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# Instabilité elliptique sous champ magnétique et Dynamo d'ondes inertielles

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UNIVERSITÉ DE PROVENCE AIX-MARSEILLE I  
Institut de Recherche sur les Phénomènes Hors Équilibre

# THÈSE

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par

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le 20 janvier 2009

## **Instabilité elliptique sous champ magnétique & Dynamo d'ondes inertielles**

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# Chapitre 1

## Introduction

### 1.1 L'effet dynamo

La première observation du champ magnétique de la Terre date des premières boussoles utilisées par les Chinois, au deuxième siècle de notre ère. Ce n'est qu'à partir du sixième siècle qu'elles ont été utilisées dans la navigation. L'origine de cette force attractive était très loin d'être comprise, et il a fallu attendre le développement de la théorie du champ électromagnétique finalisée par Maxwell pour bien comprendre les mouvements de l'aiguille aimantée dans le champ terrestre. Au tout début du dix-septième siècle, une première explication physique pour le champ magnétique de la Terre et du soleil a été proposée par William Gilbert [39]. Il compare le champ magnétique de la Terre avec celui d'un aimant de forme sphérique. Il propose alors que l'intérieur de la Terre soit aimantée. Cette hypothèse a rapidement été écartée avec les premières observations des variations lentes du champ magnétique. L'arrivée de la sismologie au début du vingtième siècle révèle la structure interne de la Terre et montre qu'une zone liquide et métallique existe à l'intérieur de celle-ci. C'est alors Larmor [74], qui propose en 1919 pour la première fois, que les champs magnétiques de la Terre ou du soleil peuvent trouver leurs origines dans les mouvements d'un fluide conducteur. En effet, les mouvements d'un fluide conducteur sont capables de générer un champ magnétique, de la même manière qu'une force électromotrice est générée dans un générateur électrique à travers les mouvements des bobinages de métaux solides. C'est le début des études sur le phénomène que l'on a aujourd'hui appelé l'effet dynamo.

Supposons un métal liquide incompressible, de densité  $\rho$ , de perméabilité  $\mu$  et de conductivité électrique  $\sigma_e$ . Les équations de la magnétohydrodynamique (MHD) décrivant la réaction d'un écoulement  $\mathbf{u}$ , d'une pression  $p$  et d'un champ magnétique  $\mathbf{b}$  à un forçage  $\mathbf{f}$  dans le volume s'écrivent

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{\nabla p}{\rho} + \frac{1}{\rho\mu} (\mathbf{b} \cdot \nabla) \mathbf{b} + \nu \Delta \mathbf{u} + \mathbf{f} \quad \nabla \cdot \mathbf{u} = 0 \quad (1.1)$$

$$\partial_t \mathbf{b} + (\mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u} + \frac{1}{\sigma_e \mu} \Delta \mathbf{b} \quad \nabla \cdot \mathbf{b} = 0 \quad (1.2)$$

On reconnaît la force de Lorentz (ou de Laplace) dans l'équation pour la quantité de mouvement. L'équation pour la dynamique du champ magnétique s'appelle l'équation d'induction, et se dérive des équations de Maxwell accompagnées de la loi constitutive d'Ohm et la loi de Gauss. L'approximation MHD filtre les ondes électromagnétiques, qui se propagent beaucoup plus vite que tous les écoulements ou toutes les ondes possédant un intérêt en géo- ou astrophysique. Ces équations sont très bien adaptées pour décrire la dynamique des métaux liquides, un peu moins pour les plasmas où elles restent cependant une bonne approximation. Nous reconnaissons que l'équation d'induction est linéaire dans la variable  $\mathbf{b}$  du champ magnétique. Le problème de la *dynamo cinématique* cherche à trouver des solutions instables de cette équation pour un écoulement  $\mathbf{u}$  donné et fixe. On voit que ce problème linéaire à la structure d'une équation de diffusion-advection pour un champ vectoriel, accompagné d'un terme source qui correspond à un étirement du champ magnétique  $(\mathbf{b} \cdot \nabla) \mathbf{u}$ . Une dynamo n'est possible seulement si la diffusion ne domine pas la dynamique du champ magnétique. La diffusion magnétique étant typiquement  $10^5$  plus forte que la diffusion visqueuse, il faut donc avoir des écoulements très intenses, ou des systèmes très grands comme des planètes par exemple pour réaliser une dynamo. Imaginons alors que l'on a identifié une dynamo, et qu'on laisse la possibilité au champ de rétro-agir sur le champ de vitesse. La croissance du champ magnétique rend la force de Lorentz de plus en plus importante et celle-ci modifie l'écoulement de manière à pouvoir faire saturer l'instabilité. Si ceci est possible, on parle d'une dynamo entretenue.



Il s'est avéré que la compréhension de la dynamo et la reproduction d'une dynamo fluide en laboratoire représentait un sérieux défi. Nous renvoyons le lecteur non-initié aux nombreux ouvrages existants sur le sujet de la dynamo comme le livre de K. Moffatt [102], les articles de revue [118] et [38] et également le livre récent issu des cours donnés à l'école d'été des Houches [18]. Il est important de savoir que ce n'est que récemment que des expériences utilisant du sodium liquide ont été reproduite en laboratoire, [33], [34], [128], [103], [5].

## 1.2 Le champ magnétique des corps célestes

Retournons maintenant au problème initial : le champ magnétique des corps célestes. Depuis le seizième siècle, le nombre de mesures ponctuelles du champ magnétique de la Terre n'a cessé de croître. La boussole étant un instrument de navigation, il n'est pas étonnant que les premiers magnétomètres étaient embarqués à bord des bateaux. Aujourd'hui il y a des stations de mesure qui suivent l'évolution du champ magnétique Terrestre en détail, et grâce aux satellites, nous avons une couverture quasi-totale de la Terre. Ces nombreuses données ont permis de mettre en évidence des variations sur des périodes assez courtes dans l'histoire de la Terre, [31], [46]. Non-seulement la Terre est observée de près, mais le soleil nous offre peut être encore un meilleur objet d'étude : grâce aux observations dans le domaine des hautes fréquences, nous pouvons même visualiser les lignes du champ magnétique émanant du soleil. L'existence de tâches solaires est connue depuis très longtemps, et la mise en évidence du cycle du champ magnétique du soleil est spectaculaire pour un corps fluide de cette taille. Pour terminer, on peut dire que c'est également grâce aux programmes spatiaux d'observation, que nous savons maintenant que la plupart des corps célestes du système solaire, possèdent des champs magnétiques eux-aussi.

Pour faire un dynamo planétaire, il semble que nous ayons besoin de peu d'ingrédients. En première instance, il faut un volume de fluide conducteur d'une taille suffisamment grande. Les corps de type terrestres (Mercure, Venus, Terre, Mars, Io) ont un intérieur partiellement liquide dont la composition est estimée riche en Fe et S, ce qui donne au fluide sa conductivité électrique. Dans les géantes gazeuses (Jupiter, Saturne), les éléments légers  $H$  et  $He$  existent dans un état métallique à cause des fortes pressions. Pour les Géants de glace (Neptune, Uranus), c'est encore différent, car la conductivité électrique est plutôt assuré par un manteau principalement composé  $H_2O$ ,  $NH_3$  et  $CH_4$ . En deuxième instance, une rotation rapide est nécessaire pour organiser les écoulements qui se superposent à cette rotation. Enfin il faut un mécanisme qui excite ces écoulements secondaires. Regardons ces deux aspects de plus près.

### 1.2.1 Rotation rapide

La rotation rapide des écoulements planétaires domine largement les perturbations qui s'y superposent. Typiquement pour la Terre, on a estimé que l'amplitude de ces perturbations est de l'ordre de  $10^{-7}$  fois plus petit que l'écoulement de rotation [18], si l'on considère que les variations lentes du champ magnétique correspondent à l'échelle de vitesse de l'écoulement. Dans ce cas, la rotation dominante influence fortement les perturbations à travers la force de Coriolis,  $2\Omega\hat{\mathbf{z}} \times \mathbf{u}$  où  $\Omega\hat{\mathbf{z}}$  est le vecteur de la vitesse de rotation le long de l'axe  $\hat{\mathbf{z}}$ . Les propriétés des écoulements en rotation rapide sont discutés en grand détail dans le livre de Greenspan, [45]. Rappelons ici quelques aspects importants.

On sépare les perturbations de la rotation rapide en deux classes : les écoulements géostrophiques et les ondes inertielles. Les écoulements géostrophiques correspondent aux perturbations stationnaires dans le repère tournant. La force de Coriolis équilibre les gradients de pression dans le fluide. En prenant le rotationnel de l'équation décrivant cet équilibre géostrophique, on obtient le théorème de Taylor-Proudman : les écoulements géostrophiques sont indépendants de la coordonnée le long de l'axe de rotation. En présence de parois courbées, il faut relâcher la contrainte de Taylor-Proudman, car une indépendance parfaite de la coordonnée  $z$  ne permet pas de respecter les conditions aux limites de confinement. Ceci donne lieu à la classe des écoulements quasi-géostrophiques. Ces écoulements ont une petite dépendance en  $z$ , mais restent majoritairement organisés en colonnes. Le petit relâchement de la contrainte de Taylor-Proudman est accompagné d'une apparition d'une fréquence faible. Les écoulements quasi-géostrophiques varient lentement dans le temps par rapport à la période de rotation rapide du fluide  $\Omega^{-1}$ . Ces écoulements sont également appelées les ondes de Rossby.

Si on continue à relâcher la contrainte de Taylor-Proudman (on autorise des variations le long de  $z$ ), on s'aperçoit que la variation dans le temps de l'écoulement devient de plus en plus rapide. L'inertie du fluide ( $\rho\partial_t\mathbf{u}$ ) devient importante et atteint le même ordre de grandeur que la force de Coriolis, si l'écoulement varie sur une période de rotation. Dans cette limite, les écoulements ne sont plus du tout indépendants de  $z$ , mais même essentiellement tri-dimensionnels. On parle d'ondes inertielles. Les ondes de Rossby, ne sont donc en fait que des ondes inertielles lentes. Les écoulements géostrophiques permettent aux particules de fluide de se

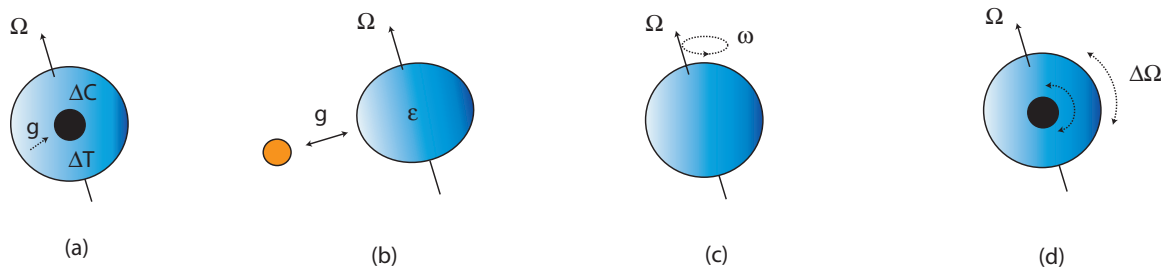


FIG. 1.1 – Différents mécanismes pour exciter des écoulements dans les corps célestes : (a) Convection, (b) Instabilité de marée, (c) Précession, (d) Libration.

déplacer sur des grandes distances sur des échelles de temps lentes, alors qu'une onde inertielle fait osciller les particules de fluide autour d'une position d'équilibre sur des échelles de temps rapide. Ceci ne veut pas dire que les ondes inertielles ne peuvent pas avoir une grande longueur d'onde.

En géométrie infinie on peut donner des expressions exactes pour les relations de dispersion des ondes planes, [21] ou des ondes tournantes [24]. Reste le problème que les ondes ont des vitesses de propagation rapide (typiquement reliées à la vitesse de rotation), et que les effets d'un confinement se font rapidement ressentir dans tout le fluide. Plusieurs propriétés générales des ondes inertielles confinées sont discutées dans le livre de Greenspan [45]. Lord Kelvin a été le premier à trouver une famille d'ondes inertielles dans un cylindre en rotation rapide [53]. Le confinement des ondes mène à un spectre discret de fréquences, d'une manière identique aux vibrations d'une corde élastique encadrée. Des solutions analytiques et des fréquences propres pour les ondes inertielles sont également connues en géométrie sphérique et sphéroïdale, [110], [9], [20], [144], [146]. D'une manière générale, peu de solutions sont connues pour les ondes inertielles dans d'autres géométries, mais on pourrait toujours penser que le confinement des ondes mène à un spectre discret de modes globaux. Il s'avère en fait que ceci n'est pas toujours aussi simple [115], [114], à cause d'une particularité mathématique : dans des volumes de fluide de forme générale, le problème des ondes inertielles est mal posé et l'écoulement forcé plus complexe. Les ondes se focalisent sur des caractéristiques, et les mouvements de fluide deviennent très localisés dans des couches de cisaillement internes. De toute manière, quelle que soit la géométrie, il s'avère que le fluide confiné en rotation est alors sensible à des forçages extérieurs oscillants avec des fréquences particulières. Ceci ouvre la voie pour l'excitation des ondes inertielles par des mécanismes de résonance, tels que la précession, la libration ou l'instabilité elliptique.

L'état de base d'un corps céleste est également plus complexe qu'une simple rotation. La présence d'une stratification fait intervenir la poussée d'Archimède, qui peut être déstabilisante et entraîner de la convection, ou stabilisante permettant alors l'existence d'ondes internes. Si on imagine que les perturbations du fluide se développent sur un état de base où un champ magnétique dominant est déjà présent, la force de Lorentz apparaîtra. Un tel champ magnétique fort permet l'existence d'ondes d'Alfvén [3]. Si le champ magnétique et la force de Coriolis interviennent au même niveau, on parle alors d'ondes Magnéto-Coriolis, qui ne sont en fait qu'un mélange des deux familles d'ondes inertielles et d'Alfvén [79], [21]. Si une stratification de densité est prise en compte, on parle d'ondes MAC (Magneto-Archimedes-Coriolis) [8], [7]. Ces ondes sont plus compliquées dans leur structure spatiale, et dans leur spectre, mais les mécanismes d'excitation tel que la précession et l'instabilité elliptique continuent d'exister. Une description détaillée des différentes familles d'ondes est donnée dans le 8-ième chapitre du livre issu de l'école d'été des Houches [18].

## 1.2.2 Mécanismes d'excitation

Sur la figure 1.1, nous avons juxtaposé quelques possibles sources de mouvement dans les corps célestes. Nous séparons la convection (thermique ou thermosolutale) Fig.1.1-(a) des instabilités inertielles due aux effets de marées Fig.1.1-(b), de la précession Fig.1.1-(c) et de la libration Fig.1.1-(d). Dans le cas de la convection, la source d'énergie qui mène aux mouvements, résulte d'un déséquilibre gravitationnel. Les instabilités inertielles quant à elles puisent leur énergie dans la rotation de la planète.

### La convection

A l'heure actuelle, la convection thermosolutale Fig. 1.1-(a), est la mieux placée pour expliquer l'origine des champ magnétiques planétaires (en tout cas pour la Terre). Combinant les effets déstabilisants d'un gradient thermique, d'un gradient de composition dans le champ de gravité de la Terre, la convection met en place des écoulements sous la forme de rouleaux bidimensionnels à cause de la rotation rapide [117], [17], [15], [19]. Les colonnes dérivent lentement dans le temps sous l'action des parois inclinées, ce qui donne lieu à des vents thermiques, équivalents aux ondes de Rossby. La dynamo cinématique de ces écoulements a d'abord été démontré théoriquement avec des approches de type champ moyen [11], [12]. Avec l'arrivée des fortes capacités de calcul numérique, les premières simulations des dynamos entretenues des écoulements convectifs ont trouvé des champs magnétiques majoritairement dipolaires avec une dynamique complexe et des renversements, tout comme pour le champ magnétique terrestre [40]. Les simulations de la géodynamo demandent des ressources informatiques importantes, et il est nécessaire de remarquer que des simulations avec des jeux de paramètres loin de la réalité arrivent finalement bien à représenter l'essentiel de la géodynamo. Ces premiers travaux ont trouvé un grand nombre de successeurs, et plusieurs articles de revue sont actuellement disponibles [30] [116], [23], [14]. Aujourd'hui on se rapproche de paramètres de plus en plus réalistes. Notons cependant que les toutes dernières simulations à la plus grande résolution [51], semblent indiquer que l'image que nous avons de rouleaux de convection alignés avec l'axe de rotation, fait place à une convection sous forme de panaches. Ces écoulements continuent à être des écoulements très efficaces pour produire des dynamos, et ils ont également été observés dans des expériences hydrodynamiques de convection en présence de rotation rapide [129], [130] loin du seuil de l'instabilité convective. La convection est aujourd'hui le mécanisme dominant pour expliquer l'origine du champ magnétique, mais pas nécessairement pour toutes les planètes. On devine aujourd'hui par exemple que le champ magnétique terrestre existe depuis plus longtemps qu'une convection suffisamment forte pour entretenir la dynamo [125]. Il est donc intéressant et nécessaire de poursuivre l'étude vers d'autres mécanismes d'excitation.

### La précession

La précession désigne le mouvement où l'axe de rotation principale,  $\Omega$  tourne lui-même autour d'un autre axe  $\omega$  ( Fig. 1.1-(c)). Elle trouve son origine dans l'aplatissement de la Terre due à la rotation rapide de celle-ci et dans le couple créé par l'interaction gravitationnelle avec le soleil. Pour la Terre, l'angle entre les deux axes de rotation est grand,  $\theta = 23.5^\circ$ , et la précession est très lente : une période de révolution vaut environ 25800 années. Ce temps est cependant assez court du point de vue géophysique. L'intérêt pour ce phénomène remonte à la fin du dix-neuvième siècle [109], [123], [110]. Dans les années 1950, la précession a été proposée pour la première fois comme source de mouvement pour la géodynamo [10]. Un bon nombre d'études hydrodynamiques ont alors suivies, [6], [127], [117], [16], ainsi qu'une première étude expérimentale [88]. Un écoulement fortement turbulent a été observé dans ces expériences. Si la même chose se passait dans la Terre, une dynamo de précession serait alors probable. Ces idées ont été mises en doute par des estimations sur les bilans énergétiques pour un écoulement laminaire [119], [85]. Plus tard des nouvelles séries d'expériences ont été réalisées [135], [137], [136], [105], [91], [92]. Une borne supérieure a alors été dérivée sur la dissipation turbulente pour un écoulement de précession [60] et prouve qu'il n'y a pas d'arguments pour exclure cette instabilité comme source d'énergie dans le noyau interne. Il n'est seulement pas encore expliqué comment ces écoulements variants sur des échelles de temps aussi rapides (le jour), seraient capables d'exciter une dynamo qui croît et varie sur des temps diffusifs (million d'années). Une étude expérimentale d'un écoulement de Sodium liquide en précession a été faite en 1971 [36]. Une dynamo n'a pas été observée, mais le champ imposé a été amplifié d'un facteur 3, signe d'un mécanisme constructif précurseur à la création d'un effet dynamo. Une expérience de plus grande taille pourrait probablement créer une dynamo. Plus récemment, la question de la dynamo de précession a trouvée une série de réponses. Il a en effet été démontré numériquement que la précession peut exciter des dynamos [132], [133] [141]. Il faut cependant noter que le même problème que pour les écoulements de convection subsiste : il est difficile d'explorer les domaines des paramètres proches de ceux de la Terre.

### La libration

La libration est un mouvement oscillatoire de petite amplitude autour d'une position d'équilibre du corps céleste en rotation. Le fluide subit des petites accélérations / décélérations, et des ondes inertielles peuvent être excitées de la même manière que dans les expériences [2]. La libration a été très peu étudiée dans le contexte géophysique en tout cas comme source possible de dynamo, même si ce mouvement est clairement observé sur Mercure par exemple, [44], [106].

### Les marées - Instabilité elliptique

Sous les effets combinés de la rotation et de la force attractive d'un corps céleste avoisinant, un corps subit des petites déformations elliptiques. En surface, cette déformation mène aux marées océaniques, qui suivent un cycle à la fréquence proche de deux fois par jour. Sur la surface de la Terre on passe deux fois par jour par le bourrelet induit par les marées, et on comprend donc qu'il s'agit globalement d'une déformation elliptique de la Terre. De la même manière que ces marées agissent sur les enveloppes de surface, la déformation peut se faire sentir à l'intérieur de la planète dans des zones de métaux liquides. L'instabilité elliptique offre alors un moyen aux ondes inertielles (ou aux ondes MAC) de croître par un phénomène de résonance paramétrique. Le mécanisme de l'instabilité elliptique est un peu plus complexe que celui d'une résonance classique et comme ce mécanisme est au centre de cette thèse, nous introduisons ce phénomène en détail dans la section suivante.

## 1.3 Instabilité elliptique

L'instabilité elliptique trouve son origine loin des applications géophysiques. La revue de Kerswell (2002) donne une bonne vue d'ensemble, mais les thèses de Waleffe (1989), de Eloy (2000) et de Lacaze (2004) ont décrit et développé le sujet. Commençons par expliquer l'origine du terme "instabilité elliptique". De façon très générale ceci indique qu'un écoulement bi-dimensionnel aux lignes de courant de forme elliptique est instable. Le cas le plus simple d'un tel écoulement est décrit par la fonction de courant

$$\Psi_e = \frac{\Omega}{2} \left( \frac{x^2}{1+\epsilon} + \frac{y^2}{1-\epsilon} \right) \quad (1.3)$$

Les lignes du plan où  $\Psi_e$  est constante sont effectivement des ellipses, possédant une excentricité  $\epsilon$ .  $\Omega$  est ici une vitesse de rotation. L'écoulement elliptique qui dérive de cette fonction de courant s'écrit

$$\mathbf{U}_e = \frac{\Omega}{1-\epsilon^2} \left[ (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}}) - \epsilon (y \hat{\mathbf{x}} + x \hat{\mathbf{y}}) \right] \quad (1.4)$$

A faible excentricité  $\epsilon$ , l'écoulement elliptique est une superposition d'une rotation dominante et d'un étirement transverse. Cet écoulement simple est instable vis à vis de perturbations tridimensionnelles. Comme la rotation est dominante, les ondes inertielles jouent un rôle important. L'instabilité elliptique a suscité beaucoup d'intérêts pour la simple raison que les ingrédients : rotation plus étirement transverse, sont fréquemment rencontrés dans les écoulements. Elle offre un mécanisme d'instabilité permettant pour un bon nombre d'écoulements (couches de mélange, sillages, jets, ...) de transiter vers la turbulence.

Les premiers travaux sur des tourbillons isolés dans un champ d'étirement datent des années 70. L'instabilité de Crow, [25], pour des paires de tourbillons contra-rotatifs, comme en apparaissent dans les sillages des avions, est une première manifestation de l'effet d'un étirement sur ce système de tourbillons en interaction. Une déformation de grande longueur d'onde (grande devant la taille du coeur du vortex) est observée dans ces tourbillons de sillages. L'instabilité de Crow n'est cependant pas universelle : le même phénomène n'apparaît pas pour des paires de tourbillons co-rotatifs [49]. Plus universelle, est l'instabilité due à l'étirement, de courte longueur d'onde, où courte signifie de même échelle que le coeur du vortex. Cette instabilité est également vue dans des expériences de sillages [81], [99], [120] et offre la possibilité à un écoulement de grande échelle de se déstabiliser en générant tout de suite beaucoup de petites échelles. Les premières études sur des vortex étirés dans un plan transverse sont dues à Moore & Saffman (1975) et Tsai & Widnall (1976) qui démontrent par des approches globales l'existence d'une instabilité de courte longueur d'onde. La même instabilité est redécouverte plus tard par Pierrehumbert (1986), et une théorie non-visqueuse et locale est proposée peu après par Bayly (1986). "Locale" voulait dire ici, que l'on s'intéresse à l'instabilité de petites particules de fluide advectées par l'écoulement. La caractéristique remarquable de cette instabilité non visqueuse est qu'elle mène tout de suite à des écoulements de petite échelle, permettant donc un transfert direct de l'énergie entre l'écoulement à grande

échelle et les fines échelles tridimensionnelles. Utilisant les solutions de Craik & Criminale (1968), Landman & Saffman (1987) ont ensuite introduit les effets visqueux, ce qui introduit une coupure dans les petites échelles mais conserve l'idée essentielle du mécanisme non-visqueux. Le lien entre les théories locales et globales, et une interprétation physique simple du mécanisme de l'instabilité est dû à Waleffe (1989,1990), qui s'aperçoit que la perturbation instable de courte longueur d'onde possède une vorticit e pr ef erentiellement align ee avec les directions de l' etirement. Le m echanisme d'amplification de l'instabilit e elliptique est l' etirement de la vorticit e. Si on revient aux effets de la rotation rapide sur les  coulements, discut es plus haut, on comprend que seules les ondes inertielles peuvent b en eficier de cette amplification. Si l'analyse est pouss ee un peu plus loin, on voit que le champ d' etirement introduit un couplage qui permet   des paires d'ondes inertielles r esonantes de se coupler de fa on constructive. L'instabilit e elliptique peut donc formellement se mod eliser comme une r esonance param etrique d'ondes inertielles.

Les exp eriences sur le sujet de l'instabilit e elliptique ont d ebut e en Russie. Gledzer montre que dans un sph ero ide (1974), puis dans un cylindre elliptique (1975), mis en rotation solide et brusquement arr et e, un  coulement elliptique se d eveloppe et se d estabilise. D'autres exp eriences plus syst ematiques ont suivies [22], produisant des r esultats en accord avec la th eorie [43]. Dans ces exp eriences, l'instabilit e elliptique se d eveloppe en m eme temps que l'instabilit e centrifuge dans la couche limite, ce qui rend les visualisations difficiles et l' etude de l' volution non-lin eaire difficile. Malkus (1989) a alors imagin e un montage exp erimental qui permet de mieux isoler l'instabilit e elliptique. Le principe de ce dispositif est le suivant. Un cylindre issu d'un mat eriel d eformable est mis en rotation solide. Deux rouleaux oppos es viennent comprimer le cylindre rendant sa section transverse elliptique. L'excentricit e est transmise sur les lignes de courant et un  coulement d'ellipticit e homog ene est obtenu. Cette exp erience a mis en  vidence la complexit e de l' volution non lin eaire de l'instabilit e elliptique et la transition brutale vers la turbulence ("inertial wave collapse") -  galement vues dans d'autres exp eriences de syst emes d'ondes inertielles forc es, [95] et dans les  coulements de pr ecession [88], [91], [92]. L' coulement bifurqu e ne semble pas saturer et de grandes amplitudes peuvent  tre atteintes. Plus tard, le dispositif de Malkus a  t e reconstruit sous une forme plus petite, mieux adapt ee pour  tudier l'instabilit e elliptique proche du seuil de l'instabilit e. Ceci a permis   Eloy et al. (2001 - 2003) d'identifier des  tats satur es de l'instabilit e elliptique, et de comparer avec succ es les th eories lin eaires et faiblement non-lin eaires aux exp eriences proche du seuil. De plus, ces exp eriences ont mis en  vidence l'apparition d'instabilit es secondaires, qui trouvent leur origine dans les interactions triadiques entre ondes inertielles [57]. Actuellement il est propos e que ces instabilit es soient responsables de l'explosion abrupte des  coulements d'ondes inertielles et un rapprochement avec les sc enarios de transition vers le chaos de Ruelle & Takens a  t e envisag e [55].

Revenons maintenant au cas g eophysique des corps c elestes. On peut attribuer deux r oles diff erents   l'instabilit e elliptique. Soit l'instabilit e ne va uniquement perturber que l' tat de base. Dans ce cas il est n ecessaire d'inclure les effets du champ magn etique et de la stratification. Soit on lui attribue un r ole plus important, et on dit que l'instabilit e offre une source d' nergie   la dynamo. La premi ere direction a men e   une g en eralisation du m echanisme lin eaire de l'instabilit e pour les ondes MAC. Les effets suppl ementaires de la stratification de densit e [100], [54], du champ magn etique dans la limite de la MHD id eale [54], [77], des effets thermiques [75] et d'une force de Coriolis suppl ementaire [100], [75], changent   chaque fois les ondes et permettent plus de couplages. Le prix   payer est une complexit e de plus en plus importante, pour souvent un domaine d'application de plus en plus restreint. L'instabilit e elliptique se montre particuli erement g en erique pour que celle-ci continue d'appara tre dans ces diff erentes situations.

Nous trouvons deux  tudes sp ecifiquement destin ees   la compr ehension du r ole de l'instabilit e elliptique sur Terre et sur Io, lune de Jupiter. L'excentricit e de la Terre au niveau du CMB (core-mantle-boundary) est estim ee de l'ordre  $O(10^{-8})$ , ce qui est tr es petit. En m eme temps, la Terre est assez grande pour que la diffusion magn etique et la diffusion visqueuse soient moins importantes. En raisonnant sur les ondes Magn eto-Coriolis (MC) en g eom etrie sph erique, Kerswell d emontre que l'instabilit e elliptique n'est pas exclue dans la Terre [58]. La d eformation de Io est bien plus forte   cause de la proximit e de Jupiter. On estime son excentricit e de l'ordre  $O(10^{-3})$ . Les effets des mar ees sont alors tr es importants, comme en t emoigne le volcanisme intense   sa surface, [96]. L'instabilit e elliptique dans le noyau liquide de Io n'est donc pas exclue [57], et peut effectivement procurer une source importante de mouvements. Les r esultats plut ot positifs de ces th eories lin eaires nous permettent alors d'aller plus loin et de suivre la deuxi eme direction : est-ce que l'instabilit e elliptique peut mener   une dynamo. Mais dans ce cas, un probl eme majeur appar ait : on ne connait que tr es peu l' tat non lin eaire qui suit l'instabilit e initiale. Aucune simulation ou exp erience d ecrivent ce r egime quantitativement. Seule une  tude est r epertori ee qui s'int eresse   l' volution non-lin eaire de l'instabilit e elliptique [94]. En cons equence, on ne sait pas quel  coulement est excit e loin du seuil, et il est donc difficile de mesurer l'impact de l'instabilit e elliptique par rapport aux autres m echanismes d'excitation tels que la convection ou la pr ecession.

## 1.4 Plan de la thèse

Il est clair que par rapport à la convection, les mécanismes des instabilités inertielles sont moins bien compris. Même à un niveau purement hydrodynamique, la caractérisation des écoulements de précession ou de ceux provoqués par l'instabilité elliptique sont peu connus. Ceci constitue une des motivations de cette thèse.

L'idée d'étudier le régime non-linéaire de l'instabilité elliptique découle littéralement de la discussion précédente. Un de nos buts était de comprendre comment la dynamique non-linéaire change si on augmente progressivement la distance par rapport au seuil de l'instabilité. Est-ce qu'on peut suivre de cette manière le rôle des instabilités secondaires, et proposer des explications au phénomène de l'effondrement des écoulements d'ondes inertielles ("inertial wave collapse"), toujours peu compris ?

Pour répondre à cette question, nous restons dans le cadre fondamental du dispositif de Malkus en géométrie cylindrique. En tout début de thèse, nous avons remarqué qu'il était possible de trouver des extensions des ondes de Kelvin (1880) et des écoulements géostrophiques, solutions du problème linéaire et visqueux pour les perturbations. Nous avons identifié une relation d'orthogonalité qui permet d'utiliser ces modes comme base dans des problèmes plus complexes, par exemple non-linéaires. Ces résultats sont présentés dans le premier chapitre.

L'idée suivante a été d'utiliser ces solutions dans le problème de l'évolution non-linéaire de l'instabilité elliptique : nous projetons l'équation de Navier-Stokes sur un ensemble réduit de modes visqueux suivant la procédure classique de Galerkin. Cette procédure nous conduit à un système d'équations d'amplitude que l'on étudiera pour quelques cas concrets. Ces études permettent de mettre en avant quelques idées sur les scénarios de transition possibles et leur lien avec l'effondrement des ondes inertielles ("inertial-wave-collapse"). Cependant, nous montrerons également à cause d'une spécificité des termes de pression dans l'analyse, l'approche de Galerkin utilisée n'est valide qu'avec l'ajout d'une force supplémentaire. Nous proposons une méthode pour éviter l'ajout d'une telle force mais la découverte tardive de cette difficulté ne nous a pas permis de présenter des résultats avec le nouveau système d'équations.

Pour étudier l'évolution non-linéaire dans les expériences réalisées sur le dispositif de Malkus, différentes techniques expérimentales ont été testées préalablement. La visualisation par particules réfléchissantes anisotropes permet d'obtenir une bonne compréhension qualitative, mais devient difficile à interpréter lorsque l'écoulement est plus complexe. La Vélocimétrie par Imagerie de Particules (PIV) [26], est alors une très bonne alternative, mais la mise en place du système sur notre expérience de petite taille n'est pas simple. Les mesures par Vélocimétrie Laser Doppler (en particulier à travers d'une paroi qui se déforme) souffrent d'un niveau de bruit élevé [68]. Nous décidons alors de mettre en place une nouvelle technique de mesure : la détection magnétique des mouvements de fluide. Le principe de l'expérience est le suivant : un champ magnétique homogène est imposé le long de l'axe de rotation d'un cylindre rempli d'un métal liquide. Si le champ imposé est faible, on ne perturbe la dynamique du fluide que très faiblement. La croissance des ondes inertielles est par contre accompagnée d'un champ induit que l'on mesure à l'extérieur. Ceci nous donne une information quantitative sur l'écoulement. Des mesures récentes utilisant le même principe ont permis de mettre en évidence les ondes inertielles dans un écoulement de Couette sphérique [52]. Cette technique expérimentale est présentée dans le troisième chapitre, ainsi que l'analyse nécessaire pour le calcul du champ induit par les ondes inertielles. On caractérise également ce qui se passe lorsque le champ est plus fort : comment agit la force de Lorentz sur les ondes inertielles et comment prendre en compte la correction magnétique dans la théorie linéaire de l'instabilité elliptique sous champ magnétique. Nous présentons quelques résultats expérimentaux pour le cylindre, et démontrons que la détection magnétique est un bon outil pour suivre l'évolution non-linéaire dans un cylindre.

La même méthode expérimentale a été appliquée dans un premier temps en géométrie sphérique. Dans le quatrième et le cinquième chapitre nous reprenons deux articles concernant deux études finalisées. La première étude est réalisée sous champ faible. Nous discutons l'instabilité elliptique dans son contexte géophysique, et calculons l'expression du champ induit. Nous montrons qu'il est possible de suivre l'instabilité à travers le champ induit dans les expériences. Le deuxième article sur le sujet, discute comment un champ fort arrive à influencer l'instabilité elliptique. Nous comparons les résultats expérimentaux à un modèle non-linéaire de basse dimension.

Nombreux sont les articles sur l'instabilité elliptique ou la précession qui prônent la possibilité d'une dynamo forcée par ces instabilités. Mais le sujet en n'a réellement commencé à se développer avec les résultats récents des simulations numériques de Tilgner (2005-2007) et de Wu & Roberts (2008). Comme les ondes inertielles y jouent un rôle important, il nous est apparu utile d'étudier la possibilité de génération de dynamo par les ondes inertielles. Ce sujet est également relativement absent dans la littérature, car seul Moffatt (1970-1972), et Soward (1975) lui ont accordé une attention particulière en géométrie infinie. Ce manque d'étude est probablement reliée à la complexité de ces écoulements qui sont fortement tri-dimensionnelles et rapidement variables dans le temps. Dans le cinquième chapitre, nous nous intéressons donc à la dynamo cinématique des ondes inertielles confinées dans un cylindre.

En annexe, nous présentons quelques généralités sur les composantes  $\pm$ -cylindriques que nous avons beaucoup utilisé durant cette thèse. La deuxième annexe résume une étude théorique et expérimentale sur l'instabilité elliptique dans une coquille cylindrique. Le dernier annexe discute l'excitation magnétique des ondes d'Alfvén en géométrie cylindrique, une étude motivée par une expérience actuellement en cours à LGIT/ICM de Grenoble.

## Chapitre 2

# Perturbations visqueuses d'une colonne de fluide en rotation

Dans ce chapitre nous discutons les perturbations visqueuses d'un écoulement en rotation solide confiné dans un cylindre. Après avoir rappelé les solutions non-visqueuses et les estimations asymptotiques pour leur taux de dissipation visqueuse, nous présentons une nouvelle procédure pour calculer la correction visqueuse de l'écoulement suite au pompage des couches limites visqueuses. Ensuite nous présentons la détermination analytique des modes visqueux de façon analytique. Ces nouvelles solutions correspondent aux extensions visqueuses des ondes inertielles et de l'écoulement géostrophique, et prennent en compte la couche limite et le pompage en même temps. Nous trouvons également une relation d'orthogonalité.

### 2.1 Introduction

In this chapter we analyse the linear perturbations, which can develop on a base state of solid body rotation in cylindrical shaped containers. We find viscous equivalents of Kelvin waves and geostrophic flow, and compare their structure, damping and frequency to known asymptotic formula, obtained by boundary layer theory. An orthogonality relation is constructed, which enables us to use these viscous solutions as expansion functions in the context of nonlinear models, discussed in the following chapter.

#### Problem definition

A fluid with density  $\rho$ , kinematic viscosity  $\nu$ , is rotating at an angular velocity  $\Omega$  in a cylindrical container with radius  $R$  and top and bottom plates at  $z = 0, H$ . This fluid domain will be noted  $V$ . We are interested in small perturbations which can grow on the base state of rapid rotation. The proper scales for time, velocity and pressure fields to non-dimensionalise the problem are

$$[t] = \Omega^{-1} \quad , \quad [\mathbf{u}] = \Omega R \quad , \quad [p] = (\Omega R)^2 \quad (2.1)$$

In the rotating frame of reference, the linear non-dimensionalised perturbation equations write

$$\partial_t \mathbf{u} + 2 \hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + E \Delta \mathbf{u} \quad , \quad \nabla \cdot \mathbf{u} = 0 \quad (2.2)$$

where  $\mathbf{u}$  and  $p$  are the velocity and pressure fields. We recognize the Ekman number,  $E = \nu/\Omega R^2$  in front of the diffusive term, which is assumed small in the following. The aspect ratio  $h = H/R$  of the cylinder is the second non-dimensional parameter in this problem. We note this linear viscous problem with the velocity four-vector  $\mathbf{U} = [\mathbf{u}, p]^T$ , as  $\mathcal{L} \mathbf{U} = 0$ . The velocity field needs to respect impermeability or no-slip boundary conditions on the boundary with external normal  $\mathbf{n}$ .

$$\text{Impermeable : } \mathbf{u} \cdot \mathbf{n} = 0|_{\delta V} \quad , \quad \text{No-slip : } \mathbf{u} = 0|_{\delta V} \quad (2.3)$$

This problem admits wave-like solutions, the so-called inertial waves and 2D transverse solutions corresponding to the geostrophic flow. We search for global eigenmode solutions as

$$\begin{bmatrix} \mathbf{u}' \\ p' \end{bmatrix} = \begin{bmatrix} \mathbf{u}(\mathbf{r}) \\ p(\mathbf{r}) \end{bmatrix} e^{i\lambda t} \quad (2.4)$$



where  $\lambda$  is the complex frequency. The purpose here is to determine the spatial structure of perturbations of this type in the presence of viscosity. We start by recalling the non-viscous limit, for which an important literature exists, [53], [45]. In absence of external forcing or viscosity, all the perturbations are neutral,  $Im(\lambda) = 0$ . Viscosity adds a damping, both in the volume or through thin boundary layers on the surface. We discuss the lowest order viscous corrections which have been found in perturbative approaches. The viscous problem provides equivalents of the inertial waves and geostrophic flow which satisfy no-slip boundary conditions on the cylindrical side-wall and impermeability conditions on top and bottom plates. Frequency and spatial structure of the viscous solutions are compared to those given by the asymptotic formula. The analysis benefits from the use of cylindrical  $\pm$ -components of the velocity field. Defined as

$$u_{\pm} = \frac{1}{\sqrt{2}} (u_r \pm iu_{\phi}) \quad (2.5)$$

they diagonalise both Coriolis-force and the diffusive term in cylindrical coordinates. General properties of  $\pm$ -components and useful vector analysis formula are discussed in Appendix A.1.

## 2.2 Non-viscous limit

We solve the non-viscous problem,  $E \rightarrow 0$ , in a cylinder with impermeable sidewalls. In  $\pm$ -components the perturbation equations write explicitly as

$$\begin{aligned} i(\lambda \pm 2) u_{\pm} &= -\frac{1}{\sqrt{2}} (\partial_r \pm \frac{i}{r} \partial_{\phi}) p \\ i \lambda u_z &= -\partial_z p \end{aligned} \quad (2.6)$$

$$\frac{1}{\sqrt{2}} (\partial_r - \frac{i}{r} \partial_{\phi} + \frac{1}{r}) u_+ + \frac{1}{\sqrt{2}} (\partial_r + \frac{i}{r} \partial_{\phi} + \frac{1}{r}) u_- + \partial_z u_z = 0$$

Note that  $u_+$  and  $u_-$  are no longer coupled by the Coriolis-force. Substituting the velocity field components determined by the first three equations in the incompressibility constraint, we find the decoupled problem for the pressure field :

$$(-\lambda^2 \Delta + 4 \partial_{zz}^2) p = 0 \quad (2.7)$$

This is the Poincaré equation. For non-zero frequency values in the range  $\lambda \in [-2, 2]$ , this equation is hyperbolic and possesses wave-like solutions : the **inertial waves**. The inertial term is needed to balance Coriolis-force effects and pressure gradients, and waves vary on the typical time-scale of the rapid rotation. The smaller  $\lambda$ , the larger the vertical structure of the flow must be to balance the terms in the Poincaré equation. In the limit of vanishing  $\lambda$ , the solutions do not have any vertical structure, and correspond to geostrophic flows. This is in agreement with the Taylor-Proudman theorem, which states that stationary perturbations have no axial dependence. In the geostrophic balance, the Coriolis force exactly equilibrates pressure gradients. A third type of solutions, which are **singular** may also exist due to the hyperbolic nature of the Poincaré-equation. In the cylinder, such singular modes can be generated by the singularities at the corners. They are localized on characteristic surfaces which start from these regions. In a cylinder with fixed aspect ratio  $h$ , a geometrical relation defines the eigenfrequencies for which characteristics are closed.

### Inertial waves

In cylindrical geometry, equation (2.7) is separable and admits an infinite number of solutions. The problem in cylindrical geometry was first solved by Lord Kelvin in 1887, [53]. The solutions have a trigonometric structure with respect to vertical and azimuthal coordinates, and the radial structure of the pressure field can be expressed as a Bessel-function :

$$p = A J_m(kr) e^{im\phi} \cos(lz) \quad (2.8)$$

The integer  $m$  is the azimuthal wavenumber, the real numbers  $l$  and  $k$  respectively are axial and radial wavenumbers. The Bessel function of the second type is not present since the axis  $r = 0$  is included in the fluid domain. Sinusoidal vertical structure is possible, but will not be compatible with the boundary conditions that will be imposed below. This pressure field solves the Poincaré equation provided that the wavenumbers  $(k, m, l)$ , are linked to the frequency  $\lambda$  by

$$k^2 = \frac{4 - \lambda^2}{\lambda^2} l^2 \quad (2.9)$$

With the first three equations of (2.6), one subsequently finds expressions for the velocity field components, which all behave like  $\exp(im\phi)$ . It is interesting to notice that in  $\pm$ - components the differential operators organise into particular operators

$$\left(\partial_r \pm \frac{i}{r}\partial_\phi\right) \rightarrow \left(\partial_r \mp \frac{m}{r}\right) \quad (2.10)$$

which will act on Bessel-functions in a simple way :

$$\left(\partial_r \pm \frac{m}{r}\right) J_m(kr) = \pm k J_{m\mp 1}(kr) \quad (2.11)$$

Applying these operators on the pressure function (2.8), we obtain that the radial structure of  $u_\pm$  behaves like  $J_{m\pm 1}(kr)$ , which gives some sense to the  $\pm$  notation. As a result, the independant velocity field components are easily recovered and can be written as

$$\begin{aligned} u_\pm &= C \frac{1}{\sqrt{2}} (2 \mp \lambda) J_{m\pm 1}(kr) \cos(lz) e^{im\phi} \\ u_z &= C k\lambda/l J_m(kr) \sin(lz) e^{im\phi} \\ p &= i C k\lambda^2/l^2 J_m(kr) \cos(lz) e^{im\phi} \end{aligned} \quad (2.12)$$

The arbitrary amplitude  $A = i C k\lambda^2/l^2$ , was redefined to simplify the notation. As a last step in the analysis, we need to impose the boundary conditions. Impermeability on the bottom plate is automatically satisfied since  $u_z \sim \sin(lz)$ , which explains the choice  $p \sim \cos(lz)$  made above. The same condition on the top plate quantifies the vertical wavenumber :  $l = n\pi/h, n \in \mathbb{N}$ . Imposing impermeability at the cylindrical sidewall,  $u_r = 0|_{r=1}$  gives

$$(2 + \lambda)J_{m-1}(k) + (2 - \lambda)J_{m+1}(k) = 0 \quad (2.13)$$

With  $\lambda$  defined by (2.9) as a function of  $k$  and  $n$ , this relation quantifies the possible radial wavenumbers  $k = k_x, x \in \mathbb{N}_0$ , and thereby determines the frequency or dispersion relation of the Kelvin waves.

The implicit dispersion relation is solved numerically, and it appears that radial wavenumbers  $k_x$  are different for modes with different  $m$  or  $l$ . At fixed  $m$  and  $l$ ,  $k_1$  is the smallest possible radial wavenumber corresponding to the mode with the largest radial structure. In the following, we will often label the quantified inertial waves using the three integers  $[x, m, n]^\pm$  and a suffix which corresponds to the sign of the frequency  $\lambda$ .  $x$  is the radial label,  $m$  is the azimuthal wavenumber and  $n$  the number of half vertical wavelengths in the spatial structure.  $n = 0$ , for geostrophic modes without vertical structure. In Fig. 2.1, we have plotted the frequency as a function of the vertical wavenumber  $l$  for modes with  $m = 1$  and the four smallest radial wavenumbers  $k_x, x = 1, \dots, 4$ . The vertical dashed lines show the result of vertical quantification  $n = 1, 2, \dots$ , once the aspect ratio  $h$  of the cylinder is fixed. Only the modes intersecting the vertical dashed lines, are admitted in cylinders with this aspect ratio. For each set  $[x, m, n]$  there is a mode with positive frequency (+) and a mode with negative frequency (−) traveling in and against the rotation sense of the cylinder, at different speed if  $m \neq 0$ . In the large azimuthal wavenumber limit, this difference disappears, and for high  $l$  all frequencies accumulate at  $\lambda_\infty = \pm 2$ . We will note  $\omega = \lambda - m$ , the fixed frame frequency, and since  $\lambda \in [-2, 2]$  it is clear that only the helical modes,  $m = 1$ , can be stationary in the fixed frame. These particular modes with  $\omega = 0, \lambda = 1$  are the so-called spinover modes which play an important role in the elliptical instability. They exist as (unforced) modes only for particular cylinder aspect ratios. We will display the spatial structure of some inertial waves further.

In Fig. 2.1, it is observed that all branches converge to  $\lambda = 0$ , for  $l \rightarrow 0$ . This is associated with the degeneracy of  $z$ -independant solutions in the cylinder. The solutions split up in two distinct classes. Two-dimensional transverse flows, with  $u_z = 0$  are the so-called geostrophic flows. Substituting the  $u_\pm$ -expression in the incompressibility constraint, we find any pressure function is a solution, which explains the degenerate nature of geostrophic flow. A priori, we can expand the flow on any set of orthogonal base functions. However, it is convenient to choose geostrophic modes which satisfy no-slip conditions on the cylindrical side-wall.

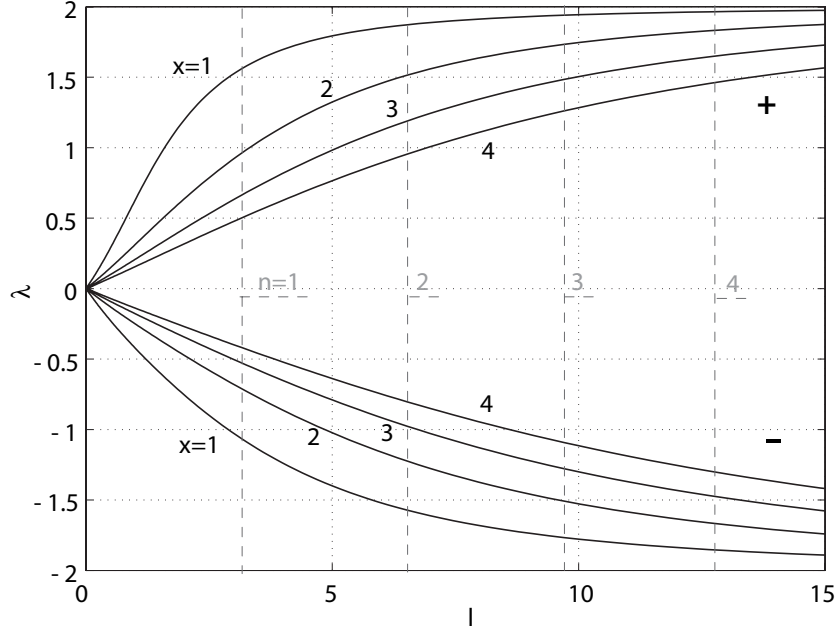


FIG. 2.1 – Frequency  $\lambda$  as a function of the vertical wavenumber  $l$  for Kelvin-waves  $[x, m, n]^\pm$ , with azimuthal wavenumber  $m = 1, \dots, 4$ . For each positive frequency branch (+), there is negative branch (-). The vertical dashed lines indicate how waves are vertically quantified, when the aspect ratio  $h$  is fixed.

For  $m \geq 0$ , the hydrodynamic fields of geostrophic modes can be written as

$$\begin{aligned}
 u_+ &= i\sqrt{2} J_{m+1}(k_x r) e^{im\phi}, \\
 u_- &= i\sqrt{2} (J_{m-1}(k_x r) - J_{m-1}(k_x) r^{m-1}) e^{im\phi}, \\
 p &= -\left(\frac{4}{k_x} J_m(k_x r) - \frac{2}{m} J_{m-1}(k_x) r^m\right) e^{im\phi}.
 \end{aligned} \tag{2.14}$$

These expressions satisfy no-slip boundary conditions on the cylindrical surface if  $J_{m+1}(k_x) = 0$ . These flows do not have boundary layers at the lateral surface. For  $m < 0$ , the mode profiles are linked to those for positive  $m$  as it will be seen below. The geostrophic modes are noted  $[x, m, 0]$ . The second class z-independant solutions are pure axial flows. These solutions satisfy the equations whatever the axial flow, but they are not allowed due to the presence of horizontal top and bottom plates.

### Real flow

The perturbation problem (2.2) is real, so that each complex solution  $[x, m, n]^\pm$  with frequency  $\lambda$  and amplitude  $A$ , is accompanied by a complex conjugate solution whose labels are  $[x, -m, n]^\mp$ , with frequency  $-\lambda$  and amplitude  $A^*$ . For the spatial structure of the fields this means

$$u_j^{(m)} = u_j^{(-m)*}, \quad j = r, \phi, z, \quad p^{(m)} = p^{(-m)*}, \quad u_\pm^{(m)} = u_\mp^{(-m)*}. \tag{2.15}$$

The complex nature of the  $\pm$ -components, makes  $\pm$ -components are interchanged and complex conjugates for positive and negative  $m$ -modes. Axisymmetric geostrophic flows ( $\lambda = m = 0$ ) are not accompanied by a complex conjugate solution. For these modes, the flow is purely azimuthal,  $u_+ = -u_-$ , and the complex factor (i) in (2.14) assures their reality.

### Singular modes

The hyperbolicity of the Poincaré equation, allows singularities to propagate along the characteristic surfaces without being disturbed. Oscillating perturbations with frequency  $\lambda$ , propagate on conical surfaces with colatitude  $\theta = \arctan r/z$  determined by

$$\cos \theta = \frac{\lambda}{2} \quad (2.16)$$

whatever the azimuthal structure of the perturbation. In a cylindrical container, the only places where some discontinuities may be generated are the corner regions. Fixing the aspect ratio  $h$  of the cylinder, a geometrical relation determines the eigenfrequency values for which characteristics close upon themselves. The simplest singular mode, spreads out of the corner region on the top plate, and reaches the corner region of the bottom plate without reflecting on one of the walls. This happens in a cylinder with aspect ratio  $h$  such that  $\tan \theta = 2/h$ . The same mode exists in any cylinder with aspect ratio,  $h_* = qh$ , with  $q \in \mathbb{Q}_0^+$  a positive rational number. Inverting these relations, we find the singular mode frequencies in a cylinder with fixed aspect ratio  $h$  as

$$\frac{\lambda}{2} = \left(1 + \frac{q^2}{h^2}\right)^{-1/2} \quad (2.17)$$

In real viscous fluids, the singular modes exist as shear layers. They have been observed in the experiments in cylindrical geometry of Mc Ewan, [95], in which inertial waves and shear layered modes were forced through imposed oscillations of the top-plate. It appears that the singular modes, can also be interpreted from linear theory of as infinite sums of regular inertial waves. Kerswell, [59] also studied how shear layers can be generated by boundary layer discontinuities. The question whether singular modes exist as free waves in cylinders with rigid boundaries is still an open issue. Singular modes are very often observed in containers other than spheroids such as spherical shells, [115], [114]. In these containers the non-viscous problem is mathematically ill-posed : few are no regular modes exist. Note that these particular modes appear also in stratified fluids, [87]. In the context of elliptical instability in cylinders, the singular modes have never been observed.

### Adjoint modes and orthogonality

Inertial waves in cylinders can be excited by resonant forcing, through the boundaries as in the experiment of Mc Ewan, [95], through volume forcing as it is the case for precession, [91], [92], [64], [65], [35], [98] or by elliptical instability [139], [27] or other related parametric instabilities, [112], [113]. Perturbation theories describing these phenomena have been developed to describe linear and weakly nonlinear stages of the dynamics. At the different orders of these perturbation problems, solvability conditions are usually obtained which requires the definition of a scalar product together with an adjoint problem. With  $\mathbf{U} = [\mathbf{u}, p]^T$  the velocity pressure quadrivector containing the spatial structure of a mode with frequency  $\lambda$ , the standard scalar product

$$\langle \mathbf{U}_1 | \mathbf{U}_2 \rangle = \int_{V_i} \left( \mathbf{u}_1^* \cdot \mathbf{u}_2 + p_1^* p_2 \right) dV \quad (2.18)$$

is well adapted to the inertial wave problem. Shortnoting the linear system of equations (2.6), as  $\mathcal{L} \mathbf{U} = 0$ , we define the adjoint problem as

$$\langle \mathbf{U}^A | \mathcal{L} \mathbf{U} \rangle = \langle \mathcal{L}^A \mathbf{U}^A | \mathbf{U} \rangle + BT \quad (2.19)$$

With the scalar product (2.18), the adjoint operator is to be determined by partial integration. Boundary conditions on the adjoint fields  $\mathbf{U}_A$  are such that the boundary terms  $BT$  are zero whatever the direct mode. By definition they satisfy these boundary conditions and solve the problem  $\mathcal{L}^A \mathbf{U}^A = 0$ , and respect these boundary conditions. This leads to the well-known results that the non-viscous problem is self-adjoint

$$\mathbf{U}^A = \mathbf{U} \quad (2.20)$$

The combination of direct and adjoint equations and boundary conditions, leads to an orthogonality relation which involves direct and adjoint modes. Here the orthogonality relation only operates on the velocity field. One finds

$$(\lambda_1 - \lambda_2) (\mathbf{u}_1 | \mathbf{u}_2) = (\lambda_1 - \lambda_2) \int_{V_i} \mathbf{u}_1^* \cdot \mathbf{u}_2 dV = 0, \quad (2.21)$$

for two solutions 1 and 2 of the non-viscous problem. The frequencies  $\lambda_j$  are always real-valued, and never identical except for the geostrophic modes, for which  $\lambda = 0$ . The orthogonality relation of the non-viscous inertial waves can be used to project the dynamics on a set of inertial waves, and is directly linked to the kinetic energy.

## 2.3 Viscous perturbation problem

Real fluids satisfy no-slip boundary conditions on rigid boundaries which damp the eigenmodes. If  $E$  is small, this condition is only respected through the help of a thin layer near the boundary, and corresponding boundary layer damping and spatial corrections to the flows can be calculated using a perturbative approach. The total field  $\mathbf{U}_t = \mathbf{U} + \bar{\mathbf{U}}$ , with  $\mathbf{U}$  the velocity-pressure four vector, satisfies no-slip conditions at the boundary surface, and is the sum of an interior field  $\mathbf{U}$  and a boundary layer field  $\bar{\mathbf{U}}$  which is non-zero only near the boundary surface. The fields, the frequency and the equations are expanded in powers of  $E^{1/2}$

$$\begin{aligned}\mathbf{U} &= \mathbf{U}^{(0)} + E^{1/2} \mathbf{U}^{(1)} + \dots & \bar{\mathbf{U}} &= \bar{\mathbf{U}}^{(0)} + E^{1/2} \bar{\mathbf{U}}^{(1)} + \dots \\ \lambda &= \lambda^{(0)} + E^{1/2} \lambda^{(1)} + \dots & \mathcal{L} &= \mathcal{L}^{(0)} + E \mathcal{L}^{(2)} + \dots\end{aligned}\tag{2.22}$$

and proposed as solutions for interior and boundary layer equations. Equations in the boundary layers are obtained from  $\mathcal{L}$  by rescaling space and spatial derivatives, according to

$$\begin{aligned}\text{Bottom} & & z &= E^{1/2} \bar{z} & \partial_z &= E^{-1/2} \partial_{\bar{z}} \\ \text{Top} & & h - z &= E^{1/2} \bar{z} & \partial_z &= -E^{-1/2} \partial_{\bar{z}} \\ \text{Cylinder} & & 1 - r &= E^{1/2} \bar{r} & \partial_r &= -E^{-1/2} \partial_{\bar{r}}\end{aligned}\tag{2.23}$$

The general procedure is explained in Greenspan's book, [45] and has a very ordered structure. First the non-viscous problem is solved, which provides the inertial waves and geostrophic flow :  $\mathbf{U}^{(0)}$  and  $\lambda^{(0)}$ . No-slip conditions are not satisfied by  $\mathbf{U}^{(0)}$ . In a boundary layer, one adds the  $O(1)$  boundary correction  $\bar{\mathbf{U}}^{(0)}$  which cancels the total velocity at the boundary surface. This boundary layer correction does not fulfill the impermeability condition at  $O(E^{1/2})$ , so an interior flow  $\mathbf{U}^{(1)}$  is generated to cancel the total flow at this order. In practice we obtain a solvability condition, which defines the lowest order viscous frequency correction  $\lambda^{(1)}$ . Once this value is known, the interior flow correction  $\mathbf{U}^{(1)}$  can be calculated. The analysis can be similarly continued to higher orders.

We go through the different existing estimates of the viscous frequency correction, and calculate the interior flow at  $O(E^{1/2})$  driven by the boundary layer pumping.

### 2.3.1 Viscous frequency corrections

#### Wedemeyer's estimate

In the context of cylinders the boundary layer analysis was first done by Wedemeyer [140], in a procedure, which treats top and bottom layers separately from the cylindrical boundary layer without worrying about the corner regions. The lowest order frequency correction of a non-viscous mode with wavenumbers  $(k, m, l)$  and non-viscous frequency  $\lambda^{(0)}$ , on a cylinder with aspect ratio  $h$  was found as

$$\lambda_W^{(1)} = \frac{i}{4\sqrt{2}} \frac{(4 - \lambda^{(0)^2})(m^2 + l^2)}{(m^2 + l^2 - m\lambda^{(0)}/2)} \left[ \begin{aligned} & (1+i)\sqrt{\lambda^{(0)}} \\ & + \frac{(1+i)}{h} \frac{(2-\lambda^{(0)})}{\sqrt{2+\lambda^{(0)}}} \\ & + \frac{(1-i)}{h} \frac{(2+\lambda^{(0)})}{\sqrt{2-\lambda^{(0)}}} \end{aligned} \right]\tag{2.24}$$

The first term between the parentheses results from the boundary layer on the cylindrical sidewall. The second and third terms with the aspect ratio  $h$  in the denominator are produced by boundary layers on top and bottom plates. For the geostrophic modes,  $l = \lambda_0 = 0$ , the formula reduces to  $\lambda_1^W = 2i/h$ . In the Wedemeyer estimate, the corner region is not properly modelled.

#### Kudlick's estimate

Kudlick, [66], extended the boundary layer theory to containers of arbitrary shape. The radius of curvature of the boundary surface, is supposed to be large compared to  $E^{1/2}$ , the typical size of the boundary layer. Applying

his results to cylindrical fluid domains with sufficiently rounded corner regions, he finds the estimate

$$\lambda_K^{(1)} = \frac{i}{4\sqrt{2}} \frac{4 - \lambda^{(0)2}}{(m^2 + l^2 - m\lambda^{(0)}/2)} \left[ \begin{array}{l} (1+i)\lambda^{1/2}(m^2 + l^2) \\ + \frac{(1+i)}{h} \frac{(2-\lambda^{(0)})}{\sqrt{2+\lambda^{(0)}}} \left(m^2 + l^2 - \frac{2m\lambda^{(0)}}{2-\lambda^{(0)}}\right) \\ + \frac{(1-i)}{h} \frac{(2+\lambda^{(0)})}{\sqrt{2-\lambda^{(0)}}} \left(m^2 + l^2 - \frac{2m\lambda^{(0)}}{2+\lambda^{(0)}}\right) \end{array} \right] \quad (2.25)$$

The terms  $\sim -\frac{2m\lambda^{(0)}}{2\pm\lambda^{(0)}}$  in the top and bottom plate contributions, carry the only differences with respect to the Wedemeyer estimate. This formula is often preferred in the literature.

### Volume damping

The surface damping does not introduce a strong viscous damping for the small wavelengths. This is assured by volume damping,

$$\lambda_V^{(2)} = i(k^2 + l^2) \quad (2.26)$$

Formally of second order in the perturbation series, this term quickly dominates over the surface damping for waves with small spatial structures, when  $k$  or  $l \gg E^{-1/4}$ .

### Combination

Kerswell and Barenghi, [56] studied the damping of inertial waves in cylinders numerically, and found that the estimate of Kudlick systematically performs better than Wedemeyer's estimate. The volume damping is important and quickly dominates over surface damping at moderate  $E$  values, even for the flows with the largest structures. The formula

$$\hat{\lambda} = E^{1/2} \lambda_K^{(1)} + E \lambda_V^{(2)} \quad (2.27)$$

for the total viscous frequency shift provides a good estimate in cylinders and was previously used by Eloy, [26] and Meunier [98], in the modeling of multipolar instability and precessional forcing. In recent work, Zhang [145] finds new estimates for the decay of inertial waves in cylinders, which better compare to numerical results, at the cost of greater complexity. In the present work, we will continue to use (2.27).

### 2.3.2 Boundary layer driven interior flow at $O(E^{1/2})$

Weakly nonlinear theories have been developed for different inertial wave systems in cylindrical geometry. In general these theories are rather complex due to Greenspan's theorem [45], which states that nonlinear interaction of inertial waves, through the advection operator  $\mathbf{u} \cdot \nabla \mathbf{u}$ , does not generate geostrophic flow at first order of nonlinearity. Since this geostrophic flow plays an important role in weakly nonlinear processes, it is necessary to go to higher order, to include viscous effects or other more complicated nonlinear interactions. In particular it is important to calculate the  $O(E^{1/2})$  corrections to the inertial waves, which are generated by the boundary layer.

Here we propose an alternative procedure to compute the viscous corrections in the volume (the so-called Ekman pumping). It is first important to note that the equations in the volume are not modified at  $O(E^{1/2})$ . This means that the solutions in the volume have the same structure as those of the non-viscous problem previously discussed up to  $O(E^{1/2})$ . More precisely the solutions can be searched in the following form

$$\begin{bmatrix} u_{\pm} \\ u_z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(2 \mp \lambda) & J_{m\pm 1}(kr) & (\cos(lz) + E^{1/2} A \sin(lz)) \\ k\lambda/l & J_m(kr) & (\sin(lz) - E^{1/2} A \cos(lz)) \end{bmatrix} e^{im\phi} \quad (2.28)$$

The  $O(E^{1/2})$  correction in (2.28) has the structure of an inertial wave but does not satisfy the impermeability conditions on the top and bottom walls : it contributes to the Ekman pumping. The wavenumbers  $k$ ,  $l$  and the frequency  $\lambda$  are also expanded as

$$\begin{bmatrix} \lambda \\ k \\ l \end{bmatrix} = \begin{bmatrix} \lambda^{(0)} \\ k^{(0)} \\ l^{(0)} \end{bmatrix} + E^{1/2} \begin{bmatrix} \lambda^{(1)} \\ k^{(1)} \\ l^{(1)} \end{bmatrix} + \dots \quad (2.29)$$

If we then expand (2.28) in Taylor-series, we find the inertial waves at  $O(1)$ . At  $O(E^{1/2})$  we obtain corrections of the form

$$\begin{aligned}
 u_{\pm}^{(1)} = & \left[ \begin{array}{ll} \mp \lambda^{(1)} \frac{1}{\sqrt{2}} & J_{m\pm 1}(k^{(0)} r) \cos(l^{(0)} z) \\ +k^{(1)} \frac{1}{\sqrt{2}} (2 \mp \lambda^{(0)}) & r J'_{m\pm 1}(k^{(0)} r) \cos(l^{(0)} z) \\ -l^{(1)} \frac{1}{\sqrt{2}} (2 \mp \lambda^{(0)}) & J_{m\pm 1}(k^{(0)} r) z \sin(l^{(0)} z) \\ +A \frac{1}{\sqrt{2}} (2 \mp \lambda^{(0)}) & J_{m\pm 1}(k^{(0)} r) \sin(l^{(0)} z) \end{array} \right] e^{im\phi} \\
 & \tag{2.30}
 \end{aligned}$$

$$\begin{aligned}
 u_z^{(1)} = & \left[ \begin{array}{ll} \left( \lambda^{(1)} \frac{k^{(0)}}{l^{(0)}} + k^{(1)} \frac{\lambda^{(0)}}{l^{(0)}} - l^{(1)} \frac{k^{(0)} \lambda^{(0)}}{l^{(0)2}} \right) & J_m(k^{(0)} r) \sin(l^{(0)} z) \\ +k^{(1)} \frac{k^{(0)} \lambda^{(0)}}{l^{(0)}} & r J'_m(k^{(0)} r) \sin(l^{(0)} z) \\ +l^{(1)} \frac{k^{(0)} \lambda^{(0)}}{l^{(0)}} & J_m(k^{(0)} r) z \cos(l^{(0)} z) \\ -A \frac{k^{(0)} \lambda^{(0)}}{l^{(0)}} & J_m(k^{(0)} r) \cos(l^{(0)} z) \end{array} \right] e^{im\phi}
 \end{aligned}$$

The  $O(E^{1/2})$ -solution depends on four constants  $k^{(1)}, l^{(1)}, \lambda^{(1)}, A$  to be determined by four linear conditions. Incompressibility of the lowest order boundary layer corrections  $\bar{\mathbf{U}}^{(0)}$ , produce an  $O(E^{1/2})$  boundary pumping normal to the boundary surface. The interior flow must oppose to this boundary layer pumping, which fixes three conditions on the constants. A fourth condition is needed and found in the relation (2.9) between wavenumbers  $k$  and  $l$  and frequency  $\lambda$  of (2.9), which is Taylor expanded with  $k, l$  and  $\lambda$  of the form (2.29). This determines the four constants uniquely to

$$\begin{aligned}
 A &= \frac{l^{(1)} h}{2} = -\frac{l^{(0)}}{2\lambda^{(0)}} \left[ \frac{2 - \lambda^{(0)}}{\Gamma_+} - \frac{2 + \lambda^{(0)}}{\Gamma_-} \right] \\
 \lambda^{(1)} &= \lambda_W^{(1)} \quad , \quad k^{(1)} = \frac{k^{(0)}}{l^{(0)}} l^{(1)} - \frac{k^{(0)2} + l^{(0)2}}{k^{(0)} \lambda^{(0)}} \lambda^{(1)}
 \end{aligned} \tag{2.31}$$

with  $\Gamma_{\pm} = \frac{1 \pm i}{\sqrt{2}} \sqrt{\lambda \pm 2}$ . We find  $\lambda^{(1)}$  as the Wedemeyer-estimate (2.24). The solution is identical to the one obtained with a longer procedure [45], which does not make use of (2.29). In the first two panels of Fig. 2.2, we visualised the velocity field  $\mathbf{u}^{(0)}$  of the spinover mode  $[1, 1, 2]^+$ , superimposed on the solid body rotation base state. The arrows indicate r-z components of the velocity and the color levels show the angular velocity. In a plane  $\phi = 0$ , the angular velocity is zero, so that we only see the solid body rotation base state. The r-z components are maximal. At  $\phi = \pi/2$ , we find the inverse situation and recognise the bended rotation axis of the spinover mode. The corresponding viscous corrections  $\mathbf{u}^{(1)}$  are plotted in the two last panels. At  $\phi = 0$ , the terms  $\sim z \sin(lz)$  produce the cross-like angular velocity field component. For geostrophic flow a slightly different approach is needed. We cannot just take the limit  $l^{(0)}, \lambda^{(0)} \rightarrow 0$  in the equations above. The first step is to extend the 2D-field profiles of (2.14) to modes with non-zero frequency and with an  $O(E^{1/2})$  vertical structure. We find the flow profiles for non-zero  $\lambda$  and  $l$  as

$$\begin{aligned}
 \begin{bmatrix} u_+ \\ u_- \\ u_z \end{bmatrix} = & i \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} (2 - \lambda) & J_{m+1}(kr) & \cos(lz) \\ \frac{1}{\sqrt{2}} (2 + \lambda) & \left( J_{m-1}(kr) - J_{m-1}(k) r^{m-1} \right) & \cos(lz) \\ k \lambda / l & J_m(kr) & \sin(lz) \end{bmatrix} + E^{1/2} A \begin{bmatrix} 0 \\ 0 \\ J_m(kr) \end{bmatrix} \right\} e^{im\phi} \\
 & \tag{2.32}
 \end{aligned}$$

Note that the contribution proportional to  $A$  is a purely vertical 1D flow, which was not allowed in the non-viscous theory due to the top and bottom plates.

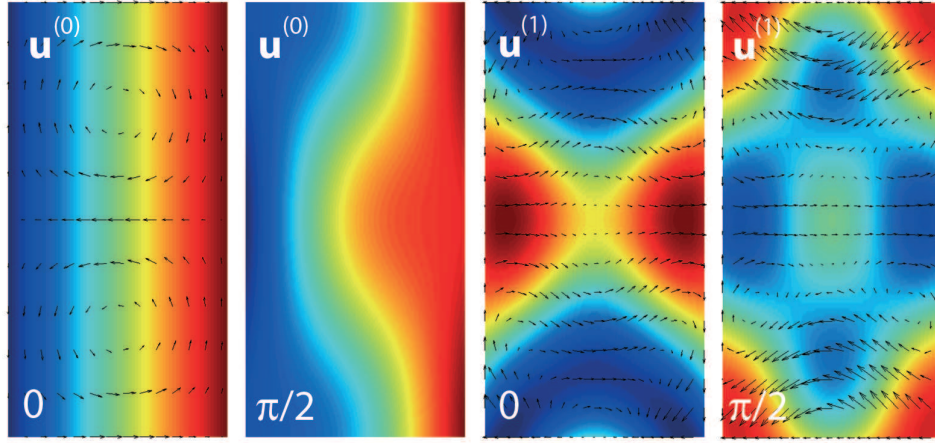


FIG. 2.2 – Visualisation of the velocity field profiles of the spinover mode  $\mathbf{u}^{(0)}$ ,  $(1, 1, 2, +)$  and the lowest order correction  $\mathbf{u}^{(1)}$  in two different vertical planes,  $\phi = 0, \pi/2$ .

The expansion of the wavenumbers and frequency are now

$$\begin{bmatrix} \lambda \\ k \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ k^{(0)} \\ 0 \end{bmatrix} + E^{1/2} \begin{bmatrix} \lambda^{(1)} \\ k^{(1)} \\ l^{(1)} \end{bmatrix} + \dots \quad (2.33)$$

Expanding the spatial profiles in Taylor series in  $E^{1/2}$  we easily obtain the first order geostrophic mode correction

$$\begin{aligned} u_+^{(1)} &= i \left[ \begin{array}{l} -\lambda^{(1)} \quad \frac{1}{\sqrt{2}} \quad J_{m+1}(k^{(0)} r) \\ +k^{(1)} \quad \frac{1}{\sqrt{2}} \quad r J_{m+1}'(k^{(0)} r) \end{array} \right] e^{im\phi} \\ u_-^{(1)} &= i \left[ \begin{array}{l} -\lambda^{(1)} \quad \frac{1}{\sqrt{2}} \quad (J_{m-1}(k^{(0)} r) - J_{m-1}(k^{(0)}) r^{m-1}) \\ +k^{(1)} \quad \frac{1}{\sqrt{2}} \quad (r J_{m+1}'(k^{(0)} r) - J_{m+1}'(k^{(0)}) r^{m-1}) \end{array} \right] e^{im\phi} \\ u_z^{(1)} &= i \left[ \begin{array}{l} \lambda^{(1)} \quad k^{(0)} z \quad J_m(k^{(0)} r) \\ +A \quad \quad \quad J_m(k^{(0)} r) \end{array} \right] e^{im\phi}. \end{aligned} \quad (2.34)$$

The geostrophic eigenmodes have no radial boundary layer, so that there is no radial pumping. The proper boundary condition on the cylindrical surface is  $u_r^{(1)}|_{r=1} = 0$ . On the top and bottom plates, we must compensate the boundary layer pumping. This leads to

$$A = -i k^{(0)} \quad , \quad \lambda^{(1)} = i \frac{2}{h} \quad , \quad k^{(1)} = 0 \quad (2.35)$$

Again, we recover Wedemeyer's estimate for the frequency correction. The vertical wavenumber correction,  $l^{(1)}$  is not determined at this order, but neither does it intervene in equations (2.34) for the  $O(E^{1/2})$  interior flow. The alternative procedure we have used here, provides a shortcut in similar perturbation problems, where the equations are not changed at the order we are interested in.



## 2.4 Viscous solutions

In the previous sections, we have computed non-viscous solutions and their leading order viscous correction. We will now consider directly the viscous problem, without assuming small Ekman numbers. The eigenmodes will be assumed to satisfy no-slip boundary conditions on the cylindrical surface, but only no-stress boundary conditions on the top and bottom plates. Eigenmodes can then be searched with the same simple Fourier structure in the azimuthal and vertical coordinate as the non-viscous modes. We shall explain why it is difficult to solve the problem with no-slip boundary conditions everywhere.

### Diffusive Inertial Waves

The viscous problem was previously decoupled using Cartesian components, and it leads to a modified Poincaré equation for the pressure, [45]

$$\left( (E\Delta - i\lambda)^2 \Delta + 4\partial_{zz}^2 \right) p = 0 \quad (2.36)$$

This problem is still separable in cylindrical geometry. For a wave of azimuthal wavenumber  $m$ , and axial wavenumber  $l$ , the problem is reduced to a sixth order one-dimensional problem along the radial direction. The resulting equation is rewritten as

$$\prod_{j=1}^3 (\Delta - z_j) p = 0 \quad (Ez_j - i\lambda)^2 z_j - 4l^2 = 0 \quad (2.37)$$

where  $z_j$  are three solutions of a characteristic polynomial, which further determines three radial wavenumbers  $K_j$  by  $K_j^2 = z_j + l^2$ . In the general case all three  $K_j$  are different, and the general solution for the pressure is written as a linear combination of the three independent solutions which are regular at the origin.

$$p = \sum_{j=1}^3 a_j I_m(K_j r) \cos(lz) e^{im\phi} \quad (2.38)$$

We prefer to use modified Bessel-functions  $I_m$ . Explicit expressions for the velocity field components are easily obtained, when using  $\pm$ -components of the velocity field. The viscous perturbation equations are

$$\left( E\Delta_{\pm} - i(\lambda \pm 2) \right) u_{\pm} = \frac{1}{\sqrt{2}} \left( \partial_r \pm \frac{i}{r} \partial_{\phi} \right) p, \quad \left( E\Delta - i\lambda \right) u_z = \partial_z p \quad (2.39)$$

where the operator  $\Delta$  is the scalar Laplacian, and

$$\Delta_{\pm} = \Delta \pm \frac{2i}{r^2} \partial_{\phi} - \frac{1}{r^2} \quad (2.40)$$

On the right-hand side of (2.39), we recognise the recurrence relation operators, acting on the pressure field. For each modified Bessel function component of the pressure, this gives

$$\left( \partial_r \pm \frac{m}{r} \right) I_m(K_j r) = K_j I_{m \mp 1}(K_j r) \quad (2.41)$$

In this way, the pressure gradient forces Helmholtz problems in each of the velocity field components. Note that  $I_{m \pm 1}(K_j r) e^{im\phi} \cos(lz)$ , has the structure of the eigenfunctions of  $\Delta_{\pm}$ , and similarly for the  $z$ -part of the equation, so that the velocity field components are easily recovered. If we introduce the notation  $c_j = -a_j l^2 (Ez_j - i\lambda) / K_j \sqrt{2}$ , we finally obtain for the velocity field components of the viscous eigenmodes

$$\begin{aligned} u_{\pm} &= \sum_{j=1}^3 c_j \left( 1 \pm \frac{2i}{Ez_j - i\lambda} \right) I_{m \pm 1}(K_j r) \cos(lz) e^{im\phi} \\ u_z &= \sum_{j=1}^3 c_j \left( -\frac{\sqrt{2} K_j}{l} \right) I_m(K_j r) \sin(lz) e^{im\phi} \end{aligned} \quad (2.42)$$

This homogeneous solution of the problem depends on three arbitrary coefficients  $c_j$ . The flows must satisfy no-slip boundary conditions,  $u_{\pm, z} = 0|_{r=1}$  on the cylindrical side wall. This leads to an algebraic system of equations for the coefficients  $c_j$ . Non-trivial solutions only exist when the determinant of this system is exactly zero, which defines the dispersion relation of the viscous eigenmodes. The frequency  $\lambda$  is now a complex unknown,

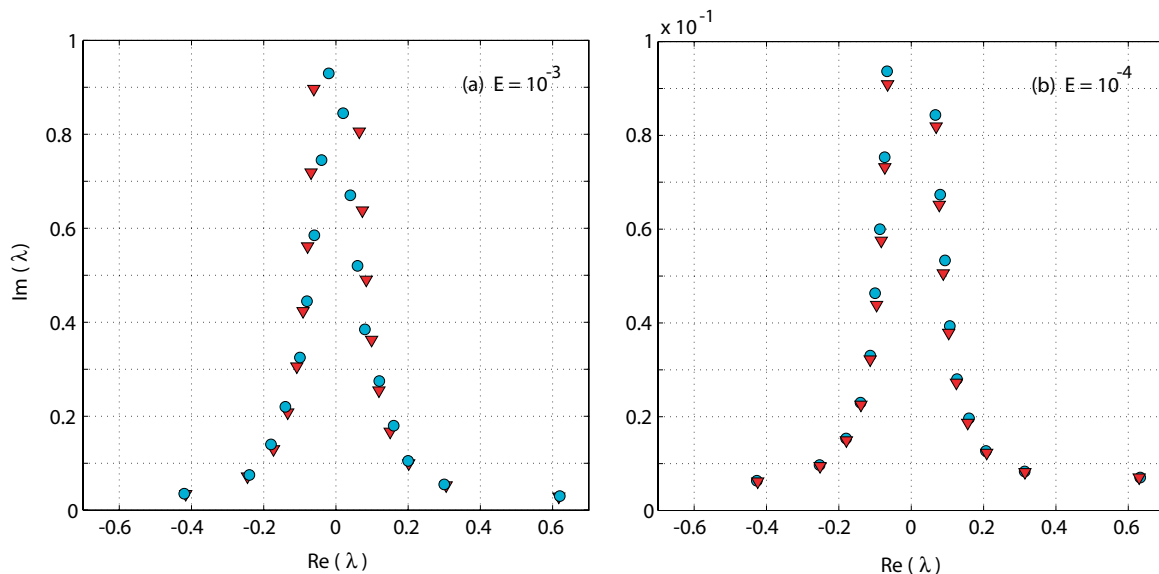


FIG. 2.3 – Complex frequency  $\lambda$  of the first 18 viscous eigenmodes with the largest radial structures obtained by the viscous procedure ( $\circ$ ) and the asymptotical analysis for small  $E$  ( $\nabla$ ). The cylinder has aspect ratio  $h = 3.9796$ , wavenumbers  $m = 1$ ,  $l = 1$  are fixed. (a)  $E = 10^{-3}$ , (b)  $E = 10^{-4}$ .

so are the wavenumbers  $K_j$ . We solve the dispersion relation numerically by using an optimisation procedure. Fixing  $m$ ,  $l$ ,  $E$ , an initial estimate for the frequency  $\hat{\lambda}$  produces three wavenumber estimates  $\hat{K}_j$  as solutions of the characteristic polynomial. The determinant is evaluated, and depending on its gradient in the complex  $\lambda$ -space, the estimation  $\hat{\lambda}$  is changed such that the solution converges towards a zero of the determinant. Initial estimates of  $\lambda$  can be based on the non-viscous analysis or on the asymptotical formula (2.27), where the contribution from top and bottom plates has been discarded. Estimates can also be obtained graphically by plotting the inverse norm of the determinant in the complex  $\lambda$ -plane. For low  $E$ , the wavenumbers  $K_j$  and the coefficients  $c_j$  have very different orders of magnitude. This comes from the fact that the non-viscous part of the flow and the viscous boundary layer have very different space-scales at low  $E$ . For the problem to be numerically solvable up to  $E = O(10^{-8})$ , we need to introduce some weighting in the coefficients  $c_j$  before evaluating the determinant. Finding the zeros of the dispersion relation is also possible through an optimisation on one of the complex wavenumbers.

In Fig. 2.3, we have plotted the complex frequency  $\lambda$  of the eighteen least damped viscous eigenmodes, ( $\circ$ ), for an example. As expected the damping rate of the modes increase as their frequency decreases, because their radial structure gets smaller. The comparison with the asymptotical results obtained by considering only the side-wall boundary layer is good, especially for the modes with the larger radial structures (large  $\text{Re}(\lambda)$ ), and for small  $E$ . Similar trends are observed for other values of  $m$ ,  $l$  and  $E$ . We have also analysed the variation of the eigenfrequencies with respect to  $E$  (see Fig. 2.4). The viscous results (full lines) compare very well with asymptotical estimates (dashed lines) given by (2.27), over a large range of  $E$ , for both the real and the imaginary part. We can also observe that the viscous damping variations exhibits the volumic dissipation scaling in  $E$  for large Ekman number but the surfacic dissipation scaling in  $E^{1/2}$  for small  $E$ . Such a plot confirms that volumic damping is important for moderate Ekman number ( $E > 10^{-4}$ ) and has to be included in the asymptotic estimates. Note also the departure of the asymptotical theory from the viscous results at large  $E$ . This can be related to a viscous transition of the mode which occurs at large  $E$ . The real part of the frequency drops down sharply to zero, and the boundary layer estimate is no longer valid. The viscous eigenmodes change in character to become almost evanescent (exactly evanescent for  $m = 0$ ). Similar pictures are found for the other modes, with other radial structures,  $l$  and  $m$ . The smaller the spatial scale of the modes, the quicker the viscous transition arises. This transition is illustrated in Fig. 2.5. In this figure, we show the evolution of the frequency with  $l$ , for the viscous modes with the largest radial structures for  $m = 1$  and  $E = 2.5 \times 10^{-4}$ . At this Ekman number,  $\text{Re}(\lambda)$  behaves as the non-viscous frequency, and is almost indistinguishable from the non-viscous

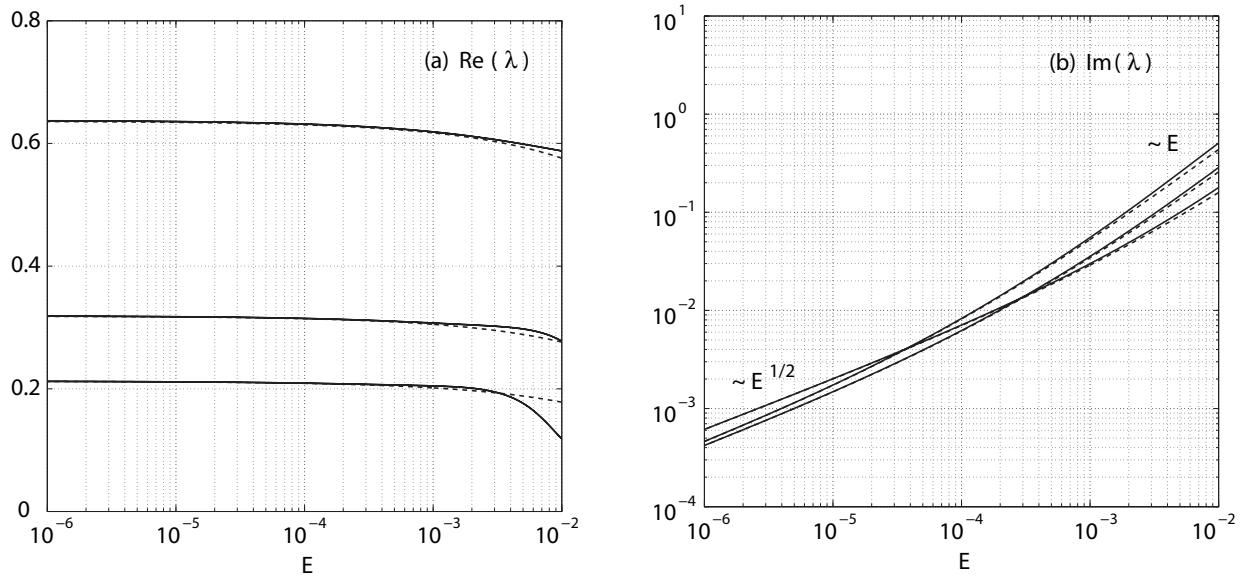


FIG. 2.4 – Evolution of the complex eigenfrequencies  $\lambda$  with Ekman number for the first 3 viscous eigenmodes with wavenumbers  $m = 1$ ,  $l = 1$  and positive frequencies.

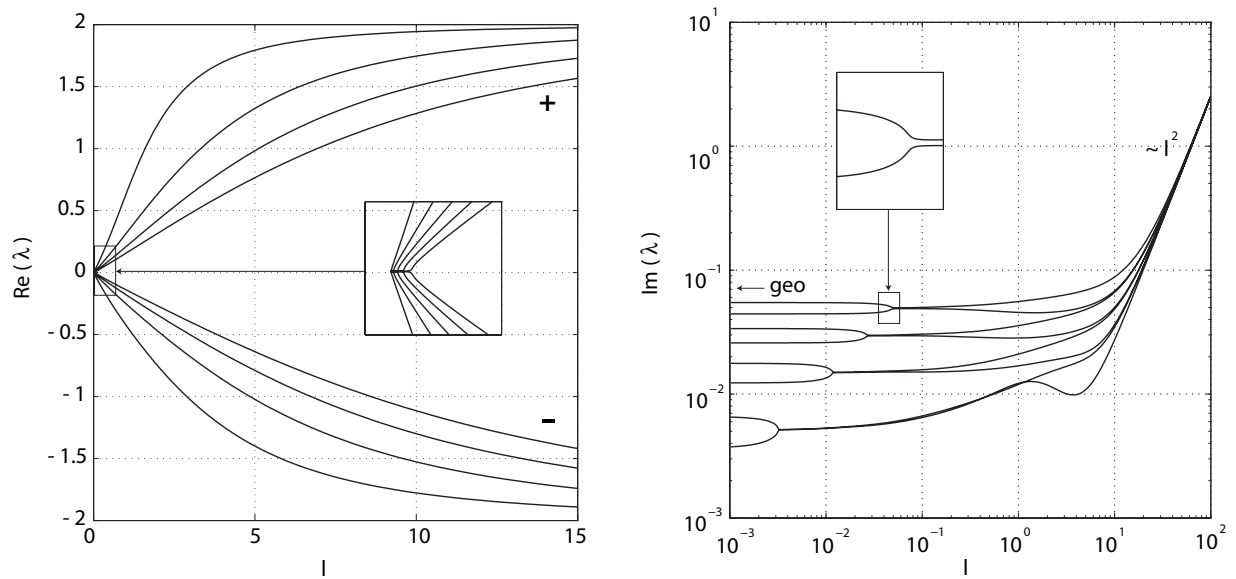


FIG. 2.5 – Variation of the six least stable complex eigenfrequencies  $\lambda$  with respect to the vertical wavenumber  $l$  for  $m = 1$ ,  $E = 2.5 \times 10^{-4}$ .

inertial wave frequency. For very small  $l$  only, we observe that the frequency drops down more rapidly than the non-viscous branches, which all accumulate at  $l = 0$ . If we increase  $E$ , or decrease the scale of the radial structures, this transition happens for larger  $l$  values. The plot of  $Im(\lambda)$  in Fig. 2.5 shows the complex interplay of the different modes. For large  $l$ , the classical  $l^2$ -scaling for the viscous damping is recovered. For small  $l$  in very long cylinders, we can see that the viscous transition is actually associated with the close interaction of two distinct modes which then separate as  $l$  decreases. These new branches converge to almost two-dimensional (the geostrophic flows) and almost one dimensional flows (vertical flows).

In Fig. 2.6 we display and compare the vertical vorticity and horizontal velocity fields for the viscous and non-viscous modes, (a)  $[1, 0, 3]^+$ , (b),  $[1, 1, 2]^+$ , (c),  $[1, 2, 3]^-$ . We can see that the central region of the viscous mode is mainly non-viscous, but that viscous modes for  $m \neq 0$  also possesses a strong boundary layer region of vorticity which is  $\pi/m$  out of phase with respect to the vorticity of the bulk. Visualisations were made in  $z = 0$ , where these field-components are maximal. The plotted modes can exist in cylinders with aspect ratio  $h = 3.9796$ , and correspond to  $[x, m, n]^\pm$ : the center region of the viscous modes is composed of the non-viscous mode. A lot of vorticity is concentrated in the boundary layers for  $m \neq 0$ , about  $\pi/m$  out of phase with the non-viscous part of the flow. In the same figure, we show the angular velocity of the same three non-viscous modes in vertical planes. The inertial wave with amplitude 0.2 is superposed on the base state of solid body rotation of amplitude 1. We recognise the spatial structure of the second mode (b), typical for spinover modes. Note that visualisations of the viscous eigenmodes are practically undistinguishable from the non-viscous ones.

### Diffusive geostrophic modes

The viscous geostrophic modes have no vertical structure as their non-viscous counterpart thanks to the no-stress boundary condition on top and bottom plates. They are purely transverse ( $u_z = 0$ ) and their pressure satisfies :

$$(E\Delta_\perp - i\lambda)\Delta_\perp p = 0 \quad (2.43)$$

where  $\Delta_\perp$  is the two-dimensional transverse Laplacian. This problem has bounded solutions of the form

$$p = [a J_m(kr) + b r^m] e^{im\phi} \quad (2.44)$$

where  $k$  is defined from

$$\lambda = i E k^2 \quad (2.45)$$

The velocity field is obtained from

$$(E\Delta_{\pm,\perp} - i(\lambda \pm 2)) u_\pm = \frac{1}{\sqrt{2}} \left( \partial_r \pm \frac{i}{r} \partial_\phi \right) p \quad (2.46)$$

and find the flows of the non-viscous case (2.14). There is a small viscous correction in the pressure expression

$$p = - \left( \frac{4}{k_x} J_m(k_x r) - \frac{2 - \lambda}{m} J_{m-1}(k_x) r^m \right) e^{im\phi} \quad (2.47)$$

As in the non-viscous case, the radial wavenumbers  $k$  are the positive zeros of  $J_{|m|+1}(k) = 0$ . In the viscous problem. The viscous geostrophic modes are evanescent in the rotating frame with a damping rate given by (2.45). All the geostrophic modes have then different complex eigenfrequencies. Viscosity has broken the degeneracy that was present in the non-viscous problem. Note also that contrarily to the other viscous modes, no  $O(E^{1/2})$  boundary layer and associated damping is present for these modes. The viscous geostrophic mode has a weak  $O(E)$  damping rate, entirely controlled by volumic effects.

### Other modes ?

The analytic expressions of the viscous eigenmodes of (2.42) is no longer valid when the characteristic polynomial has multiple roots. Inspection of the characteristic polynomial permit us to rule out the case of triple roots. Double roots  $K_1, K_2 = K_3$  can appear in three particular cases

$$K_2^2 = K_1^2/4 = l^2 + l^{2/3} E^{-2/3} \begin{cases} 1 \\ -\frac{\sqrt{3}}{2} \pm \frac{i}{2} \end{cases} \quad \lambda = 3 l^{2/3} E^{1/3} \begin{cases} -i \\ i \frac{\sqrt{3}}{2} \pm \frac{1}{2} \end{cases} \quad (2.48)$$

The spatial structure of the velocity fields, involves two Bessel functions and one bi-Bessel functions  $r I'_m(K_2 r)$ , and is again written as a sum of three linearly independent parts, preceded by three constants  $c_j$ . The boundary

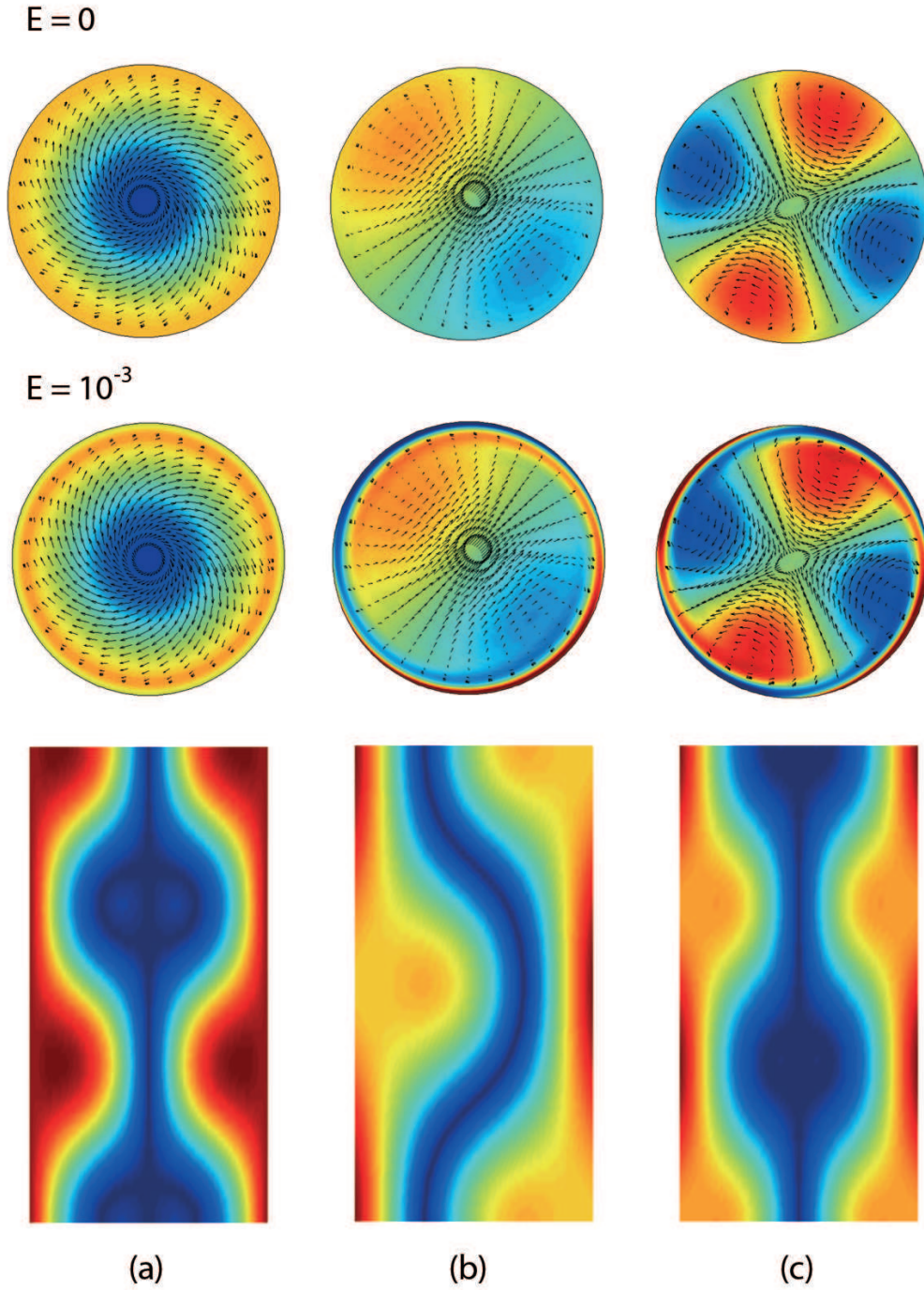


FIG. 2.6 – Horizontal sections of the vertical part of the vorticity (color) and transverse flow (arrows) in the plane  $z = 0$ , for the modes (a)  $[1, 0, 3]^+$ , (b)  $[1, 1, 2]^+$  and (c)  $[1, 2, 3]^-$  in a cylinder with  $h = 3.9796$ . The vertical cuts show the total angular velocities of the non-viscous modes with amplitude 0.2, superposed on the solid body rotation base state with amplitude 1. The non-viscous profiles are undistinguishable from the viscous profiles.

conditions lead to a homogenous system of three equations for the three coefficients  $c_j$ . Non-trivial solutions may then be obtained for very particular relations between the parameters  $m$ ,  $l$  and  $E$ . If such solutions exist, they might have a link with the singular modes for which the  $E^{1/3}$  scaling for frequency and radial scale were proposed, [126], [59]. Moreover, we do not expect the unstable case  $\lambda = -3il^{2/3}E^{1/3}$  to be ever possible, as solid body rotation is known to be globally stable [50].

### 2.4.1 Adjoint modes and orthogonality

The standard scalar product,  $\langle \cdot | \cdot \rangle$ , defined in (2.18) is well adapted to define adjoint modes in both the non-viscous and viscous problem. If we write the (direct) linear equations as  $\mathcal{L}\mathbf{U} = 0$ , we obtain for the adjoint modes  $\mathcal{L}^A\mathbf{U}^A = 0$ , with  $\mathcal{L}^A = \mathcal{L}^*$  and the same boundary conditions as for the direct problem : impermeability on top and bottom plates and no-slip on the cylindrical surface. The problem is then no-longer self-adjoint, but the adjoint modes are very close to the direct modes, they have the same vertical and azimuthal structure and radial structures satisfying

$$u_{\pm,z}^A(r) = u_{\pm,z}^*(r) \quad , \quad p^A(r) = -p^*(r) \quad (2.49)$$

By construction an orthogonality relation is found when combining direct and adjoint problems

$$(\lambda_1 - \lambda_2)(\mathbf{u}_1^{A*} | \mathbf{u}_2) = (\lambda_1 - \lambda_2) \int_V \mathbf{u}_1^{A*} \cdot \mathbf{u}_2 \, dV = 0 \quad (2.50)$$

This viscous orthogonality relation applies to both geostrophic modes and inertial waves. We use it in a following chapter to write down amplitude equations for the nonlinear evolution of the elliptical instability.

### 2.4.2 No-slip boundaries everywhere ?

We have tried to find viscous solutions satisfying no-slip conditions on the entire boundary surface, including top and bottom plates. Unlike previously and for this purpose we have placed the origin of the coordinate system at the center of the cylinder. Since the equations are symmetrical with respect to the transformation :  $z \rightarrow -z$ , the modes are split up into pair and impair symmetry classes. Modes with anti-symmetrical vertical velocity have  $u_z \sim \sin(l_j z)$  whatever  $l_j \in \mathbb{C}$ . This offers the advantage that the no-slip boundary condition only needs to be applied at for example the top plate  $z = h/2$ . It is natural to search viscous solutions in the same form as before :

$$\begin{bmatrix} u_{\pm} \\ u_z \end{bmatrix} = \sum_{ij}^N c_{ij} \begin{bmatrix} \left(1 \pm \frac{2i}{Ez_{ij} - i\lambda}\right) I_{m\pm 1}(K_i r) \cos(l_j z) \\ \frac{\sqrt{2}K_i}{l_j} I_m(K_i r) \sin(l_j z) \end{bmatrix} e^{im\phi} \quad (2.51)$$

with  $z_{ij} = K_i^2 - l_j^2$ , allowing  $K_i$  and  $l_j$  to be complex solutions of

$$(Ez_{ij} - i\lambda)^2 z_{ij} - 4l_j^2 = 0 \quad (2.52)$$

We immediately see from (2.52), that there are at most 3  $K_i$  for each  $l_j$  and  $3l_j$  for each  $K_j$ . This means that the number N of functions is bounded by  $3N_r$  and  $3N_z$ , if  $N_r$  and  $N_z$  are the number of different  $K_i$  and  $l_j$  respectively. However, the number of equations obtained from the boundary conditions is  $3N_r + 3N_z$ , which is larger than the total number N of coefficients  $c_{ij}$ . A finite sum of the form 2.51 can therefore not be a solution of the viscous problem in general.

## 2.5 Conclusion

In this chapter, we have discussed the different linear perturbations of the flow in a rotating cylinder. The properties of the non-viscous perturbations have been recalled together with their viscous corrections obtained by asymptotical expansions. We have also presented an alternative and new procedure to calculate these viscous corrections more easily. We have analysed the viscous perturbation problem with no-slip conditions on the cylindrical boundary and no-stress conditions on top and bottom plates. We have shown that the complex eigenfrequencies of the viscous modes compare very well with the asymptotic estimate obtained at low to moderate Ekman numbers. Increasing  $E$ , a viscous transition has been evidenced, which cannot be properly modeled by the asymptotical theory. By definition, the spatial structure of the viscous eigenmodes models both the non-viscous interior and the boundary layer contributions. An orthogonality relation between the viscous eigenmodes has been found, which will be used in the following chapters. We have not been able to find simple viscous solutions satisfying no-slip boundary conditions on the entire boundary surface. It is possible to derive viscous eigenmodes in other cylindrical problems, such as the cylindrical shell, discussed in Appendix B. Additional or other effects, such as those of an imposed magnetic field can also be introduced : this will be the concern of Sec. 4.2.2 and Appendix C.

## Chapitre 3

# Evolution non-linéaire de l'instabilité elliptique

Dans ce chapitre nous discutons les modèles non-linéaires que nous avons conçus pour décrire l'évolution non-linéaire de l'instabilité elliptique dans un cylindre. Nous expliquons la méthode de projection, qui utilise les modes visqueux du chapitre précédent. A la fin de la thèse, nous nous sommes aperçus d'une difficulté supplémentaire dans la modélisation. Les modifications non-linéaires de la pression interviennent activement dans le problème de l'évolution non-linéaire de l'instabilité elliptique, et elles n'ont pas été incluses dans les modèles initiaux. Nous proposons une solution à ce problème et discutons quelques résultats essentiels des modèles initiaux qui pourraient réapparaître dans les modèles corrigés. Nous trouvons deux scénarios de transition différentes, dans trois exemples particuliers étudiés en détail.

### 3.1 Introduction

In this chapter we discuss the nonlinear evolution of the elliptical instability in the Malkus' setup. This fluid system is sketched in Fig. 3.1. A deformable cylindrical container filled with a fluid is spun up to solid body rotation. With a pair of opposed rollers the cylindrical container is deformed elliptically, with eccentricity  $\epsilon$ . The boundary deformation is transmitted to the streamlines of fluid, and a flow with elliptical streamlines of homogeneous eccentricity  $\epsilon$  is obtained with good approximation for small enough  $\epsilon$ . This system can be realised experimentally and was the subject of previous studies [89], [139], [94], [26]. This set-up is an ideal system for analysis, since experimental observations can be combined with advanced theoretical analysis. The linear stages of the instability have been described in great detail [26], and linear instability phase-diagrams, including both finite-size effects and viscous corrections are now available. Eloy identified weakly nonlinear saturated states for both the spinover mode and also for other couplings in experiments close to the instability threshold. Further away from threshold, a violent nonlinear evolution and transition to turbulence is observed in the experiments, which has been called the inertial wave collapse. Fig. 3.2, shows an experimental Kalliroscope visualisations of a growth-collapse-relaminarisation cycle in the Malkus set-up obtained by L. Lacaze. The pictures are tilted over 90 degrees, so that the rotation axis is horizontal in each panel. The flow is illuminated in the plane of maximum stretching at  $+\pi/4$  radians away from the rollers. The Kalliroscope flakes align preferentially with the local strain field, and if little strain is present, almost steady particles reflect a diffuse light : at small mode amplitudes the rotation axis of the flow is visualised. In the first three pictures, we follow the slow linear growth of the spinover mode, which bends the rotation axis into a wavy S-shaped pattern, with two wavelengths along the rotation axis. In the fourth panel, the mode has grown to such large amplitudes, that small fluctuations start to appear in the flow, which rapidly contaminate the entire flow, leading to a fine-scale pattern of fine-scale turbulent fluctuations. Then, the disordered flow relaminarises to a solid body rotation, and the process starts again, on a background of fluctuations. The increased contrast of the vortex core in the last panel indicates large shear, which means that the core is rotating at a very different rate than the fluid in its immediate vicinity : after the first cycle, there is strong background of differential rotation. Similar phenomena have been reported in approximatively all experimental studies on directly forced inertial wave systems (precession, rotating lid) [88], [95], [91], [92], [64], [65], [98], [73] and all studies on elliptical instability [42], [41], [22], [89], [26]. The inertial wave collapse has long been puzzling, and several secondary instability mechanisms have been proposed. The first ideas attributed an important role to the large azimuthal circulation observed in the experiments. Centrifugal instability was



one possibility, but traces of this were never observed in the experiments. Malkus & Waleffe [90] proposed a less stringent criterium, based on the idea that strong azimuthal circulations can change the properties of the inertial waves locally. In certain places of space, the equation for the inertial waves would jump from a hyperbolic to an elliptical character, which would then be the origin of the collapse. The rotational background would not be able to sustain any longer the waves excited by the elliptical instability. Kerswell proposed another scenario [57]. He found that inertial waves were secondarily unstable with respect to instabilities associated with triad-resonances of inertial waves. Similar ideas also followed from local stability analysis of inertial waves [82], [29], [84]. The importance of secondary instability by triad resonances in the elliptical instability was then studied by Mason & Kerswell (1999), but their numerical solver was not capable to describe the inertial wave collapse. The experimental work of Eloy using the Malkus set-up [27], has revealed saturated states of secondary instability in and similar phenomena seem to occur in elliptically deformed spheroids [71]. More recently, resonant triads of waves received particular attention in the case of precession [98], [73], where it seems to play an important role in the generation of small scales in the flow.

In this chapter we wish to continue the study of the inertial wave collapse in the context of elliptical instability. Our goal is to understand the transition of elliptically unstable flows to turbulence, and more precisely to study the role played by triad resonance and zonal circulations in the transition. We use the same cylindrical geometry as in the Malkus set-up, and start our analysis as Mason & Kerswell. However, the perturbations are here expanded on the viscous eigenmodes obtained in the previous chapter, such that the Navier-Stokes equations are reduced to a system of equations for the amplitudes of these modes. Unfortunately, we have not been careful enough in the projection procedure and misdetermined the role of the pressure terms. We found that the system requires the addition of a supplementary body-force. This difficulty was encountered at the very end of this PHD-thesis. We have found a way to perform correctly the projection, but to the lack of time, we have not been able to simulate the new system of equations. The results which are discussed are then based on the initial model. In the first section we define the problem and introduce the elliptico-polar projection of the Navier-Stokes equations [138], [94]. We show that the complexity of the equations is greatly reduced when cylindrical  $\pm$ -components of the fields are used. In the second section, we expose the initial approach. We then explain the problem we encountered and propose a solution to it. We then discuss the most important results which were found with the initial model.

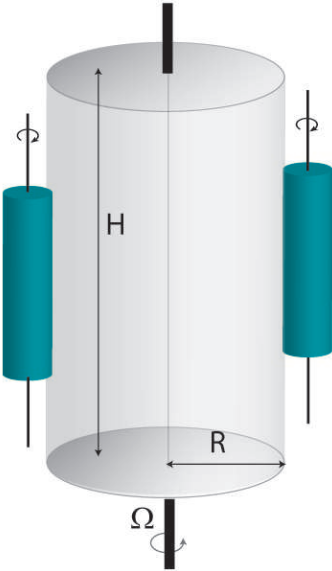


FIG. 3.1 – Sketch of the Malkus set-up.

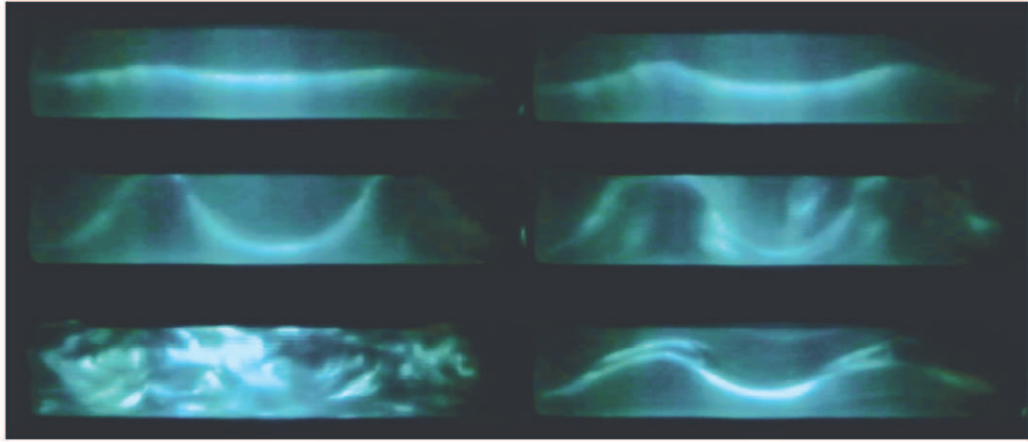


FIG. 3.2 – Kalliroscope visualisations of growth relaminarisation cycles, in the non linear development of elliptical instability, (courtesy of L. Lacaze).

## 3.2 Formulation using elliptico-polar coordinates

The fluid we study is a newtonian incompressible liquid with density  $\rho$  and kinematic viscosity  $\nu$ . We note  $\Omega$  the angular velocity at which the flow is spun up. The cylindrical container has radius  $R$  and top and bottom plates at  $z = 0, H$ . The imposed elliptical deformation has eccentricity  $\epsilon$ . We use the inertial scales to adimensionalise the problem

$$[t] = \Omega^{-1} \quad , \quad [\mathbf{u}] = \Omega R \quad , \quad [p] = (\Omega R)^2 \quad (3.1)$$

so that the Ekman number  $E = \nu/\Omega R^2$  and the aspect ratio  $h = H/R$  together with the eccentricity  $\epsilon$ , are the non-dimensional parameters, which describe the problem. We assume that the deformed cylinder has a homogeneous eccentricity over its entire height. The boundary  $\delta V$  of the fluid domain is then composed of an elliptical tube,

$$\frac{x^2}{1+\epsilon} + \frac{y^2}{1-\epsilon} = 1, \quad z \in [0, h]. \quad (3.2)$$

and of horizontal top and bottom plates at  $z = 0, h$ . The elliptical base-flow can be written as

$$\mathbf{U}_e = - \left( \sqrt{\frac{1+\epsilon}{1-\epsilon}} y \hat{\mathbf{x}} + \sqrt{\frac{1-\epsilon}{1+\epsilon}} x \hat{\mathbf{y}} \right) + O(\epsilon^2) \quad (3.3)$$

where  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$  denote cartesian unit-vectors. This flow is an exact 2D solution of the Navier-Stokes equations. The total velocity is written as  $\mathbf{u}_{tot} = \mathbf{U}_e + \mathbf{u}$ , and we are interested in the perturbation problem, which needs to be solved in a domain with elliptical boundaries. Orthogonal elliptic coordinates could be used, but they transform the simple elliptical base-flow into an awkward expression, making even the linear stage of the elliptical instability a complex problem. The non-orthogonal elliptico-polar coordinates  $(s, \phi, z)$  has proven to be more helpful in many works on the elliptical instability [41], [138], [94]. Defined as

$$x = s \sqrt{1+\epsilon} \cos \phi \quad , \quad y = s \sqrt{1-\epsilon} \sin \phi \quad (3.4)$$

this non-orthogonal transformation maps the elliptical boundary (3.2) on the circular coordinate surface,  $s = 1$ . The elliptico-polar coordinates are accompanied by the base (non-normalised) vectors

$$\begin{aligned} \tilde{\mathbf{s}} &= \sqrt{1+\epsilon} \cos \phi \hat{\mathbf{x}} + \sqrt{1-\epsilon} \sin \phi \hat{\mathbf{y}} \\ \tilde{\phi} &= -\sqrt{1+\epsilon} \sin \phi \hat{\mathbf{x}} + \sqrt{1-\epsilon} \cos \phi \hat{\mathbf{y}} \end{aligned} \quad (3.5)$$

which are used to project out independent components of the velocity-field. Using this transformation, the elliptical baseflow, transforms into a simple solid body like rotation,  $\mathbf{U}_e = s \tilde{\phi}$ . The perturbation field is written as

$$\mathbf{u} = u_s \tilde{\mathbf{s}} + u_\phi \tilde{\phi} + u_z \hat{\mathbf{z}} \quad (3.6)$$

The non-orthogonality brings in a number of more or less complex terms, which arise through the spatial derivatives in the Navier-Stokes equations. It is noticed that the relations,  $\partial_\phi \tilde{\mathbf{s}} = \tilde{\phi}$  and  $\partial_\phi \tilde{\phi} = -\tilde{\mathbf{s}}$ , are similar to those with cylindrical coordinates. In this way couplings due to non-orthogonality do not arise in the nonlinear advection, nor in the incompressibility constraint

$$\mathbf{u} \cdot \nabla = u_s \partial_s + u_\phi \frac{1}{s} \partial_\phi + u_z \partial_z \quad (3.7)$$

$$\nabla \cdot \mathbf{u} = \left( \partial_s + \frac{1}{s} \right) u_s + \frac{1}{s} \partial_\phi u_\phi + \partial_z u_z \quad (3.8)$$

and only enter the problem through the pressure gradient, and the diffusive Laplacian. Mason & Kerswell [94], derives exact equations of the Navier Stokes for the elliptico-polar field components. We have seen previously that the use of  $\pm$ -vector components, here defined as

$$u_\pm = \frac{1}{\sqrt{2}} (u_s \pm i u_\phi) \quad (3.9)$$

permits to diagonalise the Laplacian operator and the Coriolis force. In the following, we use the rescaled velocity-pressure four-vector  $\mathbf{U} = [u_+, u_-, \tilde{u}_z, \tilde{p}]^T$ . Tilded variables have been introduced to express the problem more elegantly. They are defined by

$$\tilde{z} = \frac{\tilde{u}_z}{u_z} = (1-\epsilon^2)^{-1/2} \quad \tilde{p} = \frac{\tilde{E}}{E} = (1-\epsilon^2)^{-1} \quad (3.10)$$

The tildes are dropped in the following. Since the elliptical deformation is stationary in the laboratory frame, it is convenient to write the perturbation problem in this frame. Formally this can be written :

$$\mathcal{L} \mathbf{U} = \epsilon \mathcal{E} \mathbf{U} + \mathcal{N}_{\mathbf{u}} \mathbf{U} \quad (3.11)$$

where the operator  $\mathcal{L}$ , contains the time derivatives, coriolis force and viscous part. This operator is identical to the one used in the previous chapter

$$\mathcal{L} = \begin{bmatrix} \partial_t + \partial_\phi + 2i - E\Delta_+ & 0 & 0 & \frac{1}{\sqrt{2}}(\partial_s + \frac{i}{s}\partial_\phi) \\ 0 & \partial_t + \partial_\phi - 2i - E\Delta_- & 0 & \frac{1}{\sqrt{2}}(\partial_s - \frac{i}{s}\partial_\phi) \\ 0 & 0 & \partial_t + \partial_\phi - E\Delta & \partial_z \\ \frac{1}{\sqrt{2}}(\partial_s - \frac{i}{s}\partial_\phi + \frac{1}{s}) & \frac{1}{\sqrt{2}}(\partial_s + \frac{i}{s}\partial_\phi + \frac{1}{s}) & \partial_z & 0 \end{bmatrix} \quad (3.12)$$

The linear elliptical interaction operator  $\mathcal{E}$  is composed of two parts, stepping up and down azimuthal wave-number by two. Each of the  $\mathcal{E}_\pm$  has a single leading  $O(1)$  term acting on the pressure field, accompanied by 3 smaller  $O(E)$  elliptico-viscous interactions on the diagonal.

$$\mathcal{E} = e^{i2\phi} \mathcal{E}_+ + e^{-i2\phi} \mathcal{E}_- \quad (3.13)$$

$$\begin{aligned} \mathcal{E}_+(1,1) &= -\frac{E}{2}(\partial_s + \frac{i}{s}\partial_\phi - \frac{2}{s})(\partial_s + \frac{i}{s}\partial_\phi - \frac{1}{s}) \\ \mathcal{E}_+(2,2) &= -\frac{E}{2}(\partial_s + \frac{i}{s}\partial_\phi)(\partial_s + \frac{i}{s}\partial_\phi + \frac{1}{s}) \\ \mathcal{E}_+(3,3) &= -\frac{E}{2}(\partial_s + \frac{i}{s}\partial_\phi - \frac{1}{s})(\partial_s + \frac{i}{s}\partial_\phi) \\ \mathcal{E}_+(2,4) &= \frac{1}{\sqrt{2}}(\partial_s + \frac{i}{s}\partial_\phi) \end{aligned} \quad (3.14)$$

$$\begin{aligned} \mathcal{E}_-(1,1) &= -\frac{E}{2}(\partial_s - \frac{i}{s}\partial_\phi)(\partial_s - \frac{i}{s}\partial_\phi + \frac{1}{s}) \\ \mathcal{E}_-(2,2) &= -\frac{E}{2}(\partial_s - \frac{i}{s}\partial_\phi - \frac{2}{s})(\partial_s - \frac{i}{s}\partial_\phi - \frac{1}{s}) \\ \mathcal{E}_-(3,3) &= -\frac{E}{2}(\partial_s - \frac{i}{s}\partial_\phi - \frac{1}{s})(\partial_s - \frac{i}{s}\partial_\phi) \\ \mathcal{E}_-(1,4) &= \frac{1}{\sqrt{2}}(\partial_s - \frac{i}{s}\partial_\phi) \end{aligned} \quad (3.15)$$

We recognise that it is mainly the elliptico-viscous terms which cause the elliptico-polar equations to be complex. The nonlinear interaction operator writes

$$\mathcal{N}_{\mathbf{u}} = - \begin{bmatrix} \nabla_{\mathbf{u}} & \mathbf{0} \\ 0 & 0 \end{bmatrix} \quad (3.16)$$

so that  $\mathcal{N}_{\mathbf{u}} \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u}$ . In  $\pm$ -components the field deformation tensor is reorganised as

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{2}}(\partial_s - \frac{i}{s}\partial_\phi + \frac{1}{s})u_+ & \frac{1}{\sqrt{2}}(\partial_s + \frac{i}{s}\partial_\phi - \frac{1}{s})u_+ & \partial_z u_+ \\ \frac{1}{\sqrt{2}}(\partial_s - \frac{i}{s}\partial_\phi - \frac{1}{s})u_- & \frac{1}{\sqrt{2}}(\partial_s + \frac{i}{s}\partial_\phi + \frac{1}{s})u_- & \partial_z u_- \\ \frac{1}{\sqrt{2}}(\partial_s - \frac{i}{s}\partial_\phi)u_z & \frac{1}{\sqrt{2}}(\partial_s + \frac{i}{s}\partial_\phi)u_z & \partial_z u_z \end{bmatrix} \quad (3.17)$$

We want to solve this problem with no-slip boundary conditions on the cylindrical surface, and impermeability (or no-stress) conditions on the top and bottom plates. We shall use a Galerkin approach by expanding the field in terms of linear viscous eigenmodes.

### 3.3 Projection on the viscous eigenmodes

In Ch. 2, we have identified the linear viscous eigenmodes, which satisfy the no-slip constraint on the radial surface and no-stress boundary conditions on top and bottom plates. A property of orthogonality has also been obtained which will enable us to use the spatial structure of the eigenmodes as base-fields for finite sized expansions of the hydrodynamic field in more complex problems, such as the nonlinear evolution of the elliptical instability. In practice, we expanded the flow as a sum of viscous eigenmodes

$$\mathbf{U}(\mathbf{r}, t) = \sum_{j=1}^N (a_j(t) \mathbf{U}_j(\mathbf{r}) + \zeta_j a_j^*(t) \mathbf{U}_j^*(\mathbf{r})) \quad (3.18)$$

The  $N$  fields  $\mathbf{U}_j$  contain the spatial profiles of the viscous eigenmodes, for all azimuthal wavenumbers  $m \geq 0$ . We note the waves as previously  $[x, m, n]^\pm$ , recalling that  $x \in \mathbb{N}_0$  is the radial label, counting the number of lobes in the radial structure,  $n \in \mathbb{N}$  is the number of half vertical wavelengths in the spatial structure, and equal to zero for geostrophic modes. The sign  $\pm$  here stands for the sign of the rotating frame frequency  $Re(\lambda)$ . Reality of the flow imposes that each complex field  $\mathbf{U}_j$  is accompanied by its complex conjugate,  $\mathbf{U}_j^*$ . Therefore  $\zeta_j = 1$  for all modes except for the axisymmetric geostrophic modes, for which  $\zeta_j = 0$ . For the nonlinear analysis, it is important to normalise the velocity fields in the wave-package. For this purpose, we use the non-dimensional 'kinetic' energy

$$K = \int_0^1 s ds \int_0^{2\pi} d\phi \int_0^h \mathbf{u}^* \cdot \mathbf{u} dz \quad (3.19)$$

The energy  $K_0$  of a solid-body rotation in a cylinder with aspect ratio  $h$  is in particular given by

$$K_0 = \frac{\pi h}{2} \quad (3.20)$$

We normalise the eigenmodes such that they have a kinetic energy  $|a_j|^2 K_0$ . This implies that the radial parts of the integrals are to be normalised as

$$\int_0^1 \mathbf{u}^*(s) \cdot \mathbf{u}(s) s ds = \begin{cases} 1/4 & , \text{ geostrophic} \\ 1/2 & , \text{ inertial} \end{cases} \quad (3.21)$$

We insert (7.10) in (3.11) with this normalisation and use the orthogonality condition

$$\left( \mathbf{u}_j^A \mid \mathbf{u}_k \right) = \int_V \mathbf{u}_j^{A*} \cdot \mathbf{u}_k dV = \delta_{jk} \left( \mathbf{u}_j^A \mid \mathbf{u}_j \right) \quad (3.22)$$

This leads to a nonlinear system of equations for the amplitudes  $a_j$  of the modes

$$\begin{aligned} \dot{a}_j &= i \omega_j a_j + \beta \sum_k \left( E_{jk}^{(1)} a_k + \zeta_k E_{jk}^{(2)} a_k^* \right) \\ &+ \sum_{k, l} \left( N_{jkl}^{(1)} a_k a_l + \zeta_l N_{jkl}^{(2)} a_k a_l^* + \zeta_k \zeta_l N_{jkl}^{(3)} a_k^* a_l^* \right) \end{aligned} \quad (3.23)$$

We have explicitly included the  $\zeta_j$  coefficients in the amplitude equations, to prevent from counting twice interactions with the axisymmetric geostrophic modes. In (3.23),  $\omega_j = \lambda_j - m_j$  is the complex frequency of the eigenmodes in the fixed frame : these terms contain both the non-viscous frequency and viscous damping of the modes. The elliptical and nonlinear matrix elements are defined as

$$E_{jk}^{(1)} = \frac{\left( \mathbf{u}_j^A \mid \mathcal{P}_u(\mathcal{E} \mathbf{U}_k) \right)}{\left( \mathbf{u}_j^A \mid \mathbf{u}_j \right)} \quad E_{jk}^{(2)} = \frac{\left( \mathbf{u}_j^A \mid \mathcal{P}_u(\mathcal{E} \mathbf{U}_k^*) \right)}{\left( \mathbf{u}_j^A \mid \mathbf{u}_j \right)} \quad (3.24)$$

$$N_{jkl}^{(1)} = -\frac{\left( \mathbf{u}_j^A \mid (\mathbf{u}_k \cdot \nabla) \mathbf{u}_l \right)}{\left( \mathbf{u}_j^A \mid \mathbf{u}_j \right)} \quad N_{jkl}^{(2)} = -\frac{\left( \mathbf{u}_j^A \mid (\mathbf{u}_k \cdot \nabla) \mathbf{u}_l^* \right)}{\left( \mathbf{u}_j^A \mid \mathbf{u}_j \right)}$$

$$N_{jkl}^{(3)} = -\frac{\left( \mathbf{u}_j^A \mid (\mathbf{u}_k^* \cdot \nabla) \mathbf{u}_l^* \right)}{\left( \mathbf{u}_j^A \mid \mathbf{u}_j \right)} \quad (3.25)$$

With the  $\pm$ -components of the fields, one can notice that all radial and azimuthal derivatives combine in such a way that they lead to recursion relations. The action of the nonlinear and elliptical term on the fields  $\mathbf{U}_j$  can then be calculated analytically, making this Galerkin approach practical. The system of equations (3.23) is time-stepped using a standard integration scheme, starting from an initial noiselevel or any other state. Technically it is necessary to prevent the amplitudes from falling down to numerically invisible small values, and this can happen for the modes which are not elliptically excited or not harmonics of the unstable coupling, even though they may play an important role in secondary instabilities. We prevent this by adding a small stationary forcing  $f_j = 0(10^{-5} - 10^{-6})$  in the amplitude equations, which slightly changes the trivial fixed point  $\mathbf{0}$ . This has only small consequences for the weakly nonlinear dynamics, but keeps all amplitudes above  $0(10^{-7})$  at low cost. We can maximally simulate the dynamics of about  $N = 0(1000)$  modes, but we have preferred smaller  $N$  when possible.

### 3.4 A difficulty associated with pressure terms

The previous section explains the numerical procedure as it was originally conceived. It is only at the very end of this PHD-thesis, that we came across an awkward particularity of the use of elliptico-polar coordinates : the pressure field is primordial. This is of course generally the case in the Navier-Stokes equation, and is illustrated by taking the divergence of this equation, which for incompressible flow leads to

$$\Delta p + \nabla \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} = 0 \quad (3.26)$$

This problem appears also in the time-splitting algorithm, which solves the Navier-Stokes equation. At each time-step, the pressure field is to be corrected in such a way that incompressibility is guaranteed. Using velocity field modes with build-in incompressibility, and a projection method as ours, this actually never poses a problem, as long as the pressure does not intervene in other couplings, which is always the case in orthogonal coordinate systems. The pressure adjusts in agreement with the equation above, and is an unknown variable which receives no further attention. In the elliptical coupling (3.14 - 3.15), the pressure field intervenes dominantly, and if it changes nonlinearly, then these will have consequences here. These terms are not included in the previous model and are a priori necessary. In the present framework, this means that they have been artificially suppressed by the addition of a body-force. This problem was left unnoticed for a long time, for a number of reasons.

Since the missing terms are purely of nonlinear origin, there are no significant differences between the linear asymptotical theory for elliptical instability and the linear growth rates we find in the previous model at small  $\epsilon$ . There are no traces in the matrix elements of interacting inertial wave triads, which were also successfully compared to non-viscous values. Most of the initial testing was concentrated at this level. At a next level we calculated nonlinear timeseries, for the parameter set  $\epsilon = 0.033$ , in a cylinder with aspect ratio  $h = 3.9796$ , for a number of 40 to 640 modes in the model. We find the growth-collapse-relaminarisation cycles of Fig. 3.3, and compare them to the result of Mason and Kerswell [94]. We identify the same modes as those present in their work, the spinover mode coupling  $(-1, 1, 1)$  involving the mode (1)  $[1, 1, 2]^+$ , the large scale geostrophic flow (2)  $[1, 0, 0]$ , and the competing elliptical coupling  $(0, 2, 1)$ , which links the inertial waves (3)  $[1, 0, 3]^-$  and (4)  $[1, 2, 3]^+$ , rapidly oscillating at frequency  $\omega \simeq 1$ , (other mode contribution are not shown). The flow goes through growth-collapse-relaminarisation cycles. Growth happens on a long time-scale and the collapse takes only few rotations. We observe a competition between both elliptical couplings, and even though the spinover mode grows less rapidly, it reaches higher amplitudes. Up to this point, our model compares well with the one of Mason & Kerswell (1999), but there is a bothering particularity : the geostrophic flow that is excited **accelerates** the flow. Cyclonic regimes in inertial wave systems have been reported in the literature on the experiments of forced inertial wave systems, [95] , [65], and their origin continues to be puzzling [57]. But the accelerated geostrophic flow continues to be bothering especially since Mason and Kerswell, combined their numerical result with detailed weakly nonlinear calculations. Even if the numerical procedure which 'forced' saturation in their simulations is tricky, their results agree well with the weakly nonlinear calculations. We therefore insisted to find the origin of the differences between both models, and realised that it lies in the nonlinear modifications of the pressure term, as mentioned above.

We propose the following solution. We want to hold on to the representation using viscous eigenmodes, and at first it seems that the only possible solution is to calculate the nonlinear pressure correction. This is a lengthy procedure and not a good idea, as there is a much better alternative. Let us write the elliptico-polar system of equations in a slightly modified form

$$\partial_t \mathbf{u} + \mathbf{L} \mathbf{u} + (\mathbf{I} - \epsilon \mathbf{E}_1) \nabla p = \epsilon E \mathbf{E}_2 \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} \quad \nabla \cdot \mathbf{u} = 0 \quad (3.27)$$

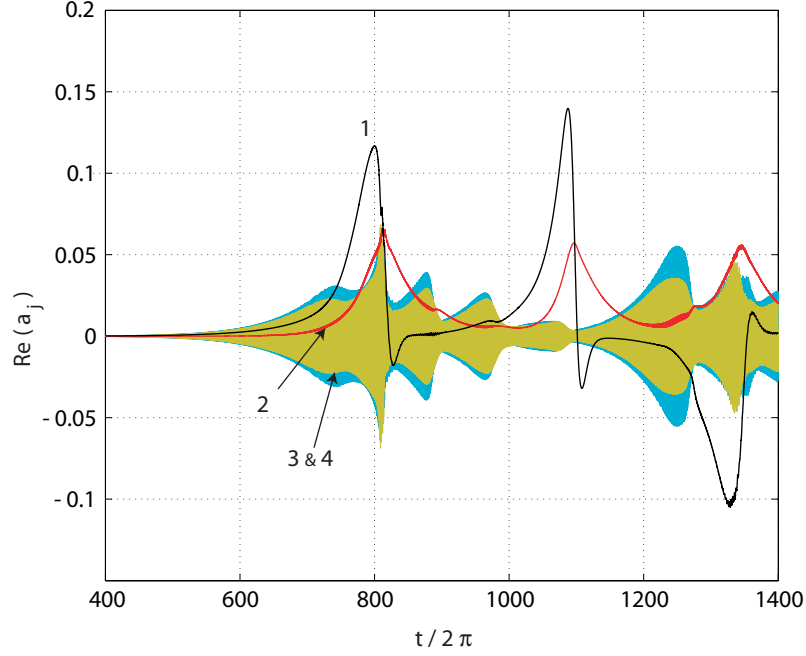


FIG. 3.3 – Timeseries for the real part of the amplitudes of the modes (1)  $[1, 1, 2]^+$ , (2)  $[1, 0, 0]$ , (3)  $[1, 0, 3]^-$  and (4)  $[1, 2, 3]^+$ , calculated with the nonlinear models for elliptical instability.  $\epsilon = 0.033$ ,  $E = 2.5 \times 10^{-4}$ ,  $h = 3.9796$ .

The elliptical  $\mathbf{L}$ , contains the mean flow advection ( $\partial_\phi$ ), the Coriolis force and the viscous terms. The operator  $\mathbf{E}_2$  gathers the elliptico-viscous terms, and the operator in front of the pressure gradient gathers the ordinary pressure term and the dominant elliptical coupling :

$$(\mathbf{I} - \epsilon \mathbf{E}_1) = \begin{bmatrix} 1 & -\epsilon e^{-i2\phi} & 0 \\ -\epsilon e^{i2\phi} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.28)$$

This matrix has an inverse which can be readily calculated :

$$(\mathbf{I} - \epsilon \mathbf{E}_1)^{-1} = \frac{1}{1 - \epsilon^2} \begin{bmatrix} 1 & \epsilon e^{-i2\phi} & 0 \\ \epsilon e^{i2\phi} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.29)$$

Thus, if we multiply (3.27) by  $(\mathbf{I} - \epsilon \mathbf{E}_1)^{-1}$ , we obtain

$$(\mathbf{I} - \epsilon \mathbf{E}_1)^{-1} (\partial_t + \mathbf{L}) \mathbf{u} + \nabla p = \epsilon E (\mathbf{I} - \epsilon \mathbf{E}_1)^{-1} \mathbf{E}_2 \mathbf{u} - (\mathbf{I} - \epsilon \mathbf{E}_1)^{-1} (\mathbf{u} \cdot \nabla) \mathbf{u} \quad (3.30)$$

which suppresses the interaction between pressure gradient and elliptical terms. However, this equation possesses new interactions on all the other terms. Since the inverse operator is explicitly calculated, the representation stays exact. The pressure gradient may now vary nonlinearly without further concern, since it no longer appears in any other coupling. This system of equations can now be projected on the viscous eigenmodes set, to obtain a set of slightly more complex amplitude equations than the previous ones, in implicit form. The elliptical modification of the nonlinear advection contains the main part of the terms which were forgotten in the original version. Actual implementation and results of this new model are postponed, due to the very late stage at which the problem was identified and solved. In the next section we present results which have been obtained from the analysis of (3.23).

## 3.5 Nonlinear evolution of the elliptical instability described by system (3.23)

In this section, we describe the nonlinear evolution of the elliptical instability described by (3.23). As stressed above, this is not compatible with the correct expression for the pressure and requires the addition of a body force to be valid. Unfortunately, we do not know the precise effect of such terms but we think that the analysis which is performed in the following will be qualitatively the same without this force. By studying three different configurations in detail, we put forward two possible nonlinear scenarios of transition.

### 3.5.1 Nonlinear dynamics of the spinover mode at moderate $E$ : Ruelle-Takens transition scenario

The first configuration we consider is a cylinder of length  $h = 3.9796$  at moderate Ekman number  $E = 10^{-3}$ . In a cylinder twice as long, for  $\epsilon = 0.1$  and  $Ek \simeq 10^{-3}$ , Eloy et al. (2003) observed saturating flows. For slightly lower Ekman number, secondary instabilities saturating at low amplitude were also reported (Mason & Kerswell 1999). These secondary instabilities can often be related to triad resonances of inertial waves (Kerswell 1998) : the elliptically excited wave(s) destabilise other pairs of inertial waves through again a mechanism of parametric instability. It has been proposed that triad resonances should play an important role in the collapse of the flow. In this first example, we are going to show that we can obtain saturated flows and saturated secondary instabilities, using the mentioned parameter set, but also that the presence of many secondary instabilities does not necessarily lead to the collapse of the flow.

Instead of varying  $E$  as it was done in the experiment, we here increase progressively  $\epsilon$  from the linear instability threshold. This is more practical as the large number of matrix elements in (3.23) only needs to be calculated once. In the present simulation, we expand the flow on a total number of 137 modes. The least damped modes with the largest spatial structure are used. More specifically we have used the 10 axisymmetric geostrophic modes  $[x, 0, 0]$  with  $x \leq 10$ , 15 other geostrophic modes  $[x, m, 0]$  with  $x \leq 5$  and  $m = 1, 2, 3$ , and then all inertial waves  $[x, m, n]^{\pm}$  with  $x = 1, 2, 3, 4$ ,  $m = 0, 1, 2, 3$  and  $n = 1, 2, 3, 4$ . The spinover mode  $[1, 1, 2]^+$  is here accompanied by 42 of its spatial harmonics (H), which are the modes  $[x, m, n]$  with

$$n \text{ pair} \quad \& \quad m, n/2 \quad \text{same parity} \quad (3.31)$$

These harmonics include the axisymmetric geostrophic modes and they are forced by the non-linear interaction of the elliptically destabilised mode (here the spinover mode) with itself. They are primordial in the weakly non-linear evolution (saturation process). Harmonical modes can also appear through secondary instabilities of the growing spinover mode. It is also important to have other modes in the expansion, which are not spatial harmonics (NH). They are not excited in the weakly nonlinear evolution of the flow, but they may appear through secondary instabilities, such as in triad or higher order resonances. We first show some time-series calculated by numerical integration of (3.23), and then provide a fixed-point analysis. We have tested the results by performing simulations with 692 modes and similar time-series have been obtained, yet at significantly greater cost.

In Fig. 3.4, we have plotted the real parts  $Re(a_j)$  of the amplitudes of the five most important modes, corresponding to the spinover mode and four of its harmonics for different values of  $\epsilon$ . The spectra of the fluctuations of the spinover mode are shown in figure 3.5. For  $\epsilon = 0.085$ , the spinover mode is the only unstable mode. We see in Fig. 3.4-(a) that this mode grows and saturates at a finite amplitude. Its harmonics have followed the same evolution. The system has converged to a fixed point which remains stable until  $\epsilon$  exceeds 0.095. For  $\epsilon = 0.100$ , rapid and small fluctuations are observed in the time series (Fig. 3.4-(b)). Figure 3.5(a) tells us that these fluctuations are sinusoidal with a dominant frequency  $\omega = 2.501$ . A slight increase of the eccentricity  $\epsilon$  leads to the appearance of other frequencies, multiples of  $\omega = 0.596$ , in the time series (Fig. 3.5-(b)), but the oscillations remain regular up to  $\epsilon \approx 0.1055$ . For this value of  $\epsilon$ , chaotic fluctuations are observed for small time but after  $O(500)$  rotation, the flow reaches another regular cycle (Fig. 3.5-(c)). For  $\epsilon = 0.1060$ , the flow does not relaminarise and a large number of discrete peaks is now present in the spectrum [figure 3.5(d)]. For larger values of  $\epsilon$ , the spectrum becomes broadband and the frequency peaks progressively disappear (Fig. 3.5-(e,f)). The fluctuations also become more intense as it can be observed in figures 3.4-(c,d).

The main features of the dynamics just described are that there is a progressive destabilisation of the spin-over mode but no growth-collapse-relaminarisation cycles. We are now going to provide a more precise analysis of the transition in terms of the fixed point and its stability. The saturated timeseries of Fig. 3.4-(a) indicate the existence of a fixed point. We recover this fixed point from the system (3.23) through an optimisation procedure, and once this point is identified, we can follow how the fixed point varies with  $\epsilon$  over



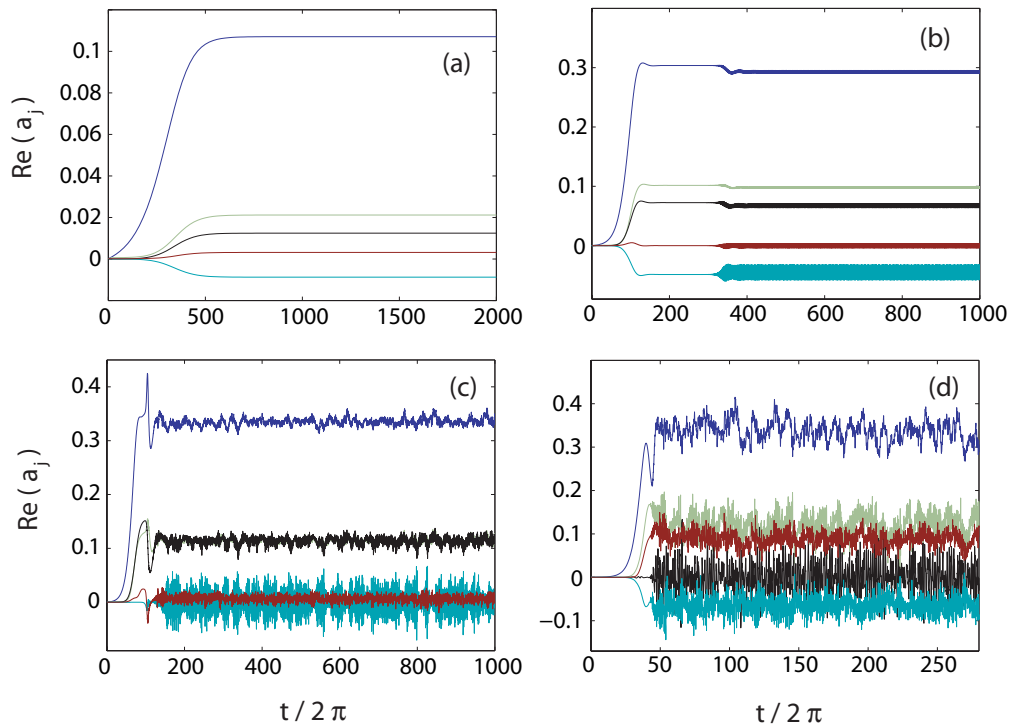


FIG. 3.4 – Temporal evolution of the amplitude of the dominant modes for various eccentricities in a cylinder of aspect ratio  $h = 3.979$  and Ekman number  $E = 10^{-3}$ . (a)  $\epsilon = 0.085$ , (b)  $\epsilon = 0.100$ , (c)  $\epsilon = 0.110$ , (d)  $\epsilon = 0.150$

the entire studied  $\epsilon$ -span. This leads to the results shown in Fig. 3.6. We see on this plot that the fixed point characteristics are dominated by the spinover mode  $[1, 1, 2]^+$  for all  $\epsilon$ . The bifurcation is supercritical and close to the linear instability threshold, we recover the scaling in  $(\epsilon - \epsilon_c)^{1/2}$  for the spinover mode and in  $\epsilon - \epsilon_c$  for the first harmonics in agreement with the weakly nonlinear hypothesis (Eloy et al. 2003). Near  $\epsilon \approx 0.105$ , we observe that some of the harmonics change their dominance and that the spinover component of the fixed point departs from the mentioned scaling. This transition occurs at the same time as the chaos in the time series. For larger  $\epsilon$ , the fixed point characteristics do not change much and the spinover component tends to a fixed amplitude. In the same plot, we included the mean value (black dots) and the standard deviations (errorbars) of the fluctuating spinover mode amplitude, in the timeseries we integrated from (3.23) at different  $\epsilon$ . Interestingly, the mean spinover mode component remains close to the fixed point, even for the regime where the fluctuations are the largest ( $\epsilon = 0.15$ ). This will allow us to propose some modeling of this disordered state.

In order to understand the transition observed in the time series, we have analysed the stability of the fixed point. Such an analysis is close to the one performed by Kerswell (1999), for a single inertial wave, but takes here into account the harmonical flow necessary for the weakly nonlinear saturation of the spinover mode. The results of the analysis are displayed in Fig. 3.7. In this figure, we have plotted the growth rate  $Re(\sigma)$  and frequency  $Im(\sigma)$  of the first 15 destabilized secondary instability modes as  $\epsilon$  is increased. The unstable modes can be separated in two classes whether they are composed of spatial harmonics of the spinover mode (H) or not (NH). Harmonical modes (H) correspond to the type 1 modes of Kerswell (1999), whereas Non-Harmonical modes (NH) are of type 2 and 3. In figure 3.7, (H) modes are (2), (9), (10) and (12). All the other unstable modes are (NH). The stability analysis shows that the fixed point is stable up to  $\epsilon = 0.0948$ , at which the mode (1) is destabilised. This first mode is associated with a triad resonance between the modes  $[1, 2, 1]^-$ ,  $[1, 3, 3]^+$  and the spinover mode. This is in agreement with the time series, where no unstable modes are observed for  $\epsilon = 0.085 - 0.090$ . At  $\epsilon = 0.095$ , its frequency 2.257 is very close to the natural frequencies of the two resonant modes  $[1, 2, 1]^-$ ,  $[1, 3, 3]^+$  which are 2.277 and 2.241 (inviscid estimates) respectively. This mode was also obtained in the numerical simulations by Mason & Kerswell (1999) and in the experiments by Eloy et al. (2003). For  $\epsilon = 0.1$ , the most unstable mode is no longer mode (1), but the (H) mode (2) with a growth rate  $\sigma_2 = 0.0196 \pm 2.557i$ . The spectral structure reveals that it is composed of the harmonics  $[1, 3, 2]^+$ ,  $[2, 1, 2]^-$ ,

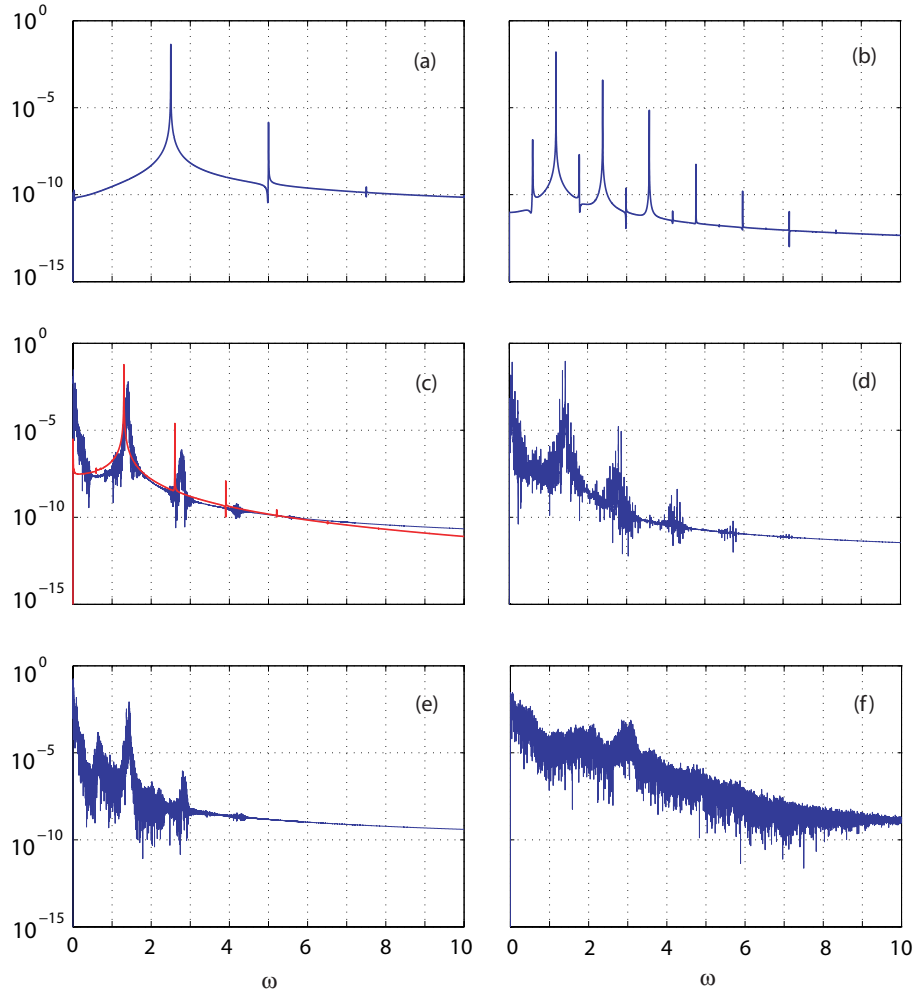


FIG. 3.5 – Frequency spectra of the fluctuations of the spinover mode amplitude for various eccentricities in a cylinder of aspect ratio  $h = 3.979$  and Ekman number  $E = 10^{-3}$ . (a)  $\epsilon = 0.1000$ , (b)  $\epsilon = 0.1050$ , (c)  $\epsilon = 0.1055$ , (d)  $\epsilon = 0.1060$ , (e)  $\epsilon = 0.1065$ , (f)  $\epsilon = 0.1200$ .

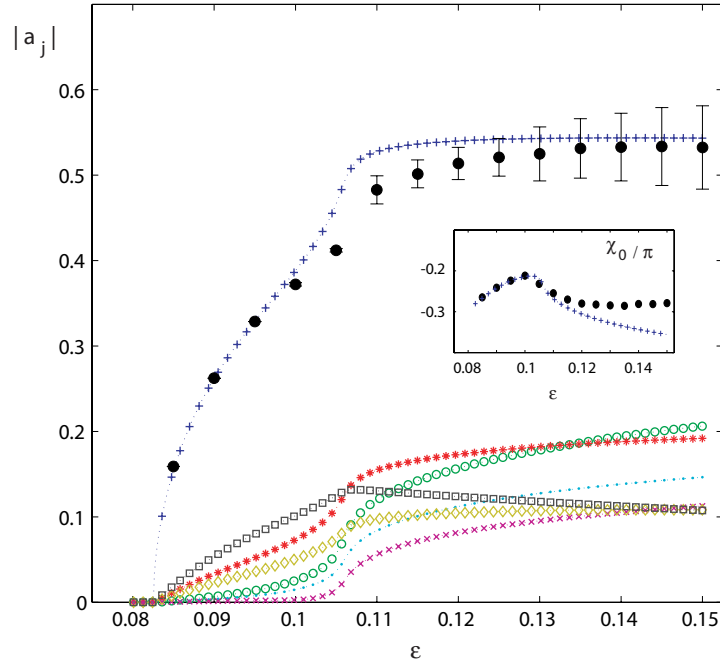


FIG. 3.6 – Amplitudes of the dominant modes defining the fixed point for  $h = 3.9796$  and  $E = 10^{-3}$ . The symbols are the spinover mode (+)  $[1, 1, 2]^+$ , and its harmonics ( $\square$ )  $[1, 0, 0]$ , ( $\diamond$ )  $[1, 2, 0]$ , ( $\circ$ )  $[1, 2, 4]^+$ , ( $*$ )  $[1, 0, 4]^+$ , ( $\cdot$ )  $[1, 3, 2]^+$ , ( $\times$ )  $[2, 2, 4]^+$ . The mean amplitude of the spinover mode, together with the standard deviation of the fluctuations are also indicated with the black dots and the error bars. The small panel compares the phase of the spinover mode amplitude (+), to the phase of the mean amplitude (dot), as a function of  $\epsilon$ .

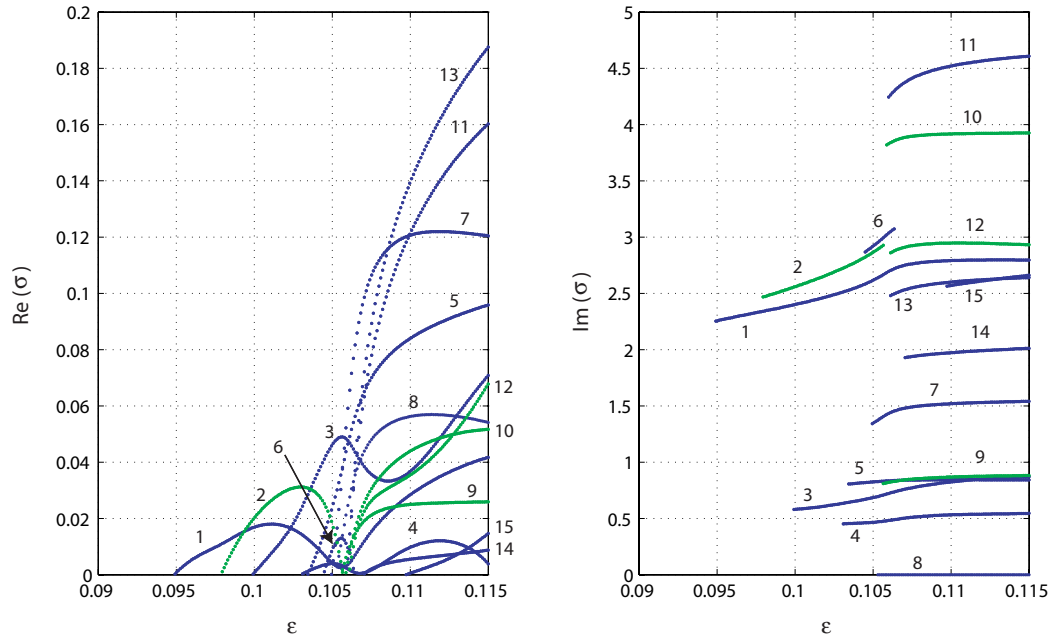


FIG. 3.7 – Secondary instability at the fixed point for  $h = 3.9796$ ,  $E = 10^{-3}$ . Growth rate  $Re(\sigma)$  and frequency  $Im(\sigma)$  of the first 15 unstable modes are plotted against eccentricity  $\epsilon$ . Green curves are used for modes involving spinover mode harmonics (H) in their spatial structure. Blue curves only involve non-harmonical modes (NH).

and  $[1, 2, 0]$ . The trace of this mode is visible on the time series of Fig. 3.4 : it corresponds to the oscillations of the harmonic  $[1, 2, 0]$ . The spinover and the other harmonics also exhibit oscillations but their amplitudes are smaller suggesting that they are slaved to the resonant triad. This is also suggested by the measure of the frequency  $\omega = 2.501$  of the small oscillations of the spinover mode, which is close to the frequency of mode (2). At  $\epsilon = 0.105$ , six modes are unstable. The most unstable (NH) mode (3) has a growth rate  $\sigma_3 = 0.0458 \pm 0.6828i$ . Its spatial structure is composed of several modes. The dominant components are (i)  $[1, 1, 1]^+$ , (ii)  $[2, 1, 3]^+$ , (iii)  $[1, 1, 3]^+$  and (iv)  $[1, 0, 1]^-$ . The spinover mode connects (i) & (iv), (ii) & (iv), and (iii) & (iv), whereas (ii) & (iii) are coupled by the elliptical deformation. This mode illustrates how complex the spatial structure of the secondary instability modes can become as  $\epsilon$  increases and shows the limitation of the analysis in terms of mode coupling. The mode (3) has been seen in the time series, and its frequency is transmitted to the spectra of the fluctuations of the spinover mode : it corresponds to the peak at  $\omega = 0.596$  in Fig. 3.5-(b). For larger  $\epsilon$ , there is a dramatic increase of the number of unstable modes, exactly 13 for  $\epsilon = 0.110$ , and all have large growth rates of the same order of magnitude as the elliptical growth rate,  $O(\epsilon)$ . This happens at the same time as the changes we previously observed in the fixed point equilibrium. These modes add several incommensurable frequencies in the flow and, following the Ruelle-Takens scenario, they are responsible of the transition to chaos in the time series (see Fig. 3.4). A more detailed survey of the spatial structure of all the other unstable modes is not often informative, as waves couple through both ellipticity and nonlinearity at the same time, but it helps to identify numerical artefacts which can appear due to limited resolution.

It is important to point out that this scenario does not lead here to the collapse of the flow. Even far from threshold at  $\epsilon = 0.15$ , the mean dynamics of the flow remains close to the unstable fixed point. This offers some interesting perspectives. If we define the mean (time average) amplitudes  $\bar{a}_j$  and fluctuating parts  $a'_j$ , it is possible to calculate a posteriori from the time series the correlation matrix  $C_{kl} = \overline{a'_k a'_l}$  of the fluctuating parts, and find a turbulent forcing  $f_j = \sum_{k,l} N_{jkl} C_{kl}$  of the mean. Adding this forcing a posteriori in the system (3.23) and searching for the fixed points, we recover the mean state with a very good precision. This is not surprising, but the dependency of the turbulent forcing on different parameters such as  $\epsilon$ ,  $E$  and wavenumbers, may provide valuable information for wave-turbulence modelling.

To further study the transition scenario, we have also tested various reduced models with increasing complexity. In one of the most interesting model, the dynamics is reduced to the spinover mode and its 42 harmonics. By doing so, we do not modify the characteristics of the fixed point but suppress (NH) secondary instabilities. What we have observed is that the chaotic transition is postponed to  $\epsilon \simeq 0.117$ , and is fully controlled by only three unstable modes as in the most simple Ruelle-Takens scenario.

### 3.5.2 Nonlinear dynamics of the spinover mode at small $E$ : limit cycle and collapse

At lower  $E$ , Eloy et al. (2003) observed a more violent nonlinear dynamics with growth-collapse-relaminarisation cycles (Fig. 3.2). Trying to reproduce such a dynamics, we have reduced the Ekman number to  $E = 2.5 \times 10^{-4}$  and changed the eccentricity from  $\epsilon = 0.04$  to 0.07. We have not modified the aspect ratio  $h = 3.9796$ . We have also used the same wave-package and kept the same small stationary forcing  $f = 10^{-5}$  as in the previous section. But, contrary to the previous case, we shall see that the weakly nonlinear dynamics does not stay close to fixed points but evolves on periodic orbits, which are perturbed and get intermittent through the action of secondary instabilities such as wave-triads. We first discuss some characteristic time-series, and then identify the fixed points and their stability. We then find a way to force the weakly nonlinear slaving of the harmonics to the spinover mode. This filters out all rapidly oscillating secondary instabilities, revealing the weakly nonlinear limit of the dynamics.

Typical time series are displayed in figure 3.8 together with their phase portraits in figure 3.9. For  $E = 2.510^{-4}$ , the linear instability threshold is at  $\epsilon_c = 0.03658$  for the spinover mode, and at  $\epsilon_c = 0.04524$  for the competing  $(0, 2, 1)$  mode. In Fig. 3.8-(a), we can observe how a secondary instability comes into play even close to the threshold. Unlike the previous case, the secondary instability does not saturate at low amplitude, but tears down the spinover mode amplitude, thereby stopping the secondary instability itself. Then, the spinover mode grows again and a secondary instability growth-relaminarisation cycle starts. For  $\epsilon = 0.044$ , the dynamics changes drastically (see Fig. 3.8-(b) and 3.9-(c)). The rapid secondary instability has disappeared and the spinover mode with its harmonics gets on a very regular limit cycle, with a low frequency  $\omega = 0.0105$ . The slowly oscillating pattern is found to be associated with the elliptic instability mode  $(0, 2, 1)$ , which is linearly stable for this eccentricity. Its appearance is probably due to the nonlinear modifications of the base flow by the spinover mode. This could explain the amplitude modulation of this mode as the spinover amplitude varies. The competition of both modes was also observed in Eloy's experiments and in the numerical simulation of Mason & Kerswell (1999). For  $\epsilon = 0.046$ , the limit cycle has reached a sufficiently large amplitude to be close to a

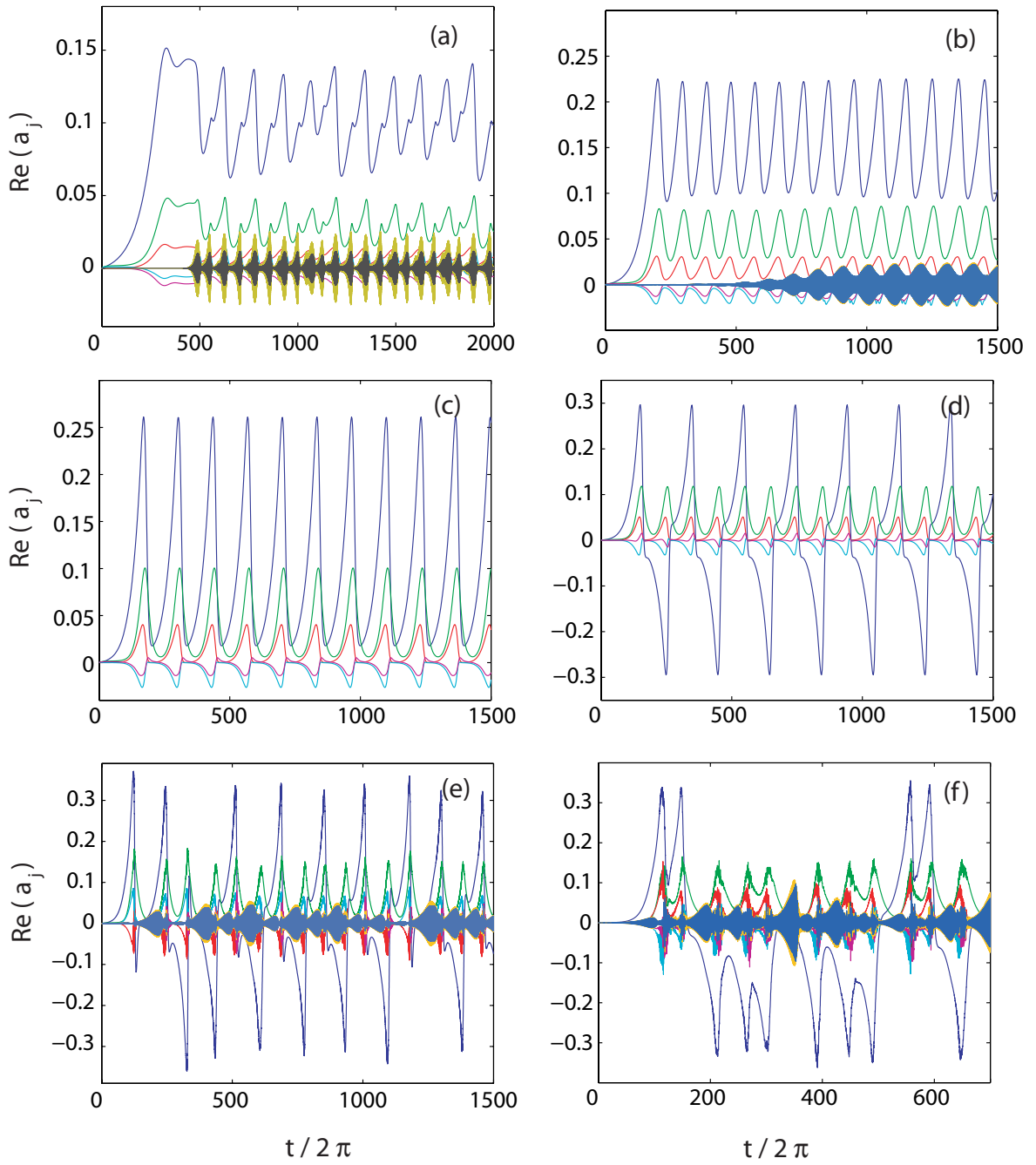


FIG. 3.8 – Temporal evolution of the real part of the amplitude of the dominant modes in a cylinder of aspect ratio  $h = 3.9796$  and Ekman number  $E = 2.5 \times 10^{-4}$  for different eccentricities  $\epsilon$ . (a)  $\epsilon = 0.04$ , (b)  $\epsilon = 0.044$ , (c)  $\epsilon = 0.046$ , (d)  $\epsilon = 0.048$ , (e)  $\epsilon = 0.052$ , (f)  $\epsilon = 0.06$ .

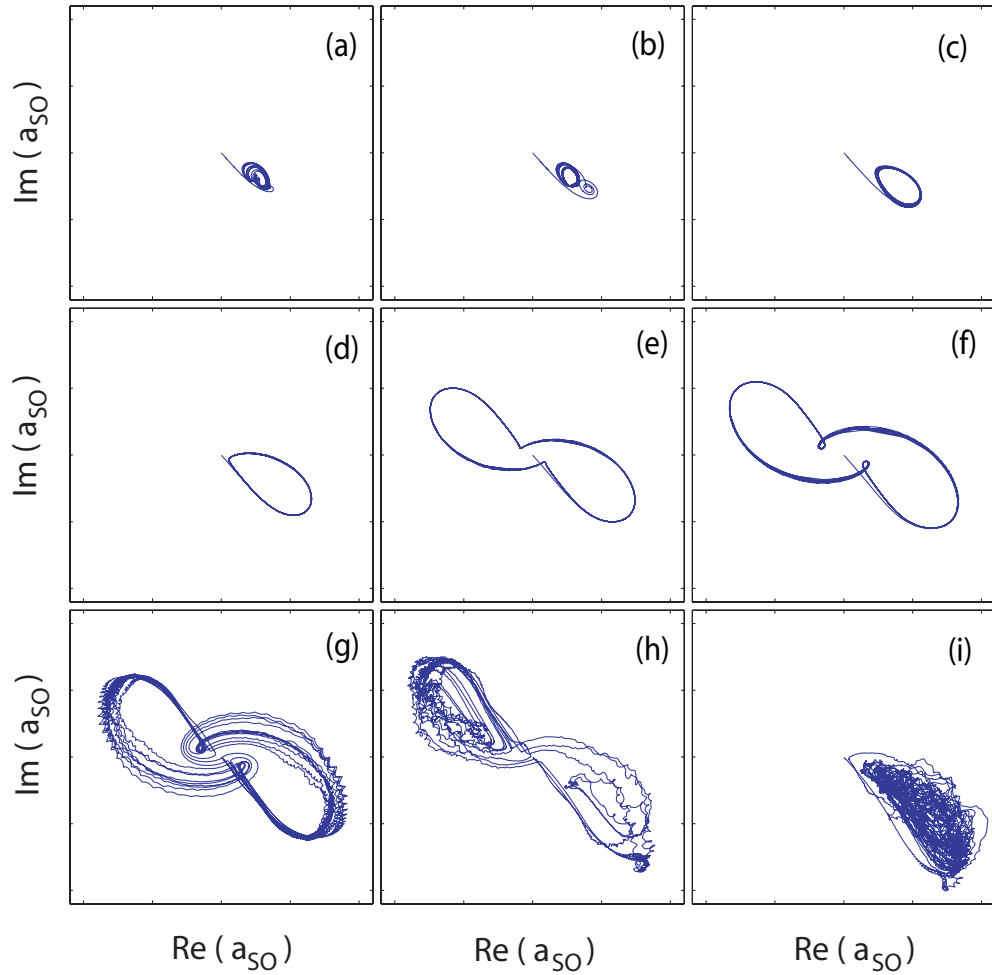


FIG. 3.9 – Phase portraits of the evolution of the complex spinover mode amplitude at different  $\epsilon$  in a cylinder of aspect ratio  $h = 3.9796$  and Ekman number  $E = 2.5 \times 10^{-4}$ . (a)  $\epsilon = 0.04$ , (b)  $\epsilon = 0.042$ , (c)  $\epsilon = 0.044$ , (d)  $\epsilon = 0.046$ , (e)  $\epsilon = 0.048$ , (f)  $\epsilon = 0.050$ , (g)  $\epsilon = 0.052$ , (h)  $\epsilon = 0.060$ , (i)  $\epsilon = 0.070$ .

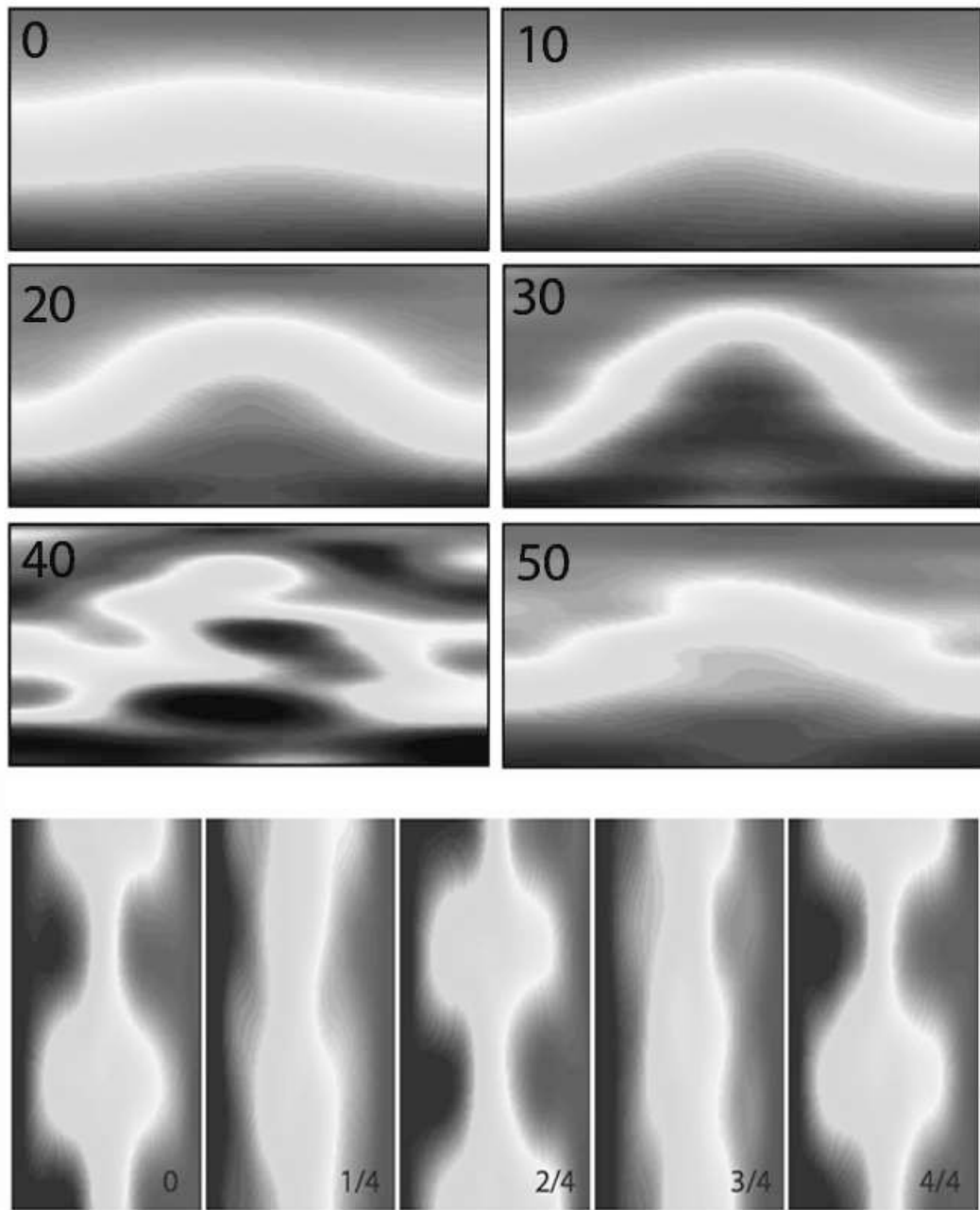


FIG. 3.10 – Azimuthal velocity field (magnitude in gray scale) in vertical planes along the stretching direction,  $\phi = -\pi/4$  for the flow parameters  $h = 3.9796$ ,  $E = 2.5 \times 10^{-4}$ ,  $\epsilon = 0.06$ . (Top) First growth-relaminarisation cycle. Each picture is separated by exactly 10 revolutions. (Bottom) The competing  $(0, 2, 1)$  coupling may become dominant during short periods. One revolution is visualised.

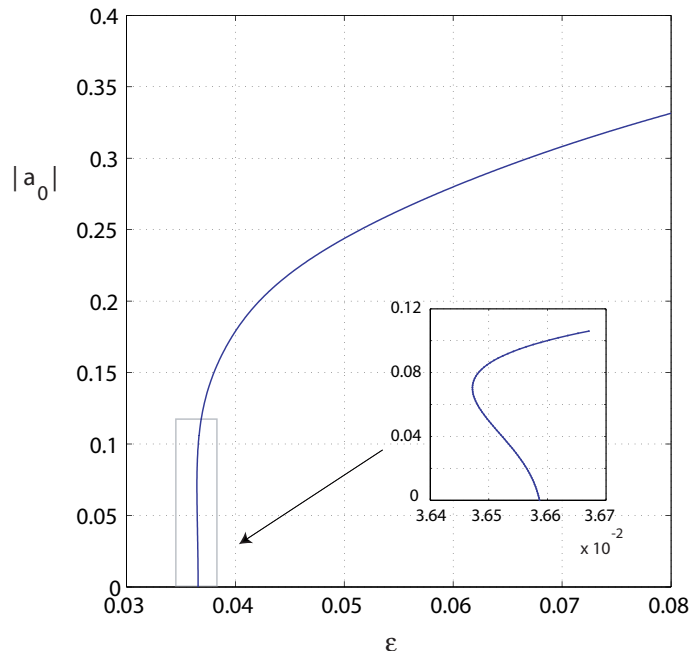


FIG. 3.11 – Amplitude of the spinover mode at the non trivial fixed points as a function of the eccentricity for the parameters  $h = 3.9796$ ,  $E = 2.5 \times 10^{-4}$ .

homoclinic cycle with the trivial fixed point at the origin (see Fig. 3.9-(d)). A homoclinic bifurcation occurs for a slightly larger eccentricity  $\epsilon_*$ . Above this critical value, the period of the limit cycle is doubled, and we observe regular reversals exploring both symmetric parts of phase space ( see Fig. 3.8-(d) and 3.9-(e-f)). Similar dynamics has been observed in the weakly nonlinear model of Racz & Scott (2008b), for the closely related parametric instability of the sinusoidal compression of a rotating cylinder of gas. During one cycle, growth takes  $O(100)$  rotations, which is much longer than the downfall only needing  $O(10)$  rotations. These cycles are similar to the weakly nonlinear non-viscous dynamics, [139], where growth and downfall happen on the same time-scales. However the asymmetry observed here is due to viscous damping. During these cycles, the trivial fixed point  $\mathbf{0}$  is closely approached on each turn. This makes the periodicity of the cycles very fragile as any noise generated by rapid secondary instabilities or by external sources, will render the nonlinear cycles intermittent. This is exactly what happens for  $\epsilon = 0.052$  (see Fig. 3.8-(e) and 3.9-(g)). Rapid oscillations are indeed observed during a brief moment of large spinover mode amplitude when the reversals start to be decorrelated. For  $\epsilon = 0.06$ , the same qualitative picture is obtained, but reversals are less regular (see Fig. 3.8-(f) and 3.9-(h)). Small scale fluctuations now accompany the collapse of the spinover mode amplitude. This is illustrated in Fig. 3.10-(a) where we visualise the first growth-collapse cycle of the time series shown in Fig. 3.8-(f). Competing linear instabilities involving the mode  $(0, 2, 1)$  also pop up during the dynamics [see Fig. 3.10-(b)]. For  $\epsilon = 0.070$  and further, almost no reversals are observed [see figure 3.9(i)]. The collapse gets more violent and relaminarisations are often only partial. If more modes are considered in the Galerkin decomposition, more small scales are observed during the collapse, but the main aspect of the emerging picture is conserved. However, for larger  $\epsilon$ , the model can blow up due to limited resolution.

As in the previous section, we have calculated, for each  $\epsilon$ , the characteristics of the “spin-over” fixed point. The evolution of the spin-over amplitude at the fixed point with respect to  $\epsilon$  is shown in Fig. 3.11. We can see that the bifurcation is slightly subcritical. Two non trivial points are indeed found in the interval  $0.03647 < \epsilon < 0.03660$ . This observation is very different from the previous example, and goes against the common idea that elliptical instability is always supercritical, although Malkus already suggested this possibility. Here, it explains why the spinover mode reaches large amplitudes close to the threshold, and why secondary instabilities are favored. We have calculated the characteristics of the secondary instability modes. They are provided in Fig. 3.12. The first mode (1), which is destabilised just above the linear threshold (see Fig. 3.12) does correspond to the secondary instability mode observed in Fig. 3.8-(a). It corresponds to a resonant triad of the modes  $[1, 2, 3]^-$  and  $[1, 3, 1]^+$



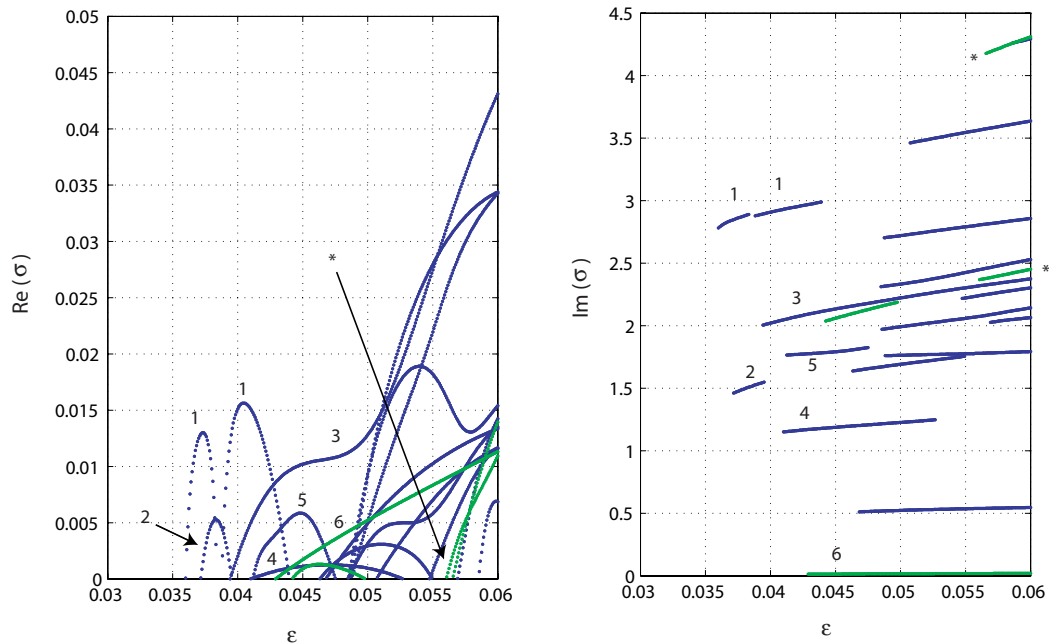


FIG. 3.12 – Secondary instability at the fixed point for  $h = 3.9796$ ,  $E = 2.5 \times 10^{-4}$ . Growth rate ( $Re(\sigma)$ ) and frequency ( $Im(\sigma)$ ) of the first 15 unstable modes versus the eccentricity  $\epsilon$ . Green curves are used for modes involving spinover mode harmonics (H) in their spatial structure. Blue curves only involve non-harmonical modes (NH).

with the spinover mode. Second and third modes are similar resonant triads with the spinover mode; the mode (2) involving  $[1, 1, 1]^-$  and  $[2, 2, 3]^+$ , and the mode (3) involving  $[1, 2, 1]^-$  and  $[1, 3, 3]^+$ . The third mode already appeared in the previous case at higher  $E$  and its spatial structure again contains a large contribution on the mode  $[1, 1, 3]^-$ . Close to the threshold we can compare the linear stability results with the time series. For  $\epsilon = 0.04$ , the mode (1) dominates, with a complex growth rate  $\sigma_1 = 0.0151 \pm 2.901i$ , and is clearly observed in the time series. Mode (2) is stabilised and mode (3) has a very low growth rate,  $\sigma_3 = 0.0020 \pm 2.0209i$ . For  $\epsilon = 0.042$ , the growth rates of the two unstable modes have changed:  $\sigma_1 = 0.0121 \pm 2.950i$  and  $\sigma_3 = 0.0074 \pm 2.07i$ . Moreover, there are two other unstable (NH) modes with  $\sigma_5 = 0.0029 \pm 1.771i$  and  $\sigma_4 = 0.0003 \pm 1.16i$ . This last mode corresponds to the elliptical instability mode  $(0, 2, 1)$  which is unstable thanks to the small stationary forcing (without forcing, this mode is marginally stable). For  $\epsilon = 0.044$ , mode (1) has become stable and is replaced by a new slowly oscillating (H) mode, labelled (6). This mode essentially involves the spinover mode (90%) and the geostrophic modes  $[1, 0, 0]$ ,  $[1, 2, 0]$  in its spatial decomposition. It has a very small growth rate  $\sigma_6 = 0.0009 \pm 0.0128i$ , but its frequency compares well with that observed in the slow oscillations of the time series. The limit cycle which has been observed in figure 3.9-(b) is associated with this mode. As soon as the cycle establishes, the dynamics never goes back to the fixed point anymore. Its linear stability properties thus become less relevant to analyse the subsequent dynamics.

In their weakly nonlinear analysis of the parametric instability in a sinusoidally compressed rotating flows Racz & Scott (2008b) recently found nonlinear cycles similar to ours. However, no such cycles were obtained in the weakly nonlinear analysis of Eloy et al. (2003) and Mason & Kerswell (1999). But in both studies, the system parameters were different. For this reason, it is worth addressing whether such a limit cycle could be obtained by a weakly nonlinear dynamics. In our approach, we can mimic the weakly nonlinear analysis by reducing the amplitude equations to those of the spinover mode and the axisymmetric geostrophic harmonics.

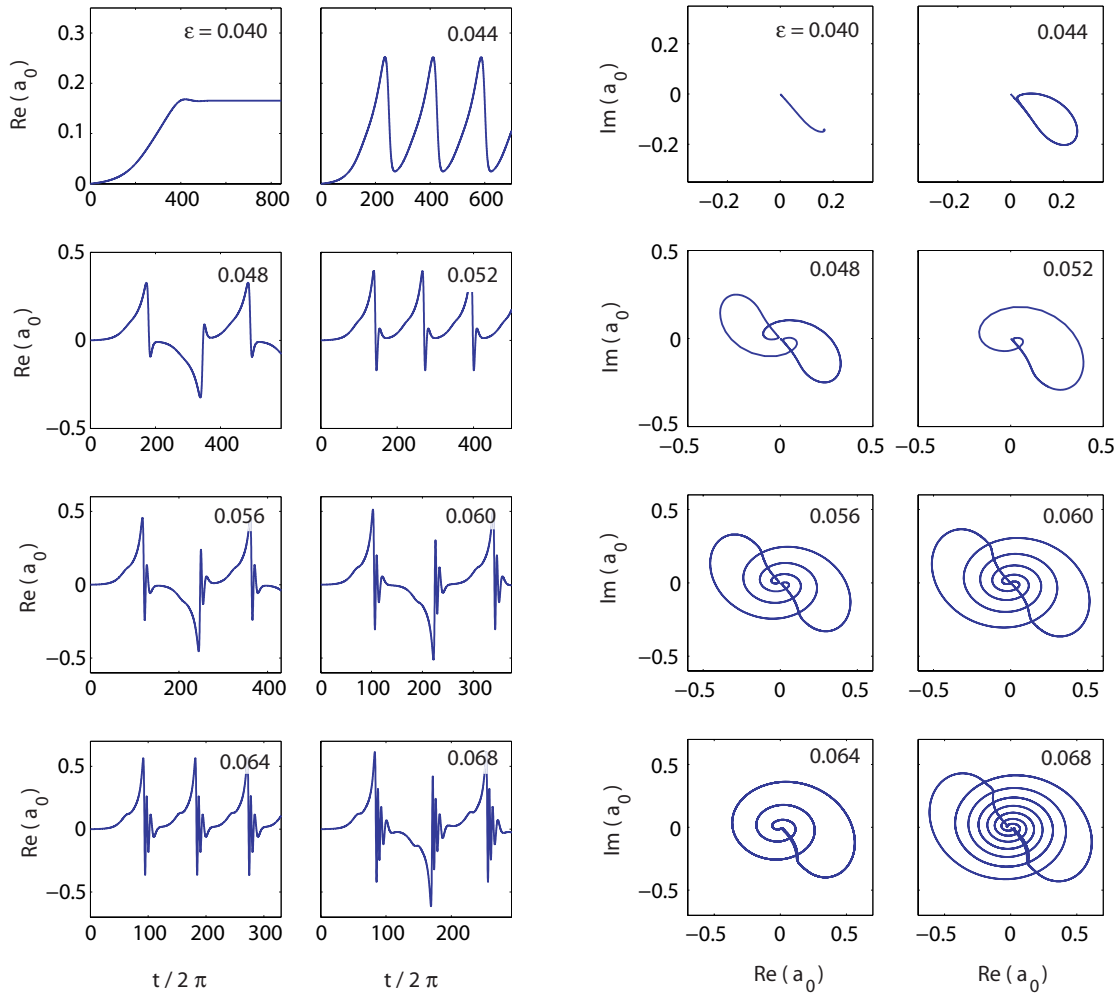


FIG. 3.13 – Weakly non linear evolution of the spinover mode for different  $\epsilon$ . Time series of the real part of spinover amplitude (left), and phase portraits of the spinover mode amplitude in the complex plane (right).

The other first order harmonics (labelled ( $j$ )) are indeed not resonantly excited and are slaved to the spinover mode (labelled (0)) via the relations :

$$a_j = \begin{cases} -i(N_{j00}^{(2)}/\omega_j) |a_0|^2 & , \quad m_j = 0 \\ -i(N_{j00}^{(1)}/\omega_j) a_0^2 & , \quad m_j = 2 \end{cases} \quad (3.32)$$

We can then use these expressions to calculate the elliptical and nonlinear retro-action of the slaved harmonics on the spinover mode and the axisymmetric geostrophic modes. All the higher order harmonics are neglected. This procedure filters out all rapid time-scales and reveals the underlying weakly nonlinear structure. In Fig. 3.13, we display the time series of the spinover mode and the corresponding phase portraits obtained by this procedure for different  $\epsilon$ . We can observe that for small  $\epsilon$  ( $\epsilon < 0.05$ ) the dynamics of the full system is well reproduced (compare Fig. 3.9 and 3.13). For large  $\epsilon$ , the slow amplitude growth is followed by a rapid oscillatory regime during which the dynamics spirals back to the trivial fixed point at the origin. The cause of this spiraling behavior is the generation of an important geostrophic flow which tends to advect the spinover mode away from the maximum stretching direction. Its strength increases with  $\epsilon$  together with the number of spinning. When the amplitude of the geostrophic modes has sufficiently decreased, the elliptical instability becomes active again, and a new linear growth of the spinover mode can start in one of the principal stretching directions  $-\pi/4$  or  $3\pi/4$ . By this mechanism, the dynamical system successively goes through homoclinic bifurcations, which alternates the dynamics between periodic non-reversing cycles and doubly-periodic reversing cycles. This behavior is very similar to what Racz & Scott (2008b) have also observed.

### 3.5.3 Nonlinear dynamics of an oscillating instability mode

So far, the analysis focused on the nonlinear dynamics of the spinover mode, which is only one of the possible unstable modes. If we change the aspect ratio to  $h = 3.0557$  and choose  $E = 4 \times 10^{-4}$  and  $\epsilon = 0.1$ , the spinover is no longer the most unstable mode. In that case, the dominant mode is the oscillatory mode  $(1, 3, 1)$ , which appears with  $3/2$  vertical wavelengths in the cylinder, in agreement with the prediction by Eloy et al. (2003). We have analysed the nonlinear dynamics of this mode using 299 modes. The wave package is larger than in the previous case because we have to consider the leading order harmonics of both modes involved in the elliptical mode  $(1, 3, 1)$ .

Fig. 3.14 shows the time evolution of the real part amplitude of the most important modes. The modes  $[1, 1, 3]^-$  and  $[1, 3, 3]^+$  of the elliptical coupling  $(1, 3, 1)$  fill the figure with rapid oscillations which grow in amplitude. The frequency of these oscillation is  $\omega \simeq 2$ . It is worth mentioning that the geostrophic flow is here nonlinearly forced at negative amplitude, which is in the opposite direction of the previous examples for the spinover mode. As for the nonlinear evolution of the spinover mode, the slow linear growth is followed by a rapid spiraling behaviour towards the trivial fixed point. Here we seen this through the phase reversals after the first down-fall. Closer to the threshold, the oscillatory instability mode has been found to saturate. But, for larger  $\epsilon$ , we have observed slow growth-collapse cycles and intermittency as in the previous section for the spinover mode.

It is interesting to see that the nonlinear dynamics of the oscillatory mode follows the same transition scenario as the spinover mode for small Ekman numbers. It is also possible that for higher  $E$ , we recover the same picture as for the spinover mode for moderate  $E$ . However, this is difficult to check because the oscillatory mode is less adapted to a detailed analysis of the fixed point and its stability than the spinover mode. This is due to a weak dependency with respect to  $\epsilon$  and  $E$  of the linear mode frequency, that was absent for the spinover mode, and which is not easy to filter out from the nonlinear dynamics.

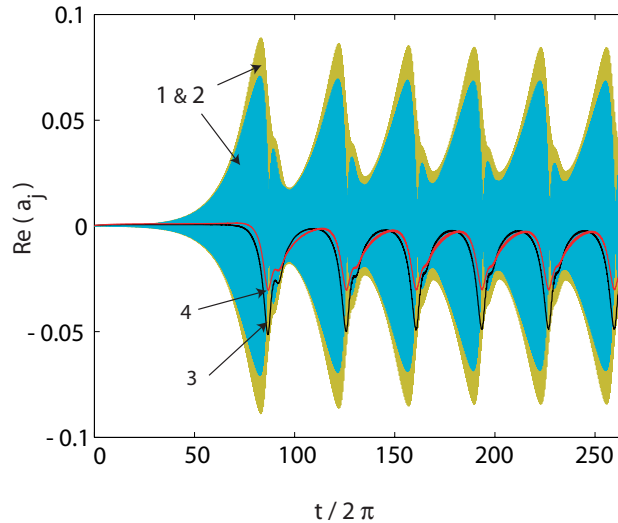


FIG. 3.14 – Temporal evolution of the real part of the amplitude of the dominant modes, (1)  $[1, 1, 3]^-$ , (2)  $[1, 3, 3]^+$  of the oscillatory elliptical coupling  $(1, 3, 1)$ , and nonlinearly excited geostrophic modes (3)  $[1, 0, 0]$  and (4)  $[1, 2, 0]$ , in a cylinder of aspect ratio  $h = 3.0557$  and Ekman number  $E = 4 \times 10^{-4}$  for  $\epsilon = 0.1$ .

### 3.6 Conclusion

In this chapter, we have presented a detailed study of the nonlinear evolution of the elliptical instability for a case where a body force has been added to correct the pressure terms. Several interesting results have been obtained from this analysis. We have recalled the role of the detuning on the nature of the bifurcation and demonstrated that the elliptical instability could become subcritical in the presence of viscous detuning. We have identified two different scenarios of transition. We have shown by analysing the spin-over mode close to the threshold that a Ruelle-Takens scenario of transition to chaos is active and that this transition does not lead to the inertial wave collapse. A different scenario of transition has been observed with the spinover mode for smaller Ekman numbers and for a oscillatory mode. In this scenario, weakly nonlinear limit cycles, similar to those Racz & Scott (2008b), get destabilised by secondary instabilities, and the flow goes to successive growth relaminarisation cycles. This scenario is probably closer to what could correspond to the inertial wave collapse. Those two scenarios have been observed in the experiments. For this reason, we are confident that the resolution of the equations deduced from (3.30), will give us qualitatively similar results. However, its analysis remains to be performed in order to do quantitative comparisons with the experiments.



## Chapitre 4

# Instabilité elliptique sous champ magnétique en géométrie cylindrique

Dans ce chapitre, nous discutons comment le champ magnétique peut servir pour détecter les écoulements excités par l'instabilité elliptique dans un cylindre. Nous calculons le champ magnétique induit par une onde inertielle et nous évaluons comment la force de Lorentz peut changer les ondes inertielles. Avec cette information nous trouvons la correction que le champ magnétique introduit sur le taux de croissance linéaire de l'instabilité elliptique sous champ magnétique. Ensuite nous présentons le dispositif expérimental IMAGINE. Nous montrons quelques résultats expérimentaux.

### 4.1 Introduction

In this chapter, we discuss how magnetic field induction can be used in cylindrical geometry, to study the nonlinear evolution of the elliptical instability. We extended the Malkus set-up to use liquid metals, and the column is rotating in an imposed vertical magnetic field, see Fig. 4.1. The inertial waves driven by the elliptical instability are accompanied by an induced field, that we measure outside the cylinder. This enables us to produce time-series which contain valuable information on both the linear and the nonlinear stages of the elliptical instability.

The principle of magnetic field induction is rather new, compared to the more classic experimental procedures. Experimental data for directly forced (precession - oscillating lid) or parametrically forced (elliptical instability) inertial wave systems are currently available in the form of Kalliroscope or reflective flake visualisations, [95], [88], [91], [92] [27], electrolytic dyeline experiments, [92], Laser Doppler Velocimetry, [64], [65], Particle Image Velocimetry, [27] [98], [73]. In the context of elliptical instability in cylinders, the linear theory has been successfully compared with experiments. Weakly nonlinear saturated states and secondary instabilities have also been described and observed. However, the question of the origin of the inertial wave-collapse stays open, despite the fact that all mentioned experimental studies report on this phenomenon. Magnetic field induction provides a unique way to study both long and short term behavior of the flow in a quantitative way. For this reason we first need to know the field induced by an inertial wave. Magnetic flow detection is not always non-intrusive, since the Lorentz force can add a supplementary non-negligible coupling if the imposed field is strong enough. In the following section, our aim will be to include this magnetic field effect in the theory of the elliptical instability. The third section of this chapter provides a description of the experimental set-up "IMAGINE", specifically designed to carry out magnetic field detection experiments. A small series of results are then presented, which demonstrate that magnetic field detection is a valuable experimental technique to study the nonlinear evolution of elliptical instability. The final section concludes and discusses some interesting perspectives of the IMAGINE experiment.

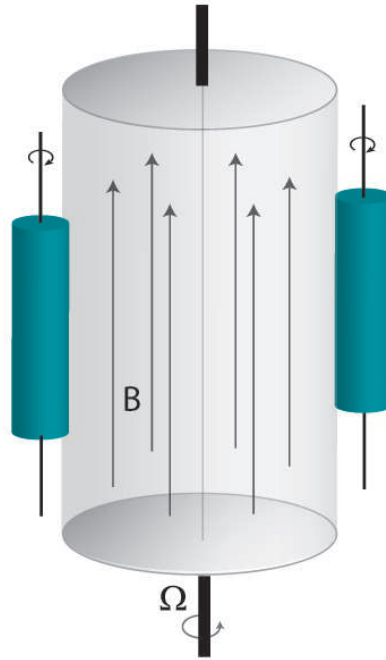


FIG. 4.1 – Elliptically deformed rotating column of fluid in the presence of a vertical magnetic field.

## 4.2 Theoretical analysis

### 4.2.1 Problem definition

The flow we study is based on Malkus' set-up, Fig. 4.1, and we use the same notations and scales as introduced in Ch. 3. We note the radius of the cylinder  $R$ , and its height  $H$ . The rotation speed is  $\Omega$  and  $\epsilon$  is the eccentricity of the elliptical deformation. The fluid is now a liquid metal, with density  $\rho$ , viscosity  $\nu$ , electrical conductivity  $\sigma_e$  and permeability  $\mu \simeq \mu_0$ . The elliptical deformation is created as before with a pair of opposed rollers. A homogenous magnetic field is imposed along the rotation axis  $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ , with a pair of Helmholtz coils. The two-dimensional elliptical base-flow does not interact with the imposed magnetic field, so that the base flow and the imposed magnetic field form an exact MHD base state. Our first goal is to understand how a strong magnetic field modifies the linear growth rates of elliptical instability.

In order to write down the perturbation equation in a convenient way, let us first look at some values of the non-dimensional parameters relevant to the MHD problem. Together with the aspect ratio  $h$  of the cylinder, the eccentricity  $\epsilon$  and the Ekman number  $E = \nu/\Omega R^2$ , the MHD problem depends on two additional non-dimensional parameters : the magnetic Ekman number  $E_m$  and the Elsasser number  $\Lambda$

$$E_m = (\sigma_e \mu_0 \Omega R^2)^{-1} \quad , \quad \Lambda = \frac{\sigma_e B_0^2}{\rho \Omega} . \quad (4.1)$$

$E_m$  specifies the ratio between the magnetic field diffusion and the magnetic field advection ; the Elsasser number measures the ratio of the Lorentz force to the Coriolis force. The spatial extend of our experimental equipment is limited, so that  $E_m$  is always superior to 100 : magnetic field diffusion is thus always dominant. It is also usefull to define the Lundquist number  $S = \Lambda E_m = \sigma_e \mu_0 \sqrt{B/\rho \mu_0} R$ . Here this parameter is  $O(0.1)$ . As it will be seen in Appendix C, the magnetic field  $B_0$  is therefore too small to admit Alfvén waves. In this double limit of small  $S$  and high  $E_m$ , the quasi-static description of the dynamics of the liquid metal applies ([18], p. 52). The convenient scale of the magnetic field perturbations is  $E_m^{-1} B_0$ . In the following analysis, we will assume that all the parameters  $\epsilon$ ,  $E_m^{-1}$ ,  $E$  and  $\Lambda$  are small.

Using the non-orthogonal elliptico-polar coordinate system of the previous chapter, the relevant equations for the perturbation fields,  $\mathbf{u}$ ,  $\mathbf{b}$ ,  $p$  inside the cylinder are then written, at leading orders, as

$$\begin{aligned} (\partial_t + \partial_\phi)\mathbf{u} + 2\hat{\mathbf{z}} \times \mathbf{u} + \nabla p &= \epsilon \mathbf{E}_1 \nabla p + E \Delta \mathbf{u} + \Lambda \partial_z \mathbf{b} + \dots \\ \Delta \mathbf{b} + \partial_z \mathbf{u} &= 0 + \dots \end{aligned} \quad \mathbf{r} \in V_i. \quad (4.2)$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0$$

On the left-hand side of the equations, we have gathered the dominant  $O(1)$  terms. On the right-hand side, we can identify the perturbation terms associated with the non-viscous elliptical coupling which acts through the pressure term, the viscous term and the Lorentz-force correction. We have neglected all the other terms, which can only add higher order contributions, such as the elliptico-viscous terms of order  $\epsilon E$ , or the elliptical deformation of the magnetic field. The perturbation terms in the second equation of (4.2) have not been written down because as it will be seen below they do not intervene in the dynamics of the elliptical instability. The velocity field satisfies a no-slip boundary condition on the surface  $\delta V$  of the cylinder,  $\mathbf{u} = 0|_{\delta V}$ . The magnetic field perturbations in  $V_i$  must match an external potential field :

$$\Delta \Phi = 0 \quad , \quad \mathbf{b} = \nabla \Phi \quad , \quad \mathbf{r} \in V_e. \quad (4.3)$$

For a finite cylinder, the condition of matching of the magnetic field with a 3D bounded potential field creates some difficulties, which requires the use of boundary element or boundary integral methods [48], [143], [142]. In order to avoid this difficulty, we shall limit the exterior region  $V_e$  to a layer  $z \in [0, h]$ , by adding infinite magnetic permeability boundary plates at  $z = 0$  and  $h$ . The boundary condition, will then be that the magnetic field is normal to these surfaces,  $\hat{\mathbf{z}} \times \mathbf{b} = 0|_{z=0,h}$ . This domain reduction permits to express the external potential flow as waves along the axial direction of the same form as inside the cylinder. The boundary condition  $\mathbf{b}|_{s=1-} = \mathbf{b}|_{s=1+}$  can then be implemented analytically. In the following we will shortnote (4.2) and (4.3) formally as

$$\mathcal{L} \mathbf{X} = \epsilon \mathcal{D} \mathbf{X} + E \mathcal{V} \mathbf{X} + \Lambda \mathcal{M} \mathbf{X}, \quad (4.4)$$

with  $\mathbf{X} = [\mathbf{u}, p, \mathbf{b}, \Phi]^T$ . The operator  $\mathcal{L}$  contains the left-hand side of (4.2) and (4.3).  $\mathcal{D}$  is the elliptical coupling,  $\mathcal{V}$  the viscous operator, and  $\mathcal{M}$  the Lorentz force operator.

We shall start the asymptotic theory by solving the unperturbed problem  $\mathcal{L} \mathbf{X} = 0$ . Here we shall also consider the case  $\Lambda = O(1)$  where the Lorentz force retroacts on the flow at leading order, in order to find how the Lorentz force changes the global modes of the system, and why this is a dynamically irrelevant limit in the case of elliptical instability at small  $\epsilon$ .

### 4.2.2 Magnetic field induced by an inertial wave & Lorentz force retroaction

Consider solutions of the form  $\mathbf{X} \sim e^{i\omega t}$  of the linear problem  $\mathcal{L} \mathbf{X} = 0$ . The hydrodynamical equation is decoupled from the induction equation and the solutions to this problem are the inertial or Kelvin waves [53] of structure (see also Ch. 2)

$$\begin{bmatrix} u_\pm \\ u_z \\ p \end{bmatrix} = A \begin{bmatrix} (2 \mp \lambda) / \sqrt{2} & J_{m\pm 1}(ks) \cos(lz) \\ k\lambda/l & J_m(ks) \sin(lz) \\ i k \lambda^2 / l^2 & J_m(ks) \cos(lz) \end{bmatrix} e^{i(m\phi + \omega t)}. \quad (4.5)$$

$\lambda = m + \omega$  is the rotating frame frequency, and  $k$ ,  $m$  and  $l$ , the radial, azimuthal and vertical wavenumber respectively. The dispersion relation of the Kelvin waves fixes  $\lambda$  as a function of the other parameters. Using  $\pm$ -components for the fields, which diagonalise the Laplacian operator, and due to the presence of the infinite permeability plates at  $z = 0$  and  $h$ , there is no difficulty to calculate respectively the interior and exterior magnetic field solution explicitly

$$\begin{aligned} \begin{bmatrix} b_\pm \\ b_z \end{bmatrix} &= \frac{Al}{k^2 + l^2} \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} (2 \mp \lambda) & J_{m\pm 1}(ks) & \sin(lz) \\ -k\lambda/l & J_m(ks) & \cos(lz) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} d_\pm & I_{m\pm 1}(ls) & \sin(lz) \\ \frac{1}{\sqrt{2}} (d_+ + d_-) & I_m(ls) & \cos(lz) \end{bmatrix} \right\} e^{i(m\phi + \omega t)} \quad , \quad \mathbf{r} \in V_i \end{aligned} \quad (4.6)$$

$$\begin{bmatrix} \Phi \\ b_\pm \\ b_z \end{bmatrix} = \frac{Al}{k^2 + l^2} c \begin{bmatrix} \frac{1}{l} & K_m(lr) & \sin(lz) \\ -\frac{1}{\sqrt{2}} & K_{m\pm 1}(lr) & \sin(lz) \\ & K_m(lr) & \cos(lz) \end{bmatrix} e^{i(m\phi + \omega t)} \quad , \quad \mathbf{r} \in V_e$$



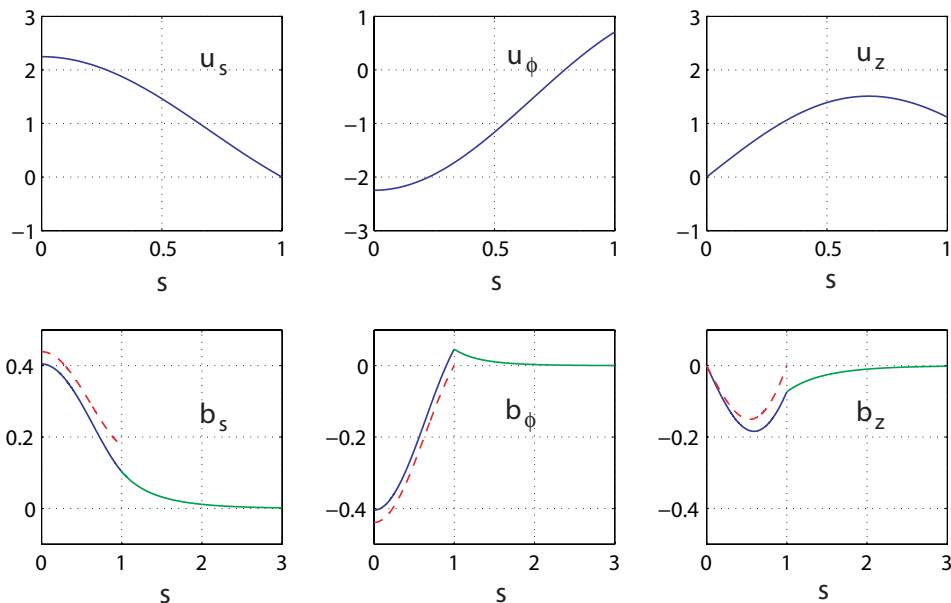


FIG. 4.2 – Radial structure of the velocity field  $\mathbf{u}$  of the spinover mode and induced magnetic field  $\mathbf{b}$ . The effect of the boundary condition : (full) potential field at the exterior, (dashed) infinite permeability at container surface. Profiles are similar but the differences stretch out to the entire fluid domain.

The induced magnetic field in the cylinder is composed of two parts : a first part is proportional to  $\partial_z \mathbf{u}$  and a second part is necessary to match with the external potential field. The fact that it is composed of modified Bessel functions indicates that this part of the field mainly adds contributions close to the lateral surface of the cylinder. The three constants  $c$  and  $d_{\pm}$ , are determined by matching interior and exterior solutions at  $r = 1$ . Note the factor  $Al(k^2 + l^2)^{-1}$  in front of the total induced magnetic field : inertial waves with finer structure induce less field, a physical manifestation of the increased diffusion of smaller-scaled structures. Due to the idealised ferromagnetic boundary conditions on top and bottom plates, the vertical structure of the velocity and magnetic field components are in exact quadrature. The azimuthal structures of  $\mathbf{u}$  and  $\mathbf{b}$  are by contrast in phase. In Fig. 4.2, we have plotted the radial structures (full lines) of  $\mathbf{u}$  and  $\mathbf{b}$  of the spinover mode  $(k, m, l, \omega) = (2.73, 1, 1.58, 0)$ . The amplitude is such that the kinetic energy of the spinover mode is identical to that of the solid-body rotation. Only the radial component of the velocity field cancels at  $s = 1$  as expected from the non-viscous boundary condition. Two cases have been considered for the magnetic field, corresponding to two different boundary conditions. It can be remarked that the magnetic field has similar profiles in the cylinder for both cases. In the following, we will note  $\mathbf{Q} = [\mathbf{u}, p, \mathbf{b}, \Phi]^T(\mathbf{r})$  the spatial structure of the wave defined in (4.5) and (4.6)

When  $\Lambda = O(1)$ , the hydrodynamics is no longer decoupled from the induction equation. We can still solve this problem analytically in the non-viscous limit, using similar techniques as those used for the viscous eigenmodes. The radial structures of the fields could *a priori* involve more than one Bessel function, but at the end, when boundary conditions are applied to the field expression, one obtains that these additional functions generally cancel out. It turns out that the spatial structure of the velocity and magnetic field perturbations is identical to the ones above for both the inertial wave and its magnetic field. The only modification appears in the eigenfrequency  $\tilde{\lambda}$  which becomes complex valued and related to the non-viscous frequency  $\lambda$  of the inertial waves as follows :

$$\tilde{\lambda} = \lambda + i \frac{\Lambda l^2}{k^2 + l^2} = \lambda + i \Lambda \frac{\lambda^2}{4} \quad (4.7)$$

The above formula, has been obtained in the previous chapter ( eq. (??) for  $\lambda = 1$ ) for the particular case of elliptically coupled waves. It is interesting to note that the magnetic damping is entirely controlled by the ratio  $k/l$ , and that the smaller the radial structures get, the lesser they are damped. On the contrary, damping increases when the vertical structure of the flow is smaller. This anisotropy of Joule damping is typical of flows

described by the quasi-static limit of MHD, and it clearly differs from the isotropic viscous volume damping  $E(k^2 + l^2)$ . Expression (4.7) shows that when  $\Lambda = O(1)$ , the modes are generally strongly damped. This has an important consequence concerning the elliptical instability. The elliptic instability will be active only if  $\Lambda$  is small and at most of order  $\epsilon$ . This justifies the hypothesis of small  $\Lambda$  we made above.

The solution we have derived satisfies non-viscous boundary conditions and supposes ideal ferromagnetic end-plates. In reality, we will work with a viscous fluid, and the soft iron endplates are not in immediate contact with the fluid. We need to study if this influences the problem significantly. Some inspection learns that the viscous boundary layers will not be affected by the magnetic field if  $\Lambda \ll 1$ . Neither do the viscous boundaries add significant contributions to the induced magnetic field. With purely viscous boundary layers, this means that we can still use the asymptotical estimates for the surface damping of the flow [140], [66] (see also sec. 2.3.1).

We have seen in Fig. 4.2 the effect of the radial conditions in the structure of the magnetic field, and how the differences spread out in the cylinders interior. Modifying top and bottom boundary conditions is expected to have a similar effect on the external structure of each mode in the cylinder. However, this difference is difficult to quantify because as already mentioned above, as soon as the external region is of infinite extend in the radial and axial directions, the potential field has a complex structure which cannot be matched to a single sinuous mode.

### 4.2.3 Magnetic field effects on the elliptical instability

The hydrodynamical theory of elliptical instability in the Malkus set-up is well established, [138], [26]. It is known that the instability can be described as a parametric instability of inertial waves, where the elliptical deformation is responsible for the inertial wave coupling. Viscous corrections and geometrical detuning add independent corrections to the non-viscous growth rate formula. We shall see below that we will also be able to add the magnetic damping contribution to the growth rate formula. Before carrying this analysis, we need to check that the elliptical deformation of the magnetic field does not add new couplings.

In the previous section, we have discussed the unperturbed problem, which provides a family of solutions  $\mathbf{Q}_j e^{i\omega_j t}$ , for the inertial waves and their induced fields. In order to analyse the coupling between such waves, we search for an asymptotic solution of the system (4.4) of the form :

$$\mathbf{X} = (A_1 \mathbf{Q}_1 + A_2 \mathbf{Q}_2) e^{i\bar{\omega}t + \epsilon\sigma' t} + \epsilon \mathbf{Y} + \dots \quad (4.8)$$

where  $A_j$  are the amplitudes of a two fields  $\mathbf{Q}_j$ , and  $\bar{\omega} = (\omega_1 + \omega_2)/2$  the mean frequency of the two modes and  $\sigma = \epsilon\sigma'$  the growth rate. Following the parametric resonance idea, two waves of wavenumbers and frequency  $(k_j, m_j, l_j, \omega_j)$ ,  $j = 1, 2$ , can be coupled through the elliptical flow of characteristics  $(k, m, l, \omega) = (\cdot, \pm 2, 0, 0)$ , if the following resonance conditions are fulfilled :

$$m_1 + 2 = m_2 \quad , \quad l_1 = l_2 \quad , \quad \omega_1 = \bar{\omega} + \delta \quad , \quad \omega_2 = \bar{\omega} - \delta \quad (4.9)$$

We have noted  $\delta = \epsilon\delta'$  the frequency detuning. This detuning has to be small for the coupling to be destabilising. Owing to the dispersion relation of the Kelvin modes, this filters out a discrete number of interesting paires of modes. Eloy ?? noticed that the additional radial constraint  $k_1 \simeq k_2$  systematically produces couplings with higher growth rates. paires of modes satisfying this additional constraint are called principal modes and noted  $(m, m + 2, x)$ , with  $x$  the radial label of the inertial waves. Wavenumber-frequency values  $(l_*, \omega_*)$  of principal modes are reported in Eloy et al. [26]. In cylinders with fixed aspect ratios, this means that resonance conditions are only perfectly met when  $h_* = n\pi/l_*$  with  $n \in \mathbb{N}$  the number of half wavelengths in the vertical structure of the mode. The larger  $\epsilon$  is, the larger the interval around  $h_*$  is for which elliptical instability can still appear. We propose the asymptotic ansatz as a solution and further write  $E' = E/\epsilon$ ,  $\Lambda' = \Lambda/\epsilon$ , to reduce the number of parameters relevant in the asymptotical series. At first order in  $\epsilon$ , we obtain :

$$\begin{aligned} \mathcal{L} \mathbf{Y} + ((\sigma - i\delta)A_1 \mathbf{Q}_1 - (\sigma + i\delta)A_2 \mathbf{Q}_2) e^{i\bar{\omega}t} \\ = (\mathcal{D} + E' \mathcal{V} + \Lambda' \mathcal{M})(A_1 \mathbf{Q}_1 + A_2 \mathbf{Q}_2) e^{i\bar{\omega}t} \end{aligned} \quad (4.10)$$

For modes which satisfy the resonance conditions, the linear problem  $\mathcal{L} \mathbf{Y}$  is secularly forced, and for this system of equations to be solvable, the slow growth rate  $\sigma'$  is only allowed to have well determined values. We need a scalar product to pose the solvability condition, for example

$$\langle \mathbf{Q}_1 | \mathbf{Q}_2 \rangle = \int_{V_i} (\mathbf{u}_1^* \cdot \mathbf{u}_2 + p_1^* p_2 + \mathbf{b}_{i,1}^* \cdot \mathbf{b}_{i,2}) dV + \int_{V_e} \Phi_1^* \Phi_2 dV \quad (4.11)$$

A priori, the choice of the scalar product is arbitrary, but it is advised to choose it in a way that the adjoint problem is simple. The adjoint problem is defined by

$$\langle \mathbf{Q}^A | \mathcal{L} \mathbf{Q} \rangle = \langle \mathcal{L}^A \mathbf{Q}^A | \mathbf{Q} \rangle + \text{BT} \quad (4.12)$$

By partial integration, we find the adjoint operator  $\mathcal{L}^A$  and adjoint boundary conditions are such that the boundary terms BT are zero. The adjoint modes  $\mathbf{Q}^A$  satisfy the problem  $\mathcal{L}^A \mathbf{Q}^A = 0$  together with adjoint boundary conditions. More details on this procedure are provided in the Sec. 7.3.2. We find that

$$\mathbf{u}^A = \mathbf{u} \quad , \quad \mathbf{b}^A = 0 \quad , \quad \Phi^A = 0 \quad . \quad (4.13)$$

The hydrodynamic field is thus self-adjoint as in the pure hydrodynamical case, and the magnetic field part of the adjoint modes is zero. This implies that interactions are not possible via the magnetic field and explains why the correction terms in the second equation of (4.2) have been discarded. This particularity finds its origin in the fact that hydrodynamics is decoupled from the magnetics in the quasi-static limit of MHD : the magnetic field is slaved to the velocity field, and adds no new time-scale to the problem. The solvability condition of the elliptical problem is obtained by multiplying (4.10) by both  $\mathbf{Q}^{1*}$  and  $\mathbf{Q}^{2*}$ , and taking the integral over the volume. This leads to the algebraic system of equations

$$\begin{aligned} (\sigma - i\delta + d_1^M + d_1^V) A_1 &= \epsilon E_{12} A_2 \\ (\sigma + i\delta + d_2^M + d_2^V) A_2 &= \epsilon E_{21} A_1 \end{aligned} \quad (4.14)$$

The elliptical matrix elements are identical to the non-viscous values found by Waleffe [138]. With the inertial waves defined by (4.5) and we find them here as  $E_{12} = e_{12}/n_1$  and  $E_{21} = e_{21}/n_2$ , with

$$e_{12} = +\frac{i}{2} (2 - \lambda_1) (4 - \lambda_2^2) I \quad (4.15)$$

$$e_{21} = -\frac{i}{2} (2 + \lambda_2) (4 - \lambda_1^2) I \quad (4.16)$$

$$n_1 = \frac{2}{k_1} (2 + \lambda_1) J_m(k_1) J_{m-1}(k_1) + 4 \left( J_m^2(k_1) - J_{m+1}(k_1) J_{m-1}(k_1) \right) \quad (4.17)$$

$$n_2 = \frac{2}{k_2} (2 + \lambda_2) J_{m+2}(k_2) J_{m+1}(k_2) + 4 \left( J_{m+2}^2(k_2) - J_{m+3}(k_2) J_{m+1}(k_2) \right) \quad (4.18)$$

and

$$I = \frac{1}{k_2^2 - k_1^2} \left( k_1 J_m(k_1) J_{m+1}(k_2) - k_2 J_{m+1}(k_1) J_m(k_2) \right) \quad k_1 \neq k_2 \quad (4.19)$$

$$= \frac{1}{2} \left( J_m(k_1)^2 - J_{m+1}(k_1) J_{m-1}(k_1) \right) \quad k_1 = k_2 \quad (4.20)$$

The viscous corrections  $d_j^V$  in (4.14), contain a surface contribution and the volumic damping. The  $O(E)$  volume damping, seems to be of lower order than the surface term, but it actually quickly dominates the surfacic damping for modes with small-scale structures, as we saw in Ch. 2. The  $O(E^{1/2})$  surface damping is due to boundary layer pumping. Computing the correction associated with this pumping leads to the asymptotic estimate of Wedemeyer, but as Eloy [26] we prefer to use the Kudlick-estimate [66], which is slightly better. The viscous term can be written down explicitly :

$$\begin{aligned} d^V &= E^{1/2} \frac{4-\lambda^2}{4\sqrt{2}(m^2+l^2-m\lambda/2)} \left[ (1+i) \lambda^{1/2} (m^2 + l^2) \right. \\ &\quad \left. + \frac{(1+i)}{h} \frac{(2-\lambda)}{\sqrt{2+\lambda}} \left( m^2 + l^2 - \frac{2m\lambda}{2-\lambda} \right) + \frac{(1-i)}{h} \frac{(2+\lambda)}{\sqrt{2-\lambda}} \left( m^2 + l^2 - \frac{2m\lambda}{2+\lambda} \right) \right] \\ &\quad + E (k^2 + l^2) \end{aligned} \quad (4.21)$$

The magnetic terms  $d_j^M$  are produced by the terms  $\mathcal{M}$  of (4.10). They are associated with the magnetic damping computed in the previous section (see 4.7). We obtain :

$$d^M = \Lambda \frac{\lambda^2}{4} \quad (4.22)$$

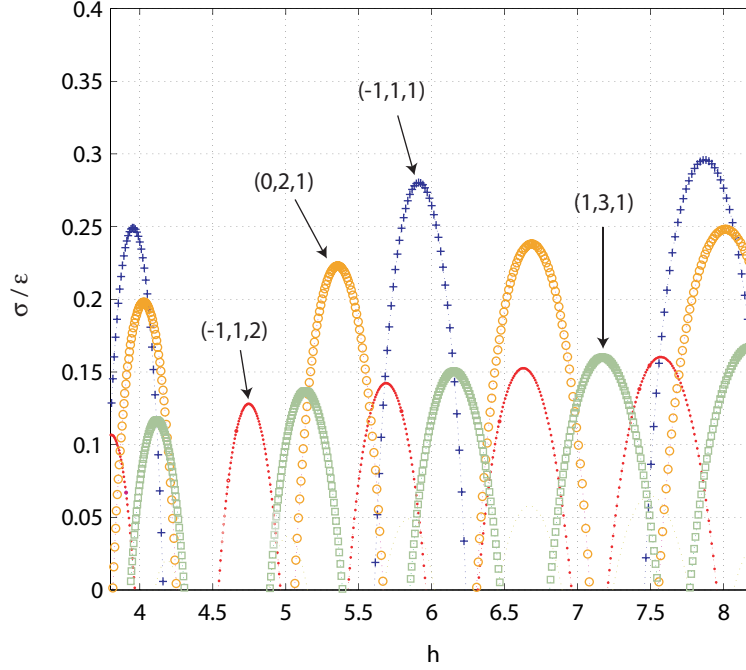


FIG. 4.3 – Growth rate of elliptical instability modes  $(m, m + 2, x)$  in a cylinder as a function of the aspect ratio  $h$ , for  $\epsilon = 0.1$ ,  $E = 5 \times 10^{-4}$ . In the presence of a magnetic field, Joule damping uniformly lowers the growth rates of all elliptically unstable couplings by an amount of  $-\Lambda/4$ .

The elliptical matrix elements vary slightly with detuning. Yet they can be calculated at perfect resonance, since the  $O(\delta)$  corrections in the spatial profiles, induced by detuning appear as higher order corrections in the asymptotical theory. The growth rate is finally given by a formula of the form :

$$\sigma = -\frac{d_1 + d_2}{2} + \sqrt{\epsilon^2 E_{12} E_{21} - \left(\delta + i\frac{d_1 - d_2}{2}\right)^2}, \quad d_j = d_j^M + d_j^V \quad (4.23)$$

This formula can be used to produce linear phase diagrams for the elliptical instability. Fig. 4.3 provides an example of such diagrams for  $\epsilon = 0.1$ ,  $E = 5 \times 10^{-4}$ . Without magnetic field, the growth rates of the different set, the spinover mode  $(-1, 1, 1)$  labeled (+) in Fig. 4.3, produces the largest resonance fingers centered around the aspect ratios  $h_* = n\pi/1.58 \simeq 2n$ , with  $n$  the number of half vertical wavelengths. The other most important modes are  $(\circ)$ ,  $(0, 2, 1)$ , centered around  $h_* = n\pi/2.33 \simeq 1.35n$ ,  $(\square)$ ,  $(1, 3, 1)$  around  $h_* = n\pi/3.04 \simeq n$  and  $(\bullet)$ ,  $(-1, 1, 2)$  around  $h_* = n\pi/3.29 \simeq 0.95n$ . In presence of an imposed magnetic field, the mean magnetic damping is exactly  $(d_1^M + d_2^M) = \Lambda/4$ , for all modes in the diagram. Therefore, Joule damping does **not** change the relative positions of the resonance fingers in the phase diagram, and just translates the picture of Fig. 4.3 over a distance  $\Lambda/4\epsilon$ . For each resonant coupling, there exists a critical Elsasser number  $\Lambda_c$ , above which it becomes stable.

The elliptical instability threshold can then be modified by varying the imposed magnetic field. Increasing  $B_0$  is similar to decreasing  $\epsilon$ . From an experimental point of view, this new control parameter is much easier to vary.

### 4.3 The IMAGINE set-up : experiments

The experimental set-up called IMAGINE for "Instabilités MAGnéto INertielles" is based on the original idea of Malkus, and is sketched in Fig. 4.4. This set-up was specifically designed to study the magnetic field effects.

#### 4.3.1 Mechanical details, Materials & The imposed field.

The deformable cylindrical container (1) in Fig. 4.4 has an inner radius  $R = 25 \pm 0.5 \text{ mm}$ . The container is composed of a rigid PVC tube on the exterior, the same as those used by Eloy,[27] to guarantee its rigidity. A thin cylindrical silicone sock of thickness  $2 \text{ mm}$  has also been added to avoid leaks. We moulded this sock by extrusion using the PVC tubes as a mould, which can be easily cut away after casting of the silicone. Uniform thickness of the silicone was assured by parallel guiding along the revolution axis. The parallel guiding leaves a pinned  $3 \text{ mm}$  thick silicone disk at the bottom of the sock, which we use as an anti-leaking shaft : a Teflon disk quenches the silicone disk, and rigidly fixes the sock to the bottom piece which is clamped to the motor shaft. The exterior surface of the silicone sock needs to be coated with a thin paper, or a dry lubricant to be able to slide into the exterior PVC cylinder, which is then also attached on the bottom piece. The cylindrical liquid metal domain is closed on top by a teflon plug resting on the sock, and the height of the cylinder can be adjusted. We mount the cylinder in between two Helmholtz coils ((3) in fig. 4.4) fixed to the two parallel soft-iron plates ((4) in fig. 4.4). The rotation axis of the bottom piece, turns freely in a set of ceramic bearings (to avoid any induction by rotating metal bearings) rigidly mounted on the iron plates. The ferromagnetic cavity is closed back again by a small ferromagnetic cylinder which is fixed to the iron plate. The elliptical deformation is imposed with two opposed Teflon rollers ( Fig. 4.4-(2) ) mounted on ceramic bearings. Contrarily to Eloy's set-up, the elliptical deformation can be imposed after the fluid has been spun up to rigid rotation. The ensemble is fixed to a granitic table, under which a 600 W AC-motor (Motovario) is clamped on the rotation axis. The rotation speed is digitally controlled by the power source (SSO drives - AC integrator 690+ series).

The vertical magnetic field is produced by the two Helmholtz coils. The coils are powered by a stabilised DC-supply, (Lambda), producing currents up to 40 A. In the experiments, we want a homogeneous magnetic field over a rather long zone. Using the FEMM-freeware package for axisymmetric magnetostatics, we have studied the geometry of the imposed magnetic field. We have found that the use of a pair of parallel cylindrically shaped soft-iron ( $\mu_{Fe}/\mu_0 \simeq 700$ ) plates with thickness  $20 \text{ mm}$  increases the intensity and homogeneity of the imposed field. The plates also provide shielding from the magnetic fields produced by the AC-motor under the table. The constraint of the homogeneity of the magnetic field field limits the length of the cylinder to  $\max(H) \simeq 180 \text{ mm}$ . The liquid metal we use in the experiments is Galinstan, a gallium-indium-tin eutectic, liquid at room temperature, with  $\rho = 6440 \text{ kg m}^{-3}$ ,  $\nu = 9.510^{-5} \text{ m}^2 \text{ s}^{-1}$ ,  $\sigma_e = 2.9 \pm 0.5 \times 10^6 \text{ S m}^{-1}$ , and  $\mu = \mu_0 = 4\pi \times 10^{-7} \text{ T A}^{-1} \text{ m}$ . Previous experiments found that the conductivity of Galinstan may be subject to some corrections.

#### Magnetic field detection

To measure the magnetic field associated with the elliptical instability, we have used a pair of Hall -probes with a maximum sensibility of  $1/300 \text{ mV}/\mu\text{T}$  (FW-Bell -Sypris - Model 7030). The sensibility of the Hall probes is not perturbed by large magnetic fields transverse to the probe. A first probe measures the imposed axial magnetic field  $B_0$ . A second one measures the radial field component and is positioned in the vicinity of the surface of the rotating cylinder. In horizontal planes, a maximal radial field component is expected in the direction of the radial compression (C) (see fig. 4.4-(5)), that is  $-\pi/4$  away from the direction of the short axis of the ellipse. In the vertical direction, the radial field extrema are expected at the nodes of  $\cos(l_*z)$ . The probes should not be positioned close to the corner regions. Using (4.6), a crude estimate of the field amplitude can be given by

$$\frac{b}{B_0} \simeq A \frac{l}{l^2 + k^2} E_m^{-1} \simeq 0.1 \epsilon^{1/2} E_m^{-1} \simeq 10^{-4} \quad (4.24)$$

where we have taken  $\epsilon^{1/2}$  for the mode amplitude  $A$ , and 0.1 for the factor  $l/(l^2 + k^2)^{-1}$ . With  $\epsilon \simeq 0.1$ ,  $E_m \simeq 100$  typically, this leads to the ratio of  $O(10^{-4})$ , in agreement with experimental observations. The induced field we are interested in is much smaller than the imposed field. With a typical field strength of  $B_0 = 10 \text{ mT}$ , we find  $b \simeq 1 \mu\text{T}$ , which means that the Hall-probes are used at the limit of their sensibility. In practice we need to search for the signal in surrounding magnetic fields of different origin. The most important source of stationary field is due to the inhomogeneity of the field, which is not guaranteed up to this level of precision. This problem is solved by carefully positioning the magnetic probes, so that the imposed field is dominantly transverse to the probes. Small stationary contributions are left out with an electronic offset. The signal is

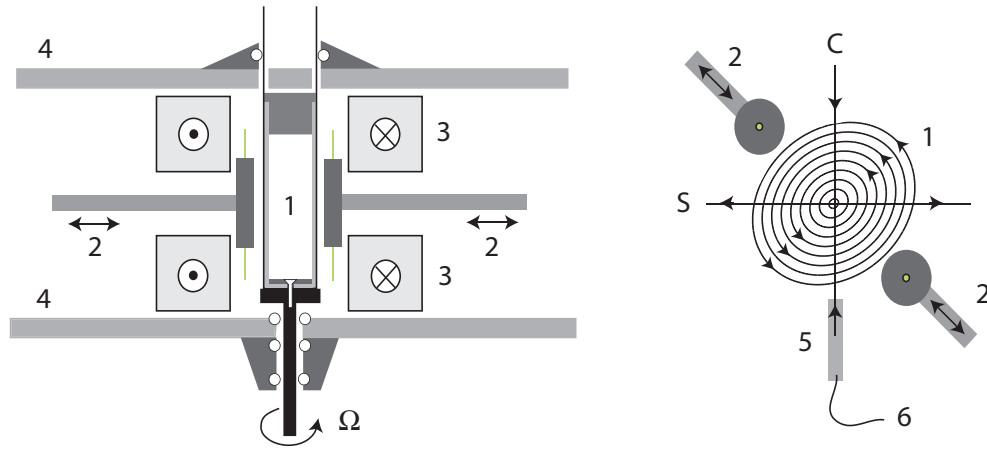


FIG. 4.4 – Sketch of the IMAGINE experimental set-up. Left : side-view; Right : top-view. (1) Elliptically deformed rotating column of Galinstan liquid metal. (2) Pair of adjustable opposed rollers for elliptical deformation of the column. (3) Helmholtz coils. (4) Iron plates for magnetic shielding and increased homogeneity. (5) Magnetic field probe. (6) Signal processing.

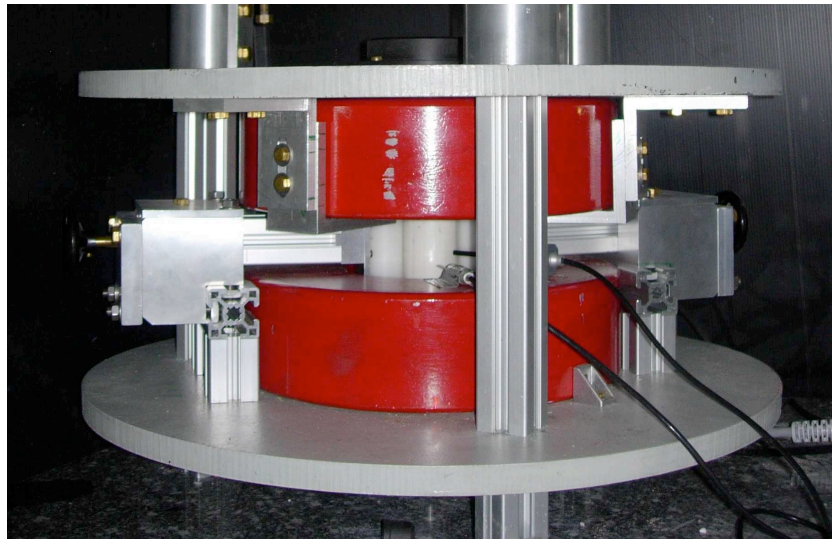


FIG. 4.5 – Photograph of the IMAGINE experimental set-up.

then filtered with a low-pass filter  $f_c = 4\text{ Hz}$ , amplified with a factor 10 – 100 and recorded with a Labview data-acquisition unit. The magnetic field shielding offered by the iron plates, reduces magnetic field noises from the nearby AC-motor, and the choice for isolating and non-ferromagnetic materials reduces numerous sources of time-dependant noises. Careful manufacturing prevents erroneous signals due to the mechanical misalignment. Only very close to the linear instability threshold, when the induced field amplitudes are expected to be small we had some difficulties in identifying the signal. The field amplitudes are generally larger than in the spherical case of Ch. 5. We have been able to detect and follow the magnetic field induced by the elliptical instability in this experimental set-up.

The time series we have obtained, can serve as an input for a spectral analysis, to identify the dominant frequencies are dominant on the short time-scale of the inertial wave oscillations. Modes are identified by their frequency as in previous experiments using Kalliroscope visualisations [26]. The linear growth rate of the elliptical instability can be measured, as well as the nonlinear evolution of the amplitude. The long term nonlinear evolution of the elliptical instability is of specific interest, since relatively few quantitative information is currently available.

### 4.3.2 Experimental results & Discussion

In this section we discuss some magnetic field recordings that were obtained with the IMAGINE set-up. We fixed the height of the cylinder to  $H = 153 \pm 1\text{ mm}$ , which corresponds to an aspect ratio of  $h = 6.12 \pm 0.12$ , and the eccentricity to  $\epsilon = 0.110 \pm 0.005$ . The magnetic field strength was constant at about  $B_0 = 16.8\text{ mT}$ , the rotation speed was varied in  $\Omega \in 2\pi \times [0.5, 2]$ . The radial Hall probe was fixed close to the mid-plane of the cylinder, facing the compressed direction. Fig. 4.6-(a), shows a recording that was obtained with  $E = 4.55 \times 10^{-4}$ ,  $\Lambda = 4.04 \times 10^{-2}$ . The blue line shows the signal, the red line is a mean value obtained in a moving window of 2 rotation periods. In the short time-scale evolution of the left-panel, we have observed an oscillation with frequency close to 2, and approximatively zero mean. The long term dynamics has not revealed a complex nonlinear evolution, as we can see in the right panel of Fig. 4.6-(a), but the mean slightly increases.

We double the rotation speed in a second series of experiments, so that  $E = 2.27 \times 10^{-4}$ ,  $\Lambda = 2.02 \times 10^{-2}$ , and get the timeseries of Fig. 4.6-(b). We still identify the dominant  $\omega \simeq 2$  frequency in the short term dynamics, but the most striking feature, is the clear manifestation of a regular long-term nonlinear cycle. Each cycle is composed of the slow growth of the dominant oscillation, which takes about 25 revolutions. This growth is abruptly ended and the oscillation loses almost all of its amplitude. The cycles are often accompanied with oscillations of the mean-value. We measured a slow frequency  $\omega_{nl} = 0.015$  for the cycles, which were repeated for more than 20 times before we ended the recording. The frequency spectrum of one cycle is shown in Fig. 4.7.

The rotation speed is doubled again. We now have  $E = 1.13 \times 10^{-4}$ ,  $\Lambda = 1.01 \times 10^{-2}$  and the timeseries of Fig. 4.6-(c). The  $\omega \simeq 2$  oscillation is still present but now accompanied by a strong variation of the mean flow. There is still a clear evidence of slow nonlinear cycles which repeat at slow frequency  $\omega_{nl} \simeq 0.020$ . The mean field can reverse from cycle to cycle.

In order to interpret the experimental observations, we first calculate the linear growth rates of different modes for each of the previously discussed cases, (a), (b) and (c) of Fig. 4.6. In Fig. 4.8, we plotted the linear growth rates  $\sigma' = \sigma/\epsilon$  as a function of the aspect ratio  $h \in [5.9, 6.3]$ . The small error on the radius of the cylinder introduces significant changes on the aspect ratio  $h = 6.12 \pm 0.12$ . The full lines correspond to the spinover mode coupling  $(-1, 1, 1)$ . For the mode having  $n = 3$  half wavelengths in its vertical structure, the resonance finger is centered around  $h \simeq 5.93$ . The growth rates of the competing  $(1, 3, 1)$ -coupling follow the dashed lines in Fig. 4.8. A mode with  $n = 6$  vertical structures is centered around  $h \simeq 6.18$ , according to the phase diagram. All the other unstable modes are shown with the dotted lines. In Fig. 4.8-(a) and close to  $h \simeq 6.12$ , we see that only the two modes  $(-1, 1, 1)$  and  $(1, 3, 1)$  have large growth rates. For this small numbers of unstable modes, a simple saturated weakly nonlinear state is not excluded. The magnetic field damping has a significant effect for this parameter set (a). As the rotation speed is doubled from Fig. 4.8-(a) to (b) and from (b) to (c), we see how the two modes  $(-1, 1, 1)$  and  $(1, 3, 1)$  continue to be the dominant couplings, but also how many other modes get destabilised. We can expect a gradual transition to more complex dynamics, as more modes are linearly destabilised.

Each resonant mode, has a particular magnetic field signature. For the spinover mode, we expect an induced field with a maximum radial component in the direction of compression in the mid-plane of the cylinder, close to were the probe was positioned. Since spinover modes are stationary in the laboratory frame, they only contribute to the mean induced fields. The red lines in Fig. 4.6 then give some information about the different spinover modes  $(-1, 1, x)$  in the flow. The  $(1, 3, 1)$  mode, has a radial component of the induced field which vanishes

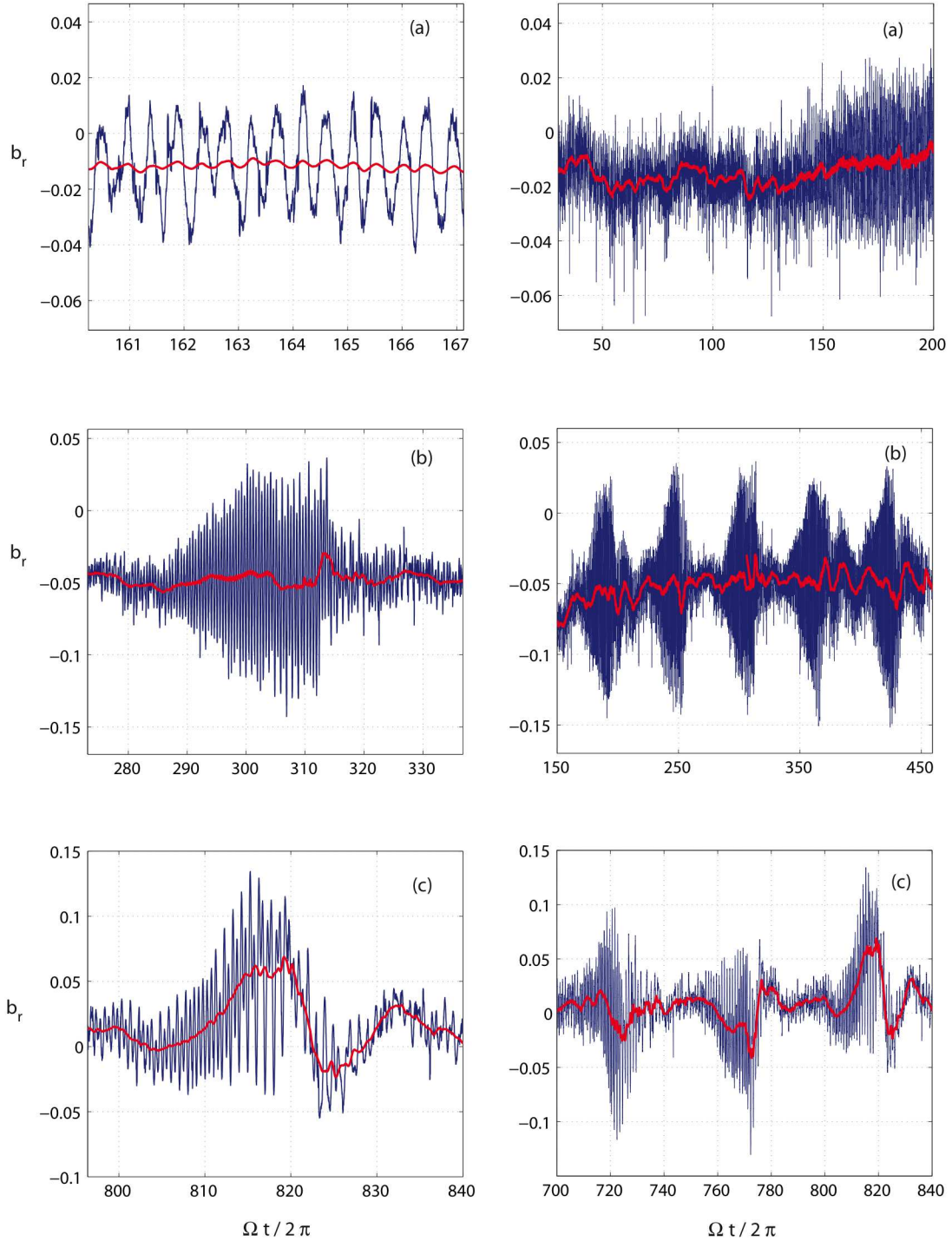


FIG. 4.6 – Experimental time series of the radial component of the magnetic field induced by the flow driven by elliptical instability in a cylinder of aspect ratio  $h = 6$ . Eccentricity is fixed at  $\epsilon = 0.11$ . The imposed magnetic field is fixed, the rotation speed is doubled from (a)  $E = 4.55 \times 10^{-4}$ ,  $\Lambda = 4.04 \times 10^{-2}$  to (b)  $E = 2.27 \times 10^{-4}$ ,  $\Lambda = 2.02 \times 10^{-2}$  to (c)  $E = 1.13 \times 10^{-4}$ ,  $\Lambda = 1.01 \times 10^{-2}$ .



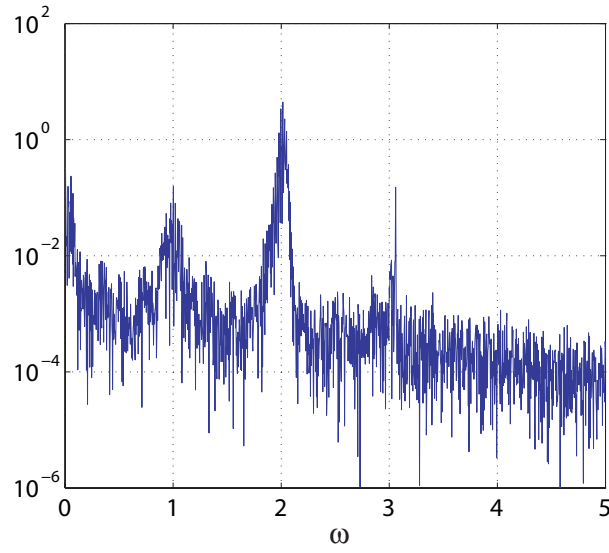


FIG. 4.7 – Frequency spectrum of the timeseries of one nonlinear cycle from the timeseries of Fig. 4.6-(b). The frequency  $\omega \simeq 2$  clearly corresponds to the dominant peak.

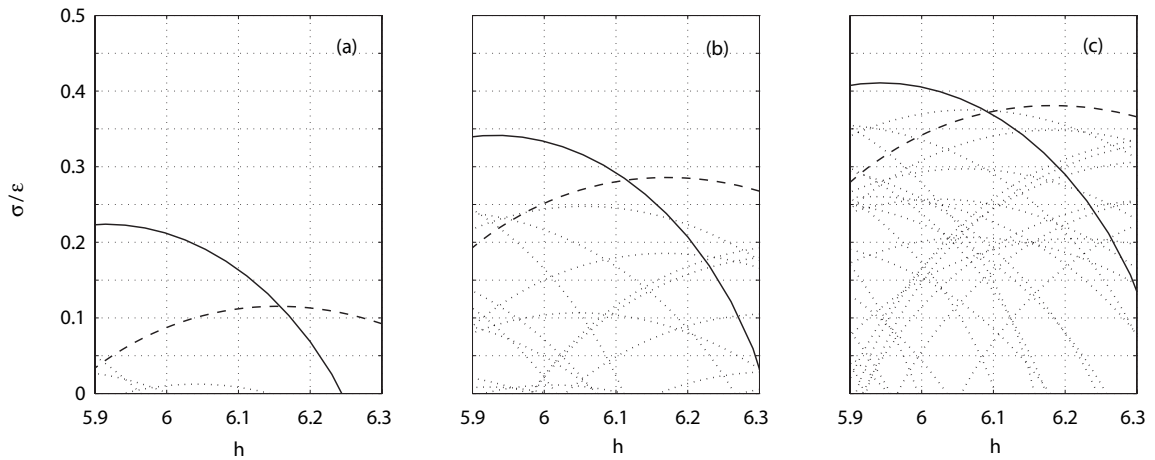


FIG. 4.8 – Theoretical linear growth rates of elliptical instability as a function of the aspect ratio for different instability modes for the parameter sets of the cases (a), (b), (c) of Fig. 4.6. (full lines)  $(-1, 1, 1)$ , (dashed line)  $(1, 3, 1)$ , (dotted lines) other couplings.

in the midplane. The probe is not positioned at this exact midplane, so that this field is not invisible in the experiment. The magnetic field signature of  $(1, 3, x)$ -modes oscillates around a zero mean at frequency  $\omega \simeq 2$ , which is very close to the frequency we measured in the timeseries. We therefore have the following interpretation for the results of Fig. 4.6. Relatively close to the threshold in Fig. 4.6-(a), we observe a dominant signal which arises from an unstable  $(1, 3, 1)$  coupling. The growth rate is small and the oscillations saturate. It is unclear whether the slow growth of the red line correspond to a spinover mode coupling growing on a longer time-scale or whether this is due to the artificial variations induced by the electronics. In Fig. 4.6-(b), the  $(1, 3, 1)$ -coupling is again dominant, but grows and falls back down, following a regular nonlinear cycle. The spinover mode is present, but only at small amplitude. In Fig. 4.6-(c), there is no doubt that both the spinover mode and the  $(1, 3, 1)$  mode are present. The flow goes through growth-relaminarisation cycles, with intermittent reversals of the spinover mode. With  $h \simeq 6.12 \pm 0.12$  we are close to what is expected from the linear theory. It seems that the mode  $(1, 3, 1)$  dominates close to the threshold, which may indicate that the experiment was performed with a slightly higher aspect ratio than initially thought.

### 4.3.3 Conclusion

In this chapter, we have studied the magnetic field effects on the elliptical instability in cylinders in the quasi-static limit of MHD. With infinite permeability boundaries at  $z = 0$  and  $h$  we have been able to find analytical solutions for the magnetic field induced by an inertial wave. We have shown that the Lorentz force does not change the spatial structure of the fields, but simply damps the inertial waves according to a simple law. We have used these fields to calculate the Lorentz-force corrections on the elliptical instability growth rates, and obtained a very simple answer : the magnetic field damping acts uniformly on all couplings. As in the spherical case of the previous chapter, the magnetic field damping inhibits the elliptical instability when  $\Lambda \gg \epsilon$ .

The experimental set-up IMAGINE was built to test this theory, but also to study the nonlinear evolution of the elliptical instability by measuring the induced magnetic field. We have presented a series of results, exhibiting the magnetic field signatures of the inertial waves expected from theory. We have demonstrated that the transition to complex dynamics can be monitored with the magnetic field induced by the inertial waves. We want to attract particular attention to the regular nonlinear cycles of Fig. 4.6-(b), which were also observed in the nonlinear models for elliptical instability ( see Fig. 3.14 ), for the  $(1, 3, 1)$ -mode in a cylinder with aspect ratio  $h = 3.056$ .

The IMAGINE experiment offers many perspectives. First of all, as for the spherical set-up, it would be interesting to confirm the linear theory in the presence of strong magnetic fields. Secondly and perhaps more important, the experiment offers a unique way to study the nonlinear evolution of the elliptical instability. It should be possible in well-controlled experiments to determine whether the inertial wave-collapse observed in elliptical instability is induced by a weakly nonlinear cycle. With careful calibration and more magnetic probes, more detailed information on inertial wave amplitudes could also be obtained.



## Chapitre 5

# Le champ magnétique induit par l'instabilité elliptique dans un sphéroïde

Cette section reprend une publication parue dans la revue *Geophysical and a Astrophysical Fluid Dynamics*, **100**, pp. 299-318, sous le titre "Magnetic field induced by elliptical instability in a rotating spheroid" par L. Lacombe, W. Herreman, M. Le Bars, S. Le Dizès & P. Le Gal. Nous discutons de façon générale comment l'instabilité elliptique dans un sphéroïde peut avoir un rôle dans le noyau liquide des corps célestes elliptiquement déformés par les effets de marées. Le champ magnétique induit par un mode de spinover est calculé. Nous présentons une série de mesures utilisant le champ magnétique pour suivre l'écoulement de métal liquide dans une expérience de l'instabilité elliptique en géométrie sphérique, ce qui nous permet de mesurer les taux de croissance de l'instabilité. Mon apport à cet article a été le calcul du champ magnétique induit par un mode de spinover.

The tidal or elliptical instability of rotating fluid flows is generated by the resonant interaction of inertial waves. In a slightly elliptically deformed rotating sphere, the most unstable linear mode is called the spin-over mode and is a solid body rotation versus an axis aligned with the maximum strain direction. In the non viscous case, this instability corresponds to the median moment of inertia instability of solid rotating bodies. This analogy is furthermore illustrated by an elliptical top experiment, which shows the expected inviscid heteroclinic behaviour. In geophysics, the elliptical instability may appear in the molten liquid cores of rotating planets, which are slightly deformed by tidal gravitational effects of close bodies. It may then participate to the general outer core dynamics and possibly to the geodynamo process. In this context, Kerswell and Malkus (Kerswell, R.R. and Malkus, W.V.R., Tidal instability as the source for Io's magnetic signature, *Geophys. Res. Lett.*, 1998, **25**, 603-606) showed that the puzzling magnetic field of the Jovian satellite Io may indeed be induced by the elliptically unstable motions of its liquid core that deflect Jupiter magnetic field. Our magnetohydrodynamics experiment is a toy-experiment of this geophysical situation and demonstrates for the first time the possibility of an induction of a magnetic field by the flow motions due to the elliptical instability. A full analytical calculation of the magnetic dipole induced by the spin-over is presented. Finally, exponential growths of this induced magnetic field in a slightly deformed rotating sphere filled with Galinstan liquid metal are measured for different rotating rates. Their growth rates compare well with theoretical predictions in the limit of a vanishing Lorentz force.

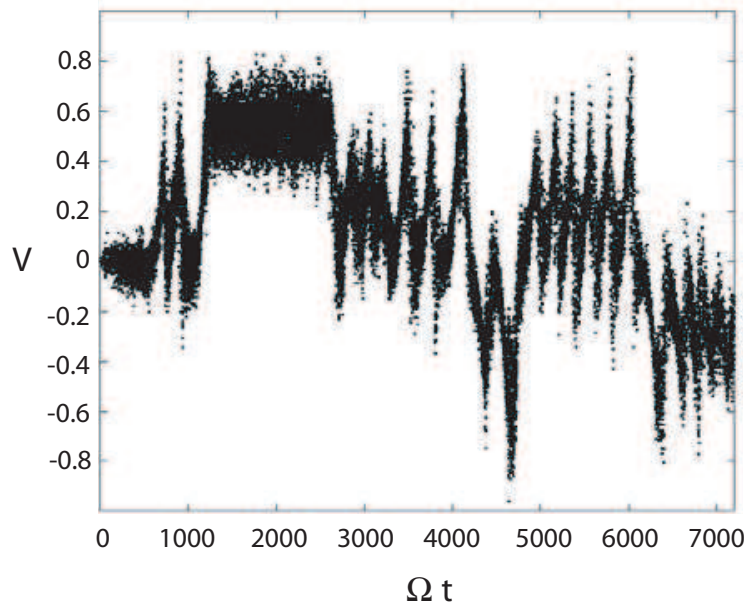


FIG. 5.1 – Doppler anemometry measurements of the radial velocity somewhere in the rotating deformed cylinder. Intermittent cycle of elliptical instability in the cylinder (where the phase of the unstable mode changes between two cycles) would imply a reversal of the magnetic field in MHD; one can also notice that the characteristic times of the dynamics range from the rotation period to thousands of rotation periods.

## 5.1 The elliptical instability

The elliptical instability [55] affects rotating flows when their cross-sectional streamlines are not exactly circular but slightly deformed into ellipses. It is known in particular that vortices are generically unstable when they are subject to a planar strain field. For instance, in a vortex pair, each vortex is subjected to the strain induced by the other vortex. The elliptical instability manifests in this system as axial undulations of the vortices [80]. This three-dimensional instability is also often invoked to explain the secondary three-dimensional instability of shear flows when several vortices have been created by Kelvin-Helmholtz type instabilities [108]. The instability is generated by a parametric resonance that couples together two inertial waves [139] and several experiments have already been devoted to the study of the elliptical instability in rotating deformed cylinders. In the very first experiments, rigid spinning cylinders with elliptical cross-sections or three-axial ellipsoids were considered [41]. At the abrupt stop of the spinning body, e.g. during the transitory spin-down regime, axial undulations were briefly visualized before the complete stop of the fluid. To avoid this major disadvantage, Malkus [89] invented the elastic rotating cylinder device where a transparent soft silicone cylinder is gently and permanently deformed during rotation. Malkus was able to visualise the appearance of the elliptical instability and for higher rotation rates, the complete breakdown of the flow through a kind of "resonant collapses" [95]. This experimental arrangement was then extensively used by Eloy *et al.* (2003)[28] who measured the linear and weakly non linear elliptical instability characteristics in the cylindrical geometry. In these experiments, stationary saturated regimes have been observed for high Ekman numbers. On the contrary, at low Ekman numbers, and similarly to the observations of Malkus (1989)[89], intermittent regimes oscillating between turbulent and laminar flows take place. Our laser anemometry measures presented in figure 5.1 show this amazing intermittency where the phase of the instability can even change between two cycles (see the changes of sign of the radial velocity on figure 5.1).

Following these analyses, it was recently shown that, in a rotating deformed sphere [71], the most unstable mode of the elliptical instability is a rotation around an axis perpendicular to the container rotation axis. This mode, called the spin-over mode, can be visualised using the classical Kalliroscope technique as can be seen in figure 5.2. The experimental container consists now in a hollow spherical cavity that is molded in a cylindrical elastic and transparent silicone gel cylinder. In order to cast this sphere, a ping pong ball was first

inserted in a cylinder of liquid silicone that was then cured at a temperature of 50° Celsius. The ping-pong ball was finally dissolved by ethyl acetate injected inside the ball via 2 mm in diameter pipes. A hollow sphere molded in a transparent and deformable cylinder is finally obtained. The radius of this sphere is 21.75mm. The silicone cylinder is mounted on the vertical shaft of the device already used in the cylindrical case [28] and is compressed between two vertical rollers. Note that these rollers are always in place ; the device does not offer the possibility to move them in or out when rotation is started. The distance separating these rollers gives directly the elliptical deformation  $\epsilon$  of the deformable sphere. The flow whose rotation axis is visualized by Kalliroscope flakes illuminated by a laser plane, consists as already said, in a solid body rotation with an axis given by the maximum strain direction. Because of the combination of the entrainment basic rotation around a vertical axis and of the spin-over mode, the axis of rotation of the fluid in the core of the flow is tilted as can be seen in figure 5.2. The S shape comes from the presence of viscous Ekman layers that arise from the differential rotation between the fluid and the container.

As shown by Lacaze *et al.* [71] other modes of the elliptical instability can also become linearly unstable. However, the spin-over mode growth rate is always the largest at threshold and apart from high  $\epsilon$ , higher modes are not expected to be selected by linear processes. Moreover, as it was the case in the cylindrical geometry, at low Ekman numbers  $E$  ( $E = \nu/\Omega R^2$ , where  $\nu$  is the fluid viscosity,  $\Omega$  the rotation rate and  $R$  the radius of the container), the spin-over mode is destabilised and an intermittent regime takes place. To complete the study, a solid core was finally placed in the center of the sphere : experimental and theoretical analyses then showed the persistence of the spin-over in the spherical shell with a typical growth rate  $\sigma$  given by

$$\sigma = (\epsilon \sigma_{NV} - E^{1/2} \sigma_V) \Omega \quad (5.1)$$

where  $\epsilon$  is the ellipticity,  $\sigma_{NV}$  and  $\sigma_V$  are functions of the inner to outer spheres ratio  $\eta$  ( $\sigma_{NV}=0.5$  and  $\sigma_V=2.62$  for  $\eta=0$ ,  $\sigma_{NV}=0.5$  and  $\sigma_V=1.8$  for  $\eta=1/3$ ) [69].

## 5.2 The elliptical top : a model for the the spin-over mode of the inviscid flow

In fact, the mechanism of appearance of this rotation in the core of the fluid sphere (the spin-over), is nothing else than the instability of rotation of a solid around its median axis of inertia. This phenomenon is often illustrated in physics classes by the throw of a spinning box of matches which deviates from the rotation initially imposed around its median axis of inertia. In a more quantitative way, one can show that the non viscous and nonlinear problem of the solid rotation of the fluid in the laboratory frame of reference is completely equivalent to the problem of the stability of the rotation of a solid ellipsoid in the rotating reference frame. In both cases, the equations of motions are the Euler equations and read :

$$\begin{aligned} (2 - \epsilon)\dot{\Omega}_1 &= -\epsilon(1 + \Omega_3)\Omega_2, \\ (2 + \epsilon)\dot{\Omega}_2 &= -\epsilon(1 + \Omega_3)\Omega_1, \\ \dot{\Omega}_3 &= \epsilon\Omega_1\Omega_2, \end{aligned} \quad (5.2)$$

where the  $\Omega_i$  are the projections on the three principal axes of the instantaneous rotation vector disturbance and where  $\Omega$  was normalised to one. The dot represents time derivative and the median axis of inertia is the axis  $e_3$  which is the unstable axis of rotation. The dilation and the compression of the ellipsoid were carried out according to the first and the second axis with an amplitude  $\epsilon$ . The solutions of this system of equations are heteroclinic trajectories which pass near the unstable fixed points of the poles. These trajectories are illustrated in figure 5.4.

In order to experimentally illustrate the dynamics of such a spinning solid ellipsoid that would imitate the motion of an inviscid fluid contained in a rotating sphere undergoing the effects of a tide, a spinning top having three different moments of inertia was realised. The two poles of its median axis of inertia are marked by two colour stickers which are used to observe the axis of rotation during spinning. Figure 5.3 shows the two first cycles of the heteroclinic dynamics as described previously : we can observe an alternation of the poles marked by the coloured stickers. Of course, even if friction in this case of solid dynamics is much less than in the case of a fluid (only one contact point with the table), the spinning top ends up deviating from this heteroclinic trajectory and is attracted by the rotation around one of the axes of greater or smaller moment of inertia : both centres, seen in figure 5.4 are now transformed into two stable focus because of energy dissipation.

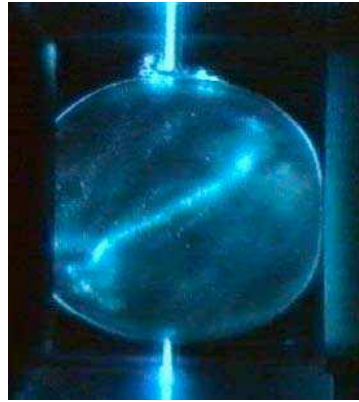


FIG. 5.2 – Kalliroscope visualisation of the spin-over mode of the elliptical instability in a rotating sphere filled with water. The observation plane is parallel to the spin-over axis. The sphere rotates around a vertical axis while it is slightly compressed by two rollers. The tilted axis of flow rotation is clearly visible by the use of Kalliroscope flakes. This inclination of the flow rotation axis comes from the combination of the main rotation imposed by the container and by the unstable spin over mode which is a solid body rotation around an horizontal axis.

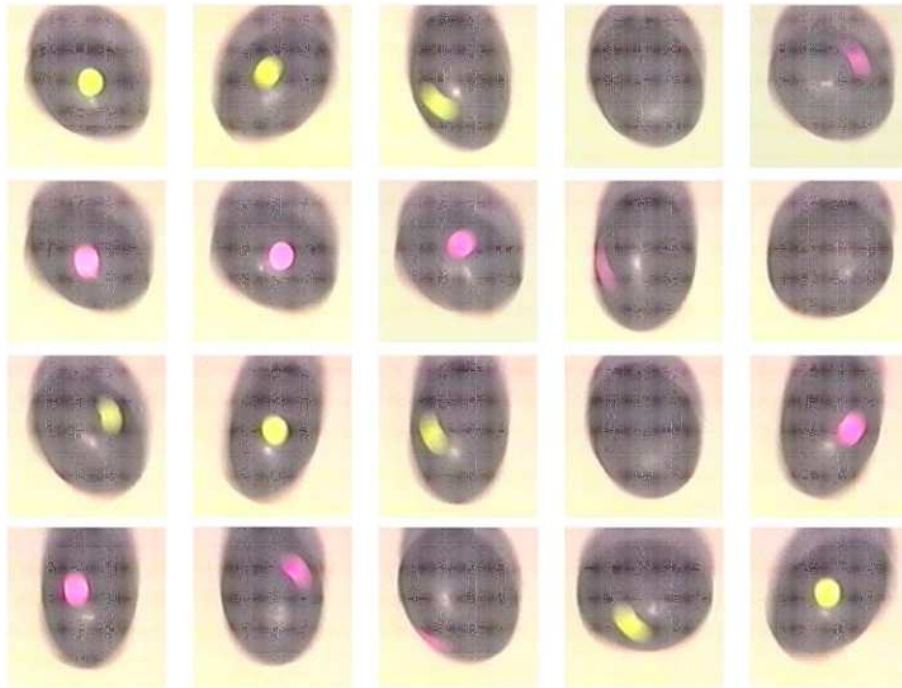


FIG. 5.3 – Series of chronological images showing two heteroclinic cycles of the axis of rotation of the elliptical spinning top.

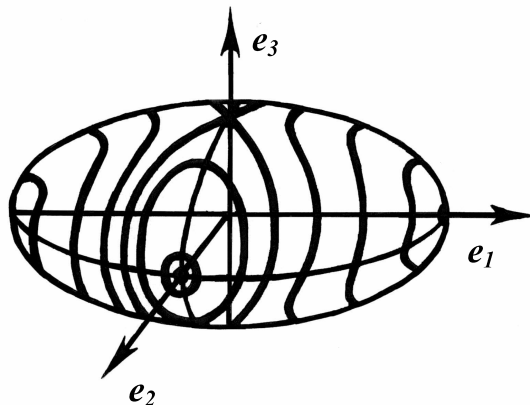


FIG. 5.4 – The heteroclinic trajectories, which pass by the poles of the ellipsoid, illustrate the instability of the rotation of a solid versus its median axis of inertia.

In a similar way, the motions in the viscous fluid ellipsoid, due to the elliptical instability, do not describe the inviscid heteroclinic cycles, at least for the Ekman numbers considered in this study. The angle of the tilted axis of rotation saturates after a slight overshoot. On the other hand, it is relatively easy to follow the angle of this axis as a function of time and deduce the instability growth rate for various Ekman numbers. A theoretical analysis which takes into account the nonlinear and viscous effects was also carried out [71]. The equations of motion are then modified in the following way :

$$\begin{aligned}\dot{\Omega}_1 &= -\frac{\epsilon}{(2-\epsilon)}(1+\Omega_3)\Omega_2 + \nu_{SO}\Omega_1, \\ \dot{\Omega}_2 &= -\frac{\epsilon}{(2-\epsilon)}(1+\Omega_3)\Omega_1 + \nu_{SO}\Omega_2, \\ \dot{\Omega}_3 &= \epsilon\Omega_1\Omega_2 + \nu_{EC}\Omega_1 + \nu_{NL}(\Omega_1^2 + \Omega_2^2),\end{aligned}\tag{5.3}$$

where the different coefficients have been calculated following Greenspan (1968)[45] :  $\nu_{SO} = -2.62E^{1/2}$  (for the linear damping rate of the spin-over),  $\nu_{EC} = -2.85E^{1/2}$  (for the linear damping rate of the axial rotation) and  $\nu_{NL} = 1.42E^{1/2}$  (for the damping in the boundary layer due to the non linear interaction of the spin-over with itself). The viscous and nonlinear dynamics confirms the selection of the spin-over mode whose growth rate  $\sigma$  have been reproduced from Lacaze *et al.* (2004) in figure 5.5. As can be seen, a good agreement is obtained between theory and experiment. Note that the extrapolation presented in figure 5.6 is obtained from these analyses.

### 5.3 The elliptical instability in geophysics

If up to now, most of the studies about the elliptical instability concern shear flows and vortices in fluids, another fundamental interest in the study of elliptical instabilities is related to the understanding of geophysical and astrophysical flows. In particular, despite a wealth of studies, planetary magnetic fields remain a challenging interdisciplinary research topic at the international level (see for example the recent review of Busse [13]). The main point is to understand how planets - and especially the Earth - can generate and maintain a magnetic field over durations of time much longer than the typical magnetic diffusion time. The commonly accepted answer is that such fields are due to motions in an electrically conducting fluid within the planets. But the appropriate energy source as well as the organisation of these motions remain (among others) open questions. On Earth, the prevalent and most reasonable hypothesis is that the present dynamo is powered by thermo-chemical convection within the conducting core driven by the crystallisation of the inner core : multiple studies have thus been performed (see for instance the review of Busse and references therein [14]). Following the Earth's



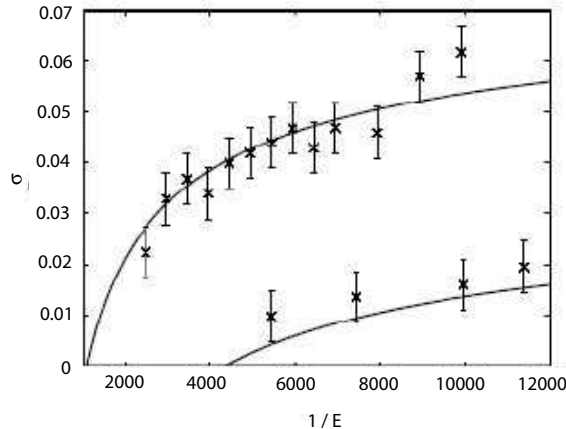


FIG. 5.5 – Comparison between the linear analysis of elliptical instability in the sphere and experimental measurements of the growth rates  $\sigma$  of the spin-over for two cases of ellipticity ( $\epsilon = 0.08$  and  $0.16$ ).

example, it is commonly and tacitly assumed that the generation of a magnetic field in other planetary cores is also due to convection. However, other mechanisms have been suggested as for instance inertial instabilities due to precession and/or tides [137], [57], [105], [132]), which could strongly influence the dynamics of the Earth's liquid core, and even generate or induce a magnetic field in other planets or in the early Earth, i.e. before the beginning of the inner core crystallisation one billion years ago [67]. Actually, on no account planetary dynamo systematically means core's convection. For instance, the recent flybys of the Galilean satellites by the Galileo mission have brought the surprising conclusions that Io may have an intrinsic magnetic field, and that Ganymede has a permanent dipole due to a recent or present dynamo. In both cases however, the traditional convective scenario derived from the Earth's case is not entirely satisfying : convection is impossible in Io's core because of the steadily tidal heating of its mantle [134], and sustaining convection over billions of years is very surprising in Ganymede's core because of its small size [125]. Taking into account the importance of tides in Galilean satellites (deformation of several hundred meters in Io for instance), [57] have thus suggested that these magnetic fields could be due to tidal instabilities.

Inertial instabilities due to precession and tides come from the same physical mechanism. As in laboratory rotating flows, precession or tides in rotating planetary liquid cores can also excites pairs of inertial waves selected according to the geometry of the container and of the forcing. The linear theory then shows a divergence of the amplitude of the selected modes, and the structure of the flow observed experimentally changes radically. Because of this parametric resonance, inertial effects can be of primary importance even in flows at (estimated) low Rossby numbers, as for example in planets. Then, this parametric resonance would radically change the usual vision of motions in planetary cores, for instance invalidating the Proudman-Taylor constrain, often called upon on Earth.

In this context, the studies in the sphere [71]) and in the spherical shell [69]) are especially interesting from a geophysical point of view. Indeed, we can now quantitatively estimate the presence of an elliptical instability in a planetary core given by  $\sigma > 0$  (see equation 5.5) : as shown in figure 5.6, an elliptical instability is proved on Io and possible on Earth [58]. Besides, we can also quantitatively demonstrate that the typical timescale of the inertial instabilities (given by  $1/\sigma$ ) is much larger than the rotation rate [1]. Finally, the intermittency cycles and the phase reversals observed in the hydrodynamics at low Ekman numbers (see figure 5.1) could translate into magnetic field excursions and reversals in planets. All these points thus suggest that tidal instabilities could create realistic magnetic fields at the planetary scale.

However, the previous conclusions are only based on hydrodynamics ground. A systematic study of the magnetohydrodynamics of the inertial instabilities is now necessary. Regarding geophysical applications, two different problems arise : first, from a purely MHD point of view, one can wonder what the characteristics of the induced field are (intensity, dipolar/quadripolar, orientation) and how the hydrodynamical instability is in turn modified by the induced field ; then, one can wonder whether tidal instabilities can indeed initiate and power a dynamo in the parameters range relevant to the planets, as recently demonstrated numerically [132] in the closely related case of precession.

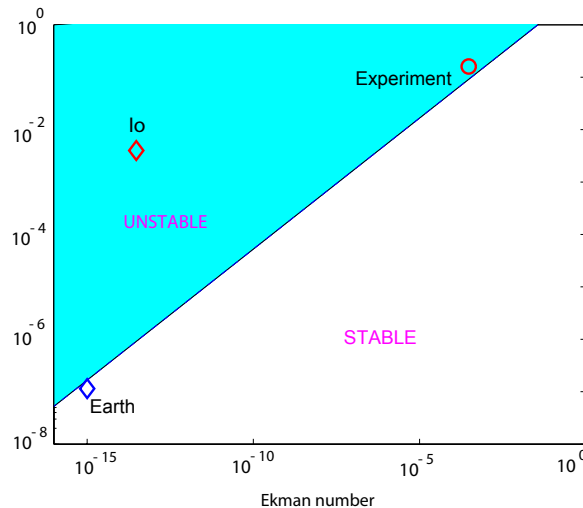


FIG. 5.6 – Stability of a flow in a sphere in the presence of an elliptical deformation, depending on the Ekman number and on the imposed ellipticity. Data for Io come from Kerswell and Malkus (1998) and are derived from Anderson *et al.* (1996). The straight line is given by  $\sigma = 0$  (see equation 5.1)

In the present paper, our main objective is to answer the first question, which is interesting on its own, but can also be considered as a first step toward the second one. We thus aim at systematically quantifying the characteristics of the magnetic field induced by tidal instabilities in various configurations. Theoretical and experimental studies have been performed concurrently and provide complementary results. In the first part of section 4, analytical results are presented that precisely determine the influence of an imposed magnetic field on a simplified spin-over mode (i.e. no Ekman layers are taken into account) : the shape of the induced magnetic field and its intensity are determined over very large parameters range, especially regarding the magnetic Reynolds number. Experimental results are described in the second part of section 4 : measurements of the magnetic field induced by the elliptical instabilities are presented for magnetic Reynolds numbers around  $10^{-2}$ .

## 5.4 Magnetic field induction by elliptical instability.

We have seen that the spin-over mode is spontaneously generated by the elliptical instability in a slightly deformed rotating spheroid of non-magnetic fluid. This simple hydrodynamic eigenmode of the uniformly rotating sphere, is unaffected when dealing with liquid metals in the limit of small magnetic fields : magnetohydrodynamics reduces to hydrodynamics and the induction equation when the Elsasser number is small. A liquid metal analogue of the former experiment in presence of a small external magnetic field should thus be able to excite the same spin-over mode. However, this time, it would be accompanied by an induced magnetic field, resulting from the interaction of the flow with the imposed field. This magnetic signature would also be visible from the exterior where we aim to detect it. In this section, our first goal is to provide an analytic description of the magnetic field induced by the spin-over mode inside and outside the sphere. Our second goal is to provide experimental measurements of this induced magnetic field.

In the experimental configuration, a small homogeneous vertical magnetic field of 14 Gauss is created by two Helmholtz coils (see figure 5.9). The imposed magnetic field  $\mathbf{H}_0 = H_0 \mathbf{e}_z$  is parallel to the axis of rotation of the sphere. The sphere rotates at the angular rotation  $\Omega$  and is filled with a Gallium-Indium-Tin eutectic (Galinstan). This metal has the advantage over Gallium to be liquid at room temperature. Its density is  $\rho = 6.44 \cdot 10^3 \text{ kg m}^{-3}$ , its kinematic viscosity is  $\nu = 0.93 \cdot 10^{-6} \text{ m}^2 \text{ s}^{-1}$  and its magnetic diffusivity is  $\nu_m = 0.36 \text{ m}^2 \text{ s}^{-1}$ .

A small elliptical deformation is applied to the sphere by using the rollers in order to excite the spin-over mode. In the experiment, the ellipticity of the streamline is approximately  $\epsilon = 0.095$ . If we neglect the viscous Ekman layers, the spin-over mode corresponds to a solid body rotation of angular rotation  $\Omega_{SO}$  around a horizontal axis, say  $Ox$ . The total flow in presence of this mode is thus approximately a solid-body rotation around an axis parallel to  $\Omega_{SO} \mathbf{e}_x + \Omega \mathbf{e}_z$ . The associated velocity field is noted  $\mathbf{U}_T$ .

The magnetic field induced by this flow in presence of the imposed field  $\mathbf{H}_0$  can be calculated exactly. If we write the total magnetic field (inside and outside the sphere) as  $\mathbf{H}_T = \mathbf{H}_0 + \mathbf{H}$ , the induced field  $\mathbf{H}$  is found to satisfy, in a fixed frame, the equations :

In the sphere ( $r < 1$ ) :

$$\frac{\partial \mathbf{H}_i}{\partial t} + \nabla \times (\mathbf{U}_T \times \mathbf{H}_i) - \frac{1}{Re_m} \nabla^2 \mathbf{H}_i = -\nabla \times (\mathbf{U}_T \times \mathbf{H}_0) , \quad (5.4)$$

$$\nabla \cdot \mathbf{H}_i = 0 ; \quad (5.5)$$

Outside the sphere ( $r > 1$ )

$$\mathbf{H}_e = \nabla \Psi , \quad (5.6)$$

$$\nabla^2 \Psi = 0 . \quad (5.7)$$

The indexes  $i$  and  $e$  refer to internal and external fields respectively. The magnetic fields have been non-dimensionalized by  $H_0$  (such that  $\mathbf{H}_0 = \mathbf{e}_z$ ), spatial variables by the sphere radius  $R$ , and time by  $\Omega^{-1}$ . The magnetic Reynolds number is defined by

$$Re_m = \frac{\Omega R^2}{\nu_m} ,$$

where  $\nu_m = (\sigma_m \mu_m)^{-1}$  with  $\sigma_m$  the fluid conductivity, and  $\mu_m$  its magnetic permeability.

Inside the sphere, the dynamics of the magnetic field is controlled by the induction equation, whereas the field is the gradient of a scalar potential field  $\Psi$  outside the sphere (no source of magnetic field). Magnetic fields are always solenoidal as prescribed by (5.5) and (5.7). In addition, the total magnetic field has to be continuous at the spherical boundary, a consequence of  $\mu \simeq \mu_0$  and the usual boundary conditions for the magnetic induction and field.

To obtain the induction field, it is useful to consider the problem for the full magnetic field in the rotating frame where the fluid is at rest. In this frame, the induction equation inside the sphere reduces to a diffusion equation. Outside the sphere, the equations are unchanged, but the uniform external field  $\mathbf{H}_0$  becomes oscillating with a frequency  $\sqrt{1 + \Omega_{SO}^2}$  (with our normalisation). This oscillating behaviour prescribes the time-dependency of the field inside the sphere. The diffusion equation then becomes a Helmholtz equation which can be solved in the spherical geometry using classical techniques [104]. The idea is then to return to a fixed frame in which the induced field is stationary. The expression can be simplified by considering a tilted frame, such that its vertical axis  $\mathbf{e}'_z$  is oriented along with the rotation axis of the total flow (see figure 5.10). In this fixed frame, the imposed magnetic field  $\mathbf{H}_0$  can be written as  $\mathbf{H}_0 = -\sin \theta_0 \mathbf{e}'_x + \cos \theta_0 \mathbf{e}'_z$  where the inclination angle  $\theta_0$  is connected to the normalised spin-over rotation rate by

$$\sin \theta_0 = \frac{\Omega_{SO}}{(1 + \Omega_{SO}^2)^{1/2}} . \quad (5.8)$$

If we express the cylindrical components ( $H'_\rho, H'_\phi, H'_z$ ) of the induced magnetic field in terms of spherical variables ( $r', \theta', \phi'$ ), we obtain an expression composed of two different parts :

$$\begin{aligned} \mathbf{H}'_i(r', \theta', \phi') = \sin \theta_0 |A(r')| & \begin{pmatrix} \cos[\phi' + \chi_A(r')] \\ \sin[\phi' + \chi_A(r')] \\ 0 \end{pmatrix} \\ + \sin \theta_0 |B(r')| & \begin{pmatrix} \frac{1}{4}(3 \cos 2\theta' - 1) \cos(\phi' + \chi_B(r')) \\ -\frac{1}{2} \sin(\phi' + \chi_B(r')) \\ -\frac{3}{4}(\sin 2\theta') \cos(\phi' + \chi_B(r')) \end{pmatrix} , \end{aligned} \quad (5.9)$$

$$\mathbf{H}'_e(r', \theta', \phi') = \mathbf{H}'_i(1, \theta', \phi') r'^{-3} , \quad (5.10)$$

where  $|A|$ ,  $|B|$ ,  $\chi_A$  and  $\chi_B$  are defined by

$$A(r') = |A(r')| e^{i\chi_A(r')} = 1 - \frac{j_0(kr')}{j_0(k)} , \quad (5.11)$$

$$B(r') = |B(r')| e^{i\chi_B(r')} = \frac{j_2(kr')}{j_0(k)} . \quad (5.12)$$

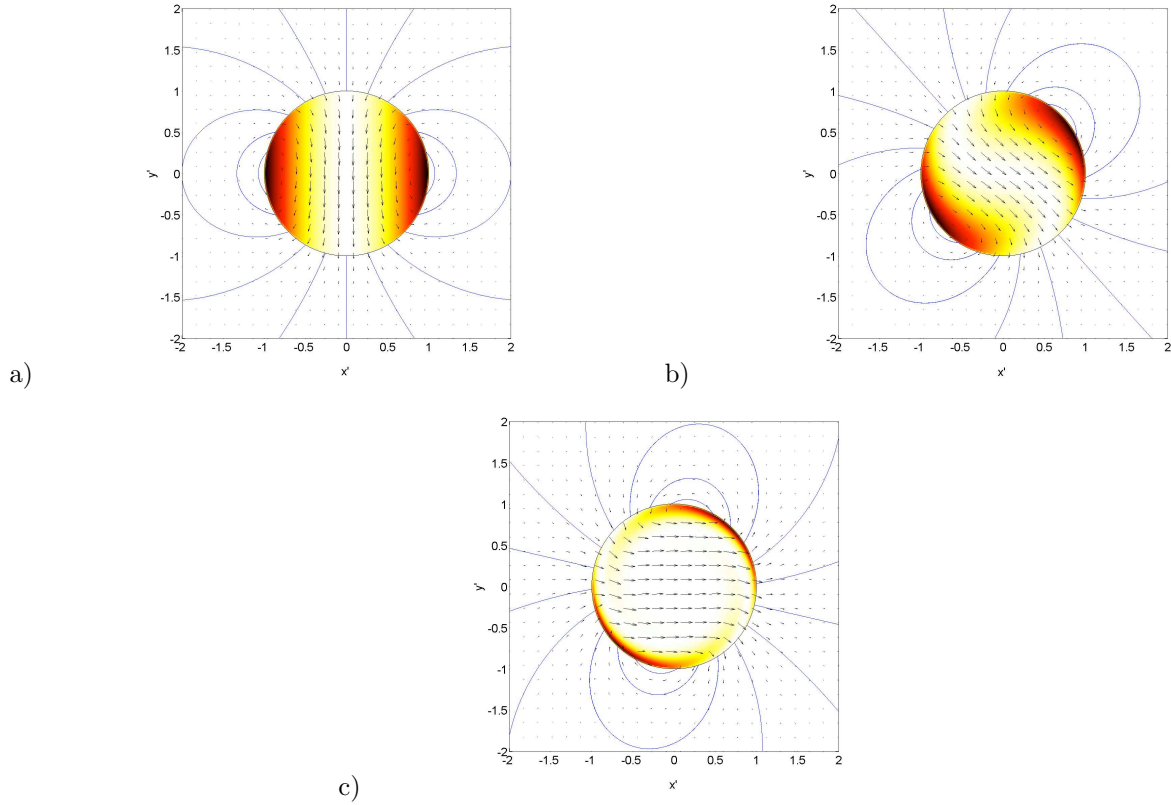


FIG. 5.7 – Arrows represent the induced magnetic field in the (tilted) equatorial plane ( $\theta' = \pi/2$ ). Field lines outside the sphere are solid lines. Joule dissipation  $|\nabla \times \mathbf{H}'|^2$  inside the sphere is indicated by the gray scale. (a)  $Re'_m = 0.01$ , (b)  $Re'_m = 10$ , (c)  $Re'_m = 100$ . Note that the field is perpendicular to the spin-over axis at low  $Re'_m$  but tends to be parallel at higher  $Re'_m$ .

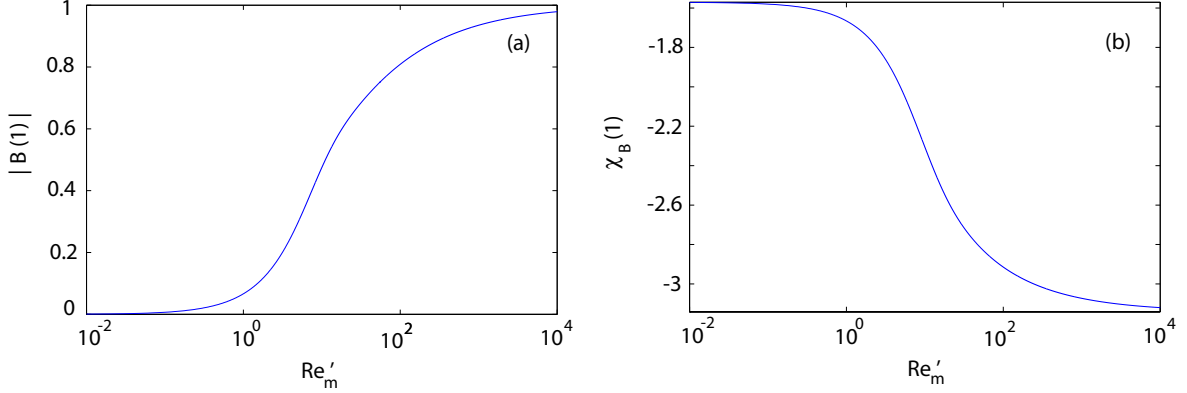


FIG. 5.8 – Variation of the dipole magnitude  $|B(1)|$  (a) and orientation angle  $\chi_B(1)$  (b) with respect to  $Re'_m$ .

In these formulas,  $j_0$  and  $j_2$  are spherical Bessel functions and the parameter  $k$  is given by  $k = \sqrt{-iRe_m}(1 + \Omega_{SO}^2)^{1/4}$ . With these expressions, one can clearly see that the  $A$ -part of the solution only exists inside the sphere and disappears at  $r' = 1$ , and that it has no vertical component. It can be interpreted as the field needed to compensate the transverse part of the imposed field in the tilted frame. The  $B$ -part exhibits almost opposite behaviour since it vanishes at the origin, becomes more and more important as we approach the border, and extends to the outside. This part of the solution is necessary since the  $A$ -part by its own cannot be matched to the external potential field. The structure of the magnetic field is a dipole outside the sphere. However, inside the sphere, the field lines are deformed by the rotation of the fluid. The magnetic Reynolds number appears only via a renormalised magnetic Reynolds number  $Re'_m = Re_m \sqrt{1 + \Omega_{SO}^2}$ , which takes into account the rotation of the spin-over mode.

A few symmetries can be pointed out from the above expressions. The equatorial plane  $\theta' = \pi/2$  is a mirror-symmetry plane for the internal as well as the external field. In this plane,  $\mathbf{H}'$  has no vertical  $\mathbf{e}'_z$  component. Field lines in this plane are plotted in figure 5.7 for three different values of  $Re'_m$ .

The intensity of the Joule dissipation  $|\nabla \times \mathbf{H}'|^2$  within the liquid metal is also indicated by the gray scale. For small  $Re'_m$ , the dipole field points in the  $-\mathbf{e}'_y$  direction, both inside and outside the sphere. Field amplitudes are small, and dissipation seems to be rather well distributed over the entire circle. For moderate  $Re'_m$ , ( $Re'_m = O(1)$ ), we observe that the external field starts to change its orientation toward the  $x'$ -axis. Inside the sphere, we can see that dissipation begins to be enhanced close to the boundary, and field amplitudes increase. For higher  $Re'_m$ , the external field is a dipole field aligned with the  $x'$ -axis. Inside the sphere, dissipation concentrates in a fine region close to the border, and a rather uniform magnetic field establishes in the  $x'$ -direction. The field magnitude never ceases to grow with increasing  $Re'_m$  but never becomes bigger than  $H_0 \Omega_{SO} / \Omega$  (see figure 5.8-a). The tilting of the external dipole as  $Re'_m$  increases can be understood from the full field expression. Indeed, the external field possesses a symmetry axis along the direction  $\phi' = -\chi_B(1)$  in the equatorial plane  $\theta' = \pi/2$ . The evolution of  $|B(1)|$  and  $\chi_B(1)$  with respect to  $Re'_m$  is plotted in figure 5.8.

It is also interesting to discuss the effect of variation of  $\Omega_{SO}$  on the magnetic field. The amplitude of the spin-over mode appears in the rescaled magnetic Reynolds number and in the definition of the angle  $\theta_0$ . Increasing  $\Omega_{SO}$  is therefore expected to have two effects : it increases the tilting of the frame, and increases the rescaled magnetic Reynolds number which has been shown also to tilt the dipole axis of the external field. We now consider two asymptotic limits  $Re_m \rightarrow \infty$  and  $Re_m \rightarrow 0$  for which the expressions for the magnetic field become particularly simple. The first limit ( $Re_m \rightarrow \infty$ ) corresponds to ideal MHD. This limit permits to show how magnetic boundary layers develop. The second limit ( $Re_m \rightarrow 0$ ) is close to the experimental conditions. For  $Re_m \rightarrow \infty$ , we find

$$A(r') \sim 1 - \exp\left(-\frac{(1-i)}{\sqrt{2}} Re_m^{1/2} (1 + \Omega_{SO}^2)^{1/4} (1 - r')\right), \quad (5.13)$$

$$B(r') \sim -\exp\left(-\frac{(1-i)}{\sqrt{2}} Re_m^{1/2} (1 + \Omega_{SO}^2)^{1/4} (1 - r')\right). \quad (5.14)$$

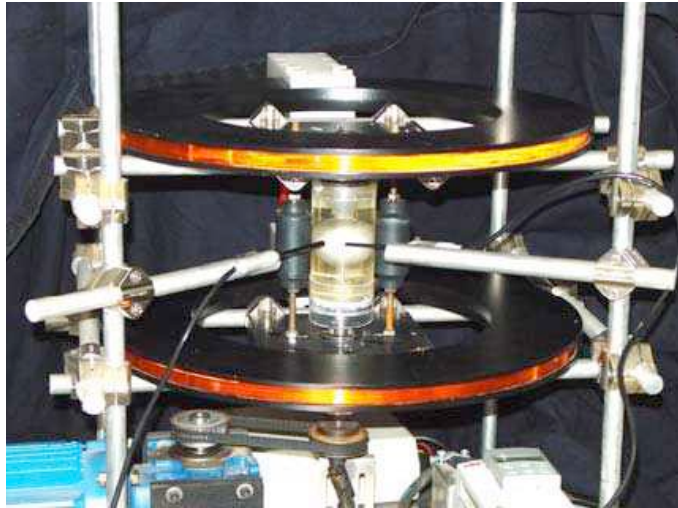


FIG. 5.9 – Experimental set up with the pair of Helmholtz coils surrounding the rotating sphere and the Hall effect probes.

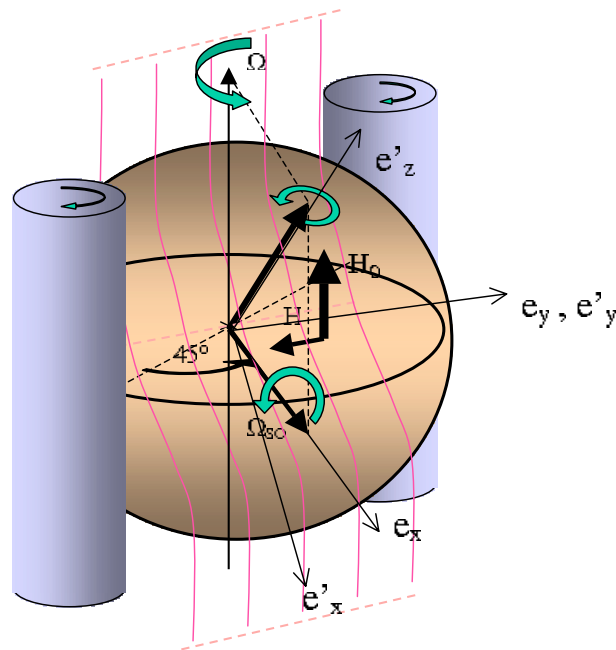


FIG. 5.10 – Schematic representation of the distorted magnetic field lines associated with the  $\Omega$  effect generated by the spin-over mode of the elliptical instability in the rotating deformed sphere.

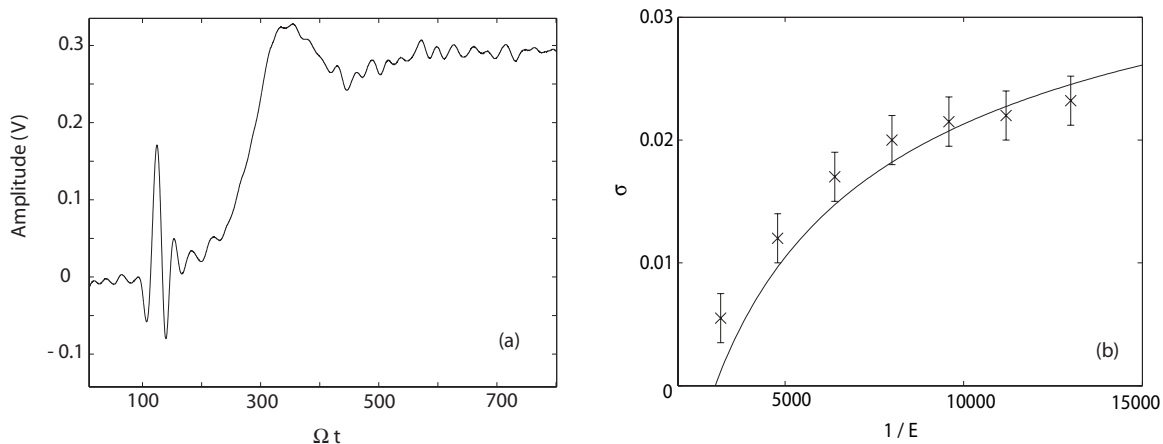


FIG. 5.11 – (a) Temporal exponential growth of the induced magnetic field (50 mV/mGauss). The vertical imposed magnetic field had a value of 14 Gauss at the external position of the probe and the radial induced field reaches 6 mGauss. The oscillations for  $100 < \Omega t < 200$  are induced by the starting of the electric motor and cannot be avoided in our present experimental set-up. (b) Comparison of the measured growth rates  $\sigma$  of the induced magnetic field with the theoretical predictions for  $\epsilon = 0.095$ . The error bars have been obtained from the error calculation of the exponential fits of the experimental curves.

From these expressions, we see that within the sphere, for large  $Re_m$ ,  $A$  is constant and  $B$  is exponentially small. The resulting magnetic field  $\mathbf{H}_i$  is thus uniform and oriented with the spin-over rotation axis, which is in agreement with figure 5.7(c). It is only in a fine region of order  $Re_m^{-1/2}$  near the inner boundary of the sphere ( $r' \approx 1$ ), that  $A$  and  $B$  vary. This magnetic boundary layer permits to match the uniform magnetic field with a decreasing potential field outside the sphere. It is worth mentioning that the  $B$ -part of the field, which is generated in the boundary layer, remains present outside the sphere far from the boundary. This demonstrates the non-trivial nature of the large magnetic Reynolds number limit.

For  $Re_m \rightarrow 0$ , we obtain

$$A(r') \sim iRe_m \sqrt{1 + \Omega_{SO}^2} \frac{(1 - r'^2)}{10} + O(Re_m^2), \quad (5.15)$$

$$B(r') \sim -iRe_m \sqrt{1 + \Omega_{SO}^2} \frac{r'^2}{35} + O(Re_m^2), \quad (5.16)$$

which shows that the induced magnetic field is small. Moreover, as both  $A$  and  $B$  are purely imaginary at leading order, both external and internal fields have now an axis of symmetry in the  $\mathbf{e}'_y$  direction. This direction also corresponds to  $\mathbf{e}_y$ . The structure of the induced magnetic field can be understood in this limit as a torsion of the imposed magnetic field lines by the spin-over mode as schematically illustrated in figure 5.10. This effect corresponds to the so-called  $\Omega$ -effect in the MHD literature.

Indeed, along the  $Oy$  line, the external field possesses a particularly simple expression :

$$\mathbf{H}_e = -\frac{\Omega_{SO} Re_m}{35r^3} \mathbf{e}_y. \quad (5.17)$$

This expression is valid whatever the amplitude of the spin-over mode (as long as  $Re'_m$  remains small). In the experiment, the measurements are performed outside the sphere along the line  $Oy$ . The magnetic Reynolds number of the flow is of the order of  $10^{-2}$  and the Elsasser number which compares the Lorentz force with inertial forces is of order  $10^{-4}$ . The magnetic field is therefore not expected to modify the hydrodynamic evolution and (5.17) can be used for the expression of the magnetic field at the probe. What is important to note is that the induced magnetic field at the probe is proportional to  $\Omega_{SO}$  whatever the amplitude of the spin-over mode. The slow evolution of spin-over mode at instability threshold, in particular its exponential growth and saturation, is therefore expected to be captured by the magnetic field measurement. This is demonstrated in figure 5.11a) where the exponential growth of the induced field is recorded by a Hall effect probe positioned outside the sphere, in the equatorial plane and aligned with the  $Oy$  axis. The vertical imposed magnetic field had a value

of 14 Gauss at the external position of the probe and the radial induced field reaches 6 mGauss. This compares well with formula 5.17, which gives an estimated value of 4.2 mGauss, taking  $\Omega_{SO} = 1$ ,  $Re_m = 10^{-2}$  and  $r = 1$ . Different growth rates  $\sigma$  can thus be measured when the rotation rate  $\Omega$  (and thus the Ekman number) of the flow is changed. Figure 5.11b) presents the evolution of  $\sigma$  with the Ekman number. The solid curve is drawn from the theory where the Lorentz force was not taken into account and for an ellipticity value of  $\epsilon = 0.095$ . Note that there is no free parameter in equation 5.1 and that the agreement between the theory and the experiment is very good. The error bars are estimated from the exponential fit calculation of the growth curves. Note that their size is of the same order of magnitude than those obtained from the visualization method (see figure 5.1). In this case, the errors come from the difficulty we had to determine the position of the axis of rotation. In the present magnetic field growth rates measurement, some strong oscillations perturb the induction signal as can be seen on figure 5.11-a). These magnetic perturbations cause the major lack of accuracy in the growth rates determination. They are induced when starting the electric motor of our device and they appear to be difficult to avoid in the present set-up where the imposed field and consequently the induced field are very weak. We expect to improve the signal to noise ratio by using stronger magnetic fields in the future.

## 5.5 Conclusion.

We have presented a theoretical and experimental study of the hydrodynamics and magnetohydrodynamics of the elliptical or tidal instability in a spherical geometry. This geometry is of great interest for geophysical applications. Experiments as well as a theoretical analysis have been performed in a deformable rotating sphere : we have characterized the growth of the most unstable mode called the spin-over, which forces the fluid to rotate along an axis perpendicular to the axis of entrainment, and gives a "S" shape to the axis of rotation of the fluid. We have then presented an analogy of this fluid flow instability with the median moment of inertia instability of spinning solid bodies in the inviscid limit. An elliptical top was even built in order to illustrate the expected inviscid dynamics. Finally, the induction of a magnetic field by the spin-over has been studied theoretically as well as experimentally. As expected, we have succeeded in recording the exponential growth of a magnetic dipole, induced by the spin-over close to instability threshold. The measured growth rates have been shown to compare well with the hydrodynamics estimates where the Lorentz force is not taken into account. In the experiments, the magnetic Reynolds number was quite limited but we have shown theoretically that the orientation of the induced dipole progressively aligns with the spin-over direction when  $Re_m$  tends to infinity. This work was initially inspired by the work of Kerswell and Malkus (1998) about the magnetic field of Io, the inner most moon of Jupiter. But we expect that inertial instabilities will also play a major role in other planetary fluid cores as well as in astrophysical flows like accretion disks (Lebovitz and Zweibel, 2004)[77] and binary stars... Therefore, in addition to a more global knowledge of the magnetohydrodynamics of inertial instabilities, we expect that the present studies will contribute to quantify the relative importance of each physical mechanism in the energy balance of such natural flows, and especially in the energy balance of planetary dynamos.





## Chapitre 6

# Effets d'un champ magnétique sur l'instabilité elliptique dans un sphéroïde

Ce chapitre reprend une publication soumise à la revue *Physics of Fluids* sous le titre "On the effects of an imposed magnetic field on the elliptical instability in rotating spheroids" par W. Herreman, M. Le Bars & P. Le Gal. Nous étudions les effets d'un champ magnétique fort sur l'instabilité elliptique en géométrie sphérique de façon théorique et expérimentale. Une étude de stabilité locale adaptée à Io est présentée et suggère que l'instabilité survit à la dissipation par effet Joule. Nous discutons la signature du champ magnétique si l'instabilité qui se développe ne génère pas de dynamo, mais déforme simplement le champ imposé par Jupiter. Mon apport à cet article a été l'étude théorique et expérimentale du mode de spinover.

The effects of an imposed magnetic field on the development of the elliptical instability in a rotating spheroid filled with a conducting fluid are considered. Theoretical and experimental studies of the spin-over mode, as well as a more general short-wavelength Lagrangian approach, demonstrate that the linear growth rate of the instability and the square amplitude of the induced magnetic field fall down linearly with the square of the imposed magnetic field. Application of the results to the Galilean moon Io confirms the fundamental role played by the elliptical instability at the planetary scale.

### 6.1 Introduction

The elliptical instability is a generic instability of rotating flows with elliptical streamlines. It has for instance been observed in wakes [80], [97], in elliptically deformed containers, [89], [28], [71], [72], and more generally in the transition to turbulence of strained vortices [108]. Since its discovery in the mid-1970s, it has received considerable attention, theoretically, experimentally and numerically (see for instance the review by Kerswell [55]).

Flows with elliptical streamlines arise as a superposition of rotation and a small strain field, and the instability mechanism has been identified as a parametric resonance of pairwise inertial waves coupled by this strain [4], [139]. In the geophysical context of liquid planetary cores [88],[58], [57], the strain comes from the tidal deformations due to gravitational interaction between neighboring celestial bodies. The elliptical instability (also called tidal instability in this context), as well as the closely related precessional instability, may leave traces in the gravitational and magnetic fields of planets [1], [57], and may even provide alternative sources to power the geodynamo [88], [132]. Even if the hydrodynamic of the elliptical instability is today well known, its planetary consequences are still controversial and necessitate a full understanding of the magnetohydrodynamic (MHD) of the elliptical instability, which remains a mostly open question (e.g. [70]). Understanding the MHD of the elliptical instability is also important in metallurgic applications, especially regarding its role in the transition from two to three-dimensional MHD-turbulence [131].

In the present paper, we consider an elliptically deformed rotating sphere filled with a conducting fluid (figure 6.1) and we study both theoretically and experimentally the effects of an imposed magnetic field parallel to the rotation axis on the development of the elliptical instability. This situation is reminiscent of planetary

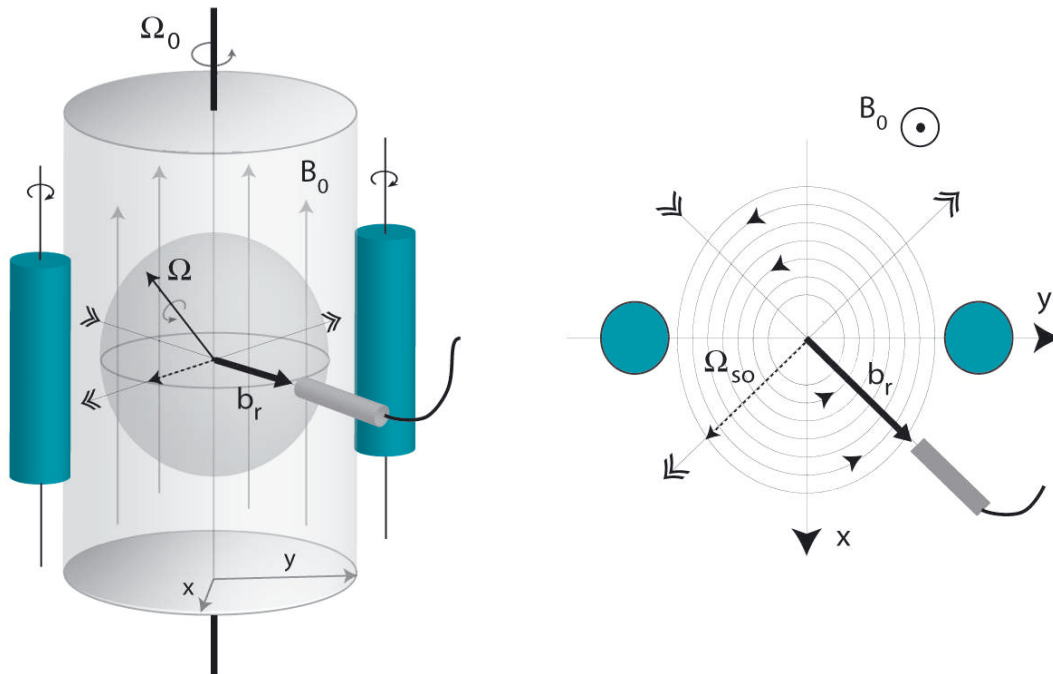


FIG. 6.1 – Sketch of the set-up, side and top-view (see also Lacaze et al. [70]). A liquid metal in a deformable spheroidal cavity rotates at  $\Omega_0$ . A strong magnetic field  $B_0$  is imposed along the rotation axis. Fixed rollers induce an elliptical deformation of the streamlines. Also shown here is the horizontal projection of the spin-over mode, corresponding to a transverse solid body rotation  $\Omega_H$  in the stretched direction (dashed arrow), which tilts the rotation axis of the fluid to  $\Omega$ . In the limit of low magnetic Reynolds number,  $\Omega_H$  induces in the compressed direction a dipolar magnetic field  $b_r \sim \Omega_H$ , which is measured by a Hall probe.

configurations where a tidally deformed moon with a liquid iron core rotates in the magnetic field of its planet, as for instance the Galilean moon Io in the vicinity of Jupiter. Our purpose is to answer the two following questions. How is the elliptical instability damped by the magnetic field? And what is the amplitude of the magnetic field induced by the elliptical instability?

This article, which completes and extends the previous works of Lacaze et al. [70] and Thess & Zikanov [131], is organized as follow. We first focus on the so-called spin-over mode, which corresponds to the simplest mode of the elliptical instability in spheroids, excited at the smallest values of the Reynolds number above threshold. We derive a nonlinear and viscous model of its development under an imposed magnetic field valid for low values of the magnetic Reynolds number, based on the hydrodynamical model of Lacaze et al. [71] and including the magnetic damping term determined by Thess & Zikanov [131]. These results are validated experimentally using an extended version of the set-up of Lacaze et al. [70], with stronger imposed magnetic fields. These results are then extended to the large magnetic Reynolds number, large Reynolds number limit relevant to planetary applications, using a short-wavelength Lagrangian theory [83]. An analytical expression of the growth rate of the elliptical instability is determined and results are finally applied to the case of Io, highlighting the importance of the elliptical instability at the planetary scale.

## 6.2 Set-up and state of the art

We consider the experimental system sketched in figure 6.1. A spherical cavity with radius  $R$ , molded in a deformable silicone block, is filled with a liquid metal, with permeability  $\mu$ , conductivity  $\sigma_e$ , kinematic viscosity  $\nu$ , and density  $\rho$ . It is set in rotation at a constant angular velocity  $\mathbf{\Omega}_0 = \Omega_0 \hat{\mathbf{z}}$  around the vertical  $z$ -axis, and a homogenous magnetic field  $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$  is externally imposed along the same axis with a pair of Helmholtz coils. Following the original idea of Malkus [89], a pair of fixed and opposed rollers compresses the transverse section

of the deformable container, giving it an elliptical cross-section with long axis  $R\sqrt{1+\epsilon}$  along  $x$  and short axis  $R\sqrt{1-\epsilon}$  along  $y$ ,  $\epsilon$  being the eccentricity of the elliptical deformation. Previous experimental studies (e.g. [71]) have demonstrated that the flow in the volume effectively exhibits elliptical streamlines due to the boundary deformation and approaches the theoretical elliptical base-flow

$$\mathbf{U}_{\mathbf{b}} = \left( -\Omega_0 \sqrt{\frac{1+\epsilon}{1-\epsilon}} y, \Omega_0 \sqrt{\frac{1-\epsilon}{1+\epsilon}} x, 0 \right). \quad (6.1)$$

Together with a homogenous magnetic field along the axis, this flow defines an exact base state of the MHD equations on which perturbations may grow due to the elliptical instability. In our experiments, the magnetic Reynolds number  $Rm = \Omega_0 R^2/\eta$ , where  $\eta$  is the magnetic field diffusion  $\eta = (\sigma_e \mu)^{-1}$ , is small ( $Rm = O(10^{-2})$ ) and magnetic field diffusion is always dominant over magnetic field advection. In this limit, the scales

$$\begin{aligned} [\mathbf{r}] &= R & [t] &= \Omega_0^{-1} & [\mathbf{u}] &= \Omega_0 R \\ [p] &= \rho(\Omega_0 R)^2 & [\mathbf{b}] &= Rm B_0 \end{aligned} \quad (6.2)$$

respectively for space, time, velocity, pressure and magnetic field, are well adapted to non-dimensionalize the perturbation problem. In addition to  $Rm$ , the relevant non-dimensional parameters are the Ekman number  $E = \nu/\Omega_0 R^2$ , which measures the importance of diffusive effects over inertial terms, and the Elsasser number  $\Lambda = \sigma_e B_0^2/\rho\Omega$ , measuring the ratio of Lorentz force effects over the inertial forces. In our experiments,  $E = O(10^{-4})$  and  $\Lambda \leq O(1)$  typically.

The hydrodynamic stability of elliptical flow in spheroids was previously studied in [71], which formalizes an asymptotic theory in terms of inertial wave coupling. In the limit  $\Lambda \sim \epsilon$ ,  $Rm \rightarrow 0$ , this theory can be extended to include the magnetic field effect perturbatively. We will not go as far, as the combination of the results of [71], [70] and [131], allows us to describe the linear and nonlinear dynamics of the dominant spin-over mode, which is the only mode accessible to purely hydrodynamical experiments using the present device in a spherical geometry with a fixed strain field [?]. The spin-over mode is mainly a solid body rotation around an inclined axis, whose horizontal projection  $\boldsymbol{\Omega}_{\mathbf{H}}$  is aligned with the axis stretched by the strain field, at polar angle in the vicinity of  $-45^\circ$  in the  $(x, y)$  plane (see figure 6.1). A low-dimensional model was derived in [71] in close agreement with the experiments, which describes the nonlinearly purely fluid evolving spin-over mode as a solid body rotation. Even though  $E$  is small in the experiment, viscosity plays an essential role. Indeed, it postpones the elliptical instability to a critical eccentricity and allows the nonlinear dynamical system to have stable non-trivial fixed points.

Stays the question whether the spin-over mode remains the most unstable mode in presence of a magnetic field, which seems hard to answer without a more complex global analysis of the elliptical instability. In [131] (see also section 6.5), the local growth rates of elliptical instability in an unbounded domain were calculated using Floquet theory. For the limit of small  $\epsilon$  we are interested in, asymptotic arguments as in [139] imply a growth rate linear in  $\epsilon$  and a magnetic damping  $\gamma_M = \Lambda\lambda^2/4$ , where  $\lambda$  is the wave frequency in the rotating frame. Since all elliptically interesting waves have  $\lambda \simeq 1$  [71], [?], local theory indicates that the magnetic field damping acts similarly on all couplings, no matter what their spatial structure is. Since in our device the spin-over mode is always the most unstable mode in the hydrodynamical experiments, we expect that it remains the case when a magnetic field is imposed.

Thess and Zikanov [131] also extended the non-linear, inviscid model of the spin-over dynamics to include the magnetic field effects in the low  $Rm$  limit. They found that the magnetic field introduces a Joule damping, which only operates on the rotations with axis transverse to the imposed magnetic field, identically to the Joule damping of solid conductors rotating in a strong magnetic field, commonly used in magnetic brakes. In our experimental set-up, both viscous and magnetic field effects are important. The eccentricity is small  $\epsilon \simeq 0.1$ , which means that  $\Lambda$  is at most of order  $O(10^{-1})$  in the experiments where we observe the instability. The important consequence is that the magnetic field in this case is always too small to change the viscous boundary layer into a Hartmann-layer, so that there is no need of a more complex boundary layer analysis. This also implies that the viscous terms in the nonlinear system of [71] may be used here. Finally, there will be no significant contributions to the external and internal magnetic fields due to the boundary layer, which would make the field deviate from the field induced by the non-viscous spin-over mode, calculated in [70].

Notice that all these suppositions will be confirmed a-posteriori by the good agreement between the following theory and our experimental results.

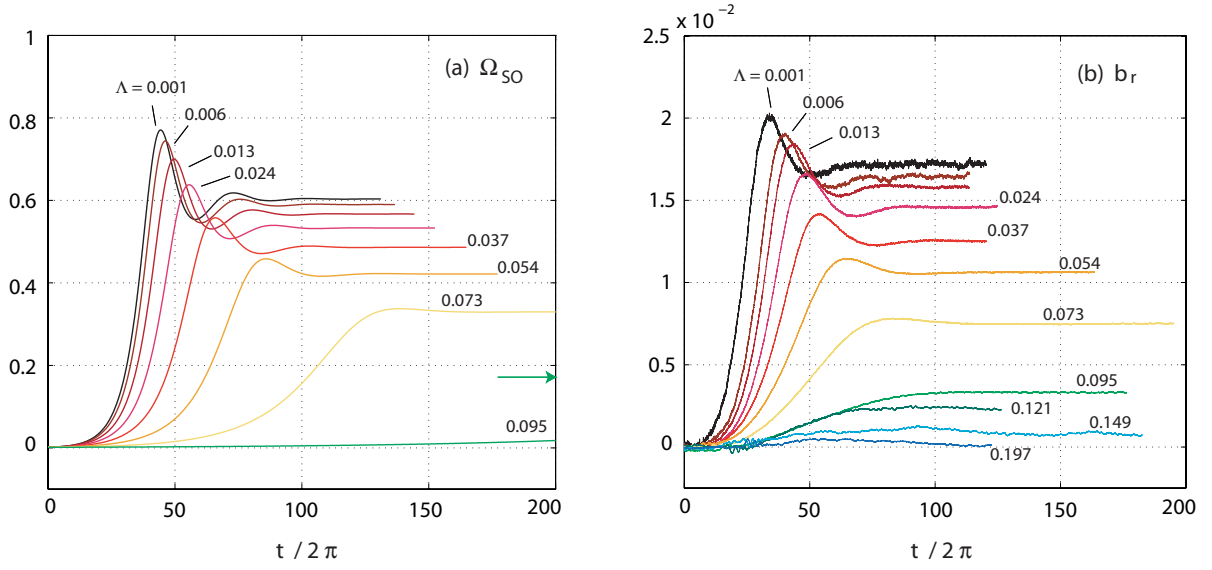


FIG. 6.2 – (a) Theoretical nonlinear temporal evolution of the horizontal projection of the spin-over mode amplitude  $\Omega_H = (\Omega_1^2 + \Omega_2^2)^{1/2}$ , for various values of the Elsasser number  $\Lambda$ . Ekman number and eccentricity are fixed,  $E = 8.53 \times 10^{-5}$ ,  $\epsilon = 0.10$ . Calculations started from the initial state  $\Omega_1 = 10^{-3}$ ,  $\Omega_2 = -10^{-3}$ ,  $\Omega_3 = 0$ , which is the linearly unstable spin-over mode with small amplitude. The arrow on the right side, indicate the saturation level of the slowly growing spin-over mode horizontal amplitude at  $\Lambda = 0.095$ . The critical Elsasser number is  $\Lambda^c = 0.103$  for this parameter set. (b) Typical recorded magnetic field-signals for varying Elsasser number. Ekman number and eccentricity are fixed  $E = 8.53 \times 10^{-5}$ ,  $\epsilon = 0.10 \pm 0.005$ . The experiments agree with the theoretical profiles of figure 6.2(a).

### 6.3 Analytical study of the spin-over mode

Combining the results of [71] and [131], the nonlinear evolution of the spin-over mode can be modeled in the laboratory frame of reference as a solid body rotation with angular velocity  $\mathbf{\Omega} = (\Omega_1(t), \Omega_2(t), \Omega_3(t))$ , which evolves according to the nonlinear system

$$\dot{\Omega}_1 = -\frac{\epsilon}{2-\epsilon}(1+\Omega_3)\Omega_2 - (\gamma_{so} + \Lambda/4)\Omega_1, \quad (6.3)$$

$$\dot{\Omega}_2 = -\frac{\epsilon}{2+\epsilon}(1+\Omega_3)\Omega_1 - (\gamma_{so} + \Lambda/4)\Omega_2, \quad (6.4)$$

$$\dot{\Omega}_3 = \epsilon\Omega_1\Omega_2 - \gamma_3\Omega_3 + \nu_{nl}(\Omega_1^2 + \Omega_2^2). \quad (6.5)$$

On the right hand sides, we first recognize the destabilizing terms from the non-viscous system. The damping of the spin-over mode is controlled by the viscous linear boundary layer  $\gamma_{so} = 2.62\sqrt{E}$  in the horizontal directions and  $\gamma_3 = 2.85\sqrt{E}$  around the vertical axis. Supplementary nonlinear terms arise through the boundary layer,  $\nu_{nl} = 1.42\sqrt{E}$ . All these coefficients are explicitly detailed in Lacaze et al. [71] and find their origin in the classical analysis of Greenspan [45]. The magnetic field only adds a linear term corresponding to the Joule-damping  $\Lambda/4$  in the directions perpendicular to the imposed field. The terms due to the viscous frequency detuning are left out from the model as in [71], since they only introduce negligible differences in the limit of small Ekman numbers we are interested in.

Linearising the system around the trivial fixed point  $\mathbf{0}$ , we calculate the linear growth rate of the spin-over mode

$$\sigma = \frac{\epsilon}{\sqrt{4-\epsilon^2}} - 2.62\sqrt{E} - \Lambda/4. \quad (6.6)$$

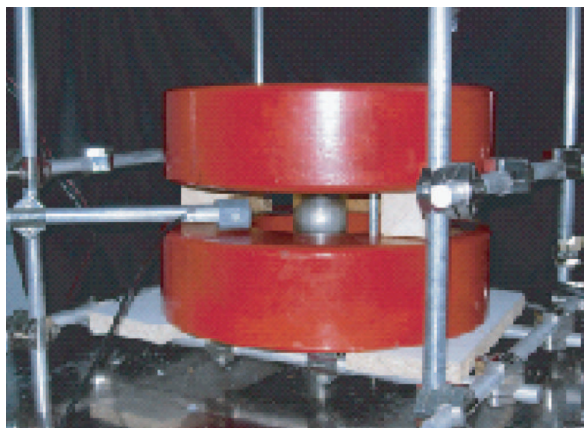


FIG. 6.3 – Picture of the experimental set-up. Large watercooled Helmholtz coils provide a homogenous magnetic field up to  $B_0 = 0.1 T$ . Induced fields are measured with a radial Hall probe.

In agreement with [131], the magnetic damping lowers the growth rate of the spin-over mode linearly with  $\Lambda$ , and the system becomes stable above a critical Elsasser number

$$\Lambda^c = 4\left(\frac{\epsilon}{\sqrt{4 - \epsilon^2}} - 2.62\sqrt{E}\right). \quad (6.7)$$

Some time-series for the horizontal projection of the spin-over mode amplitude,  $\Omega_H = \sqrt{\Omega_1^2 + \Omega_2^2}$ , found by numerical integration of the nonlinear system (6.3)-(6.5), are shown in figure 6.2(a). After an exponential growth, the flow always goes towards a stable non-zero fixed point which is a stable focus. Before saturation the spin-over mode horizontal amplitude displays a small overshoot which originates from the spiral trajectory around this focus. For increasing magnetic field amplitudes, both the linear growth rate and the saturation amplitude decrease. The non-zero fixed points of (6.3) can be calculated explicitly. The square of the spin-over mode amplitude at saturation writes

$$\Omega_H^2 = 4 \frac{\gamma^3}{\epsilon} \frac{\sigma}{\epsilon - 4\nu_{nl}/\sqrt{4 - \epsilon^2}}. \quad (6.8)$$

Note that  $\Lambda$  only appears in this formula through the growth rate. According to [70], the field induced by the non-viscous spin-over mode at low  $Rm$  is a dipole with axis transverse to the imposed field, in quadrature with the rotation axis of the spin-over mode. On the dipole axis outside the spheroid, the field is purely radial and decays as

$$b_r = \frac{|\Omega_H|}{35} \frac{1}{r^3}. \quad (6.9)$$

Combining (6.8) and (6.9), we expect a linear decrease of the square of the induced field amplitude with  $\Lambda$  at fixed  $\epsilon$  and  $E$ . The polar angle in the  $(x, y)$  plane of the saturated spin-over axis is determined by

$$\bar{\phi}_{so} = \pm \arctan\left(\frac{\Omega_2}{\Omega_1}\right) = \pm \arctan\left(-\sqrt{\frac{2 - \epsilon}{2 + \epsilon}}\right) \quad (6.10)$$

so that the vorticity of the saturated spin-over mode is not exactly aligned with the direction of maximum stretching at  $-45^\circ$  of the long axis of the spheroid (e.g.  $\bar{\phi}_{so} = -42.1^\circ$  for  $\epsilon = 0.10$ ).

Lacaze et al. [71] tested experimentally the purely hydrodynamical version of this theory and found good agreement for both linear and nonlinear stages. However, the nonlinear overshoot was not observed and the authors noticed that the experimental saturation amplitudes of the spin-over mode agree better with the maximum theoretical amplitudes attained during the overshoot than with the theoretical saturation amplitudes. We now extend these experimental results by taking the magnetic field into account.

## 6.4 Experiments

Our experimental set-up is an extension of the one presented in [70] (see figures 6.1 and 6.3). The experimental parameters are  $R = 22.75 \text{ mm}$ ,  $\Omega_0 \in [0, 10\pi] \text{ rad s}^{-1}$  and  $\epsilon = 0.10$ . The imposed field  $B_0$  ranges between 0 and  $0.13 \text{ T}$  (up to 100 times larger than in the previous set-up [70]). It is realized with a set of watercooled copper Helmholtz coils, powered by a stabilized DC-supply. The liquid metal we use is Galinstan, a gallium-indium-tin eutectic liquid at room temperature, with  $\rho = 6440 \text{ kg m}^{-3}$ ,  $\nu = 9.5 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$ ,  $\sigma_e = 2.9 \times 10^6 \text{ S m}^{-1}$  and  $\mu = \mu_0 = 4\pi \times 10^{-7} \text{ T A}^{-1} \text{ m}$ . According to [70], the field induced by the non-viscous spin-over mode at low  $Rm$  is a dipole with axis transverse to the imposed field, in quadrature with the rotation axis of the spin-over mode. It is measured in the experiment by a radial Hall-probe mounted in the equator plane of the spheroid, facing the compressed direction at a polar angle of  $45^\circ$ . The probe is  $26.5 \pm 0.5 \text{ mm}$  away from the center of the sphere. The hall probe and the Gaussmeter have a maximum sensibility of  $s = 300 \text{ } \mu\text{T/mV}$ . Since the induced fields are of order  $O(10^{-4} B_0)$ , the probes are used at the limit of their sensibility. This implies careful positioning, thorough prefiltering and amplification of the recorded signals. In practice, the electric signal produced by the Hall-probe is put to zero before each experimental run. The recorded signals are prefiltered with a low-pass filter at  $f_c = 2 \text{ Hz}$ , and amplified by a factor 50. The signals are transferred to the data-acquisition unit on the laboratory computer.

Figure 6.2(b) shows the experimentally recorded radial components of the induced magnetic fields for different  $\Lambda$ . The shapes and relative positions of the experimental records compare well with the theoretical profiles of figure 6.2(a), with an exponential growth (see also figure 6.4) and a slight overshoot preceding a saturation at a constant level. As expected, there is a gradual decrease of growth rates and saturation amplitudes with  $\Lambda$ . Notice again that the amplitude overshoot was not observed in the purely hydrodynamical experiments (see [71]). At  $\Lambda = 0.121$  and higher, we continue to observe a non-zero induced magnetic field, but it becomes increasingly difficult to determine a true exponential growth. These fields probably come in our experimental set-up from misalignment between the axis of rotation of the sphere, the axis of the rollers inducing the elliptical deformation, and the axis of the imposed field.

The signals also give us quantitative informations on the growth rates and saturation amplitudes. As shown in figure 6.5, the growth rate decreases as  $\Lambda/4$ , following the analytical result given by (6.6). Also shown are the theoretical dashed curves for  $\epsilon = 0.095$  and  $\epsilon = 0.105$  representing the uncertainty in  $\epsilon$ . As can be observed, the experimental data are in complete agreement with the theory within this 5% error range, without any adjustment parameter. Figure 6.6 shows that the square of the induced field  $b_r^2$  behaves as the growth rate, in close agreement with the theory (see formula 6.8 and 6.9). Figure 6.6 also provides an experimental measurement of the critical Elsasser number  $\Lambda^c \simeq 0.096$ , close to the theoretical value  $\Lambda^c = 0.103$ .

Using (6.9), we systematically translate the magnetic field measurements to spin-over-mode amplitude and show in figure 6.7 the variations of the ratio  $\zeta_1$  of experimental saturation amplitudes to theoretical saturation amplitudes. The ratio  $\zeta_1$  significantly decreases with the Elsasser number, where we expected a constant value close to 1. This discrepancy between theoretical and experimental saturation values was already observed in the absence of magnetic field in [71]. Several explanations can be provided. From a theoretical point of view, all nonlinear viscous corrections as well as possible secondary instabilities are not included in our model but may become important, especially far from the linear instability threshold. And from an experimental point of view, one can notice that measurements at a fixed  $-45^\circ$  angle only take into account a fraction of the spin-over amplitude when  $\bar{\phi}_{so} \neq -45^\circ$ , this effect being also more important far from the linear instability threshold.

Lacaze et al. [71] also remarked that the ratio  $\zeta_2$  of experimental saturation amplitudes to the maximal theoretical amplitudes attained during the overshoot remains constant over a rather large range of Ekman number. As shown in figure 6.7, this remain valid over a large  $\Lambda$ -interval. We expect that this behavior is not a coincidence, but that it could be revealed by a more sophisticated model, beyond the scope of this paper. However, and contrary to the hydrodynamical experiments by [71] where  $\zeta_2 \sim 1$ , theoretical predictions always underestimate the experimental measurements by a factor 1.42 in our case. Possible explanations are error in the positioning of the probe (the field decreases rapidly in  $r^{-3}$ ), but most probably uncertainties in the value of the electrical conductivity  $\sigma_e$  of Galinstan (values in the literature typically range between  $2.3$  and  $3.5 \times 10^6 \text{ S m}^{-1}$ ). Note also that the elliptical deformation of the spheroid as well as the misalignment between the spin-over axis and the axis of maximum strain are not taken into account in equation (6.9).

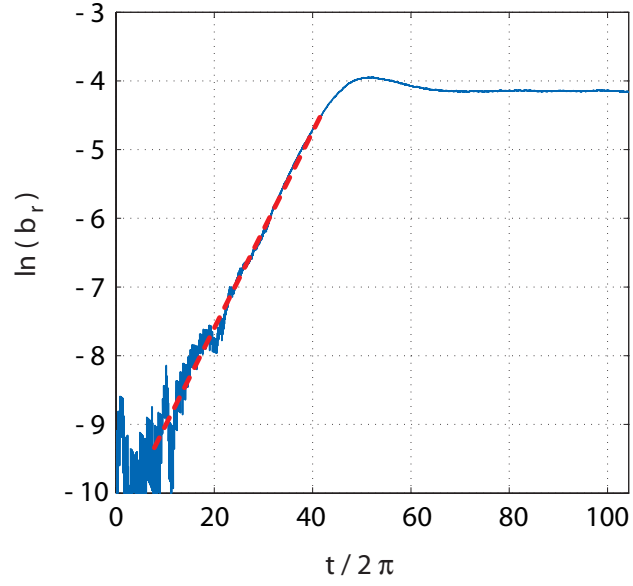


FIG. 6.4 – Logarithm of the induced magnetic field signal (full line) at  $E = 8.53 \times 10^{-5}$ ,  $\epsilon = 0.100 \pm 0.005$ ,  $\Lambda = 0.01$ . The slope of the linear fit (dashed line) provides the initial linear growth rate.

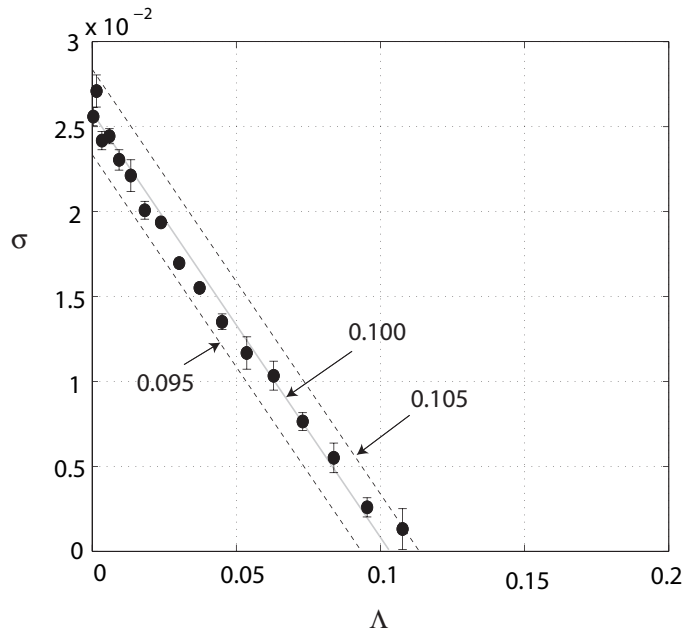


FIG. 6.5 – Linear growth rates  $\sigma$  as a function of Elsasser number  $\Lambda$ .  $E = 8.5 \times 10^{-5}$ ,  $\epsilon = 0.100 \pm 0.005$ . The experimental measurements ( $\bullet$ ) are in good agreement with the theoretical values for  $\epsilon = 0.10$  (soft line). Also shown are the theoretical (dashed) curves for  $\epsilon = 0.095$  and  $\epsilon = 0.105$ , representing the uncertainty in  $\epsilon$ ; note however that there is no adjusting parameter in the comparison between theory and experiment. Growth was no longer exponential beyond  $\Lambda \geq 0.121$ .



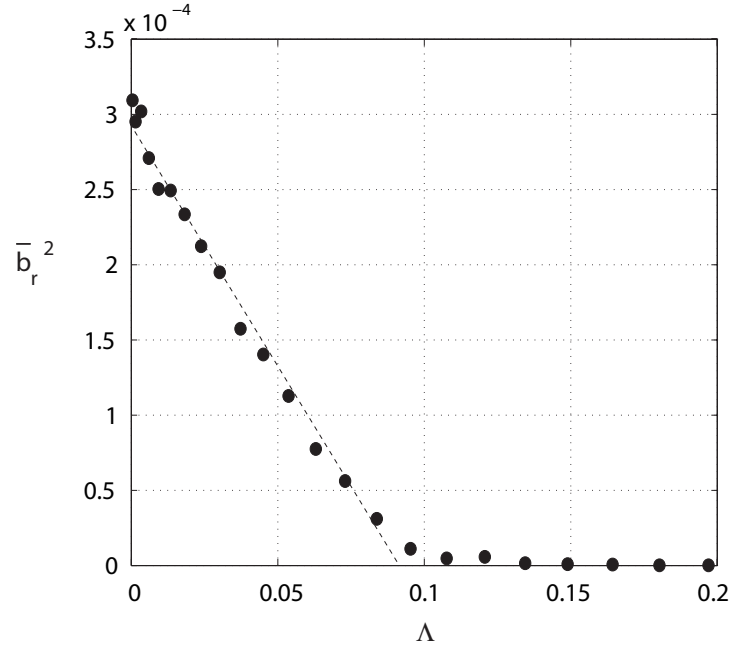


FIG. 6.6 – Square of the measured saturation-amplitudes of the magnetic field  $b_{r, sat}^2$ , as a function of Elsasser number  $\Lambda$ .  $E = 8.5 \times 10^{-5}$ ,  $\epsilon = 0.10$ . Experimental measurements ( $\bullet$ ) and linear fit (dashed line). The saturation amplitudes are in agreement with the weakly nonlinear scaling, predicting a linear dependence on  $\Lambda$ .

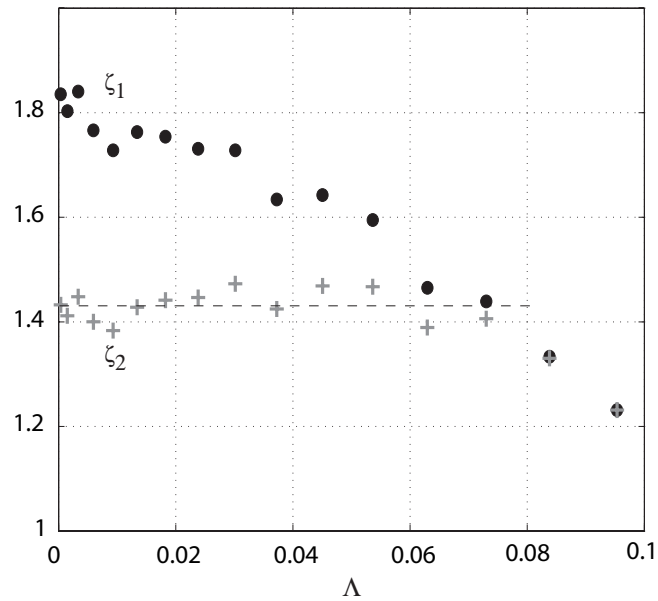


FIG. 6.7 – Ratios of the experimentally observed saturation amplitude, to the theoretical saturation amplitudes  $\zeta_1$  ( $\bullet$ ) and to the maximum amplitude attained during the overshoot  $\zeta_2$  (+), as a function of the Elsasser number  $\Lambda$ .  $E = 8.5 \times 10^{-5}$ ,  $\epsilon = 0.10$ .

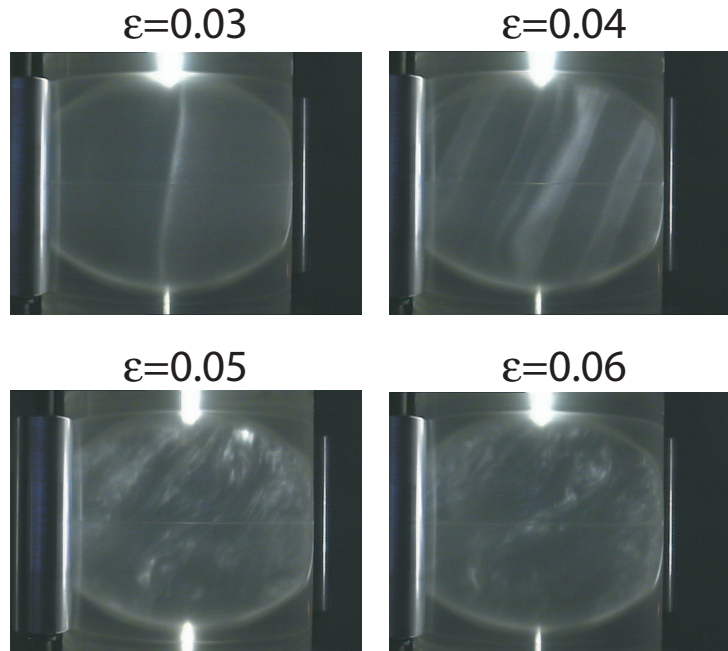


FIG. 6.8 – Kalliroscope visualization of the elliptical instability for a fixed Ekman number  $E = 10^{-5}$  and increasing values of  $\epsilon$  (purely hydrodynamical experiment). As suggested by (6.6), the relevant parameter to describe the dynamics of the elliptical instability is  $\alpha = E^{1/2}/\epsilon$ . Decreasing  $\alpha$  from 0.11 to 0.053, the flow becomes more and more complex, especially at small scale, but the spin-over mode remains present at large scale. The same behavior is expected to remain valid at the planetary scale, for instance in Io’s core where  $\alpha \sim 0.0036$ .

## 6.5 From laboratory models to geophysical applications

Magnetic induction by inertial waves is of particular interest in geo- and astrophysical applications. For instance, Kerswell and Malkus [57] have suggested that Io’s magnetic field is induced from Jupiter’s magnetic field by tidally driven inertial waves resonance, without dynamo action. However, our previous results derived in the limit of dominant magnetic diffusion (i.e. low  $Rm$ ) and for the laminar spin-over mode (i.e. at rather large  $E$ ) cannot apply directly to planetary configurations, corresponding to the limit of small  $E$ , large  $Rm$  and probably large wavenumbers. As can be seen in the visualizations of figure 6.8, the flow can then become increasingly complex, especially at small scale, and an extension of our analysis is necessary. Fortunately, a more general expression of the growth rate of the tidal instability, independent of the geometry of the flow, can be derived using the so-called local approach. Our goal here is not to give the exact expression of the growth rate of the various modes explicitly excited in a given planet but to determine an analytical expression able to describe the power dependence of the growth rate on all dimensionless numbers and to determine an order of magnitude of the various prefactors.

The local approach is based on the inviscid short-wavelength Lagrangian theory developed in [4] and [24], then generalized in [32] and [83]. There, perturbations are assumed to be sufficiently localized in order to be advected along flow trajectories and are searched as local plane waves of the form

$$(\mathbf{u}, p) = \left( \mathbf{u}(t), p(t) \right) e^{i\mathbf{k}(t) \cdot \mathbf{x}}, \quad (6.11)$$

where  $\mathbf{k}(t)$  is the time-dependent wave vector and  $\mathbf{x}$  the position vector. This method has been applied to the elliptical instability by Le Dizès [76]. Here, we extend his results by taking into account the induction equation and the Lorentz force in the limit of small Elsasser number (e.g.  $\Lambda \leq O(\epsilon)$ ) in the presence of an imposed

vertical magnetic field  $B_0$ , looking for a perturbed magnetic field under the same wave form

$$\mathbf{b} = \mathbf{b}(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}. \quad (6.12)$$

Details of the analysis are given in the appendix. Notice that in the following, we do not consider the limit of small magnetic Reynolds number anymore; hence, the magnetic field is made dimensionless using the amplitude of the imposed field  $B_0$  rather than  $Rm B_0$  as in the previous sections. MHD equations are solved analytically using a perturbative expansion in eccentricity  $\epsilon$ , supposing that the Lorentz force is of order  $\epsilon$ . In this context, equations for fluid motions at order 0 are similar to the purely hydrodynamical case. Through the Lorentz force, the magnetic field induces a correction in the fluid equation at order 1, hence a correction in the growth rate of the instability.

Using the 2D base flow  $\mathbf{U}_b$  given by (6.1), which corresponds to a stationary tidal deformation, we find a non viscous growth rate

$$\sigma_{nv} = \frac{9}{16}\epsilon - \frac{k^4\Lambda}{4(Rm^2 + k^4)}, \quad (6.13)$$

where  $k$  is a constant equal to the norm of the wave vector  $\mathbf{k}(t)$  at leading order in  $\epsilon$  (see equation 6.29 in appendix). The viscous damping rate resulting from the boundary layer can be estimated following [71], and induces a supplementary correction of the order  $O(E^{1/2})$ . Notice that in the limit of small magnetic Reynolds number, one immediately finds the linear Joule damping  $-\Lambda/4$  determined in section 6.3. The present result generalizes the validity of this scaling to all possible excited modes of the elliptical instability. Note also that the numerical factor before  $\epsilon$  is different from (6.6), but remains of the same order of magnitude in the relevant limit of small  $\epsilon$ .

The previous result can still not be directly applied to the case of Io, where the elliptical deformation is not stationary. Indeed, as explained for instance in [57], Io is almost synchronized in its revolution around Jupiter, but orbital resonances with Europa and Ganymede force it to follow a slightly elliptical orbit of eccentricity 0.004. As a result, the tidal bulge raised by Jupiter, of magnitude  $\epsilon \sim 6 \times 10^{-3}$ , does not rotate exactly at the same velocity as Io's spin, but oscillates back and forth across Io's body with a typical angular velocity  $1 - \beta \cos(t)$ , where  $\beta \sim 0.008$  is twice the eccentricity of Io's orbit and where time is made dimensionless using Io's spin velocity. In this case, the base flow in Io's core at first order in  $\beta\epsilon$  writes (see Appendix)

$$\mathbf{U}_b = \left( \begin{array}{l} -y + \beta\epsilon \cos(t)(\sin(2t)x - \cos(2t)y), \\ x - \beta\epsilon \cos(t)(\cos(2t)x + \sin(2t)y), 0 \end{array} \right), \quad (6.14)$$

and the growth rate of the elliptic instability writes

$$\sigma = \frac{17}{64} \sqrt{(\beta\epsilon)^2 - \frac{576}{289} \frac{\Lambda^2 Rm^2 k^4}{(Rm^2 + 4k^4)^2}} - \frac{3}{4} \frac{k^4 \Lambda}{Rm^2 + 4k^4}. \quad (6.15)$$

Formula (6.15) is closely related to (6.13), where the eccentricity  $\epsilon$  in the case of a stationary tidal deformation has been replaced by the product of the tidal bulge times the amplitude of the perturbation  $\beta\epsilon$ . In particular at small  $Rm$ , we once again end up at first order with a Joule damping linear in  $\Lambda$ . As mentioned before, surface viscous effects induce a correction to this formula of order  $O(E^{1/2})$  that could be explicitly determined. This is not done here, since the interest of formula (6.15) is to determine the relevant power law dependence on all dimensionless parameters (i.e.  $\epsilon, Rm, \Lambda, E$ ) as well as the order of magnitude of the various prefactors. In the following, we use for illustration the explicit values shown in (6.15) as well as the viscous correction  $8.8E^{1/2}$  determined in [57] for the first excited resonance in Io's configuration, but all our conclusions remain valid using prefactors of the same order of magnitude.

Formula (6.15) allows us to compute the order of magnitude of the growth rate of the elliptical instability in Io's core. We take as typical values an imposed magnetic field by Jupiter  $B_0 = 1850nT$ , and for Io's core  $R = 900km$ ,  $2\pi/\Omega_0 = 1.77days$ , and  $\nu = 10^{-6}m^2s^{-1}$ ,  $\sigma_e = 4 \times 10^5 Sm^{-1}$ ,  $\rho = 12000kgm^{-3}$ , consistent with a Fe/Fe-S composition. Then,  $Rm = 1.7 \times 10^7$ ,  $E = 3.0 \times 10^{-14}$ ,  $\Lambda = 2.8 \times 10^{-6}$  and (6.15) implies that none of the elliptical modes is significantly affected by Joule damping. The typical growth rate of the tidal instability in Io is about  $0.014years^{-1}$ , suggesting rapid large-scale variations in its core flows. Supposing that the spin-over mode still has an important component in Io's core (see figure 6.8 and appendix), its saturation amplitude would be about  $\Omega_H = 0.096\Omega_0$  according to (6.8). The corresponding induced magnetic field would be a dipole

aligned with the spin-over axis and of typical amplitude  $b_r \sim \sin(0.096) B_0 = 178nT$ , as derived from [70] in the relevant limit of large magnetic Reynolds number. We expect this field to fluctuate on rapid times ranging between the rotation period of 1.77 days and the typical time given by the growth rate of the instability of 72 years. The question then remains whether this field is measurable outside the core. Taking as in the Earth a typical mantle conductivity of  $0.1 - 1 S.m^{-1}$  and considering the short period signal of about 1 day, we find a skin length of  $200 - 600 km$ . In the Earth, this low value compared to the typical depth of the mantle means that signals coming from the elliptical instability will be totally filtered. This will not be the case in Io. Hence, continuous field measurements of the ambient field in the vicinity of Io would allow to discriminate between its internal and atmospheric origins, an issue raised since the first punctual measurements provided by the Galileo mission (e.g. [62]).

## 6.6 Conclusion

In this paper, we have studied the effects of an imposed magnetic field on the elliptical instability in spheroids. By combining theoretical elements of previous works [70], [131], we have extended the nonlinear system governing the dynamics of the spin-over mode to include simultaneously the magnetic and viscous damping. We have shown theoretically and confirmed experimentally that the linear growth rate of the instability as well as the square amplitude of the induced magnetic dipole fall down linearly with the Elsasser number (i.e. with the square of the imposed magnetic field), with good agreement regarding predicted and measured prefactors. These conclusions have then been extended to all possible resonances of the elliptical instability using a short-wavelength Lagrangian approach. Applied to the specific case of Io in the magnetic field of Jupiter, we conclude that despite the viscous and Joule damping, a tidal instability is more than probable in the Jovian moon's core and induces in the core a relatively important field of about 10% of the ambient value. In addition to the magnetospheric interactions with Jupiter [121], we thus conclude from purely magnetohydrodynamical considerations that the elliptical instability provides a significant and non-stationary contribution to the magnetic field measured in the vicinity of Io, as first suggested by [57]. Continuous measurements in Io's vicinity should allow to discriminate between internal and external magnetic signatures.

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## 6.7 Appendix : WKB analysis of Io's tidal instability

The now classical application of the short-wavelength "WKB" theory to inviscid fluids was developed in [4] and [24], generalized in [32] and [83], and summarized for the elliptical instability in [76]. It consists in looking for a perturbed solution of the full equations of motion under the form of a plane wave along the streamlines of the base flow. In our case, we thus look for a perturbed solution of the non-dimensional system

$$\nabla \cdot \mathbf{u} = 0, \quad (6.16)$$

$$d_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{\Lambda}{Rm} (\nabla \times \mathbf{b}) \times \mathbf{b}, \quad (6.17)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (6.18)$$

$$d_t \mathbf{b} + (\mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u} + \frac{1}{Rm} \nabla^2 \mathbf{b}, \quad (6.19)$$

under the form

$$\mathbf{u} = \mathbf{U}_b + \mathbf{u}(t) e^{i\mathbf{k}(t) \cdot \mathbf{x}}, \quad (6.20)$$

$$\mathbf{p} = \mathbf{P}_b + \mathbf{p}(t) e^{i\mathbf{k}(t) \cdot \mathbf{x}}, \quad (6.21)$$

$$\mathbf{b} = \mathbf{B}_0 + \mathbf{b}(t) e^{i\mathbf{k}(t) \cdot \mathbf{x}}, \quad (6.22)$$

along the streamlines of the base flow described by

$$\frac{d\mathbf{x}}{dt} = \mathbf{U}_b, \quad (6.23)$$

where  $\mathbf{U}_b$  stands for the two-dimensional base flow,  $\mathbf{P}_b$  for the corresponding pressure field,  $\mathbf{B}_0 = (0, 0, 1)$  for the (non-dimensional) imposed vertical magnetic field and  $\mathbf{x}$  for the position vector. The linearised MHD equations then write

$$\mathbf{k} \cdot \mathbf{u} = 0, \quad (6.24)$$

$$d_t \mathbf{u} + i\mathbf{u}(d_t \mathbf{k} \cdot \mathbf{x}) + i(\mathbf{U}_b \cdot \mathbf{k})\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{U}_b = -i\mathbf{k}p + \frac{\Lambda}{Rm}(\mathbf{i}\mathbf{k} \times \mathbf{b}) \times \mathbf{B}_0, \quad (6.25)$$

$$\mathbf{k} \cdot \mathbf{b} = 0, \quad (6.26)$$

$$d_t \mathbf{b}(t) + i\mathbf{b}(d_t \mathbf{k} \cdot \mathbf{x}) + i(\mathbf{U}_b \cdot \mathbf{k})\mathbf{b} = (\mathbf{b} \cdot \nabla)\mathbf{U}_b + (\mathbf{B}_0 \cdot \mathbf{k})\mathbf{u} - \frac{k^2}{Rm}\mathbf{b}, \quad (6.27)$$

The velocity and induction equations can be decoupled in space and time to give an equation for the wave vector only

$$d_t \mathbf{k} \cdot \mathbf{x} + \mathbf{U}_b \cdot \mathbf{k} = 0. \quad (6.28)$$

Linearized equations are then solved analytically using a perturbative expansion in the small parameter (i.e. the eccentricity in our case), supposing that the Elsasser number is of order 1. In this context, equations for fluid motions at order 0 are similar to the purely hydrodynamical equations, and the Lorentz force only induces a correction at order 1. Technically, the easiest way to solve the MHD equations in our case is to use the vertical velocity  $u_z$  and the vertical vorticity  $W_z = \partial_x u_y - \partial_y u_x = i(k_x u_y - k_y u_x)$  of the perturbed field as unknowns, as well as the vertical component  $b_z$  of the perturbed magnetic field and the corresponding magnetic vertical vorticity  $C_z = i(k_x b_y - k_y b_x)$ . The resolution is then straightforward.

The study of the 2D base flow  $\mathbf{U}_b$  given by (6.1), which corresponds to a stationary tidal deformation, closely follows the results already presented by Le Dizès [76]. From (6.28), one immediately finds the wave vector

$$\mathbf{k}(t) = k \left( \frac{\sin(a)}{\sqrt{A}} \cos(\chi t), \sin(a)\sqrt{A} \sin(\chi t), \cos(a) \right), \quad (6.29)$$

where  $k$  is a constant,  $A = \sqrt{(1 + \epsilon)/(1 - \epsilon)}$  is the ellipticity,  $\chi = \sqrt{1 - \epsilon^2}$ , and  $a$  is the angle between the flow rotation axis and the wavevector. Equations for fluid motions at order 0 give the expression of  $\mathbf{u}$  at order 0 with a temporal frequency  $f$

$$f = \pm 2 \cos(a), \quad (6.30)$$

whereas the linearised induction equation immediately gives the expression of  $\mathbf{b}$  at order 0. According to [76], an elliptical instability is possible if the forcing terms due to the elliptical deformation oscillate at the same frequency as the inertial wave, which means in our case  $f = 1$ . Then, at order 1 in  $\epsilon$ , inertial waves resonate implying an exponential growth rate of the elliptical instability [76]

$$\sigma_{nv} = \frac{9}{16}\epsilon - \frac{k^4 \Lambda}{4(Rm^2 + k^4)}. \quad (6.31)$$

This result is not directly applicable to the case of Io, where the elliptical deformation is not stationary. Indeed, Io is almost synchronized with Jupiter. It means that, in an isolated Jupiter-Io system, Io's spinning and orbital periods would be exactly equal : Io would always present the same face to Jupiter, and the tidal bulge would rotate at exactly the same frequency as Io. However, well-known orbital resonances with the other Galilean satellites maintain a 0.004 eccentricity in Io's orbit. The equality of orbital and spinning velocities is only true on average : in reality the orbital angular velocity - hence the tidal bulge angular velocity - varies periodically with the orbital radius around this mean. Focusing on the first harmonic of this oscillation, the orbital angular velocity in the absolute frame of reference writes  $1 - \beta \cos(t)$ , where  $\beta \sim 0.008$  is twice the eccentricity of Io's orbit and where time is made dimensionless using Io's (constant) spin velocity. As demonstrated by Kerswell and Malkus [57], the fluid's laminar response in Io's core driven by this tidally distorted mantle motion corresponds in the bulge frame to the simple elliptical flow

$$\mathbf{U}_b^{B.F.} = \left( -(1 + \epsilon)\beta \cos(t)Y, (1 - \epsilon)\beta \cos(t)X, 0 \right), \quad (6.32)$$

which is an exact nonlinear solution to the incompressible Navier-Stokes equations of motion for any finite viscosity in the spheroid of equation  $X^2/(1 + \epsilon) + Y^2/(1 - \epsilon) + Z^2 = 1$ . A simple change of frame then gives

the base flow in the absolute frame of reference at first order in  $\beta$

$$\mathbf{U}_b = \left( \begin{array}{l} -y + \beta\epsilon \cos(t)(\sin(2t)x - \cos(2t)y), \\ x - \beta\epsilon \cos(t)(\cos(2t)x + \sin(2t)y), 0 \end{array} \right). \quad (6.33)$$

For a given initial position  $(R, 0)$ , streamlines are described by

$$x(t) = R \cos(t) + \frac{\beta\epsilon R}{2}(1 - \cos(2t)), \quad (6.34)$$

$$y(t) = R \sin(t) - \frac{\beta\epsilon R}{2} \sin(2t), \quad (6.35)$$

(note that the results of the WKB theory do not depend on the chosen initial position along a closed trajectory). The solution to the wave vector equation along this streamline then writes

$$\mathbf{k}(t) = k \left( \begin{array}{l} \sin(a) \cos(t + \phi) + \beta\epsilon/2(\cos(2t - \phi) - \cos(\phi)), \\ \sin(a) \sin(t) + \beta\epsilon/2(\sin(2t - \phi) + \sin(\phi)), \cos(a) \end{array} \right), \quad (6.36)$$

where  $k$ ,  $a$  and  $\phi$  are constant. At order zero in  $\beta\epsilon$ , the system can be reduced to a single equation for  $u_z$

$$\frac{d^2 u_z^0}{dt^2} + 4 \cos^2(a) u_z^0 = 0, \quad (6.37)$$

whose solution writes

$$u_z(t) = c_1 e^{ift} + c_2 e^{-ift}, \quad (6.38)$$

where  $c_1, c_2$  are constant and  $f$  is the frequency determined as a function of the wave vector, i.e.  $f = 2 \cos(a)$ . At order 1, we allow a long term variation of the solution at order zero, i.e.

$$u_z(t) = (c_1 e^{ift} + c_2 e^{-ift}) e^{\beta\epsilon\sigma t}, \quad (6.39)$$

where  $\sigma$  is the growth rate of the instability. The system then reduces to the same type of equation as (6.37), with a forcing term directly coming from the first order terms in the base flow. According to (6.33), terms of type  $\cos(t) \sin(2t)$  and  $\cos(t) \cos(2t)$  arise here, and the equation at order 1 schematically reads

$$\frac{d^2 u_z^1}{dt^2} + 4 \cos^2(a) u_z^1 = F(e^{it/2}, e^{-it/2}, e^{3it/2}, e^{-3it/2}). \quad (6.40)$$

Solvability conditions then imply

$$\int_0^{2\pi/f} F(e^{it/2}, e^{-it/2}, e^{3it/2}, e^{-3it/2}) e^{ift} dt = 0, \quad (6.41)$$

$$\int_0^{-2\pi/f} F(e^{it/2}, e^{-it/2}, e^{3it/2}, e^{-3it/2}) e^{-ift} dt = 0, \quad (6.42)$$

which validate the whole asymptotic approach by ensuring that forcing terms are not secular. This system with unknown  $c_1, c_2$  admits a non trivial solution if and only if  $f = 1/2$  or  $f = 3/2$ , in which case the growth rate  $\sigma$  is determined by the nullity of the determinant. It is then maximized over all values of wavevector phase  $\phi$ . The maximum is obtained for  $f = 1/2$  and  $\phi = \pi/4$  and writes

$$\sigma = \frac{17}{64} \sqrt{(\beta\epsilon)^2 - \frac{576}{289} \frac{\Lambda^2 R m^2 k^4}{(R m^2 + 4k^4)^2}} - \frac{3}{4} \frac{k^4 \Lambda}{R m^2 + 4k^4}. \quad (6.43)$$

Note that the resonance condition  $f = 1/2$  corresponds to the resonance condition for the closely related precession instability, where the spin-over mode is also known to be excited [88]. This validates the evaluation of the spin-over induction performed at the end of section 6.5.



## Chapitre 7

# Effet dynamo d'une onde inertielle confinée en géométrie cylindrique

Dans ce chapitre, nous présentons les premiers résultats d'une étude sur la dynamo cinématique d'une onde inertielle dans un cylindre en rotation rapide. Nous développons une méthode numérique pour les modes spinover, pour lequel le problème de la dynamo se réduit à un problème aux valeurs propres. Nous identifions des dynamos, mais seulement si les effets visqueux sont pris en compte dans l'écoulement. Nous combinons l'approche numérique avec une théorie asymptotique qui identifie le mécanisme de la dynamo et les paramètres pertinents dans le régime asymptotique.

### 7.1 Introduction

De façon générale, le problème de la dynamo cinématique consiste à trouver des champs magnétiques  $\mathbf{b}$  croissants des équations

$$\partial_t \mathbf{b} + (\mathbf{U} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{U} + \eta \Delta \mathbf{b} \quad , \quad \nabla \cdot \mathbf{b} = 0 \quad (7.1)$$

pour un champ de vitesse  $\mathbf{U}$  donné. Nous reconnaissons l'équation d'induction et la loi de Gauss. Le problème de la dynamo cinématique ressemble à un problème de diffusion-advection pour le champ magnétique dans un champ de vitesse  $\mathbf{U}$ , avec un terme source  $(\mathbf{b} \cdot \nabla) \mathbf{U}$ , associé à l'étirement du champ magnétique. Cette terminologie provient de l'analogie avec la vorticit  qui satisfait la m me  quation. Comme pour la vorticit , sans diffusion magn tique, les lignes du champ magn tique sont gel es dans le fluide. La diffusion magn tique est donc souvent un ingr dient important dans la production d'un effet dynamo. Nous renvoyons le lecteur au livre de Moffatt [102], ou   celui  dit    l'issue de l' cole d' t  des Houches [18] pour une introduction   la dynamo dans un contexte g n ral.

L' quation (7.1) est invariante si on se place dans un rep re tournant. En pr sence de rotation rapide, il est donc possible de soustraire cette rotation en se plaçant dans le rep re tournant. Le probl me de la dynamo cinématique, ne sent alors les effets de la rotation rapide qu'  travers l' coulement qui est fortement influenc  par la rotation. Les ondes de Rossby ou les  coulements de type colonne, tels que les colonnes de Busse, ont re u beaucoup d'attention dans la communaut  de la g odynamo, en vue notamment de comprendre l'origine du champ magn tiques des plan tes. Dans ce contexte, nous renvoyons le lecteur aux travaux de Busse [11], [12] et aux nombreuses simulations de la g odynamo [40], [30], [23], [14], [51], qui confirment que la convection est capable de g n rer un  coulement ayant un effet dynamo. Mais tous les corps c lestes avec un champ magn tique   grande  chelle ne rentrent pas dans les crit res de la convection, et des alternatives comme la dynamo de pr cession ou elliptique ont  t  envisag es [88], [60]. Ces m canismes d'excitation restent plus difficiles   mod liser, et   simuler num riquement, et ce n'est que r cemment que la possibilit  d'une dynamo a  t  d montr e [132], [133], [141]. Les ondes inertielles jouent un r le important dans ces m canismes d'excitation. Oscillant typiquement sur l' chelle de temps du jour, et   la structure spatiale essentiellement tridimensionnelle et non-triviale, m me le probl me de la dynamo cinématique est suffisamment complexe pour ne pas avoir re u encore beaucoup d'attention dans la litt rature. Les premi res  tudes qui cherchent sp cifiquement   identifier une dynamo d'onde inertielles, ont  t  r alis es en domaine infini par Moffatt [101], et Soward [124]. Ils montrent qu'une superposition al atoire d'ondes propagatives, en absence de sym trie selon un plan de r flexion perpendiculaire   l'axe de rotation, poss de une h licit  non n gligeable, ce qui donc autorise la pr sence d'une



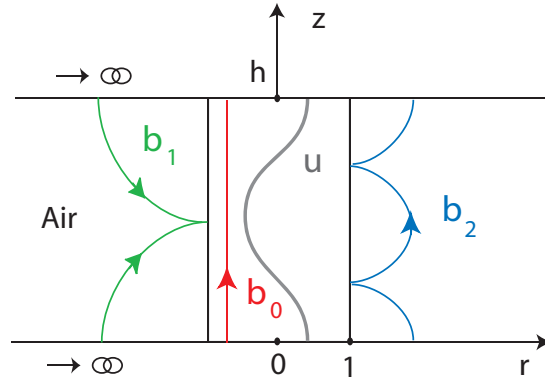


FIG. 7.1 – Schéma du domaine cylindrique que nous considérons pour la modélisation du dynamo cinématique. Dans la zone intérieure de métal liquide, un écoulement d'onde inertielle est présente. Le champ magnétique s'échappe vers un domaine extérieur, (air) où le champ est potentiel. Nous mettons des parois de perméabilité infinie en  $z = 0$  et  $z = h$ , pour rendre l'analyse possible.

dynamo par un effet "alpha". Malheureusement, les géométries planétaires intéressantes ont toutes un plan de symétrie axial et des parois sur lesquelles se réfléchissent les ondes de manière à supprimer l'hélicité globale de l'écoulement. La présence de ces symétries rajoutée à la rapidité de l'échelle temporelle de variation des ondes inertielles rend le problème de la dynamo cinématique des ondes inertielles particulièrement difficile. Cependant, plusieurs expériences montrent qu'un effet dynamo peut bien être présent. Gans [36] rapporte en effet qu'un écoulement de sodium dans un cylindre en précession est capable d'amplifier un champ imposé par un facteur trois. Il est probable qu'une expérience de plus grande taille, permettrait de créer une dynamo. Les expériences du groupe de Lathrop [52] et du LGIT, ont montré que des ondes inertielles ou des ondes magnéto-Coriolis apparaissent dans un écoulement de métal liquide dans une configuration de Couette sphérique. Il n'est pas impossible que ces ondes puissent intervenir dans les dynamos envisagées.

Nous proposons, dans ce chapitre, une étude de la dynamo d'une onde inertielle particulière. Nous allons considérer le mode de spinover en géométrie cylindrique. Nous connaissons l'expression exacte de ce mode. De plus, le mode de spinover est stationnaire dans le repère fixe : l'effet dynamo peut donc être étudié comme un problème aux valeurs propres. Dans le cas général, les ondes inertielles sont périodiques et une analyse de Floquet est donc nécessaire. Dans une première partie, nous allons montrer qu'un mode de spinover est capable de produire un effet dynamo. Dans la deuxième partie, nous utiliserons une approche asymptotique pour analyser le mécanisme d'instabilité, ce qui nous aidera à comprendre le rôle de la viscosité et de la diffusion magnétique.

## 7.2 Définition du problème

Le système est schématisé sur la Fig. 7.1. Le métal liquide est contenu dans un cylindre, de hauteur  $H$  et de rayon  $R$ . Le métal liquide a une conductivité  $\sigma_e$ , une perméabilité  $\mu$ , une viscosité cinématique  $\nu$ , et une diffusivité magnétique  $\eta = (\sigma_e \mu)^{-1}$ . Le cylindre et le métal liquide sont en rotation rapide autour de l'axe de révolution du cylindre,  $\hat{\mathbf{z}}$ , à la vitesse angulaire  $\Omega$ . Nous superposons à cette rotation rapide, une onde inertielle  $\mathbf{U}$  de faible amplitude et nous voulons savoir si cet écoulement peut exciter un champ magnétique par effet dynamo. Nous supposons que le champ magnétique se raccorde à une zone extérieure isolante, sur les bords latéraux du cylindre. De plus, nous allons supposer que les bords supérieurs et inférieurs du cylindre en  $z = 0$  et  $z = H$  ont une perméabilité infinie, pour que l'analyse théorique et numérique reste abordable. Par la suite, nous noterons l'intérieur du cylindre  $V_i$ , et l'extérieur  $V_e$ .

Pour adimensionner le problème, nous utilisons les échelles.

$$[\mathbf{r}] = R \quad , \quad [t] = \Omega^{-1} \quad , \quad [\mathbf{u}] = U. \quad (7.2)$$

Remarquons que l'échelle de temps correspond à celui de la variation typique des ondes inertielles dans le repère tournant et non à un temps de diffusion magnétique. L'échelle du champ magnétique est arbitraire dans le problème linéaire de la dynamo cinématique.

L'écoulement dépendra alors du temps de la manière suivante :

$$\mathbf{U} = \mathbf{u}(\mathbf{r}) e^{i\bar{\lambda}t} + cc. , \quad (7.3)$$

avec  $\bar{\lambda} \in [-2, 2]$ . L'équation d'induction dans le repère tournant s'écrit

$$\partial_t \mathbf{b} = Ro \left[ \left( -(\mathbf{u} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{u} \right) e^{i\bar{\lambda}t} + cc. \right] + E_m \Delta \mathbf{b} , \quad \mathbf{r} \in V_i \quad (7.4)$$

où les nombres de Rossby  $Ro$  et d'Ekman magnétique  $E_m$ , sont définis par  $Ro = U/\Omega R$  et  $E_m = \eta/\Omega R^2$ . De plus, le champ magnétique doit respecter la loi de Gauss  $\nabla \cdot \mathbf{b} = 0$  et se raccorder continûment à un champ potentiel à l'extérieur :

$$\mathbf{b} = \nabla \Phi , \quad \Delta \Phi = 0 , \quad \mathbf{r} \in V_e \quad (7.5)$$

Il est important de souligner que dans le contexte géophysique, les nombres de Rossby et d'Ekman magnétique sont très petits,  $E_m \simeq 10^{-9}$  et  $Ro \simeq 10^{-7}$ , (Ch. 3 de [18]). Cela permettra de justifier l'approche asymptotique qui sera développée dans la section 7.4.

A partir de maintenant, nous allons considérer une onde inertielle particulière : le mode de spinover. Ce mode est stationnaire dans le repère fixe (du laboratoire). Pour la résolution numérique, nous nous placerons dans ce repère. Le problème se réduit alors à un problème aux valeurs propres. Nous rappelons qu'il existe plusieurs modes de spinover labellisé par l'indice  $j$ . Dans le repère fixe, le champ de vitesse s'écrit  $\mathbf{U}_j^{so} = \mathbf{u}_j + cc.$ , et dans l'approximation non-visqueuse  $\mathbf{u}_j$  s'écrit

$$\begin{bmatrix} u_{j+} \\ u_{j-} \\ u_{jz} \end{bmatrix} = \frac{1}{N_j} \begin{bmatrix} 1/\sqrt{2} & J_2(\bar{k}_j r) \cos(\bar{l}_j z) \\ 3/\sqrt{2} & J_0(\bar{k}_j r) \cos(\bar{l}_j z) \\ \bar{k}_j / \bar{l}_j & J_1(\bar{k}_j r) \sin(\bar{l}_j z) \end{bmatrix} e^{i\phi} . \quad (7.6)$$

Les modes sont normés par  $N_j$  qui est défini par

$$N_j^2 = \frac{2}{\pi h} \int_{V_i} \mathbf{u}_j^* \cdot \mathbf{u}_j dV , \quad (7.7)$$

où  $h = H/R$  est le rapport d'aspect du cylindre. De cette façon un mode d'amplitude unitaire a une énergie cinétique égale à celle de la rotation solide. Les nombres d'onde radiaux  $\bar{k}_j$  sont les zéros de la relation de dispersion

$$J_2(\bar{k}_j) + 3 J_0(\bar{k}_j) = 0 \quad (7.8)$$

Le nombre d'onde vertical est déterminé par  $\bar{l}_j = \bar{k}_j / \sqrt{3}$ . Ce nombre fixe les rapports d'aspects des cylindres dans lesquels le mode de spinover peut exister :  $h = \bar{n}\pi / \bar{l}_j$ ,  $\bar{n} \in \mathbb{N}$ . Les trois premiers modes de spinover (ceux avec les plus grandes structures spatiales) ont les nombres d'onde radiaux suivants :  $\bar{k}_1 = 2.74$ ,  $\bar{k}_2 = 5.69$ ,  $\bar{k}_3 = 8.76$ . Ces modes existent respectivement dans les cylindres de rapports d'aspect  $h_1 = 1.98 \bar{n}$ ,  $h_2 = 0.955 \bar{n}$ ,  $h_3 = 0.621 \bar{n}$ . Les modes de spinover peuvent être excités par l'instabilité elliptique si le rapport d'aspect prend une de ces valeurs. Dans la suite, nous nous concentrons sur le premier mode de spinover dans un cylindre au rapport d'aspect  $h = 3.9796$ . Le mode a une longueur d'onde entière dans sa structure spatiale vertical ( $\bar{n} = 2$ ). Des visualisations du champ de vitesse ont été montrées sur la figure Fig. 2.6. Nous allons profiter de la connaissance des expressions analytiques du mode de spinover, lorsque celui-ci respecte une condition de non-glissement sur la paroi cylindrique. Les profils azimutaux et verticaux des modes visqueux restent identiques aux modes non-visqueux. La dépendance radiale des modes est composée de trois fonctions de Bessel modifiée. Les coefficients et les nombres d'ondes dépendent de façon non-triviale du nombre d'Ekman,  $E$ , et sont généralement complexes. Dans l'analyse, nous fixerons  $E$  et nous calculerons les structures spatiales des modes de spinover pour les mêmes  $\bar{l}$ . Un detuning visqueux des fréquences  $\omega_j^v$  est inévitable, c'est à dire  $Re(\omega^v) \neq 0$ , mais comme précédemment nous pouvons argumenter que si le mode est excité par l'instabilité elliptique, il sera effectivement stationnaire. Les profils seront normalisés à l'aide de  $N_j^{-1}$  comme pour les modes non-visqueux.

## 7.3 Solution numérique de la dynamo cinématique de l'écoulement du mode de spinover.

Ecrivons l'équation d'induction dans le repère fixe :

$$(\partial_t + \partial_\phi) \mathbf{b} = Ro \left[ \left( -(\mathbf{u} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{u} \right) + cc. \right] + E_m \Delta \mathbf{b} . \quad (7.9)$$

Les équations pour le champ magnétique à l'extérieur ne changent pas. Le terme d'advection dû à la rotation apparaît explicitement, mais la dépendance temporelle de l'écoulement disparaît. Nous n'avons pas changé la notation des champs, car la structure spatiale du champ ne change pas lors du changement de repère. Pour résoudre le problème de dynamo numériquement, nous projetons l'équation (7.9) sur la base des modes propres de l'équation de diffusion. Une méthode similaire a été utilisée précédemment en [93], [107]. En pratique, nous décomposons  $\mathbf{b}$  sous la forme

$$\mathbf{b} = \sum_{x=1}^{N_r} \sum_{m=-N_\phi}^{N_\phi} \sum_{n=0}^{N_z} a_{xnm}(t) \mathbf{B}_{xnm}(\mathbf{r}) , \quad (7.10)$$

où les  $\mathbf{B}_{xnm}(\mathbf{r})$  correspondent aux profils spatiaux des modes propres du problème de diffusion pure (et qui se raccordent à un champ potentiel à l'extérieur). Ces modes sont calculés ci-dessous. Pour que le champ magnétique total soit réel, chaque mode non-axisymétrique est accompagné par son complexe conjugué, ce qui implique

$$a_{xmn} = (a_{x-mn})^* . \quad (7.11)$$

Pour les modes axisymétriques, des précautions serions prises pour assurer la réalité des champs. L'utilisation des modes propres diffusifs comme champs de base nécessite la construction d'une relation d'orthogonalité, en utilisant les modes adjoints. La projection du problème conduit à représentation matricielle du problème et la réalité des champs sera utilisée pour réduire la dimensionnalité. Les valeurs propres de cette matrice sont ensuite calculées numériquement. Nous donnons ci-dessous quelques détails sur l'implémentation de la méthode avant de passer aux résultats.

### 7.3.1 Les modes de diffusion pure du champ magnétique

En absence d'écoulement, le champ magnétique à l'intérieur du domaine de métal liquide satisfait une équation de diffusion et doit se raccorder sur un champ potentiel à l'extérieur du domaine de métal liquide :

$$\partial_t \mathbf{B} = E_m \Delta \mathbf{B} \quad \nabla \cdot \mathbf{B} = 0 \quad , \quad \mathbf{r} \in V_i , \quad (7.12)$$

$$\mathbf{B} = \nabla \Phi \quad \Delta \Phi = 0 \quad , \quad \mathbf{r} \in V_e . \quad (7.13)$$

Comme pour les modes visqueux, nous utiliserons les champ  $b_\pm = (b_r \pm ib_\phi)/\sqrt{2}$ . En géométrie cylindrique, ce problème est séparable, et nous pouvons trouver les solutions propres analytiquement. Les conditions aux limites imposent au champ magnétique d'être perpendiculaire aux parois horizontales de perméabilité infinie et continu à travers la paroi cylindrique :

$$\hat{z} \times \mathbf{B} = 0|_{z=0,h} \quad , \quad \mathbf{B}|_{r=1-} = \mathbf{B}|_{r=1+} . \quad (7.14)$$

Les modes magnétiques diffusifs dans un cylindre seront discrétisés. Comme pour les ondes inertielles, on les désignera à l'aide d'un triplet  $[x, m, n]$  où  $m \in \mathbb{Z}$  est le nombre d'onde azimutal,  $x \in \mathbb{N}_0$ , le nombre de structures dans la direction radiale, et  $n \in \mathbb{N}_0$ , le nombre de demi longueurs d'onde dans la direction verticale. Pour les modes sans structure verticale, nous aurons  $n = 0$ . Dans le repère tournant, les modes propres  $[x, m, n]$  sont purement évanescents, quelque soit leur structure spatiale. Le taux de dissipation est exactement  $\gamma_{xn} = E_m (k_x^2 + l_n^2)$ . La structure spatiale des modes propres de diffusion avec structure verticale ( $n \neq 0$ ) s'écrit

$$\begin{bmatrix} (B_{xmn})_\pm \\ (B_{xmn})_z \end{bmatrix} = \frac{1}{N_{xmn}} \begin{bmatrix} d_\pm & J_{m\pm 1}(k_x r) \sin(l_n z) \\ d_z & J_m(k_x r) \cos(l_n z) \end{bmatrix} e^{im\phi} , \quad (7.15)$$

à l'intérieur du cylindre. La loi de Gauss impose une contrainte supplémentaire. Utilisant les relations de récurrence des fonctions de Bessel (voir Annexe A.2), cette contrainte se réduit à une relation entre les constantes  $d_{\pm}$  et  $d_z$ ,

$$\left( \frac{k_x}{\sqrt{2}} (d_+ - d_-) - l_n d_z \right) J_m(k_x r) e^{im\phi} \sin(l_n z) = 0 \quad \Rightarrow \quad d_z = \frac{1}{\sqrt{2}} \frac{k_x}{l_n} (d_+ - d_-). \quad (7.16)$$

A l'extérieur du cylindre, le potentiel et le champ magnétique qui en dérive, sont donnés par

$$\begin{bmatrix} \Phi_{xmn} \\ (B_{xmn})_{\pm} \\ (B_{xmn})_z \end{bmatrix} = \frac{1}{N_{xmn}} \begin{bmatrix} \delta / l_n & K_m(l_n r) \sin(l_n z) \\ -\delta / \sqrt{2} & K_{m\pm 1}(l_n r) \sin(l_n z) \\ \delta & K_m(l_n r) \cos(l_n z) \end{bmatrix} e^{im\phi}. \quad (7.17)$$

Avec  $l_n = n\pi/h$ , la condition aux limites sur les surfaces horizontales est automatiquement respectée. La continuité du champ sur la surface cylindrique entre domaine intérieur et extérieur exige

$$\begin{bmatrix} J_{m+1}(k_x) & 0 & \frac{1}{\sqrt{2}} K_{m+1}(l_n) \\ 0 & J_{m-1}(k_x) & \frac{1}{\sqrt{2}} K_{m-1}(l_n) \\ \frac{1}{\sqrt{2}} \frac{k_x}{l_n} J_m(k_x) & -\frac{1}{\sqrt{2}} \frac{k_x}{l_n} J_m(k_x) & -K_m(l_n) \end{bmatrix} \begin{bmatrix} d_+ \\ d_- \\ \delta \end{bmatrix} = 0, \quad (7.18)$$

d'où l'on obtient la relation de dispersion qui détermine les nombres radiaux  $k_x$  des modes propres, ainsi que les constantes  $d_+$ ,  $d_-$ ,  $\delta$ , en fonction d'une seule constante arbitraire. La solution numérique de cette relation de dispersion implicite, est obtenue à l'aide d'un logiciel programmé sous Matlab. Nous normalisons les modes en utilisant

$$N_j^2 = \int_V \mathbf{B}_j'^* \cdot \mathbf{B}_j' dV, \quad (7.19)$$

où  $V$  est tout l'espace. La normalisation n'a cependant pas vraiment un sens énergétique comme pour les ondes inertielles. Ici, cette opération est nécessaire pour faciliter l'interprétation, et pour éviter des problèmes numériques. Pour les modes axisymétriques,  $m = 0$ , nous trouvons deux types de solutions à la main. Les modes axisymétriques purement toroïdaux ont des nombres d'onde radiaux qui vérifient  $J_1(k_x) = 0$ . Le champ associé est purement azimutal,  $d_+ = d_- = i d$ ,  $d \in \mathbb{R}$ . Ce mode est invisible à l'extérieur car  $\delta = 0$ . La deuxième classe des modes axisymétriques est purement poloïdale  $d_+ = -d_- = d$ , avec  $d \in \mathbb{R}$ . Pour ces modes la composante azimutale du champ s'annule exactement. Les nombres d'ondes  $k_x$  sont fixé par la relation de dispersion

$$l_n J_1(k_x) K_0(l_n) + k_x J_0(k_x) K_1(l_n) = 0, \quad (7.20)$$

et  $\delta = -d \sqrt{2} J_1(k_x) / K_1(l_n)$ . Ce n'est que pour les modes axisymétriques qu'il y a une séparation parfaite entre modes toroïdaux et modes poloïdaux. Il existe d'autres solutions du champ magnétique sans structure verticale. A cause des conditions aux limites aux parois de perméabilité infinie, ces champ ne peuvent être que purement verticaux, avec la structure spatiale

$$(B_{xm0})_z = \frac{1}{N_{xm0}} J_m(k_x r) e^{im\phi}, \quad (B_{xm0})_{\pm} = \Phi_{xm0} = 0. \quad (7.21)$$

Ce champ vertical ne peut que se raccorder à un champ nul à l'extérieur. Le nombre d'onde  $k_x$  est le  $x$ -ième zéro le plus petit de  $J_m(k_x)$ . Le taux de dissipation des modes sans structure verticale est  $\gamma_{xm0} = E_m k_x^2$ . L'utilisation de parois horizontales de conductivité infinie aurait conduit à une classe de modes purement horizontaux, qui ne sont pas possible ici. Il est important de signaler que la structure spatiale des modes magnétiques ne dépend pas du nombre d'Ekman magnétique. Les modes ne sont donc calculés qu'une seule fois. La dépendance vis-à-vis de  $E_m$  n'apparaît que dans la formule explicite du taux de dissipation. Dans le repère fixe, les modes magnétique de diffusion ont une fréquence complexe  $\omega_{xmn} = -m + i \gamma_{xmn}$ .

### 7.3.2 Produit scalaire des modes diffusifs

L'expression (7.10) vérifie les conditions aux limites. Nous allons insérer cette expression dans l'équation (7.9) et projeter l'équation résultante. Nous ne rencontrerons pas les difficultés rencontrées pour le problème de l'instabilité elliptique et qui étaient reliées à la présence d'un couplage par la pression. Nous suivons la procédure classique et introduisons un produit scalaire, qui agit sur toutes les variables du problème. Nous noterons  $\mathbf{X}_j = [\mathbf{B}_j, \Phi_j]^T$  le quadrivecteur contenant les structure spatiales des modes magnétiques diffusifs ( $\mathbf{B}_j$  pour les champs dans  $V_i$ ,  $\Phi$  pour le potentiel dans  $V_e$ ). Nous utiliserons le produit scalaire entre deux modes,  $\mathbf{X}_1$  et  $\mathbf{X}_2$  :

$$\langle \mathbf{X}_1 | \mathbf{X}_2 \rangle = \int_{V_i} \mathbf{B}_1^* \cdot \mathbf{B}_2 \, dV + \int_{V_e} \Phi_1^* \Phi_2 \, dV . \quad (7.22)$$

Il est facile de montrer que pour ce produit scalaire, l'opérateur  $\mathcal{L}_j$  obtenu à partir des equations (7.12) et (7.13) est auto-adjoint. Les modes adjoints auront donc la même fonction dans leur structure spatiale que les modes directes :

$$\begin{bmatrix} (B_{xmn})_{\pm}^A \\ (B_{xmn})_z^A \\ \Phi_{xmn}^A \end{bmatrix} = \frac{1}{N_{xmn}^A} \begin{bmatrix} (d_{xmn})_{\pm}^A & J_{m\pm 1}(k_x r) & \sin(l_n z) \\ (d_{xmn})_z^A & J_m(k_x r) & \cos(l_n z) \\ (\delta_{xmn})^A / l_n & K_m(k_x r) & \sin(l_n z) \end{bmatrix} e^{im\phi} , \quad (7.23)$$

$$(B_{xm0})_z^A = \frac{1}{N_{xmn}^A} J_m(k_x r) e^{im\phi} \quad , \quad (B_{xm0})_{\pm} = \Phi_{xm0}^A = 0 , \quad (7.24)$$

avec  $N_{xmn}^A$  le facteur normalisant qui sera définie plus bas. Les conditions aux limites adjointes déterminent les coefficients  $d_{j\pm}^A$ ,  $d_{jz}^A$  et  $\delta_j^A$ . Ces conditions sont différentes de celles du problème direct, et ceci vient du fait que la loi de Gauss doit être imposée à travers une condition au limite dans le problème. Nous ne pouvons pas simplement inclure cette loi dans  $\mathcal{L}_j$ , comme en hydrodynamique, car l'absence d'une variable de pression fait que le système est surdéterminé. Par contre, nous pouvons remarquer que la loi de Gauss se réduit dans le problème direct à une condition sur les constantes  $d_{\pm}, d_z$  qui peut donc être vu comme une quatrième condition sur la surface cylindrique (en plus de la continuité du champ à travers la surface). Pour déterminer les conditions aux limites adjointes, nous devons annuler le terme de bord que l'on obtient dans l'opération  $\langle \mathbf{B}_j^A | \mathcal{L}_j \mathbf{B}_j \rangle - \langle \mathcal{L}_j^A \mathbf{B}_j^A | \mathbf{B}_j \rangle$ , en prenant en compte ces quatre conditions sur  $\mathbf{X}_j$ . Ce terme de bord s'écrit :

$$\begin{aligned} & B_{j+}^{A*} (\partial_r B_{j+}) - (\partial_r B_{j+}^{A*}) B_{j+} + B_{j-}^{A*} (\partial_r B_{j-}) - (\partial_r B_{j-}^{A*}) B_{j-} \\ & + B_{jz}^{A*} (\partial_r B_{jz}) - (\partial_r B_{jz}^{A*}) B_{jz} - \Phi_j^{A*} (\partial_r \Phi_j) + (\partial_r \Phi_j^{A*}) \Phi_j = 0 \Big|_{r=1} , \end{aligned} \quad (7.25)$$

et on obtient après quelques manipulations en  $r = 1$  :

$$\begin{aligned} & B_{+j}^A - B_{-j}^A = 0 \quad , \quad B_{zj}^A = 0 , \\ & \Phi_j^A + \frac{1}{\sqrt{2}} (\partial_r B_{+j}^A + (m_j + 1) B_{+j}^A) + \frac{1}{\sqrt{2}} (\partial_r B_{-j}^A - (m_j - 1) B_{-j}^A) = 0 . \end{aligned} \quad (7.26)$$

Les constantes apparaissant dans les expressions (7.23) et (7.24) peuvent maintenant être déterminées :

$$\begin{aligned} & (d_{xmn})_{\pm}^A = J_{m\mp 1}(k_x) \quad , \quad (d_{xmn})_z^A = 0 , \\ & (\delta_{xmn})^A = -\frac{1}{\sqrt{2}} k_x \left( J_{m-1}(k_x) - J_{m+1}(k_x) \right) J_m(k_x) . \end{aligned} \quad (7.27)$$

L'expression exacte du champ adjoint à l'extérieur se déduit toujours du potentiel en appliquant le gradient. Nous remarquons que la partie verticale des champs adjoints est toujours exactement égale à zéro sauf pour les modes sans structure verticale. Utilisant équations directes et adjointes et les conditions limites respectives, nous trouvons la relation d'orthogonalité des modes diffusifs sous la forme

$$(\tilde{\gamma}_j - \tilde{\gamma}_k) \int_{V_i} \mathbf{B}_j^{A*} \cdot \mathbf{B}_k \, dV = 0 . \quad (7.28)$$

Deux modes n'ayant pas la même fréquence sont donc orthogonaux vis à vis d'un produit scalaire réduit au cylindre intérieur, que l'on notera avec des crochets ronds  $(\cdot | \cdot)$ . Les modes adjoints sont normalisés de sorte que l'on ait :

$$\left( \mathbf{B}_j^A \mid \mathbf{B}_k \right) = \delta_{jk} \quad (7.29)$$

Cette relation nous permet d'utiliser les modes diffusifs comme champs de base.

### 7.3.3 La projection du problème

La projection de l'équation ( (7.9) après avoir insérée (7.10) ), sur chacun des modes adjoints, conduit à un système linéaire pour les amplitudes  $a_j$  des modes à nombre azimutal  $m_j \geq 0$

$$\dot{a}_j = \left( -i m_j - E_m \tilde{\gamma}_j \right) a_j + Ro \left[ \sum_j \left( D_{jk}^{(1)} + D_{jk}^{(2)} \right) a_k + D_{jk}^{(3)} a_k^* \right], \quad (7.30)$$

où  $\tilde{\gamma}_j = k_j^2 + l_j^2$ . Les matrices d'interaction entre les modes diffusives, sont respectivement définies comme

$$\begin{aligned} D_{jk}^{(1)} &= \left( \mathbf{B}_j^A \mid \mathcal{D}_u \mathbf{B}_k \right), \\ D_{jk}^{(2)} &= \left( \mathbf{B}_j^A \mid \mathcal{D}_{u^*} \mathbf{B}_k \right), \\ D_{jk}^{(3)} &= \left( \mathbf{B}_j^A \mid \mathcal{D}_u \mathbf{B}_k^* \right), \end{aligned} \quad (7.31)$$

$$\text{où } \mathcal{D}_u \bullet = -(\mathbf{u} \cdot \nabla) \bullet + (\bullet \cdot \nabla) \mathbf{u}. \quad (7.32)$$

Ces matrices sont creuses à cause de la structure trigonométrique dans les directions verticale et azimutale des modes magnétiques et du champ de vitesse. Plus spécifiquement, pour le premier mode de spinover, de nombres d'onde  $\bar{k}_1$ ,  $\bar{m} = \pm 1$ ,  $\bar{n} = 2$  et dans un cylindre de rapport d'aspect  $h = 3.9796$ , les seuls éléments non-nuls sont ceux obtenus lorsque :

$$\begin{aligned} D_{jk}^{(1)} \neq 0 &\Leftrightarrow m_j = m_k + 1 \quad , \quad n_j = n_k \pm 2, \\ D_{jk}^{(2)} \neq 0 &\Leftrightarrow m_j = m_k - 1 \quad , \quad n_j = n_k \pm 2, \\ D_{jk}^{(3)} \neq 0 &\Leftrightarrow m_j = 0, m_k = 1 \geq 0 \quad , \quad n_j = n_k \pm 2. \end{aligned} \quad (7.33)$$

Pour une troncature  $(N_r, N_\phi, N_z)$ , nous pouvons estimer le nombre d'éléments à calculer à  $(N_r^2 \times N_\phi \times 2N_z)$  pour  $D^{(1)}$  et  $D^{(2)}$ , et  $(N_r^2 \times 2 \times 2N_z)$  pour  $D^{(3)}$ . Les interactions entre les différents modes magnétiques ne peuvent se faire qu'au travers du mode de spinover  $[1, 1, 2]^+$ . On peut donc séparer les modes magnétiques diffusifs en trois classes distinctes

$$\begin{aligned} \text{Type I : } & n \text{ pair} \quad \& \quad m, n/2 \text{ même parité,} \\ \text{Type II : } & n \text{ pair} \quad \& \quad m, n/2 \text{ parité différente,} \\ \text{Type III : } & n \text{ impair} \quad , \end{aligned} \quad (7.34)$$

de sorte que les interactions se fassent au sein de chacun de ces classes. En pratique, nous commençons par calculer les nombres d'ondes  $k_x$  solutions de la relation de dispersion des modes magnétiques diffusifs, ainsi que les coefficients nécessaires pour évaluer la structure spatiale de tous les modes utilisés dans la décomposition. Nous remplissons alors les matrices d'interaction progressivement en utilisant les contraintes (7.33). La structure spatiale des modes est ensuite calculée sur une grille d'environ 1900 points de  $r \in [0, 1]$ , qui est raffinée dans la couche limite visqueuse proche de la paroi. La partie de l'intégrale sur  $r$ , dans les produits  $(\cdot | \cdot)$  est calculée numériquement utilisant la méthode de Boole. La représentation numérique des profils se révèle environ 200 fois plus rapide que l'utilisation d'une méthode de quadrature adaptative (quadl en Matlab). L'utilisation de 1900 points pour décrire la structure radiale nous permet d'obtenir une précision supérieure à  $10^{-10}$ . Les calculs des

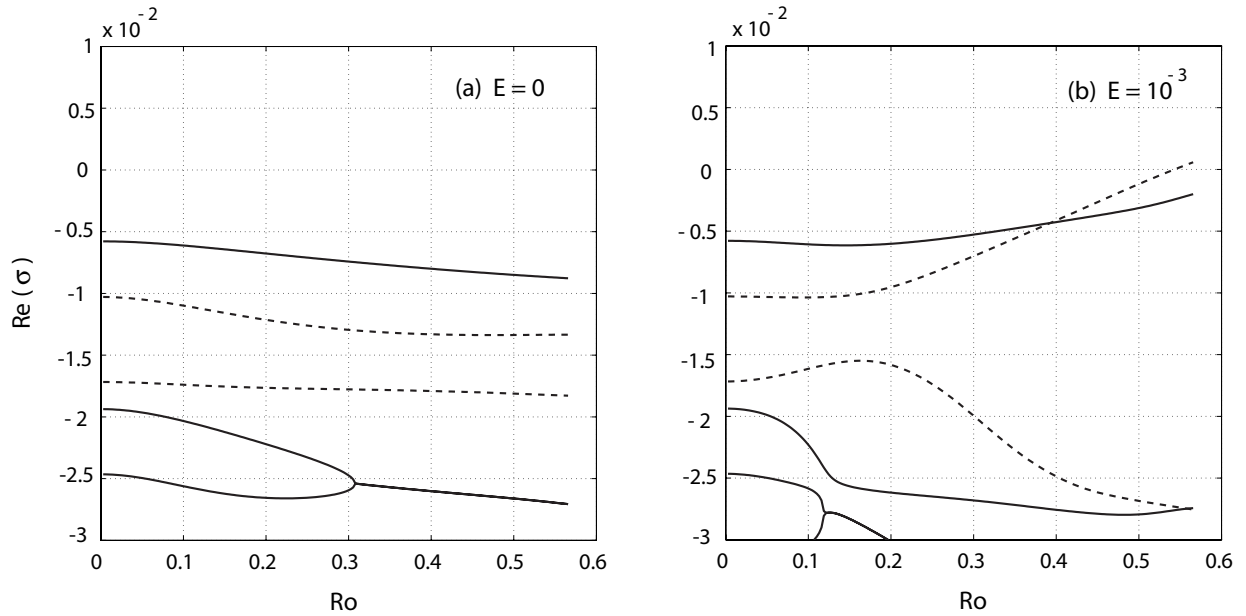


FIG. 7.2 – Taux de croissance  $Re(\sigma)$  des modes propres de l'équation d'induction avec  $E_m = 10^{-3}$ , pour un écoulement de mode de spinover, (a) non-visqueux ( $E = 0$ ) et (b) visqueux ( $E = 10^{-3}$ ). Les lignes continues sont des modes de Type I, les lignes pointillées sont des modes de Type II.

éléments d'interaction ont été testés en utilisant trois méthodes différentes pour évaluer les éléments. Un bon indicateur de la précision des calculs est obtenu à l'aide de la relation d'orthonormalité (7.29). Avec la formule

$$\int_0^1 J_m(k_x r)^2 r dr = \frac{1}{2} \left( J_m(k_x)^2 - J_{m+1}(k_x) J_{m-1}(k_x) \right), \quad (7.35)$$

nous arrivons à calculer de doubles produits d'intégrales de fonctions de Bessel de façon analytique. Ceci permet de tester la précision sur les nombres d'ondes  $k_x$  et les coefficients  $d_{\pm}$ ,  $d_z$  et  $\delta$ . La comparaison des résultats avec ceux trouvés avec la représentation numérique et l'intégration numérique avec la méthode de Boole, nous donne une indication sur la précision de l'intégration numérique. Les calculs sont actuellement réalisés avec Matlab, et des matrices creuses  $10000 \times 10000$  comprenant typiquement  $1.8 \times 10^5$  éléments non nuls peuvent être calculées en un temps de l'ordre de deux heures. Après le calcul des matrices d'interaction, reste à déterminer les valeurs propres du système linéaire. La présence des amplitudes,  $a_j^*$ , nécessite un traitement particulier. Nous séparons les amplitudes en partie réelle et imaginaire  $a_j = Re(a_j) + i Im(a_j)$ , sachant que la partie imaginaire est toujours nulle pour les modes axisymétriques. Nous faisons de même pour les matrices d'interaction et combinons partie réelle et imaginaire du problème. Ceci conduit à un problème aux valeurs propres réel pour une matrice de taille  $(2N_{tot} - N_{axi}) \times (2N_{tot} - N_{axi})$ , avec  $N_{axi}$  le nombre de modes axisymétriques. Nous calculons enfin les valeurs propres les moins atténuées en utilisant une méthode d'Arnoldi.

### 7.3.4 Résultats

La résolution utilisée pour le calcul est  $N_r = 50$ ,  $N_{\phi} = 11$ ,  $N_z = 29$ . Cette résolution s'est avérée suffisante pour bien décrire les modes de Type I et II, mais pas ceux de Type III, dont l'analyse est reportée à une étude ultérieure. Les modes propres du problème de dynamo cinématique sont analysés en augmentant progressivement  $Ro$  pour des valeurs fixées de  $E$  et de  $E_m$ . Typiquement nous avons utilisé :  $E_m = 10^{-3}$  ou  $E_m = 10^{-4}$  et  $E = 0$  (cas non-visqueux) ou  $E = 10^{-3}$ . Sur la Fig. 7.2, nous avons tracé le taux de croissance  $Re(\sigma)$  des modes les plus instables de Type I et de Type II, en fonction de  $Ro$ , pour le cas non-visqueux ( $E = 0$ ) et un cas visqueux ( $E = 10^{-3}$ ) pour un nombre d'Ekman magnétique  $E_m = 10^{-3}$ . Pour  $Ro = 0$ , il n'y a pas d'interaction avec le mode de spinover : les modes propres ne sont rien d'autres que les modes magnétiques diffusifs dont on connaît le taux d'atténuation explicitement. On s'attend donc à avoir les mêmes modes dans les cas visqueux et non-visqueux pour des petits  $Ro$ . On remarque cependant, que rapidement les variations du taux de croissance

n'ont pas la même tendance dans le deux cas. Dans le cas non-visqueux, tous les modes sont plus atténués lorsque  $Ro$  augmente et aucun ne devient donc instable : il n'y a pas de dynamo dans ce cas. Dans le cas visqueux, c'est clairement différent : le taux de croissance de certains modes augmente alors que celui d'autres modes diminue. Pour les paramètres que nous avons considérés, un mode de Type II devient instable pour  $Ro_c \simeq 0.537$ , alors qu'un mode de Type I n'est pas loin de le devenir. Ces premiers résultats nous permettent de tirer la première conclusion. Pour le paramètre d'Ekman magnétique que nous avons considéré, la viscosité est un facteur déstabilisant nécessaire pour obtenir une dynamo. Le taux de croissance reste petit dans les unités  $[t] = \Omega^{-1}$  de la variation temporelle des ondes inertielles. Si, par contre, nous adimensionnons le temps par le temps de diffusion magnétique  $[t] = \eta/R^2$ , le taux de croissance est  $O(1)$ . Tous les modes sont stationnaires ( $Im(\sigma) = 0$ ) dans le repère fixe, tout comme l'écoulement. Nous avons regardé l'effet de la diffusion magnétique en diminuant  $E_m$  à la valeur  $E_m = 10^{-4}$  pour les modes de type II dans le cas visqueux. Nous avons obtenu des courbes pour les taux de croissance tout à fait similaires à celles obtenues pour  $E_m = 10^{-3}$ , mais avec des taux de croissance 10 fois inférieurs, et un seuil de la dynamo plus bas à  $Ro_c = 0.167$ . La diffusion magnétique baisse donc le seuil et la valeur absolue des taux de croissance en même temps : l'effet de la diffusion magnétique est déstabilisant, mais semble imposer son échelle de temps caractéristique, comme il l'est souvent le cas pour des dynamos de type 'lente'.

La structure spatiale tridimensionnelle du mode de type II, au seuil de sa déstabilisation à  $Ro_c = 0.537$  dans le cas  $E_m = 10^{-3}$ ,  $E = 10^{-3}$  est visualisée sur la Fig. 7.3. Sur la première image (a), nous montrons par une iso-surface d'énergie cinétique la structure spatiale de l'écoulement de fluide total, composé de la rotation de base et du mode de spinover à  $E = 10^{-3}$  et  $Ro = Ro_c$ . Nous reconnaissons la forme typique d'un tube en S. L'image est également représentative du tube de vorticit  de l'écoulement. L'image (b) montre une iso-surface d'énergie magnétique du mode dynamo. Nous pouvons observer une forte concentration d'énergie magnétique au voisinage de la paroi du cylindre, à l'endroit où les couches limites visqueuses sont les plus intenses. Sur les images (c) et (d), nous avons visualisé les lignes de champ magnétique en "lâchant" des rubans de champ magnétique<sup>1</sup> dans des zones proches de la paroi (c) et à l'intérieur du fluide (d). Nous pouvons considérer que les lignes de champ circulent principalement autour du tube de vorticit , comme l'écoulement de fluide. Au coeur du fluide, le champ reste fortement enroulé autour de l'axe de rotation totale du fluide. Par endroit, les lignes de champ arrivent à s'échapper à l'extérieur, ou à se connecter entre les 2 demi longueurs d'onde de l'écoulement. Sur la fig. 7.4, nous montrons une carte du champ à la surface  $r = 1$  du cylindre. Les couleurs représentent l'intensité du champ radial. Le champ vectoriel visualise les composantes verticales et azimutales du champ. Il est intéressant de souligner que le champ sur la surface est de petite échelle par rapport aux grandes structures que nous avons observées à l'intérieur du cylindre. Notons cependant que l'amplitude du champ surfacique est environ 50 fois plus petite que le champ à l'intérieur. La composition spectrale des modes de dynamo de type II, pour les deux cas que l'on a considérés est donnée dans les tableaux 7.1. Nous pouvons remarquer, que dans les deux cas, la structure spatiale du champ magnétique reste fortement concentrée sur les modes de grande structure, même si la couche limite joue un rôle important. Pour  $E_m = 10^{-3}$ , les modes axisymétriques sont accompagnés principalement de modes  $m = 1$ , et quelques contributions plus faibles de  $m = 2$  et 3 sont présentes à faible amplitude. Nous arrivons à représenter 86 % du mode dynamo avec ces 10 premiers modes propres de diffusion. La contribution des cinq premiers modes représente presque 75% du mode. Pour  $E_m = 10^{-4}$  le mode dynamo est légèrement différent. Il se concentre plus fortement sur les modes toroïdaux et axisymétriques. Le premier mode non-axisymétrique  $m = 1$  n'ajoute que 5.9% à la structure spatiale du mode. Pour de petits  $E_m$  et  $Ro$  le mode dynamo semble donc plus simple car composé principalement de modes axisymétriques toroïdaux. Cette observation constitue une des motivations de la théorie asymptotique réalisée dans la prochaine section pour des faibles  $Ro$  et  $E_m$ .

Les premiers calculs présentés ci-dessus ont mis en évidence la possibilité d'une dynamo cinématique du mode de spinover. Nous retiendrons que l'écoulement non-visqueux ne peut pas exciter de dynamo, et que la viscosité est nécessaire et déstabilisante. La couche limite est une région de champ fort, ce qui n'est surprenant à la vue du fort cisaillement qui y règne. Cependant, le champ ne reste pas concentré dans la couche limite mais s'étend dans le volume. Nous avons également vu que le champ magnétique suit bien l'écoulement total du fluide. La décomposition spectrale du mode déstabilisé a finalement aussi montré le rôle important de la partie axisymétrique et toroïdale du champ.

<sup>1</sup>Les rubans suivent les lignes de champ magnétique, et montrent également comment le champ s'enroule.



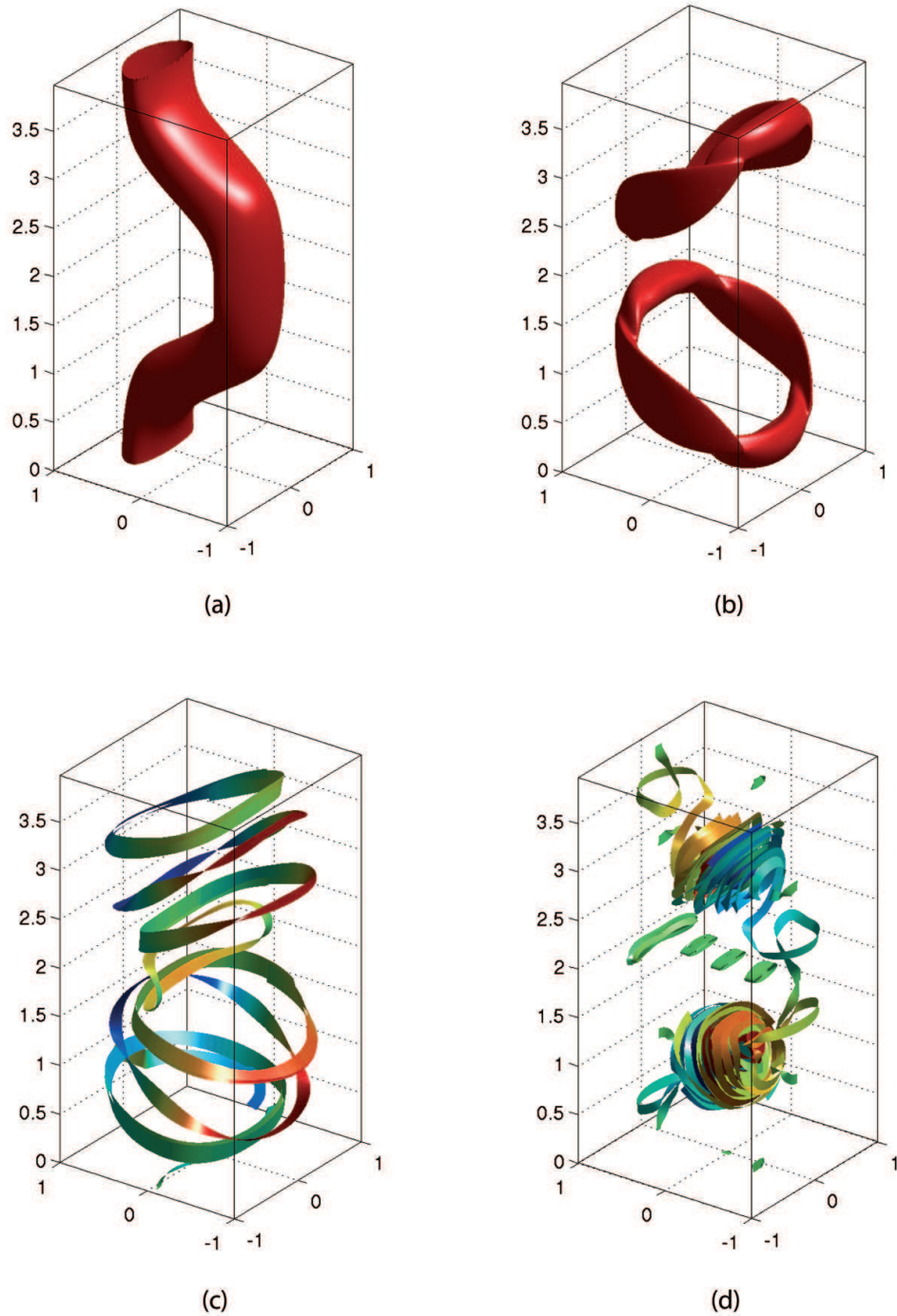


FIG. 7.3 – Visualisation de l'écoulement et du mode dynamo de Type II pour les paramètres  $Ro = Ro_c = 0.537$ ,  $E = 10^{-3}$ ,  $E_m = 10^{-3}$ . (a) Iso-surface d'énergie cinétique de l'écoulement total. (b) Iso-surface d'énergie magnétique, du mode dynamo de type II. (c) Rubans de champ magnétique proche de la paroi, dans les zone de champ intenses et (d) dans le centre du cylindre.

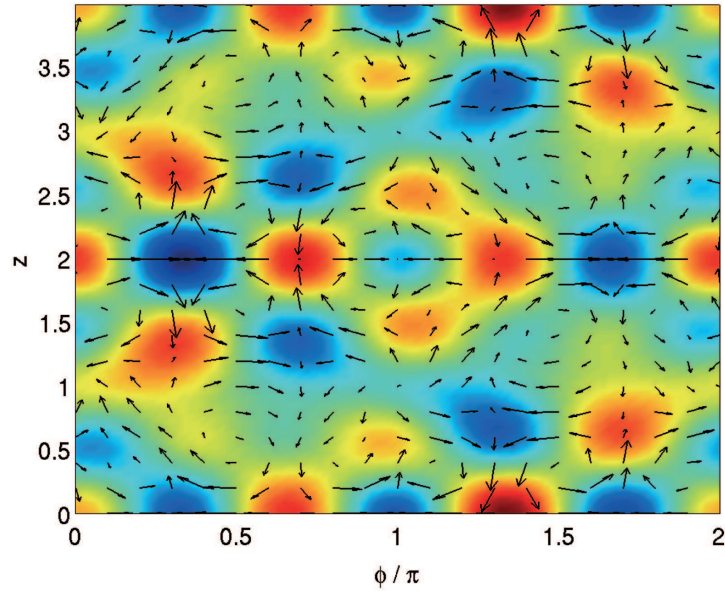


FIG. 7.4 – Visualisation du champ magnétique à la surface du cylindre,  $r = 1$ ,  $\phi \in [0, 2\pi]$ ,  $z \in [0, 3.9796]$ , du mode dynamo de Type II, au seuil de l'instabilité pour les paramètres  $Ro = Ro_c = 0.537$ ,  $E = 10^{-3}$ ,  $E_m = 10^{-3}$ .

$E_m = 10^{-3}$ $Ro_c = 0.537$					$E_m = 10^{-4}$ $Ro_c = 0.167$				
m	n	x	Tor	%	m	n	x	Tor	%
1	4	1		25.1	0	2	2	x	47.1
0	2	2	x	21.4	0	2	4	x	18.4
1	0	1		17.2	0	2	6	x	7.4
0	2	4	x	6.2	1	4	1		5.9
1	4	2		5.0	1	4	3		3.6
1	0	2		3.2	1	0	1		3.5
0	2	6	x	2.5	0	2	8	x	3.1
2	6	1		1.9	1	4	2		3.1
1	4	4		1.8	0	2	10	x	1.3
1	4	3		1.8	0	6	1		1.1

TAB. 7.1 – Composition des dynamos de Type II pour un mode de spinover visqueux, à  $E = 10^{-3}$ , au seuil de l'instabilité, pour différents  $E_m$ . Les modes axisymétriques et toroïdaux sont marqués avec  $x$ .

## 7.4 Analyse asymptotique de la dynamo cinématique des ondes inertielles dans la limite $Ro \ll 1$ , $E_m \ll 1$ .

Dans l'approche asymptotique, il sera préférable de raisonner dans le repère tournant et de considérer, dans un premier temps, un écoulement d'ondes inertielles général. Dans le repère tournant, ce champ magnétique vérifie l'équation

$$\partial_t \mathbf{b} = Ro \left[ \left( -(\mathbf{u} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{u} \right) e^{i\bar{\lambda}t} + cc. \right] + E_m \Delta \mathbf{b}, \quad (7.36)$$

où  $\mathbf{U} = \mathbf{u}(\mathbf{r}) e^{i\bar{\lambda}t} + cc.$  est l'onde inertielle, en plus de la loi de Gauss et des conditions de raccordement à la frontière du cylindre. Comme précédemment, on décompose le champ magnétique sur les modes magnétiques diffusifs :

$$\mathbf{b} = \sum_{x=1}^{N_r} \sum_{m=-N_\phi}^{N_\phi} \sum_{n=0}^{N_z} c_{xnm} \mathbf{B}_{xnm}(\mathbf{r}). \quad (7.37)$$

On obtient en projetant sur les modes adjoints, un système d'équations pour  $\mathbf{c}$ , le vecteur des amplitudes des modes,

$$\partial_t \mathbf{c} = -E_m \tilde{\Gamma} \mathbf{c} + Ro \left[ \left( e^{i\bar{\lambda}t} \mathbf{D}^{(1)} + e^{-i\bar{\lambda}t} \mathbf{D}^{(2)} \right) \mathbf{c} + e^{i\bar{\lambda}t} \mathbf{D}^{(3)} \mathbf{c}^* \right], \quad (7.38)$$

où on a noté  $\tilde{\Gamma}$ , la matrice avec  $\tilde{\gamma}_j$  sur la diagonale, et  $\mathbf{D}^{(\cdot)}$ , les matrices d'interaction définies en (7.31). L'analyse que l'on propose par la suite est une analyse asymptotique du système (7.38), dans la limite où  $Ro$  et  $E_m$  tendent vers 0. Comme nous le verrons plus bas, les effets constructifs pour le champ magnétique ne peuvent se faire qu'à l'ordre 2 en  $Ro$ . Ceux-ci équilibreront donc la dissipation magnétique  $O(E_m)$  que lorsque  $Ro$  sera de l'ordre de  $E_m^{1/2}$ . Pour cette raison, nous supposerons par la suite que

$$Ro = \tilde{Ro} E_m^{1/2} \quad \text{avec} \quad \tilde{Ro} = O(1). \quad (7.39)$$

L'analyse asymptotique est donc faite avec un seul paramètre  $E_m^{1/2}$ . Les différentes quantités sont développées en puissance de  $E_m^{1/2}$

$$\mathbf{c} = \sum_{\beta=0}^{\infty} E_m^{\beta/2} \mathbf{c}^{(\beta)}. \quad (7.40)$$

Il est également nécessaire d'introduire l'échelle de temps lent sur laquelle va se faire la croissance de la dynamo,  $T = t/E_m$ . Avec ce développement, ces échelles et la loi d'échelle (7.39), on peut maintenant résoudre le système (7.38) de façon itérative. Nous construisons deux théories différentes qu'on appellera "théorie non-diffusive" et "théorie diffusive". Dans le premier cas, nous allons traiter l'opérateur  $E_m \tilde{\Gamma}$  comme étant  $O(E_m)$ . Ceci suppose que  $\tilde{\Gamma}$  est  $O(1)$  pour tous les modes. La dissipation accrue des modes de petite échelle n'est donc pas prise en compte. Dans la théorie diffusive, nous prendrons en compte la diffusion magnétique à l'ordre zéro de la théorie asymptotique, c'est à dire nous supposerons que  $E_m \tilde{\Gamma} = O(1)$ . Cette hypothèse n'est a priori valable que pour les petites échelles pour lesquelles  $\tilde{\gamma}_j = O(E_m^{-1})$ . Nous verrons que les théories conduisent au même résultat si la dissipation accrue des petites échelles magnétiques n'est pas importante dans le mécanisme de la dynamo. La théorie asymptotique ressemble à une théorie qui modélise un effet alpha, mais avec une séparation d'échelle sur le temps plutôt que sur l'espace.

### 7.4.1 Théorie non-diffusive

Dans la théorie non-diffusive, à l'ordre 1 on obtient,

$$\partial_t \mathbf{c}^{(0)} = 0 \quad \Leftrightarrow \quad \mathbf{c}^{(0)} = \mathbf{c}^{(0)}(T). \quad (7.41)$$

A l'ordre  $O(E_m^{1/2})$ , on obtient

$$\partial_t \mathbf{c}^{(1)} = \tilde{Ro} \left[ \left( e^{i\bar{\lambda}t} \mathbf{D}^{(1)} + e^{-i\bar{\lambda}t} \mathbf{D}^{(2)} \right) \mathbf{c}^{(0)} + e^{i\bar{\lambda}t} \mathbf{D}^{(3)} \mathbf{c}^{(0)*} \right], \quad (7.42)$$

qui donne immédiatement

$$\mathbf{c}^{(1)} = -\frac{i}{\bar{\lambda}} \left[ \left( e^{i\bar{\lambda}t} \mathbf{D}^{(1)} - e^{-i\bar{\lambda}t} \mathbf{D}^{(2)} \right) \mathbf{c}^{(0)} + e^{i\bar{\lambda}t} \mathbf{D}^{(3)} \mathbf{c}^{(0)*} \right]. \quad (7.43)$$

A l'ordre  $O(E_m)$ , on obtient les effets diffusifs et l'interaction de la solution forcée obtenue à l'ordre  $O(E_m^{1/2})$  avec l'écoulement :

$$\partial_T \mathbf{c}^{(0)} + \partial_t \mathbf{c}^{(2)} = \tilde{R}o^2 \left[ \left( e^{i\bar{\lambda}t} \mathbf{D}^{(1)} + e^{-i\bar{\lambda}t} \mathbf{D}^{(2)} \right) \mathbf{c}^{(1)} + e^{i\bar{\lambda}t} \mathbf{D}^{(3)} \mathbf{c}^{(1)*} \right] - \tilde{\Gamma} \mathbf{c}^{(0)} \quad (7.44)$$

Ce qui nous intéresse ici sont les termes du membre de droite qui ne dépendent pas de l'échelle de temps rapide  $t$ , et qui sont donc résonants. Ces termes changent la dépendance temporelle de  $\mathbf{c}^{(0)}$ , sur l'échelle de temps lent  $T$ .

$$\partial_T \mathbf{c}^{(0)} = \tilde{R}o^2 \left( \mathbf{P} \mathbf{c}^{(0)} + \mathbf{Q} \mathbf{c}^{(0)*} \right) - \tilde{\Gamma} \mathbf{c}^{(0)}, \quad (7.45)$$

avec

$$\mathbf{P} = \frac{i}{\bar{\lambda}} \left( \mathbf{D}^{(1)} \mathbf{D}^{(2)} - \mathbf{D}^{(2)} \mathbf{D}^{(1)} + \mathbf{D}^{(3)} \mathbf{D}^{(3)*} \right), \quad (7.46)$$

$$\mathbf{Q} = \frac{i}{\bar{\lambda}} \left( -\mathbf{D}^{(2)} \mathbf{D}^{(3)} + \mathbf{D}^{(3)} \mathbf{D}^{(1)*} \right). \quad (7.47)$$

En revenant aux variables initiales, on obtient donc

$$\partial_t \mathbf{c} = Ro^2 \left( \mathbf{P} \mathbf{c} + \mathbf{Q} \mathbf{c}^* \right) - E_m \tilde{\Gamma} \mathbf{c}. \quad (7.48)$$

L'équation (7.48) ne révèle pas comment le problème aux valeurs propres a été réduit, et pour cela il faut regarder les interactions à l'ordre 2 de plus près. Remarquons tout d'abord que les interactions résonantes opèrent à l'intérieur des classes de modes du même nombre azimutal  $m$ . Ceci vient du fait que les ondes inertielles sont toujours de la forme

$$\mathbf{U} = \mathbf{u}(\mathbf{r}, \mathbf{z}) e^{i(\bar{\lambda}t + \bar{m}\phi)} + c.c., \quad (7.49)$$

et que les opérateurs  $\mathcal{D}_u$  et  $\mathcal{D}_{u^*}$  qui apparaissent dans  $\mathcal{D}^{(j)}$ ,  $j = 1, 2, 3$  agissent de manière opposée sur un mode magnétique  $\mathbf{B}_{xnm}$  en modifiant respectivement son nombre azimutal de  $-\bar{m}$  et de  $+\bar{m}$ . Pour ce qui concerne la structure axiale, on obtient que les couplages résonants ont lieu entre des modes aux nombres verticaux,  $n + 2\bar{n}\zeta$ ,  $\zeta \in \mathbb{N}$ . Les ondes inertielles dès qu'on prend en compte la viscosité du fluide, ont une décomposition radiale riche : l'action successive des opérateurs  $\mathcal{D}_u$  et  $\mathcal{D}_{u^*}$  sur un mode  $\mathbf{B}_{xnm}$  génère donc beaucoup de nombres d'onde radiaux. En résumé, on peut réduire le problème général (7.48) en sous-problèmes pour chacune des classes de modes de la forme  $(m, n + 2\bar{n}\zeta, \cdot)$  :

$$\partial_t \mathbf{c}_* = Ro^2 \left( \mathbf{P}_* \mathbf{c}_* + \mathbf{Q}_* \mathbf{c}_*^* \right) - E_m \tilde{\Gamma}_* \mathbf{c}_*. \quad (7.50)$$

## 7.4.2 Théorie diffusive

Dans cette théorie, on suppose que  $E_m \tilde{\Gamma} = O(1)$ . Ainsi à l'ordre zéro, on obtient

$$\left( \partial_t + E_m \tilde{\Gamma} \right) \mathbf{c}^{(0)} = 0. \quad (7.51)$$

Vu l'origine de la matrice diagonale  $\tilde{\Gamma}$ , on obtient pour chacun des éléments de  $\mathbf{c}^{(0)}$

$$\mathbf{c}_j^{(0)} = \mathbf{c}_j^{(0)}(T) e^{-E_m \tilde{\gamma}_j t}. \quad (7.52)$$

Le fait que la solution à l'ordre zéro soit atténuée, rend l'analyse asymptotique à priori caduque. Rien ne nous empêche, cependant, de continuer l'analyse formelle. A l'ordre  $O(E_m^{1/2})$ , nous obtenons :

$$\left( \partial_t + E_m \tilde{\Gamma} \right) \mathbf{c}^{(1)} = \tilde{R}o \left[ \left( e^{i\bar{\lambda}t} \mathbf{D}^{(1)} + e^{-i\bar{\lambda}t} \mathbf{D}^{(2)} \right) \mathbf{c}^{(0)} + e^{i\bar{\lambda}t} \mathbf{D}^{(3)} \mathbf{c}^{(0)*} \right], \quad (7.53)$$

qui se résout formellement sous la forme :

$$\mathbf{c}^{(1)} = \left[ \left( \mathbf{\Lambda} \mathbf{D}^{(1)} e^{i\bar{\lambda}t} + \mathbf{\Lambda}^* \mathbf{D}^{(2)} e^{-i\bar{\lambda}t} \right) \mathbf{c}^{(0)} + \mathbf{\Lambda} \mathbf{D}^{(3)} e^{i\bar{\lambda}t} \mathbf{c}^{(0)*} \right], \quad (7.54)$$

$$\text{où } \mathbf{\Lambda}_{kj} = \left( i\bar{\lambda} + E_m(\tilde{\gamma}_k - \tilde{\gamma}_j) \right)^{-1}. \quad (7.55)$$

La différence principale avec la théorie non-diffusive est visible dans l'expression (7.55) de la matrice  $\Lambda$ . Les taux de dissipation réels apparaissent maintenant à côté du terme purement imaginaire  $O(1)$  qui était présent dans la théorie non-diffusive. Si les structures magnétiques des modes  $j$  et  $k$  sont de grande échelle spatiale, ou lorsque elles sont similaires, la dissipation magnétique est faible devant ce terme complexe et la solution sera très proche de celle de la théorie non-diffusive. Pour les structures magnétiques plus fines et différentes, le terme  $E_m(\tilde{\gamma}_k - \tilde{\gamma}_j)$  peut devenir important, et changer la phase de  $\Lambda_{kj}$ . C'est ce changement de phase qui est important de prendre en compte. A l'ordre  $E_m$ , on obtient des termes lents comme dans la théorie non-diffusive, qui incluent ces corrections. Si on revient aux variables initiales, on retrouve (7.48), mais avec des expressions de  $\mathbf{P}$  et  $\mathbf{Q}$  différentes :

$$\mathbf{P} = \mathbf{D}^{(1)} \Lambda^* \mathbf{D}^{(2)} + \mathbf{D}^{(2)} \Lambda \mathbf{D}^{(1)} + \mathbf{D}^{(3)} \Lambda^* \mathbf{D}^{(3)*}, \quad (7.56)$$

$$\mathbf{Q} = \mathbf{D}^{(2)} \Lambda \mathbf{D}^{(3)} + \mathbf{D}^{(3)} \Lambda^* \mathbf{D}^{(1)*}. \quad (7.57)$$

Il est important de signaler que l'on retrouve la théorie non-diffusive lorsqu'on suppose que  $E_m \Gamma$  n'est plus  $O(1)$  dans les opérateurs  $\mathbf{P}$  et  $\mathbf{Q}$ . La théorie diffusive est moins propre dans le sens que l'on est obligé de revenir aux variables initiales pour écrire une équation finale. Pour l'instant, c'est la seule manière que nous avons trouvée pour prendre en compte la diffusion magnétique à petite échelle.

### 7.4.3 Application à l'écoulement de spinover

Dans cette partie, nous considérons plus spécifiquement le cas du mode de spinover dans un cylindre de rapport d'aspect  $h = 3.9796$  afin de comparer les prédictions asymptotiques aux résultats numériques de la section 7.3.4. Les résultats numériques ont fait apparaître que le mode de dynamo le plus instable est de type II et qu'il est plutôt axisymétrique pour des petits  $E_m$ . Nous limitons donc l'analyse aux modes de type II axisymétriques. Nous rappelons que les modes axisymétriques sont soit poloïdaux (P), soit toroïdaux (T). On décomposera donc le vecteur  $\mathbf{c}_*$  des amplitudes des modes en 2 sous-vecteurs P et T :

$$\mathbf{c}_* = \begin{bmatrix} P \\ T \end{bmatrix}. \quad (7.58)$$

Les amplitudes des modes axisymétriques sont réels. On peut donc écrire l'opérateur d'interaction

$$\mathbf{P}_* \mathbf{c}_* + \mathbf{Q}_* \mathbf{c}_*^* = \mathbf{M}_* \mathbf{c}_* = \begin{bmatrix} \text{P2P} & \text{T2P} \\ \text{P2T} & \text{T2T} \end{bmatrix} \begin{bmatrix} P \\ T \end{bmatrix}, \quad (7.59)$$

ce qui permet de traiter les différents transferts entre modes poloïdaux et toroïdaux séparément.

#### Théorie non-diffusive

Pour un mode de spinover **non-visqueux** nous trouvons le résultat important que

$$\text{P2P} = \text{T2T} = \text{T2P} = 0 \quad \text{P2T} \sim 1. \quad (7.60)$$

Les seuls éléments non-nuls sont ceux qui transforment un champ poloïdal en champ toroïdal. Ces éléments sont  $O(1)$  pour les interactions entre modes ayant des structures radiales proches. La forme de la matrice d'interaction garantit que les valeurs propres obtenus à partir du système (7.48) sont celles de la matrice diagonale  $-E_m \tilde{\Gamma}_*$  : aucune dynamo n'est donc possible avec un mode de spinover non-visqueux. Ce premier résultat est en accord avec les calculs numériques qui ne prédisaient aucune dynamo pour l'écoulement de spinover non-visqueux. Ceci est également en accord avec les résultats généraux de Moffatt [101] sur les écoulements d'ondes inertielles. On obtient cependant des informations sur les transferts : le champ poloïdal est transformé en champ toroïdal avant d'être dissipé. Ce transfert peut être physiquement compris comme étant un effet similaire à l'effet  $\Omega$ . Le mode de spinover déforme les lignes d'un champ poloïdal initial, de manière à les faire entrer dans l'écoulement, générant une augmentation forte de la partie toroïdale du champ.

La viscosité permet de résoudre une partie des problèmes, car grâce aux légers déphasages introduits dans les profils spatiaux des ondes, les blocs P2P, T2T et T2P de  $\mathbf{M}_*$  ne sont à priori plus exactement nuls. Nous avons analysé les éléments de chacun de ces blocs en fonction du nombre d'Ekman et obtenu les résultats représentés sur la Fig. 7.6-(a). Ceci met en évidence les lois d'échelles suivantes pour des petits  $E$

$$\text{P2P} \sim E^{1/2} \quad \text{T2T} \sim E^{1/2} \quad \text{T2P} \simeq 0 \quad \text{P2T} \sim 1. \quad (7.61)$$

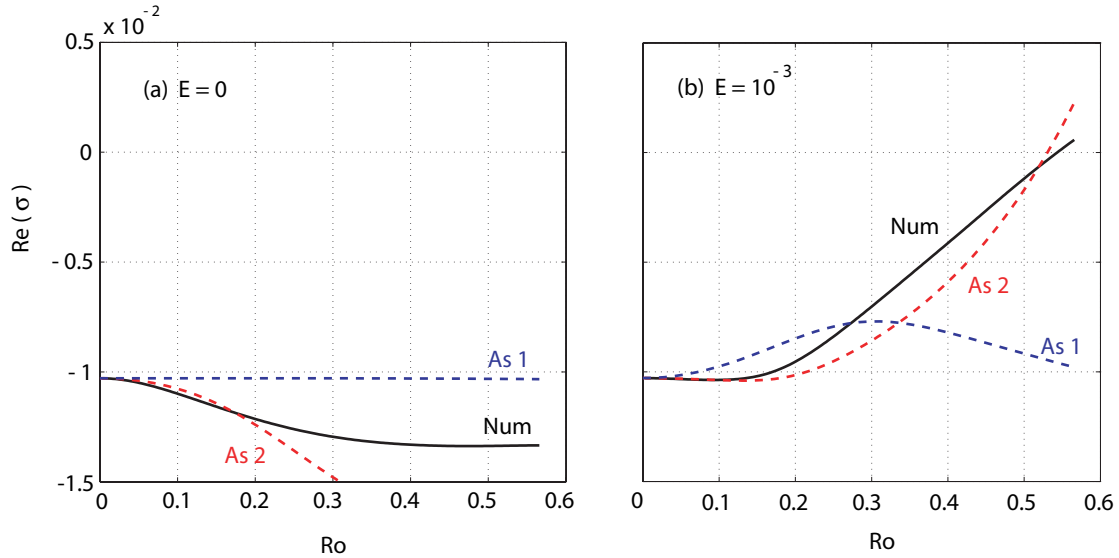


FIG. 7.5 – Taux de croissance  $Re(\sigma)$  pour le mode de type II le moins atténué, en fonction de  $Ro$  pour un écoulement de mode spinover (a) non-visqueux, et (b) visqueux  $E = 10^{-3}$ . Comparaison entre le calcul aux valeurs propres (Num) et les théories asymptotiques non-diffusives (As 1) et diffusives (As 2).

Les éléments P2T restent toujours grands et proches de leurs valeurs non-visqueuses. Pour les parties T2T et P2P, la loi d'échelle  $E^{1/2}$  suggère qu'elle pourrait provenir des couches limites visqueuses. Le bloc T2P reste proche de zéro. Cependant, cela n'exclut pas un mécanisme de dynamo où les couplages constructifs se feraient entre modes toroïdaux uniquement ou entre modes poloïdaux uniquement. Les résultats asymptotiques pour  $E = 10^{-3}$  sont montrés en Fig. 7.5-(b). Nous observons effectivement une augmentation du taux de croissance pour des petits  $Ro$  qui peut être attribuée aux interactions constructives mentionnées ci-dessus. Cependant cette croissance n'est pas suffisante pour obtenir une dynamo et la prédiction asymptotique se détache rapidement des résultats numériques.

La forme de la matrice d'interaction et les loi d'échelles (7.61), nous permettent d'extrapoler les résultats vers d'autres régimes de paramètres. La nullité du bloc  $T2P$ , permet de séparer le problème des valeurs propres en deux sous-problèmes :

$$\partial_t P = Ro^2 E^{1/2} \widetilde{P2P} P - E_m \tilde{\Gamma}_P P, \quad (7.62)$$

$$\partial_t T = Ro^2 E^{1/2} \widetilde{T2T} T - E_m \tilde{\Gamma}_T T. \quad (7.63)$$

Dans ces équations, on a utilisé les lois d'échelles

$$P2P = E^{1/2} \widetilde{P2P}, \quad T2T = E^{1/2} \widetilde{T2T}, \quad (7.64)$$

et  $\tilde{\Gamma}_P$  et  $\tilde{\Gamma}_T$  désignent respectivement la matrice  $\tilde{\Gamma}$  réduite aux modes poloïdaux et toroïdaux. Utilisant le temps long  $T = E_m t$ , les valeurs propres de chacun des sous-systèmes ne dépendent donc plus que d'un seul paramètre  $\chi = Ro^2 E^{1/2} E_m^{-1}$ . Le fait que nous n'ayons pas trouvé une dynamo asymptotique non-diffusive à cet ordre est alors général pour tous les valeurs de  $\chi \in [0, 150]$ , toujours sous la contrainte que  $Ro \ll 1$ ,  $E \ll 1$ ,  $E_m \ll 1$ . Au delà de  $\chi \simeq 150$ , la théorie asymptotique est confrontée à des instabilités numériques. Cependant, il est fort probable que l'observation s'étende dans la limite  $\chi \rightarrow \infty$ , ce qui exclurait la possibilité d'une dynamo rapide, où la diffusion magnétique n'interviendrait plus. Dans la gamme de paramètres de la simulation numérique, la discussion ci-dessus nous indique qu'un autre ingrédient doit être pris en compte : c'est la diffusion magnétique qui va permettre un couplage entre les modes toroïdaux et poloïdaux.

### Théorie diffusive

Pour les modes axisymétriques, la matrice d'interaction de la théorie diffusive devient

$$\mathbf{M}_* = 2 \operatorname{Re} \left( \mathbf{D}_*^{(2)} \mathbf{\Lambda} \mathbf{D}_*^{(1)} \right). \quad (7.65)$$

La matrice  $\mathbf{\Lambda}$  introduit  $E_m$ , ainsi que des termes dépendants de la structure spatiale des champs magnétiques,  $\tilde{\gamma}_j$  dans les interactions entre modes. Nous avons notamment vu que ce terme introduit un déphasage. De façon simple, nous pouvons dire que les interactions qui mettent en jeu que les plus grandes échelles sont contrôlées par la partie imaginaire de  $\mathbf{D}_*^{(2)} \mathbf{D}_*^{(1)}$ , comme précédemment dans le cas de la théorie non-diffusive. Les interactions impliquant des petites échelles seront plutôt contrôlées par la partie réelle de  $\mathbf{D}_*^{(2)} \mathbf{D}_*^{(1)}$ , or il s'avère que les parties réelles et imaginaires de cet opérateur ont des ordres de grandeur très différents, d'où l'importance des petites échelles magnétiques dans ce problème.

Regardons d'abord la variation de la valeur absolue des éléments matriciels en fonction de  $E_m$ . Le cas de l'écoulement du mode de spinover **non-visqueux** représenté sur la Fig. 7.6-(b), nous montre que les valeurs absolues des différents blocs de  $\mathbf{M}_*$  varient selon les lois d'échelles

$$\text{P2P} \sim \text{T2T} \sim E_m \quad \text{T2P} \sim E_m^2 \quad \text{P2T} \sim 1. \quad (7.66)$$

La diffusion magnétique assure donc que tous les transferts sont non-nuls. Le "cusp" dans la courbe de T2P est lié à un changement de signe de l'élément que l'on considère. Les autres éléments que nous regardons, ne changent pas de signe. Les valeurs des éléments matriciels dépendent de la structure spatiale des deux modes magnétiques qu'ils relient. De manière générale, pour tous les éléments, les modes à structure spatiale proche interagissent le plus fortement. Sur la Fig. 7.6-(c), nous montrons comment les éléments P2P évoluent lorsque la structure spatiale des deux modes en interaction se raffine, c'est à dire lorsque  $\gamma_j$  augmente. Plus la structure spatiale est fine, plus l'interaction est forte. La même chose est observée pour les couplages T2T et T2P. Les éléments P2T sont très peu affectés. Sur la Fig. 7.5-(a), la courbe (As2) montre l'évolution des valeurs propres obtenues par la théorie diffusive. On observe que cette courbe suit assez bien la courbe issue de la simulation pour des petits  $Ro$ . Les écarts deviennent plus importants à grand  $Ro$ , ce qui en soi n'est pas étonnant vu que  $Ro$  était supposé petit dans la théorie.

Pour un écoulement **visqueux**,  $E = 10^{-3}$ , nous observons des différences dans les lois d'échelles des différentes parties de la matrice d'interaction. Nous voyons que les termes T2T, P2P suivent une loi en  $E_m$  à grand  $E_m$ . Diminuant  $E_m$ , il y a un changement de signe (cusp), et les éléments convergent vers les valeurs limites (\* - panneau (a) de la figure), contrôlées par la viscosité. Pour une structure spatiale du champ magnétique plus fine, les courbes sont similaires mais décalées vers les plus petits  $E_m$ . Nous remarquons, que le terme T2P suit maintenant à peu près une loi d'échelle en  $E_m$  contrairement au cas non visqueux. Deux changements de signes séparent cette zone de la loi non visqueuse en  $E_m^2$ . Le terme P2T reste principalement déterminé par la partie non-visqueuse de l'écoulement, et varie peu pour les modes en interaction avec les plus grandes structures spatiales. Au total, nous pouvons proposer les lois d'échelles suivantes pour la configuration diffusive et visqueuse :

$$\text{P2P} \sim \text{T2T} \sim A E_m + C E^{1/2} \quad , \quad \text{T2P} \sim D E_m \quad , \quad \text{P2T} \sim 1, \quad (7.67)$$

où le coefficient  $A$  varie comme  $\gamma_j^2$  tant que  $\gamma_j < E_m^{-1}$  et le coefficient  $C$  est à peu près constant sur une grande gamme d'échelle. Le terme T2P permet un couplage entre les modes toroïdaux et poloïdaux. Bien que la dépendance de ce coefficient soit différente en présence de viscosité, nous ne pensons pas que ce type de couplage soit responsable de la dynamo observée numériquement. Nous préférons attribuer l'effet constructif au coefficient  $A E_m$  de T2T et de P2P qui devient  $O(1)$  pour les modes de petites échelles (lorsque  $\gamma_j \sim O(E_m^{-1/2})$ ) et qui en interaction avec les termes visqueux  $C E^{1/2}$  de ces mêmes coefficients produisent des valeurs propres de la matrice d'interaction  $\mathbf{M}_*$  qui ne dépendent plus de  $E_m$ . Les résultats asymptotiques sont comparés aux résultats numériques sur la figure 7.5-(b). Nous voyons en particulier que la théorie diffusive pour un écoulement visqueux à  $E = 10^{-3}$ , arrive assez bien à reproduire l'évolution du mode dynamo issue du calcul numérique jusqu'au seuil. Pour cette raison, nous pensons que la théorie asymptotique, capte l'essentiel de la physique, et que les différences entre les deux courbes peuvent être attribuées à un effet d'ordre supérieur en  $Ro$ .

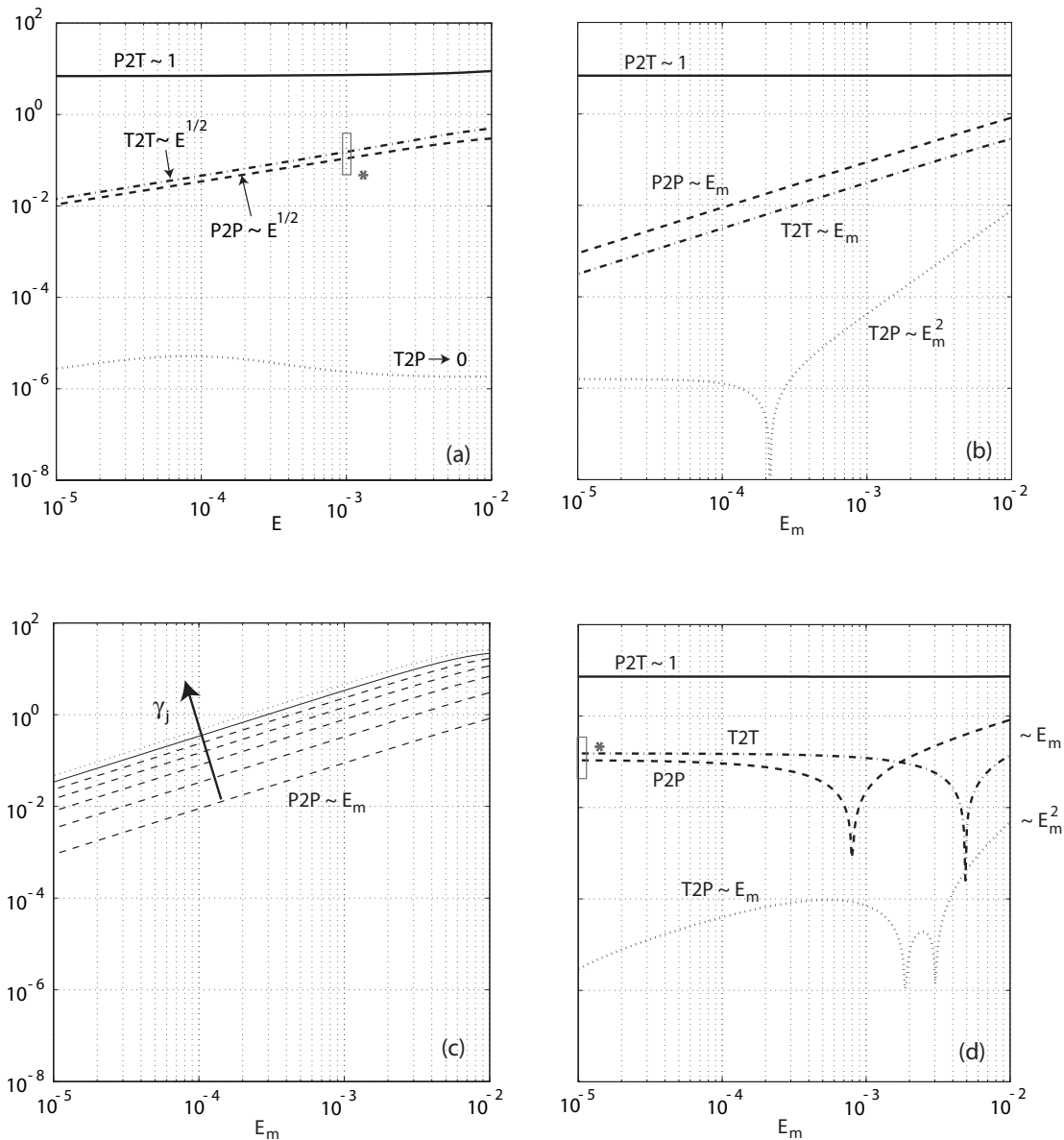


FIG. 7.6 – Variation typique de la valeur absolue des éléments matriciels de  $\mathbf{M}_*$  pour les modes magnétiques de type II. (a) Théorie non-diffusive et écoulement visqueux : variation avec  $E$ . Théorie diffusive et écoulement non-visqueux : (b) variation avec  $E_m$  et (c) la structure spatiale de plus en plus fine,  $\gamma_j \uparrow$ . (d) Théorie diffusive et écoulement visqueux,  $E = 10^{-3}$  fixe : variation avec  $E_m$ .



## 7.5 Discussion & conclusion

Dans cette partie, nous avons étudié la possibilité d'une dynamo cinématique induite par une onde inertielle dans une géométrie cylindrique. Nous avons développé une analyse asymptotique qui permet de traiter ce problème pour toute onde dans les limites  $Ro$  et  $E_m$  tendant vers zéro. Nous avons regardé plus spécifiquement le cas du mode de spinover pour lequel nous avons mis en évidence numériquement la présence d'une dynamo associée à un mode stationnaire dans le repère fixe, à dominante axisymétrique. Pour ce type de dynamo (mode axisymétrique sur le mode de spinover), la théorie a permis d'obtenir plusieurs résultats asymptotiques intéressants. Nous avons tout d'abord mis en évidence l'absence d'un mécanisme constructif de la dynamo non-visqueuse à l'ordre  $Ro^2$ . Ni la théorie non-diffusive, ni les apports de la diffusion magnétique ne permettent d'obtenir une dynamo. En présence de viscosité, des couplages supplémentaires issus de la couche limite visqueuse surfacique apparaissent qui ne suffisent pas pour trouver une dynamo. La diffusion magnétique est nécessaire. Nous avons pour cela développé une analyse asymptotique "ad hoc", dans laquelle les effets de diffusion, et notamment ceux associés aux couches d'Hartman sont importants. Dans cette théorie, les effets de diffusion magnétique sont présents dès le premier ordre. Les matrices d'interaction ainsi obtenues ont fait apparaître de nouveaux mécanismes de couplage notamment un couplage entre les modes toroïdaux et poloïdaux qui était absent dans la théorie non-diffusive. Les comparaisons avec les résultats numériques ont montré que cet effet était important pour bien prédire le seuil de la dynamo pour  $E_m = 10^{-3}$  et  $E = 10^{-3}$ .

Essayons maintenant d'interpréter les résultats obtenus dans un contexte géophysique plus général en revenant tout d'abord sur le mécanisme physique de la dynamo. L'étude non-diffusive a mis en évidence l'importance de la viscosité pour créer une dynamo. On peut comprendre le mécanisme physique mis-en oeuvre à l'aide du raisonnement suivant. Sans diffusion magnétique, le champ magnétique satisfait l'équation

$$\partial_t \mathbf{b} - \nabla \times (\mathbf{u} \times \mathbf{b}) \simeq 0, \quad (7.68)$$

qui est la même que celle satisfaite par un élément de fluide. Le champ magnétique va donc suivre les lignes de courant de l'écoulement total. On a vu cela dans les visualisations du champ magnétique obtenu par le calcul numérique : le champ magnétique est majoritairement toroïdal et suit l'inclinaison de l'axe de rotation. Pour comprendre le rôle de la diffusion visqueuse, nous remarquons que les lignes de courants sont refermées sur elles-mêmes, si l'écoulement est non-visqueux. Partant d'un champ initiale arbitraire, l'écoulement déforme et étire les lignes de champ jusqu'au moment où elles suivent les lignes de courants. Après ce transitoire, le champ magnétique n'est plus étiré et disparaît suite à la diffusion magnétique. Pour continuer à étirer les lignes de champ, il faut du mélange ou que les lignes de courants ne soient plus fermées sur elles mêmes. C'est exactement, ce qu'apportent les effets visqueux en paroi. Grâce aux couches limites qui se développent à la paroi, un pompage radial d'ordre  $E^{1/2}$  vient mélanger l'écoulement d'une manière non-triviale, et les trajectoires des particules de fluide ne sont plus refermées sur elles-mêmes. Dans la configuration que l'on a étudiée, ces effets de paroi, bien que petits, permettent de régénérer les modes poloïdaux ou les modes toroïdaux et d'obtenir ainsi l'effet constructif nécessaire pour la dynamo. Il n'est pas impossible que cet effet soit plus important dès que les couches limites sur les bords supérieurs et inférieurs du cylindre sont prises en compte ou si on considère un autre mode inertiel que le mode de spinover. En particulier, si un terme de couplage entre les modes toroïdaux et poloïdaux d'ordre  $E^{1/2}$  était présent, l'effet constructif serait probablement plus important.

Nous avons remarqué que l'introduction d'un nombre Reynolds magnétique

$$Rm = \frac{Ro^2}{E_m} = \frac{U \frac{U}{\Omega}}{\eta} \quad (7.69)$$

basé sur les caractéristiques locales de l'onde inertielle ( $U$  est l'échelle de vitesse réelle de l'écoulement et  $U/\Omega$  correspond au rayon typique des trajectoires circulaires des particules de fluide dans le repère tournant) permet d'unifier les résultats numériques. Sur la figure 7.7, nous avons représenté le taux de croissance normalisé  $\sigma/E_m$  des deux modes de type II, dont celui conduisant à la dynamo, en fonction de  $Rm$  pour les deux valeurs de  $E_m$  que l'on a considéré. Nous pouvons constater que les deux courbes sont très proches et que les deux valeurs d' $E_m$  fournissent la même valeur critique de  $Rm$  pour le seuil de la dynamo, ( $Rm = 288$  à  $E_m = 10^{-3}$ ;  $Rm = 279$  à  $E_m = 10^{-4}$ ). Ces observations tendent à montrer que pour un nombre d'Ekman fixé, le taux de croissance des modes, normalisé par le temps diffusif, ne dépend que du nombre de Reynolds magnétique  $Rm$ . Ce résultat serait également en accord avec l'hypothèse avancée précédemment dans la théorie asymptotique diffusive : le taux de croissance des ondes propres les plus instables de la matrice d'interaction  $\mathbf{M}_*$  ne dépendrait pas de  $E_m$  dans la limite des petits Ekman magnétiques. En conséquence, le seuil de la dynamo serait obtenu pour un Reynolds magnétique critique  $Rm_c$  qui ne dépendrait que du nombre d'Ekman.

$$Rm_c \simeq Rm_c(E) \quad (7.70)$$

La forme de la matrice d'interaction nous pousse à proposer une loi d'échelle pour  $Rm_c(E)$  de la forme

$$Rm_c(E) \sim E^{1/2} \quad (7.71)$$

mais nous n'avons pas validé ce résultat numériquement. De nouvelles simulations avec d'autres nombres d'Ekman seraient nécessaires pour confirmer une telle loi. Dans la limite géophysique pour laquelle le nombre d'Ekman est généralement très petit ( $E \simeq 10^{-15}$  pour la Terre), cette loi conduirait à un nombre de Reynolds magnétique critique très grand ( $Rm_c \simeq 10^7$  pour la Terre) difficilement atteignable. Mais cette extrapolation met trop rapidement en doute les dynamos d'ondes inertielles dans des systèmes géophysiques, car d'autres mécanismes de mélange seront probablement plus actifs que le pompage visqueux, en présence de parois courbées ou dans les géométries de coquilles sphériques. Nous pensons que la même analyse peut être faite pour des ondes inertielles (visqueuses) dans d'autres géométries, mais seulement à l'aide de moyens numériques plus avancés, compte tenu de la structure plus complexe des ondes inertielles [115]. Pour illustrer le rôle important des parois inclinées, il est instructif de regarder l'écoulement quasi-géostrophique qui serait engendré par un mécanisme de convection en rotation rapide. La dynamo de ces écoulements a été démontrée par Busse [11], [12] et a récemment été étudiée numériquement [122]. Un tel écoulement se décomposerait, dans le repère tournant, en une partie bi-dimensionnelle  $\mathbf{u}_{2D}$  que nous supposons stationnaire (lentement variable), les rouleaux de convection, et une correction tri-dimensionnelle  $\epsilon \mathbf{u}_{3D}$  contenant notamment les effets induits par les parois. L'équation d'induction s'écrirait alors :

$$\partial_t \mathbf{b} - \nabla \times (\mathbf{u}_{2D} \times \mathbf{b}) = \epsilon \nabla \times (\mathbf{u}_{3D} \times \mathbf{b}) + Rm^{-1} \Delta \mathbf{b} \quad (7.72)$$

où le nombre de Reynolds magnétique est définie  $Rm = UR/\eta$  avec  $U$  l'échelle de l'écoulement  $\mathbf{u}_{2D}$  et  $R/U$  l'échelle de temps. A petit  $\epsilon$  et  $Rm$ , nous pouvons faire une analyse asymptotique de la même manière que pour les ondes inertielles. A l'ordre 0 en  $\epsilon$ , on aurait un problème 2D qui ne peut jamais donner une dynamo. Ce n'est qu'à l'ordre  $O(\epsilon)$ , qu'une dynamo pourrait être obtenue, et dans ce cas nous trouverions

$$Rm \sim \epsilon^{-1} \quad (7.73)$$

Il est bien connu que l'amplitude de  $\epsilon$  est très fortement liée à l'inclinaison des parois, phénomène connu sous le nom de l'effet du plan beta [45], [12], [122]. Dans un cylindre les parois inférieures et supérieures sont horizontales, et la seule source de la tri-dimensionnalité sont de nouveau les pompes visqueuses et donc  $\epsilon \sim E^{1/2}$ . En géométrie cylindrique, la dynamo quasi-géostrophique est confrontée au même problème que la dynamo des ondes inertielles. Si par contre les parois supérieures et inférieures sont inclinées,  $\epsilon$  est obligatoirement plus important, car lié à un effet non-visqueux dit  $\beta$ . La valeur de  $\epsilon$  peut donc devenir plus importante que  $E^{1/2}$ , si les parois sont inclinées. Revenant au problème de la dynamo, nous comprenons alors que  $Rm_c \sim \epsilon^{-1}$ , pourrait alors être beaucoup plus facilement atteignable grâce aux parois inclinées. Un phénomène similaire n'est pas exclu pour les ondes inertielles. Il serait de toute manière intéressant de faire de telles études asymptotiques basées sur une séparation d'échelles de temps, au lieu d'une séparation d'échelles spatiales, dans d'autres géométries que le cylindre et de les comparer aux nombreux résultats existants sur le sujet.

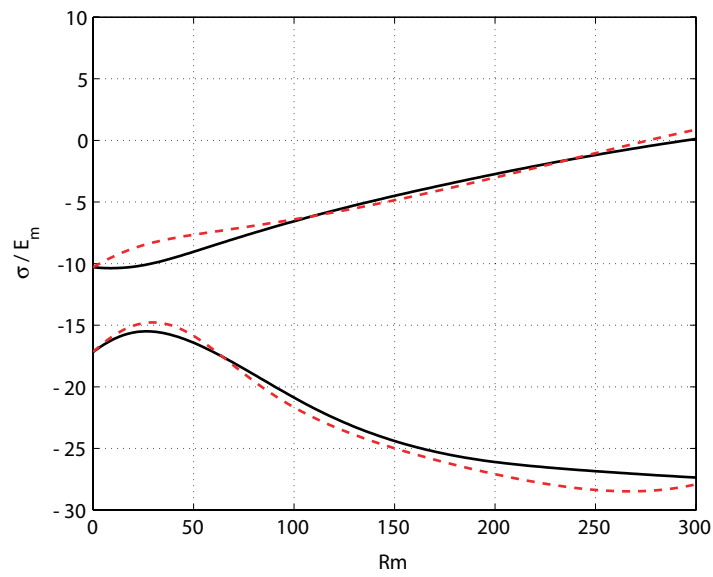


FIG. 7.7 – Taux de croissance, normalisé par un temps diffusif, des 2 modes de type II les plus instables en fonction de  $Rm = Ro^2/E_m$  pour  $E_m = 10^{-3}$  (ligne entière) et  $E_m = 10^{-4}$  (ligne pointillé) pour un nombre d'Ekman  $E = 10^{-3}$ .

# Chapitre 8

## Conclusion

Dans cette étude, nous nous sommes intéressés à l'instabilité elliptique d'un écoulement confiné en géométrie sphéroïdale et cylindrique. Nous avons mis en place une série d'expériences se servant du champ magnétique comme technique de mesure. Nous avons caractérisé l'action de la force de Lorentz sur l'instabilité lorsque le champ imposé est fort. Un modèle pour l'évolution non-linéaire de l'instabilité elliptique a été proposé. La possibilité d'une dynamo cinématique dans un écoulement d'ondes inertielles a également été étudiée.

### 8.1 Résumé

#### **Perturbations visqueuses d'un cylindre de fluide en rotation rapide**

Dans le deuxième chapitre, nous avons trouvé les perturbation "visqueuses" d'un cylindre en rotation rapide. Ces solutions sont les extensions visqueuses des écoulements géostrophiques et des ondes inertielles non-visqueuses connues dans la littérature. Nous avons identifié un produit scalaire orthogonalisant, ce qui nous a permis d'utiliser les modes visqueux comme base pour les perturbations de la colonne de fluide.

#### **Evolution non linéaire de l'instabilité elliptique**

Ces modes ont ensuite été utilisés pour décrire le problème de l'évolution non-linéaire de l'instabilité dans le troisième chapitre. Le but de cette étude a été de modéliser et de suivre la transition progressive d'une dynamique faiblement non-linéaire proche du seuil vers des états plus complexes. Nous avons été confrontés à une difficulté supplémentaire qui trouve son origine dans les variations non-linéaires de la pression, non prises en compte dans l'approche initiale. Cette approche initiale produit cependant des comportements de fluide qualitativement en accord avec des observations expérimentales, et deux scénarios de transition issus de ces modèles initiaux ont été discutés en détail. Nous avons montré que le collapse semble trouver son origine dans un cycle faiblement non-linéaire qui devient instable.

#### **Instabilité elliptique sous champ magnétique dans un cylindre**

Dans le quatrième chapitre, nous avons étudié la dynamique d'un métal liquide dans le dispositif de Malkus placé sous un champ magnétique homogène imposé le long de l'axe. Nous avons montré que les écoulements excités par l'instabilité elliptique induisent un champ magnétique que l'on arrive à détecter à l'extérieur du cylindre et que cette mesure fournit certaines informations sur l'écoulement à l'intérieur du cylindre. Par ailleurs, une analyse théorique nous a permis d'identifier la signature magnétique de chaque onde inertielle. La rétroaction du champ magnétique a également été prise en compte. Nous avons en particulier montré que la force de Lorentz ne change pas la structure spatiale de l'onde inertielle, ni celle du champ induit, mais qu'elle introduit une dissipation magnétique qui atténue les ondes inertielles. Ce résultat nous a permis de calculer la correction magnétique de l'instabilité elliptique. Nous avons notamment montré qu'il existe un seuil pour l'amplitude du champ magnétique, au delà duquel l'instabilité elliptique disparaît.

Le dispositif expérimental IMAGINE a été présenté, et nous avons exposé les choix faits pour optimiser le dispositif, permettant de mesurer les petites variations du champ magnétique induit qui nous intéressent. Nous avons analysé une série de mesures dans lesquelles nous avons identifié le mode  $(1, 3, 1)$  de l'instabilité elliptique, en accord la théorie de stabilité linéaire. Nous avons mis en évidence l'existence de cycles lents qui deviennent intermittents, lorsque l'on s'éloigne du seuil de l'instabilité.

### Instabilité elliptique sous champ magnétique dans un sphéroïde

Dans le cinquième chapitre, nous avons discuté du rôle que pourrait tenir l'instabilité elliptique dans le contexte géophysique d'un corps céleste elliptiquement déformé. Nous avons considéré le cas spécifique où un mode de spinover est excité, c'est à dire lorsque l'écoulement est soumis à une rotation supplémentaire autour d'un axe incliné par rapport à l'axe de rotation initiale. Nous avons calculé la signature magnétique de cet écoulement, ce qui nous a permis d'optimiser le placement de la sonde magnétique. Nous avons mesuré le taux de croissance de l'instabilité elliptique à différentes vitesses de rotation et montré que les résultats étaient en accord avec la théorie linéaire. Dans cette première étude, le champ magnétique était trop faible pour pouvoir influencer l'instabilité elliptique de façon notable.

Dans une deuxième étude, présentée dans le sixième chapitre, des expériences sous un champ magnétique plus intense ont permis de mettre en évidence l'atténuation de l'instabilité elliptique par effet Joule. Grâce au champ magnétique fort, des mesures de très bonne qualité ont pu être obtenues. Les résultats issues de ces mesures ont été comparés de manière très satisfaisante avec un modèle de basse dimension, basé sur les résultats de Lacaze (2008a) et Thess (2007). Nous avons montré que l'instabilité elliptique pouvait disparaître avec un champ magnétique suffisamment fort.

### Dynamo cinématique des ondes inertielles

Dans le dernier chapitre, nous avons étudié l'effet dynamo d'une onde inertielle confinée dans une boîte cylindrique. Nous avons développé une procédure numérique qui permet de projeter l'équation d'induction sur les modes propres de l'équation de diffusion magnétique. Pour le cas spécifique du mode de spinover, le problème a pu être réduit ainsi à un problème aux valeurs propres. Nous avons identifié des dynamos, mais seulement lorsque les effets visqueux sur le mode de spinover sont pris en compte. Nous avons montré que le mode magnétique instable est principalement composé d'un champ magnétique axisymétrique toroïdal, dans la limite où la diffusion magnétique et l'amplitude de l'onde inertielle deviennent faibles. Cela nous a permis d'analyser le mécanisme responsable de la dynamo par une approche asymptotique. Le rôle déstabilisant de la viscosité a ainsi pu être mis en évidence. Elle nous a permis également de montrer le rôle essentiel de la diffusion magnétique sur les structures de petite échelle pour pouvoir obtenir une dynamo.

## 8.2 Perspectives

Une étude expérimentale détaillée sur l'instabilité sous champ magnétique est actuellement en cours sur le banc expérimental IMAGINE. La figure 8.1, montre quelques résultats préliminaires, où l'évolution non-linéaire d'un mode de spinover, en choisissant un cylindre de rapport d'aspect  $h = 5.9$ . Suffisamment proche du seuil, la dynamique sature (voir Fig. 8.1-(a)). En augmentant la distance au seuil (on double la vitesse de rotation), des oscillations lentes apparaissent, qui initialement semblent de moins grande amplitude. Leur amplitude devient plus grande, plus loin du seuil, et finalement apparaît un régime où on peut obtenir des renversements. L'expérience IMAGINE est très bien adaptée pour l'étude de la dynamique non-linéaire. Elle produit des signaux de haute qualité, qui permettent un traitement quantitatif. L'atténuation magnétique permet de contrôler la vigueur de l'instabilité, et de cette manière également la transition. Ceci pourrait s'avérer très utile dans l'étude expérimentale, qui est donc la perspective la plus importante de cette thèse.

Les résultats expérimentaux permettront de tester les idées issues des modèles non-linéaires de l'instabilité elliptique où la correction due à la pression devra être incluse.

L'étude de la dynamo des ondes inertielles n'a pas non plus touché à sa fin. Dans un premier temps il sera nécessaire de vérifier les lois d'échelles proposées. Il serait utile de comparer les dynamos des ondes inertielles à celles des écoulements quasi-géostrophiques, pour répondre à la question : "Quel écoulement est le plus efficace?". Nous pensons également qu'il serait intéressant de quitter la géométrie cylindrique, pour éviter les problèmes trop spécifiques à cette géométrie (notamment l'absence d'un effet de plan  $\beta$ ). Une étude similaire en géométrie sphérique est d'ailleurs envisagée.

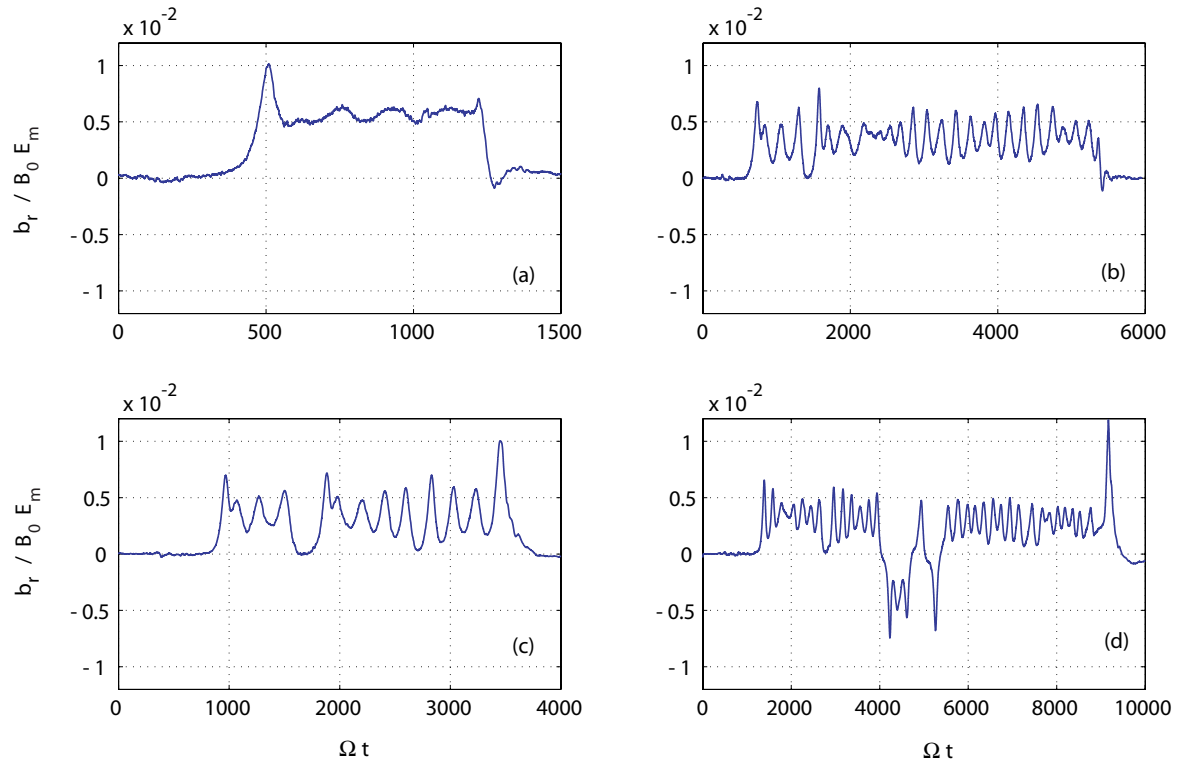


FIG. 8.1 – Evolution temporelle du champ magnétique induit par un mode de spinover  $(-1, 1, 1)$ , dans un cylindre de rapport d'aspect  $h = 5.9$ ,  $\epsilon = 0.071$ , à différents nombres d'Ekman et nombres d'Elzasser. (a)  $E = 2.37 \times 10^{-4}$ ,  $\Lambda = 0.0161$ , (b)  $E = 1.19 \times 10^{-4}$ ,  $\Lambda = 0.008$ , (c)  $E = 0.596 \times 10^{-5}$ ,  $\Lambda = 0.004$  (d)  $E = 0.298 \times 10^{-5}$ ,  $\Lambda = 0.002$ .



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# Annexe A

## Cylindrical $\pm$ -vector field components.

### A.1 Introduction

In this appendix, we summarise the properties of the cylindrical  $\pm$ -vector components, which have been found useful in solving the eigenvalue problems in cylindrical geometry. With  $(r, \phi, z)$ , the cylindrical coordinates adapted to this geometry and  $(f_r, f_\phi, f_z)$  the cylindrical components of a vector field  $\mathbf{f}$ , they are defined as

$$f_{\pm} = \frac{1}{\sqrt{2}}(f_r \pm i f_\phi) \quad , \quad \begin{cases} f_r &= \frac{1}{\sqrt{2}}(f_+ + f_-) \\ f_\phi &= \frac{-i}{\sqrt{2}}(f_+ - f_-) \end{cases} \quad (\text{A.1})$$

This simple transform has some particular properties, which were previously noticed and used in different contexts. This transform appears in the analysis of the Ponomarenko-dynamo, [111], and was first used in the context of inertial waves by Kerswell, [59] in the analysis of shear layers expelled by boundary layer discontinuities.

### A.2 Vector calculus in $\pm$ -components

Using the linear combinations defining the  $\pm$ -components we find the expressions of some useful vector formula. We note  $\mathbf{f}$  and  $\mathbf{g}$  two vector fields and  $\psi$  is a scalar field. We give an overview of the basic operations in vector analysis.

#### Scalar product

$$\mathbf{f} \cdot \mathbf{g} = f_+ g_+ + f_- g_- + f_z g_z \quad (\text{A.2})$$

#### Vector product

$$\begin{aligned} (\mathbf{f} \times \mathbf{g})_{\pm} &= \mp i (f_{\pm} g_z - f_z g_{\pm}) \\ (\mathbf{f} \times \mathbf{g})_z &= i (f_+ g_- - f_- g_+) \end{aligned} \quad (\text{A.3})$$

#### Gradient

$$\begin{aligned} (\nabla \psi)_{\pm} &= \frac{1}{\sqrt{2}} (\partial_r \pm \frac{i}{r} \partial_\phi) \psi \\ (\nabla \psi)_z &= \partial_z \psi \end{aligned} \quad (\text{A.4})$$

### Divergence

$$\nabla \cdot \mathbf{f} = \frac{1}{\sqrt{2}} (\partial_r - \frac{i}{r} \partial_\phi + \frac{1}{r}) f_+ + \frac{1}{\sqrt{2}} (\partial_r + \frac{i}{r} \partial_\phi + \frac{1}{r}) f_- + \partial_z f_z \quad (\text{A.5})$$

### Curl

$$\begin{aligned} (\nabla \times \mathbf{f})_\pm &= \mp i \left[ \frac{1}{\sqrt{2}} (\partial_r \pm \frac{i}{r} \partial_\phi) f_z - \partial_z f_\pm \right] \\ (\nabla \times \mathbf{f})_z &= -i \left[ \frac{1}{\sqrt{2}} (\partial_r - \frac{i}{r} \partial_\phi + \frac{1}{r}) f_+ \right. \\ &\quad \left. - \frac{1}{\sqrt{2}} (\partial_r + \frac{i}{r} \partial_\phi + \frac{1}{r}) f_- \right] \end{aligned} \quad (\text{A.6})$$

### Vector field Laplacian

$$(\Delta \mathbf{f})_\pm = \Delta_\pm f_\pm = \left( \partial_{rr}^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} (\partial_\phi^2 \pm 2i \partial_\phi - 1) + \partial_{zz}^2 \right) f_\pm \quad (\text{A.7})$$

$$\begin{aligned} (\Delta \mathbf{f})_z &= \Delta f_z = \left( \partial_{rr}^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\phi^2 + \partial_{zz}^2 \right) f_z \\ \Delta_\pm &= (\partial_r \pm \frac{i}{r} \partial_\phi) (\partial_r \mp \frac{i}{r} \partial_\phi + \frac{1}{r}) + \partial_{zz}^2 \\ \Delta &= (\partial_r \pm \frac{i}{r} \partial_\phi + \frac{1}{r}) (\partial_r \mp \frac{i}{r} \partial_\phi) + \partial_{zz}^2 \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} (\partial_r \pm \frac{i}{r} \partial_\phi) \Delta &= \Delta_\pm (\partial_r \pm \frac{i}{r} \partial_\phi) \\ (\partial_r \mp \frac{i}{r} \partial_\phi + \frac{1}{r}) \Delta_\pm &= \Delta (\partial_r \mp \frac{i}{r} \partial_\phi + \frac{1}{r}) \end{aligned} \quad (\text{A.9})$$

### Field-deformation-tensor

$$\nabla \mathbf{f} = \begin{bmatrix} \frac{1}{\sqrt{2}} (\partial_r - \frac{i}{r} \partial_\phi + \frac{1}{r}) f_+ & \frac{1}{\sqrt{2}} (\partial_r + \frac{i}{r} \partial_\phi - \frac{1}{r}) f_+ & \partial_z f_+ \\ \frac{1}{\sqrt{2}} (\partial_r - \frac{i}{r} \partial_\phi - \frac{1}{r}) f_- & \frac{1}{\sqrt{2}} (\partial_r + \frac{i}{r} \partial_\phi + \frac{1}{r}) f_- & \partial_z f_- \\ \frac{1}{\sqrt{2}} (\partial_r - \frac{i}{r} \partial_\phi) f_z & \frac{1}{\sqrt{2}} (\partial_r + \frac{i}{r} \partial_\phi) f_z & \partial_z f_z \end{bmatrix} \quad (\text{A.10})$$

### Coriolis force diagonalisation

$$(2 \hat{\mathbf{z}} \times \mathbf{u})_\pm = 2i u_\pm \quad (\text{A.11})$$

### A.3 Interest of $\pm$ -components

We illustrate the use and interest of  $\pm$ -components on a simple example. The Laplacian problem for a scalar field  $\psi$

$$\Delta\psi = 0 \tag{A.12}$$

is separable in cylindrical geometry. The homogeneous solution for the problem is a linear combination of solutions which depend on two wavenumbers, the azimuthal wavenumber  $m$  and the vertical wavenumber  $l$ . The solution writes explicitly

$$\psi = \sum_{m,l} \left( C I_m(lr) + D K_m(lr) \right) e^{im\phi+ilz} \tag{A.13}$$

with  $m \in \mathbb{Z}$  due to periodicity and  $l \in \mathbb{C}$ . By applying successively gradient and divergence operators we recover the Laplacian problem  $\nabla \cdot \nabla\psi = 0$ . It is interesting to have a look at the structure of the gradient of separable fields of this kind. With (A.4), we find

$$(\nabla\psi)_{\pm} = \frac{1}{\sqrt{2}} \left( \partial_r \mp \frac{m}{r} \right) \psi \quad , \quad (\nabla\psi)_z = il\psi \tag{A.14}$$

in the  $\pm$ -parts, we recognise the operators, which appear in the recursion relation operators of Bessel functions

$$\begin{aligned} \left( \partial_r \pm \frac{m}{r} \right) J_m(kr) &= \pm k J_{m\mp 1}(kr) \\ \left( \partial_r \pm \frac{m}{r} \right) Y_m(kr) &= \pm k Y_{m\mp 1}(kr) \\ \left( \partial_r \pm \frac{m}{r} \right) I_m(kr) &= k I_{m\mp 1}(kr) \\ \left( \partial_r \pm \frac{m}{r} \right) K_m(kr) &= k K_{m\mp 1}(kr) \end{aligned} \tag{A.15}$$

so that we find the following result for the gradient.

$$\begin{aligned} (\nabla\psi)_{\pm} &= \frac{l}{\sqrt{2}} \left( C I_{m\pm 1}(lr) + D K_{m\pm 1}(lr) \right) e^{im\phi+ilz} \\ (\nabla\psi)_z &= il \left( C I_m(lr) + D K_m(lr) \right) e^{im\phi+ilz} \end{aligned} \tag{A.16}$$

The use of  $\pm$ -parts of the gradient vector field, has simplified the structure of the field-gradient, by stepping up and down the argument  $m$  of the Bessel-functions, respectively to  $m \pm 1$  in the  $\pm$ -parts. The vertical component still has argument  $m$ . Applying now the divergence operator (A.5) on this field,

$$\frac{1}{\sqrt{2}} \left( \partial_r + \frac{m+1}{r} \right) (\nabla\psi)_+ + \frac{1}{\sqrt{2}} \left( \partial_r - \frac{m-1}{r} \right) (\nabla\psi)_- + il (\nabla\psi)_z = 0 \tag{A.17}$$

we necessarily need to return to a single functional dependance of space. It is therefore no surprise that the operators before ( $\pm$ )-parts in the divergence constraint above are also step operators which get the arguments of  $I_{m\pm 1}$  and  $K_{m\pm 1}$  back to  $m$ , to find

$$\left( \frac{l^2}{2} + \frac{l^2}{2} - l^2 \right) \left( C I_m(lr) + D K_m(lr) \right) e^{im\phi+ilz} = 0 \tag{A.18}$$

as expected. Once this is remarked, one rewrites the Laplacian terms of (A.7) as a succession of step operators, as in (A.8) and reveals the commutation relations (A.8) which simplify decoupling of more complex problems such as those solved in this work. Similar reasoning can be done for scalar Helmholtz problems, and even in the related vector field problems involving the Coriolis force, the hierarchy of  $\pm$ -parts of vector fields behaving like  $I_{m\pm 1}$  and z-parts behaving like  $I_m$  is conserved. In general eigenmodes of the velocity and pressure fields,  $\mathbf{u}$  and  $p$ , are composed of terms like

$$\begin{aligned} u_{\pm} &= c_{\pm} I_{m\pm 1}(kr) e^{im\phi+ilz} \\ u_z &= c_z I_m(kr) e^{im\phi+ilz} \\ p &= d I_m(kr) e^{im\phi+ilz} \end{aligned} \tag{A.19}$$



where  $k \in \mathbb{C}$  is a complex radial wavenumber,  $m \in \mathbb{Z}$  the azimuthal wavenumber and  $l \in \mathbb{R}$  a real-valued vertical wavenumber. In cylindrical shells, Bessel functions of the second type ( $K_m$ ) also intervene. The spatial structure of the eigenmodes is composed of twice as less terms as when  $(r, \phi)$ -components were used. This may seem irrelevant, but in further analysis asymptotic theories are builded on these eigenmodes, and matrix elements are to be evaluated. For this we use scalar products of the kind

$$\int_V \mathbf{f}^* \cdot \mathbf{g} \, dV \quad (\text{A.20})$$

With eq. (A.2), and half as much terms in the field-components, we need to calculate four times less terms, when  $\pm$ -components are used instead of  $(r, \phi)$ -ones. Great was the surprise to notice that the nonlinear advection also benefits from the use of  $\pm$ -components. For a field like form (A.19), one evaluates the field deformation tensor as

$$\nabla \mathbf{f} = \begin{bmatrix} c_+ \frac{k}{\sqrt{2}} I_m(kr) & c_+ \frac{k}{\sqrt{2}} I_{m+2}(kr) & c_+(il) I_{m+1}(kr) \\ c_- \frac{k}{\sqrt{2}} I_{m-2}(kr) & c_- \frac{k}{\sqrt{2}} I_m(kr) & c_-(il) I_{m-1}(kr) \\ c_z \frac{k}{\sqrt{2}} I_{m+1}(kr) & c_z \frac{k}{\sqrt{2}} I_{m-1}(kr) & c_z(il) I_m(kr) \end{bmatrix} e^{i(m\phi+lz)} \quad (\text{A.21})$$

All radial and azimuthal derivatives are combined in such a way that they transform into exact recursion relation operators on fields like (A.19). This is a very significant gain compared to what would have been necessary for the same field but using  $(r, \phi)$ -components, and enables us to keep analytical insight in advection terms and corresponding matrix elements of the style

$$\int_V \mathbf{f}^* \cdot (\mathbf{g} \cdot \nabla) \mathbf{h} \, dV \quad (\text{A.22})$$

for which the number of terms, quickly increases when  $(r, \phi)$ -components of the fields are preferred. The simplified structure of  $\pm$ -components of vector fields and the use of recursion relations are the main reason why we were able to find analytical profiles for the diffusive eigenmodes of the previous chapters. Both linear and nonlinear theories benefit from the fact that much less terms need to be calculated. When there is no possible confusion, it is useful to introduce step operators  $\hat{a}_\pm$ , to simplify notation. They are defined as

$$\begin{aligned} \hat{a}_\pm I_m(kr) &= \frac{k}{\sqrt{2}} I_{m\pm 1}(kr) \\ \hat{a}_\pm J_m(kr) &= \mp \frac{k}{\sqrt{2}} J_{m\pm 1}(kr) \end{aligned} \quad (\text{A.23})$$

whatever  $m$ . Gradient, divergence, curl, Laplacian and field-deformation tensor of field with structures as in (A.19) then simply write

### Gradient

$$(\nabla \psi) = \hat{a}_\pm \psi \quad , \quad (\nabla \psi)_z = \partial_z \psi \quad (\text{A.24})$$

### Divergence

$$\nabla \cdot \mathbf{f} = \hat{a}_- f_+ + \hat{a}_+ f_- + \partial_z f_z \quad (\text{A.25})$$

### Curl

$$\begin{aligned} (\nabla \times \mathbf{f})_\pm &= \mp i(\hat{a}_\pm f_z - \partial_z f_\pm) \\ (\nabla \times \mathbf{f})_z &= i(\hat{a}_+ f_- - \hat{a}_- f_+) \end{aligned} \quad (\text{A.26})$$

### Laplacian

$$\Delta_{\pm,z} = 2\hat{a}_+\hat{a}_- + \partial_{zz}^2 = 2\hat{a}_-\hat{a}_+ + \partial_{zz}^2 \quad (\text{A.27})$$

### Field deformation tensor

$$\nabla \mathbf{f} = \begin{bmatrix} \hat{a}_- f_+ & \hat{a}_+ f_+ & \partial_z f_+ \\ \hat{a}_- f_- & \hat{a}_+ f_- & \partial_z f_- \\ \hat{a}_- f_z & \hat{a}_+ f_z & \partial_z f_z \end{bmatrix} \quad (\text{A.28})$$

## A.4 Reality of the fields in $\pm$ components

For the purpose of analysis it is often useful to use exponentials such as  $\exp(im\phi + ilz + i\lambda t)$  for independent velocity and magnetic field contributions, but at the end the fields need to be real and respect certain boundary conditions or symmetries, so that the solutions are to be constructed as linear combinations. In the vertical structure of the fields, we will always prefer the use of real functions  $\cos(lz)$  or  $\sin(lz)$ , since boundary conditions or symmetries will then automatically be more easily builded in. The cylindrical  $(r, \phi, z)$  components of the velocity and magnetic fields are real, so that each candidate field-solutions  $\mathbf{f}^{(+)}$ ,  $\psi^{(+)}$  with positive  $m$  and frequency  $\lambda$ , are accompanied by the fields  $\mathbf{f}^{(-)}$ ,  $\psi^{(-)}$  with azimuthal wavenumber  $-m$  and frequency  $-\lambda^*$ . The reality of the fields then requires

$$\psi^{(-)} = \psi^{(+)*} \quad , \quad f_j^{(-)} = f_j^{(+)*} \quad , \quad j = r, \phi, z \quad (\text{A.29})$$

With the  $\pm$ -components defined by a complex transformation, this does not translate similarly to these components for the vector field. Using the definitions (A.1), eq. (A.29) is compatible with

$$\begin{aligned} f_{\pm}^{(-)} &= f_{\mp}^{(+)*} \\ a_{\pm} f_{\pm}^{(-)} &= a_{\mp} f_{\pm}^{(+)*} \\ a_{\pm} f_z^{(-)} &= a_{\mp} f_z^{(+)*} \end{aligned} \quad (\text{A.30})$$

These properties are used to simplify the linear or nonlinear matrix element calculations, which involve the complex conjugate fields. Axisymmetric geostrophic modes are real and have to be treated separately.



## Annexe B

# Elliptical instability in a cylindrical shell

### B.1 Introduction

The instability of elliptical flow is a well known phenomenon. The subject has been reviewed by Kerswell (2002). Through a mechanism of parametric resonance, couples of inertial waves are excited and may grow to large amplitudes. In the geophysical context of a tidally deformed planet, elliptical instability may provide an alternative to convection driven flows driving a planetary dynamo, [1], [89], [58], [57]. The role played by an inner solid core in a spherical shell, has been studied by Lacaze, [72], who found that the spinover modes continue to be the most unstable. Theoretically this system is untrivial to analyse as the structure of the main part of the inertial waves becomes rather complicated, [114]. In cylindrical shells, the inertial modes keep their simple regular expressions, which helps in answering to certain questions. Spinover modes in cylinders have non zero velocity on the axis, and this can no longer be the case in cylindrical shells, which may be inhibiting instability. It turned out that the contrary was true for the non-viscous growth rates at perfect resonance, [54]. Spinover modes were slightly destabilised, whereas other couplings were more stable. Some experiments by Malkus were mentioned, [57], [54], but no real specifications have been published since.

In this letter we will introduce and discuss viscous effects and detuning, which may change this picture. We will obtain some detailed phase diagrams, and compare them with experimental Kalliroscope measurements.

### B.2 Analysis

In experiments and geophysical situations, the ellipticity  $\epsilon$  of the rotating flows is always very small. This allows asymptotic theories to be builded in which elliptical instability is modelled as a small  $0(\epsilon)$ , but effective coupling between resonant couples of solutions of the unperturbed  $0(1)$  problem. The first step of analysis finds these solutions and identifies resonant paires of them, often in a non viscous limit. For a solid body rotation base state these solutions are known as the inertial waves, and in cylindrically shaped containers, spheres and spheroids, the non viscous boundary value problem is well-posed and regular solutions can be found. An analytical study of the effect of an inner core is only possible in cylindrical shells. The  $0(1)$  solutions are then known as the Kelvin waves, [53], and are parametrised by a set of 3 numbers  $(k, m, l)$  respectively the radial, azimuthal and axial wavenumber and a frequency  $\omega$ . The dispersion relation depends on two supplementary geometrical parameters  $\alpha = R_i/(R_e - R_i)$ ,  $\beta = H/(R_e - R_i)$ , with  $R_i$ ,  $R_e$ ,  $H$ , respectively the radius of the inner and outer cylindrical boundary surfaces and the height of the cylinder.

An elliptical coupling is perfectly resonant if two modes, 1 and 2, have  $\omega_1 = \omega_2$ ,  $l_1 = l_2$  and  $m_1 + 2 = m_2$ . A fourth condition,  $k_1 \simeq k_2$ , is less stringent, but systematically produces couplings with bigger growth rates, [26], due to more overlapping radial structures. This filters out the central couplings, noted as  $(m_1, m_1 + 2, x)$ , with  $x \in \mathcal{N}$  labeling the degree of radial complexity. Their coordinates  $(l_*, \omega_*)$  are calculated numerically, and depend slightly on  $\alpha$ . The frequency of the central couplings always stays close to  $\omega_* \simeq -(m_1 + m_2)/2$ , the wavenumber coordinate  $l_*$  varies more rapidly as is illustrated in Fig. B.1-(a), for the first couplings with the smallest wavenumbers.  $m_1 = -1, 0, 1$ ,  $m_2 = m_1 + 2$ ,  $x = 1$ . For very small inner cylinders,  $\alpha \rightarrow 0$ , we find to the coordinates of the case without inner cylinder - from now on briefly noted as  $\alpha = 0$ . As  $\alpha$  grows, a rather

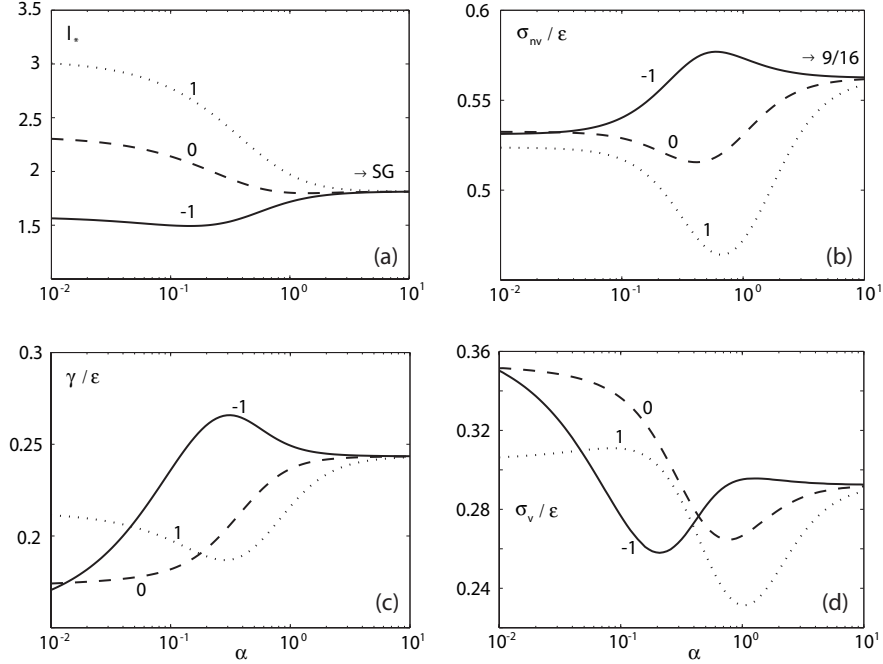


FIG. B.1 – Influence of  $\alpha$  on the elliptical instability in a cylindrical shell of infinite length, for the couplings  $(m_1, m_1 + 2, 1)$ ,  $m_1 = -1, 0, 1$ . (a) Variation of the vertical wavenumber coordinates. (b) Non viscous growth rates at perfect resonance. (c) Viscous damping at perfect resonance,  $\epsilon = 0.085$ ,  $Re = 4000$ . (d) Growth rates corrected by viscosity,  $\epsilon = 0.085$ ,  $Re = 4000$ .

rapid convergence towards the small gap limit  $l_* = x \pi / \sqrt{3}$  is observed. In this limit, azimuthal gradients are negligible as long as  $m \ll \pi\beta$ . As a consequence radial and vertical structure and  $\lambda = m + \omega$  are independent of  $m$ , and the modes are trigonometric functions simplifying further analysis.

The next step in analysis explicitly evaluates the couplings at perfect resonance and introduces the long time scale  $\tau = \epsilon^{-1} t$ , on which the inertial waves are coupled by ellipticity. By posing a solvability condition linear growth rates are expressed as closed integrals. This was done by Kerswell, [54]. In Fig. B.1-(b), we reproduced some results for the same couplings. The most important observation is that non viscous instability of elliptical flow in cylindrical shells, cannot be inhibited by the inner core. The growth rates slightly vary with  $\alpha$ , and for intermediate values, the spinover modes  $m_1 = -1$  are more unstable than the other ones. In the small gap limit all couplings have the same growth rate, exactly  $\sigma = 9/16 \epsilon$ , the asymptotic value for unbounded elliptical flow. For experimental comparison, one needs to not take into account viscosity and detuning effects. Using a set of viscous equivalents of the kelvin modes, instead of a more usual boundary layer theory approach, we calculated attenuation rates and frequency corrections due to viscosity. These new analytic solutions will be detailed elsewhere, and are in harmony with the boundary layer approach. In Fig. B.1-(c), we plotted corresponding attenuation rates  $\gamma$ , for the same couplings. With the parameter set  $\epsilon = 0.085$ ,  $Re = 4000$  typical for our experiments, attenuation rates are of the same order of magnitude as the growth rates of Fig. B.1-(b). For intermediate  $\alpha$ , the spinover modes  $m_1 = -1$  are slightly more damped than the others, probably because of the increased velocity near the inner core. This means that viscosity may change the idea that spinover modes are always the most unstable couplings in cylindrical shells, which is illustrated in Fig. B.1-(d), where elliptical growth rates, viscous attenuation and slight viscous detuning have been combined into one picture, for the same  $\epsilon$  and  $Re$ . At intermediate  $\alpha$  the spinover mode may be up to 25 % less unstable, than concurring couplings. As the gap gets smaller, the hierarchy comes back to normal, with the spinover modes on top, but small differences in the respective growth rates.

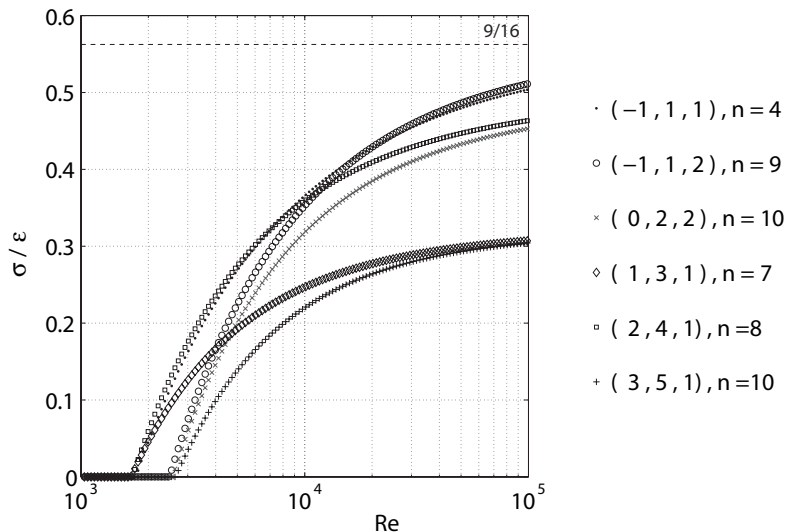


FIG. B.2 – Linear growth rates of the elliptical instability,  $\epsilon = 0,085$ , in a cylindrical shell,  $\alpha = 0.23$ ,  $\beta = 4 * 2.09$ , as a function of Reynolds number, for the first 6 unstable modes  $(m_1, m_1 + 2, x)$ , with  $n/2$  vertical wavelengths.

In all preceding pictures,  $\beta$  was an absent parameter, making these results only applicable in very lengthy  $\beta \rightarrow \infty$  fluid domains. Closed containers have Ekman layers on top and bottom plates and are impermeable, which imposes a geometrical constraint, allowing only kelvin modes with vertical wavenumbers  $l = n\pi/\beta$ ,  $n \in \mathbb{Z}$ . In the context of elliptical instability, this means that  $\beta$  and thus  $l$  need to be tuned within a band of  $0(\epsilon)$  around the resonant  $l_*$  of Fig. B.1-(a). For small  $\alpha$ , the different  $l_*$  are not harmonical and quite distant, and so it is often possible to tune in into one single mode, and to be detuned for others. But for bigger  $\alpha$ , all  $l_*$  get closer to each other, and become harmonical in the small gap limit. In this same limit growth rates and viscous dissipation are identical for all couplings with the same  $x$ , implying a broad band instability for what concerns the azimuthal structure. Progressively increasing  $\alpha$ , we expect that a larger number of modes will get unstable, and this more closer to the instability threshold. Most probably, this will also reduce the interval in which weakly non linear saturated states may be observed.

It would be interesting to experimentally study the small gap limit, but current apparatus does not permit to go beyond  $\alpha \simeq 2$ . Instead we will report on two studied cases,  $\alpha = 0.23$ , and  $\alpha = 1.22$ . With the first choice, we are in the domain where the spinover mode is not the most unstable coupling, but geometrical detuning effects will still be important. With the second choice, both geometrical detuning and viscous dissipation start to be less selective and spinover modes should be the unstable ones. Ellipticity is fixed at  $\epsilon = 0.085$ .

In Fig. B.2, we plotted viscous growth rates for the 6 first unstable couplings, with  $m_1 < 3$ . The choices of  $\beta$  were initially motivated by non-viscous analysis. With  $\alpha = 0.23$ ,  $\beta = 4 * 2.09$  we tune in into the stationary spinover mode  $(-1, 1, 1)$ , with coordinates  $(l_*, w_*) = (\pi/2.09, 0)$ .  $n = 4$  half vertical wavelengths are to be expected in the flow pattern. This coupling is effectively one of the first ones to appear, but it is likely the  $(1, 3, 1)$  can be excited by elliptic instability, even close to the threshold  $Re_c = 1690$ . In the same way we can tune in into perfect resonance conditions for the non viscous  $(0, 2, 1)$ ,  $n = 5$  coupling when  $\alpha = 0.23$  and  $\beta = 5 * 1.57$ . In this case this coupling was the most unstable one for  $Re \in [1300, 3000]$ . A third experimentally studied case, is  $\alpha = 0.23$ ,  $\beta = 6 * 1.24$ , where the  $(1, 3, 1)$  coupling seems to be the preferred one for  $Re \in [1100, 5000]$ . For the large inner cylinder  $\alpha = 1.22$  we choose  $\beta = 8 * 1.81$ , to tune in into the mode  $(-1, 1, 1)$ . This mode has the biggest growth rate over large interval,  $[1470, 10000]$ . We used these kind of diagrams to compare them with experimental results.

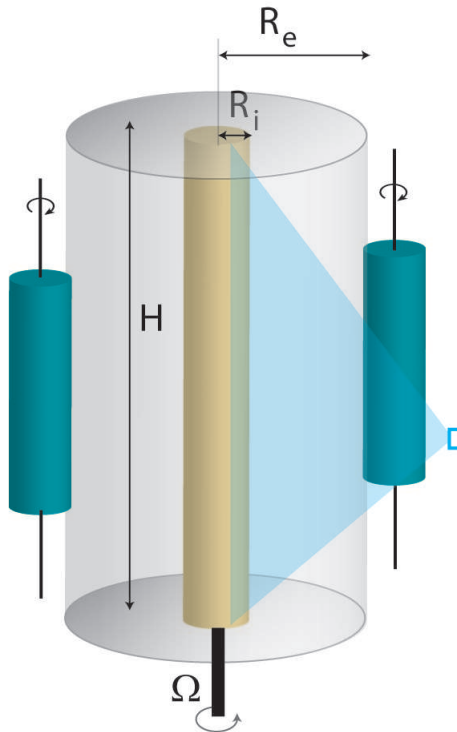


FIG. B.3 – Experimental device to study the elliptical instability. A laser sheet is used to visualize the flow seeded by Kalliroscope particles.

## B.3 Experimental results

### B.3.1 Experimental set-up

Our experimental device, presented in Fig. B.3, is based on the system invented by Malkus (1989), and formerly used by Eloy (2000). A cylindrical tube made of a transparent deformable material, of radius  $R_e = 27.5 \text{ mm}$ , rotates between two rollers which slightly compress its lateral wall. This deforms the initially circular section of the external cylinder wall into an ellipse, with fixed ellipticity  $\epsilon = 0.085 \pm 0.05$ , and it has been shown that this effectively produces flows with elliptical streamlines in cylinders. In our shells, the inner cylinder cannot be deformed and imposes circular streamlines in its vicinity. This means that ellipticity will not be homogeneous along the radial direction, which is not properly modelled by the former theory. Secondly the ellipticity is not homogeneous along the vertical direction, as the rollers cannot deform the fluid domain over its entire length. Qualitatively it is expected that a smaller effective ellipticity will be operational. As  $\alpha$  grows, both external and internal boundary surfaces become comparable in size, and the effective ellipticity should become thus the smaller.

The cylindrical shells we study have  $R_i = 5.15, 15.0 \text{ mm}$ , corresponding to  $\alpha = 0.23, 1.22$ . The height  $H$  of the cylinder can be varied in the interval  $[130, 200] \text{ mm}$ . We have studied different configurations, but here we will restrict our description to the cases  $\alpha = 0.23, \beta = 4 \times 2.09, 5 \times 1.57, 6 \times 1.24$  and  $\alpha = 1.22, \beta = 8 \times 1.81$ . With these choices we tune in into different resonances with respectively 4, 5, 6 and 8 half vertical wavelengths. The working fluid is water seeded with anisotropic kalliroscope particles,  $\nu = 10^6 \text{ m}^2 \text{ s}^{-1}$ , illuminated by a vertical laser sheet, along the direction of maximum strain, as maximum growth is expected in this direction. Visualisation was often improved by looking in planes which did not contain the rotation axis. The images are recorded with a standard digital camera at the maximum rate of 25 frames  $\text{s}^{-1}$ . The rotation rates are limited to  $\Omega_{max} \simeq 8\pi \text{ s}^{-1}$ . The maximum Reynolds number then is,  $Re_{max} = 1.9 \times 10^4 / (1 + \alpha)$ . We were not able to go beyond  $\alpha = 2$ , with the current experimental dispositive.

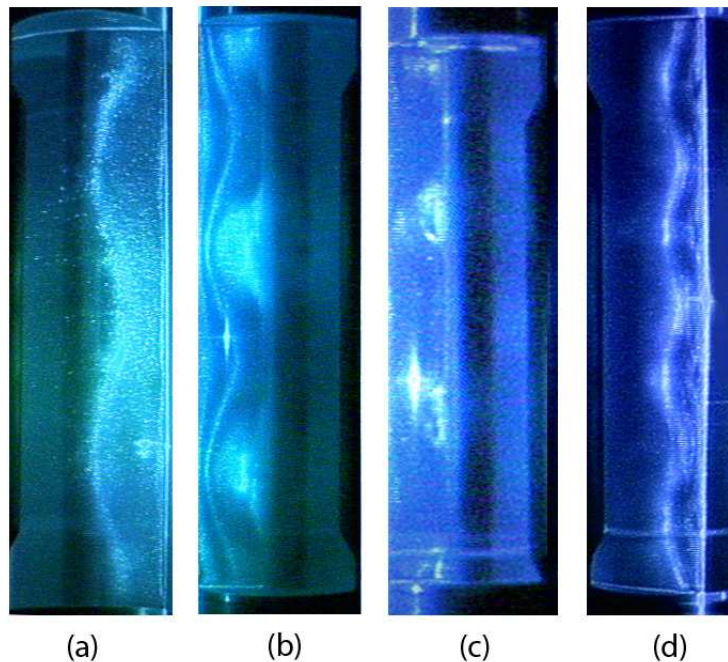


FIG. B.4 – Visualization of the flow pattern due to elliptic instability in cylindrical shells of different aspect ratios and Reynolds numbers. a)  $\alpha = 0.23$ ,  $\beta = 4 * 2.09$ ,  $Re = 2575$ . b)  $\alpha = 0.23$ ,  $\beta = 5 * 1.57$  and  $Re = 1931$ , c)  $\alpha = 0.23$ ,  $\beta = 6 * 1.24$ ,  $Re = 5857$ , d)  $\alpha = 1.22$ ,  $\beta = 8 * 1.81$ ,  $Re = 3960$ .

### B.3.2 Observations and discussion

Every experiment starts from rest. The fluid is then spun up smoothly in presence of the rollers. This transient usually gives rise to a short-wavelength instability that always decays away as it was observed by Eloy [26]. After this transient an elliptical flow is effectively realized. Fig. B.4 presents snap-shots of some observed flow-patterns, which grow on the elliptical base flow. We can only illuminate on one side of the shell with the laser sheet. Dark parts have been cutted in the pictures. We identify modes on the basis of their axial wavelengths and frequencies, and measure the instability thresholds  $Re_c$ . The error is determined by the systematic variation of the rotation frequency of the base flow, and corresponds to a  $\Delta\Omega \simeq 5 \text{ rpm}$ .

For the configuration of Fig. B.4-(a),  $\alpha = 0.23$ ,  $\beta = 4 * 2.09$ , we located the threshold of elliptical instability at  $Re_c = 1930 \pm 300$ . The picture was taken at  $Re = 2575$ . We observe a stationary S-shaped structure with 4 half wavelengths as it is expected by the choice of  $\beta$ . In the early stages of the experiment this mode saturates, but soon an oscillating flow with a frequency close to 1 grows on a longer time scale and superimposes on the first mode. This does not clearly modify the initial number of vertical wavelengths. Both modes coexist and no breakdown or significant variation is observed afterwards. This behavior was reproduced in separate experiments. The linear instability threshold is slightly higher than the value we found theoretically with  $\epsilon = 0.085$ , but has the right order of magnitude. Using similar experimental apparatus, Eloy, [26], found the same deviations without inner cylinder. Slightly above the instability threshold, the systematic presence of the  $(-1, 1, 1)$  mode is in good agreement with Fig. B.2 and even seems to be more robust than would be expected from the linear phase diagram. No other couplings were primarily excited in this geometry. Experiments at higher  $Re$  give similar growth of the spinover mode, and but the flows become chaotic with partial bursts and relaminarisations.

For the same Reynolds number,  $Re = 1930$ , but with  $\alpha = 0.23$ ,  $\beta = 5 * 1.57$ , a pulsating flow pattern with 5 half wavelengths and frequency close to 1 grew to larger amplitude, Fig. B.4-(b). For this parameter set we did not measure the instability threshold. According to the theoretical results, the fastest growing mode at this Reynolds number, is indeed a  $(0, 2, 1)$  with 5 half wavelengths. The amplitude of this pulsation oscillates slowly around a limit for the lower Reynolds numbers. An experiment at  $Re = 4190$ , relatively far beyond the threshold, still preferred the same mode, but the oscillations are less structured. In this experiment we did not detect rapid small scale fluctuations. In Fig. B.4-(c),  $\alpha = 0.23$ ,  $\beta = 6 * 1.24$ , and we are at high Reynolds



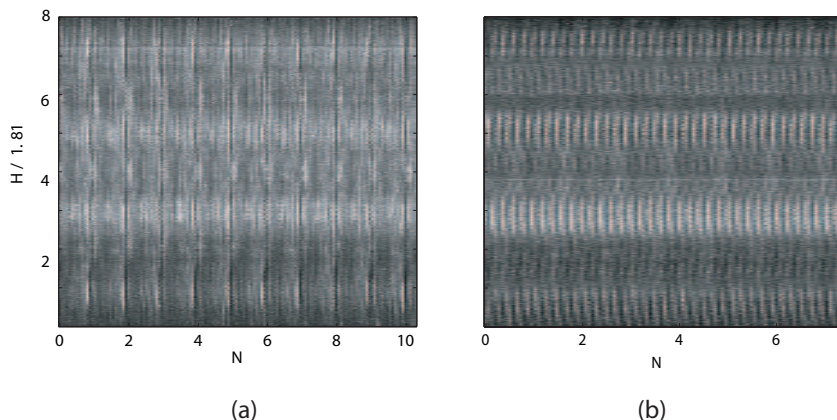


FIG. B.5 – Spatiotemporal diagram of the non linear development of elliptic instability in a cylindrical shell.  $\epsilon = 0.085$ ,  $\alpha = 1.22$ ,  $\beta = 8 \times 1.81$ ,  $Re = 3960$ .

number  $Re = 5857$ . A fast oscillating,  $\omega \simeq 2$ , flow with mainly 6 axial half-wavelengths, appears. We detected no instability beneath  $Re = 2275 \pm 300$ , which is significantly higher than the theoretical values. Beyond this threshold, the  $(1, 3, 1)$  coupling is indeed preferred and in agreement with linear arguments. An experiment at  $Re = 8786$ , still preferred the same coupling, but the oscillation is less organised and small scale fluctuations intermittently appear.

In the cylindrical shell using a large diameter inner cylinder,  $\alpha = 1.22$ ,  $\beta = 8 \times 1.81$ , we obtained no instability up until  $Re_c = 3600 \pm 179$ , which is very high compared to the theoretical results. This may indicate that the effective ellipticity of the flow is much lower than  $\epsilon = 0.085$ . Relatively close to this threshold, for  $Re = 3960$ , we were able to detect the four wavelengths of the  $(-1, 1, 1)$  mode for which we have tuned  $\beta$  for, Fig. B.4-(d). The amplitude of this mode saturated, before giving oscillating flow patterns. This is shown in Fig. B.5, which displays two spatiotemporal diagrams recorded along a vertical line in the kalliroscope pattern for separate experiments with the same parameter setting.  $N$  is the number of revolutions, height is divided by the wavenumber of the dominant spinover mode. In both figures, four horizontal bands dominate the picture, and this represents the  $n = 8$ , half vertical wavelengths of the spinover mode, which continues to be present. On top of this we see oscillations with frequency close to 1 in B.5.(a). They are also present in the second picture B.5.(b), but here some quicker oscillations with frequency close to 5 are more important. Both situations have been reproduced in other experiments for the same parameter set. In experiments further from the threshold,  $Re = 4320$ , we found similar quickly oscillating patterns.

Linear theory was successful in its prediction of the most unstable modes, but less for what concerns the instability thresholds, often far away from the theoretical values. This may be explained by small saturation amplitudes near the threshold, who would then be difficult to measure using kalliroscope imaging. For the big inner cylinder, the difference is flagrant, and is a clear indication that an effective ellipticity, much smaller than the externally imposed value is operating. Weakly non linear saturated states, were effectively observed in different cases, but were often perturbed by other modes. The pulsating flow with frequency close to 1, was reproduced in both cylindrical shells, and observed even close to the threshold. Quicker small scale oscillations were seen close to the threshold with  $\alpha = 1.22$ , but only much further away when  $\alpha = 0.23$ . This may be interpreted as an indication of a bigger number of unstable modes near the threshold for big  $\alpha$ , but one must be aware that  $\beta$  is also necessarily bigger in these case, due to our limited experimental  $H$ -window. For big  $\beta$  geometrical detuning becomes irrelevant, and we approach infinite length phase diagram such as Fig. B.1-(d), in which many modes are always unstable with similar growth rates close to the threshold. The pulsating flow with frequency 1 may be an elliptically excited  $(0, 2, x)$  mode. Fig. B.2, indicates that a  $(0, 2, 2)$ ,  $n = 10$  resonance may be unstable beyond  $Re_c = 2540$ , but these small wavelengths are not observed in the kalliroscope patterns. Geostrophic  $m = 1$  flow is another option, but then it is not clear how this oscillatory flow may be created by low order non linear interactions of the spinover mode with itself. Secondary instabilities of the spinover mode or other more complicated scenarios in which secondary and elliptical instabilities compete [94], are then other plausible scenarios.

## B.4 Conclusion

In this letter we studied elliptical instability in a cylindrical shell. Where non viscous theory suggested that the spinover modes are always the most unstable couplings in infinitely long shells, this is more delicately weighted when viscous dissipation is included : the spinover modes are slightly more damped than the others for small  $\alpha$ . When the inner core gets bigger, all result converge to the small gap limit for all  $m \ll \pi\beta$ , and no preferences for a specific coupling result from linear viscous theory. The untrivial consequence is the larger amount of unstable modes, having similar growth rates, will probably reduce the interval behind the threshold, in which weakly non linear states are dominant. It is expected that flows involving a small number of modes, will be more difficult to realise as the gap gets smaller.

Growth rate diagrams for finite length cylinders have been produced and compared with experimental visualisations using Kalliroscope flakes. The most unstable modes are correctly predicted by linear theory, measures of the instability thresholds were less succesful. For the big inner cylinder, there are clear indications that an effective ellipticity much smaller than the one imposed at the boundaries was realised in the flow. It was harder to isolate only one mode when the inner cylinder was larger, and this even close to the threshold. We want to see this as a manifestation of the smaller gap, which destabilises a larger number of modes, but the higher values of  $\beta$  will also contribute in making the instability more broadband.

We believe that elliptic instability, precessionally induced motion, or any other excitation mecanism in thin shells, might be ideal candidates to study inertial wave turbulence, as much of the small gap analysis is simple, but the dynamics necessarily complex.



## Annexe C

# Ondes d'Alfvén en géométrie cylindrique & Excitation magnétique

Dans ce chapitre nous présentons une étude préliminaire sur l'excitation magnétique des ondes d'Alfvén en géométrie cylindrique. Nous montrons qu'il est possible de construire des solutions analytiques en géométrie cylindrique pour les ondes d'Alfvén qui incluent les effets de la diffusion magnétique. Nous étudions la possibilité de l'excitation magnétique des ondes d'Alfvén par une source de champ magnétique à l'extérieur du cylindre. Nous donnons les ordres de grandeurs pour la réponse du système à ce forçage, et nous montrons qu'en même temps qu'un forçage résonant, une instabilité par résonance paramétrique est également possible.

### C.1 Introduction

Les ondes d'Alfvén sont des ondes hydromagnétiques qui résultent d'un équilibre entre l'inertie du fluide et la force de Lorentz. Découvertes par Alfvén [3] en 1942, elles ont été étudiées et observées pour la première fois dans les expériences de Lundquist [86] et celles de Lehnert [78]. Les ondes d'Alfvén axisymétriques et purement toroïdales sont excitées mécaniquement et se propagent le long des lignes de champ. La dissipation magnétique est non-négligable, et fait que la réponse résonante du fluide reste de petite amplitude. Des champs magnétiques plus intenses, ou des expériences de grande taille sont nécessaires pour favoriser l'observation des ondes d'Alfvén dans les métaux liquides. Par ailleurs, les plasmas semblent en fait être de meilleurs milieux pour étudier les ondes d'Alfvén. De nombreux exemples existent de détection des ondes d'Alfvén dans les plasmas, et les résultats ont été appliqués aux observations dans l'espace. L'article de revue de Gekelman [37] offre une bonne vue d'ensemble des travaux sur l'excitation des ondes d'Alfvén dans ce contexte. Il est intéressant de remarquer que les ondes d'Alfvén peuvent également exister dans les plasmas quantiques de gaz d'électrons et sous forme de trous dans les cristaux de Bismuth [47].

Dans ce chapitre, nous considérons l'excitation magnétique des ondes d'Alfvén en géométrie cylindrique. Cette étude a été motivée par une visite de l'équipe géodynamo du LGIT à Grenoble, qui réalise actuellement une étude expérimentale de ce problème. Dans leur dispositif expérimental, un cylindre de rayon  $R = 0.05 m$ , de hauteur  $H = 0.1 m$  est rempli de Galinstan. La conductivité électrique de ce métal liquide est estimé à  $\sigma_e = 2.89 \times 10^6 S m^{-1}$  et sa densité est  $\rho = 6440 kg m^{-3}$ . Le cylindre est positionné entre deux bobines supraconductrices capables de produire un champ magnétique intense de l'ordre de  $B = 10 T$ , homogène et aligné avec l'axe de révolution du cylindre. La vitesse des ondes d'Alfvén est alors de l'ordre de  $U_A = B/\sqrt{\rho\mu} \simeq 110 m s^{-1}$ . Si nous utilisons  $R = 0.05 m$  comme échelle spatiale typique, nous évaluons le nombre de Lundquist à  $S = \sigma_e \rho U_A R \simeq 20$ , ce qui implique que la dynamique n'est pas dominée par la diffusion, mais que la diffusion continue à être importante. Une bobine excitatrice est montée à l'extérieur du cylindre, et disposée actuellement en dessous de la plaque horizontale du bas du cylindre. Cette bobine peut générer des impulsions magnétiques, qui se propagent dans le fluide selon la direction verticale du champ, sous la forme d'un paquet d'ondes d'Alfvén. La bobine peut également être alimentée afin de produire un champ oscillant. Dans ce cas, une réponse résonante est attendue lorsque la fréquence de forçage est proche d'une fréquence propre d'un mode d'Alfvén du cylindre. Dans cette section, nous examinerons brièvement ce régime forcé d'un point de vue théorique. Pour cela nous utiliserons une théorie linéaire dans laquelle la réponse magnétique est supposé de petite amplitude. Dans un premier temps nous allons analyser les modes propres du cylindre, notamment les ondes dites d'Alfvén. Nous utiliserons une méthode similaire à celle utilisée pour obtenir les modes inertiels

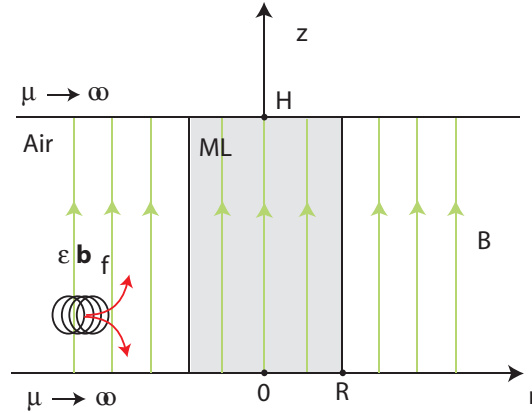


FIG. C.1 – Le problème des ondes d'Alfvén en géométrie cylindrique. Un cylindre rempli de métal liquide, est placé sous un champ magnétique fort et homogène  $B$ , imposé le long de l'axe vertical. A l'extérieur du cylindre nous supposons de l'air et des parois horizontales de perméabilité infinie, qui confinent le champ entre les deux plaques, et rendent les calculs plus simples. Nous considérerons le cas d'une bobine excitatrice montée sur le coté du cylindre

visqueux pour prendre en compte les effets de diffusion magnétique. Nous verrons que ces effets diffusifs, pour des bas nombres de Lundquist,  $S = 20 - 50$ , rendent les ondes d'Alfvén fortement atténuées et dispersives. Ensuite nous analyserons comment ces modes peuvent être excités par un champ magnétique oscillant placé à l'extérieur du cylindre. Nous montrerons que ce problème se résout en deux étapes. Dans une première étape, on déterminera comment le champ oscillant produit à l'extérieur agit à l'intérieur du cylindre. Dans une seconde étape, on évaluera l'ordre de grandeur de l'amplitude des ondes excitées. Nous verrons que dans le même régime  $S \gg 1$ , où la vitesse de propagation des ondes d'Alfvén est suffisamment élevée pour surmonter la dissipation magnétique, un effet de forçage magnétique se localise au voisinage des parois et que le problème du forçage résonant, ne se découple pas d'un problème de couplage résonant, ce qui rend le phénomène de l'excitation d'ondes d'Alfvén plus complexe.

## C.2 Les ondes d'Alfvén

Le système modèle que l'on considère est schématisé sur la Fig. C.1. Un champ magnétique fort d'amplitude  $B$  est imposé le long de l'axe du cylindre, de hauteur  $H$  et de rayon  $R$ . Le cylindre est rempli avec un métal liquide de densité  $\rho$ , permeabilité  $\mu$ , conductivité électrique  $\sigma_e$ . La viscosité cinématique est notée  $\nu$ , la diffusivité magnétique  $\eta = (\sigma\eta)^{-1}$ . Nous notons  $V_i$  l'intérieur du cylindre rempli de métal liquide au repos. Nous supposons que l'extérieur est composé d'air. De plus, des parois horizontales de perméabilité infinie confinent le champ dans une couche plane, comme indiqué sur la Fig. C.1. Ces parois sont nécessaires pour pouvoir développer la théorie la plus simple. Le domaine extérieur  $r \in [1, +\infty[$ ,  $\forall \phi \in [0, 2\pi]$ ,  $z \in [0, h]$ , sera noté  $V_e$ . Nous sommes intéressés par les perturbations hydromagnétiques linéaires de cet état de base. Nous adimensionalisons le problème avec les échelles adaptées aux ondes d'Alfvén

$$\begin{aligned}
 [\mathbf{b}] &= B, & [\mathbf{u}] &= U_A = B(\rho\mu)^{-1/2}, & [\mathbf{r}] &= R \\
 [t] &= R(\rho\mu)^{1/2}B^{-1}, & [p] &= B^2(\rho\mu)^{-1}
 \end{aligned}
 \tag{C.1}$$

où on reconnaît la vitesse d'Alfvén  $U_A$ . Dans le domaine de métal liquide  $V_i$ , les équations pour les perturbations linéaires  $\mathbf{u}$ ,  $\mathbf{b}$  et  $p$  du champ de vitesse, en absence forçage s'écrivent

$$\begin{aligned}
 \partial_t \mathbf{u} &= -\nabla p + \partial_z \mathbf{b} + Pm S^{-1} \Delta \mathbf{u} & \nabla \cdot \mathbf{u} &= 0 \\
 \partial_t \mathbf{b} &= \partial_z \mathbf{u} + S^{-1} \Delta \mathbf{b} & \nabla \cdot \mathbf{b} &= 0
 \end{aligned}
 \tag{C.2}$$

A l'extérieur le champ est potentiel,

$$\Delta\Phi = 0 \quad \mathbf{b} = \nabla\Phi \quad (\text{C.3})$$

Le problème dépend de trois paramètres sans dimension

$$Pm = \frac{\nu}{\eta} \quad , \quad S = \sigma_e \mu \sqrt{B^2/\rho\mu} R \quad , \quad h = \frac{H}{R} \quad (\text{C.4})$$

qui sont respectivement le nombre de Prandtl magnétique, le nombre de Lundquist et le rapport d'aspect. Les champs  $\mathbf{u}$  et  $\mathbf{b}$  vérifient les conditions aux limites suivantes. Les parois sont supposées imperméables :  $u_z = 0|_{z=0,h}$ ,  $\mathbf{u}_r = 0|_{r=1}$ . Le champ magnétique est normal aux parois de perméabilité infinie,  $b_r = b_\phi = 0|_{z=0,h}$ . Le champ magnétique est continu à travers la surface métal-liquide air,  $[\mathbf{b} = 0]|_{r=1}$ , mais dans la limite idéale, i. e. où la diffusion magnétique est négligeable, il faut réduire cette condition à la continuité de la partie normale du champ  $b_r = 0|_{r=1}$ . Dans la suite nous supposons que  $Pm = 0$ , c'est à dire nous considérons le cas non-visqueux. Nous discuterons d'abord la limite idéale, puis nous introduirons la diffusion magnétique dans un second temps.

### Limite non-dissipative

La limite non-dissipative  $S^{-1} \rightarrow 0$  réduit de beaucoup la complexité du problème, notamment parce que la pression n'intervient pas. Nous introduisons les variables d'Elzasser,  $\mathbf{x}^\pm = \mathbf{u} \pm \mathbf{b}$ , et prenons la somme et la différence des deux équations de (C.2) tout en prenant en compte le résultat que nous montrerons plus loin ( $p = 0$ ) pour trouver finalement que

$$(\partial_t - \partial_z) \mathbf{x}^\pm = 0 \quad \Leftrightarrow \quad \mathbf{x}^\pm = \mathbf{f}^\pm(r, \phi, z \mp t) \quad (\text{C.5})$$

En géométrie infinie, ces équations en plus de la loi de Gauss et l'incompressibilité qui imposent  $\nabla \cdot \mathbf{f}^\pm = 0$  définissent les ondes d'Alfvén de façon unique. Une perturbation initiale se propage le long des lignes de champ sans déformation : les ondes d'Alfvén sont non-dispersives. Lorsque les ondes sont confinées, il faut qu'elles respectent les conditions aux limites du problème. Au lieu de résoudre le problème en variables d'Elzasser, nous cherchons les solutions sous la forme  $[\mathbf{u}, \mathbf{b}](\mathbf{r}) e^{i\lambda t}$ , ce qui mène à

$$(\partial_{zz}^2 + \lambda^2) \mathbf{u} = 0 \quad , \quad i\lambda \mathbf{b} = \partial_z \mathbf{u} \quad (\text{C.6})$$

Les différentes composantes du champ ne sont pas couplées entre elles. En géométrie cylindrique, on peut décomposer les profils azimutaux des ondes d'Alfvén sur des fonctions  $e^{im\phi}$  avec  $m$  le nombre d'onde azimutal entier, et les profils verticaux sur  $\cos(lz)$ ,  $\sin(lz)$  avec  $l$  le nombre d'onde axial qui est réel. Les conditions en  $z = 0$  et  $z = h$  imposent alors que les champs ont des structures verticales particulières :

$$u_\pm \sim b_z \sim \cos(lz) \quad , \quad u_z \sim b_\pm \sim \sin(lz) \quad \text{avec} \quad l = n\pi/h, n \in \mathbb{N}_0 \quad (\text{C.7})$$

et que  $\lambda = \pm l$ . Les profils radiaux des champs à l'intérieur sont tels que la loi de Gauss et l'incompressibilité sont garanties. Le champ magnétique à l'extérieur est trouvé comme

$$b_\pm = -\frac{D}{\sqrt{2}} K_{m\pm 1}(lr) \sin(lz) e^{im\phi} \quad , \quad b_z = D K_m(lr) \cos(lz) e^{im\phi} \quad (\text{C.8})$$

où  $K_m$  est la fonction de Bessel modifiée du deuxième type. Dans le cylindre, les conditions aux limites radiales sur la vitesse  $u_r = 0|_{r=1}$  et (C.6) impliquent de plus que  $b_r = 0|_{r=1}$  pour les ondes d'Alfvén. Ceci impose à cause de la continuité du champ radial, que  $D = 0$  et donc  $\mathbf{b} = 0$  à l'extérieur.

Au départ de l'analyse, nous avons supposé que  $p = 0$ . Cette hypothèse est courante pour les ondes d'Alfvén en géométrie infinie, mais pour un écoulement confiné, il est possible qu'elle nous fasse manquer une partie de la solution. Examinons donc la structure spatiale des champs associés à une pression non-nulle, de la forme  $p e^{i\lambda t}$ . Tout d'abord, remarquons que la divergence de l'équation de la quantité de mouvement impose  $\Delta p = 0$ . La solution de nombre d'onde azimutal  $m$  et axial  $l$  s'écrit alors dans le cylindre

$$p = I_m(lr) e^{im\phi} \left( \delta \cos(lz) + \gamma \sin(lz) \right) \quad (\text{C.9})$$

où  $I_m$  est la fonction de Bessel modifiée du premier type.

Si nous éliminons  $\mathbf{b}$  en combinant les équations du départ, on obtient de plus

$$(-l^2 + \lambda^2) \mathbf{u} = i \lambda \nabla p \quad (\text{C.10})$$

Ainsi si  $\lambda^2 \neq l^2$ ,  $u_z$  et  $u_\phi$  sont proportionnel à  $p$ . Les conditions aux limites en  $z = 0$  et  $h$  sur  $u_z$  impose donc  $\gamma = 0$  et  $l = n\pi/h$ . La condition radiale impose également que  $\partial_r p = 0|_{r=1}$ , c'est à dire  $\delta I'_m(l) = 0$  ce qui entraîne  $\delta = 0$ , d'où  $p = 0$ .

Les fréquences propres  $\lambda = \pm l$  sont donc entièrement déterminées par la structure verticale et quantifiées par le rapport d'aspect du cylindre. Ces valeurs propres sont très dégénérées, car la structure azimutale et radiale peut être quelconque tant qu'elle satisfait la loi de Gauss, l'incompressibilité et les conditions limites aux parois du cylindre. Cette dégénérescence est une conséquence du fait que les ondes d'Alfvén sont non-dispersives dans la limite de la MHD idéale, et elle sera levée par les effets diffusifs introduits dans la prochaine section.

### Solutions non-visqueuses et diffusives

Il existe des équivalents des ondes d'Alfvén en présence de diffusion magnétique (toujours dans la limite non-visqueuse,  $Pm = 0$ ). Cette limite se rapproche du cas expérimental pour lequel,  $Pm \simeq 10^{-6}$ , et  $S \simeq 20$ . Nous cherchons les solutions de (C.2) sous la forme  $[\mathbf{u}, \mathbf{b}, p](\mathbf{r}) e^{i\lambda t}$ . Les profils spatiaux des modes dans le cylindre respectent les équations

$$\begin{aligned} i \lambda \mathbf{u} &= -\nabla p + \partial_z \mathbf{b} \\ i \lambda \mathbf{b} &= \partial_z \mathbf{u} + S^{-1} \Delta \mathbf{b} \end{aligned} \quad (\text{C.11})$$

en plus de l'incompressibilité du fluide  $\nabla \cdot \mathbf{u} = 0$  et la loi de Gauss  $\nabla \cdot \mathbf{b} = 0$ . La divergence de la première équation de (C.2), mène toujours à  $\Delta p = 0$ , et la substitution de la première équation de (C.11) dans la seconde mène à

$$\left[ i \lambda (S^{-1} \Delta - i \lambda) + \partial_{zz}^2 \right] \mathbf{b} = \partial_z \nabla p \quad (\text{C.12})$$

Comme précédemment, on peut considérer la partie de la solution associée à une pression nulle. Cette partie vérifie

$$\left[ i \lambda (S^{-1} \Delta_{\pm} - i \lambda) + \partial_{zz}^2 \right] b_{\pm} = 0 \quad , \quad \left[ i \lambda (S^{-1} \Delta - i \lambda) + \partial_{zz}^2 \right] b_z = 0 \quad (\text{C.13})$$

où  $b_{\pm} = (b_r \pm i b_\phi) / \sqrt{2}$  et  $\Delta_{\pm}$  a été défini dans la section (2) et dans l'annexe (A.2). Avec la structure verticale (C.7) imposée par les conditions aux limites, nous trouvons que cette partie de la solution est donc

$$b_{\pm} = d_{\pm} I_{m\pm 1}(Kr) e^{im\phi} \sin(lz) \quad , \quad b_z = d_z I_m(Kr) e^{im\phi} \cos(lz) \quad (\text{C.14})$$

avec

$$K^2 = l^2 + i \lambda S - i \frac{l^2}{\lambda} S \quad (\text{C.15})$$

La loi de Gauss permet d'éliminer une constante, par exemple celle du champ vertical :

$$d_z = \frac{K}{l\sqrt{2}}(d_+ + d_-) \quad (\text{C.16})$$

La partie associée à la pression se résout comme précédemment.  $p$  est toujours donnée par (C.9). Les conditions en  $z = 0$  et  $h$  imposent toujours que  $\gamma = 0$  par contre. Les conditions en  $r = 1$  n'imposent plus que  $\delta = 0$ . Ces conditions doivent être imposées sur la solution totale qui s'écrit maintenant à l'intérieur du cylindre

$$\begin{aligned} \begin{bmatrix} b_{\pm} \\ b_z \\ u_{\pm} \\ u_z \\ p \end{bmatrix} &= \begin{bmatrix} d_{\pm} & I_{m\pm 1}(Kr) & \sin(lz) \\ K/l\sqrt{2} & (d_+ + d_-) & I_m(Kr) & \cos(lz) \\ -il/\lambda & d_{\pm} & I_{m\pm 1}(Kr) & \cos(lz) \\ iK/l\sqrt{2} & (d_+ + d_-) & I_m(Kr) & \sin(lz) \\ 0 & & & \end{bmatrix} e^{im\phi} \\ &+ \delta' \begin{bmatrix} 1/\sqrt{2} & I_{m\pm 1}(lr) & \sin(lz) \\ -i\lambda/l\sqrt{2} & I_m(lr) & \cos(lz) \\ i\lambda/l & I_{m\pm 1}(lr) & \cos(lz) \\ (\lambda^2 - l^2)/l & I_m(lr) & \sin(lz) \\ & I_m(lr) & \cos(lz) \end{bmatrix} e^{im\phi} \end{aligned} \quad (\text{C.17})$$

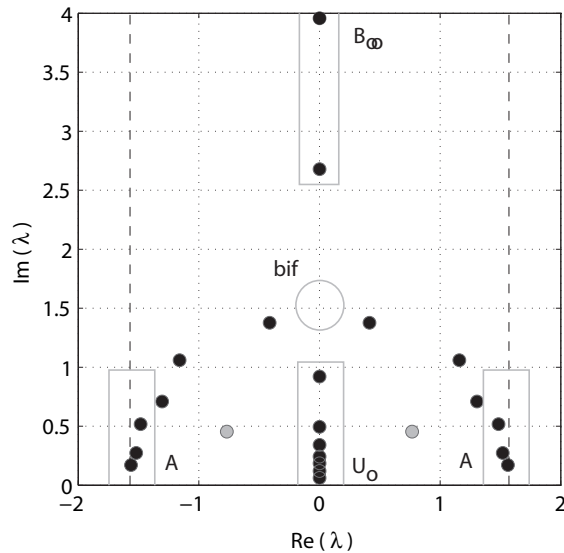


FIG. C.2 – Fréquence complexe  $\lambda$  des ondes d'Alfvén avec différentes structures radiales, nombre d'onde azimutal  $m = 0$ , nombre d'onde vertical  $l = \pi/2$  et nombre de Lundquist  $S = 50$ . En zone (A), les modes sont des ondes d'Alfvén, en ( $U_0$ ) des modes hydrodynamiques purs, et en ( $B_\infty$ ) des modes magnétiques purs. (bif) est la zone de bifurcation. Les modes de couleur gris, correspondent à des solutions spéciales.

Le champ magnétique à l'extérieur est par contre toujours donné par (C.8). Les quatre conditions aux limites à notre disposition sur la surface latérale du cylindre, sont l'imperméabilité de la paroi et la continuité du champ magnétique. Ces conditions aux limites nous donnent un système algébrique homogène pour les constantes  $d_+$ ,  $d_-$ ,  $\delta'$  et  $D$ . La relation de dispersion des ondes d'Alfvén est alors obtenue en annulant le déterminant de ce système linéaire. La relation est implicite, mais peut facilement se résoudre numériquement par une méthode d'optimisation dans le plan complexe des fréquences  $\lambda$ . En pratique, nous fixons les paramètres  $S^{-1}$ ,  $m$  et  $l$ , et cherchons les modes de différentes structures radiales. Des estimations pour les fréquences propres sont obtenues graphiquement par la même méthode que celle utilisée pour les modes visqueux (voir Ch. 2.42).

La figure Fig. C.2 montre comment la fréquence complexe  $\lambda$  varie avec la structure radiale des modes, pour quelques modes particuliers de nombre d'onde azimutal  $m = 0$  et nombre axial  $l = \pi/2$ , pour un nombre de Lundquist  $S = 50$ . Les modes de plus grandes structures radiales sont des ondes d'Alfvén (zones A) qui sont proches de l'équipartition : autant d'énergie se trouve dans la partie magnétique que dans le champ de vitesse du mode. La fréquence de ces modes est proche de la limite idéale  $Re(\lambda) \simeq \pm\pi/2$ , (lignes pointillées). Ces modes sont cependant fortement atténués par la diffusion magnétique. Les modes de structures radiales plus fines, se trouvent plus proche de l'axe imaginaire, et sont proche d'un demi arc de cercle centré à  $\lambda = 0$ , et de rayon  $\pi/2$ . Pour ces modes, nous sommes toujours très proches de l'équipartition. Une diminution de la valeur absolue de leur fréquence est accompagnée par une augmentation sensible du taux de dissipation. Pour des structures plus fines encore, les modes quittent l'arc de cercle et deviennent stationnaires. Pour ces modes, le couplage entre le champ de vitesse et le champ magnétique est négligeable, et ils se séparent en deux classes distinctes. Dans la zone ( $U_0$ ), les modes sont à dominante hydrodynamique. Ils sont très peu atténués par la dissipation magnétique, et cela d'autant moins que leur structure radiale est fine. L'autre classe de modes ( $B_\infty$ ), est constituée de modes purement magnétiques, qui sont fortement atténués par la diffusion magnétique. Plus la structure radiale est fine, plus le taux de dissipation est important. Entre les zones (A) et ( $U_0$ ), nous avons également trouvé un mode particulier, indiqué en gris sur la Fig. C.2, qui n'est ni une véritable onde d'Alfvén, ni un mode purement hydrodynamique. L'existence de ce mode n'est pas prédite par la théorie non-diffusive.

La figure Fig. C.3 montre comment varient les profil radiaux du champ de vitesse et du champ magnétique des différents modes d'Alfvén axisymétriques  $m = 0$  de nombre axial  $l = \pi/2$ . Le mode (a)  $\lambda = 1.52 + 0.27i$  est un mode purement toroïdal ( $u_r = u_z = 0$ ,  $b_r = b_z = 0$ ). Il s'agit ici d'une onde, similaire à celles décrites et observées expérimentalement par Lundquist [86] et Lehnert [78]. Ce mode ne possède pas de champ magnétique



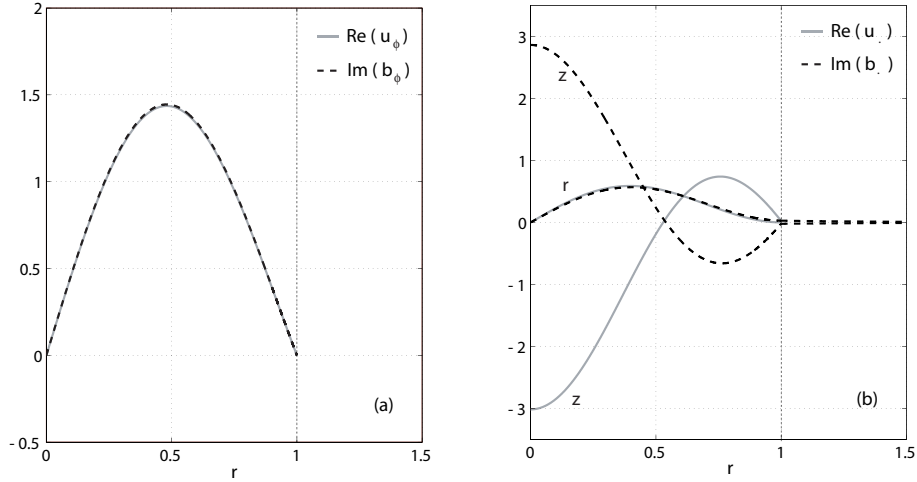


FIG. C.3 – Structure radiale du champ de vitesse et du champ magnétique, des deux premiers modes d’Alfvén axisymétriques ( $m = 0$ ) de nombre axial  $l = \pi/2$ , pour un nombre de Lundquist  $S = 50$ . (a)  $\lambda = 1.52 + 0.27i$  est un mode toroïdal, (b)  $\lambda = 1.56 + 0.17i$  est un mode poloïdal.

à l’extérieur du cylindre. Le deuxième mode (b)  $\lambda = 1.56 + 0.17i$ , est purement poloïdal ( $u_\phi = b_\phi = 0$ ). Pour ce mode, nous observons que  $Re(u_r) \sim Im(b_r)$ , et  $Re(u_z) \sim -Im(b_z)$ , et nous trouvons de faibles contributions pour le champ magnétique extérieur.

Le cas considéré sur la Fig. C.2 nous permet de comprendre comment les modes sont organisés en fonction de leur fréquence et de leur structure spatiale. Les modes d’Alfvén ont une composante de champ magnétique importante et une grande structure spatiale : ce sont les modes de la région A. Si on augmente le nombre de Lundquist  $S$ , on s’aperçoit que les ondes d’Alfvén apparaissent à partir des deux branches magnétiques ( $B_\infty$ ) et hydrodynamique ( $U_0$ ) à travers une bifurcation sur l’axe imaginaire. Dans le cas de la Fig. C.2, cette bifurcation a lieu dans la zone (bif), au voisinage de  $\lambda \sim il$  avec ici  $l = \pi/2$ . Après la bifurcation, les modes suivent le demi arc de cercle presque jusqu’aux points  $\lambda \simeq \pm l$ . Pour les autres nombres d’onde azimutaux, des diagrammes similaires sont obtenus. La symétrie de réflexion du diagramme par rapport à l’axe imaginaire reste présente : les ondes d’Alfvén apparaissent par paire avec des fréquences complexes conjuguées.

Sur la Fig. C.4, nous montrons quelques résultats plus quantitatifs concernant les variations de la fréquence avec  $S$  et avec le nombre d’onde axial  $l$  pour un cas axisymétrique  $m = 0$ . Nous avons tracé l’évolution des cinq premières paires d’ondes d’Alfvén. La Fig. C.4-(a) met en évidence la bifurcation des ondes d’Alfvén à partir de l’axe imaginaire. Chaque paire de modes d’Alfvén (mode non-stationnaire) apparaît à un nombre  $S_c$  particulier et évolue ensuite pour les grands  $S$  vers les modes idéaux de fréquence  $Re(\lambda) \simeq \pm\pi/2$ . Fig. C.4-(c) montre que l’atténuation des modes d’Alfvén suit une loi d’échelle

$$Im(\lambda) \sim S^{-1} \quad (C.18)$$

ce qui indique une dissipation dans le volume. Une paire de modes particulière ne suit pas la même évolution (les modes colorés en gris en Fig. C.2). Leur fréquences tendent vers zéro et leurs taux d’atténuation suivent un scaling en  $S^{-1/3}$ . Cette loi d’échelle particulière indique qu’un tout autre mécanisme de dissipation est à l’oeuvre pour ce mode. Sur les figures C.4-(b) et (d), nous voyons que la fréquence varie comme

$$Re(\lambda) \sim l \quad , \quad Im(\lambda) \sim l^2 \quad (C.19)$$

en accord avec le résultat issu de la théorie idéale et la loi d’échelle classique d’une dissipation dans le volume. Le mode particulier a de nouveau une évolution spécifique en fonction de  $l$ . Les résultats obtenus ci-dessus dans le cas non-visqueux peuvent facilement s’étendre au cas visqueux. Cependant, il n’y a pas de différences significatives pour le nombre de Prandtl  $Pm \simeq 10^{-5} - 10^{-6}$  des métaux liquides utilisés dans les expériences.

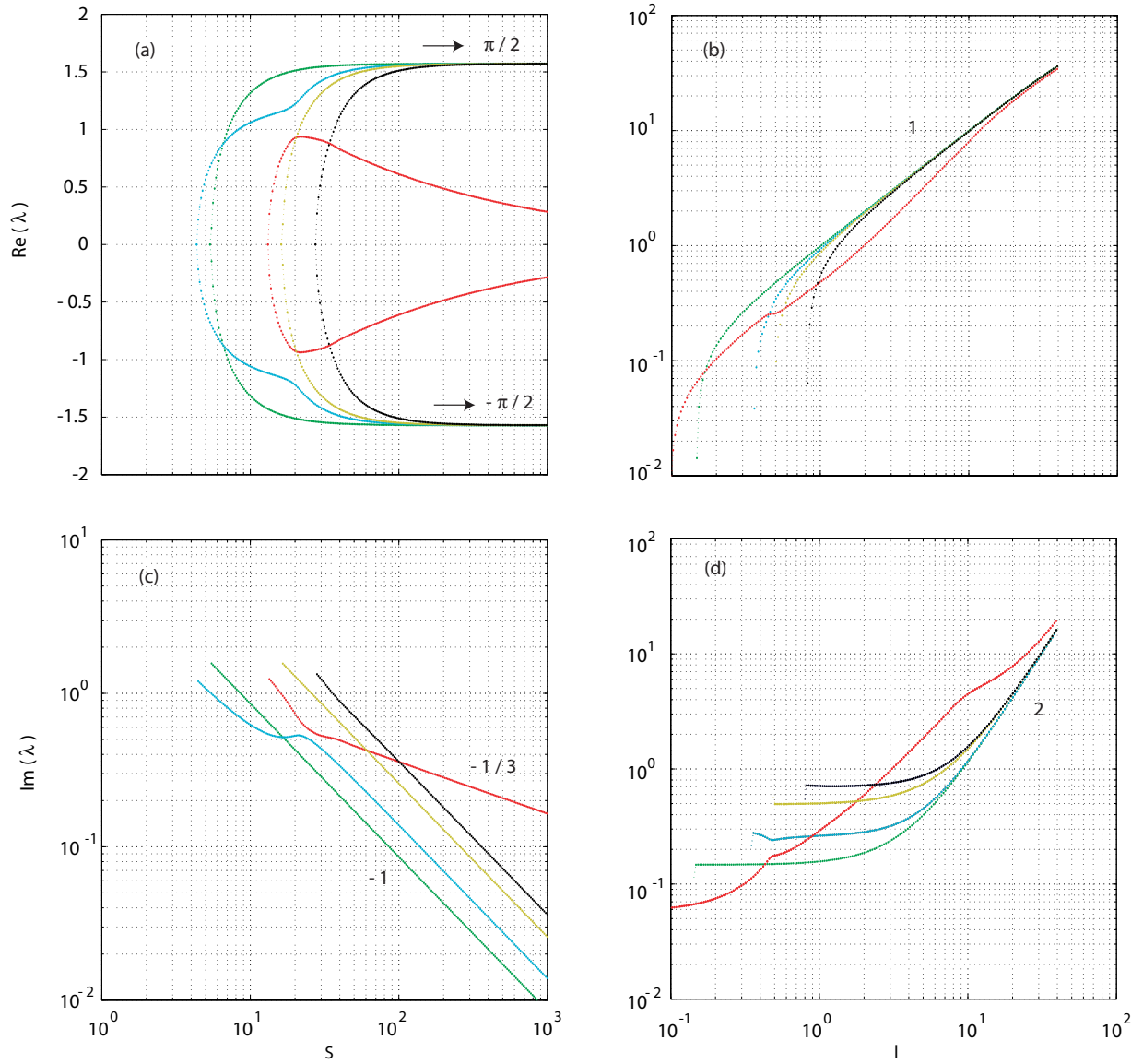


FIG. C.4 – Variation de la fréquence complexe  $\lambda$  des ondes d'Alfvén à nombre d'onde azimutal  $m = 0$  avec le nombre de Lundquist  $S$ , (a) et (c) pour  $l = \pi/2$  et avec le nombre d'onde axiale  $l$  (b) et (d) pour  $S = 50$ .

### C.3 Excitation magnétique des ondes d'Alfvén

Considérons une bobine excitatrice placée à l'extérieur du cylindre. Pour l'instant, on suppose la bobine à coté du cylindre comme sur la Fig. C.1. Pour modéliser l'action de cette bobine excitatrice sur le système, nous ne pouvons pas simplement ajouter un forçage oscillatoire dans les équations du domaine intérieur. Le champ magnétique doit d'abord pénétrer l'intérieur du cylindre avant de pouvoir agir sur les ondes d'Alfvén. La linéarité du problème nous permet alors de couper le problème en deux parties. Nous pouvons chercher dans un premier temps, comment un champ magnétique oscillant produit par la bobine excitatrice, pénètre dans le cylindre en **absence** d'écoulement,  $\mathbf{u} = 0$ . Ce champ sera noté  $\epsilon \mathbf{B}_f$ , et le petit paramètre  $\epsilon$  indique que ce champ perturbatif est nécessairement beaucoup plus petit que le champ fort  $\hat{\mathbf{z}}$  imposé le long de l'axe du cylindre. A l'intérieur du cylindre, le champ  $\epsilon \mathbf{B}_f$ , respecte l'équation de diffusion. A l'extérieur, le champ  $\mathbf{B}_f$  sera composé d'une partie provenant de la bobine associée à une densité de courant

$$\mathbf{J}_f = \mathbf{j}_f(\mathbf{r}) e^{i\lambda_f t} + cc. \quad (\text{C.20})$$

et d'une autre partie potentielle qui permettra de raccorder domaine extérieur et domaine intérieur. Dans les variables sans dimension, l'échelle de la densité de courant est  $[\mathbf{J}] = BR/\mu$ . Le champ  $\mathbf{B}_f$  sera de la même forme

$$\mathbf{B}_f = \mathbf{b}_f(\mathbf{r}) e^{i\lambda_f t} + cc. \quad (\text{C.21})$$

Pour le calculer, il est utile d'introduire le potentiel vecteur

$$\mathbf{A}_f = \mathbf{a}_f(\mathbf{r}) e^{i\lambda_f t} + cc. \quad (\text{C.22})$$

relié au champ magnétique par  $\mathbf{B}_f = \nabla \times \mathbf{A}_f$ . A l'intérieur du cylindre, la partie  $\mathbf{a}_f$  du potentiel vecteur satisfait alors une équation de diffusion

$$(i\lambda - S^{-1}\Delta) \mathbf{a}_f = 0 \quad , \quad \nabla \cdot \mathbf{a}_f = 0 \quad , \quad \mathbf{r} \in V_i \quad (\text{C.23})$$

A l'extérieur, elle satisfait

$$\Delta \mathbf{a}_f = \mathbf{j}_f \quad , \quad \nabla \cdot \mathbf{a}_f = 0 \quad , \quad \mathbf{r} \in V_e \quad (\text{C.24})$$

On néglige ici la conductivité du cuivre des bobines. Le potentiel vecteur doit être continu à la frontière du cylindre. A priori, ce problème nécessite une intégration numérique pour un champ de courant donné. Les résultats précédents et la forme des équations nous donnent cependant quelques informations sur l'efficacité du forçage. Nous avons tout d'abord vu que les ondes d'Alfvén n'étaient pas trop amorties uniquement si  $S$  était suffisamment grand et que dans ce cas, leur fréquence valait  $Re(\lambda) = l = n\pi/h$ . Si nous voulons exciter les ondes d'Alfvén par un phénomène de résonance, il faudra que  $\lambda_f \simeq Re(\lambda)$ . Ces deux contraintes ont des conséquences sur le problème de diffusion (C.23). Le champ oscillant ne pourra pénétrer dans le cylindre que sur une épaisseur d'ordre  $d_{skin} \sim (\lambda_f S)^{-1/2}$ , ce qui implique à priori que le forçage ne va pas être très efficace. Essayons de quantifier cela de manière plus précise.

Supposons à partir de maintenant que le champ magnétique  $\epsilon \mathbf{B}_f$  soit connu dans tout le domaine et décomposons le champ magnétique total en  $\mathbf{B}_{tot} = \hat{\mathbf{z}} + \epsilon \mathbf{B}_f + \mathbf{b}$  où  $\mathbf{b}$  sera le champ associé aux ondes d'Alfvén. On notera  $\mathbf{u}$  l'écoulement total. On suppose que ces champs vérifient les équations de la MHD. Comme  $\mathbf{B}_f$  a été défini pour se raccorder avec le champ de la bobine, nous pouvons oublier cette source extérieure dans la suite. Le champ  $\mathbf{B}_f$  n'est pas une solution des équations MHD, et introduit en conséquence des termes de plus dans les équations pour les perturbations linéaires :

$$\begin{aligned} (\partial_t - Pm S^{-1}) \mathbf{u} - \partial_z \mathbf{b} + \nabla p &= \epsilon \left\{ \partial_z \mathbf{b}_f e^{i\lambda_f t} + [(\mathbf{b}_f \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{b}_f] e^{i\lambda_f t} + cc. \right\} \\ (\partial_t - S^{-1} \Delta) \mathbf{b} - \partial_z \mathbf{u} &= \epsilon \left\{ \nabla \times (\mathbf{u} \times \mathbf{b}_f) e^{i\lambda_f t} + cc. \right\} \end{aligned} \quad (\text{C.25})$$

On a négligé ici les termes  $O(\epsilon^2)$ . A gauche, nous reconnaissons le problème linéaire des ondes d'Alfvén. A droite, nous voyons que la Force de Lorentz, issue de l'interaction du champ fort et le champ excitateur  $\mathbf{B}_f$ , est à l'origine d'un forçage oscillant dans les équations pour le mouvement du fluide. A côté du forçage nous constatons que le champ introduit également des couplages. Ce système d'équations peut à priori être analysé théoriquement par une méthode asymptotique. Une telle théorie est envisageable et nécessite l'introduction des modes adjoints pour remonter aux équations d'amplitude des différentes modes. Ce travail va au delà de la présente étude préliminaire et dans laquelle nous souhaitons seulement faire apparaître quelques points

importants. Supposons tout d'abord que  $\epsilon$  et  $S^{-1}$  sont petits et  $Pm = 0$  pour simplifier le problème. Si  $\lambda_f$  n'est pas proche d'une fréquence propre d'une onde d'Alfvén  $\lambda$ , il n'y aura pas de résonance, et on s'attend à avoir une réponse de petite amplitude oscillant à la fréquence  $\lambda_f$ . Cette réponse harmonique peut être calculée en gardant uniquement le terme  $\epsilon \partial_z \mathbf{b}_f e^{i\lambda_f t}$  dans le membre de droite de l'équation (C.25). Vu que  $\mathbf{b}_f$  n'est localisé que dans une fine couche d'épaisseur  $S^{-1/2}$ , on s'attend donc à avoir une réponse  $O(\epsilon S^{-1/2}/(\lambda_f - \lambda))$ , où  $\lambda$  est la fréquence du mode d'Alfvén la plus proche de  $\lambda_f$ . Notons qu'une réponse n'oscillant pas nécessairement à  $\lambda_f$  pourrait également être possible si une instabilité paramétrique était provoquée par les autres termes du membre de droite de l'équation (C.25). Cela se produit si deux modes d'Alfvén de fréquences  $\lambda_1$  et  $\lambda_2$  vérifient

$$\lambda_1 + \lambda_2 \simeq \lambda_f \quad (\text{C.26})$$

Vu que les  $\lambda_j$  sont de la forme  $n_j \pi/h$ ,  $\lambda_f$  serait dans ce cas aussi une fréquence d'Alfvén. En conclusion, si  $\lambda_f$  n'est pas proche d'une fréquence des ondes d'Alfvén, la réponse est simple et harmonique et aucune résonance paramétrique n'est possible.

Si  $\lambda_f$  est proche d'une fréquence d'Alfvén, et sous la contrainte que le champ forcé ait une structure verticale similaire à celui de l'onde d'Alfvén que l'on souhaite exciter, la réponse forcée va devenir de plus grande amplitude, à priori d'ordre  $O(\epsilon S^{1/2})$ , vu que  $Im(\lambda) = O(S^{-1})$ . De la même manière, des couplages résonants vont être possibles et on peut essayer d'évaluer les conditions pour lesquelles une instabilité paramétrique liée aux couplages pourrait devenir active. Bien que le champ soit localisé dans une région  $O(S^{-1/2})$  près du bord, les couplages feront intervenir des termes du type  $\partial_r \mathbf{b}_f$  qui seront d'ordre  $O(S^{1/2})$ . On peut alors penser que le couplage global pourrait être d'ordre  $O(\epsilon)$ . Cela impliquerait un seuil pour cette instabilité paramétrique de l'ordre de  $\epsilon_c \sim S^{-1}$ , soit  $\epsilon_c \simeq 2\%$  avec  $S = 50$  dans le montage actuel. Ceci impliquerait un champ oscillant de  $\mathbf{b}_f \sim 0.2T$  ce qui reste tout de même très grand.

Les ordres de grandeurs mentionnés ci-dessus ont été obtenus pour les cas d'un forçage sur le côté du cylindre avec des parois de perméabilité infinie en haut et en bas du cylindre. Dans le cas de l'expérience réalisée à l'IMG Grenoble, les parois horizontales ne sont pas présentes et le forçage est effectué sous la plaque horizontale inférieure. Cela introduit quelques différences qualitatives qu'il peut être intéressant de signaler. Tout d'abord, en ce qui concerne les modes : si les parois verticales et horizontales ne sont pas présentes, on ne peut plus découpler les structures radiales et axiales des modes comme on l'a fait précédemment. Par contre, des modes similaires pourraient être encore obtenus si les parois verticales étaient de perméabilité infinie (et se prolongeaient à l'infini). Dans ce cas, l'excitation magnétique par une bobine au dessus de la plaque inférieure serait également différente. Le champ magnétique  $\mathbf{B}_f$  à l'intérieur du cylindre serait maintenant localisé au voisinage du bord inférieur dans une couche d'épaisseur  $S^{-1/2}$ . Cela impliquerait notamment des forts gradients axiaux de  $\mathbf{B}_f$

$$\partial_z \mathbf{B}_f \sim \mathbf{B}_f S^{1/2} \quad (\text{C.27})$$

et donc par voie de conséquence un terme de forçage plus grand que précédemment. Une réponse forcée d'ordre  $O(\epsilon/(\lambda_f - \lambda))$  pourrait ainsi être obtenue conduisant à une amplitude  $O(\epsilon S)$  au plus proche de la résonance. L'évaluation du terme de couplage éventuellement responsable d'une instabilité paramétrique ne serait quant à elle pas modifiée, ce qui impliquerait à nouveau un seuil  $\epsilon_c \sim S^{-1}$ . Contrairement au cas précédent, une réponse forcée d'amplitude importante pourrait ainsi être obtenue avant le seuil de l'instabilité.

## C.4 Conclusion

Dans ce chapitre nous avons discuté le problème de l'excitation magnétique des ondes d'Alfvén en géométrie cylindrique. Nous avons trouvé une nouvelle classe de solutions pour les ondes d'Alfvén libres, qui incorpore les effets de la diffusion magnétique. Les ondes ont été trouvées dans le cadre d'un modèle qui suppose des parois horizontales de perméabilité infinie qui confinent le champ magnétique dans une couche plane. De la même manière, il est possible de trouver des ondes simples dans un cas où la paroi verticale serait de perméabilité infinie. En absence de toute paroi idéalisée, nous ne pouvons plus trouver des ondes à la structure simple. Cependant, la limite de la MHD idéale nous apprend que les ondes garderont une structure proche de leurs équivalents diffusifs à des nombres de Lundquist pas trop faibles.

Pour l'excitation magnétique des ondes d'Alfvén, nous avons proposé une modélisation en deux étapes. Une première étape identifie comment le champ oscillant provenant de la bobine excitatrice agit sur l'intérieur du cylindre. Dans la deuxième étape, nous faisons apparaître les termes de forçage du fluide suite à cette excitation magnétique. Nous avons mis en évidence que le champ magnétique oscillant ne peut agir que sur une couche

fine d'épaisseur  $S^{-1/2}$  située près des bords du cylindre. Nous montrons qu'en même temps que le forçage, le couplage des ondes d'Alfvén est inévitable, et une résonance paramétrique en plus de la réponse résonante du forçage résonant ne peut pas être exclue. L'effet de peau est responsable d'une localisation forte du forçage, qui peut désormais être suffisamment grand à cause des gradients qui apparaissent dans le forçage et dans les couplages. L'emplacement de la bobine excitatrice permet alors de différencier l'importance des différents termes. Si la bobine est montée sur le côté du cylindre, les couplages joueront un rôle plus important que le forçage et une résonance paramétrique pourra apparaître avant que la réponse au forçage résonant devienne grande. Dans le cas où la bobine est montée au dessous du cylindre, le forçage est plus important, et une réponse grande pourra apparaître avant que les couplages deviennent trop importants.

Une théorie linéaire plus détaillée pour l'excitation magnétique des ondes d'Alfvén est envisageable et le régime de forçage impulsif, où se sont concentrées la plus grande partie des expériences passées, peut être modélisée de la même manière en coupant la résolution du problème en deux étapes. Il est important que des plus grand nombres de Lundquist soient atteints, car pour  $S \simeq 20 - 30$ , la diffusion magnétique reste trop forte pour permettre une réponse résonante importante à un forçage oscillant.

Finalement, notons qu'en présence d'une rotation solide rapide, les ondes d'Alfvén et les ondes inertielles se combinent pour donner lieu aux ondes magnéto-Coriolis. Des solutions analytiques pour ces ondes en présence de diffusion visqueuse et de diffusion magnétique peuvent alors être trouvées en suivant la même procédure que pour les modes visqueux.



## Instabilité elliptique sous champ magnétique & Dynamo d'ondes inertielles

Sous l'effet combiné de la rotation rapide et de l'interaction gravitationnelle avec une lune avoisinante, un corps céleste est elliptiquement déformé (marées). Dans les zones liquides à l'intérieur de la planète, cette déformation rend les lignes de courant elliptiques. Cet écoulement elliptique peut être instable, et des ondes inertielles peuvent croître à des amplitudes importantes. Ce mécanisme offre donc un alternatif à la convection pour exciter des écoulements dans les intérieurs planétaires, et donc aussi pour la génération du champ magnétique par effet dynamo. En première instance, il est important de mieux comprendre comment l'écoulement elliptique évolue après sa déstabilisation initiale. Nous proposons un modèle théorique, et nous étudions les scénarios de transition vers des écoulements de plus en plus complexes dans un système modèle en géométrie cylindrique. Nous mettons en place une expérience qui vise à étudier le même problème, utilisant un métal liquide comme fluide, et mettant l'ensemble sous champ. Nous montrons qu'il est possible de se servir du champ magnétique induit comme méthode de détection des écoulements excités par l'instabilité. A champ magnétique imposé fort, la force de Lorentz devient non-négligeable et nous montrons comment celle-ci agit sur l'instabilité elliptique. Des études en géométrie cylindrique et sphéroïdales sont présentées. Le problème de la dynamo elliptique est de grande importance à l'échelle géophysique. Par une approche numérique, nous trouvons que les ondes inertielles, peuvent exciter une dynamo. Nous proposons une modélisation théorique pour le mécanisme de la dynamo d'ondes inertielles.

**Mots clés :** MHD, rotation, instabilité, dynamo, galinstan, ondes inertielles

## Elliptical instability under an imposed magnetic field & Inertial wave dynamo

Through rapid rotation and gravitational interaction with a neighbouring moon, a celestial body is elliptically deformed (tides). This elliptical deformation is transmitted to the streamlines of the flow in the liquid parts of the body and the resulting elliptical flow can be unstable. Inertial waves are excited and may grow to significant amplitudes. This mechanism therefore offers an alternative to convection driven flows and the associated magnetic field generation or dynamo. First of all, it is necessary to better understand how the flow behaves nonlinearly. We propose a theoretical model in cylindrical geometry, and study different transition scenarios which separate simple weakly nonlinear dynamics from more complex nonlinear states. We combine this approach with liquid metal experiments under an imposed magnetic field, we show that magnetic field induction can be used to monitor the flow which grows by elliptical instability in both spheroids and cylinders. When the imposed magnetic field is strong enough, the Lorentz force acts on the flow and we show how it changes the elliptical instability. The subject of magnetic field generation (dynamo) by tidally driven flows, is not accessible in current experimental facilities, but the topic is of great interest at geophysical scales. We solve the kinematic dynamo problem for inertial waves confined in cylinders, and find that inertial waves can excite a dynamo. We derive a theoretical model, which reveals the instability mechanism and the asymptotically relevant parameters.

**Keywords :** MHD, rotation, instability, dynamo, galinstan, inertial waves