Étude de la quadrangulation infinie uniforme

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Mémoire de thèse :

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29 septembre 2009

Résumé

LES CARTES PLANAIRES sont des plongements dans la sphère de graphes planaires. Dans cette thèse, nous nous intéressons aux quadrangulations, qui sont les cartes ayant toutes leurs faces de degré 4. Cet intérêt pour les quadrangulations vient de l'existence d'une bijection entre celles-ci et certains arbres planaires étiquetés. Cette bijection encode des propriétés métriques des quadrangulations et a donc l'avantage de transposer certaines études sur les quadrangulations vers les arbres.

L'objet central de cet thèse est la quadrangulation infinie de loi uniforme. Cette carte a été définie de deux manières indépendantes. La première méthode, naturelle du point de vue des cartes, est de prendre la limite locale de grandes quadrangulations aléatoires de loi uniforme parmi les quadrangulations de même taille. La seconde méthode repose sur la bijection avec les arbres. On y construit dans un premier temps un arbre infini de loi uniforme, puis on transporte la loi de cet arbre sur l'ensemble des quadrangulations infinies avec la bijection.

L'objet du chapitre 2 de ce mémoire est de démontrer que ces deux constructions aboutissent au même objet. Ce fait n'est à priori pas évident car la bijection entre les arbres et les quadrangulations n'est pas continue pour la topologie de la convergence locale. Le résultat s'obtient alors en étudiant des propriétés combinatoires de cette bijection et les sommets ayant de petites étiquettes dans les générations élevées d'un arbre sous la loi uniforme.

Le chapitre 3 utilise ensuite cette équivalence des deux points de vue pour calculer les limites d'échelle de certaines fonctionnelles de la quadrangulation infinie uniforme. En effet, des quantités comme le volume des boules autour d'un point distingué de la quadrangulation infinie uniforme peuvent se calculer grâce à une étude de l'arbre infini uniforme. Ce chapitre est articulé autour de la preuve de la convergence des fonctions de contour de l'arbre infini uniforme vers un processus stochastique lié au serpent brownien.

Avant ces contributions originales, nous donnons dans le chapitre 1 un survol de la littérature sur le sujet et une présentation précise des objets probabilistes et combinatoires utilisés pendant cette thèse.

Abstract

PLANAR MAPS are proper embeddings of finite connected graphs in the two-dimensional sphere. In this thesis, we are interested in quadrangulations, which are planar maps with all faces of degree 4. The special interest in quadrangulations comes from the existence of a nice bijection between these objects and a certain class of decorated trees. This bijection encodes metric properties of the quadrangulation and thus allows one to transform certain problems about quadrangulations into similar quastions about trees.

Our main object of study is the so-called uniform infinite quadrangulation. This infinite random map has been defined in two different ways. The first definition consists in taking the local limit of large random (finite) quadrangulations whose laws are uniform over the set of all quadrangulations with a given size. The second definition takes advantage of the bijection between quadrangulations and trees. The starting point is a suitably defined uniform infinite random tree whose law is mapped to the set of all infinite quadrangulations using an extended version of the bijection.

In the second chapter of this manuscript, we prove that the two previous definitions lead to the same object. This fact is not trivial as, in the infinite setting, the bijection between quadrangulations and trees is not continuous for the topology of local convergence. The proof depends on studying some combinatorial properties of the bijection and estimating the distribution of vertices with small labels in high generations of the uniform infinite tree.

In chapter 3, we use the equivalence of the two definitions to compute scaling limits for the uniform infinite quadrangulation. Indeed, quantities such as the volume of balls around a distinguished point of the quadrangulation can be evaluated by studying the uniform infinite tree. The main technical ingredient of this chapter is the convergence of the rescaled contour functions of the uniform infinite tree towards a stochastic process linked to the Brownian snake.

Before these original results, in the first chapter of this manuscript, we give a survey of the existing literature on this subject and we introduce the probabilistic and combinatorial objects that are studied in this thesis.

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Sommaire

Introduction

Ce travail de thèse est consacré à l'étude de cartes planaires aléatoires. Nous proposons dans ce chapitre un survol de la littérature sur le sujet et une présentation des résultats de cette thèse. Nous commencerons par définir les cartes au paragraphe 1.1 avant d'en explorer les aspects combinatoires au paragraphe 1.2. La thématique centrale de ce mémoire de thèse, les limites de grandes cartes aléatoires, est ensuite présentée aux paragraphes 1.3 et 1.4.

Les contributions originales de ce mémoire sont présentées dans les paragraphes 1.3.5 et 1.4.4.

1.1 Cartes planaires

Les cartes sont des objets assez intuitifs, mais il importe de prendre quelques précautions pour en donner une définition rigoureuse. Ainsi, parmi les différents points de vue possibles, on peut définir les cartes de manière purement combinatoire par des classes de conjugaisons associées à certaines paires de permutations, ou encore de manière plus algébrique par des revêtements ramifiés de la sphère. Ces définitions ont l'avantage d'être compactes, mais elles nous seront peu utiles pour notre propos. En effet, ce manuscrit est consacré à l'étude de certains aspects métriques des cartes, difficilement reconnaissables dans des définitions formelles. De même, notre étude se limitera aux cartes planaires, et nous ne parlerons pas de cartes en genre quelconque. Nous pouvons cependant citer les ouvrages [LZ04, MT01] pour le lecteur voulant se renseigner sur ces aspects des cartes.

Le point de vue que nous privilégions est le suivant : une carte planaire est le plongement propre d'un graphe connexe fini dans la sphère $\mathbb{S}^2$, considéré à homéomorphisme conservant l'orientation près. Notons que le graphe sous-jacent d'une carte est seulement supposé connexe et fini ; il peut donc avoir des arêtes multiples et des boucles. Nous allons toujours considérer des cartes planaires enracinées, pour lesquelles une arête orientée appelée racine de la carte est distinguée. De manière informelle, une carte est donc un graphe dessiné dans le plan de façon à ce que ses arêtes ne se rencontrent qu'au niveau de ses sommets. Ceci permet de comprendre qu'une carte contient plus d'information que son graphe sous-jacent : un même graphe peut donner plusieurs cartes différentes comme illustré en Figure 1.1.

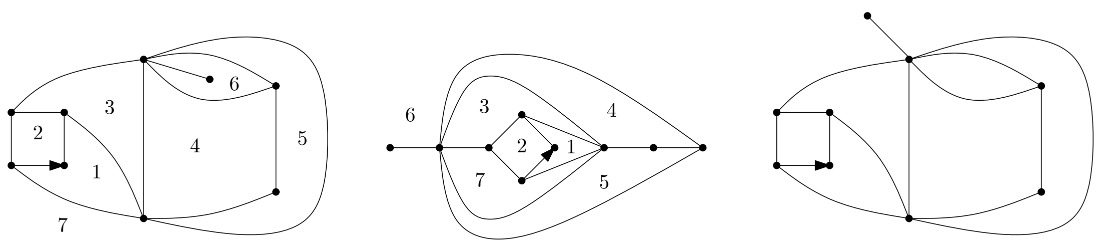

FIG. 1.1 – Exemples de cartes planaires : les deux premières sont identiques et sont des quadrangulations (les faces sont numérotées pour faciliter l'identification). La troisième, bien qu'ayant le même graphe sous-jacent que les premières, est une carte différente (sa face externe est de degré 6).

Considérons une carte planaire m. Une face de m est une composante connexe de $\mathbb{S}^2 \setminus m$, et le degré de cette face est le nombre d'arêtes orientées de m que l'on rencontre en faisant le tour de son bord. Le fait de considérer les arêtes orientées dans la définition du degré d'une face est fondamental, et il arrive effectivement que des arêtes non-orientées soient comptées deux fois dans le degré d'une face (voir Figure 1.1).

Parmi les sous-ensembles remarquables de cartes, une attention particulière est portée aux triangulations, dont toutes les faces sont de degré 3, ainsi qu'à leur généralisation, les p-angulations, dont les faces sont toutes de degré p.

Avant d'aller plus loin, nous pouvons tenter de retracer les grandes lignes de l'histoire de la théorie des cartes en mathématiques mais aussi en physique, afin de mieux comprendre l'interêt suscité par ce domaine de recherche. Les cartes planaires sont apparues en théorie des graphes, avec le théorème des quatre couleurs. C'est en effet en voulant démontrer ce théorème que Tutte, dans les années soixante, a véritablement fondé l'étude énumérative des cartes. Dans sa série de travaux [Tut62a, Tut62b, Tut62c, Tut63], il développe une méthode pour résoudre les équations satisfaites par les séries génératrices de certaines familles de cartes planaires. Cette méthode, baptisée méthode quadratique, continue encore à être fertile comme le prouvent les articles récents de Gao, Wanless et Wormald [GW02, GWW01].

Les travaux de Tutte ont aussi trouvé un écho important en mécanique statistique dans les années soixante dix grâce à leur liens avec les intégrales de matrices. Les premiers articles parus dans cette veine ont été publiés par t'Hooft [tH74] ainsi que par Brézin, Itzykson, Parisi et Zuber [BIPZ78]. Cela a donné naissance à de nombreux travaux en théorie des représentations et en géométrie algébrique comme on peut par exemple le voir avec le livre [LZ04].

L'intérêt des probabilistes pour les cartes est encore plus récent et prend sa substance dans la question suivante : à quoi ressemble une géométrie planaire typique ? Avant d'expliquer un peu plus en détail cette question qui commence l'article d'Angel et Schramm [AS03], nous devons dire que c'est encore une question qui trouve son origine dans la physique. Avec la théorie de la gravité quantique en 2 dimensions, les physiciens essayent de développer une théorie quantique de la gravité en généralisant à des surfaces le concept d'intégrale de Feynman sur les chemins. Là où l'intégrale de Feynman se fait par rapport à des mesures sur des chemins, l'équivalent en gravité quantique se fait par rapport à des mesures sur des surfaces. Ces mesures sont encore au pire mal définies, et au mieux mal comprises.

Les cartes s'avèrent alors particulièrement utiles en tant que modèles simples de géométries sphériques. En effet, on peut toujours munir l'ensemble des sommets d'une carte de la distance de graphe d_{gr}, et la carte est alors un espace métrique discret. Les cartes donnent alors un ensemble de modèles de géométries sphériques, et on peut étudier les mesures sur cet ensemble comme le suggère en particulier le livre [ADJ97].

La démarche est alors classique en théorie des probabilités : on fait grandir un objet combinatoire aléatoire tout en le changeant d'échelle pour trouver une convergence vers un objet aléatoire continu. L'exemple caractéristique de cette démarche est la construction du mouvement brownien plan, qui est la limite d'échelle de la marche aléatoire simple sur $\mathbb{Z}^2$. La loi du mouvement brownien plan est donc en quelque sorte la «mesure uniforme sur les chemins continus du plan», mesure qui apparaît dans l'intégrale de Feynman. Nous parlerons de ce que cette démarche a amené comme résultats pour les cartes un peu plus tard dans ce chapitre.

Les objets qui sont au centre des travaux que nous présentons ici sont issus d'une approche similaire. Nous nous intéressons en effet à des limites de cartes, mais cette fois-ci sans changement d'échelle. L'approche réside dans l'étude de la convergence de voisinages arbitraires de la racine de grandes cartes planaires aléatoires. Comme les distances ne sont pas renormalisées, les objets limites, appellés limites locales de grandes cartes planaires, sont des objets discrets mais infinis. Le but de cette thèse est d'étudier certaines propriétés de ces cartes planaires infinies.

1.2 Un peu de combinatoire

1.2.1 Arbres spatiaux

Nous allons voir un peu plus tard que les cartes planaires ont des liens étroits avec une autre grande classe d'objets combinatoires, les arbres planaires. Tout au long de ce manuscrit, nous allons donc faire un usage intensif des arbres. Le but de cette partie est de décrire le formalisme, maintenant standard, des arbres planaires tel qu'il a été introduit par Neveu [Nev86]. On note

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

où $\mathbb{N} = \{1, 2, \ldots\}$ et $\mathbb{N}^0 = \{\emptyset\}$ par convention. Un élément $u \in \mathcal{U}$ est donc une suite finie d'entiers naturels non nuls (ou encore, u est un mot fini d'entiers). Si u et v sont deux éléments de $\mathcal{U}$, on note uv la concaténation de u et v. Le concept central dans la théorie des arbres planaires telle que nous la décrivons est la notion de filiation : si un mot v est de la forme uj avec $j \in \mathbb{N}$, on dit que u est le parent de v ou, réciproquement, que v est un enfant de u. De manière générale, si v est de la forme uw avec $u, w \in \mathcal{U}$, on dit que u est un ancêtre de v et que v est un descendant de u.

Un arbre planaire enraciné τ est un sous ensemble fini de $\mathcal{U}$ satisfaisant les propriétés suivantes :

1. $\emptyset \in \tau$ (le mot $\emptyset$ est la racine de l'arbre τ),
2. si $v \in \tau$ et que $v \neq \emptyset$, alors nécessairement le parent de v appartient τ,
3. pour tout $u \in \mathcal{U}$, il existe un entier $k_u(\tau) \geqslant 0$ tel que $uj \in \tau$ si et seulement si $j \leqslant k_u(\tau)$.

Si $v \in \tau$, on dit que v est un sommet de τ et sa génération est sa longueur en tant que mot d'entiers. De plus, les arêtes de l'arbre τ sont les paires (u, v) lorsque $u, v \in \tau$ et que u est le parent de v. Dans toute la suite, nous n'allons considérer que des arbres qui seront plans et enracinés ; nous omettrons donc ces qualificatifs par souci de simplicité.

Le nombre total d'arêtes d'un arbre τ, noté $|\tau|$, est appelé la taille de τ. On note aussi $h(\tau)$ la génération maximale d'un sommet de τ, encore appelée la hauteur de τ.

Pour pouvoir étudier les cartes planaires grâce à des arbres, nous allons devoir ajouter à ceux-ci des étiquettes. Nous appellerons arbre étiqueté toute paire $\theta = (\tau, (\ell(u))_{u \in \tau})$ constituée d'un arbre plan τ et d'une fonction d'étiquetage ℓ, qui à chaque sommet v de l'arbre associe un entier $\ell(v) \in \mathbb{Z}$ appelé étiquette de v, de manière à ce que pour tous sommets $u, v \in \tau$, si v est un enfant de u, alors $|\ell(u) - \ell(v)| \leqslant 1$.

Si $\theta = (\tau, \ell)$ est un arbre étiqueté, on notera $|\theta| = |\tau|$ la taille de θ et $h(\theta) = h(\tau)$ sa hauteur. Si de plus $\ell(\emptyset) = l$ et que $\ell(u) \geqslant 1$ pour tout sommet u de τ, on dira que θ est un arbre l-bien-étiqueté. Pour $l = 1$, on parlera plus simplement d'arbres bien-étiquetés.

Il sera pratique par la suite de représenter les arbres planaires par leur fonctions de contours. Considérons un arbre spatial $\theta = (\tau, \ell)$. Nous pouvons définir à partir de θ un couple (C, V) de fonctions sur $[0, 2|\theta|]$ de la manière qui suit. Imaginons une particule qui parcourt l'arbre à vitesse 1 en partant de $\emptyset$ et dans le sens des aiguilles. Chaque arête est visitée deux fois et il faut donc un temps $2|\theta|$ à la particule pour faire le tour de l'arbre. Pour chaque entier $k \in [0, 2|\theta|]$, on définit $C(k)$ comme la génération du sommet visité par la particule au temps k et $V(k)$ comme l'étiquette de ce sommet. On complète ensuite la définition de (C, V) en interpolant linéairement. Le couple de fonctions (C, V) définit alors de manière unique l'arbre θ. La fonction C est appellée fonction de contour généalogique de θ et la fonction V est appellé fonction de contour spatial de θ. La Figure 1.2 donne un exemple de ce codage.

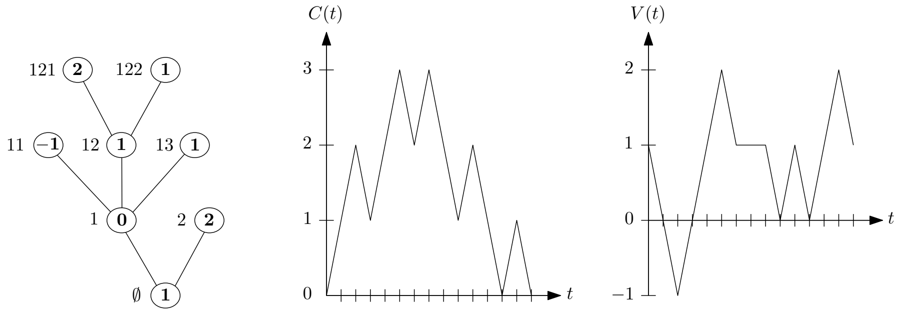

FIG. 1.2 – Un arbre spatial et ses deux fonctions de contour (C, V).

1.2.2 Énumération

Comme nous l'avons expliqué au début de ce chapitre, le point de départ des problèmes que nous nous posons est de choisir un objet (une carte quelconque, une quadrangulation, un arbre...) de manière aléatoire parmi tous les objets semblables de même taille. Dans tous les cas, il nous faudra donc savoir compter ces objets et faire un peu de combinatoire énumérative. En particulier, l'utilisation de séries génératices est récurrent dans cette thèse. Sans vouloir faire un cours de combinatoire analytique — renvoyons le lecteur curieux à l'ouvrage [FS09] par exemple — nous allons en illustrer l'utilisation sur deux exemples qui s'avèreront fondamentaux par la suite.

Commençons par déterminer le nombre a_n d'arbres plans enracinés à n arêtes. Pour cela, on définit la série formelle

$$A(x) = \sum_{n \geqslant 0} a_n x^n.$$

L'approche générique pour déterminer les a_n est alors d'identifier la série A grâce à une décomposition récursive des arbres. En effet, une telle décomposition se traduit généralement en une équation fonctionnelle satisfaite par A. Dans le cas présent, on peut découper un arbre de taille $n + 1$ suivant sa première arête : l'arbre se décompose alors en deux arbres de taille totale n (voir la Figure 1.3).

En prenant en compte toutes les tailles possibles pour les deux sous-arbres obtenus, on obtient la relation de récurrence

$$a_0 = 1 \quad \text{et} \quad a_{n+1} = \sum_{k=0}^{n} a_k \, a_{n-k}.$$

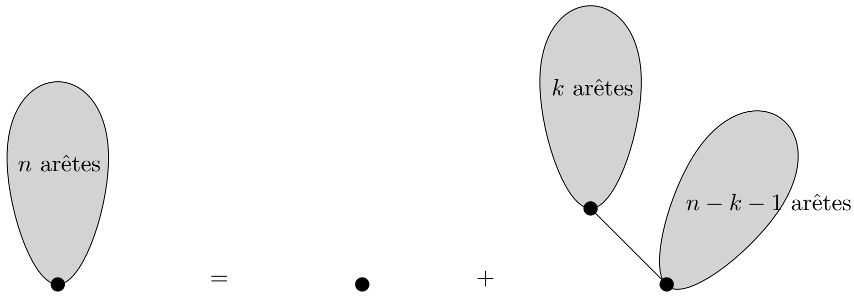

FIG. 1.3 – Décomposition récursive des arbres planaires

Il est alors facile de voir que cette relation de récurrence est équivalente à l'équation suivante pour A :

$$A(x) = 1 + xA(x)^2.$$

Cette équation fonctionelle admet deux solutions, dont une n'est pas une série formelle ayant un terme en z^{-1}. On a donc l'expression suivante pour la série génératrice A :

$$A(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Un développement en série entière donne alors la formule

$$a_n = \frac{1}{n+1}\binom{2n}{n}$$

des nombres de Catalan.

Notre second exemple sera celui des cartes planaires enracinées. Nous allons donc identifier la fonction génératrice

$$f(x) = \sum_{n\geqslant 0} b_n x^n$$

où, pour tout n, b_n désigne le nombre de cartes enracinées à n arêtes. Comme pour les arbres, nous allons faire une décomposition récursive des cartes en effacant leur arête racine (voir la Figure 1.4). Se présentent alors deux cas. Si l'arête racine est un isthme, alors l'effacer coupe la carte en deux composantes connexes. Sinon, effacer l'arête racine donne une seule composante connexe qui est donc une carte.

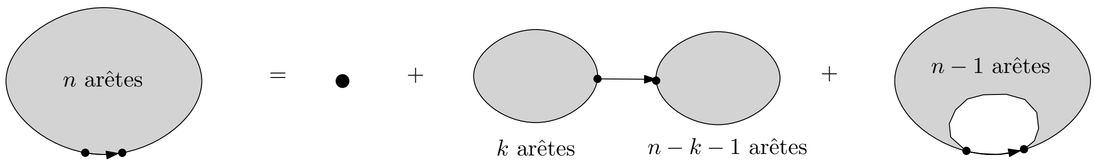

FIG. 1.4 – Décomposition récursive des cartes planaires

On se rend alors compte que pour traduire cette décomposition en une équation fonctionnelle pour f, il manque un paramètre. Il est en effet nécessaire de connaitre le degré de la face externe de la carte (la face directement à droite de la racine) pour pouvoir aller plus loin. Nous allons donc avoir besoin d'introduire une autre variable dans f et de considérer la fonction

$$F(x,y) = \sum_{n \geqslant 0} c_{n,m}\, x^n y^m$$

avec $c_{n,m}$ désignant le nombre de cartes à n arêtes et dont la face externe est de degré m pour tout couple d'entiers (n,m). La variable y est souvent appellée variable catalytique par les combinatoriciens.

En termes de fonctions génératrices, le premier cas de la décomposition de la Figure 1.4 correspond à $xy^2F(x,y)^2$. Pour le deuxième cas, on peut faire la construction inverse en partant d'une carte avec une face externe de degré $k \geqslant 0$, et en lui ajoutant une arête entre son sommet racine et un des sommets de son bord, ce qui donne une carte avec une arête de plus et une face externe de degré compris entre 1 et $k+1$. La décomposition récursive donne donc l'équation suivante pour F :

$$\begin{aligned} F(x,y) &= 1 + xy^2F(x,y)^2 + x\sum_{k \geqslant 0} \left[y^k\right] F(x,y) \left(y + y^2 + \ldots + y^{k+1}\right) \\ &= 1 + xy^2F(x,y)^2 + xy\frac{xF(x,y) - F(x,1)}{x-1} \end{aligned} \tag{1.1}$$

où $\left[y^k\right] F(x,y)$ désigne le terme en y^k de $F(x,y)$ considéré comme une série formelle en y. Il est intéressant de noter que $f(x) = F(x,1)$ est la série génératrice des cartes planaires indexées par leur nombre d'arêtes que nous recherchions initialement.

L'équation (1.1) ne se résoud pas aussi simplement que son analogue pour les arbres. La méthode quadratique de Tutte [Tut63] permet cependant de la résoudre, ainsi qu'une méthode plus générale dûe à Bousquet-Mélou et Jehanne [BMJ06]. On arrive alors à calculer le nombre de cartes enracinées à n arêtes qui est :

$$b_n = \frac{2}{n+2} 3^n \frac{1}{n+1}\binom{2n}{n}.$$

1.2.3 Bijection de Schaeffer

Le fait que les nombres de Catalan apparaissent dans les formules d'énumération de cartes enracinées peut s'interpréter par l'existence de bijections entre les cartes et différentes sortes d'arbres. Cori et Vauquelin [CV81] sont les premiers à avoir fait un pas dans cette direction. Schaeffer a ensuite développé dans sa thèse [Sch98] toute une panoplie de bijections entre les cartes et les arbres. Nous allons décrire ici celle qui servira tout au long de ce document. Cette bijection, qui fait le lien entre les quadrangulations et les arbres bien étiquetés, est appellée bijection de Schaeffer.

Comme souvent pour de telles bijections, un exemple sera le plus parlant. Prenons un arbre bien étiqueté $\theta = (\tau, \ell)$, de taille n et considérons un plongement de τ dans le plan. Les régions du plan délimitées par deux arêtes consécutives autour d'un sommet sont appelées des coins. La première étape consiste à numéroter ces coins par leur ordre d'apparition dans le contour horaire, en commençant par le coin racine (voir Figure 1.5). Notons $c_1, c_2, \ldots, c_k$ la suite des coins de l'arbre dans cet ordre. Fixons un point du plan qui n'appartient pas à l'arbre, que nous appelons ∂, donnons-lui l'étiquette 0 et numérotons son coin c_0. Pour chaque coin c_i de l'arbre, on construit alors une arête avec les règles suivantes :

1. Si $i = 1$, on relie c_1 à c_0. Ceci donne une arête $(\partial, \emptyset)$, que nous orientons de ∂ vers $\emptyset$.
2. Si l'étiquette du sommet correspondant à c_i est l, on relie c_i et le dernier coin d'étiquette $l - 1$ parmi $c_0, \ldots, c_{i-1}$.

La carte définie par le sommet ∂, l'ensemble des sommets de l'arbre, et les arêtes déssinées par ce procédé est une quadrangulation de racine $(\partial, \emptyset)$ et ayant n faces. La Figure 1.5 donne une illustration de la bijection de Schaeffer. Cette bijection a connu beaucoup de généralisations, en particulier pour les cartes biparties par Bouttier, Di Francesco et Guitter [BDFG04].

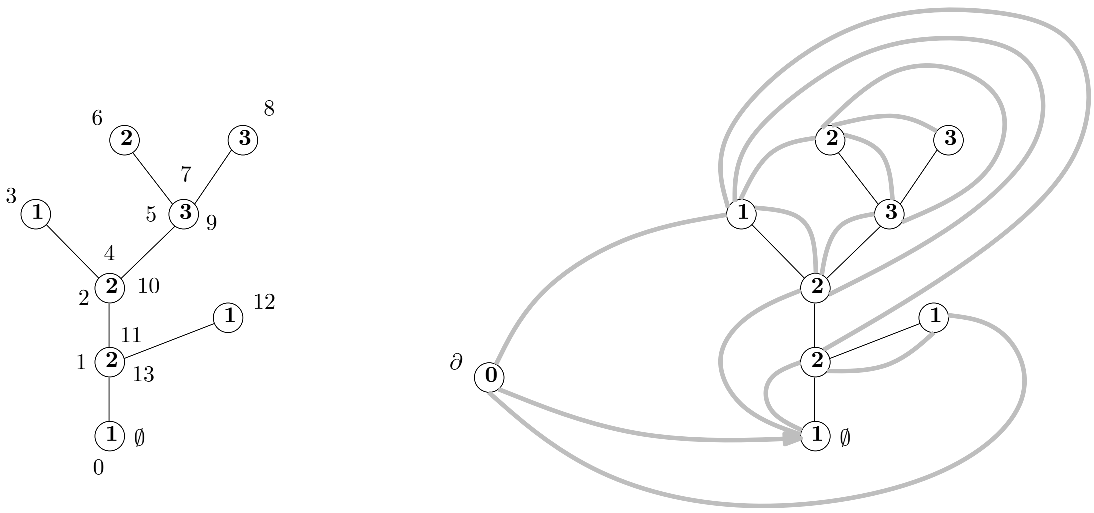

Fig. 1.5 – Numérotation des coins d'un arbre bien étiqueté et construction de la quadrangulation associée par la bijection de Schaeffer. On retrouve la quadrangulation de la Figure 1.1.

La bijection de Schaeffer s'est avérée remarquable d'efficacité dans l'étude métrique des cartes grâce à la propriété suivante : dans une carte, la distance de graphe entre un sommet quelconque v et le sommet racine ∂ correspond à l'étiquette $\ell(v)$ dans l'arbre associé. La connaissance des étiquettes d'un arbre donne donc beaucoup de renseignements sur les distances dans la carte associée. Il faut cependant souligner que les étiquettes de l'arbre ne portent pas toute l'information relative aux distances dans la carte : il ne suffit pas de connaitre l'étiquette de deux sommets pour en déduire leur distance dans la carte si aucun de ces sommets n'est le sommet racine ∂.

1.3 Limites locales

Nous sommes maintenant prêts à présenter les objets étudiés dans ce manuscrit, à savoir les limites locales de cartes et d'arbres. Comme la majorité de cette thèse est spécifique aux quadrangulations, nous allons les priviligier dans la présentation qui suit.

Cependant, nous devons souligner qu'une partie des résultats que nous donnons ont dans un premier temps été prouvés pour des triangulations.

1.3.1 Topologie sur les quadrangulations

Pour tout $n \geqslant 1$, notons $\mathbf{Q}_n$ l'ensemble des quadrangulations enracinées à n faces. L'ensemble des quadrangulations finies est donc l'ensemble

$$\mathbf{Q}_f = \bigcup_{n \geqslant 1} \mathbf{Q}_n.$$

L'idée derrière la topologie locale sur cet ensemble est la suivante : deux quadrangulations sont proches l'une de l'autre si elles sont identiques autour de leur racines. De manière plus précise, pour toute quadrangulation enracinée q et tout $r > 0$, on considèrera les «sous-cartes» de q notées $B_{\mathbf{Q},r}(q)$ composées de l'union des faces de q qui ont un sommet à distance strictement plus petite que r de la racine de q, comme illustré en Figure 1.6. Ces cartes sont des quadrangulations avec frontière dans le sens où toutes leur faces sont de degré 4, exepté un certain nombre de degré pair quelconque, qui sont les faces frontières. Il est d'ailleurs important de bien comprendre que les boules $B_{\mathbf{Q},r}(q)$ peuvent avoir plusieurs faces frontières.

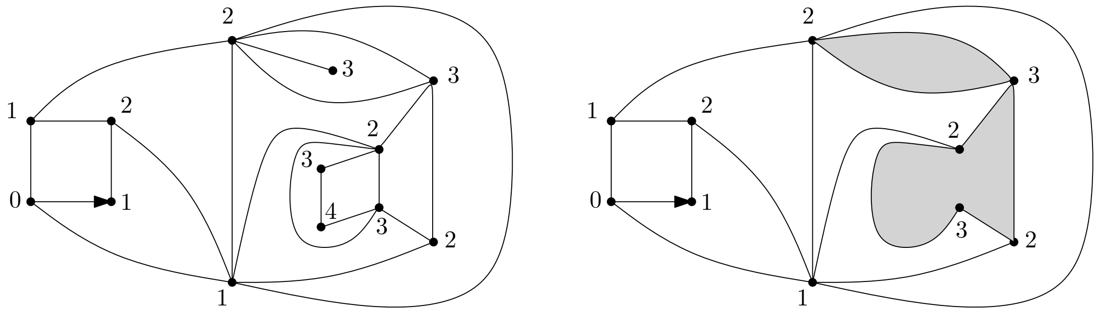

FIG. 1.6 – Une quadrangulation q et sa boule $B_{\mathbf{Q},2}(q)$ de rayon 2. Les distances à la racine des sommets sont indiquées et les faces extérieures sont grisées.

La topologie locale sur $\mathbf{Q}_f$ est alors donnée par la distance $d_{\mathbf{Q}}$, définie par

$$d_{\mathbf{Q}}(q, q') = \left(1 + \sup\{r : B_{\mathbf{Q},r}(q) = B_{\mathbf{Q},r}(q')\}\right)^{-1}$$

pour tout couple de quadrangulations $q, q' \in \mathbf{Q}_f$, avec la convention $\sup \emptyset = 0$. Appelons $(\mathbf{Q}, d_{\mathbf{Q}})$ la complétion de l'espace métrique $(\mathbf{Q}_f, d_{\mathbf{Q}})$. Ce que nous appelons quadrangulations infinies sont alors les éléments de $\mathbf{Q}_\infty = \mathbf{Q} \setminus \mathbf{Q}_f$.

Avec cette métrique, toutes les quadrangulations finies sont des points isolés tandis que les quadrangulations infinies sont les points d'accumulation des quadrangulations finies. Sans rentrer dans les détails, il est assez intuitif de voir qu'avec cette définition, les quadrangulations infinies peuvent être interprétées comme les classes d'équivalence des plongements de graphes localement finis dans la sphère, à homéomorphisme conservant l'orientation près. Nous renvoyons le lecteur aux articles [AS03] et [BS01] pour une discussion plus précise sur ce point de vue qui comporte quelques pièges.

Bien que nous ayons détaillé uniquement le cas des quadrangulations, définir cette métrique sur d'autres classes de cartes comme les p-angulations ne pose pas de problème particulier. Les méthodes de travail sont alors similaires en tout point.

Une métrique équivalente à $d_{\mathbf{Q}}$ peut être définie facilement en considérant les arêtes de la carte plutôt que les faces dans la définition des boules $B_{\mathbf{Q}}$. La boule de rayon r d'une carte q serait alors l'union de ses arêtes ayant un sommet à distance strictement plus petite que r de la racine. Cette seconde métrique est cependant plus utile pour étudier des cartes pour lesquelles les faces n'ont pas de rôle particulier ou pour lesquelles le degré des sommets est important. C'est par exemple une métrique beaucoup plus naturelle pour étudier les cartes planaires sans contraintes qui peuvent avoir des faces de degré arbitraire, et donc contenant des sommets arbitrairement loin de la racine.

1.3.2 Quadrangulation infinie uniforme

Pour tout entier n notons ν_n la mesure de probabilité uniforme sur l'ensemble $\mathbf{Q}_n$ des quadrangulations à n faces. On peut considérer ces mesures comme des mesures sur $\mathbf{Q}$ en donnant un poids nul au complémentaire de $\mathbf{Q}_n$. Dans [Kri06], Krikun prouve que cette suite de mesures converge faiblement vers une mesure de probabilité ν pour la topologie locale. Une quadrangulation de loi ν est appelée quadrangulation infinie uniforme.

Pour prouver ce résultat, une des approches possibles est de trouver une mesure de probabilité λ telle que pour toute quadrangulation $q^\star \in \mathbf{Q}$ et tout $r > 0$ on ait la convergence

$$\nu_n \left(q \in \mathbf{Q} : B_{\mathbf{Q},r}(q) = B_{\mathbf{Q},r}(q^\star)\right) \underset{n\to\infty}{\longrightarrow} \lambda \left(q \in \mathbf{Q} : B_{\mathbf{Q},r}(q) = B_{\mathbf{Q},r}(q^\star)\right). \tag{1.2}$$

On sait alors que la suite des mesures ν_n converge faiblement vers λ.

Une des difficultés pour prouver la convergence (1.2) vient du fait que $q^\star \setminus B_{\mathbf{Q},r}(q^\star)$ a plusieurs composantes connexes. Dans $B_{\mathbf{Q},r}(q^\star)$, il y a donc plusieurs morceaux à ajouter pour obtenir une quadrangulation q. Cependant, avec une probabilité proche de 1, si q est une quadrangulation à n faces avec n très grand, une seule des composantes connexes de $q \setminus B_{\mathbf{Q},r}(q)$ devient infinie. Ceci incite à considérer les enveloppes plutôt que les boules dans la convergence (1.2). C'est la démarche adoptée par Krikun [Kri06], qui définit l'enveloppe d'une quadrangulation de la manière suivante : si $q \in \mathbf{Q}$ et $r > 0$, l'enveloppe $\widehat{B}_{\mathbf{Q},r}(q)$ est le complémentaire dans q de la plus grande composante connexe de $q \setminus B_{\mathbf{Q},r}(q)$ (si il se trouve qu'il existe plusieurs composantes connexes de taille maximale, on en choisit une avec une règle déterministe). La Figure 1.7 donne un exemple d'enveloppe.

Il est alors assez simple de voir que pour toute quadrangulation q et tout $r > 0$, l'enveloppe $\widehat{B}_{\mathbf{Q},r}(q)$ est une quadrangulation à bord simple dans le sens où une seule de ses faces n'est pas nécessairement de degré 4, et que cette face est bordée par un cycle d'arêtes sans pincements γ_r, joignant entre eux des sommets à distance r et $r+1$ de la racine. Le lecteur peut se reporter à la Figure 1.7 pour une illustration de ceci.

Notons $C(n, m)$ le nombre de quadrangulations à n faces internes et de bord simple de longueur $2m$ pour lesquelles un sommet sur deux du bord est de degré 2. Avec des techniques analogues à celles que nous avons évoqué au paragraphe 1.2.2, Krikun [Kri06] calcule $C(n, m)$ et son équivalent quand n tend vers l'infini :

$$C(n, m) \sim \beta(m) n^{-5/2} 12^n$$

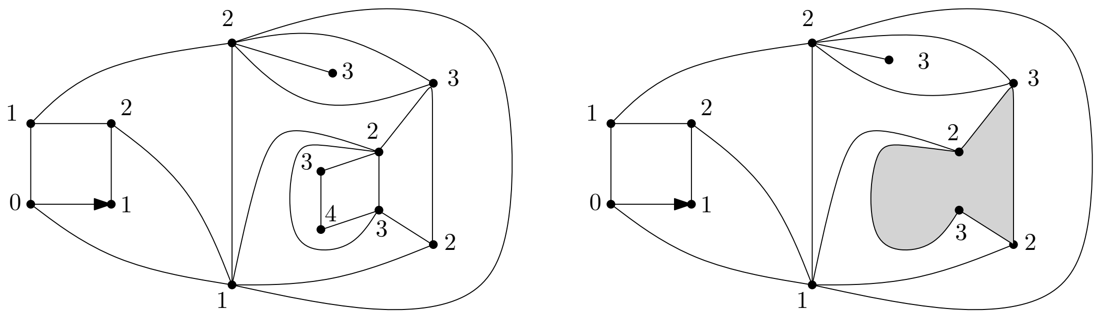

FIG. 1.7 – Exemple d'enveloppe de rayon 2.

où $\beta(m)$ est une constante connue dépendant uniquement de m. Il est alors possible de montrer que si $q^\star$ et $r > 0$ sont fixées, on a

$$\nu_n\Big(q \in \mathbf{Q} : \widehat{B}_{\mathbf{Q},r}(q) = \widehat{B}_{\mathbf{Q},r}(q^\star)\Big) = \frac{C(n-n'+m,m)}{C(n,1)}$$

si l'enveloppe $\widehat{B}_{\mathbf{Q},r}(q^\star)$ est une quadrangulation à bord simple avec $n-n'+m$ faces et un bord de longueur $2m$. On a donc la convergence suivante :

$$\nu_n\Big(q \in \mathbf{Q} : \widehat{B}_{\mathbf{Q},r}(q) = \widehat{B}_{\mathbf{Q},r}(q^\star)\Big) \underset{n\to\infty}{\longrightarrow} \frac{\beta(m)}{\beta(1)} 12^{n'-m+1}. \tag{1.3}$$

Pour finir de prouver (1.2), Krikun prouve alors que la partie droite de (1.3) définit une mesure de probabilité ν sur $\mathbf{Q}$ qu'il appelle loi de la quadrangulation infinie uniforme.

Une des propriétés prouvées par Krikun dans [Kri06] qui retiendra notre attention un peu plus tard concerne la longeur des frontières des enveloppes. Plus précisément, il montre que si q est une quadrangulation de loi ν, alors pour tout $r > 0$, $q \setminus B_{\mathbf{Q},r}(q)$ a ν-presque sûrement une unique composante connexe infinie, et que la variable aléatoire $|\gamma_r|/r^2$ converge en loi vers une loi gamma de paramètre $3/2$. La figure 1.8 montre une vue de profil d'une quadrangulation et permet de mieux comprendre comment est placé le cycle γ_r au sein de celle-ci.

Comme nous l'avons dit, le cas des triangulations a été évoqué avant les travaux de Krikun par Angel et Schramm [AS03], qui introduisent une triangulation infinie uniforme du plan. La démonstration est similaire dans sa partie combinatoire, mais à la place de prouver que l'analogue de la mesure limite (1.3) est bien une mesure de probabilité, Angel et Schramm démontrent que la suite des mesures uniformes sur les triangulations de taille n est tendue en étudiant les degrés des sommets, ce qui permet aussi de conclure à la convergence de cette suite pour la topologie faible.

Dans un deuxième article, Angel [Ang03] étudie certaines propriétés de cette loi. Il démontre en particulier grâce à une technique d'échantillonnage de la triangulation infinie uniforme que les boules de rayon r autour de la racine ont un volume qui croît presque surement comme r^4 à des termes logarithmiques près. Angel démontre de plus dans cetarticler qu'un analogue au cycle γ_r existe et qu'il est aussi de taille r^2. Finalement, Angel étudie la percolation par sites sur une triangulation infinie uniforme et montre que

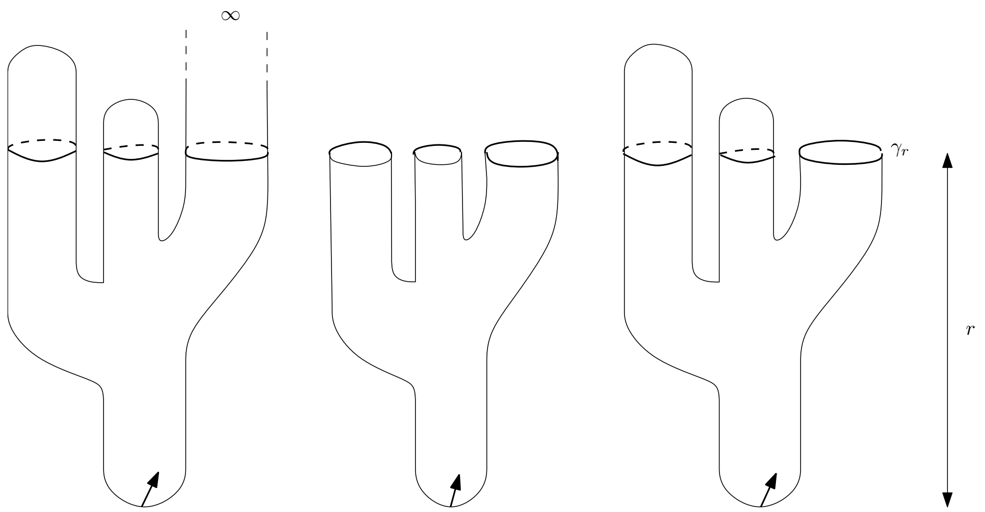

FIG. 1.8 – Une quadrangulation q vue de profil ainsi sa boule $B_{\mathbf{Q},r}(q)$ et son enveloppe $\widehat{B}_{\mathbf{Q},r}(q)$.

la probabilité critique est presque surement 1/2. Angel étudie d'autres propriétés de la percolation par sites sur la triangulation infinie uniforme dans la prépublication [Ang05].

Avant de s'intéresser d'avantage aux quadrangulations, il semble utile de mentionner l'article de Benjamini et Schramm [BS01], qui démontre que les limites locales de graphes sont récurrentes si les graphes ont des sommets de degré uniformément bornés. Cette hypothèse fait bien sûr défaut dans le cas des limites de triangulations ou de quadrangulations, mais [AS03] donne des estimations des degrés des sommets pour les triangulations.

1.3.3 Arbre bien étiqueté infini uniforme

Afin d'étudier les propriétés métriques de la quadrangulation infinie uniforme, il est intéressant de chercher à étendre la bijection de Schaeffer aux arbres infinis et aux quadrangulations infinies. Pour cela, nous devons d'abord préciser ce que nous entendons par arbres infinis. L'approche est la même que pour les quadrangulations ; nous définissons une métrique locale sur les arbres spatiaux et nous considérons l'espace métrique complété.

Pour tout $n \geqslant 0$, notons $\mathbb{T}_n$ l'ensemble des arbres plans bien étiquetés à n arêtes et $\mathbb{T}_f$ l'ensemble des arbres bien étiquetés finis. La distance que nous allons considérer est alors définie par

$$d_{\mathbb{T}}(\theta, \theta') = \left(1 + \sup \{h : B_{\mathbb{T},h}(\theta) = B_{\mathbb{T},h}(\theta')\}\right)^{-1}$$

pour tout couple d'arbres $\theta, \theta' \in \mathbb{T}_f$, et où pour tout entier h, $B_{\mathbb{T},h}(\theta)$ est le sous arbre de θ constitué de ses sommets de génération plus petite de h avec les mêmes étiquettes. Notons alors $(\mathbb{T}, d_{\mathbb{T}})$ l'espace complété de $(\mathbb{T}_f, d_{\mathbb{T}})$. Comme pour les quadrangulations, il est légitime d'appeler les éléments de $\mathbb{T}_\infty = \mathbb{T} \setminus \mathbb{T}_f$ les arbres bien étiquetés infinis.

Ce cadre est encore plus simple que celui des quadrangulations car il y a une description très simple de $\mathbb{T}$. En effet, pour retouver les éléments de $\mathbb{T}$, il suffit d'enlever l'hypothèse qu'un arbre est un sous ensemble fini de $\mathcal{U}$ dans la définition des arbres donnée au paragraphe 1.2.1 pour prendre tous les sous ensembles de $\mathcal{U}$ satisfaisant les conditions de la définition. On peut par exemple voir de cette manière que tous les arbres spatiaux, qu'ils soient finis ou infinis, ont un nombre fini de sommets à chaque génération.

Un arbre infini peut avoir des sous arbres linéaires infinis qui partent de la racine, nous appellerons ces sous arbres des troncs. La notation suivante nous sera utile : pour tout $l \geqslant 1$, $N_l(\theta)$ désigne le nombre de sommets de θ qui ont l pour étiquette. On notera aussi

$$\mathscr{S} = \{\theta \in \mathbb{T}_\infty : \forall l \geqslant 1,\ N_l(\theta) < \infty \text{ et } \theta \text{ a un unique tronc}\} \cup \mathbb{T}_f$$

l'ensemble des arbres ayant au plus un tronc et pour lesquels la fonction d'étiquetage ℓ prend chaque entier un nombre fini de fois comme valeur.

Chassaing et Durhuus [CD06] ont prouvé qu'il existe un arbre bien étiqueté infini uniforme au sens de cette topologie. Plus précisément, si pour tout n μ_n désigne la probabilité uniforme sur l'ensemble $\mathbb{T}_n$, alors il existe une mesure de probabilité sur $\mathbb{T}$, notée μ, telle que μ_n converge faiblement vers μ pour la topologie associée à $d_{\mathbb{T}}$. Un arbre bien étiqueté de loi μ est appelé arbre bien étiqueté infini uniforme.

Dans le même article, Chassaing et Durhuus démontrent plusieurs propriétés de cette mesure et donnent une description explicite d'un arbre bien étiqueté infini uniforme. Par exemple, ils démontrent que $\mu(\mathscr{S}) = 1$, ce qui sera crucial par la suite. Ils démontrent ensuite qu'un arbre θ de loi μ peut se décrire de la manière suivante :

1. θ a un unique tronc et, lorsqu'on les ordonne en fonction de leur hauteur, les étiquettes des sommets du tronc forment une chaîne de Markov inhomogène $(X_n)_{n \geqslant 0}$ dont les probabilités de transition sont connues ;
2. conditionellement à $\{X_n = x_n \, \forall n \geqslant 0\}$, les arbres $(L_n)_{n \geqslant 0}$ et $(R_n)_{n \geqslant 0}$ des descendants du sommet du tronc de hauteur n situés respectivement à gauche et à droite du tronc sont tous indépendants entre eux et sont des arbres de Galton-Watson de loi de reproduction géométrique de paramètre $1/2$, dont les étiquettes sont choisies uniformément parmi les cas possibles sans contrainte de positivité (avec x_n pour la racine), puis sont conditionnées à rester strictement positives.

Un tel arbre est représenté schématiquement par la Figure 1.9.

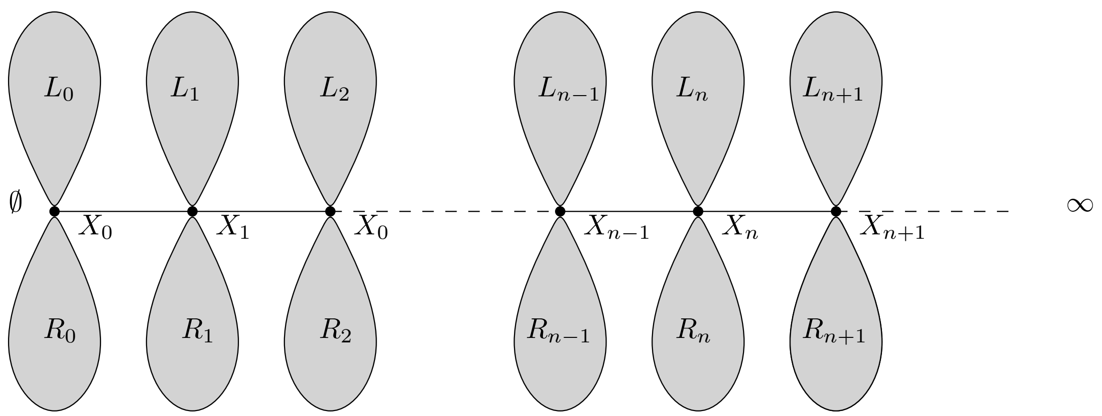

FIG. 1.9 – Exemple d'arbre de loi μ.

1.3.4 Une autre quadrangulation infinie uniforme

Il n'existe pas de généralisation de la bijection de Schaeffer entre $\mathbb{T}$ et $\mathbf{Q}$; cependant il existe bien une extension de la bijection de Schaeffer à l'ensemble $\mathscr{S}$. Cette extension, que nous noterons Φ, est introduite dans le même article de Chassaing et Durhuus et est similaire au cas des arbres finis. Prenons donc un arbre $\theta = (\tau, \ell) \in \mathscr{S}$ et décrivons comment lui associer une quadrangulation. Si l'arbre est fini, on lui associe bien entendu une quadrangulation par la bijection de Schaeffer. Si l'arbre est infini, il a un unique tronc. Nous pouvons plonger τ dans $\mathbb{S}^2$ de manière à ce que toutes les suites de points (p_n) de sommets de τ deux à deux distincts aient pour unique point d'accumulation un point fixé $\triangle \in \mathbb{S}^2 \setminus \tau$.

Pour construire $\Phi(\tau)$, commençons par ajouter un sommet ∂ dans le complémentaire de $\tau \cup \{\triangle\}$. Pour chaque chaque coin c de chaque sommet v de τ, on construit une arête avec les règles suivantes :

1. Si $\ell(v) = 1$, on construit une arête entre c and ∂ (voir Figure 1.10, cas de gauche).
2. Si c est du côté droit du tronc, que $\ell(v) \geqslant 2$, et si de plus il existe un coin d'étiquette $\ell(v) - 1$ visité après c lors du parcours du côté droit du tronc, on ajoute une arête entre c et le premier tel coin (voir Figure 1.10, cas de gauche).
3. Si c est du côté droit du tronc, que $\ell(v) \geqslant 2$, et si de plus il n'existe pas de coin d'étiquette $\ell(v) - 1$ après c lors du parcours du côté droit du tronc, on ajoute une arête entre c et le coin d'étiquette $\ell(v) - 1$ visité en dernier lors d'un parcours du côté gauche du tronc (voir Figure 1.10, cas du milieu).
4. Enfin, si c est du côté gauche du tronc et que $\ell(v) \geqslant 2$, on ajoute une arête entre c et le coin d'étiquette $\ell(v) - 1$ qui est visité en dernier avant c pendant le parcours du côté gauche du tronc (voir Figure 1.10, cas de droite).

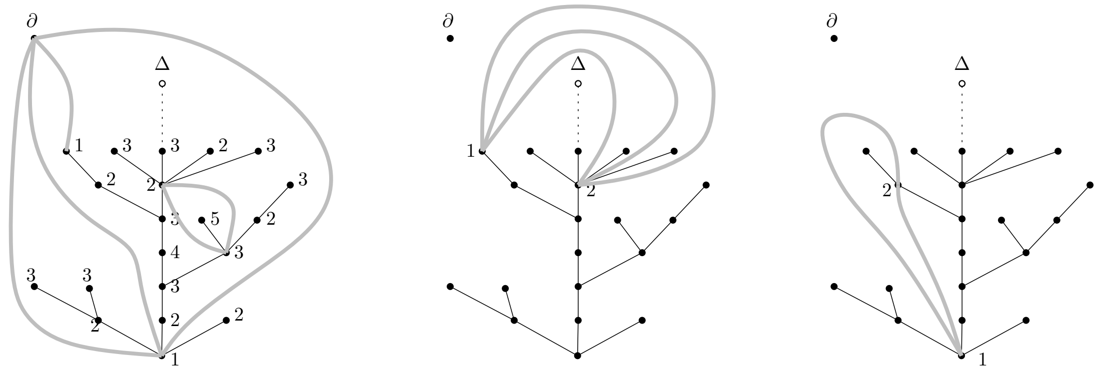

Fig. 1.10 – Extension de la bijection de Schaeffer à des arbres infinis.

Chassaing et Durhuus prouvent dans [CD06] que cette construction donne bien une quadrangulation, dont la racine est l'arête orientée entre ∂ et le premier coin de $\emptyset$. Ils prouvent de plus que cette correspondance a la même propriété que la bijection de Schaeffer, à savoir que les étiquettes d'un arbre θ sont les distances à la racine dans la quadrangulation $\Phi(\theta)$.

Grâce à cette extension de la bijection de Schaeffer, Chassaing et Durhuus transportent la loi de l'arbre infini uniforme sur l'ensemble des quadrangulations. Pour cela, ils équipent l'ensemble $\Phi(\mathscr{S})$ de la distance d_Φ qui rend l'application Φ isométrique de $\mathscr{S}$ sur $\Phi(\mathscr{S})$. Les mesures images des lois uniformes μ_n sur les ensembles $\mathbb{T}_n$ sont alors les lois uniformes ν_n sur les ensembles $\mathcal{Q}_n$ et elles convergent vers la mesure image de μ par Φ pour la topologie faible sur l'espace des mesures de probabilité sur $(\Phi(\mathscr{S}), d_\Phi)$.

La mesure μ_Φ, image de μ par Φ est donc en ce sens une mesure uniforme sur les quadrangulations infinies, et une quadrangulation aléatoire infini de loi μ_Φ a l'avantage d'être codée par un arbre infini de loi uniforme grâce à l'extension de la bijection de Schaeffer Φ. Ceci permet par exemple à Chassaing et Durhuus de démontrer que le volume des boules de rayon r centrées en la racine d'une quadrangulation infinie aléatoire de loi μ_Φ grandit en espérance comme r^4 grâce à un travail sur les étiquettes de l'arbre infini uniforme.

1.3.5 Les deux quadrangulations infinies uniformes ont la même loi

Il est naturel de penser que les deux notions de quadrangulation infinie uniforme que nous avons présentées sont en fait le même objet. Cette question fait l'objet du Chapitre 2 de ce manuscrit qui prouve que c'est effectivement le cas. Nous décrivons ici les grandes lignes de ce chapitre.

Nous avons pour l'instant défini deux espaces métriques de quadrangulations infinies. L'espace $(\mathbf{Q}, d_{\mathbf{Q}})$ des limites locales de quadrangulations, et l'espace $(\Phi(\mathscr{S}), d_\Phi)$ des images de limites locales d'arbres par la bijection de Schaeffer. Pour que la question de l'égalité des mesures ν et μ_Φ ait un sens, il faut déjà que ces deux mesures soient définies sur le même espace mesurable. Il est facile d'identifier $\Phi(\mathscr{S})$ à un sous ensemble de $\mathbf{Q}$. L'espace mesurable que nous considérons est donc $\mathbf{Q}$ muni de la tribu borélienne associée à $d_{\mathbf{Q}}$. Le mesure μ_Φ est alors une mesure sur cette espace si l'application Φ est mesurable de $\mathbb{T}$ dans $\mathbf{Q}$ pour la tribu borélienne associée à $d_{\mathbf{Q}}$: c'est le Lemme 1 du chapitre 2.

Comme la mesure $\mu_n \circ \Phi^{-1}$, image par Φ de la loi uniforme sur les arbres bien étiquetés à n arêtes, est la loi uniforme sur les quadrangulations à n faces ν_n, pour prouver l'égalité entre μ_Φ et ν, il suffit de prouver que la suite des mesures $\mu_n \circ \Phi^{-1}$ converge vers μ_Φ pour la topologie faible sur l'espace des mesures de probabilité sur $(\mathbf{Q}, d_{\mathbf{Q}})$.

La principale difficulté vient alors du fait que Φ n'est pas continue pour $d_{\mathbf{Q}}$. En effet, si θ_n est une suite d'arbres bien étiquetés finis qui converge vers un arbre infini θ_∞ pour la distance sur les arbres $d_{\mathbb{T}}$, alors la suite des quadrangulations $q_n = \Phi(\theta_n)$ ne converge pas nécessairement vers la quadrangulation $q_\infty = \Phi(\theta_\infty)$. Il se peut en effet que les arbres θ_n aient des sommets d'étiquette plus petite que $l > 0$ fixé dans des génération plus grandes que $k(n)$ avec $k(n) \to \infty$ quand $n \to \infty$.

Cependant, lorsque θ_n est de loi uniforme sur les arbres bien étiquetés à n arêtes, le problème précédent n'arrive qu'avec une probabilité uniformément faible. C'est la Proposition 6 du chapitre 2, démontrée en étudiant le comportement asymptotique des étiquettes des arbres θ_n dans des générations élevées. La convergence voulue découle alors d'un lemme combinatoire sur la bijection de Schaeffer : si deux arbres étiquetés sont identiques jusqu'à la hauteur H, alors les quadrangulations associées ont des boules de rayon r autour de la racine identiques pour r valant l'étiquette minimale des sommets au dessus de la hauteur H des deux arbres (Proposition 4 du chapitre 2).

Le fait de savoir que $\mu_\Phi = \nu$ est très intéressant car il permet d'avoir les avantages des deux points de vue. En effet, le point de vue des arbres est pratique pour étudier les propriétés métriques de la quadrangulation infinie uniforme, tandis que le point de vue plus direct des limites locales de quadrangulations donne une topologie bien plus naturelle. Nous pouvons par exemple déduire des résultats de Chassaing et Durhuus [CD06] que si q est une quadrangulation infinie de loi ν, alors le volume des boules de rayon r autour de la racine de q croît comme r^4 quand $r \to \infty$.

1.4 Limites d'échelle

1.4.1 Limite d'échelle d'arbres uniformes

Considérons un arbre aléatoire τ_n de loi uniforme sur l'ensemble des arbres plans enracinés à n arêtes. Sa fonction de contour C_n est alors une marche aléatoire de $2n$ pas conditionnée à rester positive et à terminer en 0. Notons $\mathbf{e}$ l'excursion brownienne normalisée. Un cas particulier des résultats d'Aldous [Ald93] montre que

$$\left(\frac{1}{\sqrt{2n}} C_n(2nt)\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{} (\mathbf{e}_t)_{t\in[0,1]}\,, \tag{1.4}$$

la convergence ayant lieu au sens de la convergence en distribution sur l'espace des fonctions continues sur $[0, 1]$ à valeurs réelles (en fait, ce cas particulier est une forme conditionnelle du théorème de Donsker).

Ce résultat ne fait pas intervenir directement des arbres ; il trouve cependant tout son sens si on essaye de considérer l'excursion $\mathbf{e}$ comme la fonction de contour d'un arbre aléatoire continu. C'est le point de vue des arbres réels, et l'arbre aléatoire dont la fonction de contour est $\mathbf{e}$ est l'arbre continu brownien d'Aldous [Ald93]. Il est alors possible de démontrer que dans un certain sens, l'arbre τ_n converge en distribution vers l'arbre d'Aldous. Pour une introduction complète sur ce sujet, nous invitons la lecteur à consulter l'article de survol [LG05].

Dans le cas des arbres spatiaux bien étiquetés, il est possible de coupler la convergence des processus de contour avec une convergence des processus de contour spatial. Avant d'énoncer un résultat pour les arbres bien étiquetés, nous allons nous placer dans le cadre un peu plus simple des arbres étiquetés, sans condition de positivité sur les étiquettes.

Pour tout entier n, soit donc $\theta_n = (\tau_n, \ell_n)$ un arbre étiqueté, de loi uniforme sur l'ensemble des arbres étiquetés à n arêtes et pour lesquels $\ell(\emptyset) = 0$. Conditionnellement à τ_n, les étiquettes de θ_n forme une marche aléatoire branchante de pas uniformes dans $\{-1, 0, 1\}$. Si on considère un sommet typique de θ_n, on s'attend donc à ce que sa hauteur soit de l'ordre de $\sqrt{n}$, et que son étiquette soit de l'ordre de $\sqrt{\sqrt{n}}$.

Notons (C_n, V_n) les fonctions de contour de θ_n. Un cas particulier des résultats de Janson et Marckert [JM05] permet d'étendre la convergence (1.4) de la manière suivante :

$$\left(\frac{1}{\sqrt{2n}} C_n(2nt), \left(\frac{9}{8n}\right)^{1/4} V_n(2nt)\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{} (\mathbf{e}_t, Z_t)_{t\in[0,1]}\,, \tag{1.5}$$

au sens de la convergence en distribution sur l'espace des couples de fonctions continues sur $[0,1]$ à valeurs réelles, et où, conditionnellement à $\mathbf{e}$, Z est un processus gaussien centré de covariance

$$\mathrm{Cov}\,(Z_s, Z_t) = \inf_{s\wedge t\leqslant u\leqslant s\vee t} \mathbf{e}_u.$$

La convergence (1.5) est déjà obtenue dans [CS04].

Le processus $(\mathbf{e}, Z)$, qui n'est pas un processus de Markov mais possède des trajectoires höldériennes d'exposant dans $(0, 1/4)$, est appelé la tête du serpent Brownien. Cette appelation vient du fait que ce processus est une fonctionnelle très simple du serpent brownien de Le Gall, qui est un processus de Markov à valeurs trajectoires. Nous donnons une présentation succincte du serpent brownien dans le prochain paragraphe et renvoyons le lecteur à la monographie [LG99] pour une référence plus complète.

La convergence (1.5) est aussi vraie pour les arbres bien étiquetés avec des processus conditionnés. On notera donc $\left(\overline{\mathbf{e}}_t, \overline{Z}_t\right)_{t\in[0,1]}$ le processus $(\mathbf{e}_t, Z_t)_{t\in[0,1]}$ conditioné par l'événement $\{Z_t \geqslant 0,\ \forall t \in [0,1]\}$. Il faut cependant prendre certaines précautions pour effectuer ce conditionnement car l'événement $\{Z_t \geqslant 0,\ \forall t \in [0,1]\}$ est de probabilité nulle. Ce processus, naturellement appellé tête du serpent brownien conditionné à rester positif a été introduit par Le Gall et Weill [LGW06]. Il est aussi démontré dans cet article que si θ_n est de loi uniforme sur $\mathbb{T}_n$ et que (C_n, V_n) est sa paire de fonctions de contours, alors on a la convergence

$$\left(\frac{1}{\sqrt{2n}} C_n(2nt), \left(\frac{9}{8n}\right)^{1/4} V_n(2nt)\right)_{t\in[0,1]} \underset{n\to\infty}{\longrightarrow} \left(\overline{\mathbf{e}}_t, \overline{Z}_t\right)_{t\in[0,1]} \tag{1.6}$$

en distribution sur l'espace des couples de fonctions continues sur $[0,1]$ à valeurs réelles.

1.4.2 Serpent brownien

Nous venons de voir que les limites d'échelle d'arbres étiquetés uniformes se décrivent à l'aide d'un processus stochastique appelé serpent brownien. Ce processus est un processus de Markov à valeurs dans l'espace $\mathcal{W}$ des trajectoires arrêtées sur $\mathbb{R}$. Ainsi, un élément de $\mathcal{W}$ est une application continue $\mathrm{w} : [0, \zeta] \to \mathbb{R}$, où $\zeta = \zeta_{(\mathrm{w})} \geqslant 0$ est appelé la durée de vie de w. La tête de w est le réel $\hat{\mathrm{w}} = \mathrm{w}(\zeta)$, et le support de w est noté $\mathrm{w}[0, \zeta_{(\mathrm{w})}]$. Pour tout $x \in \mathbb{R}$, on note $\mathcal{W}_x$ l'ensemble des trajectoires qui partent de x. Le chemin de $\mathcal{W}_x$ de durée de vie $\zeta_{(\mathrm{w})} = 0$ est identifié avec le point x. L'espace $\mathcal{W}$ est polonais pour la distance

$$d(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\geqslant 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

On munit l'espace canonique $\Omega = C(\mathbb{R}_+, \mathcal{W})$ des fonctions continues sur $\mathbb{R}_+$ de la topologie de la convergence uniforme sur tout compact. Le processus canonique sur Ω est noté $W_s(\omega) = \omega(s)$ pour tout $\omega \in \Omega$ et on note $\zeta_s = \zeta_{(W_s)}$ la durée de vie de W_s.

Soit $\mathrm{w} \in \mathcal{W}$. La loi du serpent brownien partant de w est la mesure de probabilité $\mathbb{P}_{\mathrm{w}}$ sur Ω définie de la manière suivante. Premièrement, le processus $(\zeta_s)_{s\geqslant 0}$ sous $\mathbb{P}_{\mathrm{w}}$ est un mouvement brownien réfléchit sur $[0, \infty[$ et partant de $\zeta_{(\mathrm{w})}$. Deuxièmement, conditionnellement à $(\zeta_s)_{s\geqslant 0}$, la loi de $(W_s)_{s\geqslant 0}$ que nous notons $\Theta^{\zeta}_{\mathrm{w}}$ est définie avec les conditions suivantes :

1. $W_0 = \mathrm{w}$, $\Theta^\zeta_{\mathrm{w}}$-presque sûrement
2. Sous $\Theta^\zeta_{\mathrm{w}}$, $(W_s)_{s\geqslant 0}$ est un processus de Markov inhomogène en temps. De plus, si $0 \leqslant s \leqslant s'$,
   - $W_{s'}(t) = W_s(t)$ pour tout $t \leqslant m(s,s') = \inf_{[s,s']} \zeta_r$, $\Theta^\zeta_{\mathrm{w}}$-presque sûrement ;
   - le processus $(W_{s'}(m(s',s)+t) - W_{s'}(m(s,s')))_{0\leqslant t\leqslant \zeta_{s'}-m(s,s')}$ est un mouvement brownien partant de 0 et indépendant de W_s.

Heuristiquement, le serpent brownien au temps s est une trajectoire aléatoire dont la durée de vie ζ_s évolue comme un mouvement brownien réfléchi sur $[0,\infty[$. Quand ζ_s décroît, la trajectoire est éffacée par son bout, et quand ζ_s croît, la trajectoire est étendue en ajoutant des morceaux de trajectoires browniennes indépendantes à son bout.

Notons $n(\mathrm{d}e)$ la mesure d'Itô sur les excursions positives (le lecteur peut se reporter au Chapitre 12 du livre [RY99] pour une introduction à la théorie des excursions). C'est une mesure σ-finie sur l'espace $C(\mathbb{R}_+,\mathbb{R}_+)$ des fonctions continues de $\mathbb{R}_+$ dans $\mathbb{R}_+$. Notons

$$\sigma(e) = \inf\{s > 0 : e(s) = 0\}$$

la durée de vie d'une excursion e. Pour tout $s > 0$, on note $n_{(s)}$ la mesure conditionnée $n(\cdot \mid \sigma = s)$. La normalisation de la mesure d'Itô est fixée par l'égalité

$$n = \int_0^\infty \frac{\mathrm{d}s}{2\sqrt{2\pi s^3}} n_{(s)}.$$

Pour tout $x \in \mathbb{R}$, la mesure d'excursion $\mathbb{N}_x$ du serpent brownien partant de x est définie par

$$\mathbb{N}_x = \int_{C(\mathbb{R}_+,\mathbb{R}_+)} n(\mathrm{d}e)\Theta^e_x. \tag{1.7}$$

Avec un léger abus de notation, on note aussi $\sigma(\omega) = \inf\{s > 0 : \zeta_s(\omega) = 0\}$ pour tout $\omega \in \Omega$. Nous pouvons alors considérer les mesures conditionnées

$$\mathbb{N}^{(s)}_x = \mathbb{N}_x(\cdot \mid \sigma = s) = \int_{C(\mathbb{R}_+,\mathbb{R}_+)} n_{(s)}(\mathrm{d}e)\Theta^e_x.$$

Le processus limite (e,Z) qui intervient dans (1.5) est alors le processus $(e,\widehat{W})$ sous la mesure $\mathbb{N}^{(1)}_0$, ce qui justifie l'appellation tête du serpent brownien.

1.4.3 Carte brownienne

La convergence (1.5) a permis à Chassaing et Schaeffer [CS04] de démontrer les premiers résultats sur les limites d'échelle de cartes qui concernent le rayon et le profil des grandes quadrangulations aléatoires. Si q est une carte, nous noterons l'ensemble de ces sommets $V(q)$. Soit q_n de loi uniforme sur $\mathbf{Q}_n$. Le rayon de q_n est la distance maximale entre un sommet de q_n et son sommet racine ∂ :

$$R(q_n) = \sup_{v\in V(q_n)} d_{gr}(\partial, v).$$

Considérons l'arbre aléatoire $\theta_n = (\tau_n, \ell_n)$ associé à q_n par la bijection de Schaeffer, on a alors

$$R(q_n) = \sup_{v\in\theta_n} \ell_n(v).$$

Chassaing et Schaeffer ont démontré grâce à (1.5) la convergence en loi suivante :

$$\left(\frac{9}{8n}\right)^{1/4} R(q_n) \underset{n\to\infty}{\longrightarrow} \sup_{t\in[0,1]} \overline{Z}_t. \tag{1.8}$$

De la même façon, l'article de Chassaing et Schaeffer étudie la convergence du profil de q_n, qui est la mesure sur $\mathbb{R}_+$ définie par

$$\mathcal{I}_{q_n}(k) = \sharp \{v \in V(q_n) : d(\partial, v) = k\}$$

pour $k \in \mathbb{R}_+$. Le profil changé d'échelle défini par

$$\mathcal{I}^{(n)}_{q_n}(A) = \frac{1}{n+1}\mathcal{I}_{q_n}\left(\left(\frac{9n}{8}\right)^{1/4} A\right)$$

pour tout borélien A de $\mathbb{R}_+$ converge alors en loi pour la topologie faible vers la mesure aléatoire définie par

$$\langle \mathcal{I}, g\rangle = \int_0^1 \mathrm{d}t\, g\left(\overline{Z}_t\right) \tag{1.9}$$

pour toute fonction mesurable positive g. Cette mesure coïncide à une translation près avec la mesure appelée ISE pour *Integrated Super-Brownian Excursion*, qui intervient dans différents domaines de la mécanique statistique [DS98, HS00, Sla02].

Un des points délicats dans les preuves de Chassaing et Schaeffer concerne la positivité des étiquettes des arbres θ_n. Ils réussissent cependant à utiliser la convergence (1.5) grâce à des arguments combinatoires. En prouvant la convergence des processus conditionnés (1.6), Le Gall [LG06] fournit une preuve plus directe de ces résultats.

Ces résultats suggèrent, comme dans la cas des limites d'arbres, l'existence d'un espace métrique aléatoire continu qui serait la limite des quadrangulations q_n avec leur distances de graphe renormalisées. Le cadre naturel pour étudier la convergence d'espaces métriques est la distance de Gromov-Hausdorff d_{GH}, qui est une distance sur l'ensemble $\mathbb{M}$ des classes d'isométries d'espaces métriques compacts rendant cet espace séparable et complet. Bien que cette notion mériterait une présentation détaillée, nous renvoyons le lecteur au livre [BBI01] pour ceci. Le problème fondamental est alors le suivant : existe-t-il un espace métrique aléatoire compact (m, D) tel qu'on ait

$$\left(q_n, n^{-1/4} d_{gr}\right) \underset{n\to\infty}{\longrightarrow} (m, D) \tag{1.10}$$

au sens de la convergence en loi dans $(\mathbb{M}, d_{GH})$. L'espace métrique limite (m, D) serait alors appelé carte brownienne par analogie avec le mouvement brownien et le théorème de Donsker. Ce problème, posé par Schramm [Sch07], n'a pas encore reçu de réponse complète. Il y a eu cependant de nombreuses avancées dans sa résolution ces dernières années.

Un des résultats les plus aboutis concernant la convergence (1.10) est dû à Le Gall [LG07], qui montre que la suite des lois des espaces métriques aléatoires $\left(q_n, n^{-1/4} d_{gr}\right)$ est tendue dans l'espace des mesures de probabilité sur $(\mathbb{M}, d_{GH})$. Da manière plus précise, le résultat de Le Gall [LG07] est démontré pour les suites de $2p$-angulations uniformes. Pour finir de démontrer la convergence, il reste à démontrer qu'il n'y a qu'une seule limite possible pour les sous-suites de $\left(q_n, n^{-1/4}d_{gr}\right)$. Dans ce sens, Le Gall [LG07] démontre que la topologie des espaces limites possibles est unique et est déterminée par un quotient de l'arbre continu d'Aldous par une relation d'équivalence faisant intervenir la tête du serpent brownien Z. Un espace quotient similaire avait déjà été introduit par Marckert et Mokkadem [MM03] dans le même but. Cette topologie a été identifiée par Le Gall et Paulin [LGP08] puis par Miermont [Mie08] comme celle de la sphère $\mathbb{S}^2$. Des travaux récents se concentrent maintenant sur l'étude de la métrique des espaces limites possibles qui reste à identifier. Dans ce sens Le Gall [LG09] démontre plusieurs propriétés remarquables des géodésiques dans les espaces limites. Le Gall prouve aussi dans l'article [LG07] que la dimension de Hausdorff des espaces limites est 4. Les articles de Bouttier et Guitter [BG08a, BG08b, BG09] sont aussi à citer pour les résultats qu'ils apportent sur les propriétés métriques des quadrangulations.

De même, plusieurs travaux se concentrent sur l'universalité de la carte brownienne, à savoir qu'elle est indépendante du type de cartes choisi pour la convergence, qui peuvent être des p-angulations quelconques ou encore des cartes avec une distribution de Boltzmann sur le degré des faces. On pourra consulter dans ce sens les travaux de Marckert & Miermont [MM07], Miermont & Weill [MW08] et Weill [Wei07].

1.4.4 Volume infini

Comme nous l'avons vu à la fin du paragraphe 1.3, si q est une quadrangulation infinie de loi uniforme, alors les boules de rayon r autour de la racine de q ont un volume qui croît comme r^4 quand $r \to \infty$. On peut alors se demander si le volume de ces boules converge vers une variable aléatoire dans l'échelle r^4. De même, on peut tenter de changer d'échelle le profil de la quadrangulation q pour que celui-ci converge vers une mesure aléatoire continue sur $\mathbb{R}_+$. De tels résultats sont intéressants car ils suggèrent l'existence d'un espace métrique aléatoire continu qui serait la limite des cartes renormalisées au sens de Gromov-Hausdorff, mais dans un cadre non compact. Dans le Chapitre 3 de ce manuscrit, nous étudions les limites d'échelles de certaines fonctionnelles de la quadrangulation infinie uniforme comme son rayon et son profil.

L'outil clé pour démontrer ce genre de convergence est l'étude asymptotique de l'arbre bien étiqueté infini uniforme. Nous savons en effet que cet arbre décrit la quadrangulation infinie uniforme, et en particulier la distance à la racine de ses sommets grâce à la bijection de Schaeffer étendue et aux résultats du Chapitre 2. Comme pour le cas des arbres finis, nous allons nous intéresser aux fonctions de contour d'un arbre sous la loi μ et décrire leurs limites d'échelle à l'aide du serpent brownien. Il faut cependant se rappeler que, sous μ, un arbre est infini avec un unique tronc. Il faut donc deux paires de fonctions de contour pour le décrire : une paire (C_L, V_L) pour le côté gauche du tronc et une paire (C_R, V_R) pour le côté droit comme illustré en Figure 1.11.

La description de Chassaing et Durhuus d'un arbre infini uniforme que nous avons donné au paragraphe 1.3.3 permet ensuite de démontrer la convergence de ces processus de contour changés d'échelle. Le premier résultat que nous donnons dans ce sens est la Proposition 5 du chapitre 2, qui énonce la convergence de la chaîne de Markov des

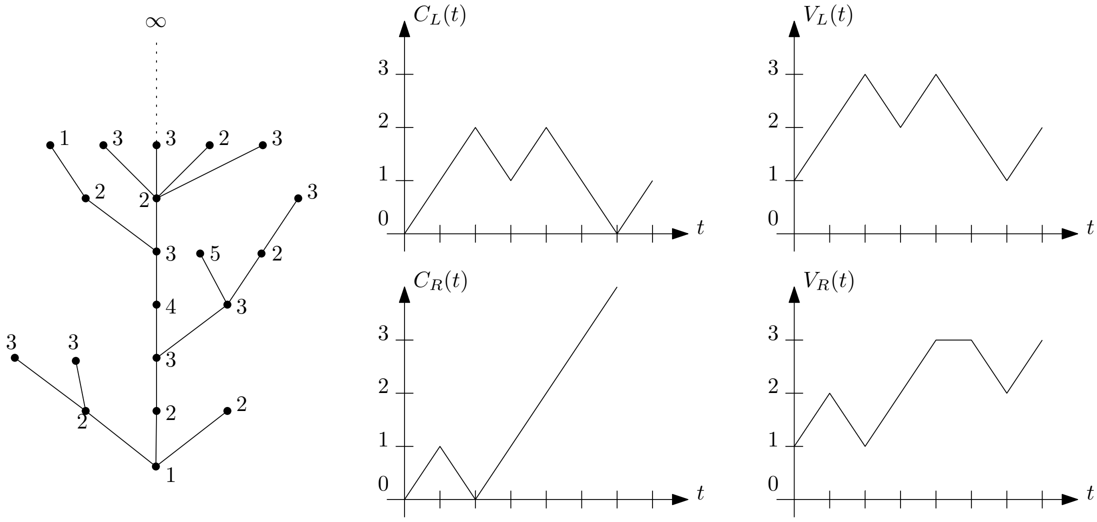

FIG. 1.11 – Un arbre de $\mathscr{S}$ et ses deux paires de fonction de contour.

étiquettes le long du tronc vers un processus de Bessel ρ de dimension 9 issu de 0 :

$$\left(\sqrt{\frac{3}{2n}}X_{\lfloor nt\rfloor}\right)_{t\geqslant 0} \underset{n\to\infty}{\longrightarrow} (\rho_t)_{t\geqslant 0}$$

au sens de la convergence en distribution dans l'espace de Skorohod $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

Il est de plus possible grâce à la description de Chassaing et Durhuus d'interpréter les fonctions de contour de l'arbre infini uniforme comme la concaténation de fonctions de contour d'arbres de Galton-Watson indépendants (avec un conditionnement sur les étiquettes), avec un biais pour tenir compte de la hauteur sur le tronc pour les processus généalogiques C_L et C_R ou un biais pour tenir compte de l'étiquette X_n pour V_L et V_R.

Le processus stochastique qui décrit la limite d'échelle des fonctions de contour de l'arbre infini uniforme a été introduit par Le Gall et Weill [LGW06] : c'est le serpent brownien éternel conditionné à rester positif. Conditionnellement au processus de Bessel ρ, soit

$$\mathcal{P} = \sum_{i\in I} \delta_{(h_i,\omega_i)}$$

un processus ponctuel de Poisson sur $\mathbb{R}_+ \times \Omega$ et d'intensité

$$2\mathbf{1}_{\{\mathcal{R}(\omega)\subset]-\rho_t,\infty[\}}\mathrm{d}t\mathbb{N}_0(\mathrm{d}\omega) \tag{1.11}$$

où $\mathcal{R}(\omega) = \left\{\widehat{W}_s, s \geqslant 0\right\}$. On construit alors le serpent éternel conditionné W^∞ comme une fonctionnelle mesurable $F(\rho, \mathcal{P})$ de la paire $(\rho, \mathcal{P})$. Pour simplifier les notations, notons

$$\sigma_i = \sigma(\omega_i), \quad \zeta_s^i = \zeta_s(\omega_i), \quad W_s^i = W_s(\omega_i)$$

pour tout $i \in I$ et tout $s \geqslant 0$. Pour tout $u \geqslant 0$, on définit

$$\tau_u = \sum_{i\in I} \mathbf{1}_{\{h_i\leqslant u\}}\sigma_i.$$

Alors, pour tout $s \geqslant 0$, il existe un unique u tel que $\tau_{u-} \leqslant s \leqslant \tau_u$, et de plus :

– Soit il existe un unique $i \in I$ tel que $u = h_i$. On définit alors

$$\zeta_s^\infty = u + \zeta^i_{s-\tau_{u-}},$$
$$W_s^\infty(t) = \begin{cases} \rho_t & \text{si } t \leqslant u, \\ \rho_u + W^i_{s-\tau_{u-}}(t-u) & \text{si } u < t \leqslant \zeta_s^\infty. \end{cases}$$

– Soit il n'existe pas de tel i et on définit

$$\zeta_s^\infty = u,$$
$$W_s^\infty(t) = \rho_t, \quad t \leqslant u.$$

Ce procédé définit un processus continu $W^\infty = F(\rho, \mathcal{P})$ à valeurs dans $\mathcal{W}$ qui est le serpent brownien éternel conditionné à rester positif adossé au processus de Bessel ρ. La tête au temps s de ce serpent est $\widehat{W}_s^\infty = W_s^\infty(\zeta_s^\infty)$.

Conditionnellement à ρ, donnons-nous deux processus de Poisson $\mathcal{P}_L, \mathcal{P}_R$, d'intensité donnée par (1.11) et tels que $\mathcal{P}_L$ et $\mathcal{P}_R$ soient indépendants. La paire de processus $\left(W^L = F(\rho, \mathcal{P}_L), W^R = F(\rho, \mathcal{P}_R)\right)$ est une paire de serpents browniens éternels correlés et adossés au processus de Bessel ρ. Nous prouvons au Théorème 9 du chapitre 3 que les fonctions de contour de l'arbre bien étiqueté infini uniforme changées d'échelle convergent en loi vers cette paire de serpents de la manière suivante :

$$\left(\left(\frac{1}{n} C_L(n^2 s), \sqrt{\frac{3}{2n}} V_L(n^2 s) \right)_{s \geqslant 0}, \left(\frac{1}{n} C_R(n^2 s), \sqrt{\frac{3}{2n}} V_R(n^2 s) \right)_{s \geqslant 0} \right)$$
$$\xrightarrow[n\to\infty]{(\mathrm{d})} \left(\left(\zeta_s^{(L)}, \widehat{W}_s^{(L)} \right)_{s \geqslant 0}, \left(\zeta_s^{(R)}, \widehat{W}_s^{(R)} \right)_{s \geqslant 0} \right). \tag{1.12}$$

en distribution, avec $\zeta_s^{(L)} = \zeta_{\left(W_s^{(L)}\right)}$, respectivement $\zeta_s^{(R)} = \zeta_{\left(W_s^{(R)}\right)}$, pour tout $s \geqslant 0$.

La convergence des fonctions de contour (1.12) permet alors de montrer l'analogue de la convergence en loi du profil (1.9) d'une quadrangulation pour la quadrangulation infinie uniforme. C'est l'objet du paragraphe 3.4 du chapitre 3 où on démontre par exemple la convergence suivante dans le Théorème 10 : la suite $\left(\lambda_q^{(n)}\right)_{n \geqslant 1}$ des profils changés d'échelle de la quadrangulation infinie uniforme q définis par

$$\lambda_q^{(n)}(A) = \frac{1}{n^2} \lambda_q \left(\sqrt{\frac{2n}{3}} A \right)$$

pour tout A borélien de $\mathbb{R}_+$ converge en distribution vers la mesure aléatoire $\mathcal{I}^\infty$ définie par

$$\langle \mathcal{I}^\infty, g \rangle = \int_0^\infty \mathrm{d}s \left(g\left(\widehat{W}_s^{(L)}\right) + g\left(\widehat{W}_s^{(R)}\right) \right).$$

Une des conséquences de ce résultat est la convergence en loi du volume des boules de rayon r autour de la racine dans l'échelle r^4 :

$$\frac{1}{n^4} \left| B_{\mathbf{Q}, nr}(q) \right| \xrightarrow[n\to\infty]{(d)} \frac{9}{4} r^4 \int_0^\infty \mathrm{d}s \left(\mathbf{1}_{[0,1]}\left(\widehat{W}_s^L\right) + \mathbf{1}_{[0,1]}\left(\widehat{W}_s^R\right) \right).$$

La mesure $\mathcal{I}^\infty$, analogue à la mesure ISE, est étudiée dans le paragraphe 3.4, notamment à travers sa transformée de Laplace par la Proposition 11 :

$$E\left[\exp -\lambda \mathcal{I}\left([0,1]\right)\right] = E\left[\exp -4\int_0^\infty \mathrm{dt}\ \left(u_\lambda(\rho_t) - \frac{3}{2\rho_t^2}\right)\right]$$

où la fonction u_λ est continue sur $]0,\infty[$ et solution de l'équation diférentielle

$$\frac{1}{2}u'' = 2u^2 - \lambda$$

sur $]0,1[$ avec la condition limite $u(0)=\infty$. De plus

$$u(x) = \frac{3}{2(x-a)^2}$$

pour tout $x \geqslant 1$, avec $a = 1 - \sqrt{\frac{3}{2u(1)}}$.

Il est aussi possible de faire un lien entre le cycle γ_r de Krikun, frontière de l'enveloppe $B_{\mathbf{Q},r}(q)$ de la quadrangulation infinie uniforme, et l'arbre infini uniforme grâce à un lemme combinatoire démontré dans la Proposition 12. Ce lemme permet en effet d'identifier simplement les sommets d'un cycle $\overline{\gamma}_r$, analogue de γ_r pour la quadrangulation infinie uniforme à laquelle on a ajouté les arêtes de l'arbre infini uniforme qui lui est associé (ces arêtes sont les diagonales de certaines faces de la quadrangulation). Le cycle $\overline{\gamma}_r$ est légèrement différent de γ_r, mais on s'attend à ce qu'il ait le même comportement que γ_r dans sa limite d'échelle.

Des techniques de calcul sur les mesures de sortie du serpent brownien permettent alors de démontrer le résultat suivant (Proposition 13) : pour tout $\lambda > 0$, on a la convergence suivante

$$E\left[\exp -\lambda\frac{|\overline{\gamma}_r|}{r^2}\right] \underset{r\to\infty}{\longrightarrow} E\left[\exp -6\int_{h_1(\rho)}^\infty \mathrm{d}t\left(\frac{1}{\left(\rho_t - (1-(1+\lambda)^{-1/2})\right)^2} - \frac{1}{\rho_t^2}\right)\right], \qquad (1.13)$$

où $h_1(\rho)$ est le dernier temps d'atteinte de 1 du processus de Bessel ρ.

1.5 Conclusion et perspectives

En conclusion à ce chapitre, donnons quelques pistes qui semblent intéressantes à explorer et qui complèteraient bien le présent travail. Le premier problème naturel qui se pose au vu de nos résultats est celui de l'universalité. Il est en effet tentant de définir des $2p$-angulations infinies uniformes de manière similaire à la quadrangulation infinie uniforme. À l'instar de notre présentation, deux approches sont possibles : une directe par convergence locale, et une indirecte avec une possible extension de la bijection de Bouttier, Di Francesco et Guitter. Il semble raisonnable de conjecturer que ces constructions sont possibles et compatibles, étendant alors les résultats des chapitres 2 et 3 aux $2p$-angulations.

Une autre question naturelle est le lien entre la quadrangulation infinie uniforme et la carte brownienne. Peut-on en effet changer d'échelle la distance de graphe de la quadrangulation infinie uniforme, par exemple par un facteur $n^{-\alpha}$ avec $\alpha > 0$, de manière à ce que l'espace métrique (non compact) que constitue la quadrangulation infinie uniforme munie de cette distance changée d'échelle converge en distribution vers un espace métrique aléatoire non compact au sens de la distance de Gromov-Hausdorff sur les espaces mériques localement compacts pointés introduite dans le livre [Gro07]. On peut conjecturer que cet espace métrique aléatoire est aussi la limite d'échelle des quadrangulations uniformes à n faces lorsqu'on les munit de la distance $n^{-\alpha}d_{gr}$ avec $0 < \alpha < 1/4$. Cet espace métrique limite serait alors une version non compacte de la carte brownienne.

Nous avons évoqué dans ce chapitre que la percolation par sites sur la triangulation infinie uniforme a été étudiée par Angel [Ang03]. L'étude de la percolation par arêtes sur plusieurs types de cartes infinies uniformes fait l'objet d'un travail en cours avec Pierre Nolin. Ce travail repose tout d'abord sur une bijection entre les cartes à n arêtes et les quadrangulations à n faces, qui se comporte bien pour la topologie de la convergence locale si on la définit en fonction des arêtes pour les cartes et en fonction des faces pour les quadrangulations. De plus cette bijection se spécialise entre les p-angulations à n arêtes et certaines quadrangulations à n faces.

L'étude de la percolation par arêtes sur une carte se base alors sur un procédé d'épluchage par faces de la quadrangulation associée. Ce procédé a été introduit dans le livre [ADJ97] et a permis à Angel [Ang03] de calculer p_c pour une triangulation infinie uniforme. Dans notre cas, cette technique permet d'explorer les arêtes de la carte une par une de façon markovienne et se prête donc bien à l'étude de la percolation par arêtes.

2

Sommaire

The two uniform infinite planar quadrangulations have the same law

IL EXISTE deux façons naturelles de définir une quadrangulation infinie uniforme du plan. La première méthode est de construire une limite locale de quadrangulations de la sphère de grande taille. Une autre construction possible est de définir un arbre bien étiqueté infini et de loi uniforme par des limites locales d'arbres finis bien étiquetés de grande taille, et de construire à partie de cet objet une quadrangulation grâce à une extension de la bijection de Schaeffer. Nous rappelons ici ces deux constructions comme elles se trouvent dans la littérature, et nous démontrons ensuite que ces deux quadrangulations infinies ont la même loi.

Ce chapitre est une version légèrement modifiée de l'article [Mé08], accepté pour publication dans la revue Annales de l'Institut Henri Poincaré (Probabilités et Statistiques).

2.1 Introduction

Planar maps are proper embeddings of connected graphs in the two-dimensional sphere $\mathbb{S}^2$. Their combinatorial properties have been studied by Tutte [Tut63] and many others. Planar maps have recently drawn much attention in the theoretical physics literature as models of random surfaces, especially in the setting of the theory of two-dimensional quantum gravity (see in particular the book [ADJ97]). A powerful tool to study these objects is the encoding of planar maps in terms of labelled trees, which was first introduced by Cori and Vauquelin in [CV81] and was much developed in Schaeffer's thesis [Sch98] (see also Bouttier, Di Francesco and Guitter [BDFG04] for a generalized version of this encoding). This correspondence between planar maps and trees makes it possible to derive certain asymptotics of large random planar maps in terms of continuous random trees (see the work of Chassaing and Schaeffer [CS04]) and to define a Brownian map (see Marckert and Mokkadem [MM03]) which is a continuous random metric space conjectured to be the scaling limit of various classes of planar maps (see the papers by Marckert and Miermont [MM07], Le Gall [LG07], Le Gall and Paulin [LGP08]). This approach has led to new asymptotic properties of large planar maps.

Another point of view is to study properties of random infinite planar maps, more precisely to study probability measures on certain classes of infinite planar maps, which are uniform in some sense. This has been done by Angel and Schramm [AS03] who introduced a uniform infinite triangulation of the plane, later studied by Angel [Ang03, Ang05] and Krikun [Kri04].

In the present paper, we are interested in infinite random planar quadrangulations. Recall that Schaeffer's bijection (see e.g. [CS04]) yields a one-to-one correspondence between rooted planar quadrangulations with n faces and well-labelled trees with n edges. Then, there are two natural ways to define a uniform infinite quadrangulation of the sphere: one as the local limit of uniform finite quadrangulations as their size goes to infinity, and one going through Schaeffer's bijection and using local limits of uniform well-labelled trees. The first approach is developed in Krikun [Kri06] while the second one is developed in Chassaing and Durhuus [CD06]. The topologies in which the uniform finite quadrangulations converge to the infinite object differ in the two cases: in the Chassaing–Durhuus paper, the topology on quadrangulations is induced by Schaeffer's bijection and the natural topology of local convergence of rooted trees, while the topology used in Krikun's paper is the natural topology of local convergence of rooted planar maps. Therefore, the two uniform infinite random quadrangulations defined in these papers are a priori two different objects. The goal of the paper is to show that these two definitions coincide. This result is stated in Theorem 4 below. Note that our work also gives an alternative approach to Theorem 1 of Krikun [Kri06]: independently of the results of [Kri06], Theorem 4 shows that the uniform probability measure on the space of all rooted planar quadrangulations with n faces converges as $n \to \infty$ to a probability measure on the space of infinite quadrangulations, in the sense of the metric used in [Kri06].

Let us briefly explain the main point of our argument. Consider a sequence of (deterministic or random) finite well-labelled trees θ_n, that converges as $n \to \infty$ towards an infinite well-labelled tree θ_∞, in the sense that, for every $k \geqslant 1$, the restriction of θ_n to the first k generations is equal to the same restriction of θ_∞, when n is sufficiently large.

Let Q_n be the quadrangulation associated with θ_n via Schaeffer's bijection and let Q_∞ be the infinite quadrangulation associated with θ_∞ via the extension of Schaeffer's bijection that is presented in Subsection 2.2.3 below (θ_∞ needs to satisfy certain properties so that this makes sense). Then it is not always true that Q_∞ is the local limit of Q_n as $n \to \infty$. The problem comes from the fact that θ_n may have small labels at generations larger than $k(n)$ with $k(n) \to \infty$. Note that this problem may occur even if one knows that θ_∞ has finitely many labels smaller than K, for every integer K (the latter property holds for the uniform infinite well-labelled tree thanks to the estimates of [CD06], see Proposition 3 below). Nonetheless, in the case when θ_n is uniformly distributed over all well-labelled trees with n edges, the preceding phenomenon does not occur: for every fixed $R > 0$, the probability that θ_n has a label less than R above generation S tends to 0 as $S \to \infty$, uniformly in n. This uniform estimate is stated in Proposition 6 below.

We can combine this estimate with the following combinatorial argument. If two well-labelled trees coincide up to generation S, then the associated quadrangulations are also the same within distance R from the root, where R is essentially the minimum label above generation S in either tree. See Proposition 4 below for a precise statement.

The paper is organized as follows: Section 2 gives some notation and an extension of Schaeffer's bijection to the infinite case; Section 3 presents the two different definitions of the uniform infinite quadrangulation; and Section 4 contains the key estimates that allow us to prove that these definitions actually lead to the same object.

2.2 Preliminaries

2.2.1 Spatial trees

Throughout this work we will use the standard formalism on planar trees as found in [Nev86]. Let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where by convention $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. An element u of $\mathcal{U}$ is thus a finite sequence of positive integers. If $u, v \in \mathcal{U}$, uv denotes the concatenation of u and v. If v is of the form uj with $j \in \mathbb{N}$, we say that u is the *parent* of v or that v is a *child* of u. More generally, if v is of the form uw for $u, w \in \mathcal{U}$, we say that u is an *ancestor* of v or that v is a *descendant* of u. A *rooted planar tree* τ is a subset of $\mathcal{U}$ such that

1. $\emptyset \in \tau$ ($\emptyset$ is called the *root* of τ),
2. if $v \in \tau$ and $v \neq \emptyset$, the parent of v belongs to τ
3. for every $u \in \mathcal{U}$ there exists $k_u(\tau) \geqslant 0$ such that $uj \in \tau$ if and only if $j \leqslant k_u(\tau)$.

The edges of τ are the pairs (u, v), where $u, v \in \tau$ and u is the father of v. $|\tau|$ denotes the number of edges of τ and is called the size of τ. $h(\tau)$ denotes the maximal generation of a vertex in τ and is called the height of τ. We denote by $\mathbf{T}_n$ the set of all rooted planar trees of size n and by $\mathbf{T}_\infty$ the set of all infinite rooted planar trees. Then $\mathbf{T} = \cup_{n=0}^{\infty} \mathbf{T}_n$ is the set of all finite rooted planar trees and $\overline{\mathbf{T}} = \mathbf{T} \cup \mathbf{T}_\infty$ is the set of all rooted (finite or

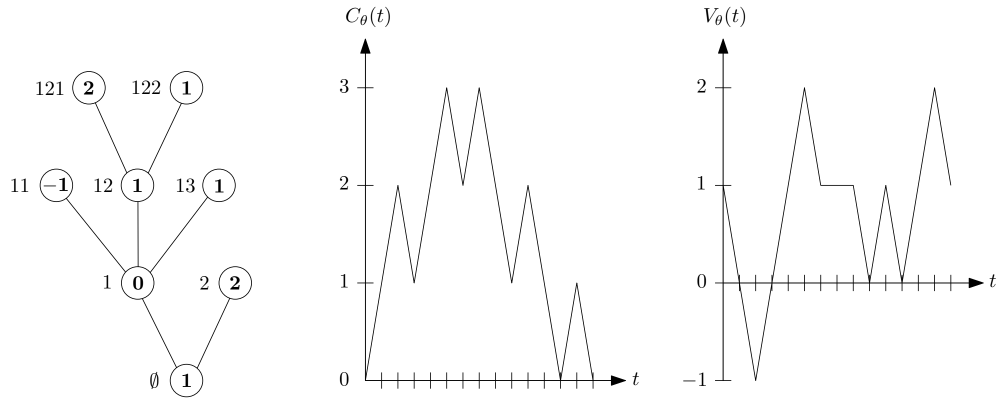

Figure 2.1: A spatial tree and its pair of contour functions (C, V).

infinite) planar trees. A *spine* of a tree τ is an infinite linear sub-tree of τ starting from its root.

A *rooted labelled tree* (or spatial tree) is a pair $\theta = (\tau, (\ell(u))_{u\in\tau})$ that consists of a planar tree τ and a collection of integer labels assigned to the vertices of τ such that if $u, v \in \tau$ and v is a child of u, then $|\ell(u) - \ell(v)| \leqslant 1$. For every $l \in \mathbb{Z}$, we denote by $\overline{\mathbf{T}}^{(l)}$ the set of all spatial trees for which $\ell(\emptyset) = l$, by $\mathbf{T}^{(l)}_\infty$ the set of all such trees with an infinite number of edges, by $\mathbf{T}^{(l)}_n$ the set of all such trees with n edges and by $\mathbf{T}^{(l)}$ the set of all such trees with finitely many vertices. Similarly as before, $\mathbf{T}^{(l)} = \bigcup_{n=0}^{\infty} \mathbf{T}^{(l)}_n$.

If $\ell(\emptyset) = l$ and in addition $\ell(u) \geqslant 1$ for every vertex u of τ, we say that θ is an l-well-labelled tree. The corresponding sets of spatial trees are denoted by $\overline{\mathbb{T}}^{(l)}$, $\mathbb{T}^{(l)}$, $\mathbb{T}^{(l)}_\infty$ and $\mathbb{T}^{(l)}_n$. For $l = 1$ we will simply say well-labelled tree and denote the corresponding sets by $\overline{\mathbb{T}}$, $\mathbb{T}$, $\mathbb{T}_\infty$ and $\mathbb{T}_n$.

A finite spatial tree $\omega = (\tau, \ell)$ can be coded by a pair (C, V), where $C = (C(t))_{0\leqslant t\leqslant 2|\tau|}$ is the contour function of τ and $V = (V(t))_{0\leqslant t\leqslant 2|\tau|}$ is the spatial contour function of ω (see Figure 2.1). To define these contour functions, let us consider a particle which, starting from the root, traverses the tree along its edges at speed one. When leaving a vertex, the particle visits the first non-visited child of this vertex if there is such a child, or returns to the parent of this vertex. Since all edges will be crossed twice, the total time needed to explore the tree is $2|\tau|$. For every $t \in [0, 2|\tau|]$, $C(t)$ denotes the distance from the root of the position of the particle. In addition if $t \in [0, 2|\tau|]$ is an integer, $V(t)$ denotes the label of the vertex that is visited at time t. We then complete the definition of V by interpolating linearly between successive integers. See Figure 2.1 for an example. A spatial tree is uniquely determined by its pair of contour functions.

To conclude this section, let us introduce some relevant notation. If $\omega = (\tau, \ell)$ is a labelled tree, $|\omega| = |\tau|$ is the size of ω, $h(\omega) = h(\tau)$ is the height of ω and, for $S \geqslant 0$, $g_S(\omega)$ is the set of all vertices of ω at generation S. Finally, for every $l \in \mathbb{N}$, we let $N_l(\omega)$ denote the number of vertices of ω that have label l. We define $\mathscr{S}$ as the set of all trees of $\overline{\mathbb{T}}$ that have at most one spine, and for which labels takes each integer value a finite number of times:

$$\mathscr{S} = \{\omega \in \mathbb{T}_\infty : \forall l \geqslant 1,\ N_l(\omega) < \infty \text{ and } \omega \text{ has a unique spine}\} \cup \mathbb{T}. \tag{2.1}$$

2.2.2 Planar maps and quadrangulations

Consider a proper embedding of a finite connected graph in the sphere $\mathbb{S}^2$ (loops and multiple edges are allowed). A (finite) *planar map* is an equivalent class of such embedded graphs with respect to orientation preserving homeomorphisms of the sphere. A planar map is *rooted* if it has a distinguished oriented edge, and the origin of the root is called the root vertex. In what follows, planar maps are always rooted even if this is not mentioned explicitly. The set of vertices will always be equipped with the graph distance. The faces of the map are the connected components of the complement of the union of its edges. A finite planar map is a *quadrangulation* if all its faces have degree 4.

For every integer $n \geqslant 1$ we let $\mathbf{Q}_n$ denote the set of all (rooted) quadrangulations with n faces and $\mathbf{Q} = \bigcup_{n \geqslant 1} \mathbf{Q}_n$ denote the set of finite quadrangulations. Each set $\mathbf{Q}_n$ is in bijective correspondence with the set $\mathbb{T}_n$ by Schaeffer's bijection [CV81, Sch98]. There is no bijection between infinite well-labelled trees and infinite quadrangulations, but Schaeffer's correspondence has been extended to $\mathscr{S}$ in [CD06]. To discuss this extention, we first have to define precisely what we mean by an infinite quadrangulation. To this end we recall some definitions of [AS03, CD06] in a slightly different form.

Throughout this work, we consider only infinite graphs such that the degree of every vertex is finite. Consider a proper embedding of an infinite graph in the plane $\mathbb{R}^2$. We say that this embedding is locally finite if every compact subset of $\mathbb{R}^2$ intersects only finitely many edges.

Definition 1. An infinite planar map $\mathcal{M}$ is an equivalent class of locally finite embeddings of an infinite graph in $\mathbb{R}^2$, with respect to orientation preserving homeomorphisms of the plane.

The faces of an infinite planar map $\mathcal{M}$ are the bounded connected components of the complement of the union of its edges. With this definition, every edge of $\mathcal{M}$ is not necessarily adjacent to a face; for example infinite trees have only one "face" of infinite degree, which is not a face in the sense of the previous definition. This motivates the next definition.

Definition 2. A *regular* infinite planar map is an infinite planar map such that every connected component of the complement of the union of its edges is bounded.

In a *regular* infinite planar map, every edge is either shared by two faces or appears twice in the border of a face.

Remark. With the previous definitions, an infinite tree can be embedded as an infinite planar map in $\mathbb{R}^2$, but not as a regular infinite planar map.

Definition 3. An *infinite planar quadrangulation* is a regular infinite planar map having every face bordered by four-sided polygons. A rooted infinite quadrangulation is an infinite quadrangulation with a distinguished oriented edge (v_0, v_1) called the root of the quadrangulation; v_0 is called the root vertex of the quadrangulation. We denote by $\overline{\mathbb{Q}}$ the set of all (finite or infinite) rooted planar quadrangulations and we have the self-evident decomposition $\overline{\mathbb{Q}} = \mathbf{Q} \cup \mathbf{Q}_\infty$.

2.2.3 Schaeffer's correspondence

We are now going to describe the extension of Schaeffer's correspondence to the set $\mathscr{S}$. We refer to Section 6 of [CD06] for details and proofs.

With every infinite well-labelled tree $\omega \in \mathscr{S}$ we will associate an infinite planar quadrangulation $\Phi(\omega)$. We identify $\mathbb{S}^2$ with the set $\mathbb{R}^2 \cup \{\infty\}$, and we fix an infinite tree $\omega \in \mathscr{S}$. We can also fix an embedding of ω into $\mathbb{R}^2$ as in Definition 1 above. We root ω at the edge between vertices $\emptyset$ and 1. Let F_0 denote the complement of the union of edges of ω in $\mathbb{S}^2$.

Definition 4. A *corner* of F_0 is a sector between two consecutive edges around a vertex. The label of a corner is the label of the corresponding vertex.

A vertex of degree d defines d corners and a tree $\omega \in \mathscr{S}$ has a finite number $C_k(\omega) \geqslant N_k(\omega)$ of corners with label k. The map $\Phi(\omega)$ is defined in three steps.

Step 1 (see Figure 2.2)**.** A vertex v_0 with label 0 is added in $F_0 \setminus \{\infty\}$ and one edge is added between this vertex and each of the $C_1(\omega)$ corners with label 1. The new root is taken to be the edge that connects v_0 to the corner before the root edge of ω.

Remark. Notice that the construction in step 1 is possible because ω has at most one spine.

After step 1, a uniquely defined rooted infinite planar map $\mathcal{M}_0$ with $C_1(\omega) - 1$ faces is obtained (in the sense of the definitions of Section 2.2.2, in particular, the faces are bounded subsets of $\mathbb{R}^2$). Notice that each face of $\mathcal{M}_0$ has a unique corner with label 0 and two corners with label 1. Such a face is bordered by the two edges joining v_0 to the two corners with label 1 and, in the case where the two corners with label 1 correspond to two different vertices, the unique injective path in the tree between these two vertices with label 1.

It is natural to consider the complement of $\mathcal{M}_0$ and its faces as an additional face of infinite degree. Let us denote this face by F_∞. It possesses a unique corner with label 0 and two corners with label 1 lying on each side of the spine of ω. In addition, these two corners are the last visited corners with label 1 during a contour of the left side and right side of ω. F_∞ is thus delimited by the two edges joining these vertices and v_0, and the unique injective path in the tree joining these two vertices. The spine of ω lies in this face, except for finitely many vertices.

The second step takes place independently in each face of $\mathcal{M}_0$, including F_∞. Let F be a face of $\mathcal{M}_0$ and let c_0 be its corner with label 0. If F has finitely many vertices — and therefore finitely many corners — we number its corners from 0 to $k-1$ in clockwise order along the border, starting with c_0. If F is the infinite face, we number its corners on the right side of the spine with non-negative integers in clockwise order, starting right after c_0. Similarly, we number its corners on the left side of the spine with negative integers

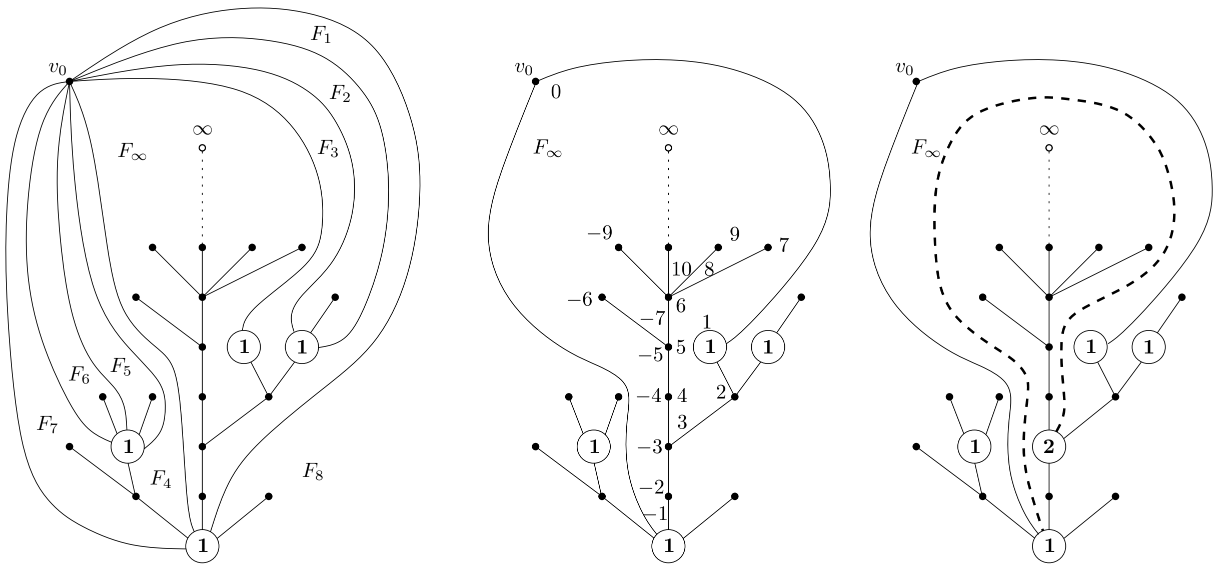

Figure 2.2: Left: step 1, edges are added between v_0 and corners with label 1. Middle: step 2, numbering of a few corners in F_∞. Right: step 2, a chord between the two sides of the spine.

in counterclockwise order, starting right after c_0. See, for example Figure 2.2. Let $\ell(i)$ denote the label of the i-th corner, so that $\ell(0) = 0$ and $\ell(1) = \ell(k-1) = 1$ for a finite face whereas $\ell(1) = \ell(-1) = 1$ for F_∞ (note that the function ℓ depends of the considered face).

In each face, let us define the successor function for all corners except the corners with label 0 or 1 by

$$s(i) = \begin{cases} \min\{j > i : \ell(j) = \ell(i) - 1\} & \text{if } i < 0, \\ \min\{j > i : \ell(j) = \ell(i) - 1\} & \text{if } i > 0 \text{ and } \{j > i : \ell(j) = \ell(i) - 1\} \neq \emptyset, \\ \min\{j \leqslant 0 : \ell(j) = \ell(i) - 1\} & \text{if } i > 0 \text{ and } \{j > i : \ell(j) = \ell(i) - 1\} = \emptyset. \end{cases}$$

For a finite face, only the second case occurs, while for F_∞ the second property of Definition 2.1 ensures that $\{j \leqslant 0 : \ell(j) = \ell(i) - 1\}$ is finite.

Step 2. In every face, for each corner i with label $\ell(i) \geqslant 2$ and such that $|s(i) - i| \neq 1$ a chord $(i, s(i))$ is added inside the face.

Proposition 1 ([CD06], Property 6.1)**.** *Step 2 can be done in such a way that the various chords $(i, s(i))$ do not intersect.*

Remark. The condition $|s(i) - i| \neq 1$ means that the chord $(i, s(i))$ does not already exist in ω. In F_∞, a chord $(i, s(i))$ can connect two corners that lie on different sides of the spine (see e.g. Figure 2.2). This happens in the third case occurring in the definition of $s(i)$. In that case, the corner i is visited after the last occurrence of the label $\ell(i) - 1$ during the contour of the right side of the spine.

Step 2 defines a uniquely determined regular planar map $\mathcal{M}_1$ whose faces are described by the following proposition:

Proposition 2 ([CD06], Property 6.2)**.** *The faces of $\mathcal{M}_1$ are either triangular with labels l, $l+1$, $l+1$ or quadrangular with labels l, $l+1$, $l+2$, $l+1$.*

Step 3. All edges of $\mathcal{M}_1$ with the same label on both ends are deleted.

After this last step, a unique infinite quadrangulation $\Phi(\omega)$ is obtained (see [CD06] for details). In addition, labels of vertices in the tree ω coincide with distances from the root of the corresponding vertices in $\Phi(\omega)$. Furthermore, the function Φ is one-to-one.

2.3 Uniform infinite quadrangulations

This section presents two different ways to define a uniform infinite random quadrangulation of the plane.

2.3.1 Direct approach

In [Kri06], the uniform infinite quadrangulation is defined as the law of the local limit of uniformly distributed finite random quadrangulations. This limit is taken with respect to the following topology: for $Q \in \mathbf{Q}$ and $R \geqslant 0$, we denote by $B_{\mathbf{Q},R}(Q)$ the union of the faces of Q that have a vertex at distance strictly smaller than R from the root vertex. We may view $B_{\mathbf{Q},R}(Q)$ as a finite rooted planar map. The set $\mathbf{Q}$ is equipped with the distance

$$d_{\mathbf{Q}}(Q_1, Q_2) = (1 + \sup\{R :\ B_{\mathbf{Q},R}(Q_1) = B_{\mathbf{Q},R}(Q_2)\})^{-1},$$

where the equality $B_{\mathbf{Q},R}(Q_1) = B_{\mathbf{Q},R}(Q_2)$ is in the sense of equality between two finite rooted planar maps.

Let $(\overline{\mathbf{Q}}, d_{\mathbf{Q}})$ be the completion of the metric space $(\mathbf{Q}, d_{\mathbf{Q}})$. Elements of $\overline{\mathbf{Q}}$ that are not finite quadrangulations are called infinite rooted quadrangulations in the sense of Krikun.

Note that this definition is not equivalent to Definition 3. For example, the quadrangulation Q_n of Figure 2.3 converges as n goes to infinity in $(\overline{\mathbf{Q}}, d_{\mathbf{Q}})$ to an infinite quadrangulation Q in Krikun's sense that is not an infinite planar map in the sense of Definition 1: any proper embedding of Q in $\mathbb{R}^2$ is not locally finite.

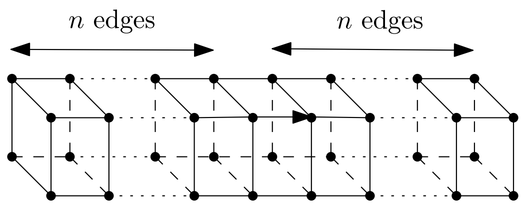

Figure 2.3: A quadrangulation that converges in Krikun's sense to an infinite quadrangulation that is not an infinite planar map.

Theorem 1 ([Kri06], Theorem 1)**.** *For every $n \geqslant 1$ let ν_n be the uniform probability measure on $\mathbf{Q}_n$. The sequence $(\nu_n)_{n\in\mathbb{N}}$ converges to a probability measure ν in the sense of weak convergence in the space of all probability measures on $(\overline{\mathbf{Q}}, d_{\mathbf{Q}})$. Moreover, ν is supported on the set of infinite rooted quadrangulations (in the sense of Krikun).*

Remark. One can extend the function $Q \in \mathbf{Q} \mapsto B_{\mathbf{Q},R}(Q)$ to a continuous function $B_{\overline{\mathbf{Q}},R}$ on $\overline{\mathbf{Q}}$. $B_{\overline{\mathbf{Q}},R}(Q)$ is naturally interpreted as the union of faces of Q that have a vertex at distance strictly smaller than R from the root.

2.3.2 Indirect approach

Another possible approach to define a uniform infinite random quadrangulation is to start from a uniform infinite well-labelled tree and to consider the image of its law under Schaeffer's correspondence. This method has been developed in [CD06], to which we refer for details and for proofs of what follows in this section. Let us equip $\overline{\mathbb{T}}$ with the distance

$$d_{\mathbb{T}}(\omega, \omega') = \left(1 + \sup\{S : \, B_{\mathbb{T},S}(\omega) = B_{\mathbb{T},R}(\omega')\}\right)^{-1},$$

where $B_{\mathbb{T},S}(\omega)$ is the subtree of ω up to generation S. The metric space $(\overline{\mathbb{T}}, d_{\mathbb{T}})$ is complete.

We have the following result:

Theorem 2 ([CD06], Theorem 3.1)**.** *Let μ_n be the uniform probability measure on the set of all well-labelled trees with n edges. The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to a probability measure μ supported on $\mathbb{T}_\infty$. This limit law is called the law of the uniform infinite well-labelled tree.*

One of the key steps to prove this result is to show the convergence

$$\mu_n\left(\omega \in \overline{\mathbb{T}} : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\right) \underset{n\to\infty}{\longrightarrow} \mu\left(\omega \in \overline{\mathbb{T}} : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\right)$$

for every integer $S > 0$ and every well-labelled tree $\omega^\star$ of height S. This is done by explicit computations. Let $\omega^\star$ be a well-labelled tree of height S , and assume that $\omega^\star$ has exactly k vertices at generation S, with respective labels $l_1, \dots, l_k$. Then,

$$\mu_n\left(\omega \in \overline{\mathbb{T}} : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\right) = \frac{1}{D_n} \sum_{n_1+\cdots+n_k = n-|\omega^\star|} \prod_{j=1}^{k} D_{n_j}^{(l_j)}, \tag{2.2}$$

$$\mu\left(\omega \in \overline{\mathbb{T}} : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\right) = \frac{1}{12^{|\omega^\star|}} \sum_{i=1}^{k} d_{l_i} \prod_{j\neq i} w_{l_j}, \tag{2.3}$$

where, for every $l \geqslant 1$, $D_n^{(l)}$ is the cardinal of $\mathbb{T}_n^{(l)}$, $D_n^{(1)} = D_n$ and

$$w_l = 2\frac{l(l+3)}{(l+1)(l+2)}, \tag{2.4}$$

$$d_l = \frac{2w_l}{560}(4l^4 + 30l^3 + 59l^2 + 42l + 4). \tag{2.5}$$

Proposition 3 ([CD06], Theorem 4.3 and Theorem 5.9)**.** *The measure μ is supported on $\mathscr{S}$. Furthermore,*

$$\mathbb{E}_\mu\left[N_l\right] = O(l^3) \quad \textit{as } l \to \infty.$$

A tree with law μ has almost surely a unique spine; [CD06] gives a precise description of the law of the labels of this spine and of the subtrees attached to each of its vertices. For every $l > 0$, let $\rho^{(l)}$ be the measure on $\mathbf{T}^{(l)}$ defined by $\rho^{(l)}(\omega) = 12^{-|\omega|}$ for every $\omega \in \mathbf{T}^{(l)}$. Then $\frac{1}{2}\rho^{(l)}$ is the law of the Galton–Watson tree with geometric offspring distribution with parameter $\frac{1}{2}$ and with random labels generated according to the following rules. The root has label l and the label of every other vertex is chosen uniformly in $\{m-1, m, m+1\}$ where m is the label of its parent. Furthermore, these choices are made independently for every vertex. Proposition 2.4 of [CD06] proves that $\rho^{(l)}(\mathbb{T}^{(l)}) = w_l$, therefore the measure $\widehat{\rho}^{(l)}$ defined on $\mathbb{T}^{(l)}$ by $\widehat{\rho}^{(l)}(\omega) = w_l^{-1}\rho^{(l)}(\omega) = w_l^{-1}12^{-|\omega|}$ for every $\omega \in \mathbb{T}^{(l)}$ is a probability measure. The following result will be useful for our purposes.

Theorem 3 ([CD06], Theorem 4.4)**.** *Let ω be a random tree distributed according to μ and let $u_0, u_1, u_2, \dots$ be the sequence of the vertices of its spine listed in genealogical order. For every $n \geqslant 0$, let Y_n be the label of u_n.*

1. *The process $(Y_n)_{n\geqslant 0}$ is a Markov chain taking values in $\mathbb{N}$ with transition kernel Π defined by:*

$$\begin{aligned} \Pi(l, l-1) &= \frac{(w_l)^2}{12 d_l} d_{l-1} := q_l & \text{if } l \geqslant 2, \\ \Pi(l, l) &= \frac{(w_l)^2}{12} := r_l & \text{if } l \geqslant 1, \\ \Pi(l, l+1) &= \frac{(w_l)^2}{12 d_l} d_{l+1} := p_l & \text{if } l \geqslant 1. \end{aligned}$$

2. *Conditionally given $(Y_n)_{n\geqslant 0} = (y_n)_{n\geqslant 0}$, the sequence $(L_n)_{n\geqslant 0}$ of subtrees of ω attached to the left side of the spine and the sequence $(R_n)_{n\geqslant 0}$ of subtrees attached to the right side of the spine form two independent sequences of independent labelled trees distributed according to the measures $\widehat{\rho}^{(y_n)}$.*

We can now map the law of the uniform infinite random tree on the set of quadrangulations using Schaeffer's correspondence. Let us equip $\Phi(\mathscr{S})$ with the distance d_Φ so that Φ is an isometry from $\mathscr{S}$ onto $\Phi(\mathscr{S})$. We denote by $\mu_{\Phi,n}$ and μ_Φ the respective image measures of μ_n and μ under Φ. The measure μ_Φ is well defined because μ is supported on $\mathscr{S}$.

Since Φ is a bijection between $\mathbb{T}_n$ and $\mathbf{Q}_n$, $\mu_{\Phi,n} = \nu_n$ is the uniform probability measure on the set of quadrangulations with n faces. As a direct consequence of Theorem 2, the sequence $(\mu_{\Phi,n})_{n\in\mathbb{N}}$ converges weakly to μ_Φ in the space of all probability measures on $(\Phi(\mathscr{S}), d_\Phi)$. Thus, in some sense, μ_Φ can also be viewed as a uniform probability measure on the space of infinite quadrangulations.

Remark. The topology induced by d_Φ on the set $\Phi(\mathscr{S})$ is rather different than the one that would be induced by $d_{\mathbf{Q}}$. Indeed it may happen that two trees ω and ω' are close for the metric $d_{\mathbb{T}}$, but the quadrangulations $\Phi(\omega)$ and $\Phi(\omega')$ are very different for $d_{\mathbf{Q}}$. For example, the linear tree ω_n with $2n-1$ vertices and with labels given by the sequence $1, 2, \dots, n-1, n, n-1, \dots, 2, 1$ converges as n goes to infinity to the infinite linear tree ω with labels given by the sequence $1, 2, \dots$. As a consequence, the quadrangulation $\Phi(\omega_n)$ converges to the infinite quadrangulation $\Phi(\omega)$ in $(\Phi(\mathscr{S}), d_\Phi)$ as n goes to infinity. On the other hand, for every $n \geqslant 1$, the quadrangulation $\Phi(\omega_n)$ has two vertices at distance 1 from its root whereas $\Phi(\omega)$ has only one vertex at distance 1 from its root and therefore the sequence $(\Phi(\omega_n))_{n\in\mathbb{N}}$ does not converge to $\Phi(\omega)$ in $(\overline{\mathbf{Q}}, d_{\mathbf{Q}})$.

It is then a natural question to ask whether the two notions of uniform infinite quadrangulation that we have introduced coincide.

2.4 Equality of the two uniform infinite quadrangulations

In this section, we will show that the two definitions of the uniform infinite quadrangulation coincide. The first problem comes from the fact that we have two different notions of infinite quadrangulations: elements of $\Phi(\mathscr{S})$, which are regular planar maps on one hand, and elements of the completion $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ on the other hand. This problem can be solved by identifying $\Phi(\mathscr{S})$ with a subset of $\overline{\mathbf{Q}}$, allowing us to consider μ_Φ as a measure on $\overline{\mathbf{Q}}$ supported on $\Phi(\mathscr{S})$.

More precisely, let $R > 0$ and $\omega \in \mathscr{S}$. Define $B_R(\Phi(\omega))$ as the union of all faces of $\Phi(\omega)$ that have a vertex at distance strictly smaller than R from the root. Since the tree ω has only finitely many vertices with labels smaller than $R+1$, there are finitely many such faces and $B_R(\Phi(\omega))$ is a finite map. Therefore $\mathbb{S}^2 \setminus B_R(\Phi(\omega))$ has finitely many connected components; and the boundaries of these components are finite length cycles of $\Phi(\omega)$.

Let γ be such a cycle. Each edge of γ is adjacent to two faces of $\Phi(\omega)$. One has a vertex at distance strictly smaller than R from the root, and the other one has only vertices at distance at least R from the root. The quadrangulation being bipartite, each edge of γ connects a vertex at distance R from the root with a vertex at distance $R+1$ from the root. Therefore, by adding to $B_R(\Phi(\omega))$ an extra vertex in the connected component of $\mathbb{S}^2 \setminus B_R(\Phi(\omega))$ bounded by γ and an edge between this vertex and each vertex of γ at distance $R+1$ from the root, and repeating this operation for every connected component of $\mathbb{S}^2 \setminus B_R(\Phi(\omega))$, we obtain a finite quadrangulation. The sequence of finite quadrangulations obtained in this way for every $R > 0$ converges to $\Phi(\omega)$ as R goes to infinity, in the sense of Krikun, showing that for every tree $\omega \in \mathscr{S}$, $\Phi(\omega)$ can be identified with an element of $\overline{\mathbf{Q}}$.

To be able to consider μ_Φ as a measure on $\overline{\mathbf{Q}}$, we now need to verify that the mapping $\Phi : \mathscr{S} \to \overline{\mathbf{Q}}$ is measurable with respect to the Borel σ-field of $\left(\overline{\mathbf{Q}}, d_{\mathbf{Q}}\right)$. The following lemma is proved in Section 2.4.1:

Lemma 1. *Fix $R > 0$ and $\omega_0 \in \mathscr{S}$. The set $A = \{\omega \in \mathscr{S} : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\}$ is measurable with respect to the Borel σ-field of $(\mathscr{S}, d_{\overline{\mathbb{T}}})$.*

Fix $Q^\star \in \overline{\mathbf{Q}}$. Lemma 1 implies that

$$\Phi^{-1}\left(\left\{Q \in \overline{\mathbf{Q}} : d_{\mathbf{Q}}(Q, Q^\star) \leqslant \frac{1}{R+1}\right\}\right) = \Phi^{-1}\left(\left\{Q \in \overline{\mathbf{Q}} : B_{\overline{\mathbf{Q}},R}(Q) = B_{\overline{\mathbf{Q}},R}(Q^\star)\right\}\right)$$

is measurable with respect to the Borel σ-field of $(\mathscr{S}, d_{\overline{\mathbb{T}}})$, proving that $\Phi : \mathscr{S} \to \overline{\mathbf{Q}}$ is measurable. Therefore, we may and will see μ_Φ as a probability measure on $\left(\overline{\mathbf{Q}}, d_{\mathbf{Q}}\right)$.

We are now ready to state our main result:

Theorem 4. *The sequence $(\mu_{\Phi,n})_{n\in\mathbb{N}}$ converges weakly to μ_Φ in the space of all probability measures on $\left(\overline{\mathbf{Q}}, d_{\mathbf{Q}}\right)$. Therefore μ_ϕ viewed as a probability measure on $\left(\overline{\mathbf{Q}}, d_{\mathbf{Q}}\right)$ coincides with ν.*

Since $\mu_{\Phi,n} = \nu_n$ and ν is defined as the limit of the sequence (ν_n) in the space of all probability measures on $\left(\overline{\mathbf{Q}}, d_{\mathbf{Q}}\right)$, the second assertion is a direct consequence of the first one.

To establish the first assertion, we have to show that for every $Q^\star \in \overline{\mathbf{Q}}$ and $R > 0$ one has

$$\mu_{\Phi,n}\Big(Q \in \overline{\mathbf{Q}} : B_{\overline{\mathbf{Q}},R}(Q) = B_{\overline{\mathbf{Q}},R}(Q^\star)\Big) \underset{n\to\infty}{\longrightarrow} \mu_\Phi\Big(Q \in \overline{\mathbf{Q}} : B_{\overline{\mathbf{Q}},R}(Q) = B_{\overline{\mathbf{Q}},R}(Q^\star)\Big).$$

The remaining part of this work is devoted to the proof of this convergence.

2.4.1 A property of Schaeffer's correspondence

For every integers $S > 0$ and $R > 0$ we let

$$\Omega_S(R) = \{\omega \in \overline{\mathbb{T}} : \omega \text{ has a label} \leqslant R+1 \text{ strictly above generation } S\}. \tag{2.6}$$

In the first two statements of this section, S and R are two fixed positive integers.

Proposition 4. *Let ω be a tree of $\mathscr{S}$ which does not belong to $\Omega_S(R)$ (i.e. ω is such that the label of every vertex at a generation strictly greater than S is at least $R+2$). Then $B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(B_{\overline{\mathbb{T}},S}(\omega)))$.*

Proof. The proof follows step by step the construction of $\Phi(\omega)$ in Section 2.2.3. Fix an embedding of ω as an infinite planar map.

In the first step, an infinite planar map $\mathcal{M}_0(\omega)$ is obtained from ω by adding an extra vertex v_0 with label 0 and edges between v_0 and corners with label 1. Similarly, we can construct a planar map $\mathcal{M}_0(B_{\overline{\mathbb{T}},S}(\omega))$. The extra edges in these two maps are uniquely determined by corners with label 1, and these corners are determined by $B_{\overline{\mathbb{T}},S}(\omega)$ (no vertex at a generation greater than S has a label less than 2). We consider the unique "infinite face" of $\mathcal{M}_0$ as an extra face. The maps $\mathcal{M}_0(\omega)$ and $\mathcal{M}_0(B_{\overline{\mathbb{T}},S}(\omega))$ then have the same number of faces, say p, which in addition have the same boundaries, composed by the two edges joining v_0 to the corners with label 1 and, in the case when these corners with label 1 belong to different vertices, the unique injective path in the tree between these two vertices. Let $F_1(\omega), \dots, F_p(\omega)$ and $F_1(B_{\overline{\mathbb{T}},S}(\omega)), \dots, F_p(B_{\overline{\mathbb{T}},S}(\omega))$ denote the faces of $\mathcal{M}_0(\omega)$ and $\mathcal{M}_0(B_{\overline{\mathbb{T}},S}(\omega))$ respectively, listed in such a way that, for every i, the faces $F_i(\omega)$ and $F_i(B_{\overline{\mathbb{T}},S}(\omega))$ have the same boundary.

In the second step, edges $(c, s(c))$ are added inside each face for every corner c, finally giving two regular planar maps $\mathcal{M}_1(\omega)$ and $\mathcal{M}_1(B_{\overline{\mathbb{T}},S}(\omega))$. Let us consider a face $F_i(\omega)$ of $\mathcal{M}_0(\omega)$ and the corresponding face $F_i(B_{\overline{\mathbb{T}},S}(\omega))$. The corners of these faces are numbered $(c_{i,j})_{j\in J_i}$ for $F_i(\omega)$ and $(c'_{i,j})_{j\in J'_i}$ for $F_i(B_{\overline{\mathbb{T}},S}(\omega))$, the numbering being in clockwise order for a finite face and counterclockwise order for corners on the left side of the spine of the tree, clockwise order for corners on the right side of the spine of the tree, in the case of the infinite face.

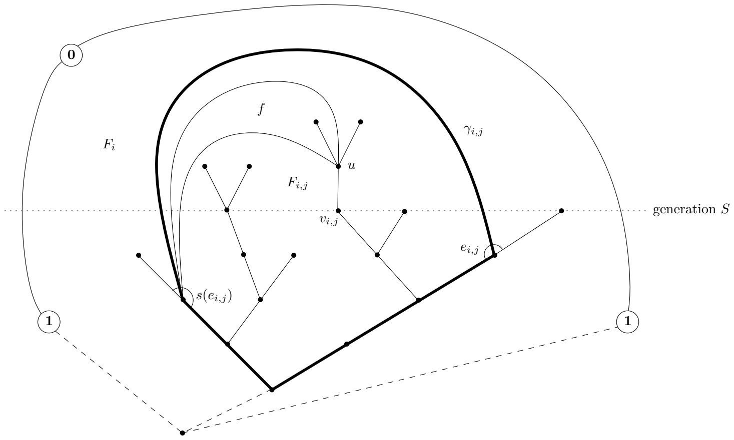

Figure 2.4: A face F_i and a cycle $\gamma_{i,j}$ associated with a vertex $v_{i,j}$ at generation S.

For every $i \in \{1, \ldots, p\}$, let $v_{i,1}, \ldots, v_{i,k_i}$ be the vertices of $F_i(\omega)$ at generation S that have at least one child. These vertices are also vertices of $F_i(B_{\overline{\mathbb{T}},S}(\omega))$ at generation S and their labels are greater than $R+1$. For every $j \leqslant k_i$, let $e_{i,j}$ be the last corner before $v_{i,j}$ in $F_i(\omega)$ and with label $R+1$. This corner is the same in $F_i(\omega)$ and $F_i(B_{\overline{\mathbb{T}},S}(\omega))$. The same edge $(e_{i,j}, s(e_{i,j}))$ joining $e_{i,j}$ to the first corner following $e_{i,j}$ with label R is thus added to $F_i(\omega)$ and $F_i(B_{\overline{\mathbb{T}},S}(\omega))$ (note that this corner is also the first corner with label R following every corner of $v_{i,j}$, see Figure 2.4).

Therefore for every $j \in \{1, \ldots k_i\}$ the same cycle $\gamma_{i,j}$ composed by the edge $(e_{i,j}, s(e_{i,j}))$ and the genealogical path between $e_{i,j}$ and $s(e_{i,j})$ appears in both $F_i(\omega)$ and $F_i(B_{\overline{\mathbb{T}},S}(\omega))$ (see Figure 2.4). For $j \neq j'$ the (strict) interiors of the cycles $\gamma_{i,j}$ and $\gamma_{i,j'}$ are either disjoint, or one of them is contained in the other one. Here we define the interior of a cycle as the connected component of the complement of this cycle which does not contain v_0.

Let us now show that if a face f of $\Phi(\omega)$ intersects the interior of a cycle $\gamma_{i,j}$, the labels of vertices of f are greater than or equal to R. We first deal with the case when f has a vertex u that belongs to the interior of the cycle $\gamma_{i,j}$. If the label of u is greater than or equal to $R+2$ the conclusion is obvious. If not, the label of u is $R+1$ and for f to have a vertex with label $R-1$, u must be connected to vertices with label R by two edges: the only possible choice for a vertex with label R is $s(e_{i,j})$ and the last vertex of f would then belong to the domain bounded by the union of the two edges connecting u to $s(e_{i,j})$ (see Figure 2.4) so that its label could not be $R-1$. The case when no vertex of f belongs to the interior of the cycle $\gamma_{i,j}$ is treated in a similar manner.

The previous discussion shows that faces of $\Phi(\omega)$, respectively of $\Phi(B_{\overline{\mathbb{T}},S}(\omega))$, that intersect the interior of a cycle $\gamma_{i,j}$, are not taken into account in the definition of $B_{\overline{\mathbf{Q}},R}(\Phi(\omega))$, respectively of $B_{\overline{\mathbf{Q}},R}(\Phi(B_{\overline{\mathbb{T}},S}(\omega)))$.

Let us denote by $\Phi_1(\omega)$ the planar map obtained after the second step of the construction of $\Phi(\omega)$ (this map is denoted by $\mathcal{M}_1$ in Section 2.2.3). Let us consider the map $\widetilde{\Phi}_1(\omega)$ obtained by removing every edge and vertex of $\Phi_1(\omega)$ lying in the interior of a cycle $\gamma_{i,j}$. By construction, every vertex of ω with generation strictly greater than S belongs to the interior of a cycle $\gamma_{i,j}$. It follows that

$$\widetilde{\Phi}_1(\omega) = \widetilde{\Phi}_1\left(B_{\overline{\mathbb{T}},S}(\omega)\right). \tag{2.7}$$

Finally, let $\Phi_2(\omega)$ denote the map obtained by removing every edge of $\widetilde{\Phi}_1(\omega)$ connecting two vertices of $\widetilde{\Phi}_1(\omega)$ with the same label less than or equal to R. Every face of $\Phi(\omega)$ that is taken into account in the ball $B_{\overline{\mathbf{Q}},R}(\Phi(\omega))$ is also a quadrangular face of $\Phi_2(\omega)$. Conversely, every quadrangular face of $\Phi_2(\omega)$ having a vertex with a label strictly smaller than R is also a face of $B_{\overline{\mathbf{Q}},R}(\Phi(\omega))$. In other words, the ball $B_{\overline{\mathbf{Q}},R}(\Phi(\omega))$ is the union of the quadrangular faces of $\Phi_2(\omega)$ having a vertex with a label strictly smaller than R. From (2.7) we have

$$\Phi_2(\omega) = \Phi_2\left(B_{\overline{\mathbb{T}},S}(\omega)\right),$$

and the previous observations allow us to conclude that

$$B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega)\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi\left(B_{\overline{\mathbb{T}},S}(\omega)\right)\right)$$

which completes the proof. □

Corollary 1. *Let $\omega_0 \in \mathscr{S}$. There exists a countable collection $\left(\omega_i^{S,R}\right)_{i\in I}$ of trees in $\mathscr{S} \cap \Omega_S(R)^c$ verifying for all $i \in I$*

$$B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega_i^{S,R})\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega_0)\right)$$

and such that for every $\omega \in \mathscr{S} \cap \Omega_S(R)^c$ the following assertions are equivalent:

1. $B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega)\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega_0)\right)$;
2. *there exists $i \in I$ such that $B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i^{S,R})$.*

Proof. The collection $\left(\omega_i^{S,R}\right)_{i\in I}$ that consists of all finite trees ω' having at most S generations and such that $B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega')\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega_0)\right)$ is countable and has the desired properties. Indeed, if $\omega \in \mathscr{S} \cap \Omega_S(R)^c$ and if there exists $i \in I$ such that $B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i^{S,R})$, Proposition 4 ensures that $B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega)\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega_i^{S,R})\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega_0)\right)$. Conversely, if $B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega)\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega_0)\right)$ and $\omega \in \mathscr{S} \cap \Omega_S(R)^c$, then $\omega' = B_{\overline{\mathbb{T}},S}(\omega)$ verifies $B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega)\right) = B_{\overline{\mathbf{Q}},R}\left(\Phi(\omega')\right)$ by Proposition 4 and ω' belongs to the collection $\left(\omega_i^{S,R}\right)_{i\in I}$. □

We conclude this section with the proof of Lemma 1.

Proof of Lemma 1. Fix $R > 0$. For every $S > 0$ the set $\Omega_S(R)$ is open and closed in $\overline{\mathbb{T}}$. In addition one has

$$\begin{aligned} A &= \bigcup_{S>0} \left(\left\{\omega \in \mathscr{S} :\, B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\right\} \cap \Omega_S(R)^c\right) \\ &= \bigcup_{S>0} \bigcup_{i\in I_{S,R}} \left(\left\{\omega \in \mathscr{S} :\, B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i^{S,R})\right\} \cap \Omega_S(R)^c\right) \end{aligned}$$

where $\left(\omega_i^{S,R}\right)_{i\in I_{S,R}}$ is the collection given by Corollary 1. This shows that the set A is measurable. □

2.4.2 Asymptotic behavior of labels on the spine

Recall that the sequence $(Y_k)_{k\geqslant 0}$ of the successive labels of vertices of the spine is a Markov chain with transition matrix Π given by Theorem 3. In this section we study the asymptotic behavior of this Markov chain.

Lemma 2. *The Markov chain $(Y_k)_{k\geqslant 0}$ is transient. In addition, for every $\varepsilon > 0$ there exists $\alpha > 0$ such that for k large enough one has*

$$\mathbb{P}\left[Y_j \geqslant \alpha k, \forall j \geqslant 0 \,|\, Y_0 = k\right] \geqslant 1 - \varepsilon.$$

Proof. The Taylor expansion $\frac{q_k}{p_k} = 1 - \frac{8}{k} + O\left(\frac{1}{k^2}\right)$ ([CD06], Lemma 5.5) implies that there exists $C > 0$ such that

$$\prod_{i=2}^{k} \frac{q_i}{p_i} \underset{k\to\infty}{\sim} Ck^{-8}.$$

A standard argument for birth and death processes then ensures that Y is transient. Furthermore, for every $k > j \geqslant 1$,

$$\mathbb{P}_k\left[T_j = \infty\right] = \frac{\sum_{i=j}^{k-1} \frac{q_i}{p_i}\frac{q_{i-1}}{p_{i-1}} \cdots \frac{q_{j+1}}{p_{j+1}}}{\sum_{i=j}^{\infty} \frac{q_i}{p_i}\frac{q_{i-1}}{p_{i-1}} \cdots \frac{q_{j+1}}{p_{j+1}}}$$

where T_j is the hitting time of j. Therefore one has, for $\alpha < 1$,

$$\begin{aligned}\mathbb{P}_k\left[T_{[\alpha k]} = \infty\right] &= \frac{\frac{1}{k}\sum_{i=[\alpha k]}^{k-1} \frac{q_i}{p_i}\frac{q_{i-1}}{p_{i-1}} \cdots \frac{q_{[\alpha k]+1}}{p_{[\alpha k]+1}}}{\frac{1}{k}\sum_{i=[\alpha k]}^{\infty} \frac{q_i}{p_i}\frac{q_{i-1}}{p_{i-1}} \cdots \frac{q_{[\alpha k]+1}}{p_{[\alpha k]+1}}}\\ &\underset{k\to\infty}{\longrightarrow} \frac{\int_\alpha^1 \left(\frac{\alpha}{t}\right)^8 dt}{\int_\alpha^\infty \left(\frac{\alpha}{t}\right)^8 dt} = 1 - \alpha^7.\end{aligned}$$

The desired result follows. $\square$

Proposition 5. *Let Z be a nine-dimensional Bessel process started at 0. Then*

$$\left(\frac{1}{\sqrt{n}} Y_{[nt]}\right)_{t\geqslant 0} \underset{n\to\infty}{\longrightarrow} \left(Z_{\frac{2}{3}t}\right)_{t\geqslant 0}$$

in the sense of convergence in distribution in the space $D(\mathbb{R}_+, \mathbb{R}_+)$.

Proof. The convergence in the proposition is a direct consequence of a more general result by Lamperti [Lam62] which we now recall. Let $(X_n)_{n\geqslant 0}$ be a time-homogeneous Markov chain on $\mathbb{R}_+$ verifying:

1. for every $K > 0$ one has uniformly in $x \in \mathbb{R}_+$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{P}\left(X_i \leqslant K \,|\, X_0 = x\right) = 0;$$

2. for every $k \in \mathbb{N}$ the following moments exist and are bounded as functions of $x \in \mathbb{R}_+$

$$m_k(x) = \mathbb{E}\left[(X_{n+1} - X_n)^k \middle| X_n = x\right];$$

3. there exist $\beta > 0$ and $\alpha > -\beta/2$ such that

$$\lim_{x\to\infty} m_2(x) = \beta,$$
$$\lim_{x\to\infty} x\, m_1(x) = \alpha.$$

Let us define the process $\left(x_t^{(n)}\right)_{t\in\mathbb{R}_+}$ by $x_t^{(n)} = n^{-1/2}X_i$ if $t = \frac{i}{n}$, $i = 0, 1, 2, \ldots$, and linear interpolation on intervals of the form $\left[\frac{i-1}{n}, \frac{i}{n}\right]$. Lamperti's theorem states that $\left(x_t^{(n)}\right)_{t\in\mathbb{R}_+}$ converges in distribution to the diffusion process $(x_t)_{t\in\mathbb{R}_+}$ with generator

$$L = \frac{\alpha}{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\beta}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

In our case, we consider the Markov chain $\widetilde{Y}$ whose transition matrix is given by $\widetilde{\Pi}(x, y) = \Pi([x], [y])$ if $y = x+1, x-1$ or x. Assertion 1 easily follows from Lemma 2 and Assertion 2 is trivial. In addition one has $p_n = \frac{1}{3} + \frac{4}{3n} + O(n^{-2})$ and $q_n = \frac{1}{3} - \frac{4}{3n} + O(n^{-2})$ ([CD06], Lemma 5.5) giving:

$$\lim_{x\to\infty} m_2(x) = \frac{2}{3},$$
$$\lim_{x\to\infty} x\, m_1(x) = \frac{8}{3}.$$

and therefore Assertion 3 holds with $\alpha = 8/3$ and $\beta = 2/3$.

The rescaled chain Y thus converges in law to the diffusion process with generator

$$L = \frac{2}{3}\left(\frac{4}{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right),$$

which was the desired result. □

2.4.3 Asymptotic properties of small labels

Thanks to Corollary 1, the proof of Theorem 4 will reduce to showing that the μ_n-measure of certain balls in the space of trees converges to the corresponding μ-measure. Still we need to show that the error made by disregarding trees that belong to $\Omega_S(R)$ is small when S is large. In this section we fix $R, \varepsilon > 0$ and we write $\Omega_S = \Omega_S(R)$ to simplify notation.

Lemma 3. *There exists an integer $S^\star > 0$ such that $\mu(\Omega_S) \leqslant \varepsilon$ for every $S > S^\star$.*

Proof. Let $\Omega = \bigcap_{S=1}^{\infty} \Omega_S$. If $\omega \in \Omega$ then ω has infinitely many vertices with labels in $\{1, \ldots, R+1\}$ and there exists $l \in \{1, \ldots, R+1\}$ such that $N_l(\omega) = \infty$. Since μ is supported on $\mathscr{S}$, one has $\mu(\Omega) = 0$ and therefore $\mu(\Omega_S) \to 0$ as $S \to \infty$. □

The main ingredient of the proof of Theorem 4 is Proposition 6, which gives an analog of Lemma 3 when μ is replaced by μ_n, with *uniformity* in n. To establish this estimate, we will need an upper bound for the probability that there exists a vertex at generation S with label smaller than S^α, where $\alpha < 1/2$ is fixed (see Lemma 5 below). Let us first give an easy preliminary lemma:

Lemma 4. *Fix $S > 0$. There exist positive integers $N_1(S)$ and $K_\varepsilon(S)$ such that for every $n > N_1(S)$:*

$$\mu_n\left(\omega : \left|B_{\overline{\mathbb{T}},S}(\omega)\right| > K_\varepsilon(S)\right) < \varepsilon.$$

Proof. This result is a direct consequence of the convergence of the measures μ_n to μ. Indeed, as $\left|B_{\overline{\mathbb{T}},S}(\omega)\right|$ is finite for every tree ω, one can choose $K_\varepsilon(S)$ large enough such that

$$\mu\left(\omega : \left|B_{\overline{\mathbb{T}},S}(\omega)\right| > K_\varepsilon(S)\right) < \varepsilon.$$

The convergence of μ_n to μ then gives $N_1(S)$ such that the inequality of the lemma is true for $n > N_1(S)$. □

There exists a finite number of well-labelled trees with height exactly S and having at most K_ε edges. Let us denote this number by $M_\varepsilon(S)$.

For every $S > 0$ and $\alpha \in \left[0, \frac{1}{2}\right[$ we let

$$A_\alpha(S) = \left\{\omega \in \overline{\mathbb{T}} : \omega \text{ has a vertex at generation } S \text{ with a label } \leqslant S^\alpha\right\}.$$

Lemma 5. *Fix $\alpha < \frac{1}{2}$. For every sufficiently large integer S, there exists $N_2(S)$ such that, for every $n > N_2(S)$, one has*

$$\mu_n\left(A_\alpha(S)\right) < \varepsilon.$$

Proof. We first observe that it is enough to prove the bound $\mu\left(A_\alpha(S)\right) < \varepsilon$ when S is large. Indeed the set $A_\alpha(S)$ is closed in $\overline{\mathbb{T}}$, and thus we have $\limsup_n \mu_n(A_\alpha(S)) \leqslant \mu(A_\alpha(S))$.

Recall the notation $\rho^{(l)}$ and $\widehat{\rho}^{(l)}$ introduced in Section 2.3.2. For $H > 0$ and $l > 0$ one has

$$\widehat{\rho}^{(l)}\left(h(\omega) > H\right) = \frac{1}{w_l}\sum_{\substack{\omega\in\mathbb{T}^{(l)}\\ h(\omega)>H}} 12^{-|\omega|} \leqslant \frac{1}{w_l}\sum_{\substack{\omega\in\mathbf{T}^{(l)}\\ h(\omega)>H}} 12^{-|\omega|} = \frac{1}{w_l}\rho^{(l)}\left(h(\omega) > H\right).$$

Therefore

$$\widehat{\rho}^{(l)}\left(h(\omega) > H\right) \leqslant \frac{2}{w_l}\mathbb{P}_{GW(1/2)}\left[h(\omega) > H\right],$$

where $\mathbb{P}_{GW(1/2)}$ is the law of a Galton–Watson tree whose offspring distribution is geometric with parameter 1/2. Theorem 1 (page 19) of [AN72] gives

$$\lim_{H\to\infty} H\,\mathbb{P}_{GW(1/2)}\left[h(\omega) > H\right] = 1.$$

From the explicit formula for w_l we have $\frac{2}{w_l} \leqslant \frac{3}{2}$ for every $l \geqslant 0$. Hence there exists $H_1 > 0$ such that for $H > H_1$

$$\widehat{\rho}^{(l)}\left(h(\omega) > H\right) \leqslant \frac{2}{H}.$$

Fix $\eta \in \left]0, \frac{1}{2}\right[$. Recall that $g_S(\omega)$ is the set of vertices of ω at generation S and that, for every integer k, L_k and R_k are the subtrees of ω attached respectively to the left side and to the right side of the k-th vertex of the spine of ω. For $S > (1-\eta)^{-1}H_1$, Theorem 3 and the previous bound give

$$\mu\left[g_S(\omega) \cap \bigcup_{0 \leqslant k \leqslant [\eta S]-1} (L_k \cup R_k) \neq \emptyset\right] \leqslant 2 \sum_{k=1}^{[\eta S]-1} \frac{2}{S-k} \leqslant 4\frac{\eta}{1-\eta} \leqslant 8\eta,$$

and therefore

$$\mu\left(A_\alpha(S)\right) \leqslant 8\eta + \mu\left(\exists s \in g_S(\omega) \cap \bigcup_{k=[\eta S]}^{S} (L_k \cup R_k) \,:\, \ell(s) \leqslant S^\alpha\right).$$

Applying the Markov property at time $[\eta S]$ to the Markov chain Y and then using Proposition 5 and Lemma 2 we find $\delta > 0$ and S_1 such that for $S > S_1$ one has

$$\mu\left(Y_k \geqslant [\delta\sqrt{S}],\, \forall k \geqslant [\eta S]\right) \geqslant 1-\eta.$$

We now have

$$\mu\left(A_\alpha(S)\right) \leqslant 9\eta + \mu\left(\left\{\exists s \in g_S(\omega) \cap \bigcup_{k=[\eta S]}^{S} (L_k \cup R_k) : \ell(s) \leqslant S^\alpha\right\} \cap \left\{\forall k \geqslant [\eta S], Y_k \geqslant [\delta\sqrt{S}]\right\}\right). \tag{2.8}$$

Let us fix a collection $(y_k)_{[\eta S] \leqslant k \leqslant S}$ such that $y_k \geqslant [\delta\sqrt{S}]$ for every k. Theorem 3 gives

$$\begin{aligned}&\mu\left(\exists s \in g_S(\omega) \cap \bigcup_{k=[\eta S]}^{S} (L_k \cup R_k) \,:\, \ell(s) \leqslant S^\alpha \middle| Y_k = y_k,\, [\eta S] \leqslant k \leqslant S\right)\\ &\quad\leqslant 2\sum_{k=[\eta S]}^{S} \widehat{\rho}^{(y_k)}\left(\exists s \in g_{S-k}(\omega) \,:\, \ell(s) \leqslant S^\alpha\right) = 2\sum_{k=0}^{S-[\eta S]} \widehat{\rho}^{(y_{S-k})}\left(\exists s \in g_k(\omega) \,:\, \ell(s) \leqslant S^\alpha\right).\end{aligned} \tag{2.9}$$

If $0 \leqslant k \leqslant S - [\eta S]$, one has:

$$\begin{aligned}\widehat{\rho}^{(y_{S-k})}\left(\exists s \in g_k(\omega) \,:\, \ell(s) \leqslant S^\alpha\right) &\leqslant \widehat{\rho}^{(y_{S-k})}\left(\inf_{s\in\omega} \ell(s) \leqslant S^\alpha\right)\\ &= \frac{1}{w_{y_{S-k}}} \sum_{\substack{\omega \in \mathbb{T}^{(y_{S-k})}\\ \inf_{s\in\omega} \ell(s) \leqslant S^\alpha}} 12^{-|\omega|}\\ &= \frac{1}{w_{y_{S-k}}} \sum_{\substack{\omega \in \mathbf{T}^{(y_{S-k})}\\ 0<\inf_{s\in\omega} \ell(s) \leqslant S^\alpha}} 12^{-|\omega|}\\ &= \frac{1}{w_{y_{S-k}}} \rho^{(y_{S-k})}\left(0 < \inf_{s\in\omega} \ell(s) \leqslant S^\alpha\right).\end{aligned}$$

But

$$\rho^{(y_{S-k})}\left(\inf_{s\in\omega}\ell(s)>0\right)=w_{y_{S-k}}$$

and

$$\rho^{(y_{S-k})}\left(\inf_{s\in\omega}\ell(s)>S^\alpha\right)=\rho^{(y_{S-k}-[S^\alpha])}\left(\inf_{s\in\omega}\ell(s)>0\right)=w_{y_{S-k}-[S^\alpha]}.$$

We thus have

$$\widehat{\rho}^{(y_{S-k})}\left(\exists s\in g_k(\omega):\ \ell(s)\leqslant S^\alpha\right)\leqslant\frac{1}{w_{y_{S-k}}}\left(w_{y_{S-k}}-w_{y_{S-k}-[S^\alpha]}\right)=1-\frac{w_{y_{S-k}-[S^\alpha]}}{w_{y_{S-k}}}.$$

Using our assumption $y_{S-k}\geqslant[\delta\sqrt{S}]$, a Taylor expansion gives

$$1-\frac{w_{y_{S-k}-[S^\alpha]}}{w_{y_{S-k}}}=4S^\alpha y_{S-k}^{-3}+o\left(S^\alpha y_{S-k}^{-3}\right)\leqslant\frac{4}{\delta^3}S^{\alpha-\frac{3}{2}}+o\left(S^{\alpha-\frac{3}{2}}\right),$$

and the right-hand side of (2.9) is smaller than $\frac{8}{\delta^3}S^{\alpha-1/2}+o\left(S^{\alpha-1/2}\right)$. From (2.8) we now get

$$\mu\left(A_\alpha(S)\right)\leqslant 9\eta+\frac{8}{\delta^3}S^{\alpha-\frac{1}{2}}+o\left(S^{\alpha-\frac{1}{2}}\right).$$

Hence $\mu(A_\alpha(S))<10\eta$ as soon as S is large enough. This completes the proof. □

Proposition 6. *For every sufficiently large integer S, there exists an integer $N(S)$ such that, for every $n>N(S)$, one has*

$$\mu_n(\Omega_S)\leqslant\varepsilon.$$

Proof. In this proof $\alpha\in\left]\frac{1}{3},\frac{1}{2}\right[$ is fixed. For $S>0$, Lemma 4 gives $K_\varepsilon(S)>0$ and $N_1(S)>0$ such that if $n>N_1(S)$ then $\mu\left(\omega:\ |B_{\overline{\mathbb{T}},S}(\omega)|>K_\varepsilon(S)\right)<\varepsilon$. Let us also recall that the number of well-labelled trees with height S and size smaller than $K_\varepsilon(S)$ is denoted by $M_\varepsilon(S)$. Lemma 5 shows that, for S large enough, there exists $N_2(S)$ such that $\mu_n(A_\alpha(S))<\varepsilon$ for every $n>N_2(S)$. Therefore, for S large enough and for $n>N_1(S)\vee N_2(S)$ one has

$$\begin{aligned}\mu_n(\Omega_S)=&\sum_{\substack{\omega^\star\notin A_\alpha(S)\\|\omega^\star|\leqslant K_\varepsilon(S),\,h(\omega^\star)=S}}\mu_n\left(\{\omega:\ B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\}\cap\Omega_S\right)\\&+\mu_n(A_\alpha(S))+\mu_n\left(\omega:\ |B_{\overline{\mathbb{T}},S}(\omega)|>K_\varepsilon(S)\right)\\\leqslant&\ 2\varepsilon+\sum_{\substack{\omega^\star\notin A_\alpha(S)\\|\omega^\star|\leqslant K_\varepsilon(S),\,h(\omega^\star)=S}}\mu_n\left(\{\omega:\ B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\}\cap\Omega_S\right).\end{aligned}\tag{2.10}$$

Fix a tree $\omega^\star\notin A_\alpha(S)$ with height S and size smaller than $K_\varepsilon(S)$. We assume that S is large enough so that $S^\alpha>R+1$. We denote by k the number of vertices of $\omega^\star$ at generation S and by $l_1,\ldots,l_k$ the labels of these vertices. By considering the subtrees of ω originating from vertices at generation S, one obtains:

$$\mu_n\left(\{\omega:\ B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\}\cap\Omega_S\right)\leqslant\frac{1}{D_n}\sum_{n_1+\cdots+n_k=n-|\omega^\star|}\sum_{i=1}^{k}D_{n_i}^{(l_i)}(R)\prod_{j\neq i}D_{n_j}^{(l_j)}\tag{2.11}$$

where $D_n^{(l)}(R)$ is the number of trees in $\mathbb{T}_n^{(l)}$ with at least one vertex with a label less than or equal to $R+1$ (compare (2.11) with formula (2.2)). Since $\omega \notin A_\alpha(S)$, we have $l_i > S^\alpha > R+1$ and thus $D_{n_i}^{(l_i)}(R) = D_{n_i}^{(l_i)} - D_{n_i}^{(l_i-R-1)}$ for $i = 1, \dots, k$. The bound (2.11) then gives

$$\begin{aligned}
&\mu_n \left(\{\omega : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\} \cap \Omega_S \right) \\
&\quad \leqslant \frac{1}{D_n} \sum_{n_1+\cdots+n_k=n-|\omega^\star|} \sum_{i=1}^{k} (D_{n_i}^{(l_i)} - D_{n_i}^{(l_i-R-1)}) \prod_{j\neq i} D_{n_j}^{(l_j)} \\
&\quad = \frac{k}{D_n} \sum_{n_1+\cdots+n_k=n-|\omega^\star|} \prod_{j=1}^{k} D_{n_j}^{(l_j)} - \frac{1}{D_n} \sum_{i=1}^{k} \sum_{n_1+\cdots+n_k=n-|\omega^\star|} D_{n_i}^{(l_i-R-1)} \prod_{j\neq i} D_{n_j}^{(l_j)}.
\end{aligned}$$

By Theorem 2, $\mu_n \left(\omega : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\right) \to \mu \left(\omega : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\right)$ as $n \to \infty$. Using this convergence and identities (2.2) and (2.3), we get the existence of an integer $N(\omega^\star, S)$ such that for $n > N(\omega^\star, S)$ one has

$$\frac{1}{D_n} \sum_{n_1+\cdots+n_k=n-|\omega^\star|} \prod_{j=1}^{k} D_{n_j}^{(l_j)} \leqslant 12^{-|\omega^\star|} \sum_{t=1}^{k} d_{l_t} \prod_{s\neq t} w_{l_s} + \frac{\varepsilon}{K_\varepsilon(S) M_\varepsilon(S)}$$

and for $i = 1, \dots, k$

$$\begin{aligned}
&\frac{1}{D_n} \sum_{n_1+\cdots+n_k=n-|\omega^\star|} D_{n_i}^{(l_i-R-1)} \prod_{j\neq i} D_{n_j}^{(l_j)} \\
&\qquad \geqslant 12^{-|\omega^\star|} \left(d_{l_i-R-1} \prod_{j\neq i} w_{l_j} + \sum_{t\neq i} d_{l_t} w_{l_i-R-1} \prod_{j\neq t,i} w_{l_j} \right) - \frac{\varepsilon}{K_\varepsilon(S) M_\varepsilon(S)}.
\end{aligned}$$

We now have for every $n > N(\omega^\star, S)$:

$$\begin{aligned}
&\mu_n \left(\{\omega : B_{\overline{\mathbb{T}},S}(\omega) = \omega^\star\} \cap \Omega_S \right) \\
&\quad \leqslant \frac{2\varepsilon}{M_\varepsilon(S)} + k 12^{-|\omega^\star|} \sum_{t=1}^{k} d_{l_t} \prod_{s\neq t} w_{l_s} - 12^{-|\omega^\star|} \sum_{i=1}^{k} \left(d_{l_i-R-1} \prod_{j\neq i} w_{l_j} + \sum_{t\neq i} d_{l_t} w_{l_i-R-1} \prod_{j\neq t,i} w_{l_j} \right) \\
&\quad = \frac{2\varepsilon}{M_\varepsilon(S)} + 12^{-|\omega^\star|} \sum_{t=1}^{k} (d_{l_t} - d_{l_t-R-1}) \prod_{s\neq t} w_{l_s} \\
&\qquad\qquad + 12^{-|\omega^\star|} \sum_{t=1}^{k} d_{l_t} \left(\sum_{i\neq t} (w_{l_i} - w_{l_i-R-1}) \prod_{s\neq t,i} w_{l_s} \right). \qquad (2.12)
\end{aligned}$$

Define

$$\begin{aligned}
d(\omega^\star) &= \max_{i=1\dots k} \left(1 - \frac{d_{l_i-R-1}}{d_{l_i}} \right), \\
w(\omega^\star) &= \max_{i=1\dots k} \left(1 - \frac{w_{l_i-R-1}}{w_{l_i}} \right).
\end{aligned}$$

From the bound (2.12), we get

$$\begin{aligned}
&\mu_n\left(\{\omega :\, B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\}\cap\Omega_S\right)\\
&\leqslant \frac{2\varepsilon}{M_\varepsilon(S)}+d(\omega^\star)\,12^{-|\omega^\star|}\sum_{t=1}^{k}d_{l_t}\prod_{s\neq t}w_{l_s}+kw(\omega^\star)\,12^{-|\omega^\star|}\sum_{t=1}^{k}d_{l_t}\prod_{s\neq t}w_{l_s}\\
&=\frac{2\varepsilon}{M_\varepsilon(S)}+\left(d(\omega^\star)+kw(\omega^\star)\right)\mu\left(\omega : B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\right) \qquad (2.13)
\end{aligned}$$

where we used (2.3) in the last equality.

Let us now define $N^\star(S)=\max_{|\omega^\star|\leqslant K_\varepsilon(S)}N(\omega^\star,S)\vee N_1(S)\vee N(S)$. For S large enough and for $n>N^\star(S)$ we obtain using (2.10) and (2.13):

$$\mu_n(\Omega_S)\leqslant 4\varepsilon+\sum_{\substack{\omega^\star\notin A_\alpha(S)\\ |\omega^\star|\leqslant K_\varepsilon(S),\,h(\omega^\star)=S}}\left(d(\omega^\star)+|g_S(\omega^\star)|w(\omega^\star)\right)\mu\left(\omega : B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\right).$$

A Taylor expansion gives $w(\omega^\star)\leqslant 4(5R+2)S^{-3\alpha}+o\left(S^{-3\alpha}\right)$ where the remainder is uniform over $\omega^\star\notin A_\alpha(S)$. In addition, $\sup_{\omega^\star\notin A_\alpha(S)}d(\omega^\star)\to 0$ as $S\to\infty$. This allows us to find $S^\star$ such that for $S>S^\star$ and $n>N^\star(S)$:

$$\begin{aligned}
\mu_n(\Omega_S)&\leqslant 4\varepsilon+\sum_{\substack{\omega^\star\notin A_\alpha(S)\\ |\omega^\star|\leqslant K_\varepsilon(S),\,h(\omega^\star)=S}}\left(\varepsilon+|g_S(\omega^\star)|4(5R+2)S^{-3\alpha}\right)\mu\left(\omega : B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\right)\\
&\leqslant 5\varepsilon+4(5R+2)S^{-3\alpha}\sum_{\substack{\omega^\star\notin A_\alpha(S)\\ |\omega^\star|\leqslant K_\varepsilon(S),\,h(\omega^\star)=S}}|g_S(\omega^\star)|\,\mu\left(\omega : B_{\overline{\mathbb{T}},S}(\omega)=\omega^\star\right)\\
&\leqslant 5\varepsilon+4(5R+2)S^{-3\alpha}\mathbb{E}_\mu\left[|g_S(\omega)|\right]. \qquad (2.14)
\end{aligned}$$

The description of μ given in Theorem 3 allows us to estimate $\mathbb{E}_\mu\left[|g_S(\omega)|\right]$. Indeed we have for every integer $H>0$ and $k\geqslant 1$

$$\mathbb{E}_{\widehat{\rho}^{(k)}}\left[|g_H(\omega)|\right]\leqslant\frac{1}{w_k}\mathbb{E}_{\rho^{(k)}}\left[|g_H(\omega)|\right]=\frac{2}{w_k}\mathbb{E}_{GW(1/2)}\left[|g_H(\omega)|\right]=\frac{2}{w_k}\leqslant 2.$$

It follows that

$$\mathbb{E}_\mu\left[|g_S(\omega)|\right]\leqslant 4S+1.$$

Recalling that $\alpha>\frac{1}{3}$, we get that for every S large enough and for $n>N^\star(S)$,

$$\mu_n(\omega_S)\leqslant 6\varepsilon.$$

This completes the proof. □

2.4.4 Proof of the main result

In this section we fix $Q^\star\in\overline{\mathbf{Q}}$ and $R>0$. As in the previous section, we write $\Omega_S=\Omega_S(R)$ to simplify notation. From the remarks following Theorem 4, the proof reduces to verifying the convergence

$$\mu_n\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega))=B_{\overline{\mathbf{Q}},R}(Q^\star)\Big)\underset{n\to\infty}{\longrightarrow}\mu\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega))=B_{\overline{\mathbf{Q}},R}(Q^\star)\Big). \qquad (2.15)$$

First of all, we need to reformulate the problem in terms of trees. Since $Q^\star \in \overline{\mathbf{Q}}$, we know that there exists a finite quadrangulation $Q_0 \in \mathbf{Q}$ such that $d_{\mathbf{Q}}(Q_0, Q^\star) < \frac{1}{R+1}$ and therefore $B_{\overline{\mathbf{Q}},R}(Q_0) = B_{\overline{\mathbf{Q}},R}(Q^\star)$. Then there exists $\omega_0 \in \mathbb{T}$ such that $\Phi(\omega_0) = Q_0$. The convergence (2.15) can now be restated as

$$\mu_n\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big) \underset{n\to\infty}{\longrightarrow} \mu\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big). \quad (2.16)$$

We fix $\varepsilon > 0$ in the remaining part of this proof.

We need to characterize the trees ω for which $\Phi(\omega)$ has the same ball of radius R as $\Phi(\omega_0)$. As we have already mentioned at the end of Section 2.3, the main difficulty comes from the fact that two trees that are very similar in $\overline{\mathbb{T}}$ can give very different quadrangulations if they have vertices with small labels in high generations. We can remedy this problem thanks to Proposition 6.

Note that ω_0 is a finite tree. Let S_0 denote the height of ω_0. According to Lemma 3 and Proposition 6 we can choose $S_1 > S_0$ such that if $S \geqslant S_1$ and $n \geqslant N(S)$ then $\mu(\Omega_S) < \varepsilon$ and $\mu_n(\Omega_S) < \varepsilon$.

Let $S > S_1$ and let $(\omega_i)_{i\in I}$ be the collection of trees given by Corollary 1, such that, for every $\omega \in \mathscr{S} \cap \Omega_S^c$, the equality $B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))$ holds if and only if there exists $i \in I$ such that $B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)$. If $A \triangle B$ denotes the symmetric difference between two sets A and B, we have

$$\begin{aligned}&\mu\left(\left\{\omega \in \overline{\mathbb{T}} :\ B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\right\} \triangle \bigcup_{i\in I}\left\{\omega \in \overline{\mathbb{T}} :\ B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\right\}\right)\\ &\quad \leqslant \mu\left(\Omega_S\right) < \varepsilon.\end{aligned}$$

We deduce from this last bound that

$$\begin{aligned}&\left|\mu_n\Big(\omega :\ B_{\overline{\mathbb{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big) - \mu\Big(\omega :\ B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big)\right|\\ &\leqslant \left|\mu\Big(\bigcup_{i\in I}\left\{\omega :\ B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\right\}\Big) - \mu_n\Big(\omega :\ B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big)\right|\\ &\quad + \left|\mu\Big(\bigcup_{i\in I}\left\{\omega :\ B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\right\}\Big) - \mu\Big(\omega :\ B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big)\right|\\ &\leqslant \left|\mu\Big(\bigcup_{i\in I}\left\{\omega :\ B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\right\}\Big) - \mu_n\Big(\omega :\ B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big)\right| + \varepsilon.\end{aligned}$$

The set $\bigcup_{i\in I}\left\{\omega \in \overline{\mathbb{T}} :\ B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\right\}$ is both open and closed in $\overline{\mathbb{T}}$, and thus

$$\mu_n\left(\bigcup_{i\in I}\left\{\omega :\ B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\right\}\right) \underset{n\to\infty}{\longrightarrow} \mu\left(\bigcup_{i\in I}\left\{\omega :\ B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\right\}\right).$$

Therefore there exists $N'(S) > 0$ such that for $n > N'(S)$:

$$\begin{aligned}
&\left|\mu_n\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big) - \mu\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big)\right| \\
&\leqslant 2\varepsilon + \left|\mu_n\Big(\bigcup_{i\in I}\Big\{\omega : B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\Big\}\Big) - \mu_n\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big)\right| \\
&= 2\varepsilon + \Bigg|\mu_n\Big(\bigcup_{i\in I}\Big\{\omega : B_{\overline{\mathbb{T}},S}(\omega) = B_{\overline{\mathbb{T}},S}(\omega_i)\Big\} \cap \Omega_S\Big) \\
&\qquad\qquad - \mu_n\Big(\Big\{\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big\} \cap \Omega_S\Big)\Bigg|
\end{aligned}$$

by the choice of the collection $(\omega_i)_{i\in I}$.

We also know that $\mu_n(\Omega_S) < \varepsilon$ for $n > N(S)$, and it follows that, for $n > N(S) \vee N'(S)$,

$$\left|\mu_n\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big) - \mu\Big(\omega : B_{\overline{\mathbf{Q}},R}(\Phi(\omega)) = B_{\overline{\mathbf{Q}},R}(\Phi(\omega_0))\Big)\right| \leqslant 3\varepsilon.$$

This completes the proof of Theorem 4. □

Acknowledgments. The author would like to thank Jean-François Le Gall for many helpful discussions about this work.

3

Sommaire

Scaling limits for the uniform infinite planar quadrangulation

Dans le chapitre précédent , nous avons défini et étudié la quadrangulation infinie uniforme du plan. Nous nous intéressons ici à certaines propriétés asymptotiques de cet objet, comme les limites d'échelle du profil des distances mesurées à partir de la racine de la carte. Ceci donnne en particulier le comportement asymptotique des volumes de grandes boules centrées en la racine de la quadrangulation.

L'outil clé pour l'obtention de ces résultats est la déscription de la limite d'échelle des fonctions de contour de l'arbre bien étiqueté infini uniforme en termes de serpent brownien conditionné et éternel. Les limites d'échelles pour la quadrangulation infinie sont ensuite obtenues grâce à la version étendue de la bijection de Schaeffer.

Ce chapitre est un travail effectué en collaboration avec Jean-François Le Gall [LGM09].

3.1 Introduction

Planar maps are proper embeddings of connected graphs in the two-dimensional sphere. They have been studied intensively for their combinatorial properties since the work of Tutte [Tut63]. They have also been studied in the theoretical physics literature for their connections with matrix integrals (see [BIPZ78]), and more recently as models of random surfaces, especially in the setting of the theory of two-dimensional quantum gravity (see in particular the book by Ambjørn, Durhuus and Jonsson [ADJ97]).

The idea of studying scaling limits of certain classes of large planar maps appeared in the pioneering work of Chassaing and Schaeffer [CD06], which exploits a powerful correspondence between rooted quadrangulations and labeled trees due to Schaeffer [Sch98]. Scaling limits of random trees have been studied extensively in the work of Aldous [Ald93], Le Gall [LG99] and others. Up to some point, the bijections between trees and maps make it possible to understand scaling limits for random maps from the known results for random trees. This approach has led to many interesting results about the geometry of large random planar maps. In particular, Le Gall [LG07] proved that sequential limits of rescaled random bipartite planar maps, in the sense of the Gromov-Hausdorff distance, are given by the so-called Brownian map, which had been discussed earlier by Marckert and Mokkadem [MM03]. The Brownian map, which is a quotient space of Aldous' continuum random tree [Ald93] for an equivalence relation defined in terms of Brownian labels assigned to the vertices of the tree, was proved to be homeomorphic to the two-dimensional sphere by Le Gall and Paulin [LGP08] (see also Miermont's alternative proof in [Mie08]).

On the other hand, the study of various properties of random infinite planar maps was initiated by Angel and Schramm [AS03], who defined an infinite random triangulation of the plane whose law is uniform in the sense that it is the local limit of uniformly distributed triangulations with a given number of faces. This object was studied by Angel [Ang03, Ang05] and Krikun [Kri04].

In the setting of quadrangulations, there are two natural ways to define a uniform infinite quadrangulation of the sphere. The first approach, which was developped by Krikun [Kri06], is to take a local limit of uniform finite quadrangulations as their size goes to infinity. The second one, due to Chassaing and Durhuus [CD06], is based on an extended version of Schaeffer's bijection, and the so-called uniform infinite well-labeled tree, which is the local limit of uniformly distributed well-labeled trees whose size tend to infinity. We proved in the previous work [Mé08] that both definitions are equivalent. Thus the same method as in the case of finite random maps can be used to derive geometric properties of the uniform infinite quadrangulation. For example it is proved in [CD06] that the volume of balls of radius r around the root vertex of the uniform infinite quadrangulation grows like r^4 as r goes to infinity. This follows from a similar statement for the labels of the uniform infinite well-labeled tree.

The purpose of the present work is to study scaling limits of certain functionals of the uniform infinite quadrangulation by using the connection with trees. Our main tool will be a limit theorem for the contour functions of the uniform infinite well-labeled tree of Chassaing and Durhuus [CD06]. Before we state this result, let us recall a similar result known for (finite) well-labeled trees. Let $\mathbf{e} = (\mathbf{e}_s)_{s\in[0,1]}$ be a normalized Brownian excursion. Let $\widehat{W} = \left(\widehat{W}_s\right)_{s\in[0,1]}$ be a real-valued process such that, conditionally given $\mathbf{e}$, the process $\widehat{W}$ is centered and Gaussian with covariance given by

$$\operatorname{cov}\left(\widehat{W}_s, \widehat{W}_{s'}\right) = \inf_{s\leq t\leq s'} \mathbf{e}(t)$$

for every $0 \leq s \leq s' \leq 1$. We may view $\widehat{W}$ as the terminal point process of the one-dimensional Brownian snake driven by the normalized Brownian excursion $\mathbf{e}$ (see [LG99]). Let $\left(\overline{\mathbf{e}}, \overline{W}\right)$ be distributed as the pair $\left(\mathbf{e}, \widehat{W}\right)$ conditioned on the event

$$\left\{\widehat{W}_s \geq 0 \text{ for every } s \in [0,1]\right\}.$$

See [LG06] for a precise definition of $\left(\overline{\mathbf{e}}, \overline{W}\right)$. Let θ_n be uniformly distributed over the set of all well-labeled trees with n edges. We denote by $C_n = (C_n(t))_{t\in[0,2n]}$ and $V_n = (C_n(t))_{t\in[0,2n]}$ the contour function and the spatial contour function of θ_n respectively (see subsection 3.2.1 below). Then we have the convergence in distribution

$$\left(\frac{1}{\sqrt{2n}}C_n(2nt), \left(\frac{8}{9}\right)^{1/4}\frac{V_n(2nt)}{n^{1/4}}\right)_{t\in[0,1]} \underset{n\to\infty}{\longrightarrow} \left(\overline{\mathbf{e}}, \overline{W}\right)$$

as a special case of Theorem 2.1 of [LG06], which is itself a conditional version of invariance principles for discrete snakes proved in [JM05].

Our limit theorem for the uniform infinite well-labeled tree has a slightly different form because one needs two pairs of contour functions to characterize the tree. Indeed this tree has almost surely a unique infinite branch, called the spine, and thus it can be coded by the pairs of contour functions $\left(C^{(L)}, V^{(L)}\right)$ and $\left(C^{(R)}, V^{(R)}\right)$ of the left side and right side of the spine (see subsection 3.2.1). The scaling limit of each of these pair of functions is, in some sense, a Brownian snake with an infinite lifetime, which is conditioned to stay positive. Such a process has been introduced by Le Gall and Weill in [LGW06] and is called the eternal conditioned Brownian snake (see Section 3.3.1 for details). We denote by $\left(\zeta^{(L)}, \widehat{W}^{(L)}\right)$ and $\left(\zeta^{(R)}, \widehat{W}^{(R)}\right)$ two copies of the eternal conditioned Brownian snake that are independent conditionally given the labels on the spine (see subsection 3.3.1 for a precise definition). Our main result about asymptotics for the uniform infinite well-labeled tree is the convergence in distribution

$$\left(\left(\frac{1}{n}C^{(L)}(n^2t), \sqrt{\frac{3}{2n}}V^{(L)}(n^2t)\right)_{t\geq 0}, \left(\frac{1}{n}C^{(R)}(n^2t), \sqrt{\frac{3}{2n}}V^{(R)}(n^2t)\right)_{t\geq 0}\right)$$
$$\underset{n\to\infty}{\longrightarrow} \left(\left(\zeta_t^{(L)}, \widehat{W}_t^{(L)}\right)_{t\geq 0}, \left(\zeta_t^{(R)}, \widehat{W}_t^{(R)}\right)_{t\geq 0}\right).$$

A precise statement is given in Theorem 9. Together with the extended version of Schaeffer's bijection, this result can be used to derive various asymptotics for the uniform infinite quadrangulation in terms of the eternal conditioned Brownian snake. For example, the rescaled profile of distances and volumes of balls around the root of the quadrangulation are discussed in Theorem 10. We also study, in a slightly different model, the size of the set of vertices that separate the root from infinity at a given large distance from the root (this set has been studied earlier by Krikun [Kri06]). See Proposition 13 for a precise statement.

The paper is organized as follows. Section 3.2 gives a number of preliminaries concerning trees, finite or infinite quadrangulations, and the extended version of Schaeffer's bijection. We also present the uniform infinite well-labeled tree and quadrangulation as defined in [CD06, Kri06] and recall some basic facts about the Brownian snake and Bessel processes. Section 3.3 gives the proof of the convergence of the rescaled contour functions of the uniform infinite well-labeled tree to a pair of correlated eternal conditioned Brownian snakes. Section 3.4 contains our applications to scaling limits for the uniform infinite quadrangulation.

Notation. If I is an interval of the real line, and E is a metric space, the notation $C(I, E)$ stands for the space of all continuous functions from I into E. This space is equipped with the topology of uniform convergence on compact sets. If E is a Polish space, $\mathbb{D}(E)$ stands for the space of all càdlàg functions from $[0, \infty[$ into E, which is equipped with the usual Skorokhod topology.

3.2 Preliminaries

3.2.1 Trees and quadrangulations

Spatial trees

Throughout this work we will use the standard formalism for planar trees as found in [Nev86]. Let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\emptyset\}$ by convention. An element u of $\mathcal{U}$ is thus a finite sequence of positive integers. If $u, v \in \mathcal{U}$, uv denotes the concatenation of u and v. If v is of the form uj with $j \in \mathbb{N}$, we say that u is the *parent* of v or that v is a *child* of u. More generally, if v is of the form uw for $u, w \in \mathcal{U}$, we say that u is an *ancestor* of v or that v is a *descendant* of u. A *rooted planar tree* τ is a (finite or infinite) subset of $\mathcal{U}$ such that

1. $\emptyset \in \tau$ ($\emptyset$ is called the *root* of τ),

2. if $v \in \tau$ and $v \neq \emptyset$, the parent of v belongs to τ

3. for every $u \in \mathcal{U}$ there exists $k_u(\tau) \geq 0$ such that $uj \in \tau$ if and only if $j \leq k_u(\tau)$.

The edges of τ are the pairs (u, v), where $u, v \in \tau$ and u is the father of v. The integer $|\tau|$ denotes the number of edges of τ and is called the size of τ. The integer $H(\tau)$ denotes the maximal generation of a vertex in τ and is called the height of τ. We denote by $\mathcal{T}$ the set of all planar trees. A *spine* of a tree τ is an infinite linear sub-tree of τ starting from its root.

A *rooted labeled tree* (or spatial tree) is a pair $\theta = (\tau, (\ell(u))_{u\in\tau})$ that consists of a planar tree τ and a collection of integer labels assigned to the vertices of τ, such that if

$u, v \in \tau$ and v is a child of u, then $|\ell(u) - \ell(v)| \leq 1$. For every $l \in \mathbb{Z}$, we denote by $\overline{\mathbf{T}}^l$ the set of all spatial trees for which $\ell(\emptyset) = l$, by $\mathbf{T}^l_\infty$ the set of all such trees with an infinite number of edges, by $\mathbf{T}^l_n$ the set of all such trees with n edges and by $\mathbf{T}^l$ the set of all such trees with finitely many edges. If $\theta = (\tau, \ell)$ is a labeled tree, $|\theta| = |\tau|$ is the size of θ and $H(\theta) = H(\tau)$ is the height of θ.

Suppose that $l \geq 1$. If $\ell(\emptyset) = l$ and in addition $\ell(u) \geq 1$ for every vertex u of τ, we say that θ is an l-well-labeled tree. The corresponding sets of spatial trees are denoted by $\overline{\mathbb{T}}^l$, $\mathbb{T}^l$, $\mathbb{T}^l_\infty$ and $\mathbb{T}^l_n$. For $l = 1$ we will simply say well-labeled tree and denote the corresponding sets by $\overline{\mathbb{T}}$, $\mathbb{T}$, $\mathbb{T}_\infty$ and $\mathbb{T}_n$.

A finite spatial tree $\theta = (\tau, \ell)$ can be coded by a pair of functions (C_θ, V_θ), where $C_\theta = (C_\theta(t))_{0 \leq t \leq 2|\theta|}$ is the contour function of τ and $V_\theta = (V_\theta(t))_{0 \leq t \leq 2|\theta|}$ is the spatial contour function of θ (see Figure 3.1). To define these contour functions, let us consider a particle which, starting from the root, traverses the tree along its edges at speed one. When leaving a vertex, the particle moves towards the first non visited child of this vertex if there is such a child, or returns to the parent of this vertex. Since all edges will be crossed twice, the total time needed to explore the tree is $2|\theta|$. For every $t \in [0, 2|\theta|]$, $C_\theta(t)$ denotes the distance from the root of the position of the particle at time t. In addition if $t \in [0, 2|\theta|]$ is an integer, $V_\theta(t)$ denotes the label of the vertex that is visited at time t. We then complete the definition of V_θ by interpolating linearly between successive integers. See Figure 3.1 for an example. A finite spatial tree is uniquely determined by its pair of contour functions. It will sometimes be convenient to define the functions C_θ and V_θ for every $t \geq 0$, by setting $C_\theta(t) = 0$ and $V_\theta(t) = V_\theta(0)$ for every $t \geq 2|\theta|$.

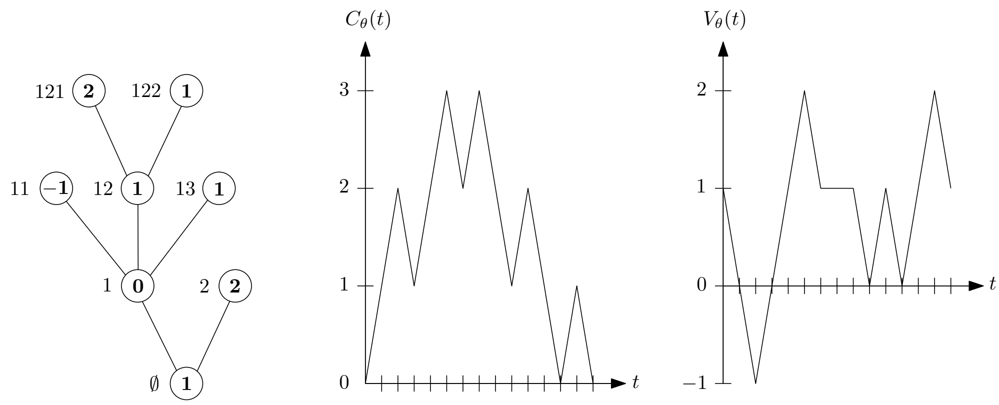

Figure 3.1: A spatial tree θ and its pair of contour functions (C_θ, V_θ).

Fix $l \in \mathbb{Z}$. For every pair of trees $\theta, \theta' \in \overline{\mathbf{T}}^l$ define

$$d_{\overline{\mathbf{T}}^l}(\theta, \theta') = \left(1 + \sup\{h :\ B_h(\theta) = B_h(\theta')\}\right)^{-1}$$

where, for $h \geq 0$, $B_h(\theta)$ is the element of $\overline{\mathbf{T}}^l$ consisting of all vertices of θ up to generation h, with the same labels. One easily checks that $d_{\overline{\mathbf{T}}^l}$ is a distance on $\overline{\mathbf{T}}^l$.

To conclude this section, let us introduce some notation. If $\theta \in \overline{\mathbf{T}}^l$, for every $k \in \mathbb{N}$, we let $N_k(\theta)$ denote the number of vertices of θ that have label k. We then define $\mathscr{S}$ as the set of all trees in $\overline{\mathbb{T}}$ that have at most one spine, and whose labels take each integer value only finitely many times:

$$\mathscr{S} = \{\theta \in \mathbb{T}_\infty : \forall l \geq 1,\ N_l(\theta) < \infty \text{ and } \theta \text{ has a unique spine}\} \cup \mathbb{T}.$$

A tree $\theta \in \mathscr{S}$ can obviously be coded by two pairs of contour functions, $\left(C_\theta^{(L)}, V_\theta^{(L)}\right) : \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+$ and $\left(C_\theta^{(R)}, V_\theta^{(R)}\right) : \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+$, each pair coding one side of the spine. The definition of these contour functions should be clear from Figure 3.2. Note that the functions $C_\theta^{(L)}$, $V_\theta^{(L)}$, $C_\theta^{(R)}$ and $V_\theta^{(R)}$ tend to infinity at infinity.

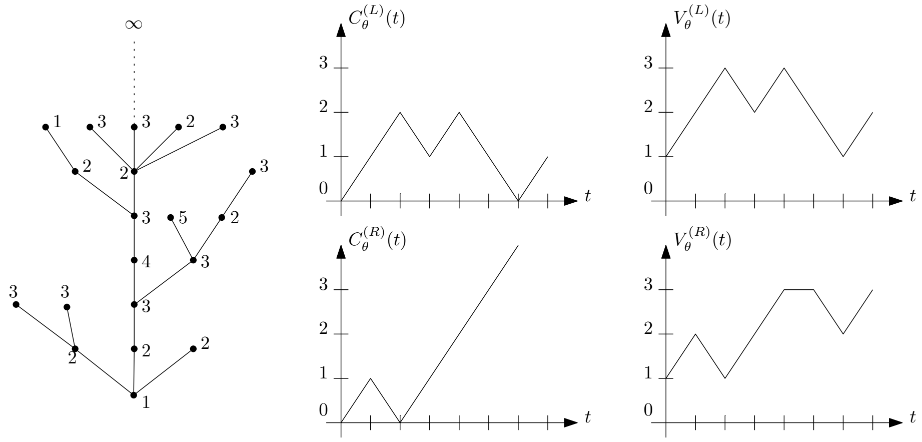

Figure 3.2: A infinite well-labeled tree θ and its contour functions $\left(C_\theta^{(L)}, V_\theta^{(L)}\right)$, $\left(C_\theta^{(R)}, V_\theta^{(R)}\right)$.

Planar maps and quadrangulations

Consider a proper embedding of a finite connected graph in the sphere $\mathbb{S}^2$ (loops and multiple edges are allowed). A *finite planar map* is an equivalent class of such embeddings with respect to orientation preserving homeomorphisms of the sphere. A planar map is *rooted* if it has a distinguished oriented edge, which is called the root edge. The origin of the root edge is called the root vertex. Faces of the map are the connected components of the complement of the union of its edges. A finite planar map is a *quadrangulation* if all its faces have degree 4, that is 4 adjacent edges. A planar map is a *quadrangulation with a boundary* if all its faces have degree 4, except for a number of distinguished faces which can be arbitrary even-sided polygons.

Let us introduce infinite quadrangulations using the approach of Krikun [Kri06]. For every integer $n \geq 1$ we denote by $\mathbf{Q}_n$ the set of all rooted quadrangulations with n faces. For every pair $q, q' \in \mathbf{Q}_f = \bigcup_{n\geq 1} \mathbf{Q}_n$, we define

$$d_{\mathbf{Q}}(q, q') = \left(1 + \sup\{r : B_{\mathbf{Q},r}(q) = B_{\mathbf{Q},r}(q')\}\right)^{-1}$$

where, for $r \geq 1$, $B_{\mathbf{Q},r}(q)$ is the planar map given as the union of the faces of q that have at least one vertex at distance strictly smaller than r from the root and $\sup \emptyset = 0$ by convention. Note that $B_{\mathbf{Q},r}(q)$ is a quadrangulation with a boundary. Then $(\mathbf{Q}_f, d_{\mathbf{Q}})$ is a metric space. Denote by $(\mathbf{Q}, d_{\mathbf{Q}})$ the completion of this space. We call *infinite quadrangulations* the elements of $\mathbf{Q}$ that are not finite quadrangulations and we denote the set of all such quadrangulations by $\mathbf{Q}_\infty$.

Note that one can extend the function $q \in \mathbf{Q}_f \mapsto B_{\mathbf{Q},r}(q)$ to a continuous function $B_{\mathbf{Q},r}$ on $\mathbf{Q}$. The ball $B_{\mathbf{Q},r}(q)$ is naturally interpreted as the union of faces of q that have a vertex at distance strictly smaller than r from the root.

Let q be a rooted quadrangulation. We will always denote by $V(q)$ the vertex set of q, and by ∂ its root vertex. We will equip $V(q)$ with the graph distance: if v and v' are two vertices, $d(v, v')$ is the minimal number of edges on a path from v to v'. The profile λ_q of the quadrangulation q is the integer-valued measure on $\mathbb{Z}_+$ defined by

$$\lambda_q(k) = |\{a \in V(q) : d(\partial, a) = k\}|$$

for every $k \in \mathbb{Z}_+$.

3.2.2 Schaeffer's correspondence

The relations between quadrangulations and spatial trees come from the following key result [CV81, Sch98]. There exists a bijection Φ_n, called Schaeffer's bijection, from $\mathbb{T}_n$ onto $\mathbf{Q}_n$ that enjoys the following property: if $\theta = (\tau, \ell) \in \mathbb{T}_n$, then, for every integer $k \geq 1$ one has

$$|\{a \in V(\Phi_n(\theta)) : d(\partial, a) = k\}| = |\{v \in \theta : \ell(v) = k\}| .$$

Schaeffer's bijection has been extended in the infinite setting in [CD06]. There exists a one-to-one mapping Φ from $\mathscr{S}$ into $\mathbf{Q}$ that enjoys similar properties. If $\theta = (\tau, \ell) \in \mathscr{S}$, then, for every integer $k \geq 1$ one has

$$|\{a \in V(\Phi(\theta)) : d(\partial, a) = k\}| = |\{v \in \theta : \ell(v) = k\}| .$$

Let us describe the mapping Φ (see [CD06], Section 6.2. for details). Fix a tree $\theta = (\tau, \ell) \in \mathscr{S}$ and consider an embedding of τ in the sphere $\mathbb{S}^2$, such that every sequence $p = (p_n)_{n\in\mathbb{N}}$ of points of $\mathbb{S}^2$ belonging to distinct edges of τ, has a unique accumulation point $\triangle \in \mathbb{S}^2$. A corner of τ is a sector between two consecutive edges around a vertex and its label is the label of the corresponding vertex.

First, add a vertex ∂ in the complement of $\tau \cup \{\triangle\}$. Then, for every vertex v of τ and every corner c of v, an edge is added according to the following rules:

- If $\ell(v) = 1$ draw an edge between the corner c and ∂ (see Figure 3.3, left).
- If c is on the right side of the spine, if $\ell(v) \geq 2$, and if there exists a corner with label $\ell(v) - 1$ that is visited after c in the contour of the right side of the spine; draw an edge between c and the first such corner (see Figure 3.3, left).

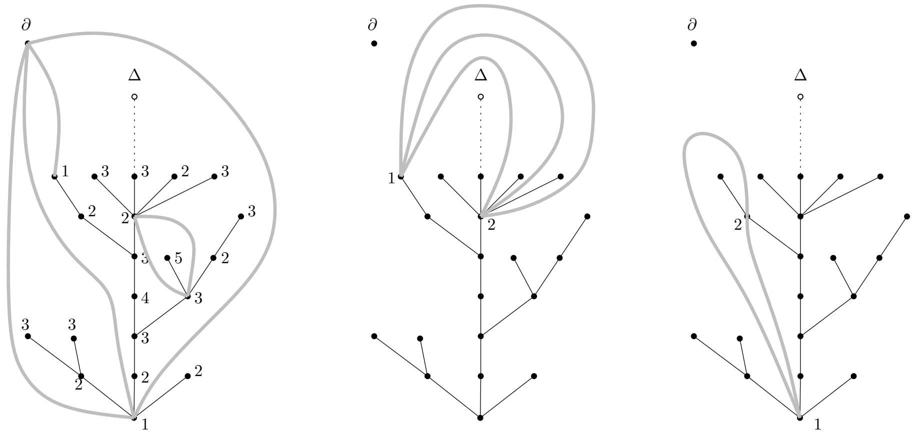

Figure 3.3: Construction of a few edges in Schaeffer's correspondence.

- If c is on the right side of the spine, if $\ell(v) \geq 2$, and if there is no corner with label $\ell(v) - 1$ that is visited after c in the contour of the right side of the spine; draw an edge between c and the corner on the left side of the spine with label $\ell(v) - 1$ that is the last one to be visited during the contour of the left side of the spine (see Figure 3.3, middle).
- If c is on the left side of the spine and if $\ell(v) \geq 2$, draw an edge between c and the corner with label $\ell(v) - 1$ that is the last one to be visited before c during the contour of the left side of the spine (see Figure 3.3, right).

The construction can be made in such a way that edges do not intersect. The resulting planar map whose vertices are the vertices of τ and the extra vertex ∂, and whose edges are all the edges obtained by the preceding device, is an infinite planar quadrangulation, which is rooted at the oriented edge between ∂ and the first corner of $\emptyset$. Moreover, for each vertex v of τ, the distance $d(\partial, v)$ between the root vertex ∂ and v in the map $\Phi(\theta)$ coincides with the label $\ell(v)$.

3.2.3 Uniform infinite well-labeled tree and quadrangulation

The main purpose of this paper is to study properties of the random infinite quadrangulation whose law is uniform on the set of infinite quadrangulations, in the sense of the following theorem:

Theorem 5 ([Kri06]). *For every $n \geq 1$ let ν_n be the uniform probability measure on $\mathbf{Q}_n$. The sequence $(\nu_n)_{n\in\mathbb{N}}$ converges to a probability measure ν, in the sense of weak convergence in the space of all probability measures on $(\mathbf{Q}, d_{\mathbf{Q}})$. Moreover, ν is supported on the set of infinite rooted quadrangulations. A random quadrangulation distributed according to ν will be called a uniform infinite quadrangulation.*

This probability measure is connected with the law of the uniform infinite well-labeled tree, which appears in the next theorem. We write $d_{\overline{\mathbb{T}}}$ for the restriction of the distance $d_{\overline{\mathbf{T}}^1}$ to $\overline{\mathbb{T}}$.

Theorem 6 ([CD06]). *For every $n \geq 1$, let μ_n be the uniform probability measure on the set of all well-labeled trees with n edges. The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to a probability measure μ in the sense of weak convergence in the space of all probability measures on $\left(\overline{\mathbb{T}}, d_{\overline{\mathbb{T}}}\right)$. Moreover, μ supported on the set $\mathscr{S} \subset \mathbb{T}_\infty$. A random tree distributed according to μ will be called a uniform infinite well-labeled tree.*

It was proved in the previous work [Mé08] that ν is the image of μ under the mapping Φ (the extended Schaeffer's correspondence) described in subsection 3.2.2. This is stated in the next theorem.

Theorem 7 ([Mé08]). *For every Borel subset A of $\mathbf{Q}$ one has*

$$\nu(A) = \mu\left(\Phi^{-1}(A)\right).$$

Informally, we may say that the uniform infinite quadrangulation is coded by the uniform infinite tree.

Let $\theta \in \mathscr{S}$. If v is the vertex at generation n in the spine of θ, we denote the label of v by $X_n(\theta)$. The trees attached to v respectively on the left side and the right side of the spine are denoted by $L_n(\theta)$ and $R_n(\theta)$.

For every integer $l > 0$ we denote by ρ_l the law of the Galton-Watson tree with geometric offspring distribution with parameter 1/2, labeled according to the following rules. The root has label l and every other vertex has a label chosen uniformly in $\{m - 1, m, m+1\}$ where m is the label of its parent, these choices being made independently for every vertex. Then, for every tree $\theta \in \mathbf{T}^l$, $\rho_l(\theta) = \frac{1}{2}12^{-|\theta|}$. By a standard result (see e.g. [Ott49]):

$$\rho_l\left(|\theta| = n\right) = \rho_1\left(|\theta| = n\right) = \frac{n^{-3/2}}{2\sqrt{\pi}} + O\left(n^{-5/2}\right) \tag{3.1}$$

$$\rho_l\left(|\theta| \geq n\right) = \rho_1\left(|\theta| \geq n\right) = O\left(n^{-1/2}\right) \tag{3.2}$$

as n goes to infinity.

Proposition 2.4 of [CD06] shows that

$$\rho_l\left(\mathbb{T}^l\right) = \frac{l(l+3)}{(l+1)(l+2)} =: \frac{w_l}{2}.$$

We can thus define a probability measure $\widehat{\rho}_l$ on $\mathbb{T}^l$ by

$$\widehat{\rho}_l(\theta) = 2w_l^{-1}\rho_l(\theta)$$

for every $\theta \in \mathbb{T}^l$. We will often use the bound $\widehat{\rho}_l \leq 2\rho_l$, which holds for every $l > 0$, from the bound $w_l \geq \frac{4}{3}$.

The following theorem recalls a few important properties of μ.

Theorem 8. *Let θ be a random spatial tree distributed according to μ. Write $X_n = X_n(\theta)$ for every $n \geq 0$.*

1. *The process* $X = (X_n)_{n\geq 0}$ *is a Markov chain with transition kernel* Π *defined by*

$$\begin{aligned} \Pi(l,l-1) &= \frac{(w_l)^2}{12d_l} d_{l-1} =: q_l && \text{if } l \geq 2, \\ \Pi(l,l) &= \frac{(w_l)^2}{12} =: r_l && \text{if } l \geq 1, \\ \Pi(l,l+1) &= \frac{(w_l)^2}{12d_l} d_{l+1} =: p_l && \text{if } l \geq 1 \end{aligned}$$

where

$$\begin{aligned} w_l &= 2\frac{l(l+3)}{(l+1)(l+2)}, \\ d_l &= \frac{2w_l}{560}(4l^4 + 30l^3 + 59l^2 + 42l + 4). \end{aligned}$$

2. *The sequence of processes* $\left(\sqrt{\frac{3}{2n}} X_{\lfloor nt \rfloor}\right)_{t\geq 0}$ *converges in distribution in the Skorokhod sense to a nine-dimensional Bessel process started at* 0.

3. *Conditionally given* $(X_n)_{n\geq 0} = (x_n)_{n\geq 0}$*, the sequence* $(L_n)_{n\geq 0}$ *of subtrees of* θ *attached to the left side of the spine and the sequence* $(R_n)_{n\geq 0}$ *of subtrees attached to the right side of the spine form two independent sequences of independent labeled trees distributed according to the measures* $\widehat{\rho}_{x_n}$.

Proof. Assertions 1. and 3. are proved in [CD06]. Assertion 2. is derived in [Mé08]. □

An infinite quadrangulation q is said to have (only) one end if, for every finite subgraph g of q, $q \setminus g$ has a unique infinite connected component. It is proved in [Kri06] that ν is supported on the set of all quadrangulations with one end. This also follows from Theorem 7: It is an easy exercise to verify that a quadrangulation of the form $q = \Phi(\theta)$, with $\theta \in \mathscr{S}$, has only one end.

Fix a quadrangulation q with one end, and $r > 0$. Recall that $B_{\mathbf{Q},r}(q)$ is the ball of radius r centered at the root vertex of q (see section 2.1.2 for the precise definition of this ball). Since q has exactly one end, $q \setminus B_{\mathbf{Q},r}(q)$ has only one infinite connected component. Following Krikun [Kri06], we define the r-hull of q as the union of $B_{\mathbf{Q},r}(q)$ and all finite components of $q \setminus B_{\mathbf{Q},r}(q)$. The r-hull of q, which we will denote by $\widehat{B}_r(q)$, is a finite quadrangulation with a boundary and, by construction, it has only one boundary face. This face is delimited by an even-sided polygon that we denote by $\gamma_r(q)$. The cycle $\gamma_r(q)$ contains only edges of q that connect two vertices at distance respectively r and $r+1$ from the root vertex of q and it separates the root of q from infinity in q. See Figure 3.4 for a graphical representation.

3.2.4 The Brownian snake

In this section we recall some facts about the Brownian snake that we will use later. We refer to [LG99] for a more complete presentation of the Brownian snake.

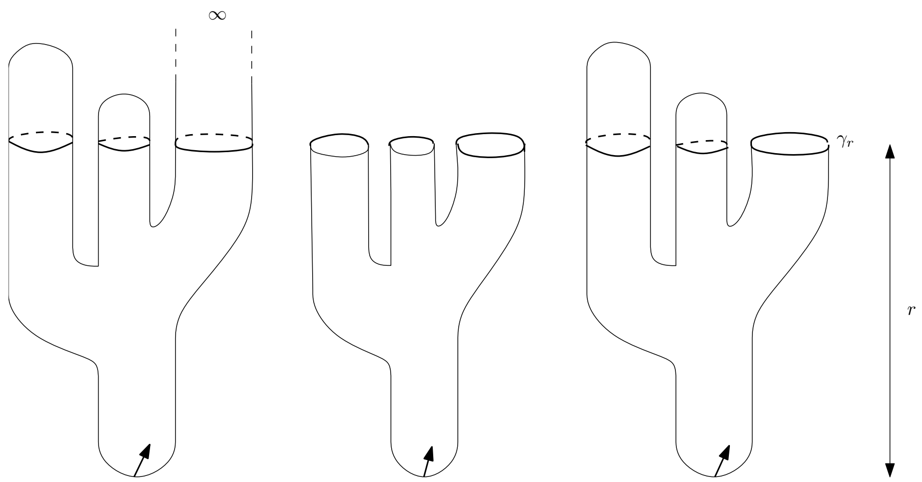

Figure 3.4: An infinite quadrangulation with one end q, its ball $B_r(q)$ and its hull $\widehat{B}_r(q)$.

The Brownian snake is a Markov process taking values in the space $\mathcal{W}$ of all finite paths in $\mathbb{R}$. An element of $\mathcal{W}$ is simply a continuous mapping $\mathrm{w} : [0, \zeta] \to \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})} \geq 0$ depends on w and is called the lifetime of w. The final point (or tip) of w will be denoted by $\widehat{\mathrm{w}} = \mathrm{w}(\zeta)$. The range of w is denoted by $\mathrm{w}[0, \zeta_{(\mathrm{w})}]$. If $x \in \mathbb{R}$, we denote the subset of paths with initial point x by $\mathcal{W}_x$. The trivial path in $\mathcal{W}_x$ such that $\zeta_{(\mathrm{w})} = 0$ is identified with the point x. The set $\mathcal{W}$ is a Polish space for the distance

$$d(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The canonical space $\Omega = C(\mathbb{R}_+, \mathcal{W})$ of continuous functions from $\mathbb{R}_+$ into $\mathcal{W}$ is equipped with the topology of uniform convergence on every compact subset of $\mathbb{R}_+$. The canonical process on Ω is denoted by $W_s(\omega) = \omega(s)$ for $\omega \in \Omega$ and we write $\zeta_s = \zeta_{(W_s)}$ for the lifetime of W_s.

Let $\mathrm{w} \in \mathcal{W}$. The law of the (one-dimensional) Brownian snake started from w is the probability $\mathbb{P}_\mathrm{w}$ on Ω which can be characterized as follows. First, the process $(\zeta_s)_{s \geq 0}$ is under $\mathbb{P}_\mathrm{w}$ a reflected Brownian motion in $[0, \infty[$ started from $\zeta_{(\mathrm{w})}$. Secondly, the conditional distribution of $(W_s)_{s \geq 0}$ knowing $(\zeta_s)_{s \geq 0}$, which is denoted by Θ^ζ_w, is characterized by the following properties:

1. $W_0 = \mathrm{w}$, Θ^ζ_w a.s.

2. The process $(W_s)_{s \geq 0}$ is time-inhomogeneous Markov under Θ^ζ_w. Moreover, if $0 \leq s \leq s'$,

   - $W_{s'}(t) = W_s(t)$ for every $t \leq m(s, s') = \inf_{[s,s']} \zeta_r$, Θ^ζ_w a.s.
   - $(W_{s'}(m(s', s) + t) - W_{s'}(m(s, s')))_{0 \leq t \leq \zeta_{s'} - m(s,s')}$ is independent of W_s and distributed under Θ^ζ_w as a Brownian motion started at 0.

Informally, the value W_s of the Brownian snake at time s is a random path with a random lifetime ζ_s evolving like a reflected Brownian motion in $[0, \infty[$. When ζ_s decreases, the path is erased from its tip, and when ζ_s increases, the path is extended by adding "little pieces" of Brownian paths at its tip.

We denote by $\mathbf{n}(\mathrm{d}e)$ the Itô measure of positive excursions. This is a σ-finite measure on the space $C(\mathbb{R}_+, \mathbb{R}_+)$ of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$. We write

$$\sigma(e) = \inf\{s > 0 : e(s) = 0\}$$

for the duration of an excursion e. For $s > 0$, $\mathbf{n}_{(s)}$ denotes the conditioned measure $\mathbf{n}(\cdot \mid \sigma = s)$. Our normalization of the Itô measure is fixed by the relation

$$\mathbf{n} = \int_0^\infty \frac{\mathrm{d}s}{2\sqrt{2\pi s^3}} \mathbf{n}_{(s)}. \tag{3.3}$$

If $x \in \mathbb{R}$, the excursion measure $\mathbb{N}_x$ of the Brownian snake started at x is defined by

$$\mathbb{N}_x = \int_{C(\mathbb{R}_+,\mathbb{R}_+)} \mathbf{n}(\mathrm{d}e)\Theta_x^e.$$

With a slight abuse of notation we will also write $\sigma(\omega) = \inf\{s > 0 : \zeta_s(\omega) = 0\}$ for $\omega \in \Omega$. We can then consider the conditioned measures

$$\mathbb{N}_x^{(s)} = \mathbb{N}_x(\cdot \mid \sigma = s) = \int_{C(\mathbb{R}_+,\mathbb{R}_+)} \mathbf{n}_{(s)}(\mathrm{d}e)\Theta_x^e.$$

The first moment formula for the Brownian snake states that

$$\mathbb{N}_x\left(\int_0^\sigma \mathrm{d}s\, F(W_s)\right) = \int_0^\infty \mathrm{d}r\, E_x\left[F\left((B_t)_{t\in[0,r]}\right)\right] \tag{3.4}$$

for every nonnegative measurable function F on $\mathcal{W}$. Here and later, $(B_t)_{t\geq 0}$ stands for a linear Brownian motion that starts from x under the probability measure P_x.

The Brownian snake enjoys the following Poissonian representation ([LG99], Lemma 5 page 80). Fix $\mathrm{w} \in \mathcal{W}$ and argue under the probability measure $\mathbb{P}_\mathrm{w}$. Set

$$T_0 = \inf\{s \geq 0 : \zeta_s = 0\}.$$

Denote by $(\alpha_j, \beta_j)_{j\in J}$ the excursion intervals of $(\zeta_s - \inf_{[0,s]} \zeta_r)_{s\geq 0}$ away from 0 before time T_0. For every $j \in J$ define $W^{(j)} \in C(\mathbb{R}_+, \mathcal{W})$ by setting for every $s \geq 0$

$$W_s^{(j)}(t) = W_{(\alpha_j+s)\wedge\beta_j}(\zeta_{\alpha_j} + t)$$

for $0 \leq t \leq \zeta_{\left(W^{(j)}\right)} := \zeta_{(\alpha_j+s)\wedge\beta_j} - \zeta_{\alpha_j}$. Then, under $\mathbb{P}_\mathrm{w}$, the point measure

$$\sum_{j\in J} \delta_{\left(\zeta_{\alpha_j}, W^{(j)}\right)}$$

is a Poisson point measure on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ with intensity

$$2\mathbf{1}_{\left[0,\zeta_{(\mathrm{w})}\right]}(t)\, \mathrm{d}t\, \mathbb{N}_{\mathrm{w}(t)}(\mathrm{d}\omega).$$

The range $\mathcal{R} = \mathcal{R}(\omega)$ is defined by $\mathcal{R} = \{\widehat{W}_s : s \geq 0\}$. The following lemma gives certain properties of the range of the Brownian snake under $\mathbb{N}_x$.

Lemma 6. *1. For every $x > 0$,*

$$\mathbb{N}_x \left(\mathcal{R} \cap]-\infty, 0] \neq \emptyset\right) = \frac{3}{2x^2}.$$

2. For every $x > 0$ and $\lambda > 0$,

$$\mathbb{N}_x \left(1 - \mathbf{1}_{\{\mathcal{R}\cap]-\infty,0]=\emptyset\}} e^{-\lambda\sigma}\right) = \sqrt{\frac{\lambda}{2}} \left(3(\coth(2^{1/4}x\lambda^{1/4}))^2 - 2\right).$$

Proof. The first assertion is proved in Section VI.1 of [LG99] and the second one is Lemma 7 in [Del03]. □

We conclude this section by introducing the exit measure and the special Markov property of the Brownian snake. We refer to [LG99] for the general theory of exit measures and to [LG95] (Theorem 2.4) for the special Markov property. Let $I =]a, b[$ and $x \in \mathbb{R}$ with $-\infty \leq a < x < b \leq +\infty$. For every $\mathrm{w} \in \mathcal{W}_x$ set

$$\tau_I(\mathrm{w}) = \inf \left\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) \notin I\right\}$$

where $\inf \emptyset = \infty$. The process L^I, defined for $s \in [0, \sigma]$ by

$$L^I_s = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^s \mathrm{d}r \mathbf{1}_{\{\tau_I(W_r)<\zeta_r<\tau_I(W_r)+\varepsilon\}},$$

is a continuous increasing process called the exit local time from I of the Brownian snake. The exit measure Y^I from I is defined under $\mathbb{N}_x$ by the formula

$$\langle Y^I, g\rangle = \int_0^\sigma g\left(\widehat{W}_s\right) \mathrm{d}L^I_s.$$

It is $\mathbb{N}_x$-a.e. supported on $\partial I = \{a, b\}$. The following first moment formula holds:

$$\mathbb{N}_x \left(\int_0^\sigma \mathrm{d}L^I_s\, F(W_s)\right) = \mathbb{E}_x \left[\mathbf{1}_{\{\tau_I<\infty\}} F\left((B_t)_{t\in[0,\tau_I]}\right)\right] \tag{3.5}$$

for every nonnegative measurable function F on $\mathcal{W}$ (here τ_I stands for the first exit time of B from I).

The random set

$$\{s \geq 0 : \tau_I(W_s) < \zeta_s\}$$

is open $\mathbb{N}_x$-a.e., and can be written as a disjoint union of open intervals $]a_j, b_j[$, $j \in J$. Moreover, $\mathbb{N}_x$-a.e., for every $j \in J$, and every $s \in]a_j, b_j[$,

$$\tau_I(W_s) = \tau_I(W_{a_j}) = \tau_I(W_{b_j}) = \zeta_{a_j} = \zeta_{b_j}$$

and the paths W_s, $s \in [a_j, b_j]$ coincide up to their exit time from I. We let τ_j be the common value of $\tau_I(W_s)$ for $s \in [a_j, b_j]$ and $y_j = \widehat{W}_{a_j} = W_s(\tau_j)$ for every $s \in [a_j, b_j]$.

For every $j \in J$, define $W^j \in C(\mathbb{R}_+, \mathcal{W}_{y_j})$ by

$$W^j_s(t) = W_{(a_j+s)\wedge b_j}(\tau_j + t), \qquad 0 \leq t \leq \zeta_{(W^j_s)} := \zeta_{(\alpha_j+s)\wedge\beta_j} - \zeta_{\alpha_j}.$$

The processes $(W^j)_{j\in J}$ are the excursions of W outside I. We set

$$\gamma_s = \inf\left\{t : \int_0^t \mathrm{d}u \mathbf{1}_{\{\zeta_u \le \tau_I(W_u)\}} > s\right\}$$

and we define a continuous process W' by $W'_s = W_{\gamma_s}$. By definition, the σ-field $\mathcal{E}^I$ is generated by W' and the class of all $\mathbb{N}_x$-negligible sets. From Proposition 2.3 in [LG95], we get that Y^I is $\mathcal{E}^I$-measurable. The special Markov property (see [LG95], Theorem 2.4) states that conditionally given $\mathcal{E}^I$, the random measure

$$\sum_{j\in J} \delta_{W^j}$$

is under $\mathbb{N}_x(\cdot \mid \mathcal{R}\cap I^c \neq \emptyset)$ a Poisson measure with intensity

$$\int Y^I(\mathrm{d}y)\, \mathbb{N}_y\,(\cdot)\,.$$

3.2.5 Convergence towards the Brownian snake

In this section, we recall a standard result of convergence towards the Brownian snake that will be used in the subsequent proofs. Let $\mathcal{F} = (\theta_1, \theta_2, \ldots)$ be a sequence of independent spatial trees distributed according to the probability measure ρ_1. We denote by $C^{\mathcal{F}} = (C^{\mathcal{F}}(t))_{t\ge 0}$ the contour function of the forest $\mathcal{F}$, which is obtained by concatenating the contour functions of the trees $\theta_1, \theta_2, \ldots$. Similarly, $V^{\mathcal{F}} = (V^{\mathcal{F}}(t))_{t\ge 0}$ is obtained by concatenating the spatial contour functions of the trees $\theta_1, \theta_2, \ldots$. Note that this concatenation creates no problem because the labels of the roots of $\theta_1, \theta_2, \ldots$ are all equal to 1.

In the next statement, $(W_t)_{t\ge 0}$ is the Brownian snake under the probability measure $\mathbb{P}_0$ and $(\zeta_t)_{t\ge 0}$ is the associated lifetime process.

Proposition 7. *The sequence of processes*

$$\left(\frac{1}{n} C^{\mathcal{F}}(n^2 t), \sqrt{\frac{3}{2n}}\, V^{\mathcal{F}}(n^2 t)\right)_{t\ge 0}$$

converge in distribution to the process $\left(\zeta_t, \widehat{W}_t\right)_{t\ge 0}$ *in the sense of weak convergence of the laws on the space* $C(\mathbb{R}_+, \mathbb{R}^2)$.

The convergence of contour functions in the proposition follows from the more general Theorem 1.17 of [LG05] (in our particular case, it is just a straightforward application of Donsker's theorem). The joint convergence with the spatial contour process can then be obtained as an easy application of the techniques in [JM05].

3.2.6 Bessel processes

Throughout this work, for every $d > 0$, $\mathbb{P}_z^{(d)}$ denotes the law of a d-dimensional Bessel process started at $z \ge 0$. Thus $\mathbb{P}_z^{(d)}$ is a probability measure on the space $C(\mathbb{R}_+, \mathbb{R}_+)$ and we denote by $(\mathcal{F}_t)_{t\ge 0}$ the canonical filtration on this space. The canonical process on $C(\mathbb{R}_+, \mathbb{R}_+)$ is denoted by $\mathbf{R} = (\mathbf{R}_t)_{t\ge 0}$. Let $\mathbb{Q}_z$ be the law of $(B_{t\wedge S_0})_{t\ge 0}$ under P_z, where $S_0 := \inf\{t \ge 0 : B_t = 0\}$ (recall that B is a linear Brownian motion started at z under P_z).

Proposition 8. *Fix $z > 0$. Let $d \geq 2$ and $\kappa = d/2 - 1$. Then, for every $t > 0$:*

$$\left.\frac{\mathrm{d}\mathbb{P}_z^{(d)}}{\mathrm{d}\mathbb{Q}_z}\right|_{\mathcal{F}_t} = \left(\frac{\mathbf{R}_t}{z}\right)^{\kappa+1/2} \exp\left(-\frac{1}{2}(\kappa^2 - 1/4)\int_0^t \frac{\mathrm{d}s}{\mathbf{R}_s^2}\right).$$

See Proposition 2.1 of [PY81] for a proof. We shall deal with the case $d = 9$, and then $\kappa = 7/2$.

3.3 Scaling limit of the uniform infinite well-labeled tree

3.3.1 The eternal conditioned Brownian snake

Our goal is to investigate the scaling limit of the uniform infinite well-labeled tree. Before stating our results, we discuss a process that arises in this scaling limit: the eternal conditioned Brownian snake. This process, which was introduced in [LGW06], can be viewed as a Brownian snake conditioned to stay positive and to live forever. Let $Z = (Z_t)_{t\geq 0}$ be a nine-dimensional Bessel process started at 0. Conditionally given Z, let

$$\mathcal{P} = \sum_{i\in I} \delta_{(h_i,\omega_i)}$$

be a Poisson point process on $\mathbb{R}_+ \times \Omega$ with intensity

$$2\,\mathbf{1}_{\{\mathcal{R}(\omega)\subset]-Z_t,\infty[\}}\mathrm{d}t\,\mathbb{N}_0(\mathrm{d}\omega) \tag{3.6}$$

where we recall that $\mathcal{R}(\omega)$ denotes the range of the snake. We then construct our conditioned snake W^∞ as a measurable functional $F(Z, \mathcal{P})$ of the pair $(Z, \mathcal{P})$. We now describe this functional. To simplify notation, we put

$$\sigma_i = \sigma(\omega_i), \quad \zeta_s^i = \zeta_s(\omega_i), \quad W_s^i = W_s(\omega_i)$$

for every $i \in I$ and $s \geq 0$. For every $u \geq 0$, we set

$$\tau_u = \sum_{i\in I} \mathbf{1}_{\{h_i\leq u\}}\sigma_i.$$

Then, if $s \geq 0$, there is a unique u such that $\tau_{u-} \leq s \leq \tau_u$, and:

- Either there is a (unique) $i \in I$ such that $u = h_i$ and we set

$$\zeta_s^\infty = u + \zeta_{s-\tau_{u-}}^i,$$
$$W_s^\infty(t) = \begin{cases} Z_t & \text{if } t \leq u, \\ Z_u + W_{s-\tau_{u-}}^i(t-u) & \text{if } u < t \leq \zeta_s^\infty; \end{cases}$$

- Or there is no such i, then $\tau_{s-} = u = \tau_s$ and we set

$$\zeta_s^\infty = u,$$
$$W_s^\infty(t) = Z_t, \quad t \leq u.$$

These prescriptions define a continuous process $W^\infty = F(Z,\mathcal{P})$ with values in $\mathcal{W}$. We denote by $\overline{\mathbb{N}}_0^\infty$ the law of W^∞, which is a probability measure on $\Omega = C(\mathbb{R}_+,\mathcal{W})$. As usual the head of W^∞ at time s is $\widehat{W}_s^\infty = W_s^\infty(\zeta_s^\infty)$.

The preceding construction can be reinterpreted by saying that the pair $(\zeta_s^\infty, \widehat{W}_s^\infty)_{s\geq 0}$ is obtained by concatenating (in the appropriate order given by the values of h_i) the functions

$$\left(h_i + \zeta_s^i, Z_{h_i} + \widehat{W}_s^i\right)_{0\leq s\leq \sigma_i}.$$

In particular, it is easy to verify that, a.s. for every $u \geq 0$,

$$\tau_u = \sup\{s \geq 0 : \zeta_s^\infty \leq u\}.$$

This simple observation will be useful later.

Suppose that conditionally given Z, $\mathcal{P}'$ is another Poisson measure with the same intensity as $\mathcal{P}$, and that $\mathcal{P}$ and $\mathcal{P}'$ are independent conditionally given Z. Then let $W = F(Z,\mathcal{P})$ and $W' = F(Z,\mathcal{P}')$ with the same functional F as above. We say that (W,W') is a pair of correlated eternal conditioned Brownian snakes (driven by the Bessel process Z).

3.3.2 Convergence of the rescaled uniform infinite well-labeled tree

Throughout this subsection, we consider a uniform infinite well-labeled tree, and we use the notation introduced in Theorem 8: In particular X_n, $n \in \mathbb{Z}_+$ are the labels along the spine, and L_i and R_i, $i \in \mathbb{Z}_+$, are the subtrees attached respectively to the left side and to the right side of spine. Recall that the left side (resp. right side) of the spine can be coded by the contour functions $\left(C^{(L)}, V^{(L)}\right)$ (resp. $\left(C^{(R)}, V^{(R)}\right)$). The main result of this section gives the joint convergence of these suitably rescaled random functions towards a pair of correlated eternal conditioned Brownian snakes.

Theorem 9. *Let Z be a nine-dimensional Bessel process and let $\left(W^{(L)}, W^{(R)}\right)$ be a pair of correlated eternal conditioned Brownian snakes driven by Z. We have the joint convergence in distribution:*

$$\begin{aligned}\left(\left(\frac{1}{n}C^{(L)}(n^2 s), \sqrt{\frac{3}{2n}}V^{(L)}(n^2 s)\right)_{s\geq 0}, \left(\frac{1}{n}C^{(R)}(n^2 s), \sqrt{\frac{3}{2n}}V^{(R)}(n^2 s)\right)_{s\geq 0}\right)&\\ \xrightarrow[n\to\infty]{\text{(d)}} \left(\left(\zeta_s^{(L)}, \widehat{W}_s^{(L)}\right)_{s\geq 0}, \left(\zeta_s^{(R)}, \widehat{W}_s^{(R)}\right)_{s\geq 0}\right).& \qquad (3.7)\end{aligned}$$

where $\zeta_s^{(L)} = \zeta_{\left(W_s^{(L)}\right)}$, resp. $\zeta_s^{(R)} = \zeta_{\left(W_s^{(R)}\right)}$, for every $s \geq 0$.

Before proving Theorem 9, we will establish a few preliminary results. For $\theta \in \mathbb{T}^l$ and every $t \geq 0$, we set

$$\left(C_\theta^{(n)}(t), V_\theta^{(n)}(t)\right) = \left(\frac{1}{n}C_\theta(n^2 t), \sqrt{\frac{3}{2n}}V_\theta(n^2 t)\right),$$

where (C_θ, V_θ) is the pair of contour functions of θ. To simplify notation, we will often write $\left(C_\theta^{(n)}, V_\theta^{(n)}\right) = \Gamma_\theta^{(n)}$. In addition, we also write

$$\mathcal{R}\left(\Gamma_\theta^{(n)}\right) = \{V_\theta^{(n)}(t) : t \geq 0\}.$$

Proposition 9. *Let φ be a bounded continuous function from $C(\mathbb{R}_+, \mathbb{R})^2 \times \mathbb{R}_+$ into $\mathbb{R}_+$. Assume that there exists $\eta > 0$ such that $\varphi(f, g, s) = 0$ if $s \leq \eta$. Fix $z > 0$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that $\sqrt{\frac{3}{2n}}x_n \to z$ as n goes to ∞. We have the following convergence:*

$$n\widehat{\rho}_{x_n}\left(\varphi\left(C_\theta^{(n)}, V_\theta^{(n)}, \frac{2|\theta|}{n^2}\right)\right) \underset{n\to\infty}{\longrightarrow} 2\mathbb{N}_z\left(\varphi\left(\zeta, \widehat{W}, \sigma\right)\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\right).$$

Proof. Fix $K > \eta$. Then, for every integer $n > 0$:

$$\begin{aligned}
&n\widehat{\rho}_{x_n}\left(\varphi\left(\Gamma_\theta^{(n)}, \frac{2|\theta|}{n^2}\right)\right) \\
&= 2nw_{x_n}^{-1}\rho_{x_n}\left(\mathbf{1}_{\left\{\mathcal{R}\left(\Gamma_\theta^{(n)}\right)\subset]0,\infty[\right\}}\varphi\left(\Gamma_\theta^{(n)}, \frac{2|\theta|}{n^2}\right)\right) \\
&= 2nw_{x_n}^{-1}\sum_{k=\lfloor\eta n^2/2\rfloor}^{\lfloor Kn^2\rfloor}\rho_{x_n}\left(|\theta| = k\right)\rho_{x_n}\left(\mathbf{1}_{\left\{\mathcal{R}\left(\Gamma_\theta^{(n)}\right)\subset]0,\infty[\right\}}\varphi\left(\Gamma_\theta^{(n)}, \frac{2|\theta|}{n^2}\right)\middle| |\theta| = k\right) \\
&\quad + 2nw_{x_n}^{-1}\rho_{x_n}\left(|\theta| > Kn^2\right)\rho_{x_n}\left(\mathbf{1}_{\left\{\mathcal{R}\left(\Gamma_\theta^{(n)}\right)\subset]0,\infty[\right\}}\varphi\left(\Gamma_\theta^{(n)}, \frac{2|\theta|}{n^2}\right)\middle| |\theta| > Kn^2\right)
\end{aligned} \tag{3.8}$$

The first term in the right-hand side of (3.8) can be written as

$$\begin{aligned}
&2n^3w_{x_n}^{-1}\int_{\frac{\lfloor\eta n^2/2\rfloor}{n^2}}^{\frac{\lfloor Kn^2\rfloor+1}{n^2}}\mathrm{d}s\,\rho_{x_n}\left(|\theta| = \lfloor sn^2\rfloor\right) \\
&\qquad\times\rho_{x_n}\left(\mathbf{1}_{\left\{\mathcal{R}\left(\Gamma_\theta^{(n)}\right)\subset]-0,\infty[\right\}}\varphi\left(\Gamma_\theta^{(n)}, \frac{2\lfloor sn^2\rfloor}{n^2}\right)\middle| |\theta| = \lfloor sn^2\rfloor\right)
\end{aligned} \tag{3.9}$$

In order to investigate the behavior of the quantity (3.9) as $n \to \infty$, we use a result about the convergence of discrete snakes. Fix $y > 0$ and let $(y_k)_{k\in\mathbb{N}}$ be a sequence of positive integers such that $\left(\frac{9}{8k}\right)^{1/4}y_k \to y$ as n goes to ∞. Let $(\mathbf{W}_t)_{t\in[0,1]}$ be a Brownian snake excursion with law $\mathbb{N}_y^{(1)}$. Then $(\mathbf{e}_t)_{t\in[0,1]} = (\zeta_{(\mathbf{W}_t)})_{t\in[0,1]}$ is a normalized Brownian excursion. Theorem 4 of [CS04] (see also Theorem 2 of [JM05]) implies that the law of the pair

$$\left(\left(\frac{C_\theta(2kt)}{\sqrt{2k}}\right)_{t\in[0,1]}, \left(\left(\frac{9}{8}\right)^{1/4}\frac{V_\theta(2kt)}{k^{1/4}}\right)_{t\in[0,1]}\right)$$

under $\rho_{y_k}(\cdot\,||\theta| = k)$ converges as k goes to infinity to the law of $\left(\mathbf{e}_t, \widehat{\mathbf{W}}_t\right)_{t\in[0,1]}$ in the sense of weak convergence of probability measures on $C\left([0,1], \mathbb{R}\right)^2$. This entails that, for every $s > 0$,

$$\rho_{x_n}\left(\mathbf{1}_{\left\{\mathcal{R}\left(\Gamma^{(n)}_\theta\right)\subset]0,\infty[\right\}}\varphi\left(\Gamma^{(n)}_\theta,\frac{2\lfloor sn^2\rfloor}{n^2}\right)\middle|\,|\theta|=\lfloor sn^2\rfloor\right)$$
$$\underset{n\to\infty}{\longrightarrow}\mathbb{N}^{(1)}_{(2s)^{-1/4}z}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\sqrt{2s}\zeta_{(./2s)},(2s)^{1/4}\widehat{W}_{(./2s)},2s\right)\right).$$

To justify the latter convergence, we use the property

$$\mathbb{N}^{(1)}_{(2s)^{-1/4}z}\left(\inf_{t\in\mathbb{R}_+}\widehat{W}_t=0\right)=0,$$

which follows from the fact that the law of the infimum of a Brownian snake driven by a normalized Brownian excursion $\mathbf{e}$ has no atoms: see the beginning of the proof of Lemma 7.1 in [LG06].

A scaling argument then gives

$$\mathbb{N}^{(1)}_{(2s)^{-1/4}z}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\sqrt{2s}\zeta_{(./2s)},(2s)^{1/4}\widehat{W}_{(./2s)},2s\right)\right)=\mathbb{N}^{(2s)}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\zeta,\widehat{W},2s\right)\right)$$

and thus we have proved

$$\rho_{x_n}\left(\mathbf{1}_{\left\{\mathcal{R}\left(\Gamma^{(n)}_\theta\right)\subset]0,\infty[\right\}}\varphi\left(\Gamma^{(n)}_\theta,\frac{2\lfloor sn^2\rfloor}{n^2}\right)\middle|\,|\theta|=\lfloor sn^2\rfloor\right)\underset{n\to\infty}{\longrightarrow}\mathbb{N}^{(2s)}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\zeta,\widehat{W},2s\right)\right).\tag{3.10}$$

From the explicit formula for w_l, we have $w_l\geq 4/3$ for every $l>0$. Using also (3.1) we get the following bound for n large enough: for every $s\in[\eta,K]$,

$$2n^3w_{x_n}^{-1}\rho_{x_n}\left(|\theta|=\lfloor sn^2\rfloor\right)$$
$$\times\,\rho_{x_n}\left(\mathbf{1}_{\left\{\mathcal{R}\left(\Gamma^{(n)}_\theta\right)\subset]0,\infty[\right\}}\varphi\left(\Gamma^{(n)}_\theta,\frac{2|\theta|}{n^2}\right)\middle|\,|\theta|=\lfloor sn^2\rfloor\right)\leq\frac{3}{2\sqrt{\pi\eta^3}}\|\varphi\|_\infty,\tag{3.11}$$

where $\|\varphi\|_\infty$ is the supremum of $|\varphi|$.

We can use (3.1), (3.10), (3.11) (to justify dominated convergence) and the fact that $w_{x_n}\to 2$ as $n\to\infty$ to see that the quantity (3.9) converges as $n\to\infty$ to

$$\int_\eta^K\frac{\mathrm{d}s}{2\sqrt{\pi s^3}}\mathbb{N}^{(2s)}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\zeta,\widehat{W},2s\right)\right)=\int_0^K\frac{\mathrm{d}s}{2\sqrt{\pi s^3}}\mathbb{N}^{(2s)}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\zeta,\widehat{W},2s\right)\right).$$

Since this holds for every $K>\eta$, we get by using (3.3) that

$$\liminf_{n\to\infty}n\widehat{\rho}_{x_n}\left[\varphi\left(\Gamma^{(n)}_\theta,\frac{2|\theta|}{n^2}\right)\right]\geq 2\mathbb{N}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\zeta,\widehat{W},\sigma\right)\right).$$

Similar arguments, using also the estimate (3.2), lead to

$$\limsup_{n\to\infty}n\widehat{\rho}_{x_n}\left[\varphi\left(\Gamma^{(n)}_\theta,\frac{2|\theta|}{n^2}\right)\right]\leq\int_\eta^K\frac{\mathrm{d}s}{2\sqrt{\pi s^3}}\mathbb{N}^{(2s)}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\zeta,\widehat{W},2s\right)\right)+\frac{C}{\sqrt{K}}\|\varphi\|_\infty.$$

with a constant C that does not depend on K. By letting $K\to\infty$, we get

$$\limsup_{n\to\infty}n\widehat{\rho}_{x_n}\left[\varphi\left(\Gamma^{(n)}_\theta,\frac{2|\theta|}{n^2}\right)\right]\leq 2\mathbb{N}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\varphi\left(\zeta,\widehat{W},\sigma\right)\right)$$

which completes the proof. □

We now state a technical lemma, which will play an important role in the proof of Theorem 9. We need to introduce some notation. For every integer $n \geq 1$ and every $h > 0$, we set

$$\tau^{(L,n,h)} = \frac{\lfloor nh \rfloor}{n^2} + \sum_{i=0}^{\lfloor nh \rfloor} 2n^{-2}|L_i|.$$

This is the time needed in the rescaled contour of the left side of the spine to explore the trees L_i, $0 \leq i \leq \lfloor nh \rfloor$. Furthermore, for every integer $k \geq 0$, we write J_k for the unique index i such that the vertex visited at time k in the contour of the left side of the spine belongs to L_i.

Lemma 7. *Let $h > 0$. For every $\kappa > 0$, we can find $\delta > 0$ sufficiently small so that, for all large integers n,*

$$P\left[\sup_{0\leq u<v\leq \tau^{(L,n,h)},\ v-u<\delta} \frac{1}{n}\left|J_{\lfloor n^2u \rfloor} - J_{\lfloor n^2v \rfloor}\right| > \kappa\right] < \kappa.$$

Proof. To simplify notation, we write $p_n(\kappa,\delta)$ for the probability that is considered in the lemma. Suppose that there exist u and v with $0 \leq u < v \leq \tau^{(L,n,h)}$ and $v - u < \delta$, such that $|J_{\lfloor n^2u \rfloor} - J_{\lfloor n^2v \rfloor}| > n\kappa$. Notice that all vertices belonging to the subtrees L_i for indices i such that $J_{\lfloor n^2u \rfloor} < i < J_{\lfloor n^2v \rfloor}$ are visited by the contour of the left side of the spine between times $\lfloor n^2u \rfloor$ and $\lfloor n^2v \rfloor$. Hence

$$2\sum_{J_{\lfloor n^2u \rfloor}<i<J_{\lfloor n^2v \rfloor}} |L_i| \leq \lfloor n^2v \rfloor - \lfloor n^2u \rfloor \leq n^2\delta + 1.$$

Since $|J_{\lfloor n^2u \rfloor} - J_{\lfloor n^2v \rfloor}| > n\kappa$, we can find an integer j of the form $j = l\lfloor n\kappa/2 \rfloor$, with $1 \leq l \leq nh/\lfloor n\kappa/2 \rfloor$, such that the inequalities $J_{\lfloor n^2u \rfloor} < i < J_{\lfloor n^2v \rfloor}$ hold for $i = j+1, j+2, \ldots, j + \lfloor n\kappa/2 \rfloor$.

It follows from the preceding considerations that

$$\begin{aligned} p_n(\kappa,\delta) &\leq P\left[\bigcup_{1\leq l\leq nh/\lfloor n\kappa/2 \rfloor}\left\{2\sum_{i=1}^{\lfloor n\kappa/2 \rfloor} |L_{l\lfloor n\kappa/2 \rfloor+i}| \leq n^2\delta + 1\right\}\right] \\ &\leq P\left[\bigcup_{1\leq l\leq nh/\lfloor n\kappa/2 \rfloor}\left(\bigcap_{i=1}^{\lfloor n\kappa/2 \rfloor}\left\{2|L_{l\lfloor n\kappa/2 \rfloor+i}| \leq n^2\delta + 1\right\}\right)\right]. \end{aligned}$$

From assertion 2. in Theorem 8 and properties of the Bessel process, we can fix $\eta > 0$ and $A > 0$ such that

$$P\left[\eta\sqrt{n} \leq X_i \leq A\sqrt{n},\ \forall i \in \{\lfloor n\kappa/2 \rfloor, \ldots, \lfloor nh \rfloor + \lfloor n\kappa/2 \rfloor\}\right] > 1 - \kappa/2.$$

It follows that

$$\begin{aligned} p_n(\kappa,\delta) &\leq \frac{\kappa}{2} + \sum_{1\leq l\leq nh/\lfloor n\kappa/2 \rfloor} P\left[\bigcap_{i=1}^{\lfloor n\kappa/2 \rfloor}\left\{2|L_{l\lfloor n\kappa/2 \rfloor+i}| \leq n^2\delta + 1, \eta\sqrt{n} \leq X_{l\lfloor n\kappa/2 \rfloor+i} \leq A\sqrt{n}\right\}\right] \\ &\leq \frac{\kappa}{2} + \frac{nh}{\lfloor n\kappa/2 \rfloor}\left(\sup_{\eta\sqrt{n}\leq x\leq A\sqrt{n}} \widehat{\rho}_x\left(2|\theta| \leq n^2\delta + 1\right)\right)^{\lfloor n\kappa/2 \rfloor} \end{aligned}$$

using the conditional distribution of the trees L_i given the labels on the spine (Assertion 3. in Theorem 8). We can find a large constant $K > 0$ such that, for every sufficiently large n,

$$\frac{\kappa}{2} + \frac{nh}{\lfloor n\kappa/2\rfloor}\left(1 - \frac{K}{n}\right)^{\lfloor n\kappa/2\rfloor} < \kappa.$$

To complete the proof of the lemma, we just have to observe that we can choose $\delta > 0$ sufficiently small so that, for all n large,

$$\inf_{\eta\sqrt{n}\le x\le A\sqrt{n}} \widehat{\rho}_x\left(2|\theta| > n^2\delta + 1\right) \ge \frac{K}{n}.$$

This is indeed a consequence of Proposition 9, together with the fact that

$$\lim_{\delta\downarrow 0} \mathbb{N}_\eta\left(\sigma > \delta, \mathcal{R}\subset]0,\infty[\right) = \mathbb{N}_\eta\left(\mathcal{R}\subset]0,\infty[\right) = +\infty.$$

□

We denote the rescaled contour functions of the spatial trees L_i (resp. R_i) by $C^{(n)}_{L_i}$ and $V^{(n)}_{L_i}$ (resp. $C^{(n)}_{R_i}$, $V^{(n)}_{R_i}$), in agreement with the notation introduced after Theorem 9. To simplify notation we also put

$$X^{(n)}_t = \sqrt{\frac{3}{2n}}X_{\lfloor nt\rfloor},\ t\ge 0.$$

Proposition 10. *Fix $\varepsilon > 0$ and $h_0 > 0$. Let $\phi : \mathbb{D}(\mathbb{R}_+) \to \mathbb{R}$ and $\psi^{(L)}, \psi^{(R)} : \mathbb{R}_+ \times C(\mathbb{R}_+,\mathbb{R})^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ be bounded continuous functions. Assume that $\psi^{(L)}$ and $\psi^{(R)}$ are Lipschitz with respect to the first variable and such that $\psi^{(L)}(h,f,g,s) = 0$ and $\psi^{(R)}(h,f,g,s) = 0$ if $h \ge h_0$ or $s \le \varepsilon$. Then*

$$\begin{aligned}
E\Bigg[\phi\left(X^{(n)}\right)\exp\left(-\sum_{i=0}^{\infty}\psi^{(L)}\left(\frac{i}{n}, C^{(n)}_{L_i}, V^{(n)}_{L_i}, \frac{2|L_i|}{n^2}\right)\right)&\\
\times\exp\left(-\sum_{i=0}^{\infty}\psi^{(R)}\left(\frac{i}{n}, C^{(n)}_{R_i}, V^{(n)}_{R_i}, \frac{2|R_i|}{n^2}\right)\right)\Bigg]&\\
\underset{n\to\infty}{\longrightarrow} \mathbb{E}^{(9)}_0\Bigg[\phi\left(\mathbf{R}\right)\exp\left(-2\int_0^\infty \mathrm{d}h\,\mathbb{N}_{\mathbf{R}_h}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\left(1-\exp-\psi^{(L)}\left(h,\zeta,\widehat{W},\sigma\right)\right)\right)\right)&\\
\times\exp\left(-2\int_0^\infty \mathrm{d}h\,\mathbb{N}_{\mathbf{R}_h}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\left(1-\exp-\psi^{(R)}\left(h,\zeta,\widehat{W},\sigma\right)\right)\right)\right)\Bigg]&.
\end{aligned}$$

Remark. We can interpret the limit in the theorem in terms of Poisson point processes. Let Z be a nine-dimensional Bessel process started at 0. Conditionally given Z, let $(\mathcal{P}^{(L)}, \mathcal{P}^{(R)})$ be a pair of independent Poisson point processes on $\mathbb{R}_+\times\Omega$ with intensity given by (3.6). Then, the exponential formula for Poisson point processes shows that the limit appearing in the proposition is equal to

$$\begin{aligned}
E\Bigg[\phi\left(Z\right)\exp\left(-\int\psi^{(L)}\left(h,\zeta_\cdot(\omega), Z_h+\widehat{W}_\cdot(\omega),\sigma(\omega)\right)\mathcal{P}^{(L)}(\mathrm{d}h,\mathrm{d}\omega)\right)&\\
\times\exp\left(-\int\psi^{(R)}\left(h,\zeta_\cdot(\omega), Z_h+\widehat{W}_\cdot(\omega),\sigma(\omega)\right)\mathcal{P}^{(R)}(\mathrm{d}h,\mathrm{d}\omega)\right)\Bigg]&
\end{aligned}$$

Proof. We have

$$\begin{aligned}
E\Bigg[\phi\left(X^{(n)}\right)\exp\left(-\sum_{i=0}^{\infty}\psi^{(L)}\left(\frac{i}{n},C_{L_i}^{(n)},V_{L_i}^{(n)},\frac{2|L_i|}{n^2}\right)\right)&\\
\times\exp\left(-\sum_{i=0}^{\infty}\psi^{(R)}\left(\frac{i}{n},C_{R_i}^{(n)},V_{R_i}^{(n)},\frac{2|R_i|}{n^2}\right)\right)\Bigg]&\\
=E\Bigg[\phi\left(X^{(n)}\right)\prod_{i=0}^{\infty}E\left[\exp-\psi^{(L)}\left(\frac{i}{n},C_{L_i}^{(n)},V_{L_i}^{(n)},\frac{2|L_i|}{n^2}\right)\Bigg|X_i\right]&\\
\times\prod_{i=0}^{\infty}E\left[\exp-\psi^{(R)}\left(\frac{i}{n},C_{L_i}^{(n)},V_{L_i}^{(n)},\frac{2|R_i|}{n^2}\right)\Bigg|X_i\right]\Bigg]&
\end{aligned}\tag{3.12}$$

using the independence of the subtrees L_i and R_i given the labels on the spine (Theorem 8).

Let us study the contribution of the left side of the spine in (3.12). By Theorem 8 again,

$$\begin{aligned}
&\prod_{i=0}^{\infty}E\left[\exp-\psi^{(L)}\left(\frac{i}{n},C_{L_i}^{(n)},V_{L_i}^{(n)},\frac{2|L_i|}{n^2}\right)\Bigg|X_i\right]\\
&\quad=\prod_{i=0}^{\infty}\widehat{\rho}_{X_i}\left(\exp-\psi^{(L)}\left(\frac{i}{n},\Gamma_\theta^{(n)},\frac{2|\theta|}{n^2}\right)\right)\\
&\quad=\exp\sum_{i=0}^{\infty}\log\widehat{\rho}_{X_i}\left(\exp-\psi^{(L)}\left(\frac{i}{n},\Gamma_\theta^{(n)},\frac{2|\theta|}{n^2}\right)\right)\\
&\quad=\exp n\int_0^{\infty}\mathrm{d}t\,\log\left(1-\widehat{\rho}_{X_{\lfloor nt\rfloor}}\left(1-\exp-\psi^{(L)}\left(\frac{\lfloor nt\rfloor}{n},\Gamma_\theta^{(n)},\frac{2|\theta|}{n^2}\right)\right)\right).
\end{aligned}\tag{3.13}$$

By the second assertion of Theorem 8 and the Skorokhod representation theorem we can find, for every $n\geq 1$ a process $\left(\widetilde{X}_k^n\right)_{k\geq 0}$ having the same distribution as $(X_k)_{k\geq 0}$, and a nine-dimensional Bessel process Z started from 0, such that almost surely, for every $a>0$, $\left(\sqrt{\frac{3}{2n}}\widetilde{X}_{\lfloor nt\rfloor}^n\right)_{0\leq t\leq a}$ converges uniformly to $(Z_t)_{0\leq t\leq a}$ as n goes to infinity. Using the Lipschitz property of $\psi^{(L)}$ in the first variable, together with the fact that $\psi^{(L)}(h,f,g,s)=0$ if $s\leq\varepsilon$, we can easily deduce from Proposition 9 that for every fixed $t>0$:

$$\begin{aligned}
&n\,\widehat{\rho}_{\widetilde{X}_{\lfloor nt\rfloor}^n}\left(1-\exp-\psi^{(L)}\left(\frac{\lfloor nt\rfloor}{n},\Gamma_\theta^{(n)},\frac{2|\theta|}{n^2}\right),\right)\\
&\qquad\underset{n\to\infty}{\longrightarrow}2\mathbb{N}_{Z_t}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\left(1-\exp-\psi^{(L)}\left(t,\zeta,\widehat{W},\sigma\right)\right)\right),\ \text{a.s.}
\end{aligned}\tag{3.14}$$

From our assumptions on $\psi^{(L)}$, we have for every $t>0$ and $n\geq 0$:

$$\begin{aligned}
&n\widehat{\rho}_{\widetilde{X}_{\lfloor nt\rfloor}^n}\left(1-\exp-\psi^{(L)}\left(\frac{\lfloor nt\rfloor}{n},\Gamma_\theta^{(n)},\frac{2|\theta|}{n^2}\right)\right)\\
&\quad=n\widehat{\rho}_{\widetilde{X}_{\lfloor nt\rfloor}^n}\left(\mathbf{1}_{\{t\leq h_0+1\}}\mathbf{1}_{\{|\theta|\geq\lfloor\varepsilon n^2\rfloor/2\}}\left(1-\exp-\psi^{(L)}\left(\frac{\lfloor nt\rfloor}{n},\Gamma_\theta^{(n)},\frac{2|\theta|}{n^2}\right)\right)\right)\\
&\quad\leq\mathbf{1}_{\{t\leq h_0+1\}}\,n\widehat{\rho}_{\widetilde{X}_{\lfloor nt\rfloor}^n}\left(|\theta|\geq\lfloor\varepsilon n^2\rfloor/2\right).
\end{aligned}$$

It then follows from (3.2) that there exists a constant $K > 0$, which does not depend on t, such that for every $t > 0$ and every $n \geq 0$ one has:

$$n\widehat{\rho}_{\widetilde{X}^n_{\lfloor nt\rfloor}}\left(1 - \exp -\psi^{(L)}\left(\frac{\lfloor nt\rfloor}{n}, \Gamma^{(n)}_\theta, \frac{2|\theta|}{n^2}\right)\right) \leq K\mathbf{1}_{\{t\leq h_0+1\}}.$$

Thus, we can use (3.14) and dominated convergence to see that the right hand side of (3.13) (with X replaced by $\widetilde{X}^n$) converges a.s. to

$$\exp -2\int_0^\infty \mathrm{d}t\, \mathbb{N}_{Z_t}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\left(1 - \exp -\psi^{(L)}\left(t, \zeta, \widehat{W}, \sigma\right)\right)\right)$$

as $n \to \infty$. A similar analysis applies to the contribution of the right side of the spine in (3.12), and we conclude that

$$\begin{aligned}
E\Bigg[\phi\left(X^{(n)}\right)\exp\left(-\sum_{i=0}^{\infty}\psi^{(L)}\left(\frac{i}{n}, C^{(n)}_{L_i}, V^{(n)}_{L_i}, \frac{2|L_i|}{n^2}\right)\right)&\\
\times \exp\left(-\sum_{i=0}^{\infty}\psi^{(R)}\left(\frac{i}{n}, C^{(n)}_{R_i}, V^{(n)}_{R_i}, \frac{2|R_i|}{n^2}\right)\right)\Bigg]&\\
\underset{n\to\infty}{\longrightarrow} E\Bigg[\phi(Z)\exp -2\int_0^\infty \mathrm{d}t\, \mathbb{N}_{Z_t}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\left(1 - \exp -\psi^{(L)}\left(t, \zeta, \widehat{W}, \sigma\right)\right)\right)&\\
\times \exp -2\int_0^\infty \mathrm{d}t\, \mathbb{N}_{Z_t}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}}\left(1 - \exp -\psi^{(R)}\left(t, \zeta, \widehat{W}, \sigma\right)\right)\right)\Bigg]&.
\end{aligned}$$

This completes the proof. □

Fix $h_0 > 0$ and $\varepsilon > 0$. Let $\mathcal{P}^{(L,n,h_0,\varepsilon)}$ be the finite point measure on $[0, h_0]\times C(\mathbb{R}_+, \mathbb{R})^2\times \mathbb{R}_+$ defined by

$$\mathcal{P}^{(L,n,h_0,\varepsilon)} = \sum_{i\geq 0}\mathbf{1}_{\left\{\frac{i}{n}\leq h_0\right\}}\mathbf{1}_{\left\{\sigma\left(C^{(n)}_{L_i}\right)\geq\varepsilon\right\}}\delta_{\frac{i}{n}}\otimes\delta_{\left(C^{(n)}_{L_i}, V^{(n)}_{L_i}\right)}\otimes\delta_{\frac{2|L_i|}{n^2}}.$$

We denote by $\mathcal{P}^{(R,n,h_0,\varepsilon)}$ the point measure defined similarly for the right side of the spine. The random variables $\mathcal{P}^{(L,n,h_0,\varepsilon)}$ and $\mathcal{P}^{(R,n,h_0,\varepsilon)}$ take values in the space

$$E := \mathcal{M}_f\left(\mathbb{R}_+\times C(\mathbb{R}_+, \mathbb{R})^2\times\mathbb{R}_+\right)$$

of all finite measures on $\mathbb{R}_+\times C(\mathbb{R}_+, \mathbb{R})^2\times\mathbb{R}_+$, which is a Polish space.

Let Z be a nine-dimensional Bessel process started at 0. As in the preceding proof we consider two point processes $\mathcal{P}^{(L)}$ and $\mathcal{P}^{(R)}$ on $\mathbb{R}_+\times\Omega$, which conditionally given Z are independent and Poisson with intensity given by (3.6). Then we define a random element $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$ of E by

$$\begin{aligned}
\int\mathcal{P}^{(L,\infty,h_0,\varepsilon)}&(\mathrm{d}h\mathrm{d}f\mathrm{d}g\mathrm{d}s)F(h, f, g, s)\\
&= \int\mathcal{P}^{(L)}(\mathrm{d}h\mathrm{d}\omega)F\left(h, \zeta(\omega), Z_h + \widehat{W}(\omega), \sigma(\omega)\right)\mathbf{1}_{\{h\leq h_0, \sigma(\omega)\geq\varepsilon\}}.
\end{aligned}$$

We similarly define $\mathcal{P}^{(R,\infty,h_0,\varepsilon)}$ from the point process $\mathcal{P}^{(R)}$.

Corollary 2. *For every fixed $\varepsilon > 0$ and $h_0 > 0$,*

$$\left(X^{(n)}, \mathcal{P}^{(L,n,h_0,\varepsilon)}, \mathcal{P}^{(R,n,h_0,\varepsilon)}\right) \underset{n\to\infty}{\longrightarrow} \left(Z, \mathcal{P}^{(L,\infty,h_0,\varepsilon)}, \mathcal{P}^{(R,\infty,h_0,\varepsilon)}\right),$$

in the sense of convergence in distribution for random variables with values in $\mathbb{D}(\mathbb{R}_+) \times E \times E$.

Proof. Let us first show that the sequence of the laws of $\mathcal{P}^{(L,n,h_0,\varepsilon)}$ is tight. We will verify that, for every $\alpha > 0$, there is a real number $M_\alpha \geq 0$ and a compact subset K_α of $[0, h_0] \times C(\mathbb{R}_+, \mathbb{R})^2 \times \mathbb{R}_+$ such that, for every integer $n \geq 1$, with probability at least $1-\alpha$, the measure $\mathcal{P}^{(L,n,h_0,\varepsilon)}$ has total mass bounded by M_α and is supported on K_α. Since the set of all finite measures supported on K_α with total mass bounded by M_α is compact, Prohorov's theorem will imply the desired tightness.

Since for every $x \geq 1$,

$$\widehat{\rho}_x(\sigma(C_\theta^{(n)}) \geq \varepsilon) \leq 2\rho_x(\sigma(C_\theta^{(n)}) \geq \varepsilon) = 2\rho_1(2|\theta| \geq \varepsilon n^2) = O(n^{-1})$$

a first moment calculation shows that we can find a constant M_α such that, for every $n \geq 1$,

$$P\left[\left|\mathcal{P}^{(L,n,h_0,\varepsilon)}\right| \geq M_\alpha\right] < \frac{\alpha}{2}.$$

A similar argument shows the existence of a constant H_α large enough so that, for every n,

$$P\left[\mathcal{P}^{(L,n,h_0,\varepsilon)}([0, h_0] \times C(\mathbb{R}_+, \mathbb{R})^2 \times]H_\alpha, \infty[) > 0\right] < \frac{\alpha}{4}.$$

We will thus take the compact set K_α of the form

$$K_\alpha = [0, h_0] \times \mathcal{K}_\alpha \times [0, H_\alpha].$$

where $\mathcal{K}_\alpha$ will be a suitable compact subset of $C(\mathbb{R}_+, \mathbb{R})^2$. To construct $\mathcal{K}_\alpha$, we rely on the convergence results for discrete snakes. We first note that, thanks to the convergence in distribution of the rescaled processes $\left(\sqrt{\frac{3}{2n}} X_{\lfloor nt \rfloor}\right)_{t\geq 0}$, we can find a constant A_α such that, for every $n \geq 1$,

$$P\left[\sup_{0\leq i\leq \lfloor h_0 n\rfloor} X_i \geq A_\alpha \sqrt{n}\right] < \alpha/8.$$

Theorem 4 of [CS04], or Theorem 2 of [JM05], implies that the collection of the distributions of the processes $(C_\theta^{(n)}, V_\theta^{(n)})$ under the probability measures $\rho_x(\cdot \mid \varepsilon n^2 \leq |\theta| \leq H_\alpha n^2)$, for $n \geq 1$ and x varying in $[0, A_\alpha\sqrt{n}]$, is tight (of course the choice of x here just amounts to a translation of the labels). In particular, we can find compact subsets $\mathcal{K}$ of $C(\mathbb{R}_+, \mathbb{R})^2$ for which

$$\rho_x\left((C_\theta^{(n)}, V_\theta^{(n)}) \notin \mathcal{K} \mid \varepsilon n^2 \leq |\theta| \leq H_\alpha n^2\right)$$

is arbitrarily small, uniformly in $x \in [0, A_\alpha\sqrt{n}]$ and $n \geq 1$. Using once again the bound $\widehat{\rho}_x \leq 2\rho_x$, we can thus find a compact subset $\mathcal{K}_\alpha$ of $C(\mathbb{R}_+, \mathbb{R})^2$ such that

$$(\lfloor nh_0 \rfloor + 1) \times \widehat{\rho}_x\left(\left\{(C_\theta^{(n)}, V_\theta^{(n)}) \notin \mathcal{K}_\alpha\right\} \cap \left\{\varepsilon n^2 \leq |\theta| \leq H_\alpha n^2\right\}\right) \leq \alpha/8,$$

for every $x \in [0, A_\alpha\sqrt{n}]$ and $n \geq 1$. From this last bound and a first moment calculation, we get

$$P\left[\left\{\sup_{0\leq i\leq \lfloor h_0 n\rfloor} X_i \leq A_\alpha\sqrt{n}\right\} \cap \left\{\mathcal{P}^{(L,n,h_0,\varepsilon)}([0,h_0]\times\mathcal{K}_\alpha^c\times[0,H_\alpha]) > 0\right\}\right] \leq \alpha/8.$$

We take $K_\alpha = [0,h_0]\times\mathcal{K}_\alpha\times[0,H_\alpha]$ as already mentioned, and by putting together the previous estimates, we arrive at

$$P\left[\left\{\left|\mathcal{P}^{(L,n,h_0,\varepsilon)}\right| \leq M_\alpha\right\} \cap \left\{\mathcal{P}^{(L,n,h_0,\varepsilon)}(K_\alpha^c) = 0\right\}\right] \geq 1-\alpha.$$

This completes the proof of tightness.

The same arguments also give the tightness of the sequence of the laws of $\mathcal{P}^{(R,n,h_0,\varepsilon)}$. Therefore, we know that the sequence of the laws of $\left(X^{(n)}, \mathcal{P}^{(L,n,h_0,\varepsilon)}, \mathcal{P}^{(R,n,h_0,\varepsilon)}\right)$ is tight.

Proposition 10, and the remark following the statement of this proposition, now show that

$$E\left[\Psi\left(X^{(n)}, \mathcal{P}^{(L,n,h_0,\varepsilon)}, \mathcal{P}^{(R,n,h_0,\varepsilon)}\right)\right] \underset{n\to\infty}{\longrightarrow} E\left[\Psi\left(Z, \mathcal{P}_L^{(\infty,h_0,\varepsilon)}, \mathcal{P}_R^{(\infty,h_0,\varepsilon)}\right)\right]$$

for all functions Ψ of the type

$$\Psi(u,m_1,m_2) = \phi(u)\exp\left(-\int\psi^{(L)}\,\mathrm{d}m_1 - \int\psi^{(R)}\,\mathrm{d}m_2\right),$$

with ϕ, $\psi^{(L)}$ and $\psi^{(R)}$ as in Proposition 10. Once we know that the sequence of the laws of $\left(X^{(n)}, \mathcal{P}^{(L,n,h_0,\varepsilon)}, \mathcal{P}^{(R,n,h_0,\varepsilon)}\right)$ is tight, this suffices to get the statement of Corollary 2 (we also use the fact that $\mathbb{N}_0(\sigma=\varepsilon)=0$). $\square$

Proof of Theorem 9. Throughout the proof, $h_0 > 0$ is fixed. We consider as previously a triplet $(Z, \mathcal{P}^{(L)}, \mathcal{P}^{(R)})$ such that Z is a nine-dimensional Bessel process started at 0, and conditionally given Z, $(\mathcal{P}^{(L)}, \mathcal{P}^{(R)})$ is a pair of independent Poisson point processes on $\mathbb{R}_+\times\Omega$ with intensity given by (3.6). We assume that the process $W^{(L)}$, resp. $W^{(R)}$ is then determined from the pair $(Z, \mathcal{P}^{(L)})$, resp. $(Z, \mathcal{P}^{(R)})$, in the way explained in subsection 3.1. In agreement with this subsection, we also use the notation

$$\tau_u^{(L)} = \sup\left\{s\geq 0 : \zeta_s^{(L)} \leq u\right\}$$

for every $u \geq 0$.

Let us fix $\varepsilon > 0$. For every $n > 0$, let $C^{(L,n,h_0,\varepsilon)}$ denote the concatenation of the functions $\left(\frac{i}{n} + C_{L_i}^{(n)}(t)\right)_{0\leq t<2n^{-2}|L_i|}$, for all integers i such that $2n^{-2}|L_i| \geq \varepsilon$ and $i \leq nh_0$. The random function $C^{(L,n,h_0,\varepsilon)}$ is defined and càdlàg on the time interval $[0, \tau^{(L,n,h_0,\varepsilon)}[$, where

$$\tau^{(L,n,h_0,\varepsilon)} = \sum_{i\leq nh_0} \mathbf{1}_{\{2n^{-2}|L_i|\geq\varepsilon\}} 2n^{-2}|L_i|.$$

We extend the function $t \to C^{(L,n,h_0,\varepsilon)}$ to $[0,\infty[$ by setting $C^{(L,n,h_0,\varepsilon)}(t) = \frac{\lfloor nh_0\rfloor}{n}$ for every $t \in [\tau^{(L,n,h_0,\varepsilon)}, \infty[$.

We denote the rescaled contour function of the left side of the spine of the uniform infinite well-labeled tree, up to and including its subtree $L_{\lfloor nh_0\rfloor}$ at generation $\lfloor nh_0\rfloor$, by

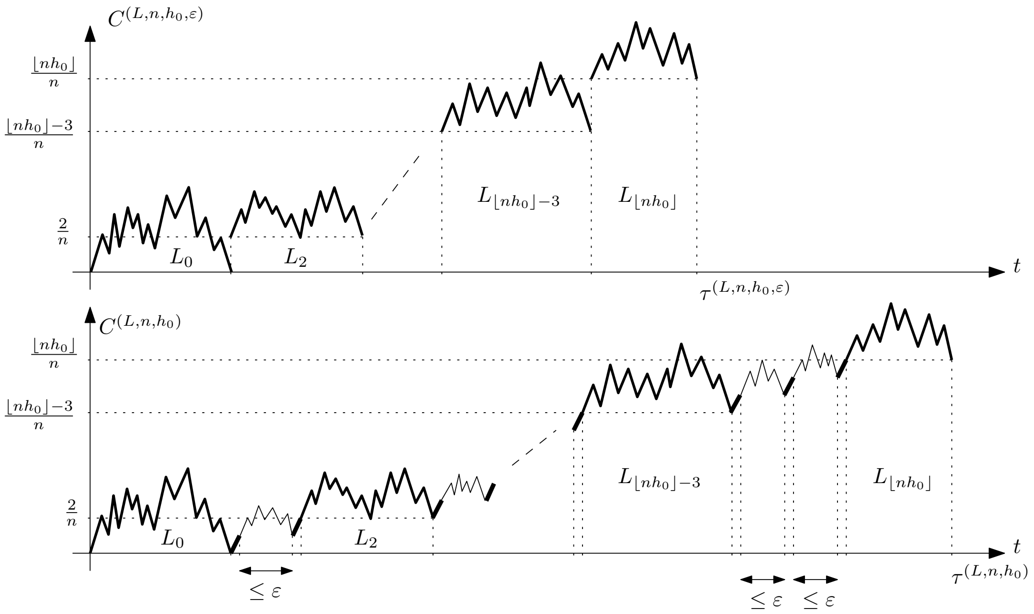

Figure 3.5: The processes $C^{(L,n,h_0)}$ and $C^{(L,n,h_0,\varepsilon)}$.

$C^{(L,n,h_0)}$. The function $t \to C^{(L,n,h_0)}(t)$ is defined and continuous over $[0, \tau^{(L,n,h_0)}]$, where as previously

$$\tau^{(L,n,h_0)} = \frac{\lfloor nh_0 \rfloor}{n^2} + \sum_{i \le nh_0} 2n^{-2}|L_i|.$$

Again, we extend $C^{(L,n,h_0)}$ to $[0,\infty[$ by setting $C^{(L,n,h_0)}(t) = \frac{\lfloor nh_0 \rfloor}{n}$ if $t \ge \tau^{(L,n,h_0)}$. Note that we have also

$$\tau^{(L,n,h_0)} = \sup\left\{t \ge 0 : \frac{1}{n}C^{(L)}(n^2 t) \le \frac{\lfloor nh_0 \rfloor}{n}\right\}$$

and that $C^{(L,n,h_0)}(t) = \frac{1}{n}C^{(L)}(n^2(t \wedge \tau^{(L,n,h_0)}))$ for every $t \ge 0$. The difference between $C^{(L,n,h_0)}$ and $C^{(L,n,h_0,\varepsilon)}$ comes from the time spent on the spine by the contour of θ and the contribution of small trees. See Figure 3.5 for an illustration of the processes $C^{(L,n,h_0)}$ and $C^{(L,n,h_0,\varepsilon)}$.

Similarly, we denote by $V^{(L,n,h_0,\varepsilon)}$ the concatenation of the functions

$$\left(V^{(n)}_{L_i}(t)\right)_{0 \le t < 2n^{-2}|L_i|}$$

for all integers i such that $2n^{-2}|L_i| \ge \varepsilon$ and $i \le nh_0$, and we extend this function to $[0,\infty[$ by setting $V^{(L,n,h_0,\varepsilon)}(t) = X^{(n)}_{\lfloor nh_0 \rfloor/n}$ for $t \ge \tau^{(L,n,h_0,\varepsilon)}$. We define the process $V^{(L,n,h_0)}$ analogously to $C^{(L,n,h_0)}$, replacing the contour function by the spatial contour function.

We define in the same way the processes $C^{(R,n,h_0,\varepsilon)}$, $V^{(R,n,h_0,\varepsilon)}$, $C^{(R,n,h_0)}$ and $V^{(R,n,h_0)}$ for the right side of the spine.

Finally, let $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$ and $\mathcal{P}^{(R,\infty,h_0,\varepsilon)}$ be the point measures on $\mathbb{R}_+ \times C(\mathbb{R}_+,\mathbb{R})^2 \times \mathbb{R}_+$ defined from $\mathcal{P}^{(L)}$ and $\mathcal{P}^{(R)}$ in the way explained before Corollary 2. We define four processes $C^{(L,\infty,h_0,\varepsilon)}$, $V^{(L,\infty,h_0,\varepsilon)}$, $C^{(R,\infty,h_0,\varepsilon)}$ and $V^{(R,\infty,h_0,\varepsilon)}$ by imitating the preceding construction but using the point measures $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$ and $\mathcal{P}^{(R,\infty,h_0,\varepsilon)}$ instead of $\mathcal{P}^{(L,n,h_0,\varepsilon)}$ and $\mathcal{P}^{(R,n,h_0,\varepsilon)}$. More explicitly, if $(h_1,(f_1,g_1),s_1)$, $(h_2,(f_2,g_2),s_2)$, etc. are the atoms of $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$ listed in such a way that $h_1 < h_2 < \cdots$, the process $C^{(L,\infty,h_0,\varepsilon)}$ is obtained by concatenating the functions $(h_1+f_1(t))_{0\le t<s_1}$, $(h_2+f_2(t))_{0\le t<s_2}$, etc., and the process $V^{(L,\infty,h_0,\varepsilon)}$ is obtained by concatenating the functions $(g_1(t))_{0\le t<s_1}$, $(g_2(t))_{0\le t<s_2}$, etc. The random functions $C^{(L,\infty,h_0,\varepsilon)}$ and $V^{(L,\infty,h_0,\varepsilon)}$ are a priori only defined on a finite interval $[0,\tau^{(L,h_0,\varepsilon)}[$, but we extend them to $[0,\infty[$ by setting

$$\left(C_t^{(L,\infty,h_0,\varepsilon)}, V_t^{(L,\infty,h_0,\varepsilon)}\right) = (h_0, Z_{h_0})$$

for every $t \ge \tau^{(L,h_0,\varepsilon)}$.

Using Corollary 2 and the Skorokhod representation theorem, we may find, for every $n \ge 1$, a triplet $\left(\widetilde{X}^{(n)}, \widetilde{\mathcal{P}}^{(L,n,h_0,\varepsilon)}, \widetilde{\mathcal{P}}^{(R,n,h_0,\varepsilon)}\right)$ having the same law as the triplet $\left(X^{(n)}, \mathcal{P}^{(L,n,h_0,\varepsilon)}, \mathcal{P}^{(R,n,h_0,\varepsilon)}\right)$ and such that

$$\left(\widetilde{X}^{(n)}, \widetilde{\mathcal{P}}^{(L,n,h_0,\varepsilon)}, \widetilde{\mathcal{P}}^{(R,n,h_0,\varepsilon)}\right) \underset{n\to\infty}{\longrightarrow} \left(Z, \mathcal{P}^{(L,\infty,h_0,\varepsilon)}, \mathcal{P}^{(R,\infty,h_0,\varepsilon)}\right) \tag{3.15}$$

almost surely. We can order the atoms of the point measures considered in (3.15) according to their first component. From the convergence (3.15), we deduce that almost surely for n large enough the measures $\widetilde{\mathcal{P}}^{(L,n,h_0,\varepsilon)}$ and $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$ have the same number of atoms, and the i-th atom of $\widetilde{\mathcal{P}}^{(L,n,h_0,\varepsilon)}$ converges as $n\to\infty$ to the i-th atom of $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$. The same property holds for the right side of the spine.

With the point measure $\widetilde{\mathcal{P}}^{(L,n,h_0,\varepsilon)}$, we can associate random functions $\widetilde{C}^{(L,n,h_0,\varepsilon)}$ and $\widetilde{V}^{(L,n,h_0,\varepsilon)}$ defined in the same way as $C^{(L,n,h_0,\varepsilon)}, V^{(L,n,h_0,\varepsilon)}$ were defined from $\mathcal{P}^{(L,n,h_0,\varepsilon)}$. Similarly, with the point measure $\widetilde{\mathcal{P}}^{(R,n,h_0,\varepsilon)}$ we associate the random functions $\widetilde{C}^{(R,n,h_0,\varepsilon)}$ and $\widetilde{V}^{(R,n,h_0,\varepsilon)}$. From the almost sure convergence of the atoms of $\widetilde{\mathcal{P}}^{(L,n,h_0,\varepsilon)}$, respectively $\widetilde{\mathcal{P}}^{(R,n,h_0,\varepsilon)}$, towards the corresponding atoms of $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$, resp. $\mathcal{P}^{(R,\infty,h_0,\varepsilon)}$, it is then an easy exercise, using the definition of the Skorokhod topology, to check that we have almost surely

$$\left(\widetilde{C}^{(L,n,h_0,\varepsilon)}, \widetilde{V}^{(L,n,h_0,\varepsilon)}\right) \underset{n\to\infty}{\longrightarrow} \left(C^{(L,\infty,h_0,\varepsilon)}, V^{(L,\infty,h_0,\varepsilon)}\right) \tag{3.16}$$

and similarly

$$\left(\widetilde{C}^{(R,n,h_0,\varepsilon)}, \widetilde{V}^{(R,n,h_0,\varepsilon)}\right) \underset{n\to\infty}{\longrightarrow} \left(C^{(R,\infty,h_0,\varepsilon)}, V^{(R,\infty,h_0,\varepsilon)}\right) \tag{3.17}$$

in the sense of the Skorokhod topology on $\mathbb{D}(\mathbb{R}^2)$.

The Skorokhod topology on $\mathbb{D}(\mathbb{R}^2)$ can be defined by a metric d_{Sk}, and we may assume that $d_{\mathrm{Sk}}((f_1,g_1),(f_2,g_2)) \le \|f_1-f_2\| + \|g_1-g_2\|$, where $\|f\| = \sup\{|f(t)| : t\ge 0\} \le \infty$. Let F be a bounded Lipschitz function on $\mathbb{D}(\mathbb{R}^2)\times\mathbb{D}(\mathbb{R}^2)$.

From (3.16) and (3.17), we have

$$\begin{aligned} &E\left[F\left(\left(C^{(L,n,h_0,\varepsilon)}, V^{(L,n,h_0,\varepsilon)}\right), \left(C^{(R,n,h_0,\varepsilon)}, V^{(R,n,h_0,\varepsilon)}\right)\right)\right] \\ &\quad = E\left[F\left(\left(\widetilde{C}^{(L,n,h_0,\varepsilon)}, \widetilde{V}^{(L,n,h_0,\varepsilon)}\right), \left(\widetilde{C}^{(R,n,h_0,\varepsilon)}, \widetilde{V}^{(R,n,h_0,\varepsilon)}\right)\right)\right] \\ &\quad \underset{n\to\infty}{\longrightarrow} E\left[F\left(\left(C^{(L,\infty,h_0,\varepsilon)}, V^{(L,\infty,h_0,\varepsilon)}\right), \left(C^{(R,\infty,h_0,\varepsilon)}, V^{(R,\infty,h_0,\varepsilon)}\right)\right)\right]. \end{aligned} \tag{3.18}$$

Our goal is to prove that

$$
\begin{aligned}
&E\left[F\left(\left(C^{(L,n,h_0)},V^{(L,n,h_0)}\right),\left(C^{(R,n,h_0)},V^{(R,n,h_0)}\right)\right)\right]\\
&\qquad\underset{n\to\infty}{\longrightarrow} E\left[F\left(\left(C^{(L,\infty,h_0)},V^{(L,\infty,h_0)}\right),\left(C^{(R,\infty,h_0)},V^{(R,\infty,h_0)}\right)\right)\right] \qquad (3.19)
\end{aligned}
$$

where

$$
(C^{(L,\infty,h_0)}(t),V^{(L,\infty,h_0)}(t)) = (\zeta^{(L)}_{t\wedge\tau^{(L)}_{h_0}},\widehat{W}^{(L)}_{t\wedge\tau^{(L)}_{h_0}})
$$

and the processes

$$
(C^{(R,\infty,h_0)}(t),V^{(R,\infty,h_0)}(t))
$$

are defined in a similar manner. As we will explain later, the statement of Theorem 9 easily follows from the convergence (3.19).

In order to derive (3.19) from (3.18), we use the next lemma.

Lemma 8. (i) *For every $\eta>0$, we have, for all $\varepsilon>0$ small enough,*

$$
\limsup_{n\to\infty} P\left[\sup_{t\ge0}\left|C^{(L,n,h_0,\varepsilon)}(t)-C^{(L,n,h_0)}(t)\right|>\eta\right]<\eta
$$

and

$$
\limsup_{n\to\infty} P\left[\sup_{t\ge0}\left|V^{(L,n,h_0,\varepsilon)}(t)-V^{(L,n,h_0)}(t)\right|>\eta\right]<\eta.
$$

(ii) *We have for every $\eta>0$,*

$$
\lim_{\varepsilon\to0} P\left[\sup_{t\ge0}\left|C^{(L,\infty,h_0,\varepsilon)}(t)-C^{(L,\infty,h_0)}(t)\right|>\eta\right]=0
$$

and

$$
\lim_{\varepsilon\to0} P\left[\sup_{t\ge0}\left|V^{(L,\infty,h_0,\varepsilon)}(t)-V^{(L,\infty,h_0)}(t)\right|>\eta\right]=0.
$$

Let us postpone the proof of Lemma 8 and complete the proof of Theorem 9. Fix $\delta>0$. From part (ii) of the lemma (and the obvious analogue of this lemma for processes attached to the right side of the spine), and our assumptions on F, we can choose $\varepsilon_0>0$ such that, for every $\varepsilon\in]0,\varepsilon_0[$,

$$
\begin{aligned}
&E\Big[\Big|F\left(\left(C^{(L,\infty,h_0)},V^{(L,\infty,h_0)}\right),\left(C^{(R,\infty,h_0)},V^{(R,\infty,h_0)}\right)\right)\\
&\quad - F\left(\left(C^{(L,\infty,h_0,\varepsilon)},V^{(L,\infty,h_0,\varepsilon)}\right),\left(C^{(R,\infty,h_0,\varepsilon)},V^{(R,\infty,h_0,\varepsilon)}\right)\right)\Big|\Big]\le\delta\,.
\end{aligned}
$$

From part (i) of the lemma, and by choosing ε even smaller if necessary, we have also

$$
\begin{aligned}
&\limsup_{n\to\infty} E\Big[\Big|F\left(\left(C^{(L,n,h_0)},V^{(L,n,h_0)}\right),\left(C^{(R,n,h_0)},V^{(R,n,h_0)}\right)\right)\\
&\quad - F\left(\left(C^{(L,n,h_0,\varepsilon)},V^{(L,n,h_0,\varepsilon)}\right),\left(C^{(R,n,h_0,\varepsilon)},V^{(R,n,h_0,\varepsilon)}\right)\right)\Big|\Big]\le\delta\,.
\end{aligned}
$$

Hence, using also (3.18),

$$
\begin{aligned}
&\limsup_{n\to\infty} E\Big[\Big|F\left(\left(C^{(L,n,h_0)},V^{(L,n,h_0)}\right),\left(C^{(R,n,h_0)},V^{(R,n,h_0)}\right)\right)\\
&\quad - F\left(\left(C^{(L,\infty,h_0)},V^{(L,\infty,h_0)}\right),\left(C^{(R,\infty,h_0)},V^{(R,\infty,h_0)}\right)\right)\Big|\Big]\le2\delta\,.
\end{aligned}
$$

Since δ was arbitrary, this completes the proof of (3.19). We have thus obtained

$$\Big(\Big(C^{(L,n,h_0)},V^{(L,n,h_0)}\Big),\Big(C^{(R,n,h_0)},V^{(R,n,h_0)}\Big)\Big)$$
$$\xrightarrow[n\to\infty]{\rm (d)} \Big(\Big(C^{(L,\infty,h_0)},V^{(L,\infty,h_0)}\Big),\Big(C^{(R,\infty,h_0)},V^{(R,\infty,h_0)}\Big)\Big). \tag{3.20}$$

However, the pair $(C^{(L,n,h_0)},V^{(L,n,h_0)})$ coincides with the process

$$\left(\frac{1}{n}C^{(L)}(n^2\cdot),\sqrt{\frac{3}{2n}}V^{(L)}(n^2\cdot)\right)$$

stopped at time $\tau^{(L,n,h_0)}$, and similarly the pair $(C^{(L,\infty,h_0)},V^{(L,\infty,h_0)})$ is nothing but the process $(\zeta^{(L)},\widehat{W}^{(L)})$ stopped at time $\tau^{(L)}_{h_0}$. The statement of Theorem 9 thus follows from the convergence (3.20). □

Proof of Lemma 8. We start by proving (ii). Write the atoms in $\mathcal{P}^{(L)}$ in the form

$$\mathcal{P}^{(L)}=\sum_{i\in I}\delta_{(h_i,\omega_i)}$$

and notice that, for every $u\geq 0$,

$$\tau^{(L)}_u=\sum_{i\in I}\mathbf{1}_{\{h_i\leq u\}}\,\sigma(\omega_i).$$

The construction of $W^{(L)}$ from the point measure $\mathcal{P}^{(L)}$ (cf subsection 3.1) shows that the pair $(\zeta^{(L)},\widehat{W}^{(L)})$ is obtained by concatenating (in the appropriate order given by the values of h_i) the functions

$$\Big(h_i+\zeta_\cdot(\omega_i),Z_{h_i}+\widehat{W}_\cdot(\omega_i)\Big).$$

On the other hand, the definition of the point measure $\mathcal{P}^{(L,\infty,h_0,\varepsilon)}$, and the construction of the pair $(C^{(L,\infty,h_0,\varepsilon)},V^{(L,\infty,h_0,\varepsilon)})$ from this point measure, show that the pair $(C^{(L,\infty,h_0,\varepsilon)},V^{(L,\infty,h_0,\varepsilon)})$ is obtained by concatenating the same functions, but only for those indices i such that $h_i\leq h_0$ and $\sigma(\omega_i)\geq\varepsilon$. In other words, if we define for every $t\geq 0$,

$$A^{(L,h_0,\varepsilon)}_t=\int_0^t \mathrm{d}s\sum_{i\in I}\mathbf{1}_{\{h_i\leq h_0,\sigma(\omega_i)\geq\varepsilon\}}\,\mathbf{1}_{\{\tau^{(L)}_{h_i-}<s<\tau^{(L)}_{h_i}\}}$$

and

$$\gamma^{(L,h_0,\varepsilon)}_t=\inf\Big\{s\geq 0: A^{(L,h_0,\varepsilon)}_s>t\Big\}\wedge\tau^{(L)}_{h_0},$$

we have

$$\Big(C^{(L,\infty,h_0,\varepsilon)}(t),V^{(L,\infty,h_0,\varepsilon)}(t)\Big)=\Big(\zeta^{(L)}_{\gamma^{(L,h_0,\varepsilon)}_t},\widehat{W}^{(L)}_{\gamma^{(L,h_0,\varepsilon)}_t}\Big), \tag{3.21}$$

for every $t\geq 0$. It is however immediate that

$$A^{(L,h_0,\varepsilon)}_t\xrightarrow[\varepsilon\to 0]{} t\wedge\tau^{(L)}_{h_0}$$

and the convergence is uniform in t by a monotonicity argument. It follows that

$$\gamma^{(L,h_0,\varepsilon)}_t\xrightarrow[\varepsilon\to 0]{} t\wedge\tau^{(L)}_{h_0}$$

again uniformly in t. Part (ii) of the lemma now follows from (3.21).

Let us turn to the proof of (i), which is more delicate. The general idea again is that the process $C^{(L,n,h_0,\varepsilon)}$ can be written as a time change of $C^{(L,n,h_0)}$ (this should be obvious from Figure 3.5), and that this time change is close to the identity when ε is small. We start by estimating the difference $\tau^{(L,n,h_0)} - \tau^{(L,n,h_0,\varepsilon)}$. For every fixed $\delta > 0$, and n large enough so that $h_0/n < \delta/2$, we have

$$\begin{aligned}
P\left(\tau^{(L,n,h_0)} - \tau^{(L,n,h_0,\varepsilon)} \geq \delta\right) &= P\left(\frac{\lfloor nh_0\rfloor}{n^2} + \sum_{i\leq nh_0} \mathbf{1}_{\{2n^{-2}|L_i|\leq\varepsilon\}} 2n^{-2}|L_i| \geq \delta\right) \\
&\leq \frac{2}{\delta} E\left[\sum_{i\leq nh_0} \widehat{\rho}_{X_i}\left(\mathbf{1}_{\{2n^{-2}|\theta|\leq\varepsilon\}} 2n^{-2}|\theta|\right)\right] \\
&\leq \frac{4\lfloor nh_0\rfloor}{\delta} \rho_1\left(\mathbf{1}_{\{2n^{-2}|\theta|\leq\varepsilon\}} 2n^{-2}|\theta|\right) \\
&\leq K(h_0,\delta)\,\varepsilon^{1/2} \qquad (3.22)
\end{aligned}$$

where the last bound is an easy consequence of (3.1), with a constant $K(h_0,\delta)$ that depends only on h_0 and δ.

We now compare $C^{(L,n,h_0,\varepsilon)}$ and $C^{(L,n,h_0)}$. Note that we can write $C^{(L,n,h_0,\varepsilon)}(t) = C^{(L,n,h_0)}(A_t)$, where the time change A_t is such that $0 \leq A_t - t \leq \tau^{(L,n,h_0)} - \tau^{(L,n,h_0,\varepsilon)}$ (a brief look at Figure 3.5 should convince the reader). It follows that

$$\begin{aligned}
&\sup_{t\geq 0}\left|C^{(L,n,h_0,\varepsilon)}(t) - C^{(L,n,h_0)}(t)\right| \\
&\quad\leq \sup_{|t_1-t_2|\leq\tau^{(L,n,h_0)}-\tau^{(L,n,h_0,\varepsilon)}} \left|C^{(L,n,h_0)}\left(t_1\wedge\tau^{(L,n,h_0)}\right) - C^{(L,n,h_0)}\left(t_2\wedge\tau^{(L,n,h_0)}\right)\right|.
\end{aligned}$$

We fix $t_1 \leq t_2 \leq \tau^{(L,n,h_0)}$ such that $t_2 - t_1 \leq \tau^{(L,n,h_0)} - \tau^{(L,n,h_0,\varepsilon)}$. If there exists $0 \leq i \leq nh_0$ such that

$$\tau^{(L,n,(i-1)/n)} + n^{-2} \leq t_1 \leq t_2 < \tau^{(L,n,i/n)} + n^{-2},$$

(with the convention $\tau^{(L,n,-1/n)} = -n^{-2}$) then this means that the times t_1 and t_2 correspond, in the time scale of the rescaled contour process, to the exploration of the same tree L_i, or perhaps of the edge of the spine above the root of L_i. In that case we can clearly bound

$$\left|C^{(L,n,h_0)}(t_1) - C_L^{(L,n,h_0)}(t_2)\right| \leq \sup_{|u-v|\leq\tau^{(L,n,h_0)}-\tau^{(L,n,h_0,\varepsilon)}} \left|C_{L_i}^{(n)}(u) - C_{L_i}^{(n)}(v)\right| + \frac{1}{n}. \qquad (3.23)$$

On the other hand, if there exists no such i, then we can find $0 \leq i < j \leq nh_0$ such that

$$\tau^{(L,n,(i-1)/n)} + n^{-2} \leq t_1 < \tau^{(L,n,i/n)} + n^{-2} \leq \tau^{(L,n,(j-1)/n)} + n^{-2} \leq t_2 < \tau^{(L,n,j/n)} + n^{-2}$$

and we have:

$$\begin{aligned}
&\left|C^{(L,n,h_0)}(t_1) - C^{(L,n,h_0)}(t_2)\right| \\
&\quad\leq \left|C_{L_j}^{(n)}\left(t_2 - \tau^{(L,n,(j-1)/n)} - n^{-2}\right) - C_{L_i}^{(n)}\left(t_1 - \tau^{(L,n,(i-1)/n)} - n^{-2}\right)\right| + \frac{j-i+1}{n},
\end{aligned}$$

where we recall the convention that $C^{(n)}_{L_i}(s) = 0$ for $s \geq 2|L_i|/n^2$. Now note that $i = J_{\lfloor n^2 t_1 \rfloor}$ and $j = J_{\lfloor n^2 t_2 \rfloor}$, with the notation introduced before Lemma 7. We obtain

$$\begin{aligned}
&\left| C^{(L,n,h_0)}(t_1) - C^{(L,n,h_0)}(t_2) \right| \\
&\leq \frac{J_{\lfloor n^2 t_2 \rfloor} - J_{\lfloor n^2 t_1 \rfloor} + 1}{n} \\
&\qquad + \max\left\{ C^{(n)}_{L_j}\left(t_2 - \tau^{(L,n,(j-1)/n)} + n^{-2}\right), C^{(n)}_{L_i}\left(t_1 - \tau^{(L,n,(i-1)/n)} + n^{-2}\right) \right\} \quad (3.24)
\end{aligned}$$

Put $\gamma_{n,\varepsilon} = \tau^{(L,n,h_0)} - \tau^{(L,n,h_0,\varepsilon)}$ to simplify notation. We can summarize the bounds (3.23) and (3.24) by writing

$$\begin{aligned}
&\sup_{t \geq 0} \left| C^{(L,n,h_0,\varepsilon)}(t) - C^{(L,n,h_0)}(t) \right| \\
&\leq \sup_{u,v \leq \tau^{(L,n,h_0)}, |v-u| \leq \gamma_{n,\varepsilon}} \frac{|J_{\lfloor n^2 v \rfloor} - J_{\lfloor n^2 u \rfloor}| + 1}{n} + \sup_{0 \leq k \leq \lfloor n h_0 \rfloor} \sup_{|v-u| \leq \gamma_{n,\varepsilon}} \left| C^{(n)}_{L_k}(v) - C^{(n)}_{L_k}(u) \right|.
\end{aligned} \quad (3.25)$$

We write $B_1(n,\varepsilon)$ and $B_2(n,\varepsilon)$ for the two terms in the sum of the right-hand side of (3.25). We will use Lemma 7 to handle $B_1(n,\varepsilon)$, but we need a different argument for $B_2(n,\varepsilon)$. If θ is a spatial tree, we write $H(\theta)$ for the height of θ (or maximal generation in θ) as in subsection 3.2.1. Then, for every $\delta > 0$ and $\kappa > 0$,

$$\begin{aligned}
&P\left[\sup_{0 \leq k \leq \lfloor n h_0 \rfloor} \sup_{|u-v| \leq \delta} \left| C^{(n)}_{L_k}(u) - C^{(n)}_{L_k}(v) \right| > \kappa \right] \\
&\quad \leq \sum_{k=0}^{\lfloor n h_0 \rfloor} P\left[\sup_{|u-v| \leq \delta} |C_{L_k}(n^2 u) - C_{L_k}(n^2 v)| > n\kappa \right] \\
&\quad = \sum_{k=0}^{\lfloor n h_0 \rfloor} E\left[\widehat{\rho}_{X_k}\left(\sup_{|u-v| \leq \delta} |C_\theta(n^2 u) - C_\theta(n^2 v)| > n\kappa \right) \right] \\
&\quad \leq 2(\lfloor n h_0 \rfloor + 1)\, \rho_1\left(\sup_{|u-v| \leq \delta} |C_\theta(n^2 u) - C_\theta(n^2 v)| > n\kappa \right) \\
&\quad = 2(\lfloor n h_0 \rfloor + 1)\, \rho_1(H(\theta) > n\kappa) \times \rho_1\left(\sup_{|u-v| \leq \delta} |C^{(n)}_\theta(u) - C^{(n)}_\theta(v)| > \kappa \,\middle|\, H(\theta) > n\kappa \right).
\end{aligned} \quad (3.26)$$

By standard results about Galton-Watson trees,

$$\sup_{n \geq 1} n \rho_1(H(\theta) \geq n) < \infty \quad (3.27)$$

and so the quantities $2(\lfloor n h_0 \rfloor + 1)\ \rho_1(H(\theta) > n\kappa)$ are bounded above by a constant $K(h_0, \kappa)$ depending only on κ and δ. On the other hand, from Corollary 1.13 in [LG05] (or as an easy consequence of Proposition 7), the law of $(C^{(n)}_\theta(t))_{t \geq 0}$ under the conditional probability measure $\rho_1(\cdot \mid H(\theta) > n\kappa)$ converges as $n \to \infty$ to the law of a Brownian excursion with height greater than κ. Consequently,

$$\limsup_{n\to\infty} \rho_1 \left(\sup_{|u-v|\leq\delta} |C_\theta^{(n)}(u) - C_\theta^{(n)}(v)| > \kappa \,\Big|\, H(\theta) > n\kappa \right) \leq \mathbf{n} \left(\sup_{|u-v|\leq\delta} |e(u) - e(v)| \geq \kappa \,\Big|\, \sup_{t\geq 0} e(t) \geq \kappa \right),$$

where $\mathbf{n}$ stands for the Itô excursion measure as in subsection 2.4. For any fixed κ, the right-hand side can be made arbitrarily small by choosing δ small enough.

To complete the argument, fix $\eta > 0$. By the preceding considerations, we can choose $\delta > 0$ small enough so that

$$\limsup_{n\to\infty} P\left[\sup_{0\leq k\leq \lfloor nh_0 \rfloor} \sup_{|u-v|\leq\delta} \left| C_{L_k}^{(n)}(u) - C_{L_k}^{(n)}(v) \right| > \frac{\eta}{2} \right] < \frac{\eta}{3}. \tag{3.28}$$

and, using Lemma 7,

$$\limsup_{n\to\infty} P\left[\sup_{u,v\leq\tau^{(L,n,h_0)}\,,\,|v-u|\leq\delta} \frac{|J_{\lfloor n^2 v\rfloor} - J_{\lfloor n^2 u \rfloor}| + 1}{n} > \frac{\eta}{2} \right] < \frac{\eta}{3}. \tag{3.29}$$

From (3.25), we get

$$P\left[\sup_{t\geq 0} \left| C^{(L,n,h_0,\varepsilon)}(t) - C^{(L,n,h_0)}(t) \right| > \eta \right] \leq P[\gamma_{n,\varepsilon} \geq \delta] + P\left[\gamma_{n,\varepsilon} < \delta\,,\, B_1(n,\varepsilon) > \frac{\eta}{2} \right] + P\left[\gamma_{n,\varepsilon} < \delta\,,\, B_2(n,\varepsilon) > \frac{\eta}{2} \right].$$

The quantities $P[\gamma_{n,\varepsilon} < \delta\,,\, B_1(n,\varepsilon) > \frac{\eta}{2}]$ and $P[\gamma_{n,\varepsilon} < \delta\,,\, B_2(n,\varepsilon) > \frac{\eta}{2}]$ are smaller than $\frac{\eta}{3}$ when n is large (independently of the choice of ε), by (3.28) and (3.29). Finally, (3.22) allows us to choose $\varepsilon > 0$ sufficiently small so that $P[\gamma_{n,\varepsilon} \geq \delta] < \frac{\eta}{3}$ for every $n \geq 1$. This completes the proof of the first assertion in (i).

The second assertion in (i) is proved in a similar way, and we only point at the differences. The same arguments we used to obtain the bound (3.25) give

$$\begin{aligned} &\sup_{t\geq 0} \left| V^{(L,n,h_0,\varepsilon)}(t) - V^{(L,n,h_0)}(t) \right| \\ &\leq \sup_{u,v\leq\tau^{(L,n,h_0)},|v-u|\leq\gamma_{n,\varepsilon}} \sqrt{\frac{3}{2n}} \left(|X_{J_{\lfloor n^2 v\rfloor}} - X_{J_{\lfloor n^2 u\rfloor}}| + 1 \right) \\ &\qquad + \sup_{0\leq k\leq\lfloor nh_0\rfloor} \sup_{|v-u|\leq\gamma_{n,\varepsilon}} \left| V_{L_k}^{(n)}(v) - V_{L_k}^{(n)}(u) \right|. \end{aligned} \tag{3.30}$$

If $\eta > 0$ is fixed, we can again use Lemma 7 together with Assertion 2. in Theorem 8 to see that we can choose $\delta > 0$ small enough so that

$$\limsup_{n\to\infty} P\left[\sup_{|v-u|\leq\delta} \sqrt{\frac{3}{2n}} \left(|X_{J_{\lfloor n^2 v\rfloor}} - X_{J_{\lfloor n^2 u\rfloor}}| + 1 \right) > \frac{\eta}{2} \right] < \frac{\eta}{3}. \tag{3.31}$$

Then, in order to estimate the second term of the right-hand side of (3.30), we replace the bound (3.26) by

$$P\left[\sup_{0\le k\le \lfloor nh_0\rfloor}\ \sup_{|u-v|\le\delta}\left|V^{(n)}_{L_k}(u)-V^{(n)}_{L_k}(v)\right|>\kappa\right]$$
$$\le 2(\lfloor nh_0\rfloor+1)\,\rho_1(V^*(\theta)>\kappa\sqrt{n})$$
$$\times\,\rho_1\left(\sup_{|u-v|\le\delta}|V^{(n)}_\theta(u)-V^{(n)}_\theta(v)|>\kappa\,\middle|\,V^*(\theta)>\frac{\kappa}{2}\sqrt{n}\right), \tag{3.32}$$

where $V^*(\theta)$ denotes the maximal absolute value of a label in θ. The analogue of (3.27) is

$$\sup_{n\ge1} n\,\rho_1(V^*(\theta)\ge\sqrt{n})<\infty. \tag{3.33}$$

This bound can be derived from the much more precise estimate given in Proposition 4 of [CS04] (together with (3.1)). Then, Proposition 7 implies that the law of $\left(V^{(n)}_\theta(t)\right)_{t\ge0}$ under the conditional probability measure $\rho_1\left(\cdot\mid V^*(\theta)>\frac{\kappa}{2}\sqrt{n}\right)$ converges as $n\to\infty$ to the law of $\left(\widehat{W}_s\right)_{s\ge0}$ under $\mathbb{N}_0\left(\cdot\mid W^*>\kappa\sqrt{3/8}\right)$, where $W^*=\max\left\{\widehat{W}_s : s\ge0\right\}$ (the precise justification of this convergence uses arguments very similar to the proof of Corollary 1.13 in [LG05]). Consequently,

$$\limsup_{n\to\infty}\rho_1\left(\sup_{|u-v|\le\delta}|V^{(n)}_\theta(u)-V^{(n)}_\theta(v)|>\kappa\,\middle|\,V^*(\theta)>\frac{\kappa}{2}\sqrt{n}\right)$$
$$\le\mathbb{N}_0\left(\sup_{|u-v|\le\delta}|\widehat{W}(u)-\widehat{W}(v)|\ge\kappa\,\middle|\,W^*>\kappa\sqrt{3/8}\right),$$

and, for any fixed $\kappa>0$, the left-hand side can be made arbitrarily small by choosing δ small. The remaining part of the proof is exactly similar to the proof of the first assertion in (i). This completes the proof of Lemma 8. □

3.4 Some asymptotic properties of the uniform infinite quadrangulation

3.4.1 Profile

Recall from Section 3.2.1 the definition of the profile λ_q of a quadrangulation q. If $q\in\mathbf{Q}$ and $n\ge1$ is an integer, the rescaled profile of λ_q is the measure $\lambda^{(n)}_q$ on $\mathbb{R}_+$ defined by

$$\lambda^{(n)}_q(A)=\frac{1}{n^2}\lambda_q\left(\sqrt{\frac{2n}{3}}A\right)$$

for any Borel subset A of $\mathbb{R}_+$. In this section we study the convergence of these rescaled measures when q is distributed according to the law of the uniform infinite quadrangulation (see Theorem 5), and we then derive some information about the limiting measure.

Theorem 10. *Suppose that q is a uniform infinite quadrangulation. The sequence of measures $\left(\lambda_q^{(n)}\right)_{n\geq 1}$ converges in distribution to the random measure $\mathcal{I}$ defined by*

$$\langle \mathcal{I}, g\rangle = \frac{1}{2}\int_0^\infty \mathrm{d}s \left(g\left(\widehat{W}_s^{(L)}\right) + g\left(\widehat{W}_s^{(R)}\right)\right)$$

where $\left(W^{(L)}, W^{(R)}\right)$ is a pair of correlated eternal conditioned Brownian snakes.

In particular we have for $r \geq 0$:

$$\frac{1}{n^4}\left|B_{\mathbf{Q},nr}(q)\right| \xrightarrow[n\to\infty]{(d)} \frac{9}{8}r^4\int_0^\infty \mathrm{d}s\left(\mathbf{1}_{[0,1]}\left(\widehat{W}_s^{(L)}\right) + \mathbf{1}_{[0,1]}\left(\widehat{W}_s^{(R)}\right)\right).$$

Remark. Both $\lambda_q^{(n)}$ and $\mathcal{I}$ are random variables with values in the space of Radon measures on $\mathbb{R}_+$, which is a Polish space for the topology of vague convergence.

Proof. As previously we consider a random infinite well-labeled tree distributed according to μ, and we use the notation $(X_i, L_i, R_i)_{i\geq 0}$ introduced in Theorem 8. For every $i \in \mathbb{N}$, we write the spatial trees L_i and R_i as $L_i = (\tau_{L_i}, \ell_{L_i})$ and $R_i = (\tau_{R_i}, \ell_{R_i})$. Since ν is the image of μ under the extended Schaeffer correspondence (see Theorem 7), the law of $\lambda_q^{(n)}$ under ν coincides with the law of the random measure $\mathcal{I}^{(n)}$ defined by

$$\langle \mathcal{I}^{(n)}, g\rangle = \frac{1}{n^2}g(0) + \frac{1}{n^2}\sum_{i\geq 0}\left(\sum_{v\in L_i} g\left(\sqrt{\frac{3}{2}}\frac{\ell_{L_i}(v)}{\sqrt{n}}\right) + \sum_{v\in R_i} g\left(\sqrt{\frac{3}{2}}\frac{\ell_{R_i}(v)}{\sqrt{n}}\right) - g\left(\sqrt{\frac{3}{2}}\frac{X_i}{\sqrt{n}}\right)\right) \tag{3.34}$$

for every continuous function g with compact support. Note that in (3.34), the vertex at height i in the spine of the tree appears once in τ_{L_i} and once in τ_{R_i}. This explains why we have to substract the term $g\left(\sqrt{\frac{3}{2}}\frac{X_i}{\sqrt{n}}\right)$ in the sum. Also note that

$$\frac{1}{n^2}\sum_{i\geq 0} g\left(\sqrt{\frac{3}{2}}\frac{X_i}{\sqrt{n}}\right) = \frac{1}{n}\int_0^\infty g\left(\sqrt{\frac{3}{2}}\frac{X_{[nt]}}{\sqrt{n}}\right)\mathrm{d}t \xrightarrow[n\to\infty]{} 0,$$

using assertion 2. of Theorem 8 and the fact that a nine-dimensional Bessel process is transient (in fact we need slighly more, such as the estimate provided by Lemma 2 of [Mé08], but we omit details).

The term in (3.34) corresponding to the left part of the spine is

$$\begin{aligned}
\langle \mathcal{I}^{(L,n)}, g\rangle &:= \frac{1}{n^2}\sum_{i\geq 0}\sum_{v\in L_i} g\left(\sqrt{\frac{3}{2}}\frac{\ell_{L_i}(v)}{\sqrt{n}}\right)\\
&= \frac{1}{n^2}\left(\sum_{i\geq 0} g\left(\sqrt{\frac{3}{2}}\frac{X_i}{\sqrt{n}}\right) + \frac{1}{2}\int_0^{2|L_i|} g\left(\sqrt{\frac{3}{2}}\frac{V_{L_i}([t]_{C_{L_i}})}{\sqrt{n}}\right)\mathrm{d}t\right)\\
&= \frac{1}{n^2}\sum_{i\geq 0} g\left(\sqrt{\frac{3}{2}}\frac{X_i}{\sqrt{n}}\right) + \frac{1}{2}\sum_{i\geq 0}\int_0^{\sigma\left(C_{L_i}^{(n)}\right)} g\left(\sqrt{\frac{3}{2}}\frac{V_{L_i}([n^2t]_{C_{L_i}})}{\sqrt{n}}\right)\mathrm{d}t
\end{aligned}$$

where if $t \in [k, k+1[$ we set $[t]_C = k$ if $C(k) \geq C(t)$ and $[t]_C = k+1$ otherwise. Using the fact that $|t - [t]_C| \leq 1$, we deduce from Theorem 9 that $\mathcal{I}^{(L,n)}$ converges to the random measure $\mathcal{I}^{(L)}$ defined by

$$\langle \mathcal{I}^{(L)}, g \rangle = \frac{1}{2} \int_0^\infty g\left(\widehat{W}_s^{(L)}\right) \mathrm{d}s.$$

The same argument shows that the random measures $\mathcal{I}^{(R,n)}$ defined analogously to $\mathcal{I}^{(L,n)}$ for the right part of the spine converge in distribution to the random measure $\mathcal{I}^{(R)}$ defined by

$$\langle \mathcal{I}^{(R)}, g \rangle = \frac{1}{2} \int_0^\infty g\left(\widehat{W}_s^{(R)}\right) \mathrm{d}s.$$

Furthermore this convergence holds jointly with that of the sequence $\mathcal{I}^{(L,n)}$. This completes the proof of the first assertion.

Noting that for every quadrangulation q

$$\frac{1}{n^4} \left|B_{\mathbf{Q},nr}(q)\right| = \left\langle \lambda_q^{(n^2)}, \mathbf{1}_{\left[0, r\sqrt{3/2}\right]} \right\rangle,$$

and using a scaling argument, the second assertion of the theorem reduces to the convergence in law of $\langle \lambda_q^{(n)}, \mathbf{1}_{[0,x]} \rangle$ to $\langle \mathcal{I}, \mathbf{1}_{[0,x]} \rangle$ for every $x > 0$. This is a straightforward consequence of the first assertion and the fact that $\mathcal{I}(\{r\}) = 0$ a.s. The latter fact is easy from a first-moment calculation. □

Let us now derive some information on $\mathcal{I}$ by computing the Laplace transform of $\mathcal{I}([0,1])$. Let the triplet $\left(Z, \mathcal{P}^{(L)}, \mathcal{P}^{(R)}\right)$ be as in the previous section. We have, for every $\lambda > 0$,

$$\begin{aligned}
&E\left[\exp -\lambda \mathcal{I}([0,1])\right] \\
&= E\left[\exp -\frac{\lambda}{2} \int_0^\infty \mathrm{d}s \left(\mathbf{1}_{\left\{\widehat{W}_s^{(L)} \leq 1\right\}} + \mathbf{1}_{\left\{\widehat{W}_s^{(R)} \leq 1\right\}}\right)\right] \\
&= E\left[E\left[\exp -\frac{\lambda}{2} \int_0^\infty \mathrm{d}s\, \mathbf{1}_{\left\{\widehat{W}_s^{(L)} \leq 1\right\}} \middle| Z\right] \times E\left[\exp -\frac{\lambda}{2} \int_0^\infty \mathrm{d}s\, \mathbf{1}_{\left\{\widehat{W}_s^{(R)} \leq 1\right\}} \middle| Z\right]\right] \\
&= E\left[E\left[\exp -\frac{\lambda}{2} \int \mathcal{P}^{(L)}(\mathrm{d}t, \mathrm{d}\omega) \int_0^{\sigma(\omega)} \mathrm{d}s \mathbf{1}_{\left\{Z_t + \widehat{\omega}_s \leq 1\right\}} \middle| Z\right]\right. \\
&\qquad \left. \times E\left[\exp -\frac{\lambda}{2} \int \mathcal{P}^{(R)}(\mathrm{d}t, \mathrm{d}\omega) \int_0^{\sigma(\omega)} \mathrm{d}s \mathbf{1}_{\left\{Z_t + \widehat{\omega}_s \leq 1\right\}} \middle| Z\right]\right] \\
&= E\left[\exp -4 \int_0^\infty \mathrm{d}t\, \mathbb{N}_{Z_t}\left(\mathbf{1}_{\{\mathcal{R} \subset]0,\infty[\}}\left(1 - \exp -\frac{\lambda}{2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\left\{\widehat{W}_s \leq 1\right\}}\right)\right)\right]. \qquad (3.35)
\end{aligned}$$

Using assertion 1. of Lemma 6, formula (3.35) can be rewritten as

$$E\left[\exp -\lambda \mathcal{I}([0,1])\right] = E\left[\exp -4 \int_0^\infty \mathrm{d}t \left(u_{\lambda/2}(Z_t) - \frac{3}{2Z_t^2}\right)\right]$$

where for every $\lambda, x > 0$,

$$u_\lambda(x) = \mathbb{N}_x\left(1 - \mathbf{1}_{\{\mathcal{R} \subset]0,\infty[\}} \exp -\lambda \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\left\{\widehat{W}_s \leq 1\right\}}\right).$$

Proposition 11. *The function u_λ is continuous on $]0,\infty[$ and solves the differential equation*

$$\frac{1}{2}u'' = 2u^2 - \lambda \tag{3.36}$$

in $]0,1[$ with the boundary condition $u_\lambda(0)=\infty$. Moreover, for every $x \geq 1$

$$u_\lambda(x) = \frac{3}{2(x-a_\lambda)^2}$$

where $a_\lambda = 1 - \sqrt{\frac{3}{2u_\lambda(1)}}$.

Proof. We follow closely the proof given for a similar result in [Del03] (Lemmas 6 and 7). Let Y denote the exit measure of the Brownian snake from $]0,\infty[$, let $\mathcal{E}$ denote the σ-field containing the information given by the paths W_s before they exit $]0,\infty[$: $\mathcal{E} = \mathcal{E}^{]0,\infty[}$ in the notation of Section 3.2.4. Let $|Y|$ denote the total mass of Y. We also denote by τ the first exit time from $]0,\infty[$. Define for $\gamma, x > 0$

$$v_{\gamma,\lambda}(x) = \mathbb{N}_x\left(1 - \exp\left(-\gamma|Y| - \lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}}\right)\right).$$

Fix $\gamma > 0$. Then we have for every $x > 0$,

$$\begin{aligned} v_{\gamma,\lambda}(x) &= \mathbb{N}_x\left(1 - \exp\left(-\gamma|Y| - \lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)=\infty\}} - \lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)<\infty\}}\right)\right) \\ &= \mathbb{N}_x\left(1 - \exp\left(-\gamma|Y| - \lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)=\infty\}}\right)\right. \\ &\qquad \left.\times\, \mathbb{N}_x\left(\exp -\lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)<\infty\}}\middle|\mathcal{E}\right)\right), \end{aligned} \tag{3.37}$$

because Y is $\mathcal{E}$-measurable, and it is easy to verify that the same holds for

$$\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)=\infty\}}.$$

The special Markov property of the Brownian snake (see section 3.2.4) gives that under $\mathbb{N}_x$:

$$\mathbb{N}_x\left(\exp -\lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)<\infty\}}\middle|\mathcal{E}\right) = \exp -\beta_\lambda|Y|$$

where

$$\beta_\lambda = \mathbb{N}_0\left(1 - \exp -\lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}}\right).$$

Formula (3.37) becomes

$$v_{\gamma,\lambda}(x) = \mathbb{N}_x\left(1 - \exp\left(-\left(\gamma+\beta_\lambda\right)|Y| - \lambda\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)=\infty\}}\right)\right).$$

Recall the definition of the exit local time $L^{]0,\infty[}$ from Section 3.2.4 and consider the continuous additive functional L of the Brownian snake, defined by

$$\mathrm{d}L_s = (\gamma+\beta_\lambda)\,\mathrm{d}L_s^{]0,\infty[} + \lambda\mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)=\infty\}}\mathrm{d}s.$$

We have

$$v_{\gamma,\lambda}(x) = \mathbb{N}_x\left(1-\exp(-L_\sigma)\right) = \mathbb{N}_x\left(\int_0^\sigma \mathrm{d}L_s \exp\left(-\int_s^\sigma \mathrm{d}L_u\right)\right). \tag{3.38}$$

Recall the notation $\mathbb{P}_{\mathrm{w}}$ from Section 3.2.4. If $(\mathcal{G}_s)_{s\geq 0}$ stands for the canonical filtration of the Brownian snake, we have for every $s>0$,

$$\mathbb{N}_x\left(\exp\left(-\int_s^\sigma \mathrm{d}L_u\right)\Big|\mathcal{G}_s\right) = \mathbb{E}_{W_s}\left(\exp - L_{T_0}\right)$$

where $T_0 = \inf\{t \geq 0 : \zeta_t = 0\}$. Also using the Poissonian representation discussed in Section 3.2.4, we have for every $\mathrm{w} \in \mathcal{W}_x$,

$$\mathbb{E}_{\mathrm{w}}\left[\exp - L_{T_0}\right] = \exp\left(-2\int_0^{\zeta(\mathrm{w})} \mathrm{d}t\, \mathbb{N}_{\mathrm{w}(t)}\left(1-\exp\left(-L_\sigma\right)\right)\right).$$

From (3.38) and the preceding remarks we get

$$\begin{aligned}
v_{\gamma,\lambda}(x) &= \mathbb{N}_x\left(\int_0^\sigma \mathrm{d}L_s \exp\left(-2\int_0^{\zeta_s} \mathrm{d}t\, v_{\gamma,\lambda}\left(W_s(t)\right)\right)\right)\\
&= (\gamma+\beta_\lambda)\ \mathbb{N}_x\left(\int_0^\sigma \mathrm{d}L_s^{]0,\infty[} \exp\left(-2\int_0^{\zeta_s} \mathrm{d}t\, v_{\gamma,\lambda}\left(W_s(t)\right)\right)\right)\\
&\quad + \lambda \mathbb{N}_x\left(\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s\leq 1\}\cap\{\tau(W_s)=\infty\}} \exp\left(-2\int_0^{\zeta_s} \mathrm{d}t\, v_{\gamma,\lambda}\left(W_s(t)\right)\right)\right).
\end{aligned} \tag{3.39}$$

Applying the first moment formulas (3.4) and (3.5) to the right-hand side of (3.39) we get

$$\begin{aligned}
v_{\gamma,\lambda}(x) &= (\gamma+\beta_\lambda)\ E_x\left[\exp\left(-2\int_0^\tau \mathrm{d}t\, v_{\gamma,\lambda}(B_t)\right)\right]\\
&\qquad + \lambda E_x\left[\int_0^\tau \mathrm{d}t\, \mathbf{1}_{\{B_t\leq 1\}} \exp\left(-2\int_0^t \mathrm{d}s\, v_{\gamma,\lambda}(B_s)\right)\right].
\end{aligned}$$

In a way similar to the proof of the Feynman-Kac formula, we substitute

$$\exp\left(-2\int_0^\tau \mathrm{d}t\, v_{\gamma,\lambda}(B_t)\right) = 1 - 2\, E_x\left[\int_0^\tau \mathrm{d}s\, v_{\gamma,\lambda}(B_s)\ \exp\left(-2\int_s^\tau \mathrm{d}t\, v_{\gamma,\lambda}(B_t)\right)\right]$$

in the right-hand side of the previous display, and apply the Markov property at time s. After a few lines of calculation, we arrive at the integral equation

$$v_{\gamma,\lambda}(x) = \gamma + \beta_\lambda - 2E_x\left[\int_0^\tau \mathrm{d}t\, v_{\gamma,\lambda}(B_t)^2\right] + \lambda E_x\left[\int_0^\tau \mathrm{d}t\, \mathbf{1}_{\{B_t\leq 1\}}\right].$$

By standard arguments (see e.g. Chapter 5 of [LG99] or Theorem 0.3 page 111 of [Dyn91]), this integral equation implies that $v_{\gamma,\lambda}$ is continuous on $[0,\infty[$ and solves the differential equation

$$\frac{1}{2}v''_{\gamma,\lambda}(x) = 2v^2_{\gamma,\lambda}(x) - \lambda\mathbf{1}_{\{x\leq 1\}}$$

on $]0,1[\cup]1,\infty[$. Furthermore, $v_{\gamma,\lambda}$ satisfies the boundary conditions

$$\begin{aligned}
&v_{\gamma,\lambda}(0) = \lim_{x\to 0} v_{\gamma,\lambda}(x) = \gamma+\beta_\lambda,\\
&\lim_{x\to\infty} v_{\gamma,\lambda}(x) = 0.
\end{aligned}$$

In particular, on $]1,\infty[$, $v_{\gamma,\lambda}$ solves the differential equation $v'' = 4v^2$ with the boundary condition $v_{\gamma,\lambda}(\infty) = 0$, hence we have for $x \geq 1$:

$$v_{\gamma,\lambda}(x) = \frac{3}{2(x - a_{\gamma,\lambda})^2}$$

where $a_{\gamma,\lambda} < 1$ is a constant depending only on γ and λ.

Since, for $x > 0$, $\{\mathcal{R} \subset]0,\infty[\} = \{Y = 0\}$ $\mathbb{N}_x$-a.e., we have

$$u_\lambda(x) = \lim_{\gamma\to+\infty} \uparrow v_{\gamma,\lambda}(x).$$

The set of positive solutions of (3.36) being closed under pointwise convergence, u_λ also solves (3.36) in $]0,1[$, with the boundary condition

$$u_\lambda(0) = \lim_{\gamma\to+\infty} v_{\gamma,\lambda}(0) = +\infty.$$

Finally, we must have $a_{\gamma,\lambda} \uparrow a_\lambda$ as $\gamma \uparrow \infty$, with a constant $a_\lambda < 1$. This gives the last assertion. □

We can compute explicitly the first moment of $\mathcal{I}([0,r])$:

$$E\left[\mathcal{I}\left([0,r]\right)\right] = 2E\left[\int_0^\infty \mathrm{d}t\, \mathbb{N}_{Z_t}\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s \leq r\}}\right)\right]$$

For $z > 0$, let

$$\varphi(z) = \mathbb{N}_z\left(\mathbf{1}_{\{\mathcal{R}\subset]0,\infty[\}} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s \leq r\}}\right)$$

Then, by the case $p = 1$ of Theorem 2.2 in [LGW06], we have

$$\begin{aligned}
\varphi(z) &= \int_0^\infty \mathrm{d}a\, E_z\left[\mathbf{1}_{\{B_a \leq r\}} \exp\left(-4\int_0^a \mathrm{d}s\, \mathbb{N}_{B_s}\left(\mathcal{R}\cap]0,\infty[\neq \emptyset\right)\right)\right] \\
&= \int_0^\infty \mathrm{d}a\, E_z\left[\mathbf{1}_{\{B_a \leq r\}} \exp\left(-6\int_0^a \frac{\mathrm{d}s}{B_s^2}\right)\right] \\
&= \int_0^\infty \mathrm{d}a\, z^4 \mathbb{E}_z^{(9)}\left[\mathbf{R}_a^{-4} \mathbf{1}_{\{\mathbf{R}_a \leq r\}}\right]
\end{aligned}$$

by Lemma 6 and Proposition 8. Using the explicit form of the Green function of the nine-dimensional Brownian motion we get

$$\varphi(z) = z^4 \mathbb{E}_z^{(9)}\left[\int_0^\infty \mathrm{d}a\, \mathbf{R}_a^{-4} \mathbf{1}_{\{\mathbf{R}_a \leq r\}}\right] = \frac{15}{(2\pi)^4} z^4 \int_{\mathbb{R}^9} \mathrm{d}y\, |y|^{-4} \mathbf{1}_{\{|y|\leq r\}} |y - z\mathbf{e}_1|^{-7},$$

where $\mathbf{e}_1$ is the first vector in the canonical basis of $\mathbb{R}^9$. Finally:

$$\begin{aligned}
E\left[\mathcal{I}\left([0,r]\right)\right] &= 2\mathbb{E}_0^{(9)}\left[\int_0^\infty \mathrm{d}t\, \varphi(\mathbf{R}_t)\right] \\
&= \frac{30}{(2\pi)^4} \int_{\mathbb{R}^9} \mathrm{d}y\, |y|^{-4} \mathbf{1}_{\{|y|\leq r\}} \mathbb{E}_0^{(9)}\left[\int_0^\infty \mathrm{d}t\, \mathbf{R}_t^4 |\mathbf{R}_t \mathbf{e}_1 - y|^{-7}\right] \\
&= \frac{450}{(2\pi)^8} \int_{\mathbb{R}^9} \mathrm{d}y\, |y|^{-4} \mathbf{1}_{\{|y|\leq r\}} \int_{\mathbb{R}^9} \mathrm{d}z\, |z|^{-3} |z - y|^{-7} \\
&= \frac{450}{(2\pi)^8} c \int_{\mathbb{R}^9} \mathrm{d}y\, |y|^{-5} \mathbf{1}_{\{|y|\leq r\}} \\
&= \frac{9!!15^2}{2^{10}\pi^3} c r^4
\end{aligned}$$

where

$$c = \int_{\mathbb{R}^9} \mathrm{d}z\, |z|^{-3} \left| z - \frac{y}{|y|} \right|^{-7}$$

does not depend on the choice of $y \in \mathbb{R}^9 \setminus \{0\}$ and 9!! is the double factorial (9!)!.

3.4.2 Points of escape to infinity

Let $\theta \in \mathscr{S}$ and let $q = \Phi(\theta)$ be the infinite quadrangulation associated with θ via the (extended) Schaeffer bijection of section 2.2. As we already noticed, q has only one end. It follows that, for every integer $r > 0$, the set $\{v \in V(q) : d(\partial, v) > r\}$ has only one infinite connected component. We denote this infinite component by $F_r(q)$, and let $\partial F_r(q)$ be the set of all vertices in $V(q)\backslash F_r(q)$ that are connected by an edge to a point of $F_r(q)$. Obviously, for every $v \in \partial F_r(q)$, we have $d(\partial, v) = r$ and moreover there exists a path from v to infinity in q that stays in $F_r(q)$ (except of course at its initial point). The set $\partial F_r(q)$ is clearly related to the boundary of the r-hull of q (cf section 2.3). Our goal here is to investigate the asymptotic behavior of the cardinality $|\partial F_r(q)|$ of $\partial F_r(q)$ when r tends to infinity and q is distributed according to the law of the uniform infinite quadrangulation.

We will in fact treat a slightly modified version of this problem. Starting from the quadrangulation $q = \Phi(\theta)$ as above, we construct a new planar map in the following way. For each face of q such that the four vertices around q have respective labels $k, k+1, k, k+1$, for some integer $k \geq 1$ (such a face is called a confluent face in [CS04]), we add a diagonal edge beween the two vertices having label $k+1$. The resulting map is denoted by $\bar{q}$. This map (or rather its analogue for a finite quadrangulation) is in fact considered in [CS04] as an intermediate step of the construction of Schaeffer's bijection: See section 3.3 in [CS04]. It is important to observe that all edges of the tree θ are still present in $\bar{q}$ (which is not the case for q): More precisely, whenever two vertices v and v' are linked by an edge of θ, they are also linked by (at least) one edge of $\bar{q}$. This easily follows from the definition of Schaeffer's bijection and our construction of $\bar{q}$: Note that each edge of θ connecting two vertices with the same label r appears as a diagonal in a confluent face with labels of vertices equal to $r-1, r, r-1, r$. See also section 3.3 of [CS04].

The map $\bar{q}$ is no longer a quadrangulation: certain faces are rectangles but some others are triangles. Still the map $\bar{q}$ is in some sense very close to $\bar{q}$. In particular $V(\bar{q}) = V(q)$, and the distances from the root in $\bar{q}$ are the same as in q, because we only added edges between vertices at the same distance from the root. Therefore the distance from the root of a vertex v in the map $\bar{q}$ is still equal to its label in the tree θ. It is expected that the asymptotic metric properties (under the law of the uniform infinite quadrangulation) of the map $\bar{q}$ should be similar to those of q, and in particular that the scaling limits, in the sense of [LG07], should be the same.

To avoid confusion, we write $\bar{d}$ for the graph distance in $\bar{q}$. By the preceding remarks, $\bar{d}(\partial, v) = d(\partial, v) = \ell(v)$ for every $v \in \theta$. Similarly as above, we let $F_r(\bar{q})$ be the unique infinite component of $\{v \in V(q) : \bar{d}(\partial, v) > r\}$, and we denote its "boundary" by $\partial F_r(\bar{q})$.

If $v \in V(q)$, $v \neq \partial$, then we denote by $g^{\theta}_{v\to\infty}$ the unique geodesic path from v (viewed as a vertex of θ) to infinity in the tree θ. Except for finitely many points at the beginning, this geodesic path of course lives on the spine of θ.

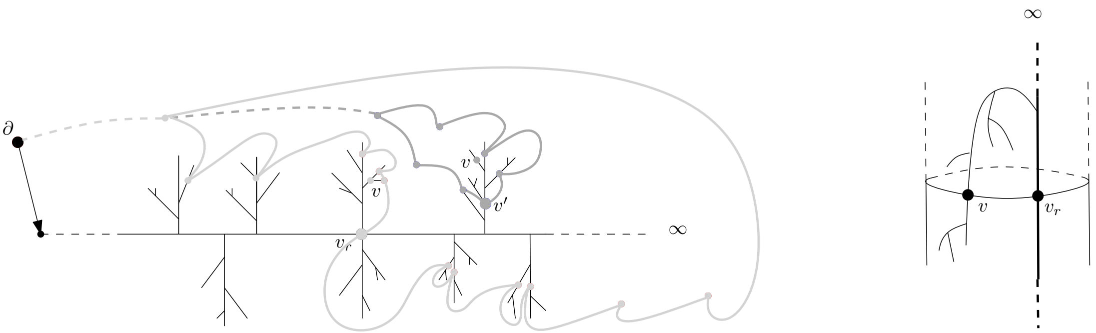

Figure 3.6: Left: cycles $\mathcal{C}$ separating vertices from infinity composed by two geodesics. Right: a vertex v with label r whose last ancestor with label r is v_r belongs to $\partial F_r(\bar{q})$.

Proposition 12. *Let $\theta \in \mathscr{S}$ and $q = \Phi(\theta)$. Let $r > 0$. A vertex v of $V(q)$ belongs to $\partial F_r(\bar{q})$ if and only if $\ell(v) = r$ and all the points of the geodesic path $g^{\theta}_{v\to\infty}$ other than v have a label strictly greater than r.*

Remark. In particular there is only one vertex of $\partial F_r(\bar{q})$ on the spine of θ, namely the vertex v_r which is the last vertex on the spine with label r.

Proof. Suppose first that v has the properties stated in the proposition. By noting that the geodesic path $g^{\theta}_{v\to\infty}$ is also an infinite path in $\bar{q}$, it immediately follows that $v \in \partial F_r(\bar{q})$. So we only need to prove the converse. We suppose that $\ell(v) = r$ but that there exists a vertex $v' \neq v$ on the geodesic path $g^{\theta}_{v\to\infty}$ such that $\ell(v') \leq r$. Note that v' belongs to the line of ancestors of v in θ or to the spine of θ. We aim at showing that $v \notin \partial F_r(\bar{q})$.

We consider the two corners c_1 and c_2 of the vertex v' defined as follows. If v' is on the ancestral line of v and not on the spine, we let c_1 be the rightmost corner of v' and c_2 be the leftmost corner of v'. If v' belongs to the spine, we let c_1 be the first corner of v' at the right of the spine, and c_2 be the first corner of v' at the left of the spine. We then construct two geodesics g_1 and g_2 from v' to ∂ in the following way. To construct g_1, we start from the corner c_1 and consider the edge from this corner that is obtained in the construction of section 2.2. This edge connects v' to a vertex v'' with label $\ell(v') - 1$. We then take the rightmost corner of v'' and consider the associated edge from this corner, and so on, considering at each step the rightmost corner of the new vertex. The concatenation of the edges generated in this process gives us the geodesic g_1. To get g_2, we proceed in the same way, with the only difference that we start from the corner c_2 instead of c_1.

The geodesics g_1 and g_2 eventually coalesce. If we remove the common part of the geodesics, the concatenation of g_1 and g_2 gives a cycle $\mathcal{C}$ of the map $\bar{q}$ (in fact also of q). It is easy to see that v belongs to the finite component of $V(q)\backslash\mathcal{C}$: Considering for instance the case when v' is not on the spine of θ, one checks that all descendants of v' in θ must belong to the finite component of $V(q)\backslash\mathcal{C}$, or possibly to the cycle $\mathcal{C}$ (but this cannot occur for v since vertices on $\mathcal{C}$ distinct from v' have a label strictly smaller than the label of v). To conclude, consider a path from v to infinity in $\bar{q}$. This path must intersect $\mathcal{C}$ and so it must contains a vertex, distinct from v, with label less than or equal to $\ell(v') \leq r$. It follows than v does not belong to $\partial F_r(\bar{q})$. □

Let us now use Proposition 12 to discuss asymptotics for the number of vertices in the boundary $\partial F_r(\bar{q})$, when q is distributed according to the law ν of the uniform infinite quadrangulation. Recall that the only vertex of $\partial F_r(\bar{q})$ that belongs to the spine is the last vertex of the spine with label r. Denote the height of this vertex in θ by H_r. Then, Proposition 12 implies that

$$|\partial F_r(\bar{q})| = 1 + \sum_{i>H_r} |Y_{L_i}(r)| + \sum_{i>H_r} |Y_{R_i}(r)| \tag{3.40}$$

where, if T is a spatial tree whose root label is strictly larger than r, $Y_T(r)$ is the set of all vertices of T with label r and such that all their ancestors have a label strictly larger than r. Recalling that the trees (L_i) and (R_i) are independent conditionally given (X_i), we get from (3.40) that

$$\begin{aligned}
&E\left[\exp -\lambda \frac{|\partial F_r(\bar{q})|}{r^2}\right] \\
&= \exp\left(-\frac{\lambda}{r^2}\right) E\left[\prod_{i>H_r} E\left[\exp\left(-\frac{\lambda}{r^2}|Y_{L_i}(r)|\right)\middle| X_i\right] E\left[\exp\left(-\frac{\lambda}{r^2}|Y_{R_i}(r)|\right)\middle| X_i\right]\right] \\
&= \exp\left(-\frac{\lambda}{r^2}\right) E\left[\prod_{i>H_r} \widehat{\rho}_{X_i}\left(\exp\left(-\frac{\lambda}{r^2}|Y_T(r)|\right)\right)^2\right] \\
&= \exp\left(-\frac{\lambda}{r^2}\right) E\left[\exp\left(2\sum_{i>H_r} \log \widehat{\rho}_{X_i}\left(\exp\left(-\frac{\lambda}{r^2}|Y_T(r)|\right)\right)\right)\right].
\end{aligned} \tag{3.41}$$

To compute the right hand side of (3.41), we need to evaluate the generating functions:

$$f_{l,r}(x) = \widehat{\rho}_l\left(x^{|Y_T(r)|}\right),\ 0 \le x \le 1$$

for $l \ge r \ge 1$. To this aim, we use the description of the measures $\widehat{\rho}_l$ given in Theorem 4.5 of [CD06] in terms of multitype Galton-Watson trees. In such multitype Galton–Watson trees, the progeny of a type l individual is determined by a sequence of independent trials with four possible outcomes, $l-1$, l, $l+1$ and e (for "extinction"), with respective probabilities, $w_{l-1}/12$, $w_l/12$, $w_{l+1}/12$ and $1/w_l$ (that add up to 1), that are sequence-stopped just before the first occurrence of e (see [CD06] for more details). This leads to the following recursive relation for $l > r$:

$$f_{l,r}(x) = \sum_{k=0}^{\infty} \sum_{i_1+i_2+i_3=k} \binom{k}{i_1, i_2, i_3} \left(\frac{w_{l-1}}{12} f_{l-1,r}(x)\right)^{i_1} \left(\frac{w_l}{12} f_{l,r}(x)\right)^{i_2} \left(\frac{w_{l+1}}{12} f_{l+1,r}(x)\right)^{i_3} \frac{1}{w_l}.$$

From this identity, we get the following recurrence relation for $l > r$:

$$w_l f_{l,r}(x) = 1 + \frac{1}{12} w_l f_{l,r}(x)\left(w_{l-1} f_{l-1,r}(x) + w_l f_{l,r}(x) + w_{l+1}.f_{l+1,r}(x)\right).$$

Similar recursion relations have been studied, e.g. in [BDFG04]. Putting

$$F(x,y) = xy\left(1 - \frac{1}{12}x - \frac{1}{12}y\right) - x - y$$

one can easily see that $F\left(w_l f_{l,r}(x), w_{l+1} f_{l+1,r}(x)\right)$ does not depend on $x \in [0,1]$ and $l \geq r$, by checking that

$$F\left(w_l f_{l,r}(x), w_{l+1} f_{l+1,r}(x)\right) - F\left(w_{l-1} f_{l-1,r}(x), w_l f_{l,r}(x)\right) = \left(w_{l+1} f_{l+1,r}(x) - w_{l-1} f_{l-1,r}(x)\right) \times \left(w_l f_{l,r}(x) - 1 - \frac{1}{12} w_l f_{l,r}(x)\left(w_{l-1} f_{l-1,r}(x) + w_l f_{l,r}(x) + w_{l+1}.f_{l+1,r}(x)\right)\right).$$

It is easy to verify that $f_{l,r}(x) \to 1$ as $l \to \infty$ and thus $w_l f_{l,r}(x) \to 2$. Since $F(2,2) = -\frac{4}{3}$, we have the relation:

$$F\left(w_l f_{l,r}(x), w_{l+1} f_{l+1,r}(x)\right) = -\frac{4}{3} \tag{3.42}$$

for $l \geq r$, with the initial condition $f_{r,r}(x) = x$. The general solution of (3.42) is given by

$$w_l f_{l,r} = 2 - \frac{4}{(l-r+a)(l-r+1+a)} \tag{3.43}$$

for $l \geq r$. From the initial condition one finds

$$a(r,x) = \frac{-1 + \sqrt{1 + 8\left(1 - \frac{w_r}{2} x\right)^{-1}}}{2}$$

for $x \in [0,1]$.

Substituting (3.43) in (3.41), one gets:

$$\exp\left(\frac{\lambda}{r^2}\right) E\left[\exp -\lambda \frac{|\partial F_r(\bar{q})|}{r^2}\right]$$
$$= E\left[\exp \sum_{i > H_r} \log \frac{2}{w_{X_i}}\left(1 - \frac{2}{\left(X_i - r + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)\left(X_i - r + 1 + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)}\right)\right]$$
$$= E\left[\exp 2r^2 \int_{H_r/r^2}^{\infty} \mathrm{d}t \log \frac{2}{w_{X_{\lfloor r^2 t \rfloor}}}\left(1 - \frac{2}{\left(X_{\lfloor r^2 t\rfloor} - r + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)\left(X_{\lfloor r^2 t\rfloor} - r + 1 + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)}\right)\right].$$

From the explicit formula for w_l (Theorem 8), one easily gets the following asymptotics

$$\frac{w_k}{2} = 1 - \frac{2}{k^2} + o\left(\frac{1}{k^2}\right)$$
$$a\left(k, e^{-\frac{\lambda}{k^2}}\right) = \frac{k}{\sqrt{1+\lambda/2}} + o(k)$$

as $k \to \infty$.

As explained in the proof of Proposition 10, for every $n \geq 1$, there exists a process $\left(\widetilde{X}_k^n\right)_{k\geq 0}$ having the same distribution as $(X_k)_{k\geq 0}$, and a nine-dimensional Bessel process Z started from 0, such that almost surely, $\left(\sqrt{\frac{3}{2n}}\widetilde{X}^n_{\lfloor nt\rfloor}\right)_{t\geq 0}$ converges to $(Z_t)_{t\geq 0}$ as n goes to infinity. For every $t > 0$ we have as $r \to \infty$

$$r^2 \log \frac{2}{w_{\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor}}} = r^2 \log\left(1 + \frac{2}{\left(\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor}\right)^2} + o\left(\frac{1}{\left(\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor}\right)^2}\right)\right) = 2\left(\frac{r}{\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor}}\right)^2 + o(1)$$

and

$$r^2 \log\left(1 - \frac{2}{\left(\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor} - r + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)\left(\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor} - r + 1 + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)}\right)$$
$$= r^2 \log\left(1 - \frac{1}{r^2}\frac{2}{\left(\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor}/r - 1 + a\left(r, e^{-\frac{\lambda}{r^2}}\right)/r\right)^2} + o\left(\frac{1}{r^2}\right)\right).$$

From the preceding asymptotics one gets the almost sure convergence, for every $t > 0$,

$$r^2 \log \frac{2}{w_{\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor}}}\left(1 - \frac{2}{\left(\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor} - r + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)\left(\widetilde{X}^{r^2}_{\lfloor r^2 t\rfloor} - r + 1 + a\left(r, e^{-\frac{\lambda}{r^2}}\right)\right)}\right)$$
$$\underset{r\to\infty}{\longrightarrow} \frac{3}{Z_t^2} - \frac{3}{\left(Z_t - \sqrt{\frac{3}{2}}\,(1-(1+\lambda/2)^{-1/2})\right)^2}$$

as $r \to \infty$.

Furthermore, $\frac{H_r}{r^2}$ is the last hitting time of $\sqrt{3/2}$ for the process $X^{(r^2)}$. Let us denote by $\widetilde{H}^{r^2}_{\sqrt{3/2}}$ the last hitting time of $\sqrt{3/2}$ for the process $\widetilde{X}^{r^2}$. Then $\widetilde{H}^{r^2}_{\sqrt{3/2}}$ and $\frac{H_{r+1}}{r^2}$ have the same law. In addition, one can also verify the almost sure convergence (we omit some details here):

$$\widetilde{H}^{r^2}_{\sqrt{3/2}} \underset{r\to\infty}{\longrightarrow} h_{\sqrt{3/2}}(Z)$$

where $h_{\sqrt{3/2}}(Z)$ is the last hitting time of $\sqrt{3/2}$ of the Bessel process Z. An argument of dominated convergence together with the scaling property of the Bessel process then gives the proposition:

Proposition 13. *Suppose that q is distributed according to the law of the uniform infinite quadrangulation. For every $\lambda > 0$, we have the convergence*

$$E\left[\exp -\lambda\frac{|\partial F_r(\bar{q})|}{r^2}\right] \to \mathbb{E}_0^{(9)}\left[\exp -6\int_{h_1(\mathbf{R})}^{\infty} \mathrm{d}t\left(\frac{1}{(\mathbf{R}_t - (1-(1+\lambda/2)^{-1/2}))^2} - \frac{1}{\mathbf{R}_t^2}\right)\right],$$

as $r \to \infty$, where $h_1(\mathbf{R})$ is the last hitting time of 1 *by the process* $\mathbf{R}$.

It would be of interest to calculate more explicitly the limit appearing in Proposition 13. The discussion at the end of this section suggests that the limiting distribution for $r^{-2}|\partial F_r(\bar{q})|$ should be related (or perhaps even equal) to the limiting distribution for the rescaled length $r^{-2}|\gamma_r(q)|$ of the cycle $\gamma_r(q)$ which was defined in section 2.3. This limiting distribution is computed in Corollary 1 of Krikun [Kri06], where it is shown that the random variables $2r^{-2}|\gamma_r(q)|$ converge to a $\Gamma(3/2)$ distribution, with Laplace transform $(1+\lambda)^{-3/2}$.

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