Modèles de polymères dirigés en milieux aléatoires

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Préface

Nous étudions plusieurs modèles de polymères dirigés en milieux aléatoires. Pour le modèle classique sur $\mathbb{Z}^d$, nous étudions la convergence de l'environnement vu par la particule dans la région de faible désordre. Nous donnons des résultats très forts pour de très hautes valeurs de la température.

Nous donnons ensuite un traitement complet de la fonction de partition pour un modèle de polymères dirigés en milieux aléatoires sur le réseau hiérarchique en diamant.

Finalement, nous étudions l'énergie libre des polymères dirigés en milieux aléatoires sur $\mathbb{Z}^d$ dans des boites très asymétriques. Nous parvenons à prouver que, dans un régime approprié, elle coïncide avec l'énergie libre d'un modèle en temps continu dans un environnement Brownien. En dimension 1, la valeur exacte de cette énergie libre est connue. Nous étudions également des polymères dirigés en dimension 1 avec un drift qui tend vers l'infini. Nous donnons la valeur exacte de l'énergie libre et l'ordre des fluctuations de la fonction de partition.

We study several models of directed polymers in random environments. For the classical model on $\mathbb{Z}^d$, we study the convergence of the environments seen by the particle in the weak disorder region. We prove strong results for very high values of the temperature.

We then give a complete treatment of the partition function of directed polymers on the diamond hierarchical lattice.

Finally, we study the free energy of directed polymers in random environments on $\mathbb{Z}^d$ in very asymmetric boxes. We prove that, in a particular regime, it coincides with the free energy of a continuous time model in a Brownian environment. For d=1, the exact value of this free energy is known. We also study one-dimensional directed polymers with a huge drift. We give the exact value of the free energy and compute the order of fluctuations of the partition function.

CHAPITRE 1

Introduction

Dans ce chapitre, nous introduisons le modèle classique des polymères dirigés en milieu aléatoire. Ce modèle provient de la physique, où il a été introduit pour tenter de modéliser les interfaces de certains systèmes soumis à des impuretés.

Après une présentation plus détaillée de ces motivations physiques, nous discuterons les résultats mathématiques fondamentaux que l'on retrouve dans la littérature. Quelques preuves complètes sont incluses, leur but étant d'illustrer les outils centraux pour l'étude du modèle : la sous-additivité, la concentration de mesure, le calcul L^2 et les calculs de moments fractionnaires.

Finalement, nous décrivons les résultats originaux qui seront présentés en détail dans les prochains chapitres.

1.1. Le modèle

Le modèle classique de polymères dirigés en milieu aléatoire se décrit à partir de deux objets :

– La marche aléatoire : $(\mathbf{s}_t)_t$ est la marche aléatoire simple symétrique sur $\mathbb{Z}^d$. Nous notons sa loi par P. Donc, $P(\mathbf{s}_{t+1} = x + e|\mathbf{s}_t = x) = 1/2d$, pour tout $t \geq 0$ et $e \in \mathbb{Z}^d$, $|e| = 1$.

– L'environnement : $\{(\eta(t,x)) : t \in \mathbb{Z}, x \in \mathbb{Z}^d\}$ est une famille de nombres réels.

Pour tout N, nous associons ensuite une *énergie* à chaque chemin $\mathbf{s}$:

$$H_N(\mathbf{s}) = \sum_{t=1}^{N} \eta(t, \mathbf{s}_t). \tag{1.1}$$

La mesure de polymère μ_N dans l'environnement η, à température inverse β se définit comme la mesure sur les chemins dont la densité par rapport à P est

$$\frac{d\mu_N^{\eta,\beta}}{dP}(\mathbf{s}) = \frac{1}{Z_N} e^{\beta H_N(\mathbf{s})}, \tag{1.2}$$

où Z_N est donnée par

$$Z_N^{\eta,\beta} = P\left[e^{\beta H_N(\mathbf{s})}\right], \tag{1.3}$$

et est habituellement appelée la fonction de partition.

Nous dotons l'environnement d'une mesure produit Q. Nous étudions les propriétés de la mesure de polymères pour des environnements typiques, c'est à dire des propriétés qui sont vraies pour Q-presque tout environnement η. Une question centrale consiste à étudier la limite

$$p(\beta) = \lim_{N\to+\infty} \frac{1}{N} Q \log Z_N^{\eta,\beta}, \tag{1.4}$$

(connue comme énergie libre *quenched*) et décider si elle coïncide avec l'énergie libre *annealed*

$$\lambda(\beta) = \lim_{N\to+\infty} \frac{1}{N} \log Q Z_N^{\eta,\beta} = \log Q(e^{\beta\eta(0,0)}). \tag{1.5}$$

Il est également possible de prouver que, sous quelques hypothèses sur la loi de l'environnement, la limite (1.4) est une limite presque sûre :

$$p(\beta) = \lim_{N\to+\infty} \frac{1}{N} \log Z_N^{\eta,\beta}, \quad Q-\text{p.s.} \tag{1.6}$$

Nous reviendrons sur ce point dans la section suivante.

Le modèle de polymères dirigés en milieu aléatoire provient de la physique où il a été introduit pour tenter de modéliser le comportement des interfaces dans le modèle d'Ising soumis à des impuretés [**49**]. Il est en étroite relation avec l'équation de Kardar-Parisi-Zhang (KPZ), une équation différentielle partielle stochastique qui modélise des phénomènes de propagation dans un milieu inhomogène. Il a ensuite été étudié mathématiquement dans [**52**], et particulièrement dans [**12**], où l'auteur a introduit des techniques de martingales qui demeurent parmi les outils les plus utilisés dans l'étude des polymères dirigés. De nombreux progrès ont été réalisés dans le traitement mathématique rigoureux de ce modèle (voir par exemple [**53, 28, 24, 26, 21, 20, 30**] et [**25**] pour un *review* récent). On sait à présent qu'il y a une transition de phase depuis une phase délocalisée à haute température où le comportement du polymère est diffusif, vers une phase localisée où l'influence de l'environnement devrait être conséquente et en mesure de produire des phénomènes non triviaux, comme la super-diffusivité. Notons qu'une caractérisation simple de cette dichotomie peut être donnée en terme de la limite d'une martingale positive associée à la fonction de partition du modèle.

On sait qu'en dimension 1 et 2, le polymère est toujours dans la zone de fort désordre, quelle que soit la température (voir [**66**], pour des résultats plus précis). Par contre, pour $d \geq 3$, il existe un intervalle non trivial de températures pour lesquelles on a faible désordre. Une version faible du Théorème Central Limite (TCL) est prouvée dans cette situation dans [**28**]. Le TCL presque sûr ou *quenched* n'est démontré que pour de très grandes valeurs de la température.

Cependant, la valeur exacte de la température critique qui sépare ces deux régions est toujours une question ouverte (bien sur, dans les cas où l'on sait qu'elle est finie). Elle est connue exactement pour le modèle analogue sur l'arbre, cas pour lequel une analyse complète de la fonction de partition est disponible (voir [**15, 41, 60**]). Dans le cas de $\mathbb{Z}^d$, pour $d \geq 3$, un simple calcul L^2 permet de donner une borne supérieure pour la température critique. On sait que cette borne ne coïncide pas avec la valeur réelle (voir [**9, 8**] et [**18**]).

Dans cette thèse nous nous intéresseront à ce modèle ainsi qu'à quelques variantes : nous remplaçons, par exemple, la marche aléatoire simple sur $\mathbb{Z}^d$ par la mesure uniforme sur les chemins dirigés dans le réseau hiérarchique en diamant. Nous considérons également des marches aléatoires très asymétriques et des modèles en temps continu.

1.2. Motivations physiques

Comme nous l'avons mentionné plus haut, le modèle de polymères dirigés en milieu aléatoire provient de la physique où il a été introduit pour tenter de modéliser le comportement des interfaces dans le modèle d'Ising soumis à des impuretés. Dans l'article fondateur [**49**], où le modèle de polymères dirigés est introduit, les auteurs considèrent le système de spins suivant : l'espace des configurations est donné par $\{-1,1\}^{\mathbb{Z}^d}$. Pour $i,j \in \mathbb{Z}^d$, nous notons $i \sim j$ si i et j sont voisins dans $\mathbb{Z}^d$. L'énergie d'une configuration $(s_i)_i$ est donnée par $H = H_0 + H_{\text{imp}}$, où

$$H_0 = J \sum_{i\sim j} s_i s_j, \qquad H_{\text{imp}} = \sum_{i\sim j} \Delta J_{ij} s_i s_j.$$

Les couplages $(\Delta J_{ij})_{ij}$ sont aléatoires (ils traduisent la présence des impuretés) et $J > 0$ est fixé ; les auteurs supposent que le désordre ne présente que des corrélations de faible portée et qu'il est suffisamment faible pour ne pas détruire le caractère ferromagnétique du système : $J + \Delta J_{ij} \geq 0$.

Le paramètre β introduit précédemment correspond à l'inverse de la température. À température nulle, le modèle d'Ising se concentre sur les deux configurations d'énergie optimale : $s_i = 1$ pout tout i, et $s_i = -1$ pour tout i. À basse température et $d \geq 2$, le modèle admet deux phases pures. Lorsque l'on force la coexistence de phases, il se forme une interface $(d-1)$-dimensionnelle séparant deux phases de magnétisations opposées. La composante H_0 de l'énergie tente de minimiser le volume de cette interface, tandis que H_{imp} tente de l'accrocher à une zone énergétiquement favorable.

Les auteurs introduisent un modèle continu pour tenter de modéliser le phénomène précédent : ils supposent que l'interface oscille parallèlement à un hyperplan $(d-1)$-dimensionnel ; soit $z(\mathbf{x})$ la position de l'interface mesurée par rapport à cet hyperplan. À un niveau macroscopique, l'énergie de l'interface est alors donnée par

$$H_c(z) = \int d^{(d-1)}\mathbf{x} \left\{ -\frac{1}{2}\sigma |\nabla z|^2 + V(\mathbf{x}, z(\mathbf{x})) \right\},$$

où le potentiel $V(\mathbf{x}, z)$ est une fonction locale du désordre et ne présente donc que des corrélations de faible portée, et $\sigma > 0$.

En se basant sur des simulations (et des calculs dans le cas $d = 2$, [**40**]), les auteurs concluent que l'interface présente des fluctuations du type

$$z(\mathbf{x}) \simeq |\mathbf{x}|^{\zeta},$$

avec $\zeta = 2/3$, tandis que, sur des segments de longueur L, l'énergie H_c présente des fluctuations de l'ordre L^{ξ}, avec $\xi = 1/3$.

Pour $d = 2$, la fonction de partition point-à-point du système peut s'écrire comme

$$Z(x, y) = \int_{z:(0,0)\to(x,y)} e^{\beta H_c(z)},$$

où l'intégrale porte sur tous les chemins dirigés de l'origine au point (x, y). Cet objet satisfait l'équation

$$\frac{\partial}{\partial x} Z(x, y) = \frac{1}{2\beta\sigma} \Delta_y Z(x, y) + \beta V(x, y) Z(x, y).$$

Cette équation est liée à l'équation de KPZ (pour Kardar-Parisi-Zhang), introduite dans [**57**] pour décrire la dynamique d'une surface $h(x, t)$:

$$\frac{\partial h}{\partial t} = \nu \Delta_x h + \frac{\lambda}{2} |\nabla_x h|^2 + V(t, x).$$

En effet, par une application formelle de la formule d'Itô, $h(x, t) = \log Z(t, x)$ satisfait cette équation avec $\lambda = 2/\beta$ et $\nu = (2\sigma\beta)^{-1}$. Pour de nombreuses applications de l'équation de KPZ, voir [**46**].

Notons que les exposants $\zeta = 2/3$ et $\xi = 1/3$ introduits plus hauts apparaissent à de nombreuses reprises dans la littérature : citons, par exemple, [**72**] dans le cas de la percolation de premier passage, [**53**] pour un modèle de type percolation de dernier passage dans le plan et le travail récent [**86**] pour des polymères dirigés en environnement log-Gamma. Ces phénomènes sont étroitement liés aux fluctuations des valeurs propres de matrices aléatoires Gaussiennes (voir [**2**] et les nombreuses références incluses dans cet ouvrage).

1.3. L'énergie libre

Nous démontrons à présent l'existence de l'énergie libre. La preuve consiste en deux partie : d'abord, nous prouvons l'existence de la limite (1.4) à l'aide de la sous-additivité. Ensuite, à l'aide d'inégalités de concentration, nous prouvons que la limite (1.6) existe, et coïncide avec (1.4).

Pour des entiers positifs N et M,

$$\begin{aligned}
Z_{N+M} &= P(e^{\beta \sum_{t=1}^{N+M} \eta(t,\mathbf{s}_t)}) \\
&= \sum_{x\in\mathbb{Z}^d} P(e^{\beta \sum_{t=1}^{N+M} \eta(t,\mathbf{s}_t)} \mathbf{1}_{\mathbf{s}_N = x}) \\
&= \sum_{x\in\mathbb{Z}^d} P\left(e^{\beta \sum_{t=1}^{N} \eta(t,\mathbf{s}_t)} \mathbf{1}_{\mathbf{s}_N = x}\right) \times P_x\left(e^{\beta \sum_{t=1}^{N} \eta(t+N,\mathbf{s}_t)}\right) \\
&= \sum_{x\in\mathbb{Z}^d} P\left(e^{\beta \sum_{t=1}^{N} \eta(t,\mathbf{s}_t)} \mathbf{1}_{\mathbf{s}_N = x}\right) \times Z_M \circ \theta_{N,x} \\
&= Z_N \sum_{x\in\mathbb{Z}^d} \mu_N\left(\mathbf{s}_N = x\right) \times Z_M \circ \theta_{N,x},
\end{aligned}$$

où l'on a utilisé la propriété de Markov pour la marche aléatoire dans la troisième égalité. Le shift $\theta_{N,x}$ est défini par $\theta_{N,x}\eta(\cdot,\cdot) = \eta(N+\cdot, x+\cdot)$. Comme $\mu_N(\mathbf{s}_N = \cdot)$ est une mesure de probabilité, nous pouvons appliquer l'inégalité de Jensen :

$$\begin{aligned}
\log Z_{N+M} &= \log Z_N + \log \sum_{x\in\mathbb{Z}^d} \mu_N\left(\mathbf{s}_N = x\right) \times Z_M \circ \theta_{N,x} \\
&\geq \log Z_N + \sum_{x\in\mathbb{Z}^d} \mu_N\left(\mathbf{s}_N = x\right) \log Z_M \circ \theta_{N,x}.
\end{aligned}$$

Par construction, $\mu_N\left(\mathbf{s}_N = x\right)$ est indépendant de $\log Z_M \circ \theta_{N,x}$, pour tout N et x. Donc, en intégrant par rapport à Q et en utilisant l'invariance par translations, on obtient

$$Q \log Z_{N+M} \geq Q \log Z_N + Q \log Z_M.$$

Ceci entraîne l'existence de la limite :

$$p(\beta) = \lim_{N\to+\infty} \frac{1}{N} Q \log Z_N.$$

En utilisant l'inégalité de Jensen, on remarque que $p(\beta) \leq \lambda(\beta) < +\infty$. La preuve de l'existence de la limite (1.6) utilise, dans le cas Gaussien, l'inégalité de concentration suivante :

THEORÈME 1.1 (Inégalité de concentration Gaussienne). Soit μ la loi normale centrée sur $\mathbb{R}^K$. Si $f : \mathbb{R}^K \to \mathbb{R}$ est une fonction Lipschitz-continue de constante L, alors

$$\mu\left(x : |f(\mathbf{x}) - \int f d\mu| \geq u\right) \leq 2 \exp\left\{-\frac{u^2}{2L^2}\right\}.$$

Il est facile de voir que la fonction $f : \mathbb{R}^K \to \mathbb{R}$, avec $K = N(N+1)/2$, donnée par

$$f(\mathbf{z}) = \frac{1}{N} \log P\left(e^{\beta \sum_{t=1}^{N} \mathbf{z}(t,\mathbf{s}_t)}\right),$$

est Lipschitzienne de constante $\beta/\sqrt{N}$, et donc

$$Q\left(x : \left| \frac{1}{N} \log Z_N - \frac{1}{N} Q \log Z_N \right| \geq u \right) \leq 2 \exp \left\{ -\frac{Nu^2}{2\beta^2} \right\}.$$

La limite presque sure

$$p(\beta) = \lim_{N \to +\infty} \frac{1}{N} \log Z_N.$$

découle alors du théorème de Borel-Cantelli. Pour des lois Q plus générales, voir [**24**] où les auteurs utilisent des inégalités de concentration pour des martingales, démontrées dans [**71**]. Voir aussi [**29**] pour des résultats plus précis.

1.4. Les techniques de martingales

Nous introduisons à présent la structure de martingale suivante : soit $\mathcal{H}_N = \sigma(\eta(t,x) : t \leq N)$. Alors,

$$W_N(\beta) = Z_N(\beta) e^{-N\lambda(\beta)}$$

est une $\mathcal{H}_N$-martingale positive. En effet,

$$\begin{aligned} Q\left(W_{N+1}(\beta) | \mathcal{H}_N\right) &= Q\left(P\left(e^{\sum_{t=1}^{N+1} \{\beta\, \eta(t,\mathbf{S}_t) - \lambda(\beta)\}} \right) \middle| \mathcal{H}_N \right) \\ &= P\left(e^{\sum_{t=1}^{N} \{\beta \eta(t,\mathbf{S}_t) - \lambda(\beta)\}} Q\left(e^{\beta\, \eta(N+1,\mathbf{S}_{N+1}) - \lambda(\beta)} \middle| \mathcal{H}_N \right) \right) \\ &= P\left(e^{\sum_{t=1}^{N} \{\beta\, \eta(t,\mathbf{S}_t) - \lambda(\beta)\}} Q\left(e^{\beta\, \eta(N+1,\mathbf{S}_{N+1}) - \lambda(\beta)} \right) \right) \\ &= W_N(\beta). \end{aligned}$$

En particulier, W_N converge vers une limite $W_{+\infty}$ qui satisfait la loi du zéro-un :

PROPOSITION 1.1. *$Q(W_{+\infty} > 0) = 0$ ou 1.*

Cette dichotomie caractérise de fait le comportement de la mesure de polymères. Ceci motive la définition suivante.

DEFINITION 1.2. Si $W_{+\infty} > 0$ Q-p.s., nous disons qu'il y a faible désordre. Dans le cas contraire, il y a désordre fort.

Notons que s'il y a désordre faible, les énergies libres quenched et annealed sont toujours égales. Cependant l'affirmation réciproque n'est pas immédiate et est en fait toujours une question ouverte (sauf dans les cas $d = 1$ et $d = 2$ où il est prouvé que les énergies libres quenched et annealed sont différentes pour tout $\beta > 0$). Quand les énergies libres quenched et annealed sont différentes, nous disons qu'il y a désordre très fort.

Nous pouvons obtenir une équation fonctionnelle pour $W_{+\infty}$: la récurrence élémentaire pour la fonction de partition :

$$Z_{N+1}(\beta) = \frac{1}{2d} \sum_{|e|=1} e^{\beta\eta(1,e)} Z_N(\beta) \circ \theta_{1,e}, \tag{1.7}$$

où $\theta_{t,x}\eta(\cdot,\cdot) = \eta(t+\cdot, x+\cdot)$, entraine

$$W_{+\infty}(\beta) = \frac{1}{2d} \sum_{|e|=1} e^{\beta\eta(1,e)-\lambda(\beta)} W_{+\infty}(\beta) \circ \theta_{1,e}. \tag{1.8}$$

Les différents termes de la somme ne sont bien sûr pas indépendants.

1.5. Le diagramme de phases

Le passage d'un régime de désordre faible à un régime de fort désordre constitue une transition de phase qui se produit quand β grandit. Nous avons le résultat suivant :

PROPOSITION 1.3. *(*[**28**], *Theorem* 1.1*) Il éxiste une valeur critique* $\beta_c \in [0. +\infty]$ *telle que*
– *Pour tout* $\beta < \beta_c$*, il y a désordre faible.*
– *Pour tout* $\beta > \beta_c$*, il y a désordre fort.*

Pour $d = 1$ et $d = 2$, on sait que $\beta_c = 0$. La proposition suivante permet de déterminer des cas pour lesquels $\beta_c \in (0, +\infty)$:

PROPOSITION 1.4. *(i) Si* $d \geq 3$ *et* β *est assez petit, alors* $\sup_N Q(W_{+\infty}^2) < +\infty$ *et il y a désordre faible.*
(ii) Si $\beta\lambda'(\beta) - \lambda(\beta) > \log 2d$*, alors il y a (très) fort désordre.*

DÉMONSTRATION. Pour démontrer le premier point, on définit

$$\pi_d = P_0^{\otimes 2}\left[\omega_t = \widetilde{\omega}_t \text{ for some } t\right].$$

On observe que $\pi_d < 1$ si et seulement si $d \geq 3$. Nous pouvons calculer le second moment de W_N à l'aide de deux copies indépendantes de marches aléatoires :

$$\begin{aligned} Q(W_N^2) &= QP^{\otimes 2}\left[\exp\left\{\beta \sum_{t=1}^N \eta(t,\omega_t) + \beta \sum_{t=1}^N \eta(t,\widetilde{\omega}_t) - 2N\lambda(\beta)\right\}\right] \\ &= e^{-2N\lambda} P^{\otimes 2}\left[\prod_{t=1}^N Q(e^{2\beta\eta(t,\omega_t)\mathbf{1}_{\omega_t=\widetilde{\omega}_t}}) \times Q(e^{\beta\eta(t,\widetilde{\omega}_t)\mathbf{1}_{\omega_t\neq\widetilde{\omega}_t}})^2\right] \\ &= P^{\otimes 2}\left[e^{(\lambda(2\beta)-2\lambda(\beta))L_N(\omega,\widetilde{\omega})}\right], \end{aligned}$$

où $L_N(\omega,\widetilde{\omega}) = \sum_{t=1}^N \mathbf{1}(\omega_t = \widetilde{\omega}_t)$. Observons que $L_{+\infty}(\omega,\widetilde{\omega})$ a une distribution géométrique de paramètre π_d, et donc $Q(W_N^2)$ est uniformément borné en N pour

$$\lambda(2\beta) - 2\lambda(\beta) < \log(1/\pi_d), \tag{1.9}$$

ce qui entraîne l'uniforme intégrabilité de la martingale $(W_N)_N$. Le point (ii) est un exemple de calcul de moments fractionnaires pour démontrer qu'il y a décroissance exponentielle. Observons d'abord

$$p(\beta)-\lambda(\beta)=\lim_{N\to+\infty}\frac{1}{N}Q(\log W_N)\leq\limsup_{N\to+\infty}\frac{1}{\theta N}\log Q(W_N^\theta), \tag{1.10}$$

grâce à l'inégalité de Hölder. Nous montrerons que, si $\beta\lambda'(\beta)-\lambda(\beta)>\log 2d$, il existe une constante $c>0$ telle que $QW_N^\theta\leq e^{-cN}$.
La récurrence (1.7) et l'inégalité élémentaire

$$\left(\sum_i x_i\right)^\theta\leq\sum_i x_i^\theta,$$

pour $0<\theta<1$ et $x_i\geq 0$ pour tout i, nous permettent de conclure

$$QW_N^\theta\leq e^{(1-\theta)\log 2d+\lambda(\theta\beta)-\theta\lambda(\beta)}QW_{N-1}^\theta,$$

Posons $g(\theta)=(1-\theta)\log 2d+\lambda(\theta\beta)-\theta\lambda(\beta)$. Cette fonction est convexe et continuement différentiable, et satisfait $g(0)=\log 2d>0=g(1)$. Pour qu'il existe $x\in(0,1)$ tel que $g(x)<0$ il suffit que $dg(\theta)/d\theta|_{\theta=1}>0$, ce qui équivaut à $\beta\lambda'(\beta)-\lambda(\beta)>\log 2d$. Ceci implique (ii).

□

Remarque 1.5. La condition (1.9) est connue comme la condition L^2 et détermine un sous-intervalle de la région de faible désordre connu comme la région L^2. Elle garantit que, pour $d\geq 3$, il existe une valeur $\beta_2>0$ telle que, pour $\beta<\beta_2$, il y a faible désordre. C'est donc une borne inférieure pour β_c. L'inégalité stricte $\beta_2<\beta_c$ est une conséquence des résultats prouvés dans [**8**].

Remarque 1.6. La deuxième condition est vérifiée pour de nombreuses lois de l'environnement, pour β assez grand, par exemple, pour des environnements non-bornés supérieurement ou des environnements bornés supérieurement qui ont une masse suffisante sur leur supremum essentiel. Cependant, il existe des environnements pour lesquels elle n'est jamais vérifiée.

1.6. Répliques et overlaps

Nous avons vu qu'il est possible de prouver qu'il y a faible désordre pour des petites valeurs de β en contrôlant certains moments exponentiels de marches aléatoires. Il existe en fait une relation très étroite entre les intersections (les *overlaps*) de deux polymères indépendants sur le même environnement et la caractérisation faible désordre / fort désordre donnée par la Définition (1.2).

Definition 1.7.

$$I_t(\beta)=\mu_t^{\otimes 2}\left(\omega_{t+1}=\widetilde{\omega}_{t+1}\right). \tag{1.11}$$

La proposition suivante traduit la dichotomie $Q(W_{+\infty} > 0) = 0$ ou 1, en terme des overlaps et donne des informations plus quantitatives dans les cas de fort désordre.

PROPOSITION 1.8. [**20, 24**]

$$\{W_{+\infty} > 0\} = \left\{ \sum_{t=1}^{+\infty} I_t < +\infty \right\}. \tag{1.12}$$

De plus, quand $Q(W_{+\infty} = 0) = 1$, *il existe des constantes* c *et* C, *telles que,*

$$c \sum_{k=1}^{N} I_k \le -\log W_N \le C \sum_{k=1}^{N} I_k. \tag{1.13}$$

Le phénomène de localisation peut-être quantifié de la façon suivante :

PROPOSITION 1.9. [**20, 24**] *Suposons que* $d = 1, 2$ *ou* $d \le 3$ *et* $\beta\lambda'(\beta) - \lambda(\beta) > \log 2d$. *Alors, il existe une constante* $c = c(\beta, d)$ *telle que*

$$\limsup_{N\to+\infty} I_N \ge c, \quad Q-\text{a.s.}. \tag{1.14}$$

Pour plus de résultats concernant la localisation, voir [**93**].

1.7. Théorème central limite en faible désordre

Pour de très grandes valeurs de la température, et $d \ge 3$, il est possible de démontrer un principe d'invariance *quenched* ou Q-presque sur :

THEORÈME 1.2. [**1, 12, 52, 88**] Dans la région L^2, pour Q-presque tout environnement, la suite de processus renormalisés,

$$\omega^{(N)}(\cdot) = \omega_{\cdot N}/\sqrt{N}, \tag{1.15}$$

converge, sous la mesure de polymères, vers un mouvement Brownien de variance $d^{-1}I$, où I est la matrice identité dans $\mathbb{R}^d$.

Cependant, dans la totalité de la région de faible désordre, seule une version en Q probabilité de ce résultat a été obtenue :

THEORÈME 1.3. [**28**] Dans la région de faible désordre, pour toute fonction F bornée et continue des trajectoires,

$$\lim_{N\to+\infty} \mu_N(F(\omega^{(N)})) = QF(B),$$

en Q-probabilité, où B est un mouvement Brownien de variance $d^{-1}I$ et $\omega^{(N)}$ est donné par (1.15).

1.8. Modèles similaires

De nombreux modèles de polymères dirigés ont été étudiés. Nous donnons ici une liste non-exhaustive et des références. Notons que de nombreuses techniques ou démarches générales sont communes à tout ces modèles, mais pour chacun d'eux, des méthodes spécifiques reposant sur la structure particulière du modèle ont pu être développées pour obtenir des informations additionnelles. Par exemple, l'utilisation de techniques de concentration pour obtenir la limite presque sûre de la fonction de partition est plus ou moins récurrente, bien qu'elle découle parfois de principe généraux (dans les modèles discrets) ou parfois de calcul stochastique, voire de calcul de Malliavin (dans les cas Gaussiens). Autre exemple, la structure particulière de l'arbre rend le traitement de la fonction de partition bien plus simple que sur $\mathbb{Z}^d$.

1.8.1. Polymères dirigés sur l'arbre. Ce modèle est défini de façon identique que sur $\mathbb{Z}^d$. La marche aléatoire simple est remplacée par une marche aléatoire dirigée sur l'arbre qui, à chaque instant, s'éloigne un pas de la racine. La structure d'arbre entraîne l'indépendance de tous les termes de la somme (1.8), où $2d$ doit être remplacé par le degré de l'arbre.

Ce modèle a été abordé à plusieurs reprises ([**39, 60, 41, 15, 78**]). La fonction de partition est bien connue. Nous exposons plus de résultats dans le Chapitre 3.

1.8.2. Polymères Browniens dans un environnement Poissonnien. Ici, P est la loi du mouvement Brownien dans $\mathbb{R}^d$ et Q, la loi d'un processus de Poisson d'intensité 1 dans $\mathbb{R}^{d+1}$ dénoté par η.
Pour $x \in \mathbb{Z}^d$, soit $V(x)$ la boule unitaire centrée en x. Pour un environnement η fixé et toute trajectoire $\mathbf{s} : [0,T] \to \mathbb{R}^d$, nous définissons l'énergie :

$$H_T^\eta(\mathbf{s}) = \eta\left(V(\mathbf{s}_t) : \, t \in [0,T]\right).$$

Finalement, la mesure de polymères μ_T^η est définie comme :

$$\frac{d\mu_T^\eta}{dP}(\mathbf{s}) = \frac{1}{Z_T^\eta} \exp\left\{\beta H_T^\eta(\mathbf{s})\right\}.$$

où Z_T^η est la fonction de partition. Ce modèle est étudié en détail dans [**26**] et [**27**].

1.8.3. Polymères en temps continu dans un environnement Gaussien. Ici, la mesure originelle P sur les chemins est une marche aléatoire en temps continu sur $\mathbb{Z}^d$, de taux de saut 1. L'environnement est une collection de mouvements Browniens indépendants $\{B(\cdot, \mathbf{x}) : \, \mathbf{x} \in \mathbb{Z}^d\}$. L'énergie d'une trajectoire $\mathbf{s}$ est définie comme

$$H_T(\mathbf{s}) = \int_0^T dB(t, \mathbf{s}_t).$$

La définition de la mesure de polymère est analogue aux cas précédents. La structure Markovienne des mouvements Browniens permet d'appliquer les techniques classiques de polymères dirigés, en particulier, les techniques de martingales.

La fonction de partition (à un retournement du temps près) satisfait l'équation de Anderson parabolique

$$dy_t = \Delta y_t dt + \frac{1}{\beta} y_t dB(t, y_t),$$

où Δ est le Laplacien discret. Ce lien fait que l'étude de ce modèle ait été amorcée tant du point de vue du modèle de Anderson [**32**] que des polymères [**17, 73**].

Nous pouvons citer également les modèles de polymères dirigés Browniens considérés dans [**17**], où P est un mouvement Brownien et Q est un processus Gaussien de corrélations

$$C(t_1, \mathbf{x}_1; t_2, \mathbf{x}_2) = \min(t_1, t_2) K(|\mathbf{x}_1 - \mathbf{x}_2|),$$

avec $K()$ satisfaisant certaines hypothèses de régularité. Pour des corrélations appropriées, il est possible de prouver que le polymère est super-diffusif en dimension 1 (voir [**7**]). Pour un modèle similaire, voir aussi [**94**].

1.8.4. Polymères dirigés en environnement Brownien : Ce modèle sera abordé rigoureusement dans la deuxième partie. Il est introduit dans [**79**]. Il s'agit, en quelque sorte, d'une version dirigée de la marche aléatoire en temps continu dans un environnement Brownien décrite plus haut. L'étude de ce modèle en dimension 1 a été particulièrement fructueuse (voir [**79**] et [**77**]). À l'aide de techniques inspirées des files d'attentes, les auteurs identifient de façon explicite la fonction de partition. Nous nous intéresserons particulièrement aux relations entre ce modèle et des modèles discrets et très asymétriques de polymères dirigés.

1.9. Présentation des résultats de la thèse

Nous résumons à présent les principaux problèmes abordés dans cette thèse. Nous supposons toujours que les environnement que nous considérons ont au moins des moments exponentiels finis.

L'ENVIRONNEMENT VU PAR LA PARTICULE :

Dans le Chapitre 2, nous étudions l'environnement vu par la particule pour les polymères dirigés en milieu aléatoire. Ce type d'objets apparaît fréquemment dans l'étude des processus en milieux aléatoires, mais n'avait pas encore été étudié jusqu'ici dans le cas des polymères.
L'environnement vu par la particule à l'instant N est un champs aléatoire obtenu à partir de l'environnement initial en se positionnant sur un point précis de la trajectoire du polymère. Rigoureusement,

$$\eta_{N,M}(t, x) = \eta(N + t, \omega_N + x), \quad t \in \mathbb{Z},\ x \in \mathbb{Z}^d,$$

est l'environnement vu par la particule depuis sa position 'à l'instant N', quand la trajectoire ω est choisie par rapport à la mesure μ_{N+M}. Il s'agit de l'environnement vu d'une position intermédiaire, ce qui généralise un peu la situation habituelle, où l'on considère l'environnement vu du point final. Il est facile de prouver que la densité de ce champ par rapport à Q est donnée par

$$\frac{dQ_N}{dQ} = \sum_x \mu^x_{-N,0}(\omega_N = 0). \tag{1.16}$$

Le résultat principal du Chapitre 2 est contenu dans le théorème suivant qui sera prouvé dans la Section 2.4 :

THEOREM 1.10. *Considérons des environnement bornés ou Gaussiens. Dans la région L^2 region,*

$$q_{N,M} \longrightarrow \overleftarrow{W}_{+\infty} \times e^{\beta\eta(0,0)-\lambda(\beta)} \times W_{+\infty}, \quad \text{as } M,\ N \to +\infty,$$

où la convergence a lieu dans $L^1(Q)$.

Ici, $\overleftarrow{W}_{+\infty}$ se définit comme $W_{+\infty}$ sur l'environnement réfléchi par rapport à l'hyperplan $t = 0$. $\overleftarrow{W}_{+\infty}$ et $W_{+\infty}$ sont indépendants et de même loi.

Les techniques employées dans notre preuve semblent être restreintes à la région L^2. Cependant, nous discutons un résultat plus faible dans la totalité de la région de faible désordre, dans la dernière section du Chapitre 2. Ce résultat, qui utilise des techniques habituelles de marches aléatoires en milieux aléatoires, établit la convergence des moyennes ergodiques de l'environnement vu par la particule, quand les trajectoires suivent une loi définie en volume infini.

LES POLYMÈRES DIRIGÉS SUR LE RÉSEAU HIÉRARCHIQUE EN DIAMANT :

Les polymères dirigés possèdent une structure de corrélation qui s'exprime de façon naturelle en terme d'intersections de marches aléatoires indépendantes (voir le calcul L^2 dans la preuve de la Proposition 1.4). Pour cette raison, il est tentant de croire que les modèles de polymères définis sur des espaces où cette structure est plus simple que sur $\mathbb{Z}^d$ devraient être plus simples à traiter. C'est le cas du modèle sur l'arbre, pour lequel un traitement complet de la fonction de partition a été donné.
L'arbre est un cas assez lointain de $\mathbb{Z}^d$: deux marches qui se séparent ne se s'intersectent jamais par la suite. Cette simple observation a des conséquences très importantes sur le comportement des polymères dirigés : sur $\mathbb{Z}^d$, les réintersections entrainent d'importantes fluctuations de l'énergie qui, à leur tour impliquent la convexité de l'énergie libre. Le réseau hiérarchique en diamant est, quant à lui, plus proche de $\mathbb{Z}^d$ en ce sens, tout en conservant une certaine simplicité de par sa structure hiérarchique.

Le réseau hiérarchique en diamant se construit par récurrence de la façon suivante :
- D_0 est composé de deux sites A et B unis par une arête.
- D_{n+1} s'obtient de D_n en remplaçant chaque arête par b branches de $s-1$ arêtes.

Cette famille de graphes apparaît naturellement en mécanique statistique dans l'étude de modèles de spins à l'aide de techniques de renormalisation. Cette procédure donne lieu à des équations récursives qui peuvent être traitées aisément (consulter les références du Chapitre 4).

Nous considérons P_n la mesure de probabilité uniforme sur les chemins dirigés de A à B. Les polymères dirigés sur le réseau hiérarchique en diamant ont été introduits par Cook et Derrida, dans [**31**]. Les auteurs considèrent un modèle avec désordre sur les arêtes : l'énergie d'un chemin donné correspond à la somme des variables d'environnement le long des arêtes visitées par le chemin. Ce modèle permet d'étudier la fonction de partition (plus particulièrement, son second moment) à l'aide d'équations récursives très simples. Cependant, il ne permet pas d'appliquer les techniques de martingales habituelles. C'est pour cette raison que nous avons choisi d'étudier le modèle analogue avec désordre par site : bien que les équations récursives obtenues soient un peu plus compliquées, nous récupérons la propriété de martingale pour la fonction de partition renormalisée.

Nos principaux résultats traitent de
- l'existence de l'énergie libre.
- la convexité stricte de l'énergie libre.
- conditions suffisantes pour l'existence de zones de faible et fort désordre.
- calculs d'exposants dans les cas où il y a fort désordre pour toute température finie ($b \leq s$).

La stricte convexité de l'énergie libre indique d'une certaine façon que le modèle que nous considérons est plus proche des polymères sur $\mathbb{Z}^d$ que les modèles sur l'arbre. En effet, elle est constante dans toute la région de fort désordre dans le cas de l'arbre mais strictement convexe pour $\mathbb{Z}^d$. Le comportement particulier des cas $b \leq s$ quant à lui rapelle les cas de petite dimension ($d = 1$ et 2).

Nous présentons brièvement les modèles de polymères dirigés sur l'arbre et sur le réseau hiérarchique en diamant avec désordre par arête dans le Chapitre 3. Le cas du désordre par site est traité de façon approfondie dans le Chapitre 4. Il s'agit d'un travail en collaboration avec Hubert Lacoin [**67**].

Polymères dirigés dans un environnement Brownien :

La deuxième partie de cette thèse établit des liens entre des modèles discrets très asymétriques et des modèles continus en environnement Brownien.

D'un côté, nous considérons le modèle habituel de polymères dirigés dans $\mathbb{Z}^d$, mais nous étudions la fonction de partition point-à-point $Z_\beta(N, \mathbf{x})$, c'est à dire,

$$Z_\beta(N, \mathbf{x}) = P_{N,\mathbf{x}} \left(\exp \beta H(\mathbf{S}) \right),$$

où $P_{N,\mathbf{x}}$ est la probabilité uniforme sur les chemins dirigés de l'origine au point $(N, \mathbf{x})$ et $H(\mathbf{S})$ est l'énergie du chemin $\mathbf{S}$. Nous allons nous intéresser essentiellement au cas où $\mathbf{x} = \mathbf{x}_N$ est tel que chacune de ses coordonnées croit comme N^a avec $a \in (0,1)$. Une situation similaire a déja été étudiée dans le cas de la percolation de dernier passage (voir [**11**]).

D'un autre côté, nous considérons des polymères dirigés en temps continu dans un environnement Brownien : soit $P^c_{N,\mathbf{x}}$ la probabilité uniforme sur les chemins $\mathbf{s} : [0,N] \to \mathbb{Z}^d$ tels que :

– $\mathbf{s}_0 = 0$ et $\mathbf{s}_N = \mathbf{x}$,
– $\mathbf{s}$ saute exactement $|\mathbf{x}|$ fois,
– les sauts de $\mathbf{s}$ sont unitaires et s'effectuent selon un des vecteurs coordonnés.

Considérons $\{B(\mathbf{y}) : \mathbf{y} \in \mathbb{Z}^d\}$ une famille de mouvements Browniens unidimensionnels indépendants. Nous associons à chaque chemin $\mathbf{s}$ une énergie :

$$H^{\mathbf{Br}}(\mathbf{s}) = \int_0^N dB_u(\mathbf{s}_u).$$

Finalement, la fonction de partition de notre modèle Brownien est définie comme

$$Z^{\mathbf{Br}}(N, \mathbf{x}) = P^c_{N,\mathbf{x}}\left(\exp \beta H^{\mathbf{Br}}(\mathbf{s})\right).$$

Il est facile de démontrer, en utilisant les techniques habituelles, que l'énergie libre existe dans le régime $\mathbf{x} = \mathbf{x}_N = N\alpha$ avec $\alpha \in \mathbb{Z}^d$. En particulier, dénotons par $f(\beta)$ l'énergie libre pour le cas $\mathbf{x}_N = (N, \cdots, N)$. En dimension 1, et $\mathbf{x} = N$, la valeur explicite de l'énergie libre est connue.

Nous pouvons maintenant énoncer le résultat principal du Chapitre 5 :

Theorem 1.11. *Soit $\beta_{N,a} = \beta N^{(a-1)/2}$ et $\boldsymbol{x}_{N,a} = (N^a, \cdots, N^a)$ avec $a \in (0,1)$. Alors,*

$$\lim_{N \to +\infty} \frac{1}{\beta_{N,a} N^{(1+a)/2}} \log Z_{\beta_{N,a}}(N, \boldsymbol{x}_{N,a}) = f(\beta)/\beta, \tag{1.17}$$

où $f(\beta)$ est l'énergie libre du modèle Brownien.

Nous considérons également une 'Poissonisation' de la fonction de partition point-à-point des polymères dirigés en dimension 1. Plus précisément, nous étudions la fonction génératrice :

$$Z^{(h)}_{\beta,N} = \sum_{1 \le n \le N} \overline{Z}_\beta(n, N-n) e^{-h \times (N-n)},$$

où, pour chaque n, $\overline{Z}_N(n, N-n)$ est la fonction de partition point-à-point

$$\overline{Z}_\beta(n, N-n) = \sum_{\mathbf{S}:(0,0)\to(n,N-n)} e^{\beta H_N(\mathbf{S})}.$$

Le rôle de h est de pénaliser les chemins qui s'éloignent de l'axe horizontal et peut donc être considéré comme un biais. Notre résultat est le suivant :

THEOREM 1.12. *Prenons $h = h_N = \gamma N^{(1-a)/2}$. Alors,*

$$\lim_{N\to+\infty} \frac{1}{N^{(1+a)/2}} \log Z_{\beta,N}^{(h_N)} = \frac{\beta^2}{\gamma}.$$

Nous discutons également les fluctuations autour de cette limite.

Part 1

Discrete Models of Directed Polymers

CHAPTER 2

The environment seen from the particle in the L^2 region

We consider the model of Directed Polymers in an i.i.d. gaussian or bounded Environment [**52, 20, 25**] in the L^2 region. We prove the convergence of the law of the environment seen by the particle.

As a main technical step, we establish a lower tail concentration inequality for the partition function for bounded environments. Our proof is based on arguments developed by Talagrand in the context of the Hopfield Model [**90**]. This improves in some sense a concentration inequality obtained by Carmona and Hu for gaussian environments [**20**]. We use this and a Local Limit Theorem [**87, 92**] to prove the L^1 convergence of the density of the law of the environment seen by the particle with respect to the product measure.

We discuss weaker results in the whole weak disorder region.

2.1. Introduction

We consider the following model of directed polymers in a random media: let $d \geq 3$ and let $\{\eta(t,x) : t \in \mathbb{Z}, x \in \mathbb{Z}^d\}$ denotes a family of real variables. We call it the environment. For $y \in \mathbb{Z}^d$, let P_y be the law of the simple symmetric nearest neighbor random walk on $\mathbb{Z}^d$ starting at y. For $L \in \mathbb{N}$, we denote the space of nearest neighbor paths of length L by Ω_L, i.e.

$$\Omega_L := \left\{ s : \{0,1,2,...,L\} \to \mathbb{Z}^d, \, |s_{t+1} - s_t| = 1, \, \forall\, t = 0,...,L-1 \right\},$$

where $|\cdot|$ is the euclidean norm. Denote by ω the canonical process on Ω_L, i.e., $\omega_t(s) = s_t$ for $s \in \Omega_L$ and $t = 0,1,...,L$.

For a fixed configuration η, $y \in \mathbb{Z}^d$, M, $N \in \mathbb{Z}$, $M < N$ and $0 < \beta < +\infty$, we can define the (quenched) law of the polymer in environment η, inverse temperature β, based on (M,y) and time horizon N (or simply the polymer measure): for all $s \in \Omega_{N-M}$,

$$\mu^y_{M,N}(\omega = s) = \frac{1}{Z^y_{M,N}} \exp\left\{ \beta \sum_{t=1}^{N-M} \eta(t+M, s_t) \right\} P^y(\omega = s), \tag{2.1}$$

where

$$Z^y_{M,N} = P^y\left[\exp\left\{ \beta \sum_{t=M+1}^{N} \eta(t+M, \omega_t) \right\} \right], \tag{2.2}$$

is usually called the partition function. $\mu^y_{M,N}$ can be though as a measure on directed paths, starting from y at time M and ending at time N. When $M = 0$ and $y = 0$, we will sometimes not write them.

We endow the space $E = \mathbb{R}^{\mathbb{Z}^{d+1}}$ with the product σ-algebra and a product measure Q. We can then see η as a random variable with values in E. In the following we will deal with laws Q which marginals are Gaussian or have bounded support. We are interested in the properties of the polymer measure for a typical configuration η.

A quantity of special interest is $\lambda(\beta) = \log Q(e^{\beta\eta(0,0)})$. We can compute it for simple laws of the environment. For example, in the case of a standard Gaussian law, we have $\lambda(\beta) = \beta^2/2$.

The model of Directed Polymers in Random Environment as been extensively studied over the last thirty years [**52, 20, 25, 30, 93**]. It turns out that the behavior of the polymer depends strongly on the temperature. Here we will be concerned by a region of very high temperature, or equivalently, we will focus on very small values of β. Let state this more precisely: we can define the normalized partition function,

$$W_N = Z_N \exp\{-N\lambda(\beta)\}, \tag{2.3}$$

It follows easily that this is a positive martingale with respect to the filtration $\mathcal{H}_N = \sigma(\eta(t,x) : t \leq N)$. Then, by classical arguments, we can prove that it converges when $N \to +\infty$ to a non-negative random variable $W_{+\infty}$ that satisfies the following zero-one law:

$$Q(W_{+\infty} > 0) = 0 \text{ or } 1.$$

In the first situation, we say that strong disorder holds. In the second, we say that weak disorder holds. The behavior is qualitatively different in each situation [**25**]. Let recall an argument showing that the weak disorder region is nontrivial, at least for $d \geq 3$ and a wide class of environments. Observe first that in order to prove weak disorder, it is enough to obtain some uniform integrability condition on the martingale $(W_N)_N$. Indeed, uniform integrability implies that $Q(W_{+\infty}) = 1$, and then the zero-one law gives us that $Q(W_{+\infty} > 0) = 1$. The easiest situation occurs when the martingale is bounded in L^2: Define

$$\pi_d = P_0^{\otimes 2}\left[\omega_t = \widetilde{\omega}_t \text{ for some } t\right].$$

We see that $\pi_d < 1$ only for $d \geq 3$. Let's perform the following elementary calculation:

$$\begin{aligned} Q(W_N^2) &= QP^{\otimes 2}\left[\exp\left\{\beta\sum_{t=1}^N \eta(t,\omega_t) + \beta\sum_{t=1}^N \eta(t,\widetilde{\omega}_t) - 2N\lambda(\beta)\right\}\right] \\ &= e^{-2N\lambda}P^{\otimes 2}\left[\prod_{t=1}^N Q(e^{2\beta\eta(t,\omega_t)\mathbf{1}_{\omega_t=\widetilde{\omega}_t}}) \times Q(e^{\beta\eta(t,\widetilde{\omega}_t)\mathbf{1}_{\omega_t\neq\widetilde{\omega}_t}})^2\right] \\ &= P^{\otimes 2}\left[e^{(\lambda(2\beta)-2\lambda(\beta))L_N(\omega,\widetilde{\omega})}\right], \end{aligned} \tag{2.4}$$

where $L_N(\omega,\widetilde{\omega}) = \sum_{t=1}^N \mathbf{1}(\omega_t = \widetilde{\omega}_t)$. We observe that $L_{+\infty}(\omega,\widetilde{\omega})$ has a geometric distribution with parameter π_d, so that $Q(W_N^2)$ is uniformly bounded in N for

$$\lambda(2\beta) - 2\lambda(\beta) < \log(1/\pi_d), \tag{2.5}$$

and uniform integrability follows. We call this the L^2 condition. In the general case, we can prove that, in fact, such a condition holds for sufficiently small values of β (we can see this directly for Gaussian environments). The range of values for which (2.5) holds is called the L^2 region.

We know [**52, 12, 1, 88**] that in the L^2 region, the polymer is diffusive. Indeed, we have the following (quenched) invariance principle:

THEOREM 2.1. [**1, 12, 52, 88**] *In the L^2 region, for Q-almost every environment, we have that, under the polymer measure,*

$$\frac{1}{\sqrt{N}}\,\omega(Nt)$$

converges in law to a Brownian motion with covariance matrix $1/d\,I$, where I is the identity matrix in dimension d.

More generally, in the full weak disorder region, a slightly weaker result holds ([**28**]). The situation is quite different and more subtle in the strong disorder region. In that case, large values of the medium attract the path of the polymer, so that a localization phenomenon arises (for more information, see [**20, 28**], and also [**93**] for milder assumptions).

In this article, we are concerned with another kind of (still related) result, namely the convergence of the law of the environment viewed by the particle. The environment viewed by the particle is a process η_N with values in E, defined by

$$\eta_N(t,x) = \eta(N+t, \omega_N + x), \quad t \in \mathbb{Z},\ x \in \mathbb{Z}^d. \tag{2.6}$$

where ω follows the law $\mu^0_{0,N}$. Let's denote by Q_N the law of this process at time N. We can see that

$$\frac{dQ_N}{dQ} = \sum_x \mu^x_{-N,0}(\omega_N = 0). \tag{2.7}$$

Indeed, take a bounded measurable function $f : E \to \mathbb{R}$. Then, by translation invariance,

$$\begin{aligned} Q\left(\mu^0_{0,N} f(\eta_N)\right) &= Q\left(\sum_x f(\eta(N+\cdot, x+\cdot))\mu^0_{0,N}(\omega_N = x)\right) \\ &= Q\left(f(\eta)\sum_x \mu^x_{-N,0}(\omega_N = 0)\right) \\ &= Q_N\left(f(\omega)\right). \end{aligned} \tag{2.8}$$

For a wide variety of models, the convergence of Q_N has been a powerful tool for proving invariance theorems. Obviously, it is not the case here because we have already the invariance principle at hand. However, we can see this result as completing the picture in the L^2 region. In the best of our knowledge, the point of view of the particle has never been studied before in the literature for directed polymers. We can state the principal result of this article: let $\overleftarrow{\eta}(t,x) := \eta(-t,x)$, $\overleftarrow{W}_N(\eta) := W_N(\overleftarrow{\eta})$ and let $\overleftarrow{W}_{+\infty}$ be the (almost sure) limit of $\overleftarrow{W}_N$ when N tends to infinity.

2.2. Results

Theorem 2.2. *In the L^2 region,*

$$q_N := \frac{dQ_N}{dQ} \longrightarrow \overleftarrow{W}_{+\infty} \times e^{\beta\eta(0,0)-\lambda(\beta)}, \quad \text{as } N \to +\infty, \tag{2.9}$$

where the limit is in the $L^1(Q)$ sense.
In other words, Q_N converges in the total variation distance to a probability measure $Q_{+\infty}$ such that

$$\frac{dQ_{+\infty}}{dQ} = \overleftarrow{W}_{+\infty} \times e^{\beta\eta(0,0)-\lambda(\beta)}.$$

Much in the same spirit, we consider the law of the environment seen by the particle at an intermediate time N under $\mu^0_{0,N+M}(\cdot)$. Formally, this new environment is defined as the field $\eta_{N,M} \in E$ with

$$\eta_{N,M}(t,x) = \eta(N+t, \omega_N + x), \quad t \in \mathbb{Z},\, x \in \mathbb{Z}^d,$$

where ω is taken from $\mu^0_{0,N+M}$. Following the argument of (2.8), its density with respect to Q is easily seen to be

$$q_{N,M} = \sum_x \mu^x_{-N,M}(\omega_N = 0). \tag{2.10}$$

Theorem 2.3. *In the L^2 region, we have*

$$q_{N,M} \longrightarrow \overleftarrow{W}_{+\infty} \times e^{\beta\eta(0,0)-\lambda(\beta)} \times W_{+\infty}, \quad \text{as } M,\, N \to +\infty,$$

where the limit is in the $L^1(Q)$ sense.

A statement similar to Theorem 2.3 may be found in Bolthausen and Sznitman [**13**] for directed random walks in random environment. In their context, q_N is a martingale. In our case, the denominator in the definition of the polymer measure is quite uncomfortable as it depends on the whole past, so no martingale property should be expected for q_N. Let us mention that, we can cancel this denominator by multiplying each term of the sum in (2.9) by $W^x_{-N,0}$. This defines a martingale sequence that converges (almost surely!) to the same limit as q_N. In other terms, let $\mathcal{G}_N = \sigma(\eta(t,x) : -N \leq t)$. Then, there exists a unique law $Q_{+\infty}$ on E such that for any $A \in \mathcal{G}_N$,

$$Q_{+\infty}(A) = Q(\mathbf{1}_A \overleftarrow{W}_N e^{\beta\eta(0,0)-\lambda(\beta)}).$$

The density of this law coincides with the limit in Theorem 2.3 but we emphasize that our strategy of proof here is completely different.

Let us mention that, in the weak disorder region, an infinite time horizon polymer measure has been introduced in [**28**]. For each realization of the environment, it defines a Markov process with (inhomogeneous) transition probabilities given by

$$\mu_\infty^\beta(\omega_{N+1} = x + e | \omega_N = x) = \frac{1}{2d} e^{\beta\eta(N+1,x+e)-\lambda(\beta)} \frac{W_{+\infty}(N+1, x+e)}{W_{+\infty}(N,x)},$$

for all $x \in \mathbb{Z}^d$ and $|e| = 1$, where for $k \in \mathbb{Z}$ and $y \in \mathbb{Z}^d$,

$$W_{+\infty}(k,y)(\eta) = W_{+\infty}(\eta(k+\cdot, y+\cdot)).$$

We can define the process of the environment seen by the particle for this polymer measure by means of the formula (2.6), where, this time, ω_N is taken from $\mu_{+\infty}^\beta$. A simple modification of the proof of the Theorem 2.3 suffices to show that the density of the environment seen by the particle converges, as N tends to $+\infty$, to the same limiting density as in Theorem 2.3. The additional term $W_{+\infty}$ arises from the very definition of the infinite time horizon measure.

We will prove the following lower tail concentration inequality. A similar statement as already been proved by Carmona and Hu [**20**], Theorem 1.5, in the context of Gaussian environments. As we will consider bounded environments, there is no loss of generality if we assume that $\eta(t,x) \in [-1,1]$ for all $t \in \mathbb{Z}$, $x \in \mathbb{Z}^d$.

PROPOSITION 2.4. *In the L^2 region, we can find $C > 0$ such that,*

$$Q\left(\log Z_N \leq N\lambda - u\right) \leq C \exp\left\{-\frac{u^2}{16C\beta^2}\right\}, \quad \forall N \geq 1, \forall u > 0. \tag{2.11}$$

It would be interesting to extend this result to a larger part of the weak disorder region. This is indeed a major obstacle to extend our Theorems to larger values of β.

Another cornerstone in the proof of Theorem 2.2 is a Local Limit Theorem [**87, 92**] that we will describe later. Again, this result is available in the L^2 region only.

We will now introduce some obvious notation that will be useful in what follows: For $x \in \mathbb{Z}^d$ and $M < N \in \mathbb{Z}$, we write

$$W_{M,N}(x) = Z_{M,N}^x e^{-(N-M)\lambda(\beta)}. \tag{2.12}$$

Similarly, we write

$$\overleftarrow{W}_{M,N}(x) = e^{-(N-M)\lambda} P^x[\exp\{\beta \sum_{t=1}^{N-M} \eta(N-t, \omega_t)\}].$$

for the related backward expression. Another useful notation is the 'conditional' partition function: take $M < N \in \mathbb{Z}$, $x, y \in \mathbb{Z}^d$,

$$W_{M,N}(x|y) = e^{-(N-M)\lambda} P^x[\exp\{\beta \sum_{t=1}^{N-M} \eta(M+t, \omega_t)\} | \, \omega_{N-M} = y].$$

The next section presents an example from random walks in a degenerate random environment where the limiting measure is no longer absolutely continuous with respect to the law of the environment. In Section 2.4 we will prove Theorem 2.2 and Theorem 2.3, postponing the proof of Proposition 2.4 to Section 2.5.

2.3. A toy model from RWRE

Before turning to the proofs, we will study a very simple counterexample introduced by Bolthausen and Sznitman [**9**] in the context of directed random walks in a random environment. This example illustrates that, under some degenerate assumptions, the limit the law of the environment seen from the particle may fail to be absolutely continuous with respect to the original measure of the environment.

Let us first introduce the model: let $\{e_i : i = 1, \cdots, d\}$ denote the coordinate vectors in $\mathbb{R}^d$. For each $y \in \mathbb{Z}^d$, $\eta_y = \{\eta_y(e_i) : i = 1, \cdots, d\}$ is a probability measure on the coordinate vectors chosen according to a product measure Q. For a fixed environment η, the directed random walk in a random environment with starting point x is defined by:

$$\begin{aligned} P_x^\eta \left[X_0 = x \right] &= 1 \\ P_x^\eta \left[X_{t+1} = y + e \,|\, X_t = y \right] &= \eta_y(e), \qquad \forall \, i = 1, \cdots, d. \end{aligned}$$

The term *directed* comes from the fact that the walk is restricted to jump along the coordinates vectors. For example, if $d = 2$, each time, the walk can jump either one step up or one step to the right (north/east walk). Let us call E_{dir} the set of directed paths on $\mathbb{Z}^d$, i.e., the paths $\mathbf{S} = (\mathbf{S}_t)_{t \in \mathbb{Z}}$ such that, $\forall \, t \in \mathbb{Z}$, $\mathbf{S}_{t+1} - \mathbf{S}_t = e_i$ for some $i = 1, \cdots, d$.

The environment seen by the particle $(\eta_N)_N$ is a Markov chain on the space of environments, with transition operator given by (2.15) below. As in (2.7), the density of its law at time N with respect to Q is easily seen to be equal to

$$g_N = \sum_x p_N^\eta(x, 0),$$

where $p_k^\eta(x, y) = P_x^\eta \left[X_k = y \right]$ is the probability to reach y in k steps starting from x. The following properties are straightforward:

PROPOSITION 2.5. [**9**] *Let* $\mathcal{H}_N = \sigma\{\eta(x) : x_1 + \cdots + x_d \geq -N\}$. *Then,*

- g_N *is an* $\mathcal{H}_N$*-martingale under* Q.
- *There is a unique probability measure* $\overline{Q}$ *on* E_{dir} *such that, for all* $N \geq 0$, *the restriction of* $\overline{Q}$ *to* $\mathcal{H}_N$ *coincides with* $g_N \cdot Q$.
- *The law of* η_N *under* $P_0 = \int P_0^\eta(\cdot)dQ$ *is* $g_N \cdot Q$ *and it converges weakly to* $\overline{Q}$, *which is an invariant probability measure for the environment seen by the particle.*

Usually, some ellipticity condition is needed in order to ensure that the invariant law is equivalent to Q. We will consider instead the following degenerate example where such hypothesis breaks down: for each $i = 1, \cdots, d$, $Q(\eta(e_i) = 1) = 1/d$, i.e., for each site x, we chose uniformly one of the neighbors. Observe that once the environment is fixed, the walk is deterministic.

PROPOSITION 2.6. [**9**] *In this setting,* Q *and* $\overline{Q}$ *are mutually orthogonal. In fact, any probability measure that is invariant for the environment seen by the particle is orthogonal to* Q.

PROOF. We take a slightly different route compared to [**9**]. Let us denote by $g_{+\infty}$ the limit when $N \to +\infty$ of the positive martingale g_N. Observe that here g_N just counts the number of sites x such that 0 is reachable from x in N steps. Then, it is easy to see that g_N can be stochastically dominated by a critical Galton-Watson process which offspring distribution is the sum of d Bernoulli random variables of parameter d. This implies that g_N vanishes Q-almost surely for N large enough. Then, $Q(\cap_{N\geq 1}\{g_N \geq 1\}) = 0$.

Now, observe that, by definition, if $g_N = 0$, then $g_M = 0$, $\forall\, M > N$. This implies that

$$\{g_N \geq 1\} = \cap_{k\leq N} \{g_k \geq 1\}, \tag{2.13}$$

and

$$\{g_{+\infty} \geq 1\} = \cap_{N\geq 1} \{g_k \geq 1\}. \tag{2.14}$$

Then, by the monotone convergence theorem, $\overline{Q}(\cap_{N\geq 1}\{g_N \geq 1\}) = 1$. This proves the first point.

Let now $\widetilde{Q}$ be invariant for the environment seen by the particle. Define an operator R on measurable functions by

$$Rf = \sum_y p^\eta(0, y) f \circ \theta_y. \tag{2.15}$$

In particular,

$$\widetilde{Q}(g_N \geq 1) \;=\; \int R^N \mathbf{1}_{g_N \geq 1}\, d\widetilde{Q}.$$

Now,

$$\begin{aligned} R^N \mathbf{1}_{g_N \geq 1} &= \sum_y p_N^\eta(0,y) \mathbf{1}(\sum_x p_N^\eta(x,0) \circ \theta_y \geq 1) \\ &= \sum_y p_N^\eta(0,y) \mathbf{1}(\sum_x p_N^\eta(x,y) \geq 1) \\ &= \sum_y p_N^\eta(0,y) \\ &= 1. \end{aligned}$$

Then, $\widetilde{Q}(g_N \geq 1) = 1$. By (2.13), (2.14) and monotonicity, this implies that $\widetilde{Q}(g_{+\infty} \geq 1) = 1$. But $Q(g_{+\infty} = 0) = 1$, and then Q and $\widetilde{Q}$ are orthogonal. □

2.4. Proof of Theorems 2.2 and 2.3

In this section, we make no specific assumptions on the environment. We just need Proposition 2.4 and Theorem 2.1 to hold, which is the case for gaussian or bounded environments in the L^2 region. Actually, we will see that we don't need the whole strength of the invariance principle, but just an averaged version of it.

In both cases, the proof is a sequence of L^2 calculus. Let's begin describing the aforementioned Local Limit Theorem as it appears in [**92**], page 6, Theorem 2.3: take $x, y \in \mathbb{Z}^d$, $M \in \mathbb{Z}$, then, for $A > 0$, $N > M$ and $l_N = O(N^\alpha)$ with $0 < \alpha < 1/2$,

$$\begin{aligned} W_{M,N}(x|y) &= W_{M,M+l_N}(x) \times \overleftarrow{W}_{N-l_N,N}(y) \times e^{\beta\eta(N,y)-\lambda(\beta)} \\ &+ R_{M,N}(x,y), \end{aligned} \tag{2.16}$$

where

$$\lim_{N\to+\infty} \sup_{|x-y|<AN^{1/2}} Q(R^2_{M,N}(x,y)) = 0. \tag{2.17}$$

Note that by symmetry, this result is still valid when we fix y and take the limit $M \to -\infty$.

Proof of Theorem 2.2: We first restrict the sum in (2.7) to the region of validity of the local limit theorem. This is done using the quenched central limit theorem averaged with respect to the disorder:

$$q_N = \sum_{|x|<AN^{1/2}} \mu^x_{-N,0}(\omega_N = 0) + \sum_{|x|\geq AN^{1/2}} \mu^x_{-N,0}(\omega_N = 0). \tag{2.18}$$

We compute the L^1 norm of the second term of the sum making use of the invariance by translation under the law Q:

$$Q\sum_{|x|\geq AN^{1/2}}\mu^x_{-N,0}(\omega_N=0)=Q\mu^0_{0,N}(|\omega_N|\geq AN^{1/2}). \tag{2.19}$$

By Theorem 2.1, we have that

$$\lim_{A\to+\infty}\limsup_{N\to+\infty}Q\mu^x_{0,N}(|\omega_N|\geq AN^{1/2})=0.$$

We can now concentrate on the first term in (2.18). We denote by $p(\cdot,\cdot)$ (resp. $p_N(\cdot,\cdot)$) the transition probabilities of the simple symmetric random walk on $\mathbb{Z}^d$ (resp. its N-step transition probabilities). Thanks to (2.16),

$$\begin{aligned}\sum_{|x|<AN^{1/2}}\mu^x_{-N,0}(\omega_N=0) &= \sum_{|x|<AN^{1/2}}\frac{W_{-N,0}(x|0)}{W_{-N,0}(x)}p_N(x,0)\\ &= \overleftarrow{W}_{-l_N,0}(0)e^{\beta\eta(0,0)-\lambda}\sum_{|x|<AN^{1/2}}\frac{W_{-N,-N+l_N}(x)}{W_{-N,0}(x)}p_N(x,0)\\ &+ \sum_{|x|<AN^{1/2}}\frac{R_{-N,0}(x,0)}{W_{-N,0}(x)}p_N(x,0).\end{aligned} \tag{2.20}$$

We again integrate the second term of the sum (2.20). We use Cauchy-Schwarz inequality and translation invariance:

$$\begin{aligned}Q\sum_{|x|<AN^{1/2}}\frac{R_{-N,0}(x,0)}{W_{-N,0}(x)}p_N(x,0) &\leq \sup_{|x|<AN^{1/2}}\{Q(R^2_{-N,0}(x,0))^{1/2}\}\\ &\times\ Q(W^{-2}_{-N,0}(0))^{1/2}\sum_{|x|<AN^{1/2}}p_N(x,0).\end{aligned} \tag{2.21}$$

The first term in the right side tends to zero thanks to (2.17), the second one is easily seen to be bounded thanks to Proposition 2.4 and the third one is less than

$$\sum_x p_N(x,0)=\sum_x p_N(0,x)=1,$$

so the left member of (2.21) tends to zero. We are left to the study of the first summand in (2.20),

$$\begin{aligned}
&\overleftarrow{W}_{-l_N,0}(0)e^{\beta\eta(0,0)-\lambda}\sum_{|x|<AN^{1/2}}\frac{W_{-N,-N+l_N}(x)}{W_{-N,0}(x)}p_N(x,0)\\
&= \overleftarrow{W}_{-l_N,0}(0)e^{\beta\eta(0,0)-\lambda}\sum_{|x|<AN^{1/2}}p_N(x,0)\\
&+ \overleftarrow{W}_{-l_N,0}(0)e^{\beta\eta(0,0)-\lambda}\sum_{|x|<AN^{1/2}}\left\{\frac{W_{-N,-N+l_N}(x)}{W_{-N,0}(x)}-1\right\}p_N(x,0).
\end{aligned} \tag{2.22}$$

We see that we are done as long as we can control the convergence of the second summand in (2.22). It will be enough to prove that it converges to zero in probability. Let us denote

$$g_N := \overleftarrow{W}_{-l_N,0}(0)e^{\beta\eta(0,0)-\lambda},$$

$$h_N := \sum_{|x|<AN^{1/2}}\left\{\frac{W_{-N,-N+l_N}(x)}{W_{-N,0}(x)}-1\right\}p_N(x,0).$$

We already know that $\{g_N : N \geq 1\}$ is bounded in $L^1(Q)$. It is then enough to prove that h_N tends to zero in L^1. Indeed, using translation invariance, Cauchy-Schwarz inequality and the uniform boundedness of negative moments of $W_{-N,0}$,

$$\begin{aligned}
Q(|h_N|) &\leq \sum_{|x|<AN^{1/2}} Q\left(\left|\frac{W_{-N,-N+l_N}(x)}{W_{-N,0}(x)}-1\right|\right)p_N(x,0)\\
&\leq Q\left(\left|\frac{W_{-N,-N+l_N}(0)}{W_{-N,0}(0)}-1\right|\right)\\
&\leq Q\left(|W_{0,l_N}(0)-W_{0,N}(0)|^2\right)^{1/2}Q\left(W_{-N,0}^{-2}(0)\right)^{1/2}.
\end{aligned}$$

This clearly tends to zero. It is now a simple exercise to show that $Q(g_N|h_N| \geq \delta)$ tends to zero as N tends to infinity for all $\delta > 0$.
So far, we have proved that q_N tends to $q_{+\infty}$ in Q-probability. But, by an elementary result, we know that, for q_n, $q_{+\infty} > 0$, convergence in probability implies L^1 convergence as long as $Q(|q_N|) \to Q(|q_{+\infty}|)$ (which is clearly the case here because all these expressions are equal to 1). This finishes the proof of Theorem 2.2. □

Proof of Theorem 2.3: The details are very similar to the previous proof. We split the sum in (2.10) according to $|x| \leq AN^{1/2}$ or not. We can apply the central limit theorem to

$$Q\sum_{|x|>AN^{1/2}}\mu^x_{-N,M}(\omega_N=0) = Q\mu^0_{0,N+M}\left(|\omega_N| > AN^{1/2}\right).$$

Now, by the Markov property and the local limit theorem, we have

$$\begin{aligned}\sum_{|x|\le AN^{1/2}} \mu^x_{-N,M}(\omega_N=0) &= \sum_{|x|\le AN^{1/2}} \frac{W_{0,M}(0)\,W_{-N,0}(x|0)}{W_{-N,M}(x)}\,p_N(x,0)\\ &= W_{0,M}(0)\overleftarrow{W}_{-l_N,0}\sum_{|x|\le AN^{1/2}}\frac{W_{-N,N+l_N}(x)}{W_{-N,M}(x)}p_N(x,0)\\ &+ W_{0,M}(0)\sum_{|x|\le AN^{1/2}}\frac{R_{-N,0}(x,0)}{W_{-N,M}(x)}p_N(x,0)\end{aligned}$$

The second summand is again treated using the Cauchy-Schwarz inequality, (2.17) and the independence of $W_{0,M}(0)$ and $W^{-1}_{-N,M}(x)$. The first summand as to be written as

$$\begin{aligned} . \qquad & W_{0,M}(0)\overleftarrow{W}_{-l_N,0}(0)\sum_{|x|\le AN^{1/2}} p_N(x,0)\\ + \;& W_{0,M}(0)\overleftarrow{W}_{-l_N,0}(0)\sum_{|x|\le AN^{1/2}}\left\{\frac{W_{-N,N+l_N}(x)}{W_{-N,M}(x)}(0)-1\right\}p_N(x,0).\end{aligned} \tag{2.23}$$

The second summand of (2.23) can be handled like the one in (2.22), and using the independence of $W_{0,M}(0)$ and $\overleftarrow{W}_{-l_N,0}(0)$. □

2.5. Concentration inequalities

The proof follows closely [**90**], Section 2 (see the proof of the lower bound of Theorem 1.1 therein). Recall that we assumed that the environment is bounded by one. In the L^2 region, it is known that $QZ_N^2 \le K(QZ_N)^2$ (see (2.4)). This implies that

$$Q\left(Z_N \ge \frac{1}{2}QZ_N\right) \ge \frac{1}{4}\frac{(QZ_N)^2}{QZ_N^2} \ge \frac{1}{4K}, \tag{2.24}$$

thanks to Paley-Zigmund inequality.

The following is easily proved ([**20**], proof of Theorem 1.5): Let

$$A = \left\{Z_N \ge \frac{1}{2}QZ_N,\; \langle L_N(\omega,\widetilde{\omega})\rangle_N^{(2)} \le C\right\}.$$

where the brackets mean expectation with respect to two independent copies of the polymer measure on the same environment. Then, we can find $C>1$ such that, for all $N\ge 1$,

$$Q(A) \ge 1/C. \tag{2.25}$$

It is convenient to see Z_N as a function from $[-1,1]^{T_N}$ to $\mathbb{R}$, where $T_N = \{(t,x) : 0 \le t \le N,\, |x|_1 \le t\}$ and $|x|_1 = \sum_{i=1}^d |x_i|$ for $x=(x_1,...,x_d)$. For $u>0$, let

$B = \left\{z \in [-1,1]^{T_N} : \log Z_N(z) \leq \lambda N - \log 2 - u\right\}$. This is a convex compact subset of $[-1,1]^{T_N}$.

In order to apply theorems for concentration of product measures, we need to introduce some notation. For $x \in [-1,1]^{T_N}$, let

$$U_B(x) = \{h(x,y) : y \in B\}, \tag{2.26}$$

where $h(x,y)_i = \mathbf{1}_{x_i \neq y_i}$. Let also $V_B(x)$ be the convex envelope $U_B(x)$ when we look at it as a subset of $\mathbb{R}^{T_N}$. Finally, let $f(x,B)$ be the euclidean distance from the origin to $V_B(x)$. Let us state the following result from [**89**], Theorem 6.1:

THEOREM 2.7.

$$\int \exp\left\{\frac{1}{4}f^2(x,B)\right\} dQ(x) \leq \frac{1}{Q(B)}.$$

On the other hand, for $x \in [-1,1]^{T_N}$, $y \in B$, we have $|x_i - y_i| \leq 2h(x,y)_i$. Then for every finite sequence $(y^{(k)})_{k=1}^M \subset B$ and any sequence of non-negative number such that $\sum_{k=1}^M \alpha_k = 1$, we have that

$$|x_i - \sum_{k=1}^M \alpha_k y_i^{(k)}| \leq 2\sum_{k=1}^M \alpha_k h(x,y^{(k)})_i,$$

for every $i \in T_N$. This yields

$$||x - \sum_{k=1}^M \alpha_k y^{(k)}|| \leq 2\left(\sum_{i \in T_N}\{\sum_{k=1}^M \alpha_k h(x,y_i^{(k)})\}^2\right)^{1/2}.$$

We can now optimize over all convex combinations of elements of B (remember that it is convex), we obtain

$$d(x,B) \leq 2f(x,B),$$

where $d(x,B)$ is the euclidean distance from x to B. We use Theorem 2.7 to conclude that

$$\int_A \exp\left\{\frac{1}{16}d^2(x,B)\right\} dQ(x) \leq \frac{1}{Q(B)}.$$

We can find $\overline{x} \in A$ such that $Q(A)\exp\{\frac{1}{16}d^2(\overline{x},B)\} \leq 1/Q(B)$. Using (2.25), we find that $1/C\exp\{\frac{1}{16}d^2(\overline{x},B)\} \leq 1/Q(B)$. Let $q = Q(B)$. By compacity and some simple calculations, we conclude that we can find $\overline{z} \in B$ such that

$$d(\overline{x},\overline{z}) \leq 4\sqrt{\log\frac{C}{q}}. \tag{2.27}$$

Now,

$$
\begin{aligned}
Z_N(\overline{z}) &= Z_N(\overline{x}) \left\langle \exp\left\{\beta \sum_{t=1}^{N} (\overline{z}(t,\omega_t) - \overline{x}(t,\omega_t))\right\}\right\rangle_{\overline{x}} \\
&\geq Z_N(\overline{x}) \exp \beta \left\langle \sum_{t=1}^{N} (\overline{z}(t,\omega_t) - \overline{x}(t,\omega_t))\right\rangle_{\overline{x}},
\end{aligned}
$$

where the brackets mean expectation with respect to the polymer measure in the $\overline{x}$ environment. But using successively the Cauchy-Schwarz inequality, (2.27) and the fact that $\overline{x} \in A$, we find that

$$
\begin{aligned}
\left|\left\langle \sum_{t=1}^{N} (\overline{z}(t,\omega_t) - \overline{x}(t,\omega_t))\right\rangle_{\overline{x}}\right| &= \left|\sum_{t=1}^{N} \sum_{a} (\overline{z}(t,a) - \overline{x}(t,a) \left\langle \mathbf{1}_{\omega_t = a}\right\rangle_{\overline{x}}\right| \\
&\leq ||\overline{x} - \overline{z}|| \left\langle L_N(\omega, \widetilde{\omega})\right\rangle_{\overline{x}}^{1/2} \\
&\leq 4\sqrt{\log \frac{C}{q}} \sqrt{C}.
\end{aligned}
$$

So, using again the fact that $\overline{x} \in A$,

$$
\begin{aligned}
\log Z_N(\overline{z}) &\geq \log Z_N(\overline{x}) - 4|\beta| \sqrt{\log \frac{C}{q}} \sqrt{C} \\
&\geq \log \left(\frac{1}{2} Q Z_N\right) - 4|\beta| \sqrt{\log \frac{C}{q}} \sqrt{C}.
\end{aligned}
$$

Recalling that $\overline{z} \in B$ and after a few calculations, we conclude that

$$
q \leq C \exp\left\{-\frac{u^2}{16 C \beta^2}\right\},
$$

which finishes the proof.

2.6. The empirical distribution of the environment seen by the particle in the full weak disorder region

We deal here with the whole weak disorder region (cf. Definition 1.2). Recall the definition of the infinite volume polymer measure μ_∞ with transitions

$$\mu_\infty(\omega_{t+1} = x + e \,|\omega_t = x) = \frac{1}{2d} e^{\beta\eta(t+1,x+e)-\lambda(\beta)} \frac{W_{+\infty}(t+1, x+e)}{W_{+\infty}(t,x)}. \tag{2.28}$$

For an environment $\eta \in E$, define the environment seen by the particle as the field $\overline{\eta}_t \in E$ given by

$$\overline{\eta}_t(n,x) = \eta(t+n, \omega_n + x), \quad (n,x) \in \mathbb{Z} \times \mathbb{Z}^d,$$

where ω_n is chosen according to μ_∞. Observe that this is a time homogeneous Markov chain: denote by $R(\eta, \eta')$ the probability to jump from η to η' and define

$$\pi(\eta, e) = \begin{cases} \frac{1}{2d} e^{\beta\eta(1,e)-\lambda(\beta)} \frac{W_\infty^\eta(1,e)}{W_\infty^\eta(0,0)} & : \eta \in \mathcal{E} \\ \frac{1}{2d} & : \eta \in \mathcal{E}^c \end{cases}$$

where $\mathcal{E} = \{\eta : W_{+\infty}^\eta(1,e) \in (0, +\infty), \forall\, e\}$. Remark that, in the weak disorder region, $Q(\mathcal{E}) = 1$. Then,

$$R(\eta, \eta') = \sum_{e \in \mathcal{R}} \pi(\eta, e)\, \mathbf{1}_{\eta \circ \theta_{1,e} = \eta'},$$

where $\mathcal{R} = \{e \in \mathbb{Z}^d : |e| = 1\}$ and $\eta \circ \theta_{t,x}(\cdot,\cdot) = \eta(t+\cdot, x+\cdot)$ is the shift. Moreover, we can see that the law

$$d\overline{Q} = \overleftarrow{W}_{+\infty}(0,0) e^{\beta\eta(0,0)-\lambda(\beta)} W_{+\infty}(0,0)\, dQ,$$

is invariant for the environment seen by the particle.

Based on these observations, we can follow an approach that is typical in the field of random walks in a random environment. We say that a function $\widehat{\pi}(\cdot,\cdot) : E \times \mathcal{R} \to [0,1]$ is an **environment kernel** if:

(*i*) $\widehat{\pi}(\cdot, e)$ is measurable for all $|e| = 1$.

(*ii*) $\sum_{|e|=1} \widehat{\pi}(\cdot, e) = 1$.

Given an environment kernel, we can define a transition probability on the environment as:

$$\overline{\pi}(\eta, \eta') = \sum_{e \in \mathcal{R}} \widehat{\pi}(\eta, e)\, \mathbf{1}_{\eta \circ \theta_{1,e} = \eta'},$$

and we can define the transition probability for the **random walk in the environment** η as

$$\begin{aligned} \mu^{\widehat{\pi},\eta}(\omega_0 = 0) &= 1 \\ \mu^{\widehat{\pi},\eta}(\omega_{N+1} = x+e \mid \omega_N = x) &= \widehat{\pi}(\eta \circ \theta_{N,x}, e), \quad N \geq 1,\, e \in \mathcal{R}. \end{aligned}$$

Moreover, $\widehat{\pi}$ can be identified as the transition probability of the environment seen by the particle under the law $\mu^{\widehat{\pi},\eta}(\cdot)$.

For exemple, the function $\pi(\cdot,\cdot)$ defined above is an environment kernel. The corresponding transition probability on the environment is $R(\cdot,\cdot)$ and the law of the walk is μ_∞.

PROPOSITION 2.8. *[the Kozlov argument] Let $\widehat{\pi}$ be an environment kernel and suppose that we have an invariant probability Π for the environment seen by the particle that is absolutely continuous with respect to Q. Then Q and Π are in fact mutually absolutelly continuous and the environment seen by the particle is ergodic for Π. Moreover, Π is the unique probability measure satisfying these properties.*

A proof of this result can be found in [**14**], Theorem 1.2. The above definition of a environment kernel is taken from [**96**], Definition 1.
In particular, this implies that the law $\overline{Q}$ is ergodic for the environment seen by the particle. We use this to prove:

LEMMA 2.9. *Assume weak disorder. The finite dimensional distributions of the empirical distribution of the environment seen by the particle*

$$\frac{1}{n}\sum_{t=0}^{n-1} \delta_{\overline{\eta}_t},$$

converge to $\overline{Q}$, in the sense that Q-a.e., for all $k \geq 1$, $t_1 \leq \cdots \leq t_n$, $x_1, \cdots, x_k \in \mathbb{Z}^d$ and for every bounded continuous and compactly supported function $f : \mathbb{R}^k \to \mathbb{R}$, we have

$$\frac{1}{n}\sum_{t=0}^{n-1} f(\overline{\eta}(t_1,x_1)_t, \cdots, \overline{\eta}(t_k,x_k)_t) \to \overline{Q} f(\eta(t_1,x_1), \cdots, \eta(t_k,x_k)). \tag{2.29}$$

PROOF. Let us introduce some notation. Let $\overline{\alpha}$ denote a generic finite collection of sites:

$$\overline{\alpha} = (t_1,x_1), ..., (t_k,x_k).$$

Note that the collection of all such $\overline{\alpha}$ is countable, as it is just the family of all finite subsets of $\mathbb{Z}^d$. Denote by $|\overline{\alpha}|$ the length of $\overline{\alpha}$ (in the previous example, $|\overline{\alpha}| = k$). For $f : \mathbb{R}^k \to \mathbb{R}$, and $|\overline{\alpha}| = k$, set

$$f_{\overline{\alpha}}(\eta) = f(\eta(t_1, x_1), ..., \eta(t_k, x_k)).$$

The result is a consequence of the Kozlov argument stated in Proposition 2.8. By the ergodic theorem, for any bounded measurable function $f : \mathbb{R}^k \to \mathbb{R}$, and any $|\overline{\alpha}| = k$ we have, Q-a.s.

$$\frac{1}{n}\sum_{t=0}^{n-1} f_{\overline{\alpha}}(\overline{\eta}_t) \to \overline{Q} f_{\overline{\alpha}}(\eta).$$

As the space of continuous bounded compactly supported functions $f : \mathbb{R}^k \to \mathbb{R}$ endowed with the topology of uniform convergence is separable, we can find a dense and countable subfamily $(f^{(k,i)})_i$, such that, Q-a.s., for all i and for all $\overline{\alpha}$

$$\frac{1}{n}\sum_{t=0}^{n-1} f^{(k,i)}_{\overline{\alpha}}(\overline{\eta}_t) \to \overline{Q} f^{(k,i)}_{\overline{\alpha}}(\eta),$$

Now, take any bounded continuous compactly supported function $f : \mathbb{R}^k \to \mathbb{R}$. Take $\epsilon > 0$ small, i such that $\sup |f - f^{(k,i)}| < \epsilon$. Then

$$\begin{aligned} |\frac{1}{n}\sum_{t=0}^{n-1} f_{\overline{\alpha}}(\overline{\eta}_t) - \overline{Q}(f_{\overline{\alpha}})| &\leq |\frac{1}{n}\sum_{t=0}^{n-1} f^{(k,i)}_{\overline{\alpha}}(\overline{\eta}_t) - \overline{Q}(f^{(k,i)}_{\overline{\alpha}})| \\ &\quad + |\frac{1}{n}\sum_{t=0}^{n-1}(f_{\overline{\alpha}}(\overline{\eta}_t) - f^{(k,i)}_{\overline{\alpha}}(\overline{\eta}_t))| + |\overline{Q}(f_{\overline{\alpha}}) - \overline{Q}(f^{(k,i)}_{\overline{\alpha}})| \\ &\leq |\frac{1}{n}\sum_{t=0}^{n-1} f^{(k,i)}_{\overline{\alpha}}(\overline{\eta}_t) - \overline{Q}(f^{(k,i)}_{\overline{\alpha}})| + 2\epsilon. \end{aligned}$$

This finishes the proof of (2.29). □

THEOREM 2.10. *For a bounded environment and weak disorder, the empirical distribution of the environment seen by the particle converges weakly to the law $\overline{Q}$.*

PROOF. The tighness follows by compactness of the state space. The limit has been identified in the previous lemma. □

COROLLARY 2.11. *The energy of a μ_∞-typical path satisfies the law of large numbers:*

$$\frac{1}{n}\sum_{t=0}^{n-1} \eta(t, \omega_t) \to \lambda'(\beta),$$

for Q-a.e. η.

PROOF. This follows from the last Proposition, with $f(\eta) = \eta(0,0)$. It is straightforward to verify that $\overline{Q}f = \lambda'(\beta)$. □

REMARK 2.12. Of course, a weaker result holds for all values of β, as a consequence of the convexity of $p(\beta)$: at all values of β s.t. $p'(\beta)$ exists, we have $p'_n(\beta) \to p'(\beta)$, Q-a.s.. But

$$p'_n(\beta) = \mu_{n,\beta}\left(\frac{1}{n}H_n(\beta)\right), \tag{2.30}$$

so the quenched mean of the averaged energy along the paths converges to $p'(\beta)$ even in the strong disorder region. As the free energy is strictly convex [**23**], this is an increasing function of β, in contrast with the case of directed polymers on a tree [**78**].

REMARK 2.13. In the L^2 region, it is known that $\mu_N(H_N) - N\lambda(\beta)$ converges to a finite random variable as N tends to $+\infty$. This says that the fluctuations of the quenched mean of the energy are of order 1. In contrast, it can be shown that under μ_N, H_N satisfies a CLT for almost every realization of the environment. See [**28**], Theorem 6.2 for a proof using complex variables arguments. The improvement of these results to the whole weak disorder region is still an open question.

CHAPTER 3

Directed polymers on tree-like graphs

The aim of this chapter is to provide an introduction to Chapter 4 which contains our results on the directed polymers on the hierarchical diamond lattice.

In the first section, we compute explicitly the free energy of the directed polymers on the tree, following [**60**] and [**15**]. The techniques we present make use of the particular geometry of the tree and cannot be applied in general to the polymers on $\mathbb{Z}^d$. A noticeable difference is that the shape of the free energy is strictly convex on $\mathbb{Z}^d$ but linear above β_c on the tree. Indeed, the convexity of the free energy is related to the fluctuations of the energy of the paths when we modify them slightly (see Theorem 4.1 in the next chapter). While on $\mathbb{Z}^d$, a typical path can be modified almost all along its trajectory, leading to small variations of the energy, on the tree, this procedure can be done only close to the end point of the path, any other modification leading to a completely different path, with a completely different energy.

The hierarchical lattices are expected to mimic some of the important aspects of the geometry of $\mathbb{Z}^d$ but offer several technical simplifications that make them close to the tree. They appear very naturally in the study of spin systems through a renormalization group as they allow for exact recursive computations (see [**75, 56**] and [**10**]).

A model of directed polymers on hierarchical lattice with bond disorder has been introduced in [**31**]. We discuss it in the second section of this chapter and show some recursive computations that can be made in this case.

We will see in the next chapter, that we can define directed polymers on the hierarchical lattice with site disorder. This model is even closer to the directed polymers on $\mathbb{Z}^d$. Contrarily to the bond disorder case, it allows a martingale approach, bringing a better insight, and its free energy is, surprisingly enough, strictly convex, even though the loop structure of the graph is highly simplified.

3.1. Directed polymers on trees

Let $\mathbb{T}$ be a b-ary tree with root 0. For each $x \in \mathbb{T}$, let $|x|$ denotes its distance from the root. If $|x| = n$, we say that x if of generation n. P will be the law of the directed random walk on $\mathbb{T}$: initially, $\mathbf{S}_0 = 0$, and, for each $n \leq 0$, the walk jumps to either of its b neighbors of generation $n+1$ with probability $1/b$.
The random environment is represented by a collection of i.i.d. random variables $\{\eta(x) : x \in \mathbb{T}\}$ such that $\lambda(\beta) = \log Q e^{\beta\eta}$ is finite for all non-negative β. The energy of a path $\mathbf{S}$ of length N is

$$H_N^\eta(\mathbf{S}) = \sum_{t=0}^{N} \eta(\mathbf{S}_t).$$

The polymer measure at environment η, inverse temperature β and time horizon N, is defined as usual by

$$\frac{d\mu_N}{dP}(\mathbf{S}) = \frac{1}{Z_N^\eta} \exp \beta H_N^\eta(\mathbf{S}_t), \tag{3.1}$$

where the partition function Z_N^η is

$$Z_N^\eta = P\left(\exp \beta H_N^\eta(\mathbf{S}_t)\right).$$

The free energy is defined as the limit

$$p(\beta) = \lim_{N\to+\infty} \frac{1}{N} \log Z_N^\eta.$$

The martingale $W_N = Z_N e^{-N\lambda(\beta)}$ will again play a key role here. It satisfies a zero-one law, just as in Proposition 1.1, and the following recursion equation:

$$W_{N+1} = \frac{1}{b} \sum_{i=1}^{b} e^{\beta\eta(\epsilon_i)} W_N \circ \theta(\epsilon_i), \tag{3.2}$$

where θ is a canonical shift and the ϵ_i's are the sites of generation 1. Note that the summands are independent for different indexes, contrarily to the $\mathbb{Z}^d$ case, where different summands are strongly correlated. This large amount of independence allows for very direct second moments computations: for example, if we set $x_N = QW_N^2$, it is easy to verify that

$$x_{N+1} = \frac{1}{b} e^{\lambda(2\beta)-2\lambda(\beta)} x_N + \frac{b-1}{b},$$

from which we can deduce that, for β small enough, the second moments stay bounded. This is an analogy with the L^2 region exhibited for the directed polymers on $\mathbb{Z}^d$, but here we do not need to take track of the exponential moments of overlaps.

The following theorem summarizes the principal results concerning the free energy of this model:

THEOREM 3.1. *There exists $\beta_c \in (0, +\infty]$, such that*

$$p(\beta) = \begin{cases} \lambda(\beta) & : \beta \le \beta_c \\ \lambda(\beta_c) & : \beta > \beta_c \end{cases}$$

Furthermore, β_c is the root of the equation

$$\beta\lambda'(\beta) - \lambda(\beta) = \log b \tag{3.3}$$

if it exists; if not, $\beta_c = +\infty$.

PROOF. For $\beta \leq \beta_c$, we use the following (not so) elementary inequality: for $(x_i)_{i=1}^M$ non-negative integers and $0 < \theta < 1$,

$$\left(\sum_{i=1}^M x_i\right)^h \geq \sum_{i=1}^M x_i^h - 2(1-h)\sum_{i<j}(x_i x_j)^{h/2}. \tag{3.4}$$

For a proof, see [**60**]. By applying this to the recursion equation (3.2) and integrating with respect to Q, we obtain

$$b^h Q W_{N+1}^h \geq b e^{\lambda(\beta;h)} Q W_N^h - 2(1-h)d(d-1)e^{2\lambda(\beta;h/2)}Q(W_N^{h/2})^2.$$

with $\lambda(\beta;s) = \lambda(s\beta) - s\lambda(\beta)$. After straightforward manipulations together with the supermartingale inequality $QW_{N+1}^h \leq QW_N^h$, we can take the limit $h \uparrow 1$, leading to

$$2b^{-1}d(d-1)Q(W_N^{1/2})^2 \geq \log b - \beta\lambda'(\beta) + \lambda(\beta).$$

The positivity of the RHS insures that the LHS do not vanish when $N \to +\infty$. By the zero-one law, this implies that $W_{+\infty} > 0$ a.s., and then the equality $p(\beta) = \lambda(\beta)$.

We now turn to $\beta > \beta_c$. The argument presented in Proposition 1.4 in the introduction can be easily adapted to prove that, in this case, the martingale W_N vanishes when $N \to +\infty$. The rest of the proof is performed by using convexity arguments.
Define β_c as the solution of Equation (3.3) (or $+\infty$ if there is no solution). We need two easy analytical facts that are proved in Lemma 4 and Lemma 5 of [**15**] respectively: First, the function $h(\beta) = \lambda(\beta)/\beta$ satisfies the following:

- Either β_c is finite and h is strictly decreasing in $(0, \beta_c)$ and strictly increasing in $(\beta_c, +\infty)$.
- Either $\beta_c = +\infty$ and h is strictly decreasing in $(0, +\infty)$.

We will assume in the following that $\beta_c < +\infty$. The second fact we will use is that, for any collection of M real numbers $x_1, \cdots, x_M$, the function

$$g(\beta) = \frac{1}{\beta}\log\sum_{i=1}^M e^{\beta x_i},$$

is decreasing and convex in β. Using this second fact, we see that, for any $\epsilon > 0$, $\beta > \beta_c$,

$$\frac{1}{\beta N}\log Z_N(\beta) \leq \frac{1}{(\beta_c - \epsilon)N}\log Z_N(\beta_c - \epsilon),$$

and then, using what we already know for $\beta < \beta_c$,

$$\limsup_{N\to+\infty} \frac{1}{\beta N} \log Z_N(\beta) \le h(\beta_c - \epsilon). \tag{3.5}$$

Now, the convexity property stated before, can be used to show that

$$\frac{1}{\beta N} \log Z_N(\beta) \ge \frac{d}{d\beta} \frac{1}{\beta N} \log Z_N(\beta)|_{\beta=\beta_c-\epsilon} \times (\beta - \beta_c + \epsilon) + \frac{1}{(\beta_c - \epsilon)N} \log Z_N(\beta_c - \epsilon).$$

Again, the convexity of the function $1/(\beta N) \log Z_N(\beta)$ and its convergence to the differentiable function $h(\beta)$, which together ensure the convergence of the derivatives as well, imply that

$$\liminf_{N\to+\infty} \frac{1}{\beta N} \log Z_N(\beta) \ge h'(\beta_c - \epsilon)(\beta - \beta_c + \epsilon) + h(\beta_c - \epsilon).$$

Now, β_c is a global minimum for h, and then $\lim_{\epsilon\downarrow 0} h'(\beta_c - \epsilon) = 0$. These facts, together with (3.5), end the proof.

□

3.2. Directed polymers on the hierarchical diamond lattice

We examine a model of directed polymers in a random environment on a hierarchical diamond lattice, introduced by Cook and Derrida in [**31**]. These lattices have been introduced in the study of spin systems in order to perform some exact calculations (see [**75, 56**]). They arise naturally while performing renormalization group analysis. The reader can consult [**61, 62**] for a detailed analysis of more general hierarchical lattices and [**10**] for an example of recursive computations that can be performed on such lattices (there, for the Ising model).
The model of Cook and Derrida has the particularity of displaying bond disorder. This fact is convenient for moment computations, but is problematic when we try to analyze some deeper properties. For instance, we loose the martingale property for the normalized partition function that is a cornerstone in the classical approach to directed polymers. Even worst, the limit of the normalized partition functions is a constant in the L^2 region, a fact that is in disagreement with the basic properties of the corresponding model on $\mathbb{Z}^d$.

In the next chapter, we will consider a directed polymer model on the diamond hierarchical lattice with site disorder, a model that is, in our though, closer to the original model on $\mathbb{Z}^d$.

We will devote the rest of this section to introduce the bond disorder model and review some easy facts.

The diamond hierarchical lattice D_n can be constructed recursively:

- D_0 is one single edge linking two vertices A and B.
- D_{N+1} is obtained from D_N by replacing each edge by b branches of $s-1$ edges.

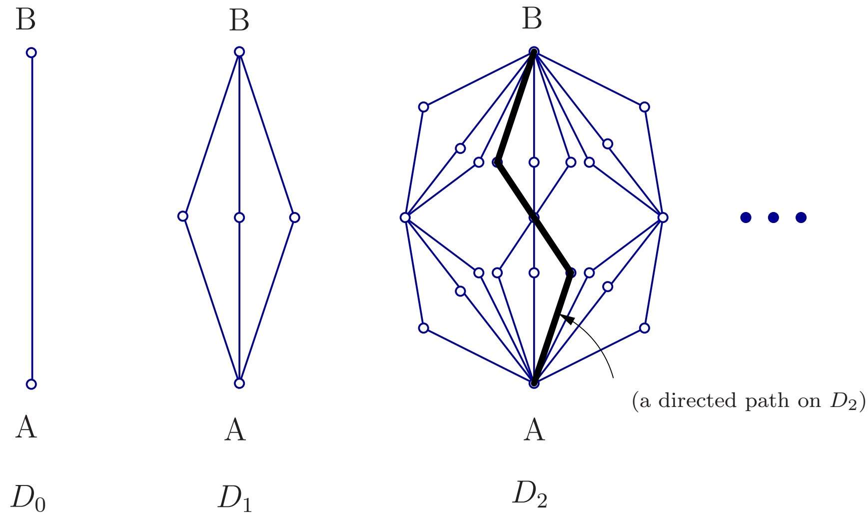

FIGURE 1. We present here the recursive construction of the first three levels of the hierarchical lattice D_N, for $b = 3$, $s = 2$.

We restrict to $b \geq 2$ and $s \geq 2$. The case $b = 1$ (resp. $s = 1$) is not interesting as it just corresponds to a family of edges in serie (resp. in parallel)

We consider now an i.i.d. family of random variables $\mathcal{E}_N = \{\eta(e) : e \in E_N\}$ indexed by E_N, the set of edges of D_N. For different values of N, we suppose that these families are independent. Consider Ω_N the space of directed paths in D_N linking A to B. For each $\mathbf{S} \in \Omega_N$ (to be understood as an ordered sequence of edges in D_N), we define the Hamiltonian

$$H_N^\eta(\mathbf{S}) := \sum_{t=1}^{s^N} \eta(\mathbf{S}_t).$$

For $\beta > 0$, $N \geq 1$, we define the (quenched) polymer measure on Ω_N which picks a path ω at random with law

$$\mu_{\beta,N}^\eta(\omega = \mathbf{S}) := \frac{1}{Z_N(\beta)} \exp(\beta H_N^\eta(\mathbf{S})),$$

where

$$Z_N(\beta) = Z_N(\beta, \omega) := \sum_{\mathbf{S} \in \Omega_n} \exp(\beta H_N^\eta(\mathbf{S})),$$

is the partition function, and β is the inverse temperature parameter.

It is easy to observe that the partition function satisfies the following recursive equation:

$$Z_{N+1} = \sum_{i=1}^{b} \prod_{l=1}^{s} Z_N(i,l), \tag{3.6}$$

where $(Z_N(i,l))_{i,l}$ is a family of i.i.d. copies of Z_N. This comes from the fact that we can view D_{N+1} as made from sb copies of D_N pasted on the edges of D_1. From this, we can infer a recursive relation for the second moments:

$$Q(Z_{N+1}^2) = b\,Q(Z_N^2)^s + b(b-1)Q(Z_N)^{2s}.$$

We now want to obtain informations on $Q(W_N^2)$ where $W_N = Z_N/QZ_N$. From (3.6),

$$QZ_{N+1} = b\,Q(Z_N)^s.$$

Putting this back in the second moment computation, we obtain (recalling that $QW_N = 1$)

$$Q(W_{N+1}^2) = \frac{1}{b}\,Q(W_N^2)^s + \frac{b-1}{b}.$$

So, similarly to the tree case, we are reduced to study the behavior of the recursive equations:

$$x_{N+1} = \frac{1}{b}x_N^s + \frac{b-1}{b},$$

with initial condition $x_0 = Q(W_0^2) = \exp\{\lambda(2\beta) - 2\lambda(\beta)\} \geq 1$.
If $b > s$, this dynamical system has two fixed-points: 1 and another one $\overline{x} > 1$. If $x_0 \in (1, \overline{x}]$, then $x_N \to 1$ as $N \to +\infty$, but, if $x_0 \in (\overline{x}, +\infty)$, then $x_N \to +\infty$. The L^2 region, defined as the set of β's such that QW_N^2 stays bounded, is exactly the set of β's such that $x_0 \in [1, \overline{x}]$. Observe that in this case, $W_N \to 1$, Q-a.s. as $N \to +\infty$.
If $b \leq s$, there is a single fixed point at 1 and $x_N \to +\infty$ for all initial conditions $x_0 > 1$. As a matter of fact, it is possible to show that strong disorder always holds for $b \leq s$ (see Section 4.8).

CHAPTER 4

Directed polymers on hierarchical diamond lattices with site disorder

We study a polymer model on hierarchical lattices very close to the one introduced and studied in [**36, 31**]. For this model, we prove the existence of free energy and derive the necessary and sufficient condition for which very strong disorder holds for all β, and give some accurate results on the behavior of the free energy at high temperature. We obtain these results by using a combination of fractional moment method and change of measure over the environment to obtain an upper bound, and second moment method to get a lower bound. We also get lower bounds on the fluctuation exponent of $\log Z_N$, and study the infinite polymer measure in the weak disorder phase.

4.1. Introduction and presentation of the model

The model of directed polymers in random environment appeared first in the physics literature as an attempt to modelize roughening in domain wall in the $2D$-Ising model due to impurities [**49**]. Then, it reached the mathematical community in [**52**], and in [**12**], where the author applied martingale techniques that have became the major technical tools in the study of this model since then. A lot of progress has been made recently in the mathematical understanding of directed polymer model (see for example [**53, 28, 24, 26, 21, 20, 30**] and [**25**] for a recent review). It is known that there is a phase transition from a delocalized phase at high temperature, where the behavior of the polymer is diffusive, to a localized phase, where the influence of the medium is relevant and is expected to produce nontrivial phenomena, such as superdiffusivity. These two different situations are usually referred to as weak and strong disorder, respectively. A simple characterization of this dichotomy is given in terms of the limit of a certain positive martingale related to the partition function of this model.

It is known that in low dimensions ($d = 1$ or 2), the polymer is in the strong disorder phase at all temperature (see [**30, 66**], for more precise results), whereas for $d \geq 3$, there is a nontrivial region of temperatures where weak disorder holds. A weak form of invariance principle is proved in [**28**].

However, the exact value of the critical temperature which separates the two regions (when it is finite) remains an open question. It is known exactly in the case of directed polymers on the tree, where a complete analysis is available (see [**15, 41, 60**]). In the case of $\mathbb{Z}^d$, for $d \geq 3$, an L^2 computation yields an upper bound on the critical temperature, which is however known not to coincide with this bound (see [**9, 8**] and [**18**]).

We choose to study the same model of directed polymers on diamond hierarchical lattices. These lattices present a very simple structure allowing to perform a lot of computations together with a richer geometry than the tree (see Remark 4.3 for more details). They have been introduced in physics in order to perform exact renormalization group computations for spin systems ([**75, 56**]). A detailed treatment of more general hierarchical lattices can be found in [**61**] and [**62**]. For an overview of the extensive literature on Ising and Potts models on hierarchical lattices, we refer the reader to [**10, 34**] and references therein. Whereas statistical mechanics model on trees have to be considered as mean-field versions of the original models, the hierarchical lattice models are in many sense very close to the models on $\mathbb{Z}^d$; they are a very powerful tool to get an intuition for results and proofs on the more complex $\mathbb{Z}^d$ models (for instance, the work on hierarchical pinning model in [**42**] lead to a solution of the original model in [**38**]. In the same manner, the present work has been a great source of inspiration for [**66**]).

Directed polymers on hierarchical lattices (with bond disorder) appeared in [**31, 35, 36, 37**] (see also [**82**] for directed first-passage percolation). More recently, these lattice models raised the interest of mathematicians in the study of random resistor networks ([**95**]), pinning/wetting transitions ([**42, 65**]) and diffusion on a percolation cluster ([**48**]).

We can also mention [**47**] where the authors consider a random analogue of the hierarchical lattice, where at each step, each bond transforms either into a series of two bonds or into two bonds in parallel, with probability p and $p-1$ respectively.

Our aim in this paper is to describe the properties of the quenched free energy of directed polymers on hierarchical lattices with site disorder at high temperature:

- First, to be able to decide, in all cases, if the quenched and annealed free energy differ at low temperature.
- If they do, we want to be able to describe the phase transition and to compute the critical exponent for the free energy.

We choose to focus on the model with site disorder, whereas [**44, 31**] focus on the model with *bond disorder* where computations are simpler. We do so because we believe that this model is closer to the model of directed polymer in $\mathbb{Z}^d$ (in particular, because of the inhomogeneity of the Green Function), and because there exists a nice recursive construction of the partition functions in our case, that leads to a martingale property. Apart from that, both models are very similar, and we will shortly talk about the bond disorder model in section 4.8.

The diamond hierarchical lattice D_N can be constructed recursively:

- D_0 is one single edge linking two vertices A and B.
- D_{N+1} is obtained from D_N by replacing each edge by b branches of $s-1$ edges.

We can, improperly, consider D_N as a set of vertices, and, with the above construction, we have $D_N \subset D_{N+1}$. We set $D = \bigcup_{N\geq 0} D_N$. The vertices introduced at the N-th iteration are said to belong to the N-th generation $V_N = D_N \setminus D_{N-1}$. We easily see that $|V_N| = (bs)^{N-1}b(s-1)$.
We restrict to $b \geq 2$ and $s \geq 2$. The case $b = 1$ (resp. $s = 1$) is not interesting as it just corresponds to a family of edges in series (resp. in parallel).

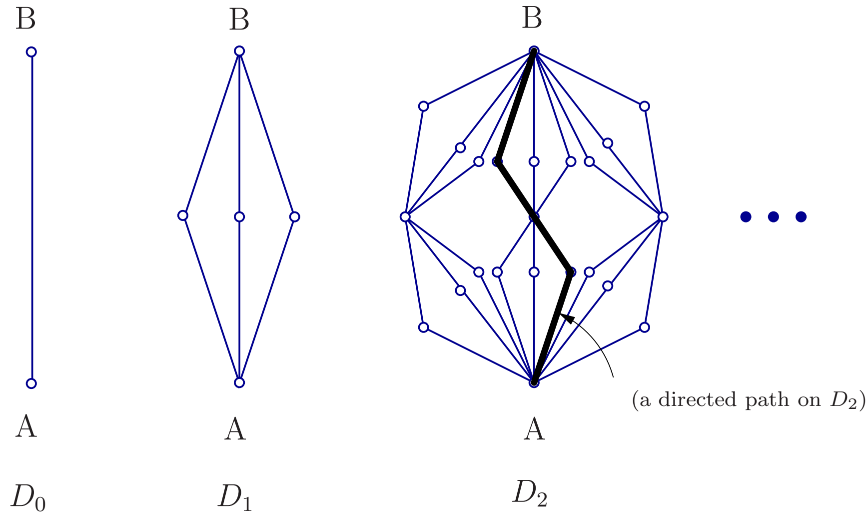

FIGURE 1. We present here the recursive construction of the first three levels of the hierarchical lattice D_N, for $b = 3$, $s = 2$.

We introduce disorder in the system as a set of real numbers $\eta = (\eta_z)_{z\in D\setminus\{A,B\}}$ associated to the vertices in $D \setminus \{A, B\}$. Consider Ω_N the space of directed paths in D_N linking A to B. For each $\mathbf{S} \in \Omega_N$ (to be understood as a sequence of connected vertices in D_N, $(\mathbf{S}_0 = A, \mathbf{S}_1, \ldots, \mathbf{S}_{s^N} = B)$), we define the Hamiltonian

$$H_N^\eta(\mathbf{S}) := \sum_{t=1}^{s^N-1} \eta(\mathbf{S}_t). \tag{4.1}$$

For $\beta > 0$, $N \geq 1$, we define the (quenched) polymer measure $\mu_N^{\beta,\eta}$ on Ω_N which picks a path ω at random with law

$$\mu_N^{\beta,\eta}(\omega = \mathbf{S}) := \frac{1}{Z_N(\beta)} \exp(\beta H_N^\eta(\mathbf{S})), \tag{4.2}$$

where

$$Z_N(\beta) = Z_N(\beta, \eta) := \sum_{\mathbf{S}\in\Omega_N} \exp(\beta H_N^\eta(\mathbf{S})), \tag{4.3}$$

is the partition function, and β is the inverse temperature parameter.

In the sequel, we will focus on the case where $\eta = (\eta_z,\ z \in D\setminus\{A, B\})$ is a collection of i.i.d. random variables and denote the product measure by Q. Let η_0 denote a one dimensional marginal of Q, we assume that η_0 has expectation zero, unit variance, and that

$$\lambda(\beta) := \log Q e^{\beta\eta_0} < \infty \quad \forall \beta > 0. \tag{4.4}$$

As usual, we define the quenched free energy (see Theorem 4.1) by

$$p(\beta) := \lim_{N\to+\infty} \frac{1}{s^N} Q \log Z_N(\beta), \tag{4.5}$$

and its annealed counterpart by

$$f(\beta) := \lim_{N\to+\infty} \frac{1}{s^N} \log QZ_N(\beta). \tag{4.6}$$

This annealed free energy can be exactly computed. We will prove

$$f(\beta) := \lambda(\beta) + \frac{\log b}{s-1}. \tag{4.7}$$

This model can also be stated as a random dynamical system: given two integer parameters b and s larger than 2, $\beta > 0$, consider the following recursion

$$\begin{aligned} W_0 &\overset{\mathcal{L}}{=} 1 \\ W_{N+1} &\overset{\mathcal{L}}{=} \frac{1}{b} \sum_{i=1}^{b} \prod_{j=1}^{s} W_N^{(i,j)} \prod_{i=1}^{s-1} A_N^{(i,j)}, \end{aligned} \tag{4.8}$$

where equalities hold in distribution, $W_N^{(i,j)}$ are independent copies of W_N, and $A_N^{(i,j)}$ are i.i.d. random variables, independent of the $W_N^{(i,j)}$ with law

$$A \overset{\mathcal{L}}{=} \exp(\beta\eta - \lambda(\beta)).$$

In the directed polymer setting, W_N can be interpreted as the normalized partition function

$$W_N(\beta) = W_N(\beta, \eta) = \frac{Z_N(\beta, \eta)}{QZ_N(\beta)}. \tag{4.9}$$

Then, (4.8) turns out to be an almost sure equality if we interpret $W_N^{(i,j)}$ as the partition function of the j-th edge of the i-th branch of D_1.

The sequence $(W_N)_{N\geq 0}$ is a martingale with respect to $\mathcal{F}_N = \sigma(\eta_z : z \in \cup_{i=1}^N V_i)$ and as $W_N > 0$ for all N, we can define the almost sure limit $W_\infty = \lim_{N\to+\infty} W_N$. Taking limits in both sides of (4.8), we obtain a functional equation for W_∞.

4.2. Results

Our first result is about the existence of the free energy.

THEOREM 4.1. *For all β, the limit*

$$\lim_{N\to+\infty} \frac{1}{s^N} \log Z_N(\beta), \tag{4.10}$$

exists a.s. and is a.s. equal to the quenched free energy $p(\beta)$. In fact for any $\varepsilon > 0$, one can find $N_0 = N_0(\varepsilon, \beta)$ such that

$$Q\left(|\log Z_N - Q\log Z_N| > s^N\varepsilon\right) \leq \exp\left(-\frac{\varepsilon^{2/3}s^{N/3}}{4}\right), \quad \textit{for all } N \geq N_0 \tag{4.11}$$

Moreover, $p(\cdot)$ is a strictly convex function of β.

REMARK 4.2. The inequality (4.11) is the exact equivalent of [**24**, Proposition 2.5], and the proof given there can easily be adapted to our case. It applies concentration results for martingales from [**71**]. It can be improved in order to obtain the same bound as for Gaussian environments stated in [**20**] (see [**22**] for details). However, it is believed that it is not of the optimal order, similar to the case of directed polymers on $\mathbb{Z}^d$.

REMARK 4.3. The strict convexity of the free energy is an interesting property. It is known that it holds also for the directed polymer on $\mathbb{Z}^d$ but not on the tree. In the later case, the free energy is strictly convex only for values of β smaller than the critical value β_c (to be defined latter) and it is linear on $[\beta_c, +\infty)$. This fact is related to the particular structure of the tree that leads to major simplifications in the "correlation structure" of the model (see [**15**]). The strict convexity, in our setting, arises essentially from the property that two paths on the hierarchical lattice can re-intersect after being separated at some step. This underlines once more, that $\mathbb{Z}^d$ and the hierarchical lattice have a lot of features in common, which they do not share with the tree.

Next, we establish the martingale property for W_N and the zero-one law for its limit.

LEMMA 4.4. *$(W_N)_N$ is a positive $\mathcal{F}_N$-martingale. It converges Q-almost surely to a non-negative limit W_∞ that satisfies the following zero-one law:*

$$Q\left(W_\infty > 0\right) \in \{0, 1\}. \tag{4.12}$$

Recall that martingales appear when the disorder is displayed on sites, in contrast with disorder on bonds as in [**31, 35**].

Observe that

$$p(\beta) - f(\beta) = \lim_{N\to+\infty} \frac{1}{s^N} \log W_N(\beta),$$

so, if we are in the situation $Q(W_\infty > 0) = 1$, we have that $p(\beta) = f(\beta)$. This motivates the following definition:

DEFINITION 4.5. If $Q(W_\infty > 0) = 1$, we say that weak disorder holds. In the opposite situation, we say that strong disorder holds.

REMARK 4.6. Later, we will give a statement(Proposition 4.21) that guarantees that strong disorder is equivalent to $p(\beta) \neq f(\beta)$, a situation that is sometimes called very strong disorder. This is believed to be true for polymer models on $\mathbb{Z}^d$ or $\mathbb{R}^d$ but it remains an unproved and challenging conjecture in dimension $d \geq 3$ (see [**21**]).

The next proposition lists a series of partial results that in some sense clarify the phase diagram of our model.

PROPOSITION 4.7. (*i*) *There exists* $\beta_0 \in [0, +\infty]$ *such that strong disorder holds for* $\beta > \beta_0$ *and weak disorder holds for* $\beta \leq \beta_0$.

(*ii*) *If* $b > s$, $\beta_0 > 0$. *Indeed, there exists* $\beta_2 \in (0, \infty]$ *such that for all* $\beta < \beta_2$, $\sup_N Q(W_N^2(\beta)) < +\infty$, *and therefore weak disorder holds.*

(*iii*) *If* $\beta\lambda'(\beta) - \lambda(\beta) > \frac{2\log b}{s-1}$, *then strong disorder holds.*

(*iv*) *In the case where* η_z *are Gaussian random variables,* (*iii*) *can be improved for* $b > s$: *strong disorder holds as soon as* $\beta > \sqrt{\frac{2(b-s)\log b}{(b-1)(s-1)}}$.

(*v*) *If* $b \leq s$, *then strong disorder holds for all* β.

REMARK 4.8. On can check that the formula in (*iii*) ensures that $\beta_0 < \infty$ whenever the distribution of η_z is unbounded.

REMARK 4.9. An implicit formula is given for β_2 in the proof and this gives a lower bound for β_0. However, when $\beta_2 < \infty$, it never coincides with the upper bound given by (*iii*) and (*iv*), and therefore knowing the exact value of the critical temperature when $b > s$ remains an open problem.

We now provide more quantitative information for the regime considered in (*v*):

THEOREM 4.10. *When* $s > b$, *there exists a constant* $c = c_{s,b}$ *such that for any* $\beta \leq 1$ *we have*

$$\frac{1}{c}\beta^{\frac{2}{\alpha}} \leq f(\beta) - p(\beta) \leq c\beta^{\frac{2}{\alpha}}$$

where $\alpha = \frac{\log s - \log b}{\log s}$.

THEOREM 4.11. *When* $s = b$, *there exists a constant* $c = c_s$ *such that for any* $\beta \leq 1$ *we have*

$$\exp\left(-\frac{c}{\beta^2}\right) \leq f(\beta) - p(\beta) \leq c\exp\left(-\frac{1}{c\beta}\right)$$

In the theory of directed polymer in random environment, it is believed that, in low dimension, the quantity $\log Z_N$ undergoes large fluctuations around its average (as opposed to what happens in the weak disorder regime where the fluctuation are of order 1). More precisely: it is believed that there exists an exponent $\chi > 0$ such that

$$\log Z_N - Q \log Z_N \asymp M^{\xi} \text{ and } \mathrm{Var}_Q \log Z_N \asymp M^{2\chi}, \tag{4.13}$$

where M is the length of the system ($= N$ on $\mathbb{Z}^d$ and s^N one our hierarchical lattice). In the non-hierarchical model this exponent is of major importance as it is closely related to the *volume exponent* ξ that gives the spatial fluctuation of the polymer chain (see e.g. [**53**] for a discussion on fluctuation exponents). Indeed it is conjectured for the $\mathbb{Z}^d$ models that

$$\chi = 2\xi - 1. \tag{4.14}$$

This implies that the polymer trajectories are superdiffusive as soon as $\chi > 0$. In our hierarchical setup, there is no such geometric interpretation but having a lower bound on the fluctuation allows to get a significant localization result.

Proposition 4.12. *When $b < s$, there exists a constant c such that for all $N \geq 0$ we have*

$$\mathrm{Var}_Q \left(\log Z_N\right) \geq c(s/b)^N \beta^2. \tag{4.15}$$

Moreover, for any $\varepsilon > 0$, $N \geq 0$, and $a \in \mathbb{R}$,

$$Q\left\{\log Z_N \in [a, a + \varepsilon(s/b)^{N/2}]\right\} \leq \frac{8\varepsilon}{\beta}. \tag{4.16}$$

This implies that if the fluctuation exponent χ exists, $\chi \geq \frac{\log s - \log b}{2 \log s}$. We also have the corresponding result for the case $b = s$

Proposition 4.13. *When $b = s$, there exists a constant c such that for all $N \geq 0$, we have*

$$\mathrm{Var}_Q \left(\log Z_N\right) \geq cN\beta^2. \tag{4.17}$$

Moreover for any $\varepsilon > 0$, $N \geq 0$, and $a \in \mathbb{R}$,

$$Q\left\{\log Z_N \in [a, a + \varepsilon\sqrt{N}]\right\} \leq \frac{8\varepsilon}{\beta}. \tag{4.18}$$

From the fluctuations of the free energy we can prove the following: For $\mathbf{S} \in \Omega_N$ and $M < N$, we define $\mathbf{S}|_M$ to be the restriction of $\mathbf{S}$ to D_M.

Corollary 4.14. *If $b \leq s$ and $M \in \mathbb{N}$ is fixed, we have*

$$\lim_{N\to\infty} \sup_{\boldsymbol{S} \in \Omega_M} \mu_N(\omega|_M = \boldsymbol{S}) = 1, \tag{4.19}$$

where the convergence holds in probability.

Intuitively this result means that if one looks on a large scale, the law of μ_N is concentrated in the neighborhood of a single path. Considering Ω_N with a natural metric (two path $\mathbf{S}$ and $\mathbf{S}'$ in Ω_N are at distance 2^{-M} if and only if $\mathbf{S}|_M \neq \mathbf{S}'|_M$ and $\mathbf{S}|_{M-1} = \mathbf{S}|_{M-1}$) makes this statement rigorous.

REMARK 4.15. With some (very minor) additional effort, one can prove that the convergence in (4.19) holds almost surely, at least in the case $b < s$. However, this is just a consequence of the fact that the size of our system grows very fast (exponentially), so that Borel-Cantelli Lemma easily works, and has not deep meaning. The result could also be improved by taking $m(N)$ going to infinity with N, but we are far from being able to give an optimal rate for the divergence of $M(N)$ with our actual techniques.

REMARK 4.16. Proposition 4.7(v) brings the idea that $b \le s$ for this hierarchical model is equivalent to the $d \le 2$ case for the model in $\mathbb{Z}^d$ (and that $b > s$ is equivalent to $d > 2$). Let us push further the analogy: let $\omega^{(1)}$, $\omega^{(2)}$ be two paths chosen uniformly at random in Ω_N (denote the uniform-product law by $P^{\otimes 2}$), their expected site overlap is of order $(s/b)^N$ if $b < s$, of order N if $b = s$, and of order 1 if $b > s$. If one denotes by $M = s^N$ the length of the system, one has

$$P^{\otimes 2}\left[\sum_{t=0}^{M} \mathbf{1}_{\{\omega_t^{(1)}=\omega_t^{(2)}\}}\right] \asymp \begin{cases} M^{\alpha} & \text{if } b < s, \\ \log M & \text{if } b = s, \\ 1 \text{ if } b > s, \end{cases} \tag{4.20}$$

(where $\alpha = (\log s - \log b)/\log s$). Comparing this to the case of random walk on $\mathbb{Z}^d$, we can infer that the case $b = s$ is just like $d = 2$ and that the case $d = 1$ is similar to $b = \sqrt{s}$ ($\alpha = 1/2$). One can check in comparing [**66**, Theorem 1.4, 1.5, 1.6] with Theorem 4.10 and 4.11, that this analogy is relevant.

The paper is organized as follow

- In section 4.3 we prove some basic statements about the free energy, Lemma 4.4 and the first part of Proposition 4.7.
- In section 4.4, we prove (ii) of Proposition 4.7 and the upper-bound inequalities Theorems 4.10 and 4.11.
- In section 4.5, we prove (iii) and (iv) of Proposition 4.7 and the lower-bound inequalities for Theorems 4.10 and 4.11.
- In section 4.6 we prove Propositions 4.12 and 4.13 and Corollary 4.14.
- In section 4.7 we define and investigate the properties of the infinite volume polymer measure in the weak disorder phase.
- In section 4.8 we shortly discuss about the bond disorder model.

4.3. Martingale tricks and free energy

We first look at to the existence of the quenched free energy

$$p(\beta) = \lim_{N\to+\infty} \frac{1}{N} Q\left(\log Z_N(\beta)\right),$$

and its relation with the annealed free energy. Much in the same spirit than (4.8), we can find a recursion for Z_N:

$$Z_{N+1}(\beta) = \sum_{i=1}^{b} Z_N^{(i,1)}(\beta) \cdots Z_N^{(i,s)}(\beta) \times e^{\beta\eta_{i,1}} \cdots e^{\beta\eta_{i,s-1}}. \tag{4.21}$$

The case $\beta = 0$ is somehow instructive. It gives the number of paths in Ω_N and is handled by the simple recursion

$$Z_{N+1}(0) = b\,(Z_N(0))^s\,,$$

which follows by taking $\beta = 0$ in (4.21).This easily yields

$$|\Omega_N| = Z_N(\beta = 0) = b^{\frac{s^N-1}{s-1}}. \tag{4.22}$$

The existence of the quenched free energy follows by monotonicity: again from (4.21), we have

$$Z_{N+1} \geq Z_N^{(1,1)} Z_N^{(1,2)} \cdots Z_N^{(1,s)} \times e^{\beta\eta_{1,1}} \cdots e^{\beta\eta_{1,s-1}},$$

so that (recall the η's are centered random variables)

$$\frac{1}{s^{N+1}} Q \log Z_{N+1} \geq \frac{1}{s^N} Q \log Z_N.$$

The annealed free energy provides an upper bound

$$\begin{aligned} \frac{1}{s^N} Q \log Z_N &\leq \frac{1}{s^N} \log Q Z_N \\ &= \frac{1}{s^N} \log e^{\lambda(\beta)(s^N-1)} Z_N(\beta = 0) \\ &= \left(1 - \frac{1}{s^N}\right)\left(\lambda(\beta) + \frac{\log b}{s-1}\right) \\ &= \left(1 - \frac{1}{s^N}\right) f(\beta). \end{aligned}$$

We now prove the strict convexity of the free energy. The proof is essentially borrowed from [**23**], but it is remarkably simpler in our case.

Proof of the strict convexity of the free energy. We do our proof in the simpler case of Bernoulli environment ($\eta_z = \pm 1$ with probability p, $1-p$; note that our assumptions on the variance and expectation for η are violated but centering and rescaling η does not change the argument). See the remark following this proof for the generalization to arbitrary environment.
An easy computation yields

$$\frac{\partial^2}{\partial \beta^2} Q \log Z_N = Q \mathrm{Var}_{\mu_N} H_N(\omega).$$

We will prove that for each $K > 0$, there exists a constant C such that, for all $\beta \in [0, K]$ and $N \geq 1$,

$$Q \mathrm{Var}_{\mu_N} H_N(\omega) \geq C s^N. \tag{4.23}$$

For $\mathbf{S} \in \Omega_N$ and $M < N$, we define $\mathbf{S}|_M$ to be the restriction of $\mathbf{S}$ to D_M. By the conditional variance formula,

$$\begin{aligned} \mathrm{Var}_{\mu_N} H_N &= \mu_N \left(\mathrm{Var}_{\mu_N}(H_N(\omega) \,|\, \omega_{|_{N-1}})\right) + \mathrm{Var}_{\mu_N} \left(\mu_N(H_N(\omega) \,|\, \omega_{|_{N-1}})\right) \\ &\geq \mu_N \left(\mathrm{Var}_{\mu_N}(H_N(\omega) \,|\, \omega_{|_{N-1}})\right) \end{aligned} \tag{4.24}$$

Now, for $l = 0, ..., s^{N-1} - 1$, $\mathbf{S} \in \Omega_N$, define

$$H_N^{(l)}(\mathbf{S}) = \sum_{t=ls+1}^{(l+1)s-1} \eta(\mathbf{S}_t),$$

so the right-hand side of (4.24) is equal to

$$\mu_N \mathrm{Var}_{\mu_N} \left(\sum_{l=0}^{s^{N-1}-1} H_N^{(l)}(\omega) | \omega_{|_{N-1}} \right) = \sum_{l=0}^{s^{N-1}-1} \mu_N \mathrm{Var}_{\mu_N} \left(H_N^{(l)}(\omega) | \omega_{|_{N-1}} \right),$$

by independence. Summarizing,

$$\mathrm{Var}_{\mu_N} H_N \geq \sum_{l=1}^{s^{N-1}-1} \mu_N \mathrm{Var}_{\mu_N} \left(H_N^{(l)}(\omega) | \omega_{|_{N-1}} \right). \tag{4.25}$$

The rest of the proof consists in showing that each term of the sum is bounded from below by a positive constant, uniformly in l and N. For any $x \in D_{N-1}$ such that the graph distance between x and A is ls in D_N (i.e. $x \in D_{N-1}$), we define the set of environments

$$\mathcal{M}(N, l, x) = \left\{ \omega : \left| \{ H_N^{(l)}(\mathbf{S}, \eta) \,:\, \mathbf{S} \in \Omega_N, \mathbf{S}_{ls} = x \} \right| \geq 2 \right\}.$$

These environments provide the fluctuations in the energy needed for the uniform lower bound we are searching for. One second suffices to convince oneself that $Q(\mathcal{M}(N, l, x)) > 0$, and does not depend on the parameters N, l or x. Let $Q(M)$ denote improperly the common value of $Q(\mathcal{M}(N, l, x))$. Now, from (4.25),

$$Q \left[\mathrm{Var}_{\mu_N} H_N\right] \geq Q \sum_{l=1}^{s^{N-1}-1} \sum_{x \in D_{N-1}} \mathbf{1}_{\mathcal{M}(N,l,x)} \mu_N \left[\mathrm{Var}_{\mu_N} \left(H_N^{(l)}(\omega) | \omega_{|_{N-1}} \right) \mathbf{1}_{\omega_{ls} = x} \right]. \tag{4.26}$$

On the event $\mathcal{M}(N, l, x) \cap \{\omega_{ls} = x\}$, the variance with respect to $\mu_N \left(\ldots | \omega_{|_{N-1}} \right)$ only depends on the environment inside the little diamond based on x, i.e., the isomorphic embedding of D_1 inside D_N, where the image of A is x (let us call it $D_1(N, l, x)$). Based on this observation, it is easy to show that the $\mu_N \left(\ldots | \omega_{|_{N-1}} \right)$ probability of $\{ |H_N^{(l)}(\omega) - \mu_N(H_N^{(l)}(\omega) | \omega_{|_{N-1}})| \geq 1 \}$ is at least $\frac{1}{b} \exp(-2\beta(s-1))$, so that

$$\text{(4.27)} \qquad \operatorname{Var}_{\mu_N}\left(H_N^{(l)}(\omega)|\omega_{|_{N-1}}\right) \geq \frac{1}{b}\exp(-2\beta(s-1)).$$

Putting this back to (4.26) yields

$$Q\left[\operatorname{Var}_{\mu_N} H_N\right] \;\geq\; \frac{1}{b}\exp(-2\beta(s-1))\,Q\left[\sum_{l=1}^{s^{N-1}-1}\sum_{x\in D_{N-1}} \mathbf{1}_{\mathcal{M}(N,l,x)}\mu_N(\omega_{ls}=x)\right].$$

Define now $\mu_N^{(l,x)}$ as the polymer measure in the environment obtained from η by setting $\eta_y = 0$ for all sites $y \in D_1(N,l,x)$. One can check that for all N, and all path $\mathbf{S}$,

$$\exp(-2\beta(s-1))\mu_N^{(l,x)}(\omega=\mathbf{S}) \leq \mu_N(\omega=\mathbf{S}) \leq \exp(2\beta(s-1))\mu_N^{(l,x)}(\omega=\mathbf{S}).$$

We note that under Q, $\mu_N^{(l,x)}(\omega_{ls}=x)$ and $\mathbf{1}_{\mathcal{M}(N,l,x)}$ are independent random variables, so that

$$\begin{aligned}
Q\left[\operatorname{Var}_{\mu_N} H_N\right] \;&\geq\; \frac{1}{b}\exp(-4\beta(s-1))Q\left[\sum_{l=0}^{s^{N-1}-1}\sum_{x} \mathbf{1}_{\mathcal{M}(N,l,x)}\mu_N^{(l,x)}(\omega_{ls}=x)\right]\\
&=\; \frac{1}{b}\exp(-4\beta(s-1))\sum_{l=1}^{s^{N-1}}\sum_{x\in D_{N-1}} Q(\mathcal{M}(N,l,x))Q\left[\mu_N^{(l,x)}(\omega_{ls}=x)\right]\\
&\geq\; \frac{1}{b}\exp(-6\beta(s-1))\sum_{l=1}^{s^{N-1}}\sum_{x\in D_{N-1}} Q(\mathcal{M}(N,l,x))Q\left[\mu_N(\omega_{ls}=x)\right]\\
&=\; \frac{1}{b}\exp(-6\beta(s-1))Q(\mathcal{M})s^{N-1}.
\end{aligned}$$

This lower bound is uniform as long as β stays in a compact. □

Remark 4.17. For arbitrary environments, we can always find $K > K'$ and a positive constant L such that the event

$$\begin{aligned}
\mathcal{M}(N,l,x) = \big\{\omega : \exists \mathbf{S},\mathbf{S}' \,\text{with}\, \mathbf{S}_{ls} = \mathbf{S}'_{ls} = x \text{ s.t. } H_N^{(l)}(\mathbf{S},\eta) \geq K,\\
H_N^{(l)}(\mathbf{S}',\eta) \leq K',\, \text{and}\, |\eta_y| \leq L,\, \forall\, y \in D_1(N,l,x)\big\},
\end{aligned}$$

has positive Q-probability. The proof can then be achieved with this definition of $\mathcal{M}(N,l,x)$ (with lower bound $\frac{K-K'}{2b}\exp(-6\beta L(s-1))$ at the end).

We now establish the martingale property for the normalized free energy.

Proof of Lemma 4.4. Set $z_N = Z_N(\beta=0)$. We have already remarked that this is just the number of (directed) paths in D_N, and its value is given by (4.22). Observe that $\mathbf{S} \in \Omega_N$ visits $s^N(s-1)$ sites of $N+1$-th generation. The restriction of paths in

D_{N+1} to D_N is obviously not one-to-one as for each path $\mathbf{S}' \in \Omega_N$, there are b^{s^N} paths in Ω_{N+1} such that $\mathbf{S}|_N = \mathbf{S}'$. Now,

$$\begin{aligned}
Q\left(Z_{N+1}(\beta)|\mathcal{F}_N\right) &= \sum_{\mathbf{S}\in\Omega_{N+1}} Q\left(e^{\beta H_{N+1}(\mathbf{S})}|\mathcal{F}_N\right) \\
&= \sum_{\mathbf{S}'\in\Omega_N}\sum_{\mathbf{S}\in\Omega_{N+1}} Q\left(e^{\beta H_{N+1}(\mathbf{S})}|\mathcal{F}_N\right)\mathbf{1}_{\mathbf{S}|_N=\mathbf{S}'} \\
&= \sum_{\mathbf{S}'\in\Omega_N}\sum_{\mathbf{S}\in\Omega_{N+1}} e^{\beta H_N(\mathbf{S}')}e^{s^N(s-1)\lambda(\beta)}\mathbf{1}_{\mathbf{S}|_N=\mathbf{S}'} \\
&= \sum_{\mathbf{S}'\in\Omega_N} e^{\beta H_N(\mathbf{S}')}e^{s^N(s-1)\lambda(\beta)}\sum_{\mathbf{S}\in\Omega_{N+1}}\mathbf{1}_{\mathbf{S}|N=\mathbf{S}'} \\
&= e^{s^N(s-1)\lambda(\beta)}b^{s^N}\sum_{\mathbf{S}'\in\Omega_N} e^{\beta H_N(g')} \\
&= Z_N(\beta)\frac{z_{N+1}e^{s^{N+1}\lambda(\beta)}}{z_N e^{s^N\lambda(\beta)}}.
\end{aligned}$$

This proves the martingale property. For (4.12), let us generalize a little the preceding restriction procedure. As before, for a path $\mathbf{S} \in \Omega_{N+k}$, denote by $\mathbf{S}|_N$ its restriction to Ω_N. Denote by $I_{N,N+k}$ the set of time indexes that have been removed in order to perform this restriction and by $M_{N,N+k}$ its cardinality. Then

$$Z_{N+k} = \sum_{\mathbf{S}\in\Omega_N} e^{\beta H_N(\mathbf{S})}\sum_{\mathbf{S}'\in\Omega_{N+k},\mathbf{S}'|_N=\mathbf{S}}\exp\left\{\beta\sum_{t\in I_{N,N+k}}\eta(\mathbf{S}'_t)\right\}.$$

Consider the following notation, for $\mathbf{S} \in \Omega_N$,

$$\widetilde{W}_{N,N+k}(\mathbf{S}) = c_{N,N+k}^{-1}\sum_{\mathbf{S}'\in\Omega_{N+k},\mathbf{S}'|_N=\mathbf{S}}\exp\left\{\beta\sum_{t\in I_{N,N+k}}\eta(\mathbf{S}'_t) - M_{N,N+k}\lambda(\beta)\right\},$$

where $c_{N,N+k}$ stands for the number paths in the sum. With this notations, we have,

$$W_{N+k} = \frac{1}{z_N}\sum_{\mathbf{S}\in\Omega_N} e^{\beta H_N(\mathbf{S})-(s^N-1)\lambda(\beta)}\widetilde{W}_{N,N+k}(\mathbf{S}), \tag{4.28}$$

and, for all N,

$$\{W_\infty = 0\} = \left\{\widetilde{W}_{N,N+k}(\mathbf{S}) \to 0,\ \text{as}\, k \to +\infty,\ \forall\,\mathbf{S}\in\Omega_N\right\}. \tag{4.29}$$

The event in the right-hand side is measurable with respect to the disorder of generation not earlier than N. As N is arbitrary, the left-hand side of (4.29) is in the tail σ-algebra and its probability is either 0 or 1. □

This, combined with FKG-type arguments (see [**28**, Theorem 3.2] for details), proves part (i) of Proposition 4.7. Roughly speaking, the FKG inequality is used to insure that there is no reentrance phase.

4.4. Second moment method and lower bounds

This section contains all the proofs concerning coincidence of annealed and quenched free energy for $s > b$ and lower bounds on the quenched free energy for $b \leq s$ (i.e. half of the results from Proposition 4.7 to Theorem 4.11.) First, we discuss briefly the condition on β that one has to fulfill to have W_N bounded in $\mathbb{L}_2(Q)$. Then for the cases when strong disorder holds at all temperature ($b \leq s$), we present a method that combines control of the second moment up to some scale N and a percolation argument to get a lower bound on the free energy.

First, we investigate how to get the variance of W_N (under Q). From (4.8) we get the induction for the variance $v_N = Q\left[(W_N - 1)^2\right]$:

$$v_{N+1} = \frac{1}{b}\left(e^{(s-1)\gamma(\beta)}(v_N+1)^s - 1\right), \tag{4.30}$$

$$v_0 = 0. \tag{4.31}$$

where $\gamma(\beta) := \lambda(2\beta) - 2\lambda(\beta)$.

4.4.1. The L^2 domain: $s < b$. If $b > s$, and $\gamma(\beta)$ is small, the map

$$g: \; x \mapsto \frac{1}{b}\left(e^{(s-1)\gamma(\beta)}(x+1)^s - 1\right)$$

possesses a fixed point. In this case, (4.30) guaranties that v_N converges to some finite limit. Therefore, in this case, W_N is a positive martingale bounded in $\mathbb{L}^2$, and therefore converges almost surely to $W_\infty \in \mathbb{L}^2(Q)$ with $QW_\infty = 1$, so that

$$p(\beta) - \lambda(\beta) = \lim_{N\to\infty} \frac{1}{s^N} \log W_N = 0,$$

and weak disorder holds. One can check that g has a fixed point if and only if

$$\gamma(\beta) \leq \frac{s}{s-1}\log\frac{s}{b} - \log\frac{b-1}{s-1}$$

4.4.2. Control of the variance: $s > b$. For $\epsilon > 0$, let n_0 be the smallest integer such that $v_{n_0} \geq \varepsilon$.

LEMMA 4.18. *For any $\varepsilon > 0$, there exists a constant c_ε such that for any $\beta \leq 1$*

$$n_0 \geq \frac{2|\log\beta|}{\log s - \log b} - c_\varepsilon.$$

PROOF. Expanding (4.30) around $\beta = 0$, $v_N = 0$, we find a constant c_1 such that, whenever $v_N \leq 1$ and $\beta \leq 1$,

$$v_{N+1} \leq \frac{s}{b}(v_N + c_1\beta^2)(1 + c_1 v_N). \tag{4.32}$$

Using (4.32), we obtain by induction

$$v_{n_0} \leq \prod_{i=0}^{n_0-1}(1+c_1v_i)\left[c_1\beta^2\left(\sum_{i=0}^{n_0-1}(s/b)^i\right)\right].$$

From (4.30), we see that $v_{i+1} \geq (s/b)v_i$. By definition of n_0, $v_{n_0-1} < \epsilon$, so that $v_i < \varepsilon(s/b)^{i-n_0+1}$. Then

$$\prod_{i=0}^{n_0-1}(1+c_1v_i) \leq \prod_{i=0}^{n_0-1}(1+c_1\varepsilon(s/b)^{i-n_0+1}) \leq \prod_{k=0}^{\infty}(1+c_1\varepsilon(s/b)^{-k}) \leq 2,$$

where the last inequality holds for ε small enough. In that case we have

$$\varepsilon \leq v_{n_0} \leq 2c_1\beta^2(s/b)^{n_0},$$

so that

$$n_0 \geq \frac{\log(\varepsilon/2c_1\beta^2)}{\log(s/b)}.$$

□

4.4.3. Control of the variance: $s = b$.

LEMMA 4.19. *There exists a constant c_2 such that, for every $\beta \leq 1$,*

$$\forall N \ \leq \ \frac{c_2}{\beta}, \quad v_N \leq \beta.$$

PROOF. By (4.32) and induction we have, for any N such that $v_{N-1} \leq 1$ and $\beta \leq 1$,

$$v_N \leq N\beta^2\prod_{i=0}^{N-1}(1+c_1v_i).$$

Let n_0 be the smallest integer such that $v_{n_0} > \beta$. By the above formula, we have

$$v_{n_0} \leq n_0\beta^2(1+c_1\beta)^{n_0}$$

Suppose that $n_0 \leq (c_2/\beta)$, then

$$\beta \leq v_{n_0} \leq c_2c_1\beta(1+c_1\beta)^{c_2/\beta}.$$

If c_2 is chosen small enough, this is impossible. □

4.4.4. Directed percolation on D_N. For technical reasons, we need to get some understanding on directed independent bond percolation on D_N. Let p be the probability that an edge is open. The probability of having an open path from A to B in D_N follows the recursion

$$p_0 = p,$$
$$p_N = 1-(1-p_{N-1}^s)^b.$$

On can check that the map $x \mapsto 1-(1-x^s)^b$ has a unique unstable fixed point on $(0,1)$; we call it p_c. Therefore if $p > p_c$, with a probability tending to 1, there will be an open path linking A and B in D_N. If $p < p_c$, A and B will be disconnected in D_N with probability tending to 1. If $p = p_c$, the probability that A and B are linked in

D_N by an open path is stationary. See [**48**] for a deep investigation of percolation on hierarchical lattices.

4.4.5. From control of the variance to lower bounds on the free energy. Given b and s, let $p_c = p_c(b, s)$ be the critical parameter for directed bond percolation.

PROPOSITION 4.20. *Let N be an integer such that $v_N = Q(W_N - 1)^2 < \frac{1-p_c}{4}$ and β such that $p(\beta) \leq (1 - \log 2)$. Then*

$$f(\beta) - p(\beta) \leq s^{-N}$$

PROOF. If N is such that $Q\left[(W_N - 1)^2\right] < \frac{1-p_c}{4}$, we apply Chebycheff inequality to see that

$$Q(W_N < 1/2) \leq 4v_N < 1 - p_c.$$

Now let be $M \geq N$. D_M can be seen as the graph D_{M-N} where the edges have been replaced by i.i.d. copies of D_N with its environment (see fig. 2). To each copy of D_N we associate its renormalized partition function; therefore, to each edge e of D_{M-N} corresponds an independent copy of W_N, $W_N^{(e)}$. By percolation (see fig. 3), we will have, with a positive probability not depending on N, a path in D_{M-N} linking A to B, going only through edges which associated $W_N^{(e)}$ is larger than 1/2.

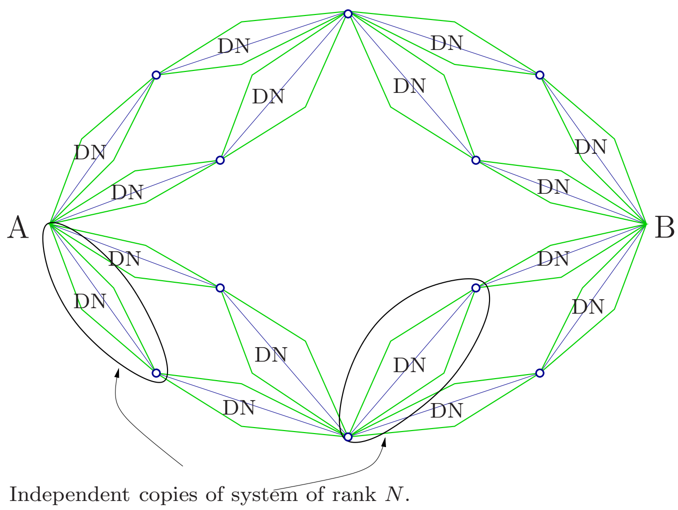

FIGURE 2. On this figure, we scheme how D_{N+M} with its random environment can be seen as independent copies of D_N arrayed as D_M. Here, we have $b = s = 2$ $M = 2$, each diamond corresponds to a copy of D_N (we can identify it with an edge and get the underlying graph D_2). Note that we also have to take into account the environment present on the vertices denoted by circles.

When such paths exist, let ω_0 be one of them (chosen in a deterministic manner, e.g. the lowest such path for some geometric representation of D_N). We look at the

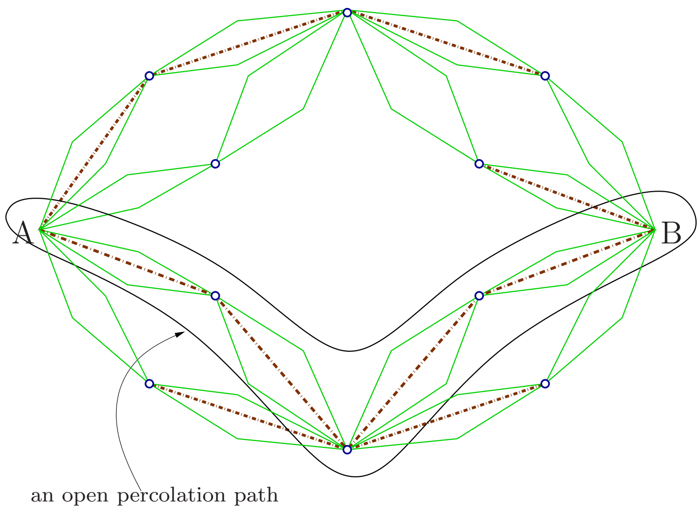

FIGURE 3. We represent here the percolation argument we use. In the previous figure, we have replaced by an open edge any of the copies of D_N which satisfies $W_N \geq 1/2$. As it happens with probability larger than p_c, it is likely that we can find an open path linking A to B in D_{N+M}, especially if M is large.

contribution of these family of paths in D_M to the partition function. We have

$$W_M \geq (1/2)^{s^{M-N}} \exp\left(\sum_{z\in\omega_0} \beta\eta_z - \lambda(\beta)\right)$$

Again, with positive probability (say larger than 1/3), we have $\sum_{z\in\omega_0} \eta_z \geq 0$ (the η_z are i.i.d. centered random variable and are independent of the $W_N^{(e)}$ along the path, therefore (by the central limit theorem), the probability that $\sum_{z\in\omega_0} \eta_z \geq 0$ can get arbitrary close to 1/2 if M is large enough). Therefore with positive probability we have

$$\frac{1}{s^M} \log W_M \geq -\frac{1}{s^N}(\log 2 + \lambda(\beta)).$$

As $1/s^M \log W_M$ converges in probability to the free energy this proves the result. □

PROOF OF THE RIGHT-INEQUALITY IN THEOREMS 4.10 AND 4.11. .
The results now follow by combining Lemma 4.18 or 4.19 for β small enough, with Proposition 4.20. □

4.5. Fractional moment method, upper bounds and strong disorder

In this section we develop a way to find an upper bound for $p(\beta) - f(\beta)$, or just to find out if strong disorder holds. The main tool we use are fractional moment estimates and measure changes.

4.5.1. Fractional moment estimate. In the sequel we will use the following notation. Given a fixed parameter $\theta \in (0,1)$, define

$$u_N := QW_N^\theta, \tag{4.33}$$

$$a_\theta := QA^\theta = \exp(\lambda(\theta\beta) - \theta\lambda(\beta)). \tag{4.34}$$

PROPOSITION 4.21. *The sequence $(f_N)_N$ defined by*

$$f_N := \theta^{-1} s^{-N} \log\left(a_\theta b^{\frac{1-\theta}{s-1}} u_N\right)$$

is decreasing and we have

$$\lim_{N\to\infty} f_N \geq p(\beta) - f(\beta).$$

(i) In particular, if for some $N \in \mathbb{N}$, $u_N < a_\theta^{-1} b^{\frac{\theta-1}{s-1}}$, strong disorder holds.

(ii) Strong disorder holds in particular if $a_\theta < b^{\frac{\theta-1}{s-1}}$.

PROOF. Applying the inequality $(\sum a_i)^\theta \leq \sum a_i^\theta$ (which holds for any $\theta \in (0,1)$ and any collection of positive numbers a_i) to (4.8) and averaging with respect to Q gives

$$u_{N+1} \leq b^{1-\theta} u_N^s a_\theta^{s-1}.$$

From this we deduce that the sequence

$$s^{-N} \log\left(a_\theta b^{\frac{1-\theta}{s-1}} u_N\right)$$

is decreasing. Moreover we have

$$p(\beta) - f(\beta) = \lim_{N\to\infty} \frac{1}{s^N} Q \log W_N \leq \lim_{N\to\infty} \frac{1}{\theta s^N} \log QW_N^\theta = \lim_{N\to\infty} f_N.$$

As a consequence very strong disorder holds if $f_N < 0$ for any f_N. As a consequence, strong disorder and very strong disorder are equivalent. □

4.5.2. Change of measure and environment tilting. The result of the previous section assures that we can estimate the free energy if we can bound accurately some non-integer moment of W_N. Now we present a method to estimate non-integer moment via measure change, it has been introduced to show disorder relevance in the case of wetting model on a hierarchical lattice [**42**] and used since in several different contexts since, in particular for directed polymer models on $\mathbb{Z}^d$ [**66**]. Yet, for the directed polymer on hierarchical lattice, the method is remarkably simple to apply, and it seems to be the ideal context to present it. Let $\widetilde{Q}$ be any probability measure such that Q and $\widetilde{Q}$ are mutually absolutely continuous. Using Hölder inequality we observe that

$$QW_N^\theta = \widetilde{Q} \frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}} W_N^\theta \leq \left[\widetilde{Q}\left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\right)^{\frac{1}{1-\theta}}\right]^{(1-\theta)} \left(\widetilde{Q} W_N\right)^\theta. \tag{4.35}$$

Our aim is to find a measure $\widetilde{Q}$ such that the term $\left[\widetilde{Q}\left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\right)^{\frac{1}{1-\theta}}\right]^{(1-\theta)}$ is not very large (i.e. of order 1), and which significantly lowers the expected value of W_N. To do so we look for $\widetilde{Q}$ which lowers the value of the environment on each site, by exponential tilting. For $b < s$ it sufficient to lower the value for the environment uniformly of every site of $D_N \setminus \{A, B\}$ to get a satisfactory result, whereas for the $b = s$ case, on has to do an inhomogeneous change of measure. We present the change of measure in a united framework before going to the details with two separate cases. For a simpler presentation, we restrict here to the case of Gaussian environment (the η_z are i.i.d. centered standard Gaussian under the measure Q). The method easily adapt to general environment as shown at the end of the section.

Recall that V_i denotes the sites of $D_i \setminus D_{i-1}$, and that the number of sites in D_N is

$$|D_N \setminus \{A, B\}| = \sum_{i=1}^{N} |V_i| = \sum_{i=1}^{N} (s-1)b^i s^{i-1} = \frac{(s-1)b((sb)^N - 1)}{sb - 1} \tag{4.36}$$

We define $\widetilde{Q} = \widetilde{Q}_{N,s,b}$ to be the measure under which the environment on the site of the i-th generation for $i \in \{1, \dots, N\}$ are standard Gaussian variables with mean $-\delta_i = -\delta_{i,N}$, where $\delta_{i,N}$ is to be defined. The density of $\widetilde{Q}$ with respect to Q is given by

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q}(\eta) = \exp\left(-\sum_{i=1}^{N}\sum_{z\in V_i}(\delta_{i,N}\eta_z + \frac{\delta_{i,N}^2}{2})\right).$$

As each path in D_N intersects V_i on $s^{i-1}(s-1)$ sites, this change of measure lowers the value of the Hamiltonian (4.1) by $\sum_{i=1}^{N} s^{i-1}(s-1)\delta_{i,N}$ on any path. Therefore, both terms can be easily computed,

$$\widetilde{Q}\left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\right)^{\frac{1}{1-\theta}} = \exp\left\{\frac{\theta}{2(1-\theta)}\sum_{i=1}^{N}|V_i|\delta_{i,N}^2\right\}. \tag{4.37}$$

$$\left(\widetilde{Q}W_N\right)^{\theta} = \exp\left\{-\beta\theta\sum_{i=1}^{N}s^{i-1}(s-1)\delta_{i,N}\right\}. \tag{4.38}$$

Replacing (4.38) and (4.37) back into (4.35) gives

$$u_N \le \exp\left\{\theta\sum_{i=1}^{N}\left(\frac{|V_i|\delta_{i,N}^2}{2(1-\theta)} - \beta s^{i-1}(s-1)\delta_{i,N}\right)\right\}. \tag{4.39}$$

When $\delta_{i,N} = \delta_N$ (i.e. when the change of measure is homogeneous on every site) the last expression becomes simply

$$u_N \leq \exp\left\{\theta\left(\frac{|D_N \setminus \{A,B\}|\delta_N^2}{2(1-\theta)} - (s^N-1)\beta\delta_N\right)\right\}. \tag{4.40}$$

In either case, the rest of the proof consists in finding convenient values for $\delta_{i,N}$ and N large enough to ensure that (i) from Proposition 4.21 holds.

4.5.3. Homogeneous shift method: $s > b$. Proof of the left inequality in Theorem 4.10 for Gaussian environments:

Let $0 < \theta < 1$ be fixed (say $\theta = 1/2$) and $\delta_{i,N} = \delta_N := (sb)^{-N/2}$. Observe from (4.36) that $|D_N \setminus \{A,B\}|\delta_N^2 \leq 1$, so that (4.40) implies

$$u_N \leq \exp\left(\frac{\theta}{2(1-\theta)} - \theta\beta(s/b)^{N/2}\frac{s-1}{s}\right).$$

Taking $N = \frac{2(|\log\beta| + \log c_3)}{\log s - \log b}$, we get

$$u_N \leq \exp\left(\frac{\theta}{2(1-\theta)} - \frac{\theta c_3 s}{s-1}\right).$$

Choosing $\theta = 1/2$ and c_3 sufficiently large, we have

$$f_N = s^{-N}\log a_\theta b^{\frac{1-\theta}{s-1}} u_N \leq -s^{-N}, \tag{4.41}$$

so that Proposition 4.21 gives us the conclusion

$$p(\beta) - f(\beta) \leq -s^{-N} = -(\beta/c_3)^{\frac{2\log s}{\log s - \log b}}.$$

□

4.5.4. Inhomogeneous shift method: $s = b$. One can check that the previous method does not give good enough results for the marginal case $b = s$. One has to do a change of measure which is a bit more refined and for which the intensity of the tilt in proportional to the Green function on each site. This idea was used first for the marginal case in pinning model on hierarchical lattice (see [**65**]).

Proof of the left inequality in Theorem 4.11 for Gaussian environments: This time, we set $\delta_{i,N} := N^{-1/2}s^{-i}$. Then (recall (4.36)), (4.39) becomes

$$u_N \leq \exp\left(\frac{\theta}{2(1-\theta)}\frac{s-1}{s} - \theta\beta N^{-1/2}\frac{s-1}{s}\right).$$

Taking $\theta = 1/2$ and $N = (c_4/\beta)^2$ for a constant c_4 large enough, we get that $f_N \leq -s^N$ and applying Proposition 4.21, we obtain

$$p(\beta) - f(\beta) \leq -s^{-N} = -s^{-(c_4/\beta)^2} = -\exp\left(-\frac{c_4^2\log s}{\beta^2}\right).$$

□

4.5.5. Bounds for the critical temperature. From Proposition 4.21, we have that strong disorder holds if $a_\theta < b^{(1-\theta)/(s-1)}$. Taking logarithms, this condition reads

$$\lambda(\theta\beta) - \theta\lambda(\beta) < (1-\theta)\frac{\log b}{s-1}.$$

We divide both sides by $1-\theta$ and let $\theta \to 1$. This proves part (*iii*) of Proposition 4.7. For the case $b > s$, this condition can be improved by the inhomogeneous shifting method; here, we perform it just in the case of Gaussian environment. Recall that

$$u_N \leq \exp\left\{\theta \sum_{i=1}^{N} \left(\frac{|V_i|\delta_{i,N}^2}{2(1-\theta)} - \beta s^{i-1}(s-1)\delta_{i,N}\right)\right\}. \tag{4.42}$$

We optimize each summand in this expression taking $\delta_{i,N} = \delta_i = (1-\theta)\beta/b^i$. Recalling that $|V_i| = (bs)^{i-1}b(s-1)$, this yields

$$\begin{aligned} u_N &\leq \exp\left\{-\theta(1-\theta)\frac{\beta^2}{2}\frac{s-1}{s}\sum_{i=1}^{N}\left(\frac{s}{b}\right)^i\right\} \\ &\leq \exp\left\{-\theta(1-\theta)\frac{\beta^2}{2}\frac{s-1}{s}\frac{s/b-(s/b)^{N+1}}{1-s/b}\right\}. \end{aligned}$$

Because N is arbitrary, in order to guaranty strong disorder it is enough to have (cf. first condition in Proposition 4.21) for some $\theta \in (0,1)$

$$\theta(1-\theta)\frac{\beta^2}{2}\frac{s-1}{s}\frac{s/b}{1-s/b} > (1-\theta)\frac{\log b}{s-1} + \log a_\theta.$$

In the case of Gaussian variables $\log a_\theta = \theta(\theta-1)\beta^2/2$. This is equivalent to

$$\frac{\beta^2}{2} > \frac{(b-s)\log b}{(b-1)(s-1)}.$$

This last condition is an improvement of the bound in part (*iii*) of Proposition 4.7.

4.5.6. Adaptation of the proofs for non-Gaussian variables. Proof of the left inequality in Theorem 4.10 and 4.11 for general environments: To adapt the preceding proofs to non-Gaussian variables, we have to investigate the consequence of exponential tilting on non-Gaussian variables. We sketch the proof in the inhomogeneous case $b = s$, we keep $\delta_{i,N} := s^{-i}N^{-1/2}$.

Consider $\widetilde{Q}$ with density

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q}(\eta) := \exp\left(-\sum_{i=1}^{N}\sum_{z\in V_i}\left(\delta_{i,N}\eta_z + \lambda(-\delta_{i,N})\right)\right),$$

(recall that $\lambda(x) := \log Q \exp(x\eta)$). The term giving cost of the change of measure is, in this case,

$$\begin{aligned}\left[\widetilde{Q}\left(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\right)^{\frac{1}{1-\theta}}\right]^{(1-\theta)} &= \exp\left((1-\theta)\sum_{i=1}^{N}|V_i|\left[\lambda\left(\frac{\theta\delta_{i,N}}{1-\theta}\right)+\frac{\theta}{1-\theta}\lambda(-\delta_{i,N})\right]\right)\\ &\leq \exp\left(\frac{\theta}{(1-\theta)}\sum_{i=1}^{N}|V_i|\delta_{i,N}^2\right) \leq \exp\left(\frac{\theta}{(1-\theta)}\right)\end{aligned}$$

Where the inequality is obtained by using the fact the $\lambda(x) \sim_0 x^2/2$ (this is a consequence of the fact that η has unit variance) so that if β is small enough, one can bound every $\lambda(x)$ in the formula by x^2.

We must be careful when we estimate $\widetilde{Q}W_N$. We have

$$\widetilde{Q}W_N = \exp\left(\sum_{i=1}^{N}(s-1)s^{i-1}\left[\lambda(\beta-\delta_{i,N})-\lambda(\beta)-\lambda(-\delta_{i,N})\right]\right)QW_N.$$

By the mean value theorem

$$\lambda(\beta-\delta_{i,N})-\lambda(\beta)-\lambda(-\delta_{i,N})+\lambda(0) = -\delta_{i,N}\left(\lambda'(\beta-t_0)-\lambda'(-t_0)\right) = -\delta_{i,N}\beta\lambda''(t_1),$$

for some $t_0 \in (0,\delta_{i,N})$ and some $t_1 \in (\beta,-\delta_{i,N})$. As we know that $\lim_{\beta\to 0}\lambda''(\beta) = 1$, when δ_i and β are small enough, the right-hand side is less than $-\beta\delta_{i,N}/2$. Hence,

$$\widetilde{Q}W_N \leq \exp\left(-\sum_{i=1}^{N}(s-1)s^{i-1}\frac{\beta\delta_{i,N}}{2}\right).$$

We get the same inequalities that in the case of Gaussian environment, with different constants, which do not affect the proof. The case $b < s$ is similar. □

4.6. Fluctuation and localization results

In this section we use the shift method we have developed earlier to prove fluctuation results

4.6.1. Proof of Proposition 4.12. The statement on the variance is only a consequence of (4.16) as it will be show at the end of the section.

Recall that the random variable η_z here are i.i.d. centered standard Gaussian variables, and that the product law is denoted by Q. We have to prove

$$Q\left\{\log Z_N \in [a, a+\beta\varepsilon(s/b)^{N/2}]\right\} \leq 4\varepsilon \quad \forall\varepsilon > 0, N \geq 0, a \in \mathbb{R} \tag{4.43}$$

Assume there exist real numbers a and ε, and an integer N such that (4.43) does not hold, i.e.

$$Q\left\{\log Z_N \in [a, a+\beta\varepsilon(s/b)^{N/2})\right\} > 4\varepsilon. \tag{4.44}$$

Then one of the following holds

$$Q\left\{\log Z_N \in [a, a+\beta\varepsilon(s/b)^{N/2})\right\} \cap \left\{\sum_{z\in D_N} \eta_z \geq 0\right\} > 2\varepsilon,$$
$$Q\left\{\log Z_N \in [a, a+\beta\varepsilon(s/b)^{N/2})\right\} \cap \left\{\sum_{z\in D_N} \eta_z \leq 0\right\} > 2\varepsilon. \tag{4.45}$$

We assume that the first line is true. We consider the events related to Q as sets of environments $(\eta_z)_{z\in D_N\setminus\{A,B\}}$. We define

$$A_\varepsilon = \left\{\log Z_N \in [a, a+\beta\varepsilon(s/b)^{N/2})\right\} \cap \left\{\sum_{z\in D_N} \eta_z \geq 0\right\}, \tag{4.46}$$

and

$$A_\varepsilon^{(i)} = \left\{\log Z_N \in [a - i\beta\varepsilon(s/b)^{N/2}, a-(i-1)\beta\varepsilon(s/b)^{N/2})\right\}. \tag{4.47}$$

Define $\delta = \frac{s^{N/2}}{(s^N-1)b^{N/2}}$. We define the measure $\widetilde{Q}_{i,\varepsilon}$ with its density:

$$\frac{\mathrm{d}\widetilde{Q}_{i,\varepsilon}}{\mathrm{d}Q}(\eta) := \exp\left(\left[i\varepsilon\delta \sum_{z\in D_N} \eta_z\right] - \frac{i^2\varepsilon^2\delta^2|D_N\setminus\{A,B\}|}{2}\right). \tag{4.48}$$

If the environment $(\eta_z)_{z\in D_N}$ has law Q then $(\widehat{\eta}_z^{(i)})_{z\in D_N}$ defined by

$$\widehat{\eta}_z^{(i)} := \eta_z + \varepsilon i\delta, \tag{4.49}$$

has law $\widetilde{Q}_{i,\varepsilon}$. Going from η to $\widehat{\eta}^{(i)}$, one increases the value of the Hamiltonian by $\varepsilon i(s/b)^{N/2}$ (each path cross s^N-1 sites). Therefore if $(\widehat{\eta}_z^{(i)})_{z\in D_N} \in A_\varepsilon$, then $(\eta_z)_{z\in D_N} \in A_\varepsilon^{(i)}$. From this we have $\widetilde{Q}_{i,\varepsilon} A_\varepsilon \leq QA_\varepsilon^{(i)}$, and therefore

$$QA_\varepsilon^{(i)} \geq \int_{A_\varepsilon} \frac{\mathrm{d}\widetilde{Q}_{i,\varepsilon}}{\mathrm{d}Q} Q(\,\mathrm{d}\eta) \geq \exp(-(\varepsilon i)^2/2) Q(A_\varepsilon). \tag{4.50}$$

The last inequality is due to the fact that the density is always larger than $\exp(-(\varepsilon i)^2/2)$ on the set A_ε (recall its definition and the fact that $|D_N\setminus\{A,B\}|\delta^2 \leq 1$). Therefore, in our setup, we have

$$QA_\varepsilon^{(i)} > \varepsilon, \quad \forall i \in [0, \varepsilon^{-1}]. \tag{4.51}$$

As the $A_\varepsilon^{(i)}$ are disjoints, this is impossible. If we are in the second case of (4.45), we get the same result by shifting the variables in the other direction.

From Chebycheff inequality, we know that for every $x > 0$

$$Q\{|\log Z_N - Q\log Z_N| > x\} \leq \frac{\mathrm{Var}_Q \log Z_N}{x^2}. \tag{4.52}$$

Therefore

$$\mathrm{Var}_Q \log Z_N \geq x^2(1 - Q\{|\log Z_N - Q\log Z_N| \leq x\}). \tag{4.53}$$

For $x = \beta(s/b)^{N/2}/10$ using (4.43) with $\varepsilon = 1/5$ and $a = Q\log Z_N - \beta(s/b)^{N/2}/10$, we get

$$\mathrm{Var}_Q \log Z_N \geq \frac{\beta^2(s/b)^N}{500}. \tag{4.54}$$

□

4.6.2. Proof of Proposition 4.13. Let us suppose that there exist N, ε and a such that

$$Q\left\{\log Z_N \in [a, a+\beta\varepsilon\sqrt{N})\right\} > 8\varepsilon. \tag{4.55}$$

We define $\delta_{i,N} = \delta_i := \varepsilon s^{1-i}(s-1)^{-1}N^{-1/2}$. Then one of the following inequalities holds (recall the definition of V_i)

$$\begin{aligned} &Q\left\{\log Z_N \in [a, a+\beta\varepsilon\sqrt{N})\right\} \cap \left\{\sum_{i=1}^N \delta_i \sum_{z\in V_i} \eta_z \geq 0\right\} > 4\varepsilon, \\ &Q\left\{\log Z_N \in [a, a+\beta\varepsilon\sqrt{N})\right\} \cap \left\{\sum_{i=1}^N \delta_i \sum_{z\in V_i} \eta_z \leq 0\right\} > 4\varepsilon. \end{aligned} \tag{4.56}$$

We assume that the first line holds and define

$$A_\varepsilon = \left\{\log Z_N \in [a, a+\beta\varepsilon\sqrt{N})\right\} \cap \left\{\sum_{i=1}^N \delta_i \sum_{z\in V_i} \eta_z \geq 0\right\} \tag{4.57}$$

And

$$A_\varepsilon^{(j)} = \left\{\log Z_N \in [a - j\beta\varepsilon\sqrt{N}, a-(j-1)\beta\varepsilon\sqrt{N})\right\} \tag{4.58}$$

$$j\beta\sum_{i=1}^N \delta_i(s-1)s^{i-1} = j\beta\varepsilon\sqrt{N}. \tag{4.59}$$

Therefore, an environment $\eta \in A_\varepsilon$ will be transformed in an environment in $A_\varepsilon^{(j)}$.

We define $\widetilde{Q}_{j,\varepsilon}$ the measure whose Radon-Nikodym derivative with respect to Q is

$$\frac{\mathrm{d}\widetilde{Q}_{i,\varepsilon}}{\mathrm{d}Q}(\eta) := \exp\left(\left[j\sum_{i=1}^N \delta_i \sum_{z\in V_i}\eta_z\right] - \sum_{i=1}^N \frac{j^2\delta_i^2|V_i|}{2}\right). \tag{4.60}$$

We can bound the deterministic term.

$$\sum_{i=1}^N \frac{j^2\delta_i^2|V_i|}{2} = \frac{j^2\varepsilon^2}{N}\sum_{i=1}^N \frac{s}{2(s-1)} \leq j^2\varepsilon^2. \tag{4.61}$$

For an environment $(\eta_z)_{z\in D_N\setminus\{A,B\}}$, define $(\widehat{\eta}_z^{(j)})_{z\in D_N\setminus\{A,B\}}$ by

$$\widehat{\eta}_z^{(j)} := \eta_z + j\varepsilon\delta_i, \quad \forall z \in V_i. \tag{4.62}$$

If $(\eta_z)_{z\in D_N\setminus\{A,B\}}$ has Q, then $(\widehat{\eta}_z^{(j)})_{z\in D_N\setminus\{A,B\}}$ has law $\widetilde{Q}_{j,\varepsilon}$. When one goes from η to $\widehat{\eta}^{(j)}$, the value of the Hamiltonian is increased by

$$\sum_{i=1}^{N} j\delta_i s^{i-1}(s-1) = \varepsilon\sqrt{N}.$$

Therefore, if $\widehat{\eta}^{(j)} \in A_\varepsilon$, then $\eta \in A_\varepsilon^{(j)}$, so that

$$QA_\varepsilon^{(j)} \geq \widetilde{Q}_{j,\varepsilon} A_\varepsilon.$$

Because of the preceding remarks

$$QA_\varepsilon^{(j)} \geq \widetilde{Q}_{j,\varepsilon} A_\varepsilon = \int_{A_\varepsilon} \frac{\mathrm{d}\widetilde{Q}_{i,\varepsilon}}{\mathrm{d}Q} Q(\,\mathrm{d}\eta) \geq \exp\left(-j^2\varepsilon^2\right) QA_\varepsilon. \tag{4.63}$$

The last inequality comes from the definition of A_ε which gives an easy lower bound on the Radon-Nikodym derivative. For $j \in [0, (\varepsilon/2)^{-1}]$, this implies that $QA_\varepsilon^{(j)} > 2\varepsilon$. As they are disjoint events this is impossible. The second case of (4.56) can be dealt analogously. The lower bound on the variance is obtained just as in the $b < s$ case. □

4.6.3. Proof of Corollary 4.14. Let $\mathbf{S} \in \Omega_N$ be a fixed path. For $M \geq N$, define

$$Z_M^{(\mathbf{S})} := \sum_{\{\mathbf{S}'\in\Omega_M : \mathbf{S}|_N = \mathbf{S}\}} \exp\left(\beta H_M(\mathbf{S}')\right). \tag{4.64}$$

With this definition we have

$$\mu_M(\omega|_N = \mathbf{S}) = \frac{Z_M^{(\mathbf{S})}}{Z_M}. \tag{4.65}$$

To show our result, it is sufficient to show that for any constant K and any distinct $\mathbf{S}, \mathbf{S}' \in \Omega_N$

$$\lim_{M\to\infty} Q\left(\frac{\mu_M(\omega|_N = \mathbf{S})}{\mu_M(\omega|_N = \mathbf{S}')} \in [K^{-1}, K]\right) = 0. \tag{4.66}$$

For $\mathbf{S}$ and $\mathbf{S}'$ distinct, it is not hard to see that

$$\log\left(\frac{\mu_M(\omega|_N = \mathbf{S})}{\mu_M(\omega|_N = \mathbf{S}')}\right) = \log Z_M^{(\mathbf{S})} - \log Z_M^{(\mathbf{S}')} =: \log Z_{M-N}^{(0)} + X, \tag{4.67}$$

where $Z_{M-N}^{(0)}$ is a random variable whose distribution is the same as the one of Z_{M-N}, and X is independent of $Z_{M-N}^{(0)}$. We have

$$\begin{aligned} Q&\left(\log\left(\frac{\mu_M(\omega|_N = \mathbf{S})}{\mu_M(\omega|_N = \mathbf{S}')}\right) \in [-\log K, \log K]\right) \\ &= Q\left[Q\left(\log Z_{M-N}^{(0)} \in [-\log K - X, \log K - X] \,\middle|\, X\right)\right] \\ &\leq \max_{a\in\mathbb{R}} Q\left(\log Z_{M-N} \in [a, a + 2\log K]\right). \end{aligned} \tag{4.68}$$

Proposition 4.12 and 4.13 show that the right–hand side tends to zero. □

4.7. The weak disorder polymer measure

Comets and Yoshida introduced in [**28**] an infinite volume Markov chain at weak disorder that corresponds in some sense to the limit of the polymer measures μ_N when N goes to infinity. We perform the same construction here. The notation is more cumbersome in our setting.

Recall that Ω_N is the space of directed paths from A to B in D_N. Denote by P_N the uniform probability on Ω_N. For $\mathbf{S} \in \Omega_N$, $0 \le t \le s^N - 1$, define $W_\infty(\mathbf{S}_t, \mathbf{S}_{t+1})$ by performing the same construction that leads to W_∞, but taking $\mathbf{S}_t$ and $\mathbf{S}_{t+1}$ instead of A and B respectively. On the classical directed polymers on $\mathbb{Z}^d$, this would be equivalent to take the $(t, \mathbf{S}_t)$ as the initial point of the polymer.

We can now define the weak disorder polymer measure for $\beta < \beta_0$. We define Ω as the projective limit of Ω_N (with its natural topology), the set of paths on $D := \bigcup_{N \ge 1} D_N$. As for finite paths, we can define, for $\bar{\mathbf{S}} \in \Omega$, its projection onto Ω_N, $\bar{\mathbf{S}}|_N$. We define, for each $N \le 1$ and each $\mathbf{S} \in \Omega_N$,

$$\mu_\infty(\bar{\omega}|_N = \mathbf{S}) := \frac{1}{W_\infty} \exp\{\beta H_N(\mathbf{S}) - (s^N - 1)\lambda(\beta)\} \prod_{i=0}^{s^N-1} W_\infty(\mathbf{S}_i, \mathbf{S}_{i+1})\, P_N(\bar{\omega}|_N = \mathbf{S}). \tag{4.69}$$

Let us stress the following:

- Note that the projection on the different Ω_N's are consistent (so that our definition makes sense)

$$\mu_\infty(\bar{\omega}|_N = \mathbf{S}) = \mu_\infty\left((\bar{\omega}|_{N+1})|_N = \mathbf{S}\right).$$

- Thanks to the martingale convergence for both the numerator and the denominator, for any $\mathbf{S} \in \Omega_N$,

$$\lim_{k \to +\infty} \mu_{k+N}(\omega|_N = \mathbf{S}) = \mu_\infty(\bar{\omega}|_N = \mathbf{S}).$$

  Therefore, μ_∞ is the only reasonable definition for the limit of μ_N.

It is an easy task to prove the law of large numbers for the time-averaged quenched mean of the energy. This follows as a simple consequence of the convexity of $p(\beta)$.

Proposition 4.22. *At each point where $\beta \mapsto p(\beta)$ admits a derivative,*

$$\lim_{N \to +\infty} \frac{1}{s^N} \mu_N(H_N(\omega)) \to p'(\beta), \quad Q - a.s..$$

Proof. It is enough to observe that

$$\frac{\partial}{\partial \beta} \log Z_N = \mu_n(H_N(\omega)),$$

then use the convexity to pass to the limit. □

We can also prove a quenched law of large numbers under our infinite volume measure μ_∞, for almost every environment. The proof is very easy, as it involves just a second moment computation.

PROPOSITION 4.23. *At weak disorder,*

$$\lim_{N\to+\infty} \frac{1}{s^N} H_N(\bar{\omega}|_N) = \lambda'(\beta), \quad \mu_\infty - a.s., \, Q - a.s.$$

PROOF. We will consider the following auxiliary measure (size biased measure) on the environment

$$\overline{Q}(f(\eta)) = Q(f(\eta)W_\infty).$$

So, Q-a.s. convergence will follow from $\overline{Q}$-a.s. convergence. This will be done by a direct computation of second moments. Let us write $\Delta = Q(\eta^2 e^{\beta\eta-\lambda(\beta)})$, and let P_N denote the uniform measure on $|\Omega_N|$.

$$\begin{aligned}
& \overline{Q}\left(\mu_\infty(|H_N(\bar{\omega}|_N)|^2)\right) \\
= \; & Q\left[P_N\left(|H_N(\omega)|^2 \exp\left(\beta H_N(\omega) - (s^N-1)\lambda(\beta)\right) \prod_{i=0}^{s^N-1} W_\infty(\omega_i, \omega_{i+1})\right)\right] \\
= \; & Q\left[P_N\left(|H_N(\omega)|^2 \exp\left(\beta H_N(\omega) - (s^N-1)\lambda(\beta)\right)\right)\right] \\
= \; & Q\left[P_N\left(|\sum_{t=1}^{s^N} \eta(\omega_t)|^2 \exp\left(\beta H_N(\omega) - (s^N-1)\lambda(\beta)\right)\right)\right] \\
= \; & Q\left[\sum_{t=1}^{s^N-1} P_N\left(|\eta(\omega_t)|^2 \exp\left(\beta H_N(\omega) - (s^N-1)\lambda(\beta)\right)\right)\right] \\
& + Q\left[\sum_{1\le t_1\neq t_2\le s^N-1} P_N\left(\eta(\omega_{t_1})\eta(\omega_{t_2}) \exp\left(\beta H_N(\omega) - (s^N-1)\lambda(\beta)\right)\right)\right] \\
= \; & (s^N-1)\Delta + (s^N-1)(s^N-2)(\lambda'(\beta))^2,
\end{aligned}$$

where we used independence to pass from line two to line three. So, recalling that $\overline{Q}(\mu_\infty(H_N(\bar{\omega}|_N))) = (s^N-1)\lambda'(\beta)$, we have

$$\overline{Q}\left(\mu_\infty(|H_N(\bar{\omega}|_N) - (s^N-1)\lambda'(\beta)|^2)\right) = (s^N-1)\left(\Delta - \lambda'(\beta)^2\right). \tag{4.70}$$

Then

$$\overline{Q}\mu_\infty\left(\left|\frac{H_N(\bar{\omega}|_N) - (s^N-1)\lambda'(\beta)}{s^N}\right|^2\right) \le \frac{1}{s^N}\left(\Delta - (\lambda'(\beta)^2)\right),$$

so the result follows by Borel-Cantelli. □

4.8. Some remarks on the bond–disorder model

In this section, we shortly discuss, without going through the details, how the methods we used in this paper could be used (or could not be used) for the model of directed polymer on the same lattice with disorder located on the bonds.

In this model to each bond e of D_N we associate i.i.d. random variables η_e. We consider each set $\mathbf{S} \in \Omega_N$ as a set of bonds and define the Hamiltonian as

$$H_N^{\omega}(\mathbf{S}) = \sum_{e \in \mathbf{S}} \eta_e, \tag{4.71}$$

The partition function Z_N is defined as

$$Z_N := \sum_{\mathbf{S} \in \Omega_N} \exp(\beta H_N(\mathbf{S})). \tag{4.72}$$

One can check that is satisfies the following recursion

$$\begin{aligned} Z_0 &\stackrel{\mathcal{L}}{=} \exp(\beta\eta) \\ Z_{N+1} &\stackrel{\mathcal{L}}{=} \sum_{i=1}^{b} Z_N^{(i,1)} Z_N^{(i,2)} \dots Z_N^{(i,s)}. \end{aligned} \tag{4.73}$$

where equalities hold in distribution and $Z_N^{(i,j)}$ are i.i.d. distributed copies of Z_N. Because of the loss of the martingale structure and the homogeneity of the Green function in this model (which is equal to b^{-N} on each edge), Lemma 4.4 does not hold, and we cannot prove part (*iv*) in Proposition 4.7, Theorem 4.11 and Proposition 4.13 for this model. Moreover we have to change $b \leq s$ by $b < s$ in (*v*) of Proposition 4.7. Moreover, the method of the control of the variance would give us a result similar to 4.11 in this case

PROPOSITION 4.24. *When b is equal to s, on can find constants c and β_0 such that for all $\beta \leq \beta_0$*

$$0 \leq f(\beta) - p(\beta) \leq \exp\left(-\frac{c}{\beta^2}\right). \tag{4.74}$$

However, we would not be able to prove that the annealed and quenched free energy differ at high temperature for $s = b$ using our method. The techniques used in [**43**] or [**66**] for dimension 2 are able to tackle this problem, and show marginal disorder relevance in this case as well.

Part 2

Directed Polymers in a Brownian Environment and Asymmetric Discrete Models

The model of Brownian Percolation has been introduced as an approximation of discrete last-passage percolation models close to the axis. It allowed to compute some explicit limits and prove fluctuation theorems for these. We review the development of these ideas, and we outline some proofs. We emphasize the relations between the Brownian percolation and models exhibiting determinantal structure, like non-colliding random walks and random matrices.
Next, we establish analogous relations between asymmetric discrete directed polymers in random environments and a continuous-time directed polymers model in a Brownian environment introduced in [**79**] and studied in [**77**] in the one-dimensional case. The key ingredient is a strong approximation argument developed by Kómlos, Major and Túsnady. We give a complete treatment of the partition function in the multi-dimensional case.
Finally, we give an explicit formula for the free energy of a $1+1$-dimensional directed polymer in a random environment with a drift tending to infinity and find the exact order of the fluctuations.

CHAPTER 5

Overview

The Brownian Percolation model was introduced by Glynn and Whitt in [**45**], where the authors studied the asymptotic of passage times for customers in an infinite network of M/M/1 queues in tandem. This continuous model was easier to handle than the original discrete problem, mostly because of the scaling properties of the Brownian motion.

Let state the problem more precisely in its original setting: Let $\Omega_{N,M}$ be the set of directed paths from $(0,0)$ to (N,M), i.e., the paths with steps equal to $(0,1)$ or $(1,0)$. Let $\{\eta(x) : x \in \mathbb{Z}^2\}$ be a collection of (centered) i.i.d. random variables with finite exponential moments $e^{\lambda(\beta)} = Q(e^{\beta\eta}) < +\infty$, which will be referred as the environment variables, or just as the environment.

Define

$$T(N,M) = \max_{\mathbf{s}\in\Omega_{N,M}} H(\mathbf{S}). \tag{5.1}$$

where $H(\mathbf{S}) = \sum_{(t,x)\in\mathbf{S}} \eta(t,x)$ will be called the energy of the path $\mathbf{S}$. This is usually referred to as a last-passage percolation problem. It can be interpreted as the departure time of the M-th customer from the N-th queue in a series of queues in tandem. The variable $\eta(k,n)$ has then to be understood as the service time of the k-th customer in the n-th queue.

A regime of special interest occurs when

$$M = O(N^a)$$

for some $a \in (0,1)$. Glynn and Whitt [**45**] proved that

$$\lim_{N\to+\infty} \frac{T(N, \lfloor xN^a \rfloor)}{N^{(1+a)/2}} = c\sqrt{x}, \tag{5.2}$$

where the constant is independent of a and of the distribution of the service times, given that they satisfy some mild integrability conditions. The proof used a strong approximation of sums of i.i.d. random variables by Brownian motions (see [**63, 64**]) in order to approximate $T(N, \lfloor xN^a \rfloor)$ by the corresponding maximal energy along continuous-time paths in a Brownian environment (see below for precise definitions). Then, scaling arguments lead to (5.2). Based on simulations, they conjectured that $c = 2$.
The proof of this conjecture was first given by Seppäläinen in [**84**]. It uses a coupling between queues in tandem and TASEP. Later proofs used an interesting relation between the Brownian model and eigenvalues of random matrices. For a shorter proof

using ideas from queueing theory and Gaussian concentration, see [**50**]. We will review the ideas of these proofs in the next Chapter.

Let us now state the following as a summary of the previous discussion:

THEOREM 5.1.

$$\lim_{N\to+\infty} \frac{T(N,\lfloor xN^a \rfloor)}{N^{(1+a)/2}} = 2\sqrt{x}, \tag{5.3}$$

in probability.

Some fluctuation results are also available (see [**4, 11**]). The limiting law is identified as the Tracy-Widom distribution. This is closely related to the link between Brownian percolation and random matrices we have mentioned. See also [**51**] for large deviations results at the Tracy-Widom scale. As usual in this type of models, the upper deviations are much larger than the lower ones (see [**54**] for last-passage percolation, [**33**] and [**85**] for the related model of increasing subsequences in the plane and [**6**] for directed polymers. See also [**69**] for a general discussion on the subject, including random matrices). This can be explained heuristically by noticing that, in order to increase the values of the max, it is enough to increase the values of the environment along a single path. Decreasing the value of the max requires to decrease the values of the whole environment.

We will be mostly concerned with non-zero temperature analogs to the LPP problem, namely directed polymers in random environment. Let $P_{N,M}$ be the uniform probability measure on $\Omega_{N,M}$. For a given realization of the environment, we define on $\Omega_{N,M}$ the polymer measure at inverse temperature β as

$$\mu^{\beta}_{N,M}(\omega = \mathbf{S}) = \frac{1}{Z_\beta(N,M)} e^{\beta H(\mathbf{S})} P_{N,M}(\omega = \mathbf{S}), \quad \forall\, \mathbf{S} \in \Omega_{N,M}, \tag{5.4}$$

where $Z_\beta(N,M)$ is a normalizing constant called the (point-to-point) partition function, given by

$$Z_\beta(N,M) = P_{N,M}\left(e^{\beta H(\omega)}\right). \tag{5.5}$$

It is easy to show the existence of the limit of the free energy in the regime considered above for the LPP. Indeed, for $M = O(N^a)$ for some $a \in (0,1)$, the following limit holds for almost every realization of environment:

$$\lim_{N\to+\infty} \frac{1}{N^{(1+a)/2}} \log Z_\beta(N,N^a) = 2\beta. \tag{5.6}$$

The proof is straightforward as it applies directly the corresponding result for last-passage percolation. Just note that

$$-\log|\Omega_{N,N^a}| + \beta T(N,N^a) \le \log Z_\beta(N,N^a) \le \beta T(N,N^a), \tag{5.7}$$

observe that $\log|\Omega_{N,N^a}| = O(N^a \log N)$, divide by $N^{(1+a)/2}$ and let N goes to $+\infty$.

To obtain a non trivial regime, we have to ensure that the normalizing term is of the same order than $|\Omega_{N,N^a}|$. This will be done by increasing the temperature with N (equivalently, decreasing β). Although this is not the usual situation in statistical mechanics, it allows us to recover a well known model of continuous-time directed polymer in a Brownian environment (see below for a precise definition). Until now, no precise relation between discrete models and this Brownian model has been given in the literature.

Let us introduce more precisely the Brownian setting: let $(B^{(i)}_\cdot)_i$ be an i.i.d. sequence of one-dimensional Brownian motions. Let $\Omega^c_{N,M}$ be the set of increasing sequences $0 = u_0 < u_1 < \cdots < u_M < u_{M+1} = N$. This can be identified as the set of piecewise constant paths with M positive jumps of size 1 in the interval $[0, N]$. Note that $|\Omega^c_{N,M}| = N^M/M!$, where $|\cdot|$ stands here for the Lebesgue measure. Denote by $P^c_{N,M}$ the uniform probability measure on $\Omega^c_{N,M}$. For $\mathbf{u} \in \Omega^c_{M,N}$, define

$$\mathbf{Br}(N, M)(\mathbf{u}) = \mathbf{Br}(\mathbf{u}) = \sum_{i=0}^{M}(B^{(i)}_{u_{i+1}} - B^{(i)}_{u_i}), \tag{5.8}$$

$$L(N, M) = \max_{\mathbf{u}\in\Omega^c_{N,M}} \mathbf{Br}(\mathbf{u}), \tag{5.9}$$

$$Z^{\mathbf{Br}}_\beta(N, M) = P^c_{N,M}\left(e^{\beta\mathbf{Br}(\mathbf{u})}\right). \tag{5.10}$$

The functional (5.9) is the aforementioned Brownian percolation problem from queueing theory. Observe that it has the interesting property that

$$L(N, M) = \sqrt{N}L(1, M),$$

in law. This is due to the scaling properties of Brownian motions. It is now a well known fact that $L(1, M)$ has the same law as the larger eigenvalue of a Gaussian Unitary random matrix (GUE, see [**5, 79**] among other proofs). As a consequence,

$$N^{1/6}\left(L(1, N) - 2N^{1/2}\right) \longrightarrow F_2, \tag{5.11}$$

where F_2 denotes the Tracy-Widom distribution [**91**]. It describes the fluctuations of the top eigenvalue of the GUE and its distribution function can be expressed as

$$F_2(s) = \exp\left\{-\int_s^{+\infty}(x - s)u(s)^2dx\right\},$$

where u is the unique solution of the Painlevé II equation

$$u'' = 2u^3 + xu,$$

with asymptotics

$$u(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}} \exp\left\{-\frac{2}{3}x^{3/2}\right\}.$$

The distribution function F_2 is non-centered and its asymptotics behavior is as follows:

$$F_2(s) \sim e^{\frac{1}{12}s^3}, \text{ as } s \to -\infty, \quad 1 - F_2(s) \sim e^{-\frac{4}{3}t^{3/2}} \text{ as } t \to +\infty.$$

See [**2**] for more details about the Tracy-Widom distribution and random matrices in general. In the discrete setting, it is shown in [**11**] that, for $M = N^a$ with $0 < a < 3/7$,

$$\frac{T(N, N^a) - 2N^{(1+a)/2}}{N^{1/2-a/6}} \longrightarrow F_2.$$

The proof uses similar approximations than the seminal work of Glynn and Whitt. See also [**4**] for similar results.

The third display (5.10) is the partition function of the previously mentioned continuous-time directed polymer in Brownian environment. The free energy of this polymer model is explicit. Its exact value was first conjectured in [**79**] based on a generalized version of the Burke's Theorem and detailed heuristics. The proof was then completed in [**77**]:

THEOREM 5.2 (Moriarty-O'Connell). [**77**]

$$\lim_{N\to+\infty} \frac{1}{N} \log Z_\beta^{\mathbf{Br}}(N, N) = f(\beta), \tag{5.12}$$

where

$$f(\beta) = \begin{cases} -(-\Psi)^*(-\beta^2) - 2\log|\beta| & : \beta \neq 0 \\ 0 & : \beta = 0 \end{cases} \tag{5.13}$$

where $\Psi(m) \equiv \Gamma'(m)/\Gamma(m)$ *is the restriction of the digamma function to* $(0, +\infty)$, Γ *is the Gamma function*

$$\Gamma(m) = \int_0^{+\infty} t^{m-1} e^{-t} dt,$$

and $(-\Psi)^*$ *is the convex dual of the function* $-\Psi$*:*

$$(-\Psi)^*(u) = \inf_{m\geq 0} \{mu + \Psi(m)\}.$$

We now search for a 'regime' in which the limiting free energy of the discrete model is the same as the Brownian one. It turns out that a way to achieve this is to increase the temperature in the asymmetric discrete model, as N tends to $+\infty$. So the Moriarty-O'Connell polymer can be viewed as an approximation of a discrete polymer close to an axis at a very high temperature.

THEOREM 5.3 (The Moriarty-O'Connell regime). *Let* $\beta_{N,a} = \beta N^{(a-1)/2}$,

$$\lim_{N\to+\infty} \frac{1}{\beta_{N,a} N^{(1+a)/2}} \log Z_{\beta_{N,a}}(N, N^a) = f(\beta)/\beta \tag{5.14}$$

In Section 8.3, we will give a proof of a d dimensional version of this fact. Unfortunately, we are no longer able to compute explicitly the free energy for the Moriarty-O'Connell model when $d \geq 2$.

The rest of this work is organized as follows:

- In Chapter 6, we review several aspects of queues in tandem and last passage percolation and give sketches of several proofs of Theorem 5.1. We emphasize the link with the Brownian percolation problem.
- In Chapter 7, we introduce the generalized Brownian queue and discuss its relation with the Brownian percolation.
- In Chapter 8 we discuss the links between asymmetric directed polymers and directed polymers in a Brownian environment. We prove the continuity of the point-to-point partition function for discrete models in Section 8.1. In Section 8.2, We discuss the existence of the free energy for the directed polymers in Brownian environments. We give the proof of Theorem 5.3 in Section 8.3 and discuss a more asymmetric situation in Section 8.4.
- Finally, we study a model of directed polymers with a huge drift in Chapter 9.

CHAPTER 6

Last Passage Percolation and Queues in Tandem

The aim of this chapter is to give a proof of Theorem 5.1 in the context of queueing theory, following [**45**]. We then discuss several proofs of the fact that $c = 2$. We also discuss briefly the fluctuation results from [**11**].

6.1. The models

We consider a queueing network consisting of a series of N single-server queues. Each queue has an infinite waiting space and follows the first-in-first-out service discipline. Initially we place M labeled customers in the first queue. We denote by $V(i, j)$ the service time of the i-th customer at queue j. We assume that these are all i.i.d. with mean and variance equal to 1. Denote by $D(N, M)$ the departure time of client M from queue N.

We are interested in the asymptotics of $D(N, M)$ when $N \to +\infty$ and $M = M_N$ depends on N. The simpler situation occurs when $M_N = 1$ for all N. Then $D(1, N)$ is just a sum of i.i.d. random variables and, under our hypothesis, it satisfies the usual law of large numbers and central limit theorem.

When $M = \lfloor xN \rfloor$ for some $1 < x < +\infty$, it is easy to see by subadditivity that

$$\frac{D(N, \lfloor xN \rfloor)}{N} \to \gamma,$$

as $N \to +\infty$ for some $\gamma = \gamma(x) \in (0, +\infty]$. Some integrability condition has to be introduced in order to insure that $\gamma(x) < +\infty$ (see [**74**]). However, the explicit form of $\gamma(\cdot)$ as a function of x depends strongly on the distribution of the service times and is in general unknown. There are some specific distributions for which this value has been discovered (see [**80**] for a review of available results).
We will be concerned by a very specific asymptotics, namely $M = O(N^a)$ for some $a \in (0, 1)$. This corresponds to early departures from a large number of queues. Contrarily to the case $k = O(N)$, we will see some universal phenomenons arising. More precisely, assume that there exists positive constants K and λ such that $Q(V(0, 0) > x) \leq Ke^{-\lambda x}$ for all $x \geq 0$.

Theorem 6.1. *There exists a constant c,*

$$\frac{D(M, \lfloor xN^a \rfloor) - N}{N^{(1+a)/2}} \to c\sqrt{x},$$

in probability. Moreover the constant c is universal, i.e., it does not depend on the particular law of the service times as long as it admits some exponential moments.

This queueing system is closely related to the last passage percolation problem. Observe that

$$D(N,M) = V(N,M) + \max\{D(N-1,M), D(N,M-1)\},$$

and then, it follows by iteration that

$$D(N,M) = \sup_{\mathbf{S}\in\Omega_{N,M}} \sum_{(i,j)\in\mathbf{S}} V(i,j), \tag{6.1}$$

where, as before, $\Omega_{N,M}$ represents the set of up/right paths from $(0,0)$ to (N,M). We then see that Theorem 6.1 is the queueing theoretic equivalent to Theorem 5.1.

6.2. Approximation by Brownian Queues

The key to prove the law of large number for $D(N,M)$ is to couple each row $\{V(i,j) : j \geq 0\}$ with a Brownian motion $B^{(i)}(\cdot)$. Then the original problem reduces to prove the analogue asymptotics for a Brownian last passage percolation problem for which we can take advantage of the scaling properties of the Brownian motion.
Recall the following definitions: let $(B_\cdot^{(i)})_i$ be an i.i.d. sequence of one-dimensional Brownian motions. Let $\Omega^c_{N,M}$ be the set of increasing sequences $0 = u_0 < u_1 < \cdots < u_M < u_{M+1} = N$. This can be identified as the set of piecewise constant paths with M positive jumps of size 1 in the interval $[0,N]$. Note that $|\Omega^c_{N,M}| = N^M/M!$, where $|\cdot|$ stands here for the Lebesgue measure. Denote by $P^c_{N,M}$ the uniform probability measure on $\Omega^c_{N,M}$. For $\mathbf{u} \in \Omega^c_{M,N}$, define

$$\mathbf{Br}(N,M)(\mathbf{u}) = \mathbf{Br}(\mathbf{u}) = \sum_{i=0}^{M} (B^{(i)}_{u_{i+1}} - B^{(i)}_{u_i}),$$

$$L(N,M) = \max_{\mathbf{u}\in\Omega^c_{N,M}} \mathbf{Br}(\mathbf{u}),$$

The following theorem corresponds to Theorem 4.1 in [**45**]. It gives the approximation of $D(N,M)$ in terms of $L(1,M)$. It can be proved using strong embeddings (see Section 8.3). As we use similar techniques in the proof of our Theorem 8.8, we will refer the reader to the original proof in [**45**] (see also [**11**]).

THEOREM 6.2. *(Glynn-Whitt) If there exists positive constants K and λ such that $Q(V(0,0) > x) \leq Ke^{-\lambda x}$ for all $x \geq 0$, then the service times $\{V(i,j) : i,j \geq 0\}$ can be coupled to the Brownian motions $\{B^{(i)}(\cdot) : i \geq 0\}$ in such a way that*

$$\max_{\substack{1\leq i\leq N \\ 1\leq j\leq N^a}} \left\{ D(i,j) - i - \sqrt{N} L(i/N, j) \right\} = O(N^a \log N). \tag{6.2}$$

The constant c appears in the following lemma:

LEMMA 6.3. *There exists a constant c such that*

$$\lim_{N\to+\infty}\frac{1}{N}L(N,\lfloor xN\rfloor)=c\sqrt{x},\quad a.s. \tag{6.3}$$

and $L(1,\lfloor xN\rfloor)/\sqrt{N}$ converges in probability to $c\sqrt{x}$.

PROOF. By subadditivity, we have

$$\lim_{N\to+\infty}\frac{1}{N}L(N,\lfloor xN\rfloor)=\gamma(x),\quad a.s.$$

for some $\gamma(x)$. The second affirmation is a consequence of the first one and Brownian scaling, since

$$\frac{1}{N}L(N,Nx)=\frac{1}{\sqrt{N}}L(1,Nx),$$

where the equality holds in law. Now, for $x>0$, we have

$$\lim_{N\to+\infty}\frac{1}{\sqrt{N}}L(1,xN)=\sqrt{x}\lim_{N\to+\infty}\frac{1}{\sqrt{xN}}L(1,xN)=\sqrt{x}\gamma(1).$$

The result follows by taking $c=\gamma(1)$. For more technical details, we refer the reader to [**45**], Theorem 7.1. □

We are now ready to complete the proof of Theorem 6.1:

$$\frac{D(N,xN^a)-N}{N^{(1+a)/2}}=\frac{D(N,xN^a)-N-\sqrt{N}L(1,\lfloor xN^a\rfloor)}{N^{(1+a)/2}}+\sqrt{N}\frac{L(1,\lfloor xN^a\rfloor)}{N^{(1+a)/2}}.$$

The first summand converges to 0 in probability, thanks to (6.2), while the second one converges to $c\sqrt{x}$, according to Lemma 6.3.

The next subsection is devoted to the proof that $c=2$.

6.3. A few proofs of Theorem 5.1

We recall that the first proof of the fact that $c=2$ in Theorem 5.1 was given by Seppäläinen [**84**]. It uses a link between queues in tandem and particle systems, namely Totally Asymmetric Exclusion process (TASEP) and zero-range process. This proof is discussed in the first subsection.

In the second subsection, we present a proof done entirely in the Brownian setting. It uses a variational formula which is reminiscent of the queueing theoretic interpretation. We discuss this approach with more details in the third subsection.

Finally, we sketch a proof relating Brownian percolation and random matrices in the fourth subsection.

6.3.1. The proof of Seppäläinen. We outline the proof in [**84**]. Consider the following representation of queues in tandem: labeled servers are placed in order at sites of $\mathbb{Z}$. The distance between servers $k-1$ and k is interpreted as the number of clients waiting at server k. Whenever one of those clients (the rightmost!) is served, it exchanges positions with server k. Forgetting the clients for a while, observe that each server tries to jump to the left with rate 1. If the left site is occupied by another server, the jump is suppressed (this situation is equivalent to an empty queue). The dynamics of these servers is usually known as TASEP, in full, totally asymmetric exclusion process (with jumps to the left).

On the one hand, we will prove a variational formula relating the position of server $\lfloor N^a x \rfloor$ at time Nt with passage times of customers. This, combined with the result of Glynn and Whitt will bring the c into the game. On the other hand, we will use well known equilibrium properties of the TASEP to identify the value of c.

Let us define carefully the TASEP. Let $\sigma(k) \in \mathbb{Z}$ denote the initial position of server $k \in \mathbb{Z}$. To each site $z \in \mathbb{Z}$, we associate a rate 1 Poisson clock. At each event time, if there is a server at z and no server at $z-1$, the server at z jumps to $z-1$. If the site $z-1$ is occupied by another server, the jump is suppressed. Note that $\eta(k) = \sigma(k) - \sigma(k-1) - 1$ is the number of clients waiting at server k. We denote by $\sigma(k,t)$ the position of server k at time t under the dynamics defined above. We restrict ourselves to situations where $\sigma(k) \le \sigma(k+1)+1$ for all $k \in \mathbb{Z}$. We allow initial conditions with $\sigma(k) = -\infty$ for all $k < k_0$ for some k_0. This amounts to consider an infinite number of clients waiting at queue k_0.

The fundamental result in [**84**] is the following: consider a sequence of possibly random initial conditions σ^N satisfying the following for some $a \in (0,1)$:

$(A1)$ $\forall y \in \mathbb{R}$,

$$\lim_{N\to+\infty} N^{-(1+a)/2}\sigma^N(\lfloor N^a y \rfloor) = v_0(y),$$

in probability for some nondecreasing function $v_0 : \mathbb{R} \to [-\infty, +\infty)$.

$(A2)$ $\exists\, \beta_0 > 0$ and $C_0 > 0$ such that

$$\lim_{N\to+\infty} P\left(\sigma^N(k) \le \beta_0 N^{(1-a)/2} k,\ \forall\, k \le -C_0 N^a\right) = 1.$$

Theorem 6.4 (Seppäläinen). *Under conditions* $(A1)$ *and* $(A2)$*, we have*

$$\lim_{N\to+\infty} \frac{\sigma^N(\lfloor N^a x \rfloor, Nt) + Nt}{N^{(1+a)/2}} = v(x,t), \tag{6.4}$$

in probability, for all $x \in \mathbb{R}$ *and* $t > 0$*, where*

$$v(x,t) = \sup_{y<x} \left\{ v_0(y) + 2\sqrt{t(x-y)} \right\}. \tag{6.5}$$

We will sketch the proof of this Theorem, first with c instead of 2 in Formula (6.5). Then, arguments of a different kind will lead to the exact value by simple comparison. Observe that it is enough to find a single convenient sequence $(\sigma^N)_N$ allowing us to compute c. Let us mention that the one we will use is in fact random, even if the configuration of interest (equivalent to infinitely many customers waiting at the first queue and no clients elsewhere) is not.
But first, let us explain how Theorem 6.4 can be used to show that $c = 2$: $\sigma^N(\lfloor N^a x \rfloor) - \sigma^N(\lfloor N^a x \rfloor, Nt)$ is the number of customers served by server $\lfloor N^a x \rfloor$ in the time interval $[0, Nt]$. By Theorem 6.4, this is roughly

$$Nt - N^{(1+a)/2}\left(v(x,t) - v_0(x)\right) + o(N^{(1+a)/2}). \tag{6.6}$$

As the service process has rate 1, let us note once and for all that the length of a given interval and the number of event times in that interval are asymptotically equivalent. For example, the service available at server $\lfloor N^a x \rfloor$ in the time interval $[0, Nt]$ is asymptotically equivalent to Nt, and then, by (6.6) the corresponding idle time is equivalent to

$$N^{(1+a)/2}\left(v(x,t) - v_0(x)\right) + o(N^{(1+a)/2}).$$

Now, the initial condition we are interested in corresponds to $\sigma^N(i) = -\infty$ for $i \leq 0$ and $\sigma^N(i) = i - 1$ for $i \geq 1$, i.e., infinitely many customers waiting at queue 1. This in turn corresponds to the choice

$$v_0(x) = \begin{cases} -\infty & : x \leq 0 \\ 0 & : x > 0 \end{cases},$$

which, thanks to (6.5), implies that

$$v(x,t) = \begin{cases} -\infty & : x \leq 0 \\ 2\sqrt{tx} & : x > 0 \end{cases}$$

In conclusion, we have shown that, under the particular situation described above, the idle time of server $\lfloor N^a x \rfloor$ in the time interval $[0, Nt]$ is asymptotically equivalent to

$$N^{(1+a)/2} 2\sqrt{tx} + o(N^{(1+a)/2}),$$

for $x > 0$. We need to relate this to a statement on passage times of customers: note that $T(\lfloor N^a x \rfloor, Nt)$ is roughly the hitting time of server $\lfloor N^a x \rfloor$ to the site $-\lfloor Nt \rfloor$. Note that, without the exclusion rule, this time will be of order $N^a x + Nt \sim Nt$. Then, $T(\lfloor N^a x \rfloor, Nt) - Nt$ corresponds to the suppressed jumps of server $\lfloor N^a x \rfloor$, i.e., its idle time, and we can conclude.

Now let us sketch the proof of Theorem 6.4. The key point is a variational formula relating the position of servers to the passage times of customers in the queueing system.

Let $\xi(j,t)$ denote the number of customers that have left server j by time t. With the passage time notation, $\xi(j,t) = \min\{k \geq 0 : T(k+1,j) > t\}$. Using the Glynn and Whitt's asymptotic on passage times, we can show that

$$\lim_{N\to+\infty} \frac{\xi(\lfloor N^a y\rfloor, Nt) - Nt}{N^{(1+a)/2}} = -c\sqrt{ty}. \tag{6.7}$$

Indeed, for $\epsilon > 0$, the condition

$$\frac{\xi(\lfloor N^a y\rfloor, Nt) - Nt}{N^{(1+a)/2}} \leq -c\sqrt{ty} + \epsilon, \tag{6.8}$$

is equivalent to $\xi(\lfloor N^a y\rfloor, Nt) \leq Ns_N$ with $s_N = t - N^{(a-1)/2}(c\sqrt{yt} - \epsilon)$. By construction, this last condition is satisfied if $T(Ns_N, N^a y) \geq Nt$. We will show that this holds with probability tending to 1 as $N \to +\infty$. The Glynn and Whitt approximation (Theorem 6.2) can be restated in the following convenient way: for each fixed $\kappa \in \mathbb{N}$, there is a finite random variable Y such that

$$|T(Nx, N^a y) - Nx - \sqrt{\kappa N} L(x/\kappa, N^a y)| \leq Y N^a \log N,$$

for all $0 \leq x \leq \kappa$ and $0 \leq y \leq N$. In particular,

$$T(Ns_N, N^a y) \geq Ns_N + \sqrt{\kappa N} L(s_N/\kappa, N^a y) + Y N^a \log N. \tag{6.9}$$

Now, the scaling properties of the Brownian percolation functional imply that

$$\sqrt{\kappa N} L(s_N/\kappa, N^a y) = c\sqrt{yt} N^{(1+a)/2} + o(N^{(1+a)/2}).$$

This in turn implies that the right-hand-side of 6.9 is larger than Nt with probability tending to 1 as $N \to +\infty$, which proves that (6.8) happens with probability tending to 1. The reverse inequality follows just in the same way.

In order to relate the position of the servers to the quantities $\xi(\cdot,\cdot)$, we will construct a special family of initial configurations χ^l for the TASEP: each site $x \geq l$ is occupied by a server and all sites $x < l$ are empty. This means that there is an infinite number of clients waiting at the server initially at l and no clients elsewhere. Servers are labeled in such a way that $\chi^l(j,0) = l + j - 1$ is the position of server j at time 0. We let all these processes evolve using the Poisson clocks we have introduced before. $\chi^l(j,t)$ will denote the position of server j at time t for the TASEP with initial conditions χ^l. These are related to passage times through the formula

$$\chi^l(j,t) = l + j - 1 - \xi^l(j,t). \tag{6.10}$$

The variational formula we have announced before is stated in the following Lemma, which corresponds to Lemma 3.1 in [**84**] and whose proof is deferred. Consider an initial configuration σ:

LEMMA 6.5.

$$\sigma(k,t) = \sup_{i\le k} \chi^l(k-i+1,t). \tag{6.11}$$

We can now continue with the proof of Theorem 6.4: by the preceding Lemma and (6.10),

$$\frac{\sigma^N(\lfloor N^a x\rfloor, Nt)+Nt}{N^{(1+a)/2}} = \sup_{i\le N^a x}\left\{\frac{\sigma^N(i)}{N^{(1+a)/2}} + \frac{\lfloor N^a x\rfloor - i}{N^{(1+a)/2}} - \frac{\xi^{\sigma^N(i)}(\lfloor N^a x\rfloor - i, Nt)}{N^{(1+a)/2}}\right\}.$$

Now, some technical work is needed in order to apply (6.7) to the right hand side of this equality (see Lemma 5.2 and 5.3 in [**84**] for details). The assumption $(A2)$ allows us to restrict conveniently the range of the i's. Once this is done, we can take the limit and we obtain

$$\frac{\sigma^N(\lfloor N^a x\rfloor, Nt)+Nt}{N^{(1+a)/2}} \sim \sup_{y\le x}\left\{v_0(y) + c\sqrt{t(x-y)}\right\},$$

which ends the proof of Theorem 6.4, except for the exact value of c. Until here, we made no additional assumption on the initial conditions other than $(A1)$ and $(A2)$. We will now consider some convenient sequence of random initial conditions $(\sigma^N)_N$: $P(\sigma^N(0)=0)=1$ and each $\eta^N(i)=\sigma^N(i)-\sigma^N(i-1)$ is $\text{Geom}(1-\nu_N)$ distributed, with

$$\nu_N = \frac{N^{(1-a)/2}}{1+N^{(1-a)/2}}.$$

In this way, $E(\sigma^N(k)) = N^{(1-a)/2}k$, and thanks to large deviations properties of the geometric distribution, assumptions $(A1)$ and $(A2)$ will hold with $v_0(y)=y$, $\beta_0<1$ and $C_0>0$. We can then conclude that

$$\lim_{N\to+\infty}\frac{\sigma^N(0,Nt)+Nt}{N^{(1+a)/2}} = \sup_{y\le 0}\left\{y+c\sqrt{-ty}\right\} = \frac{c^2 t}{4}. \tag{6.12}$$

These initial conditions are in fact invariant probabilities for the process we are considering (see [**3**], where this is stated in term of zero-range processes). In the queueing theoretic language, this means that the number of customers that leave the server 0 is a rate ν_N Poisson process (this is the celebrated Burke's theorem that we will discuss later), i.e., $-\sigma(0,Nt)$ is Poisson distributed with intensity $Nt\nu_N$ and then, after a few algebraic computations,

$$\lim_{N\to+\infty}\frac{\sigma^N(0,Nt)+Nt}{N^{(1+a)/2}} = t.$$

This, together with (6.12) implies that $c=2$.

We are now left with the proof of Lemma 6.5. Take any positive time t. It is easy to see that we can find $M_l << 0 << M_r$ such that that the clocks at M_l and M_r do not ring in the interval $[0, t]$. This means that the dynamics in the box $[M_l, M_r]$ is not influenced by events occurring outside of it in this time interval. Now, as a.s., there is finitely many event times in the box $[M_l, M_r]$ during $[0, t]$, we can proceed by induction.

So, suppose that τ is an event time for the clock at $M \in [M_l, M_r]$ and that (6.11) holds for all $t < \tau$ for all servers k such that $\sigma(k, t) \in [M_l, M_r]$.
We have to consider a few different cases. If there is no server at M at time τ, nothing happens and (6.11) still holds.
So, we can suppose that $\sigma(k, \tau-) = M$ for some k, where $\tau-$ is the event time prior to τ (so nothing happens in the interval $[\tau-, \tau]$). Now, either the server k jumps to the left or not. This depend on whether the site $M - 1$ is occupied by the server $k - 1$ or not.
Suppose that $\sigma(k-1, \tau-) = M-1$. By the induction hypothesis, (6.11) holds for $k-1$ and $\tau-$, and the supremum is attained for some i. Then,

$$M = \chi^{\sigma(i)}(k - i, \tau-) + 1 \leq \chi^{\sigma(i)}(k - i + 1, \tau-) \leq \sigma(k, \tau-) = M$$

This implies that $\sigma(k, \tau-) = \chi^{\sigma(i)}(k - i + 1, \tau)$ and $\chi^{\sigma(i)}(k - i + 1)$ cannot jump at τ. So (6.11) holds for k and τ, and the supremum is attained at i.

Now, suppose that server k does jump to position $M - 1$ at time τ. We must have $\sigma(k - 1, \tau-) \leq M - 2$. We will prove that $\chi^{\sigma(i)}(k - i + 1)$ also jumps for any i such that $\chi^{\sigma(i)}(k - i + 1, \tau-) = M$.
It is obviously true for $k = i$ because $\chi^{(\sigma(i))}(1)$ is the leftmost server of that process. For $k \leq i - 1$, we can use the induction hypothesis:

$$\chi^{\sigma(i)}(k - i, \tau-) \leq \sigma(k - 1, \tau-) \leq M - 2.$$

This means that any server $\chi^{\sigma(i)}(k)$ at M at time τ will jump at τ because it is not obstructed. This in turn implies that (6.11) still holds for k and τ and ends the proof.

As a general remark, we would like to notice that this strategy of proof will appear again in a purely Brownian setting in the proof by Hambly-Martin-O'Connell. More precisely, the exact value of c is obtained by comparing two expressions, one of them appearing as a variational formula involving queues in tandem, and the other one as a consequence of some law of large numbers related to Burke's theorem. This approach seems to be very general and is carefully exposed in [**80**]. This approach is a restatement in the setting of queueing theory of the ideas appearing in the original proof of Theorem 5.1 by Timo Seppäläinen.

6.3.2. The proof of Hambly-Martin-O'Connell. The starting point of this proof is a formula given in [**79**] (Formula 24 on page 294):

$$\sup_{t>0} \{B(-tN, 0) - mNt + L([-Nt, 0], N)\} = \sum_{k=1}^{N} q_k(0), \tag{6.13}$$

where, $L([-Nt,0],N)$ is defined just as $L(Nt,N)$ but using the Brownian motions $B^{(k)}$ in the interval $[-Nt,0]$. $B(-tN,0) = B_{-tN} - B_0$ with B a Brownian motion on the line, independent of $\{B^{(k)} : k \geq 1\}$, and $(q_k(0))_k$ is an i.i.d. family of exponential random variables of parameter m. We will discuss this formula with some details in the following subsection. By the law of the large numbers applied to the right side of (6.13), we find that

$$\sup_{t>0}\left\{\frac{1}{N}B(-tN,0) - mt + \frac{1}{N}L([-Nt,0],N)\right\} \to \frac{1}{m},$$

as N tends to $+\infty$. We are tempted to take the limit inside the sup and claim that

$$\sup_{t>0}\left\{-mt + c\sqrt{t}\right\} = \frac{1}{m}.$$

Then, it would be enough to find the maximum in the left hand side to conclude that $c = 2$. But a little uniformity is needed: the following proposition is the core of the proof of Theorem 8 in [**50**].

Proposition 6.6 (Hambly-Martin-O'Connell).

$$\sup_{t>0}\left\{\frac{1}{N}B_{tN} + \frac{1}{N}L([-Nt,0],N) - c\sqrt{t}\right\} \to 0, \tag{6.14}$$

as $N \to +\infty$.

Let us sketch this proof: it is easy to see that the first summand inside the sup tends to 0 uniformly. The difficult part consists in controlling $\frac{1}{N}L(Nt,N) - c\sqrt{t}$. The key is the following concentration inequality which corresponds to Lemma 7 in [**50**]:

$$Q\left(\sup_{t>0}\left|\frac{(1/N)L(Nt,N) - c\sqrt{t}}{1+t}\right| > y\right) \leq C\exp\{-CN(y-\varepsilon_N)^2\}, \tag{6.15}$$

where $(\varepsilon_N)_N$ is some sequence tending to 0. Define

$$V_N(t) = \frac{(1/N)L(Nt,N) - c_N t}{1+t},$$

where $c_N = QL(Nt,N)/N\sqrt{t} \uparrow c$. The event in the left hand side of (6.15) implies that $\sup\{V_N(t) : t > 0\} > x$ for $x = y - (c - c_N)$. Now, we discretize the line in intervals of length δ, for some well-chosen parameter $\delta > 0$. Then,

$$\begin{aligned} Q\left(\sup_{t>0} V_N(t) > x\right) &\leq Q\left(\sup_j V_N(\delta j) > x/3\right) \\ &+ Q\left(\sup_j V_N(\delta j) \leq x/3, \sup_{t>0} V_N(t) > x\right). \end{aligned}$$

The first summand is easily handled by standard Gaussian concentration arguments (see Lemma 5 in [**50**]). The second term says that the function $V_N(\cdot)$ performs large oscillations in some interval $[j\delta,(j+1)\delta]$. But, for $t\in(t_0,t_1)$,

$$L(Nt_0,N)+B_t^{(N)}-B_{t_0}^{(N)}\le L(Nt,N)\le L(Nt_1,N)-(B_t^{(N)}-B_{t_1}^{(N)}).$$

Thus, for $h>0$, by standard properties of Brownian motion,

$$\begin{aligned}
& Q\left(\exists t\in(t_0,t_1):\ L(Nt,N)\notin[L(Nt_0,N)-h,L(Nt_1,N)+h]\right)\\
\le\ & Q\Big(\exists t\in(t_0,t_1):\ B_t^{(N)}-B_{t_0}^{(N)}<-h,\ B_{t_1}^{(N)}-B_t^{(N)}<-h\Big)\\
\le\ & 2Q\left(M_{t_1-t_0}\ge h\right)\\
=\ & 4Q\left(B_{t_1-t_0}\ge h\right)\\
\le\ & 4\frac{\sqrt{t_1-t_0}}{h}\exp\left\{-\frac{h^2}{2(t_1-t_0)}\right\},
\end{aligned}$$

where $M_t=\sup\{B_s:s\in(0,t)\}$. Up to some technicalities (for instance, the need for a sequence (ε_N)), this yields (6.15).

6.3.3. A remark about Brownian queues. The formula (6.13) is reminiscent of the queueing theoretical interpretation of the last passage percolation problem. It has been used to compute similar limits in several discrete cases. The interested reader can consult the review [**80**] for a detailed bibliography and precise applications.

We introduce first the single Brownian queue: let B and C be independent one-dimensional Brownian motions indexed by $\mathbb{R}$ and $m>0$. The arrival and service processes during the time interval $(s,t]$ are defined respectively by

$$B_{(s,t)},\qquad m(t-s)-C_{(s,t)}.$$

We introduce the queue length and departure process as:

$$\begin{aligned}
q(t) &= \sup_{-\infty<s\le t}\left\{B_{(s,t)}+C_{(s,t)}-m(t-s)\right\},\\
d(t) &= B_t+q(0)-q(t).
\end{aligned}$$

The following theorem is the analogue of the classical Burke's theorem in our setting:

Theorem 6.7. [**79**] (1) $\{d(t):t\in\mathbb{R}\}$ *is a standard Brownian motion.*
(2) *For each* $t\in\mathbb{R}$, $\{d(s):s\le t\}$ *is independent of* $\{q(s):s\ge t\}$.

We omit the proof as it is very similar to the generalized Brownian queue case, a model that we will discuss in details in the next chapter. See [**79**] for both results.

Now, consider B, $B^{(k)}$, $k\ge1$ independent one-dimensional Brownian motions indexed by $\mathbb{R}$. We introduce an infinite series of Brownian queues in tandem by defining the arrival process at the first queue as B and the service time at the k-th queue as

$mt - B^{(k)}$, for some positive m. The arrival process at the $k+1$-th queue is the departure process from the k-th queue, d_k. We can define these processes by recursion, simultaneously with the queue length at queue k, q_k:

$$q_1(t) = \sup_{s \le t} \left\{ B(s,t) + B^{(1)}(s,t) - m(t-s) \right\},$$

$$d_1(s,t) = B(s,t) + q_1(s) - q_1(t)$$

$$q_{k+1}(t) = \sup_{s \le t} \left\{ d_k(s,t) + B^{(k+1)}(s,t) - m(t-s) \right\},$$

$$d_{k+1}(s,t) = d_k(s,t) + q_{k+1}(s) - q_{k+1}(t)$$

Summing $q_1(0)$ and $q_2(0)$, we obtain

$$q_1(0) + q_2(0) = \sup_{s \le 0} \left\{ B(s,0) + ms + L([s,0],2) \right\}.$$

The formula (6.13) follows easily by recurrence. The fact that each $q_k(0)$ has an exponential distribution follows from basic properties of Brownian motion. The independence follows from Theorem 6.7.

6.3.4. The proof of O'Connell-Yor. This proof is less self-contained than the previous one. The key point is to prove that the law of $L(1,N)$ is the same as the law of the largest eigenvalue of a GUE random matrix. The argument is harder, but as the eigenvalues of random matrices are well studied objects, this will give a large amount of information about $L(1,N)$. We will inherit, among other things, the very precise results on fluctuations and large deviations available for random matrices.

Let $(\epsilon_i)_i$ be centered real Gaussian random variables with $Q(\epsilon_i^2) = 2$, and let $(\epsilon_{i,j})_{i<j}$ be complex Gaussian random variables such that $Q\epsilon_{i,j} = Q\epsilon_{i,j}^2 = 0$, $Q|\epsilon_{i,j}|^2 = 1$. All these random variables are assumed to be independent. Let M_N be a hermitian $N \times N$ matrix with entries $[M_N]_{i,i} = \epsilon_i$ and $[M_N]_{i,j} = \epsilon_{i,j}$ for $i < j$. This construction defines a probability law on the space of complex hermitian $N \times N$ matrices known as the Gaussian Unitary Ensemble (GUE). Let $\lambda_1^{(N)} < \cdots < \lambda_N^{(N)}$ be the N different eigenvalues of M_N taking values in $W = \{\mathbf{x} : x_1 < \cdots < x_N\}$. It is now a classical result that, for $\mathbf{x} \in W$,

$$Q\left((\lambda_1^{(N)}, \cdots, \lambda_N^{(N)}) \in d\mathbf{x} \right) = C_N \Lambda(\mathbf{x})^2 \exp\{-\sum_i x_i^2\}, \tag{6.16}$$

where $\Lambda(\mathbf{x}) = \Pi_{i<j}|x_j - x_i|$ and C_N is an explicit constant. It is known that the empirical distribution of these eigenvalues converges to the so called *semi-circular* law, which support is exactly the interval $[-2,2]$, and that $\lambda^{(N)}/N \to 2$ as $N \to +\infty$.

The function Λ will be used to construct an h-transform $\widehat{P}_{\mathbf{x}}$ of a certain family of N paths starting from $\mathbf{x} \in W$. As Λ is strictly positive on W and vanishes on its boundary,

$\widehat{P}_{\mathbf{x}}$ will be interpreted as the law of these N paths conditioned on non-colliding. On one hand, we will show that, as $\mathbf{x} \to 0$, the law $\widehat{P}_{\mathbf{x}}$ will converge to a law $\widehat{P}_{0+}$ which coincides with (6.16). On the other hand, we will identify the law of $L(1, N)$ as the law of the highest path under $\widehat{P}_{0+}$. The first part of the proof uses results on non-colliding paths relying on their determinantal structure. The second is a piece of queueing theory.

Let $P_{\mathbf{x}}$ be the law of a d-dimensional Brownian motion B starting at $\mathbf{x} \in W$. Let $(\mathcal{F}_t)_t$ be the natural filtration of B. We define the law $\widehat{P}_{\mathbf{x}}$ as follows: for $A \in \mathcal{F}_t$,

$$\widehat{P}_{\mathbf{x}}(A) = P_{\mathbf{x}}\left(\frac{\Lambda(B_t)}{\Lambda(\mathbf{x})}\mathbf{1}_{A,T>t}\right), \tag{6.17}$$

where T is the exit time from W. The Karlin-McGregor formula (see, for example, [**2**], Lemma 4.2.54) giving the law of non-intersecting Markov processes in terms of determinants gives the following useful characterization of the law $\widehat{P}_{\mathbf{x}}$:

$$P_{\mathbf{x}}(B_t \in d\mathbf{y},\, T > t) = det\,[p_t(x_i, y_j)]_{i,j}\,,$$

where $p_t(\cdot,\cdot)$ is the transition function of the one-dimensional Brownian motion. Careful asymptotics (see [**58**]) allow us to take the limit $W \ni \mathbf{x} \to 0$:

$$\lim_{W \ni \mathbf{x} \to 0} \widehat{P}_{\mathbf{x}}(B_t \in d\mathbf{y}) = C_N(t)\Lambda(\mathbf{y})^2 \exp\{-\sum_i y_i^2\},$$

with $C_N(1) = C_N$. This allows us to define in an appropriate way the law of N non-colliding random motions starting from a common point (obviously, the non-colliding condition is restricted to strictly positive instants): for $A \in \sigma(B_u : u \geq t)$,

$$\widehat{P}_{0+}(A) = P_0\left(C_N(t)\Lambda(B_t)^2 \widehat{P}_{B_t}(A \circ \theta_{-t})\right). \tag{6.18}$$

At $t = 1$, this coincides with the expression (6.16) fot the eigenvalues of random matrices.

For the second part of the proof, we need to introduce a family of mappings of continuous paths following O'Connell and Yor [**79**]:

$$\begin{aligned}(f \otimes g)(t) &= \inf_{0 \leq s \leq t}[f(s) + g(t) - g(s)], \\ (f \odot g)(t) &= \sup_{0 \leq s \leq t}[f(s) + g(t) - g(s)].\end{aligned} \tag{6.19}$$

Now, define

$$\Gamma(f, g) = (f \otimes g,\, g \odot f)\,,$$

and, by recurrence, $\Gamma_2 = \Gamma$

$$\Gamma_{k+1}(f_1, \cdots, f_k) = (f_1 \otimes \cdots \otimes f_k, \\ \Gamma_{k-1}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \cdots, f_k \odot (f_1 \otimes \cdots \otimes f_{k-1}))).$$

Observe that $\Gamma_k : D(\mathbb{R}_+)^k \to D(\mathbb{R}_+)^k$ is continuous in the Skorohod topology, and, if we consider the f_i's as independent one-dimensional Brownian motions, its first coordinate corresponds to $-L(1, N)$.

We will first work in the context of Poisson processes. Later, careful scaling limits will lead to analogue results for Brownian motion.

Proposition 6.8. *Take $0 < \mu_1 < \cdots < \mu_N < +\infty$, and for each $1 \le i \le N$, let $N^{(\mu_i)}$ be a Poisson process on the line with intensity μ_i and initial condition $N_0^{(\mu_i)} = i$. We assume that all these processes are independents. Denote by P their joint law and by $\widehat{P}$ the h-transform with respect to the function Λ. Then, the law of $(N^{(\mu_1)}, \cdots, N^{(\mu_N)})$ under $\widehat{P}$ coincides with the law of $\Gamma_N(N^{(\mu_1)}, \cdots, N^{(\mu_N)})$ under P.*

The key point of the proof is the celebrated Burke theorem stating that, in equilibrium, the law of the departure process from a $M/M/1$ queue coincides with the arrival process. Let us briefly sketch the proof just in the case $N = 2$: consider $\lambda < \mu$ and A and S two Poisson processes of intensity λ and μ respectively. Denote by $A(s, t]$ the number of event times in the interval $(s, t]$ and $A_t = A(0, t]$ (and similarly for all the processes appearing in the following). We interpret A as the arrival process and S as the service process. Introduce

$$Q_t = \sup_{s \le t} \{A(s, t] - S(s, t]\},$$

$$D(s, t] = A(s, t] + Q_S - Q_t,$$

the queue length and the departure process respectively. Introduce also

$$T(s, t] = S(s, t] - Q_s + Q_t.$$

Burke's theorem says that D is a Poisson process with intensity λ. It is possible to prove, using reversibility, that T is a Poisson process with intensity μ, independent of D. Moreover, it is possible to prove that $\{(A_t, S_t) : t \ge 0\}$ conditioned to $\{A_t \le S_t, \forall\, t \ge 0\}$ has the same law than $\{(D_t, T_t) : t \ge 0\}$ given that $Q_0 = 0$. But, on the event $Q_0 = 0$, it is easy to see that $(D_t, T_t) = \Gamma(A, S)_t$ and this last expression is independent of Q_0. This leads to the conclusion that the joint law of A and S given that $A \le S$ is the same than the unconditional law of $\Gamma(A, S)$, which is what we wanted to prove.

Now, two limits have to be taken in order to recover an analogue of Proposition 6.8 for the Brownian motion: first, we have to take the limit $(\mu_1, \cdots, \mu_N) \to (\mu, \cdots, \mu)$ for any positive and finite μ, and then a diffusive limit in order to recover Brownian paths from the Poisson ones. We won't discuss these technical issues here, but we would like to mention that, one of the keys of the proof is again a consequence of the Karlin-McGregor formula, and states that, for N Poisson processes of equal rates,

$$\widehat{P}\left(N_t = \mathbf{y}\right) = C_N(t)\Lambda(\mathbf{y})^2 P\left(x^* + N_t = \mathbf{y}\right),$$

where $x^* = (1, \cdots, N)$ (see [**59**]). Let us now state the result in the Brownian setting:

PROPOSITION 6.9. *Let $(B^{(1)}, \cdots, B^{(N)})$ be a family of independent one-dimensional Brownian motions. Let P denote their joint law and $\widehat{P}$, the h-transform with respect to the function Λ. Then, the law of $(B^{(1)}, \cdots, B^{(N)})$ under $\widehat{P}_{0+}$ coincides with the law of $\Gamma_N(B^{(1)}, \cdots, B^{(N)})$ under P_0.*

Now, this implies that $-L(1, N)$ has the same law as the lowest path among N non-colliding Brownian motions with respect to the law $\widehat{P}_{0+}$. By symmetry, the law of $L(1, N)$ is the same as the highest path. But, by (6.16) and (6.18), we know that, at $t = 1$, this law is the same as the law of the larger eigenvalue of a GUE random matrix.

6.4. Fluctuations for the passage times

Recall from Theorem 6.2 that

$$D(N, N^a) - N - \sqrt{N} L(1, N^a) = O(N^a \log N).$$

In particular, if $a < 3/7$, this last quantity is $o(N^{1/2 - a/6})$. This implies that

$$\frac{D(N, N^a) - N - 2N^{(1+a)/2}}{N^{1/2-a/6}} = N^{a/6}\left(L(1, N^a) - 2N^{a/2}\right) + o(1).$$

But we know that,

$$N^{1/6}\left(L(1, N) - 2N^{1/2}\right) \longrightarrow F_{TW},$$

where F_{TW} denotes the Tracy-Widom distribution and the convergence holds in law. Replacing N by N^a, we obtain:

THEOREM 6.10. *([**9**],[**11**]) If there exist positive constants K and λ such that $Q(V(0,0) > x) \leq K e^{-\lambda x}$ for all $x \geq 0$, then, for all $a < 3/7$,*

$$\frac{D(N, N^a) - N - 2N^{(1+a)/2}}{N^{1/2-a/6}} \longrightarrow F_{TW},$$

where F_{TW} denotes the Tracy-Widom distribution and the convergence holds in law.

REMARK 6.11. The hypothesis on the integrability of the service times can be considerably relaxed. It is in fact possible to prove the Theorem assuming the existence of polynomial moments up to order p for some $p > 2$ (see [**11**]). The KMT coupling is weaker under these hypothesis, and we have to restrict to $a < 6/7(1/2 - p/6)$. We recover our bound $a < 3/7$ by taking $p \to +\infty$.

Remark 6.12. It is very unlikely that the condition $a < 3/7$ is optimal. It seems to be a technical limitation of this method of proof (see the remarks in Section 3 of [**11**]).

CHAPTER 7

Generalized Brownian queues and DP in Brownian environments

In this chapter, we discuss an 'exponential' version of the results presented in Section 6.3.3 and applications to one-dimensional directed polymers in a Brownian environment. As in the last passage percolation, it is easy to show the existence of the free energy by subadditivity. We discuss a way to identify exactly the limit, following ideas from [**79**] and [**77**].

7.1. Generalized Brownian Queues

Let B and C be independent Brownian motions indexed by $\mathbb{R}$ and let $m > 0$. Define, for $t > 0$,

$$\begin{aligned} r(t) &= \log \int_{-\infty}^{t} ds \exp\{B_{(s,t)} + C_{(s,t)} - m(t-s)\}, \\ f(t) &= B_t + r(0) - r(t) \end{aligned}$$

These can be understood as the queue length and the departure process in a generalized Brownian queue, where the arrival and service processes in the interval (s,t) are intended to be $B_{(s,t)}$ and $C_{(s,t)} - m(t-s)$ respectively. A nice feature of this model is that it shares many interesting properties with the Brownian queue introduced before. It is naturally related to the model of directed polymers in a Brownian environment, just as the Brownian queue is related to the Brownian last passage percolation.

Let us also introduce

$$g(t) = C_t + r(0) - r(t).$$

The following theorem is Burke's theorem in this generalized setting:

THEOREM 7.1. [**79**] (1) *f and g are independent standard Brownian motions.*
(2) *For each $t \in \mathbb{R}$, $\{(f(s), g(s)) :\ -\infty < s \le t\}$ is independent of $\{r(s) :\ s \ge t\}$.*

The core of the proof is contained in Proposition 7.2 below. In the following, X denotes a Brownian motion with a positive drift m and variance σ. First, we introduce two new quantities:

$$\begin{aligned} A_t &= \int_{-\infty}^{t} ds \exp\{2(X_s - X_t)\}, \\ \widehat{X} &= X_t + \log(A_t/A_0). \end{aligned}$$

Proposition 7.2. (1) $\widehat{X}$ *has the same law as* X.
(2) *For all* $t \in \mathbb{R}$, $\{\widehat{X} : s \leq t\}$ *is independent of* $\{A_s : s \geq t\}$.

The proof consists in a series of Lemmas. The first one follows by direct computations.

Lemma 7.3.

$$A_t = \int_t^{+\infty} e^{2(\widehat{X}_t - \widehat{X}_s)} ds, \tag{7.1}$$

and

$$X_t = \frac{1}{2}\int_0^t \frac{ds}{A_s} + \frac{1}{2}\log\frac{A_0}{A_t}. \tag{7.2}$$

Note that the second statement of Proposition 7.2 follows from (7.1).

Lemma 7.4. *The process* A_t *is stationary an reversible.*

Proof. The stationarity is an immediate consequence of the definition of A. For the proof of the reversibility, it will be convenient to specialize to $\sigma = \sqrt{2}$. Note that, in this case, $\exp\{r_0\}$ has the same law as A_0. The general case follows by scaling. We represent X as $X = \sqrt{2}b_t + mt$, with b a standard Brownian motion indexed by $\mathbb{R}$. Let

$$Y_t = A_t^{-1} = \frac{e^{\sqrt{2}b_t + mt}}{\int_{-\infty}^t e^{\sqrt{2}b_s + ms} ds}.$$

By Itô's formula, Y satisfies the stochastic differential equation

$$dY_t = Y_t(m + 1 - Y_t)dt + \sqrt{2}Y_t db_t.$$

Its infinitesimal generator is then

$$Lf(z) = z^2 f''(z) + z(m+1-z)f'(z),$$

whose adjoint is given by

$$L^* p(z) = \left(z^2 p(z)\right)'' - \left(z(m+1-z)p(z)\right)'.$$

We obtain the density of the invariant probability measure for Y by solving the ordinary differential equation $L^* p = 0$ on $(0, +\infty)$. The solution is given by

$$p(z) = \Gamma(m)^{-1} z^{m-1} e^{-z}, \quad z > 0. \tag{7.3}$$

We can check by direct computation that L is self-adjoint with respect to this measure, and then Y is reversible, as well as $A = Y^{-1}$. □

Remark 7.5. It follows that $\exp\{-r_0\}$ is a Gamma random variable with parameter m. As a consequence, $Q(r_0) = -\Gamma'(m)/\Gamma(m)$.

We can now proceed with the proof of Proposition 7.2. As already noticed, part (2) follows from formula (7.1). Let us denote by $\overleftarrow{X}$ the time-reverse of X: $\overleftarrow{X}_t = X_{-t}$. By formula (7.2) and stationarity,

$$\begin{aligned}
\overleftarrow{X}_t &= \frac{1}{2}\int_0^{-t} \frac{ds}{A_s} + \frac{1}{2}\log(A_0/A_{-t}) \\
&=_d \frac{1}{2}\int_t^{0} \frac{ds}{A_s} + \frac{1}{2}\log(A_t/A_0) \\
&= -\frac{1}{2}\int_0^{t} \frac{ds}{A_s} - \frac{1}{2}\log(A_0/A_t) \\
&= -X_t,
\end{aligned}$$

where the second equality holds in distribution.Then $\overleftarrow{X} = -X$ in law. On the other hand, using the reversibility of A, we have

$$\begin{aligned}
\widehat{X}_t &= \frac{1}{2}\int_0^{t} \frac{ds}{A_s} + \frac{1}{2}\log(A_t/A_0) \\
&=_d \frac{1}{2}\int_0^{t} \frac{ds}{\overleftarrow{A}_s} + \frac{1}{2}\log(\overleftarrow{A}_t/\overleftarrow{A}_0) \\
&= \frac{1}{2}\int_0^{t} \frac{ds}{A_{-s}} + \frac{1}{2}\log(A_{-t}/A_0) \\
&= -\frac{1}{2}\int_0^{-t} \frac{ds}{A_s} + \frac{1}{2}\log(A_{-t}/A_0) \\
&= -\overleftarrow{X}_t.
\end{aligned}$$

Summarizing, we also have $\widehat{X} = -\overleftarrow{X}$ in law, an then $\widehat{X} = X$ in law.

We are now ready to complete the proof of Theorem 7.1: introduce two independent Brownian motions defined in terms of B and C as

$$b^{(1)} = \frac{B-C}{\sqrt{2}}, \qquad b^{(2)} = \frac{B+C}{\sqrt{2}}.$$

Then, we can write

$$\begin{aligned}
f_t &= \frac{1}{\sqrt{2}} b_t^{(1)} + \left(\frac{1}{\sqrt{2}} b_t^{(2)} - r_t + r_0\right), \\
g_t &= -\frac{1}{\sqrt{2}} b_t^{(1)} + \left(\frac{1}{\sqrt{2}} b_t^{(2)} - r_t + r_0\right).
\end{aligned}$$

By Proposition 7.2, $\frac{1}{\sqrt{2}}b^{(2)} - r_t + r_0$ is a Brownian motion independent of $\{r_s : s \geq t\}$. It is also independent of $b^{(1)}$ by definition of r_t (it depends only on increments $B_s - B_t$ and $C_s - C_t$). Hence, f and g are both Brownian motions independent of $\{r_s : s \geq t\}$.

Furthermore, f and g are independent by orthogonality of the representations in the last display.

□

Now, consider B, $B^{(k)}$, $k \geq 1$ independent one-dimensional Brownian motions indexed by $\mathbb{R}$. We introduce an infinite series of generalized Brownian queues in tandem by defining the arrival process at the first queue as B and the service time at the k-th queue as $mt - B^{(k)}$, for some positive m. The arrival process at the $k+1$-th queue is the departure process from the k-th queue, d_k. We can define these processes by recursion, simultaneously with the queue length at queue k, q_k:

$$\begin{aligned} r_1(t) &= \log \int_{s \leq t} ds \exp\left\{B(s,t) + B^{(1)}(s,t) - m(t-s)\right\}, \\ f_1(s,t) &= B(s,t) + r_1(s) - r_1(t) \\ r_{k+1}(t) &= \log \int_{s \leq t} ds \exp\left\{f_k(s,t) + B^{(k+1)}(s,t) - m(t-s)\right\}, \\ f_{k+1}(s,t) &= f_k(s,t) + r_{k+1}(s) - r_{k+1}(t) \end{aligned}$$

Summing $q_1(0)$ and $q_2(0)$, we obtain

$$r_1(0) + r_2(0) = \log \int_{u \leq 0} du \exp(B_{(u,0)+mu}) \int_{u<s<0} ds \exp\left\{B^{(1)}(u,s) + B^{(2)}(s,0)\right\}.$$

By recurrence, we obtain the analogue of the formula (6.13):

$$\begin{aligned} \sum_{k=1}^{N} r_k(0) &= \log \int_{u \leq 0} du \exp(B_{(u,0)+mu}) \\ &\times \int_{u<s_1<\cdots<s_{N-1}<0} ds_1 \cdots ds_{N-1} \exp\left\{B^{(1)}(u,s_1) + \cdots + B^{(N)}(s_{N-1},0)\right\}. \end{aligned}$$

Remark $r_1(0)$, $r_2(0)$,... is an i.i.d. sequence, thanks to Theorem 7.1. Their common distribution is $-\log G_m$ where G_m is a Gamma random variable with parameter m. The following is a consequence of the law of large numbers:

PROPOSITION 7.6.

$$\begin{aligned} \lim_{N \to +\infty} \frac{1}{N} \log \int_{u<s_1<\cdots<s_{N-1}<0} du ds_1 \cdots ds_{N-1} \exp\{mu + B^{(1)}(u,s_1) + \cdots + B^{(N)}(s_{N-1},0)\} \\ = -\Gamma'(m)/\Gamma(m) =: -\Psi(m), \end{aligned}$$

almost surely.

PROOF. We have to get rid of the term $B_{(0,u)}$. By the law of large numbers, $B_{u,0}/u \to 0$ when $u \to -\infty$, and then, for all $\epsilon > 0$, there exists a finite random variable K such that $|B_{(u,0)}| \leq \epsilon u + K$ for all $u < 0$. This implies that the lower and upper limits of the quantity considered above lie in the interval $[\Psi(m-\epsilon), \Psi(m+\epsilon)]$. We use the continuity of Ψ to conclude. □

7.2. The Moriarty-O'Connell-Yor Model

Recall the following definitions: let $(B^{(i)}_\cdot)_i$ be an i.i.d. sequence of one-dimensional Brownian motions. Let $\Omega^c_{N,M}$ be the set of increasing sequences $0 = s_0 < s_1 < \cdots < s_M < s_{M+1} = N$. This can be identified as the set of piecewise constant paths with M positive jumps of size 1 in the interval $[0, N]$. Note that $|\Omega^c_{N,M}| = N^M/M!$, where $|\cdot|$ stands here for the Lebesgue measure. For a path $\mathbf{s} \in \Omega^c_{M,N}$, define

$$\mathbf{Br}(N,M)(\mathbf{u}) = \mathbf{Br}(\mathbf{u}) = \sum_{i=0}^{M} (B^{(i)}_{u_{i+1}} - B^{(i)}_{u_i}), \tag{7.4}$$

Now, the polymers measure is defined on $\Omega^c_{N,M}$ by

$$\frac{d\mu^{\mathbf{Br}}}{dP^c_{N,M}}(\mathbf{s}) = \frac{1}{Z^{\mathbf{Br}}_\beta(N,M)} e^{\beta \mathbf{Br}(\mathbf{s})}$$

where

$$Z^{\mathbf{Br}}_\beta(N,M) = P^c_{N,M}\left(e^{\beta \mathbf{Br}(\mathbf{u})}\right). \tag{7.5}$$

is the partition function of the model. This model has been introduced in [**79**]. The authors predicted the value of the free energy based on precise heuristics. The proof was then completed in [**77**]: recall the definition of the Gamma function,

$$\Gamma(m) = \int_0^{+\infty} x^{m-1} e^{-x}\, dx.$$

THEOREM 7.7 (Moriarty-O'Connell). [**77**]

$$\lim_{N\to+\infty} \frac{1}{N} \log Z^{\mathbf{Br}}_\beta(N,N) = f(\beta),$$

where

$$f(\beta) = \begin{cases} -(-\Psi)^*(-\beta^2) - 2\log|\beta| & : \beta \neq 0 \\ 0 & : \beta = 0 \end{cases}$$

where $\Psi(m) \equiv \Gamma'(m)/\Gamma(m)$ *is the restriction of the digamma function to* $(0,+\infty)$*, and* $(-\Psi)^*$ *is the convex dual of the function* $-\Psi$*.*

We will give a general idea of the proof avoiding technical details. First, for $x < 0$, define

$$\gamma_N(x) = \frac{1}{N} \log \int_{xN < s_1 < \cdots < s_{N-1} < 0} \exp\{B^{(1)}_{(xN,s_1)} + \cdots + B^{(N)}_{(s_{N-1},0)}\}.$$

By subadditivity, there exists a function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ such that, for each $x < 0$, $\gamma(x) = \lim_{N\to+\infty} \gamma_N(x)$, Q-a.s.. With these notations, Proposition 7.6 can be restated as

$$\lim_{N\to+\infty}\frac{1}{N}\log\int_{x<0}dx\exp N\{mx+\gamma_N(x)\} = -\Psi(m).$$

If some ideal hypothesis are satisfied, we can apply Laplace method in order to conclude that

$$\sup_{x<0}\{mx+\gamma(x)\} = -\Psi(m) = (-\gamma)^*(m).$$

Inverting the transform, we get $\gamma = -(-\Psi)^*$. The Theorem follows by Brownian scaling, as

$$\begin{aligned} f(\beta) &= \lim_{N\to+\infty}\frac{1}{N}\log Z_\beta^{\mathbf{Br}}(N,N) \\ &= \gamma(-\beta^2)-2\gamma(\beta). \end{aligned}$$

There are of course some missing details. In particular, we have to be careful while applying the Laplace method. We would have to prove the two following statements that can be found in Lemma 8 and 9 from [**79**] respectively:

(i) Q-a.s., $\lim_{N\to+\infty}\gamma_N(x)=\gamma(x)$, for all $x<0$.

(ii) The function γ is concave.

The rigorous proof of the Theorem is then completed in [**79**].

CHAPTER 8

Asymetric Directed Polymers and the Brownian Model

The central part of this Chapter is the proof of a multidimensional version of Theorem 4.69 (Section 8.3). In Section 8.1, we show a continuity property for the point-to-point partition function that will be useful in the study of very asymmetric polymers (Section 8.4). We also show the existence of the free energy for the Brownian model in Section 8.2.

8.1. Continuity of the point-to-point partition function for the discrete model

We prove here the continuity of the point-to-point free energy seen as a function from the octant $\{\mathbf{x} \in \mathbb{R}^d : x_i \geq 0\}$ to $\mathbb{R}$. Only the continuity at the boundary of the octant requires a proof, as the continuity in the interior is an easy consequence of the concavity properties of the free energy (which itself follows from sub-additivity). For $\mathbf{y} \in \mathbb{R}_+^d$, define

$$Z_N^\beta(\mathbf{y}) = \sum_{\mathbf{s} \in \Omega_{N\mathbf{y}}} \exp \beta H(\mathbf{s}),$$

where $\Omega_{N\mathbf{y}}$ is the set of directed paths from the origin to $N\mathbf{y}$, which, by notational abuse, denotes the point in $\mathbb{Z}^d$ which i-th coordinate is $\lfloor N y_i \rfloor$. Note that the dimension here is d and not $d+1$ as usual. We will be interested in directions of the form $\mathbf{y}_h = (h, \mathbf{x})$ with $\mathbf{x} \in \mathbb{R}_+^{d-1}$ (i.e. $\mathbf{x} \in \mathbb{R}^{d-1}$, $x_i > 0$), and $h \geq 0$. In this case, we just denote the partition function by $Z_N(h, \mathbf{x})$. We also define the point-to-point free energy:

$$\psi^\beta(\mathbf{y}) = \lim_{N \to +\infty} \frac{1}{N} \log Z_N^\beta(\mathbf{y}),$$

and we adopt the convenient notation $\psi(h, \mathbf{x})$ for $\psi(\mathbf{y}_h)$ (we also dropped the dependence in β). ψ is a function from the octant $\{\mathbf{x} \in \mathbb{R}^d : x_i \geq 0\}$ to $\mathbb{R}$.

Proposition 8.1.

$$\lim_{h \downarrow 0} \psi(h, \boldsymbol{x}) = \psi(0, \boldsymbol{x}).$$

Proof. Each path from the origin to $N(h, \mathbf{x})$ can be decomposed into Nh segments with constant first coordinate: for each path, there is a collection of points $(\mathbf{m}_i)_{i \leq Nh}$ with $\mathbf{m}_i \in \mathbb{Z}_+^{d-1}$ and such that for each $0 \leq i < Nh$, there is a segment of the path linking $(i, \mathbf{m}_i)$ and $(i, \mathbf{m}_{i+1})$. So the partition function can be decomposed itself as

$$Z_N(h,\mathbf{x}) = \sum_{(\mathbf{m}_i)_i} \Pi_i Z(i;\mathbf{m}_i,\mathbf{m}_{i+1}), \tag{8.1}$$

where, for each i, $Z(i;\mathbf{m}_i,\mathbf{m}_{i+1})$ is a sum over directed paths linking $(i,\mathbf{m}_i)$ and $(i,\mathbf{m}_{i+1})$. The collection of possible points $(\mathbf{m}_i)_i$ runs over a set $J^N_{h,\mathbf{x}}$ which cardinality satisfies $\log|J^N_{h,\mathbf{x}}| = N\phi(h,\mathbf{x}) + o(N)$ for some $\phi(h,\mathbf{x}) \to 0$ as $h \to 0$ (see remark below). We will analyze each summand of the right hand side of (8.1) separately:

$$\begin{aligned} Q\left(\log \Pi_i Z(i;\mathbf{m}_i,\mathbf{m}_{i+1})\right) &= \sum_i Q\left(\log Z(i;\mathbf{m}_i,\mathbf{m}_{i+1})\right) \\ &= \sum_i Q\left(\log Z(0;\mathbf{m}_i,\mathbf{m}_{i+1})\right) \\ &\le Q\left(\log Z_N(0,\mathbf{x}) + \beta \sum \eta(i,\mathbf{m}_{i+1})\right) \\ &\le N\,\psi(0,\mathbf{x}). \end{aligned} \tag{8.2}$$

The second equality follows by translation invariance; in the third line, we use the fact that the partition functions do not consider the environment at the starting point; the last inequality follows by subadditivity, as

$$\psi(\mathbf{y}) = \sup_N \frac{1}{N} Q \log Z_N(\mathbf{y}),$$

and the fact that $Q\eta = 0$. Now, the concentration inequality implies that

$$Q\left(\left|\log \Pi_i Z(i;\mathbf{m}_i,\mathbf{m}_{i+1}) - Q \log \Pi_i Z(i;\mathbf{m}_i,\mathbf{m}_{i+1})\right| \ge \epsilon N\right) \le e^{-c\epsilon^2 N}. \tag{8.3}$$

for ϵ small enough (see [**29**] Proposition 3.2.1-b). Using (8.1), we can see that, if

$$\log Z_N(h,\mathbf{x}) \ge N\,\psi(0,\mathbf{x}) + \epsilon N,$$

for some $\epsilon > 0$, then, for some $(\mathbf{m}_i)_i \in J^N_{h,\mathbf{x}}$, it must happen that

$$\log \Pi_i Z(i;\mathbf{m}_i,\mathbf{m}_{i+1}) \ge N\,\psi(0,\mathbf{x}) + \epsilon N - \log|J^N_{h,\mathbf{x}}|.$$

By (8.2), this means that the quantity in the left hand side deviates more than $\epsilon N - \log|J^N_{h,\mathbf{x}}|$ from its mean. By the asymptotics on $|J^N_{h,N}|$, for h small enough, we will have that $\log|J^N_{h,\mathbf{x}}| < \epsilon N/2$, and then the inequality (8.3) applies. Then,

$$Q\left(\log Z_N(h,\mathbf{x}) \ge N\,p(0,\mathbf{x}) + \epsilon N\right) \le \exp\left\{N\phi(h,\mathbf{x}) - c\epsilon^2 N + o(N)\right\}.$$

By taking h even smaller if necessary, the right hand side of this inequality becomes summable. By Borel-Cantelli we will then have that

$$\log Z_N(h,\mathbf{x}) \leq N\,\psi(0,\mathbf{x}) + \epsilon N,$$

Q-almost surely for N large enough. Dividing both sides by N and taking the limit $N \to +\infty$, we conclude that

$$\psi(h,\mathbf{x}) \leq \psi(0,\mathbf{x}) + \epsilon,$$

for h small enough. We now have to check the reverse inequality. But it follows easily that

$$\log Z_N(h,\mathbf{x}) \geq \log Z_N(0,\mathbf{x}) + \beta \sum_{i=0}^{hN} \eta(N\mathbf{x}, i).$$

Integrating with respect to Q, recalling that the η's are centered, dividing by N and taking the limit $N \to +\infty$ gives that $\psi(h,\mathbf{x}) \geq \psi(0,\mathbf{x})$. □

REMARK 8.2. The function ϕ can be made explicit: as

$$\log |J^N_{h,\mathbf{x}}| = \prod_{i=2}^{d} \binom{\lfloor Nx_i \rfloor + \lfloor Nh \rfloor}{\lfloor Nh \rfloor},$$

by Stirling formula, we have $\log |J^N_{h,\mathbf{x}}| = N\phi(h,\mathbf{x}) + o(N)$, with

$$\phi(h,\mathbf{x}) = \sum_{\substack{2\leq i\leq d \\ x_i>0}} \left(h \log \frac{x_i + h}{h} + x_i \log \frac{x_i + h}{x_i} \right).$$

THEOREM 8.3. *The point-to-point free energy is continuous on* $\mathbb{R}^d_+$.

PROOF. The continuity in the interior of $\mathbb{Z}^d_+$ is a consequence of the concavity properties arising from the subadditivity. See the proof of Theorem 8.6 where this is explained in the continuous setting. The continuity at the boundary follows from repeated use of the preceding Proposition. □

REMARK 8.4. In the one-dimensional case, a very precise asymptotic for the last-passage percolation is available. It implies that $\psi(1,h) = 2\sqrt{h} + o(\sqrt{h})$ as $h \downarrow 0$ (see [**74**], Theorem 2.3).

REMARK 8.5. This scheme of proof will reappear later in the proof of a certain continuity at the borders property for very asymmetric directed polymers, in the regime where the limit is the Brownian free energy.

8.2. Directed Polymers in a Brownian Environment

We will now generalize the Brownian setting introduced before to larger dimensions.

Let $\mathbf{x} \in \mathbb{Z}^d$ such $x_i \geq 1$ for all $i = 1, ..., d$. Let $M = \sum_{i=1}^{d} x_i$. This is basically the length of a nearest-neighbor path from the origin $\mathbf{0}$ to $\mathbf{x}$. Let $\Omega^c_{t,\mathbf{x}}$ be the set of right-continuous paths $\mathbf{s}$ such that:

(i) $\mathbf{s}_0 = \mathbf{0}$ and $\mathbf{s}_t = \mathbf{x}$.

(ii) $\mathbf{s}$ performs exactly M jumps, according to the coordinate vectors.

So the *skeleton* of $\mathbf{s}$ can be thought of as a discrete nearest-neighbor path form the origin to $\mathbf{x}$. $\mathbf{s}$ itself can be viewed as a directed path in $\mathbb{R}^+ \times \mathbb{Z}^d$ starting from the origin at time 0 and reaching the site $(t, \mathbf{x})$ at time t. Let $P^c_{t,\mathbf{x}}$ be the uniform measure on $\Omega^c_{t,\mathbf{x}}$.

Now consider a family $\{B(\mathbf{y}) : \mathbf{y} \in \Lambda_{\mathbf{x}}\}$ of independent Brownian motions, where $\Lambda_{\mathbf{x}} = \{\mathbf{y} \in \mathbb{Z}^d : 0 \leq y_i \leq x_i, \forall\, i = 1, ..., d\}$. Define the energy of a path $\mathbf{s}$ in the following way: let $0 = t_0 < t_1 < \cdots < t_M < t$ be the jumps times of $\mathbf{s}$ and put $t_{M+1} = t$, then

$$\mathbf{Br}(\mathbf{s}) = \mathbf{Br}(t, \mathbf{x})(\mathbf{s}) = \sum_{k=1}^{M+1} \left(B_{t_k}(\mathbf{s}_{t_k}) - B_{t_{k-1}}(\mathbf{s}_{t_k}) \right). \tag{8.4}$$

The partition function of the directed polymers in Brownian environment at inverse temperature β is

$$Z^{\mathbf{Br}}_{\beta}(t, \mathbf{x}) = P^c_{t,\mathbf{x}} \left(\exp \beta \mathbf{Br}(\mathbf{s}) \right). \tag{8.5}$$

We first prove the existence of the free energy in the linear regime. Take $\alpha \in \mathbb{R}^d$ with strictly positive entries.

THEOREM 8.6. *Let αN be the point of $\mathbb{Z}^d$ whose i-th coordinate is equal to $\lfloor \alpha_i N \rfloor$. Then the following deterministic limit*

$$p(\beta, \alpha, d) = \lim_{N \to +\infty} \frac{1}{N} \log Z^{\mathbf{Br}}_{\beta}(N, \alpha N) \tag{8.6}$$

exists Q-a.s.. Moreover, the function $\alpha \mapsto p(\beta, \alpha, d)$ is continuous on its domain.

PROOF. First, fix α. The proof uses subadditivity. To lighten notation, denote $|\Omega_N|$ for $|\Omega_{N,\alpha N}|$. We consider unnormalized versions of the partition function:

$$\begin{aligned}
\int_{\Omega_{N+M}} e^{\beta \mathbf{Br}(N+M, \mathbf{x}_{N+M})(\mathbf{s})} &\geq \int_{\Omega_{N+M}} e^{\beta \mathbf{Br}(N+M, \mathbf{x}_{N+M})(\mathbf{s})} \mathbf{1}_{\mathbf{s}_N = \alpha N} \\
&= \int_{\Omega_N} e^{\beta \mathbf{Br}(N, \alpha N)(\mathbf{s})} \times \left(\int_{\Omega_M} e^{\beta \mathbf{Br}(M, \alpha M)(\mathbf{s})} \right) \circ \theta_{N, \alpha N},
\end{aligned}$$

where the shift $\theta_{k,\mathbf{x}}$ means that we use the Brownian motions

$$\overline{B}^{(\mathbf{y})}(\cdot) = B^{(\mathbf{y}+\mathbf{x})}(\cdot + k),$$

to define **Br**. By subadditivity, it follows that there exists a deterministic function $\overline{p}(\beta,\alpha,d)$ such that

$$\overline{p}(\beta,\alpha,d)=\lim_{N\to+\infty}\frac{1}{N}\log\int_{\Omega_N}e^{\beta\mathbf{Br}(N,\alpha N)(\mathbf{s})},$$

Q-almost surely. Apply this with $\beta=0$ and the theorem follows with $p(\beta,\alpha,d)=\overline{p}(\beta,\alpha,d)/\overline{p}(0,\alpha,d)$. Now, take α_1 and α_2 in $\mathbb{R}^d$ with strictly positive coordinates, and $\lambda\in(0,1)$. Then,

$$Z_\beta^{\mathbf{Br}}\left(N,N(\lambda\alpha_1+(1-\lambda)\alpha_2)\right)\geq Z_\beta^{\mathbf{Br}}\left(N,\lambda\alpha_1 N\right)\times Z_\beta^{\mathbf{Br}}\left(N,(1-\lambda)\alpha_2 N\right)\circ\theta_{\lambda N,\lambda\alpha_1 N}.$$

Taking logarithms in both sides, dividing by N and taking limits, leads to,

$$\overline{p}\left(\beta,\lambda\alpha_1+(1-\lambda)\alpha_2,d\right)\geq\lambda\overline{p}\left(\beta,\alpha_1,d\right)+(1-\lambda)\overline{p}\left(\beta,\alpha_2,d\right)$$

So $\alpha\mapsto\overline{p}(\beta,\alpha,d)$ is concave, and then continuous. As $p(\beta,\alpha,d)=\overline{p}(\beta,\alpha,d)/\overline{p}(0,\alpha,d)$, it is also continuous. □

Remark 8.7. Note that as we have true subadditivity, we can avoid the use of concentration. However, we can state the following result:

$$Q\left(\left|\log Z_\beta^{\mathbf{Br}}(N,\alpha N)-Q\log Z_\beta^{\mathbf{Br}}(N,\alpha N)\right|>uN\right)\leq C\exp\left\{-\frac{Nu^2}{C\beta^2}\right\}.$$

(8.7)

This can be proved as Formula (9) in [**83**], using ideas from Malliavin Calculus.

8.3. Asymmetric Directed Polymers in a Random Environment

In this section, we will study discrete asymmetric models of directed polymers in random environment. We will show that when there is enough asymmetry and we let decrease β at the right rate, the free energy of this model will coincide with the free energy of the continuous-time model in Brownian environment.

Let $\mathbf{x}\in\mathbb{Z}^d$ such $x_i\geq 1$ for all $i=1,...,d$ and $N\geq 1$. Let $M=\sum_{i=1}^d x_i$ be the distance between the origin and x in $\mathbb{Z}^d$. Let $\Omega_{N,\mathbf{x}}$ be the set of directed paths from the origin in $\mathbb{Z}^{d+1}$ to (N,x) that is

$$\Omega_{N,\mathbf{x}}=\{\mathbf{S}:\{0,...,N+M\}\to\mathbb{Z}^{d+1}:\mathbf{S}_0=0,\,\mathbf{S}_{N+M}=(N,\mathbf{x}),\\ \forall t,\,\mathbf{S}_{t+1}-\mathbf{S}_t\in\{e_i\,:\,i=1,...,d\}\}.$$

Consider a collection of i.i.d. random variables $\{\eta(k,\mathbf{x})\,:\,k\in\mathbb{Z},\,\mathbf{x}\in\mathbb{Z}^d\}$. We will assume that $Q(e^{\beta\eta})<+\infty$ for all $\beta\geq 0$. For a fixed realization of the environment, define the energy of a path $\mathbf{S}\in\Omega_{N,\mathbf{x}}$ as

$$H(\mathbf{S}) = \sum_{t=1}^{N+M} \eta(\mathbf{S}_t). \tag{8.8}$$

The polymer measure at inverse temperature β is now defined as the measure on $\Omega_{N,\mathbf{x}}$ such that

$$\frac{d\mu_{N,\mathbf{x}}}{dP_{N,\mathbf{x}}}(\mathbf{S}) = \frac{1}{Z_\beta(N,\mathbf{x})} \exp \beta H(\mathbf{S}),$$

where $Z_\beta(N,\mathbf{x})$ is the point-to-point partition function

$$Z_\beta(N,\mathbf{x}) = P_{N,\mathbf{x}} \left(\exp \beta H(\mathbf{S}) \right).$$

We will be interested in the limit as N grows to infinity and $\mathbf{x} = \mathbf{x}_N$, with $|\mathbf{x}_N| \to +\infty$ with N in an appropriate way. Take $\alpha \in \mathbb{R}^d$ with strictly positive coordinates. Let αN^a be the point in $\mathbb{Z}^d$ which i-th coordinate is equal to $\lfloor \alpha_i N^a \rfloor$. The following theorem is the generalization to $\mathbb{Z}^d$ of Theorem 5.3.

THEOREM 8.8. *Let $\beta_{N,a} = \beta N^{(a-1)/2}$. Then,*

$$\lim_{N \to +\infty} \frac{1}{\beta_{N,a} N^{(1+a)/2}} \log Z_{\beta_{N,a}}(N, \alpha N^a) = p(\beta, \alpha, d)/\beta, \tag{8.9}$$

Q-almost surely, where $p(\beta,\alpha,d)$ is the free energy of the continuous-time directed polymer in a Brownian environment as in (8.6).

REMARK 8.9. To lighten notation, in the following, C will denote a generic constant whose value can vary from line to line. Also, we can consider $\alpha = (1, \cdots, 1)$ for simplicity and introduce the notations $\Omega_{N,a} = \Omega_{N,N^a}$ and $P_{N,a} = P_{N,N^a}$, and similarly for their continuous counterparts.

8.3.1. Proof of Theorem 8.8. The proof is carried on in 4 Steps. Much of the computations in Steps 1 and 2 are inspired by [**11, 51**], while the scaling argument in Step 3 is already present in [**45**].

8.3.1.1. *First Step: approximation by a Gaussian environment.* The central ingredient of this part of the proof is a strong approximation technique by Komlós, Major and Tusnády: let $\{\eta_t : t \geq 0\}$ denote an i.i.d. family of random variables, with $Q(\eta_0) = 0$, $Q(\eta_0^2) = 1$ and $Q(e^{\beta \eta_0}) < +\infty$ for all $0 \leq \beta \leq \beta_0$ for some $\beta_0 > 0$. Let $\{g_t : t \geq 0\}$ denote an i.i.d. family of standard normal variables. Denote

$$S_N = \sum_{t=0}^{N} \eta_t, \quad T_N = \sum_{t=0}^{N} g_t.$$

THEOREM 8.10 (KMT approximation). [**64**] *The sequences* $\{\eta_t : t \geq 0\}$ *and* $\{g_t : t \geq 0\}$ *can be constructed in such a way that, for all* $x > 0$ *and every* N,

$$Q\left\{\max_{k\leq N} |S_k - T_k| > K_1 \log N + x\right\} \leq K_2 e^{-K_3 x}, \tag{8.10}$$

where K_1, K_2 *and* K_3 *depend only on the distribution of* η, *and* K_3 *can be taken as large as desired by choosing* K_1 *large enough. Consequently,* $|S_N - T_N| = O(\log N)$, *Q-a.s..*

Now consider our environment variables $\{\eta(t, \mathbf{x}) : t \in \mathbb{Z}, \mathbf{x} \in \mathbb{Z}^d\}$. Use Theorem 8.10 to couple each 'row' $\eta(\cdot, \mathbf{x})$ with standard normal variables $g(\cdot, x)$ such that

$$Q\left\{\max_{k\leq N} |S(k, \mathbf{x}) - T(k, \mathbf{x})| > C \log N + \theta\right\} \leq K_2 e^{-K_2\theta}, \quad \forall\theta > 0,$$

where $S(k, \mathbf{x}) = \sum_{t=0}^{k} \eta(t, \mathbf{x})$ and $T(k, \mathbf{x}) = \sum_{t=0}^{k} g(t, \mathbf{x})$.

Now, we need to decompose each path $\mathbf{S} \in \Omega_{N,a}$ into its 'jump' times $\mathbf{T} = (T_i)_i$ and its position between jump times $\mathbf{L} = (L_i)_i$. We say that T is a jump time if one of the coordinates of $\mathbf{S}$ other than the first changes between instants $T - 1$ and T. We can order the jump times of $\mathbf{S}$: $T_0 = 0 < \cdots < T_{dN^a} < T_{dN^a+1} = N$. We can then define L_i as the point $\mathbf{y} \in \mathbb{Z}^d$ such that $\mathbf{S}_{T_i} = (T_i, \mathbf{y})$. We can rewrite the Hamiltonian (8.8) as

$$H(\mathbf{S}) = \sum_{i=0}^{dN^a} \Delta H(\mathbf{S}, i),$$

where

$$\Delta H(\mathbf{S}, i) = \sum_{k=T_i}^{T_{i+1}-1} \eta(k, L_i).$$

Define $g(\mathbf{S})$ and $\Delta g(\mathbf{S}, i)$ just in the same way by replacing the variables η by the Gaussians g. Then,

$$|H(\mathbf{S}) - g(\mathbf{S})| \leq \sum_{i=0}^{dN^a} |\Delta H(\mathbf{S}, i) - \Delta g(\mathbf{S}, i)|.$$

Let θ_N be an increasing function to be determined later and $\Lambda_{N,a} = \{\mathbf{y} \in \mathbb{Z}^d : 0 \leq y_i \leq \lfloor N^a \rfloor\}$:

$$\begin{aligned} & Q\left\{|H(\mathbf{S}) - g(\mathbf{S})| > 2dN^a\theta_N, \text{ for some } \mathbf{S} \in \Omega_{N,a}\right\} \\ \leq\ & Q\left\{\sum_{i=1}^{dN^a} |\Delta H(\mathbf{S}, i) - \Delta g(\mathbf{S}, i)| > 2dN^a\theta_N, \text{ for some } \mathbf{S} \in \Omega_{N,a}\right\} \end{aligned}$$

$$\begin{aligned}
&\leq Q\left\{\max_{i\leq dN^a}|\Delta H(\mathbf{S},i)-\Delta g(\mathbf{S},i)|>2\theta_N,\,\text{for some}\,\mathbf{S}\in\Omega_{N,a}\right\}\\
&\leq Q\left\{\max_{k\leq N}|S(k,\mathbf{x})-g(k,\mathbf{x})|,\,\text{for some}\,\mathbf{x}\in\Lambda_{N,a}\right\}\\
&\leq |\Lambda_{N,a}|Q\left\{\max_{k\leq N}|S_k-T_k|>\theta_N\right\},
\end{aligned}$$

In order to apply Theorem 8.10, we have to take $\theta_N = K_1\log N + \epsilon_N$, and to apply Borel-Cantelli, as $|\Lambda_{N,a}| \leq N^{da}$, it is enough to take $\epsilon_N = c\log N$ with c large enough to make $N^a e^{-K_3\epsilon_N}$ summable. Then, Q-a.s., $|H(\mathbf{S}) - g(\mathbf{S})| \leq CN^a\log N$ for all $\mathbf{S}\in\Omega_{N,a}$, for N large enough. This shows that

$$P_{N,a}\left(e^{\beta_{N,a}H(\mathbf{S})}\right) = P_{N,a}\left(e^{\beta_{N,a}g(\mathbf{S})}\right)O(e^{C\beta_{N,a}N^a\log N}).$$

Recall that $\beta_{N,a} = \beta N^{(a-1)/2}$, so that $\beta_{N,a}N^a\log N = O(N^{(3a-1)/2}\log N)$. As $0 < a < 1$, we have $\beta_{N,a}N^a\log N << \beta_{N,a}N^{(a+1)/2} = \beta N^a$, and then

$$\log Z_{\beta_{N,a}}(N,\alpha N^a) = \log Z^g_{\beta_{N,a}}(N,\alpha N^a) + o(\beta_{N,a}N^{(1+a)/2}),$$

where the superscript g means that the environment is Gaussian.
We can conclude that, if the limit free energy exists Q-a.s. for Gaussian environment variables, it exists for all environment variables having some finite exponential moments, and the limit is the same.

8.3.1.2. *Second Step: approximation by continuous-time polymers in Brownian environment.* Having replaced our original disorder variables by Gaussians, we can take them as unitary increments of independent one-dimensional Brownian motions. We then just have to control their fluctuations to replace the discrete paths by continuous paths in a Brownian environment. This is what will be done in the following paragraphs.

We first need to establish a correspondence between continuous paths an discrete ones.

Take $\mathbf{s}\in\Omega^c_{N,a}$, and recall the definition (8.4) for the Brownian Hamiltonian $\mathbf{Br}(\mathbf{s})$ and that $0 = t_0 < t_1 < \cdots < t_{dN^a+1} = N$ denote the jump times of $\mathbf{s}$. Let $l_i = \mathbf{s}_{t_i}$. The path $\mathbf{s}$ can be discretized by defining the following Gaussian Hamiltonian:

$$H^g(\mathbf{s}) = \sum_{i=0}^{dN^a}\left(B^{(l_i)}_{\lfloor t_{i+1}\rfloor} - B^{(l_i)}_{\lfloor t_i\rfloor-1}\right). \tag{8.11}$$

This is equivalent to consider $g(\mathbf{S})$ where $\mathbf{S}\in\Omega_{N,a}$ is defined through its jump times T_i and successive positions L_i by

$$\begin{aligned} T_i &= \lfloor t_i \rfloor, \\ L_k &= l_i, \quad \forall\, T_i \le k < T_{i+1}, \end{aligned}$$

(Recall that the Gaussian variables obtained in the previous step are now *embedded* in the Brownian motions). In this way,

$$P^c_{N,a}\left(\exp \beta H^{\mathbf{Br}}(\mathbf{s})\right) = P_{N,a}\left(\exp \beta g(\mathbf{S})\right).$$

We have now to approximate the previous expression by $Z^{\mathbf{Br}}_\beta(N, N^a)$. Take $\mathbf{s} \in \Omega^c_{N,a}$:

$$\begin{aligned} |H^g(\mathbf{s}) \;&-\; \mathbf{Br}(\mathbf{s})| = \left| \sum_{i=0}^{dN^a} \left(B^{(l_i)}_{\lfloor t_{r+1} \rfloor} - B^{(l_i)}_{\lfloor t_r \rfloor - 1} \right) - \sum_{i=0}^{dN^a} \left(B^{(l_i)}_{t_{i+1}} - B^{(l_i)}_{t_i} \right) \right| \\ &\le \sum_{i=0}^{dN^a} \left| B^{(l_i)}_{\lfloor t_{r+1} \rfloor} - B^{(r)}_{t_{r+1}} \right| + \sum_{i=0}^{dN^a} \left| B^{(l_i)}_{\lfloor t_i \rfloor} - B^{(l_i)}_{t_i - 1} \right| \\ &\le 2 \sum_{i=0}^{dN^a} \sup_{\substack{0 \le s,\, t \le N+1 \\ |s-t|<2}} |B^{(l_i)}_s - B^{(l_i)}_t|. \end{aligned}$$

This can be handled with basic properties of Brownian motion: denote by x_N an increasing function to be determined,

$$\begin{aligned} & Q\left(\sum_{i=0}^{dN^a} \sup_{\substack{0 \le s,\, t \le N+1 \\ |s-t|<2}} |B^{(l_i)}_s - B^{(l_i)}_t| > dN^a x_N, \text{ for some } \mathbf{s} \in \Omega_{N,a} \right) \\ \le\;& Q\left(\max_{1 \le i \le dN^a} \sup_{\substack{0 \le s,\, t \le N+1 \\ |s-t|<2}} |B^{(l_i)}_s - B^{(l_i)}_t| > x_N \text{ for some } \mathbf{s} \in \Omega_{N,a} \right) \\ \le\;& Q\left(\max_{\mathbf{x} \in \Lambda_{N,a}} \sup_{\substack{0 \le s,\, t \le N+1 \\ |s-t|<2}} |B^{(\mathbf{x})}_s - B^{(\mathbf{x})}_t| > x_N \right) \\ \le\;& |\Lambda_{N,a}| Q\left(\sup_{\substack{0 \le s,\, t \le N+1 \\ |s-t|<2}} |B_s - B_t| > x_N \right) \\ \le\;& CN^{da} \sum_{i=0}^{N-2} Q\left(\sup_{i \le t \le i+3} B_t - \inf_{i \le t \le i+3} B_t > x_N \right) \end{aligned}$$

$$
\begin{aligned}
&\leq \; CN^{da+1}Q\left(\sup_{0\leq t\leq 3}|B_t| > \frac{x_N}{2}\right)\\
&\leq \; CN^{da+1}Q\left(B_3 > \frac{x_N}{2}\right)\\
&\leq \; CN^{da+1}e^{-Cx_N^2}.
\end{aligned}
$$

With $x_N = \log N$ and recalling (8.11) from Step 1, we see that Q-a.s., for N large enough,

$$P_{N,a}\left(e^{\beta_{N,a}H(N,\alpha N^a)}\right) = P^c_{N,a}\left(e^{\beta_{N,a}Br(N,\alpha N^a)}\right) \times O(e^{\beta_{N,a}N^a \log N}).$$

Again, this will imply that

$$\log Z_{\beta_{N,a}}(N,\alpha N^a) = \log Z^{\mathrm{Br}}_{\beta_{N,a}}(N,\alpha N^a) + o(\beta_{N,a}N^{(1+a)/2}), \tag{8.12}$$

8.3.1.3. *Third Step: scaling.* Observe that, for a fixed path $\mathbf{s} \in \Omega_{N,a}$,

$$\mathbf{Br}(N,\alpha N^a)(\mathbf{s}_\cdot) = \sqrt{N}\mathbf{Br}(1,\alpha N^a)(\mathbf{s}_{\cdot/N}) = N^{(1-a)/2}\mathbf{Br}(N^a,\alpha N^a)(\mathbf{s}_{\cdot\times N^{a-1}}),$$

where the equalities hold in law. Note also that $\mathbf{s}_{\cdot\times N^{a-1}} \in \Omega_{N^a,\alpha N^a}$. It follows that

$$
\begin{aligned}
Z^{\mathrm{Br}}_{\beta_{N,a}}(N,\alpha N^a) &= P^c_{N,a}\left(\exp \beta_{N,a}\mathbf{Br}(N,\alpha N^a)(\mathbf{s})\right)\\
&= P^c_{N^a,\alpha N^a}\left(\exp \beta_{N,a}N^{(1-a)/2}\mathbf{Br}(N^a,\alpha N^a)(\mathbf{s}_{\cdot\times N^{a-1}})\right)\\
&= P^c_{N^a,\alpha N^a}\left(\exp \beta\mathbf{Br}(N^a,\alpha N^a)(\mathbf{s}_{\cdot\times N^{a-1}})\right).
\end{aligned}
$$

But the last expression is simply $Z^{\mathbf{Br}}_\beta(N^a,\alpha N^a)$ so that, by Theorem 8.6,

$$\lim_{N\to+\infty}\frac{1}{N^a}\log Z^{\mathrm{Br}}_{\beta_{N,a}}(N,\alpha N^a) = p(\beta,\alpha,d). \tag{8.13}$$

From (8.12) and (8.13), we can deduce that the limit (8.9) holds in law.

8.3.1.4. *Final Step: concentration.* So far, we proved convergence in law for the original problem. But we can write a convenient concentration inequality for the free energy with respect to his average, in the Gaussian case. So, a.s. convergence holds for Gaussian, and, according to step 1, for any environment.

The classical concentration inequality for Gaussian random variables can be stated as follows:

THEOREM 8.11. *Consider the standard normal distribution μ on $\mathbb{R}^K$. If $f : \mathbb{R}^K \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L, then*

$$\mu\left(x : |f(\boldsymbol{x}) - \int f d\mu| \geq u\right) \leq 2\exp\{-\frac{u^2}{2L^2}\}.$$

For a detailed exposition of concentration of measures, see for example, the lecture notes of Ledoux [**68**]. We now proceed exactly in the same way as in the proof of Proposition 1.14 of [**20**]: define

$$F(z) = \frac{1}{N^a} \log P_{N,a} \left(e^{\beta_{N,a} \sum_{t=1}^{N+dN^a} z(\mathbf{S}_t)} \right).$$

It is easy to prove that F is a Lipschitz continuous function with Lipschitz constant $CN^{-a/2}$. By Gaussian concentration, this yields

$$Q\left\{ \left| \frac{1}{N^a} \log Z_{\beta_{N,a}}(N, \alpha N^a) - \frac{1}{N^a} Q \log Z_{\beta_{N,a}}(N, \alpha N^a) \right| > u \right\} \leq 2 \exp - \frac{N^a u^2}{2C^2}.$$

(8.14)

This ends the proof of the theorem. □

8.4. Very asymmetric cases

We now consider an even more asymmetric case: let $\mathbf{a} = (a_1, \cdots, a_d)$ with $0 \leq a_i \leq a$ for all i but $a_i = a$ for exactly $d - l$ values of i, $1 \leq l < d$, and consider paths from the origin to points of type $\alpha N^{\mathbf{a}}$ with coordinates $\alpha_i N^{a_i}$, $\alpha_i > 0$.

THEOREM 8.12. *Let α' be the vector of $\mathbb{R}^{d-l}$ which coordinates are those of α for the indexes i such that $a_i = a$. Then,*

$$\lim_{N \to +\infty} \frac{1}{\beta_{N,a} N^{(1+a)/2}} \log Z_{\beta_{N,a}}(N, \alpha N^{\boldsymbol{a}}) = p(\beta, \alpha', d - l)/\beta.$$

PROOF. The idea of the proof is exactly that same than in the proof of Proposition 8.1. We will consider the simple case $d = 2$ and a final point of type (N^a, N^b) with $b < a$. We then have to prove convergence to $p(\beta, 1, 1)/\beta$. The general case follows easily. We can think as h as $h = h_N = N^{(b-a)}$.
From the proof of Theorem 8.8, we have to remember that

$$\log \overline{Z}_{\beta_{N,a}}(N, \alpha N^a) = \log Z_\beta^{\mathbf{Br}}(N^a, \alpha N^a) + o(N^a)$$

Denote by $Z(N, M, L)$ (resp. $\overline{Z}(N, M, L)$) the normalized (resp. non-normalized) partition function over discrete paths from the origin to (N, M, L). We perform the same decomposition than before:

$$\begin{aligned} Z_{\beta_{N,a}}(N, N^a, N^b) &= \frac{\overline{Z}_{\beta_{N,a}}(N, N^a, N^b)}{\overline{Z}_0(N, N^a, N^b)} \\ &= \frac{1}{\overline{Z}_0(N, N^a, N^b)} \sum_{(\mathbf{m}_i)_i} \Pi_i \overline{Z}(i; \mathbf{m}_i, \mathbf{m}_{i+1}) \end{aligned}$$

Here, $0 \leq i \leq N^b - 1$ and $\mathbf{m}_{N^b} = N^a$. Recalling Remark 8.2, the cardinality of the set J_N of the possible configurations of $(\mathbf{m}_i)$ satisfies $|J_N| \sim \exp\{cN^{(a+b)/2} \log N\}$. For a fixed $\mathbf{m}_i$, recalling that the environment variables are centered,

$$
\begin{aligned}
Q\left(\log \Pi_i \overline{Z}_{\beta_{N,a}}(i;\mathbf{m}_i,\mathbf{m}_{i+1})\right) &= Q\left(\log \Pi_i \overline{Z}_{\beta_{N,a}}(0;\mathbf{m}_i,\mathbf{m}_{i+1})\right) \\
&= \log \overline{Z}_0(N,N^a,0) + Q\left(\log \frac{\Pi_i \overline{Z}_{\beta_{N,a}}(0;\mathbf{m}_i,\mathbf{m}_{i+1})}{\overline{Z}_0(N,N^a,0)}\right) \\
&\leq \log \overline{Z}_0(N,N^a,0) + Q\left(\log Z_{\beta_{N,a}}(N,N^a,0)\right) \\
&\leq \log \overline{Z}_0(N,N^a,0) + Q\left(\log Z^{\mathbf{Br}}_{\beta}(N^a,N^a)\right) + o(N^a) \\
&\leq \log \overline{Z}_0(N,N^a,0) + N^a\, p(\beta,1,1) + o(N^a).
\end{aligned}
$$

Now, if

$$
\log Z_{\beta_{N,a}}(N,N^a,N^b) > N^a\left(p(\beta,1,1)+\epsilon\right),
$$

there must exist some $(\mathbf{m}_i)_i$ such that

$$
\log \Pi_i \overline{Z}_{\beta_{N,a}}(i;\mathbf{m}_i,\mathbf{m}_{i+1}) > \log \overline{Z}_0(N,N^a,N^b) + N^a\left(p(\beta,1,1)+\epsilon\right) - \log |J_N|.
$$

Using the fact that $\overline{Z}_0(N,N^a,N^b) > \overline{Z}_0(N,N^a,0)$, (8.14) and the union bound, we find that

$$
Q\left(Z_{\beta_{N,a}}(N,N^a,N^b) > N^a\left(p(\beta,1,1)+\epsilon\right)\right) \leq |J_N| \exp\{-c\epsilon^2 N^a\},
$$

for ϵ small enough. As $\log |J_N| = o(N^a)$, the RHS of the last display is summable. The result follows by Borel-Cantelli. ☐

CHAPTER 9

One-dimensional directed polymers with a huge drift

We now turn to the problem of computing the free energy of a directed polymers model with a drift that grows with N. Let

$$Z^{(h)}_{\beta,N} = \sum_{1\le n\le N} \overline{Z}_\beta(n, N-n)e^{-h\times(N-n)}, \tag{9.1}$$

where, for each n, $\overline{Z}_N(n, N-n)$ is the (non-normalized) point-to-point partition function

$$\overline{Z}_\beta(n, N-n) = \sum_{\omega\in\Omega_{n,N-n}} e^{\beta H_N(\omega)}. \tag{9.2}$$

This can also be seen as a generating function or a Poissonization of the point-to-point partition function. Recall that, when $N-n = O(N^a)$, Theorem 5.1 implies that

$$\lim_{N\to+\infty} \frac{1}{\sqrt{n(N-n)}} \log \overline{Z}_\beta(n, N-n) = 2\beta,$$

as, in this regime, $\log|\Omega_{n,N-n}|$ is of much smaller order than $\sqrt{N(N-n)}$ (see also (5.6)). The role of the drift h in (9.1) is to penalize the paths for which the final point is far from the horizontal axis. It has to be calibrated in order to favor final points such that $N-n = O(N^a)$.

In the first section, we will compute the free energy. We will study the fluctuations of the partition function in the second section.

9.1. The free energy

THEOREM 9.1. *Take $h = h_N = \gamma N^{(1-a)/2}$. Then,*

$$\lim_{N\to+\infty} \frac{1}{N^{(1+a)/2}} \log Z^{(h_N)}_{\beta,N} = \frac{\beta^2}{\gamma},$$

for all environment laws such that $Q(e^{\beta\eta}) < +\infty$ for all $\beta > 0$.

Let us first sketch the proof:

SKETCH OF PROOF:
We parametrize the terminal points conveniently:

$$N = n(1+u).$$

Thus $n = N/(1+u)$ and $N - n = Nu/(1+u)$. We can then rewrite (9.1) as

$$Z_N^{(h)} = \sum_u \overline{Z}_\beta \left(\frac{N}{1+u}, \frac{Nu}{1+u} \right) \exp \left\{ -\gamma N^{(1-a)/2} \times \frac{Nu}{(1+u)} \right\}. \tag{9.3}$$

Now, for u in an interval $I_N = [N^{\kappa_0}, N^{\kappa_1}]$, we will have

$$\overline{Z}_\beta \left(\frac{N}{1+u}, \frac{Nu}{1+u} \right) = \exp \left\{ 2\beta \frac{N\sqrt{u}}{1+u} + o(1), \right\} \tag{9.4}$$

uniformly in u. Then,

$$Z_{\beta,N}^{(h)} \sim \sum_{u \in I_N} \exp N \left\{ 2\beta \frac{\sqrt{u}}{1+u} - \gamma N^{(1-a)/2} \times \frac{u}{(1+u)} \right\}. \tag{9.5}$$

Define the function

$$f_N(u) = 2\beta \frac{\sqrt{u}}{1+u} - \gamma N^{(1-a)/2} \times \frac{u}{(1+u)}.$$

It attains its global maximum at a point $u_N^* \sim (\beta^2/\gamma^2) N^{a-1}$ (in short, we will omit the dependence in N), with $f_N(u^*) \sim \beta^2 N^{(a-1)/2}/\gamma$. So, by Laplace method, we will have

$$Z_{\beta,N}^{(h)} = \exp \{ N f(u_N^*) + o(1) \} = \exp \left\{ N^{(1+a)/2} \beta^2/\gamma + o(1) \right\},$$

which would finish the proof.

REMARK 9.2. The proof is split in three steps. The first one gives the lower bound in the Theorem, minoring the whole sum by one term, given by a u very close to the minimizer. This is the easy part.
The second step will consist mainly in proving the uniformity in (9.4) (but replacing $=$ by $\leq$). This will be done by applying uniformly the KMT approximation in the whole interval I_N, and then applying some deviation inequality for the Brownian percolation.
The third step will be to prove that the u's outside I_N do not contribute to the sum.

PROOF OF THEOREM 9.1: FIRST STEP: We will now provide the lower bound: recall the notation in (9.3) and observe that for the value u^*, the asymptotics of n and $N - n$ fit the situation studied in 5.6. An easy computation yields:

$$\begin{aligned}
& \liminf_{N \to +\infty} \frac{1}{N^{(1+a)/2}} \log Z_{\beta,N}^{(h)} \\
\geq \; & \lim_{N \to +\infty} \frac{1}{N^{(1+a)/2}} \log \overline{Z}_\beta \left(\frac{N}{1+u^*}, \frac{Nu^*}{1+u^*} \right) \exp \left\{ -\gamma N^{(1-a)/2} \times \frac{Nu^*}{(1+u^*)} \right\} \\
= \; & \frac{\beta^2}{\gamma}.
\end{aligned}$$

Second step: Let $\epsilon > 0$ and take $\kappa_1 = (a-1)/2 - \epsilon$ in order to define $I_N = [N^{\kappa_0}, N^{\kappa_1}]$. Here, $\kappa_0 > -1$ is introduced to discard small values of u that have to be treated separately. Note that, in this interval, $N - n \sim Nu \le N^{1+\kappa_1} o(N^{(a+1)/2})$.

We first couple the environment variables $\{\eta(t,x) : 1 \le t \le N,\ 1 \le x \le N^{\kappa_1}\}$ row by row with Brownian motions as in the proof of Theorem 5.2. This yields

$$\overline{Z}_\beta\left(\frac{N}{1+u}, \frac{Nu}{1+u}\right) = \overline{Z}_\beta^{Br}\left(\frac{N}{1+u}, \frac{Nu}{1+u}\right) \times O(e^{N^{1+\kappa_1}\log N}),$$

uniformly for $u \in I_N$, where $\overline{Z}_\beta^{\mathbf{Br}}(N,M)$ denotes the unnormalized partition function of the Brownian model. Again, (5.6) holds for $\overline{Z}_\beta^{\mathbf{Br}}(N,M)$ with $M = O(N^a)$, as $|\Omega^c_{N,M}|$ is small compared to $\sqrt{MN}$:

$$\log|\Omega^c_{N,M}| \sim \log\frac{N^M}{(M)!} = O(N^a \log N). \tag{9.6}$$

We now search for a convenient upper bound for the (normalized) Brownian partition function:

$$\begin{aligned} & Q\left\{Z_N^{\mathbf{Br}}\left(\frac{N}{1+u}, \frac{Nu}{1+u}\right) > \exp\beta\frac{N\sqrt{u}}{1+u}(2+\epsilon_N)\right\} \\ \le\ & Q\left\{\max_\omega \mathbf{Br}\left(\frac{N}{1+u}, \frac{Nu}{1+u}\right) > \frac{N\sqrt{u}}{1+u}(2+\epsilon_N)\right\} \\ \le\ & Q\left\{\max_\omega \mathbf{Br}\left(1, \frac{Nu}{1+u}\right) > \sqrt{\frac{Nu}{1+u}}(2+\epsilon_N)\right\} \\ \le\ & C\exp\left\{-\frac{1}{C}\frac{N\sqrt{u}}{1+u}\epsilon_N^{3/2}\right\}. \end{aligned}$$

The last inequality follows from Ledoux [**69**], Section 2.1. Taking $\epsilon_N = N^{-\theta}$ with $\theta > 0$ small enough, and applying Borel-Cantelli, we conclude that, for N large enough,

$$Z_N^{\mathbf{Br}}\left(\frac{N}{1+u}, \frac{Nu}{1+u}\right) \le \exp\left\{2\beta\frac{N\sqrt{u}}{1+u} + o(1)\right\},$$

for all $u \in I_N$. Now, thanks to (9.6), this is still true with $\overline{Z}^{\mathbf{Br}}$ instead of $Z^{\mathbf{Br}}$. We then get

$$\overline{Z}_N\left(\frac{N}{1+u}, \frac{Nu}{1+u}\right) \le \exp\left\{2\beta\frac{N\sqrt{u}}{1+u} + o(1)\right\},$$

uniformly for $u \in I_N$. Once the Third Step is achieved, this uniform bound and Laplace Method will finish the proof.

THIRD STEP: We are now interested in values $u \leq N^{\kappa_0}$ and $u \geq N^{\kappa_1}$. Again we have to split the proof in three.

Let us first focus on small values of u. Recall that, in this region, by the KMT coupling, we can work directly with Gaussians. Take $\theta' > 0$.

$$\begin{aligned}
Q\left\{\overline{Z}^g\left(\frac{N}{1+u},\frac{Nu}{1+u}\right) > e^{\beta N^{\theta'}}\right\} &\leq Q\left\{T^g\left(\frac{N}{1+u},\frac{Nu}{1+u}\right) > N^{\theta'}\right\} \\
&\leq Q\left\{\exists \mathbf{s} \in \Omega_{\frac{N}{1+u},\frac{Nu}{1+u}} : H(\mathbf{s}) > N^{\theta'}\right\} \\
&\leq |\Omega_{\frac{N}{1+u},\frac{Nu}{1+u}}| \exp\{-N^{2\theta'-1}\} \\
&\leq \exp\{cN^{(1+\kappa_0)\log N} - N^{2\theta'-1}\}.
\end{aligned}$$

So, choosing κ_0 small enough and $1+\kappa_0/2 < \theta' < (1+a)/2$, we get, by Borel-Cantelli and by a computation analogous to (9.6), that for N large enough,

$$\overline{Z}^g\left(\frac{N}{1+u},\frac{Nu}{1+u}\right) = o\left(e^{N^{(1+a)/2}}\right),$$

for all $u \leq N^{\kappa_0}$.

For $N^{(a-1)/2-\epsilon} \leq u \leq N^{(a-1)/2+\epsilon}$, we have to couple the environment row by row with Gaussians until $N-n = N^{(1+a)/2+\epsilon}$ (just conserve the coupling already done in Step 2 and add the missing rows). This will yield an error uniformly of order $N^{(1+a)/2+\epsilon}\log N$. The point is that for ϵ small enough, the drift will be large compared with the point-to-point partition functions and the error in the approximation. In fact,

$$h \times (N-n) \geq \gamma N^{1-\epsilon}.$$

Recall that we are working with Gaussians, denote $\Omega_{N,u} = \Omega_{\frac{N}{1+u},\frac{Nu}{1+u}}$,

$$Q\left\{\max_{\omega\in\Omega_{N,u}} H_N(\omega) > N^{\theta'}\right\} \leq \exp(1+a)N^{(1+a)/2+\epsilon}\log N - N^{2\theta'-1},$$

and, by Borel-Cantelli (taking, of course, $(1+a)/2+\epsilon < 2\theta'-1$),

$$\overline{Z}^g_\beta\left(\frac{N}{1+u},\frac{Nu}{1+u}\right)e^{h\times(N-n)} \leq \exp\left\{(1+a)N^{(1+a)/2+\epsilon}\log N + \beta N^{\theta'} - \gamma N^{1-\epsilon}\right\},$$

where, as usual, the overline denotes that the partition function is unnormalized and the superscript g stands for Gaussian environment. To insure that the drift is larger than the other terms, we have to take $\theta' < 1-\epsilon$ and $(1+a)/2+\epsilon < 1-\epsilon$, both holding for $\epsilon < (1-a)/4$ and θ small enough. Now, this is also enough to neglect the error in the approximation as it is of order $N^{(1+a)/2+\epsilon}$ too. The first condition we have encountered, namely $(1+a)/2+\epsilon < 2\theta'-1$ is satisfied for $\epsilon < (1-a)/6$ and $\theta' < 1-\epsilon$, so that, choosing ϵ and θ according to these last restrictions gives that

$$\overline{Z}_\beta\left(\frac{N}{1+u},\frac{Nu}{1+u}\right)e^{-h\times(N-n)}\to 0, \tag{9.7}$$

as $N\to+\infty$ uniformly for $N^{(1-a)/2-\epsilon}\le u\le N^{(1-a)/2+\epsilon}$.

We are then left to the values $u>N^{(a-1)/2+\epsilon}$. This is an easy task: we can dominate each point-to-point partition function by the whole partition function (without drift!):

$$Z_N=Z_{\beta,N}=\sum_{\omega\in\Omega_N}e^{\beta H(\omega)},$$

where Ω_N is the set of directed nearest-neighbor paths of length N. Z_N grows at most as e^{CN} for some constant $C>\lambda(\beta)+\log 2d$, as we can see from

$$Q(Z_N\ge e^C N)\le e^{-CN}QZ_N=e^{(\lambda(\beta)+\log 2d-C)N}$$

and Borel-Cantelli. Now, for the range of u's we are considering, the drift satisfies,

$$h(N-n)>N^{1+\epsilon'},$$

for large N, whenever $\epsilon'<\epsilon$, and then (9.7) holds in this interval as well.

□

9.2. Fluctuations of the partition function

We now discuss the fluctuation of $\log Z_{\beta,N}^{(h_N)}$. For technical reasons, we have to restrict to $a<1/5$ for variable with finite exponential moments, and to $a<1/2$ for Gaussian variables (see Remark 9.7 at the end of this section).

9.2.1. Moderate deviations. We start proving the two following deviation inequalities:

PROPOSITION 9.3. *For all $a<1/5$ ($a<1/2$ for Gaussian variables), there exists a constant $C>0$ such that, for all $N\ge 1$ and $\epsilon\ge 0$,*

$$Q\left\{\log Z_{\beta,N}^{(h_N)}\ge\frac{\beta^2}{\gamma}N^{(1+a)/2}(1+\epsilon)\right\}\le C\exp\left\{-\frac{N^a}{C}\epsilon^{3/2}\right\}, \tag{9.8}$$

and for $0\le\epsilon\le 1$,

$$Q\left\{\log Z_{\beta,N}^{(h_N)}\le\frac{\beta^2}{\gamma}N^{(1+a)/2}(1-\epsilon)\right\}\le C\exp\left\{-\frac{N^{2a}}{C}\epsilon^{3}\right\}. \tag{9.9}$$

These are consequences of similar non-asymptotics deviation inequalities for the top eigenvalue of GUE random matrices that we recall here in the context of Brownian percolation (see [**70**], Theorem 1 and [**69**], Chapter 2, for a complete discussion of this topic):

PROPOSITION 9.4. *There exists a constant $C > 0$ such that, for all $N \geq 1$ and $\epsilon \geq 0$,*

$$Q\left\{L(1,N) \geq 2\sqrt{N}(1+\epsilon)\right\} \leq C\exp\left\{-\frac{N}{C}\epsilon^{3/2}\right\}, \tag{9.10}$$

and for $0 \leq \epsilon \leq 1$,

$$Q\left\{L(1,N) \leq 2\sqrt{N}(1-\epsilon)\right\} \leq C\exp\left\{-\frac{N^2}{C}\epsilon^{3}\right\}. \tag{9.11}$$

A simple application of the KMT approximation leads to the following lemma:

LEMMA 9.5. *For $M = O(N^a)$ with $a < 1/5$ ($a < 1/2$ if the environment is Gaussian), there exists a constant $C > 0$ such that, for all $\epsilon \geq 0$,*

$$Q\left\{T(N,M) \geq 2\sqrt{NM}(1+\epsilon)\right\} \leq C\exp\{-M\epsilon^{3/2}/C\}, \tag{9.12}$$

and for $0 \leq \epsilon \leq 1$,

$$Q\left\{T(N,M) \leq 2\sqrt{NM}(1-\epsilon)\right\} \leq C\exp\{-M^2\epsilon^{3}/C\}. \tag{9.13}$$

PROOF. Let us prove the inequality for small deviations on the left of the mean:

$$\begin{aligned} Q\left\{T(N,M) \leq 2\sqrt{NM}(1-\epsilon)\right\} &\leq Q\left\{L(N,M) \leq 2\sqrt{NM}(1-\epsilon/2)\right\} \\ &+ \quad Q\left\{|T(N,M) - L(N,M)| > \frac{\epsilon}{2}\sqrt{NM}\right\}. \end{aligned}$$

The first summand decreases as $C\exp\{-M^2\epsilon^3/C\}$ thanks to (9.10). By the same analysis we performed in the two first step of the proof of Theorem 8.8, we see that the second summand decreases as $C\exp\{-C\sqrt{N/M}\}$, which is of much smaller order whenever $a < 1/5$. When the environment is Gaussian, the first step of the proof of Theorem 8.8 is useless and an inspection of the second step of that proof shows that the second summand in the preceding display decreases as $C\exp\{-CN^{1-a}\}$.

The other inequality can be proved following the same lines. □

We turn now to the proof of Proposition 9.3.

PROOF OF THE INEQUALITY 9.9: This follows by lowering the partition function by one term: recall that $u^* \sim \beta^2/\gamma N^{a-1}$, and define $n^* = N/(1+u^*)$. Then,

$$\begin{aligned} & Q\left\{\log Z_{\beta,N}^{(h_N)} \leq \frac{\beta^2}{\gamma}N^{(1+a)/2}(1-\epsilon)\right\} \\ \leq \quad & Q\left\{\beta T(n^*, N-n^*) - h_N \times (N-n^*) \leq \frac{\beta^2}{\gamma}N^{(1+a)/2}(1-\epsilon)\right\}, \end{aligned}$$

Observe that

$$h_N \times (N - n^*) = \frac{\beta^2}{\gamma} N^{(1+a)/2}.$$

We are then reduced to estimate the quantity

$$Q\left\{T(n^*, N - n^*) \leq \frac{2\beta}{\gamma} N^{(1+a)/2}(1 - \epsilon/2)\right\}. \tag{9.14}$$

which can be handled with (9.13). □

Proof of the inequality 9.8: This proof is more involved as it requires to control all the terms in the sum defining $Z_{\beta,N}^{(h_N)}$. As the result is non-asymptotics, we do not need to give a special treatment to the terms for which $N - n$ is not of the relevant order (namely $O(N^a)$). We use the convenient parametrization $N - n = vN^a$ for some $v \geq 0$. To lighten notation, let us denote

$$q(\epsilon, v) = Q\left\{\beta T(N, vN^a) - \gamma v N^{(1+a)/2} \geq \frac{\beta^2}{\gamma} N^{(1+a)/2}(1 + \epsilon)\right\}.$$

Several cases have to be analyzed separately:

Case $v \leq \beta^2/(2\gamma)^2$: We use the fact that, for these values of v, $T(N, vN^a)$ is stochastically dominated by $T(N, \beta^2/(2\gamma)^2 N^a)$. Then, neglecting the term γv,

$$\begin{aligned} q(v, \epsilon) &\leq Q\left\{T(N, \beta^2/(2\gamma)^2 N^a) \geq \frac{\beta}{\gamma} N^{(1+a)/2}(1 + \epsilon)\right\} \\ &\leq C \exp\left\{-\frac{\beta^2 N^a}{4\gamma^2 C}\epsilon^{3/2}\right\}. \end{aligned}$$

Case $\beta^2/(2\gamma)^2 \leq v \leq 4\beta^2/\gamma^2$: We make use of the fact that $2\beta\sqrt{v} - \gamma v \leq \beta^2/\gamma$ for all $\beta \geq 0$ and that, for these values of v, we have $1/\sqrt{v} \geq \gamma/(2\beta)$. Then,

$$\begin{aligned} q(\epsilon, v) &\leq Q\left\{\beta T(N, vN^a) \geq 2\beta\sqrt{v} N^{(1+a)/2} + \frac{\beta^2 \epsilon}{\gamma} N^{(1+a)/2}\right\} \\ &\leq Q\left\{\beta T(N, vN^a) \geq 2\beta\sqrt{v} N^{(1+a)/2}\left(1 + \frac{\beta\epsilon}{2\gamma\sqrt{v}}\right)\right\} \\ &\leq Q\left\{\beta T(N, vN^a) \geq 2\beta\sqrt{v} N^{(1+a)/2}\left(1 + \frac{\epsilon}{4}\right)\right\} \\ &\leq C \exp\left\{-\sqrt{v}\frac{N^a}{C}\epsilon^{3/2}\right\} \\ &\leq C \exp\left\{-\frac{\beta^2 N^a}{\gamma^2 C}\epsilon^{3/2}\right\}, \end{aligned}$$

thanks to the lower bound we assumed on v.

CASE $v \geq 4\beta^2/\gamma^2$: Let $K > 4$ be such that $K\beta^2/\gamma^2 < v \leq (K+1)\beta^2/\gamma^2$. Then, recalling that $T(N, v_1 N^a)$ dominates $T(N, v_2 N^a)$ stochastically whenever $v_1 \geq v_2$, and using the simple fact that $K+1 > 2\sqrt{K+1}$,

$$\begin{aligned}
q(\epsilon, v) &\leq Q\left\{\beta T(N, vN^a) \geq \frac{\beta^2}{\gamma} N^{(1+a)/2}(1+\epsilon) + K\frac{\beta^2}{\gamma} N^{(1+a)/2}\right\} \\
&= Q\left\{T(N, vN^a) \geq \frac{\beta}{\gamma} N^{(1+a)/2}(K+1+\epsilon)\right\} \\
&\leq Q\left\{T(N, vN^a) \geq 2\frac{\sqrt{K+1}\beta}{\gamma}\left(1 + \frac{\epsilon}{2\sqrt{K+1}}\right) N^{(1+a)/2}\right\} \\
&\leq Q\left\{T\left(N, (K+1)\beta^2/\gamma^2 N^a\right) \geq 2\frac{\sqrt{K+1}\beta}{\gamma}\left(1 + \frac{\epsilon}{2\sqrt{K+1}}\right) N^{(1+a)/2}\right\} \\
&\leq C\exp\left\{-\frac{(K+1)\beta^2 N^a}{C\gamma^2}\left(\frac{\epsilon}{2\sqrt{K+1}}\right)^{3/2}\right\} \\
&\leq C\exp\left\{-\frac{1}{2}\frac{\beta^2 N^a}{\gamma^2 C}\epsilon^{3/2}\right\},
\end{aligned}$$

□

9.2.2. Fluctuation bounds. These bounds have a certain flavor of variance bounds without being exactly such.

THEOREM 9.6. *For all $a < 1/5$ ($a < 1/2$ for a Gaussian environment), there exists a constant $C > 0$ such that, for all $N \geq 1$,*

$$\frac{1}{C} N^{1-a/3} \leq Q\left\{\left(\log Z_{\beta,N}^{h_N} - \frac{\beta^2}{\gamma} N^{(1+a)/2}\right)^2\right\} \leq C N^{1-a/3}.$$

PROOF. To lighten notations, let us denote

$$X_N = \log Z_{\beta,N}^{(h_N)}, \qquad x_N = \frac{\beta^2}{\gamma} N^{(1+a)/2}.$$

The upper bound follows from the previous deviation inequalities by a direct computation:

$$
\begin{aligned}
Q(X_N - x_N)^2 &= \int_0^{+\infty} Q\left\{(X_N - x_N)^2 \geq t\right\} dt \\
&\leq \int_0^{+\infty} Q\left\{X_N - x_N \geq \sqrt{t}\right\} dt + \int_0^{x_N} Q\left\{X_N - x_N \leq -\sqrt{t}\right\} dt \\
&= 2N^{1+a} \int_0^{+\infty} u\, Q\left\{X_N \geq \frac{\beta^2}{\gamma}(1+u)\right\} du \\
&+ \quad 2N^{1+a} \int_0^{\beta^2/\gamma} u\, Q\left\{X_N \leq \frac{\beta^2}{\gamma}(1-u)\right\} du
\end{aligned}
$$

Let us bound the first integral. The second one can be treated in the same way. We apply (9.8) from Proposition 9.3 and split the interval of integration:

$$
\begin{aligned}
\int_0^{+\infty} uQ\left\{X_N \geq \frac{\beta^2}{\gamma}(1+u)\right\} du &\leq C \int_0^{+\infty} ue^{-\frac{N^a}{C}u^{3/2}} du \\
&= C \int_0^1 ue^{-\frac{N^a}{C}\epsilon^{3/2}} du \,+\, C \int_1^{+\infty} ue^{-\frac{N^a}{C}\epsilon^{3/2}} du.
\end{aligned}
\tag{9.15}
$$

The second integral in this last display is easily seen to decrease as $\exp\{-N^a\}$. For the first integral, observe that the integrand can be bounded by $CN^{-2a/3}$ in $[0, N^{-2a/3}]$ and decreases exponentially fast outside this interval. Then,

$$
\int_0^1 ue^{-\frac{N^a}{C}u^{3/2}} du \leq CN^{-4a/3}.
$$

Putting this back into (9.15), we found

$$
\int_0^{+\infty} uQ\left\{X_N \geq \frac{\beta^2}{\gamma}(1+u)\right\} du \leq Ce^{N^{1-a/3}}.
$$

As we already mentioned, the deviations on the left of the mean can be treated similarly. This gives the upper bound. For the lower bound, observe that

$$
\beta T(n^*, N-n^*) - h_N \times (N-n^*) \sim \beta T(N, \frac{\beta^2}{\gamma^2}N^a) - \frac{2\beta^2}{\gamma}N^{(1+a)/2}.
$$

Then, applying Jensen's inequality,

$$
\begin{aligned}
Q\left\{\left(\log Z_{\beta,N}^{(h_N)} - \frac{\beta^2}{\gamma}N^{(1+a)/2}\right)^2\right\} &\geq \left(Q\left\{\log Z_{\beta,N}^{(h_N)} - \frac{\beta^2}{\gamma}N^{(1+a)/2}\right\}\right)^2 \\
&\geq \left(Q\left\{\beta T(N, \beta^2/\gamma^2 N^a) - 2\beta^2/\gamma N^{(1+a)/2}\right\}\right)^2
\end{aligned}
$$

Now, recall [**11**] that

$$\frac{T(N,\beta^2/\gamma^2 N^a) - 2\beta^2/\gamma N^{(1+a)/2}}{N^{(\frac{1}{2}-\frac{a}{6})}}$$

converges in law to a Tracy-Widom. Then, recalling that the Tracy-Widom law has a strictly positive expected value,

$$\left(Q\left\{\beta T(N,\beta^2/\gamma^2 N^a) - 2\beta^2/\gamma N^{(1+a)/2}\right\}\right)^2 \geq cN^{1-a/3},$$

for some $c > 0$. This ends the proof. □

REMARK 9.7. Again, the condition $a < 1/5$ seems to be a technical limitation due to our use of the KMT approximation. For a more extensive discussion on asymptotics and non-asymptotics small deviations for asymmetric last-passage percolation, see [**51**].

REMARK 9.8. The limit law of the properly centered and rescaled partition function should be the GUE Tracy-Widom law from random matrix theory. A proof of this fact would need to refine the analysis performed in the first section of this chapter to reduce the relevant values of u's to an interval $[cN^{1-a} - \epsilon_N, cN^{1-a} - \epsilon_N]$ with $c = \beta^2/\gamma^2$ and $\epsilon_N \to 0$ fast enough. This can be done without much effort, but, in order to identify the limit law as the Tracy-Widom, we also need a joint control of expressions of the form $L(N - cN^a - sN^{2a/3}, cN^a + sN^{2a/3})$ for s ranging over a large interval. The result we are searching for can be expressed as follows: for $s \in \mathbb{R}$

$$N^{1/6}\left\{L(N - sN^{2/3}, N + sN^{2/3}) - 2N\right\} \to \mathrm{Ai}(s) - s^2$$

where $\mathrm{Ai}(\cdot)$ is a continuous version the Airy process. This is a stationary process which marginals are the Tracy-Widom law. See [**55**] for a related result and a precise description of the Airy process.

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