Groupes de tresses et catégorification

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Contents

Introduction

Le contexte

Cette thèse porte sur des développements récents en théorie des nœuds et des tresses. Graal des théoriciens des nœuds, la classification des nœuds à isotopie près consiste à déterminer systématiquement si deux nœuds ou plus généralement deux entrelacs de $\mathbb{R}^3$ peuvent s'obtenir par déformation continue l'un de l'autre. Une classification partielle est obtenue en construisant des invariants d'isotopie que l'on évalue sur les entrelacs à distinguer.

Ces invariants sont souvent construits à partir de diagrammes d'entrelacs, projections génériques des entrelacs sur un plan réel $\mathbb{R}^2$. Lorsqu'on déforme un entrelacs, son diagramme subit des modifications. Ces déformations peuvent avoir différentes incidences sur le diagramme. Les unes sont relativement triviales, ce sont des isotopies planaires. Les autres sont des successions de trois mouvements types : les mouvements de Reidemeister. Reidemeister a prouvé [Rei32] dans les années 1920 que pour classifier les entrelacs à isotopie près, il suffit de classifier les diagrammes planaires à mouvements de Reidemeister et à isotopie planaire près.

On s'assure alors que l'objet algébrique que l'on associe aux entrelacs (nombre, groupe, polynôme...) est un invariant en vérifiant que sa valeur reste inchangée lorsque l'on applique des mouvements de Reidemeister à leurs diagrammes. Si un invariant prend des valeurs différentes pour deux entrelacs, alors ces entrelacs ne sont pas isotopes. On peut citer, parmi les invariants classiques, le minimum du nombre de croisements sur l'ensemble des diagrammes représentant l'entrelacs, le groupe fondamental du complémentaire de l'entrelacs dans $\mathbb{R}^3$ ou encore le polynôme d'Alexander (voir par exemple [CF77]).

Dans les années 1920, Artin a introduit le premier [Art25] la notion de tresse telle qu'on l'étudie aujourd'hui. Une tresse b à n brins est une collection de n chemins lisses $b_k : [0,1] \to \mathbb{R}^2 \times [0,1]$ où $k = 1, \ldots, n$, tels que :

$$b_k(0) = ((k,0),0) \text{ et } b_k(1) = ((s(k),0),1)$$

où s est une permutation de $\{1, \ldots, n\}$, et

$$b_k(t) = (x_k^t, t) \text{ avec } x_k^t \neq x_l^t \text{ si } k \neq l$$

pour tout $t \in [0,1]$. Chaque chemin b_k est un brin de la tresse b.

Les tresses et les entrelacs sont des objets mathématiques intimement liés ; on étudie les tresses à isotopie près et on les représente par des diagrammes. On peut obtenir un entrelacs orienté à partir d'une tresse par l'opération de clôture, qui consiste à joindre les extrémités supérieures des brins aux extrémités inférieures correspondantes. Inversement, Alexander [Ale23] a prouvé en 1923 que chaque entrelacs orienté peut s'obtenir à isotopie près comme la clôture d'une tresse.

Doté de la loi de composition consistant à concaténer deux tresses puis à redimensionner la coordonnée verticale, l'ensemble des tresses à n brins devient un groupe, noté $\mathcal{B}_n$. Ce groupe est engendré par $n-1$ générateurs $\sigma_1, \dots, \sigma_{n-1}$ satisfaisant les relations suivantes, appelées relations de tresses :

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{pour} \quad |i-j| > 1$$

et

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{pour} \quad i = 1 \dots n-2.$$

En 1984 Vaughan Jones [Jon85] [Jon87] a découvert de nouvelles représentations des groupes de tresses. Il en a déduit la construction d'un nouvel invariant des nœuds orientés, le polynôme de Jones, à valeurs dans l'anneau $\mathbb{Z}[q, q^{-1}]$ des polynômes de Laurent à coefficients entiers.

Cette découverte a entraîné des développements importants en théorie des nœuds et la construction d'une myriade d'autres invariants. Citons les invariants quantiques ainsi nommés en raison de leur lien avec la théorie, également nouvelle, des groupes quantiques et le polynôme de HOMFLY-PT, généralisation à deux variables du polynôme de Jones, découvert simultanément [FYH+85] [PT88] par les mathématiciens Freyd et Yetter, Hoste, Lickorish et Millett, Ocneanu, Przytycki et Traczyk. Le polynôme de HOMFLY–PT, qui à tout entrelacs orienté L associe un polynôme à deux variables $P(L)$, est uniquement déterminé par sa normalisation, c'est-à-dire sa valeur sur le nœud trivial

$$P\left(\bigcirc \right) = \frac{a - a^{-1}}{q - q^{-1}},$$

et la relation d'écheveau suivante :

$$aP\left(\nwarrow\!\!\!\!\nearrow \right) - a^{-1}P\left(\nwarrow\!\!\!\!\nearrow \right) = (q - q^{-1})P\left( )(\right).$$

Le polynôme de Jones P_2 correspond à la spécialisation en $a = q^2$ du polynôme de HOMFLY-PT. Le polynôme de Jones admet alors une description purement combinatoire.

La notion de catégorification a été introduite par Crane [Cra95] [CF94]. C'est une démarche consistant à remplacer des objets ensemblistes par des analogues catégoriques. Suivant le cadre dans lequel on travaille, la notion de catégorification peut revêtir différents sens, non sans lien les uns avec les autres.

Dans une optique d'étude d'invariants d'entrelacs, le procédé de catégorification consiste par exemple à voir un nombre entier positif comme la dimension d'un espace vectoriel et un nombre entier quelconque comme la caractéristique d'Euler d'une homologie. Plus généralement, on exprime un polynôme de Laurent à coefficients positifs comme la dimension graduée (ou quantique) d'un espace vectoriel gradué. La graduation sert ici à coder l'information de degré de la variable du polynôme, alors que dans le cas de l'homologie la graduation sert à coder l'information de signe. Ainsi, pour les cas qui nous intéressent, à savoir les polynômes de Laurent quelconques, la catégorification nécessite au moins un espace vectoriel bigradué ; une graduation codant le signe, l'autre le degré.

Une manière plus algébrique d'appréhender la notion de catégorification consiste à voir un groupe (resp. un anneau) comme le groupe de Grothendieck (resp. l'anneau de Grothendieck) scindé d'une catégorie abélienne (resp. monoïdale).

En 2000 Khovanov [Kho00] a construit un nouvel invariant, de nature homologique, en attachant à tout entrelacs L une famille finie de groupes d'homologie bigraduée $(Kh^{i,j}(L))_{i,j\in\mathbb{Z}}$ telle que le polynôme de Jones $P_2(L)$ s'exprime comme la caractéristique d'Euler graduée de cette homologie :

$$P_2(L)(q) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \mathrm{dim} Kh^{i,j}(L).$$

L'homologie de Khovanov catégorifie donc le polynôme de Jones.

La construction de cette homologie est basée sur une expression combinatoire du polynôme de Jones par somme d'états donnée par Kauffman dans [Kau87] et utilise l'algèbre de Frobenius $V_2 = \mathbb{Q}[x]/x^2$. L'homologie de Khovanov est un invariant d'entrelacs plus puissant que le polynôme de Jones [BN02] car elle permet de distinguer des entrelacs qui ont le même polynôme de Jones.

En 2004 Khovanov et Rozansky [KR08a] ont construit une variante de l'homologie de Khovanov, elle aussi invariante par les mouvements de Reidemeister, qui catégorifie une famille d'invariants polynomiaux, les polynômes $\mathfrak{sl}_n$, spécialisations en $a = q^n$ du polynôme de HOMFLY-PT.

Puis dans [KR08b], en utilisant les mêmes outils, à savoir les factorisations matricielles, ces auteurs ont associé à tout entrelacs $\bar{b}$, clôture d'une tresse b, une homologie trigraduée $(HKR_i^{j,k}(\bar{b}))_{i,j,k\in\mathbb{Z}}$ qui est un invariant d'entrelacs orientés qui catégorifie le polynôme de HOMFLY-PT :

$$P(\bar{b})(a,q) = \sum_{i,j,k\in\mathbb{Z}} (-1)^i a^j q^k \mathrm{dim}_{\mathbb{Q}} HKR_i^{j,k}(\bar{b}).$$

En 2004 Rouquier [Rou06] a associé à chaque tresse $b \in \mathcal{B}_n$ un complexe de bimodules gradués $F(b)$ tel que des complexes associés à des tresses isotopes sont équivalents à homotopie près. Cette construction est appelée catégorification du groupe de tresses $\mathcal{B}_n$.

Les bimodules utilisés ont été introduits par Soergel dans ses travaux en théorie des représentations [Soe92] [Soe95] [Soe07] ; ce dernier prouve que la catégorie formée par ces bimodules catégorifie l'algèbre de Hecke. On considère l'algèbre polynomiale à n indéterminées et graduée $\mathbb{Q}[x_1, \ldots, x_n]$, sur laquelle le groupe symétrique S_n agit naturellement par permutation des variables. Soit R la sous-algèbre de $\mathbb{Q}[x_1, \ldots, x_n]$, préservée par l'action de S_n, définie par $R = Q[X_1, \ldots, X_{n-1}]$ avec $X_i = x_i - x_{i+1}$. Pour $i = 1, \ldots, n-1$, définissons les R-bimodules $B_i = R \otimes_{R^{\tau_i}} R$ où $\tau_i = (i, i+1)$ et R^{τ_i} est la sous-algèbre de R des éléments invariants sous l'action de τ_i. L'algèbre R agit à droite et à gauche sur ces bimodules par multiplication. Les bimodules de Soergel sont, par définition, des sommes directes de facteurs directs de produits tensoriels des bimodules B_i (avec une graduation éventuellement décalée).

L'algèbre de Hecke $\mathcal{H}_n$ est un quotient de dimension finie de l'algèbre du groupe $\mathcal{B}_n$ dépendant d'un paramètre q. Elle est engendrée par $n-1$ générateurs $T_1, \ldots, T_{n-1}$, soumis aux relations de tresses ainsi qu'aux relations $T_i^2 = (q^2-1)T_i + q^2$, $i = 1, \ldots, n-1$. Soergel a prouvé l'existence d'un isomorphisme d'anneaux de $\mathcal{H}_n$ dans l'anneau de Grothendieck scindé de la catégorie des bimodules de Soergel, qui envoie $1 + T_i$ sur B_i et q sur $R\{1\}$ (R décalé de 1). La théorie de Soergel s'inscrit dans un cadre plus général comme nous le verrons par la suite. Soergel a émis une conjecture (démontrée, entre autres, dans le cas de l'algèbre de Hecke $\mathcal{H}_n$ associée au groupe de tresses $\mathcal{B}_n$) reliant les éléments de la base de Kazhdan-Lusztig de $\mathcal{H}_n$ aux indécomposables de la catégorie des bimodules de Soergel. Cette conjecture implique, partiellement ou totalement, certaines conjectures majeures en théorie de Kazhdan-Lusztig.

Au vu des liens entre $\mathcal{H}_n$ et $\mathcal{B}_n$, il semble naturel de chercher à obtenir une catégorification du groupe de tresses $\mathcal{B}_n$ en utilisant les bimodules de Soergel. C'est ce que fait Rouquier en associant à chaque générateur $\sigma_i \in \mathcal{B}_n$ le complexe de cochaînes

$$F(\sigma_i) : 0 \longrightarrow R\{2\} \xrightarrow{\mathrm{rb}_i} B_i \longrightarrow 0,$$

où B_i est en degré 0 et le morphisme de R–bimodules rb_i envoie 1 sur $X_i \otimes 1 + 1 \otimes X_i$. À σ_i^{-1} est associé le complexe de cochaînes

$$F(\sigma_i^{-1}) : 0 \longrightarrow B_i\{-2\} \xrightarrow{\mathrm{br}_i} R\{-2\} \longrightarrow 0,$$

où $B_i\{-2\}$ est en degré 0 et le morphisme de R–bimodules br_i est la multiplication. Enfin, à un mot de tresses correspond le complexe obtenu en prenant le produit tensoriel au-dessus de R des complexes associés aux générateurs apparaissant dans l'expression du mot. Le résultat de Rouquier dit que dans la catégorie à homotopie près, les complexes ne dépendent pas de l'écriture du mot mais seulement de l'élément de $\mathcal{B}_n$.

En 2006 Khovanov a établi dans [Kho07] que l'homologie de Khovanov-Rozansky trigraduée $(HKR_i^{j,k}(\bar{b}))_{i,j,k\in\mathbb{Z}}$ peut être retrouvée à partir du complexe de Rouquier $F(b)$ en utilisant l'homologie de Hochschild.

Les résultats

Dans cette thèse, nous étendons le résultat de Rouquier à différents groupes de tresses généralisés. L'étude d'endofoncteurs de la catégorie $\mathcal{O}$ par Mazorchuk et Stroppel dans [MS07] fournit également des catégorifications de certains groupes de tresses généralisés. Cela constitue un pendant du point de vue de la théorie des représentations de l'approche adoptée dans cette thèse.

Dans le Chapitre 1, nous nous intéressons aux monoïdes des tresses singulières.

Le monoïde des tresses singulières $\mathcal{SB}_n$, également appelé monoïde de Baez-Birman, [Bae92] et [Bir93], est engendré par les générateurs σ_i, σ_i^{-1}, et ρ_i, pour $i = 1, \dots, n-1$ soumis aux relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ pour } |i-j| > 1,$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ pour } i = 1, \dots, n-2,$$
$$\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 \text{ pour } i = 1, \dots, n-1,$$
$$\rho_i\rho_j = \rho_j\rho_i \text{ pour } |i-j| > 1,$$
$$\sigma_i\rho_j = \rho_j\sigma_i \text{ pour } |i-j| \neq 1,$$
$$\sigma_i\sigma_{i+1}\rho_i = \rho_{i+1}\sigma_i\sigma_{i+1} \text{ pour } i = 1, \dots, n-2,$$
$$\sigma_{i+1}\sigma_i\rho_{i+1} = \rho_i\sigma_{i+1}\sigma_i \text{ pour } i = 1, \dots, n-2.$$

Grosso modo une tresse singulière est une tresse classique dotée d'un nombre fini de singularités (des points doubles transverses).

En suivant les idées de Rouquier et des résultats implicites de [KR08b], [Ras06] et [Wag07], on associe à chaque tresse singulière $b \in \mathcal{SB}_n$ un complexe de R–bimodules gradués $F(b)$ tel que des complexes associés à des tresses singulières isotopes sont équivalents à homotopie près. Les bimodules utilisés dans cette catégorification de $\mathcal{SB}_n$ sont ici encore des bimodules de Soergel. Plus précisément, les complexes associés aux générateurs σ_i et σ_i^{-1} sont ceux introduits par Rouquier tandis qu'au générateur ρ_i, on associe le complexe concentré en degré 0

$$F(\rho_i) : 0 \longrightarrow B_i \longrightarrow 0.$$

Dans le Chapitre 2, nous étudions le cas des groupes de tresses virtuelles.

Les entrelacs virtuels ont été introduits par Kauffman dans [Kau99] ; ils peuvent être représentés, comme les entrelacs usuels, par des diagrammes planaires possédant un nouveau type de croisements, appelés croisements virtuels. De tels croisements apparaissent par exemple losque l'on projette sur un plan un entrelacs vivant dans une surface épaissie (cf. [KK00], [Kup03]). Tout entrelacs virtuel peut être obtenu comme clôture d'une tresse virtuelle.

Les tresses virtuelles à n brins forment un groupe, noté $\mathcal{VB}_n$, dont une présentation par générateurs et relations a été donnée par Vershinin dans [Ver01]. Le groupe $\mathcal{VB}_n$ est engendré par $2(n-1)$ générateurs $\sigma_1, \ldots, \sigma_{n-1}$ et $\zeta_1, \ldots, \zeta_{n-1}$ où les σ_i satisfont les relations de tresses, les ζ_i satisfont les relations du groupe symétrique, et les relations "mixtes" suivantes sont vérifiées :

$$\sigma_i \zeta_j = \zeta_j \sigma_i \quad \text{pour} \quad |i-j| > 1$$

et

$$\sigma_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \sigma_{i+1} \quad \text{pour} \quad i = 1 \ldots n-2.$$

Pour parvenir à catégorifier $\mathcal{VB}_n$, nous introduisons un autre type de bimodules, appelés bimodules tordus. On considère, pour tout $\omega \in S_n$ le R–bimodule R_ω, égal à R en tant que R–module à gauche et où l'action à droite de $a \in R$ est la multiplication par $\omega(a)$. On comprend bien la structure des produits tensoriels entre deux bimodules tordus et entre les bimodules tordus et les bimodules de Soergel B_i. Cela permet de construire une catégorification par des complexes de R-bimodules à homotopie près du groupe $\mathcal{VB}_n$. Pour ce faire, on associe aux générateurs σ_i les complexes de Rouquier et à tout générateur ζ_i le complexe concentré en degré 0

$$F(\zeta_i) : 0 \longrightarrow R_{\tau_i} \longrightarrow 0.$$

Le groupe de tresse $\mathcal{B}_n$ s'inscrit dans un cadre plus général qui est celui de groupes de tresses $\mathcal{B}_\mathcal{W}$ associés à un groupe de Coxeter $\mathcal{W}$. Un groupe de Coxeter $\mathcal{W}$ de type fini est la donnée d'un ensemble fini $\mathcal{S}$ et d'une matrice symétrique indexée par $\mathcal{S}$ dont les entrées $m(s,t)$ sont des entiers supérieurs à 2, excepté les termes diagonaux égaux à 1. Le groupe $\mathcal{W}$ est engendré par les éléments de l'ensemble fini $\mathcal{S}$, soumis aux relations

$$\underbrace{sts\cdots}_{m(s,t)\ \text{termes}} = \underbrace{tst\cdots}_{m(s,t)\ \text{termes}}, \text{ si } s \neq t \text{ et } s^2 = 1 \text{ pour tout } s \in \mathcal{S}\,.$$

Si aucune relation ne lie les éléments s et t, on pose $m(s,t) = \infty$. Le groupe de tresses généralisé $\mathcal{B}_\mathcal{W}$ associé à $\mathcal{W}$ est défini par la même présentation par générateurs et relations que $\mathcal{W}$, excepté les relations $s^2 = 1$ que l'on omet. En l'occurrence, $\mathcal{B}_n$ correspond au groupe de Coxeter S_n qui est un groupe de Coxeter de type A_{n-1}.

En réalité, la catégorification que propose Rouquier dans [Rou06] est non seulement celle de $\mathcal{B}_n$ mais aussi celle de tout groupe de tresses associé à un groupe de Coxeter de type fini. En effet la théorie qu'a proposée Soergel est inscrite dans ce contexte plus général. Les bimodules de Soergel peuvent être définis dans ce cadre plus large en considérant l'action d'un groupe de Coxeter de type fini $\mathcal{W}$ quelconque sur une algèbre polynomiale R au lieu simplement de celle du groupe symétrique. La catégorie de ces bimodules fournit alors une catégorification de l'algèbre de Hecke $\mathcal{H}(\mathcal{W}, \mathcal{S})$ associée à $\mathcal{W}$.

Il est ainsi naturel de se demander si la catégorification du groupe de tresses virtuelles de type A peut se généraliser à d'autres groupes de Coxeter. Pour cela il faudrait commencer par définir ce qu'est un groupe de tresses virtuelles associé à un groupe de Coxeter autre que S_n.

Dans le Chapitre 3, nous nous concentrons sur un groupe de Coxeter de type B_n, *i.e.* pour lequel $\mathcal{S} = \{s_o, \ldots, s_{n-1}\}$,

$$m(s_i, s_j) = 2 \text{ si } |i - j| > 1,$$

$$m(s_i, s_{i+1}) = 3 \text{ si } 1 \leq i \leq n-2$$

et

$$m(s_0, s_1) = 4.$$

Pour un tel groupe de Coxeter, tom Dieck [tD94] a donné une interprétation diagrammatique des éléments du groupe de tresses associé $\mathcal{B}_{B_n}$ à l'aide de diagrammes de tresses possédant une certaine propriété de symétrie. En se plaçant dans ce cadre, nous donnons une définition d'un groupe de tresses virtuelles $\mathcal{VB}_{B_n}$ associé à un groupe de Coxeter $\mathcal{W}_{B_n}$ de type B_n. Puis nous établissons une catégorification de ce groupe qui étend celle de $\mathcal{B}_{B_n}$ donnée par Rouquier. Pour construire les complexes de R–bimodules apparaissant dans cette catégorification, on reproduit le schéma suivi en type A, si ce n'est que les bimodules de Soergel B_{s_i} et les bimodules tordus R_{s_j} utilisés sont maintenant associés à des éléments s_i, s_j de $\mathcal{W}_{B_n}$.

Deux appendices techniques complètent cette thèse. Dans l'Appendice A, nous établissons l'injectivité d'un certain morphisme défini au Chapitre 1. Dans l'Appendice B, nous donnons une preuve de la catégorification de Rouquier du groupe de tresses $\mathcal{B}_n$; en particulier, nous donnons des équivalences d'homotopies explicites.

La suite du présent texte est rédigée en anglais. Les chapitres peuvent être lus indépendamment les uns des autres.

Chapter 1

Categorification of the singular braid monoids

A singular link (or graph link) is a smooth immersion of circles in $\mathbb{R}^3$ whose image has finitely many singularities, that are all ordinary double points. Singular links can be represented by planar diagrams. Such a diagram displays usual positive and negative crossings plus singular crossings which correspond to the double points of the link. Just as in the classical setting, singular links can be isotoped to closures of singular braids. Out of the singular braids with n strands one can form a monoid, called the Baez-Birman monoid (see [Bae92], [Bir93]) and denoted by $\mathcal{SB}_n$. It can be described by generators and relations that generalize the ones appearing in the classical presentation of the usual braid group $\mathcal{B}_n$ with n strands. Our aim in this Chapter is, using this presentation, to categorify $\mathcal{SB}_n$ in the sense of Rouquier [Rou06]. More precisely, to any word ω in the generators of $\mathcal{SB}_n$ we associate a bounded cochain complex $F(\omega)$ of Soergel bimodules such that if two words ω and ω' represent the same element of $\mathcal{SB}_n$, then the corresponding cochain complexes $F(\omega)$ and $F(\omega')$ are homotopy equivalent.

Note that Mazorchuk and Stroppel constructed in [MS07] a categorification of an action of the singular braid monoid using certain categories of representation of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. This gives a representation-theoretic counterpart to the present approach.

1.1. Soergel bimodules

1.1.1. Definition. Let us discuss some bimodules introduced by Soergel in his work on representation theory [Soe92], [Soe95], [Soe07]. He considered a category of bimodules which categorifies the Hecke algebra in the sense that its Grothendieck ring is isomorphic to the Hecke algebra. He defined these bimodules in the general context of Coxeter groups, but here we will only be interested in the special case of the symmetric group.

Let n be a positive integer and R the subalgebra of $\mathbb{Q}[x_1,\dots,x_n]$ generated by $x_i - x_j$ for $1 \le i,j \le n$. We have

$$R = \mathbb{Q}[x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n] = \mathbb{Q}[x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n].$$

The symmetric group S_n acts on $\mathbb{Q}[x_1,\dots,x_n]$ by $\omega(x_i) = x_{\omega(i)}$ for all $i = 1,\dots,n$ and $\omega \in S_n$. This action preserves R. Let R^H be the subalgebra of elements of R fixed by a subgroup H of S_n. In particular R^{τ_i} is the subalgebra of R of elements fixed by the transposition $\tau_i = (i, i+1)$. As an algebra,

$$\begin{aligned} R^{\tau_i} = \mathbb{Q}[x_1 - x_2, \dots, (x_1 - x_i) + (x_1 - x_{i+1}),& \\ (x_1 - x_i)(x_1 - x_{i+1}), x_1 - x_{i+2}, \dots, x_1 - x_n&]. \end{aligned}$$

Let us also consider the R–bimodules $B_{\tau_i} = R \otimes_{R^{\tau_i}} R$ for $i = 1,\dots,n-1$, which we will denote by B_i for simplicity of notations. As a R–bimodule, R is spanned by 1 and B_i is spanned by $1 \otimes 1$. We introduce a grading on R, R^{τ_i} and B_i by setting $\deg(x_k) = 2$ for all $k = 1,\dots,n$.

Two R–bimodule morphisms between these objects will be relevant to us, namely $\mathrm{br}_i : B_i \to R$ and $\mathrm{rb}_i : R\{2\} \to B_i$ defined by

$$\mathrm{br}_i(1 \otimes 1) = 1 \quad \text{and} \quad \mathrm{rb}_i(1) = (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1}).$$

The curly brackets indicate a shift of the grading: if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a $\mathbb{Z}$–graded bimodule and p an integer, then the $\mathbb{Z}$–graded bimodule $M\{p\}$ is defined by $M\{p\}_i = M_{i-p}$ for all $i \in \mathbb{Z}$. The maps br_i and rb_i are degree–preserving morphisms of graded R–bimodules.

By definition, Soergel bimodules are direct summands of tensor products of B_i's, possibly shifted.

1.1.2. Structure. To shorten notation, we will subsequently use the new variable $X_i = x_i - x_{i+1}$. Two particular families of elements of R will appear in the sequel, namely

$$u_i = \frac{X_i}{2} + X_{i+1} \tag{1.1}$$

and

$$v_i = \frac{X_{i+1}}{2} + X_i. \tag{1.2}$$

Note that u_i belongs to R^{τ_i} while v_i belongs to $R^{\tau_{i+1}}$, as a consequence of

$$\begin{aligned} \tau_i(X_i) &= -X_i, \\ \tau_i(X_{i+1}) &= X_i + X_{i+1}, \\ \tau_{i+1}(X_i) &= X_i + X_{i+1}, \\ \tau_{i+1}(X_{i+1}) &= -X_{i+1}. \end{aligned}$$

The algebra R possesses a structure of right and left free R^{τ_i}–module of rank 2 with basis $\{1, X_i\}$. Indeed $X_i^2 \in R^{\tau_i}$ and the generators X_k of R can be uniquely expressed in this basis as follows:

$$\begin{aligned} X_k &= X_k + 0.X_i \quad \text{if } k \neq i-1, i, i+1, && (1.3)\\ X_{i-1} &= v_{i-1} - \frac{X_i}{2}, && (1.4)\\ X_i &= 0 + X_i, && (1.5)\\ X_{i+1} &= u_i - \frac{X_i}{2}. && (1.6) \end{aligned}$$

This gives an isomorphism of graded R^{τ_i}–modules $R \cong R^{\tau_i} \oplus R^{\tau_i}\{2\}$, from which we deduce $B_i \cong R \oplus R\{2\}$. This leads us to explain why rb_i is a morphism of R–bimodules. To see that $\mathrm{rb}_i(p)$ is well-defined for all p in R, we have to check that $p\,\mathrm{rb}_i(1) = \mathrm{rb}_i(1)p$. But any p in R is equal to $a + bX_i$ with a and b in R^{τ_i}, so that

$$\begin{aligned} p\,\mathrm{rb}_i(1) &= (a + bX_i)(X_i \otimes 1 + 1 \otimes X_i)\\ &= aX_i \otimes 1 + bX_i^2 \otimes 1 + a \otimes X_i + bX_i \otimes X_i\\ &= X_i \otimes a + 1 \otimes bX_i^2 + 1 \otimes aX_i + X_i \otimes bX_i\\ &= (X_i \otimes 1 + 1 \otimes X_i)(a + bX_i)\\ &= \mathrm{rb}_i(1)p. \end{aligned}$$

1.1.3. Tensoring Soergel bimodules. It appears in Soergel's work that the defining relations of the Hecke algebra satisfied by a certain set of generators, namely the Kazhdan–Lusztig basis elements, lift to isomorphisms between the tensor products of the corresponding Soergel bimodules, the B_i's. Let us state these isomorphisms between Soergel bimodules and give some explanations of these statements.

From now on we will implicitly identify the two isomorphic R–bimodules $R \otimes_R R$ and R and omit to write the algebra over which we are tensoring when it is not misleading.

Proposition 1.1. [Soe92] *The application $\phi^i : B_i \otimes_R B_i \to B_i \oplus B_i\{2\}$ given by*

$$\phi^i(1 \otimes 1 \otimes 1) = (1 \otimes 1, 0) \quad \textit{and} \quad \phi^i(1 \otimes X_i \otimes 1) = (0, 1 \otimes 1)$$

is an isomorphism of graded R–bimodules.

The two elements $1 \otimes 1 \otimes 1$ and $1 \otimes X_i \otimes 1$ span $B_i \otimes_R B_i$ as a R–bimodule, since R is a free R^{τ_i}–module of rank 2 with basis $\{1, X_i\}$. So B_i injects in two different ways into $B_i \otimes_R B_i$, either by ψ_1^i which sends $1 \otimes 1$ to $1 \otimes 1 \otimes 1$ or by ψ_2^i which sends $1 \otimes 1$ to $1 \otimes X_i \otimes 1$. The corresponding surjections are respectively $\phi_1^i : B_i \otimes_R B_i \to B_i$ which sends $1 \otimes 1 \otimes 1$ to $1 \otimes 1$ and $1 \otimes X_i \otimes 1$ to 0 and $\phi_2^i : B_i \otimes_R B_i \to B_i\{2\}$ which sends $1 \otimes X_i \otimes 1$ to $1 \otimes 1$ and $1 \otimes 1 \otimes 1$ to 0 ; so $\phi_1^i \circ \psi_1^i = \mathrm{id}_{B_i}$ and $\phi_2^i \circ \psi_2^i = \mathrm{id}_{B_i\{2\}}$.

Proposition 1.2. [Soe92] *If $|i-j|>1$, then there is an isomorphism of graded R–bimodules $\gamma^{i,j}: B_i\otimes_R B_j \to B_j \otimes_R B_i$.*

We have already seen that R has a structure of free R^{τ_i}–module of rank 2 with basis $\{1, X_i\}$ and of free R^{τ_j}–module of rank 2 with basis $\{1, X_j\}$. Let $<\tau_i,\tau_j>$ be the subgroup of S_n generated by τ_i and τ_j; R is then a left and right free $R^{<\tau_i,\tau_j>}$–module of rank 4 with basis $\{1, X_i, X_j, X_iX_j\}$. Indeed these four elements are linearly independent over $R^{<\tau_i,\tau_j>}$: let p, q, r, and s in $R^{<\tau_i,\tau_j>}$ be such that

$$p+qX_i+rX_j+sX_iX_j=0\ ;$$

then applying respectively the transpositions τ_i and τ_j to this equality, we obtain

$$p-qX_i+rX_j-sX_iX_j=0 \tag{1.7}$$

and

$$p+qX_i-rX_j-sX_iX_j=0. \tag{1.8}$$

By adding the two Equalities (1.7) and (1.8), we obtain $p=sX_iX_j$, which implies $p=s=0$ since $sX_iX_j\notin R^{\tau_i}$. By substracting these two equalities, we obtain $qX_i=rX_j$, which implies $q=r=0$ since $rX_j\notin R^{\tau_j}$. Moreover the family $\{1, X_i, X_j, X_iX_j\}$ is obviously a spanning set. Finally this family is closed under products (since $X_i^2, X_j^2\in R^{\tau_i}\cap R^{\tau_j}$), thus is a basis of R over $R^{<\tau_i,\tau_j>}$.

Then the isomorphism of graded R–bimodules $\gamma^{i,j}$ is completely defined by the image of $1\otimes 1\otimes 1$. Let us set $\gamma^{i,j}(1\otimes 1\otimes 1)=1\otimes 1\otimes 1$. For any $p\in R$ there exists a unique $(a,b,c,d)\in(R_i\cap R_j)^4$ such that $p=a+bX_i+cX_j+dX_iX_j$, so that

$$\gamma^{i,j}(1\otimes p\otimes 1)=a\otimes 1\otimes 1+b\otimes 1\otimes X_i+cX_j\otimes 1\otimes 1+dX_j\otimes 1\otimes X_i.$$

Remarks 1.3.
- *Actually, if $i\neq j$, then $B_i\otimes_R B_j$ is spanned by $1\otimes 1\otimes 1$ as a R–bimodule. In fact R is a free R^{τ_i}–module with basis $\{1, X_i\}$ (let us respectively choose the basis $\{1, X_{i+1}\}$ if $j=i-1$ or the basis $\{1, X_{i-1}\}$ if $j=i+1$), so every element in the middle of the tensor product $R\otimes_{R^{\tau_i}} R\otimes_{R^{\tau_j}} R$ can be slid to the left leaving at most X_i behind (respectively X_{i+1} or X_{i-1}) which belongs to R^{τ_j} thus can be slid to the right.*
- *We cannot define an isomorphism of R–bimodules $B_i\otimes_R B_{i+1}\to B_{i+1}\otimes_R B_i$ sending $1\otimes 1\otimes 1$ to $1\otimes 1\otimes 1$. Indeed this map is not well-defined as a morphism of R–bimodules: for example, the elements*

$$(X_i+2X_{i-1})\otimes 1\otimes X_{i-1}$$

and

$$X_{i-1}(X_i+X_{i-1})\otimes 1\otimes 1+1\otimes 1\otimes X_{i-1}^2$$

are equal in $B_i\otimes_R B_{i+1}$ but not in $B_{i+1}\otimes_R B_i$.

Let us denote by $B_{i,i+1}$ the R–bimodule $R \otimes_{R^{<\tau_i,\tau_{i+1}>}} R$.

Proposition 1.4. [Soe92] *For $i = 1, \ldots, n-2$, there are isomorphisms of graded R–bimodules*

$$B_i \otimes_R B_{i+1} \otimes_R B_i \cong B_{i,i+1} \oplus B_i\{2\}$$

and

$$B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \cong B_{i,i+1} \oplus B_{i+1}\{2\}.$$

We make here the involved morphisms explicit enough in order to use them in further computations.

First let us describe the first isomorphism of Proposition 1.4. It is easy to observe that B_i injects into $B_i \otimes_R B_{i+1} \otimes_R B_i$, since this injection can be expressed via the injective morphisms already introduced, in the following way:

$$B_i\{2\} \xrightarrow{\psi_1^i} B_i \otimes_R B_i\{2\} \xrightarrow{\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}} B_i \otimes_R B_{i+1} \otimes_R B_i.$$

The corresponding surjection is

$$B_i \otimes_R B_{i+1} \otimes_R B_i \xrightarrow{\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}} B_i \otimes_R B_i \xrightarrow{-\phi_2^i} B_i\{2\}.$$

In fact, composing these two R–bimodules morphisms, we get the identity of $B_i\{2\}$:

$$\begin{aligned}
& -\phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \psi_1^i(1 \otimes 1) \\
= \; & -\phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\
= \; & -\phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id})(1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1) \\
= \; & -\phi_2^i(1 \otimes 2X_{i+1} \otimes 1) \\
= \; & -\phi_2^i(2u_i \otimes 1 \otimes 1 - 1 \otimes X_i \otimes 1) \\
= \; & 1 \otimes 1.
\end{aligned}$$

Let us now consider the morphism of R–bimodules

$$\iota^i : B_{i,i+1} \to B_i \otimes_R B_{i+1} \otimes_R B_i$$

defined by $\iota^i(1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1$. It is obviously well–defined since $R^{<\tau_i,\tau_{i+1}>} \cong R^{\tau_i} \cap R^{\tau_{i+1}}$. Viewing ι^i as a morphism of left R–modules, it can be proved that it is an injection. The corresponding surjection π^i from $B_i \otimes_R B_{i+1} \otimes_R B_i$ onto $B_{i,i+1}$ sends $1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1$ to 0 and $1 \otimes 1 \otimes 1 \otimes 1$ to $1 \otimes 1$. Observe also that $-\phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}) \circ \iota^i = 0$. So we have the following split short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & B_{i,i+1} \arrow[r, "\iota^i"] & B_i \otimes_R B_{i+1} \otimes_R B_i \arrow[r, "-\phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id})"] & B_i\{2\} \arrow[r] \arrow[l, bend right, "(\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \psi_1^i"'] & 0
\end{tikzcd}$$

Thus $B_i \otimes_R B_{i+1} \otimes_R B_i$ is equal to the direct sum of the images of the injections $(\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \psi_1^i$ and ι^i.

In the sequel it will be useful to know explicitly how an element of $B_i \otimes_R B_{i+1} \otimes_R B_i$ can be split in two terms belonging to the images of these two injections. In order to make this explicit, we have to express any element of $R \otimes_{R^{\tau_{i+1}}} R$ as a (right and left) linear combination of the elements $1 \otimes 1$ and $X_{i+1} \otimes 1 + 1 \otimes X_{i+1}$ of $R \otimes_{R^{\tau_{i+1}}} R$ with coefficients in R^{τ_i}. We only need to compute it for the four elements $1 \otimes 1$, $1 \otimes X_{i+1}$, $X_{i+1} \otimes 1$ and $X_{i+1} \otimes X_{i+1}$. Indeed, R is a free R^{τ_i}–module with basis $\{1, X_{i+1}\}$ as $X_{i+1} = u_i - \frac{X_i}{2}$. So let us start with searching an expression of the four elements $1 \otimes 1$, $1 \otimes X_{i+1}$, $X_{i+1} \otimes 1$ and $X_{i+1} \otimes X_{i+1}$ of the form

$$\sum_{k=1}^{m} p_k \,(1 \otimes 1)\, q_k + r_k \,(X_{i+1} \otimes 1 + 1 \otimes X_{i+1})\, s_k \tag{1.9}$$

with $p_k, q_k, r_k, s_k \in R^{\tau_i}$. It is obvious for $1 \otimes 1$. Let us turn to $1 \otimes X_{i+1}$; we have the following equality:

$$\begin{aligned} 4u_i(1 \otimes 1) - 3(X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) &= 2v_i \otimes 1 - 3 \otimes X_{i+1} \\ &= 1 \otimes (2v_i - 3X_{i+1}) \\ &= -6(1 \otimes X_{i+1}) + 4(1 \otimes 1)u_i, \end{aligned}$$

from which we deduce

$$1 \otimes X_{i+1} = (1 \otimes 1)\frac{2u_i}{3} - \frac{2u_i}{3}(1 \otimes 1) + \frac{1}{2}(X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \tag{1.10}$$

which is the desired expression for $1 \otimes X_{i+1}$. For $X_{i+1} \otimes 1$, we proceed similarly; we have

$$\begin{aligned} (1 \otimes 1)4u_i - 3(X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) &= 1 \otimes 2v_i - 3X_{i+1} \otimes 1 \\ &= (2v_i - 3X_{i+1}) \otimes 1 \\ &= -6(X_{i+1} \otimes 1) + 4u_i(1 \otimes 1). \end{aligned}$$

So, we obtain an expression of the form (1.9) for $X_{i+1} \otimes 1$:

$$X_{i+1} \otimes 1 = \frac{2u_i}{3}(1 \otimes 1) - (1 \otimes 1)\frac{2u_i}{3} + \frac{1}{2}(X_{i+1} \otimes 1 + 1 \otimes X_{i+1}). \tag{1.11}$$

Instead of directly searching an expression for $X_{i+1} \otimes X_{i+1}$ of the form (1.9), it is easier to deduce it from one of $X_i \otimes X_i$ because $X_i^2 \in R^{\tau_i}$. Since $X_i = 2u_i - 2X_{i+1}$, we first get from Equations (1.10) and (1.11):

$$1 \otimes X_i = \frac{4u_i}{3}(1 \otimes 1) + (1 \otimes 1)\frac{2u_i}{3} - (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \tag{1.12}$$

and

$$X_i \otimes 1 = (1 \otimes 1)\frac{4u_i}{3} + \frac{2u_i}{3}(1 \otimes 1) - (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}). \tag{1.13}$$

Moreover,

$$\begin{aligned} 2u_i(X_i \otimes 1) + 3X_i^2(1 \otimes 1) &= 2X_i \otimes 2v_i \\ &= (X_i \otimes 1)2u_i + 3X_i \otimes X_i. \end{aligned}$$

It follows

$$3(X_i \otimes X_i) = 2u_i(X_i \otimes 1) - (X_i \otimes 1)2u_i + 3X_i^2(1 \otimes 1). \tag{1.14}$$

In order to obtain a more symmetric expression, we can also compute

$$\begin{aligned} (1 \otimes X_i)2u_i + (1 \otimes 1)3X_i^2 &= 2v_i \otimes 2X_i \\ &= 2u_i(1 \otimes X_i) + 3X_i \otimes X_i. \end{aligned}$$

So

$$3(X_i \otimes X_i) = -2u_i(1 \otimes X_i) + (1 \otimes X_i)2u_i + (1 \otimes 1)3X_i^2. \tag{1.15}$$

By adding the two expressions (1.14) and (1.15) of $3(X_i \otimes X_i)$ and using Equations (1.12) and (1.13), we obtain

$$\begin{aligned} X_i \otimes X_i &= \frac{4}{9}u_i(1 \otimes 1)u_i + \left(\frac{X_i^2}{2} - \frac{2u_i^2}{9}\right)(1 \otimes 1) \\ &\quad + (1 \otimes 1)\left(\frac{X_i^2}{2} - \frac{2u_i^2}{9}\right). \end{aligned} \tag{1.16}$$

Finally, as $X_{i+1} = u_i - X_i/2$, we obtain

$$\begin{aligned} X_{i+1} \otimes X_{i+1} &= \left(\frac{X_i^2}{8} - \frac{13u_i^2}{18}\right)(1 \otimes 1) + (1 \otimes 1)\left(\frac{X_i^2}{8} - \frac{13u_i^2}{18}\right) \\ &\quad + \frac{4}{9}u_i(1 \otimes 1)u_i + u_i(X_{i+1} \otimes 1 + 1 \otimes X_{i+1}). \end{aligned} \tag{1.17}$$

The second isomorphism of Proposition 1.4 can be described in exactly the same way. The bimodule B_{i+1} injects in $B_{i+1} \otimes B_i \otimes B_{i+1}$ in the following way:

$$B_{i+1}\{2\} \xrightarrow{\psi_1^{i+1}} B_{i+1} \otimes_R B_{i+1}\{2\} \xrightarrow{\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}} B_{i+1} \otimes_R B_i \otimes_R B_{i+1}.$$

Composing this injection with the corresponding surjection

$$B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \xrightarrow{\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id}} B_{i+1} \otimes_R B_{i+1} \xrightarrow{-\phi_2^{i+1}} B_{i+1}\{2\},$$

we obtain the identity of $B_{i+1}\{2\}$.

The injection $\iota^{i+1} : B_{i,i+1} \to B_{i+1} \otimes B_i \otimes B_{i+1}$ sends $1 \otimes 1$ to $1 \otimes 1 \otimes 1 \otimes 1$. The corresponding surjection π^{i+1} from $B_{i+1} \otimes B_i \otimes B_{i+1}$ onto $B_{i,i+1}$ sends $1 \otimes (X_i \otimes 1 + 1 \otimes X_i) \otimes 1$ to 0 and $1 \otimes 1 \otimes 1 \otimes 1$ to $1 \otimes 1$. Below we give the expressions of elements of $R \otimes_{R^{\tau_i}} R$ as linear (right and left) combinations of $1 \otimes 1$ and $X_i \otimes 1 + 1 \otimes X_i$ with coefficients in $R^{\tau_{i+1}}$:

$$1 \otimes X_i = (1 \otimes 1)\frac{2v_i}{3} - \frac{2v_i}{3}(1 \otimes 1) + \frac{1}{2}(X_i \otimes 1 + 1 \otimes X_i), \tag{1.18}$$

$$X_i \otimes 1 = \frac{2v_i}{3}(1 \otimes 1) - (1 \otimes 1)\frac{2v_i}{3} + \frac{1}{2}(X_i \otimes 1 + 1 \otimes X_i), \tag{1.19}$$

$$1 \otimes X_{i+1} = \frac{4v_i}{3}(1 \otimes 1) + (1 \otimes 1)\frac{2v_i}{3} - (X_i \otimes 1 + 1 \otimes X_i), \tag{1.20}$$

$$X_{i+1} \otimes 1 = (1 \otimes 1)\frac{4v_i}{3} + \frac{2v_i}{3}(1 \otimes 1) - (X_i \otimes 1 + 1 \otimes X_i), \tag{1.21}$$

$$\begin{aligned} X_{i+1} \otimes X_{i+1} &= \frac{4}{9}v_i(1 \otimes 1)v_i + \left(\frac{X_{i+1}^2}{2} - \frac{2v_i^2}{9}\right)(1 \otimes 1) \\ &\quad +(1 \otimes 1)\left(\frac{X_{i+1}^2}{2} - \frac{2v_i^2}{9}\right), \end{aligned} \tag{1.22}$$

$$\begin{aligned} X_i \otimes X_i &= \left(\frac{X_{i+1}^2}{8} - \frac{13v_i^2}{18}\right)(1 \otimes 1) + (1 \otimes 1)\left(\frac{X_{i+1}^2}{8} - \frac{13v_i^2}{18}\right) \\ &\quad +\frac{4}{9}v_i(1 \otimes 1)v_i + v_i(X_i \otimes 1 + 1 \otimes X_i). \end{aligned} \tag{1.23}$$

1.2. The singular braid monoids

Let $n \geq 2$. The singular braid monoid $\mathcal{SB}_n$, also called Baez-Birman monoid (see [Bae92] and [Bir93]) is defined as follows.

Definition 1.5. *The singular braid monoid $\mathcal{SB}_n$ is the monoid generated by $3(n-1)$ generators σ_i, σ_i^{-1} and ρ_i for $i = 1, \ldots, n-1$ satisfying the following relations:*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \textit{if } |i-j| > 1, \tag{1.24}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \textit{if } 1 \leq i \leq n-2, \tag{1.25}$$

$$\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 \quad \textit{if } 1 \leq i \leq n-1, \tag{1.26}$$

$$\rho_i\rho_j = \rho_j\rho_i \quad \textit{if } |i-j| > 1, \tag{1.27}$$

$$\sigma_i\rho_j = \rho_j\sigma_i \quad \textit{if } |i-j| \neq 1, \tag{1.28}$$

$$\sigma_i\sigma_{i+1}\rho_i = \rho_{i+1}\sigma_i\sigma_{i+1} \quad \textit{if } 1 \leq i \leq n-2, \tag{1.29}$$

$$\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1} \quad \textit{if } 1 \leq i \leq n-2. \tag{1.30}$$

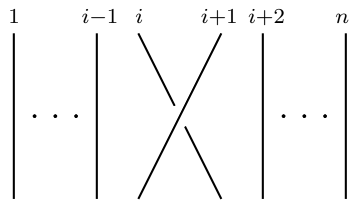


Figure 1.1: The positive braid σ_i

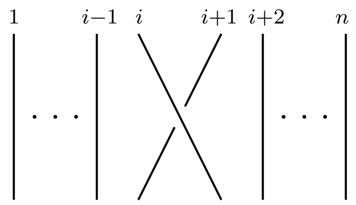


Figure 1.2: The negative braid σ_i^{-1}

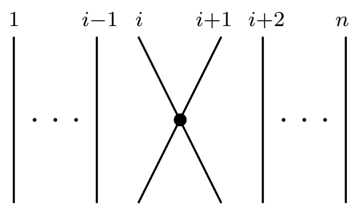


Figure 1.3: The singular braid ρ_i

Diagrammatically, the generator σ_i (respectively σ_i^{-1}) is represented by the generator of the classical braid group $\mathcal{B}_n$, namely a positive (respectively negative) crossing between the ith and $(i+1)$th strands; see Figures 1.1 and 1.2. The singular generator ρ_i is represented by a transverse double point between the ith and $(i+1)$th strands; see Figure 1.3. So a singular braid is nothing else than a classical braid with a finite number of singular crossings allowed.

The multiplication law in $\mathcal{SB}_n$ consists in concatenating the corresponding elementary braid diagrams. The relations defining $\mathcal{SB}_n$ correspond to planar isotopy and the classical and singular Reidemeister moves depicted in Figures 1.4 and 1.5.

Figure 1.4: Classical Reidemeister II–III moves

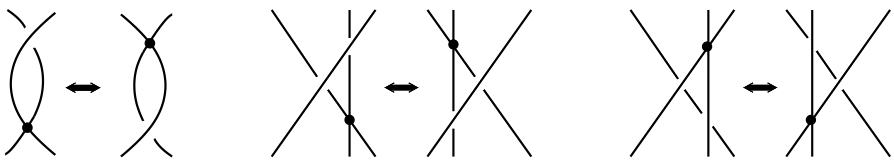


Figure 1.5: Singular Reidemeister moves

1.3. Categorification of the singular braid monoids

Some homological link invariants have been more or less explicitly generalized to graph links (which are closures of singular braids) in [KR08b], [Ras06] and [Wag07], but mostly in the context of matrix factorizations. In the framework of Soergel bimodules, we are proposing a categorification of the singular braid monoids, following ideas of [Rou06] and [Kho07].

1.3.1. Rouquier's categorification of the braid groups. We first explain Rouquier's construction. To each braid generator $\sigma_i \in \mathcal{B}_n$ we assign the cochain complex $F(\sigma_i)$ of graded R-bimodules

$$F(\sigma_i) : 0 \longrightarrow R\{2\} \xrightarrow{\mathrm{rb}_i} B_i \longrightarrow 0, \tag{1.31}$$

where B_i sits in cohomological degree 0. To σ_i^{-1} we assign the cochain complex $F(\sigma_i^{-1})$ of graded R-bimodules

$$F(\sigma_i^{-1}) : 0 \longrightarrow B_i\{-2\} \xrightarrow{\mathrm{br}_i} R\{-2\} \longrightarrow 0, \tag{1.32}$$

where $B_i\{-2\}$ sits in cohomological degree 0. To the unit element 1 of $\mathcal{B}_n$ we assign the complex of graded R-bimodules

$$F(1) : 0 \longrightarrow R \longrightarrow 0, \tag{1.33}$$

where R sits in cohomological degree 0; the complex $F(1)$ is a unit for the tensor product over R of complexes so tensoring any complex of graded R-bimodules with $F(1)$ leaves the complex unchanged. Finally to any word $\sigma = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$ where $\varepsilon_1, \dots, \varepsilon_k = \pm 1$, we assign the complex of graded R-bimodules $F(\sigma) = F(\sigma_{i_1}^{\varepsilon_1}) \otimes_R \cdots \otimes_R F(\sigma_{i_k}^{\varepsilon_k})$.

Rouquier established in [Rou06] that if ω and ω' are words representing the same element of $\mathcal{B}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of graded R-bimodules. This result is called the categorification of the braid group $\mathcal{B}_n$.

1.3.2. Categorification of $\mathcal{SB}_n$. Our aim is to extend Rouquier's categorification of the braid group $\mathcal{B}_n$ to the singular braid monoid $\mathcal{SB}_n$. The cochain complexes associated to the generators σ_i and σ_i^{-1} of $\mathcal{SB}_n$ coming from $\mathcal{B}_n$ are Rouquier's complexes of graded R–bimodules $F(\sigma_i)$ and $F(\sigma_i^{-1})$ defined by (1.31) and (1.32). We have to assign complexes to the generators ρ_i of $\mathcal{SB}_n$ corresponding to singular crossings such that all these complexes satisfy the same relations as the generators of $\mathcal{SB}_n$ up to homotopy equivalence. Let us assign to the element ρ_i the cochain complex concentrated in degree 0:

$$F(\rho_i) : 0 \longrightarrow B_i \longrightarrow 0. \tag{1.34}$$

Just as in Section 1.3.1 we assign to the unit element 1 of $\mathcal{SB}_n$ the complex $F(1)$ of (1.33), and to a singular braid word we assign the tensor product over R of the complexes associated to the generators involved in the expression of the word.

Our main result is the following.

Theorem 1.6. *If ω and ω' are words representing the same element of $\mathcal{SB}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of R–bimodules.*

Proof. By definition of $\mathcal{SB}_n$ and in view of Rouquier's result, it is enough to check that there are homotopy equivalences between the complexes associated to the braid words appearing in Relations (1.27)–(1.30).

First of all let us define the endomorphism μ_b of a R–bimodule B to be the multiplication by the element b of B.

Relation (1.27)

An isomorphism between the zero–length complexes $F(\rho_i\rho_j)$ and $F(\rho_j\rho_i)$ corresponding to Relation (1.27) arises immediately from Proposition 1.2.

Relation (1.28)

Let us now deal with Relation (1.28). We first study the case $|i - j| > 1$. Here we will also use the R–bimodules isomorphism $\gamma^{i,j}$ defined in

Proposition 1.2:

$$\begin{tikzcd}
F(\sigma_i\rho_j) : 0 \arrow[r] & B_j\{2\} \arrow[r, "\mathrm{rb}_i \otimes \mathrm{id}"] \arrow[d, "\mathrm{id}"] & B_i \otimes_R B_j \arrow[r] \arrow[d, "\gamma^{i,j}"] & 0 \\
F(\rho_j\sigma_i) : 0 \arrow[r] & B_j\{2\} \arrow[r, "\mathrm{id} \otimes \mathrm{rb}_i"] & B_j \otimes_R B_i \arrow[r] & 0.
\end{tikzcd}$$

We easily check that the vertical maps and their inverse commute with the differentials and thus provide an isomorphism of complexes of R–bimodules $F(\sigma_i\rho_j) \cong F(\rho_j\sigma_i)$ for all $|i-j| > 1$. This is a stronger result than the one announced since isomorphy implies homotopy equivalence.

Now let us turn to the case $i = j$. We define the following morphisms of complexes and homotopies:

$$\begin{tikzcd}[row sep=huge]
F(\sigma_i\rho_i) : 0 \arrow[r] & B_i\{2\} \arrow[r, "\mathrm{rb}_i \otimes \mathrm{id}"] \arrow[d, bend left, "0"] & B_i \otimes_R B_i \arrow[r] \arrow[d, bend left, "f"] \arrow[l, dashed, bend right, "-\phi_2^i"'] & 0 \\
F(\rho_i\sigma_i) : 0 \arrow[r] & B_i\{2\} \arrow[r, "\mathrm{id} \otimes \mathrm{rb}_i"] \arrow[u, bend left, "0"] & B_i \otimes_R B_i \arrow[r] \arrow[u, bend left, "g"] \arrow[l, dashed, bend left, "-\phi_2^i"] & 0,
\end{tikzcd}$$

where

$$\begin{aligned}
f &= -\mu_{X_i\otimes 1\otimes 1} \circ \psi_1^i \circ \phi_2^i + \psi_1^i \circ \phi_1^i, \\
g &= -\mu_{1\otimes 1\otimes X_i} \circ \psi_1^i \circ \phi_2^i + \psi_1^i \circ \phi_1^i.
\end{aligned}$$

Let us recall that the map $\psi_1^i : B_i \to B_i \otimes_R B_i$ sends $1 \otimes 1$ to $1 \otimes 1 \otimes 1$; $\psi_2^i : B_i\{2\} \to B_i \otimes_R B_i$ sends $1 \otimes 1$ to $1 \otimes X_i \otimes 1$; $\phi_1^i : B_i \otimes_R B_i \to B_i$ sends $1 \otimes 1 \otimes 1$ to $1 \otimes 1$ and $1 \otimes X_i \otimes 1$ to 0 and $\phi_2^i : B_i \otimes_R B_i \to B_i\{2\}$ sends $1 \otimes 1 \otimes 1$ to 0 and $1 \otimes X_i \otimes 1$ to $1 \otimes 1$. The vertical maps commute with the differentials. Indeed,

$$\begin{aligned}
& f \circ (\mathrm{rb}_i \otimes \mathrm{id})(1 \otimes 1) \\
= \;& \left(-\mu_{X_i\otimes 1\otimes 1} \circ \psi_1^i \circ \phi_2^i + \psi_1^i \circ \phi_1^i\right) \circ (\mathrm{rb}_i \otimes \mathrm{id})(1 \otimes 1) \\
= \;& \left(-\mu_{X_i\otimes 1\otimes 1} \circ \psi_1^i \circ \phi_2^i + \psi_1^i \circ \phi_1^i\right) (X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1) \\
= \;& \psi_1^i \circ \phi_1^i(X_i \otimes 1 \otimes 1) - \mu_{X_i\otimes 1\otimes 1} \circ \psi_1^i \circ \phi_2^i(1 \otimes X_i \otimes 1) \\
= \;& \psi_1^i(X_i \otimes 1) - \mu_{X_i\otimes 1\otimes 1} \circ \psi_1^i(1 \otimes 1) \\
= \;& X_i \otimes 1 \otimes 1 - \mu_{X_i\otimes 1\otimes 1}(1 \otimes 1 \otimes 1) \\
= \;& 0,
\end{aligned}$$

and similarly $g \circ (\mathrm{id} \otimes \mathrm{rb}_i)(1 \otimes 1) = 0$. Finally let us check that the morphisms of complexes form a homotopy equivalence. In cohomological degree -1, we obviously have

$$\phi_2^i \circ (\mathrm{rb}_i \otimes \mathrm{id}) = \phi_2^i \circ (\mathrm{id} \otimes \mathrm{rb}_i) = \mathrm{id}_{B_i\{2\}} .$$

Let us turn to the cohomological degree 0. It is sufficient to do the computations for the generators $1 \otimes 1 \otimes 1$ and $1 \otimes X_i \otimes 1$ of $B_i \otimes_R B_i$. Let us start by proving that $g \circ f$ is homotopic to the identity of $B_i \otimes_R B_i$:

$$\begin{aligned} & \left(g \circ f + (\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^i\right)(1 \otimes 1 \otimes 1) \\ = \;& \psi_1^i \circ \phi_1^i \circ \psi_1^i \circ \phi_1^i (1 \otimes 1 \otimes 1) + 0 \\ = \;& 1 \otimes 1 \otimes 1 \end{aligned}$$

and

$$\begin{aligned} & \left(g \circ f + (\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^i\right)(1 \otimes X_i \otimes 1) \\ = \;& -\psi_1^i \circ \phi_1^i \circ \mu_{X_i \otimes 1 \otimes 1} \circ \psi_1^i \circ \phi_2^i (1 \otimes X_i \otimes 1) + (\mathrm{rb}_i \otimes \mathrm{id})(1 \otimes 1) \\ = \;& -X_i \otimes 1 \otimes 1 + X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 \\ = \;& 1 \otimes X_i \otimes 1. \end{aligned}$$

At last, let us check that $f \circ g$ is homotopic to the identity of $B_i \otimes_R B_i$:

$$\begin{aligned} & \left(f \circ g + (\mathrm{id} \otimes \mathrm{rb}_i) \circ \phi_2^i\right)(1 \otimes 1 \otimes 1) \\ = \;& \psi_1^i \circ \phi_1^i \circ \psi_1^i \circ \phi_1^i (1 \otimes 1 \otimes 1) + 0 \\ = \;& 1 \otimes 1 \otimes 1 \end{aligned}$$

and

$$\begin{aligned} & \left(f \circ g + (\mathrm{id} \otimes \mathrm{rb}_i) \circ \phi_2^i\right)(1 \otimes X_i \otimes 1) \\ = \;& -\psi_1^i \circ \phi_1^i \circ \mu_{1 \otimes 1 \otimes X_i} \circ \psi_1^i \circ \phi_2^i (1 \otimes X_i \otimes 1) + (\mathrm{id} \otimes \mathrm{rb}_i)(1 \otimes 1) \\ = \;& -1 \otimes 1 \otimes X_i + 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i \\ = \;& 1 \otimes X_i \otimes 1. \end{aligned}$$

Relation (1.29)

We now deal with Relation (1.29). The complexes involved in this relation are $F(\sigma_i \sigma_{i+1} \rho_i)$ which is egal to

$$\begin{tikzcd}
 & & B_i \otimes_R B_i\{2\} \arrow[dr, "\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}"] & & \\
0 \arrow[r] & B_i\{4\} \arrow[ur, "\mathrm{rb}_i \otimes \mathrm{id}"] \arrow[dr, "-\mathrm{rb}_{i+1} \otimes \mathrm{id}"'] & \bigoplus & B_i \otimes_R B_{i+1} \otimes_R B_i \arrow[r] & 0 \\
 & & B_{i+1} \otimes_R B_i\{2\} \arrow[ur, "\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id}"'] & &
\end{tikzcd}$$

and $F(\rho_{i+1} \sigma_i \sigma_{i+1})$ which is egal to

$$\begin{tikzcd}
 & & B_{i+1} \otimes_R B_i\{2\} \arrow[dr, "\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1}"] & & \\
0 \arrow[r] & B_{i+1}\{4\} \arrow[ur, "\mathrm{id} \otimes \mathrm{rb}_i"] \arrow[dr, "-\mathrm{id} \otimes \mathrm{rb}_{i+1}"'] & \bigoplus & B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \arrow[r] & 0 \\
 & & B_{i+1} \otimes_R B_{i+1}\{2\} \arrow[ur, "\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}"'] & &
\end{tikzcd}$$

We define a morphism of complexes from $F(\sigma_i\sigma_{i+1}\rho_i)$ to $F(\rho_{i+1}\sigma_i\sigma_{i+1})$ as follows:

$$\begin{tikzcd}
 & B_i \otimes_R B_i\{2\} \arrow[dr, "\mathrm{id}\otimes \mathrm{rb}_{i+1}\otimes \mathrm{id}"] \arrow[ddd, bend right, "f_{1,1}"'] \arrow[ddddd, bend left, "f_{1,2}"] & \\
B_i\{4\} \arrow[ur, "\mathrm{rb}_i\otimes \mathrm{id}"] \arrow[dr, "-\mathrm{rb}_{i+1}\otimes \mathrm{id}"'] \arrow[ddd, "0"] & & B_i \otimes_R B_{i+1} \otimes_R B_i \arrow[ddd, "\iota^{i+1}\circ\pi^i"] \\
 & B_{i+1} \otimes_R B_i\{2\} \arrow[ur, "\mathrm{rb}_i\otimes \mathrm{id}\otimes \mathrm{id}"'] \arrow[d, bend left, "\mathrm{id}"] \arrow[ddd, bend right, "f_{2,2}"'] & \\
 & B_{i+1} \otimes_R B_i\{2\} \arrow[dr, "\mathrm{id}\otimes \mathrm{id}\otimes \mathrm{rb}_{i+1}"] & \\
B_{i+1}\{4\} \arrow[ur, "\mathrm{id}\otimes \mathrm{rb}_i"] \arrow[dr, "-\mathrm{id}\otimes \mathrm{rb}_{i+1}"'] & & B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \\
 & B_{i+1} \otimes_R B_{i+1}\{2\} \arrow[ur, "\mathrm{id}\otimes \mathrm{rb}_i\otimes \mathrm{id}"'] &
\end{tikzcd}$$

where

$$\begin{aligned}
f_{1,1} &= (\mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \phi_2^i, \\
f_{1,2} &= \psi_1^{i+1} \circ \mathrm{rb}_{i+1} \circ \mathrm{br}_i \circ \phi_2^i, \\
f_{2,2} &= \psi_1^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i).
\end{aligned}$$

Let us recall that the map $\iota^{i+1} : B_{i,i+1} \to B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$ sends $1 \otimes 1$ to $1 \otimes 1 \otimes 1 \otimes 1$ and that the map $\pi^i : B_i \otimes_R B_{i+1} \otimes_R B_i \to B_{i,i+1}$ sends $1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1$ to 0 and $1 \otimes 1 \otimes 1 \otimes 1$ to $1 \otimes 1$. Let us check that these vertical maps are well-defined morphisms of complexes, i.e., that they commute with the differentials. Most of the following computations are using results given in the proof of Proposition 1.4.

First let us focus on the differentials of degree -2. We check here that the map from $B_i\{4\}$, the cochain bimodule of degree -2 of $F(\sigma_i\sigma_{i+1}\rho_i)$, to $B_{i+1} \otimes_R B_i\{2\}$, the first factor of the cochain bimodule of degree -1 of $F(\rho_{i+1}\sigma_i\sigma_{i+1})$, is equal to zero:

$$\begin{aligned}
& \left(f_{1,1} \circ (\mathrm{rb}_i \otimes \mathrm{id}) + \mathrm{id} \circ (-\,\mathrm{rb}_{i+1} \otimes \mathrm{id})\right)(1 \otimes 1) \\
= \;& f_{1,1}(X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1) - X_{i+1} \otimes 1 \otimes 1 - 1 \otimes X_{i+1} \otimes 1 \\
= \;& 0 + (\mathrm{rb}_{i+1} \otimes \mathrm{id})(1 \otimes 1) - X_{i+1} \otimes 1 \otimes 1 - 1 \otimes X_{i+1} \otimes 1 \\
= \;& 0.
\end{aligned}$$

We also have to prove that the map from $B_i\{4\}$, the cochain bimodule of degree -2 of $F(\sigma_i\sigma_{i+1}\rho_i)$, to $B_{i+1} \otimes_R B_{i+1}\{2\}$, the second factor of the cochain bimodule of degree -1 of $F(\rho_{i+1}\sigma_i\sigma_{i+1})$, is equal to zero. Indeed,

$$\begin{aligned}
& (f_{1,2} \circ (\mathrm{rb}_i \otimes \mathrm{id}) + f_{2,2} \circ (-\,\mathrm{rb}_{i+1} \otimes \mathrm{id}))\,(1 \otimes 1) \\
= \;& f_{1,2}(X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1) + f_{2,2}(-X_{i+1} \otimes 1 \otimes 1 - 1 \otimes X_{i+1} \otimes 1) \\
= \;& 0 + \psi_1^{i+1} \circ \mathrm{rb}_{i+1}(1) + \psi_1^{i+1}(-X_{i+1} \otimes 1 - 1 \otimes X_{i+1}) \\
= \;& 0.
\end{aligned}$$

Let us turn to the differentials of degree -1. First we focus on the maps from $B_i \otimes_R B_i\{2\}$, the first factor of the cochain bimodule of degree -1 of $F(\sigma_i\sigma_{i+1}\rho_i)$, to $B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$, the cochain bimodule of degree 0 of $F(\rho_{i+1}\sigma_i\sigma_{i+1})$. As $B_i \otimes_R B_i$ is spanned by $1 \otimes 1 \otimes 1$ and $1 \otimes X_i \otimes 1$, we only have to compute the images of both these two elements:

$$((\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1}) \circ f_{1,1} + (\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}) \circ f_{1,2})\,(1 \otimes 1 \otimes 1) \quad = \quad 0$$

since ϕ_2^i vanishes on $1 \otimes 1 \otimes 1$; and

$$\begin{aligned}
& \iota^{i+1} \circ \pi^i \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\
= \;& \iota^{i+1} \circ \pi^i(1 \otimes X_{i+1} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_{i+1} \otimes 1) \\
= \;& 0.
\end{aligned}$$

Let us now, for sake of simplicity, split the computations of the images of $1 \otimes X_i \otimes 1$ into smaller parts: we first look at its image V_1 under the composed

$$B_i \otimes_R B_i\{2\} \to B_{i+1} \otimes_R B_i\{2\} \to B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$$

then its image V_2 under

$$B_i \otimes_R B_i\{2\} \to B_{i+1} \otimes_R B_{i+1}\{2\} \to B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$$

and at last its image V_3 under

$$B_i \otimes_R B_i\{2\} \to B_i \otimes_R B_{i+1} \otimes_R B_i \to B_{i+1} \otimes_R B_i \otimes_R B_{i+1}.$$

The sum $V_1 + V_2$ of the two first elements that we obtain has to be equal to the third one V_3. But we will not compare them directly in $B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$, we will compare their projections in $B_{i,i+1}$ and in $B_{i+1}\{2\}$ using the decomposition $B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \cong B_{i,i+1} \oplus B_{i+1}\{2\}$. The first step is to compute the projections of the two first elements V_1 and V_2. The first element V_1 is equal to

$$\begin{aligned}
& (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1}) \circ f_{1,1}(1 \otimes X_i \otimes 1) \\
= \;& (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1})(X_{i+1} \otimes 1 \otimes 1 + 1 \otimes X_{i+1} \otimes 1) \\
= \;& X_{i+1} \otimes 1 \otimes X_{i+1} \otimes 1 + X_{i+1} \otimes 1 \otimes 1 \otimes X_{i+1} \\
& +1 \otimes X_{i+1} \otimes X_{i+1} \otimes 1 + 1 \otimes X_{i+1} \otimes 1 \otimes X_{i+1}.
\end{aligned} \tag{1.35}$$

In order to apply easily the projections onto the two factors of the decomposition $B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \cong B_{i,i+1} \oplus B_{i+1}\{2\}$, we can use Equations (1.20), (1.21) and (1.22) to rewrite the right hand side of (1.35) as a right and left linear combination of $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes (X_i \otimes 1 + 1 \otimes X_i) \otimes 1$ with coefficients in R. Then applying the surjection π^{i+1} onto $B_{i,i+1}$, we obtain

$$\begin{aligned}&\frac{4}{9}v_i \otimes v_i + X_{i+1} \otimes X_{i+1} + X_{i+1} \otimes \frac{2v_i}{3} + \frac{2v_i}{3} \otimes X_{i+1} \\ &\quad + \left(\frac{X_{i+1}^2}{2} - \frac{2v_i^2}{9} + \frac{4v_iX_{i+1}}{3}\right) \otimes 1 + 1 \otimes \left(\frac{X_{i+1}^2}{2} - \frac{2v_i^2}{9} + \frac{4v_iX_{i+1}}{3}\right);\end{aligned} \tag{1.36}$$

and applying the surjection $-\phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id})$ onto B_{i+1}, we obtain

$$-X_{i+1} \otimes 1 - 1 \otimes X_{i+1}. \tag{1.37}$$

Now we have to compute the second element V_2. It is equal to

$$\begin{aligned} & (\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}) \circ f_{1,2}(1 \otimes X_i \otimes 1) \\ = \; & (\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id})(X_{i+1} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_{i+1}) \\ = \; & X_{i+1} \otimes X_i \otimes 1 \otimes 1 + X_{i+1} \otimes 1 \otimes X_i \otimes 1 \\ & +1 \otimes X_i \otimes 1 \otimes X_{i+1} + 1 \otimes 1 \otimes X_i \otimes X_{i+1}. \end{aligned} \tag{1.38}$$

As before, we can look at the image of the right hand side of Equation (1.38) under π^{i+1}, we obtain 0; and under $-\phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id})$, we obtain

$$X_{i+1} \otimes 1 + 1 \otimes X_{i+1}. \tag{1.39}$$

The second step is to compare the results that we obtained in the first step, namely the values of the projections of the elements V_1 and V_2, to the projections of the third element V_3. Remember that we have to prove that $V_1 + V_2 = V_3$. We can already observe that, by adding (1.37) and (1.39), we obtain 0, which is the expected result since the map $-\phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id}) \circ \iota^{i+1} \circ \pi^i$ is equal to zero. Finally we have to compare the former result (1.36) to

$$\begin{aligned} \pi^{i+1}(V_3) &= \pi^{i+1} \circ \iota^{i+1} \circ \pi^i \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id})(1 \otimes X_i \otimes 1) \\ &= \pi^i \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id})(1 \otimes X_i \otimes 1) \\ &= \pi^i(1 \otimes X_iX_{i+1} \otimes 1 \otimes 1 + 1 \otimes X_i \otimes X_{i+1} \otimes 1) \\ &= \pi^i \left(-\frac{X_i^2}{2} \otimes 1 \otimes 1 \otimes 1 + u_i \otimes X_i \otimes 1 \otimes 1 \right. \\ &\quad \left. +1 \otimes X_i \otimes 1 \otimes u_i - \frac{1}{2} \otimes X_i \otimes X_i \otimes 1\right) \\ &= \frac{16u_i}{9} \otimes u_i + \left(\frac{7u_i^2}{9} - \frac{3X_i^2}{4}\right) \otimes 1 \\ &\quad +1 \otimes \left(\frac{13u_i^2}{9} - \frac{X_i^2}{4}\right). \end{aligned} \tag{1.40}$$

Here we have successively used Equations (1.6), (1.13) and (1.16) to perform the previous computations. Now we have to prove that the expressions (1.36) and (1.40) are equal. So let us substract (1.40) from (1.36) and rewrite the difference by replacing u_i and v_i by their expressions (1.1) and (1.2):

$$\begin{aligned}
&\frac{4}{9}v_i \otimes v_i + X_{i+1} \otimes X_{i+1} + X_{i+1} \otimes \frac{2v_i}{3} + \frac{2v_i}{3} \otimes X_{i+1} \\
&\quad + \left(\frac{X_{i+1}^2}{2} - \frac{2v_i^2}{9} + \frac{4v_iX_{i+1}}{3}\right) \otimes 1 + 1 \otimes \left(\frac{X_{i+1}^2}{2} - \frac{2v_i^2}{9} + \frac{4v_iX_{i+1}}{3}\right) \\
&\qquad - \frac{16u_i}{9} \otimes u_i - \left(\frac{7u_i^2}{9} - \frac{3X_i^2}{4}\right) \otimes 1 - 1 \otimes \left(\frac{13u_i^2}{9} - \frac{X_i^2}{4}\right) \\
&\qquad = \frac{4X_i}{9} \otimes X_i + \frac{16X_{i+1}}{9} \otimes X_{i+1} + \frac{8X_i}{9} \otimes X_{i+1} + \frac{8X_{i+1}}{9} \otimes X_i \\
&+ \left(\frac{10X_{i+1}^2}{9} - \frac{2X_i^2}{9} + \frac{10X_iX_{i+1}}{9}\right) \otimes 1 + 1 \otimes \left(\frac{10X_{i+1}^2}{9} - \frac{2X_i^2}{9} + \frac{10X_iX_{i+1}}{9}\right) \\
&\qquad - \frac{4X_i}{9} \otimes X_i - \frac{16X_{i+1}}{9} \otimes X_{i+1} - \frac{8X_i}{9} \otimes X_{i+1} - \frac{8X_{i+1}}{9} \otimes X_i \\
&\quad - \left(\frac{7X_{i+1}^2}{9} - \frac{5X_i^2}{9} + \frac{7X_iX_{i+1}}{9}\right) \otimes 1 - 1 \otimes \left(\frac{13X_{i+1}^2}{9} + \frac{X_i^2}{9} + \frac{13X_iX_{i+1}}{9}\right) \\
&\quad = \left(\frac{X_{i+1}^2}{3} + \frac{X_i^2}{3} + \frac{X_iX_{i+1}}{3}\right) \otimes 1 - 1 \otimes \left(\frac{X_{i+1}^2}{3} + \frac{X_i^2}{3} + \frac{X_iX_{i+1}}{3}\right) = 0
\end{aligned}$$

since $(X_{i+1}^2 + X_i^2 + X_iX_{i+1}) \in R^{<\tau_i,\tau_{i+1}>}$. Indeed,

$$\begin{aligned}
& \tau_i(X_{i+1}^2 + X_i^2 + X_iX_{i+1}) \\
= \ & (X_i + X_{i+1})^2 + (-X_i)^2 - X_i(X_i + X_{i+1}) \\
= \ & X_{i+1}^2 + X_i^2 + X_iX_{i+1}
\end{aligned}$$

and

$$\begin{aligned}
& \tau_{i+1}(X_{i+1}^2 + X_i^2 + X_iX_{i+1}) \\
= \ & (-X_{i+1})^2 + (X_i + X_{i+1})^2 - (X_i + X_{i+1})X_{i+1} \\
= \ & X_{i+1}^2 + X_i^2 + X_iX_{i+1}.
\end{aligned}$$

This completes the proof of

$$\iota^{i+1} \circ \pi^i \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) = (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1}) \circ f_{1,1} + (\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}) \circ f_{1,2}.$$

To finish with the differentials of degree -1, we still have to compare the maps from $B_{i+1} \otimes_R B_i\{2\}$, the second factor of the cochain bimodule of degree -1 of $F(\sigma_i\sigma_{i+1}\rho_i)$, to $B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$, the cochain bimodule of degree 0 of $F(\rho_{i+1}\sigma_i\sigma_{i+1})$. As the R–bimodule $B_{i+1} \otimes_R B_i$ is spanned by the element $1 \otimes 1 \otimes 1$, so we only need to compute the images of this element; first V_1' under the composed

$$B_{i+1} \otimes_R B_i\{2\} \to B_{i+1} \otimes_R B_i\{2\} \to B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$$

then V_2' under

$$B_{i+1} \otimes_R B_i\{2\} \to B_{i+1} \otimes_R B_{i+1}\{2\} \to B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$$

and finally V_3' under

$$B_{i+1} \otimes_R B_i\{2\} \to B_i \otimes_R B_{i+1} \otimes_R B_i \to B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$$

The sum $V_1' + V_2'$ has to be equal to V_3'. Once again we will not compare them directly in $B_{i+1} \otimes_R B_i \otimes_R B_{i+1}$, we will compare their projections in $B_{i,i+1}$ and in $B_{i+1}\{2\}$ using the decomposition $B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \cong B_{i,i+1} \oplus B_{i+1}\{2\}$. The first element V_1' is equal to

$$\begin{aligned} & (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1}) \circ \mathrm{id}(1 \otimes 1 \otimes 1) \\ = \ & (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1})(1 \otimes 1 \otimes 1) \\ = \ & 1 \otimes 1 \otimes X_{i+1} \otimes 1 + 1 \otimes 1 \otimes 1 \otimes X_{i+1}. \end{aligned} \tag{1.41}$$

Using (1.20) to rewrite the right hand side of Equation (1.41) and applying the surjection π^{i+1} onto $B_{i,i+1}$, we obtain

$$\frac{4v_i}{3} \otimes 1 + 1 \otimes \left(\frac{2v_i}{3} + X_{i+1}\right).$$

Applying the surjection $-\phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id})$ onto B_{i+1}, we obtain

$$-1 \otimes 1. \tag{1.42}$$

Then we compute the second element V_2'; it is equal to

$$\begin{aligned} (\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}) \circ f_{2,2}(1 \otimes 1 \otimes 1) &= (\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\ &= 1 \otimes X_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_i \otimes 1 \end{aligned}$$

which is sent by π^{i+1} to 0 and by $-\phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id})$ to

$$1 \otimes 1. \tag{1.43}$$

Adding (1.42) and (1.43), we get 0, which is the expected result. In fact the element V_3' has to be sent to zero by $-\phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id})$ since the composed $-\phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id}) \circ \iota^{i+1} \circ \pi^i$ is equal to zero. Finally, let us compute $\pi^{i+1}(V_3')$. We obtain

$$\begin{aligned} \pi^{i+1}\left(V_3'\right) &= \pi^{i+1} \circ \iota^{i+1} \circ \pi^i \circ (\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\ &= \pi^i \circ (\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\ &= \pi^i(X_i \otimes 1 \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 \otimes 1) \\ &= \left(X_i + \frac{2u_i}{3}\right) \otimes 1 + 1 \otimes \frac{4u_i}{3} \\ &= \left(\frac{4X_i + 2X_{i+1}}{3}\right) \otimes 1 + 1 \otimes \left(\frac{2X_i + 4X_{i+1}}{3}\right) \\ &= \frac{4v_i}{3} \otimes 1 + 1 \otimes \left(\frac{2v_i}{3} + X_{i+1}\right) \\ &= \pi^{i+1} \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1})(1 \otimes 1 \otimes 1) \\ &= \pi^{i+1}\left(V_1'\right) + \pi^{i+1}\left(V_2'\right). \end{aligned}$$

This completes the proof that the morphisms of complexes commute with the differentials of degree -1.

We also define the following morphism of complexes from $F(\rho_{i+1}\sigma_i\sigma_{i+1})$ to $F(\sigma_i\sigma_{i+1}\rho_i)$:

$$
\begin{tikzcd}
 & & B_i \otimes_R B_i\{2\} \arrow[drr, "\mathrm{id}\otimes \mathrm{rb}_{i+1}\otimes \mathrm{id}"] & & \\
B_i\{4\} \arrow[urr, "\mathrm{rb}_i\otimes \mathrm{id}"] \arrow[drr, "-\mathrm{rb}_{i+1}\otimes \mathrm{id}"'] & & & & B_i \otimes_R B_{i+1} \otimes_R B_i \\
 & & B_{i+1} \otimes_R B_i\{2\} \arrow[urr, "\mathrm{rb}_i\otimes \mathrm{id}\otimes \mathrm{id}"'] & & \\
 & & B_{i+1} \otimes_R B_i\{2\} \arrow[drr, "\mathrm{id}\otimes \mathrm{id}\otimes \mathrm{rb}_{i+1}"] \arrow[uu, bend left=60, "g_{1,1}"] \arrow[u, bend right, "\mathrm{id}"'] & & \\
B_{i+1}\{4\} \arrow[uuu, "0"] \arrow[urr, "\mathrm{id}\otimes \mathrm{rb}_i"] \arrow[drr, "-\mathrm{id}\otimes \mathrm{rb}_{i+1}"'] & & & & B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \arrow[uuu, "\iota^i\circ\pi^{i+1}"'] \\
 & & B_{i+1} \otimes_R B_{i+1}\{2\} \arrow[urr, "\mathrm{id}\otimes \mathrm{rb}_i\otimes \mathrm{id}"'] \arrow[uuuuu, bend right=40, "g_{2,1}"'] \arrow[uuu, bend left=60, "g_{2,2}"] & &
\end{tikzcd}
.
$$

where

$$
\begin{aligned}
g_{1,1} &= \psi_1^i \circ (\mathrm{br}_{i+1}\otimes \mathrm{id}),\\
g_{2,1} &= \psi_1^i \circ \mathrm{rb}_i \circ \mathrm{br}_{i+1} \circ \phi_2^{i+1},\\
g_{2,2} &= (\mathrm{id}\otimes \mathrm{rb}_i)\circ \phi_2^{i+1}.
\end{aligned}
$$

Let us recall that the map $\iota^i : B_{i,i+1} \to B_i \otimes_R B_{i+1} \otimes_R B_i$ sends $1\otimes 1$ to $1\otimes 1\otimes 1\otimes 1$ and that the map $\pi^{i+1} : B_{i+1}\otimes_R B_i \otimes_R B_{i+1} \to B_{i,i+1}$ sends $1\otimes(X_i\otimes 1 + 1\otimes X_i)\otimes 1$ to 0 and $1\otimes 1\otimes 1\otimes 1$ to $1\otimes 1$. In order to check that the vertical maps commute with the differentials, we proceed exactly as above.

Finally, we define the homotopies of $F(\sigma_i\sigma_{i+1}\rho_i)$:

$$
\begin{tikzcd}
 & B_i \otimes_R B_i\{2\} \arrow[dr, "\mathrm{id}\otimes \mathrm{rb}_{i+1}\otimes \mathrm{id}"] \arrow[dl, dashed, bend right, "-\phi_2^i"'] & \\
B_i\{4\} \arrow[ur, "\mathrm{rb}_i\otimes \mathrm{id}"] \arrow[dr, "-\mathrm{rb}_{i+1}\otimes \mathrm{id}"'] & & B_i \otimes_R B_{i+1} \otimes_R B_i \arrow[ul, dashed, bend right=60, "h"'] \\
 & B_{i+1} \otimes_R B_i\{2\} \arrow[ur, "\mathrm{rb}_i\otimes \mathrm{id}\otimes \mathrm{id}"'] &
\end{tikzcd}
$$

where

$$h = \psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}).$$

And the homotopies of $F(\rho_{i+1}\sigma_i\sigma_{i+1})$:

$$\begin{tikzcd}
& B_{i+1} \otimes_R B_i\{2\} \arrow[dr, "\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_{i+1}"] & \\
B_{i+1}\{4\} \arrow[ur, "\mathrm{id}\otimes\mathrm{rb}_i"] \arrow[dr, "-\mathrm{id}\otimes\mathrm{rb}_{i+1}"'] & & B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \arrow[dl, dashed, bend left, "k"] \\
& B_{i+1} \otimes_R B_{i+1}\{2\} \arrow[ul, dashed, bend left, "\phi_2^{i+1}"] \arrow[ur, "\mathrm{id}\otimes\mathrm{rb}_i\otimes\mathrm{id}"'] &
\end{tikzcd}$$

where

$$k = \psi_1^{i+1} \circ \phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id}).$$

Let us now prove the homotopy equivalence between the two complexes $F(\sigma_i\sigma_{i+1}\rho_i)$ and $F(\rho_{i+1}\sigma_i\sigma_{i+1})$. We start by proving that the composition $F(\sigma_i\sigma_{i+1}\rho_i) \to F(\rho_{i+1}\sigma_i\sigma_{i+1}) \to F(\sigma_i\sigma_{i+1}\rho_i)$ of the morphisms defined previously is an endomorphism of $F(\sigma_i\sigma_{i+1}\rho_i)$ that is homotopy equivalent to the identity.

The endomorphism of $B_i\{4\}$, the cochain bimodule of degree -2 of $F(\sigma_i\sigma_{i+1}\rho_i)$, has to be homotopy equivalent to the identity:

$$(0 - \mathrm{id})(1 \otimes 1) \quad = \quad -1 \otimes 1$$

while

$$\begin{aligned}
-\phi_2^i \circ (\mathrm{rb}_i \otimes \mathrm{id})(1 \otimes 1) \quad &= \quad -\phi_2^i(X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1) \\
&= \quad 0 - 1 \otimes 1;
\end{aligned}$$

therefore,

$$0 - \mathrm{id} = -\phi_2^i \circ (\mathrm{rb}_i \otimes \mathrm{id}).$$

The endomorphism of $B_i \otimes_R B_i\{2\}$, the first factor of the cochain bimodule of degree -1 of $F(\sigma_i\sigma_{i+1}\rho_i)$, has to be homotopy equivalent to the identity. We check this for the two generators $1 \otimes 1 \otimes 1$ and $1 \otimes X_i \otimes 1$ of $B_i \otimes_R B_i\{2\}$. On one hand, we have

$$\begin{aligned}
& (g_{1,1} \circ f_{1,1} + g_{2,1} \circ f_{1,2} - \mathrm{id})\,(1 \otimes 1 \otimes 1) \\
= \quad & \big(\psi_1^i \circ (\mathrm{br}_{i+1} \otimes \mathrm{id}) \circ (\mathrm{rb}_{i+1} \otimes \mathrm{id})\phi_2^i \\
& +\psi_1^i \circ \mathrm{rb}_i \circ \mathrm{br}_{i+1} \circ \phi_2^{i+1} \circ \psi_1^{i+1} \circ \mathrm{rb}_{i+1} \circ \mathrm{br}_i \circ \phi_2^i - \mathrm{id}\big)\,(1 \otimes 1 \otimes 1) \\
= \quad & 0 + 0 - 1 \otimes 1 \otimes 1
\end{aligned}$$

and

$$\begin{aligned}
& (g_{1,1} \circ f_{1,1} + g_{2,1} \circ f_{1,2} - \mathrm{id})\,(1 \otimes X_i \otimes 1) \\
= \quad & \big(\psi_1^i \circ (\mathrm{br}_{i+1} \otimes \mathrm{id}) \circ (\mathrm{rb}_{i+1} \otimes \mathrm{id})\phi_2^i \\
& +\psi_1^i \circ \mathrm{rb}_i \circ \mathrm{br}_{i+1} \circ \phi_2^{i+1} \circ \psi_1^{i+1} \circ \mathrm{rb}_{i+1} \circ \mathrm{br}_i \circ \phi_2^i - \mathrm{id}\big)\,(1 \otimes X_i \otimes 1) \\
= \quad & 2X_{i+1} \otimes 1 \otimes 1 + 0 - 1 \otimes X_i \otimes 1.
\end{aligned}$$

On the other hand, we have

$$\begin{array}{rl} & \big(h \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) - (\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^i\big)\,(1 \otimes 1 \otimes 1) \\ = & \big(\psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \\ & -(\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^i\big)\,(1 \otimes 1 \otimes 1) \\ = & -1 \otimes 1 \otimes 1 + 0 \end{array}$$

and

$$\begin{array}{rl} & \big(h \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) - (\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^i\big)\,(1 \otimes X_i \otimes 1) \\ = & \big(\psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \\ & -(\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^i\big)\,(1 \otimes X_i \otimes 1) \\ = & \psi_1^i \circ \phi_2^i (1 \otimes 2X_i X_{i+1} \otimes 1) - X_i \otimes 1 \otimes 1 - 1 \otimes X_i \otimes 1 \\ = & \psi_1^i \circ \phi_2^i (2u_i \otimes X_i \otimes 1 - X_i^2 \otimes 1 \otimes 1) - X_i \otimes 1 \otimes 1 - 1 \otimes X_i \otimes 1 \\ = & 2u_i \otimes 1 \otimes 1 - X_i \otimes 1 \otimes 1 - 1 \otimes X_i \otimes 1 \\ = & (2u_i - X_i) \otimes 1 \otimes 1 - 1 \otimes X_i \otimes 1; \end{array}$$

so we obtain the expected equality

$$g_{1,1} \circ f_{1,1} + g_{2,1} \circ f_{1,2} - \mathrm{id} = h \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) - (\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^i.$$

The endomorphism of $B_{i+1} \otimes_R B_i\{2\}$, the second factor of the cochain bimodule of degree -1 of $F(\sigma_i \sigma_{i+1} \rho_i)$, has to be homotopy equivalent to the identity. It is immediate that

$$g_{2,2} \circ f_{2,2} + \mathrm{id} \circ \mathrm{id} - \mathrm{id} = (\mathrm{id} \otimes \mathrm{rb}_i) \circ \phi_2^{i+1} \circ \psi_1^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i) + \mathrm{id} \circ \mathrm{id} - \mathrm{id} = 0$$

since $\phi_2^{i+1} \circ \psi_1^{i+1} = 0$.

The endomorphism of $B_i \otimes_R B_{i+1} \otimes_R B_i$, the cochain bimodule of degree 0 of $F(\sigma_i \sigma_{i+1} \rho_i)$, has to be homotopy equivalent to the identity. We check this for the two generators $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1$ of $B_i \otimes_R B_{i+1} \otimes_R B_i$. On one hand, we have

$$\big(\iota^i \circ \pi^{i+1} \circ \iota^{i+1} \circ \pi^i - \mathrm{id}\big)\,(1 \otimes 1 \otimes 1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes 1 = 0$$

and

$$\begin{aligned} &\big(\iota^i \circ \pi^{i+1} \circ \iota^{i+1} \circ \pi^i - \mathrm{id}\big)\,(1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1) \\ &\qquad\qquad = 0 - 1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1. \end{aligned}$$

On the other hand, we have

$$\begin{array}{rl} & ((\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ h)\,(1 \otimes 1 \otimes 1 \otimes 1) \\ = & \big((\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id})\big)\,(1 \otimes 1 \otimes 1 \otimes 1) \\ = & 0 \end{array}$$

and

$$\begin{aligned}
& ((\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ h)\,(1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1) \\
= \ & \big((\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id})\big) \\
& (1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1) \\
= \ & \big((\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \psi_1^i \circ \phi_2^i\big)\,(1 \otimes 2X_{i+1} \otimes 1) \\
= \ & \big((\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \psi_1^i\big)\,(-1 \otimes 1) \\
= \ & (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id})(-1 \otimes 1 \otimes 1) \\
= \ & -1 \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes 1;
\end{aligned}$$

so we obtain the expected equality

$$\iota^i \circ \pi^{i+1} \circ \iota^{i+1} \circ \pi^i - \mathrm{id} = (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ h.$$

The morphism from $B_i \otimes_R B_i\{2\}$, the first factor of the cochain bimodule of degree -1 of $F(\sigma_i \sigma_{i+1} \rho_i)$, to $B_{i+1} \otimes_R B_i\{2\}$, the second factor of the cochain bimodule of degree -1 of $F(\sigma_i \sigma_{i+1} \rho_i)$, has to be homotopy equivalent to zero. We check this for the two generators $1 \otimes 1 \otimes 1$ and $1 \otimes X_i \otimes 1$ of $B_i \otimes_R B_i\{2\}$. On one hand, we have

$$\begin{aligned}
& (\mathrm{id} \circ f_{1,1} + g_{2,2} \circ f_{1,2} - 0)\,(1 \otimes 1 \otimes 1) \\
= \ & \big(\mathrm{id} \circ (\mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \phi_2^i \\
& + (\mathrm{id} \otimes \mathrm{rb}_i) \circ \phi_2^{i+1} \circ \psi_1^{i+1} \circ \mathrm{rb}_{i+1} \circ \mathrm{br}_i \circ \phi_2^i - 0\big)\,(1 \otimes 1 \otimes 1) \\
= \ & 0
\end{aligned}$$

and

$$\begin{aligned}
& (\mathrm{id} \circ f_{1,1} + g_{2,2} \circ f_{1,2} - 0)\,(1 \otimes X_i \otimes 1) \\
= \ & \big(\mathrm{id} \circ (\mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \phi_2^i \\
& + (\mathrm{id} \otimes \mathrm{rb}_i) \circ \phi_2^{i+1} \circ \psi_1^{i+1} \circ \mathrm{rb}_{i+1} \circ \mathrm{br}_i \circ \phi_2^i - 0\big)\,(1 \otimes X_i \otimes 1) \\
= \ & X_{i+1} \otimes 1 \otimes 1 + 1 \otimes X_{i+1} \otimes 1 + 0 - 0.
\end{aligned}$$

On the other hand, we have

$$(\mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \phi_2^i (1 \otimes 1 \otimes 1) = 0$$

and

$$(\mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \phi_2^i = X_{i+1} \otimes 1 \otimes 1 + 1 \otimes X_{i+1} \otimes 1;$$

so we obtain the expected equality

$$\mathrm{id} \circ f_{1,1} + g_{2,2} \circ f_{1,2} - 0 = -(\mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ \phi_2^i.$$

The morphism from $B_{i+1} \otimes_R B_i\{2\}$, the second factor of the cochain bimodule of degree -1 of $F(\sigma_i \sigma_{i+1} \rho_i)$, to $B_i \otimes_R B_i\{2\}$, the first factor of the cochain bimodule of degree -1 of $F(\sigma_i\sigma_{i+1}\rho_i)$, has to be homotopy equivalent to zero:

$$\begin{aligned} & (g_{1,1} \circ \mathrm{id} + g_{2,1} \circ f_{2,2} - 0)\,(1 \otimes 1 \otimes 1) \\ = \;& \psi_1^i \circ (\mathrm{br}_{i+1} \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\ = \;& 1 \otimes 1 \otimes 1 \end{aligned}$$

while

$$\begin{aligned} & h \circ (\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\ = \;& \psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}) \circ (\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id})(1 \otimes 1 \otimes 1) \\ = \;& \psi_1^i \circ \phi_2^i (X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1) \\ = \;& 1 \otimes 1 \otimes 1; \end{aligned}$$

therefore,

$$g_{1,1} \circ \mathrm{id} + g_{2,1} \circ f_{2,2} - 0 = h \circ (\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id}).$$

Thus this endomorphism of the complex $F(\sigma_i\sigma_{i+1}\rho_i)$ is homotopy equivalent to the identity of $F(\sigma_i\sigma_{i+1}\rho_i)$.

We similarly check that the composition $F(\rho_{i+1}\sigma_i\sigma_{i+1}) \to F(\sigma_i\sigma_{i+1}\rho_i) \to F(\rho_{i+1}\sigma_i\sigma_{i+1})$ of the morphisms defined previously is an endomorphism of $F(\rho_{i+1}\sigma_i\sigma_{i+1})$ that is homotopy equivalent to the identity.

Relation (1.30)

The last Relation (1.30) can be treated in the same way as (1.29). So we will merely make explicit the homotopy equivalence between the complexes $F(\rho_i\sigma_{i+1}\sigma_i)$ and $F(\sigma_{i+1}\sigma_i\rho_{i+1})$. The morphisms of complexes between $F(\rho_i\sigma_{i+1}\sigma_i)$ and $F(\sigma_{i+1}\sigma_i\rho_{i+1})$ are summarized in the following diagram:

$$\begin{tikzcd}
& B_i \otimes_R B_{i+1}\{2\} \arrow[dr, "\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_i"] \arrow[ddd, bend right, "f_{1,1}"'] \arrow[ddddd, bend right=60, "\mathrm{id}"'] & \\
B_i\{4\} \arrow[ur, "\mathrm{id}\otimes\mathrm{rb}_{i+1}"] \arrow[dr, "-\mathrm{id}\otimes\mathrm{rb}_i"'] \arrow[ddd, shift right, "0"'] & & B_i \otimes_R B_{i+1} \otimes_R B_i \arrow[ddd, shift right, "\iota^{i+1}\circ\pi^i"'] \\
& B_i \otimes_R B_i\{2\} \arrow[ur, "\mathrm{id}\otimes\mathrm{rb}_{i+1}\otimes\mathrm{id}"'] \arrow[d, shift left, "f_{2,1}"] \arrow[ddd, bend right, "f_{2,2}"'] & \\
& B_{i+1} \otimes_R B_{i+1}\{2\} \arrow[dr, "\mathrm{id}\otimes\mathrm{rb}_i\otimes\mathrm{id}"] \arrow[u, shift left, "g_{1,2}"] \arrow[uuu, bend right, "g_{1,1}"'] & \\
B_{i+1}\{4\} \arrow[ur, "\mathrm{rb}_{i+1}\otimes\mathrm{id}"] \arrow[dr, "-\mathrm{rb}_i\otimes\mathrm{id}"'] \arrow[uuu, shift right, "0"'] & & B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \arrow[uuu, shift right, "\iota^{i}\circ\pi^{i+1}"'] \\
& B_i \otimes_R B_{i+1}\{2\} \arrow[ur, "\mathrm{rb}_{i+1}\otimes\mathrm{id}\otimes\mathrm{id}"'] \arrow[uuu, bend right, "g_{2,2}"'] \arrow[uuuuu, bend right=60, "\mathrm{id}"'] &
\end{tikzcd}$$

where

$$\begin{aligned}
f_{1,1} &= \psi_1^{i+1} \circ (\mathrm{br}_i \otimes \mathrm{id}), \\
f_{2,1} &= \psi_1^{i+1} \circ \mathrm{rb}_{i+1} \circ \mathrm{br}_i \circ \phi_2^i, \\
f_{2,2} &= (\mathrm{id} \otimes \mathrm{rb}_{i+1}) \circ \phi_2^i, \\
g_{1,1} &= (\mathrm{rb}_i \otimes \mathrm{id}) \circ \phi_2^{i+1}, \\
g_{1,2} &= \psi_1^i \circ \mathrm{rb}_i \circ \mathrm{br}_{i+1} \circ \phi_2^{i+1}, \\
g_{2,2} &= \psi_1^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1}).
\end{aligned}$$

The homotopies for $F(\rho_i \sigma_{i+1} \sigma_i)$ are

$$\begin{tikzcd}
& B_i \otimes_R B_{i+1}\{2\} \arrow[dr, "\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_i"] & \\
B_i\{4\} \arrow[ur, "\mathrm{id}\otimes\mathrm{rb}_{i+1}"] \arrow[dr, "-\mathrm{id}\otimes\mathrm{rb}_i"'] & & B_i \otimes_R B_{i+1} \otimes_R B_i \arrow[dl, dashed, bend left, "h"] \\
& B_i \otimes_R B_i\{2\} \arrow[ur, "\mathrm{id}\otimes\mathrm{rb}_{i+1}\otimes\mathrm{id}"'] \arrow[ul, dashed, bend left, "\phi_2^i"] &
\end{tikzcd}$$

where

$$h = \psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id}).$$

The homotopies for $F\left(\sigma_{i+1}\sigma_i\rho_{i+1}\right)$ are

$$\begin{tikzcd}
 & B_{i+1} \otimes_R B_{i+1}\{2\} \arrow[dr, "\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}"] \arrow[dl, dashed, bend right, "-\phi_2^{i+1}"'] & \\
B_{i+1}\{4\} \arrow[ur, "\mathrm{rb}_{i+1} \otimes \mathrm{id}"] \arrow[dr, "-\,\mathrm{rb}_i \otimes \mathrm{id}"'] & & B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \arrow[ul, dashed, bend right, "k"'] \\
 & B_i \otimes_R B_{i+1}\{2\} \arrow[ur, "\mathrm{rb}_{i+1} \otimes \mathrm{id} \otimes \mathrm{id}"'] &
\end{tikzcd}$$

where

$$k = \psi_1^{i+1} \circ \phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id}).$$

This completes the proof of Theorem 1.6. $\square$

Chapter 2

Categorification of the virtual braid groups

Virtual links have been introduced by Kauffman in [Kau99] as a geometric counterpart of Gauss diagrams. A virtual link diagram is a generic oriented immersion of circles into the plane, with the usual positive and negative crossings plus a new kind of crossings called virtual. Such crossings appear for instance when one projects a generic link in a thickened surface onto a plane (see [KK00] or [Kup03]). Many invariants for classical links can be extended to virtual links. Classical oriented links can be represented by closed braids; likewise virtual links can be represented by the closures of virtual braids. Now virtual braids with n strands form a group, denoted $\mathcal{VB}_n$, which can be described by generators and relations, generalizing the generators and relations of the usual braid group with n strands $\mathcal{B}_n$. The aim of this Chapter is, using this presentation, to categorify $\mathcal{VB}_n$ in the sense of Rouquier [Rou06]. More precisely, to any word ω in the generators of $\mathcal{VB}_n$ we associate a bounded cochain complex $F(\omega)$ of Soergel bimodules such that if two words ω and ω' represent the same element of $\mathcal{VB}_n$, then the corresponding cochain complexes $F(\omega)$ and $F(\omega')$ are homotopy equivalent.

Note that, using Mazorchuk and Stroppel's study of Arkhipov's twisting functor in [MS07], one can give a representation theoretic approach to the categorification of virtual braids.

2.1. Virtual braids

Following [Ver01] and [Man04], we recall the definition of the virtual braid group $\mathcal{VB}_n$ with n strands.

Definition 2.1. *The virtual braid group $\mathcal{VB}_n$ is the group generated by $2(n-1)$ generators $\sigma_1, \dots, \sigma_{n-1}$ and $\zeta_1, \dots, \zeta_{n-1}$ satisfying the braid group relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if } |i-j| > 1, \tag{2.1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{if } 1 \le i \le n-2, \tag{2.2}$$

the permutation group relations

$$\zeta_i\zeta_j = \zeta_j\zeta_i, \quad \textit{if } |i-j| > 1, \tag{2.3}$$

$$\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1}, \quad \textit{if } 1 \leq i \leq n-2, \tag{2.4}$$

$$\zeta_i^2 = 1, \quad \textit{if } 1 \leq i \leq n-1, \tag{2.5}$$

and the mixed relations

$$\sigma_i\zeta_j = \zeta_j\sigma_i, \quad \textit{if } |i-j| > 1, \tag{2.6}$$

$$\sigma_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\sigma_{i+1}, \quad \textit{if } 1 \leq i \leq n-2. \tag{2.7}$$

The classical braid group $\mathcal{B}_n$ (see [KT08] for a definition) naturally embeds in $\mathcal{VB}_n$ as a subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$.

The braid group $\mathcal{VB}_n$ can be depicted diagrammatically. To each generator σ_i (resp. σ_i^{-1}) we associate the elementary braid diagram consisting of a single positive (resp. negative) crossing between the ith and $i+1$st strand as shown in Figure 2.1 (resp. Figure 2.2). To each generator ζ_i we associate the elementary virtual braid diagram with a single virtual crossing between the ith and $i+1$st strand of Figure 2.3.

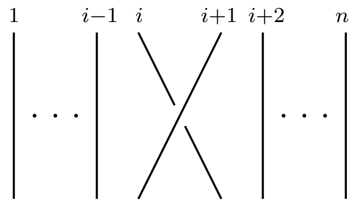


Figure 2.1: The positive braid σ_i

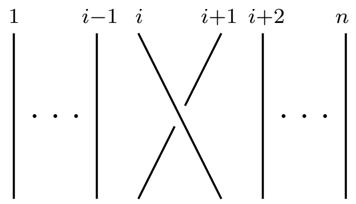


Figure 2.2: The negative braid σ_i^{-1}

The multiplication law of the group $\mathcal{VB}_n$ consists in concatenating these elementary braids. We use the convention that braids multiply from bottom to top: if D (resp. D') is a virtual braid diagram representing an element β (resp. β') of $\mathcal{VB}_n$, then the product $\beta\beta'$ is represented by the diagram obtained by putting D' on top of D and gluing the lower endpoints of D' to the upper endpoints of D.

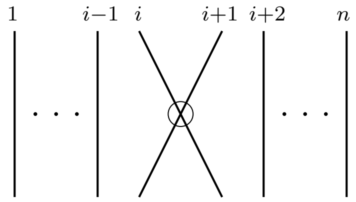


Figure 2.3: The virtual braid ζ_i

The braid group relations, the permutation group relations and the mixed relations have a diagrammatical interpretation. They correspond to planar isotopies and the generalized Reidemeister moves depicted in Figures 2.4, 2.5 and 2.6.

Figure 2.4: Classical Reidemeister II–III moves

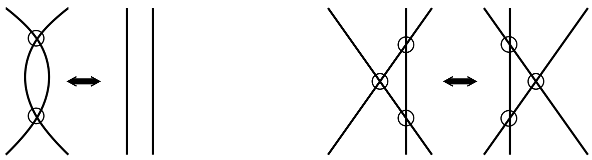


Figure 2.5: Virtual Reidemeister moves

2.2. Rouquier's categorification of the braid groups

In this section we recall how Rouquier [Rou06] categorified the braid group $\mathcal{B}_n$.

2.2.1. Soergel bimodules. We first discuss some bimodules introduced by Soergel [Soe92], [Soe95] in his work on representation theory.

Let R be the subalgebra of $\mathbb{Q}[x_1, \ldots, x_n]$ defined by

$$R = \mathbb{Q}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n] = \mathbb{Q}[x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n].$$

The symmetric group S_n acts on $\mathbb{Q}[x_1, \ldots, x_n]$ by $\omega(x_i) = x_{\omega(i)}$ for all $x_i \in R$ and $\omega \in S_n$. This action preserves R. Let R^ω be the subalgebra of elements of R fixed by ω. In particular R^{τ_i} is the subalgebra of R of

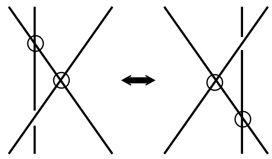


Figure 2.6: The mixed Reidemeister move

elements fixed by the transposition $\tau_i = (i, i+1)$. As an algebra,

$$R^{\tau_i} = \mathbb{Q}[x_1 - x_2, \dots, (x_1 - x_i) + (x_1 - x_{i+1}), \\ (x_1 - x_i)(x_1 - x_{i+1}), x_1 - x_{i+2}, \dots, x_1 - x_n].$$

Let us also consider the R–bimodules $B_\omega = R \otimes_{R^\omega} R$ for any $\omega \in S_n$. The R–bimodules B_{τ_i} will be denoted by B_i for simplicity of notation. We introduce a grading on R, R^{τ_i} and B_i by setting $\deg(x_k) = 2$ for all $k = 1, \dots, n$.

Two R-bimodule morphisms between these objects will be relevant to us, namely $\mathrm{br}_i : B_i \to R$ and $\mathrm{rb}_i : R\{2\} \to B_i$ defined by

$$\mathrm{br}_i(1 \otimes 1) = 1 \quad \text{and} \quad \mathrm{rb}_i(1) = (x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1}).$$

The curly brackets indicate a shift of the grading: if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a $\mathbb{Z}$-graded bimodule and p an integer, then the $\mathbb{Z}$–graded bimodule $M\{p\}$ is defined by $M\{p\}_i = M_{i-p}$ for all $i \in \mathbb{Z}$. The maps br_i and rb_i are degree-preserving morphisms of graded R-bimodules.

2.2.2. Categorification of the braid groups. Following [Kho07] and [Rou06], to each braid generator $\sigma_i \in \mathcal{B}_n$ we assign the cochain complex $F(\sigma_i)$ of graded R–bimodules

$$F(\sigma_i) : 0 \longrightarrow R\{2\} \xrightarrow{\mathrm{rb}_i} B_i \longrightarrow 0, \tag{2.8}$$

where B_i sits in cohomological degree 0. To σ_i^{-1} we assign the cochain complex $F(\sigma_i^{-1})$ of graded R–bimodules

$$F(\sigma_i^{-1}) : 0 \longrightarrow B_i\{-2\} \xrightarrow{\mathrm{br}_i} R\{-2\} \longrightarrow 0, \tag{2.9}$$

where $B_i\{-2\}$ sits in cohomological degree 0. To the unit element $1 \in \mathcal{B}_n$ we assign the complex of graded R-bimodules

$$F(1) : 0 \longrightarrow R \longrightarrow 0, \tag{2.10}$$

where R sits in cohomological degree 0; the complex $F(1)$ is a unit for the tensor product of complexes so tensoring any complex of graded R-bimodules with $F(1)$ leaves the complex unchanged. Finally to any word

$\sigma = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$ where $\varepsilon_1, \dots, \varepsilon_k = \pm 1$, we assign the complex of graded R-bimodules $F(\sigma) = F(\sigma_{i_1}^{\varepsilon_1}) \otimes_R \cdots \otimes_R F(\sigma_{i_k}^{\varepsilon_k})$.

Rouquier proved the following result, which can be called a categorification of the braid group $\mathcal{B}_n$.

Theorem 2.2. [Rou06] *If ω and ω' are words representing the same element of $\mathcal{B}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of R–bimodules.*

2.3. Categorification of the virtual braid groups

Our aim is to extend Rouquier's categorification to the virtual braid groups $\mathcal{VB}_n$. The cochain complexes associated to the generators σ_i of $\mathcal{VB}_n$ coming from $\mathcal{B}_n$ will be the same as Rouquier's complexes above. We have to assign complexes to the generators ζ_i of $\mathcal{VB}_n$ corresponding to virtual crossings such that all these complexes satisfy the same relations as the generators of $\mathcal{VB}_n$ up to homotopy equivalence.

2.3.1. Twisted bimodules. In order to achieve this categorification, we consider the R-bimodule R_ω for each permutation $\omega \in S_n$. As a left R-module, R_ω is equal to R while the right action of $a \in R$ is the multiplication by $\omega(a)$. Note that $R_{\text{id}} = R$ as an R-bimodule. The following lemma is obvious.

Lemma 2.3. *For all $\omega, \omega' \in S_n$ the map $\psi : R_\omega \otimes_R R_{\omega'} \to R_{\omega\omega'}$ defined for all $a, b \in R$ by $\psi(a \otimes b) = a\omega(b)$ is an isomorphism of R-bimodules.*

The bimodules we will mostly use are the bimodules $R_i = R_{\tau_i}$. The reason why we consider these bimodules is that, by Lemma 2.3, they possess the following interesting property:

$$R_i \otimes_R R_i \cong R$$

for all $i = 1, \dots, n-1$.

Lemma 2.4. *For all permutations $\omega, \omega' \in S_m$ the R-bimodules $R_\omega \otimes_R B_{\omega'}$ and $B_{\omega\omega'\omega^{-1}} \otimes_R R_\omega$ are isomorphic.*

Proof. First note that there are natural isomorphisms of R-bimodules:

$$R_\omega \otimes_R B_{\omega'} \cong R_\omega \otimes_{R^{\omega'}} R \quad \text{and} \quad B_{\omega\omega'\omega^{-1}} \otimes_R R_\omega \cong R \otimes_{R^{\omega\omega'\omega^{-1}}} R_\omega.$$

Now consider the map $\psi : R_\omega \otimes_{R^{\omega'}} R \to R \otimes_{R^{\omega\omega'\omega^{-1}}} R_\omega$ defined for all $a, b \in R$ by

$$\psi(a \otimes b) = a \otimes \omega(b).$$

This map is well defined: for any $c \in R^{\omega'}$, we just have to check that $a \otimes cb$ and $a\omega(c) \otimes b$ have the same image under ψ. This is true because $c \in R^{\omega'}$ implies that $\omega(c) \in R^{\omega\omega'\omega^{-1}}$. Moreover the map ψ is obviously a morphism of R-bimodules.

Similarly, the map $\varphi : R \otimes_{R^{\omega\omega'\omega^{-1}}} R_\omega \to R_\omega \otimes_{R^{\omega'}} R$ defined for all $a, b \in R$ by

$$\varphi(a \otimes b) = a \otimes \omega^{-1}(b)$$

is a well defined morphism of R-bimodules as well.

Finally, ψ and φ are easily seen to be inverse of each other. □

Now let us assign a complex of graded R-bimodules to each generator of the virtual braid group. To the elements σ_i and σ_i^{-1} we assign the complexes $F(\sigma_i)$ and $F(\sigma_i^{-1})$ defined by (2.8) and (2.9). To the element ζ_i we assign the complex concentrated in degree 0:

$$F(\zeta_i) : 0 \longrightarrow R_i \longrightarrow 0. \tag{2.11}$$

Just as in Section 2.2.2 we assign to the unit element 1 of $\mathcal{VB}_n$ the complex $F(1)$ of (2.10), and to a virtual braid word we assign the tensor product over R of the complexes associated to the generators involved in the expression of the word.

Remark 2.5. *Consider $\omega = \zeta_{i_1} \dots \zeta_{i_k}$ a word in $\{\zeta_1, \dots, \zeta_{n-1}\}$ and let $\tilde{\omega} = \tau_{i_1} \dots \tau_{i_k}$ be the corresponding element of S_n. It follows from Lemma 2.3 that the complex $F(\omega)$ is isomorphic to $0 \longrightarrow R_{\tilde{\omega}} \longrightarrow 0$.*

2.3.2. Categorification of $\mathcal{VB}_n$. We now state our main result.

Theorem 2.6. *If ω and ω' are words representing the same element of $\mathcal{VB}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of R–bimodules.*

Proof. By definition of $\mathcal{VB}_n$ and in view of Theorem 2.2, it is enough to check that there are homotopy equivalences between the complexes associated to the braid words appearing in both sides of Relations (2.3)-(2.7). Actually we will prove a stronger result: these complexes are isomorphic.

Permutation group relations. Let $\omega = \omega'$ be a permutation group relation. Since the bimodules $R_{\tilde{\omega}}$ and $R_{\tilde{\omega'}}$ are equal, the complexes $F(\omega)$ and $F(\omega')$ are isomorphic in view of Remark 2.5.

Mixed relations. Let us first deal with Relation (2.6). We have to prove that for $|i - j| > 1$ the complexes $F(\zeta_j\sigma_i)$ and $F(\sigma_i\zeta_j)$ are isomorphic. We have

$$F(\zeta_j\sigma_i) : 0 \longrightarrow R_j \otimes_R R\{2\} \xrightarrow{\mathrm{id}\otimes \mathrm{rb}_i} R_j \otimes_R B_i \longrightarrow 0$$

and

$$F(\sigma_i\zeta_j) : 0 \longrightarrow R \otimes_R R_j\{2\} \xrightarrow{\mathrm{rb}_i \otimes \mathrm{id}} B_i \otimes_R R_j \longrightarrow 0.$$

First observe that $F(\zeta_j\sigma_i)$ is naturally isomorphic to the following complex of bimodules

$$0 \longrightarrow R_j\{2\} \xrightarrow{d} R_j \otimes_{R^{\tau_i}} R \longrightarrow 0$$

whose differential d sends each $a \in R_j\{2\}$ to

$$a\big(\tau_j(x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big) = a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big).$$

Similarly, $F(\sigma_i\zeta_j)$ is isomorphic to

$$0 \longrightarrow R_j\{2\} \xrightarrow{d'} R \otimes_{R^{\tau_i}} R_j \longrightarrow 0$$

whose differential d' sends each $a \in R_j\{2\}$ to

$$a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big).$$

Since the transpositions τ_i and τ_j commute, the proof of Lemma 2.4 provides us the isomorphism of R-bimodules $\psi : R_j \otimes_{R^{\tau_i}} R \to R \otimes_{R^{\tau_i}} R_j$. Using the invariance of $(x_i - x_{i+1})$ under the action of τ_j, we easily check that the following vertical maps and their inverse commute with the differentials.

$$\begin{tikzcd}
0 \arrow[r] & R_j\{2\} \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & R_j \otimes_{R^{\tau_i}} R \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & R_j\{2\} \arrow[r, "d'"] & R \otimes_{R^{\tau_i}} R_j \arrow[r] & 0
\end{tikzcd}$$

Thus the complexes $F(\zeta_j\sigma_i)$ and $F(\sigma_i\zeta_j)$ are isomorphic for all $|i-j| > 1$.

We finally deal with Relation (2.7). We have to show that the complexes $F(\zeta_{i+1}\zeta_i\sigma_{i+1})$ and $F(\sigma_i\zeta_{i+1}\zeta_i)$ are isomorphic for $i = 1, \cdots, n-2$. The complex $F(\zeta_{i+1}\zeta_i\sigma_{i+1})$ is equal to

$$0 \longrightarrow R_{i+1} \otimes_R R_i \otimes_R R\{2\} \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_{i+1}} R_{i+1} \otimes_R R_i \otimes_R B_{i+1} \longrightarrow 0$$

and the complex $F(\sigma_i\zeta_{i+1}\zeta_i)$ is equal to

$$0 \longrightarrow R \otimes_R R_{i+1} \otimes_R R_i\{2\} \xrightarrow{\mathrm{rb}_i\otimes\mathrm{id}\otimes\mathrm{id}} B_i \otimes_R R_{i+1} \otimes_R R_i \longrightarrow 0.$$

There exist a natural isomorphism between $F(\zeta_{i+1}\zeta_i\sigma_{i+1})$ and the complex

$$0 \longrightarrow R_{\tau_{i+1}\tau_i}\{2\} \xrightarrow{d} R_{\tau_{i+1}\tau_i} \otimes_{R^{\tau_{i+1}}} R \longrightarrow 0$$

whose differential d sends each $a \in R_{\tau_{i+1}\tau_i}\{2\}$ to

$$a\big(\tau_{i+1}\tau_i(x_{i+1} - x_{i+2}) \otimes 1 + 1 \otimes (x_{i+1} - x_{i+2})\big) = \\ a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_{i+1} - x_{i+2})\big).$$

Similarly, $F(\sigma_i\zeta_{i+1}\zeta_i)$ is isomorphic to

$$0 \longrightarrow R_{\tau_{i+1}\tau_i}\{2\} \xrightarrow{d'} R \otimes_{R^{\tau_i}} R_{\tau_{i+1}\tau_i} \longrightarrow 0$$

whose differential d' sends each $a \in R_{\tau_{i+1}\tau_i}\{2\}$ to

$$a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big).$$

Applying Lemma 2.4 and using the relation $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ and the involutivity of τ_{i+1} yields the isomorphism

$$\psi : R_{\tau_{i+1}\tau_i} \otimes_{R^{\tau_{i+1}}} R \to R \otimes_{R^{\tau_i}} R_{\tau_{i+1}\tau_i}$$

given for all $a \in R_{\tau_{i+1}\tau_i}$, $b \in R$ by

$$\psi(a \otimes b) = a \otimes \tau_{i+1}\tau_i(b).$$

Let us complete the proof by checking that the following vertical maps commute with the differentials (one can similarly check it for their inverse).

$$\begin{tikzcd}
0 \arrow[r] & R_{\tau_{i+1}\tau_i}\{2\} \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & R_{\tau_{i+1}\tau_i} \otimes_{R^{\tau_{i+1}}} R \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & R_{\tau_{i+1}\tau_i}\{2\} \arrow[r, "d'"] & R \otimes_{R^{\tau_i}} R_{\tau_{i+1}\tau_i} \arrow[r] & 0
\end{tikzcd}$$

For any element a in $R_{\tau_{i+1}\tau_i}\{2\}$ we compute both its image under $d' \circ \mathrm{id}$ and $\psi \circ d$. We obtain

$$d' \circ \mathrm{id}(a) = a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big)$$

and

$$\begin{aligned}
\psi \circ d(a) &= \psi\big(a(x_i - x_{i+1}) \otimes 1 + a \otimes (x_{i+1} - x_{i+2})\big)\\
&= a(x_i - x_{i+1}) \otimes 1 + a \otimes \tau_{i+1}\tau_i(x_{i+1} - x_{i+2})\\
&= a\big((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1})\big)
\end{aligned}$$

This shows that the complexes $F(\zeta_{i+1}\zeta_i\sigma_{i+1})$ and $F(\sigma_i\zeta_{i+1}\zeta_i)$ are isomorphic for all $i = 1, \ldots, n-2$. □

Remarks 2.7.

• *Let us call virtualisation moves the moves consisting in squeezing a classical crossing between two virtual crossings, as shown in Figure 2.7. We observe that the complexes $F(\zeta_i\sigma_i^{\varepsilon}\zeta_i)$ and $F(\sigma_i^{\varepsilon})$ are isomorphic for all $i = 1, \ldots, n-1$ and $\varepsilon \in \{-1, 1\}$. This is essentially due to the involutivity of τ_i, which implies (*cf. *Lemma 2.3 and Lemma 2.4) that the bimodules $R_i \otimes_R B_i \otimes_R R_i$ and B_i are isomomorphic. Thus our categorification of $\mathcal{VB}_n$ does not detect the virtualisation moves.*

• *Adding the relation $\zeta_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\zeta_{i+1}$ to the presentation of $\mathcal{VB}_n$, one obtains a presentation of the group of welded braids with n strands defined in [FRR97]. Noting that the only morphism between R_ω and $R_{\omega'}$ is the trivial*

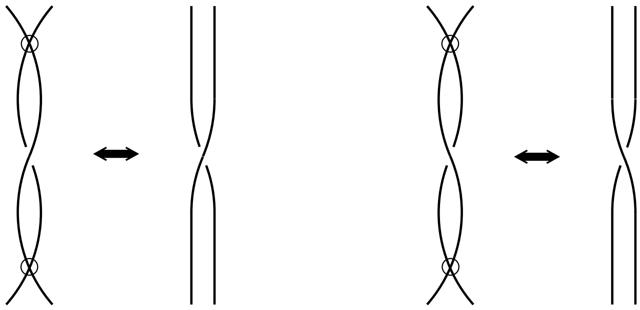


Figure 2.7: Virtualisation moves

one if $\omega \neq \omega'$*, one can check that the complexes* $F\left(\zeta_i \sigma_{i+1} \sigma_i\right)$

$$\begin{tikzcd}
 & R_i \otimes_R B_{i+1}\{2\} \arrow[dr, "\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_i"] & \\
R_i \otimes_R R\{4\} \arrow[ur, "\mathrm{id} \otimes \mathrm{rb}_{i+1}"] \arrow[dr, "-\mathrm{id} \otimes \mathrm{rb}_i"'] & & R_i \otimes_R B_{i+1} \otimes_R B_i \\
 & R_i \otimes_R B_i\{2\} \arrow[ur, "\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}"'] &
\end{tikzcd}$$

and $F\left(\sigma_{i+1} \sigma_i \rho_{i+1}\right)$

$$\begin{tikzcd}
 & B_{i+1} \otimes_R R_{i+1}\{2\} \arrow[dr, "\mathrm{id} \otimes \mathrm{rb}_i \otimes \mathrm{id}"] & \\
R \otimes_R R_{i+1}\{4\} \arrow[ur, "\mathrm{rb}_{i+1} \otimes \mathrm{id}"] \arrow[dr, "-\mathrm{rb}_i \otimes \mathrm{id}"'] & & B_{i+1} \otimes_R B_i \otimes_R R_{i+1} \\
 & B_i \otimes_R R_{i+1}\{2\} \arrow[ur, "\mathrm{rb}_{i+1} \otimes \mathrm{id} \otimes \mathrm{id}"'] &
\end{tikzcd}$$

are not equivalent up to homotopy.

Chapter 3

Virtual braid groups of type B and their categorification

Our aim in this Chapter is two-fold: we first define a virtual braid group of type B_n and next construct a categorification of this group.

Virtual knots and braids have been introduced by Kauffman in [Kau99]; they can be represented by planar diagrams that are like usual link or braid diagrams with one extra type of crossings, called virtual crossings. Such crossings appear for instance when one projects a generic link in a thickened surface onto a plane (see [KK00] or [Kup03]).

Out of the virtual braids with n strands one can form a group $\mathcal{VB}_n$, which generalizes the usual Artin braid group $\mathcal{B}_n$. The group $\mathcal{VB}_n$ has a presentation with $2(n-1)$ generators

$$\sigma_1, \ldots, \sigma_{n-1}, \zeta_1, \ldots, \zeta_{n-1}\,,$$

where $\sigma_1, \ldots, \sigma_{n-1}$ satisfy the usual braid relations, $\zeta_1, \ldots, \zeta_{n-1}$ satisfy the standard defining relations of the symmetric group S_n, and all generators together satisfy the following "mixed relations"

$$\sigma_i \zeta_j = \zeta_j \sigma_i, \quad \text{if } |i-j| > 1,$$

and

$$\sigma_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \sigma_{i+1}, \quad \text{if } 1 \leq i \leq n-2.$$

As is well known, the braid group $\mathcal{B}_n$ can be generalized in the framework of Coxeter groups. Recall that given any Coxeter system $(\mathcal{W}, \mathcal{S})$ one defines a generalized braid group $\mathcal{B}_{\mathcal{W}}$ by taking the same generators $s \in \mathcal{S}$ and the same relations as for the Coxeter group $\mathcal{W}$, except the relations $s^2 = 1$, which one drops. When $\mathcal{W}$ is the symmetric group S_n, i.e. of type A_{n-1} in the classification of Dynkin diagrams, then $\mathcal{B}_{\mathcal{W}} = \mathcal{B}_n$.

A natural question is: can one similarly attach a generalized virtual group $\mathcal{VB}_{\mathcal{W}}$ to any Coxeter system $(\mathcal{W}, \mathcal{S})$? The idea for defining such a group $\mathcal{VB}_{\mathcal{W}}$ would be, as in the type A case, to use two copies of the generating set $\mathcal{S}$, and to require that the first copy satisfies the relations defining $\mathcal{B}_{\mathcal{W}}$, the second one satisfies the relations defining $\mathcal{W}$, and all together satisfy some kind of mixed relations.

The problem is to come up with an appropriate and meaningful set of mixed relations. In this Chapter, we do not solve the problem in the general case, but we solve it for all Coxeter groups of type B. To this end we use a geometric description of the generalized braid group $\mathcal{B}_{\mathcal{W}}$ of type B due to tom Dieck [tD94]; this description is in terms of symmetric braid diagrams. We define a generalized virtual braid group $\mathcal{VB}_{B_n}$ of type B_n by considering symmetric virtual braid diagrams up to some natural equivalence.

In a second part we categorify each newly-defined group $\mathcal{VB}_{B_n}$. Here we mean categorification in the sense of Rouquier [Rou06]. More precisely, to any word w in the generators of $\mathcal{VB}_{B_n}$ we associate a bounded cochain complex $F(w)$ of Soergel bimodules such that if two words w and w' represent the same element of $\mathcal{VB}_{B_n}$, then the corresponding cochain complexes $F(w)$ and $F(w')$ are homotopy equivalent. Soergel bimodules have become important because they come up in the Khovanov-Rozansky link homology (see, e.g., [Kho07]). Our categorification extends Rouquier's categorification of generalized braid groups and the previous categorification of the virtual braid group $\mathcal{VB}_n$ of Chapter 2.

The Chapter is organized as follows. In Section 3.1 we recall the definition of the virtual braid groups $\mathcal{VB}_n$ and of an invariant of virtual braids due to Manturov, which we see as a homomorphism of $\mathcal{VB}_n$ into the automorphism group of a free group. In Section 3.2 we recall the definition of the generalized braid group of type B_n and tom Dieck's graphical description in terms of symmetric braid diagrams.

We propose a definition of a generalized virtual braid group of type B_n in Section 3.3. We show that each of its elements can be represented by a symmetric virtual braid diagram. Using Manturov's invariant, we show that certain relations do not hold in this group although they look natural.

Section 3.4 is devoted to the Soergel bimodules of type B_n. In Section 3.5 we associate a cochain complex of Soergel bimodules to each generator of our virtual braid group of type B_n, and we show that this leads to a categorification of this group in the sense of Rouquier.

3.1. Virtual braids

We first recall the definition of virtual braid groups and of Manturov's invariant for virtual braids.

3.1.1. The virtual braid groups. Let n be an integer ≥ 2. Following [Man04], [Ver01], we define the *virtual braid group* $\mathcal{VB}_n$ as the group generated by $\sigma_1, \dots, \sigma_{n-1}$ and $\zeta_1, \dots, \zeta_{n-1}$, and the following relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i-j| > 1, \tag{3.1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{if } 1 \leq i \leq n-2, \tag{3.2}$$

$$\zeta_i \zeta_j = \zeta_j \zeta_i, \quad \text{if } |i-j| > 1, \tag{3.3}$$

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1}, \quad \text{if } 1 \leq i \leq n-2, \tag{3.4}$$

$$\zeta_i^2 = 1, \quad \text{if } 1 \leq i \leq n-1, \tag{3.5}$$

$$\sigma_i \zeta_j = \zeta_j \sigma_i, \quad \text{if } |i-j| > 1, \tag{3.6}$$

$$\sigma_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \sigma_{i+1}, \quad \text{if } 1 \leq i \leq n-2. \tag{3.7}$$

Relations (3.1) and (3.2) are called *braid relations* and Relations (3.3)–(3.5) *permutation relations*. Relations (3.6) and (3.7) are called *mixed relations* because they involve both generators σ_i and ζ_i.

Elements of $\mathcal{VB}_n$ can be represented by *virtual braid diagrams with n strands*. Such a diagram is a planar braid diagram with *virtual crossings*, as in Figure 3.1, in addition to the usual positive and negative crossings.

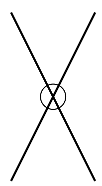


Figure 3.1: A virtual crossing

The generator σ_i can be represented by the usual braid diagram with a single positive crossing between the ith and the $i+1$st strand, whereas the generator ζ_i is represented by the virtual braid diagram with a single virtual crossing between the ith and the $i+1$st strand; see Figure 3.2.

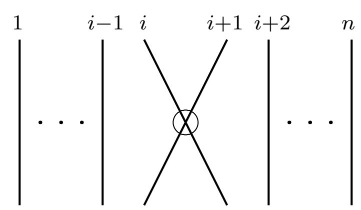


Figure 3.2: The braid diagram ζ_i

We use the following convention: if D (resp. D') is a virtual braid diagram representing an element β (resp. β') of $\mathcal{VB}_n$, then the product $\beta\beta'$ is represented by the diagram obtained by putting D' on top of D and gluing the lower endpoints of D' to the upper endpoints of D.

Relations (3.1), (3.3), (3.6) mean that we consider these planar diagrams up to planar isotopy preserving the crossings. Relation (3.2) illustrates the classical Reidemeister III move for ordinary braid diagrams.

Relations (3.4) and (3.5) mean that we consider the virtual braid diagrams up to the virtual Reidemeister II–III moves depicted in Figure 3.3. Relation (3.7) is a graphical transcription of the mixed Reidemeister move of Figure 3.4. (Note that this mixed move involves one positive crossing and two virtual ones.)

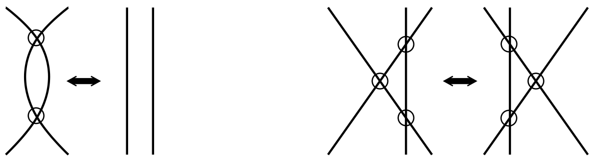


Figure 3.3: Virtual Reidemeister moves

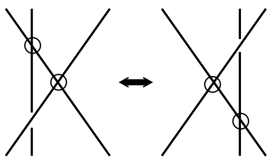


Figure 3.4: The mixed Reidemeister move

The braid group $\mathcal{B}_n$ is obtained by dropping the generators $\zeta_1, \dots, \zeta_{n-1}$ or equivalently the virtual crossings. There is a natural homomorphism $\mathcal{B}_n \to \mathcal{VB}_n$ obtained by considering each braid diagram as a virtual braid diagram without virtual crossings.

3.1.2. Manturov's invariant. Manturov [Man03] constructed an invariant of virtual braids, which he conjectured to be complete. Since we will be using it in Proposition 3.3, we have to recall its definition. We use the variant that appeared in the review [Izm04].

In this variant, Manturov's invariant can be seen as a group homomorphism $f : \mathcal{VB}_n \to \mathrm{Aut}(F_{n+1})$, where F_{n+1} is the free group on $n+1$ generators $a_1, \dots, a_n, t$. The homomorphism f is defined by the following formulas:

$$f(\sigma_i)(a_j) = \begin{cases} a_{j+1} & \text{if } j = i, \\ a_j^{-1} a_{j-1} a_j & \text{if } j = i+1, \\ a_j & \text{otherwise,} \end{cases}$$

$$f(\zeta_i)(a_j) = \begin{cases} t a_{j+1} t^{-1} & \text{if } j = i, \\ t^{-1} a_{j-1} t & \text{if } j = i+1, \\ a_j & \text{otherwise,} \end{cases}$$

and $f(\sigma_i)(t) = f(\zeta_i)(t) = t$ for all $i = 1, \ldots, n-1$.

Observe that this invariant specializes when $t = 1$ to an invariant constructed in [FRR97] for welded braids; see also [Ver01].

3.2. Braid groups of type B and symmetric braid diagrams

Let n be a positive integer. Consider the Coxeter group associated to the Dynkin diagram of type B_n; see [Hum90].

We denote the associated generalized braid group by $\mathcal{B}_{B_n}$. It has a presentation with n generators $s_0, s_1, \ldots, s_{n-1}$, and three families of relations

$$s_i s_j = s_j s_i, \quad \text{if } |i-j| > 1, \tag{3.8}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad \text{if } 1 \leq i \leq n-2, \tag{3.9}$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0. \tag{3.10}$$

In [tD94] tom Dieck gave a geometrical description of $\mathcal{B}_{B_n}$ in terms of symmetric braid diagrams with $2n$ strands.

Fix a vertical line $\{0\} \times \mathbb{R}$ in the plane $\mathbb{R}^2$. Consider the reflection in this vertical line that preserves the sign of the crossings. (Observe that if one identifies the plane $\mathbb{R}^2$ with $\{0\} \times \mathbb{R}^2 \subset \mathbb{R}^3$, this reflection can be viewed as a rotation of angle π around the vertical axis $\{0\}^2 \times \mathbb{R}$.) A *symmetric braid diagram with* $2n$ *strands* is a planar braid diagram with $2n$ strands that is symmetric under this reflection. To make things precise, we assume that the upper (resp. lower) endpoints of each symmetric braid diagram with $2n$ strands are the points $\{-n \ldots, -1, 1, \ldots, n\} \times \{1\}$ (resp. the points $\{-n \ldots, -1, 1, \ldots, n\} \times \{0\}$). We consider the symmetric braid diagrams with $2n$ strands up to planar isotopy preserving the crossings, and up to the classical Reidemeister II–III moves, as depicted in Figure 3.5. We stress the fact that neither the isotopies, nor the Reidemeister moves have to be preserved under the reflection.

Figure 3.5: Classical Reidemeister II–III moves

The equivalence classes of symmetric braid diagrams with $2n$ strands form a group, which tom Dieck [tD94] proved to be isomorphic to the generalized braid group $\mathcal{B}_{B_n}$ of type B_n. In this isomorphism, the generator s_0 is represented by the symmetric braid diagram with one positive crossing, as in Figure 3.6, and each remaining generator $s_1, \ldots, s_{n-1}$ by the symmetric braid diagram with two symmetric positive crossings, as in Figure 3.7.

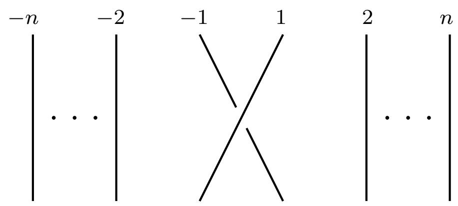


Figure 3.6: The symmetric braid diagram s_0

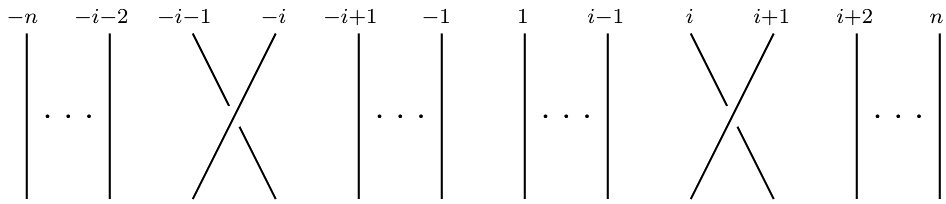


Figure 3.7: The symmetric braid diagram s_i $(i > 0)$

In terms of symmetric braid diagrams, we see that Relation (3.8) holds because the symmetric braid diagrams corresponding to each term of the relation are isotopic. Similarly, the equality in Relation (3.9) can be proved diagramatically using Reidemeister III move. As for Relation (3.10), it can be proved as shown in Figure 3.8. (Note that four Reidemeister III moves have been used.)

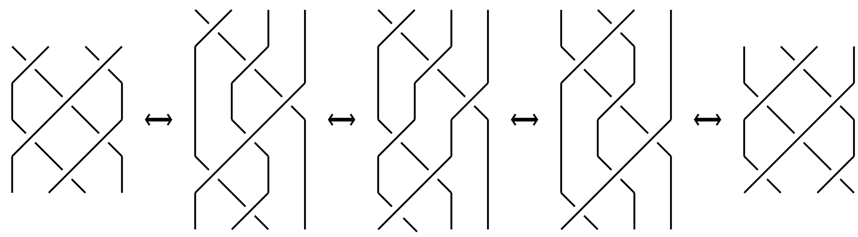


Figure 3.8: Proving Relation (3.10)

From now on, we will identify each generator s_i of $\mathcal{B}_{B_n}$ with the corresponding symmetric braid.

Forgetting the symmetry condition yields an embedding of $\mathcal{B}_{B_n}$ into the group $\mathcal{B}_{2n}$ of usual braids with $2n$ strands. Before we give a precise formula for this embedding, let us shift the indices of the $2n-1$ generators σ_i of $\mathcal{B}_{2n}$ by $-n$; in this way the indexing set for the generators becomes the set $\{-n+1, \ldots, n-1\}$. The re-indexed generators satisfy the same braid relations (3.1) and (3.2). We then define an embedding of $\mathcal{B}_{B_n}$ into the group $\mathcal{B}_{2n}$ by sending s_0 to σ_0, and each remaining s_i to $\sigma_{-i}\sigma_i = \sigma_i\sigma_{-i}$

$(i = 1, \dots, n-1)$.

Realizing $\mathcal{B}_{B_n}$ as a subgroup of $\mathcal{B}_{2n}$ allows us to translate the sequence of Reidemeister moves of Figure 3.8 into the following sequence of equalities in $\mathcal{B}_{2n}$:

$$\begin{aligned} s_0 s_1 s_0 s_1 &= \sigma_0 \sigma_{-1} \underbrace{\sigma_1 \sigma_0 \sigma_1} \sigma_{-1} = \underbrace{\sigma_0 \sigma_{-1} \sigma_0} \sigma_1 \sigma_0 \sigma_{-1} \\ &= \sigma_{-1} \sigma_0 \underbrace{\sigma_{-1} \sigma_1} \sigma_0 \sigma_{-1} = \sigma_{-1} \sigma_0 \sigma_1 \underbrace{\sigma_{-1} \sigma_0 \sigma_{-1}} \\ &= \sigma_{-1} \underbrace{\sigma_0 \sigma_1 \sigma_0} \sigma_{-1} \sigma_0 = \sigma_{-1} \sigma_1 \sigma_0 \sigma_1 \sigma_{-1} \sigma_0 \\ &= s_1 s_0 s_1 s_0 \,. \end{aligned}$$

We have put braces under the subwords to which we have applied the braid relations.

3.3. Symmetric virtual braids

We now define a group $\mathcal{VB}_{B_n}$, which will be our generalized virtual braid group of type B_n.

Definition 3.1. *The group $\mathcal{VB}_{B_n}$ has the following presentation: it is generated by $2n$ generators $s_0, s_1, \dots, s_{n-1}$ and $z_0, z_1, \dots, z_{n-1}$, where $s_0, s_1, \dots, s_{n-1}$ satisfy Relations* (3.8)*,* (3.9)*,* (3.10)*, $z_1, \dots, z_{n-1}$ satisfy the relations*

$$z_i z_j = z_j z_i, \quad \text{if } |i-j| > 1, \tag{3.11}$$

$$z_i z_{i+1} z_i = z_{i+1} z_i z_{i+1}, \quad \text{if } 1 \le i \le n-2, \tag{3.12}$$

$$z_0 z_1 z_0 z_1 = z_1 z_0 z_1 z_0, \tag{3.13}$$

$$z_i^2 = 1, \quad \text{if } 0 \le i \le n-1, \tag{3.14}$$

and all together satisfy the "mixed relations"

$$s_i z_j = z_j s_i, \quad \text{if } |i-j| > 1, \tag{3.15}$$

$$s_i z_{i+1} z_i = z_{i+1} z_i s_{i+1}, \quad \text{if } 1 \le i \le n-2, \tag{3.16}$$

$$s_0 z_1 z_0 z_1 = z_1 z_0 z_1 s_0, \tag{3.17}$$

$$z_0 s_1 z_0 z_1 = z_1 z_0 s_1 z_0, \tag{3.18}$$

$$s_0 z_1 s_0 z_1 = z_1 s_0 z_1 s_0. \tag{3.19}$$

By analogy with tom Dieck's graphical description, we represent elements of $\mathcal{VB}_{B_n}$ by *symmetric virtual braid diagrams with $2n$ strands.* These are planar virtual braid diagrams with $2n$ strands, as defined in Section 3.1.1, that are symmetric under the reflection in the vertical line $\{0\} \times \mathbb{R}$. The reflection is supposed to preserve the virtual crossings as well as the positive (resp. the negative) crossings. We consider the symmetric virtual braid diagrams with $2n$ strands up to planar isotopy preserving the crossings, up to the classical Reidemeister II–III moves of Figure 3.5, up to the virtual

Reidemeister II–III moves of Figure 3.3, and up to the mixed Reidemeister move of Figure 3.4.

We represent the generators $s_0, s_1, \dots, s_{n-1}$ by the symmetric braid diagrams of Section 3.2. The generator z_0 is represented by the symmetric virtual braid diagram with a single virtual crossing, as in Figure 3.9. Each remaining generator $z_1, \dots, z_{n-1}$ is represented by a symmetric virtual braid diagram with two symmetric virtual crossings as in Figure 3.10.

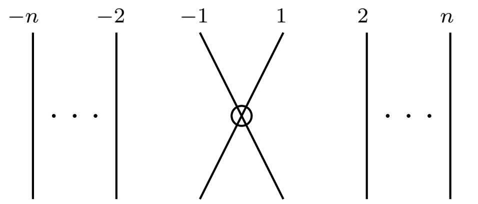


Figure 3.9: The symmetric virtual braid diagram z_0

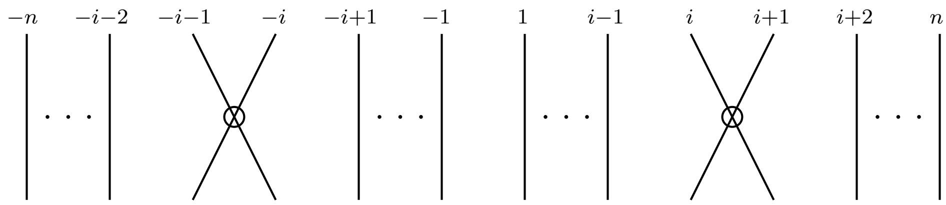


Figure 3.10: The symmetric virtual braid diagram z_i $(i > 0)$

If we consider the generators of $\mathcal{VB}_{B_n}$ as virtual braid diagrams with $2n$ strands, then we obtain the identifications

$$s_0 = \sigma_0, \qquad z_0 = \zeta_0, \tag{3.20}$$

and

$$s_i = \sigma_{-i}\sigma_i = \sigma_i\sigma_{-i}, \qquad z_i = \zeta_{-i}\zeta_i = \zeta_i\zeta_{-i} \tag{3.21}$$

for $1 \leq i \leq n-2$. Here again as in Section 3.2, we have shifted the index of the generators of the virtual braid group $\mathcal{VB}_{2n}$ by $-n$.

Proposition 3.2. *The identifications* (3.20) *and* (3.21) *induce a group homomorphism* $j : \mathcal{VB}_{B_n} \to \mathcal{VB}_{2n}$.

This shows that the relations defining the virtual braid group $\mathcal{VB}_{B_n}$ of type B are adequate. We do not claim that $j : \mathcal{VB}_{B_n} \to \mathcal{VB}_{2n}$ is an monomorphism. Should this hold, then we could claim that the defining relations are sufficient to define $\mathcal{VB}_{B_n}$ as a subgroup of $\mathcal{VB}_{2n}$.

Proof. Using the relations defining $\mathcal{VB}_{2n}$ (see Section 3.1.1), we now check that $j(s_0), j(s_1), \ldots, j(s_{n-1})$, $j(z_0), j(z_1), \ldots$, $j(z_{n-1})$ satisfy the defining relations of $\mathcal{VB}_{B_n}$.

The fact that $j(s_0), j(s_1), \ldots, j(s_{n-1})$ satisfy Relations (3.8), (3.9), (3.10) has already been dealt with in Section 3.2. One checks in a similar way that $j(z_0), j(z_1), \ldots, j(z_{n-1})$ satisfy Relations (3.11) and (3.12). Relation (3.14) is also easy to verify. One can figure out a graphical proof of Relation (3.13) by replacing any crossing in Figure 3.11 by a virtual crossing.

Let us now deal with the mixed relations. Relation (3.15) (resp. Relation (3.16)) is an immediate consequence of (3.6) (resp. of (3.7)) and of the identification above.

A graphical proof of Relation (3.17) is given in Figure 3.11, where we use the virtual Reidemeister III move of Figure 3.3 twice and the mixed Reidemeister move of Figure 3.4 twice.

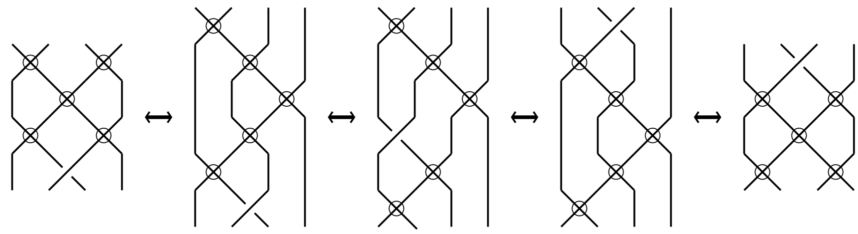


Figure 3.11: A graphical proof of Relation (3.17)

We similarly obtain Relation (3.18) using four mixed Reidemeister moves, as in Figure 3.12.

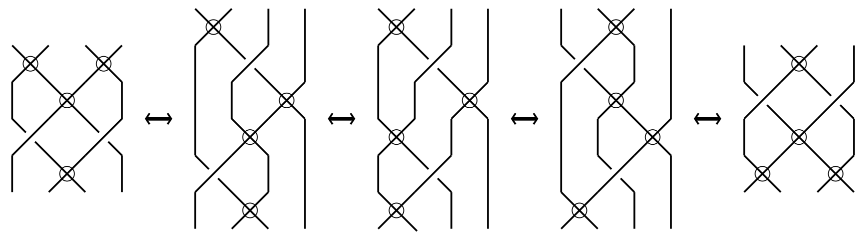


Figure 3.12: A graphical proof of Relation (3.18)

Finally a graphical proof of Relation (3.19) is given in Figure 3.13. □

We record the following; it will come up in Proposition 3.7.

Proposition 3.3. *(a) In $\mathcal{VB}_n$ we have*

$$\sigma_i\zeta_i \neq \zeta_i\sigma_i \quad \text{and} \quad \zeta_i\sigma_{i+1}\sigma_i \neq \sigma_{i+1}\sigma_i\zeta_{i+1}.$$

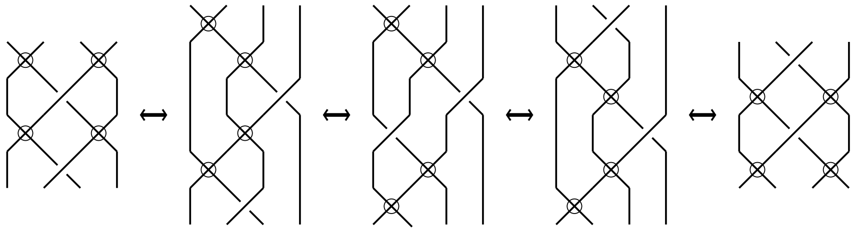


Figure 3.13: A graphical proof of Relation (3.19)

(b) In $\mathcal{VB}_{B_n}$ we have

$$z_0 s_1 z_0 s_1 \neq s_1 z_0 s_1 z_0.$$

Proof. (a) For each relation we will use Manturov's invariant f (see Section 3.1.2) to show that both sides of the relation do not have the same invariant. This will imply the desired inequality in $\mathcal{VB}_n$. For the leftmost relation, we have

$$f\left(\zeta_i\sigma_i\right)\left(a_i\right) = t^{-1}a_i t \quad \text{and} \quad f\left(\sigma_i\zeta_i\right)\left(a_i\right) = t a_{i+1}^{-1} a_i a_{i+1} t^{-1},$$

which are different.

For the righmost relation, we have

$$f(\zeta_i\sigma_{i+1}\sigma_i) : \begin{cases} t & \mapsto & t, \\ a_i & \mapsto & a_{i+2}, \\ a_{i+1} & \mapsto & a_{i+2}^{-1} t a_{i+1} t^{-1} a_{i+2}, \\ a_{i+2} & \mapsto & a_{i+2}^{-1} t^{-1} a_i t a_{i+2}, \\ a_j & \mapsto & a_j \qquad\qquad \text{if } j \neq i, i+1, i+2 \end{cases}$$

and

$$f(\sigma_{i+1}\sigma_i\zeta_{i+1}) : \begin{cases} t & \mapsto & t, \\ a_i & \mapsto & a_{i+2}, \\ a_{i+1} & \mapsto & t a_{i+2}^{-1} a_{i+1} a_{i+2} t^{-1}, \\ a_{i+2} & \mapsto & t^{-1} a_{i+2}^{-1} a_i a_{i+2} t, \\ a_j & \mapsto & a_j \qquad\qquad \text{if } j \neq i, i+1, i+2. \end{cases}$$

So $f(\zeta_i\sigma_{i+1}\sigma_i)$ and $f(\sigma_{i+1}\sigma_i\zeta_{i+1})$ are not equal in $\mathrm{Aut}\,(F_{n+1})$.

(b) The image under the homomorphism j of the leftmost relation is

$$\zeta_0\sigma_{-1}\sigma_1\zeta_0\sigma_{-1}\sigma_1 = \sigma_{-1}\sigma_1\zeta_0\sigma_{-1}\sigma_1\zeta_0 \in \mathcal{VB}_{2n}\,.$$

We again apply Manturov's invariant. But in the present context, this invariant is a group homomorphism $f : \mathcal{VB}_{2n} \to \mathrm{Aut}(F_{2n+1})$, where F_{2n+1} is the free group on the generators $a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n, t$; it is given by the same formulas as in Section 3.1.2.

A simple computation shows that

$$f(\zeta_0\sigma_{-1}\sigma_1\zeta_0\sigma_{-1}\sigma_1)(a_2) = a_2^{-1}t^{-1}a_0^{-1}ta_2a_1^{-1}t^{-1}a_{-1}ta_1a_2^{-1}t^{-1}a_0ta_2$$

and

$$f(\sigma_{-1}\sigma_1\zeta_0\sigma_{-1}\sigma_1\zeta_0)(a_2) = a_2^{-1}a_1^{-1}a_2t^{-1}a_0^{-1}a_{-1}a_0ta_2^{-1}a_1a_2\,,$$

which are different. □

Remarks 3.4.

• *The inequalities of Proposition 3.3 (a) imply that, in $\mathcal{VB}_{B_n}$ we have*

$$s_iz_i \neq z_is_i \quad \textit{if } 0 \leq i \leq n-1$$

and

$$z_is_{i+1}s_i \neq s_{i+1}s_iz_{i+1} \quad \textit{if } 1 \leq i \leq n-1.$$

• *The second inequality of Proposition 3.3 (a) means that the mixed Reidemeister move of Figure 3.14, involving one virtual crossing and two positive crossings, is not allowed. Specializing at $t = 1$ one observes that $f(\zeta_i\sigma_{i+1}\sigma_i)$ and $f(\sigma_{i+1}\sigma_i\zeta_{i+1})$ become equal in* Aut (F_n). *Adding the relation $\zeta_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\zeta_{i+1}$ to the presentation of $\mathcal{VB}_n$, one precisely obtain a presentation of the group of welded braids with n strands defined in [FRR97].*

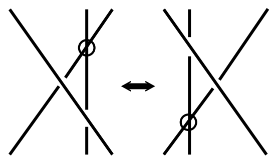


Figure 3.14: A forbidden Reidemeister move

3.4. Soergel bimodules

In this section we consider the Soergel bimodules of type B_n.

3.4.1. Generalities. Soergel bimodules have been introduced by Soergel [Soe92], [Soe95], [Soe07] in the general context of Coxeter groups.

Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system with generating set $\mathcal{S} = \{s_0, \dots, s_{n-1}\}$ of cardinality n. As usual, we denote the order of st, where $s, t \in \mathcal{S}$, by $m(s,t)$ and we set

$$\langle s_i, s_j\rangle = -\cos\frac{\pi}{m(s_i,s_j)} \in \mathbb{R}\,.$$

Let $R = \mathbb{R}[X_0, \dots, X_{n-1}]$ be a real polynomial algebra in n indeterminates $X_0, \dots, X_{n-1}$. It is a graded algebra.

For $i = 0, 1, \ldots, n-1$ we define a graded algebra automorphism α_i of R by

$$\alpha_i(X_j) = X_j - 2\langle s_i, s_j\rangle X_i$$

for all $j = 0, 1, \ldots, n-1$. It is easy to check that α_i is an involution and that there is a unique action of $\mathcal{W}$ on R by algebra automorphisms such that

$$s_i(P) = \alpha_i(P)$$

for all $i = 0, 1, \ldots, n-1$ and $P \in R$.

For any $w \in \mathcal{W}$ we consider the following objets:

(a) the subalgebra R^w of elements of R fixed by w;

(b) the R-bimodule $B_w = R \otimes_{R^w} R$;

(c) the R-bimodule R_w, which coincides with R as a left R-module whereas $a \in R$ acts on R_w on the right by multiplication by $w(a)$.

In the sequel we will systematically identify any bimodule $R_w \otimes_R B_{w'}$ (resp. $B_w \otimes_R B_{w'}$) with $R_w \otimes_{R^{w'}} R$ (resp. $R \otimes_{R^w} R \otimes_{R^{w'}} R$) since

$$R_w \otimes_R B_{w'} = R_w \otimes_R R \otimes_{R^{w'}} R \cong R_w \otimes_{R^{w'}} R$$

and

$$B_w \otimes_R B_{w'} = R \otimes_{R^w} R \otimes_R R \otimes_{R^{w'}} R \cong R \otimes_{R^w} R \otimes_{R^{w'}} R.$$

The following lemma is straightforward. It will be used repeatedly in Section 3.5.2.

Lemma 3.5. *For all $w, w' \in \mathcal{W}$, there are isomorphisms of R-bimodules*

$$R_w \otimes_R R_{w'} \xrightarrow{\cong} R_{ww'}$$

sending $a \otimes b$ to $aw(b)$, and

$$R_w \otimes_R B_{w'} \xrightarrow{\cong} B_{ww'w^{-1}} \otimes_R R_w$$

sending $a \otimes b$ to $a \otimes w(b)$ $(a, b \in R)$.

3.4.2. The type B_n. When the Coxeter system $(\mathcal{W}, \mathcal{S})$ is of type B_n, then

$$\begin{array}{rclcll}
\langle s_i, s_i\rangle & = & -\cos(\pi) & = & 1 & \text{if } i = 0, \ldots, n-1, \\
\langle s_i, s_j\rangle & = & -\cos(\pi/2) & = & 0 & \text{if } |i-j| > 1, \\
\langle s_i, s_{i+1}\rangle & = & -\cos(\pi/3) & = & -1/2 & \text{if } i = 1, \ldots, n-2, \\
\langle s_0, s_1\rangle & = & -\cos(\pi/4) & = & -\sqrt{2}/2. &
\end{array}$$

See [Hum90].

It follows that the automorphisms $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ act on the polynomial algebra $R = \mathbb{R}[X_0, \dots, X_{n-1}]$ by the formulas

$$\alpha_0 : \begin{cases} X_0 & \mapsto & -X_0, \\ X_1 & \mapsto & X_1 + \sqrt{2}\, X_0, \\ X_i & \mapsto & X_i & \text{if } i > 1, \end{cases}$$

$$\alpha_1 : \begin{cases} X_0 & \mapsto & X_0 + \sqrt{2}\, X_1, \\ X_1 & \mapsto & -X_1, \\ X_2 & \mapsto & X_1 + X_2, \\ X_i & \mapsto & X_i & \text{if } i > 2. \end{cases}$$

For $1 < j < n$, we have

$$\alpha_j : \begin{cases} X_j & \mapsto & -X_j, \\ X_i & \mapsto & X_i + X_j & \text{if } i = j-1, j+1, \\ X_i & \mapsto & X_i & \text{if } i \neq j-1, j, j+1. \end{cases}$$

For future use, we shall need an explicit set of algebraically independent homogeneous generators for certain subalgebras R^w of invariants. Noether's Bound Theorem gives an upper bound for the cardinality of such a set and an upper bound for the degrees of its elements (see [Neu07, Chap. 6] for a precise statement). More precisely, the degree of the generators is at most equal to the order d of w, whereas the cardinality of the generating set is bounded above by $\binom{n+d}{d}$, which for us is much too high. Therefore, we combine Noether's theorem and Chevalley's theorem; the latter gives exactly the cardinality of an algebraically independent generating set of R^w in the case of a reflection group acting on a polynomial algebra. In our case this cardinality is equal to n, which is the number of variables we consider (see [Hum90, Chap. 3] or [TY05, Chap. 31] for more details).

In the special case of type B_n, we need to find such a generating set when $w = s_0$, s_1, $s_0s_1s_0$, or $s_1s_0s_1$. Since these four elements are of order 2, the degree of the generators will not exceed 2.

Let us start with the subalgebra R^{s_0}. Using Chevalley's theorem, we know that R^{s_0} is generated by n algebraically independent homogeneous elements of degree at most 2. Among these generators, $n-2$ are obvious, these are the X_i for $i \geq 2$. Let us now investigate the case of other potential generators of homogeneous degree equal to 1. We search a condition on $(a, b) \in \mathbb{R}^2$ for an element $aX_0 + bX_1$ to be invariant under σ_0. But

$$\sigma_0 (aX_0 + bX_1) = -aX_0 + b\left(X_1 + \sqrt{2}X_0\right) = \left(\sqrt{2}b - a\right) X_0 + bX_1$$

so the pair (a, b) has to satisfy the system

$$\begin{cases} a & = & \sqrt{2}b - a \\ b & = & b \end{cases}$$

and we get the last generator of homogeneous degree equal to 1, namely $\sqrt{2}X_0 + 2X_1$. Finally the missing generator of R^{s_0} will be of homogeneous degree equal to 2, so it will take the form $aX_0^2+bX_0X_1+cX_1^2+\sum_{i=2}^{n-1} d_iX_0X_i+e_iX_1X_i$ where $a,b,c,d_i,e_i \in \mathbb{R}$. Since

$$\sigma_0\left(aX_0^2+bX_0X_1+cX_1^2+\sum_{i=2}^{n-1}d_iX_0X_i+e_iX_1X_i\right)=\left(a-\sqrt{2}b+2c\right)X_0^2$$
$$+\left(2\sqrt{2}c-b\right)X_0X_1+cX_1^2+\sum_{i=2}^{n-1}\left(\left(-d_i+\sqrt{2}e_i\right)X_0X_i+e_iX_1x_i\right),$$

such an element has to satisfy the following system to be invariant under σ_0

$$\begin{cases} a &= a-\sqrt{2}b+2c \\ b &= 2\sqrt{2}c-b \\ c &= c \\ d_i &= -d_i+\sqrt{2}e_i \\ e_i &= e_i \end{cases}$$

So the expression of an element of homogeneous degree 2 invariant under σ_0 is

$$aX_0^2+cX_1\left(\sqrt{2}X_0+X_1\right)+\sum_{i=2}^{n-1}d_iX_i\left(X_0+\sqrt{2}X_1\right).$$

But any element $X_i\left(X_0+\sqrt{2}X_1\right)$ is the product of two generators of degree 1 so it is not algebraically independent from the generators of R^{s_0} already expressed and does not provide us the missing generator. Moreover

$$2X_1\left(\sqrt{2}X_0+X_1\right)+X_0^2=\frac{1}{2}\left(\sqrt{2}X_0+2X_1\right)^2$$

so we can conclude that

$$R^{s_0}=\mathbb{R}[\sqrt{2}X_0+2X_1,X_0^2,X_2,\ldots,X_{n-1}],$$

where the n polynomials between brackets are algebraically independent.

Now let us turn to $R^{s_1s_0s_1}$. The action of $s_1s_0s_1$ on R is given by

$$\begin{cases} X_1 \mapsto -X_1-\sqrt{2}X_0\,, & \\ X_2 \mapsto \sqrt{2}X_0+2X_1+X_2\,, & \\ X_i \mapsto X_i & \text{if } i>2 \text{ or } i=0\,. \end{cases}$$

Proceeding as above, we conclude that

$$R^{s_1s_0s_1}=\mathbb{R}[X_0,X_1+X_2,X_1(\sqrt{2}X_0+X_1),X_3,\ldots,X_{n-1}].$$

Finally for R^{s_1} and $R^{s_0s_1s_0}$, we obtain

$$R^{s_1}=\mathbb{R}[2X_0+\sqrt{2}X_1,X_1+2X_2,X_1^2,X_3,\ldots,X_{n-1}]$$

and

$$R^{s_0s_1s_0}=\mathbb{R}[X_0+\sqrt{2}X_2,X_1,X_0(X_0+\sqrt{2}X_1),X_3,\ldots,X_{n-1}].$$

3.5. Categorification of the virtual braid group of type B_n

In this section we categorify our virtual braid group of type B_n.

3.5.1. Rouquier's categorification of generalized braid groups. We explain Rouquier's construction following [Rou06]. It works for any generalized braid group $\mathcal{B}_{\mathcal{W}}$ associated to a finite Coxeter group $\mathcal{W}$.

We introduce a grading on the algebra R by setting $\deg(X_k) = 2$ for all $k = 0, \ldots, n-1$. It induces a grading on Soergel bimodules. We will indicate a shift of the grading by curly brackets: if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a $\mathbb{Z}$–graded bimodule and p an integer, then the $\mathbb{Z}$–graded bimodule $M\{p\}$ is defined by $M\{p\}_i = M_{i-p}$ for all $i \in \mathbb{Z}$.

To each braid generator $s_i \in \mathcal{B}_{\mathcal{W}}$ we assign the cochain complex $F(s_i)$ of graded R-bimodules

$$F(s_i) : 0 \longrightarrow R\{2\} \xrightarrow{\mathrm{rb}_i} B_{s_i} \longrightarrow 0\,, \tag{3.22}$$

where B_{s_i} sits in cohomological degree 0. The degree-preserving R-bimodule morphism rb_i sends any $a \in R$ to $aX_i \otimes 1 + a \otimes X_i \in B_{s_i}$.

To the inverse s_i^{-1} of s_i we assign the cochain complex $F(s_i^{-1})$ of graded R-bimodules

$$F(s_i^{-1}) : 0 \longrightarrow B_{s_i}\{-2\} \xrightarrow{\mathrm{br}_i} R\{-2\} \longrightarrow 0\,, \tag{3.23}$$

where $B_{s_i}\{-2\}$ sits in cohomological degree 0 and the degree-preserving R-bimodule morphism br_i is given by multiplication; in other words, it sends $a \otimes b \in B_{s_i}$ to $ab \in R$.

To the unit element $1 \in \mathcal{B}_{\mathcal{W}}$ we assign the trivial complex of graded R-bimodules

$$F(1) : 0 \longrightarrow R \longrightarrow 0\,, \tag{3.24}$$

where R sits in cohomological degree 0. The complex $F(1)$ is obviously a unit for the tensor product of complexes.

Finally to any word $w = s_{i_1}^{\varepsilon_1} \ldots s_{i_k}^{\varepsilon_k}$ where $\varepsilon_1, \ldots, \varepsilon_k = \pm 1$, we assign the complex of graded R-bimodules $F(w) = F(s_{i_1}^{\varepsilon_1}) \otimes_R \cdots \otimes_R F(s_{i_k}^{\varepsilon_k})$.

In this context, Rouquier established that if w and w' are words representing the same element of $\mathcal{B}_{\mathcal{W}}$, then $F(w)$ and $F(w')$ are homotopy equivalent complexes of R-bimodules. This statement is what we mean by Rouquier's categorification of generalized braid groups.

3.5.2. Categorification of $\mathcal{VB}_{B_n}$. Our aim is to extend Rouquier's categorification to the virtual braid group $\mathcal{VB}_{B_n}$ of type B_n defined in Section 3.3.

The cochain complexes associated to the generators $s_i^{\pm 1}$ of $\mathcal{VB}_{B_n}$ are the ones defined by (3.22) and (3.23).

To the generator z_i we assign the complex of graded R-bimodules concentrated in degree 0

$$F(z_i) : 0 \longrightarrow R_{s_i} \longrightarrow 0\,. \tag{3.25}$$

To any word w in the generators $s_i^{\pm 1}$ and z_i of $\mathcal{VB}_{B_n}$ we assign the tensor product over R of the complexes associated to the generators involved in the expression of w.

We now state our main result.

Theorem 3.6. *If w and w' are words representing the same element of $\mathcal{VB}_{B_n}$, then $F(w)$ and $F(w')$ are homotopy equivalent complexes of R-bimodules.*

Proof. It is enough to check that there are homotopy equivalences between the complexes associated to the pair of words appearing in each defining relation of $\mathcal{VB}_{B_n}$.

The checking of this for Relations (3.8), (3.9), (3.10) is a consequence of Rouquier's work for generalized braid groups.

Relations (3.11)–(3.14) only involve the virtual generators z_i. In view of the simple form of the complex $F(z_i)$, the isomorphims of the corresponding complexes directly follow from the first isomorphism in Lemma 3.5.

Similarly, the isomorphism of complexes associated to Relations (3.15) and (3.16) can be constructed as performed in Chapter 2 in the case of the categorification of the virtual braid group of type A.

We are left with the mixed relations (3.17)–(3.19). Let us first deal with Relation (3.17). We have to prove that the complexes $F(s_0z_1z_0z_1)$ and $F(z_1z_0z_1s_0)$ are isomorphic. A simple computation shows that the complex $F(z_1z_0z_1s_0)$ is isomorphic to the following:

$$0 \longrightarrow R_{s_1s_0s_1}\{2\} \xrightarrow{d} R_{s_1s_0s_1} \otimes_R B_{s_0} \longrightarrow 0\,,$$

where

$$d(a) = a\,(X_0 \otimes 1 + 1 \otimes X_0)$$

for all $a \in R$. The complex $F(s_0z_1z_0z_1)$ is isomorphic to

$$0 \longrightarrow R_{s_1s_0s_1}\{2\} \xrightarrow{d'} B_{s_0} \otimes_R R_{s_1s_0s_1} \longrightarrow 0\,,$$

where

$$d'(a) = a\,(\alpha_1\alpha_0\alpha_1\,(X_0) \otimes 1 + 1 \otimes X_0) = a\,(X_0 \otimes 1 + 1 \otimes X_0)\,.$$

for all $a \in R$. Lemma 3.5 provides an isomorphism

$$\mu : R_{s_1s_0s_1} \otimes_R B_{s_0} \xrightarrow{\cong} B_{s_1s_0s_1s_0s_1s_0s_1} \otimes_R R_{s_1s_0s_1}\,;$$

it is given by $\mu(a \otimes b) = a \otimes \alpha_1\alpha_0\alpha_1(b)$, where $a, b \in R$. Now by Relation (3.10) the latter bimodule is equal to $B_{s_0} \otimes_R R_{s_1s_0s_1}$. This allows us to build the following isomorphism of complexes between $F(z_1z_0z_1s_0)$ and $F(s_0z_1z_0z_1)$:

$$\begin{tikzcd}
0 \arrow[r] & R_{s_1s_0s_1}\{2\} \arrow[r, "d"] \arrow[d, "\mathrm{id}"'] & R_{s_1s_0s_1} \otimes_R B_{s_0} \arrow[r] \arrow[d, "\mu"'] & 0 \\
0 \arrow[r] & R_{s_1s_0s_1}\{2\} \arrow[r, "d'"] & B_{s_0} \otimes_R R_{s_1s_0s_1} \arrow[r] & 0
\end{tikzcd}$$

The vertical maps commute with the differentials. Indeed, for $a \in R_{s_1 s_0 s_1}$,

$$\begin{aligned} \mu \circ d(a) &= \mu\left(a\left(X_0 \otimes 1 + 1 \otimes X_0\right)\right) \\ &= a\left(X_0 \otimes 1 + 1 \otimes \alpha_1 \alpha_0 \alpha_1\left(X_0\right)\right) \\ &= a\left(X_0 \otimes 1 + 1 \otimes X_0\right) \\ &= d'(a)\,. \end{aligned}$$

Similar arguments allow us to construct an isomorphism of complexes between $F(z_0 s_1 z_0 z_1)$ and $F(z_1 z_0 s_1 z_0)$, which proves that (3.18) is satisfied on the level of complexes.

In order to handle Relation (3.19), it is enough to find an isomorphism between $B_{s_0} \otimes_R R_{s_1} \otimes_R B_{s_0} \otimes_R R_{s_1}$ and $R_{s_1} \otimes_R B_{s_0} \otimes_R R_{s_1} \otimes_R B_{s_0}$. Applying Lemma 3.5 again, we first observe that

$$B_{s_0} \otimes_R R_{s_1} \otimes_R B_{s_0} \otimes_R R_{s_1} \cong B_{s_0} \otimes_R B_{s_1 s_0 s_1}$$

and

$$R_{s_1} \otimes_R B_{s_0} \otimes_R R_{s_1} \otimes_R B_{s_0} \cong B_{s_1 s_0 s_1} \otimes_R B_{s_0}\,.$$

Then using the generating set of R^w for $w \in \{s_0, s_1 s_0 s_1\}$ exhibited in Section 3.4.2, we can make an isomorphism of R-bimodules

$$\varphi : B_{s_0} \otimes B_{s_1 s_0 s_1} \to B_{s_1 s_0 s_1} \otimes B_{s_0}$$

explicit by setting

$$\begin{aligned} \varphi\left(1 \otimes 1 \otimes 1\right) &= 1 \otimes 1 \otimes 1\,, \\ \varphi\left(1 \otimes X_0 \otimes 1\right) &= 1 \otimes 1 \otimes X_0\,, \\ \varphi\left(1 \otimes X_1 \otimes 1\right) &= -X_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \left(X_1 + X_2\right)\,, \\ \varphi\left(1 \otimes X_i \otimes 1\right) &= X_i \otimes 1 \otimes 1 \qquad \text{if } i > 1\,. \end{aligned}$$

The map φ is clearly surjective since any element of R can be written as a sum of products of elements of R^{s_0} and $R^{s_1 s_0 s_0}$. Let us prove that this isomorphism of R-bimodules is well-defined. We have to check that

$$\varphi\left(1 \otimes p \otimes 1\right) = p \otimes 1 \otimes 1 \tag{3.26}$$

for all $p \in R^{s_0}$; and

$$\varphi\left(1 \otimes p \otimes 1\right) = 1 \otimes 1 \otimes p \tag{3.27}$$

for all $p \in R^{s_1 s_0 s_1}$. It is enough to check (3.26) (resp. (3.27)) for p equal to the generating elements of R^{s_0} (resp. of $R^{s_1 s_0 s_1}$).

For $p = X_0$ (resp. $p = X_2$) Equality (3.27) (resp. Equality (3.26)) follows directly from the definition of φ. For $p = X_i$ with $i > 2$, Equalities (3.26) and (3.27) follow from the fact that $X_i \in R^{s_0} \cap R^{s_1 s_0 s_1}$ when $i > 2$.

It remains to deal with the elements $\sqrt{2}X_0 + 2X_1$ and X_0^2 of R^{s_0}, and with the elements $X_1 + X_2$ and $X_1\left(\sqrt{2}X_0 + X_1\right)$ of $R^{s_1 s_0 s_1}$. Equality (3.26) is obvious for X_0^2 since $X_0^2 \in R^{s_0} \cap R^{s_1 s_0 s_1}$. By definition,

$$\varphi\big(1 \otimes (\sqrt{2}X_0 + 2X_1) \otimes 1\big) = -2X_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes (\sqrt{2}X_0 + 2X_1 + 2X_2).$$

Now, $\sqrt{2}X_0 + 2X_1 + 2X_2$ belongs to $R^{s_0} \cap R^{s_1 s_0 s_1}$, so we get the expected equality

$$\varphi\big(1 \otimes (\sqrt{2}X_0 + 2X_1) \otimes 1\big) = (\sqrt{2}X_0 + 2X_1) \otimes 1 \otimes 1\,.$$

For $X_1 + X_2$, Equality (3.27) follows directly from the definition of φ; we have

$$\begin{aligned}\varphi\big(1 \otimes (X_1 + X_2) \otimes 1\big) &= -X_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes (X_1 + X_2) + X_2 \otimes 1 \otimes 1 \\ &= 1 \otimes 1 \otimes (X_1 + X_2)\ .\end{aligned}$$

The last element $p \in R^{s_1 s_0 s_1}$ for which we have to check Equality (3.27) is $p = X_1\left(\sqrt{2}X_0 + X_1\right)$. By definition,

$$\begin{aligned}\varphi\big(1 \otimes X_1(\sqrt{2}X_0 + X_1) \otimes 1\big) &= -X_2 \otimes 1 \otimes \sqrt{2}X_0 \\ &\quad +1 \otimes 1 \otimes \sqrt{2}X_0(X_1 + X_2) + X_2^2 \otimes 1 \otimes 1 \\ &\quad -2X_2 \otimes 1 \otimes (X_1 + X_2) \\ &\quad +1 \otimes 1 \otimes (X_1 + X_2)^2 \\ &= 1 \otimes 1 \otimes \left(\sqrt{2}X_0(X_1 + X_2) + (X_1 + X_2)^2\right) \\ &\quad -X_2 \otimes 1 \otimes (\sqrt{2}X_0 + 2X_1 + 2X_2) \\ &\quad +X_2^2 \otimes 1 \otimes 1\,.\end{aligned}$$

Now remark that X_2^2 can be written as follows:

$$\begin{aligned}X_2^2 = \big(X_1(\sqrt{2}X_0 + X_1) - (X_1 + X_2)^2 - \sqrt{2}X_0(X_1 + X_2)\big) \\ + (\sqrt{2}X_0 + 2X_1 + 2X_2)X_2\,.\end{aligned}$$

One can check that both

$$X_1(\sqrt{2}X_0 + X_1) - (X_1 + X_2)^2 - \sqrt{2}X_0(X_1 + X_2)$$

and $\sqrt{2}X_0 + 2X_1 + 2X_2$ belong to $R^{s_0} \cap R^{s_1 s_0 s_1}$. So replacing X_2^2 by its latter expression in the expression of $\varphi\big(1 \otimes X_1(\sqrt{2}X_0 + X_1) \otimes 1\big)$, we obtain what we expected, namely

$$\varphi\big(1 \otimes X_1(\sqrt{2}X_0 + X_1) \otimes 1\big) = 1 \otimes 1 \otimes X_1(\sqrt{2}X_0 + X_1)\,.$$

The existence of the bimodule isomorphism φ leads to an isomorphism between the complexes $F(s_0 z_1 s_0 z_1)$ and $F(z_1 s_0 z_1 s_0)$. In fact, $F(s_0 z_1 s_0 z_1)$ is isomorphic to the complex

$$\begin{tikzcd} & & B_{s_0}\{2\} \arrow[dr, "d_1^{-1}"] & & \\ 0 \arrow[r] & R\{4\} \arrow[ur, "d_1^{-2}"] \arrow[dr, "d_2^{-2}"'] & & B_{s_0} \otimes_R B_{s_1 s_0 s_1} \arrow[r] & 0 \\ & & B_{s_1 s_0 s_1}\{2\} \arrow[ur, "d_2^{-1}"'] & & \end{tikzcd}$$

whose differentials are obtained by composing the ones of $F(s_0z_1s_0z_1)$ with the isomorphism of Lemma 3.5. More precisely,

$$\begin{aligned}
d_1^{-2}(a) &= a\left(X_0\otimes 1+1\otimes X_0\right),\\
d_2^{-2}(a) &= -a\left(\alpha_1\left(X_0\right)\otimes 1+1\otimes\alpha_1\left(X_0\right)\right)\\
&= -a\left(\left(X_0+\sqrt{2}X_1\right)\otimes 1+1\otimes\left(X_0+\sqrt{2}X_1\right)\right),\\
d_1^{-1}(a\otimes b) &= a\left(1\otimes\alpha_1\left(X_0\right)\otimes 1+1\otimes 1\otimes\alpha_1\left(X_0\right)\right)b\\
&= a\left(1\otimes\left(X_0+\sqrt{2}X_1\right)\otimes 1+1\otimes 1\otimes\left(X_0+\sqrt{2}X_1\right)\right)b,\\
d_2^{-1}(a\otimes b) &= a\left(X_0\otimes 1\otimes 1+1\otimes X_0\otimes 1\right)b
\end{aligned}$$

for all $a,b,c\in R$. Similarly, the complex $F(z_1s_0z_1s_0)$ is isomorphic to

$$\begin{tikzcd}
 & & B_{s_0}\{2\} \arrow[dr, "d_1'^{-1}"] & & \\
0 \arrow[r] & R\{4\} \arrow[ur, "d_1'^{-2}"] \arrow[dr, "d_2'^{-2}"'] & & B_{s_1s_0s_1}\otimes_R B_{s_0} \arrow[r] & 0\\
 & & B_{s_1s_0s_1}\{2\} \arrow[ur, "d_2'^{-1}"'] & &
\end{tikzcd}$$

whose differentials are obtained by composing the ones of $F(z_1s_0z_1s_0)$ with the isomorphism of Lemma 3.5, namely

$$\begin{aligned}
d_1'^{-2}(a) &= -a\left(X_0\otimes 1+1\otimes X_0\right),\\
d_2'^{-2}(a) &= a\left(\alpha_1\left(X_0\right)\otimes 1+1\otimes\alpha_1\left(X_0\right)\right)\\
&= a\left(\left(X_0+\sqrt{2}X_1\right)\otimes 1+1\otimes\left(X_0+\sqrt{2}X_1\right)\right),\\
d_1'^{-1}(a\otimes b) &= a\left(\alpha_1\left(X_0\right)\otimes 1\otimes 1+1\otimes\alpha_1\left(X_0\right)\otimes 1\right)b\\
&= a\left(\left(X_0+\sqrt{2}X_1\right)\otimes 1\otimes 1+1\otimes\left(X_0+\sqrt{2}X_1\right)\otimes 1\right)b,\\
d_2'^{-1}(a\otimes b) &= a\left(1\otimes X_0\otimes 1+1\otimes 1\otimes X_0\right)b
\end{aligned}$$

for all $a,b,c\in R$. The isomorphisms between the cochain bimodules of these complexes are the identity (up to a sign), except the isomorphism between the cochain bimodules of cohomological degree 0 which is given by φ. This can be summarized by the following commutative diagram:

$$\begin{tikzcd}
 & & B_{s_0}\{2\} \arrow[dddd, bend right, "\mathrm{id}"'] \arrow[dr, "d_1^{-1}"] & & \\
0 \arrow[r] & R\{4\} \arrow[ur, "d_1^{-2}"] \arrow[dr, "d_2^{-2}"'] \arrow[dddd, "-\mathrm{id}"'] & & B_{s_0}\otimes_R B_{s_1s_0s_1} \arrow[r] \arrow[dddd, "\varphi"] & 0 \\
 & & B_{s_1s_0s_1}\{2\} \arrow[ur, "d_2^{-1}"'] \arrow[dddd, bend left, "\mathrm{id}"] & & \\
 & & & & \\
 & & B_{s_0}\{2\} \arrow[dr, "d_1'^{-1}"] & & \\
0 \arrow[r] & R\{4\} \arrow[ur, "d_1'^{-2}"] \arrow[dr, "d_2'^{-2}"'] & & B_{s_1s_0s_1}\otimes_R B_{s_0} \arrow[r] & 0 \\
 & & B_{s_1s_0s_1}\{2\} \arrow[ur, "d_2'^{-1}"'] & &
\end{tikzcd}$$

This morphism of complexes is well-defined: the identities commute with the differentials and for all $a, b \in R$,

$$\begin{aligned}
\varphi \circ d_1^{-1}(a \otimes b) &= \varphi\left(a\left(1 \otimes \left(X_0 + \sqrt{2}X_1\right) \otimes 1 + 1 \otimes 1 \otimes \left(X_0 + \sqrt{2}X_1\right)\right) b\right) \\
&= a\left(\left(X_0 + \sqrt{2}X_1\right) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \left(X_0 + \sqrt{2}X_1\right)\right) b \\
&= d_1'^{-1}(a \otimes b)
\end{aligned}$$

since $\left(X_0 + \sqrt{2}X_1\right) \in R^{s_0}$. We also have

$$\begin{aligned}
\varphi \circ d_2^{-1}(a \otimes b) &= \varphi\left(a\left(X_0 \otimes 1 \otimes 1 + 1 \otimes X_0 \otimes 1\right) b\right) \\
&= a\left(X_0 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_0\right) b \\
&= d_2'^{-1}(a \otimes b)
\end{aligned}$$

since $X_0 \in R^{s_1s_0s_1}$. This completes the proof. □

3.5.3. More isomorphic complexes. We observed in Proposition 3.3 that quite unexpectedly the relation

$$z_0s_1z_0s_1 = s_1z_0s_1z_0$$

does not hold in $\mathcal{VB}_{B_n}$. Nevertheless we have the following.

Proposition 3.7. *The complexes $F(z_0s_1z_0s_1)$ and $F(s_1z_0s_1z_0)$ are isomorphic.*

Proof. The existence of an isomorphism between these complexes follows from the isomorphim of R-bimodules

$$\psi : B_{s_1} \otimes_R B_{s_0s_1s_0} \to B_{s_0s_1s_0} \otimes_R B_{s_1}$$

defined by

$$\begin{array}{rcll}
\psi\,(1\otimes 1\otimes 1) & = & 1\otimes 1\otimes 1\,, & \\
\psi\,(1\otimes X_0\otimes 1) & = & \left(X_0+\frac{\sqrt{2}}{2}X_1\right)\otimes 1\otimes 1-1\otimes 1\otimes \frac{\sqrt{2}}{2}X_1\,, & \\
\psi\,(1\otimes X_1\otimes 1) & = & 1\otimes 1\otimes X_1\,, & \\
\psi\,(1\otimes X_2\otimes 1) & = & \left(\frac{1}{2}X_1+X_2\right)\otimes 1\otimes 1+1\otimes 1\otimes \frac{1}{2}X_1\,, & \\
\psi\,(1\otimes X_i\otimes 1) & = & X_i\otimes 1\otimes 1 & \text{if } i>2\,.
\end{array}$$

Once again this morphism is well-defined since, as is easily checked,

$$\psi\,(1\otimes p\otimes 1)=p\otimes 1\otimes 1 \tag{3.28}$$

for all $p\in R^{s_1}$; and

$$\psi\,(1\otimes p\otimes 1)=1\otimes 1\otimes p \tag{3.29}$$

for all $p\in R^{s_0s_1s_0}$. It is enough to check (3.28) (resp. (3.29)) for p equal to the generating elements of R^{s_1} (resp. of $R^{s_0s_1s_0}$).

For $p=X_1$, Equality (3.29) is immediate in view of the definition of ψ. For $p=X_1^2$ and $p=X_i$ with $i>2$, Equalities (3.28) and (3.29) follow from the fact that $p\in R^{s_1}\cap R^{s_0s_1s_0}$. For the two remaining generating elements of R^{s_0}, Equality (3.28) follows directly from the definition of ψ; namely

$$\begin{array}{rcl}
\psi\left(1\otimes\left(2X_0+\sqrt{2}X_1\right)\otimes 1\right) & = & \left(2X_0+\sqrt{2}X_1\right)\otimes 1\otimes 1 \\
 & & -1\otimes 1\otimes\sqrt{2}X_1+1\otimes 1\otimes\sqrt{2}X_1 \\
 & = & \left(2X_0+\sqrt{2}X_1\right)\otimes 1\otimes 1
\end{array}$$

and

$$\begin{array}{rcl}
\psi\,(1\otimes(X_1+2X_2)\otimes 1) & = & (X_1+2X_2)\otimes 1\otimes 1 \\
 & & -1\otimes 1\otimes X_1+1\otimes 1\otimes X_1 \\
 & = & (X_1+2X_2)\otimes 1\otimes 1\,.
\end{array}$$

For the element $X_0+\sqrt{2}X_2\in R^{s_0s_1s_0}$, we have

$$\begin{array}{rcl}
\psi\left(1\otimes\left(X_0+\sqrt{2}X_2\right)\otimes 1\right) & = & \left(X_0+\frac{\sqrt{2}}{2}X_1\right)\otimes 1\otimes 1-1\otimes 1\otimes\frac{\sqrt{2}}{2}X_1 \\
 & & +\left(\sqrt{2}X_2+\frac{\sqrt{2}}{2}X_1\right)\otimes 1\otimes 1 \\
 & & -1\otimes 1\otimes\frac{\sqrt{2}}{2}X_1 \\
 & = & \left(X_0+\sqrt{2}X_1+\sqrt{2}X_2\right)-1\otimes 1\otimes\sqrt{2}X_1\,.
\end{array}$$

Now, $X_0+\sqrt{2}X_1+\sqrt{2}X_2$ belongs to $R^{s_1}\cap R^{s_0s_1s_0}$, so we obtain

$$\psi\left(1\otimes\left(X_0+\sqrt{2}X_2\right)\otimes 1\right)=1\otimes 1\otimes\left(X_0+\sqrt{2}X_2\right)\,,$$

as expected.

The last generating element of $R^{s_0s_1s_0}$ to consider is $X_0\left(X_0+\sqrt{2}X_1\right)$. By definition,

$$\begin{aligned}
\psi\left(1\otimes X_0\left(X_0+\sqrt{2}X_1\right)\otimes 1\right) &= \left(X_0+\tfrac{\sqrt{2}}{2}X_1\right)^2\otimes 1\otimes 1\\
&\quad+\left(X_0+\tfrac{\sqrt{2}}{2}X_1\right)\otimes 1\otimes \tfrac{\sqrt{2}}{2}X_1\\
&\quad-\left(X_0+\tfrac{\sqrt{2}}{2}X_1\right)\otimes 1\otimes \tfrac{\sqrt{2}}{2}X_1\\
&\quad-1\otimes 1\otimes \tfrac{1}{2}X_1^2\\
&= \left(X_0+\tfrac{\sqrt{2}}{2}X_1\right)^2\otimes 1\otimes 1-1\otimes 1\otimes \tfrac{1}{2}X_1^2.
\end{aligned}$$

One can check that $(X_0+\frac{\sqrt{2}}{2}X_1)^2$ belongs to $R^{s_1}\cap R^{s_0s_1s_0}$. We then obtain

$$\begin{aligned}
\psi\big(1\otimes X_0(X_0+\sqrt{2}X_1)\otimes 1\big) &= 1\otimes 1\otimes \left(X_0+\tfrac{\sqrt{2}}{2}X_1\right)^2-1\otimes 1\otimes \tfrac{1}{2}X_1^2\\
&= 1\otimes 1\otimes X_0(X_0+\sqrt{2}X_1)\,.
\end{aligned}$$

This ensures that the R-bimodule isomorphism ψ is well-defined, which enables us to construct an isomorphism between the complexes $F(s_1z_0s_1z_0)$ and $F(z_0s_1z_0s_1)$. First observe that via the isomorphism of Lemma 3.5 we can express these complexes in the following way. The complex $F(s_1z_0s_1z_0)$ is isomorphic to the following one:

$$\begin{tikzcd}
 & & B_{s_1}\{2\} \arrow[dr, "d_1^{-1}"] & & \\
0 \arrow[r] & R\{4\} \arrow[ur, "d_1^{-2}"] \arrow[dr, "d_2^{-2}"'] & & B_{s_1}\otimes_R B_{s_0s_1s_0} \arrow[r] & 0\\
 & & B_{s_0s_1s_0}\{2\} \arrow[ur, "d_2^{-1}"'] & &
\end{tikzcd}$$

whose differentials are given for $a,b\in R$ by

$$\begin{aligned}
d_1^{-2}(a) &= a\left(X_1\otimes 1+1\otimes X_1\right),\\
d_2^{-2}(a) &= -a\left(\alpha_0\left(X_1\right)\otimes 1+1\otimes \alpha_0\left(X_1\right)\right)\\
&= -a\left(\left(X_1+\sqrt{2}X_0\right)\otimes 1+1\otimes\left(X_1+\sqrt{2}X_0\right)\right),\\
d_1^{-1}(a\otimes b) &= a\left(1\otimes \alpha_0\left(X_1\right)\otimes 1+1\otimes 1\otimes \alpha_0\left(X_1\right)\right)b\\
&= a\left(1\otimes\left(X_1+\sqrt{2}X_0\right)\otimes 1+1\otimes 1\otimes\left(X_1+\sqrt{2}X_0\right)\right)b,\\
d_2^{-1}(a\otimes b) &= a\left(X_1\otimes 1\otimes 1+1\otimes X_1\otimes 1\right)b.
\end{aligned}$$

Similarly, the complex $F(z_0s_1z_0s_1)$ is isomorphic to

$$\begin{tikzcd}
 & & B_{s_1}\{2\} \arrow[dr, "d_1'^{-1}"] & & \\
0 \arrow[r] & R\{4\} \arrow[ur, "d_1'^{-2}"] \arrow[dr, "d_2'^{-2}"'] & & B_{s_0s_1s_0}\otimes_R B_{s_1} \arrow[r] & 0\\
 & & B_{s_0s_1s_0}\{2\} \arrow[ur, "d_2'^{-1}"'] & &
\end{tikzcd}$$

whose differentials are given for $a, b \in R$ by

$$\begin{array}{rcl}
d_1'^{-2}(a) & = & -a\left(X_1 \otimes 1 + 1 \otimes X_1\right), \\
d_2'^{-2}(a) & = & a\left(\alpha_0\left(X_1\right) \otimes 1 + 1 \otimes \alpha_0\left(X_1\right)\right) \\
 & = & a\left(\left(X_1 + \sqrt{2}X_0\right) \otimes 1 + 1 \otimes \left(X_1 + \sqrt{2}X_0\right)\right), \\
d_1'^{-1}(a \otimes b) & = & a\left(\alpha_0\left(X_1\right) \otimes 1 \otimes 1 + 1 \otimes \alpha_0\left(X_1\right) \otimes 1\right) b \\
 & = & a\left(\left(X_1 + \sqrt{2}X_0\right) \otimes 1 \otimes 1 + 1 \otimes \left(X_1 + \sqrt{2}X_0\right) \otimes 1\right) b, \\
d_2'^{-1}(a \otimes b) & = & a\left(1 \otimes X_1 \otimes 1 + 1 \otimes 1 \otimes X_1\right) b.
\end{array}$$

Then the isomorphism between $F(s_1 z_0 s_1 z_0)$ and $F(z_0 s_1 z_0 s_1)$ is summarized by the following commutative diagram:

$$\begin{tikzcd}
 & & B_{s_1}\{2\} \arrow[dr, "d_1^{-1}"] \arrow[ddd, bend right, "\mathrm{id}"'] & & \\
0 \arrow[r] & R\{4\} \arrow[ur, "d_1^{-2}"] \arrow[dr, "d_2^{-2}"'] \arrow[ddd, "-\mathrm{id}"'] & & B_{s_1} \otimes_R B_{s_0 s_1 s_0} \arrow[r] \arrow[ddd, "\psi"] & 0 \\
 & & B_{s_0 s_1 s_0}\{2\} \arrow[ur, "d_2^{-1}"'] \arrow[ddd, bend right, "\mathrm{id}"'] & & \\
 & & B_{s_1}\{2\} \arrow[dr, "d_1'^{-1}"] & & \\
0 \arrow[r] & R\{4\} \arrow[ur, "d_1'^{-2}"] \arrow[dr, "d_2'^{-2}"'] & & B_{s_0 s_1 s_0} \otimes_R B_{s_1} \arrow[r] & 0 \\
 & & B_{s_0 s_1 s_0}\{2\} \arrow[ur, "d_2'^{-1}"'] & &
\end{tikzcd}$$

This morphism of complexes is obviously well-defined: the identities commute with the differentials and

$$\begin{array}{rcl}
\psi \circ d_1^{-1}(a \otimes b) & = & \psi\left(a\left(1 \otimes \left(X_1 + \sqrt{2}X_0\right) \otimes 1 + 1 \otimes 1 \otimes \left(X_1 + \sqrt{2}X_0\right)\right) b\right) \\
 & = & a\left(\left(X_1 + \sqrt{2}X_0\right) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \left(X_1 + \sqrt{2}X_0\right)\right) b \\
 & = & d_1'^{-1}(a \otimes b)
\end{array}$$

since $\left(X_1 + \sqrt{2}X_0\right) \in R^{s_1}$; and

$$\begin{array}{rcl}
\psi \circ d_2^{-1}(a \otimes b) & = & \psi\left(a\left(X_1 \otimes 1 \otimes 1 + 1 \otimes X_1 \otimes 1\right) b\right) \\
 & = & a\left(X_1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_1\right) b \\
 & = & d_2'^{-1}(a \otimes b)
\end{array}$$

since $X_1 \in R^{s_0 s_1 s_0}$. □

Remark 3.8. *As in the type A case (treated in Chapter 2), the complexes $F(s_i z_i)$ and $F(z_i s_i)$ are isomorphic, which contrasts with the fact that the relation $s_i z_i = z_i s_i$ does not hold in $\mathcal{VB}_{B_n}$.*

Appendix A

Injectivity of ι^i

The aim of this Appendix is to check the injectivity of the morphism of R–bimodules $\iota^i : B_{i,i+1} \to B_i \otimes_R B_{i+1} \otimes_R B_i$ defined in Chapter 1. After having recalled the definitions of the involved R–bimodules, we precise their structure as left R–modules. This allows us to write ι^i, viewed as a morphism of left R–modules, under a matrix form and to prove its injectivity.

Recall that R is the subalgebra of $\mathbb{Q}[x_1, \dots, x_n]$ generated by $x_i - x_j$ for $1 \leq i, j \leq n$. We have

$$R = \mathbb{Q}[x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n] = \mathbb{Q}[x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n].$$

Let us set $X_i = x_i - x_{i+1}$ and $y_i = x_1 - x_i$, so

$$R = \mathbb{Q}[y_2, \dots y_n] = \mathbb{Q}[X_1, \dots, X_{n-1}].$$

Let R^{τ_i} be the subalgebra of R of elements fixed by the transposition $\tau_i = (i, i+1)$. As an algebra,

$$R^{\tau_i} = \mathbb{Q}[y_2, \dots, y_{i-1}, y_i + y_{i+1}, y_i y_{i+1}, y_{i+2}, \dots, y_n].$$

Let $R^{<\tau_i, \tau_{i+1}>}$ be the subalgebra of R of elements fixed by the transpositions τ_i and τ_{i+1}. As an algebra,

$$\begin{aligned} R^{<\tau_i,\tau_{i+1}>} = \mathbb{Q}[&y_2, \dots, y_{i-1}, y_i + y_{i+1} + y_{i+2}, \\ &y_i y_{i+1} + y_i y_{i+2} + y_{i+1} y_{i+2}, y_i y_{i+1} y_{i+2}, y_{i+3} \cdots, y_n]. \end{aligned}$$

Consider the R–bimodules

$$B_{i,i+1} = R \otimes_{R^{<\tau_i,\tau_{i+1}>}} R$$

and $B_i = R \otimes_{R^{\tau_i}} R$, so

$$B_i \otimes_R B_{i+1} \otimes_R B_i \cong R \otimes_{R^{\tau_i}} R \otimes_{R^{\tau_{i+1}}} R \otimes_{R^{\tau_i}} R.$$

As an R–bimodule $B_{i,i+1}$ is spanned by $1 \otimes 1$ so one can define a morphism of R–bimodules

$$\iota^i : B_{i,i+1} \to B_i \otimes_R B_{i+1} \otimes_R B_i$$

by $\iota^i(1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1$. It is obviously well–defined since $R^{<\tau_i,\tau_{i+1}>} \cong R^{\tau_i} \cap R^{\tau_{i+1}}$.

Lemma A.1. *The morphism of R–bimodules ι^i is injective.*

Proof. Since $B_{i,i+1}$ and $B_i \otimes_R B_{i+1} \otimes_R B_i$ are free as left R–modules, we can view ι^i as a morphim of left R–modules, write it under a matrix form and then check its injectivity.

First recall that, as an R^{τ_i}–module and as an $R^{\tau_{i+1}}$–module, R is free of rank 2 with basis $\{1, X_{i+1}\}$. So $B_i \otimes_R B_{i+1} \otimes_R B_i$ is a free R–module of rank 8 with basis

$$\begin{aligned} E = \{&1 \otimes 1 \otimes 1 \otimes 1, 1 \otimes 1 \otimes 1 \otimes X_{i+1}, 1 \otimes 1 \otimes X_{i+1} \otimes 1, \\ &1 \otimes X_{i+1} \otimes 1 \otimes 1, 1 \otimes 1 \otimes X_{i+1} \otimes X_{i+1}, 1 \otimes X_{i+1} \otimes 1 \otimes X_{i+1}, \\ &\qquad 1 \otimes X_{i+1} \otimes X_{i+1} \otimes 1, 1 \otimes X_{i+1} \otimes X_{i+1} \otimes X_{i+1}\}. \end{aligned}$$

Now let us give an R–basis of $B_{i,i+1}$. We already know that R is a free R^{τ_i}–module of rank 2 with basis $\{1, X_{i+1}\}$. So we only need to find an $R^{<\tau_i,\tau_{i+1}>}$–basis of R^{τ_i}. Let us prove that such a basis is given by $\{1, y_{i+2}, y_{i+2}^2\}$. Indeed these three elements are linearly independant over $R^{<\tau_i,\tau_{i+1}>}$: let p, q and r in $R^{<\tau_i,\tau_{i+1}>}$ be such that

$$p + qy_{i+2} + ry_{i+2}^2 = 0.$$

Then applying the transposition τ_{i+1} to this equation, we obtain

$$p + qy_{i+1} + ry_{i+1}^2 = 0.$$

By substracting these two egalities, we obtain $q = -r\,(y_{i+2} + y_{i+1})$, which implies $q = r = 0$ since $r\,(y_{i+2} + y_{i+1}) \notin R^{\tau_i}$ whereas q. So p is also equal to zero. Moreover, the family $\{1, y_{i+2}, y_{i+2}^2\}$ is a spanning set of R^{τ_i} over $R^{<\tau_i,\tau_{i+1}>}$, indeed the generators of R^{τ_i} can be expressed as follows:

$$\begin{aligned} y_k &= y_k + 0.y_{i+2} + 0.y_{i+2}^2 \quad \text{if } k \neq i, i+1, i+2, \\ y_{i+2} &= 0 + y_{i+2} + 0.y_{i+2}^2, \\ y_i + y_{i+1} &= (y_i + y_{i+1} + y_{i+2}) - y_{i+2} + 0.y_{i+2}^2, \\ y_i y_{i+1} &= (y_i y_{i+1} + y_i y_{i+2} + y_{i+1} y_{i+2}) - (y_i + y_{i+1} + y_{i+2}) y_{i+2} + y_{i+2}^2. \end{aligned}$$

Finally this family is closed under products since

$$y_{i+2}^3 = y_i y_{i+1} y_{i+2} - (y_i y_{i+1} + y_i y_{i+2} + y_{i+1} y_{i+2}) y_{i+2} + (y_i + y_{i+1} + y_{i+2}) y_{i+2}^2,$$

thus is basis of R^{τ_i} over $R^{<\tau_i,\tau_{i+1}>}$.

Finally R is a free $R^{<\tau_i,\tau_{i+1}>}$–module of rank 6 and this implies that $B_{i,i+1} = R \otimes_{R^{<\tau_i,\tau_{i+1}>}} R$ is a free R–module of rank 6 with basis

$$F = \{1 \otimes 1, 1 \otimes y_{i+2}, 1 \otimes y_{i+2}^2, 1 \otimes X_{i+1}, 1 \otimes X_{i+1}y_{i+2}, 1 \otimes X_{i+1}y_{i+2}^2\}.$$

Viewed as a morphism of left R–modules, ι^i can be described by the following matrix:

$$\mathrm{Mat}_{F,E}\left(\iota^i\right) = \begin{pmatrix} 1 & y_{i+2} & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & y_{i+2} & a \\ 0 & 1/2 & y_{i+2} & 0 & 0 & 0 \\ 0 & -1/2 & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & y_{i+2} \\ 0 & 0 & 0 & 0 & -1/2 & -b \\ 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 \end{pmatrix}$$

where

$$a = y_{i+2}^2 - \frac{1}{2}\left(\frac{X_i}{2} + X_{i+1}\right)^2 + \frac{X_i^2}{8}$$

and

$$b = \frac{y_i + y_{i+1}}{2}.$$

This matrix is of rank 6; thus, ι^i is injective. □

Appendix B

Explicit homotopy equivalences for Rouquier's categorification of $\mathcal{B}_n$

The aim of this Appendix is to give a direct proof of Rouquier's categorification of $\mathcal{B}_n$. We make here explicit the homotopy equivalences between complexes associated to braid words representing the same element of $\mathcal{B}_n$.

We refer to Chapter 1 for the definitions of Soergel bimodules B_i, for $i = 1, \dots, n-1$, and of Rouquier's complexes $F(\omega)$, for $\omega \in \mathcal{B}_n$.

We begin this Appendix by recalling the definitions of all the morphisms of graded R–bimodules that will appear in the sequel.

Let us consider the R–bimodule morphisms

$$\mathrm{br}_i : B_i \to R$$

defined by $\mathrm{br}_i(1 \otimes 1) = 1$;

$$\mathrm{rb}_i : R\{2\} \to B_i$$

defined by $\mathrm{rb}_i(1) = X_i \otimes 1 + 1 \otimes X_i$;

$$\psi_1^i : B_i \to B_i \otimes_R B_i$$

defined by $\psi_1^i(1 \otimes 1) = 1 \otimes 1 \otimes 1$;

$$\psi_2^i : B_i\{2\} \to B_i \otimes_R B_i$$

defined by $\psi_2^i(1 \otimes 1) = 1 \otimes X_i \otimes 1$;

$$\phi_1^i : B_i \otimes_R B_i \to B_i$$

defined by $\phi_1^i(1\otimes 1\otimes 1)=1\otimes 1$ and $\phi_1^i(1\otimes X_i\otimes 1)=0$;

$$\phi_2^i : B_i\otimes_R B_i \to B_i\{2\}$$

defined by $\phi_2^i(1\otimes 1\otimes 1)=0$ and $\phi_2^i(1\otimes X_i\otimes 1)=1\otimes 1$;

$$\gamma^{i,j} : B_i\otimes_R B_j \to B_j\otimes_R B_i$$

defined by $\gamma^{i,j}(1\otimes 1\otimes 1)=1\otimes 1\otimes 1$ if $|i-j|>1$;

$$\iota^i : B_{i,i+1} \to B_i\otimes_R B_{i+1}\otimes_R B_i$$

defined by $\iota^i(1\otimes 1)=1\otimes 1\otimes 1\otimes 1$;

$$\pi^i : B_i\otimes_R B_{i+1}\otimes_R B_i \to B_{i,i+1}$$

defined by $\pi^i\,(1\otimes(X_{i+1}\otimes 1+1\otimes X_{i+1})\otimes 1)=0$ and $\pi^i(1\otimes 1\otimes 1\otimes 1)=1\otimes 1$.

The braid group with n strands $\mathcal{B}_n$ is the group generated by $n-1$ generators $\sigma_1,\dots,\sigma_{n-1}$, thus

$$\sigma_i\sigma_i^{-1}=1,\quad \text{if } 1\le i\le n-1. \tag{B.1}$$

These generators satisfy the following relations

$$\sigma_i\sigma_j=\sigma_j\sigma_i,\quad \text{if } |i-j|>1, \tag{B.2}$$

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\quad \text{if } 1\le i\le n-2. \tag{B.3}$$

Rouquier proved the following result, which can be called a categorification of the braid group $\mathcal{B}_n$.

Theorem B.1. [Rou06] *If ω and ω' are words representing the same element of $\mathcal{B}_n$, then $F(\omega)$ and $F(\omega')$ are homotopy equivalent complexes of R–bimodules.*

Actually Rouquier proved this theorem in the more general context of braid groups associated to a Coxeter group of finite type.

We want here to give a proof of his result in the type A case that makes explicit the homotopy equivalences.

Proof.

Relation (B.1)

The homotopy equivalence between $F\left(\sigma_i\sigma_i^{-1}\right)$ and $F(1)$ is summarized in the following diagram:

$$
\begin{tikzcd}
 & & B_i \otimes_R B_i\{-2\} \arrow[ll, dashed, bend right, "h^0"'] \arrow[rd, "\mathrm{id}\otimes \mathrm{br}_i"] \arrow[dd, bend right=40, "f"'] & \\
F(\sigma_i\sigma_i^{-1}): & B_i \arrow[ru, "\mathrm{rb}_i\otimes \mathrm{id}"] \arrow[rd, "-\mathrm{br}_i"'] & & B_i\{-2\} \arrow[lu, dashed, bend right, "h^1"'] \\
 & & R \arrow[ru, "\mathrm{rb}_i"'] \arrow[d, bend right, "\mathrm{id}"'] & \\
F(1): & 0 \arrow[r] & R \arrow[r] \arrow[u, bend right, "\mathrm{id}"'] \arrow[uuu, bend right=40, "g"'] & 0
\end{tikzcd}
$$

where

$$
\begin{aligned}
f &= \mathrm{br}_i \circ \phi_2^i, \\
g &= -\psi_1^i \circ \mathrm{rb}_i, \\
h^0 &= -\phi_2^i, \\
h^1 &= -\psi_1^i.
\end{aligned}
$$

One can easily check that the vertical maps commute with the differentials. They also define a homotopy equivalence: one directly recover the identity of $F(1)$ by the composition $F(1) \to F\left(\sigma_i\sigma_i^{-1}\right) \to F(1)$ since

$$f \circ g = 0.$$

Moreover, one has to check that, for each cochain bimodule of fixed degree M of $F\left(\sigma_i\sigma_i^{-1}\right)$, the endomorphism m of M induced by the composition $F\left(\sigma_i\sigma_i^{-1}\right) \to F(1) \to F\left(\sigma_i\sigma_i^{-1}\right)$ satisfies

$$m - \mathrm{id}_M = d \circ h + h \circ d \tag{B.4}$$

where d is the differential of $F\left(\sigma_i\sigma_i^{-1}\right)$ and h its homotopy. If M is a direct sum of several factors, the composition $F\left(\sigma_i\sigma_i^{-1}\right) \to F(1) \to F\left(\sigma_i\sigma_i^{-1}\right)$ induces on the one hand endomorphisms of each factor of M and on the other hand morphisms between the factors of M. In order to prove (B.4) one has to check that the composition is equal up to homotopy to the identity on each factor and that the induced morphisms between two different factors are equal to zero up to homotopy. So one can verify that in cohomological degree -1, we have

$$0 - \mathrm{id}_{B_i} = h^0 \circ (\mathrm{rb}_i \otimes \mathrm{id})\,;$$

in cohomological degree 0, we have

$$\begin{aligned}
g \circ f - \mathrm{id}_{B_i \otimes_R B_i\{-2\}} &= (\mathrm{rb}_i \otimes \mathrm{id}) \circ h^0 + h^1 \circ (\mathrm{id} \otimes \mathrm{br}_i)\,, \\
g \circ \mathrm{id}_R - 0 &= h^1 \circ \mathrm{rb}_i, \\
\mathrm{id}_R \circ f - 0 &= -\,\mathrm{br}_i \circ h^0;
\end{aligned}$$

and in cohomological degree 1, we have

$$0 - \mathrm{id}_{B_i\{-2\}} = (\mathrm{id} \otimes \mathrm{br}_i) \circ h^1.$$

Relation (B.2)

Here we prove that the complexes $F(\sigma_i\sigma_j)$ and $F(\sigma_i\sigma_j)$ are isomorphic, which is stronger than homotopy equivalent. The isomorphism of complexes between them is the following:

$$\begin{tikzcd}
 & & B_i\{2\} \arrow[dr, "\mathrm{id}\otimes \mathrm{rb}_j"] \arrow[ddddd, bend right, "-\mathrm{id}"'] & \\
F(\sigma_i\sigma_j): & R\{4\} \arrow[ur, "\mathrm{rb}_i"] \arrow[dr, "-\mathrm{rb}_j"'] \arrow[ddd, bend right, "\mathrm{id}"'] & & B_i \otimes_R B_j \arrow[ddd, bend right, "-\gamma^{i,j}"'] \\
 & & B_j\{2\} \arrow[ur, "\mathrm{rb}_i\otimes \mathrm{id}"'] \arrow[d, bend right, "-\mathrm{id}"'] & \\
 & & B_j\{2\} \arrow[dr, "\mathrm{id}\otimes \mathrm{rb}_i"] \arrow[u, bend right, "-\mathrm{id}"'] & \\
F(\sigma_j\sigma_i): & R\{4\} \arrow[ur, "\mathrm{rb}_j"] \arrow[dr, "-\mathrm{rb}_i"'] \arrow[uuu, bend right, "\mathrm{id}"'] & & B_j \otimes_R B_i \arrow[uuu, bend right, "-\gamma^{j,i}"'] \\
 & & B_i\{2\} \arrow[ur, "\mathrm{rb}_j\otimes \mathrm{id}"'] \arrow[uuuuu, bend right, "-\mathrm{id}"'] &
\end{tikzcd}$$

Since $\gamma^{i,j}$ is an isomorphism of R–bimodules whose inverse is $\gamma^{j,i}$, the above maps define an isomorphism of complexes.

Relation (B.3)

The complexes appearing in this relation are $F(\sigma_i\sigma_{i+1}\sigma_i)$:

$$
\begin{tikzcd}
 & B_i\{4\} \arrow[r, "\mathrm{id}\otimes\mathrm{rb}_{i+1}"] \arrow[dr, "-\mathrm{id}\otimes\mathrm{rb}_i"] & B_i\otimes_R B_{i+1}\{2\} \arrow[dr, "\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_i"] & \\
R\{6\} \arrow[ur, "\mathrm{rb}_i"] \arrow[r, "-\mathrm{rb}_{i+1}"] \arrow[dr, "\mathrm{rb}_i"'] & B_{i+1}\{4\} \arrow[ur, "\mathrm{rb}_i\otimes\mathrm{id}"] \arrow[dr, "-\mathrm{id}\otimes\mathrm{rb}_i"] & B_i\otimes_R B_i\{2\} \arrow[r, "\mathrm{id}\otimes\mathrm{rb}_{i+1}\otimes\mathrm{id}"] & B_i\otimes_R B_{i+1}\otimes_R B_i \\
 & B_i\{4\} \arrow[r, "-\mathrm{rb}_{i+1}\otimes\mathrm{id}"'] \arrow[ur, "\mathrm{rb}_i\otimes\mathrm{id}"] & B_{i+1}\otimes_R B_i\{2\} \arrow[ur, "\mathrm{rb}_i\otimes\mathrm{id}\otimes\mathrm{id}"'] &
\end{tikzcd}
$$

and $F(\sigma_{i+1}\sigma_i\sigma_{i+1})$:

$$
\begin{tikzcd}
 & B_{i+1}\{4\} \arrow[r, "\mathrm{id}\otimes\mathrm{rb}_i"] \arrow[dr, "-\mathrm{id}\otimes\mathrm{rb}_{i+1}"] & B_{i+1}\otimes_R B_i\{2\} \arrow[dr, "\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_{i+1}"] & \\
R\{6\} \arrow[ur, "\mathrm{rb}_{i+1}"] \arrow[r, "-\mathrm{rb}_i"] \arrow[dr, "\mathrm{rb}_{i+1}"'] & B_i\{4\} \arrow[ur, "\mathrm{rb}_{i+1}\otimes\mathrm{id}"] \arrow[dr, "-\mathrm{id}\otimes\mathrm{rb}_{i+1}"] & B_{i+1}\otimes_R B_{i+1}\{2\} \arrow[r, "\mathrm{id}\otimes\mathrm{rb}_i\otimes\mathrm{id}"] & B_{i+1}\otimes_R B_i\otimes_R B_{i+1} \\
 & B_{i+1}\{4\} \arrow[r, "-\mathrm{rb}_i\otimes\mathrm{id}"'] \arrow[ur, "\mathrm{rb}_{i+1}\otimes\mathrm{id}"] & B_i\otimes_R B_{i+1}\{2\} \arrow[ur, "\mathrm{rb}_{i+1}\otimes\mathrm{id}\otimes\mathrm{id}"'] &
\end{tikzcd}
$$

In order to prove that the complexes $F(\sigma_i\sigma_{i+1}\sigma_i)$ and $F(\sigma_{i+1}\sigma_i\sigma_{i+1})$ are homotopy equivalent, we will show that they are both homotopy equivalent to the complex $C^\bullet$ defined as follows:

$$
\begin{tikzcd}
 & B_{i+1}\{4\} \arrow[r, "-\mathrm{rb}_i\otimes\mathrm{id}"] \arrow[ddr, "\mathrm{id}\otimes\mathrm{rb}_i"] & B_i\otimes_R B_{i+1}\{2\} \arrow[dr, "\pi^i\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_i)"] & \\
R\{6\} \arrow[ur, "\mathrm{rb}_{i+1}"] \arrow[dr, "\mathrm{rb}_i"'] & & & B_{i,i+1} \\
 & B_i\{4\} \arrow[r, "-\mathrm{rb}_{i+1}\otimes\mathrm{id}"'] \arrow[uur, "\mathrm{id}\otimes\mathrm{rb}_{i+1}"] & B_{i+1}\otimes_R B_i\{2\} \arrow[ur, "\pi^i\circ(\mathrm{rb}_i\otimes\mathrm{id}\otimes\mathrm{id})"'] &
\end{tikzcd}
$$

where $R\{6\}$ sits in cohomological degree -3.

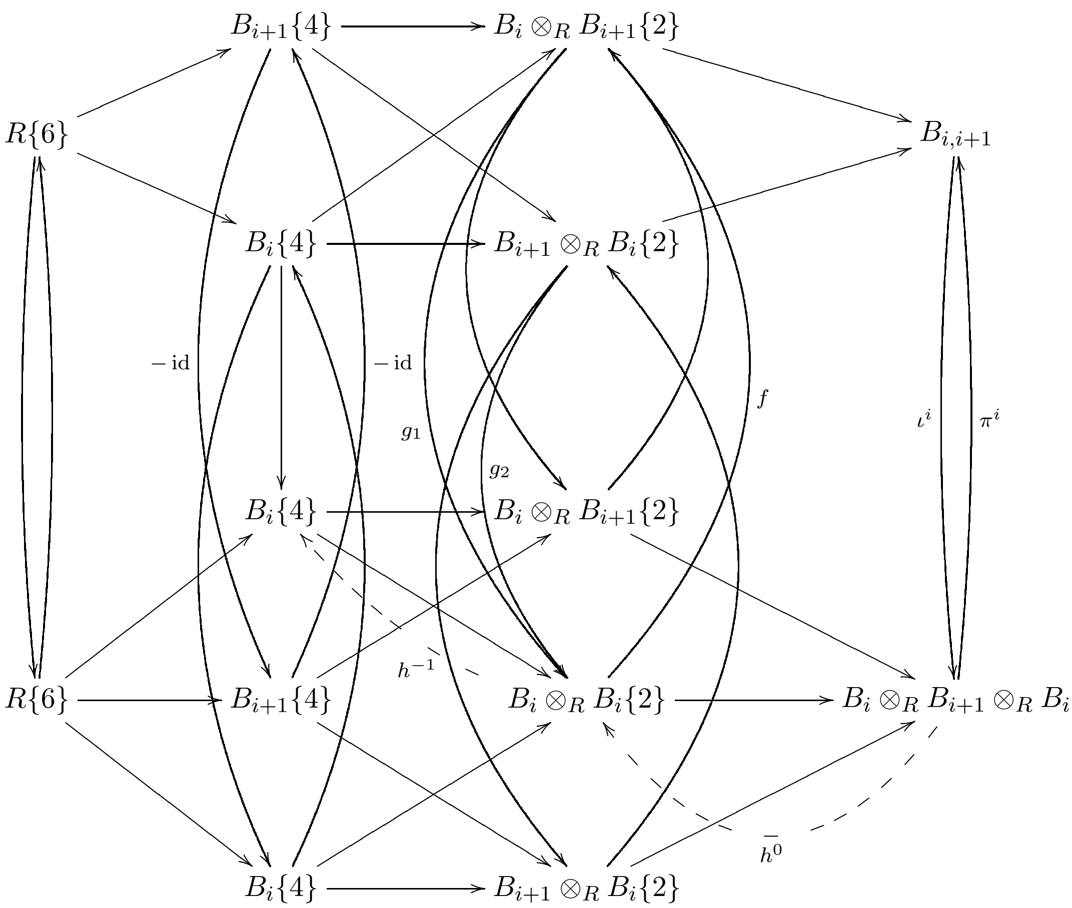
Note that the complex $C^\bullet$ is symmetric under the transposition $i \leftrightarrow i+1$ since the differential of cohomological degree -1 can be rewritten as follows:

$$\pi^i \circ (\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id}) = \pi^{i+1} \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_{i+1})$$

and

$$\pi^i \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_i) = \pi^{i+1} \circ (\mathrm{rb}_{i+1} \otimes \mathrm{id} \otimes \mathrm{id}).$$

The morphisms of complexes between $F(\sigma_i\sigma_{i+1}\sigma_i)$ and $C^\bullet$ and the homotopies are summarized in the following diagram. In order not to overload the diagrams, we omit the labels of the morphisms between the complexes when they are equal to the identity and the ones of the differentials (since they are expressed just right above).



where

$$\begin{aligned}
g_1 &= \psi_1^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1})\,, \\
g_2 &= \psi_1^i \circ (\mathrm{br}_{i+1} \otimes \mathrm{id})\,, \\
f &= (\mathrm{id} \otimes \mathrm{rb}_{i+1}) \circ \phi_2^i, \\
h^0 &= \psi_1^i \circ \phi_2^i \circ (\mathrm{id} \otimes \mathrm{br}_{i+1} \otimes \mathrm{id})\,, \\
h^{-1} &= \phi_2^i.
\end{aligned}$$

One can check that the vertical maps commute with the differentials, so they define a morphism of complexes. In order to prove that this morphism is a homotopy equivalence, first note that the composition $C^\bullet \to F(\sigma_i\sigma_{i+1}\sigma_i) \to$

$C^\bullet$ is equal to the identity of $C^\bullet$ since

$$\begin{aligned} f \circ g_1 &= 0, \\ f \circ g_2 &= 0; \end{aligned}$$

so there are no need for homotopies of $C^\bullet$ to recover the identity of $C^\bullet$. Then we have to study the composition $F(\sigma_i\sigma_{i+1}\sigma_i) \to C^\bullet \to F(\sigma_i\sigma_{i+1}\sigma_i)$. As for Relation (B.1), one has to check that, for each cochain bimodule of fixed degree M of $F(\sigma_i\sigma_{i+1}\sigma_i)$, this composition is equal to the identity up to homotopy on each factor of M and that the induced morphisms between two different factors of M are equal to zero up to homotopy. The only non-trivial equalities one have to verify are, in cohomological degree -2

$$\begin{aligned} 0 - \mathrm{id}_{B_i\{4\}} &= h^{-1} \circ (-\mathrm{id} \otimes \mathrm{rb}_i)\,, \\ \mathrm{id}_{B_i\{4\}} - 0 &= h^{-1} \circ (\mathrm{rb}_i \otimes \mathrm{id})\,; \end{aligned}$$

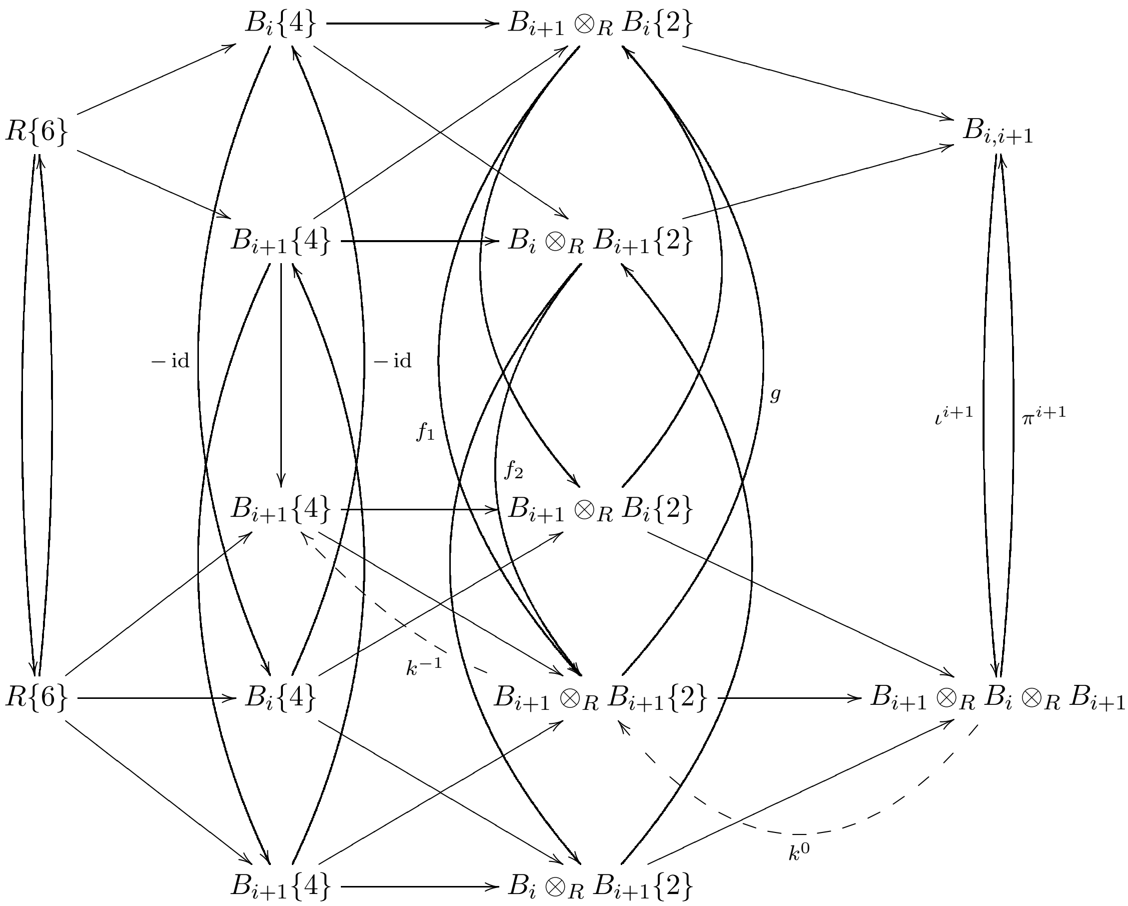
in cohomological degree -1

$$\begin{aligned} g_1 \circ \mathrm{id}_{B_i\otimes_R B_{i+1}\{2\}} - 0 &= h^0 \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{rb}_i)\,, \\ g_1 \circ f - \mathrm{id}_{B_i\otimes_R B_i\{2\}} &= h^0 \circ (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) + (-\mathrm{id} \otimes \mathrm{rb}_i) \circ h^{-1}, \\ \mathrm{id}_{B_i\otimes_R B_{i+1}\{2\}} \circ f - 0 &= (\mathrm{id} \otimes \mathrm{rb}_{i+1}) \circ h^{-1}, \\ g_2 \circ \mathrm{id}_{B_{i+1}\otimes_R B_i\{2\}} - 0 &= h^0 \circ (\mathrm{rb}_i \otimes \mathrm{id} \otimes \mathrm{id})\,; \end{aligned}$$

and in cohomological degree 0

$$\iota^i \circ \pi^i - \mathrm{id}_{B_i\otimes_R B_{i+1}\otimes_R B_i} = (\mathrm{id} \otimes \mathrm{rb}_{i+1} \otimes \mathrm{id}) \circ h^0.$$

Now let us turn to the homotopy equivalence between $F(\sigma_{i+1}\sigma_i\sigma_{i+1})$ and $C^\bullet$. It is made explicit in the following diagram. In order not to overload the diagrams, we once again omit the labels of the differentials systematically and the ones of the morphisms between the complexes when they are equal to the identity.



where

$$
\begin{aligned}
f_1 &= \psi_1^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i)\,, \\
f_2 &= \psi_1^{i+1} \circ (\mathrm{br}_i \otimes \mathrm{id})\,, \\
g &= (\mathrm{id} \otimes \mathrm{rb}_i) \circ \phi_2^{i+1}, \\
k^0 &= \psi_1^{i+1} \circ \phi_2^{i+1} \circ (\mathrm{id} \otimes \mathrm{br}_i \otimes \mathrm{id})\,, \\
k^{-1} &= \phi_2^{i+1}.
\end{aligned}
$$

One can check that the vertical maps commute with the differentials, so they define a morphism of complexes. In order to prove that this morphism is a homotopy equivalence, first note that the composition $C^\bullet \to F(\sigma_{i+1}\sigma_i\sigma_{i+1}) \to C^\bullet$ is equal to the identity of $C^\bullet$ since

$$
\begin{aligned}
g \circ f_1 &= 0, \\
g \circ f_2 &= 0.
\end{aligned}
$$

Then to prove that the composition $F(\sigma_{i+1}\sigma_i\sigma_{i+1}) \to C^\bullet \to F(\sigma_{i+1}\sigma_i\sigma_{i+1})$ is homotopy equivalent to the identity of $F(\sigma_{i+1}\sigma_i\sigma_{i+1})$, we proceed as above and the only non-trivial equalities one have to verify are, in cohomological degree -2

$$\begin{aligned} 0-\mathrm{id}_{B_{i+1}\{4\}} &= k^{-1}\circ(-\mathrm{id}\otimes \mathrm{rb}_{i+1})\,, \\ \mathrm{id}_{B_{i+1}\{4\}}-0 &= k^{-1}\circ(\mathrm{rb}_{i+1}\otimes \mathrm{id})\,; \end{aligned}$$

in cohomological degree -1

$$\begin{aligned} f_1\circ \mathrm{id}_{B_{i+1}\otimes_R B_i\{2\}}-0 &= k^0\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{rb}_{i+1})\,, \\ f_1\circ g-\mathrm{id}_{B_{i+1}\otimes_R B_{i+1}\{2\}} &= k^0\circ(\mathrm{id}\otimes\mathrm{rb}_i\otimes\mathrm{id})+(-\mathrm{id}\otimes\mathrm{rb}_{i+1})\circ k^{-1}, \\ \mathrm{id}_{B_{i+1}\otimes_R B_i\{2\}}\circ g-0 &= (\mathrm{id}\otimes\mathrm{rb}_i)\circ k^{-1}, \\ f_2\circ \mathrm{id}_{B_i\otimes_R B_{i+1}\{2\}}-0 &= k^0\circ(\mathrm{rb}_{i+1}\otimes\mathrm{id}\otimes\mathrm{id})\,; \end{aligned}$$

and in cohomological degree 0

$$\iota^{i+1}\circ\pi^{i+1}-\mathrm{id}_{B_{i+1}\otimes_R B_i\otimes_R B_{i+1}} = (\mathrm{id}\otimes\mathrm{rb}_i\otimes\mathrm{id})\circ k^0.$$

$\square$

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