HDR defense

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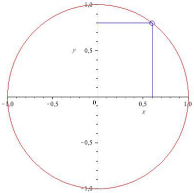
Institut de Mathématiques de Luminy, CNRS

December 2, 2009

L'approche classique : genre $0, 1, 2$ et 3



genre 0 $x^2 + y^2 = 1$

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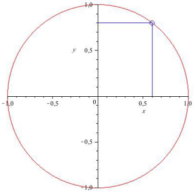

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$$

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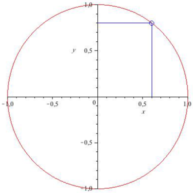

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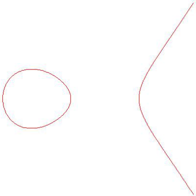

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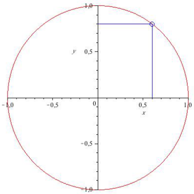

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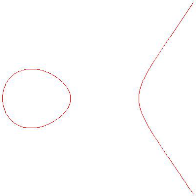

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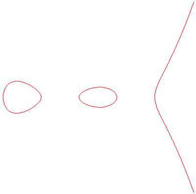

genre 2 $y^2 = x(x^2-1)(x^2-\frac{1}{4})$

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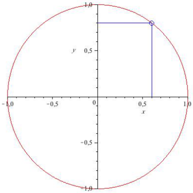

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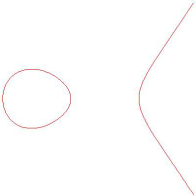

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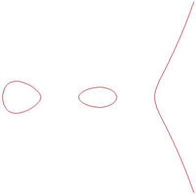

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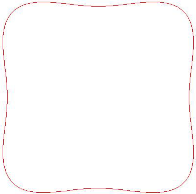
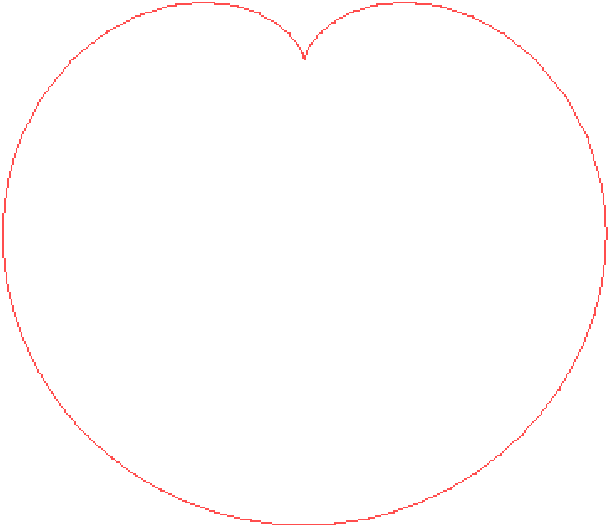


genre 3 $(x^2-1)^2 + (y^2-1)^2 = 4$

Le spirographe

Quelques quartiques de genre 0 au quotidien



La cardioïde $(x^2 + y^2 - x)^2 = x^2 + y^2$ comme épicycloïde

au réveil

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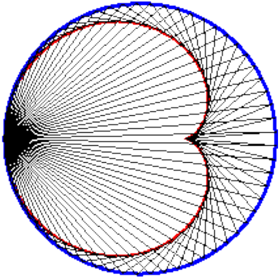
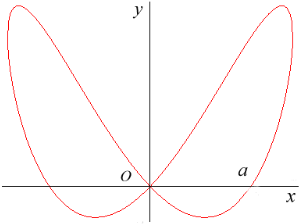
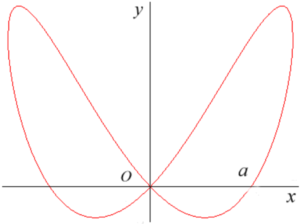
 

La cardioïde comme caustique



besace $(x^2 - y)^2 = (x^2 - y^2)$



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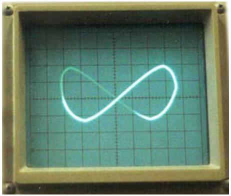
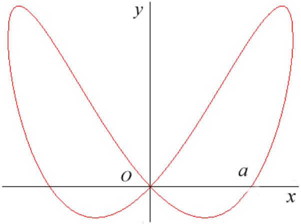


courbe de Lissajous



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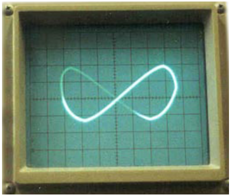

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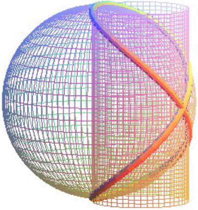
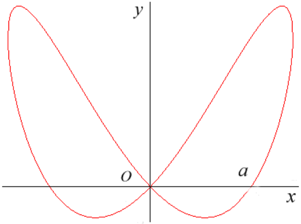


fenêtre de Vivani



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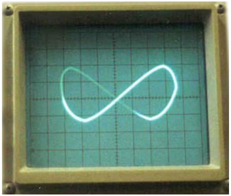

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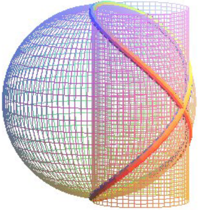

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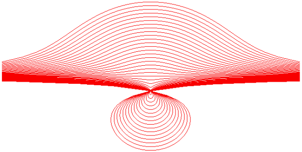
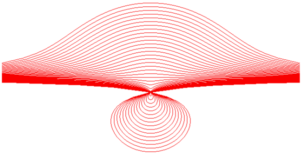
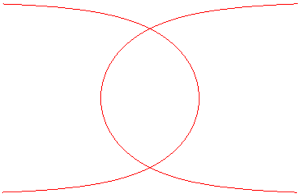
musée de la marine d'Osaka



les conchoïdes de Nicomède,





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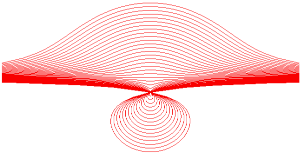
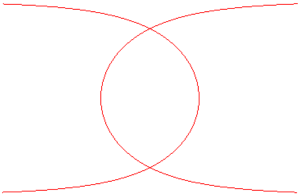
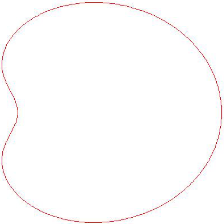
 

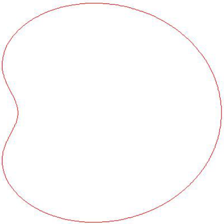
les conchoïdes de Nicomède, la trisectrice de Delanges, le bicorne, . . .

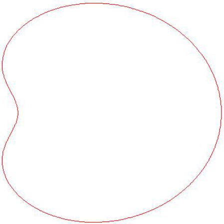
Quelques quartiques de genre 1



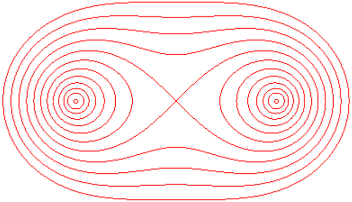
ovale de Descartes $(x^2 + y^2 - 2x + 1)^2 = 4(x^2 + y^2)$

Quelques quartiques de genre 1

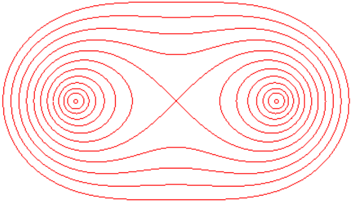




système Bélidor pour pont-levis (ici à Fort-l'Écluse)

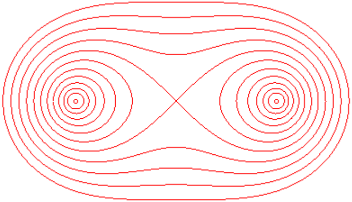


ovales de Cassini $((x-a)^2+y^2)((x+a)^2+y^2)=b^4$



ovales de Cassini

au Palais de la Découverte



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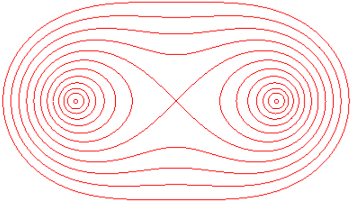
Cassini par Jean-Guillaume Moitte



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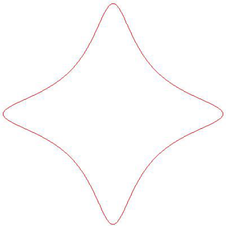
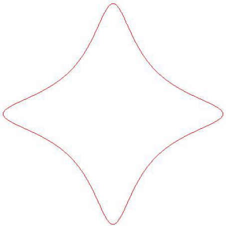


courbe d'Edwards $x^2 + y^2 = 1 - 30x^2y^2$



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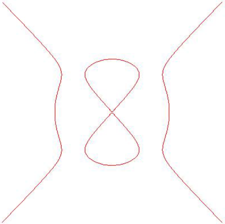
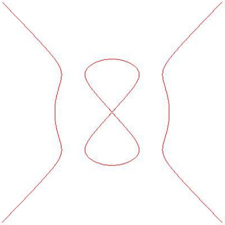
un jour dans nos cartes bleues ?

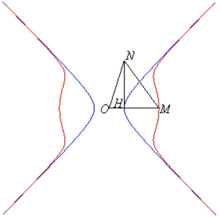
Quartique de genre 2



courbe du diable $y^4 + \frac{10}{9}x^2 = x^4 + \frac{8}{9}y^2$

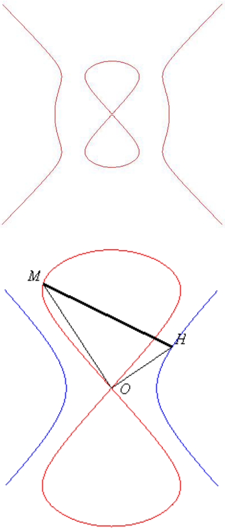
Quartique de genre 2





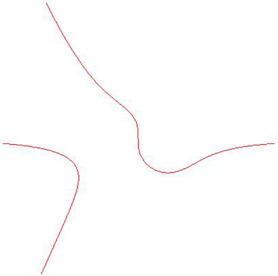
H sur l'hyperbole, $HN = 1$, $OM = ON$

Quartique de genre 2



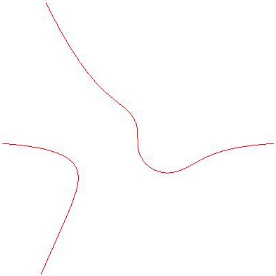
H sur l'hyperbole, rectangle en O, $HM = 1$

Quartique de genre 3



quartique de Klein
$x^3y + y^3 + x = 0$

Quartique de genre 3



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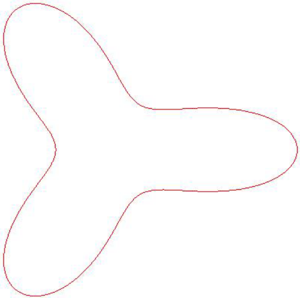
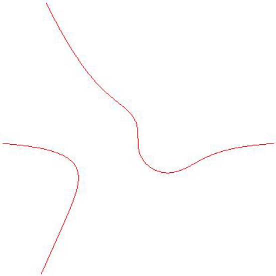


quartique de Klein (bis)
$196x^3 + 84x^2 - 588xy^2 + 84y^2 + 16 - 147x^4 - 294x^2y^2 = 147y^4$

Quartique de genre 3



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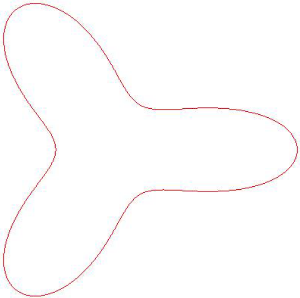

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sculpture
au MSRI

Overview of the manuscript

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    - models;

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Definition

A curve C of genus g over k is *maximal* if $\#C(k) = N_q(g)$.
It is optimal if $\#C(k) = 1 + q + gm$.

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Proposition

There does not exist an elliptic curve with trace $-m$ if and only if $n \geq 3$, n is odd and $p|m$.

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- Existence of a principal polarization in the class E^g: yes (the product polarization a_0).

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This works quite well for small values of q: see `www.manypoints.org` (van der Geer, Howe, Lauter, R.).

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Let $A = (A, a)/K$ be a p.p.a.t. defined over a field K with char $K \neq 2$. Assume that a is absolutely indecomposable. There exists a unique geometric Siegel modular form of weight 18 defined over $\mathbb{Z}$, denoted χ_{18}, such that

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- Result of (Ichikawa 1996): let $t : \mathsf{M}_3 \to \mathsf{A}_3$ be the Torelli map. There exists a Teichmüller modular form of weight 9 defined over $\mathbb{Z}$, denoted μ_9, such that $t^*(\chi_{18}) = \mu_9^2$.

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- computation by (Schiemann 1998) of such matrices. There is only one –up to equivalence–, which is indecomposable:

$$M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2+\tau \\ -1 & -2+\overline{\tau} & 3 \end{pmatrix}.$$

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(Guàrdia 2009):

$$\tilde{C} : x^4 + \frac{1}{9} y^4 + \frac{2}{3} x^2 y^2 - 190 y^2 - 570 x^2 + \frac{152}{9} y^3 - 152 x^2 y = 1083.$$

Values of $\chi = \chi_{18}((\tilde{E}^3, a_0 M), \omega_0)$

$\tilde{E}$: Gross' models with discriminant d^3 (when the class number is 1).

d	$M, \tau = (1+\sqrt{d})/2$	χ	$\#\,\mathrm{Aut}(\tilde{E}^3, a)$
-7	$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & \overline{\tau} \\ 1 & \tau & 2 \end{pmatrix}$	$(7^7)^2$	$2 \cdot 168$
-19	$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2+\tau \\ -1 & -2+\overline{\tau} & 3 \end{pmatrix}$	$(2^5 \cdot 19^7)^2 \cdot (-2)$	$2 \cdot 6$
-43	$\begin{pmatrix} 3 & 1 & 1-\overline{\tau} \\ 1 & 4 & 2 \\ 1-\tau & 2 & 5 \end{pmatrix}$	$(2^6 \cdot 43^7)^2 \cdot (-47 \cdot 79 \cdot 107 \cdot 173)$	$2 \cdot 1$
-67	$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & -2+\overline{\tau} \\ -1 & -2+\tau & 7 \end{pmatrix}$	$(2^5 \cdot 7^4 \cdot 67^7)^2 \cdot (-2 \cdot 7 \cdot 31)$	$2 \cdot 6$
-163	$\begin{pmatrix} 2 & 1 & -\overline{\tau} \\ 1 & 2 & 1-\overline{\tau} \\ -\tau & 1-\tau & 28 \end{pmatrix}$	$(2^5 \cdot 7^4 \cdot 11^4 \cdot 163^7)^2 \cdot (-2 \cdot 7 \cdot 11 \cdot 19 \cdot 127)$	$2 \cdot 6$
-15	$\begin{pmatrix} 2 & -1 & -1+\overline{\tau} \\ -1 & 2 & 1-\overline{\tau} \\ -1+\tau & 1-\tau & 3 \end{pmatrix}$	$22769095299822142340569171645771726299/4+$ $10182522603020834484863085151244322675 \cdot \sqrt{5}/4+$ $44626409093538218819956956474294768 69 \cdot \sqrt{-15}/4$ $+9978330617922886443823982755114202445 \cdot \sqrt{-3}$	$2 \cdot 24$

Algebraic interpretation of χ

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4	9	hyperelliptic curve

Worse: how to control the (parity of the) exponents ?

References for introduction material

- Mainly: `http://www.mathcurve.com/`
- Spirograph: `http://fr.wikipedia.org/wiki/Spirographe_(jeu)`
- Osaka museum: `http://fr.wikipedia.org/wiki/Fenêtre_de_Viviani`
- "Ovales de Cassini" : picture from G. Lachaud
- "Cassini statue": `http://home.nordnet.fr/~ajuhel/Obs_Paris/Cassini/cassini.html`
- The eightfold way. The beauty of Klein's quartic curve. Edited by Silvio Levy. Mathematical Sciences Research Institute Publications, **35**. Cambridge University Press, Cambridge, 1999, p. 325
- Video of Klein quartic: `http://www.gregegan.net/SCIENCE/KleinQuartic/KleinQuartic.html`