Université
Pierre et Marie
Curie

École Doctorale Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

François Charles

Cycles algébriques et cohomologie de certaines variétés projectives complexes

---

dirigée par Claire Voisin

Soutenue le 6 avril 2010 devant le jury composé de :

M. Yves André	École Normale Supérieure	Examinateur
M. Bruno Kahn	Université Paris VI	Examinateur
M. Chris Peters	Université Grenoble I	Rapporteur
M. Carlos Simpson	Université Nice-Sophia Antipolis	Examinateur
M. Christophe Soulé	IHÉS	Examinateur
Mme Claire Voisin	Université Paris VI	Directrice de thèse

Institut de Mathématiques de Jussieu
175, rue du chevaleret
75 013 Paris
École doctorale Paris centre Case 188
4 place Jussieu
75 252 Paris cedex 05

Remerciements

Je tiens d'abord à remercier chaleureusement ma directrice de thèse Claire Voisin. Avoir été son élève est un honneur. Sa générosité constante, la richesse de ses intuitions et sa grande disponibilité ont rendu ces années très enrichissantes. S'il est sans doute bien évident à qui la lit que cette thèse lui doit beaucoup, je suis loin d'avoir encore tiré parti de toutes les mathématiques qu'elle m'a transmises et sa façon de travailler restera pour moi un modèle.

Je remercie Chris Peters et Burt Totaro d'avoir accepté de relire cette thèse. Leurs commentaires et leurs remarques m'ont été précieux, et je suis reconnaissant à Burt Totaro de m'avoir invité à donner un exposé à Cambridge. C'est en outre un grand honneur pour moi d'avoir Yves André, Bruno Kahn, Carlos Simpson et Christophe Soulé dans mon jury, et c'est avec plaisir que je les remercie.

Il est impossible de mentionner dans ces remerciements tous les mathématiciens avec qui j'ai eu la chance de discuter dans ces années de thèse. Je remercie néanmoins Yves André, Jean-Louis Colliot-Thélène, Brian Conrad, Daniel Huybrechts, Bruno Kahn, Ania Otwinowska et Christian Schnell pour des discussions enrichissantes. Au début de ma thèse, j'ai eu l'occasion de passer quelques mois à l'université de Columbia, et je remercie Johan de Jong pour ce séjour très profitable scientifiquement. Je remercie Wayne Raskind pour son invitation à ASU l'an dernier et pour nos nombreuses discussions.

Je voudrais accorder une place particulière à mes professeurs. C'est sans doute à Jean-Pierre Sanchez que je dois mes premiers enthousiasmes mathématiques, et mon goût pour la recherche lui doit beaucoup. Yves Laszlo m'a appris la géométrie algébrique et m'a guidé lors de mon premier exposé. À l'ÉNS, les conseils de François Loeser et Frédéric Paulin m'ont été précieux, et je remercie toute l'équipe du DMA, notamment Zaïna et Laurence.

Je dois ici remercier tous les amis avec qui j'ai partagé mathématiques et bon moments, Gabriel, Rémy, Charles, Julien, Thomas, Mathieu, Mathieu, Cyril, Gilles, Simon et bien d'autres. Je remercie Pierre pour huit ans de mathématiques et d'amitié, Lena à qui cette thèse doit beaucoup, Benjamin pour son enthousiasme communicatif et Olivier pour sa bonne humeur et son savoir encyclopédique. Je remercie Gabrielle pour son aide lors de la préparation de la soutenance.

J'exprime toute ma gratitude à mes parents et à ma soeur pour leur soutien constant.

Table des matières

Introduction

Dans cette thèse, nous proposons plusieurs contributions à l'étude de la cohomologie des variétés projectives complexes ainsi qu'à la construction de cycles algébriques. Le mémoire se compose de trois chapitres qui, si ils sont indépendants, essaient tous de tirer parti de la nature multiple de ces variétés, à la fois variétés kähleriennes, donc objets analytiques, variétés algébriques, et enfin objets arithmétiques, étant toujours définies sur un corps de type fini sur $\mathbb{Q}$.

La première partie de ce texte, parue au journal de Crelle, [21], s'intéresse au problème de la topologie des variétés conjuguées. On y répond à une question de Grothendieck en exhibant deux variétés conjuguées dont les algèbres de cohomologie réelles ne sont pas isomorphes. Dans une deuxième partie, on aborde le problème de la construction des cycles algébriques dont l'existence est prévue par les conjectures standards, pour ensuite examiner de manière plus détaillée le cas des variétés hyperkähleriennes. Nous utilisons principalement des méthodes infinitésimales en théorie de Hodge. Enfin, dans la troisième partie, à paraitre aux International Mathematical Research Notices, [22], on s'intéresse au problème du lieu de définition des fonctions normales associées aux familles de cycles dans les variétés projectives complexes. On y prolonge des résultats récents de Brosnan et Pearlstein qui démontrent l'algébricité de ce lieu en prouvant des théorèmes de comparaison avec la cohomologie étale l-adique et en démontrant, sous certaines hypothèses de monodromie, que ces lieux sont définis sur un corps de nombres.

Les résultats que nous obtenons s'inscrivent naturellement dans le cadre de la théorie des motifs imaginée par Grothendieck dès les années 1960. Dans cette introduction, on essaie de rappeler le contexte motivique dans lequel se place ce mémoire, et l'on décrit plus en détail le contenu des parties suivantes.

0.1 Cohomologies de Weil

0.1.1 Théories cohomologiques pour les variétés projectives

L'utilisation de méthodes topologiques dans l'étude des variétés algébriques complexes remonte au moins à Riemann, puis à Poincaré et Picard. Dans les années 1920, Lefschetz met en évidence la pertinence de l'outil cohomologique et prouve plusieurs résultats en ce sens.

Soit X une variété projective lisse sur le corps des nombres complexes. À X sont associées plusieurs théories cohomologiques à valeurs dans des espaces vectoriels sur des corps différents. On peut les décrire comme suit.

L'ensemble des points complexes de X, noté $X(\mathbb{C})$, est muni d'une structure canonique de variété différentielle compacte. À cette dernière est associée une algèbre de cohomologie singulière $H^*(X(\mathbb{C}), \mathbb{Q})$, à valeurs dans $\mathbb{Q}$. Les groupes de cohomologie $H^i(X(\mathbb{C}), \mathbb{Q})$ sont donnés par la cohomologie du complexe des cochaînes singulières. La *cohomologie de Betti* de la variété projective complexe X est définie par

$$H_B^*(X) := H^*(X(\mathbb{C}), \mathbb{Q}).$$

C'est une $\mathbb{Q}$-algèbre graduée.

La variété différentielle $X(\mathbb{C})$ est munie de manière naturelle d'une structure de variété analytique complexe X^{an}. En particulier, on peut y définir le complexe de de Rham holomorphe $\Omega^*_{X^{an}}$. Les groupes d'hypercohomologie de ce complexe définissent la *cohomologie de de Rham* de X par la formule

$$H_{dR}^*(X) := \mathbb{H}^*(X^{an}, \Omega^*_{X^{an}}).$$

C'est une $\mathbb{C}$-algèbre graduée.

Au contraire de la cohomologie de Betti – nous reviendrons sur ce point – la cohomologie de de Rham peut être définie de manière purement algébrique. En effet, nous disposons sur X du complexe de de Rham algébrique $\Omega^*_{X/\mathbb{C}}$, qui est un complexe de faisceaux cohérents sur la variété X munie de la topologie de Zariski. La *cohomologie de de Rham algébrique* est l'hypercohomologie de ce complexe. On a ainsi

$$H_{dR}^*(X/\mathbb{C}) := \mathbb{H}^*(X, \Omega^*_{X/\mathbb{C}}).$$

Il résulte du principe GAGA de Serre, [61], que le morphisme d'espaces topologiques de l'ensemble des points complexes de X muni de la topologie de Zariski vers l'ensemble des points complexes de X muni de la topologie usuelle induit un isomorphisme canonique entre la cohomologie de de Rham algébrique de X et la cohomologie de de Rham de la variété analytique complexe X^{an}, c'est un théorème de Grothendieck [34]. Nous ne distinguerons pas ces deux groupes.

Enfin, si l est un nombre premier quelconque, les travaux de Grothendieck et ses collaborateurs permettent de définir la *cohomologie étale l-adique* de X. On a

$$H_l^*(X) := \varprojlim_n H^*_{ét}(X, \mathbb{Z}/l^n\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

C'est une $\mathbb{Q}_l$-algèbre graduée.

Ces théories cohomologiques vérifient toutes un certain nombres de propriétés qui en font des cohomologies de Weil, voir [45].

Définition 0.1. *Soit k un corps. Notons C_k la catégorie des variétés projectives lisses sur k dont les morphismes sont donnés par les correspondances algébriques. Une cohomologie de Weil est un foncteur contravariant*

$$H^* : C_k \to VecGr_K,$$

où K est un corps de caractéristique nulle et $VecGr_K$ est la catégorie des K-espaces vectoriels gradués en degrés positifs, qui vérifie

$$\dim_K H^2(\mathbb{P}^1_k) = 1.$$

Soit $d \in \mathbb{Z}$. Si V est un objet de $VecGr_K$, on note $V(d)$ le produit tensoriel de V par $H^2(\mathbb{P}^1_k)^{\otimes -d}$. On parle de twist à la Tate. Le foncteur H^* doit en outre être muni des données supplémentaires suivantes.

– *Pour tout couple (X, Y) d'objets de C_k, un isomorphisme canonique (formule de Künneth)*

$$H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y).$$

– *Pour tout $X \in C_k$ purement de dimension d, une application trace, K-linéaire,*

$$Tr_X : H^{2d}(X)(d) \to K,$$

qui est un isomorphisme si et seulement si X est géométriquement connexe, qui vérifie

$$Tr_{X\times Y} = Tr_X Tr_Y$$

pour tous X et Y, et telle que le produit de $H^*(X)$ induise une dualité parfaite pour tout X (dualité de Poincaré)

$$H^i(X) \otimes H^{2d-i}(X)(d) \to H^{2d}(X)(d) \to K.$$

On utilise ici la structure d'algèbre graduée sur $H^*(X)$ induite par l'inclusion diagonale $X \subset X \times X$ et la formule de Künneth.

– *Pour tout objet X de C_k, des morphismes de groupes (classes de cycle)*

$$cl^X : CH^k(X) \to H^{2k}(X)(k)$$

qui doivent être contravariants en X, compatibles au produit externe des cycles et normalisés de façon à ce que pour les zéros-cycles, la composée de la classe de cycle avec l'application trace donne l'application degré.

Nous reviendrons à plusieurs reprises sur la recherche d'une cohomologie de Weil universelle et ce qu'elle implique. Qu'il nous suffise ici de mentionner le résultat suivant :

Théorème 1. *Les cohomologies de Betti, de de Rham, et la cohomologie étale l-adique des variétés projectives complexes sont des cohomologies de Weil.*

Pour les cohomologies de Betti et de de Rham, ce sont des résultats de topologie algébrique et de calcul différentiel, en plus bien entendu de GAGA dans le cas de la cohomologie de de Rham algébrique. Dans le cas de la cohomologie l-adique, c'est une grande partie du travail effectué par Grothendieck et ses collaborateurs à la fin des années 1960.

0.1.2 Théorèmes de comparaison et topologie des variétés conjuguées

Il va de soi – mais ce n'est pas facile – que les différentes cohomologies de Weil que nous avons décrites ne sont pas sans liens. On dispose en effet des théorèmes de comparaison suivants.

Théorème 2. *(M. Artin, [1], exposé XI) Les cohomologies l-adique et de Betti d'une variété projective complexe X sont reliées par un isomorphisme fonctoriel, pour tout nombre premier l*

$$H^*_B(X) \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq H^*_l(X).$$

Théorème 3. *(Isomorphisme des périodes de Grothendieck) Les cohomologies de de Rham et de Betti d'une variété projective complexe X sont reliées par un isomorphisme fonctoriel*

$$H^*_B(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^*_{dR}(X)$$

Par fonctorialité, les isomorphismes fournis par les théorèmes de comparaison sont des isomorphismes d'algèbres.

Malgré les théorèmes de comparaison, les trois cohomologies de Weil que nous avons décrites ont des propriétés fonctorielles différentes. Partons en effet d'une variété projective lisse définie sur un corps k de caractéristique nulle pouvant se plonger dans $\mathbb{C}$.

Dans cette situation, on peut définir sans autre choix la cohomologie de de Rham algébrique $H^*_{dR}(X/k)$ en prenant comme plus haut l'hypercohomologie du complexe de de Rham algébrique sur X. On obtient de la sorte une k-algèbre de cohomologie. Cette construction est fonctorielle en k. En particulier, si σ est un plongement de k dans $\mathbb{C}$ et si σX désigne la variété complexe obtenue par changement de base, on a des isomorphismes canoniques

$$H^*_{dR}(X/k) \otimes_k \mathbb{C} \simeq H^*_{dR}(\sigma X) \simeq H^*_B(\sigma X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Si d'autre part K est un corps algébriquement clos qui contient k, on peut considérer la cohomologie l-adique $H^*_l(\overline{X})$ de la variété X_K obtenue par changement de bas. Cette dernière est indépendante du corps K et du choix d'un plongement de k dans K, d'où la notation adoptée. En effet, la définition même des groupes de cohomologie étale d'une variété projective sur un corps algébriquement clos K montre que ceux-ci ne dépendent que du schéma sous-jacent à la variété en question, et non de la flèche correspondante vers $\mathrm{Spec}(K)$.

Ces deux constructions contrastent avec le cas de la cohomologie de Betti. En effet, il est impossible de reconstruire cette cohomologie de Weil de manière algébrique. Il y a plusieurs façons d'interpréter une telle assertion. Soit X une variété projective lisse sur un corps k, et soit σ plongement de k dans $\mathbb{C}$. Il est bien évident qu'il est impossible de retrouver à partir de la seule donnée de la variété X sur k la position du $\mathbb{Q}$-espace vectoriel donné par la cohomologie de Betti dans la cohomologie de de Rham de σX. En effet, dès que X vérifie le théorème de Torelli global, cela impliquerait que la classe d'isomorphisme de la variété projective complexe σX est indépendante du plongement σ. C'est incorrect par exemple pour les courbes elliptiques.

On peut avoir des exigences plus faibles, et se demander ainsi, une variété X étant donnée sur un corps k muni d'un plongement σ dans $\mathbb{C}$, à quel point l'objet algébrique $H^*_B(\sigma X)$ – espace vectoriel gradué, algèbre graduée – dépend de σ. Cela revient à se demander à quel point la topologie de la variété différentielle $\sigma X(\mathbb{C})$ est déterminée par le schéma X.

On peut reformuler ce problème de la manière équivalente suivante. Soit X une variété projective lisse complexe. Si σ est un automorphisme du corps $\mathbb{C}$, notons X^σ la variété obtenue en conjuguant X par σ. Le problème précédent revient à comparer les topologies des variétés différentielles $X(\mathbb{C})$ et $X^\sigma(\mathbb{C})$. La situation est ambivalente. Les théorèmes GAGA de Serre impliquent que les nombres de Betti et les nombres de Hodge de $X(\mathbb{C})$ et $X^\sigma(\mathbb{C})$ sont les mêmes. De manière semblable, la théorie de Grothendieck du groupe fondamental montre que les complétés profinis du groupe fondamental de $X(\mathbb{C})$ et $X^\sigma(\mathbb{C})$ sont canoniquement isomorphes. Ces résultats sont résumés et élargis par ceux de Deligne, Morgan et Sullivan qui montrent dans [24] – voir aussi le résumé de [50] que le complété du type d'homotopie de $X(\mathbb{C})$ est invariant sous l'action des automorphismes de $\mathbb{C}$.

Néanmoins, un exemple célèbre de Serre dans [62] montre que les types d'homotopies de $X(\mathbb{C})$ et $X^\sigma(\mathbb{C})$, et même leurs groupes fondamentaux, peuvent être différents. À la suite de Serre, et plus récemment, on a pu exhiber plusieurs exemples de variétés conjuguées non homéomorphes dans [2], [7], [25], [49]. Pourtant, tous ces exemples ne distinguent $X(\mathbb{C})$ et $X^\sigma(\mathbb{C})$ que par leur type d'homotopie entier et non rationnel. Cela laisse en particulier ouverte la question suivante, posée par Grothendieck à Montréal en juillet 1970.

Peut-on trouver X et σ comme plus haut tels que les algèbres de cohomologie à coefficients rationnels de $X(\mathbb{C})$ et $X^\sigma(\mathbb{C})$ ne soient pas isomorphes ?

Dans la première partie de cette thèse, publiée dans [21], on répond par l'affirmative à cette question en démontrant le résultat plus fort suivant.

Théorème 4. *On peut trouver deux variétés projectives lisses complexes conjuguées dont les algèbres de cohomologie à coefficients réels ne sont pas isomorphes.*

La méthode de démonstration s'inspire fortement de celle qui a permis à Voisin de donner une solution négative au problème de Kodaira dans [75].

Ces considérations sont à rapprocher de la remarque de Serre qui prouve qu'en caractéristique positive, il n'existe pas de cohomologie de Weil à coefficients rationnels. De même, en caractéristique nulle, il n'existe pas de cohomologie de Weil à coefficients rationnels dont se déduiraient les cohomologies de Weil classiques par produit tensoriel.

0.2 Construction de cycles algébriques sur les variétés projectives complexes

0.2.1 Motifs purs et conjectures standards

C'est la recherche d'une cohomologie de Weil universelle, tout autant que les théorèmes de comparaison énoncés plus haut et ceux, en caractéristique positive, d'indépendance de l, qui ont amené Grothendieck dans les années 1960 à conjecturer l'existence d'une théorie cohomologique universelle à valeurs dans une catégorie abélienne qui se réaliserait en les cohomologies de Weil classiques. À partir de maintenant, nous ne considérerons plus que des variétés définies sur un corps de caractéristique nulle.

Nous ne décrirons pas ici la construction de la catégorie des motifs purs sur un corps, préférant renvoyer à par exemple [4], chapitre 4. Soit k un corps. On dispose de la catégorie $M_{num}(k)$ des motifs numériques à coefficients rationnels. Il s'agit d'une catégorie pseudo-abélienne – les projecteurs ont des noyaux – munie d'un produit tensoriel. On dispose d'un foncteur de la catégorie des variétés projectives lisses, les morphismes étant les correspondances modulo l'équivalence numérique, vers la catégorie $M_{num}(k)$.

Travaillant avec l'équivalence rationnelle ou homologique des cycles – pour une cohomologie de Weil donnée – on obtient deux autres catégories de motifs $M_{rat}(k)$ et $M_{hom}(k)$. Une cohomologie de Weil est dans ce cadre simplement donnée par un foncteur de $M_{rat}(k)$ dans la catégorie des espaces vectoriels gradués sur un corps compatibles aux différentes structures.

Dans la recherche des cohomologies de Weil, on aimerait en fait travailler directement avec les catégories de motifs modulo équivalence numérique ou homologique. En effet, on s'attend à plusieurs phénomènes. La catégorie des motifs homologiques $M_{hom}(k)$ est ce que l'on a de plus proche d'une cohomologie de Weil universelle. Néanmoins, la définition même de $M_{hom}(k)$ fait intervenir une cohomologie de Weil particulière, et il n'est pas clair que la catégorie obtenue ne dépende pas de la cohomologie de Weil choisie, la question étant de savoir si l'équivalence homologique des cycles dépend de la cohomologie de Weil choisie – en caractéristique nulle, c'est le cas pour les cohomologies de Weil classiques grâce aux théorèmes de comparaison.

Puisque l'équivalence numérique est moins fine que l'équivalence homologique – cela vient de la dualité de Poincaré – on a pour toute cohomologie de Weil un foncteur $M_{hom}(k) \to M_{num}(k)$. On s'attend en outre, c'est la traduction dans ce cadre de la pureté, à ce que la catégorie $M_{hom}(k)$ soit semi-simple. Or on a le théorème suivant de Jannsen prouvé dans [41].

Théorème 5. *La catégorie des motifs numériques $M_{num}(k)$ est abélienne semi-simple. La catégorie des motifs homologiques est semi-simple si et seulement si l'équivalence homologique et l'équivalence numérique coïncident.*

Ce résultat suggère la conjecture suivante.

Conjecture 0.2. *(conjecture $D(X)$) Soit X une variété projective lisse sur un corps k, et soit H^* une cohomologie de Weil sur k. Alors un cycle algébrique sur X est homologue à 0 si et seulement si il est numériquement équivalent à 0.*

Sous cette forme, la conjecture n'est pas très accessible. Dans l'article [35], Grothendieck énonce une autre conjecture.

Conjecture 0.3. *(conjecture $B(X)$, conjecture standard de Lefschetz) Soit X une variété projective lisse de dimension pure n sur un corps k, et soit H^* une cohomologie de Weil classique – c'est-à-dire l'une de celle que nous avons décrites dans la section précédente. Alors pour tout entier $i \leq n$, il existe une correspondance $Z \in CH^i(X \times X)$ telle que le morphisme*

$$[Z]_* : H^{2n-i}(X) \to H^i(X)$$

induit par Z soit un isomorphisme. On dit alors que la conjecture standard de Lefschetz est vraie pour X en degré i.

Ces deux conjectures font partie des conjectures standards. En caractéristique nulle, elles sont équivalentes l'une à l'autre et impliquent les autres conjectures standards. Elles ont de nombreuses conséquences décrites par exemple dans [45] et [46]. Si nous ne rendrons pas compte ici de leur importance, qu'il suffise de citer Grothendieck dans [35] :

"Alongside the resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry."

Hors le cas des variétés abéliennes connu déjà de Grothendieck et le cas du degré 1 qui découle du théorème de Lefschetz sur les classes de type $(1,1)$, ces conjectures sont tout à fait ouvertes.

0.2.2 Étude variationnelle des cycles algébriques

La seconde partie de ce mémoire examine la conjecture standard de Lefschetz pour les variétés complexes. Énoncée comme précédemment, la conjecture standard de Lefschetz prédit l'existence, pour toute variété projective lisse X, de familles (pas nécessairement plates a priori) de cycles de codimension i sur X paramétrées par X. Dans un premier temps, nous montrons que cela est équivalent à l'existence de certaines familles de cycles sur X paramétrées par des variétés quasi-projectives lisses de dimension i. Nous obtenons ainsi le résultat suivant.

Théorème 6. *Soit X une variété projective lisse de dimension pure n sur $\mathbb{C}$. Soit i un entier inférieur ou égal à n, et supposons la conjecture standard de Lefschetz vraie en degrés au plus $i-2$. Supposons qu'il existe une variété quasi-projective lisse S de dimension i et une correspondance $Z \in CH^i(S \times X)$ telle que le morphisme*

$$[Z]_* : H_B^{2n-i}(S) \to H_B^i(X, \mathbb{C})$$

induit par Z soit surjectif. Alors la conjecture standard de Lefschetz est vraie pour X en degré i.

L'ingrédient essentiel de ce résultat est l'existence d'une polarisation sur la cohomologie d'une variété projective lisse. Si la preuve est assez formelle, c'est néanmoins en remplaçant X comme espace de paramètres par une variété quasi-projective quelconque S que l'on peut introduire des méthodes variationnelles dans l'examen de la conjecture standard de Lefschetz.

L'étude des méthodes infinitésimales en théorie de Hodge et de leur application aux cycles algébriques apparaît, au moins implicitement, dès les travaux de Poincaré sur les fonctions normales. Dans les années 1980, les travaux de Griffiths et ses collaborateurs dans [18], [30] et [33] mettent en évidence la notion de variation de structure de Hodge à travers notamment le théorème de transversalité et la description de l'invariant infinitésimal des fonctions normales. Le livre [74] décrit en détail la situation.

Plusieurs résultats importants dans la théorie des cycles algébriques ont été obtenus à l'aide de ces méthodes. On peut notamment citer l'étude des lieux de Noether-Lefschetz, qui par exemple chez Bloch dans [12] débouche sur des méthodes variationnelles d'étude de la conjecture de Hodge, ou bien l'étude de l'application d'Abel-Jacobi et des groupes de Griffiths à la suite de Green, [29] ou Nori dans [53]. On peut trouver dans [72] une description plus précise de ces méthodes et de travaux qui les illustrent.

Dans notre cas, on interprète un cycle Z dans $S \times X$ comme une famille de cycles sur X paramétrée par S. La classe de cohomologie de Z est controlée non seulement par la classe des restrictions de Z aux variétés $\{s\} \times X$, qui ne dépendent pas de s, mais aussi par des éléments exprimant la variation de Z_s en $\{s\}$ décrits essentiellement en termes d'applications d'Abel-Jacobi supérieures.

En étudiant précisément cette variation, on peut, en degré 2 au moins, montrer que la conjecture de Lefschetz pour X est conséquence d'une condition sur la famille Z_s qui ne provient que de la variation au premier ordre de la famille de cycles en question. Ainsi, dans le cas où Z est donné par le caractère de Chern d'une famille de fibrés vectoriels sur X paramétrée par S, la condition du théorème précédent s'énonce uniquement en termes de l'application de Kodaira-Spencer associée. Cela permet de mener des calculs explicites dans la direction de la conjecture standard de Lefschetz.

Dans un dernier temps, on se place dans le contexte des variétés hyperkähleriennes et de leurs fibrés hyperholomorphes, systématiquement étudié par Verbitsky dans [69]. On mène les calculs dans le cas des schémas de Hilbert des points sur une surface K3, et on montre deux aspects de la géométrie de ces variétés qui permettent d'aborder la conjecture standard de Lefschetz en degré 2.

Le premier point est le fait que le théorème de Lefschetz fort hyperkählerien assure que les déformations d'un fibré hyperholomorphe stable sur une variété hyperkählerienne suffisent toujours, dès que le fibré n'est pas rigide, à produire des familles de cycles qui suffisent à démontrer la conjecture standard de Lefschetz en degré 2.

La deuxième remarque vient de l'étude des déformations des variétés hyperkähleriennes le long des espaces de twisteurs. En utilisant les travaux de Verbitsky, on donne des conditions numériques explicites qui permettent d'assurer qu'une preuve de la conjecture standard de Lefschetz en degré 2 sur une variété hyperkählerienne X fournit une preuve de cette même conjecture pour toutes les déformations projectives de X. Cela est à rapprocher, dans un contexte différent, des travaux de Bloch dans [12] qui étudie le comportement de la conjecture de Hodge par déformation.

Outre les résultats précis obtenus, ce travail essaie de mettre en évidence deux choses. D'une part, il apparaît que le programme de Verbitsky exposé dans [70], chapitre 2, de construction de variétés hyperkähleriennes via l'étude des déformations de fibrés hyperholomorphes a pour conséquence naturelle la preuve de la conjecture standard de Lefschetz en degré 2 pour certaines variétés hyperkähleriennes. D'autre part, de même que dans la première partie de cette introduction on a tenté de mettre en avant les objets de différente nature associées à une variété projective lisse, on donne un exemple de la pertinence de la géométrie kählerienne dans l'étude de questions d'origine motivique. En effet, les déformations des variétés que l'on étudie sont génériquement des variétés kähleriennes qui ne sont pas projectives.

0.3 Motifs mixtes, filtrations sur les groupes de Chow et fonctions normales

0.3.1 Problèmes de mixité et filtrations sur les groupes de Chow

Depuis le début de ce texte, nous n'avons traité que de variétés projectives lisses, et de la construction des motifs purs. Dans le cas mixte, si les variétés ne sont pas forcément propres ou lisses, des extensions non triviales apparaissent dans les groupes de cohomologie. Si nous ne nous intéressons pas ici de manière spécifique à la catégorie des motifs mixtes, les phénomènes de mixité s'observent même dans des problèmes qui ne concernent que des variétés projectives lisses. Ainsi, l'application d'Abel-Jacobi et la fonction normale qui est simplement sa version au-dessus d'une base apparaissent naturellement dans l'approche de la conjecture standard de Lefschetz que nous avons décrite plus haut.

Plus généralement, la philosophie des motifs mixtes a d'importantes applications au moins conjecturales à l'étude des groupes de Chow des variétés projectives lisses. À la suite du travail de Mumford autour de l'équivalence rationnelle des zéros-cycles sur les surfaces dans [52], Beilinson, [10] et Bloch, [11], ainsi que Murre, ont été amenés à conjecturer l'existence d'une filtration finie sur les groupes de Chow des variétés projectives lisses, compatible aux correspondances algébriques, et dont les gradués serait contrôlés par les motifs homologiques des variétés en jeu. L'existence d'une telle filtration et l'étude de ses propriétés auraient d'importantes conséquences tant géométriques qu'arithmétiques. On pourra consulter [42] pour un panorama détaillé.

Soit X une variété projective lisse sur $\mathbb{C}$. La filtration de Bloch-Beilinson F^* sur $CH^*(X)$ vérifie $F^0CH^*(X) = 0$ et $F^1CH^*(X) = CH^*(X)_{hom}$, où $CH^*(X)_{hom}$ désigne le groupe des cycles homologiquement équivalents à zéro. Ces deux termes sont bien compris. Le suivant est plus mystérieux. Conjecturalement, il est donné par le noyau de l'application d'Abel-Jacobi de Griffiths qui à un cycle de codimension i homologue à zéro associe un élément d'un tore complexe $J^i(X)$, la jacobienne intermédiaire de X. Sauf en codimension 1, cette application est très mystérieuse. Son image, qui se relie au groupe de Griffiths, comme son noyau, qui est donc conjecturalement le groupe $F^2CH^*(X)$, sont mal compris.

Le développement de la théorie des variations de structures de Hodge combiné aux conjectures de Beilinson sur les groupes de Chow des variétés définies sur les corps de nombres permettent de réduire l'étude des filtrations de Bloch-Beilinson à une compréhension fine de l'application d'Abel-Jacobi, comme démontré par Nori et Lewis, voir [47]. En effet, Beilinson prédit que si X est une variété définie sur un corps de nombres, le noyau de l'application d'Abel-Jacobi pour les cycles sur X définis sur un corps de nombres homologues à zéro est trivial modulo torsion. Si X est une variété complexe projective lisse quelconque, elle est toujours définie sur un corps de type fini sur $\mathbb{Q}$. La variété X est donc la fibre générique d'un morphisme de variétés $\mathcal{X} \to \mathcal{S}$ défini sur un corps de nombres. En faisant usage de la suite spectrale de Leray, on devrait retrouver la filtration de Bloch-Beilinson à partir de l'application d'Abel-Jacobi sur $\mathcal{X}$.

0.3.2 Fonctions normales

Soit $\pi : X \to S$ un morphisme projectif lisse de variétés quasi-projectives lisses complexes. Soit Z un cycle de codimension i dans X, homologue à zéro dans les fibres de π et plat au-dessus de S. Au morphisme π est associée une fibration en tores complexes $J^i(X/S)$ dont les fibres sont les jacobiennes intermédiaires des fibres de π. L'application d'Abel-Jacobi de Griffiths varie bien en famille et fournit une section ν_Z de $J^i(X/S)$, la *fonction normale* associée à Z.

Les remarques précédentes mettent en évidence l'importance de l'étude des fonctions normales pour la compréhension des groupes de Chow des variétés projectives lisses. Si cette fonction n'est pas d'origine algébrique, étant simplement une section holomorphe d'une famille de tores complexes, les conjectures de Bloch-Beilinson impliquent que son lieu des zéros est d'origine motivique. C'est cette question qui est le point de départ de la dernière partie de cette thèse.

Si l'application d'Abel-Jacobi de Griffiths est d'origine motivique, elle doit se réaliser non seulement dans le cadre de la théorie de Hodge mais aussi dans celui de la cohomologie étale l-adique – on revient aux préoccupations de la première partie. Pour obtenir une application d'Abel-Jacobi l-adique, on peut soit copier la définition de l'application complexe obtenue par Carlson et Beilinson dans [19] et [9] en interprétant les jacobiennes intermédiaires comme des groupes d'extension de structures de Hodge mixtes, soit en utilisant la suite spectrale de Hoschild-Serre en cohomologie étale. Bien entendu, les deux définitions coïncident, et dans les deux cas l'ingrédient essentiel est la cohomologie étale continue définie par Jannsen dans [39].

Une part importante de la troisième partie de cette thèse est consacrée à l'étude du noyau de cette application d'Abel-Jacobi l-adique, qui à torsion près doit être le même que celui de l'application d'Abel-Jacobi de Griffiths. Une motivation plus concrète à cette étude est la suivante. Revenons au cas évoqué plus haut d'un morphisme projectif lisse de variétés quasi-projectives lisses complexes $\pi : X \to S$, et soit Z un cycle de codimension i dans X, homologue à zéro dans les fibres de π et plat au-dessus de S. On s'attend à ce que le lieu des zéros de la fonction normale ν_Z soit une sous-variété algébrique de S. De manière tout à fait remarquable, ce résultat vient d'être démontré par Brosnan et Pearlstein dans [16].

Théorème 7. *(Brosnan, Pearlstein) Dans la situation précédente, le lieu des zéros de ν_Z est algébrique.*

La démonstration de ce résultat, qui trouve son origine dans le théorème de Deligne-Cattani-Kaplan dans [20] sur l'algébricité des lieux de Hodge, se fonde d'une part sur le théorème de Chow qui affirme qu'une sous-variété analytique d'une variété projective complexe est projective, et d'autre part sur une analyse fine des dégénérescences de structures de Hodge mixtes.

Si le résultat de Brosnan et Pearlstein donne des arguments forts en faveur des conjectures de Bloch-Beilinson, sa démonstration ne semble pas éclairer l'origine motivique de ces conjectures. En particulier, elle ne dit rien sur le corps de définition du lieu des zéros de la fonction normale ν_Z. Pour aborder cette question, on s'inspire des résultats de Voisin dans [76], tout en mettant l'accent sur la cohomologie étale. Il est en effet facile de montrer que si les noyaux des applications d'Abel-Jacobi étale et de Griffiths coïncident, alors le lieu des zéros de la fonction normale est union dénombrable de variétés algébriques définies sur le corps de définition de π.

De tels résultats semblent difficiles à démontrer en toute généralité, car ils reviennent à montrer que certaines classes de Hodge sont absolues. Dans ce mémoire, on prend comme point de départ le résultat de Brosnan et Pearlstein pour en déduire à la fois des théorèmes de comparaison entre les différentes applications d'Abel-Jacobi et des résultats sur le corps de définition des fonctions normales qui sont pour certains des versions mixtes des résultats de [76].

Les outils techniques viennent à la fois du monde étale et de la théorie de Hodge. On utilise de manière cruciale la version mixte du théorème des cycles invariants de Deligne due à Steenbrink et Zucker dans [64], ainsi que les conjectures de Weil. En particulier, il apparaît que l'analogue l-adique des fonctions normales se comporte de manière tout à fait semblable aux fonctions normales usuelles. On peut leur associer l'analogue des classes de Hodge d'une fonction normale, et prouver des propriétés semblables aux propriétés de ces classes de Hodge. Ce genre de résultats nous semble constituer un premier pas vers une hypothétique approche motivique de la théorie des fonctions normales.

Les résultats que nous prouvons ont des hypothèses qui ne sont pas difficiles à satisfaire dans certains cas. Comme illustration, on prouve en particulier le théorème concret suivant.

Théorème 8. *Soit $\pi : X \to S$ une famille projective lisse de variétés de Calabi-Yau complexes de dimension 3 au dessus d'une base quasi-projective lisse telle que l'application induite de S vers l'espace de modules correspondant soit finie, et soit $Z \hookrightarrow X$ une famille plate de courbes dans X qui sont homologues à zéro dans les fibres de π. Supposons que tout soit défini sur un corps k engendré par un nombre fini d'éléments sur $\mathbb{Q}$. Soit ν la fonction normale associée à Z.*

Soit T une composante irréductible du lieu des zéros de ν. Supposons que T soit de dimension strictement positive et que si t est un point général de T, la jacobienne intermédiaire $J^2(X_t)$ n'aie pas de facteur abélien. Alors T est définie sur k, et ses conjuguées sous le groupe de Galois absolu de k sont encore des composantes irréductibles du lieu des zéros de ν.

Des résultats plus généraux sont exposés en détail dans la dernière partie de cette thèse.

Chapter 1

Conjugate varieties with distinct real cohomology algebras

Résumé. En utilisant des constructions de Voisin, on construit une variété projective lisse définie sur un corps de nombre K, et deux plongements de K dans $\mathbb{C}$, tels que les deux variétés projectives complexes obtenues par ces plongements aient des algèbres de cohomologie réelle qui ne soient pas isomorphes. Ce résultat est à comparer avec le fait que les algèbres de cohomologie l-adique, où l est un nombre premier, sont toujours canoniquement isomorphes dans ce cas. Des exemples de variétés conjuguées non homéomorphes sont connus depuis Serre en 1964. Ici, notre exemple fournit en particulier un cas òu les types d'homotopie réelle sont différents. Cet article répond à une question de Grothendieck posée en 1970.

Abstract. Using constructions of Voisin, we exhibit a smooth projective variety defined over a number field K and two complex embeddings of K, such that the two complex manifolds induced by these embeddings have non isomorphic cohomology algebras with real coefficients. This contrasts with the fact that the cohomology algebras with l-adic coefficients are canonically isomorphic for any prime number l, and answers a question of Grothendieck.

1.1 Introduction

Let X be an algebraic variety defined over an algebraically closed field K of characteristic 0. If l is a prime number, the l-adic cohomology of X, $H^*(X, \mathbb{Q}_l)$, is a graded algebra over $\mathbb{Q}_l$. Its definition as an inverse limit of étale cohomology groups shows that it does not depend on the structural map $X \to \operatorname{Spec} K$.

Now suppose K is a finitely generated extension of $\mathbb{Q}$. We can consider the l-adic cohomology of $X_{\bar{K}}$, where $\bar{K}$ is an algebraic closure of K. This does not depend, up to isomorphism, on the choice of the algebraic closure. Moreover, if $\bar{L}$ is any algebraically closed field containing K, the proper base change theorem shows that the l-adic cohomology of $X_{\bar{L}}$ is canonically isomorphic to that of $X_{\bar{K}}$. In particular, $H^*(X_{\mathbb{C}}, \mathbb{Q}_l)$ is canonically defined and does not depend on an embedding $K \hookrightarrow \mathbb{C}$ up to isomorphism. Let us choose such an embedding, and assume from now on that X is smooth over K. Artin's comparison theorem of [1], Exp. XI, shows that $H^*(X_{\mathbb{C}}, \mathbb{Q}_l)$ is canonically isomorphic to the Betti cohomology of $X_{\mathbb{C}}^{an}$, the underlying complex manifold of $X_{\mathbb{C}}$, with coefficients in $\mathbb{Q}_l$. As a consequence, the latter does not depend on the embedding $K \hookrightarrow \mathbb{C}$.

The preceding discussion shows that the cohomology algebra with coefficients in $\mathbb{Q}_l$ of the complex manifold $X_{\mathbb{C}}^{an}$ does not vary under automorphisms of $\mathbb{C}$. In other words, it does only depend on the abstract scheme $X_{\mathbb{C}}$ and not on its map to $\operatorname{Spec}\mathbb{C}$.

The topology of a complex variety can nonetheless vary under automorphisms of $\mathbb{C}$. Indeed, Serre constructs in [62] two conjugate complex smooth projective varieties with different fundamental groups. Note however that the profinite completions of those are canonically isomorphic, due to Grothendieck's theory of the algebraic fundamental group. In particular, they do not have the same homotopy type.

Other constructions of conjugate varieties which are not homeomorphic can be found in [2], and more recently in [7] and [63] (the last article actually considers open varieties). See also [5] for related constructions. Nevertheless, the arguments leading to the constructions of the previous examples all make use of the integral homotopy type, by actually considering fundamental groups or Betti cohomology with coefficients in $\mathbb{Z}$.

This leads naturally to the following question, asked in [57] :

Do there exist conjugate varieties with different rational homotopy type ?

Very similarly, it had already been asked by Grothendieck (Montréal, july 1970) whether there exist conjugate varieties with distinct cohomology algebras with rational coefficients.

In this paper, we answer positively these questions. We actually show the following, which is stronger :

Theorem 1.1. *There exist smooth projective conjugate varieties whose real cohomology algebras are not isomorphic.*

To construct the example, we use the methods of [75], where Voisin shows how to use the cohomology algebra of some varieties to recover their endomorphism rings. As in [62], our example is built out of abelian varieties with complex multiplication.

Acknowledgements. I would like to thank Claire Voisin for sharing her ideas with me, and for her many very helpful remarks. This paper owes her a lot. I would also like to thank Johan de Jong for many useful discussions. This work was completed during a stay at Columbia University.

1.2 Statement of the theorem

Let k and k' be two different imaginary quadratic subfields of $\mathbb{C}$ [1], and let E (resp. E') be a complex elliptic curve with complex multiplication by $\mathcal{O}_k$ (resp. $\mathcal{O}_{k'}$). Let A be the product of E and E'.

Suppose we are given polarizations ϕ and ϕ' of E and E', that is, numerical equivalence classes of very ample line bundles. Those give a polarization ψ of A with $\psi = \phi \oplus \phi'$, which comes from some projective embedding $i : A \hookrightarrow \mathbb{P}^N$. The idea of the example is to construct a variety whose cohomology ring encodes the endomorphism ring of A and contains a distinguished line related to the polarization. This will be achieved by blowing-up some special subvarieties of $A \times A \times \mathbb{P}^N$.

Let a and a' be imaginary elements of $\mathcal{O}_k$ and $\mathcal{O}_{k'}$ which generate the fields k and k' respectively. We will denote by f (resp. f') the endomorphism $a \times 0$ (resp. $0 \times a'$) of $A = E \times E'$ – we will sometimes still denote by f (resp. f') the corresponding endomorphism of E (resp. E'). The induced homomorphisms on the first cohomology group of A have eigenvalues 0 and $a, -a$ (resp. $a', -a'$).

1. For simplicity, but at the loss of some functoriality, we fix complex embeddings of the quadratic fields.

Let x, y, z and t be points of $\mathbb{P}^N$ and u be a point of A. Let us consider the following smooth subvarieties of $A \times A \times \mathbb{P}^N$:

$$Z_1 = A \times 0 \times x,\ Z_2 = \Gamma_{\mathrm{Id}_A} \times y,\ Z_3 = \Gamma_f \times z,\ Z_4 = \Gamma_{f'} \times t,\ Z_5 = u \times \Gamma_i,$$

where Γ stands for the graph of a morphism. For a generic choice of x, y, z, t, u, those subvarieties are pairwise disjoint. Let X be the blow-up of $A \times A \times \mathbb{P}^N$ along those subvarieties. This is a smooth complex projective variety [2].

For any smooth complex variety V, one can consider the underlying complex manifold V^{an} and its real cohomology algebra $H^*(V^{an}, \mathbb{R})$. By an abuse of notation, we will denote the latter algebra by $H^*(V, \mathbb{R})$, remembering that it depends on the usual topology on V. For a scheme Y over a field K, and σ an automorphism of K, let Y^σ denote the scheme $Y \otimes_{K,\sigma} K$. If Y is smooth (resp. polarized), then so is Y^σ. Our theorem is the following.

Theorem 1.2. *Let X be constructed as above, and let σ be an automorphism of $\mathbb{C}$ which acts trivially on one of the fields k and k', but not on the other. Then the real cohomology algebras $H^*(X, \mathbb{R})$ and $H^*(X^\sigma, \mathbb{R})$ are not isomorphic.*

1.3 Proof of the theorem

Theorem 1.2 will be obtained as a consequence of propositions 1.3 and 1.5, which will be stated and proved in the next subsections.

1.3.1 Some linear algebra

Let σ be an automorphism of the field $\mathbb{C}$. In this section, we describe some linear objects attached to the polarized abelian variety (A^σ, ψ^σ) and study how they vary under the action of automorphisms of $\mathbb{C}$. What we would like to do is to recover the CM-type of E^σ and E'^σ from part of the cohomology of X^σ. Actually, we will only be able to compare, in some sense, those CM-types, which will be enough for our purpose. This is the reason why we have to work with two elliptic curves.

Let σ be an automorphism of $\mathbb{C}$. There is a canonical isomorphism of abstract schemes from A^σ to A. Though it is by no means defined over $\mathbb{C}$, it still induces an isomorphism between the endomorphism rings of the complex varieties A^σ and A. As a consequence, there is a canonical action of $k \times k'$ on A^σ. The real vector space $H^1(A^\sigma, \mathbb{R})$ thus becomes a free rank 1 $\mathbb{C} \times \mathbb{C}$-module, as $k \otimes_{\mathbb{Q}} \mathbb{R}$ and $k' \otimes_{\mathbb{Q}} \mathbb{R}$ are canonically isomorphic to $\mathbb{C}$ (recall we chose complex embeddings of k and k'). From the embedding $i^\sigma : A^\sigma \hookrightarrow \mathbb{P}^N$, we get a homomorphism $i^{\sigma *} : H^2(\mathbb{P}^N, \mathbb{R}) \to \bigwedge^2_{\mathbb{R}} H^1(A^\sigma, \mathbb{R})$.

As a consequence, with each σ comes a free rank 1 $\mathbb{C} \times \mathbb{C}$-module

$$V = H^1(A^\sigma, \mathbb{R}),$$

a 1-dimensional $\mathbb{R}$-vector space

$$L = H^2(\mathbb{P}^N, \mathbb{R})$$

and a nonzero homomorphism

$$\mu = i^{\sigma *} : L \to \bigwedge^2_{\mathbb{R}} V.$$

2. Since abelian varieties with complex multiplication are defined over number fields, we can even choose the polarizations and the points adequately so that X is also defined over a number field.

Let $\mathcal{L}$ be the set of isomorphism classes of such triples (V, L, μ), with the obvious notion of morphism. The preceding description gives us a map

$$l : \mathrm{Aut}(\mathbb{C}) \to \mathcal{L}.$$

Proposition 1.3. *Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ act trivially on one of the fields k and k', but not on the other. Then $l(\sigma) \neq l(\mathrm{Id}_{\mathbb{C}})$.*

Proof. Let V be a free rank 1 $\mathbb{C}\times\mathbb{C}$-module. Using the idempotents of $\mathbb{C}\times\mathbb{C}$, we get a canonical splitting of V as a direct sum of two complex vector spaces V_1 and V_2 of rank 1, such that $0\times\mathbb{C}$ acts trivially on V_1 and $\mathbb{C}\times 0$ acts trivially on V_2. The action of $\mathbb{C}\times 0$ on V_1 and of $0\times\mathbb{C}$ on V_2 endows those real vector spaces with a complex structure, so the underlying real vector spaces of V_1 and V_2 are canonically oriented. Let us call those complex structures and the induced orientations the *algebraic* ones.

Let L be a real vector space of rank 1. Any homomorphism μ of real vector spaces from L to $\bigwedge^2_{\mathbb{R}} V$ canonically induces homomorphisms from L to $\bigwedge^2_{\mathbb{R}} V_1$ and $\bigwedge^2_{\mathbb{R}} V_2$. Suppose that those are isomorphisms – this is the case for the triples in the image of l. We get an isomorphism between $\bigwedge^2_{\mathbb{R}} V_1$ and $\bigwedge^2_{\mathbb{R}} V_2$, which may or may not respect the algebraic orientation. Let us define the *sign* of the triple (V, L, μ) to be 1 or -1 according to wether it is the case or not. The sign only depends on the isomorphism class of the triple.

In the setting of the proposition, the sign of a triple is easy to compute. Let us indeed choose an automorphism σ of $\mathbb{C}$. The aforementioned splitting of $H^1(A^\sigma, \mathbb{R})$ corresponds to the splitting

$$H^1(A^\sigma, \mathbb{R}) = H^1(E^\sigma, \mathbb{R}) \oplus H^1(E'^\sigma, \mathbb{R}).$$

Aside from the complex structure induced by complex multiplication, the space $H^1(E^\sigma, \mathbb{R})$ has a complex structure induced by its identification with the cotangent space at 0 to the complex manifold E^σ. It does not have to agree with the one previously defined through complex multiplication. The same construction works with E'. Let us call those complex structures and the induced orientations *transcendental*. Now let $h \in H^2(\mathbb{P}^N, \mathbb{R})$ be the class of a hyperplane. The homomorphisms $H^2(\mathbb{P}^N, \mathbb{R}) \to \bigwedge^2_{\mathbb{R}} H^1(E^\sigma, \mathbb{R})$ and $H^2(\mathbb{P}^N, \mathbb{R}) \to \bigwedge^2_{\mathbb{R}} H^1(E'^\sigma, \mathbb{R})$ defined earlier both send h to elements which are positive with respect to the transcendental orientation.

As a consequence, the isomorphism $\bigwedge^2_{\mathbb{R}} H^1(E^\sigma, \mathbb{R}) \to \bigwedge^2_{\mathbb{R}} H^1(E'^\sigma, \mathbb{R})$ induced by the polarization respects the transcendental orientations. From this remark, it results that the sign of the triple $l(\sigma)$ is 1 if and only if the algebraic and transcendental orientations either coincide on both $H^1(E^\sigma, \mathbb{R})$ and $H^1(E'^\sigma, \mathbb{R})$, or if they differ on both those spaces.

Recall that a complex structure on a real vector space V is given by an automorphism I of V such that $I^2 = -\mathrm{Id}_V$. Giving I is in turn equivalent to giving a splitting of the complex vector space

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

such that $V^{1,0}$ and $V^{0,1}$ are complex conjugate of each other. Those spaces are the eigenspaces of I for the eigenvalues i and $-i$ respectively.

Now let I^σ_{alg} and I^σ_{tr} be the automorphisms of $H^1(E^\sigma, \mathbb{R})$ corresponding to the algebraic and transcendental complex structures. Since the action of complex multiplication on $H^1(E^\sigma, \mathbb{R})$ is $\mathbb{C}$-linear with respect to the transcendental complex structure (indeed, morphisms of smooth complex algebraic varieties are holomorphic), I^σ_{alg} and I^σ_{tr} commute, which implies, as their eigenspaces are one-dimensional and they have i and $-i$ has eigenvalues, that they are either equal or opposite to each other.

The splitting of $H^1(E^\sigma, \mathbb{R})$ corresponding to the transcendental complex structure is well-known, as it corresponds to the Hodge decomposition

$$H^1(E^\sigma, \mathbb{C}) = H^0(E^\sigma, \Omega_{E^\sigma}) \oplus H^1(E^\sigma, \mathcal{O}_{E^\sigma}),$$

with I^σ_{tr} acting as i on the first summand and as $-i$ on the second. Therefore, we just have to investigate the action of complex multiplication on $H^0(E^\sigma, \Omega_{E^\sigma})$.

We have an obvious isomorphism of one-dimensional complex vector spaces

$$H^0(E, \Omega_E) \to H^0(E^\sigma, \Omega_{E^\sigma}), \omega \mapsto \omega^\sigma$$

given by pullback of differential forms by the isomorphism of abstract schemes $E^\sigma \to E$. This isomorphism is σ-linear, that is, it sends $\lambda\omega$ to $\sigma(\lambda)\omega^\sigma$. Let f be the endomorphism of E we chose earlier. It generates its complex multiplication, and corresponds to a certain imaginary $a \in \mathcal{O}_k \subset \mathbb{C}$. For any global holomorphic one-form ω on E, we have

$$f^{\sigma *}\omega^\sigma = (f^*\omega)^\sigma.$$

The morphisms f^* and $f^{\sigma *}$ act as scalars on $H^0(E, \Omega_E)$ and $H^0(E^\sigma, \Omega_{E^\sigma})$ respectively. Let f^* act by multiplication by λ. The complex number λ may be either a or $\bar{a} = -a$, and it is equal to a if and only if the algebraic and transcendental complex structures on $H^1(E, \mathbb{R})$ coincide. But

$$f^{\sigma *}\omega^\sigma = (f^*\omega)^\sigma = (\lambda\omega)^\sigma = \sigma(\lambda)\omega^\sigma.$$

This proves that if σ is an automorphism of $\mathbb{C}$ fixing a, hence k, then the transcendental and algebraic complex structures on $H^1(E^\sigma, \mathbb{R})$ coincide if and only if they do on $H^1(E, \mathbb{R})$. The same goes for the algebraic and transcendental orientations. On the other hand, if σ acts as complex conjugation on k, the transcendental and algebraic orientations on $H^1(E^\sigma, \mathbb{R})$ coincide if and only if they don't on $H^1(E, \mathbb{R})$.

Since the same goes for E'^σ and k', the preceding discussion shows that the sign of $l(\sigma)$ is the same as the sign of $l(\mathrm{Id}_\mathbb{C})$ if and only if σ acts either trivially on k and k' or by complex conjugation on both. This concludes. □

Remark 1.4. Using the Hasse principle and the main theorem of complex multiplication, one can prove that the image of l has exactly two elements, and that $l(\sigma) = l(\mathrm{Id}_\mathbb{C})$ if either σ acts trivially on both k and k' or if its acts by complex conjugation on both. This is still true if we replace elliptic curves by abelian varieties (assuming the polarizations are compatible with the complex multiplication).

1.3.2 Analysis of the cohomology algebra

The goal of this section is to prove the following.

Proposition 1.5. *The variety X being defined as in the previous section, let $\mathcal{C}$ be the set of isomorphism classes of real graded algebras of the form $H^*(X^\sigma, \mathbb{R})$, where σ is an automorphism of $\mathbb{C}$, and let c be the map*

$$\mathrm{Aut}(\mathbb{C}) \to \mathcal{C}, \sigma \mapsto H^*(X^\sigma, \mathbb{R}).$$

Then the map $l : \mathrm{Aut}(\mathbb{C}) \to \mathcal{L}$ defined in 1.3.1 factors through c.

In other words, given the real cohomology algebra of X^σ, one can recover the linear algebra data described previously.

Let us fix some notations. Let $\tau : X \to A \times A \times \mathbb{P}^N$ be the blowing-down map and $\pi : X \to A \times A$ be the composite of τ with the projection on the first two factors. While $Z_1, \ldots, Z_5$ are the centers of the blow-up, let $D_1, \ldots, D_5$ denote the corresponding exceptional divisors of X and for k between 1 and 5, let

$$j_k : Z_k \hookrightarrow A \times A \times \mathbb{P}^N, \widetilde{j}_k : D_k \hookrightarrow X, \tau_k = \tau_{|D_k}$$

be the canonical morphisms. For a subvariety Z of X, let $[Z]$ denote its cohomology class. Let $h \in H^2(X, \mathbb{R})$ be induced by the cohomology class of a hyperplane in $\mathbb{P}^N$ via the natural morphism $X \to \mathbb{P}^N$. We will use the same notations, with a superscript σ, for the corresponding objects of X^σ.

Proposition 1.5 is a straightforward consequence of the following two statements.

Proposition 1.6. *Let $\mathcal{C}'$ be the set of isomorphism classes of 7-uples $(H, L_1, \ldots, L_6)$, where H is an element of $\mathcal{C}$ and $L_1, \ldots, L_6$ are lines in H. Let c' be the map*

$$\mathrm{Aut}(\mathbb{C}) \to \mathcal{C}', \sigma \mapsto (H^*(X^\sigma, \mathbb{R}), \mathbb{R}[D_1^\sigma], \ldots, \mathbb{R}[D_5^\sigma], \mathbb{R}h^\sigma).$$

If σ and σ' are automorphisms of $\mathbb{C}$, then $c(\sigma) = c(\sigma')$ if and only if $c'(\sigma) = c'(\sigma')$.

Proposition 1.7. *The map l factors through c'.*

Before going through the proofs, let us state a lemma for future reference. This is the analogue of the computations made in the proof of theorem 3 of [75], and, as in Voisin's paper, is the key to extracting information from the algebra structure on cohomology spaces. We give a proof for the reader's convenience.

Let p_1 (resp. p_2) be the restriction map from $H^1(A \times A, \mathbb{R})$ to $H^1(A, \mathbb{R})$ induced by the inclusion of the first (resp. second) factor. Let q_1 (resp. q_2) be the restriction map from $H^2(A \times \mathbb{P}^N, \mathbb{R})$ to $H^2(A, \mathbb{R})$ (resp. $H^2(\mathbb{P}^N, \mathbb{R})$) induced by the inclusion of the first (resp. second) factor. Using pullback by π, we can consider $H^*(A \times A, \mathbb{R})$ as a subspace of $H^*(X, \mathbb{R})$. Note that $H^1(A \times A, \mathbb{R}) = H^1(X, \mathbb{R})$.

Let us fix a nonzero cohomology class $\alpha \in H^2(A \times A, \mathbb{R}) \subset H^2(X, \mathbb{R})$. For k between 1 and 5, consider cup-product with $[D_k]$. It gives a homomorphism $H^1(A \times A, \mathbb{R}) = H^1(X, \mathbb{R}) \to H^3(X, \mathbb{R})$. We get similar homomorphisms by taking cup-product with h or α.

Lemma 1.8. *The images of those homomorphisms $\cup[D_1], \ldots, \cup[D_5], \cup h$, and $\cup\alpha$, are in direct sum. Furthermore, Ker$(\cup\alpha)$ is at most 2-dimensional and*

- $\mathrm{Ker}(\cup h) = 0$,
- $\mathrm{Ker}(\cup[D_1]) = \mathrm{Ker}(p_1)$,
- $\mathrm{Ker}(\cup[D_2]) = \mathrm{Ker}(p_1 + p_2)$,
- $\mathrm{Ker}(\cup[D_3]) = \mathrm{Ker}(p_1 + f^* p_2)$,
- $\mathrm{Ker}(\cup[D_4]) = \mathrm{Ker}(p_1 + f'^* p_2)$,
- $\mathrm{Ker}(\cup[D_5]) = \mathrm{Ker}(p_2)$,

The kernel of

$$\cup[D_5] : H^2(A \times \mathbb{P}^N, \mathbb{R}) \subset H^2(X, \mathbb{R}) \to H^4(X, \mathbb{R})$$

is

$$\mathrm{Ker}(q_1 + i^* q_2),$$

where the inclusion $H^2(A \times \mathbb{P}^N, \mathbb{R}) \subset H^2(X, \mathbb{R})$ is given by pullback by the composite of τ with the projection of $A \times A \times \mathbb{P}^N$ on the two last factors.

Obviously, the lemma remains true after letting any automorphism of $\mathbb{C}$ act.

Proof. Let us first prove the assertion about the images by considering the general situation of a blow-up $\tau : \widetilde{Y} \to Y$ of a complex smooth projective variety along a smooth, but not necessarily irreducible, subvariety Z, of codimension everywhere at least 2. Let E be the exceptional divisor. It is a projective bundle over Z. Let j_Z and j_E be the inclusions of Z and E in Y and $\widetilde{Y}$ respectively. It is known (see [74], 7.3.3) that there exists a homomorphism $\phi : H^*(Z, \mathbb{R}) \to H^*(E, \mathbb{R})$, given by excision and the Thom isomorphism, such that the cohomology of $\widetilde{Y}$ is the quotient in the following exact sequence of (non-graded) vector spaces

$$0 \longrightarrow H^*(Z, \mathbb{R}) \longrightarrow H^*(Y, \mathbb{R}) \oplus H^*(E, \mathbb{R}) \longrightarrow H^*(\widetilde{Y}, \mathbb{R}) \longrightarrow 0\ ,$$

where the first map is (j_{Z*}, ϕ) and the second one is $\tau^* \oplus j_{E*}$

Now let $Z_1, \ldots, Z_n$ be the irreducible components of Z, $E_1, \ldots, E_n$ the corresponding irreducible components of E, τ_{E_i} the restriction of τ to E_i, and j_{Z_i} and j_{E_i} the obvious inclusions. Let x be a degree 2 cohomology class in Y. We want to show that the images of the homomorphisms $\cup[E_i] \circ \tau^*$ and $\tau^* \circ \cup x$, restricted to degree 1 cohomology classes in Y, are in direct sum. Indeed, since τ^* is injective on cohomology because τ is birational, and since the images of $\cup h$ and $\cup\alpha$ in $H^3(A \times A \times \mathbb{P}^N, \mathbb{R})$ are in direct sum, as the Künneth formula shows, this will prove the assertion.

We have

$$\cup[E_i] \circ \tau^* = j_{E_i*} \circ \tau_{E_i}^* \circ j_{Z_i}^*,$$

so it is enough to prove that the images of the $j_{E_i*} : H^1(E_i, \mathbb{R}) \to H^3(\widetilde{Y}, \mathbb{R})$ and of $\tau^* : H^3(Y, \mathbb{R}) \to H^3(\widetilde{Y}, \mathbb{R})$ are in direct sum. But the map

$$H^3(Y, \mathbb{R}) \oplus H^1(E, \mathbb{R}) \to H^3(\widetilde{Y}, \mathbb{R})$$

has zero kernel for degree reasons, as the exact sequence above shows. This proves the assertion about the images.

Let us now consider the kernels. To compute the first two, it is enough to work on $A \times A \times \mathbb{P}^N$ since τ^* is injective on cohomology. The Künneth formula shows that h has nonzero cup-product with any nonzero element of $H^*(X, \mathbb{R})$ coming from $A \times A$.

Consider now cup-product with α. Since the real cohomology algebra of an abelian variety is the exterior algebra on the first real cohomology space, the assertion concerning $\mathrm{Ker}(\cup\alpha)$ boils down to the following lemma.

Lemma 1.9. *Let V be a finite dimensional vector space, and let α be a nonzero element of $\bigwedge^2 V$. The kernel of the homomorphism*

$$\wedge\alpha : V \to \bigwedge^3 V$$

is at most 2-dimensional.

Proof. Let us choose a basis $e_1, \ldots, e_n$ for V. The space $\bigwedge^2 V$ has a basis consisting of all the $e_i \wedge e_j$ with $i < j$. Without loss of generality, we can assume that α has a nonzero component on $e_1 \wedge e_2$ with respect to this basis. It is then clear that the elements $\alpha \wedge e_3, \ldots, \alpha \wedge e_n$ of $\bigwedge^3 V$ are linearly independent. This shows that the homomorphism $\wedge\alpha : V \to \bigwedge^3 V$ has rank at least $n - 2$, and concludes the proof. ☐

As for the computation of the other kernels, since the cohomology of a smooth variety is embedded in the cohomology of any smooth blow-up of it, the result is a straightforward consequence of the following general computation.

Consider the situation of two smooth complex projective varieties B and C, together with a morphism $f : B \to C$. Let $\tau : \widetilde{B \times C} \to B \times C$ be the blow-up of $B \times C$ along the graph Γ of f. Let E be the exceptional divisor, and let τ_E be the restriction of τ to E. Let j_Γ and j_E be the inclusions of Γ and E into $\widetilde{B \times C}$ and $B \times C$ respectively. The map

$$\cup[E] \circ \tau^* : H^*(B \times C, \mathbb{R}) \to H^{*+2}(\widetilde{B \times C}, \mathbb{R})$$

is equal to

$$j_{E*} \circ \tau_E^* \circ j_\Gamma^*.$$

It follows from [74], 7.3.3 that $j_{E*} \circ \tau_E^*$ is injective, which means that the kernel of $\cup[E] \circ \tau^*$ is equal to the kernel of j_Γ^*. Now the morphism from B to $B \times C$ with coordinates Id_B and f identifies Γ with B, and $j_\Gamma : \Gamma \hookrightarrow B \times C$ with

$$(\mathrm{Id}_B \times f) : B \to B \times C.$$

As a consequence, the kernel of j_Γ^* is equal to the kernel of the homomorphism

$$H^*(B \times C, \mathbb{R}) = H^*(B, \mathbb{R}) \otimes H^*(C, \mathbb{R}) \to H^*(B, \mathbb{R})$$

which sends an element of the form $\alpha \otimes \beta$ to $\alpha \cup f^*(\beta)$.

If b and c are complex points of B and C, let p and q be the projections from $H^*(B \times C, \mathbb{R})$ to $H^*(B, \mathbb{R})$ and $H^*(C, \mathbb{R})$ induced by the immersions $B \hookrightarrow B \times C, x \mapsto (x, c)$ and $C \hookrightarrow B \times C, x \mapsto (b, x)$. By the Künneth formula, if γ is a degree 1 cohomology class in $H^*(B \times C, \mathbb{R}) = H^*(B, \mathbb{R}) \otimes H^*(C, \mathbb{R})$, we have

$$\gamma = p(\gamma) \otimes 1 + 1 \otimes q(\gamma),$$

which shows that $j_\Gamma^*(\gamma) = 0$ if and only if $p(\gamma) + f^* q(\gamma) = 0$.

This proves that the kernel of

$$\cup[E] \circ \tau^* : H^1(B \times C, \mathbb{R}) \to H^3(\widetilde{B \times C}, \mathbb{R})$$

is

$$\mathrm{Ker}(p + f^* q).$$

It is straightforward to check that this equality remains true for degree 2 cohomology classes in case B or C has no degree 1 cohomology, since we have then $H^2(B \times C, \mathbb{R}) = H^2(B, \mathbb{R}) \oplus H^2(C, \mathbb{R})$.

□

Proof of proposition 1.6

Proof. Without loss of generality, we can suppose that σ' is the identity. Let σ be an automorphism of $\mathbb{C}$ and γ be an isomorphism from $H^*(X, \mathbb{R})$ to $H^*(X^\sigma, \mathbb{R})$. We will show that γ sends $\mathbb{R}[D_k]$ to $\mathbb{R}[D_k^\sigma]$ for any k between 1 and 5, and $\mathbb{R}h$ to $\mathbb{R}h^\sigma$. This will be achieved step by step.

The Albanese map. We use the same argument as in [75]. Recall that π is the natural map from X to $A \times A$. Pullback by π gives an isomorphism between $H^1(A\times A, \mathbb{R})$ and $H^1(X, \mathbb{R})$. The injection

$$\pi^* : H^*(A \times A, \mathbb{R}) = \bigwedge H^1(X, \mathbb{R}) \hookrightarrow H^*(X, \mathbb{R})$$

is given by this isomorphism and cup-product. As a consequence, cup-product alone allows us to recover $H^*(A\times A, \mathbb{R})$ as a subalgebra of $H^*(X, \mathbb{R})$: it is the algebra $\bigwedge H^1(X, \mathbb{R})$. This being true also after letting σ act, we get the following.

Lemma 1.10. *The isomorphism γ sends the subalgebra $H^*(A \times A, \mathbb{R})$ of $H^*(X, \mathbb{R})$ to the subalgebra $H^*(A^\sigma \times A^\sigma, \mathbb{R})$ of $H^*(X^\sigma, \mathbb{R})$.*

Image of the cohomology classes of the exceptional divisors.

Lemma 1.11. *There exists a permutation ϕ of $\{1, \ldots, 5\}$ such that for each k between 1 and 5, γ sends the line $\mathbb{R}[D_k]$ to the line $\mathbb{R}[D^\sigma_{\phi(k)}]$.*

Proof. From the Künneth formula and the computation of the cohomology of a blow-up, we get a splitting

$$H^2(X, \mathbb{Q}) = \pi^* H^2(A \times A, \mathbb{Q}) \oplus \mathbb{Q}h \oplus \bigoplus_{k=1}^{5} \mathbb{Q}[D_k].$$

Let k be an integer between 1 and 5. The isomorphism γ sends $[D_k]$ to some element of $H^2(X^\sigma, \mathbb{R})$ of the form

$$\gamma([D_k]) = \alpha^\sigma + \mu_1[D^\sigma_1] + \ldots + \mu_5[D^\sigma_5] + \nu h^\sigma,$$

with α^σ coming from $H^2(A^\sigma \times A^\sigma, \mathbb{R})$.

We now use lemma 1.8. The map

$$\cup[D^\sigma_k] : H^1(X^\sigma, \mathbb{R}) \to H^3(X^\sigma, \mathbb{R})$$

has a $2 \dim A = 4$-dimensional kernel. Furthermore, the kernel of

$$\cup\alpha^\sigma + \mu_1[D^\sigma_1] + \ldots + \mu_5[D^\sigma_5] + \nu h^\sigma : H^1(X^\sigma, \mathbb{R}) \to H^3(X^\sigma, \mathbb{R})$$

is the intersection of the kernels of the $\cup\mu_i\,[D^\sigma_i]$, $\cup\alpha^\sigma$ and $\cup\nu h^\sigma$, because the images of these homomorphisms are in direct sum. Since $\cup h^\sigma$ is injective on degree 1 cohomology, we get $\nu = 0$. Furthermore, the kernel of $\cup\alpha^\sigma : H^1(X^\sigma, \mathbb{R}) \to H^3(X^\sigma, \mathbb{R})$ is at most 2-dimensional unless $\alpha^\sigma = 0$. This proves $\alpha^\sigma = 0$, and since the kernels of the $\cup\mu_i\,[D^\sigma_i]$ are pairwise distinct 4-dimensional vector spaces, which implies that the intersection of two of them has dimension at most 3, this implies that $[D_k]$ is sent to some $\mu_i[D^\sigma_i]$. This proves the lemma. □

Lemma 1.12. *The permutation ϕ is the identity.*

Proof. For k between 1 and 5, let F_k be the subspace of $H = H^1(X, \mathbb{R})$ consisting of elements α such that $\alpha \cup [D_k] = 0$. For σ an automorphism of $\mathbb{C}$, let F^σ_k be the corresponding subspace of $H^\sigma = H^1(X^\sigma, \mathbb{R})$. We determined those spaces in lemma 1.8. From there, and from the actual definition of f and f', one sees that F_1 is the only one of the F_k that has a nonzero intersection with two other ones, namely F_3 and F_4. The same is true for the F^σ_k. As a consequence, $\phi(1) = 1$ and $\phi(\{3, 4\}) = \{3, 4\}$, thus $\phi(\{2, 5\}) = \{2, 5\}$.

For k and k' between 1 and 5 such that $F_k \cap F_{k'} = 0$, let $p_{kk'}$ (resp. $p^\sigma_{kk'}$) be the projection along F_k onto $F_{k'}$ (resp. along F^σ_k onto $F^\sigma_{k'}$). Since γ sends $\mathbb{R}[D_k]$ to $\mathbb{R}[D^\sigma_{\phi(k)}]$, it sends F_k to $F^\sigma_{\phi(k)}$. The projection $p_{kk'}$ is therefore conjugate to $p^\sigma_{\phi(k)\phi(k')}$.

Recall that we chose f and f' to be generators of the complex multiplication of A. Let f^* and f'^* be the homomorphisms they induce on the first cohomology group of A. Recall that f^* (resp. f'^*) has eigenvalues 0, a and $-a$ (resp. 0, a' and $-a'$), with a (resp. a') being a generator of k (resp. k'). Direct computation shows that, identifying F_1 with $H^1(A, \mathbb{R})$, we have the equality

$$(p_{21} \circ p_{53})_{|F_1} = 1 - f^*.$$

As a consequence, the endomorphism $(p^\sigma_{\phi(2)1} \circ p^\sigma_{\phi(5)\phi(3)})_{|F^\sigma_1}$ of F^σ_1 is conjugate to $1 - f^{\sigma *}$.

If $\phi(2) = 2$ and $\phi(5) = 5$, this imposes $\phi =$Id, as f^* and f'^* have different eigenvalues. If $\phi(2) = 5$, $\phi(5) = 2$ and $\phi(3) = 3$, then

$$(p^\sigma_{\phi(2)1} \circ p^\sigma_{\phi(5)\phi(3)})_{|F^\sigma_1} = (p^\sigma_{51} \circ p^\sigma_{23})_{|F^\sigma_1} = (1 - f^{\sigma *})^{-1},$$

where we identified F^σ_1 with $H^1(A^\sigma, \mathbb{R})$. Again, consideration of the eigenvalues proves that $(1 - f^{\sigma *})^{-1}$ is not conjugate to $1 - f^{\sigma *}$. Indeed, $\frac{1}{1-a}$ can't be equal to either $1 - a$ or $1 + a$, so this case cannot happen. Similarly, we cannot have $\phi(2) = 5$, $\phi(5) = 2$ and $\phi(3) = 4$. This proves that ϕ is the identity. □

Image of h. The only thing left to show is that γ sends the line $\mathbb{R}h$ to the line $\mathbb{R}h^\sigma$.

Since γ is an isomorphism, and because of the preceding paragraph, it sends h to some nonzero multiple of an element of $H^2(X^\sigma, \mathbb{R})$ of the form

$$h^\sigma + \alpha^\sigma + \lambda_1[D^\sigma_1] + \ldots + \lambda_5[D^\sigma_5],$$

where α^σ is the pull-back of a class in $H^2(A^\sigma \times A^\sigma, \mathbb{R})$.

The Z_k are pairwise disjoint, so the cup-product of any two different $[D_k]$ is 0. Furthermore, if H is a generic hyperplane of $\mathbb{P}^N$, then $A \times A \times H$ is disjoint from Z_1, Z_2, Z_3 and Z_4, which proves that $h \cup [D_k] = 0$ for k between 1 and 4. This is of course true after conjugation by σ.

Let k be between 1 and 4. Since $[D_k]$ is sent to a nonzero multiple of $[D^\sigma_k]$, we conclude that

$$[D^\sigma_k] \cup \alpha^\sigma + \lambda_k[D^\sigma_k]^2 = 0$$

for k between 1 and 4. We will compute this more explicitely to show that it implies that λ_k and α^σ are both zero.

Let us write $\alpha^\sigma = \tau^{\sigma *}\beta^\sigma$, where τ is the blowing-down map and β^σ is a class in $H^2(A^\sigma \times A^\sigma \times \mathbb{P}^N, \mathbb{R})$ coming from $H^2(A^\sigma \times A^\sigma, \mathbb{R})$. For k between 1 and 5, let $h_k \in H^2(D_k, \mathbb{Q})$ be the first Chern class of the normal bundle of D_k in X. It follows from [74], lemma 7.32 that the cohomology of D_k is a free module on the cohomology of Z_k, with basis $1, h_k, \ldots, h_k^{N+1}$, since the codimension of Z_k in $A \times A \times \mathbb{P}^N$ is $N + 2$. For simplicity, we will drop the σ superscript when applied to morphisms.

The self-intersection formula gives, for any positive integer a,

$$[D^\sigma_k]^a = \tilde{j}_{k*}((h^\sigma_k)^{a-1}),$$

where $\tilde{j}_k$ is the inclusion of D_k in X. As a consequence, we can compute, for any positive a and nonnegative b,

$$[D_k^\sigma]^a \cup (\tau^*\beta^\sigma)^b = \tilde{j}_{k*}((h_k^\sigma)^{a-1} \cup \tau_k^* j_k^*(\beta^\sigma)^b).$$

In particular, for k between 1 and 4, we get

$$\tilde{j}_{k*}(\tau_k^* j_k^*(\beta^\sigma) + \lambda_k h_k^\sigma) = 0.$$

Using [74], 7.3.3 as before, we see that the map

$$\tilde{j}_{k*} : H^2(D_k^\sigma, \mathbb{R}) \to H^4(X^\sigma, \mathbb{R})$$

is injective. This proves that

$$\tau_k^* j_k^*(\beta^\sigma) + \lambda_k h_k^\sigma = 0$$

in $H^2(D_k^\sigma, \mathbb{R})$, for any k between 1 and 4, which is equivalent to $\lambda_k = 0$ and $j_k^*(\beta^\sigma) = 0$. Making $k = 1$ and $k = 2$, then using lemma 1.8, this proves that β^σ is in the kernel of p_1 and p_2, hence is zero.

We thus have shown that γ sends h to some nonzero multiple of an element of $H^2(X^\sigma, \mathbb{R})$ of the form

$$h^\sigma + \lambda[D_5^\sigma].$$

We want to show that λ is zero. The next lemma concludes the proof.

Lemma 1.13. *Let λ be a real number. Then*

$$(h + \lambda[D_5])^{N+1} = 0 \Leftrightarrow \lambda = 0.$$

The same is true after conjugation by σ.

Proof. We obviously have $h^{N+1} = 0$, so let us suppose $(h + \lambda[D_5])^{N+1} = 0$. Let H be a hyperplane in $\mathbb{P}^N$. Recall that h is the pullback by the blowing-down map τ of the cohomology class g of $A \times A \times H$ in $A \times A \times \mathbb{P}^N$. Using the preceding computation, we get

$$\tilde{j}_{5*}(\sum_{i=0}^{N} \binom{N+1}{i+1} \lambda^{i+1} h_5^i \cup \tau_5^* j_5^*(g^{N-i})) = 0.$$

Using [74], 7.3.3 again, and noticing that the map

$$j_{5*} : H^*(Z_5, \mathbb{R}) \to H^*(A \times A \times \mathbb{P}^N, \mathbb{R})$$

is injective, we see that the homomorphism

$$\tilde{j}_{5*} : H^{2N}(D_5, \mathbb{Q}) \to H^{2N+2}(X, \mathbb{Q})$$

is injective. We thus get

$$\sum_{i=0}^{N} \binom{N+1}{i+1} \lambda^{i+1} h_5^i \cup \tau_5^* j_5^*(g^{N-i}) = 0,$$

hence, since the h_5^i are linearly independent over $H^*(Z_5, \mathbb{R})$ for $i < \operatorname{codim} Z_5 = N+2$, we obtain

$$\lambda^{i+1} \tau_5^* j_5^*(g^{N-i}) = 0$$

for i between 0 and N. For $i = N$, this proves that $\lambda = 0$. □

Now, since $h^{N+1} = 0$, we have

$$(h^\sigma + \lambda[D_5^\sigma])^{N+1} = 0.$$

The preceding lemma thus shows that $\lambda = 0$, which concludes the proof. □

Proof of proposition 1.7

Proof. Once we know lemma 1.8, the proof of this proposition is purely formal. Indeed, let σ be an automorphism of $\mathbb{C}$. We want to recover, given the abstract graded algebra $H^*(X^\sigma, \mathbb{R})$ together with the lines $\mathbb{R}[D_1^\sigma], \ldots, \mathbb{R}[D_5^\sigma]$ and $\mathbb{R}h^\sigma$, the space $H^1(A^\sigma, \mathbb{R})$ with the action of f^σ and f'^σ, the space $H^2(\mathbb{P}^N, \mathbb{R})$ and the restriction map

$$i^{\sigma*} : H^2(\mathbb{P}^N, \mathbb{R}) \to H^2(A^\sigma, \mathbb{R})$$

induced by the projective embedding of A^σ.

We are going to use the restriction maps p_1^σ and p_2^σ from $H^1(X^\sigma, \mathbb{R}) = H^1(A^\sigma \times A^\sigma, \mathbb{R})$ to $H^1(A^\sigma, \mathbb{R})$ induced by the inclusions of the first and the second factor respectively. From lemma 1.8, we see that the data we have allow us to recover, in the space $H^1(X^\sigma, \mathbb{R})$, the subspaces

$$\mathrm{Ker}(p_1^\sigma),\ \mathrm{Ker}(p_2^\sigma),\ \mathrm{Ker}(p_1^\sigma + p_2^\sigma),\ \mathrm{Ker}(p_1^\sigma + f^{\sigma*}p_2^\sigma),\ \mathrm{Ker}(p_1^\sigma + f'^{\sigma*}p_2^\sigma).$$

Giving those subspaces is equivalent to giving the vector space $H^1(A^\sigma, \mathbb{R})$ together with the action of f^σ and f'^σ on it. Indeed, the first two subspaces determine a splitting of $H^1(X^\sigma, \mathbb{R})$ into two subspaces isomorphic to $H^1(A^\sigma, \mathbb{R})$. The third one is the graph of a specific isomorphism between them, which allow us to identify them – actually, using the opposite of this particular isomorphism in order to get the right sign. The last two subspaces are then the graphs of the endomorphisms $-f^\sigma$ and $-f'^\sigma$ of $H^1(A^\sigma, \mathbb{R})$. We thus recover the real vector space $H^1(A^\sigma, \mathbb{R})$ with the action of f^σ and f'^σ.

The same procedure allows us to recover the other data through the second part of lemma 1.8. Indeed, the line $\mathbb{R}h$ is equal to the space $H^2(\mathbb{P}^N, \mathbb{R}) \subset H^2(X^\sigma, \mathbb{R})$. In the previous paragraph, we showed how to recover the subspace $\mathrm{Ker}(p_1^\sigma) = H^1(A^\sigma, \mathbb{R}) \subset H^1(X^\sigma, \mathbb{R})$, which allows us to recover

$$H^2(A^\sigma, \mathbb{R}) = \bigwedge^2 H^1(A^\sigma, \mathbb{R}) \subset H^2(X^\sigma, \mathbb{R}),$$

this being the image of the cohomology of A^σ under the pull-back by the projection on the second factor. We thus obtained the subspace $H^2(A^\sigma \times \mathbb{P}^N, \mathbb{R}) \subset H^2(X^\sigma, \mathbb{R})$ and its direct sum decomposition

$$H^2(A^\sigma \times \mathbb{P}^N, \mathbb{R}) = H^2(A^\sigma, \mathbb{R}) \oplus H^2(\mathbb{P}^N, \mathbb{R}).$$

Using lemma 1.8, we obtain the graph of the opposite of the homomorphism

$$i^{\sigma*} : H^2(\mathbb{P}^N, \mathbb{R}) \to H^2(A^\sigma, \mathbb{R}).$$

□

Chapter 2

Remarks on the Lefschetz standard conjecture and hyperkähler varieties

Résumé. Dans ce chapitre, on examine quelques aspects de la conjecture standard de Lefschetz sur une variété projective lisse sur $\mathbb{C}$, en s'intéressant plus particulièrement au degré 2, qui est le premier cas non trivial. On montre que la conjecture est dans ce cas équivalente à l'existence de grosses familles de déformations de fibrés vectoriels sur X. On donne un critère précis faisant intervenir l'application de Kodaira-Spencer associées à de telles familles qui permet s'il est vérifié de démontrer la conjecture de Lefschetz.

Dans une dernière partie, on examine en détail le cas des variétés hyperkähleriennes, et l'on montre comment les résultats de Verbitsky autour des espaces de modules de fibrés hyperholomorphes donnent une forme particulièrement explicite à nos résultats et comment certains de ses résultats conjecturaux impliquent la conjecture en degré 2.

Abstract. We study the Lefschetz standard conjecture on a smooth complex projective variety X. In degree 2, we reduce it to a local statement concerning local deformations of vector bundles on X. When X is hyperkähler, we give explicit criteria which imply the conjecture, using Verbitsky's theory of deformations of hyperholomorphic bundles.

2.1 Introduction

In the fundamental paper [35] of 1968, Grothendieck states a series of conjectures concerning the existence of some algebraic cycles on smooth projective algebraic varieties over an algebraically closed ground fields. Those are known as the standard conjectures. In particular, given such a variety X of pure dimension n, the Lefschetz standard conjecture predicts the existence of self-correspondences on X that give an inverse to the operations

$$H^k(X) \to H^{2n-k}(X)$$

given by the cup-product $n-k$ times with a hyperplane section for all $k \leq n$. Here $H^*(X)$ stands for any Weil cohomology theory on X, e.g. singular cohomology if X is defined over $\mathbb{C}$, or l-adic étale cohomology in characteristic different from l. If we can invert the morphism $H^k(X) \to H^{2n-k}(X)$ using self-correspondences on X, we say that the Lefschetz conjecture holds in degree k.

Let us now, and for the rest of the paper, work over $\mathbb{C}$. The Lefschetz standard conjecture then implies the other ones and has strong theoretical consequences. For instance, it implies that numerical and homological equivalence coincide, and that the category of pure motives for homological equivalence is semisimple. We refer to [45] and [46] for more detailed discussions. The Lefschetz standard conjecture for varieties which are fibered in abelian varieties over a smooth curve also implies the Hodge conjecture for abelian varieties as shown by Yves André in [3]. Grothendieck actually writes in the aforementioned paper that "alongside the resolution of singularities, the proof of the standard conjectures seems to [him] to be the most urgent task in algebraic geometry".

Though the motivic picture has tremendously developed since Grothendieck's statement of the standard conjectures, very little progress has been made in their direction. The Lefschetz standard conjecture is known for abelian varieties, see [45] and in degree 1 where it reduces to the Hodge conjecture for divisors. Aside from examples obtained by taking products and hyperplane sections, those seem to be the only two cases where a proof is known.

In this paper, we want to investigate further the geometrical content of the Lefschetz standard conjecture, and try to give insight into the specific case of hyperkähler varieties. The original form of the Lefschetz standard conjecture for a variety X predicts the existence of specific algebraic cycles in the product $X \times X$. Those cycles can be considered as family of cycles on X parametrized by X itself. Our first remark is that the conjecture actually reduces to a general statement about the existence of large families of algebraic cycles in X parametrized by any smooth quasi-projective base. For this, we use Hodge theory on X.

It turns out that for those families to give a positive answer to the conjecture, it is enough to control the local variation of the family of cycles considered. Let us give a precise statement. Let X be a smooth projective variety, S a smooth quasi-projective variety, and let $Z \in CH^k(X \times S)$ be a family of codimension k cycles in X parametrized by S. Let $\mathcal{T}_S$ be the tangent sheaf of S. Using the Leray spectral sequence for the projection onto S and constructions from Griffiths and Voisin in [33], [71], we construct a map

$$\phi_Z : \bigwedge^k \mathcal{T}_S \to H^k(X, \mathcal{O}_X) \otimes \mathcal{O}_S,$$

We then get the following result, which we state in degree 2 for simplicity, see section 2.

Theorem 2.1. *Let X be a smooth projective variety. Then the Lefschetz conjecture is true in degree 2 for X if and only if there exists a smooth quasi-projective variety S, a codimension 2 cycle Z in $CH^2(X \times S)$ and a point $s \in S$ such that the morphism*

$$\phi_{Z,s} : \bigwedge^2 \mathcal{T}_{S,s} \to H^2(X, \mathcal{O}_X)$$

considered above for $k = 2$, is surjective.

This variational approach to the existence of algebraic cycles can be compared to the study of semi-regularity maps as in [12].

In the following section, we give an explicit formula for ϕ_Z in case the cycle Z is given by the Chern classes of a family of vector bundles $\mathcal{E}$ on $X \times S$. In this situation, we show that ϕ_Z is expressed very simply in terms of the Kodaira-Spencer map. Indeed, $\mathcal{T}_{S,s}$ maps to the space $\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$. We then have a Yoneda product

$$\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s) \times \mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s) \to \mathrm{Ext}^2(\mathcal{E}_s, \mathcal{E}_s)$$

and a trace map

$$\mathrm{Ext}^2(\mathcal{E}_s, \mathcal{E}_s) \to H^2(X, \mathcal{O}_X).$$

We show that we can express $\phi_{Z,s}$ in terms of the composition

$$\phi_2(\mathcal{E}) : \bigwedge^2 \mathcal{T}_{S,s} \to H^2(X, \mathcal{O}_X)$$

of those two maps, and we get the following theorem.

Theorem 2.2. *Let X be a smooth projective variety. Then the Lefschetz conjecture is true in degree 2 for X if there exists a smooth quasi-projective variety S, a vector bundle $\mathcal{E}$ over $X \times S$, and a point $s \in S$ such that the morphism*

$$\phi_2(\mathcal{E})_s : \bigwedge^2 \mathcal{T}_{S,s} \to H^2(X, \mathcal{O}_X) \tag{2.1}$$

induced by the composition above is surjective.

The main interest of this theorem is that it makes it possible to only use first-order computations to check the Lefschetz standard conjecture, which is global in nature, thus replacing it by a local statement on deformations of $\mathcal{E}$. Of course, when one wants to ensure that there exists a vector bundle over X that has a positive-dimensional family of deformations, the computation of obstructions is needed, which involves higher-order computations. However, once a family of vector bundles is given, checking the surjectivity condition of the theorem involves only first-order deformations.

The last part of the paper is devoted to applications of the previous results to hyperkähler varieties. We will recall general properties of those and their hyperholomorphic bundles in section 4. Those varieties have $h^{2,0} = 1$, which makes the last criterion easier to check. In the case of 2-dimensional hyperkähler varieties, that is, in the case of K3 surfaces, Mukai has investigated in [51] the 2-form on the moduli space of some stable sheaves given by (2.1) and showed that it is nondegenerate. In particular, it is nonzero. Of course, the case of surfaces is irrelevant in our work, but we will use Verbitsky's theory of hyperholomorphic bundles on hyperholomorphic varieties as presented in [69]. In his work, Verbitsky extends the work of Mukai to higher dimensions and shows results implying the nondegeneracy of the the form (2.1) in some cases. Using those, we are able to show that the existence of nonrigid hyperholomorphic bundles on a hyperkähler variety is enough to prove the Lefschetz standard conjecture in degree 2. Indeed, we get the following.

Theorem 2.3. *Let X be a projective irreducible hyperkähler variety, and let $\mathcal{E}$ be a stable hyperholomorphic bundle on X. Assume that $\mathcal{E}$ has a nontrivial positive-dimensional family of deformations. Then the Lefschetz conjecture is true in degree 2 for X.*

In a slightly different direction, recall that the only known hyperkähler varieties, except in dimension 6 and 10, are the two families constructed by Beauville in [8] which are the small deformations of Hilbert schemes of points on a K3 surface or of generalized Kummer varieties. For those, the Lefschetz standard conjecture is very easy as their cohomology comes from that of a surface. We get the following.

Theorem 2.4. *Let n be a positive integer. Assume that for every K3 surface S, there exists a stable hyperholomorphic sheaf $\mathcal{E}$ with a nontrivial positive-dimensional family of deformations on the Hilbert scheme $S^{[n]}$ parametrizing subschemes of S of length n. Then the Lefschetz conjecture is true in degree 2 for any projective deformation of $S^{[n]}$. The same result holds for generalized Kummer varieties.*

Both those results could be applied taking $\mathcal{E}$ to be the tangent sheaf of the variety considered, in case it has nontrivial deformations.

Those results fit well in the – mostly conjectural – work of Verbitsky as exposed in [70] predicting the existence of large moduli spaces of hyperholomorphic bundles. Unfortunately, we were not able to exhibit bundles satisfying the hypotheses of the theorems.

Acknowledgements. It is a great pleasure to thank Claire Voisin for her help and support, as well as many enlightening discussions during the writing of this paper. I would also like to thank Daniel Huybrechts for pointing out the relevance of Verbitsky's results.

2.2 General remarks on the Lefschetz standard conjecture

This section is devoted to some general remarks on the Lefschetz standard conjecture. Although some are well-known to specialists, we include them here for ease of reference. Let us first recall the statement of the conjecture.

Let X be a smooth irreducible projective variety of dimension n over $\mathbb{C}$. Let $\xi \in H^2(X, \mathbb{Q})$ be the cohomology class of a hyperplane section of X. According to the hard Lefschetz theorem, see for instance [74], Chapter 13, for all $k \in \{0, \ldots, n\}$, cup-product with ξ^{n-k} induces an isomorphism

$$\cup\xi^{n-k} : H^k(X, \mathbb{Q}) \to H^{2n-k}(X, \mathbb{Q}).$$

The Lefschetz standard conjecture was first stated in [35], conjecture $B(X)$. It is the following.

Conjecture 2.5. *Let X and ξ be as above. Then for all $k \in \{0, \ldots, n\}$, there exists an algebraic cycle Z of codimension k in the product $X \times X$ such that the correspondence*

$$[Z]_* : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

is the inverse of $\cup\xi^{n-k}$.

If this conjecture holds for some specific k on X, we will say the Lefschetz conjecture holds in degree k for the variety X. In case X is not irreducible, we say that the Lefschetz standard conjecture holds for X if it holds for all the irreducible components of X.

Let us recall the following easy lemma, see [46], Theorem 4.1, which shows in particular that the Lefschetz conjecture does not depend on the choice of a polarization.

Lemma 2.6. *Let X and ξ be as above. Then the Lefschetz conjecture holds in degree k for X if and only if there exists an algebraic cycle Z of codimension k in the product $X \times X$ such that the correspondence*

$$[Z]_* : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

is bijective.

Proof. Let Z be as in the lemma. The morphism

$$[Z]_* \circ (\cup\xi^{n-k} \circ [Z]_*)^{-1} : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

is the inverse of $\cup\xi^{n-k} : H^k(X, \mathbb{Q}) \to H^{2n-k}(X, \mathbb{Q})$. Now by the Cayley-Hamilton theorem, the automorphism $(\cup\xi^{n-k} \circ [Z]_*)^{-1}$ of $H^{2n-k}(X, \mathbb{Q})$ is a polynomial in $(\cup\xi^{n-k} \circ [Z]_*)$. As such, it is defined by an algebraic correspondence. By composition, the morphism $[Z]_* \circ (\cup\xi^{n-k} \circ [Z]_*)^{-1}$ is defined by an algebraic correspondence, which concludes the proof. $\square$

For the next results, we will need to work with primitive cohomology classes. Let us recall some notation. Let S be a smooth connected polarized projective variety of dimension l. Let L denote cup-product with the cohomology class of a hyperplane section. For any integer k in $\{0, \ldots, l\}$, let $H^k(S, \mathbb{Q})_{prim}$ denote the primitive part of $H^k(S, \mathbb{Q})$, that is, the kernel of

$$L^{n-k+1} : H^k(S, \mathbb{Q}) \to H^{2l-k+2}(S, \mathbb{Q}).$$

The cohomology groups of S in degrees less than l then admit a Lefschetz decomposition

$$H^k(S, \mathbb{Q}) = \bigoplus_{i \geq 0} L^i H^{k-2i}(X, \mathbb{Q})_{prim}.$$

The following lemma is well-known, but we include it here for ease of reference as well as to keep track of the degrees for which we have to use the Lefschetz standard conjecture.

Lemma 2.7. *Let k be an integer, and let S be a smooth projective variety of pure dimension $n \geq k$. Consider the Lefschetz decomposition*

$$H^k(S, \mathbb{Q}) = \bigoplus_{i \geq 0} L^i H^{k-2i}(S, \mathbb{Q})_{prim},$$

where L is the cup-product by the class of a hyperplane section. Assume that the Leschetz conjecture holds for S in degrees up to $k-2$. Then the projections $H^k(S, \mathbb{Q}) \to L^i H^{k-2i}(S, \mathbb{Q})_{prim}$ are induced by algebraic correspondences.

Proof. By induction, it is enough to prove that the projection $H^k(S, \mathbb{Q}) \to LH^{k-2}(S, \mathbb{Q})$ is induced by an algebraic correspondence. Let $Z \in S \times S$ be an algebraic cycle such that

$$[Z]_* : H^{2n-k+2}(S, \mathbb{Q}) \to H^{k-2}(S, \mathbb{Q})$$

is the inverse of L^{n-k+2}. Then the composition $L \circ [Z]_* \circ L^{n-k+1}$ is the desired projection since $H^k(S, \mathbb{Q})_{prim}$ is the kernel of L^{n-k+1} in $H^k(S, \mathbb{Q})$. □

The next result is the starting point of our paper. It shows that the Lefschetz standard conjecture in degree k on X is equivalent to the existence of a sufficiently big family of codimension k algebraic cycles in X, and allows us to work on the product of X with any variety.

Proposition 2.8. *Let X be a smooth projective variety of pure dimension n, and let $k \leq n$ be an integer. Assume the Lefschetz conjecture for all smooth projective varieties in degrees up to $k-2$. Then the Lefschetz conjecture is true in degree k for X if and only if there exists a smooth projective variety S of pure dimension $l \geq k$ and a codimension k cycle Z in $X \times S$ such that the morphism*

$$[Z]_* : H^{2l-k}(S, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

induced by the correspondence Z is surjective.

Proof. Fix a polarization on S, and let L be the cup-product with the class of a hyperplane section of S. Consider the morphism $s : H^k(S, \mathbb{Q}) \to H^k(S, \mathbb{Q})$ which is given by multiplication by $(-1)^i$ on $L^i H^{k-2i}(S, \mathbb{Q})_{prim}$. By the Hodge index theorem, the pairing

$$H^k(S, \mathbb{C}) \otimes H^k(S, \mathbb{C}) \to \mathbb{C},\ \alpha \otimes \beta \mapsto \int_S \alpha \cup L^{l-k}(s(\beta))$$

turns $H^k(S, \mathbb{Q})$ into a polarized Hodge structure. Furthermore, Lemma 2.7 shows that s is induced by an algebraic correspondence.

We have a morphism $[Z]_* : H^{2l-k}(S,\mathbb{Q}) \to H^k(X,\mathbb{Q})$ which is surjective. Its dual $[Z]^* : H^{2n-k}(X,\mathbb{Q}) \to H^k(S,\mathbb{Q})$ is injective, where n is the dimension of X. Let us consider the composition

$$[Z]_* \circ L^{l-k} \circ s \circ [Z]^* : H^{2n-k}(X,\mathbb{Q}) \to H^k(X,\mathbb{Q}),$$

It is defined by an algebraic correspondence, and it is enough to show that it is a bijection. Since $H^{2n-k}(X,\mathbb{Q})$ and $H^k(X,\mathbb{Q})$ have the same dimension, we only have to prove it is injective.

Let $\alpha \in H^{2n-k}(X,\mathbb{Q})$ lie in the kernel of the composition. For any $\beta \in H^{2n-k}(X,\mathbb{Q})$, we get

$$([Z]^*\beta) \cup ((L^{l-k} \circ s)([Z]^*\alpha)) = 0.$$

Since $[Z]^*(H^{2n-k}(X,\mathbb{Q}))$ is a sub-Hodge structure of the polarized Hodge structure $H^k(S,\mathbb{Q})$, the restriction of the polarization

$$< \alpha, \beta >= \int_S \alpha \cup (L^{l-k} \circ s)(\beta)$$

on $H^k(S,\mathbb{Q})$ to this subspace is nondegenerate, which shows that α is zero. □

Remark 2.9. Using the weak Lefschetz theorem, one can always reduce to the case where S is of dimension k.

Corollary 2.10. *Let X be a smooth projective variety of pure dimension n, and let $k \leq n$ be an integer. Assume the Lefschetz conjecture for all varieties in degrees up to $k-2$ and that the generalized Hodge conjecture is true for $H^k(X,\mathbb{Q})$.*

Then the Lefschetz conjecture is true in degree k for X if and only if there exists a smooth projective variety S, of pure dimension that we will denote by l, and a codimension k cycle Z in $CH^k(X \times S)$ such that the morphism

$$H^l(S, \Omega_S^{l-k}) \to H^k(X, \mathcal{O}_X) \tag{2.2}$$

induced by the morphism of Hodge structures

$$[Z]_* : H^{2l-k}(S,\mathbb{C}) \to H^k(X,\mathbb{C})$$

is surjective.

Remark 2.11. Note that this corollary is unconditional for $k = 2$ since the generalized Hodge conjecture is just the Hodge conjecture for divisors, and the Lefschetz standard conjecture is obvious in degree 0.

Proof. Let X, S and Z be as in the statement of the corollary. Let H be the image of $H^{2l-k}(S,\mathbb{Q})$ by $[Z]_*$. Since the colevel of the image of $LH^{2l-k-2}(S,\mathbb{Q})$ by $[Z]_*$ is at least one, we have $H^{k,0} = H^k(X,\mathcal{O}_X)$. Let H' be a sub-Hodge structure of $H^k(X,\mathbb{Q})$ such that $H^k(X,\mathbb{Q}) = H \oplus H'$. Then $H'^{k,0} = 0$. As H' has no part of type $(k,0)$, the generalized Hodge conjecture then predicts that there exists a smooth projective variety X' of dimension $n-1$, together with a proper morphism $f : X' \to X$ such that H' is contained in $f_*H^{k-2}(X',\mathbb{Q})$.

If the Lefschetz conjecture is true in degree $k-2$, then it is true for $H^{k-2}(X',\mathbb{Q})$. As a consequence, we get a cycle Z' of codimension $k-2$ in some $S' \times X'$, where S' is smooth projective of pure dimension $l' \geq k-2$, such that $[Z']_* : H^{2l'-k+2}(S',\mathbb{Q}) \to H^{k-2}(X',\mathbb{Q})$ is surjective. Consider the composition

$$H^{2l'+2-k}(S' \times \mathbb{P}^1,\mathbb{Q}) \twoheadrightarrow H^{2l'-k+2}(S',\mathbb{Q}) \twoheadrightarrow H^{k-2}(X',\mathbb{Q}) \to H^k(X,\mathbb{Q}),$$

the first map being the pullback by any of the immersions $S' \to S' \times \mathbb{P}^1, s' \mapsto (s', x)$, the second one being $[Z]_*$ and the last one f_*. This composition is induced by an algebraic correspondence $Z'' \hookrightarrow S' \times \mathbb{P}^1$, and is surjective onto $f_* H^{k-2}(X', \mathbb{Q})$. It is easy to assume, after taking products with projective spaces, that S and $S' \times \mathbb{P}^1$ have the same dimension. Now since the subspaces H and $f_* H^{k-2}(X', \mathbb{Q})$ generate $H^k(X, \mathbb{Q})$, the correspondence induced by the cycle $Z + Z''$ in $(S \coprod (S' \times \mathbb{P}^1)) \times X$ satisfies the hypotheses of Proposition 2.8. □

With the notations of the previous corollary, in case, Z is flat over X, we have a family of codimension k algebraic cycles in X parametrized by S. The next theorem shows that the map (2.2), which is the one we have to study in order to prove the Lefschetz conjecture in degree k for X, does not depend on the global geometry of S, and can be computed locally on S. This will allow us to give an explicit description of the map 2.2 in terms of the deformation theory of the family Z in the next section.

Let us first recall a general cohomological invariant for families of algebraic cycles. We follow [74], 19.2.2, see also [33], [71] for related discussions. In the previous setting, Z, X and S being as before, the algebraic cycle Z has a class

$$[Z] \in H^k(X \times S, \Omega^k_{X \times S}).$$

Using the Leray spectral sequence for the projection $p : X \times S \to S$, this last group maps to $H^0(S, R^k p_* \Omega^k_{X \times S})$. Now the sheaf $\Omega^k_{X \times S}$ on $X \times S$ contains $p^* \Omega^k_S$ as a direct factor, hence a map

$$R^k p_* \Omega^k_{X \times S} \to H^k(X, \mathcal{O}_X) \otimes \Omega^k_S.$$

This construction maps the cohomology class of $[Z]$ to an element of the group

$$H^0(S, \Omega^k_S) \otimes H^k(X, \mathcal{O}_X),$$

that is, to a morphism of sheaves on S

$$\phi_Z : \bigwedge^k \mathcal{T}_S \to H^k(X, \mathcal{O}_X) \otimes \mathcal{O}_S, \tag{2.3}$$

where $\mathcal{T}_S$ is the tangent sheaf of S. If s is a complex point of S, let $\phi_{Z,s}$ be the morphism $\bigwedge^k \mathcal{T}_{S,s} \to H^k(X, \mathcal{O}_X)$ coming from ϕ_Z. From the previous discussion, it is straightforward that the definition of $\phi_{Z,s}$ is local on S. Actually, it can be shown that it only depends on the first order deformation Z_s^ϵ of Z_s in X, see [74], Remarque 19.12 under rather weak assumptions. We will recover this result in the next section by giving an explicit formula for $\phi_{Z,s}$.

The next theorem shows, using the map $\phi_{Z,s}$, that the Lefschetz conjecture can be reduced to the existence of local deformations of algebraic cycles in X.

Theorem 2.12. *Let X be a smooth irreducible projective variety. Assume as in Corollary 2.10 that the generalized Hodge conjecture is true for $H^k(X, \mathbb{Q})$ and the Lefschetz conjecture holds for smooth projective varieties in degree $k-2$.*

Then the Lefschetz conjecture is true in degree k for X if and only if there exist a smooth quasi-projective variety S, a codimension k cycle Z in $CH^k(X \times S)$ and a point $s \in S$ such that the morphism

$$\phi_{Z,s} : \bigwedge^k \mathcal{T}_{S,s} \to H^k(X, \mathcal{O}_X) \tag{2.4}$$

is surjective.

Proof. Assume the hypothesis of the theorem holds. Up to taking a smooth projective compactification of S and taking the adherence of Z, we can assume S is smooth projective. The morphism of sheaves

$$\phi_Z : \bigwedge^k \mathcal{T}_S \to H^k(X, \mathcal{O}_X) \otimes \mathcal{O}_S$$

that we constructed earlier corresponds to an element of the group

$$H^0(S, \Omega_S^k) \otimes H^k(X, \mathcal{O}_X),$$

which in turn using Serre duality corresponds to a morphism

$$H^l(S, \Omega_S^{l-k}) \to H^k(X, \mathcal{O}_X).$$

Now since Serre duality is compatible with Poincaré duality, and since the Leray spectral sequence is just a generalization of the Künneth decomposition, this morphism is by definition the morphism (2.2) of Corollary 2.10. Moreover, by construction, it is surjective as soon as $\phi_{Z,s}$ is, which concludes the proof by Corollary 2.10. □

The important part of this theorem is that it does not depend on the global geometry of S, but only on the local variation of the family Z. As such, it makes it possible to use deformation theory and moduli spaces to study the Lefschetz conjecture, especially in degree 2 where Theorem 2.12 is unconditional by Remark 2.11.

2.3 A local computation

Let X and S be smooth varieties, X being projective and S quasi-projective. Let Z be a cycle of codimension k in the product $X \times S$. As we saw earlier, for any point $s \in S$, the correspondence defined by Z induces a map

$$\phi_{Z,s} : \bigwedge^k \mathcal{T}_{S,s} \to H^k(X, \mathcal{O}_X)$$

The goal of this section is to compute this map in terms of the deformation theory of the family Z of cycles on X parametrized by S. We will formulate this result when the class of Z in the Chow group of $X \times S$ is given by the codimension k part $ch_k(\mathcal{E})$ of the Chern character of a vector bundle $\mathcal{E}$ over $X \times S$. It is well-known that we obtain all the rational equivalence classes of algebraic cycles as linear combinations of those.

Let us now recall general facts about the deformation theory of vector bundles and their Atiyah class. Given a vector bundle $\mathcal{E}$ over $X \times S$, and p being the projection of $X \times S$ to S, let $\mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E})$ be the sheafification of the presheaf $U \mapsto \mathrm{Ext}^1_{\mathcal{O}_{X\times U}}(\mathcal{E}_{|X\times U}, \mathcal{E}_{|X\times U})$ on S. The deformation of vector bundles determined by $\mathcal{E}$ is described by the Kodaira-Spencer map. This is a map of sheaves

$$\rho : \mathcal{T}_S \to \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E}),$$

where $\mathcal{T}_S$ is the tangent sheaf to S. Let s be a complex point of S. The Kodaira-Spencer map at s is given by the morphism

$$\rho_s : T_{S,s} \to \mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s),$$

which is just the specialization of the Kodaira-Spencer map ρ. The space $\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$ is the set of first-order deformations of $\mathcal{E}_s$. The elements of the group $\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$ which correspond to deformations of $\mathcal{E}_s$ that extend over a one-dimensional basis are called unobstructed. In the next section, we will use results of Verbitsky which allow us to produce unobstructed elements of $\mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$ in the hyperholomorphic setting.

Associated to $\mathcal{E}$ as well are the images in $H^k(X \times S, \Omega^k_{X\times S})$ of the Chern classes of $\mathcal{E}$, which we will denote by $c_k(\mathcal{E})$ with a slight abuse of notation. We also have the images $ch_k(\mathcal{E}) \in H^k(X \times S, \Omega^k_{X\times S})$ of the Chern character.

The link between Chern classes and the Kodaira-Spencer map is given by the Atiyah class. Recall from [37], Chapter 10, that the Atiyah class of a coherent sheaf $\mathcal{F}$ over a smooth quasi-projective variety Y is an element $A(\mathcal{F}) \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega^1_Y)$ which measures the obstruction for $\mathcal{F}$ to have an algebraic connection. It is well-known that the Chern classes of $\mathcal{F}$ can be computed from its Atiyah class, see [6], [37] :

Proposition 2.13. *For k a positive integer, let $\alpha_k \in H^k(Y, \Omega^k_Y)$ be the trace of the element $A(\mathcal{F})^k \in \mathrm{Ext}^k(\mathcal{F}, \mathcal{F} \otimes \Omega^k_Y)$ by the trace map. Then*

$$\alpha_k = k!\, ch_k(\mathcal{F}).$$

Now in the relative situation with our previous notation, the vector bundle $\mathcal{E}$ has an Atiyah class $A(\mathcal{E})$ with value in $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times S})$. The latter group maps to the group $H^0(S, \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times S}))$, which contains

$$H^0(S, \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E} \otimes p^*\Omega^1_S)) = H^0(S, \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E}) \otimes \Omega^1_S)$$

as a direct factor. We thus get a class $\beta \in H^0(S, \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E}) \otimes \Omega^1_S)$, which corresponds to a morphism of sheaves

$$\tau : \mathcal{T}_S \to \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E}).$$

For the following well-known computation, see [37] or [38], Chapter IV.

Proposition 2.14. *The map τ induced by the Atiyah class of $\mathcal{E}$ is equal to the Kodaira-Spencer map ρ.*

Those two results make it possible to give an explicit description of the map ϕ_Z of last section in case the image of Z in the Chow group of $X \times S$ is given by the codimension k part $ch_k(\mathcal{E})$ of the Chern character of a vector bundle $\mathcal{E}$ over $X \times S$. First introduce a map of sheaves coming from the Kodaira-Spencer map.

For k a positive integer, let

$$\phi_k(\mathcal{E}) : \bigwedge^k \mathcal{T}_S \to H^k(X, \mathcal{O}_X) \otimes \mathcal{O}_S$$

be the composition of the k-th alternate product of the Kodaira-Spencer map with the map

$$\bigwedge^k \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E}) \to \mathcal{E}xt^k_p(\mathcal{E}, \mathcal{E}) \to H^k(X, \mathcal{O}_X) \otimes \mathcal{O}_S,$$

the first arrow being Yoneda product and the second being the trace map.

Lemma 2.15. *We have*

$$\phi_k(\mathcal{E}) = k!\, \phi_{ch_k(\mathcal{E})},$$

where $\phi_{ch_k(\mathcal{E})}$ is the map appearing in (2.3).

Proof. We have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times S})^{\otimes k} & \longrightarrow & \mathrm{Ext}^k(\mathcal{E}, \mathcal{E} \otimes \Omega^k_{X\times S}) & \longrightarrow & H^k(X \times S, \Omega^k_{X\times S}) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(S, \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times S}))^{\otimes k} & \longrightarrow & H^0(S, \mathcal{E}xt^k_p(\mathcal{E}, \mathcal{E} \otimes \Omega^k_{X\times S})) & \longrightarrow & H^0(S, R^k p_* \Omega^k_{X\times S}) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(S, \Omega^1_S \otimes \mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E}))^{\otimes k} & \longrightarrow & H^0(S, \Omega^k_S \otimes \mathcal{E}xt^k_p(\mathcal{E}, \mathcal{E})) & \longrightarrow & H^0(S, \Omega^k_S \otimes H^k(X, \mathcal{O}_X)),
\end{array}$$

where the horizontal maps on the left are given by Yoneda product, the horizontal maps on the right side are the trace maps, the upper vertical maps come from the Leray exact sequence associated to p, and the lower vertical maps come from the projection $\Omega^1_{X\times S} \to p^*\Omega^1_S$.

By definition, and using Proposition 2.14, the element $A(\mathcal{E})^{\otimes k} \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X\times S})^{\otimes k}$ maps to

$$\phi_k(\mathcal{E}) \in \mathrm{Hom}(\bigwedge^k \mathcal{T}_S, H^k(X, \mathcal{O}_X) \otimes \mathcal{O}_S) = H^0(S, \Omega^k_S \otimes H^k(X, \mathcal{O}_X)),$$

following the left side, then the lower side of the diagram. On the other hand, Proposition 2.13 shows that it also maps to $k!\,\phi_{ch_k(\mathcal{E})}$, following the upper side, then the right side of the diagram. This concludes the proof. □

As an immediate consequence, we get the following criterion.

Theorem 2.16. *Let X be a smooth irreducible projective variety, and assume the same hypotheses as in Theorem 2.12. Then the Lefschetz conjecture is true in degree k for X if there exists a smooth quasi-projective variety S, a vector bundle $\mathcal{E}$ over $X \times S$, and a point $s \in S$ such that the morphism*

$$\phi_k(\mathcal{E})_s : \bigwedge^k \mathcal{T}_{S,s} \to H^k(X, \mathcal{O}_X) \tag{2.5}$$

induced by $\phi_k(\mathcal{E})$ is surjective.

Example. Let A be a polarized complex abelian variety of dimension g. The tangent bundle of A is canonically isomorphic to $H^1(A, \mathcal{O}_A) \otimes \mathcal{O}_A$. The trivial line bundle $\mathcal{O}_A$ on A admits a family of deformations parametrized by A itself such that the Kodaira-Spencer map $T_{A,O} \to H^1(A, \mathcal{O}_A)$ is the identity under the above identification. Now the induced deformation of $\mathcal{O}_A \oplus \mathcal{O}_A$ parametrized by $A \times A$ satisfies the criterion of Theorem 2.16, since the map $\bigwedge^2 H^1(A, \mathcal{O}_A) \to H^2(A, \mathcal{O}_A)$ given by cup-product is surjective and identifies with the map (2.5). Of course, the Lefschetz conjecture for abelian varieties is well-known, see [45].

2.4 The case of hyperkähler varieties

In this section, we describe how Verbitsky's theory of hyperholomorphic bundles on hyperkähler varieties as developed in [69] and [70] makes those a promising source of examples for theorem 2.16. Unfortunately, we were not able to provide examples, as it appears some computations of dimensions of moduli spaces in [70] were incorrect, but we will show how the existence of nontrivial examples of moduli spaces of hyperholomorphic bundles on hyperkähler varieties as conjectured in [70] implies the Lefschetz standard conjecture in degree 2.

2.4.1 Hyperholomorphic bundles on hyperkähler varieties

See [8] for general definitions and results. An irreducible hyperkähler variety is a simply connected kähler manifold which admits a closed everywhere non-degenerate two-form which is unique up to a factor. As such, an irreducible hyperkähler variety X has $H^{2,0}(X, \mathcal{O}_X) = \mathbb{C}$, and Theorem 2.16 takes the following simpler form in degree 2.

Theorem 2.17. *Let X be an irreducible projective hyperkähler variety. The Lefschetz conjecture is true in degree 2 for X if there exists a smooth quasi-projective variety S, a vector bundle $\mathcal{E}$ over $X \times S$, and a point $s \in S$ such that the morphism*

$$\phi_2(\mathcal{E})_s : \bigwedge^2 \mathcal{T}_{S,s} \to H^2(X, \mathcal{O}_X), \tag{2.6}$$

induced by the Kodaira-Spencer map and the trace map, is nonzero.

In the paper [8], Beauville constructs two families of projective irreducible hyperkähler varieties in dimension $2n$ for every integer n. Those are the n-th punctual Hilbert scheme $S^{[n]}$ of a projective $K3$ surface S and the generalized Kummer variety K_n which is the fiber at the origin of the Albanese map from $A^{[n+1]}$ to A, where A is an abelian surface and $A^{[n+1]}$ is the $n+1$-st punctual Hilbert scheme of A.

The Bogomolov-Tian-Todorov theorem, see [14], [66], [67], states that the local moduli space of deformations of an irreducible hyperkähler variety is unobstructed. Small deformations of a hyperkähler variety remain hyperkähler, and in the local moduli space of $S^{[n]}$ and K_n, the projective hyperkähler varieties form a dense countable union of hypersurfaces. The varieties $S^{[n]}$ and K_n have Picard number at least 2, whereas a very general projective irreducible hyperkähler variety has Picard number 1, hence is not of this form. Except in dimension 6 and 10, where O'Grady constructs in [54] and [55] new examples, all the known hyperkähler varieties are deformations of $S^{[n]}$ or K_n.

The Lefschetz standard conjecture is easy to prove in degree 2 for $S^{[n]}$ (resp. K_n), using the tautological correspondence with the $K3$ surface (resp. the abelian surface). In terms of Theorem 2.16, one can show that the tautological sheaf on $S^{[n]}$ (resp. K_n) associated to the tangent sheaf of S has enough deformations to prove the Lefschetz conjecture in degree 2. Since the tautological correspondence between S and $S^{[n]}$ gives an isomorphism between $H^{2,0}(S)$ and $H^{2,0}(S^{[n]})$, checking that the criterion is satisfied amounts to the following.

Proposition 2.18. *Let S be a projective $K3$ surface. Then there exists a smooth quasi-projective variety M with a distinguished point O parametrizing deformations of $\mathcal{T}_S$ and a vector bundle $\mathcal{E}$ over $M \times M$ such that $\mathcal{E}_{|\{O \times S\}} \simeq \mathcal{T}_S$, such that the map*

$$\phi_2(\mathcal{E})_O : \bigwedge^2 \mathcal{T}_{M,O} \to H^2(S, \mathcal{O}_S)$$

induced by the Kodaira-Spencer map and the trace map, is nonzero.

We will give a more general approach to this map in a moment, but we will first treat this case in an elementary way.

Proof. The trace map $\mathrm{Ext}^2(\mathcal{T}_S, \mathcal{T}_S) \to H^2(S, \mathcal{O}_S) = \mathbb{C}$ is an isomorphism. Indeed, by Serre duality, it is the dual of the inclusion $\mathbb{C} \to \mathrm{Hom}(\mathcal{T}_S, \mathcal{T}_S)$, which is an isomorphism since $\mathcal{T}_S$ is a stable vector bundle by Yau's theorem in [77].

It follows from this remark that the deformations of the tangent bundle of a $K3$ surface are unobstructed since the obstructions to deform $\mathcal{T}_S$ lie in the kernel of the trace map in $\mathrm{Ext}^2(\mathcal{T}_S, \mathcal{T}_S)$. As a result, we can choose for M a variety such that the Kodaira-Spencer map $\mathcal{T}_{M,O} \to \mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S)$ is an isomorphism, and the only thing to check is that the cup-product map

$$\mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S) \times \mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S) \to \mathrm{Ext}^2(\mathcal{T}_S, \mathcal{T}_S) \simeq \mathbb{C}$$

is nonzero.

Now we have

$$\dim(\mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S)) = 90.$$

Indeed, using Riemann-Roch or [51], Theorem 0.1, we get

$$\dim(\mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S)) = c_1(\mathcal{T}_S)^2 - 4\chi(\mathcal{T}_S) + 4\chi(\mathcal{O}_S) + 2,$$

which gives the result as $c_1(\mathcal{T}_S) = 0$, $\chi(\mathcal{T}_S) = -20$ and $\chi(\mathcal{O}_S) = 2$.

Let us choose a non-trivial extension

$$0 \to \mathcal{T}_S \to \mathcal{G} \to \mathcal{T}_S \to 0. \tag{2.7}$$

with class $\alpha \in \mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S)$. The map

$$\mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S) \to \mathrm{Ext}^2(\mathcal{T}_S, \mathcal{T}_S)$$

given by the cup-product with α is the one that comes from the long exact sequence of cohomology associated to the extension (2.7). We want to prove that it is nonzero. By Serre duality, this map is dual to the map

$$\mathrm{Hom}(\mathcal{T}_S, \mathcal{T}_S) \to \mathrm{Ext}^1(\mathcal{T}_S, \mathcal{T}_S)$$

coming from the same long exact sequence. But this last map is nonzero, since it maps the identity of $\mathcal{T}_S$ to α, which is nonzero. This concludes the proof. This is also a consequence of [51], Theorem 0.1. □

This last proof is of course very specific to Hilbert schemes and does not apply as such to other hyperkähler varieties. We feel nonetheless that it exhibits general facts about hyperkähler varieties which seem to give strong evidence to the Lefschetz conjecture in degree 2.

2.4.2 Consequences of the existence of a hyperkähler structure on the moduli space of stable hyperholomorphic bundles

In his paper [51], Mukai studies the moduli spaces of some stable vector bundles on K3 surfaces and endows them with a symplectic structure by showing that the holomorphic two-form induced by (2.5) on the moduli space is nondegenerate. Of course, this result is not directly useful when dealing with the Lefschetz standard conjecture in degree 2 as it is trivial for surfaces. Nevertheless, Verbitsky shows in [69] that it is possible to extend Mukai's result to the case of higher-dimensional hyperkähler varieties.

Before describing Verbitsky's results, let us recall some general facts from linear algebra around quaternionic actions and symplectic forms. This is all well-known, and described for instance in [8], Exemple 3, and [69], section 6. Let $\mathbb{H}$ denote the quaternions, and let V be a real vector space endowed with an action of $\mathbb{H}$ and a euclidean metric $(,)$.

Let $I \in \mathbb{H}$ be a quaternion such that $I^2 = -1$. The action of I on V gives a complex structure on V. We say that V is quaternionic hermitian if the metric on V is hermitian for all such complex structures I. Fix such an I, and choose J and K in $\mathbb{H}$ satisfying the quaternionic relations $I^2 = J^2 = K^2 = -Id, IJ = -JI = K$. We can define on V a real symplectic form η such that $\eta(x, y) = (x, Jy) + i(x, Ky)$. This symplectic form does not depend on the choice of J and K. Furthermore, η is $\mathbb{C}$-bilinear for the complex structure induced by I. Now given such I and η on V, it is straightforward to reconstruct a quaternionic action on V by taking the real and complex parts of η.

Taking V to be the tangent space to a complex variety, we can globalize the previous computations to get the following. Let X be an irreducible hyperkähler variety with given Kähler class ω. Then the manifold X is endowed with a canonical hypercomplex structure, that is, three complex structures I, J, K which satisfy the quaternionic relations $I^2 = J^2 = K^2 = -Id, IJ = -JI = K$. It is indeed possible to check that J and K obtained as before are actually integrable. Conversely, the holomorphic symplectic form on X can be recovered from I, J, K and a Kähler form on X with class ω.

If $\mathcal{E}$ is a complex hermitian vector bundle on X with a hermitian connection θ, we say that $\mathcal{E}$ is hyperholomorphic if θ is compatible with the three complex structures I, J and K. In case $\mathcal{E}$ is stable, this is equivalent to the first two Chern classes of $\mathcal{E}$ being Hodge classes for the Hodge structures induced by I, J and K, see [69], Theorem 2.5. This implies that any stable deformation of a stable hyperholomorphic bundle is hyperholomorphic. It is a consequence of Yau's theorem, see [77] that the tangent bundle of X is a stable hyperholomorphic bundle.

Let $\mathcal{E}$ be a stable hyperholomorphic vector bundle on X, and let $S = Spl(\mathcal{E}, X)$ be the reduced subscheme of the coarse moduli space of stable deformations of $\mathcal{E}$ on X. For s a complex point of S, let $\mathcal{E}_s$ be the hyperholomorphic bundle corresponding to a complex point s in S. The Zariski tangent space to S at s maps to $Ext^1(\mathcal{E}_s, \mathcal{E}_s)$ using the map from S to the coarse moduli space of stable deformations of $\mathcal{E}$. We can now define a global section η_S of $\mathcal{H}om(\mathcal{T}_S \otimes \mathcal{T}_S, \mathcal{O}_S)$, where $\mathcal{T}_S$ is the tangent sheaf to S, by the composition

$$\mathcal{T}_{S,s} \otimes \mathcal{T}_{S,s} \to \bigwedge^2 Ext^1(\mathcal{E}_s, \mathcal{E}_s) \to Ext^2(\mathcal{E}_s, \mathcal{E}_s) \to H^2(X, \mathcal{O}_X) = \mathbb{C}$$

as in the preceding section. The following is due to Verbitsky, see part (iv) of the proof in section 9 of [69] for the second statement.

Theorem 2.19. *([69], Theorem 6.3) Let $Spl(\mathcal{E}, X)$ be the reduced subscheme of the coarse moduli space of stable deformations of $\mathcal{E}$ on X. Then $S = Spl(\mathcal{E}, X)$ is endowed with a canonical hyperkähler structure. The holomorphic section of $\mathcal{H}om(\mathcal{T}_S \otimes \mathcal{T}_S, \mathcal{O}_S)$ induced by this hyperkähler structure is η_S.*

In this theorem, S does not have to be smooth. We use Verbitsky's definition of a singular hyperkähler variety as in [69], Definition 6.4.

We can now prove Theorem 2.3.

Proof of Theorem 2.3. Let X be a smooth projective irreducible hyperkähler variety, and let $\mathcal{E}$ be a stable hyperholomorphic bundle on X. Assume that $\mathcal{E}$ has a nontrivial positive-dimensional family of deformations, and let s be a smooth point of $S = Spl(\mathcal{E}, X)$ such that $\mathcal{T}_{S,s}$ is positive dimensional. We can choose a smooth quasi-projective variety S' with a complex point s' and a family $\mathcal{E}_{S'}$ of stable hyperholomorphic deformations of $\mathcal{E}$ on X parametrized by S' such that the moduli map $S' \to S$ maps s' to s and is étale at s'. Since η_S induces a symplectic form on $\mathcal{T}_{S,s}$, the map

$$\phi_2(\mathcal{E}_{S'})'_s : \bigwedge^2 \mathcal{T}_{S',s'} \to H^2(X, \mathcal{O}_X) = \mathbb{C}$$

is surjective as it identifies with $\eta_{S,s}$ under the isomorphism $\mathcal{T}_{S',s'} \xrightarrow{\sim} \mathcal{T}_{S,s}$. The result now follows from Theorem 2.16. $\square$

In order to prove Theorem 2.4, we need to recall some well-known results on deformations of hyperkähler varieties. Everything is contained in [8], Section 8 and [69], Section 1. See also [36], Section 1 for a similar discussion. Let X be an irreducible hyperkähler variety with given Kähler class ω. Let η be a holomorphic everywhere non-degenerate 2-form on X. Let q be the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{Z})$, and consider the complex projective plane P in $\mathbb{P}(H^2(X, \mathbb{C}))$ generated by $\eta, \overline{\eta}$ and ω. There exists a quadric Q of deformations of X given the elements $\alpha \in P$ such that $q(\alpha) = 0$ and $q(\alpha + \overline{\alpha}) > 0$.

Recalling that the tangent bundle of X comes with an action of the groups of quaternions of norm 1 given by the three complex structures I, J, K, which satisfy the quaternionic relations $I^2 = J^2 = K^2 = -Id, IJ = -JI = K$, this quadric Q of deformations of X corresponds to the complex structures on X of the form $aI + bJ + cK$ with a, b, c being three real numbers such that $a^2 + b^2 + c^2 = 1$ – those complex structures are always integrable. The quadric Q is called a twistor line.

In this setting, let d be the cohomology class of a divisor in $H^2(X, \mathbb{C})$, and let α be in Q. This corresponds to a deformation X_α of X. The cohomology class d corresponds to a rational cohomology class in $H^2(X_\alpha, \mathbb{C})$, and it is the cohomology class of a divisor if and only if it is of type $(1,1)$, that is, if and only if $q(\alpha, d) = 0$, where by q we also denote the bilinear form induced by q. Indeed, d is a real cohomology class, so if $q(\alpha, d) = 0$, then $q(\overline{\alpha}, d) = 0$ and d is of type $(1,1)$. It follows from this computation that d remains the class of a divisor for all the deformations of X parametrized by Q if and only if $q(\eta, d) = q(\omega, d) = 0$.

We will work with the varieties $S^{[n]}$, the case of generalized Kummer varieties being completely similar. Let us start with a K3 surface S, projective or not, and let us consider the irreducible hyperkähler variety $X = S^{[n]}$ given by the Douady space of n points in S – this is Kähler by [68]. In the moduli space M of deformations of X, the varieties of the type $S'^{[n]}$ form a countable union of hypersurfaces H_i. On the other hand, the hyperkähler variety admits deformations parametrized by a twistor line, and those cannot be included in any of the H_i. Indeed, if that were the case, the class e of the exceptional divisor of $X = S^{[n]}$ would remain algebraic in all the deformations parametrized by the twistor line. But this is impossible, as e is a class of an effective divisor, which implies that $q(\omega, e) > 0$, ω being a Kähler class, see [36], 1.11 and 1.17. Counting dimensions, we get the following.

Lemma 2.20. *Let n be a positive integer, and let X be a projective deformation of the Douady space of n points on a K3 surface. Then there exists a K3 surface S and a twistor line Q parametrizing deformations of $S^{[n]}$ such that X is a deformation of $S^{[n]}$ along Q.*

The next result of Verbitsky is the main remaining ingredient we need to prove Theorem 2.4. Recall first that if $\mathcal{E}$ is a hyperholomorphic vector bundle on an irreducible hyperkähler variety X, then by definition the bundle $\mathcal{E}$ deforms as X deforms along the twistor line.

Theorem 2.21. *([69], Corollary 10.1) Let X be an irreducible hyperkähler variety, and let $\mathcal{E}$ be a stable hyperholomorphic vector bundle on X, and let $Spl(\mathcal{E}, X)$ be the reduced subscheme of the coarse moduli space of stable deformations of $\mathcal{E}$ on X.*

Then the canonical hyperkähler structure on $Spl(\mathcal{E}, X)$ is such that if Q is the twistor line parametrizing deformations of X, Q is a twistor line parametrizing deformations of $Spl(\mathcal{E}, X)$ such that if $\alpha \in Q$, then $Spl(\mathcal{E}, X)_\alpha = Spl(\mathcal{E}_\alpha, X_\alpha)$.

This implies that the deformations of a hyperholomorphic bundle on X actually deform as the complex structure of X moves along a twistor line. We can now prove our last result.

Proof of Theorem 2.4. Let X be an irreducible projective hyperkähler variety that is a deformation of the Douady space of n points on some K3 surface. By Lemma 2.20, X is a deformation of some $S^{[n]}$ along a twistor line Q, where S is a K3 surface. Let $\mathcal{E}$ on $S^{[n]}$ be a sheaf as in the statement of the theorem. By Theorems 2.21 and 2.3, we get a bundle $\mathcal{E}'$ which still satisfies the hypothesis of Theorem 2.16. This concludes the proof. □

It is particularly tempting to use this theorem with the tangent bundle of $S^{[n]}$, which is stable by Yau's theorem and hyperholomorphic since its first two Chern classes are Hodge classes for all the complex structures induced by the hyperkähler structure of $S^{[n]}$. Unfortunately, while Verbitsky announces in [70], after the proof of Corollary 10.24, that those have some unobstructed deformations for $n = 2$ and $n = 3$, we have been able to check that if $n = 2$, the tangent bundle is actually rigid. However, we get the following result by applying the last theorem to the tangent bundle.

Corollary 2.22. *Let n be a positive integer. Assume that for every K3 surface S, the tangent bundle $\mathcal{T}_{S^{[n]}}$ of $S^{[n]}$ has a nontrivial positive-dimensional family of deformations. Then the Lefschetz conjecture is true in degree 2 for any projective deformation of the Douady space of n points on a K3 surface.*

Remark 2.23. The conditions of the corollary might be actually not so difficult to check. Indeed, Verbitsky's Theorem 6.2 of [69] which computes the obstruction to extending first-order deformations implies easily that the obstruction to deform $\mathcal{T}_{S^{[n]}}$ actually lies in $H^2(S^{[n]}, \Omega^2_{S^{[n]}})$, where we see this group as a subgroup of

$$\mathrm{Ext}^2(\mathcal{T}_{S^{[n]}}, \mathcal{T}_{S^{[n]}}) = H^2(S^{[n]}, \Omega^{\otimes 2}_{S^{[n]}})$$

under the isomorphism $\mathcal{T}_{S^{[n]}} \simeq \Omega^1_{S^{[n]}}$.

Now the dimension of $H^2(S^{[n]}, \Omega^2_{S^{[n]}})$ does not depend on n for large n, see for instance [27], Theorem 2. As a consequence, the hypothesis of the Corollary would be satisfied for large n as soon as the dimension of $\mathrm{Ext}^1(\mathcal{T}_{S^{[n]}}, \mathcal{T}_{S^{[n]}})$ goes to infinity with n.

Remark 2.24. It is quite surprising that we make use of nonprojective Kähler varieties in these results dealing with the standard conjectures. Indeed, results like those of Voisin in [73] show that there can be very few algebraic cycles, whether coming from subvarieties or even from Chern classes of coherent sheaves, on general nonprojective Kähler varieties.

Chapter 3

On the zero locus of normal functions and the étale Abel-Jacobi map

Résumé. Dans ce dernier chapitre, on s'intéresse à des questions de nature arithmétique portant sur l'application d'Abel-Jacobi. Des résultats récents de Brosnan-Pearlstein et M. Saito montrent que le lieu des zéros d'une fonction normale est une variété algébrique. On examine ici la question du corps de définition de ce lieu et l'on prouve, dans un cadre variationnel, quelques théorèmes de comparaison entre les noyaux des applications d'Abel-Jacobi complexe et l-adique. Ces questions sont liées aux conjectures de Bloch et Beilinson sur les filtrations sur les groupes de Chow, qui expliquent nos résultats de manière motivique.

Abstract. In this paper, we investigate questions of an arithmetic nature related to the Abel-Jacobi map. We give a criterion for the zero locus of a normal function to be defined over a number field, and we give some comparison theorems with the Abel-Jacobi map coming from continuous étale cohomology.

3.1 Introduction

Let $X \to S$ be a family of complex smooth projective varieties over a quasi-projective base, and let $Z \hookrightarrow X$ be a flat family of cycles of pure codimension i which are homologically equivalent to zero in the fibers of the family. For any point s of S, the Abel-Jacobi map associates to the cycle Z_s a point in the intermediate Jacobian $J^i(X_s)$ of X_s, which is a complex torus (see part 2 for details). This construction works in family, yielding a bundle of complex tori, the Jacobian fibration $J^i(X/S)$, and a normal function ν_Z, which is the holomorphic section of $J^i(X/S)$ associated to $Z \hookrightarrow X$. We can attach to ν_Z an admissible variation H of mixed Hodge structures on S, see [58], fitting in an exact sequence

$$0 \to R^{2i-1}f_*\mathbb{Z}/(torsion) \to H \to \mathbb{Z}$$

such that the zero locus of ν_Z is the locus where this exact sequence splits, that is, the projection on S of the locus of Hodge classes in H which map to 1 in $\mathbb{Z}$. In analogy with the celebrated theorem of Cattani-Deligne-Kaplan on the algebraicity of Hodge loci for variations of pure Hodge structures, Green and Griffiths have stated the following conjecture, which deals with the mixed Hodge structure appearing above. Since it has been very recently proved by Brosnan and Pearlstein, we state it as a theorem.

Theorem 3.1. *(Brosnan-Pearlstein, [16] The zero locus of the normal function ν is algebraic.*

In the same way that Deligne-Cattani-Kaplan's result gives evidence for the Hodge conjecture, this gives evidence for Bloch-Beilinson-type conjectures on filtration on Chow groups, see section 2.1. For work of M. Saito and Brosnan-Pearlstein related to the previous theorem, see [59] and [15].

Assume everything is defined over a finitely generated field k. In line with general conjectures on algebraic cycles, one would expect that not only the zero locus of ν is algebraic, but it should be defined over k, hence the interest in trying to investigate number-theoretic properties of the zero locus of normal functions. In the context of pure Hodge structures, i.e. that of Deligne-Cattani-Kaplan's theorem, Voisin shows in [76] how this question is related to the question whether Hodge classes are absolute and gives criteria for Hodge loci to be defined over number fields.

In this paper, we want to tackle such questions and also investigate the arithmetic counterpart of normal functions, namely the étale Abel-Jacobi map introduced by Jannsen using continuous étale cohomology. We give comparison results between the étale Abel-Jacobi map and Griffiths' Hodge-theoretic one. Recent work around the same circle of ideas can be found in [28]. The use of Terasoma's lemma in this context is very relevant to our work, and the results proved there are closely related to our theorem 3.2 (though up to torsion).

Let us state our main results precisely. Start with a subfield k of $\mathbb{C}$ which is generated by a finite number of elements over $\mathbb{Q}$, and let S be a quasi-projective variety over k. Let $\pi : X \to S$ be a smooth family of projective varieties of pure dimension n, and let $Z \hookrightarrow S$ be a family of codimension i algebraic cycles of X, flat over S. Assume that Z is homologically equivalent to 0 on the geometric fibers of π. In the paper [39], Jannsen defines continuous étale cohomology, which is a version of étale l-adic cohomology for varieties over fields which are not necessarily algebraically closed. There is a cycle map from Chow groups to continuous étale cohomology. For any point s of S with value in a finitely generated extension K of k, let $\overline{X_s}$ be the variety $X_s \times_K \overline{K}$, and $G_K = Gal(\overline{K}/K)$ the absolute Galois group of K. The cycle class of Z_s in the continuous étale cohomology of X_s gives a class $aj_{\acute{e}t}(Z_s) \in H^1(G_K, \tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}}(i)))$, where $\tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}})$ denotes the quotient of $H^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}})$ by its torsion subgroup, $\hat{\mathbb{Z}}$ being the profinite completion of $\mathbb{Z}$. This cohomology class is obtained by applying a Hoschild-Serre spectral sequence to continuous cohomology. Proposition 3.7 shows that the vanishing of this class is independent of the choice of K, i.e., it vanishes in $H^1(G_K, \tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}}(i)))$ if and only if it vanishes in $H^1(G_L, \tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}}(i)))$ for any finite type extension L of K. This observation appears in [60] and, according to one of the referees, is due to Nori.

It would follow from general conjectures on algebraic cycles, either a combination of the Hodge and Tate conjectures for open varieties or versions of the Bloch-Beilinson conjectures on filtrations on Chow groups, that the zero locus of the normal function associated to the family of cycles induced by $Z_{\mathbb{C}}$ on $X_{\mathbb{C}} \to S_{\mathbb{C}}$ is precisely the vanishing set of the étale Abel-Jacobi map. For the latter, assuming Beilinson's conjecture on Chow groups of varieties over number fields and Lewis' Bloch-Beilinson conjecture of [47], one would know that the kernels of both Abel-Jacobi maps are equal to the second step of the unique Bloch-Beilinson filtration on Chow groups, hence that they coincide. Unfortunately, such comparison results seem to be known only for divisors and zero-cycles, where the étale Abel-Jacobi map can be computed using the Kummer exact sequence for Picard or Albanese varieties. We are not aware of any result in other codimension. In this paper, we therefore try to tackle such comparison results. We don't prove the general case, but we prove results of two different flavors in the variational case. We obtain such results using in an essential way the algebraicity of the components of the zero locus of normal functions as proved in [16].

In the previous situation, consider the normal function ν_Z associated to the family $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$. Its zero locus is an analytic subvariety of $S(\mathbb{C})$. Our theorems are the following, where the expression "finitely generated field" denotes a field generated by a finite number of elements over its prime subfield – $\mathbb{Q}$ in our case.

Theorem 3.2.

(i) Let T be an irreducible component of the zero locus of ν_Z. Assume that $R^{2i-1}\pi_{\mathbb{C}*}\mathbb{C}$ has no nonzero global sections over T. Let k be a finitely generated field over which T is defined. Then for all point t of T with value in a finitely generated field, the étale Abel-Jacobi invariant of Z_t is zero.

(ii) Assume that for every closed point s of S with value in a number field, the étale Abel-Jacobi invariant of Z_s is zero and that $R^{2i-1}\pi_{\mathbb{C}*}\mathbb{C}$ has no nonzero global sections over $S_{\mathbb{C}}$. Then ν_Z is identically zero on $S_{\mathbb{C}}$.

Theorem 3.3. *Let T be an irreducible component of the zero locus of ν_Z. Assume that $R^{2i-1}\pi_{\mathbb{C}*}\mathbb{C}$ has no nonzero global sections over T. Then T is defined over a finite extension of the base field k and for any automorphism σ of $\mathbb{C}$ over k, the image of T by σ is an irreducible component of ν_Z.*

Remark 3.4. In this result, we are considering the image of T by σ as a subscheme of S, and we prove that it is, as a subscheme of S, a component of the zero locus of the holomorphic normal function ν_Z. This contrasts with the situation in [76], where similar results were obtained using only the reduced structure on the subschemes considered. The main difference in our setting is that the (mixed) Hodge structures we consider have at most one nonzero Hodge class, up to multiplication by a constant.

Theorem 3.5.

(i) Assume that for every closed point s of S with value in a number field, the étale Abel-Jacobi invariant of Z_s is zero and that there exists a complex point s of S such that $\nu_Z(s) = 0$. Then ν_Z is identically zero on $S_{\mathbb{C}}$.

(ii) Let T be an irreducible component of the zero locus of ν_Z. Assume that there exists a point t of T such that $aj_{ét}(Z_t)$ is zero. Then for all points t of T with value in a finitely generated field, the étale Abel-Jacobi invariant of Z_t is zero.

The lack of symmetry between both Abel-Jacobi maps in our results is frustrating. Indeed, while the local structure of zero locus of normal functions is well understood – it is an analytic variety, and its local description is well described, see [33], [74], ch. 17– ,we have very few results on the "zero locus" of the étale Abel-Jacobi map. We feel that it would be very interesting to prove an étale counterpart of the results of [16] and even of the weaker ones of [17] and [59], which deal with the case where S is a curve.

In this paper, if X is any variety, the cohomology groups of X, whether singular or étale, will always be considered modulo torsion, so as to avoid cumbersome notations. The same convention goes for higher direct images.

Acknowledgements. I would like to offer my heartfelt appreciation to Claire Voisin. It is very clear that this paper owes a great deal both to her work in [76] and to the many discussions we had and suggestions she made, notably for the example of theorem 3.13.

3.2 Preliminary results on Abel-Jacobi maps

Let X be a smooth projective variety over a field k of characteristic zero, $\overline{k}$ an algebraic closure of k and $G_k = Gal(\overline{k}/k)$ the absolute Galois group of k. In his paper [10], Beilinson constructs a conjectural filtration $F^\bullet$ on the Chow groups $CH^i(X) \otimes \mathbb{Q}$ of X with rational coefficients. It is obtained in the following way. Let $MM(k)$ be the abelian category of mixed motives over k. There should exist a spectral sequence, Beilinson's spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{MM(k)}(\mathbf{1}, \mathfrak{h}^q(X)(i)) \Rightarrow \mathrm{Hom}_{D^b(MM(k))}(\mathbf{1}, \mathfrak{h}(X)(i)[p+q])$$

where $\mathfrak{h}^q(X)$ denotes the weight q part of the image $\mathfrak{h}(X)$ of X in the category of pure motives. For $p + q = 2i$, the abutment of this spectral sequence should be canonically isomorphic with $CH^i(X)$, hence the filtration F. For weight reasons, a theorem of Deligne in [23] would imply that this spectral sequence degenerates at $E_2 \otimes \mathbb{Q}$. We get the formula

$$Gr^\nu_F CH^i(X)_{\mathbb{Q}} = \mathrm{Ext}^\nu_{MM(k)}(\mathbf{1}, \mathfrak{h}^{2i-\nu}(X)(i)) \otimes \mathbb{Q}.$$

The existence of such a filtration is also a conjecture of Bloch and Murre.

3.2.1 Étale cohomology

The previous construction should have its reflection in the various usual cohomology theories. Let us first consider étale cohomology. In the paper [39], Jannsen constructs continuous étale cohomology groups with value in the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$ for varieties over a field [1] Those enjoy good functoriality properties and they are equal to the usual étale cohomology groups in case the base field is algebraically closed. In particular, there is a Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_k, H^q(X_{\overline{k}}, \hat{\mathbb{Z}}(i)) \Rightarrow H^{p+q}(X, \hat{\mathbb{Z}}(i)) \tag{3.1}$$

as well as a cycle map

$$cl^X : CH^i(X) \to H^{2i}(X, \hat{\mathbb{Z}}(i)).$$

Those are compatible with the usual cycle map $cl^{X_{\overline{k}}}$ to $H^{2i}(X_{\overline{k}}, \hat{\mathbb{Z}}(i))$. Let $CH^i(X)_{hom}$ be the kernel of $cl^{X_{\overline{k}}}$, that is, the part of the Chow group consisting of those cycles which are homologically equivalent to zero.

Definition 3.6. *The map*

$$aj_{ét} : CH^i(X)_{hom} \to H^1(G_k, H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i))$$

induced by the spectral sequence (3.1) is called the étale Abel-Jacobi map.

This map is expected to be the image by some realization functor of the analogous map coming from Beilinson's spectral sequence. As an evidence for this, we can cite Jannsen's result in [42], lemma 2.7, stating that in the case k is finitely generated, and for any "reasonable" category of mixed motives, the filtrations on $CH^i(X) \otimes \mathbb{Q}$ induced by Beilinson's spectral sequence and by the Hochschild-Serre spectral sequence (3.1) coincide if cl^X is injective – which is also a conjecture of Bloch and Beilinson. More specifically, if any Bloch-Beilinson filtration

1. In his paper, Jannsen actually deals with $\mathbb{Z}_l$-coefficients. We define the cohomology groups with value in $\hat{\mathbb{Z}}$ as the product over all l of the cohomology groups with value in $\mathbb{Z}_l$. This is indeed a $\hat{\mathbb{Z}}$-module, which satisfies, since we work in characteristic zero, all the expected properties of an étale cohomology group. It would be easy to give a direct definition following [39].

(see [13], [42], [43]) exists on $CH^i(X) \otimes \mathbb{Q}$ and cl^X is injective, then it has to be the filtration induced by (3.1).

Our definition of the étale Abel-Jacobi map may seem to be highly dependent on the base field k, which is not convenient since we expect that for an algebraic cycle Z homologically equivalent to zero, $aj_{ét}(Z)$ should reflect geometric properties of Z related to the image of $Z_{\mathbb{C}}$ by the Abel-Jacobi map. The following proposition shows that the vanishing of $aj_{ét}(Z)$ does not actually depend of the base field. This would be false had we considered in our definition the torsion part of the cohomology of X. The fact that we want the following result to hold is the reason why we have to ignore this torsion part, which is related to arithmetic properties of algebraic cycles, as opposed to their geometric properties. It has been attributed to Nori and appears in a very similar form in [60], lemma 1.4.

Proposition 3.7. *Let X be a smooth projective variety over a finitely generated field k, and let $Z \in CH^i_{hom}(X)$. Let K be a field which is finitely generated over k. Then $aj_{ét}(Z_K) = 0$ if and only if $aj_{ét}(Z) = 0$.*

Proof. We can assume that K is Galois over k. We have an exact sequence of groups

$$1 \to G_K \to G_k \to \mathrm{Gal}(K/k) \to 1,$$

hence the following exact sequence coming from the Hochschild-Serre spectral sequence

$$0 \to H^1(\mathrm{Gal}(K/k), V^{G_K}) \to H^1(G_k, V) \to H^1(G_K, V)^{\mathrm{Gal}(K/k)}, \tag{3.2}$$

with $V = H^{2i-1}(\overline{X}, \hat{\mathbb{Z}}(i))$. The definition of the étale Abel-Jacobi map from a Leray spectral sequence shows that $aj_{ét}(Z_K)$ is obtained from $aj_{ét}(Z)$ by the last map in (3.2). On the other hand, it is a consequence of the Weil conjectures that V^{G_K} is zero (recall V is torsion-free). Indeed, let S be an irreducible quasi-projective variety defined over a number field such that K is the function field of S. Let s be a point of S with value in a number field F such that X has good reduction X_s at S, and let v be a finite place of F such that X_s has good reduction at v. The Frobenius of the residue field of F at v gives an element of G_K – up to conjugation – which by the Weil conjectures acts on V with eigenvalues of weight -1. In particular, it does not have 1 as an eigenvalue, so $V^{G_K} = 0$. This implies that the last map in (3.2) is an injection. □

The next result is due to Jannsen in [40], being inspired by results from Carlson and Beilinson we will recall later, and gives a geometric meaning to the étale Abel-Jacobi map. We recall it shortly, as it is crucial to the results of our paper.

Start with X as before, and let Z an algebraic cycle of pure codimension i in X. Let $|Z|$ be the support of Z, and U be the complement of $|Z|$ in X. By purity, we have an exact sequence of G_k-modules

$$0 \to H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i)) \to H^{2i-1}(U_{\overline{k}}, \hat{\mathbb{Z}}(i)) \to H^0(|Z|_{\overline{k}}, \hat{\mathbb{Z}}) \to 0$$

and the class of Z gives a map $\hat{\mathbb{Z}} \to H^0(|Z|_{\overline{k}}, \hat{\mathbb{Z}})$. The pull-back of the previous exact sequence by this map is an exact sequence of G_k-modules

$$0 \to H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i)) \to H_{ét} \to \hat{\mathbb{Z}} \to 0. \tag{3.3}$$

This extension gives a class $\alpha(Z) \in H^1(G_k, H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i))$.

Proposition 3.8. *We have $\alpha(Z) = aj_{ét}(Z)$.*

Let us note that this proposition immediately carries out to the variational setting for flat families of algebraic cycles. In this case, for all prime numbers l, we get an extension of locally constant $\mathbb{Z}_l$-sheaves over the base scheme which on every fiber is canonically isomorphic to the l-adic part of the extension (3.3) [2].

3.2.2 Hodge theory

The Hodge-theoretic picture is different. Indeed, the category of mixed Hodge structures has no higher extension groups as shown by Beilinson, so we cannot expect to construct directly a similar filtration on Chow groups through this means. The use of Leray spectral sequences in this setting has been studied by Nori, Saito, Green-Griffiths and others, and can be considered well-understood. Even though we cannot expect to construct a Bloch-Beilinson filtration on Chow groups using Hodge theory, at least in a naive way, we can construct a two-term filtration using Deligne cohomology. We use this approach to make the similarity with the previous discussion more obvious, but in this paper we simply use Griffiths' Abel-Jacobi map, which was defined in [31] and [32]. Griffiths' work on normal functions is of course fundamental to our results.

Let us assume for this paragraph that the base field is $\mathbb{C}$. Recall that we have Deligne cohomology groups $H^i_{\mathcal{D}}(X, \mathbb{Z}(j))$. Those are the "absolute" version of Hodge cohomology groups in the same way that continuous étale cohomology is the absolute version of étale cohomology over an algebraic closure of the base field. There is an exact sequence

$$0 \to J^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z})(i) \to 0$$

as well as a cycle map

$$cl^X : CH^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)).$$

Those are compatible with the usual cycle map to $H^{2i}(X, \mathbb{Z})(i)$.

The cohomology group $H^{2i}(X, \mathbb{Z})(i)$ is, up to a Tate twist the usual singular cohomology of the complex manifold X with its canonical Hodge structure, and $J^i(X)$ is Griffiths' i-th intermediate Jacobian, which is defined in the following way.

Integration of differential forms gives a map from the homology group $H_{2n-2i+1}(X, \mathbb{Z})$ to the group $F^{n-i+1}H^{2n-2i+1}(X, \mathbb{C})^*$, n being the dimension of X and F the Hodge filtration. The quotient group

$$F^{n-i+1}H^{2n-2i+1}(X, \mathbb{C})^*/H_{2n-2i+1}(X, \mathbb{Z})$$

is a complex torus, canonically isomorphic to

$$J^i(X) := \frac{H^{2i-1}(X, \mathbb{C})}{F^iH^{2i-1}(X, \mathbb{C}) \oplus H^{2i-1}(X, \mathbb{Z})}.$$

There is a canonical isomorphism of abelian groups between $J^i(X)$ and the extension group

$$\mathrm{Ext}^1_{MHS}(\mathbb{Z}, H^{2i-1}(X, \mathbb{Z})(i))$$

in the category of mixed Hodge structures, as noted by Carlson in [19]. One can also refer to [74], p.463.

2. In this situation, it would be more convenient to refer to the collection of $\mathbb{Z}_l$-sheaves that we get as a $\hat{\mathbb{Z}}$-sheaf. One could indeed use [26] to get a suitable category of $\hat{\mathbb{Z}}$-sheaves. We won't use this terminology.

Definition 3.9. *The map*

$$aj : CH^i(X)_{hom} \to J^i(X)$$

induced from the exact sequence above defining Deligne cohomology is called the (transcendental) Abel-Jacobi map, or Griffiths' Abel-Jacobi map.

In the light of the isomorphism above, Beilinson has shown in [9] (see also [19] for the case of divisors on curves) the following way of computing the Abel-Jacobi map, which is very similar to its étale counterpart– and has been proved earlier. Let Z an algebraic cycle of pure codimension i in X. Let $|Z|$ be the support of Z, and U be the complement of $|Z|$ in X. We have an exact sequence of mixed Hodge structures

$$0 \to H^{2i-1}(X, \mathbb{Z}(i)) \to H^{2i-1}(U, \mathbb{Z}(i)) \to H^0(|Z|, \mathbb{Z}) \to 0$$

and the class of Z gives a map $\mathbb{Z} \to H^0(|Z|, \mathbb{Z})$. The pull-back of the previous exact sequence by this map is an exact sequence of mixed Hodge structures

$$0 \to H^{2i-1}(X, \mathbb{Z}(i)) \to H \to \mathbb{Z} \to 0. \tag{3.4}$$

This extension gives a class $\alpha(Z) \in \mathrm{Ext}^1_{MHS}(\mathbb{Z}, H^{2i-1}(X, \mathbb{Z})(i)) = J^i(X)$. One can find in [9] the following.

Proposition 3.10. *We have $\alpha(Z) = aj(Z)$.*

The vanishing of the Abel-Jacobi map has a simple interpretation in these terms. Indeed, recall that if H is a mixed Hodge structure (of weight 0) with weight filtration $W_\bullet$ and Hodge filtration $F^\bullet$, a Hodge class of weight k in H is an element of $W_{2k}H \cap F^k H_{\mathbb{C}}$. In this terminology, it is straightforward to see that the extension (3.4) splits if and only if there exists a Hodge class (which has to be of weight 0) in H mapping to 1 in $\mathbb{Z}$.

Again, in case of a flat family of algebraic cycles which are homologous to zero in the fibers, we get an extension of variations of mixed Hodge structures corresponding point by point to (3.4). It satisfies Griffiths' transversality, see [58], lemma 1.3. We also get the Jacobian fibration $J^i(X/S)$, and a section ν_Z of it is obtained by applying the relative Abel-Jacobi map. The preceding remark shows that the zero locus of ν_Z is a Hodge locus for the variation of mixed Hodge structures above.

Definition 3.11. *The section of $J^i(X/S)$ attached to the cycle Z is the normal function ν_Z attached to Z.*

Normal functions have been extensively studied, see [31], [32], [58], etc. See also [74], ch. 19. It is a fundamental fact that normal functions are holomorphic. In particular, their zero locus is analytic. In this paper, our results will take into account this analytic structure. Indeed, while this zero locus is non-reduced in general, the results we get are valid without passing first to the reduced analytic subspace, as opposed for instance to [76], th. 0.6 (2).

It will be very important to us, though straightforward, that if we start with a family over a finitely generated base field, the extension of local systems coming from the étale Abel-Jacobi map and from the transcendental one, after base change to $\mathbb{C}$, are compatible in the obvious way, as Artin's comparison theorem between étale and singular cohomology readily shows.

3.3 Proof of the theorems

3.3.1 Zero loci for large monodromy groups

This section is devoted to showing how assuming the family $X \to S$ has a large monodromy can help study the vanishing locus of the Abel-Jacobi map and deduce theorem 3.2 and theorem 3.3. This kind of argument is very much inspired by [76], where it appears in the pure case as a criterion for Hodge classes to be absolute.

The main idea is the following : start with an extension $0 \to H' \to H \to \mathbb{Z} \to 0$ of variations of mixed Hodge structures on a quasi-projective variety S. If the monodromy representation on H' has no nontrivial invariant part, then any global section of H is in F^0H, the filtration F being the Hodge filtration. This remark allows us to reduce the question of the splitting of the previous exact sequence to a geometric question, and allows for comparison theorems.

In the setting of normal functions, this is equivalent to the following, which has been observed a long time ago. Under this assumption, the normal function with value in the i-th intermediate Jacobian is determined by its Hodge class, see [33]. This has been used for instance by Manin in the proof of Mordell's conjecture over function fields in [48]. Our argument does not proceed along these lines for convenience, but part of it could easily be translated using Griffiths' results and the Leray spectral sequence.

Recall the notations of the introduction. We have a smooth projective family over a quasi-projective base $\pi : X \to S$, together with a flat family of algebraic cycles $Z \to X$ of pure codimension i. Everything is defined over a finitely generated field k of characteristic zero. As far as our results are concerned, and taking into account proposition 3.7, standard spreading techniques allow us to assume without loss of generality that k is a number field. Suppose that for any geometric point s of S, Z_s is homologically equivalent to zero in X_s. Fix an embedding of k in $\mathbb{C}$. We get the normal function ν_Z on $S(\mathbb{C})$, which is a holomorphic section of the bundle of intermediate Jacobians over $S(\mathbb{C})$.

For every prime number l, we have the following exact sequence of local systems of $\mathbb{Z}_l$-sheaves on S, canonically attached to the family Z of algebraic cycles

$$0 \to R^{2i-1}\pi_*\mathbb{Z}_l(i) \to H_l \to \mathbb{Z}_l \to 0. \tag{3.5}$$

Since $\mathbb{Z}_l$ is flat over $\mathbb{Z}$, the pull-back to $S_{\mathbb{C}}$ of this sequence of sheaves is the tensor product by $\mathbb{Z}_l$ of the exact sequence

$$0 \to R^{2i-1}(\pi_{\mathbb{C}})_*\mathbb{Z}(i) \to H \to \mathbb{Z} \to 0 \tag{3.6}$$

of local systems used to compute Griffiths' Abel-Jacobi map. Those local systems are underlying variations of mixed Hodge structures. Saying that ν_Z vanishes on $S_{\mathbb{C}}$ is equivalent to saying that S is equal to the locus of Hodge classes of H which map to 1 in $\mathbb{Z}$.

We will deduce our theorems from the following result.

Theorem 3.12. *In the above setting, assume that the locally constant sheaf $R^{2i-1}(\pi_{\mathbb{C}})_*\mathbb{C}$ has no nonzero global section over $S_{\mathbb{C}}$. Then the following are equivalent :*

(i) The normal function ν associated to $Z_{\mathbb{C}}$ vanishes on $S_{\mathbb{C}}$.

(ii) For every closed point s of S with value in a finitely generated field K, the image of Z_s by the étale Abel-Jacobi map vanishes in the group $H^1(G_K, H^{2i-1}(\overline{X_s}, \widehat{\mathbb{Z}}(i)))$.

(iii) For any automorphism σ of $\mathbb{C}$, the normal function ν^σ associated to $Z^\sigma = Z_{\mathbb{C}} \times_\sigma \mathrm{Spec}(\mathbb{C})$ vanishes on S^σ.

Proof of $(i) \Rightarrow (ii)$. Fix a prime number l. Fix a point s of S with value in a finitely generated field L, and let $\overline{s}$ be a complex point of S lying over s. Under our hypothesis, we have an injective map $(H_{l,\overline{s}})^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\overline{s})} \to \mathbb{Z}_l$. This is actually an isomorphism. Indeed, Baire's theorem applied to the locus of Hodge classes of H in $S_{\mathbb{C}}$ mapping to 1 in $\mathbb{Z}$ shows that in order for $S_{\mathbb{C}}$ to be equal to this locus, which is a countable union of analytic subvarieties, there has to be a nonzero global section of H which is a Hodge class in every fiber of H – and maps to 1 in $\mathbb{Z}$. The image in $H_{\overline{s}} \otimes \mathbb{Z}_l = H_{l,\overline{s}}$ of this section lies in $(H_{l,\overline{s}})^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\overline{s})}$ and maps to 1 in $\mathbb{Z}_l$.

Now let G_L be the absolute Galois group of L. We have an exact sequence $1 \to \pi_1^{\acute{e}t}(S_{\mathbb{C}},\overline{s}) \to \pi_1^{\acute{e}t}(S \times \mathrm{Spec}(L),\overline{s}) \to G_L \to 1$, together with a splitting of this exact sequence. The full algebraic fundamental group acts on $H_{l,\overline{s}}$, and the map $H_{l,\overline{s}} \to \mathbb{Z}_l$ is equivariant with respect to the trivial action on $\mathbb{Z}_l$. It follows that the group G_L acts trivially on $(H_{l,\overline{s}})^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\overline{s})} \xrightarrow{\sim} \mathbb{Z}_l$. This being true for any l, it proves that the étale Abel-Jacobi invariant of Z_s is zero. □

Proof of $(ii) \Rightarrow (iii)$. It is enough to prove the case where σ is the identity. Fix a prime number l. Let $\overline{s}$ be a geometric point of S. Using the same notation as in the previous proof, the algebraic fundamental group $\pi_1^{\acute{e}t}(S,\overline{s})$ acts on $H_{l,\overline{s}}$. For any point s' of S with value in a field L, the absolute Galois group G_L of L maps into $\pi_1^{\acute{e}t}(S,\overline{s})$. According to a lemma by Terasoma appearing in [65], theorem 2, there exists such an L-valued point s', with L a number field, such that G_L and $\pi_1^{\acute{e}t}(S,\overline{s})$ have the same image in the linear group $\mathrm{GL}(H_{l,\overline{s}})$. Since by assumption G_L fixes an element mapping to $1 \in \mathbb{Z}_l$, we get an element of $H_{l,\overline{s}}$, mapping to $1 \in \mathbb{Z}_l$, which is fixed by the whole monodromy group. In other words, the exact sequence (3.5) splits over S. This being true for any prime number l, the exact sequence (3.5) splits over S.

This means that the local system H_l on S has a nonzero global section for any prime number l. As a consequence, $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q}) \neq 0$, and as before we get an isomorphism $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q}) \simeq \mathbb{Q}$ as local systems, the map being induced by the morphism $H \otimes \mathbb{Q} \to \mathbb{Q}$ of variations of mixed Hodge structures over $S_{\mathbb{C}}$. It is a result of Steenbrink and Zucker in [64], th. 4.1 [3] that $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ carries a canonical mixed Hodge structure which makes it a constant subvariation of mixed Hodge structures of H [4]. The isomorphism of $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ with $\mathbb{Q}$ is a morphism of mixed Hodge structures, which proves $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ consists of Hodge classes.

This shows that the exact sequence (3.6) of variations of mixed Hodge structures splits rationally. We want to prove that it splits over $\mathbb{Z}$. We just proved that a splitting of the subjacent extension of local systems over $S_{\mathbb{C}}$ gives a splitting of (3.6), so we just have to prove that the exact sequence of local systems splits.

Let $\alpha \in H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ be the class mapping to $1 \in \mathbb{Q}$. For any prime number l, the image of $\alpha \in H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ is the only class mapping to $1 \in \mathbb{Q}_l$, which shows that this image belongs to $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Z}_l)$, since the exact sequence (3.5) is split over $S_{\mathbb{C}}$. The only way for this to be true is that α belongs to $H^0(S_{\mathbb{C}}, H)$, which precisely means that the exact sequence we are considering splits. □

Proof of $(iii) \Rightarrow (i)$. This is obvious. □

3. This theorem is a generalization of Deligne's global invariant cycles theorem, which is a fundamental tool of [76].

4. The result of Steenbrink and Zucker is stated for variations of mixed Hodge structures of geometric origin – which is our case – over a one-dimensional base. The fact that $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ carries a canonical Hodge structure for S of any dimension is straightforward by restricting to a curve which is an intersection of hyperplane sections and using Lefschetz' hyperplane theorem.

Let us now use the notations of the introduction. The equivalence $(i) \Leftrightarrow (ii)$ we just proved readily implies theorem 3.2 by restriction to the component T of the zero locus of ν_Z, which is algebraic, under the assumption that the local system $R^{2i-1}(\pi_{\mathbb{C}})_*\mathbb{C}$ has no nonzero global section.

Proof of theorem 3.3. Let $\overline{k}$ be an algebraic closure of k, and let T' be the Zariski-closure of $T(\mathbb{C})$ in the k-scheme S. The previous theorem shows that the orbit of $T(\mathbb{C})$ in S under the action of the Galois group $\mathrm{Aut}(\mathbb{C}/k)$ is included in the zero locus of ν_Z. Since this orbit is dense in $T'(\mathbb{C})$ for the usual topology, it follows that ν_Z vanishes on $T'(\mathbb{C})$. By assumption, T is an irreducible component of the zero locus of ν_Z. It follows that T is an irreducible component of the algebraic variety T' defined over k, which proves that T is defined over a finite extension of k.

This shows that for any automorphism σ of $\mathbb{C}$ fixing k, the set $\sigma(T(\mathbb{C}))$ is included in the zero locus of ν_Z. Now consider the subscheme T^σ of S, which has $\sigma(T(\mathbb{C}))$ as set of complex points. We just showed that its reduced subscheme is included in the zero locus of ν_Z as an analytic subvariety, and it is irreducible. Let V be the irreducible component of the zero locus of ν_Z containing $\sigma(T(\mathbb{C}))$. We want to show that $V = T^\sigma$ as analytic varieties. Let n be a nonnegative integer. We can consider the artinian local rings $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ and $\mathcal{O}_{T^\sigma,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$, $\mathfrak{m}^n_{\sigma(t)}$ denoting indifferently the maximal ideals of both local rings.

The rings $\mathcal{O}_{T^\sigma,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ and $\mathcal{O}_{T,t}/\mathfrak{m}^n_t$ are canonically isomorphic, because the schemes T and T^σ are. On the other hand, we can explicitly describe $\mathcal{O}_{T,t}/\mathfrak{m}^n_t$ and $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$, as loci of Hodge classes, using the Gauss-Manin connection on $\mathcal{H}$ and Griffiths transversality. This is explained in [74] in the case of pure Hodge structures, and explicitly stated for $n = 1$, see lemma 17.16. Our case follows *mutatis mutandis.* As a consequence, since the Gauss-Manin connection is algebraic, see [44], we have a canonical isomorphism between $\mathcal{O}_{T,t}/\mathfrak{m}^n_t$ and $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$.

This discussion shows that $\mathcal{O}_{T^\sigma,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ and $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ are isomorphic as subrings of $\mathcal{O}_{S,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$. Since this holds for all n, and since the reduced subscheme of T^σ is included in V, we get an equality $V = T^\sigma$, which shows that T^σ is an irreducible component of the zero locus of ν_Z. □

3.3.2 Application

As in [76], there are many situations where one can easily check that the conditions of theorems 3.2 and 3.3. Let us give one example.

Theorem 3.13. *Let $\pi : X \to S$ be a smooth projective family of complex Calabi-Yau threefolds over a quasiprojective base such that the induced map from S to the corresponding moduli space is finite, and let $Z \hookrightarrow X$ be a flat family of curves in X which are homologous to 0 in the fibers of π. Assume everything is defined over a finitely generated field k. Let ν be the associated normal function.*

(i) Let T be an irreducible component of the zero locus of ν. Assume that T is of positive dimension and that for a general complex point t of T, the intermediate jacobian $J^2(X_t)$ has no abelian factor. Then T is defined over a finite extension of k, all its conjugate are irreducible components of the zero locus of ν, and for every closed point t of T with value in a finitely generated field, the étale Abel-Jacobi invariant of Z_t is zero.

(ii) Let T be a subvariety of S of positive dimension defined over a finitely generated field. Assume that for a general complex point t of T, the intermediate jacobian $J^2(X_t)$ has no abelian factor and that for any point t of T with value in a finitely generated field, the étale Abel-Jacobi invariant of Z_t is zero. Then ν vanishes on T.

Proof. In order to apply our preceding results, we only have to check that in both situations above, the local system $R^3\pi_*\mathbb{Z}$ has no global section over $T(\mathbb{C})$. First of all, since the restriction of π to T is a nontrivial family of Calabi-Yau threefolds, the Hodge structure on $H^0(T(\mathbb{C}), R^3\pi_*\mathbb{Z})$ is of type $\{(2,1),(1,2)\}$. Indeed, the infinitesimal Torelli theorem for Calabi-Yau varieties, see [74], th. 10.27, shows that the fixed part of $R^3\pi_*\mathbb{Z}$ cannot have a part of type $(3,0)$. Now this proves that the invariant part of $R^3\pi_*\mathbb{Z}$ corresponds to a constant abelian subvariety of the Jacobian fibration $J^2(X_T/T)$, which has to be zero by assumption. This shows that the local system $R^3\pi_*\mathbb{Z}$ has no global section. □

3.3.3 Hodge classes of normal functions and their étale counterpart

Let $\mathcal{H}$ be a variation of pure Hodge structures of weight -1 over an irreducible complex variety S, and let ν be a normal function on S. The Hodge class of ν is defined the following way. Let $H_{\mathbb{Z}}$ be the integral structure of $\mathcal{H}$. We have an exact sequence of sheaves on S

$$0 \to H_{\mathbb{Z}} \to \mathcal{H}/F^0\mathcal{H} \to \mathcal{J}(\mathcal{H}) \to 0,$$

$\mathcal{J}(\mathcal{H})$ being the sheaf of holomorphic sections of the Jacobian fibration. This gives a map $H^0(S, J(\mathcal{H})) \to H^1(S, H_{\mathbb{Z}})$. The normal function ν is a holomorphic section of $\mathcal{J}(\mathcal{H})$. Its image in $H^1(S, H_{\mathbb{Z}})$ is called its Hodge class and is denoted by $[\nu]$.

The homological interpretation of intermediate Jacobians suggests another way of defining Hodge classes of normal functions. Indeed, a normal function ν defines an extension of variations of mixed Hodge structures over S

$$0 \to H \to H' \to \mathbb{Z} \to 0.$$

The long exact sequence of sheaf cohomology gives a map $\delta : H^0(S,\mathbb{Z}) \to H^1(S,\mathcal{H})$. The following is straightforward, but we have been unable to find a reference.

Proposition 3.14. *We have* $[\nu] = \delta(1)$.

Proof. Let us start by briefly recalling the explicit form given in [19] of the isomorphism $\phi : \mathrm{Ext}^1_{MHS}(\mathbb{Z}, H) \simeq J(H)$ when S is a point. Choose an isomorphism of abelian groups

$$H'_{\mathbb{Z}} \simeq H_{\mathbb{Z}} \oplus \mathbb{Z}.$$

There exists an element $\alpha \in H_{\mathbb{C}}$ such that $\alpha \oplus 1 \in F^0 H'_{\mathbb{C}}$. The class of α in

$$\frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} \oplus F^0 H_{\mathbb{C}}} = J(H)$$

is well-defined and is the image of the extension

$$0 \to H \to H' \to \mathbb{Z} \to 0$$

by ϕ.

Now let us work over a general complex quasi-projective base S as before. Let us choose a covering of $S(\mathbb{C})$ by open subsets U_i (for the usual topology) such that the exact sequence

$$0 \to H_{\mathbb{Z}} \to H'_{\mathbb{Z}} \to \mathbb{Z} \to 0$$

splits over U_i. Splittings correspond to sections $\sigma_i \in H^0(U_i, H'_{\mathbb{Z}})$ mapping to 1 in $\mathbb{Z}$. The cohomology class $\delta(1)$ is represented by the cocycle $\sigma_i - \sigma_j$.

For each i and each $s \in U_i$, let $\tau_i(s)$ be the image in $H_{s,\mathbb{C}}/F^0 H_{s,\mathbb{C}}$ of an element $\alpha_s \in H_{s,\mathbb{C}}$ such that $\sigma_i(s) + \alpha_s \in F^0 H'_{s,\mathbb{C}}$. The Hodge class of the normal function ν is represented by the cocycle $\tau_i - \tau_j \in H_{\mathbb{Z}}/(H_{\mathbb{Z}} \cap F^0 H_{\mathbb{Z}}) = H_{\mathbb{Z}}$. Since $\tau_i - \tau_j = \sigma_i - \sigma_j$ through this identification, this concludes the proof. □

Let us now investigate the étale side. Let S be a quasi-projective variety over a finitely generated subfield k of $\mathbb{C}$, with extensions ν_l

$$0 \to H_l \to H_l' \to \mathbb{Z}_l \to 0$$

of $\mathbb{Z}_l$-sheaves over S for all prime numbers l. We get an extension class in $H^1(S_{\mathbb{C}}, H_l)$ by pulling back to $S_{\mathbb{C}}$, which we will denote by $[\nu_l]$. Let us denote by

$$[\nu_{\acute{e}t}] \in H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\hat{\mathbb{Z}}(i)) := \prod_l H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i))$$

the class whith components $[\nu_l]$.

The importance of the class $[\nu_{\acute{e}t}]$ relies on the following result, which shows that it controls the extensions ν_l in the same way that the Hodge class of a normal function controls the normal function itself. It is indeed an analogue of Griffiths' result in [33] which proves that a normal function with zero Hodge class is constant in the fixed part of the intermediate Jacobian.

Theorem 3.15. *In the previous setting, assume that there exists a smooth projective family $\pi : X \to S$ such that $H_l = R^{2i-1}\pi_*\mathbb{Z}_l(i)$, and that $[\nu_{\acute{e}t}] = 0$. Let s be any closed point of S, and let $\overline{s}$ be a geometric point over s. Then for every prime number l, the extension*

$$0 \to H_l \to H_l' \to \mathbb{Z}_l \to 0 \tag{3.7}$$

splits over S if and only if the extension

$$0 \to H_{l,\overline{s}} \to H_{l,\overline{s}}' \to \mathbb{Z}_l \to 0$$

of $G_{k(s)}$-modules splits, where $G_{k(s)}$ is the absolute Galois group of the residue field of s.

Proof. Using the argument of proposition 3.7, we can make a finite extension of the base field and assume that s is a k-point of S. Assume that the extension

$$0 \to H_{l,s} \to H_{l,s}' \to \mathbb{Z}_l \to 0$$

of $G_{k(s)}$-modules splits. We need to prove that the exact sequence (3.7) splits over S for any prime number l. Fix a prime number l. We have an exact sequence

$$1 \to \pi_1^{\acute{e}t}(S_{\mathbb{C}}, \overline{s}) \to \pi_1^{\acute{e}t}(S, \overline{s}) \to G_k \to 1.$$

The last arrow admits a section σ coming from the rational point s.

The vector spaces $H_{l,\overline{s}}$ and $H_{l,\overline{s}}'$ are $\pi_1^{\acute{e}t}(S, \overline{s})$-modules, and the extension (3.7) corresponds to the extension

$$0 \to H_{l,\overline{s}} \to H_{l,\overline{s}}' \to \mathbb{Z}_l \to 0$$

of $\pi_1^{\acute{e}t}(S, \overline{s})$-modules. Now by assumption, the following sequence is exact.

$$0 \to (H_{l,\overline{s}})^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\overline{s})} \to (H_{l,\overline{s}}')^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\overline{s})} \to \mathbb{Z}_l \to 0. \tag{3.8}$$

Indeed, the vanishing of $[\nu_l]$ implies that there exists a global section of $H_{l,\overline{s}}$ over $S_{\mathbb{C}}$ mapping to 1 in $\mathbb{Z}_l$, which implies the surjectivity of the last arrow.

The Galois group G_k acts on $(H_{l,\overline{s}}')^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\overline{s})}$, either through σ or through the previous exact sequence – those are the same actions. We want to prove that G_k fixes an element mapping to 1 in $\mathbb{Z}_l$. Since the exact sequence of G_k-modules

$$0 \to H_{l,\overline{s}} \to H_{l,\overline{s}}' \to \mathbb{Z}_l \to 0$$

is split by assumption, G_k acting through σ, there exists $h' \in H'_{l,\bar{s}}$, mapping to 1 in $\mathbb{Z}_l$, such that $g(h') = h'$ for any $g \in G_k$.

On the other hand, let $\mathfrak{p}$ be a finite place of k that does not divide l and such that X_s has good reduction at $\mathfrak{p}$. Fix a Frobenius element $F_{\mathfrak{p}}$ in a decomposition group of $\mathfrak{p}$. Since $F_{\mathfrak{p}}$ acts trivially on $\mathbb{Z}_l$ and has weight -1 on $(H_{l,\bar{s}})^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\bar{s})} = (H^{2i-1}(X_{\bar{s}}, \mathbb{Z}_l(i)))^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\bar{s})}$ by the Weil conjectures, there exists $h'' \in (H'_{l,\bar{s}})^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\bar{s})} \otimes \mathbb{Q}$, mapping to 1 in $\mathbb{Z}_l$, such that $F_{\mathfrak{p}}(h'') = h''$. Since $(H_{l,\bar{s}})^{F_{\mathfrak{p}}} = 0$ by the Weil conjectures again, we have $h'' = h'$, which shows that h' lies in $(H'_{l,\bar{t}})^{\pi_1^{\acute{e}t}(S_{\mathbb{C}},\bar{s})}$. This proves that the exact sequence (3.8) splits, which concludes the proof. $\square$

In case we start with a smooth projective family $\pi : X \to S$, together with a flat family of algebraic cycles $Z \hookrightarrow X$ of algebraic cycles of codimension i which are homologically equivalent to zero on the fibers of π, we get an extension of variations of mixed Hodge structures over $S_{\mathbb{C}}$ corresponding to ν_Z, and an extension ν_l of $\mathbb{Z}_l$-sheaves over S induced by the étale Abel-Jacobi map for every prime number l, with $H_l = R^{2i-1}\pi_*\mathbb{Z}_l(i)$. The pull-back of the latter to $S_{\mathbb{C}}$ is the extension of local systems induced by ν. As a consequence of Artin's comparison theorem between étale and singular cohomology in [1], exp. XI, we get the following "easy" part of the comparison theorems between Abel-Jacobi maps.

Proposition 3.16. *The class $[\nu_{\acute{e}t}]$ is the image in $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\hat{\mathbb{Z}}(i))$ of the Hodge class $[\nu] \in H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}(i))$. As a consequence, $[\nu] = 0$ if and only if $[\nu_{\acute{e}t}] = 0$.*

Proof. The first statement is a direct consequence of proposition 3.14 and Artin's theorem which identifies the cohomology group $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i))$ with $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*Z(i)) \otimes \mathbb{Z}_l$ in a functorial way. This proves that the vanishing of $[\nu]$ implies the vanishing of $[\nu_{\acute{e}t}]$. Now if $[\nu_l] = 0$, then the class $[\nu] \in H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*Z(i))$ vanishes in $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*Z(i)) \otimes \mathbb{Z}_l$, which proves that there exist an integer α prime to l such that $\alpha[\nu] = 0$. If $[\nu_{\acute{e}t}] = 0$, this is true for all l, which implies that $[\nu] = 0$. $\square$

Remark 3.17. There are of course different ways of computing the value of $[\nu_{\acute{e}t}]$. Indeed, Leray spectral sequences still exist in continuous étale cohomology, working in the category of l-adic sheaves as defined by Ekedahl in [26]. The cycle class of Z induces from the Leray spectral sequence attached to the morphism π an element in $H^1_{\acute{e}t}(S, R^{2i-1}\pi_*\mathbb{Z}_l(i))$. This cohomology class maps to an element in $H^1_{\acute{e}t}(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i)) = H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i))$. Now this class is equal to the l-adic component of $[\nu_{\acute{e}t}]$. This can either be proved directly or using proposition 3.16 and applying the corresponding well-known result for Griffiths' Abel-Jacobi map (see [74], lemma 20.20).

Remark 3.18. Let us show how what we just showed implies the fact that for cycles *algebraically equivalent to zero*, the vanishing of Griffith's Abel-Jacobi invariant is equivalent to the vanishing of the étale Abel-Jacobi invariant. This result is well-known, and for zero-cycles it is an easy consequence of the result of Raskind in the appendix of [56], but it does not appear to have been stated explicitly in the literature. We can easily reduce it to the case of divisors on curves by the following functoriality argument.

We work over a finitely generated subfield k of $\mathbb{C}$. Let $Z \hookrightarrow X$ be a flat family of cycles of codimension i over a smooth curve C such that the fiber of Z over a geometric point 0 of C is zero. Changing base to $\mathbb{C}$, the normal function $\nu_Z : C \to J^i(X)$ takes value in the algebraic part $J^i_{alg}(X)$ of the intermediate Jacobian of X. From the Kummer exact sequence on $J^i_{alg}(X)$, we get a map from the group of k-points of $J^i_{alg}(X)$ to $H^1(C, H^{2i-1}(X, \hat{\mathbb{Z}}(i))) =$

$\prod_l H^1(C, H^{2i-1}(X, \mathbb{Z}_l(i)))$ [5]. This corresponds to a collection of extensions ν'_l of sheaves on S for all l which we claim are the ν_l coming from the étale Abel-Jacobi map applied to the Z_s.

The comparison result we need comes from this claim and the Mordell-Weil theorem. Indeed, the Mordell-Weil theorem implies that the group of rational points of $J^i_{alg}(X)$ is finitely generated. On the other hand, the kernel of the map from the group of rational points of $J^i_{alg}(X)$ to $H^1(C, H^{2i-1}(X, \mathbb{Z}_l(i)))$ induced by the Kummer exact sequence is equal to the group of rational points of $J^i_{alg}(X)$ which are infinitely l-divisible. As such, this kernel is contained in the group of torsion points with torsion prime to l. It follows that the map from the group of k-points of $J^i_{alg}(X)$ to $H^1(C, H^{2i-1}(X, \hat{\mathbb{Z}}(i)))$ is injective, and that, assuming the claim, the vanishing of Griffith's Abel-Jacobi invariant is equivalent to the vanishing of the étale Abel-Jacobi invariant in our situation.

Now since for all l, the extensions ν_l and ν'_l obviously split at the point 0, we just have to prove that $[\nu'_{\acute{e}t}] = [\nu_{\acute{e}t}]$. Indeed, it will then be enough to apply theorem 3.15 to the extension $\nu_l - \nu'_l$. An easy functoriality argument reduces this to the case when X is the curve C itself, which concludes using Raskind results on 0-cycles. We could also have used functoriality for $\nu_{\acute{e}t}$ itself, but this is a little more cumbersome and is not necessary.

3.3.4 Proof of theorem 3.5

Let us now prove theorem 3.5. It is actually an application of general results about normal functions and their Hodge classes and of their étale counterparts we just proved. We go back to the notation of section 3.1.

Proof of *(i).* In the situation of the theorem, we can use Terasoma's lemma as before to see that the exact sequence ν_l of $\hat{\mathbb{Z}}$-sheaves on S associated to $Z \hookrightarrow X$ is split for every prime number l, which implies that $[\nu_{\acute{e}t}]$ is zero, and shows that the Hodge class $[\nu_Z]$ of ν_Z is zero by proposition 3.16.

According to fundamental results of Griffiths, see [33], a normal function with zero Hodge class is constant in the fixed part of the intermediate Jacobian. In our case, since ν_Z vanishes at some complex point of S, this shows that $\nu_Z = 0$. □

Proof of *(ii).* After pulling back to T, we can assume that the normal function ν_Z is identically 0 on S. It follows from proposition 3.16 that $[\nu_{\acute{e}t}] = 0$. The assumptions of theorem 3.15 are henceforth satisfied, which proves that the exact sequence (3.5) splits, and concludes the proof. □

5. Actually, we would need to change the base field to a field of definition of $J^i_{alg}(X)$ to do so, which we are allowed to do by proposition 3.7, but $J^i_{alg}(X)$ is actually defined over k.

Bibliographie

[1] *Théorie des topos et cohomologie étale des schémas. Tome 3* – Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[2] H. Abelson – « Topologically distinct conjugate varieties with finite fundamental group », *Topology* **13** (1974), p. 161–176.

[3] Y. André – « Pour une théorie inconditionnelle des motifs », *Inst. Hautes Études Sci. Publ. Math.* (1996), no. 83, p. 5–49.

[4] — , *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004.

[5] E. Artal Bartolo, J. Carmona Ruber et J. I. Cogolludo Agustín – « Effective invariants of braid monodromy », *Trans. Amer. Math. Soc.* **359** (2007), no. 1, p. 165–183 (electronic).

[6] M. F. Atiyah – « Complex analytic connections in fibre bundles », *Trans. Amer. Math. Soc.* **85** (1957), p. 181–207.

[7] I. Bauer, F. Catanese et F. Grunewald – « The absolute galois group acts faithfully on the connected components of the moduli space of surfaces of general type », 2007.

[8] A. Beauville – « Variétés Kähleriennes dont la première classe de Chern est nulle », *J. Differential Geom.* **18** (1983), no. 4, p. 755–782 (1984).

[9] A. A. Beĭlinson – « Notes on absolute Hodge cohomology », Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, p. 35–68.

[10] — , « Height pairing between algebraic cycles », K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, p. 1–25.

[11] S. Bloch et V. Srinivas – « Remarks on correspondences and algebraic cycles », *Amer. J. Math.* **105** (1983), no. 5, p. 1235–1253.

[12] S. Bloch – « Semi-regularity and de Rham cohomology », *Invent. Math.* **17** (1972), p. 51–66.

[13] — , *Lectures on algebraic cycles*, Duke University Mathematics Series, IV, Duke University Mathematics Department, Durham, N.C., 1980.

[14] F. A. Bogomolov – « Hamiltonian Kählerian manifolds », *Dokl. Akad. Nauk SSSR* **243** (1978), no. 5, p. 1101–1104.

[15] P. Brosnan et G. Pearlstein – « Zero loci of admissible normal functions with torsion singularities », 2008.

[16] — , « On the algebraicity of the zero locus of an admissible normal function », 2009.

[17] P. Brosnan et G. J. Pearlstein – « The zero locus of an admissible normal function », 2006.

[18] J. Carlson, M. Green, P. Griffiths et J. Harris – « Infinitesimal variations of Hodge structure. I », *Compositio Math.* **50** (1983), no. 2-3, p. 109–205.

[19] J. A. Carlson – « Extensions of mixed Hodge structures », Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, p. 107–127.

[20] E. Cattani, P. Deligne et A. Kaplan – « On the locus of Hodge classes », *J. Amer. Math. Soc.* **8** (1995), no. 2, p. 483–506.

[21] F. Charles – « Conjugate varieties with distinct real cohomology algebras », *J. Reine Angew. Math.* **630** (2009), p. 125–139.

[22] — , « On the zero locus of normal functions and the étale abel-jacobi map », *Int. Math. Res. Not. IMRN* (2009), p. Art. ID rnp184, 10.

[23] P. Deligne – « Théorème de Lefschetz et critères de dégénérescence de suites spectrales », *Inst. Hautes Études Sci. Publ. Math.* (1968), no. 35, p. 259–278.

[24] P. Deligne, P. Griffiths, J. Morgan et D. Sullivan – « Real homotopy theory of Kähler manifolds », *Invent. Math.* **29** (1975), no. 3, p. 245–274.

[25] R. W. Easton et R. Vakil – « Absolute Galois acts faithfully on the components of the moduli space of surfaces : a Belyi-type theorem in higher dimension », *Int. Math. Res. Not. IMRN* (2007), no. 20, p. Art. ID rnm080, 10.

[26] T. Ekedahl – « On the adic formalism », The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, p. 197–218.

[27] L. Göttsche et W. Soergel – « Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces », *Math. Ann.* **296** (1993), no. 2, p. 235–245.

[28] M. Green, P. A. Griffiths et K. H. Paranjape – « Cycles over fields of transcendence degree 1 », *Michigan Math. J.* **52** (2004), no. 1, p. 181–187.

[29] M. L. Green – « Griffiths' infinitesimal invariant and the Abel-Jacobi map », *J. Differential Geom.* **29** (1989), no. 3, p. 545–555.

[30] P. Griffiths et J. Harris – « Infinitesimal variations of Hodge structure. II. An infinitesimal invariant of Hodge classes », *Compositio Math.* **50** (1983), no. 2-3, p. 207–265.

[31] P. A. Griffiths – « Periods of integrals on algebraic manifolds. I. Construction and properties of the modular varieties », *Amer. J. Math.* **90** (1968), p. 568–626.

[32] — , « Periods of integrals on algebraic manifolds. II. Local study of the period mapping », *Amer. J. Math.* **90** (1968), p. 805–865.

[33] — , « Infinitesimal variations of Hodge structure. III. Determinantal varieties and the infinitesimal invariant of normal functions », *Compositio Math.* **50** (1983), no. 2-3, p. 267–324.

[34] A. Grothendieck – « On the de Rham cohomology of algebraic varieties », *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 29, p. 95–103.

[35] — , « Standard conjectures on algebraic cycles », Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, p. 193–199.

[36] D. Huybrechts – « Compact hyper-Kähler manifolds : basic results », *Invent. Math.* **135** (1999), no. 1, p. 63–113.

[37] D. Huybrechts et M. Lehn – *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[38] L. Illusie – *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, 1971.

[39] U. Jannsen – « Continuous étale cohomology », *Math. Ann.* **280** (1988), no. 2, p. 207–245.

[40] — , *Mixed motives and algebraic K-theory*, Lecture Notes in Mathematics, vol. 1400, Springer-Verlag, Berlin, 1990, With appendices by S. Bloch and C. Schoen.

[41] — , « Motives, numerical equivalence, and semi-simplicity », *Invent. Math.* **107** (1992), no. 3, p. 447–452.

[42] — , « Motivic sheaves and filtrations on Chow groups », Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, p. 245–302.

[43] — , « Equivalence relations on algebraic cycles », The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, p. 225–260.

[44] N. M. Katz et T. Oda – « On the differentiation of de Rham cohomology classes with respect to parameters », *J. Math. Kyoto Univ.* **8** (1968), p. 199–213.

[45] S. L. Kleiman – « Algebraic cycles and the Weil conjectures », Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968, p. 359–386.

[46] — , « The standard conjectures », Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, p. 3–20.

[47] J. D. Lewis – « A filtration on the Chow groups of a complex projective variety », *Compositio Math.* **128** (2001), no. 3, p. 299–322.

[48] J. I. Manin – « Proof of an analogue of Mordell's conjecture for algebraic curves over function fields », *Dokl. Akad. Nauk SSSR* **152** (1963), p. 1061–1063.

[49] J. S. Milne et J. Suh – « Nonhomeomorphic conjugates of connected shimura varieties », 2008.

[50] J. W. Morgan – « The rational homotopy theory of smooth, complex projective varieties (following P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan) (Invent. Math. **29** (1975), no. 3, 245–274) », Séminaire Bourbaki, Vol. 1975/76, 28ème année, Exp. No. 475, Springer, Berlin, 1977, p. 69–80. Lecture Notes in Math., Vol. 567.

[51] S. Mukai – « Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface », *Invent. Math.* **77** (1984), no. 1, p. 101–116.

[52] D. Mumford – « Rational equivalence of 0-cycles on surfaces », *J. Math. Kyoto Univ.* **9** (1968), p. 195–204.

[53] M. V. Nori – « Algebraic cycles and Hodge-theoretic connectivity », *Invent. Math.* **111** (1993), no. 2, p. 349–373.

[54] K. G. O'Grady – « Desingularized moduli spaces of sheaves on a $K3$ », *J. Reine Angew. Math.* **512** (1999), p. 49–117.

[55] — , « A new six-dimensional irreducible symplectic variety », *J. Algebraic Geom.* **12** (2003), no. 3, p. 435–505.

[56] W. Raskind – « Higher l-adic Abel-Jacobi mappings and filtrations on Chow groups », *Duke Math. J.* **78** (1995), no. 1, p. 33–57.

[57] D. Reed – « The topology of conjugate varieties », *Math. Ann.* **305** (1996), no. 2, p. 287–309.

[58] M. Saito – « Admissible normal functions », *J. Algebraic Geom.* **5** (1996), no. 2, p. 235–276.

[59] M. Saito – « Hausdorff property of the neron models of green, griffiths and kerr », 2008.

[60] C. Schoen – « On the computation of the cycle class map for nullhomologous cycles over the algebraic closure of a finite field », *Ann. Sci. École Norm. Sup. (4)* **28** (1995), no. 1, p. 1–50.

[61] J.-P. Serre – « Géométrie algébrique et géométrie analytique », *Ann. Inst. Fourier, Grenoble* **6** (1955–1956), p. 1–42.

[62] — , « Exemples de variétés projectives conjuguées non homéomorphes », *C. R. Acad. Sci. Paris* **258** (1964), p. 4194–4196.

[63] I. Shimada – « Non-homeomorphic conjugate complex varieties », 2007.

[64] J. Steenbrink et S. Zucker – « Variation of mixed Hodge structure. I », *Invent. Math.* **80** (1985), no. 3, p. 489–542.

[65] T. Terasoma – « Complete intersections with middle Picard number 1 defined over **Q** », *Math. Z.* **189** (1985), no. 2, p. 289–296.

[66] G. Tian – « Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric », Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, p. 629–646.

[67] A. N. Todorov – « The Weil-Petersson geometry of the moduli space of SU($n \geq 3$) (Calabi-Yau) manifolds. I », *Comm. Math. Phys.* **126** (1989), no. 2, p. 325–346.

[68] J. Varouchas – « Kähler spaces and proper open morphisms », *Math. Ann.* **283** (1989), no. 1, p. 13–52.

[69] M. Verbitsky – « Hyperholomorphic bundles over a hyper-Kähler manifold », *J. Algebraic Geom.* **5** (1996), no. 4, p. 633–669.

[70] M. Verbitsky et D. Kaledin – *Hyperkahler manifolds*, Mathematical Physics (Somerville), vol. 12, International Press, Somerville, MA, 1999.

[71] C. Voisin – « Une remarque sur l'invariant infinitésimal des fonctions normales », *C. R. Acad. Sci. Paris Sér. I Math.* **307** (1988), no. 4, p. 157–160.

[72] — , « Variations of Hodge structure and algebraic cycles », *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)* (Basel), Birkhäuser, 1995, p. 706–715.

[73] — , « A counterexample to the Hodge conjecture extended to Kähler varieties », *Int. Math. Res. Not.* (2002), no. 20, p. 1057–1075.

[74] — , *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés [Specialized Courses], vol. 10, Société Mathématique de France, Paris, 2002.

[75] — , « On the homotopy types of compact Kähler and complex projective manifolds », *Invent. Math.* **157** (2004), no. 2, p. 329–343.

[76] — , « Hodge loci and absolute Hodge classes », *Compos. Math.* **143** (2007), no. 4, p. 945–958.

[77] S. T. Yau – « On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I », *Comm. Pure Appl. Math.* **31** (1978), no. 3, p. 339–411.