $\star_s^d$. The bracket polynomial corresponding to $\star_s^d$ is

$$[\alpha, \beta_1][\alpha, \beta_2] \ldots [\alpha, \beta_s] (\alpha \boldsymbol{x})^{d-s} (\beta_1 \boldsymbol{x})^{d-1} \ldots (\beta_s \boldsymbol{x})^{d-1}.$$

Its symbolic expression equals

$$
\begin{aligned}
&\prod_{i=1}^{s}(\xi_\alpha\eta_{\beta_i}-\eta_\alpha\xi_{\beta_i})(\xi_\alpha x-\eta_\alpha y)^{d-s}\prod_{i=1}^{s}(\xi_{\beta_i}x-\eta_{\beta_i}y)^{d-1}\\
= \quad & \xi_\alpha^{d-s}\prod_{i=1}^{s}(\xi_\alpha\eta_{\beta_i}-\eta_\alpha\xi_{\beta_i})\prod_{i=1}^{s}\xi_{\beta_i}^{d-1}x^{(s+1)d-2s}+y\cdot\ldots\\
= \quad & (\xi_\alpha^{d}\prod_{i=1}^{s}\xi_{\beta_i}^{d-1}\eta_{\beta_i}-\xi_\alpha^{d-1}\eta_\alpha\sum_{j=1}^{s}\xi_{\beta_j}^{d}\prod_{i\neq j}\xi_{\beta_i}^{d-1}\eta_{\beta_i}\pm\ldots\\
&\qquad \ldots\pm\xi_\alpha^{d-s}\eta_\alpha^{s}\prod_{i=1}^{s}\xi_{\beta_i}^{d})x^{(s+1)d-2s}+y\cdot\ldots
\end{aligned}
$$

The U-invariant in the covariant corresponding to this symbolic expressions can now be read off to be

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}a_{d-k}a_d^k a_{d-1}^{s-k}.$$

Given $[s]=[(s_1,\ldots,s_{n-l})]\in\Sigma^d_{n;l}$, the U-invariant corresponding to $\star^d_{[s]}$ is the product of the U-invariants of the $\star^d_{s_i}$.

Example III.4.6. Let $l\geq 2$. Consider the digraph

$$\circ\xrightarrow{\;l\;}\circ\quad\underbrace{\circ\ \ldots\ \circ}_{l}.$$

One finds that it can be written as $\sum_{i=0}^{l}(-1)^i\binom{l}{i}\star_i^d\sqcup\star_{l-i}^d$.

Remark III.4.7. Proposition III.4.3 gives an effective tool to compute covariants of small co-order. For many applications it is more important to know covariants of small order, like invariants. To compute these, Proposition III.4.3 yields the following algorithm, which however is numerically very ineffective: If one wants to know all invariants of degree n of a form of even degree d, one writes down a basis $(D_1,\ldots,D_m)$ of $\mathcal{D}^d_{nd/2;nd/2}$ (fast) and computes the corresponding U-invariants $f_1,\ldots,f_m$ (fast). For each invariant f of degree n, the U-invariant $f\cdot a_d^{nd/2-n}$ can be written as linear combination of the f_i. The goal is now to find linear combinations of the f_i which contain $a_d^{nd/2-n}$ as factor. This is linear algebra, but the matrices involved become very large.

III.5. Structure of SL_2-stable ideals

In Section III.3 the main object of interest was the invariant Hilbert scheme $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))$, which turned out to be isomorphic to $\mathbf{A}^1$ for all even d. Next, we focus on $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))$. We have already stated in Corollary II.3.4 that

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))_{\mathrm{red}} = \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))^0_{\mathrm{red}} \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))_{\mathrm{red}}.$$

We now know $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))$ and turn our attention to $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))$.

Let $R^d := \mathrm{Sym}(V(d)^*)$, let $R^d_{(k)} := \mathrm{Sym}(V(d)^*)_{(k)}$ be the isotypic component of type k, and let $R^d_{(n,k)} := \mathrm{Sym}^n(V(d)^*)_{(k)}$ be the $\mathbf{G}_m \times \mathrm{SL}_2$-isotypic component of type (n,k). For an SL_2-stable ideal $J \subset R^d$ let $J_{(k)} := J \cap R^d_{(k)}$ and $J_{(n,k)} := J \cap R^d_{(n,k)}$. Now the closed points of $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))$ correspond to SL_2-stable ideals $I \subset R^d$ with $\mathrm{Spec}(R^d/I)_{\mathrm{red}} = X_d (= \overline{\mathrm{SL}_2 \cdot y^d})$ and with Hilbert function $h_{R^d/I} = h'_d$.

Definition III.5.1. Let $J^d_{odd} \subset R^d$ be the smallest SL_2-stable ideal containing all covariants of odd co-order, or equivalently all covariants of order $\equiv 2 \mod 4$. Let $J^d_{odd,+} \subset R^d$ be the smallest SL_2-stable ideal containing J^d_{odd} and $R^d_{(2,d)}$.

Let d be a multiple of 4, and let I be as above. We claim that $J^d_{odd,+} \subset I$: First, $h_{R^d/I}(k) = h'_d(k) = 0$ if $k \equiv 2 \mod 4$. Hence $J^d_{odd} \subset I$. Moreover, $I_{(d)} \subset \sqrt{I}_{(d)} = \bigoplus_{n \geq 2} R^d_{(2,d)}$ (since $X_{\mathrm{red}} = X_d$). Because $h_{R^d/I}(d) = 1$, it follows that $I_{(d)} = \sqrt{I}_{(d)} = \bigoplus_{n\geq 2} R^d_{(2,d)}$, and hence $R^d_{(2,d)} \subset I$. It now follows that $J^d_{odd,+} \subset I$, and hence that $h_{R^d/J^d_{odd,+}}(k) \geq h_{R^d/I}(k) = h'_d(k)$ for all k. We shall see in Theorem III.5.10 that I is in fact almost determined by $J^d_{odd,+}$. This opens the possibility for computations, and this is why we assume from now on that d is a multiple of 4.

The main result of this section is Theorem III.5.10.

Caveat III.5.2. We have encountered $R^d_{(n,k)}$, $I^d_{(n,k)}$, $\mathcal{D}^d_{n;l}$, and similar objects. Whereas the upper index d always refers to the degree of the corresponding form and the first index n out of two lower indices always refers to the degree of a homogeneous object, the second lower index can refer either to the order or to the co-order. We use a comma when dealing with orders (and usually denote the order by k), whereas we use a semi-colon when dealing with co-orders (and usually denote the co-order by l).

The space of d-digraphs $\mathcal{D}^d$ is a k-algebra with multiplication $D_1 \cdot D_2 = (D_1, D_2)^0 = D_1 \sqcup D_2$. An ideal $\mathcal{I} \subset \mathcal{D}^d$ is called *transvection-stable* if $(D_1, D_2)^r \in \mathcal{I}$ for all $r \geq 0$, for all $D_1 \in \mathcal{I}$, and for all $D_2 \in \mathcal{D}^d$. Recall further that if $D_1 \in \mathcal{D}^d_{n_1;l_1}$ and if $D_2 \in \mathcal{D}^d_{n_2;l_2}$, then $(D_1, D_2)^r \in \mathcal{D}^d_{n_1+n_2;l_1+l_2+r}$. In view of Section III.2 working with SL_2-stable ideals in R^d is equivalent to working with transvection-stable ideals in $\mathcal{D}^d$.

Proposition III.5.3. *Suppose that $n \geq l$.*

a) *The homomorphism of k-vector spaces*

$$\Phi\colon \mathcal{D}^d_{n;l} \oplus \mathcal{D}^d_{n+1;l+1} \longrightarrow \mathcal{D}^d_{n+2;l+2}, \quad D + D' \longmapsto (D \sqcup \circ\!\!-\!\!\circ) + (D', \circ)^1$$

is surjective.

b) *If a transvection-stable ideal $\mathcal{J} \subset \mathcal{D}^d$ contains $\mathcal{D}^d_{n;l} \oplus \mathcal{D}^d_{n+1;l+1}$, then there exists an integer N such that $\mathcal{D}^d_{n';l'} \subset \mathcal{J}$ whenever $n' \geq N$ and $l' \geq l$.*

Proof. a) Step 1. Assume that n is so large compared to l that $\{\star^d_{[s]} \mid [s] \in \Sigma^d_{n+2;l+2}\}$ spans $\mathcal{D}^d_{n+2;l+2}$ (cf. Proposition III.4.3). Then it suffices to verify that $\star^d_{[s]}$ is in the image of Φ for each $[s] \in \Sigma^d_{n+2;l+2}$.

Let $s = (s_1, \ldots, s_{n-l})$ with $[s] \in \Sigma^d_{n+2;l+2}$, and let $\min(s) := \min\{s_i \mid s_i > 0,\, 1 \leq i \leq n-l\}$. If $s_i = 1$ for some i, then $\star_{[s]} = 0$ (cf. Remarks III.4.2 and III.2.19). Hence we can assume that $\min(s) \geq 2$. We proceed by induction on $\min(s)$. If $\min(s) = 2$, then the claim is true. Otherwise, let $s_\epsilon = \min(s)$, and suppose that s is of the form $s = (s_1, \ldots, s_\epsilon, 0, \ldots, 0)$. Consider

$$s' = (s_1, \ldots, s_{\epsilon-1}, s_\epsilon - 1, 0, \ldots, 0).$$

Then $[s'] \in \Sigma^d_{n+1;l+1}$. Now $(\star^d_0, \circ)^1 = 0$, and

$$(\star^d_{s_0}, \circ)^1 = \frac{1}{d}(-(d-s_0)(\star^d_{s_0+1}) + s_0(d-1)(\star^d_{s_0-1} \sqcup \star^d_2))$$

for $s_0 \geq 2$. Hence $(\star^d_{[s']}, \circ)^1$ is a linear combination of the type

$$(\star^d_{[s']}, \circ)^1 = \lambda_0 \star^d_{[s]} + \sum_i \lambda_i \star^d_{[s^i]}$$

with $\lambda_0 \neq 0$ and with $\min(s^i) < \min(s)$ for all i. The claim now follows from the induction hypothesis.

Step 2. Until now we have assumed that n is very large compared to l. For general $n \geq l$ the claim is now a consequence of Step 1, of the fact that $(D, \circ)^1 \sqcup \bigsqcup_{i=1}^N \circ = (D \sqcup \bigsqcup_{i=1}^N \circ, \circ)^1$, and of Proposition III.4.3.

b) Remark III.4.1 and a) imply that $\mathcal{D}^d_{n';l'} \subset \mathcal{J}$ for all (n', l') with $n' - l' \geq n - l$ and with $l \leq l' \leq l + d - 1$. According to [**Ja05**], proof of Proposition 1.3, there exists $N \in \mathbf{N}$ such that all covariants of degree $n' \geq N$ are in the transvection-stable ideal $\mathcal{J}'$ generated by $\bigsqcup_{i=1}^{d+n} \circ$. Let $n' \geq N$, and let $l' \geq l$. We claim that $\mathcal{D}^d_{n';l'} \subset \mathcal{J}$. To see this, let $D \in \mathcal{D}^d_{n';l'}$. Since $D \in \mathcal{J}'$, we can write D as linear combination of transvectants

$$(((\bigsqcup_{i=1}^{n+d} \circ, \circ)^{\alpha_1}, \circ)^{\alpha_2}, \ldots, \circ)^{\alpha_s}$$

with $0 \leq \alpha_i \leq d$ and with $\sum_{j=1}^s \alpha_j = l'$. Thus, it suffices to show that each digraph of this form is in $\mathcal{J}$. First, $\bigsqcup_{i=1}^{n+d} \circ$ has co-order

0, whereas $(((\bigsqcup_{i=1}^{n+d} \circ, \circ)^{\alpha_1}, \circ)^{\alpha_2}, \dots, \circ)^{\alpha_s}$ has co-order $l' \geq l$. Observe that if D is a digraph of co-order m, then $(D, \circ)^\alpha$ is zero if $\alpha > d$, or of co-order $m + \alpha \leq m + d$ if $\alpha \leq d$. Hence, there exists j such that $(((\bigsqcup_{i=1}^{n+d} \circ, \circ)^{\alpha_1}, \circ)^{\alpha_2}, \dots, \circ)^{\alpha_j}$ is of co-order l' with $l \leq l' \leq l + d - 1$. According to the observation above, this digraph is in $\mathcal{J}$. This implies that also $(((\bigsqcup_{i=1}^{n+d} \circ, \circ)^{\alpha_1}, \circ)^{\alpha_2}, \dots, \circ)^{\alpha_s} \in \mathcal{J}$, and the claim follows. $\square$

The next lemma is formulated for an arbitrary semisimple group G.

Lemma III.5.4. *Let A be an* $\mathbf{N}$*-graded k-G-algebra such that each isotypic component $A_{(\lambda)} \subset A$ is homogeneous. Let $J \subset A$ be a G-stable ideal. Let $\lambda_1, \lambda_2 \in \Lambda^+$, and let $\lambda = \lambda_1 + \lambda_2$. Suppose that $h_{A/J}(\lambda_1) = h_{A/J}(\lambda) = 1$, and that $J_{(\lambda)} = J \cap A_{(\lambda)}$ is homogeneous. If there exist homogenous elements $a_1 \in (A_{(\lambda_1)})^U$ and $a_2 \in (A_{(\lambda_2)})^U$ such that $a_1 \cdot a_2 \notin J_{(\lambda)}$, then $J_{(\lambda_1)} = \mu_{a_2}^{-1}(J_{(\lambda)})$, where $\mu_{a_2} \colon A \to A$ is multiplication with a_2. In particular, $J_{(\lambda_1)}$ is homogeneous.*

Proof. First note that $(J_{(\lambda_1)})^U \subset \mu_{a_2}^{-1}((J_{(\lambda)})^U)$. Since $(J_{(\lambda_1)})^U$ has codimension 1 in $(A_{(\lambda_1)})^U$, it follows $\mu_{a_2}^{-1}((J_{(\lambda)})^U)$ has codimension at most 1 in $(J_{(\lambda_1)})^U$. On the other hand, $\mu_{a_2}^{-1}((J_{(\lambda)})^U) \subsetneq A_{(\lambda_1)}$ (otherwise $a_1 \cdot a_2 \in J_{(\lambda)}$). This shows that $(J_{(\lambda_1)})^U = \mu_{a_2}^{-1}((J_{(\lambda)})^U)$, and the latter space is homogeneous since $J_{(\lambda)}$ is homogeneous. $\square$

Definition III.5.5. For $n \geq 2$ and $l \in \{0, 1, \dots, d\}$ define $C_{n;l}^d := (Q^{(d)}, Q^{(d)})^l \cdot (Q^{(d)})^{n-2}$, which is a covariant of degree n and of co-order l. In addition, we let $C_{1;0}^d := Q^{(d)}$.

If l is odd, then $C_{n;l}^d = 0$, since $(Q^{(d)}, Q^{(d)})^l = 0$ according to Remark III.2.19. On the other hand, if $l \in \{0, 2, \dots, d\}$, then $(Q^{(d)}, Q^{(d)})^l \neq 0$ according to Lemma III.3.5; and using Remark III.4.1, we see that $C_{n;l}^d \neq 0$ for $l \in \{0, 2, \dots, d\}$.

Let $I(\mathcal{S}(d)) \subset R^d$ be the ideal of the minimal sheet $\mathcal{S}(d) \subset V(d)$.

Lemma III.5.6. a) $J_{odd}^d \subset I(\mathcal{S}(d))$.
b) $C_{n;l}^d \not\subset I(\mathcal{S}(d))$ *for all $n \geq 2$ and for all $l \in \{0, 2, \dots, d\}$.*

Proof. a) This follows either from the discussion in Section II.2 or directly: Suppose that $J_{odd}^d \not\subset I(\mathcal{S}(d))$. Then there exists a homogeneous covariant $V \subset R^d$ of degree n and of odd co-order l that does not vanish on $\mathcal{S}(d)$. Now V is of order $k = nd - 2l \equiv 2 \mod 4$. This covariant yields a non-zero SL_2-equivariant morphism $\varphi \colon \mathcal{S}(d) \to V(k)$. Now $x^{d/2}y^{d/2}$ is stable under $N_{\mathrm{SL}_2}(T)$, hence so is its image $\varphi(x^{d/2}y^{d/2})$. However, the only $N_{\mathrm{SL}_2}(T)$-stable point of $V(k)$ is 0 because $k \equiv 2 \mod 4$. This shows that $\varphi(\mathcal{S}(d)) = \{0\}$, a contradiction. Hence $J_{odd}^d \subset I(\mathcal{S}(d))$.

b) Suppose that $(Q^{(d)}, Q^{(d)})^l \cdot (Q^{(d)})^{n-2} = C_{n;l}^d \subset I(\mathcal{S}(d))$. Let $0 \neq f \in ((Q^{(d)}, Q^{(d)})^l)^U$. Then $f \cdot (a_d)^{n-2} \in (C_{n;l}^d)^U$, and hence $f \cdot (a_d)^{n-2} \in I(\mathcal{S}(d))$. Since $\mathcal{S}(d)$ is irreducible, the ideal $I(\mathcal{S}(d))$ is prime, and it follows that either $a_d \in I(\mathcal{S}(d))$ or $f \in I(\mathcal{S}(d))$. First, $a_d(x^d) = 1$, and so $a_d \notin I(\mathcal{S}(d))$. Moreover, $f \notin I(\mathcal{S}(d))$ according to Lemma III.3.5. This is a contradiction, and thus $C^d_{n;l} \not\subset I(\mathcal{S}(d))$. □

Lemma III.5.7. *Let* $n \geq l$. *Then*

$$R^d_{(n,nd-2l)} = \begin{cases} (J^d_{odd})_{(n,nd-2l)} & \text{if } l \text{ is odd,} \\ (J^d_{odd})_{(n,nd-2l)} \oplus C^d_{n;l} & \text{if } l \in \{0,2,\ldots,d\}. \end{cases} \qquad \text{(III.5.1)}$$

Proof. Let $\mathcal{J}^d_{odd} \subset \mathcal{D}^d$ be the transvection-stable ideal corresponding to J^d_{odd}, and let $n \in \mathbf{N}$. By definition of J^d_{odd}, we see that $(\mathcal{J}^d_{odd} \cap \mathcal{D}^d_{n;0}) = \{0\}$. Since $\dim_k(\mathcal{D}^d_{n;0}) = 1$, it follows that $\mathcal{J}^d_{odd} \cap \mathcal{D}^d_{n;0}$ has codimension 1 in $\mathcal{D}^d_{n;0}$. By definition, $\mathcal{D}^d_{n+1;1} \subset \mathcal{J}^d_{odd}$. Since Φ from Proposition III.5.3 maps homogeneous elements in $\mathcal{J}^d_{odd} \cap (\mathcal{D}^d_{n;0} \oplus \mathcal{D}^d_{n+1;1})$ to elements in $\mathcal{J}^d_{odd} \cap \mathcal{D}^d_{n+2;2}$, Proposition III.5.3 implies that $\mathcal{J}^d_{odd} \cap \mathcal{D}^d_{n+2;2}$ has codimension at most 1 in $\mathcal{D}^d_{n+2;2}$. Going on inductively, we find that $\mathcal{J}^d_{odd} \cap \mathcal{D}^d_{n+l;l}$ has codimension at most 1 in $\mathcal{D}^d_{n+l;l}$. With Lemma III.5.6 b) the claim now follows if $n \geq 2$. For $n < 2$ the claim is obviously also true. □

The d-digraph corresponding to $C^d_{n;l}$ is $(\circ,\circ)^l \sqcup \bigsqcup_{i=1}^{n-2} \circ$. For later use, the following can readily be deduced from the proof of the preceding lemma:

Remark III.5.8. Let l be an even integer, and let $l' \in \{0,2,\ldots,d\}$. Let $n \geq l'$, and let

$$D = (\circ,\circ)^{l'} \sqcup ((\circ,\circ)^2 \sqcup (\circ,\circ)^2 \sqcup \ldots \sqcup (\circ,\circ)^2) \sqcup \bigsqcup_{i=1}^{n'} \circ,$$

where the number of $(\circ,\circ)^2$ and n' are chosen such that $D \in \mathcal{D}^d_{n;l}$. Using Lemma III.5.7 and using Proposition III.5.3 as in the proof of Lemma III.5.7, one sees that for n large enough either $(\mathcal{J}^d_{odd})_{n;l} = \mathcal{D}^d_{n;l}$, or $D + (\mathcal{J}^d_{odd})_{n;l}$ spans $\mathcal{D}^d_{n;l}/(\mathcal{J}^d_{odd})_{n;l}$. In particular, it follows that for a given $D' \in \mathcal{D}^d_{n;l}$ we have

$$D' + \mathcal{J}^d_{odd} = \gamma D + \mathcal{J}^d_{odd}$$

for some $\gamma \in k$, provided that $n \in \mathbf{N}$ is large enough.

Lemma III.5.9. *There exists* $N \in \mathbf{N}$ *such that for all* $n \geq N$ *the following hold:*

$$R^d_{(n,nd-2l)} = \begin{cases} (J^d_{odd,+})_{(n,nd-2l)} & \text{if } l \notin \{0,2,\ldots,d/2-2\} \\ (J^d_{odd,+})_{(n,nd-2l)} \oplus C^d_{n;l} & \text{if } l \in \{0,2,\ldots,d/2-2\}. \end{cases}$$

Proof. Let $\mathcal{J}^d_{odd,+} \subset \mathcal{D}^d$ be the ideal corresponding to $J^d_{odd,+}$. Lemma III.5.7 and the fact that $C^d_{n;d/2} \subset J^d_{odd,+}$ imply that $\mathcal{D}^d_{n+d/2-1,d/2-1} \oplus \mathcal{D}^d_{n+d/2,d/2} \subset \mathcal{J}^d_{odd,+}$ for all $n \geq 2$. Proposition III.5.3 b) implies that there exists $N \in \mathbf{N}$ such that $\mathcal{D}^d_{n;l} \subset \mathcal{J}^d_{odd,+}$ for all $n \geq N$ and for all $l \geq d/2$.

Suppose that $C^d_{n;l} \subset J^d_{odd,+}$ for some $n \geq 2$ and for some even $l < d/2$. As in the proof of Lemma III.5.7, this implies that there exists $N' \in \mathbf{N}$ such that $\mathcal{D}^d_{n';l'} \subset J^d_{odd,+}$ for all $n' \geq N'$ and for all $l' \geq l$. Let $k := N'd - 2l$. Then $R^d_{(k)} = \bigoplus_{n \geq N'} R^d_{(n,k)} \subset J^d_{odd,+}$. This implies that $h_{R^d/J^d_{odd,+}}(k) = 0$. On the other hand, $h_{R^d/J^d_{odd,+}}(k) \geq h'_d(k) = 1$, as noted after Definition III.5.1. This is a contradiction and implies that $C^d_{n;l} \not\subset J^d_{odd,+}$ for all $n \geq N'$ and for all even $l < d/2$. The claim now follows. $\square$

Theorem III.5.10. a) *Let $I^0 \subset R^d$ be the ideal defining $X^0_d \subset V(d)$ (as defined in Definition III.3.3 and Remark III.3.4). Then I^0 is homogeneous, and*

(III.5.2)
$$R^d_{(0,0)} \oplus R^d_{(1,d)} \oplus \left(R^d_{(2,4)} \oplus \ldots \oplus R^d_{(2,d-4)}\right) \oplus \bigoplus_{n \geq 2} \left(C^d_{n;0} \oplus C^d_{n;2} \oplus C^d_{n;d/2-2}\right)$$

is an SL_2-stable homogeneous complement of I^0 in R^d.

b) *Let $I \subset R^d$ be an SL_2-stable ideal with $h_{R^d/I} = h'_d$ that defines a a closed subscheme of $V(d)$ having X_d as underlying variety. Then $I_{(k)} = I^0_{(k)}$ for all $k \neq \{4, 8, \ldots, d-4\}$.*

c) *The ideal $J^d_{odd,+}$ is contained in both I^0 and I, and has finite codimension in both I^0 and I. Moreover, $(a_d)^n I \subset J^d_{odd,+}$ and $(a_d)^n I^0 \subset J^d_{odd,+}$ for all n large enough.*

Proof. c) We have already noted right after Definition III.5.1 that $J^d_{odd,+} \subset I$. Combining Lemma III.5.9 with the facts that $J^d_{odd,+} \subset I$ and that $h_{R^d/I} = h'_d$, one sees that $I_{(n,k)} = (J^d_{odd,+})_{(n,k)}$ for all k and for all $n \geq N$ (with N as in Lemma III.5.9). If now $i \in I$, then $(a_d)^N i \in \bigoplus_{n \geq N, k \in \mathbf{N}} I_{(n,k)} = \bigoplus_{n \geq N, k \in \mathbf{N}} (J^d_{odd,+})_{(n,k)} \subset J^d_{odd,+}$, and the claim follows. (Since this holds for any I as above, it holds in particular for $I = I^0$.)

b) Consider (III.5.2). Clearly $R^d_{(0,0)} \not\subset I(\mathcal{S}(d))$ and $R^d_{(1,d)} \not\subset I(\mathcal{S}(d))$. From Lemma III.3.5 a) it follows that also $R^d_{(2,4)}, \ldots, R^d_{(2,d-4)}$ don't vanish on $\mathcal{S}(d)$. Finally, Lemma III.5.6 shows that the covariants $C^d_{n;l}$ are not in $I(\mathcal{S}(d))$.

Let k be a multiple of 4. Then there exists a unique covariant V of order k in the sum (III.5.2). Let n be its degree. Since $V(d)$ possesses no covariants of degree $n' < n$ and of order k, since $V \not\subset I(\mathcal{S}(d))$, and since $\mathcal{S}(d)$ is multiplicity-free as $\mathbf{G}_m \times \mathrm{SL}_2$-variety, it follows that

$$\mathcal{O}(\mathcal{S}(d))_{(k)} = (V + I(\mathcal{S}(d)) \oplus \bigoplus_{n' > n} \mathcal{O}(\mathcal{S}(d))_{(n',k)}.$$

Let $I_0' \subset \mathcal{O}(\mathcal{S}(d))$ be the ideal defining X_d^0 as closed subscheme of $\mathcal{S}(d)$ (cf. Remark III.3.4). The description of I_0' in Section II.2 shows that $V + I(\mathcal{S}(d) \not\subset I_0'$, or that $(I_0')_{(k)} = \bigoplus_{n'>n} \mathcal{O}(\mathcal{S}(d))_{(n',k)}$. Hence it follows that $V \not\subset I^0$. This shows that the sum (III.5.2) is contained in an SL_2-stable complement of I^0 in R^d. Since $h_{R^d/I^0} = h_d'$ and since the sum (III.5.2) is isomorphic to $\bigoplus_{k\geq 0} V(4k)$ as SL_2-module, it follows that (III.5.2) is an SL_2-stable complement of I^0 in R^d.

b) If k is not a multiple of 4, then $I_{(k)} = I^0_{(k)} = R^d_{(k)}$ anyway, and there is nothing to show. So let $k \geq d$ be a multiple of 4. According to c) there exists $K \in \mathbf{N}$ such that $I^0_{(k')} = (J^d_{odd,+})_{(k')} = I_{(k')}$ for all $k' \geq K$. We now apply Lemma III.5.4: Let $C^d_{n;l}$ be the covariant of order k in the sum (III.5.2), and choose $0 \neq r \in (C^d_{n;l})^U \subset (R^d_{(n,k)})^U$. Choose $n_0 \in \mathbf{N}$ such that $k + n_0 d \geq K$. Now $h_{R^d/I}(k) = h_{R^d/I}(k + n_0 d) = 1$. Furthermore, $I_{(k+n_0d)} = I^0_{(k+n_0d)} = (J^d_{odd,+})_{(k+n_0d)}$, and this is homogeneous in $R^d_{(k+n_0d)}$. Finally, $r \in (R^d_{(n,k)})^U$ and $(a_d)^{n_0} \in (R^d_{(n_0,n_0d)})^U$ are both homogeneous, but $r \cdot (a_d)^{n_0} \in (C^d_{n+n_0;l})^U \not\subset I$, and hence $r \cdot (a_d)^{n_0} \notin I$. Now Lemma III.5.4 implies that $I_{(k)} = \{f \in R_{(k)} \mid f \cdot (a_d)^{n_0} \in I_{(k+n_0d)}\} = I^0_{(k)}$. $\square$

Remark III.5.11. From (III.5.2) it follows immediately that $I^0_{(k)} = \bigoplus_{n\geq 3} R^d_{(n,k)}$ for all $k \in \{4, 8, \ldots, d-4\}$.

Remark III.5.12. Let $I \subset R^d$ be as in Theorem III.5.10. From Theorem III.5.10 it follows that all homogeneous covariants of $V(d)$ of degree large enough and of co-order $l \geq d/2$ are contained in I.

In order to describe I, it thus suffices to study the isotypic components of $4, 8, \ldots, d-4$. (The philosophy of reducing the moduli-problem to finitely many isotypic components can already be encountered in [**HS04**].)

III.6. Deformations of orbits in minimal sheets, Part II

Let R^d still be $\mathrm{Sym}(V(d)^*)$, and let $R^d_{(k)}$ and $R^d_{(n,k)}$ be as in the preceding section. In this section many computations have been performed with aid of a computer. These computations can be found in Appendix A. Let d still be a multiple of 4.

Lemma III.6.1. *Let $X = \mathrm{SL}_2 \cdot \gamma x^{d/2} y^{d/2} \subset V(d)$ for some $0 \neq \gamma \in k$ (with reduced structure). Then*

$$\dim_k(T_X \operatorname{Hilb}^{\mathrm{SL}_2}_{h_d'}(V(d))) = 1.$$

Proof. It is no restriction to assume that $\gamma = 1$. Proposition 1.15 in [**AB05**] applies and states that:

$$T_X \operatorname{Hilb}^{\mathrm{SL}_2}_{h_d'}(V(d)) \cong [V(d)/\mathfrak{sl}_2 \cdot x^{d/2} y^{d/2}]^{N_{\mathrm{SL}_2}(T)}.$$

To compute $\mathfrak{sl}_2 \cdot x^{d/2}y^{d/2}$ observe that

$$\begin{pmatrix} 1+\epsilon h & \epsilon e \\ \epsilon f & 1-\epsilon h \end{pmatrix} \cdot x^{d/2}y^{d/2} = x^{d/2}y^{d/2} - \epsilon \frac{d}{2}(e x^{d/2-1}y^{d/2+1} + f x^{d/2+1}y^{d/2-1})$$

(with $\epsilon^2 = 0$). Hence $\mathfrak{sl}_2 \cdot x^{d/2}y^{d/2} = k \cdot x^{d/2-1}y^{d/2+1} \oplus k \cdot x^{d/2+1}y^{d/2-1}$. Now

$$V(d) = kx^{d/2}y^{d/2} \oplus \bigoplus_{i=0}^{d/2-1} (kx^i y^{d/2-i} \oplus kx^{d/2-i}y^i)$$

is a decomposition into $N_{\mathrm{SL}_2}(T)$-stable subspaces. Hence

$$\begin{aligned}[V(d)/\mathfrak{sl}_2 \cdot x^{d/2}y^{d/2}]^{N_{\mathrm{SL}_2}(T)} &\cong (kx^{d/2}y^{d/2})^{N_{\mathrm{SL}_2}(T)} \oplus \\ &\quad \bigoplus_{i=0}^{d/2-2} (kx^i y^{d/2-i} \oplus kx^{d/2-i}y^i)^{N_{\mathrm{SL}_2}(T)} \\ &= kx^{d/2}y^{d/2}. \end{aligned}$$

□

Proposition III.6.2. *There is an isomorphism*

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))_{\mathrm{red}} \cong \mathbf{A}^1 \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))_{\mathrm{red}},$$

and $\mathbf{A}^1$ *and* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))$ *intersect in exactly one point* p, *corresponding to* X^0_d. *Furthermore,* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))$ *is smooth in* $\mathbf{A}^1 \setminus \{p\}$.

Proof. First, Corollary II.3.4 states that

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))_{\mathrm{red}} = \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))^0_{\mathrm{red}} \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))_{\mathrm{red}}$$

for an irreducible component $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))^0_{\mathrm{red}}$ of $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))_{\mathrm{red}}$. Using Theorem III.3.2, we see that $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{S}(d))^0_{\mathrm{red}} \cong \mathbf{A}^1$, and we find that

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))_{\mathrm{red}} \cong \mathbf{A}^1 \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(\mathcal{N}(d))_{\mathrm{red}}.$$

Further, the two components intersect in p. The last statement can finally be deduced from Lemma III.6.1. □

The point p corresponds to X^0_d, which is non-reduced for $d > 4$. This makes matters slightly more complicated, because we cannot use [**AB05**], Proposition 1.15, to determine the dimension of the Zariski tangent-space $T_p \,\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))$. Instead, we use [**AB05**], Proposition 1.13, which states that

$$T_I \,\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d)) \cong \mathrm{Hom}^{\mathrm{SL}_2}_{R^d}(I, R^d/I)$$

for an SL_2-stable $I \subset R^d$ with $h_{R^d/I} = h'_d$.

Before investigating $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_d}(V(d))$ for some values of d, we state another auxiliary result:

Lemma III.6.3. *Let $I \subset R^d$ be an* SL_2*-stable ideal with $h_{R^d/I} = h'_d$ and with* $\mathrm{Spec}(R^d/I)_{\mathrm{red}} = X_d$. *Further, let $\varphi \in \mathrm{Hom}_{R^d}^{\mathrm{SL}_2}(I, R^d/I)$. Then the restriction $\varphi|_{I_{(k)}}$ is determined by $\varphi|_{R^d_{(2,d)}}$ for all $k \notin \{4, 8, \ldots, d-4\}$.*

Proof. Step 1. Let $k \in \mathbf{N} \setminus \{4, 8, \ldots, d-4\}$. If k is not a multiple of 4, then $(R^d/I)_{(k)} = 0$ and $\varphi|_{I_{(k)}} = 0$.

Suppose now that $k \geq d$ is a multiple of 4, let $n \in \mathbf{N}$ such that $(n-1)d < k \leq nd$, and let $l := (nd-k)/2$. Recall from Theorem III.5.10 that

$$(R^d/I)_{(k)} = C^d_{n;l}.$$

In particular, this shows that the map $\mu_k \colon (R^d/I)^U_{(k)} \to (R^d/I)^U_{(k+d)}$ defined by $\mu_k(r + I_{(k)}) = a_d r + I_{(k+d)}$ is an isomorphism. For $r \in I_{(k)}$, there exists $n_0 \in \mathbf{N}$ such that $r(a_d)^{n_0} \in J^d_{odd,+}$, thanks to Theorem III.5.10 c). Hence $\varphi(r(a_d)^{n_0})$ is determined by $\varphi|_{J^d_{odd,+}}$. But

$$\varphi(r(a_d)^{n_0}) = (a_d)^{n_0}\varphi(r) = (\mu_{k+(n_0-1)d} \circ \ldots \circ \mu_k)(\varphi(r)).$$

Because each μ_{k+md} is an isomorphism, it follows that

$$\varphi(r) = (\mu_{k+(n_0-1)d} \circ \ldots \circ \mu_k)^{-1}(\varphi(r(a_d)^{n_0})),$$

which is determined by $\varphi|_{J^d_{odd,+}}$.

Now, $J^d_{odd,+}$ is generated as R^d-SL_2-module by covariants of odd co-orders and by $R^d_{(2,d)}$. Since any covariant of odd co-order is in the kernel of φ anyway, we conclude that $\varphi|_{J^d_{odd,+}}$ is determined by $\varphi|_{R^d_{(2,d)}}$. The claim now follows. □

Forms of degree 4. As a warm-up exercise, we start with a well-known example, which can be found in [**AB05**] Example 1 on page 99, or in [**Ja05**], Théorème 1.1. Let $d = 4$.

Proposition III.6.4. *The inclusion* $\iota \colon \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_4}(\mathcal{S}(4)) \to \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_4}(V(4))$ *is an isomorphism.*

Proof. Let $I \subset R^4$ be an SL_2-stable ideal with Hilbert function $h_{R^d/I} = h'_4$ and with $\mathrm{Spec}(R^4/I)_{\mathrm{red}} = X_4$. Then Theorem III.5.10 c) implies that $I = I^0$. Hence ι is bijective.

In view of Proposition III.6.2 the proof is completed once we have shown that $\dim_k T_{I^0} \mathrm{Hilb}^{\mathrm{SL}\,2}_{h_4}(V(4)) = \dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^4}(I^0, R^4/I^0)) \leq 1$. Instead of computing the tangent space directly with [**AB05**], Proposition 1.15, we make use of Lemma III.6.3. Applied with $I = I^0$ it states that $\varphi \in \mathrm{Hom}^{\mathrm{SL}_2}_{R^4}(I^0, R^4/I^0)$ is determined by $\varphi|_{R^4_{(2,4)}}$. Hence

$$\dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^4}(I^0, R^4/I^0)) \leq \dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^4}(R^4_{(2,d)}, R^4/I^0)).$$

With Schur's Lemma it follows that

$$\dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^4}(R^4_{(2,4)}, R^4/I^0)) = 1.$$

□

Forms of degree 8.

Theorem III.6.5. *The inclusion* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_8}(\mathcal{S}(8)) \to \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_8}(V(8))$ *is an isomorphism.*

Proof. Let $I \subset R^8$ be an SL_2-stable ideal with Hilbert function $h_{R^d/I} = h'_8$ and with $\mathrm{Spec}(R^8/I)_{\mathrm{red}} = X_8$. In view of Proposition III.6.2 it suffices to show that (a) $I = I^0$ and that (b) $\dim_k T_{I^0}\,\mathrm{Hilb}^{\mathrm{SL}\,2}_{h_8}(V(8)) \leq 1$.

a) Theorem III.5.10 c) implies that $I_{(k)} = I^0_{(k)}$ for all $k \neq 4$. Therefore, it suffices to verify that $I_{(4)} = I^0_{(4)}$ (which equals $\bigoplus_{n\geq 3} R^8_{(n,4)}$ according to Remark III.5.11). Theorem III.5.10 b) and c) imply that $R^8_{(n,k)} \subset I$ for all (n,k) with $n \geq 3$ and $k \in \{\kappa \mid 0 \leq \kappa \leq 8n-6\} \setminus \{4\}$. A computer-based calculation (see Appendix A) shows that $R^8_{(3,4)}$ is in the SL_2-stable ideal generated by $R^8_{(2,8)} \subset I$. Hence $\bigoplus_{k=0}^{18} R^8_{(3,k)} \subset I$. Since $R^8_{(n+1,4)}$ is in the SL_2-stable ideal generated by $\bigoplus_{k=0}^{12} R^8_{(n,k)}$, it follows inductively that $R^8_{(n,4)} \subset I$ for all $n \geq 4$, which shows that

$$I_{(4)} = \bigoplus_{n\geq 3} R^8_{(n,4)} = I^0_{(4)}.$$

b) Lemma III.6.3 implies that

$$\begin{aligned}\dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^8}(I^0, R^8/I^0)) \leq{}& \dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^8}(R^8_{(2,8)}, R^8/I^0)) \\ &+ \dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^8}(I^0_{(4)}, R^8/I^0)).\end{aligned} \tag{III.6.1}$$

The proof of a) reveals that $I^0_{(4)}$ is contained in the SL_2-stable ideal generated by the covariants in I^0 of order $\neq 4$. Hence (III.6.1) simplifies to

$$\dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^8}(I^0, R^8/I^0)) \leq \dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^8}(R^8_{(2,8)}, R^8/I^0)) = 1,$$

where we used Schur's lemma for the equality. $\square$

Forms of degree 12.

Theorem III.6.6. *Let* $p \in \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(\mathcal{S}(12))$ *be the point corresponding to* X^0_{12}. *The natural morphism* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(\mathcal{S}(12)) \to \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))_{\mathrm{red}}$ *is an isomorphism, and* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$ *is smooth in the complement of* p. *The invariant Hilbert scheme* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$ *is not reduced in* p.

Proposition III.6.7. *Let* $I \subset R^{12}$ *be an* SL_2*-stable ideal with Hilbert function* $h_{R^{12}/I} = h'_{12}$ *and with* $\mathrm{Spec}(R^{12}/I)_{\mathrm{red}} = X_{12}$. *Then* $I = I^0$.

Proof. Theorem III.5.10 implies that $I_{(k)} = I^0_{(k)}$ for all $k \notin \{4,8\}$. Therefore, it suffices to verify that $I_{(4)} = I^0_{(4)}$ ($= \bigoplus_{n\geq 3} R^{12}_{(n,4)}$ according to Remark III.5.11) and that $I_{(8)} = I^0_{(8)}$ ($= \bigoplus_{n\geq 3} R^{12}_{(n,8)}$ according to Remark III.5.11).

Step 1. We prove by induction on n that

$$\bigoplus_{k=0}^{12n-10} R^{12}_{(n,k)} \subset I \tag{III.6.2}$$

for all $n \geq 4$. Recall from Theorem III.5.10 that $R^{12}_{(n,k)} \subset I$ for all (n,k) with $n \geq 2$ and $k \in \{\kappa \mid 0 \leq \kappa \leq 12n-10\} \setminus \{4,8\}$. A computer-based calculation shows that $R^{12}_{(4,4)} \oplus R^{12}_{(4,8)}$ is in the ideal generated by $R^{12}_{(3,0)} \oplus R^{12}_{(3,2)} \oplus R^{12}_{(3,6)} \oplus R^{12}_{(3,10)} \oplus R^{12}_{(3,12)} \oplus R^{12}_{(3,14)} \oplus R^{12}_{(3,16)} \oplus R^{12}_{(3,18)} \subset I$. This shows that $\bigoplus_{k=0}^{38} R^{12}_{(4,k)} \subset I$. Since $R^{12}_{(n+1,4)} \oplus R^{12}_{(n+1,8)}$ is in the ideal generated by $\bigoplus_{k=0}^{20} R^{12}_{(n,k)}$, it follows inductively that $R^{12}_{(n,4)} \oplus R^{12}_{(n,8)} \subset I$ for all $n \geq 4$, which proves (III.6.2).

Step 2. A computer-based calculation shows that $R^{12}_{(3,4)}$ is in the ideal generated by $R^{12}_{(2,12)}$. This, together with $R^{12}_{(0,4)} \oplus R^{12}_{(1,4)} = \{0\}$ and with $R^{12}_{(3,4)} \cong V(4)$, shows that $I_{(4)} = \bigoplus_{n\geq 3} R^{12}_{(n,4)}$.

Step 3. We claim that

$$I_{(8)} = \bigoplus_{n\geq 3} R^{12}_{(n,8)}.$$

What do we know on $R^{12}_{(8)}$ (see Appendix A)?

a) First, $R^{12}_{(0,8)} = R^{12}_{(1,8)} = \{0\}$, $R^{12}_{(2,8)} \cong V(8)$ and $R^{12}_{(3,8)} \cong V(8) \oplus V(8)$.
b) Step 1 implies that $R^{12}_{(n,8)} \subset I$ for $n \geq 4$.
c) Finally, $R^{12}_{(3,8)}$ is in the ideal generated by $R^{12}_{(2,8)} \oplus R^{12}_{(2,12)}$.

Since $R^{12}_{(2,12)} \subset I$, it follows that $C := (R^{12}_{(2,12)}, R^{12}_{(1,12)})^8 \subset I$. Let further $C' := (R^{12}_{(2,8)}, R^{12}_{(1,12)})^6$. Then $R^{12}_{(3,8)} = C \oplus C'$, and both $C \neq 0$ and $C' \neq 0$ thanks to a) and c).

We claim that $C' \subset I$. There exists a covariant $C'' \subset R^{12}_{(2,8)} \oplus C'$ contained in I. Then $(C'', R^{12}_{(1,12)})^6 \subset I \cap (R^{12}_{(3,8)} \oplus R^{12}_{(4,8)})$. Let $0 \neq r_3 + r_4 \in ((C'', R^{12}_{(1,12)})^6)^U$ with r_3 homogeneous of degree 3 and r_4 homogeneous of degree 4. Because $r_4 \in R^{12}_{(4,8)} \subset I$ anyway, it follows that $r_3 \in I$. If $C'' = C'$, then $C' \subset I$. Otherwise, if $C'' \neq C'$, then $0 \neq r_3 \in I$. But $r_3 \in (C')^U$, and it follows that $C' \subset I$. In any case $C' \subset I$, and hence $R^{12}_{(3,8)} = C \oplus C' \subset I$. We see that $I_{(8)} = \bigoplus_{n\geq 3} R^{12}_{(n,8)}$. ☐

Proposition III.6.8. $\dim_k T_{I^0} \operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12)) \leq 2$.

Proof. The proof of Proposition III.6.7 shows that I^0 is the SL_2-stable ideal generated by the covariant $C' = (R^{12}_{(2,8)}, R^{12}_{(1,12)})^6 \subset R^{12}_{(3,8)}$ and by all covariants in I^0 of order different from 4 and 8. Hence Lemma

III.6.3 implies that

(III.6.3)

$$\begin{aligned}\dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^{12}}(I^0, R^{12}/I^0)) \leq{}& \dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^{12}}(R^{12}_{(2,12)}, R^{12}/I^0)) \\ &+ \dim_k(\mathrm{Hom}^{\mathrm{SL}_2}_{R^{12}}(C', R^{12}/I^0)) = 2,\end{aligned}$$

where we used again Schur's Lemma for the equality. □

Proposition III.6.9. $\dim_k T_{I^0} \operatorname{Hilb}^{\mathrm{SL}_2}_{h_{\mathcal{S}_{12}}}(V(12)) \geq 2$.

Proof. Step 1. We have seen in the proof of Proposition III.6.8 that the covariants of order 8 play a special role. Recall that $R^{12}_{(3,8)} = C \oplus C'$ with $C = (R^{12}_{(2,12)}, R^{12}_{(1,12)})^8$ and $C' = (R^{12}_{(2,8)}, R^{12}_{(1,12)})^6$. Decompose I^0 as $I^0 = I' \oplus C'$, where I' is the SL_2-stable ideal generated by all covariants of order $\neq 4, 8$. We claim that there is a homomorphism $0 \neq \varphi \in \mathrm{Hom}^{\mathrm{SL}_2}_{R^{12}}(I^0, R^{12}/I^0)$ with $\varphi|_{I'} = 0$. Let $0 \neq \varphi_0 \in \mathrm{Hom}^{\mathrm{SL}_2}(C', R^{12}/I^0)$, and define

$$\varphi\colon I^0 = C' \oplus I' \to R^{12}/I^0, \quad c' + i' \mapsto \varphi_0(c').$$

Then φ is SL_2-equivariant. To see that φ is a homomorphism of R^{12}-modules, let $i = c' + i' \in I$ with $c' \in C'$ and with $i' \in I'$, and let $r = \sum_{n \in \mathbf{N}} r_n \in R^{12}$ with r_n homogeneous of degree n. Then

$$\varphi(ri) = \varphi(r_0 c' + \sum_{n>0} r_n c' + ri') = \varphi(r_0 c') = r_0 \varphi(c').$$

On the other hand,

$$r\varphi(i) = r\varphi(c' + i') = r\varphi(c') = r_0\varphi(c') + \sum_{n>0} r_n \varphi(c') = r_0 \varphi(c').$$

This shows that φ is a homomorphism of R^{12}-modules.

Step 2. We construct a second tangent vector in $\mathrm{Hom}^{\mathrm{SL}_2}_{R^{12}}(I^0, R^{12}/I^0)$ that is linearly independent from φ constructed above. Whereas φ describes an infinitesimal deformation that cannot be seen on the variety underlying the Hilbert scheme, we now construct 'the' tangent vector coming from the closed subscheme $\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(\mathcal{S}(12))$ of $\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$.

Let $D = k[t]/(t^2)$. Recall that $T_{I^0}(\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12)))$ consists of those SL_2-stable ideals $J^0 \subset D \otimes R^{12}$ with the property that

a) $(D \otimes R^{12})/(J^0, t) = I^0$, and that
b) $(D \otimes R^{12})/J^0$ is flat over D.

If J^0 is such an ideal and if $u + tv \in J^0$, then the corresponding homomorphism in $\mathrm{Hom}^{\mathrm{SL}_2}_{R^{12}}(I^0, R^{12}/I^0)$ maps u to $v + I^0$.

We now construct such an ideal J^0. Let $\tilde{J}^0 \subset k[t] \otimes R^{12}$ be the ideal of $\operatorname{Univ}^{\mathrm{SL}_2}_{h'_{12}}(\mathcal{S}(12)) \subset \mathbf{A}^1 \times V(12)$. Then $J^0 := \tilde{J}^0/(t^2)$ satisfies a) and b). We claim that

$$5a_0 t + 5082 \cdot (-10a_9^2 + 15a_{10}a_8 - 6a_{11}a_7 + a_{12}a_6) \in \tilde{J}^0.$$

Even though this can be verified with the help of a computer, we give a proof here. Consider the covariant $Q^{(12)} = \sum_{i=0}^{12} a_i \binom{12}{i} x^i y^{12-i}$ and the transvectant

$$\begin{aligned}(Q^{(12)}, Q^{(12)})^6 = &\, 2(-10a_9^2 + 15a_{10}a_8 - 6a_{11}a_7 + a_{12}a_6)x^{12} \\ &+ \sum_{j=1}^{12} \varphi_j(a_i)x^j y^{12-j}.\end{aligned}$$

This shows that $-10a_9^2 + 15a_{10}a_8 - 6a_{11}a_7 + a_{12}a_6$ is a U-invariant T-eigenvector to the weight 12. The same holds for a_{12}. For $(\sigma,\tau) \in k^2$ consider the binary form

$$\begin{aligned}v_{\sigma,\tau} &= \tau x^6(\sigma x + y)^6 \\ &= \tau(x^6y^6 + 6\sigma x^7y^5 + 15\sigma^2x^8y^4 + 20\sigma^3x^9y^3 + 15\sigma^4x^{10}y^2 + 6\sigma^5x^{11}y + \sigma^6x^{12}) \\ &\in \mathrm{SL}_2 \cdot \tau x^6y^6\end{aligned}$$

Now $a_{12}(v_{\sigma,\tau}) = \tau\sigma^6$, $a_{11}(v_{\sigma,\tau}) = \frac{6}{12}\tau\sigma^5$, $a_{10}(v_{\sigma,\tau}) = \frac{15}{66}\tau\sigma^4$, $a_9(v_{\sigma,\tau}) = \frac{20}{220}\tau\sigma^3$, $a_8(v_{\sigma,\tau}) = \frac{15}{495}\tau\sigma^2$, $a_7(v_{\sigma,\tau}) = \frac{6}{792}\tau\sigma$, and $a_6(v_{\sigma,\tau}) = \frac{1}{924}\tau$. Hence

$$\begin{aligned}&(-10a_9^2 + 15a_{10}a_8 - 6a_{11}a_7 + a_{12}a_6)(v_{\sigma,\tau}) \\ &\quad = \left(-10\frac{20^2}{220^2} + 15\frac{15^2}{66\cdot 495} - 6\frac{6^2}{12\cdot 792} + \frac{1}{924}\right)\tau^2\sigma^6 = -\frac{5}{5082}\tau^2\sigma^6.\end{aligned}$$

Since $(\tau, \tau x^6y^6) \in \mathrm{Univ}_{h'_{12}}^{\mathrm{SL}_2}(\mathcal{S}(12)) \subset \mathbf{A}^1 \times V(12)$, we conclude that $f(v_{\sigma,\tau}) = 0$ for all $(\sigma,\tau) \in k^2$ for

$$f = 5ta_{12} + 5082(-10a_9^2 + 15a_{10}a_8 - 6a_{11}a_7 + a_{12}a_6).$$

Because f is a U-invariant T-eigenvector and $f(g\tau x^6y^6) = 0$ for all $g \in U^- = \left\{\begin{pmatrix}1 & \\ s & 1\end{pmatrix}\right\}$, we find that $f(g\tau x^6y^6) = 0$ for all $g \in UTU^-$ and for all $\tau \in k$. Since UTU^- is dense in SL_2 (as one readily verifies), the set $\{(\tau, g\tau x^6y^6) \mid g \in UTU^-, \tau \in k\}$ is dense in $\mathrm{Univ}_{h'_{12}}^{\mathrm{SL}_2}(\mathcal{S}(12))$, and we conclude that $f \in \tilde{J}^0$. Now f corresponds to a $\psi \in \mathrm{Hom}_{R^{12}}^{\mathrm{SL}_2}(I^0, R^{12}/I^0)$ which maps

$$5082 \cdot (-10a_9^2 + 15a_{10}a_8 - 6a_{11}a_7 + a_{12}a_6) \in R^{12}_{(2,12)}$$

to $0 \neq 5a_{16} + I^0 \in R^{12}/I^0$. Since φ and ψ are linearly independent ($\varphi|_{R^{12}_{(2,12)}} = 0 \neq \psi|_{R^{12}_{(2,12)}}$), this shows that $\dim_k(\mathrm{Hom}_{R^{12}}^{\mathrm{SL}_2}(I^0, R^{12}/I^0)) \geq 2$. □

Proof of Theorem III.6.6. Lemma II.3.2 and Propositon III.6.7 imply that the inclusion morphism $\mathrm{Hilb}_{h'_{12}}^{\mathrm{SL}_2}(\mathcal{S}(12)) \to \mathrm{Hilb}_{h'_{12}}^{\mathrm{SL}_2}(V(12))$ is bijective. Thanks to Proposition I.2.1 we now see that $\mathrm{Hilb}_{h'_{12}}^{\mathrm{SL}_2}(\mathcal{S}(12)) = \mathrm{Hilb}_{h'_{12}}^{\mathrm{SL}_2}(V(12))_{\mathrm{red}}$.

Moreover, $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$ cannot be reduced since its Zariski-tangent space at the ideal I^0 is two-dimensional. At all other closed points $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{12}}(V(12))$ is smooth according to Lemma III.6.1. □

Forms of degree 16. There is the following description for the part of the invariant Hilbert scheme coming from the nullcone:

Proposition III.6.10. *There are isomorphisms*

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{N}(16))_{\mathrm{red}} \cong \mathbf{P}^1 \cup \mathbf{P}^1,$$

and the two copies of $\mathbf{P}^1$ *intersect in a closed point* p *corresponding to* X^0_{16}.

The ultimate goal of this section consists in proving the following result:

Theorem III.6.11. a) *There are isomorphisms*

$$\begin{aligned}\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))_{\mathrm{red}} &\cong \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{S}(16)) \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{N}(16))_{\mathrm{red}} \\ &\cong \mathbf{A}^1 \cup \mathbf{P}^1 \cup \mathbf{P}^1,\end{aligned}$$

and the three irreducible components intersect in one closed point p *corresponding to* X^0_{16}.

b) *The invariant Hilbert scheme* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))$ *is smooth in the complement of* p.

c) *The action of* $\mathbf{G}_m$ *on* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))$ *induced by the* $\mathbf{G}_m$*-action on* $V(16)$ *has three closed fixed points: The point* p*, and on each copy of* $\mathbf{P}^1$ *one further closed fixed point.*

Let I' be the intersection of all SL_2-stable ideals $I \subset R^{16}$ with Hilbert function $h_{R^{16}/I} = h'_{16}$ and with $\mathrm{Spec}(R^{16}/I)_{\mathrm{red}} = X_{16}$. Define $Y := \mathrm{Spec}(R/I')$. Then the induced morphism $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y) \to \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{N}(16))$ is bijective.

Lemma III.6.12. *a) The* SL_2*-scheme* Y *is multiplicity-finite. Its Hilbert function is given by* $h_Y(k) = h'_{16}(k)$ *for all* $k \neq 8, 12$*, and* $h_Y(8) = h_Y(12) = 2$.

b) Let further Y_8 *be the closed* SL_2*-stable subscheme of* Y *defined by the ideal* $I'_8 := I' \cup \bigoplus_{n\geq 3} R^{16}_{(n,8)} \subset R^{16}$*. Similarly, let* Y_{12} *be the closed* SL_2*-stable subscheme of* Y *defined by the ideal* $I'_{12} := I' \cup \bigoplus_{n\geq 3} R^{16}_{(n,12)}$*. Then* $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y)_{\mathrm{red}} = \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_8)_{\mathrm{red}} \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_{12})_{\mathrm{red}}$ *is the decomposition into irreducible components, and the components intersect in one closed point.*

c) There are isomorphisms $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_8) \cong \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_{12}) \cong \mathbf{P}^1$.

Proof of Proposition III.6.10. Lemma III.6.12 implies that

$$\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{N}(16))_{\mathrm{red}} \cong \mathbf{P}^1 \cup \mathbf{P}^1$$

and that the two projective lines intersect in one closed point. $\square$

Proof of Lemma III.6.12. Let $I \subset R^{16}$ be any SL_2-stable ideal such that $\mathrm{Spec}(R^{16}/I)_{\mathrm{red}} = X_{16}$ and with Hilbert function $h_{R^{16}/I} = h'_{16}$. Theorem III.5.10 implies that $I_{(k)} = I^0_{(k)}$ for all $k \notin \{4, 8, 12\}$. Therefore, it suffices to examine $I_{(4)}$, $I_{(8)}$, and $I_{(12)}$.

Step 1. We prove by induction on n that

$$\bigoplus_{k=0}^{16n-14} R^{16}_{(n,k)} \subset I \tag{III.6.4}$$

for all $n \geq 4$. For the proof recall from Theorem III.5.10 that $R^{16}_{(n,k)} \subset I$ for all $n \geq 3$ and for all $k \in \{\kappa \mid 0 \leq \kappa \leq 16n - 14\} \setminus \{4, 8, 12\}$. A computer-based calculation shows that $R^{16}_{(4,4)} \oplus R^{16}_{(4,8)} \oplus R^{16}_{(4,12)}$ is in the ideal generated by $R^{16}_{(3,14)} \oplus R^{16}_{(3,16)} \oplus R^{16}_{(3,18)} \oplus R^{16}_{(3,20)} \oplus R^{16}_{(3,22)} \subset I$. This shows that $\bigoplus_{k=0}^{50} R^{16}_{(4,k)} \subset I$. Since $R^{16}_{(n+1,4)} \oplus R^{16}_{(n+1,8)} \oplus R^{16}_{(n+1,12)}$ is in the SL_2-stable ideal generated by $\bigoplus_{k=0}^{28} R^{16}_{(n,k)}$ it follows inductively that $R^{16}_{(n,4)} \oplus R^{16}_{(n,8)} \oplus R^{16}_{(n,12)} \subset I$ for all $n \geq 4$, which proves (III.6.4).

Step 2. A computer-based calculation shows that $R^{16}_{(3,4)}$ is in the ideal generated by $R^{16}_{(2,16)}$. This, together with $R^{16}_{(0,4)} \oplus R^{16}_{(1,4)} = \{0\}$ and with $R^{16}_{(3,4)} \cong V(4)$, shows that $I_{(4)} = \bigoplus_{n \geq 3} R^{16}_{(n,4)}$. This is the same as $I^0_{(4)}$ according to Remark III.5.11.

Step 3. We are left to examine $I_{(8)}$ and $I_{(12)}$, which turn out to be weirdly entangled. We claim that

$$I_{(8)} = I^0_{(8)} \left(= \bigoplus_{n \geq 3} R^{16}_{(n,8)} \right) \quad \text{or} \quad I_{(12)} = I^0_{(12)} \left(= \bigoplus_{n \geq 3} R^{16}_{(n,12)} \right). \tag{III.6.5}$$

What do we know on $R^{16}_{(8)}$ (see Appendix A)?

(1) First, $R^{16}_{(0,8)} = R^{16}_{(1,8)} = \{0\}$, $R^{16}_{(2,8)} \cong V(8)$ and $R_{(3,8)} \cong V(8) \oplus V(8)$.
(2) According to the first step, $R^{16}_{(n,8)} \subset I$ for $n \geq 4$.
(3) A computer-based computation shows that the two covariants $(R^{16}_{(2,8)}, R^{16}_{(1,16)})^8$ and $(R^{16}_{(2,16)}, R^{16}_{(1,16)})^{12}$ coincide. Denote this covariant by C.
(4) Let $C' = (R^{16}_{(2,12)}, R^{16}_{(1,16)})^{10}$. A computer-based computation shows that $R^{16}_{(3,8)} = C \oplus C'$.

Hence there are two covariants $V(8) \cong C^8_1 \subset R^{16}_{(2,8)} \oplus R^{16}_{(3,8)}$ and $V(8) \cong C^8_2 \subset R^{16}_{(2,8)} \oplus R^{16}_{(3,8)}$ such that $I_{(8)} = C^8_1 \oplus C^8_2 \oplus \bigoplus_{n \geq 4} R^{16}_{(n,8)}$. Because $C \subset I$, this yields:

(8): There exists a unique covariant

$$V(8) \cong C^8 \subset R^{16}_{(2,8)} \oplus C'$$

such that $C^8 \subset I$. Further, the choice of C^8 determines $I_{(8)}$.

This proves in particular that $h_Y(8) \leq 2$.

Bearing this in mind, we proceed with $R^{16}_{(12)}$. Similarly as above, one has:

(1) First, $R^{16}_{(0,12)} = R^{16}_{(1,12)} = \{0\}$, $R^{16}_{(2,12)} \cong V(12)$ and $R^{16}_{(3,12)} \cong V(12) \oplus V(12) \oplus V(12)$.
(2) According to the first step $R^{16}_{(n,12)} \subset I$ for $n \geq 4$.
(3) Consider $C'' = (R^{16}_{(2,8)}, R^{16}_{(1,16)})^6$, $C''' = (R^{16}_{(2,12)}, R^{16}_{(1,16)})^8$, and $C'''' = (R^{16}_{(2,16)}, R^{16}_{(1,16)})^{10}$. Then $R^{16}_{(3,12)} = C'' \oplus C''' \oplus C''''$.

Since $C'''' \subset I$, it follows that there is are two covariants $V(12) \cong C^{12}_1 \subset R_{(2,12)} \oplus C'' \oplus C'''$ and $V(12) \cong C^{12}_2 \subset R^{16}_{(2,12)} \oplus C'' \oplus C'''$ such that

$$I_{(12)} = C^{12}_1 \oplus C^{12}_2 \oplus C'''' \oplus \bigoplus_{n\geq 4} R^{16}_{(n,12)}.$$

If $C^{12}_1 \oplus C^{12}_2 \subset R^{16}_{(3,12)}$, then $I_{(12)} = \bigoplus_{n\geq 3} R^{16}_{(n,12)}$, and clearly $C''' \subset I$. On the other hand, if $C^{12}_1 \not\subset R^{16}_{(3,12)}$, then consider $(C^{12}_1, R^{16}_{(1,16)})^8$, which is a covariant in $C''' \oplus R^{16}_{(4,12)}$ not contained in $R^{16}_{(4,12)}$. Since $R^{16}_{(4,12)} \subset I$ according to (2), we conclude that $C''' \subset I$. In any of these cases $C''' \subset I$, and hence:

(12): There exists a unique covariant

$$C^{12} \subset R^{16}_{(2,12)} \oplus C''$$

such that $C^{12} \subset I$. Further, the choice of C^{12} determines $I_{(12)}$.

This proves in particular that $h_{R^{16}/I'}(12) \leq 2$.

We now shall make use of the narrow relations between $I_{(8)}$ and $I_{(12)}$. Suppose that $C^8 \not\subset C'$, and consider $I \supset (C^8, R^{16}_{(1,16)})^6 \subset C'' \oplus R^{16}_{(4,12)}$. Since $R^{16}_{(4,12)} \subset I$ and since $(C^8, R^{16}_{(1,16)})^6 \not\subset R^{16}_{(4,12)}$ by assumption, this implies that $C'' \subset I$, which in turn implies that $C^{12} = C''$ or that $I_{(12)} = \bigoplus_{n\geq 3} R^{16}_{(n,12)}$.

Vice versa, one sees in exactly the same manner that $\bigoplus_{n\geq 3} R^{16}_{(n,8)}$ if $C^{12} \not\subset C''$. This proves (III.6.5), that

$$\text{(III.6.6)} \qquad \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y)_{\mathrm{red}} = \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_8)_{\mathrm{red}} \cup \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_{12})_{\mathrm{red}},$$

and that the two latter intersect in the closed point given by the ideal I^0 with $I^0_{(8)} = \bigoplus_{n\geq 3} R^{16}_{(n,8)}$ and with $I^0_{(12)} = \bigoplus_{n\geq 3} R^{16}_{(n,12)}$. This point is the point p of Theorem III.6.11 corresponding to X^0_{16}.

Step 4. We still have to show c), that $h_{R^{16}/I'}(8) = h_{R^{16}/I'}(12) = 2$, and that the decomposition in (III.6.6) is a decomposition into irreducible components.

Suppose first that $C^{12} = C''$ or that $I_{(12)} = \bigoplus_{n\geq 3} R^{16}_{(n,12)}$. Then it is easy to see that each choice of a covariant $C^8 \subset R^{16}_{(2,8)} \oplus C'$ yields an SL_2-stable ideal with Hilbert function h'_{16}. Therefore $h_{R^{16}/I'}(8) = 2$ (recall that we already have proved that $h_{R^{16}/I'}(8) \leq 2$) and $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_{12})_{\mathrm{red}} \not\subset \mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(Y_8)_{\mathrm{red}}$.

Now Y_8 is a multiplicity-finite SL_2-scheme with $h_{Y_8}(k) = h'_{16}(k)$ for all $k \neq 12$, and with $h_{Y_8}(12) = 2$. To give a family $(\mathfrak{X} \to S) \in \mathcal{H}ilb^{\mathrm{SL}_2}_{h'_{16}}(Y_8)(S)$ is the same as to give a locally free $\mathcal{O}_S$-submodule $\mathcal{F}$ of $\mathcal{O}_S \otimes k^2$ of rank 1, because each choice of a covariant $C^8 \subset R_{(2,8)} \oplus C'$ yields an SL_2-stable ideal with Hilbert function h'_{16}. But for this problem it is well-known that the corresponding moduli-scheme is a projective line. (One can see this also as follows: Let v_2^8 and v_3^8 be fixed non-zero U-invariant vectors in $R^{16}_{(2,8)}$ and C', respectively. Then the choice of $(\lambda : \mu) \in \mathbf{P}^1(k)$ amounts to the choice of the covariant $C^8 \subset R^{16}_{(2,8)} \oplus C'$ containing $\lambda v_2^8 \oplus \mu v_3^8$ as U-invariant vector.)

The case $I_{(8)} = \bigoplus_{n\geq 3} R^{16}_{(n,8)}$ is treated similarly. This completes the proof of the lemma. □

Remark III.6.13. The proof of Lemma III.6.12 shows that the action of $\mathbf{G}_m$ on $\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(\mathcal{N}(16))$ has exactly two closed fixed points on each irreducible component $\mathbf{P}^1$ (corresponding to the homogeneous ideals) and that p is such a fixed point.

The next object of interest is the Zariski tangent-space to the invariant Hilbert scheme. Before it is computed in Lemma III.6.14, we start with an observation. We write Q for $Q^{(16)}$, and we let $C^{16}_{n;l}$ be as above. For all $l \in \{0, 2, \ldots, 16\}$ fix $0 \neq u_{2;l} \in (C^{16}_{2;l})^U$. Remark III.5.8 implies that for all n large enough

$$(\text{III.6.7}) \qquad (a_{16})^n \cdot u_{2;12} + J^{16}_{odd} = \gamma \cdot (a_{16})^{n-4} \cdot u_{2;8} \cdot (u_{2;2})^2 + J^{16}_{odd}$$

for some $0 \neq \gamma \in k$. Further, Remark III.5.8 implies that for all n large enough

$$(\text{III.6.8}) \quad (a_{16})^n(((Q,Q)^{10},Q)^{10})^U + J^{16}_{odd} \subset k \cdot (a_{16})^{n-11} u_{2;8}(u_{2;2})^6 + J^{16}_{odd}.$$

Lemma III.6.14. *Let $I \subset R^{16}$ be an SL_2-stable ideal with Hilbert function $h_{R^{16}/I} = h'_{16}$ and with $\mathrm{Spec}(R^{16}/I)_{\mathrm{red}} = X_{16}$.*

a) *If $I \neq I^0$, then*

$$\dim_k(T_I \,\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))) = 1.$$

b) *Otherwise,*

$$\dim_k(T_{I^0} \,\mathrm{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))) = 3.$$

Proof. The ideal I is generated as SL_2-stable ideal of R^{16} by covariants of orders different from 4, 8, and 12 and by the covariants C^8 and C^{12}

introduced above. Lemma III.6.3 implies that

$$\begin{aligned}\dim_k(T_I \operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V_{16})) \le{}& \dim_k(\operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(R^{16}_{(2,16)}, R^{16}/I))\\ &+ \dim_k(\operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(C^8, R^{16}/I))\\ &+ \dim_k(\operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(C^{12}, R^{16}/I)) = 3.\end{aligned} \tag{III.6.9}$$

Recall from Remark III.5.12 that all covariants of degree n of co-order ≥ 8 are contained in I if n is large enough. This will be used frequently in the sequel.

a) The proof of Lemma III.6.12 showed that $C^8 = ((Q,Q)^{10}, Q)^{10}$ or $C^{12} = ((Q,Q)^{12}, Q)^6$. Suppose first that $C^{12} = ((Q,Q)^{12}, Q)^6$ and that $C^8 \neq ((Q,Q)^{10}, Q)^{10}$. Then C^{12} is in the ideal generated by C^8 and by $J^{16}_{odd,+}$. In this case, (III.6.9) simplifies to

$$\begin{aligned}\dim_k(T_I \operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V_{16})) \le{}& \dim_k(\operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(R^{16}_{(2,16)}, R^{16}/I))\\ &+ \dim_k(\operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(C^8, R^{16}/I)) = 2.\end{aligned} \tag{III.6.10}$$

We show that for this choice of C^8 and C^{12}, one has

$$\varphi(R^{16}_{(2,16)}) = 0 \quad \text{for all } \varphi \in \operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(I, R^{16}/I). \tag{III.6.11}$$

Let $0 \neq v = v_8^2 + v_8^3 \in (C^8)^U$ with v_8^2 of degree 2 and v_8^3 of degree 3. We now assume that (III.6.11) is wrong and choose $\varphi \in \operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(I, R^{16}/I)$ with $\varphi(R^{16}_{(2,16)}) \neq 0$. This will lead to the following contradiction:

$$0 = \varphi((a_{16})^n \cdot v) = \varphi((a_{16})^n \cdot v_8^2) + \varphi((a_{16})^n \cdot v_8^3) \neq 0. \tag{III.6.12}$$

Because $C^8 \neq ((Q,Q)^{10}, Q)^{10}$, it follows that $v_8^2 \neq 0$ and that $(R^{16}/I)_{(8)} = Q_8^3 + I$ for a covariant $Q_8^3 \subset R^{16}_{(3,8)}$. Hence there exists $w_8^3 \in Q_8^3$ such that $\varphi(v) = w_8^3 + I$. This implies that

$$\varphi((a_{16})^n \cdot v) = (a_{16})^n \cdot w_8^3 + I.$$

Because $(a_{16})^n w_8^3$ has co-order 20 it is contained in I if n is large enough. We conclude that

$$\varphi((a_{16})^n \cdot v) = 0 \tag{III.6.13}$$

for all n large enough.

The assumption that $\varphi(R^{16}_{(2,16)}) \neq 0$ implies that $\varphi(C^{16}_{2;8}) = Q + I$, or equivalently that

$$\varphi(u_{2;8}) = \gamma a_{16} + I \tag{III.6.14}$$

for some $0 \neq \gamma \in k$. With (III.6.7), we find that for all n large enough there exists $0 \neq \gamma \in k$ such that

$$\begin{aligned}\varphi((a_{16})^n \cdot u_{2;12}) &= \varphi(\gamma (a_{16})^{n-4} u_{2;8} u_{2;2} u_{2;2})\\ &= \gamma (a_{16})^{n-4} u_{2;2} u_{2;2} \varphi(u_{2;8})\\ &= \gamma (a_{16})^{n-3} u_{2;2} u_{2;2} + I.\end{aligned}$$

Using Remark III.5.8 and the proof of Lemma III.5.7, it is now easy to see that $(a_{16})^{n-3}u_{2;2}u_{2;2} \notin I$ for all $n \geq 3$. Since v_8^2 is a non-zero scalar multiple of $u_{2;12}$, it follows that

$$\varphi(a_{16}^n \cdot v_8^2) \neq 0 \tag{III.6.15}$$

for all n large enough.

On the other hand, $(a_{16})^n \cdot v_8^3$ has co-order 20. Therefore, it is contained in I for n large enough. Now $v_8^3 \in ((Q,Q)^{10},Q)^{10}$, and using (III.6.8) and (III.6.14) we see that

$$\begin{aligned}\varphi((a_{16})^n \cdot v_8^3) &\subset k \cdot \varphi((a_{16})^{n-11}u_{2;8}(u_{2;2})^6) \\ &= k \cdot (a_{16})^{n-11}(u_{2;2})^6\varphi(u_{2;8}) \\ &= k \cdot \gamma(a_{16})^{n-10}(u_{2;2})^6 + I.\end{aligned}$$

Since $(a_{16})^{n-10}(u_{2;2})^6$ is a U-invariant of co-order 12, it is contained in I if n is large enough. This implies that

$$\varphi(a_{16}^n \cdot v_8^3) = 0. \tag{III.6.16}$$

Combining (III.6.13), (III.6.15), and (III.6.16) establishes the contradiction (III.6.12). Thus, $\varphi(R^{16}_{(2,16)}) = 0$, and (III.6.11) follows, and (III.6.10) simplifies to

$$\dim_k(T_I \operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V_{16})) \leq \dim_k(\operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(C^8, R^{16}/I)) = 1 \tag{III.6.17}$$

if $C^{12} = ((Q,Q)^{12},Q)^6$ and if $C^8 \neq ((Q,Q)^{10},Q)^{10}$. However, since $\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))_{\mathrm{red}} \cong \mathbf{A}^1 \cup \mathbf{P}^1 \cup \mathbf{P}^1$, the tangent space in I is at least one-dimensional, so there is equality in (III.6.17).

Suppose now that $C^{12} \neq ((Q,Q)^{12},Q)^6$ and $C^8 = ((Q,Q)^{10},Q)^{10}$. Arguing in a similar way, we find that $R^{16}_{(2,16)} \subset \ker(\varphi)$ for all $\varphi \in \operatorname{Hom}^{\mathrm{SL}_2}_{R^{16}}(I, R^{16}/I)$. Thus, $\dim_k(T_I \operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V_{16})) = 1$ also in this case.

b) To see that equality holds in (III.6.9) one shows that the tangent vectors coming from the three irreducible components $\mathbf{A}^1$, $\mathbf{P}^1$, and $\mathbf{P}^1$ are linearly independent. This is done similarly as in the proof of Proposition III.6.9. □

Proof of Theorem III.6.11. Lemma II.3.2 and Propositon III.6.10 prove a) and c). It therefore suffices to show that $\operatorname{Hilb}^{\mathrm{SL}_2}_{h'_{16}}(V(16))$ is reduced in the complement of p, which follows from Lemmata III.6.1 and III.6.14. □

CHAPTER IV

Multiplicities

Let $s \in \mathbf{N}$, and let $(\lambda_1, \ldots, \lambda_s) \in (\Lambda^+)^s$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. For $r = (r_1, \ldots, r_s) \in \mathbf{N}^s$ let $V^r := \bigoplus_{i=1}^s V(\lambda_i)^{r_i}$. In addition, we let $V := \bigoplus_{i=1}^s V(\lambda_i)$. We fix a function $h \colon \Lambda^+ \to \mathbf{N}$ with $h(\lambda_i^*) = 1$ for all i (here $\lambda^* = -w_0\lambda$, where w_0 is the longest word in the Weyl group). Recall that a subscheme of V^r is *non-degenerate* if it is not a subscheme of $\bigoplus_{i \in I} V(\lambda_i)^{r_i} \subset V^r$ for any $I \subsetneq \{1, \ldots, s\}$. Then $\mathrm{Hilb}_h^G(V^r)_0$ is defined to be the open subscheme of $\mathrm{Hilb}_h^G(V^r)$ parametrizing the non-degenrate subschemes of V^r (see [**AB05**], Definition 1.14). The goal is to describe $\mathrm{Hilb}_h^G(V^r)_0$ in terms of $\mathrm{Hilb}_h^G(V)_0$. In particular, we state a correct version of Corollary 1.17 in [**AB05**].

M. Brion pointed out to me a much shorter and more elegant way to correct his statement. We sketch this argument at the end of Section IV.1.

IV.1. Invariant Hilbert schemes for multiplicity-finite modules

Given two (finite-dimensional) G-modules W and W', we denote by $\mathrm{Hom}^G(W, W')$ the (vector space or variety of) G-equivariant linear maps from W to W'. For $r, r' \in \mathbf{N}^s$ with $r_i \leq r_i'$ for all i, let $\mathrm{Hom}^G(V^r, V^{r'})_0$ be the open dense subset of $\mathrm{Hom}^G(V^r, V^{r'})$ consisting of the injective G-equivariant homomorphisms. In Theorem IV.1.5 we show that

$$\mathrm{Hilb}_h^G(V^r)_0 \cong (\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}_h^G(V)_0) / \mathrm{GL}^G(V).$$

To start with, we investigate the $\mathrm{GL}^G(V)$-action on $\mathrm{Hilb}_h^G(V)_0$. Observe that $\mathrm{GL}^G(V) \cong \mathbf{G}_m^r$. This action is well-known, but a concrete description is useful — in particular for the proof of Lemma IV.1.4. For a scheme S we define

$$R_S^{r,r'} \colon \mathrm{Hom}^G(V^r, V^{r'})_0 \times S \times V^r \to \mathrm{Hom}^G(V^r, V^{r'})_0 \times S \times V^{r'},$$

mapping (u, s, v) to $(u, s, h(v))$. It is immediate from the definition that $R_S^{r,r'}$ maps $\mathrm{Hom}^G(V^r, V^{r'})_0 \times S \times V^r$ isomorphically onto its image.

Remark IV.1.1. Let S and T be two schemes with a morphism $S \to T$, and let $\mathfrak{X}_T \subset T \times V^r$ be a closed G-stable subscheme. For simplicity, we write Hom instead of $\mathrm{Hom}^G(V^r, V^{r'})_0$. Then

$$(\mathrm{Hom} \times S) \times_{\mathrm{Hom} \times T} R_T^{r,r'}(\mathrm{Hom} \times \mathfrak{X}_T) = R_S^{r,r'}(\mathrm{Hom} \times S \times_T \mathfrak{X}_T).$$

This follows from the definition of $R_S^{r,r'}$ and $R_T^{r,r'}$.

Given a morphism $S \to \mathrm{Hilb}_h^G(V^r)_0$ corresponding to a family $S \times V^r \supset \mathfrak{X} \to S$, we obtain a morphism $\mathrm{Hom}^G(V^r, V^{r'})_0 \times S \to \mathrm{Hilb}_h^G(V^{r'})_0$ in the following way: The projection

$$R_S^{r,r'}(\mathrm{Hom}^G(V^r, V^{r'})_0 \times \mathfrak{X}) \to \mathrm{Hom}^G(V^r, V^{r'})_0 \times S$$

defines a family in $\mathcal{H}ilb_h^G(V^{r'})_0(\mathrm{Hom}^G(V^r, V^{r'})_0 \times S)$, giving the desired morphism $\mathrm{Hom}^G(V^r, V^{r'})_0 \times S \to \mathrm{Hilb}_h^G(V^{r'})_0$. As a special case, starting with $\mathrm{Univ}_h^G(V^r)_0 \to \mathrm{Hilb}_h^G(V^r)_0$ for $\mathfrak{X} \to S$, we obtain a morphism

$$\Phi^{r,r'}\colon \mathrm{Hom}^G(V^r, V^{r'})_0 \times \mathrm{Hilb}_h^G(V^r)_0 \to \mathrm{Hilb}_h^G(V^{r'})_0.$$

If $r = r'$, then $\Phi^{r,r}$ defines an action of $\mathrm{GL}^G(V^r)$ on $\mathrm{Hilb}_h^G(V^r)_0$. The case where $r = 1 := (1, \ldots, 1)$ is of importance for the sequel. For this reason, we write Φ instead of $\Phi^{1,r'}$.

Let $S \times V^r \supset \mathfrak{X} \to S$ be a family in $\mathcal{H}ilb_h^G(V^r)_0(S)$ corresponding to a morphism $\varphi_{\mathfrak{X}}\colon S \to \mathrm{Hilb}_h^G(V^r)_0$, and let $\psi\colon S \to \mathrm{Hom}^G(V^r, V^{r'})_0$ be a morphism. Consider the following two ways to construct a morphism $S \to \mathrm{Hilb}_h^G(V^{r'})_0$:

First, consider the family

$$R_S^{r,r'}(\mathrm{Hom}^G(V^r, V^{r'})_0 \times \mathfrak{X}) \to \mathrm{Hom}^G(V^r, V^{r'})_0 \times S$$

in

$$\mathcal{H}ilb_h^G(V^{r'})(\mathrm{Hom}^G(V^r, V^{r'})_0 \times S);$$

and let $\mathfrak{X}_1$ be the pull-back of this family with respect to the morphism $(\psi \times \mathrm{id}_S)\colon S \to \mathrm{Hom}^G(V^r, V^{r'}) \times S$.

On the other hand, one can define a morphism $\varphi_2\colon S \to \mathrm{Hilb}_h^G(V^{r'})_0$ corresponding to a family $\mathfrak{X}_2 \to S$ as follows:

$$\varphi_2\colon S \xrightarrow{(\psi,\varphi_{\mathfrak{X}})} \mathrm{Hom}^G(V^r, V^{r'})_0 \times \mathrm{Hilb}_h^G(V^r)_0 \xrightarrow{\Phi^{r,r'}} \mathrm{Hilb}_h^G(V^{r'})_0.$$

If $s \in S$ is a closed point, then the fibers $(\mathfrak{X}_1)_s$ and $(\mathfrak{X}_2)_s$ both equal $\psi(s)(\mathfrak{X}_s) \subset V^{r'}$. Indeed, as a formal consequence of Remark IV.1.1 one obtains:

Lemma IV.1.2. *In the above notation,* $\varphi_1 = \varphi_2$*, or* $\mathfrak{X}_1 = \mathfrak{X}_2$*.*

Proof. For simplicity, we denote $\mathrm{Hom}^G(V^r, V^{r'})_0$ by Hom, as well as $\mathrm{Hilb}_h^G(V^r)_0$ by Hilb^r, and $\mathrm{Univ}_h^G(V^{r'})_0$ by $\mathrm{Hilb}^{r'}$. By definition,

$$\mathfrak{X}_1 = S \times_{\mathrm{Hom} \times S} R_S^{r,r'}(\mathrm{Hom} \times \mathfrak{X}),$$

and

$$\mathfrak{X}_2 = S \times_{\mathrm{Hom} \times \mathrm{Hilb}^r} R_{\mathrm{Hilb}^r}^{r,r'}(\mathrm{Hom} \times \mathrm{Univ}^r).$$

Using Remark IV.1.1, we see that

$$(\mathrm{Hom} \times S) \times_{\mathrm{Hom} \times \mathrm{Hilb}^r} R_{\mathrm{Hilb}^r}^{r,r'}(\mathrm{Hom} \times \mathrm{Univ}^r) = R_S^{r,r'}(\mathrm{Hom} \times \mathfrak{X})$$

as schemes over $\mathrm{Hom} \times S$. Therefore,

$$\begin{aligned}
\mathfrak{X}_2 &= S \times_{\mathrm{Hom} \times \mathrm{Hilb}^r} R^{r,r'}_{\mathrm{Hilb}^r}(\mathrm{Hom} \times \mathrm{Univ}^r) \\
&= (\mathrm{Hom} \times S) \times_{\mathrm{Hom} \times S} (S \times_{\mathrm{Hilb}^r \times \mathrm{Hom}} R^{r,r'}_{\mathrm{Hilb}^r}(\mathrm{Hom} \times \mathrm{Univ}^r)) \\
&= S \times_{\mathrm{Hom} \times S} R^{r,r'}_S(\mathrm{Hom} \times \mathfrak{X}) = \mathfrak{X}_1.
\end{aligned}$$ □

Consider the (contragredient) action

$$\Sigma^{r,r'} \colon \mathrm{GL}^G(V^r) \times \mathrm{Hom}^G(V^r, V^{r'})_0 \to \mathrm{Hom}^G(V^r, V^{r'})_0$$

of $\mathrm{GL}^G(V^r)$ on $\mathrm{Hom}^G(V^r, V^{r'})$ (defined by $(t \cdot \varphi)(v) = \varphi(t^{-1}v)$). Using this action, we obtain a $\mathrm{GL}^G(V)$-action σ^r on $\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}^G_h(V)_0$, defined by

$$t \cdot (\varphi, x) = (\Sigma^{1,r}(t, \varphi), \Phi^{1,1}(t, x)).$$

The morphism Φ is $\mathrm{GL}^G(V)$-invariant with respect to this action. We shall see in Lemma IV.1.3 that $\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}^G_h(V)_0$ possesses a geometric quotient by $\mathrm{GL}^G(V)$ and in Theorem IV.1.5 that Φ is the quotient morphism.

For $j_i \in \{1, \dots, r_i\}$, let $V(\lambda_i) \cong V_{j_i} \subset V(\lambda_i)^{r_i}$ be the j_i-th summand. For $j = \{j_1, \dots, j_s\}$, let $V_j \subset V^r$ be the sum of the corresponding V_{j_i}. Let $H_j := \{\varphi \in \mathrm{Hom}^G(V, V^r) \mid p_j \circ \varphi \in \mathrm{GL}^G(V)\}$, where $p_j \colon V^r \to V_j \cong V$ is the projection (the definition of H_j is independent of the choice of the G-isomorphism $V_j \cong V$). The H_j cover $\mathrm{Hom}^G(V, V^r)_0$.

Lemma IV.1.3. a) *First,* $\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}^G_h(V)_0$ *possesses a geometric quotient by* $\mathrm{GL}^G(V)$.

b) *If* $r = 1$*, then* Φ *is the quotient morphism.*

Proof. One can adapt the proof of Theorem 1.10 in [**MFK94**]. We start with an elementary observation: Let H be a group, let X be a H-scheme, and endow $H \times X$ with a H-action defined by $g \cdot (h, x) = (hg^{-1}, gx)$. Then the morphism $H \times X \to X$ mapping (g, x) to gx is a geometric quotient.

Consider the $\mathrm{GL}^G(V)$-stable subset

$$H_j \times \mathrm{Hilb}^G_h(V)_0 \subset \mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}^G_h(V)_0.$$

Now, $H_j \cong \mathrm{GL}^G(V) \times \mathrm{Hom}^G(V, V^{r-1})$. With $H = \mathrm{GL}^G(V)$ and with $X = \mathrm{Hom}^G(V, V^{r-1}) \times \mathrm{Hilb}^G_h(V)_0$, the action $\sigma^r|_{H_j \times \mathrm{Hilb}^G_h(V)_0}$ is of the above kind. Thus, there exists a geometric quotient of $H_j \times \mathrm{Hilb}^G_h(V)_0$ by $\mathrm{GL}^G(V)$. In this situation, one can proceed as in the proof of Theorem 1.10 in [**MFK94**] and glue these quotients together, which completes the proof.

b) With $H = \mathrm{GL}^G(V)$ and with $X = \mathrm{Hilb}^G_h(V)_0$, the statement follows from the above observation. □

Let

$$\pi\colon \operatorname{Hom}^G(V,V^r)_0\times\operatorname{Hilb}^G_h(V)_0\to(\operatorname{Hom}^G(V,V^r)_0\times\operatorname{Hilb}^G_h(V)_0)/\operatorname{GL}^G(V)$$

be the quotient morphism. Then Φ factors as $\Phi/\operatorname{GL}^G(V)\circ\pi$ for some morphism

$$\Phi/\operatorname{GL}^G(V)\colon(\operatorname{Hom}^G(V,V^r)_0\times\operatorname{Hilb}^G_h(V)_0)/\operatorname{GL}^G(V)\to\operatorname{Hilb}^G_h(V^r)_0.$$

We shall see in Theorem IV.1.5 that $\Phi/\operatorname{GL}^G(V)$ is an isomorphism.

Lemma IV.1.4. *Let S be a scheme, and let $\eta\colon S\to\operatorname{Hilb}^G_h(V^r)_0$ be a morphism corresponding to a family $S\times V^r\supset\mathfrak{X}\to S$.*

a) *The scheme S can be covered by open subschemes $(S_j)_{j=(j_1,\ldots,j_s)}$ such that for each j there exists a morphism $\eta_j\colon S_j\to H_j\times\operatorname{Hilb}^G_h(V)_0$ with $\varphi|_{S_j}=\Phi\circ\eta_j$.*

b) *If μ and μ' are morphisms from S to $\operatorname{Hom}^G(V,V^r)_0\times\operatorname{Hilb}^G_h(V)_0$ such that $\Phi\circ\mu=\Phi\circ\mu'$, then $\pi\circ\mu=\pi\circ\mu'$.*

Proof. To keep the notation simple, we assume that $s=1$ (and hence that $V=V(\lambda)$ and that $V^r=V(\lambda)^r$). The general case can be derived similarly.

a) We first define the subsets S_j. Let $(e^i_1,\ldots,e^i_p)$ be a basis of $V_i\subset V^r$, and let $(x^i_1,\ldots,x^i_p)$ be the dual basis of $(V_i)^*\subset(V^r)^*$. Let $M:=((V^r)^*)^U$, and suppose that $(x^1_p,\ldots,x^r_p)$ is a basis of M. Let further $\mathcal{I}\subset\mathcal{O}_{S\times V^r}$ be the ideal-sheaf defining $\mathfrak{X}\subset S\times V^r$, let $p\colon S\times V^r\to S$ be the projection, and let

$$\mathcal{J}:=(p_*(\mathcal{I})\cap\mathcal{O}_S\otimes M)\subset\mathcal{O}_S\otimes M.$$

Then $\mathcal{J}$ is a locally free $\mathcal{O}_S$-submodule of $\mathcal{O}_S\otimes M$ of rank $r-1$ (recall that $h(\lambda^*)=1$). For $j\in\{1,\ldots,r\}$, define the open subset $S_j\subset S$ by

$$S_j:=\{s\in S\mid 1_{\mathcal{O}_{S,s}}\otimes x^j_p\notin\mathcal{J}_s\}\subset S.$$

Let $\mathfrak{X}_j=S_j\times_S\mathfrak{X}$. With the $\mathcal{O}_{S_j}$-module $\mathcal{J}_j:=\mathcal{J}|_{S_j}$, we have

$$\mathcal{O}_{S_j}\otimes M=\mathcal{J}_j\oplus(\mathcal{O}_{S_j}\otimes x^j_p),$$

and hence $\mathcal{J}_j$ is a free $\mathcal{O}_{S_j}$-module of rank $r-1$. We now define a morphism $\psi_j\colon S_j\to\operatorname{GL}^G(V^r)$. For simplicity we assume that $j=1$. Let $A:=\mathcal{O}_{S_1}(S_1)$. Then $J:=\mathcal{J}_1(S_1)$ is a free A-submodule of $A\otimes M$ of rank $r-1$, beacuse $A\otimes M=J\oplus(A\otimes x^1_p)$. Let J be generated as A-module by

$$\sum_{k=1}^r\psi_{2,k}\otimes x^j_p,\ldots,\sum_{k=1}^r\psi_{r,k}\otimes x^j_p, \tag{IV.1.1}$$

where $\psi_{i,k}\in A$. Let $\psi_{1,1}:=1_A$, and let $\psi_{1,k}:=0_A$ if $k>1$. We obtain the desired morphism $\psi_1=(\psi_{i,k})_{i,k}\colon S_1\to\operatorname{GL}_r\cong\operatorname{GL}^G(V^r)$, where the isomorphism $\operatorname{GL}_r\cong\operatorname{GL}^G(V^r)$ maps $(a_{ij})_{i,j}$ to $(a_{ij}\operatorname{id}_V)_{i,j}$.

Let
$$\varphi_j\colon S_j \times V^r \to S_j \times V^r$$
be the G-equivariant isomorphism of schemes over S_j mapping (s, v) to $(s, \psi_j(s)v)$. We claim that there exists a family $\mathfrak{Y}_j \subset S_j \times V_j$ of G-schemes over S_j with Hilbert function h such that φ_j maps $\mathfrak{Y}_j$ isomorphically onto $\mathfrak{X}_j$. To see this, suppose once more that $j = 1$, and let A be as above. Consider $Z := S_1 \times V_1 \subset S_1 \times V^r$. The ideal of Z in $A \otimes \mathrm{Sym}((V^r)^*)$ is the G-stable ideal of A-algebras generated by
$$\{1 \otimes x_i^j \mid j = 2, \ldots, r,\ i = 1, \ldots, p\}.$$
Since φ_1 is given by the homomorphism of A-G-algebras
$$A \otimes \mathrm{Sym}((V^r)^*) \to A \otimes \mathrm{Sym}((V^r)^*), \quad 1 \otimes x_i^j \mapsto \sum_{k=1}^{r} \psi_{j,k} \otimes x_i^k,$$
we see that $\varphi_1(Z) \subset S_1 \times V^r$ is defined by the G-stable ideal I of A-algebras generated by
$$\{\sum_{k=1}^{r} \psi_{j,k} \otimes x_i^k \mid j = 2, \ldots, r,\ i = 1, \ldots, p\}. \tag{IV.1.2}$$
Comparing (IV.1.2) to (IV.1.1), we see that
$$I \cap (A \otimes M) = J.$$
Since I is the smallest G-stable ideal of $A \otimes \mathrm{Sym}((V^r)^*)$ containing $I \cap (A \otimes M)$, it follows that $I \subset J$, or that $\mathfrak{X}_1 \subset \varphi_1(Z)$. More generally, we see that
$$\mathfrak{X}_j \subset \varphi_j(S_j \times V_j) \subset S_j \times V^r$$
for all j. Since φ_j is an isomorphism of G-schemes over S_j, we can take $\mathfrak{Y}_j := \varphi_j^{-1}(\mathfrak{X}_j)$.

Let $\varphi_j'\colon S_j \to \mathrm{Hilb}_h^G(V_j)$ be the morphism corresponding to the family $\mathfrak{Y}_j$. Then $\varphi_j'(S_j) \subset \mathrm{Hilb}_h^G(V_j)_0 \subset \mathrm{Hilb}_h^G(V^r)_0$.

On the other hand, let $q_j\colon\ \mathrm{GL}^G(V^r) \to \mathrm{Hom}^G(V_j, V^r)$ be the natural map. By construction of ψ_j, the composition $q_j \circ \psi_j$ factors as
$$q_j \circ \psi_j\colon S_j \to H_j \subset \mathrm{Hom}^G(V_j, V^r).$$
Using Lemma IV.1.2, we see that $\eta|_{S_j}$ factors as
$$\eta|_{S_j}\colon S_j \xrightarrow{(q_j \circ \psi_j, \varphi_j')} H_j \times \mathrm{Hilb}_h^G(V_j) \xrightarrow{\Phi} \mathrm{Hilb}_h^G(V^r).$$
After identifying V_j with V via a G-isomorphism, the claim follows.

b) Let $s \in S$ be a closed point. It suffices to verify that $\pi \circ \mu = \pi \circ \mu'$ in a neighbourhood of s. In particular, we can assume that $\mu(S)$ and $\mu'(S)$ are both contained in $H_j \times \mathrm{Hilb}_h^G(V)_0$ for some suitable j. Composing μ and μ' with the morphim $H_j \times \mathrm{Hilb}_h^G(V)_0 \to \mathrm{GL}^G(V) \times \mathrm{Hilb}_h^G(V)_0$ mapping (φ, x) to $(p_j \circ \varphi, x)$ gives two morphisms $\tilde{\mu}$ and $\tilde{\mu}'$ from S to $\mathrm{GL}^G(V) \times \mathrm{Hilb}_h^G(V)_0$ with $\Phi^{1,1} \circ \tilde{\mu} = \Phi^{1,1} \circ \tilde{\mu}'$. Now, Lemma IV.1.3 b) yields a morphism $\psi\colon S \to \mathrm{GL}^G(V)$ such that $\tilde{\mu}' = \sigma^1 \circ (\psi, \tilde{\mu})$. One readily sees that $\mu' = \sigma^r \circ (\psi, \mu)$. The claim now follows. $\square$

Theorem IV.1.5. *The morphism $\Phi/\,\mathrm{GL}^G(V)$ is an isomorphism. In particular,*

$$\mathrm{Hilb}_h^G(V^r)_0 \cong (\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}_h^G(V)_0)/\,\mathrm{GL}^G(V).$$

Proof. Every morphism $\psi\colon S \to (\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}_h^G(V)_0)/\,\mathrm{GL}^G(V)$ gives a morphism $(\Phi/\,\mathrm{GL}^G(V)) \circ \psi\colon S \to \mathrm{Hilb}_h^G(V^r)_0$. On the other hand, every $\varphi\colon S \to \mathrm{Hilb}_h^G(V^r)_0$ gives morphisms $\eta_j\colon S_j \to H_j \times \mathrm{Hilb}_h^G(V)_0$ such that $\varphi|_{S_j} = \Phi \circ \eta_j$ according to Lemma IV.1.4 a). Since $\pi \circ \eta_j|_{S_j \cap S_{j'}} = \pi \circ \eta_{j'}|_{S_j \cap S_{j'}}$ for all j and j' according to Lemma IV.1.4 b), the morphism φ factors via $(\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}_h^G(V)_0)/\,\mathrm{GL}^G(V)$, and hence gives a map $\psi\colon S \to (\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}_h^G(V)_0)/\,\mathrm{GL}^G(V)$. The two assignments are clearly inverse to each other, and we obtain two inverse natural transformations between the functors of points of $\mathrm{Hilb}_h^G(V^r)_0$ and of $(\mathrm{Hom}^G(V, V^r)_0 \times \mathrm{Hilb}_h^G(V)_0)/\,\mathrm{GL}^G(V)$, and the claim follows. $\square$

Remark IV.1.6 (Brion)**.** We sketch here M. Brion's way to prove Theorem IV.1.5. The statement of Theorem IV.1.5 is equivalent to the existence of a $\mathrm{GL}^G(V^r)$-equivariant morphism

$$\varphi\colon\ \mathrm{Hilb}_h^G(V^r)_0 \to \mathrm{Hom}^G(V, V^r)_0/\,\mathrm{GL}^G(V) \cong \prod_{i=1}^{s} \mathbf{P}^{r_i - 1}$$

with fiber $\mathrm{Hilb}_h^G(V)_0$ at the base point, where the right-hand side is viewed as a homogeneous space under $\mathrm{GL}^G(V^r) \cong \prod_{i=1}^s \mathrm{GL}_{r_i}$.

To construct this morphism, proceed as follows: From

$$\mathrm{Hilb}_h^G(V^r) \times V^r \supset \mathrm{Univ}_h^G(V^r) \xrightarrow{\pi} \mathrm{Hilb}_h^G(V^r)$$

we obtain a surjective morphism of $\mathcal{O}_{\mathrm{Hilb}_h^G(V^r)}$-modules

$$\mathcal{O}_{\mathrm{Hilb}_h^G(V^r)} \otimes k[V^r] \to \mathcal{O}_{\mathrm{Univ}_h^G(V^r)},$$

and thus for all i a surjective morphism

$$\mathcal{O}_{\mathrm{Hilb}_h^G(V^r)} \otimes \mathrm{Hom}^G(V(\lambda_i^*), k[V^r]) \to \mathrm{Hom}^G(V(\lambda_i^*), \mathcal{O}_{\mathrm{Univ}_h^G(V^r)}).$$

This yields for all i a morphism

$$\mathcal{O}_{\mathrm{Hilb}_h^G(V^r)} \otimes \mathrm{Hom}^G(V(\lambda_i^*), V(\lambda_i^*)^{r_i}) \to \mathrm{Hom}^G(V(\lambda_i^*), \mathcal{O}_{\mathrm{Univ}_h^G(V^r)}),$$

which is surjective at all closed points of $\mathrm{Hilb}_h^G(V^r)_0$ by the definition of non-degeneracy. This yields a surjection

$$\mathcal{O}_{\mathrm{Hilb}_h^G(V^r)_0} \otimes k^{r_i} \to \mathrm{Hom}^G(V(\lambda_i^*), \mathcal{O}_{\mathrm{Univ}_h^G(V^r)_0}),$$

where the latter is locally free of rank 1 because $h(\lambda_i^*) = 1$. This yields a morphism

$$\varphi_i\colon\ \mathrm{Hilb}_h^G(V^r)_0 \to \mathbf{P}^{r_i},$$

and finally φ can be defined as the product of the φ_i.

IV.2. Examples

In this section we apply Theorem IV.1.5 to some situations.

Let $\lambda \in \Lambda^+$ be a dominant weight of G, and let $r \in \mathbf{N}$. Consider $V(\lambda)^r = \bigoplus_{i=1}^r V(\lambda)$, and let $h_\lambda \colon \Lambda^+ \to \mathbf{N}$ be the Hilbert function of $X_\lambda = \overline{Gv_\lambda} \subset V(\lambda)$. By $\mathrm{Bl}_x(X)$ we denote the blow-up of a variety X in a point x.

Theorem IV.2.1. *In the above notation, one has:*

a) *If both λ and 2λ are Jansou-weights, then*

$$\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)^r) \cong \mathrm{Bl}_0(\mathbf{A}^r/C_2),$$

where $C_2 = \{1, -1\}$ is a cyclic group of order 2 acting on $\mathbf{A}^k$ by $-1 \cdot x = -x$.

b) *If λ is a Jansou-weight, but 2λ is no Jansou-weight, then*

$$\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)^r) \cong \mathrm{Bl}_0(\mathbf{A}^r).$$

c) *If finally λ is no Jansou-weight, then*

$$\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)^r) \cong \mathbf{P}^{r-1}.$$

Remark IV.2.2. This remark is due to M. Brion. The isomorphism

$$\mathrm{Hilb}^G_{h_\lambda}(V^r) = (\mathrm{Hom}^G(V, V^r) \setminus \{0\} \times \mathrm{Hilb}^G_{h_\lambda}(V))/\,\mathrm{GL}^G(V)$$

from Theorem IV.1.5 induces a morphism

$$\varphi\colon\ \mathrm{Hilb}^G_{h_\lambda}(V^r) \to \mathrm{Hom}^G(V, V^r) \setminus \{0\}/\,\mathrm{GL}^G(V) \cong \mathbf{P}^{r-1}.$$

Now $\mathrm{Hilb}^G_{h_\lambda}(V^r)$ is the total space of the line bundle $\mathcal{O}_{\mathbf{P}^{r-1}}(-2)$ in case a), of $\mathcal{O}_{\mathbf{P}^{r-1}}(-1)$ in case b), and is just $\mathbf{P}^{r-1}$ in case c).

For the proof a small observation turns out to be useful. Recall that for a weight $\lambda \in \Lambda^+$ we have defined n_0 in Definition II.2.2. In particular, if λ is a Jansou-weight, then $n_0 = 2$ if 2λ is also a Jansou-weight, and $n_0 = 1$ otherwise.

Lemma IV.2.3. *Let λ be a Jansou-weight of G. Then $\mathrm{GL}^G(V(\lambda)) \cong \mathbf{G}_m$ acts on $\mathrm{Hilb}^G_{h_\lambda}(V(\lambda)) \cong \mathbf{A}^1$ (via the $\mathrm{GL}^G(V)$-action $\Phi^{1,1}$ from Section IV.1) with weight n_0.*

Proof. Let $X_\lambda \neq X \subset V(\lambda)$ be a closed G-stable subscheme corresponding to a closed point $p \in \mathbf{A}^1 \cong \mathrm{Hilb}^G_{h_\lambda}(V(\lambda))$. By definition of the action $\Phi^{1,1}$, the point $\gamma \cdot p$ corresponds to the scheme $\gamma \cdot X$. If $\gamma \in k$, then $\gamma \cdot X = X$ if and only if $\gamma^{n_0} = 1$. Hence $\gamma \cdot p = p$ if and only if $\gamma^{n_0} = 1$, and so $\mathrm{GL}^G(V(\lambda))$ acts with weight n_0 on $\mathrm{Hilb}^G_{h_\lambda}(V(\lambda))$. $\square$

Proof of Theorem IV.2.1. Let $V := V(\lambda)$, and let $V^r := V(\lambda)^r$. If λ is no Jansou-weight, then $\mathrm{Hilb}^G_{h_\lambda}(V) = \mathbf{A}^0$. Applying Theorem IV.1.5, we find:

$$\begin{aligned}\mathrm{Hilb}^G_{h_\lambda}(V^r) &= (\mathrm{Hom}^G(V, V^r) \setminus \{0\} \times \mathrm{Hilb}^G_{h_\lambda}(V))/\,\mathrm{GL}^G(V)\\ &\cong (\mathbf{A}^r \setminus \{0\} \times \mathbf{A}^0)/\mathbf{G}_m\\ &\cong \mathbf{P}^{r-1}.\end{aligned}$$

Suppose now that λ is a Jansou-weight, but that 2λ is no Jansou-weight. Then $\mathrm{Hilb}^G_{h_\lambda}(V) \cong \mathbf{A}^1$, and $\mathrm{GL}^G(V) \cong \mathbf{G}_m$ acts on $\mathrm{Hilb}^G_{h_\lambda}(V)$ with weight 1 according to Lemma IV.2.3. Consider the morphism

$$\begin{aligned}(\mathbf{A}^r \setminus \{0\}) \times \mathbf{A}^1 &\to \mathbf{A}^r \times \mathbf{P}^{r-1}\\ (t_1, \ldots, t_r, x) &\mapsto ((t_1x, \ldots, t_rx), (t_1 : \ldots : t_r)).\end{aligned}$$

If $\mathbf{G}_m$ acts on $\mathbf{A}^r$ with weight $(-1, \ldots, -1)$ and on $\mathbf{A}^1$ with weight 1, then the restriction of this morphism onto its image, which equals $\mathrm{Bl}_0(\mathbf{A}^r)$, is the quotient morphism. So,

$$\begin{aligned}\mathrm{Hilb}^G_{h_\lambda}(V^r) &= (\mathrm{Hom}^G(V, V^r) \setminus \{0\} \times \mathrm{Hilb}^G_{h_\lambda}(V))/\,\mathrm{GL}^G(V)\\ &\cong (\mathbf{A}^r \setminus \{0\} \times \mathbf{A}^1)/\mathbf{G}_m\\ &\cong \mathrm{Bl}_0(\mathbf{A}^r).\end{aligned}$$

If finally both λ and 2λ are Jansou-weights, then $\mathrm{GL}^G(V) \cong \mathbf{G}_m$ acts on $\mathrm{Hilb}^G_{h_\lambda}(V) \cong \mathbf{A}^1$ with weight 2 according to Lemma IV.2.3, and the morphism

$$\begin{aligned}(\mathbf{A}^r \setminus \{0\}) \times \mathbf{A}^1 &\to \mathrm{Bl}_0(\mathbf{A}^r/C_2) \subset \mathbf{A}^r \times \mathbf{P}^{r-1}\\ (t_1, \ldots, t_r, x) &\mapsto ((t_1^2x, \ldots, t_r^2x), (t_1 : \ldots : t_r))\end{aligned}$$

is the quotient morphism. Thus

$$\begin{aligned}\mathrm{Hilb}^G_{h_\lambda}(V^r) &= (\mathrm{Hom}^G(V, V^r) \setminus \{0\} \times \mathrm{Hilb}^G_{h_\lambda}(V))/\,\mathrm{GL}^G(V)\\ &\cong (\mathbf{A}^r \setminus \{0\} \times \mathbf{A}^1)/\mathbf{G}_m\\ &\cong \mathrm{Bl}_0(\mathbf{A}^r/C_2).\end{aligned}$$

□

Example IV.2.4. Let $G = \mathrm{SL}_2$. We use the same conventions and notation as in Chapter III. Recall that the Jansou-weights of SL_2 are 2 and 4, and the simple SL_2-modules $V(2)$ and $V(4)$ can be identified with the space of binary forms of degree 2 and 4.

Let $\lambda = 2$. Now, $\mathcal{O}(V(2))^{\mathrm{SL}_2} = k[\Delta]$, where Δ is the discriminant. Since Δ is an invariant of degree 2, the forms v and $-v$ lie in the same SL_2-orbit for each $v \in V(2)$. However, going over to multiple (say 2) copies $V(2)^2$, the discriminants Δ_{11} and Δ_{22} (defined by $\Delta_{ii}(v_1, v_2) = \Delta(v_i)$) don't generate the ring of invariants $\mathcal{O}(V(2)^2)^{\mathrm{SL}_2}$. There exists an additional invariant Δ_{12} defined by

$$\Delta_{12}(a_0x^2 + 2a_1xy + a_2y^2, b_0x^2 + 2b_1xy + b_2y^2) = a_1b_1 - a_0b_2.$$

One has

$$V(2)^2/\!\!/\mathrm{SL}_2 = \mathrm{Spec}(k[\Delta_{11}, \Delta_{12}, \Delta_{22}]) \cong \mathbf{A}^2/C^2.$$

The fact that one has to blow up $\mathbf{A}^2/C_2$ instead of $\mathbf{A}^2$ in this case is due to this phenomenon.

So far we have discussed examples for modules of the form $V^r = V(\lambda)^r$. To finish, we turn our attention to the case where $V = V(\lambda_1) \oplus \ldots \oplus V(\lambda_s)$. Once one knows $\mathrm{Hilb}_h^G(V)_0$ and its $\mathrm{GL}^G(V)$-structure, one can compute $\mathrm{Hilb}_h^G(V^r)_0$. One situation in which $\mathrm{Hilb}_h^G(V)_0$ is known is the one investigated in [**BC08**] and [**Cu08**]): Let $\lambda_1, \ldots, \lambda_s \subset \Lambda^+$ be linearly independent weights that are *saturated* in the sense that $\mathbf{Z}\Gamma \cap \Lambda^+ = \Gamma$ (see [**Pa97**] for more information on the saturation hypothesis). Let $\Gamma = \langle \lambda_1^*, \ldots, \lambda_s^* \rangle_{\mathbf{N}}$, and let $h\colon \Lambda^+ \to \mathbf{N}$ be the function with $h(\lambda) = 1$ if $\lambda \in \Gamma$ and with $h(\lambda) = 0$ else. Let further $v_\lambda = (v_{\lambda_1}, \ldots, v_{\lambda_s})$, where each v_{λ_i} is a non-zero highest weight vector in $V(\lambda_i)$. Then the closure $X_\lambda := \overline{Gv_\lambda} \subset V$ has Hilbert function h, and it is known that the G-stable non-degenerate deformations of X_λ are parametrized by an affine space, i.e. $\mathrm{Hilb}_h^G(V)_0$ is an affine space (cf. [**BC08**] and [**Cu08**]).

Let $\alpha_1, \ldots, \alpha_m$ be the set of simple roots of (G, B, T). Then the *adjoint torus* $T_{\mathrm{ad}} = T/Z(G)$ (where $Z(G)$ is the center of G) can be identified with $\mathrm{Spec}(k[\alpha_1^{\pm 1}, \ldots, \alpha_m^{\pm 1}])$. Now T_{ad} acts on $V(\lambda_i)$ by $tZ(G) \cdot v = \lambda_i(t)t^{-1}v$. This gives a T_{ad}-action on V. If V and h are as above, then T_{ad} acts on $\mathrm{Hilb}_h^G(V)_0$, and it is known (cf. [**BC08**] or [**Cu08**]) that $\mathrm{Hilb}_h^G(V)_0$ is a multiplicity-free T_{ad}-module isomorphic to $[V/\mathfrak{g} \cdot v_\lambda]^{G_{v_\lambda}}$. This action is called *normalized action* of T_{ad} on $\mathrm{Hilb}_h^G(V)_0$.

Once one knows the T_{ad}-weights of $\mathrm{Hilb}_h^G(V)_0$, one can recover the $\mathrm{GL}^G(V)$-weights $\omega_1, \ldots, \omega_p$ on $\mathrm{Hilb}_h^G(V)_0$: First note that each T_{ad}-weight ω_t lies in the monoid $\langle \alpha_1, \ldots, \alpha_m \rangle_{\mathbf{N}}$. Now each ω_t is also in $\langle \lambda_1, \ldots, \lambda_s \rangle_{\mathbf{Z}}$. If $\omega_t = \sum \nu_{it}\lambda_i$, then $\mathrm{GL}^G(V) \cong \mathbf{G}_m^s$ acts with weights $(\nu_{1t}, \ldots, \nu_{st})_t$ on $\mathrm{Hilb}_h^G(V)_0$.

Example IV.2.5. Let $G = \mathrm{SL}_2$, and let $n = 2$ or 4. Then one sees that $[V(n)/\mathfrak{sl}_2 \cdot y^n]^{\mathrm{SL}_{2y^n}} = k \cdot y^{n-2}x^2 + \mathfrak{sl}_2 \cdot y^n$. One finds that T_{ad} acts in both cases with weight 4 on $k \cdot y^{n-2}x^2$. If $n = 2$, then $4 = 2 \cdot 2$, and hence $\mathrm{GL}^{\mathrm{SL}_2}(V(2))$ acts with weight 2 on $\mathrm{Hilb}_h^{\mathrm{SL}_2}(V(2))$. On the other hand, if $n = 4$, then $4 = 1 \cdot 4$, and hence $\mathrm{GL}^{\mathrm{SL}_2}(V(4))$ acts with weight 1 on $\mathrm{Hilb}_h^{\mathrm{SL}_2}(V(4))$.

Remark IV.2.6. From [**AB05**], Lemma 2.1 it follows that the action of T_{ad} on $\mathrm{Hilb}_h^G(V)_0$ extends to an action of $\mathrm{Spec}(k[\alpha_1, \ldots, \alpha_m])$ on $\mathrm{Hilb}_h^G(V)_0$. On the other hand, the action of the i-th factor $\mathbf{G}_m$ of $\mathrm{GL}^G(V) \cong \mathbf{G}_m^s$ extends to an action of $\mathbf{A}^1$ (with respect to the usual embedding $\mathbf{G}_m \to \mathbf{A}^1$) if and only if $\nu_{it} \geq 0$ for all t. The following examples show that this can but need not hold in particular cases.

(However, recall from [**Ja05**] Proposition 1.3 that the action of $\mathbf{G}_m$ on $\mathrm{Hilb}_h^G(V(\lambda))$ always extends to an action of $\mathbf{A}^1$.)

We now provide two examples for $G = \mathrm{SL}_4$. Let $B \subset \mathrm{SL}_4$ be the Borel subgroup of upper triangular matrices, containing the maximal torus T consisting of diagonal matrices in SL_4.

Example IV.2.7. This resumes Example 2 from p. 100 in [**AB05**]. Let $G = \mathrm{SL}_4$ act on $V := V(\omega_1) \oplus V(\omega_2) \oplus V(\omega_3) \cong k^4 \oplus \bigwedge^2 k^4 \oplus (k^4)^*$, where the action of SL_4 on k^4 is the standard representation on k^4. Then one finds that T_{ad} acts with weights $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ on $\mathrm{Hilb}_h^G(V)_0$. Further, $\omega_1 = 1/4(3\alpha_1 + 2\alpha_2 + \alpha_3)$, $\omega_2 = 1/2(\alpha_1 + 2\alpha_2 + \alpha_3)$, and $\omega_3 = 1/4(\alpha_1 + 2\alpha_2 + 3\alpha_3)$.

Now $\alpha_1 + \alpha_2 = \omega_1 + \omega_2 - \omega_3$, and $\alpha_2 + \alpha_3 = -\omega_1 + \omega_2 + \omega_3$. This shows that $\mathbf{G}_m^3$ acts with weights $(1, 1, -1)$ and $(-1, 1, 1)$ on $\mathrm{Hilb}_h^G(V)_0$. (One now sees that the action of $\mathbf{G}_m^3$ on $\mathrm{Hilb}_h^G(V)_0$ extends to an action of $\mathbf{G}_m \times \mathbf{A}^1 \times \mathbf{G}_m$, but not to an action of $\mathbf{A}^3$.) So

$$\mathrm{Hilb}_h^G(V^r)_0 \cong (\mathbf{A}^{r_1} \setminus \{0\} \times \mathbf{A}^{r_2} \setminus \{0\} \times \mathbf{A}^{r_3} \setminus \{0\} \times \mathbf{A}^2)/\mathbf{G}_m^3,$$

where $\mathbf{G}_m^3$ acts on $\mathbf{A}^{r_1} \setminus \{0\}$ with weight $(-1, 0, 0)$, on $\mathbf{A}^{r_2} \setminus \{0\}$ with weight $(0, -1, 0)$, on $\mathbf{A}^{r_3} \setminus \{0\}$ with weight $(0, 0, -1)$, and on $\mathbf{A}^2$ with weights $(1, 1, -1)$ and $(-1, 1, 1)$. Consider the morphism

$$\mathbf{A}^{r_1} \setminus \{0\} \times \mathbf{A}^{r_2} \setminus \{0\} \times \mathbf{A}^{r_3} \setminus \{0\} \times \mathbf{A}^2 \to \mathbf{P}^{r_1 r_2 + r_3 - 1} \times \mathbf{P}^{r_1 + r_2 r_3 - 1}$$

mapping

$$(x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}, z_1, \ldots, z_{r_3}, u, v)$$

to

$$((\ldots : x_i y_j u : \ldots : z_1 : \ldots : z_{r_3}), (x_1 : \ldots : x_{r_1} : \ldots : y_i z_j v : \ldots)).$$

This morphism is $\mathbf{G}_m^3$-invariant and separates the $\mathbf{G}_m^3$-orbits. Its image is the invariant Hilbert scheme $\mathrm{Hilb}_h^G(V^r)_0$ and has the following geometric description: The projection of the image onto $\mathbf{P}^{r_1 r_2 + r_3 - 1}$ is covered by the open sets where $z_i \neq 0$. Such an open set is isomorphic to $C(V(r_1, r_2)) \times \mathbf{A}^{r_3 - 1}$, where $C(V(r_1, r_2))$ is the affine cone over the Segre embedding $\mathbf{P}^{r_1 - 1} \times \mathbf{P}^{r_2 - 1} \to \mathbf{P}^{r_1 r_2 - 1}$. A similar description can be given for the projection onto the second factor $\mathbf{P}^{r_1 + r_2 r_3 - 1}$.

Similarly as in Remark IV.2.2, one can show here that $\mathrm{Hilb}_h^G(V^r)_0$ is the total space of $\mathcal{O}(-1, 1, 1) \oplus \mathcal{O}(1, 1, -1)$ over $\mathbf{P}^{r_1 - 1} \times \mathbf{P}^{r_2 - 1} \times \mathbf{P}^{r_3 - 1}$.

Example IV.2.8. Take once more $G = \mathrm{SL}_4$, now acting on $V := V(\omega_1 + \omega_3) \oplus V(\omega_2) \cong \mathfrak{sl}_4 \oplus \bigwedge^2 k^4$, where SL_4 acts on $\mathfrak{sl}_4$ with the adjoint representation, and where the action on $\bigwedge^2 k^4$ is induced by the standard representation on k^4.

Denote by E_{ij} the 4×4-matrix with a 1 as unique non-zero entry at position (i, j). Choose the highest weight vectors $v_{\omega_1 + \omega_3} = E_{14} \in \mathfrak{sl}_4$, and $v_{\omega_2} = e_1 \wedge e_2 \in \bigwedge^2 k^4$. One verifies that $[V/\mathfrak{g} \cdot (v_{\omega_1 + \omega_3}, v_{\omega_2})]^{G_{(v_{\omega_1 + \omega_3}, v_{\omega_2})}}$ is spanned by $(E_{23}, 0) + \mathfrak{g} \cdot (v_{\omega_1 + \omega_3}, v_{\omega_2})$, and hence $\dim_k T_{C_\Gamma} \mathrm{Hilb}_h^G(V)_0 = 1$. Since $\langle \omega_1 + \omega_3, \omega_2 \rangle_{\mathbf{N}}$ is saturated, it follows that $\mathrm{Hilb}_h^G(V)_0 \cong \mathbf{A}^1$.

Now, T_{ad} acts on $\mathrm{Hilb}_h^G(V)_0$ with weight $2(\alpha_1+\alpha_3) = 2(\omega_1+\omega_3)-2\omega_2$, and so $\mathrm{GL}^G(V) \cong \mathbf{G}_m^2$ acts on $\mathrm{Hilb}_h^G(V)_0$ with weight $(2,-2)$; and the $\mathbf{G}_m^2$-action on $\mathrm{Hilb}_h^G(V)_0$ extends to an action of $\mathbf{A}^1 \times \mathbf{G}_m$, but not to an action of $\mathbf{A}^2$.

Indeed, if $X(t)$ is the closure of the orbit of $(E_{14}+tE_{23}, e_1 \wedge e_2)$, then one verifies that $h_{X(t)} = h$ for each $t \in k$, and that $X(t_1) = X(t_2)$ if and only if $t_1 = t_2$. Furthermore, $(s_1(E_{14} + tE_{23}), s_2 e_1 \wedge e_2) \in X(s_1^2 s_2^{-2} t)$. This shows that

$$\mathrm{Hilb}_h^G(V^r)_0 \cong (\mathbf{A}^{r_1} \setminus \{0\} \times \mathbf{A}^{r_2} \setminus \{0\} \times \mathbf{A}^1)/\mathbf{G}_m^2,$$

where $\mathbf{G}_m^2$ acts on $\mathbf{A}^{r_1} \setminus \{0\}$ with weight $(-1,0)$, on $\mathbf{A}^{r_2} \setminus \{0\}$ with weight $(0,-1)$, and on $\mathbf{A}^1$ with weight $(2,-2)$. Consider the morphism

$$\begin{aligned} \pi\colon \mathbf{A}^{r_1} \setminus \{0\} \times \mathbf{A}^{r_2} \setminus \{0\} \times \mathbf{A}^1 &\to \mathbf{P}^{r_1+r_2-1} \\ ((x_1,\dots,x_{r_1}),(y_1,\dots,y_{r_2}),z) &\mapsto (x_1^2 z : \ldots : x_{r_1}^2 z : y_1^2 : \ldots : y_{r_2}^2). \end{aligned}$$

One sees that this morphism is $\mathbf{G}_m^2$-invariant and separates the $\mathbf{G}_m^2$-orbits. Its image equals $\mathbf{P}^{r_1+r_2-1} \setminus \mathbf{P}^{r_1}$, where $\mathbf{P}^{r_1+r_2-1} \supset \mathbf{P}^{r_1-1} = \{(a_1 : \ldots : a_{r_1} : 0 : \ldots : 0)\}$. Hence

$$\mathrm{Hilb}_h^G(V^r)_0 \cong \mathbf{P}^{r_1+r_2-1} \setminus \mathbf{P}^{r_1-1}.$$

APPENDIX A

Computations

A.1. The technique

In Section III.6 many computer-based calculations were used. Here we explain how they have been performed. Recall that SL_2 acts on $V(d) = k[x,y]_d$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^i y^{d-i} = (dx - by)^i(-cx + ay)^{d-1}.$$

Then $V(d)^U = k \cdot y^d$. Moreover, in (III.2.1) we identified $\mathrm{Sym}(V(d)^*)$ with $k[a_0, a_1, \dots, a_d]$, where

$$a_i(\sum_{j=0}^{d} \binom{d}{j} \lambda_j x^j y^{d-j}) = \lambda_i.$$

For instance, one readily sees that $(V(d)^*)^U = k \cdot a_d$. Recall that in (III.2.3) we have seen that the r-th transvectant of P and P' can be computed by means of

$$(V_1, V_2)^r = \frac{(d-r)!^2}{d!^2} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \frac{\partial^r V_1}{\partial x^{r-s} \partial y^s} \frac{\partial^r V_2}{\partial x^s \partial y^{r-s}}.$$

This has been implemented in `Mathematica` by Popoviciu-Draisma ([**Po07**]) as follows:

```
<< "Combinatorica`";
l1[n_] := Table[Binomial[n, i]*a[n - i], {i, 0, n}];
w1[n_] := Table[y^(n - i)*x^i, {i, 0, n}];
w2[m_] := Table[t^(m - i)*z^i, {i, 0, m}];
Transvect[l1_, l2_, r_] :=
  CoefficientList[
   Expand[((Length[l1] - 1 - r)!*(Length[l2] - 1 - r)!*
         Sum[(-1)^i*Binomial[r, i]*
           D[D[l1.w1[Length[l1] - 1], {x, r - i}], {y, i}]*
           D[D[l2.w2[Length[l2] - 1], {t, r - i}], {z, i}], {i, 0,
           r}])/((Length[l1] - 1)!*(Length[l2] - 1)!) /. {z -> x,
       t -> y}] /. {y -> 1}, x];
```

Example A.1.1. Let $Q := Q^{(4)} = \sum_{i=0}^{4} a_i \binom{4}{i} x^i y^{4-i}$. Then we can compute $(Q,Q)^0$, $(Q,Q)^1$, $(Q,Q)^2$, $(Q,Q)^3$, and $(Q,Q)^4$ as follows:

```
Transvect[l1[4], l1[4], 0]
->     {a[4]^2, 8 a[3] a[4], 16 a[3]^2 + 12 a[2] a[4],
     48 a[2] a[3] + 8 a[1] a[4], 36 a[2]^2 + 32 a[1] a[3] + 2 a[0] a[4],
     48 a[1] a[2] + 8 a[0] a[3], 16 a[1]^2 + 12 a[0] a[2], 8 a[0] a[1],
     a[0]^2}
```

This means that $(Q,Q)^0 = \sum_{i=0}^{8} \varphi_i x^i y^{8-i}$ with $\varphi_0 = a_4^2$, $\varphi_1 = 8a_3a_4$, $\varphi_2 = 16a_3^2 + 12a_2a_4$, $\varphi_3 = 48a_2a_3 + 8a_1a_4$, $\varphi_4 = 36a_2^2 + 32a_1a_3 + 2a_0a_4$, $\varphi_5 = 48a_1a_2 + 8a_0a_3$, $\varphi_6 = 16a_1^2 + 12a_0a_2$, $\varphi_7 = 8a_0a_1$, and $\varphi_8 = a_0^2$. In particular, it follows that

$$(\mathrm{Sym}^2(V(4)^*)_{(8)})^U = k \cdot \varphi_0(a_0, \ldots, a_4) = k \cdot a_4^2.$$

Proceeding with $(Q,Q)^1$, we have:

```
Transvect[l1[4], l1[4], 1]
->     {}
```

This reflects the fact that $(Q,Q)^l = 0$ if l is odd. Continuing, we find:

```
Transvect[l1[4], l1[4], 2]
->     {-2 a[3]^2 + 2 a[2] a[4], -4 a[2] a[3] + 4 a[1] a[4], -6 a[2]^2 +
     4 a[1] a[3] + 2 a[0] a[4], -4 a[1] a[2] + 4 a[0] a[3], -2 a[1]^2 +
     2 a[0] a[2]}
Transvect[l1[4], l1[4], 3]
->     {}
Transvect[l1[4], l1[4], 4]
->     {6 a[2]^2 - 8 a[1] a[3] + 2 a[0] a[4]}
```

Hence

$$(\mathrm{Sym}^2(V(4)^*)_{(4)})^U = k \cdot (-2a_3^2 + 2a_2a_4),$$

and

$$(\mathrm{Sym}^2(V(4)^*)_{(0)})^U = k \cdot (6a_2^2 - 8a_1a_3 + 2a_0a_4).$$

Example A.1.2. We wish to find all covariants of degree 4 and of order 4 of a form of degree 4. Using either the Cayley-Sylvester-formula or using [**LiE**] we compute the decomposition of $\mathrm{Sym}^4(V(4)^*)$ into isotypic components:

```
sym_tensor(4,[4],A1)
->     1X[ 0] +2X[ 4] +2X[ 8] +1X[10] +1X[12] +1X[16]
```

This shows that $\mathrm{Sym}^4(V(4)^*) \cong V(0) \oplus V(4)^2 \oplus V(8)^2 \oplus V(10) \oplus V(12) \oplus V(16)$. In particular, $\dim_k(\mathrm{Sym}^4(V(4)^*)_{(4)})^U = 2$. Let $Q := Q^{(4)} = \sum_{i=0}^{4} a_i \binom{4}{i} x^i y^{4-i}$. We claim that

$$\mathrm{Sym}^4(V(4)^*)_{(4)} = \mathrm{span}_k((((Q,Q)^2,Q)^4,Q)^0, (((Q,Q)^2,Q)^0),Q)^4).$$

It suffices to check that the U-invariants of these two covariants are linearly independent:

```
C4 = Transvect[Transvect[ Transvect[l1[4], l1[4], 2], l1[4], 4],
 l1[4], 0][[1]]
->     -6 a[2]^3 a[4] + 12 a[1] a[2] a[3] a[4] - 6 a[0] a[3]^2 a[4] -
     6 a[1]^2 a[4]^2 + 6 a[0] a[2] a[4]^2
```

and

```
D4 = Transvect[Transvect[ Transvect[l1[4], l1[4], 2], l1[4], 0],
  l1[4], 4][[1]]
->     -(18/7) a[2]^2 a[3]^2 + 24/7 a[1] a[3]^3 + 27/35 a[2]^3 a[4] +
     6/35 a[1] a[2] a[3] a[4] - 93/35 a[0] a[3]^2 a[4] -
     9/5 a[1]^2 a[4]^2 + 93/35 a[0] a[2] a[4]^2
```

One immediately sees that these two U-invariants are linearly independent. In order to be able to verify this with the help of the computer, we proceed as follows: Consider the function `Coefficient[f`

`,CoefficientVector[d,k,n]]` that writes the coefficients of a U-invariant function f of degree n and order k of a form of degree d in a vector:

```
VerifyList[l_, deg_, sum_, n_] := (
  If[Length[l] < deg,
    For[i = l[[Length[l]]], i < n + 1, i = i + 1,
     VerifyList[Append[l, i], deg, sum, n]];
    If[l[[Length[l]]] < n, last = l[[Length[l]]];
     VerifyList[Append[Drop[l, -1], last + 1], deg, sum, n]],
    If[Sum[l[[j]], {j, 1, Length[l]}] == sum,
     next = Product[a[l[[j]]], {j, 1, Length[l]}];
     Coeff = Append[Coeff, next] ]];
  )

CoefficientVector[deg_, ord_, n_] := (
  Coeff = {}; l = {0}; sum = n*deg - (n*deg - ord)/2;
  VerifyList[l, deg, sum, n]; Return[Coeff] )
```

Now

```
Coefficient[C4, CoefficientVector[4, 4, 4]]
->     {6, -6, -6, 12, 0, -6, 0}
Coefficient[D4, CoefficientVector[4, 4, 4]]
->     {93/35, -(93/35), -(9/5), 6/35, 24/7, 27/35, -(18/7)}
MatrixRank[{Coefficient[C4, CoefficientVector[4, 4, 4]],
  Coefficient[D4, CoefficientVector[4, 4, 4]]}]
->     2
```

shows that the two U-invariants are linearly independent.

A.2. Forms of degree 8

Let $Q := Q^{(8)} = \sum_{i=0}^{8} a_i \binom{8}{i} x^i y^{8-i}$. In the proof of Theorem III.6.5 we claimed that $\mathrm{Sym}^3(V(8)^*)_{(4)}$ is contained in the SL_2-stable ideal generated by $\mathrm{Sym}^2(V(8)^*)_{(8)}$. This can be verified as follows: Using [**LiE**] we compute the decomposition of $\mathrm{Sym}^3(V(8)^*)$ into isotypic components:

```
sym_tensor(3,[8],A1)
->    1X[ 0] +1X[ 4] +1X[ 6] +2X[ 8] +1X[10] +2X[12] +1X[14] +1X[16] +
      1X[18] +1X[20] +1X[24]
```

This shows that

$$\begin{aligned}\mathrm{Sym}^3(V(8)^*) \cong & V(0) \oplus V(4) \oplus V(6) \oplus V(8)^2 \oplus V(10) \oplus V(12)^2 \\ & \oplus V(14) \oplus V(16) \oplus V(18) \oplus V(20) \oplus V(24).\end{aligned}$$

In particular, $\mathrm{Sym}^3(V(8)^*)_{(4)} \cong V(4)$. Moreover, $\mathrm{Sym}^2(V(8)^*)_{(8)} = (Q,Q)^4$ according to the Clebsh-Gordan decomposition. In order to verfiy the claim, it thus suffices to show that

$$((Q,Q)^4, Q)^6 \neq 0.$$

This can be done using `Mathematica`: The command

```
Transvect[Transvect[l1[8], l1[8], 4], l1[8], 6][[1]]
```

yields

```
->     -30/7 a[4]^2 a[6] + 48/7 a[3] a[5] a[6] - 24/7 a[2] a[6]^2 +
     12/7 a[3] a[4] a[7] - 24/7 a[2] a[5] a[7] + 24/7 a[1] a[6] a[7] -
     6/7 a[0] a[7]^2 - 30/7 a[3]^2 a[8] + 48/7 a[2] a[4] a[8] -
     24/7 a[1] a[5] a[8] + 6/7 a[0] a[6] a[8],
```

which is a non-zero U-invariant in $((Q,Q)^4,Q)^6$. Hence $((Q,Q)^4,Q)^6 \neq 0$.

A.3. Forms of degree 12

Let $Q := Q^{(12)} = \sum_{i=0}^{12} a_i \binom{12}{i} x^i y^{12-i}$. In the proof of Proposition III.6.7 we claimed that $\mathrm{Sym}^4(V(12)^*)_{(4)} \oplus \mathrm{Sym}^4(V(12)^*)_{(8)}$ is in the SL_2-stable ideal generated by

$$\bigoplus_{0\le k\le 18, k\neq 4,8} \mathrm{Sym}^3(V(12)^*)_{(k)}.$$

To see this, we first compute the multiplicities of $V(4)$ and $V(8)$ in $\mathrm{Sym}^4(V(12)^*)$ using [**LiE**]:

```
sym_tensor(4,[12],A1)
->   3X[ 0] +4X[ 4] +2X[ 6] +6X[ 8] +3X[10] +7X[12] +4X[14] +7X[16] +
     5X[18] +7X[20] +4X[22] +7X[24] +4X[26] +5X[28] +3X[30] +4X[32] +
     2X[34] +3X[36] +1X[38] +2X[40] +1X[42] +1X[44] +1X[48]
```

Hence $V(4)$ and $V(8)$ have multiplicity 4 and 6 in $\mathrm{Sym}^4(V(12)^*)$.

In a first step we compute covariants of degree 2 and 3. The covariants of degree 2 are computed by means of

```
A0 = Transvect[l1[12], l1[12], 12];
A4 = Transvect[l1[12], l1[12], 10];
A8 = Transvect[l1[12], l1[12], 8];
A12 = Transvect[l1[12], l1[12], 6];
A16 = Transvect[l1[12], l1[12], 4];
A20 = Transvect[l1[12], l1[12], 2];
A24 = Transvect[l1[12], l1[12], 0];
```

With these, we compute a series of covariants of degree 3:

```
B10 = Transvect[A12, l1[12], 7];
B12 = Transvect[A12, l1[12], 6];
B14 = Transvect[A12, l1[12], 5];
B16 = Transvect[A12, l1[12], 4];
B18 = Transvect[A12, l1[12], 3];
C12 = Transvect[A16, l1[12], 8];
D12 = Transvect[A20, l1[12], 10];
```

We now proceed with covariants of degree 4:

```
E41 = Transvect[B10, l1[12], 9];
E42 = Transvect[B12, l1[12], 10];
E43 = Transvect[B14, l1[12], 11];
E44 = Transvect[C12, l1[12], 10];
MatrixRank[{Coefficient[E41[[1]], CoefficientVector[4, 4, 12]],
  Coefficient[E42[[1]], CoefficientVector[4, 4, 12]],
  Coefficient[E43[[1]], CoefficientVector[4, 4, 12]],
  Coefficient[E44[[1]], CoefficientVector[4, 4, 12]] }]
->     4
```

shows that the transvectants E41, E42, E43, and E4 are linearly independent and hence span $\mathrm{Sym}^4(V(12)^*)_{(4)} \cong V(4)^4$. Similarly,

```
E81 = Transvect[B10, l1[12], 7];
E82 = Transvect[B12, l1[12], 8];
E83 = Transvect[B14, l1[12], 9];
E84 = Transvect[B16, l1[12], 10];
E85 = Transvect[B18, l1[12], 11];
E86 = Transvect[C12, l1[12], 8];
MatrixRank[{Coefficient[E81[[1]], CoefficientVector[4, 8, 12]],
  Coefficient[E82[[1]], CoefficientVector[4, 8, 12]],
  Coefficient[E83[[1]], CoefficientVector[4, 8, 12]],
  Coefficient[E84[[1]], CoefficientVector[4, 8, 12]],
  Coefficient[E85[[1]], CoefficientVector[4, 8, 12]],
  Coefficient[E86[[1]], CoefficientVector[4, 8, 12]]}]
->     6
```

shows that the transvectants E81, E82, E83, E84, E85, and E86 are linearly independent and hence span $\mathrm{Sym}^4(V(12)^*)_{(8)} \cong V(8)^6$.

Moreover, in the proof of Proposition III.6.7 we claim both that $\mathrm{Sym}^3(V(12)^*)_{(4)}$ is contained in the SL_2-stable ideal generated by the covariant $\mathrm{Sym}^2(V(12)^*)_{(12)}$, and that $\mathrm{Sym}^3(V(12)^*)_{(8)}$ is in the SL_2-stable ideal generated by $\mathrm{Sym}^2(V(12)^*)_{(8)} \oplus \mathrm{Sym}^2(V(12)^*)_{(12)}$. This now can be verified as follows:

```
MatrixRank[{Coefficient[Transvect[A12, l1[12], 10][[1]],
   CoefficientVector[3, 4, 12]]}]
->     1
MatrixRank[{Coefficient[Transvect[A8, l1[12], 6][[1]],
   CoefficientVector[3, 8, 12]],
  Coefficient[Transvect[A12, l1[12], 8][[1]],
   CoefficientVector[3, 8, 12]]}]
->     2
```

Comparing this with the decomposition of $\mathrm{Sym}^3(V(12)^*)$,

```
sym_tensor(3,[12],A1)
->   1X[ 0] +1X[ 4] +1X[ 6] +2X[ 8] +1X[10] +3X[12] +2X[14] +2X[16] +
     2X[18] +2X[20] +1X[22] +2X[24] +1X[26] +1X[28] +1X[30] +1X[32] +
     1X[36]
```

the claim now follows.

Finally, in the proof of Proposition III.6.9 the U-invariant -2$(-10a_9^2 + 15a_{10}a_8 - 6a_{11}a_7 + a_{12}a_6) \in (Q, Q)^6$ enters the play. Indeed:

```
A12[[1]]
->     -20 a[9]^2 + 30 a[8] a[10] - 12 a[7] a[11] + 2 a[6] a[12]
```

is the searched U-invariant.

A.4. Forms of degree 16

Let $Q := Q^{(16)} = \sum_{i=0}^{16} a_i \binom{16}{i} x^i y^{16-i}$. In Step 1 of the proof of Lemma III.6.12 we claimed that

$$\mathrm{Sym}^4(V(16)^*)_{(4)} \oplus \mathrm{Sym}^4(V(16)^*)_{(8)} \oplus \mathrm{Sym}^4(V(16)^*)_{(12)}$$

is in the SL_2-stable ideal generated by

$$\bigoplus_{0 \le k \le 22, k \ne 4,8,12} \mathrm{Sym}^3(V(12)^*)_{(k)}.$$

To see this, we first compute the multiplicities of $V(4)$, $V(8)$, and $V(12)$ in $\mathrm{Sym}^4(V(16)^*)$ using [**LiE**]:

```
sym_tensor(4,[16],A1)
->    3X[ 0] + 6X[ 4] + 2X[ 6] + 8X[ 8] + 5X[10] + 9X[12] + 6X[14] +
     11X[16] + 7X[18] +11X[20] + 8X[22] +11X[24] + 8X[26] +11X[28] +
      7X[30] +10X[32] + 7X[34] + 8X[36] + 5X[38] + 7X[40] + 4X[42] +
      5X[44] + 3X[46] + 4X[48] + 2X[50] + 3X[52] + 1X[54] + 2X[56] +
      1X[58] + 1X[60] + 1X[64]
```

Hence $V(4)$, $V(8)$ and $V(12)$ have multiplicity 6, 8 and 9, respectively, in $\mathrm{Sym}^4(V(16)^*)$.

In a first step we compute covariants of degree 2 and 3. The covariants of degree 2 are computed by means of

```
A0 = Transvect[l1[16], l1[16], 16];
A4 = Transvect[l1[16], l1[16], 14];
A8 = Transvect[l1[16], l1[16], 12];
A12 = Transvect[l1[16], l1[16], 10];
A16 = Transvect[l1[16], l1[16], 8];
A20 = Transvect[l1[16], l1[16], 6];
A24 = Transvect[l1[16], l1[16], 4];
A28 = Transvect[l1[16], l1[16], 2];
A32 = Transvect[l1[16], l1[16], 0];
```

With these, we compute a series of covariants of degree 3:

```
B16 = Transvect[A12, l1[16], 6];
B18 = Transvect[A12, l1[16], 5];
B20 = Transvect[A12, l1[16], 4];
B22 = Transvect[A12, l1[16], 3];
C14 = Transvect[A16, l1[16], 9];
C16 = Transvect[A16, l1[16], 8];
C18 = Transvect[A16, l1[16], 7];
C20 = Transvect[A16, l1[16], 6];
D14 = Transvect[A20, l1[16], 11];
D18 = Transvect[A20, l1[16], 9];
```

Proceeding with covariants of degree 4,

```
F41 = Transvect[B16, l1[16], 14];
F42 = Transvect[C14, l1[16], 13];
F43 = Transvect[C16, l1[16], 14];
F44 = Transvect[C18, l1[16], 15];
F45 = Transvect[D14, l1[16], 13];
F46 = Transvect[D18, l1[16], 15];
MatrixRank[{Coefficient[F41[[1]], CoefficientVector[4, 4, 16]],
  Coefficient[F42[[1]], CoefficientVector[4, 4, 16]],
  Coefficient[F43[[1]], CoefficientVector[4, 4, 16]],
  Coefficient[F44[[1]], CoefficientVector[4, 4, 16]],
  Coefficient[F45[[1]], CoefficientVector[4, 4, 16]],
  Coefficient[F46[[1]], CoefficientVector[4, 4, 16]]}]
->     6
```

shows that the transvectants `F41`, `F42`, `F43`, `F44`, `F45`, and `F46` are linearly independent and hence span $\mathrm{Sym}^4(V(16)^*)_{(4)} \cong V(4)^6$. Similarly,

```
F81 = Transvect[B16, l1[16], 12];
F82 = Transvect[B18, l1[16], 13];
F83 = Transvect[B20, l1[16], 14];
F84 = Transvect[C14, l1[16], 11];
F85 = Transvect[C16, l1[16], 12];
F86 = Transvect[C18, l1[16], 13];
```

```
F87 = Transvect[D14, l1[16], 11];
F88 = Transvect[D18, l1[16], 13];
MatrixRank[{Coefficient[F81[[1]], CoefficientVector[4, 8, 16]],
  Coefficient[F82[[1]], CoefficientVector[4, 8, 16]],
  Coefficient[F83[[1]], CoefficientVector[4, 8, 16]],
  Coefficient[F84[[1]], CoefficientVector[4, 8, 16]],
  Coefficient[F85[[1]], CoefficientVector[4, 8, 16]],
  Coefficient[F86[[1]], CoefficientVector[4, 8, 16]],
  Coefficient[F87[[1]], CoefficientVector[4, 8, 16]],
  Coefficient[F88[[1]], CoefficientVector[4, 8, 16]]}]
->     8
```

shows that the transvectants `F81`, ..., `F88` are linearly independent and hence span $\mathrm{Sym}^4(V(16)^*)_{(8)} \cong V(8)^8$. Finally,

```
F121 = Transvect[B16, l1[16], 10];
F122 = Transvect[B20, l1[16], 12];
F123 = Transvect[B22, l1[16], 13];
F124 = Transvect[C14, l1[16], 9];
F125 = Transvect[C16, l1[16], 10];
F126 = Transvect[C18, l1[16], 11];
F127 = Transvect[C20, l1[16], 12];
F128 = Transvect[D14, l1[16], 9];
F129 = Transvect[D18, l1[16], 11];
MatrixRank[{Coefficient[F121[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F122[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F123[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F124[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F125[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F126[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F127[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F128[[1]], CoefficientVector[4, 12, 16]],
  Coefficient[F129[[1]], CoefficientVector[4, 12, 16]]}]
->     9
```

shows that the transvectants `F121`, ..., `F129` are linearly independent and hence span $\mathrm{Sym}^4(V(16)^*)_{(12)} \cong V(12)^9$.

In Step 2 of the proof of Lemma III.6.12 we claimed that the covariant $\mathrm{Sym}^3(V(16)^*)_{(4)}$ is contained in the SL_2-stable ideal generated by $\mathrm{Sym}^2(V(16)^*)_{(16)}$. First

```
sym_tensor(3,[16],A1)
->    1X[ 0] +1X[ 4] +1X[ 6] +2X[ 8] +1X[10] +3X[12] +2X[14] +3X[16] +
      3X[18] +3X[20] +2X[22] +3X[24] +2X[26] +2X[28] +2X[30] +2X[32] +
      1X[34] +2X[36] +1X[38] +1X[40] +1X[42] +1X[44] +1X[48]
```

shows that $\mathrm{Sym}^3(V(16)^*)_{(4)} \cong V(4)$. Now with

```
MatrixRank[{Coefficient[Transvect[A16, l1[16], 14][[1]],
   CoefficientVector[3, 4, 16]]}]
->     1
```

the claim follows.

In Step 3 of the proof of Lemma III.6.12 we claimed that $C := ((Q,Q)^{12},Q)^8 = ((Q,Q)^8,Q)^{12})$, and that $V(8)^2 \cong \mathrm{Sym}^3(V(16)^*)_{(8)} = C \oplus ((Q,Q)^{10},Q)^{10}$, where $Q = \sum_{i=0}^{16} a_i \binom{16}{i} x^i y^{16-i}$. First

```
MatrixRank[{Coefficient[Transvect[A8, l1[16], 8][[1]],
   CoefficientVector[3, 8, 16]],
  Coefficient[Transvect[A16, l1[16], 12][[1]],
   CoefficientVector[3, 8, 16]]}]
```

```
->      1
```

shows that $((Q,Q)^{12},Q)^8 = ((Q,Q)^8,Q)^{12})$. Now

```
MatrixRank[{Coefficient[Transvect[A8, l1[16], 8][[1]],
   CoefficientVector[3, 8, 16]],
  Coefficient[Transvect[A12, l1[16], 10][[1]],
   CoefficientVector[3, 8, 16]]}]
->      2
```

and the fact that $V(8)^2 \cong \mathrm{Sym}^3(V(16)^*)_{(8)}$ prove the claim.

Moreover, we need to show that $V(12)^3 \cong \mathrm{Sym}^3(V(16)^*)_{(12)} = C'' \oplus C''' \oplus C''''$ with $C'' = ((Q,Q)^{12},Q)^6$, with $C''' = ((Q,Q)^{10},Q)^8$, and with $C'''' = ((Q,Q)^8,Q)^{10}$. This follows from

```
MatrixRank[{Coefficient[Transvect[A8, l1[16], 6][[1]],
   CoefficientVector[3, 12, 16]],
  Coefficient[Transvect[A12, l1[16], 8][[1]],
   CoefficientVector[3, 12, 16]],
  Coefficient[Transvect[A16, l1[16], 10][[1]],
   CoefficientVector[3, 12, 16]]}]
->     3.
```

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Glossary

The following notation is used throughout this thesis:

G	a semisimple group
B	a (fixed) Borel subgroup of G
T	a (fixed) maximal torus of G contained in B
U	the maximal unipotent subgroup of G determined by B and T
Λ^+	monoid of dominant weights (with respect to (G, B, T))
λ	dominant weight (of a fixed group G)
λ^*	$-w_0\lambda$, where w_0 is the longest word in the Weyl group of (G, B, T)
$V(\lambda)$	simple G-module with highest weight λ
$\mathcal{S}(\lambda)$	closure of the unique minimal sheet of $V(\lambda)$
$\mathcal{N}(\lambda)$	(schematic) nullcone of $V(\lambda)$, i.e. schematic fiber $V(\lambda)_0$ of the quotient morphism $V(\lambda) \to V(\lambda)/\!\!/G$
v_λ	a (fixed) non-zero highest weight vector in $V(\lambda)$
X_λ	closure of the orbit Gv_λ in $V(\lambda)$
h_λ	Hilbert function of X_λ
h'_λ	Hilbert function of a general orbit of $\mathcal{S}(\lambda)$
R	the ring of regular functions of $\mathcal{S}(\lambda)$, regarded as k-G-algebra (Caveat: R has another meaning in Chapter III.)
$R_{(\lambda)}$	the G-isotypic component of R of type λ
$R_{(n,\lambda)}$	the $\mathbf{G}_m \times G$-isotypic component of R of type (n, λ)
n_0	$\gcd\{n \mid R_{(n,0)} \neq 0\}$ (cf. Definition II.2.2). The integer n_0 equals 1 or 2, cf. Lemma II.2.3
$\mathcal{O}(X)$	if X is an affine variety, we denote by $\mathcal{O}(X)$ its ring of regular functions
$\mathrm{Hilb}_h^G(X)$	invariant Hilbert scheme to the data (G, h, X)
$\mathcal{H}ilb_h^G(X)$	invariant Hilbert functor to the data (G, h, X)
$\mathrm{Univ}_h^G(X)$	universal family for $\mathcal{H}ilb_h^G(X)$

In Chapter III we have in addition:

B	the Borel subgroup of SL_2 consisting of upper triangular matrices in SL_2
T	the maximal torus of SL_2 consisting of all diagonal matrices in SL_2

U	the maximal unipotent subgroup of SL_2 consisting of all strict upper triangular matrices in SL_2
$V(d)$	the simple SL_2-module $V(d)$ is realized as the space of binary forms $\bigoplus_{i=0}^{d} kx^i y^{d-i}$
y^d	is fixed as highest weight vector in $V(d)$
$\mathcal{S}(d)$	equals the closure of $\mathrm{SL}_2 \cdot y^d$ in $V(d)$ if d is odd, and the closure of $\{\mathrm{SL}_2 \cdot \gamma x^{d/2} y^{d/2} \mid \gamma \in k\}$ in $V(d)$ if d is even
h_d	$h_d \colon \mathbf{N} \to \mathbf{N}$ is defined by $h_d(k) = 1$ if k is divisible by d, and $h_d(k) = 0$ otherwise
h'_d	If d is odd, then $h'_d = h_d$. If d is even, but not divisible by 4, then $h'_d \colon \mathbf{N} \to \mathbf{N}$ is defined by $h'_d(k) = 1$ if k is even, and $h'_d(k) = 0$ if k is odd. If d is divisible by 4, then $h'_d \colon \mathbf{N} \to \mathbf{N}$ is defined by $h_d(k) = 1$ if k is divisible by 4, and $h_d(k) = 0$ otherwise (cf. Section III.3).
$Q^{(d)}$	$Q^{(d)} = \sum_{i=0}^{d} a_i \binom{d}{i} x^i y^{d-i}$ is the unique covariant of $V(d)$ of degree 1 (cf. Definition III.2.2)
$(Q^{(d)}, Q^{(d)})^l$	the l-th transvectant $(Q^{(d)}, Q^{(d)})^l$ is zero if l is odd, and non-zero if $l \in \{0, 2, \ldots, d\}$ (cf. Remark III.2.19 and Lemma III.3.5 a))
$i^{(d)}$	the invariant $(Q^{(d)}, Q^{(d)})^d$ of degree 2, non-zero if and only if d is even
$j^{(d)}$	the invariant $((Q^{(d)}, Q^{(d)})^{d/2}, Q^{(d)})^d$ of degree 3, non-zero if and only if d is a multiple of 4
n	usually denotes the degree of a covariant or of a digraph
k	usually denotes the order of a covariant or of a digraph
l	usually denotes the co-order of a homogeneous covariant or digraph: a covariant or digraph homogeneous of degree n and of order k has co-order $l := (nd - k)/2$
$\mathcal{D}^d$	algebra of d-digraphs
$\mathcal{D}^d_{n;l}$	vector space of d-digraphs of degree n and co-order l (cf. Section III.4
R^d	in contrast to the other chapters, R^d stands for $\mathrm{Sym}(V(d)^*)$ in Chapter III (cf. Section III.5)
$R^d_{(k)}$	is the SL_2-isotypic component of R^d of type k (cf. Section III.5)
$R^d_{(n,k)}$	is the $\mathbf{G}_m \times \mathrm{SL}_2$-isotypic component of R^d of type (n, k) (cf. Section III.5)
J^d_{odd}	is the intersection of all SL_2-stable ideals of R^d containing all covariants of odd co-order (cf. Definition III.5.1)
$J^d_{odd,+}$	is the intersection of all SL_2-stable ideals of R^d containing J^d_{odd} and $R^d_{(2,d)}$ (cf. Definition III.5.1)

$C^d_{n;l}$	the covariant $(Q^{(d)}, Q^{(d)})^l \cdot (Q^{(d)})^{n-2}$ if $n \geq 2$, or the covariant $Q^{(d)}$ if $(n;l) = (1;0)$. This covariant is of degree n and of co-order l. It is zero for all odd l (cf. Definition III.5.5).
X^0_d	the unique homogeneous SL_2-stable subscheme of $\mathcal{S}(d)$ with Hilbert function $h_{X^0_d} = h'_d$ (cf. Definition III.3.3)
I^0	the SL_2-stable ideal of R^d defining X^0_d as closed subscheme of $V(d)$ (cf. Remark III.3.4)

Curriculum Vitae

Ich, Jonas Budmiger, Bürger von Römerswil LU, bin geboren am 9. September 1981 in Binningen BL als Sohn von Thomas Budmiger und Corinna Budmiger-Böhmer.

Ausbildung

8. 1997 bis 12. 2000	Besuch des Gymnasium Oberwil BL, Maturität
10. 2001 bis 10. 2004	Studium der Mathematik und Physik an der Universität Basel, Bachelor of Science in Mathematics und Bachelor of Science in Physics
10. 2004 bis 3. 2006	Studium der Mathematik an der Universität Basel, Master of Science in Mathematics
4. 2006 bis 3. 2010	Doktorand in Mathematik an der Universität Basel, angestellt vom Schweizerischen Nationalfonds
Seit 3. 2010	Studium des Höheren Lehramts an der Pädagogischen Hochschule der Fachhochschule Nordwestschweiz
8. April 2010	Doktorkolloquium

Ich habe an mathematischen Konferenzen in Århus (Dänemark), in Oberwolfach (Deutschland) und in Pisa (Italien) teilgenommen und konnte meinen Korreferenten, Prof. Michel Brion, in Grenoble (Frankreich) besuchen.

Meine akademischen Lehrer waren Proff. N. A'Campo, C. Bandle, J. Draisma, M. Grote, H. C. Im Hof, A. Iozzi, H. Kraft, D. Masser und W. Reichel in Mathematik und Proff. B. Binggeli, Ch. Bruder, R. Buser, Ch. Schönenberger, I. Sick, G. Tamman, F. Thielemann und D. Trautmann in Astronomie und Physik.

Prof. Hanspeter Kraft hat meine Doktorarbeit geleitet.