Contributions aux problèmes d'évolution

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à ma femme Hiba

à mon fils Abdul Rahman

à mes parents

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Chapitre 1

Introduction

Dans cette thèse, nous nous intéressons à l'étude de trois équations aux dérivées partielles et d'évolution non-locales en espace (cf. (1.1.1), (1.2.1) et (1.4.1) ci-dessous) dont deux de ces trois équations (cf. (1.2.1) et (1.4.1)) sont de plus non-locales en temps. Les solutions de ces trois solutions peuvent exploser en temps fini.

Dans la théorie des équations d'évolution non-linéaire, une solution est qualifiée de globale si elle est définie pour tout temps positif. Au contraire, si une solution existe seulement sur un intervalle de temps $[0, T)$ borné, elle est dite locale. Dans ce dernier cas et quand le temps maximal d'existence est relié à une alternative d'explosion, on dit aussi que la solution explose en temps fini. Cependant, pour donner un sens à la notion d'explosion en temps fini, il faut bien préciser l'espace dans lequel on travaille et avec quelle norme on "mesure" la solution.

Dans le deuxième chapitre, nous considérons l'équation de la chaleur non-linéaire avec une puissance fractionnaire du laplacien, et obtenons notamment que, dans le cas d'exposant sur-critique, le comportement asymptotique de la solution lorsque $t \to +\infty$ est déterminé par le terme de diffusion anormale. D'autre part, dans le cas d'exposant sous-critique, l'effet du terme non-linéaire domine.

Dans le troixième chapitre, nous étudions une équation parabolique avec le laplacien fractionnaire et un terme non-linéaire et non-local en temps. On montre que la solution est globale dans le cas sur-critique pour toute donnée initiale ayant une mesure assez petite, tandis que dans le cas sous-critique, on montre que la solution explose en temps fini $T_{\max} > 0$ pour toute condition initiale positive et non-triviale. Dans ce dernier cas, on cherche le comportement de la norme L^∞ de la solution en précisant le taux d'explosion lorsque t s'approche du temps d'explosion $T_{\max}$. Nous cherchons aussi les conditions nécessaires pour l'existence locale et globale de la solution.

Le quatrième chapitre est consacré à une généralisation du troisième chapitre au cas de systèmes 2×2 avec une diffusion ordinaire. On étudie l'existence locale d'une solution ainsi qu'un résultat d'explosion de la solution avec les mêmes propriétés étudiées dans le troisième chapitre.

Finalement, dans le cinquième chapitre, nous étudions une équation hyperbolique dans $\mathbb{R}^N$, pour tout $N \geq 2$, avec un terme non-linéaire non-local en temps. Nous obtenons un résultat d'existence locale d'une solution sous des conditions restrictives sur les données initiales, la dimension de l'espace et de la croissance du terme non-linéaire. De plus on obtient, sous certaines conditions, que la solution explose en temps fini pour toute condition initiale de moyenne strictement positive.

Pour mettre en lumière les points importants et éviter qu'ils ne soient pas cachés par le technique, nous avons souvent donné dans cette introduction générale, des énoncés simplifiés des résultats. Des énoncés plus complets se trouvent dans les différents chapitres qui sont présentés ci-dessous.

1.1 Décroissance de masse pour une équation non linéaire avec le laplacien fractionnaire

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Dans la première partie, nous nous intéressons à l'équation de diffusion-réaction non-linéaire avec le laplacien fractionnaire suivante :

$$\begin{cases} u_t = -(-\Delta)^{\alpha/2}u + \lambda u^p & \text{dans } \mathbb{R}^N \times (0,\infty), \\ \\ u(0) = u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) & \text{dans } \mathbb{R}^N, \end{cases} \tag{1.1.1}$$

où le laplacien fractionnaire $(-\Delta)^{\alpha/2}$ avec $\alpha \in (0,2]$ est un opérateur pseudo-différentiel défini par la transformée de Fourier : $(-\Delta)^{\alpha/2}u(x) = \mathcal{F}^{-1}\{|\xi|^\alpha \mathcal{F}(u)(\xi)\}(x)$ pour tout $x \in \mathbb{R}^N$. Notons que $\mathcal{F}$ et $\mathcal{F}^{-1}$ sont les transformée de Fourier directe et inverse, respectivement. De plus, on suppose que $\lambda \in \{-1;+1\}$ et $p > 1$.

En fait, le laplacien fractionnaire est un cas particulier de l'opérateur de Lévy $\mathcal{L}$ qui est bien un opérateur pseudo-differentiel défini par $\mathcal{L}v(x) = \mathcal{F}^{-1}\{a(\xi)\mathcal{F}(v)(\xi)\}(x)$ pour tout $x \in \mathbb{R}^N$. Comme la fonction $e^{-ta(\xi)}$ est bien définie et positive, alors le symbole $a(\xi)$ peut se représenter, comme dans [7], par la formule de Lévy-Khintchine (cf. [5, Chapitre 1, Théorème 1], ou [25, Théorème B.2])

$$a(\xi) = ib\xi + q(\xi) + \int_{\mathbb{R}^N}(1 - e^{-i\eta\xi} - i\eta\xi \mathbb{1}_{\{|\eta|<1\}}(\eta))\Pi(\,d\eta).$$

Dans la littérature de la Physique Mathématique (voir [6, 14, 28]), les problèmes d'évolution non-linéaires avec le laplacien fractionnaire décrivent la diffusion anormale (the anamolous diffusion) ou ce qu'on appelle la diffusion $\alpha-$stable de Lévy ($\alpha-$stable Lévy diffusion).

Dans le cas d'une équation de diffusion-réaction classique (i.e. Eq. (1.1.1) avec $\alpha = 2$ et $\lambda = 1$), Fujita a montré dans [18] que pour $p < 1+2/N$, toute solution positive non-nulle explose en temps fini, alors que pour $p > 1 + 2/N$, il existe des solutions globales positives pour toutes données initiales petites en norme L^∞. Hayakawa [23] et Kobayashi, Sirao et Tanaka [36] ont étendu le résultat d'explosion de Fujita au cas $p = 1+2/N$. D'une façon analogue, des résultats d'explosion pour le problème (1.1.1) avec le laplacien fractionnaire (où l'exposant critique est $p = 1 + \alpha/N$ pour l'existence/non-existence des solutions) ont été démontrés dans [8, 20, 21, 51].

Le but de cette partie est d'étudier le comportement asymptotique des solutions de (1.1.1) pour tout $\alpha \in (0,2]$.

On commence par considérer le problème (1.1.1) avec $\lambda = -1$, et étudier le comportement, lorsque $t \to +\infty$, de la masse

$$M(t) \equiv \int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx - \int_0^t \int_{\mathbb{R}^N} u^p(x,s)\,dxds. \tag{1.1.2}$$

Remarquons que pour obtenir l'Eq. (1.1.2), il suffit d'intègrer en espace, en utilisant le théorème de Fubini, la solution intégrale suivante du problème (1.1.1) :

$$u(t) = P_\alpha(t) * u_0 - \int_0^t P_\alpha(t-s) * u^p(s)\,ds, \tag{1.1.3}$$

où la fonction $P_\alpha(t)(x) = P_\alpha(x,t)$ est la solution fondamentale de l'équation linéaire $u_t + (-\Delta)^{\alpha/2}u = 0$ représentée par :

$$P_\alpha(x,t) = t^{-N/\alpha} P_\alpha(xt^{-1/\alpha}, 1) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix.\xi - t|\xi|^\alpha}\,d\xi. \tag{1.1.4}$$

Comme, pour une donnée initiale positive, la solution de l'équation (1.1.3) est positive, alors la fonction $M(t)$ définie dans (1.1.2) est positive et décroissante. Donc, contrairement au cas de l'équation linéaire, il n'y a pas de conservation de la masse et la limite $M_\infty = \lim_{t\to+\infty} M(t)$ existe et on a deux possibilités : soit $M_\infty = 0$, soit $M_\infty > 0$. Ce phénomène décrit la compétition entre le terme de diffusion et le terme non-linéaire.

Notre premier résultat est le théorème suivant, qui étudie le cas $M_\infty > 0$ en présentant sa relation avec le comportement asymptotique de la solution lorsque $t \to +\infty$, dans le cas $p > 1 + \alpha/N$.

Théorème 1.1.1 *Supposons que $u = u(x,t)$ est une solution positive non-nulle du problème (1.1.1) avec $\alpha \in (0,2]$, $\lambda = -1$ et $p > 1 + \alpha/N$. Alors $\lim_{t\to\infty} M(t) = M_\infty > 0$.*
De plus, pour tout $q \in [1,\infty)$

$$t^{\frac{N}{\alpha}(1-\frac{1}{q})} \|u(t) - M_\infty P_\alpha(t)\|_q \to 0 \quad \textit{lorsque} \;\; t \to \infty. \tag{1.1.5}$$

La preuve de ce théorème s'appuit sur les résultats de [6, 39], en utilisant les propriétés suivantes de la solution fondamentale $P_\alpha(x,t)$:

$$P_\alpha(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad P_\alpha(x,t) \geq 0, \quad \int_{\mathbb{R}^N} P_\alpha(x,t)\,dx = 1, \tag{1.1.6}$$

et

$$\|P_\alpha(t) * v\|_r \leq Ct^{-N(1-1/r)/\alpha} \|v\|_1, \tag{1.1.7}$$

pour tout $v \in L^1(\mathbb{R}^N)$, $r \in [1,\infty]$, $x \in \mathbb{R}^N$ et $t > 0$. L'estimation (1.1.7) est obtenue en utilisant l'inégalité de Young pour la convolution et la forme auto-similaire (cf. (1.1.4)) pour P_α.

Ensuite, on montre que dans le cas $p \leq 1 + \alpha/N$, la masse $M(t)$ tend vers zéro lorsque $t \to +\infty$. Ce phénomène est interprété comme une domination du terme non-linéaire.

Théorème 1.1.2 *Supposons que* $u = u(x,t)$ *est une solution positive du problème* (1.1.1) *avec* $\alpha \in (0,2]$, $\lambda = -1$ *et* $p \leq 1 + \alpha/N$. *Alors* $\lim_{t\to\infty} M(t) = M_\infty = 0$.

L'idée de base de la démonstration du Théorème 1.1.2 repose sur la méthode de changement d'échelle dans la fonction test (the rescaled test function method) qui a été introduite par Mitidieri et Pohozaev [40, 41] pour démontrer la non-existence de la solution des équations elliptiques, paraboliques et hyperboliques non-linéaires.

Finalement, comme conséquence de notre analyse dans le Théorème 1.1.2, on peut aussi étudier l'explosion de toute solution positive non-nulle du problème (1.1.1) pour $\lambda = +1$, dans le cas où $p = 1 + \alpha/N$.

Theorem 1.1.3 *Si* $\lambda = 1$, $\alpha \in (0,2]$ *et* $p = 1 + \alpha/N$, *alors toute solution positive non-nulle de* (1.1.1) *explose en temps fini.*

1.2 Sur certaines équations d'évolution fractionnaire en temps et en espace

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Dans cette partie, on s'intérèsse à l'équation parabolique suivante avec le laplacien fractionnaire et un terme non-linéaire et non-local en temps :

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s)\, ds & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & x \in \mathbb{R}^N, \end{cases} \tag{1.2.1}$$

où $u_0 \in C_0(\mathbb{R}^N)$, $N \geq 1$, $0 < \beta \leq 2$, $0 < \gamma < 1$, $p > 1$ et $D((-\Delta)^{\beta/2}) = H^\beta(\mathbb{R}^N)$, avec

$$H^\beta(\mathbb{R}^N) = \left\{ u \in \mathcal{S}';\ (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N) \right\},$$

si $\beta \notin \mathbb{N}$, et

$$H^\beta(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N);\ (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N) \right\}$$

si $\beta \in \mathbb{N}$, l'espace de Sobolev homogène d'ordre β. Γ est la fonction gamma d'Euler et

$$C_0(\mathbb{R}^N) = \{ u \in C_0(\mathbb{R}^N) \text{ tel que } u(x) \to 0 \text{ lorsque } |x| \to \infty \}.$$

L'apparition de la constante $\Gamma(1-\gamma)$ est là juste par commodité. elle nous permet d'écrire le problème (1.2.1) sous la forme :

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = J^\alpha_{0|t}(|u|^{p-1} u) & x \in \mathbb{R}^N,\ t > 0, \\ u(0) = u_0 & x \in \mathbb{R}^N, \end{cases}$$

où $\alpha = 1-\gamma \in (0,1)$ et, pour $T>0$,

$$J^{\alpha}_{0|t}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds, \quad \text{pour tout } t\in[0,T] \text{ et tout } f\in L^q(0,T)\ (1\le q\le+\infty),$$

est l'intégrale fractionnaire à gauche de Riemann-Liouville d'ordre α (voir [32]).

Comme une motivation physique, le problème (1.2.1) nous a suggéré la possibilité d'une intéressante modèle physique dans lequel un milieu superdiffusive est couplé à un milieu classique de diffusion. Le terme non-linéaire de (1.2.1) pourrait être interprétée comme l'effet d'un milieu qui est classiquement diffusif non linéaire liée à un milieu superdiffusive. Un tel lien pourrait prendre la forme d'un milieu poreux matériaux ayant des propriétés réactives qui est partiellement isolé par contact avec un milieu classique diffusif. Pour plus d'information, voir le papier récent de Roberts et Olmstead [45].

Rappelons maintenant un résultat d'existence locale et d'unicité pour l'équation (1.2.1).
Si $u_0\in C_0(\mathbb{R}^N)$ (respectivement $u_0\in C_0(\mathbb{R}^N)\cap L^r(\mathbb{R}^N)$ avec $r\in[1,+\infty)$) alors il existe une unique solution maximale u de (1.2.1) définie sur un intervalle de temps maximal $[0,T_{\max})$, $0<T_{\max}\le+\infty$. De plus, les propriétés suivantes sont vérifiées.

− $u\in C([0,T],C_0(\mathbb{R}^N))$ (respectivement $u\in C([0,T],C_0(\mathbb{R}^N)\cap L^r(\mathbb{R}^N))$) pour tout $0<T<T_{\max}$.

− u est une solution douce de (1.2.1) (pour la notion de solution douce voir ci-dessous).

− Si $u_0\ge$ et $u_0\not\equiv 0$ alors $u(t)>0$ pour tout $0<t<T_{\max}$.

− L'alternative d'explosion suivante est vérifiée :
ou bien $T_{\max}=+\infty$,
ou bien $T_{\max}<+\infty$ et $\lim_{t\nearrow T_{\max}}\|u(t)\|_{L^\infty(\mathbb{R}^N)}=+\infty$.

Lorsque $T_{\max}<+\infty$, on dit que u explose en temps fini ; lorsque $T_{\max}=+\infty$, on dit que u est globale.

Maintenant, il est simple de vérifier que si $u(x,t)$ est une solution de (1.2.1), alors pour tout $\lambda>0$, $\lambda^{\beta(2-\gamma)/(p-1)}u(\lambda x,\lambda^\beta t)$ est encore une solution de donnée initiale $\lambda^{\beta(2-\gamma)/(p-1)}u_0(\lambda x)$. Comme

$$\|\lambda^{\beta(2-\gamma)/(p-1)}u_0(\lambda\,\cdot)\|_{L^q} = \lambda^{\beta(2-\gamma)/(p-1)-N/q}\|u_0(\cdot)\|_{L^q} \quad \text{pour tout } q\in[1,\infty], \tag{1.2.2}$$

alors l'exposant qui garde la norme de Lebesgue dans (1.2.2) invariante est :

$$q_{sc} = \frac{N(p-1)}{\beta(2-\gamma)}.$$

Donc, on peut imaginer, comme dans le résultat de Weissler dans [52], que si $q_{sc}>1$, i.e. $p>p_{sc}$

où l'exposant d'échelle (scaling exposent) p_{sc} est donné par :

$$p_{sc} = 1 + \frac{\beta(2-\gamma)}{N},$$

et si $\|u_0\|_{L^{q_{sc}}}$ est suffisamment petite, alors la solution est globale.

Cazenave, Dickstein et Weissler [10] ont montré que ce phénomène ne marche pas, i.e. que l'exposant critique n'est pas celui prévu par Fujita, pour l'équation suivante :

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma}|u|^{p-1}u(s)\,ds & x \in \mathbb{R}^N,\ t>0, \\ u(0) = u_0 & x \in \mathbb{R}^N, \end{cases} \tag{1.2.3}$$

qui est un cas particulier de l'Eq. (1.2.1).

Théorème 1.2.1 (*Cazenave, Dickstein et Weissler* [10])
Supposons que $0 \le \gamma < 1$, $p > 1$,

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(N-2+2\gamma)_+},$$

et

$$p_* = \max\left\{\frac{1}{\gamma}\ ,\ p_\gamma\right\} \le +\infty.$$

Soit $u_0 \in C_0(\mathbb{R}^N)$ *et soit* $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$ *la solution de* (1.2.3). *Si* $\gamma = 0$ *alors toute solution positive non nulle explose. Si* $0 < \gamma < 1$ *on a les résultats suivants :*

(i) Si

$$p \le p_*,$$

et $u_0 \ge 0$, $u_0 \not\equiv 0$, *alors* u *explose en temps fini.*

(ii) Si

$$p > p_*,$$

et $u_0 \in L^{\overline{q}_{sc}}(\mathbb{R}^N)$ (*où* $\overline{q}_{sc} = N(p-1)/(2(2-\gamma))$) *avec* $\|u_0\|_{L^{\overline{q}_{sc}}}$ *suffisamment petite, alors* u *existe globalement.*

Notons que dans le cas où $\gamma = 0$, Souplet a montré dans [49] que toute solution positive non nulle explose en temps fini.
On peut expliquer pourquoi ce phénomène (de Fujita) ne marche pas ici en remarquant que l'Eq. (1.2.3) peut se transformer formellement en

$$D^\alpha_{0|t} u_t - D^\alpha_{0|t} \Delta u = C|u|^{p-1}u, \tag{1.2.4}$$

où $C = \Gamma(1-\gamma)$, $\alpha = 1-\gamma \in (0,1)$ et $D^{\alpha}_{0|t}$ est la dérivée fractionnaire à gauche de Riemann-Liouville d'ordre α définie (voir [32]) par :

$$D^{\alpha}_{0|t}u(t) := \frac{d}{dt}J^{1-\alpha}_{0|t}u(t).$$

Clairement, l'équation (1.2.4) est une interpolation entre l'équation de la chaleur et l'équation des ondes. Il est connu que la méthode de changement d'échelle ne donne l'exposant de Fujita que pour des équations de type paraboliques.

Le permier résultat dans cette partie est une généralisation des résultats trouvés par Cazenave, Dickstein et Weissler [10] à l'Eq. (1.2.1).

Théorème 1.2.2 (*Explosion*)
Soit $u_0 \in C_0(\mathbb{R}^N)$ tel que $u_0 \geq 0$ et $u_0 \not\equiv 0$. Si

$$p \leq 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+} := p^* \quad ou \quad p < \frac{1}{\gamma}, \tag{1.2.5}$$

alors la solution douce de (1.2.1) explose en temps fini.
On note que dans le cas où $p = p^*$ et $\beta \in (0,2)$ on se restreindra au cas où $p > N/(N-\beta)$ avec $N > \beta$.

Théorème 1.2.3 (*Existence globale*)
Soit $u_0 \in C_0(\mathbb{R}^N) \cap L^{q_{sc}}(\mathbb{R}^N)$ et $0 < \beta \leq 2$. Si

$$p > \max\{\frac{1}{\gamma}; p^*\}, \tag{1.2.6}$$

et $\|u_0\|_{L^{q_{sc}}}$ est suffisamment petite, alors la solution u existe globalement.
On note qu'on peut prendre $|u_0(x)| \leq C|x|^{-\beta(2-\gamma)/(p-1)}$ au lieu de $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$.

La démonstration du théorème 1.2.2 se fait par contradiction et en deux étapes.
Tout d'abord, on donne les notions de solution douce et faible pour l'Eq. (1.2.1).

Définition 1.2.4 (*Solution douce*)
Soit $u_0 \in L^{\infty}(\mathbb{R}^N)$, $0 < \beta \leq 2$, $p > 1$ et $T > 0$. On dit que $u \in C([0,T], L^{\infty}(\mathbb{R}^N))$ est une solution douce du problème (1.2.1) si u satisfait à l'équation intégrale suivante

$$u(t) = P_\beta(t) * u_0 + \int_0^t P_\beta(t-s) * J^{\alpha}_{0|s}\left(|u|^{p-1}u\right)\, ds, \quad t \in [0,T], \tag{1.2.7}$$

où P_β est la solution fondamentale de l'équation linéaire $u_t + (-\Delta)^{\beta/2}u = 0$ définie par (1.1.4) et $u * v$ est la convolution en espace de u et v.

Définition 1.2.5 (*Solution faible*)
Soit $u_0 \in L^{\infty}_{Loc}(\mathbb{R}^N)$, $0 < \beta \leq 2$, $p > 1$, $T > 0$ et $u \in L^p((0,T), L^{\infty}_{Loc}(\mathbb{R}^N))$. On dit que u est une solution faible du problème (1.2.1) si

$$\int_{\Omega} u_0(x)\varphi(x,0) + \int_0^T \int_{\Omega} J^{\alpha}_{0|t}(|u|^{p-1}u)(x,t)\varphi(x,t) \;=\; \int_0^T \int_{\Omega} u(x,t)(-\Delta)^{\beta/2}\varphi(x,t)$$

$$- \int_0^T \int_\Omega u(x,t)\varphi_t(x,t), \qquad (1.2.8)$$

pour toute fonction test $\varphi \in C^1([0,T], H^\beta(\mathbb{R}^N))$ *telle que* $\Omega :=$*supp*φ *est compacte avec* $\varphi(\cdot\,,T) = 0$, *où* $\alpha := 1-\gamma \in (0,1)$.

Lemme 1.2.6 *Soit* $u_0 \in L^\infty(\mathbb{R}^N)$ *et soit* $u \in C([0,T], L^\infty(\mathbb{R}^N))$ *une solution douce de* (1.2.1), *alors* u *est une solution faible de* (1.2.1), *pour tout* $0 < \beta \leq 2$ *et tout* $T > 0$.

La démonstration du théorème 1.2.2 se mène en choisissant une fonction test convenable et en s'appuyant sur la méthode de changement d'échelle de la fonction test déjà introduite dans la Section 1.1, puis à la fin on passe à la limite lorsque $T \to +\infty$. On note qu'on a besoin d'utiliser l'inégalité suivante (voir l'Appendice dans le chapitre 3 ci-dessous)

$$(-\Delta)^{\beta/2}\psi^q \leq q\psi^{q-1}(-\Delta)^{\beta/2}\psi,$$

valable pour toute fonction positive de Schwartz ψ et pour tout $N, q \geq 1$.

La preuve du théorème 1.2.3 est basée (voir Section 3.5 dans le chapitre 3 ci-dessous) sur les techniques utilisés par Weissler dans [52] et récemment par Cazenave, Dickstein et Weissler dans [10].

Le théorème suivant est notre deuxième résultat dans cette partie. Il présente le taux d'explosion (Blow-up rate) de la solution u du problème (1.2.1) dans le cas $\beta = 2$.

Théorème 1.2.7 *Supposons que* $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ *est telle que* $u_0 \geq 0$ *et* $u_0 \not\equiv 0$. *Si* $\alpha_1 := (2-\gamma)/(p-1)$, $\beta = 2$ *et* u *est la solution douce de* (1.2.1) *qui explose en temps fini* $T_{\max} := T^*$, *alors il existe deux constantes* $c, C > 0$ *telles que*

$$c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(\cdot\,,t) \leq C(T^* - t)^{-\alpha_1}, \qquad t \in (0,T^*), \qquad (1.2.9)$$

pour tout $p \leq p^*$ *ou* $p < 1/\gamma$.

Pour démontrer la majoration dans (1.2.9), on utilise l'argument de changement d'échelle pour réduire le problème (1.2.1), dans le cas $\beta = 2$, à un problème de type Liouville i.e. défini pour tout $t \in \mathbb{R}$. Cette méthode a été utilisée pour la première fois pour les problèmes paraboliques par Hu [24]. Notons que pour démontrer le résultat, on a besoin du

Lemme 1.2.8 *Soit* φ *une solution positive de l'équation suivante :*

$$\varphi_t = \Delta\varphi + J^{1-\gamma}_{-\infty|t}(\varphi^p) \qquad \text{dans } \mathbb{R}^N \times \mathbb{R}, \qquad (1.2.10)$$

où $\gamma \in (0,1)$, $p > 1$ *et*

$$J^{1-\gamma}_{-\infty|t}(\varphi)^p(t) := \frac{1}{\Gamma(1-\gamma)} \int_{-\infty}^t (t-s)^{-\gamma}\varphi^p(s)\,ds.$$

Alors on a $\varphi \equiv 0$ *pour*

$$p \leq p^* \quad ou \quad p < \frac{1}{\gamma}. \qquad (1.2.11)$$

La preuve de ce lemme se trouve dans la section 3.5 du chapitre 3 ci-après.
D'une manière classique, on détermine la minoration dans (1.2.9), en utilisant la preuve du théorème d'existence locale de la solution qui s'appuit sur le théorème de point fixe de Banach.

Enfin, le dernier résultat dans cette partie est décomposé en deux parties. Les conditions nécessaires pour l'existence locale et globale de la solution.

Théorème 1.2.9 (*Condition nécessaire pour l'existence globale*)
Supposons que $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, $0 < \beta \leq 2$ et $p > 1$. Si u est une solution douce globale du problème (1.2.1),alors il existe une constante $C > 0$ telle que

$$\liminf_{|x|\to\infty}(u_0(x)|x|^{\frac{\beta(2-\gamma)}{p-1}}) \leq C. \tag{1.2.12}$$

Remarquons que l'estimation (1.2.12) est similaire à celle trouvée dans le théorème 1.2.3.

Théorème 1.2.10 (*Condition nécessaire pour l'existence locale*)
Soient $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, $\beta \in (0, 2]$ et $p > 1$. Si u est une solution douce locale de (1.2.1) sur $[0, T]$ avec $0 < T < +\infty$, alors on a l'estimation suivante

$$\liminf_{|x|\to\infty} u_0(x) \leq C\, T^{-\frac{2-\gamma}{p-1}}, \tag{1.2.13}$$

pour une certaine constante $C > 0$. Notons que, si $A := \liminf_{|x|\to\infty} u_0(x)$, alors on obtient une estimation similaire à celle du temps maximal $T_{\max}$ trouver dans le théorème d'existence locale (voir section 3.3 du chapitre 3) :

$$\frac{T^{2-\gamma}A^{p-1}}{C^{p-1}} \leq 1.$$

La méthode de changement d'échelle dans la fonction test, utilisée dans la preuve du théorème 1.2.2 avec un choix approprié de la fonction test, est l'idée principale pour démontrer les théorèmes 1.2.9 et 1.2.10.

1.3 Sur certains systèmes d'évolution fractionnaires en temps

Cette partie est consacrée à l'étude du système

$$\begin{cases} u_t - \Delta u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma}|v|^{p-1}v(s)\,ds & x \in \mathbb{R}^N,\ t > 0, \\ v_t - \Delta v = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t (t-s)^{-\delta}|u|^{q-1}u(s)\,ds & x \in \mathbb{R}^N,\ t > 0, \end{cases} \tag{1.3.1}$$

assujetti aux conditions initiales

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}^N, \tag{1.3.2}$$

où $u_0, v_0 \in C_0(\mathbb{R}^N)$, $\gamma, \delta \in (0,1)$, $p, q > 1$ et $D(-\Delta) = H^2(\mathbb{R}^N)$ avec $H^2(\mathbb{R}^N)$ est l'espace de Sobolev classique d'ordre 2. Notons qu'on reprend ici les mêmes notions utilisées dans la Section 1.2 ci-dessus.

Afin de présenter le résultat d'existence locale de la solution pour le problème (1.3.1) – (1.3.2), on a besoin de la

Définition 1.3.1 (*Solution douce*)
Supposons que $u_0, v_0 \in L^\infty(\mathbb{R}^N)$. *On dit que* $(u,v) \in C([0,T], L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N))$ *est une solution douce de* (1.3.1) – (1.3.2), *pour tout* $T > 0$, *si*

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta} J^\alpha_{0|s}(|v|^{p-1}v)\,ds, & \text{dans } \mathbb{R}^N \times [0,T], \\ v(t) = e^{t\Delta}v_0 + \displaystyle\int_0^t e^{(t-s)\Delta} J^\beta_{0|s}(|u|^{q-1}u)\,ds, & \text{dans } \mathbb{R}^N \times [0,T], \end{cases} \tag{1.3.3}$$

avec $\alpha = 1-\gamma$, $\beta = 1-\delta$ *et* $e^{t\Delta}$ *est le semi-groupe, engendré par l'opérateur auto-adjoint négative* Δ, *défini par* $e^{t\Delta}v = G(t)*v$ *pour tout* $v \in L^2(\mathbb{R}^N)$, *où* G *est la solution fondamentale de l'équation de la chaleur* $u_t - \Delta u = 0$ *définie par* $G(t,x) := (4\pi t)^{-N/2}e^{-|x|^2/4t}$.

Théorème 1.3.2 (*Existence locale de la solution douce*)
Supposons que $u_0, v_0 \in C_0(\mathbb{R}^N)$ (*respectivement* $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ *pour tout* $r \in [1,\infty)$) *et* $p, q > 1$. *Alors il existe un temps maximal* $T_{\max} > 0$ *et une unique solution douce*

$$(u,v) \in C([0,T_{\max}), C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$$

(*respectivement*

$$(u,v) \in C([0,T_{\max}), (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)) \times (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))))$$

du système (1.3.1) – (1.3.2). *De plus, soit* $T_{\max} = \infty$, *soit* $T_{\max} < \infty$ *et* $(\|u\|_{L^\infty((0,t)\times\mathbb{R}^N)} + \|v\|_{L^\infty((0,t)\times\mathbb{R}^N)}) \to \infty$ *lorsque* $t \to T_{\max}$. *Enfin, si* $u_0, v_0 \geq 0$, $u_0, v_0 \not\equiv 0$, *on a* $u(t), v(t) > 0$ *pour tout* $0 < t < T_{\max}$.

Le théorème 1.3.2 se démontre en utilisant le théorème de point fixe de Banach.

Notre motivation dans cette partie est de généraliser au cas de systèmes 2×2 le travail considéré par Cazenave, Dickstein et Weissler [10] concernant les résultats d'explosion, et la section 1.2 concernant le taux d'explosion, et les conditions nécessaires d'existence locale ou globale de solutions.

Tout d'abord, le résultat d'explosion de la solution en temps fini nécessite l'utilisation d'une notion de solution faible et un lemme qui dit que chaque solution douce est aussi faible.

Définition 1.3.3 (*Solution faible*)
Soit $T > 0$, $u_0, v_0 \in L^\infty_{Loc}(\mathbb{R}^N)$, $u \in L^q((0,T), L^\infty_{Loc}(\mathbb{R}^N))$ *et* $v \in L^p((0,T), L^\infty_{Loc}(\mathbb{R}^N))$. *On dit que* $U := (u,v)$ *est une solution faible du problème* (1.3.1) – (1.3.2) *si*

$$\int_{\Omega_1} u_0(x)\varphi(x,0) + \int_0^T \int_{\Omega_1} J^\alpha_{0|t}(|v|^{p-1}v)(x,t)\varphi(x,t) \;=\; -\int_0^T \int_{\Omega_1} u(x,t)\Delta\varphi(x,t)$$

$$- \int_0^T \int_{\Omega_1} u(x,t)\varphi_t(x,t), \qquad (1.3.4)$$

et

$$\begin{aligned} \int_{\Omega_2} v_0(x)\psi(x,0) + \int_0^T \int_{\Omega_2} J^{\beta}_{0|t}(|u|^{q-1}u)(x,t)\psi(x,t) &= -\int_0^T \int_{\Omega_2} v(x,t)\Delta\psi(x,t) \\ &- \int_0^T \int_{\Omega_2} v(x,t)\psi_t(x,t), \qquad (1.3.5) \end{aligned}$$

pour tout $\varphi, \psi \in C^1([0,T], H^2(\mathbb{R}^N))$ *telle que* Ω_1 :=suppφ *et* Ω_2 :=suppψ *sont compactes avec* $\varphi(\cdot\,,T) = \psi(\cdot\,,T) = 0,$ *où* $\alpha = 1-\gamma$ *et* $\beta = 1-\delta.$

Lemme 1.3.4 (*douce* → *faible*)
Soit $T > 0$, $u_0, v_0 \in L^\infty(\mathbb{R}^N)$ *et* $U := (u,v)$ *tel que* $u, v \in C([0,T], L^\infty(\mathbb{R}^N))$. *Si* U *est une solution douce de* (1.3.1) − (1.3.2), *alors* U *est une solution faible de* (1.3.1) − (1.3.2).

Théorème 1.3.5 (*Explosion de la solution*)
Supposons que $u_0, v_0 \in C_0(\mathbb{R}^N)$ *sont telles que* $u_0, v_0 \geq 0$ *et* $u_0, v_0 \not\equiv 0$. *Si*

$$\frac{N}{2} \leq \max\left\{\frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}\ ;\ \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1}\right\} \qquad (1.3.6)$$

ou

$$p < \frac{1}{\delta} \qquad et \qquad q < \frac{1}{\gamma} \qquad (1.3.7)$$

alors la solution douce (u,v) *de* (1.3.1) − (1.3.2) *explose en temps fini.*

Comme dans le théorème d'explosion pour le cas d'une équation de la partie précédente, on démontre ce théorème en utilisant le lemme 1.3.4 et un changement d'échelle pour la fonction test et en passant à la limite lorsque $T \to +\infty$.

Le deuxième résultat dans cette partie est la détermination du taux d'explosion de la solution du problème (1.3.1) − (1.3.2).

Théorème 1.3.6 *Soit*

$$\alpha_1 := \frac{(2-\gamma) + (2-\delta)p}{pq-1} \quad et \quad \alpha_2 := \frac{(2-\delta) + (2-\gamma)q}{pq-1}.$$

Supposons que $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ *sont telles que* $u_0, v_0 \geq 0$ *et* $u_0, v_0 \not\equiv 0$. *Si*

$$\frac{N}{2} \leq \max\left\{\frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}\ ;\ \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1}\right\},$$

ou

$$p < \frac{1}{\delta} \qquad et \qquad q < \frac{1}{\gamma},$$

et (u,v) *est la solution de* (1.3.1) $-$ (1.3.2) *qui explose en temps fini* $T_{\max} := T^*$, *alors il existe des constantes* $c_i, C_i > 0$, *pour* $i = 1,2$, *telles que :*

$$\begin{cases} c_1(T^*-t)^{-\alpha_1} \le \sup_{\mathbb{R}^N} u(\cdot\,,t) \le C_1(T^*-t)^{-\alpha_1}, & t\in(0,T^*),\\ c_2(T^*-t)^{-\alpha_2} \le \sup_{\mathbb{R}^N} v(\cdot\,,t) \le C_2(T^*-t)^{-\alpha_2}, & t\in(0,T^*). \end{cases} \tag{1.3.8}$$

La preuve de ce théorème repose sur celle du théorème 1.2.7. elle s'appuit sur le

Lemme 1.3.7 *Soit* (u,v) *une solution positive du système*

$$\begin{cases} u_t = \Delta u + J^{1-\gamma}_{-\infty|t}(v^p) & \textit{dans } \mathbb{R}^N\times\mathbb{R},\\ v_t = \Delta v + J^{1-\delta}_{-\infty|t}(u^q) & \textit{dans } \mathbb{R}^N\times\mathbb{R}, \end{cases} \tag{1.3.9}$$

où $\gamma,\delta\in(0,1)$ *et* $p,q>1$. *Alors, pour*

$$\frac{N}{2} \le \max\left\{\frac{(2-\delta)p+(1-\gamma)pq+1}{pq-1}\,;\,\frac{(2-\gamma)q+(1-\delta)pq+1}{pq-1}\right\}, \tag{1.3.10}$$

ou

$$p<\frac{1}{\delta} \qquad \textit{et} \qquad q<\frac{1}{\gamma}, \tag{1.3.11}$$

on a $u\equiv v\equiv 0$.

Finalement, un résultat concernant les conditions nécessaires pour l'existence locale (respectivement globale) de la solution est présené.

Théorème 1.3.8 (*Conditions nécessaires pour l'existence globale*)
Supposons que $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, $u_0, v_0 \ge 0$ *et* $p,q>1$. *Si* (u,v) *est une solution douce globale du problème* (1.3.1) $-$ (1.3.2), *alors il existe une constante* $C>0$ *telle que*

$$\liminf_{|x|\to\infty}(u_0(x)|x|^{2\alpha_1}) \le C \quad \textit{et} \quad \liminf_{|x|\to\infty}(v_0(x)|x|^{2\alpha_2}) \le C, \tag{1.3.12}$$

où $C>0$ *est un nombre réel qui peut changer d'une formule à une autre.*

Théorème 1.3.9 (*Conditions nécessaires pour l'existence locale*)
Supposons que $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, $u_0, v_0 \ge 0$ *et* $p,q>1$. *Si* (u,v) *est une solution locale du* (1.3.1) $-$ (1.3.2) *sur* $[0,T]$ *où* $0<T<+\infty$, *alors on a les estimations suivantes :*

$$\liminf_{|x|\to\infty} u_0(x) \le C\,T^{-\alpha_1} \quad \textit{et} \quad \liminf_{|x|\to\infty} v_0(x) \le C\,T^{-\alpha_2}, \tag{1.3.13}$$

pour une certaine constante $C>0$. *Notons que, si* $A := \liminf_{|x|\to\infty} u_0(x)$ *et* $B := \liminf_{|x|\to\infty} v_0(x)$, *alors on obtient une estimation similaire à celle de* $T_{\max}$ *dans la preuve du théorème* 1.3.2 *d'existence locale (voir Section* 4.3 *du chapitre* 4 *ci-dessous) :*

$$\begin{cases} C^{-(pq-1)}T^{(2-\gamma)+(2-\delta)p}A^{pq-1} \le 1,\\[2ex] C^{-(pq-1)}T^{(2-\delta)+(2-\gamma)q}B^{pq-1} \le 1. \end{cases}$$

Les démonstrations des théorèmes 1.3.8 et 1.3.9 s'appuient sur l'argument de changement d'échelle dans la fonction test introduite dans la partie 1.1.

1.4 Equation hyperbolique avec une nonlinéarité non-locale

Ici on traite une équation de type hyperbolique avec un terme non-linéaire et non-local en temps :

$$\begin{cases} u_{tt} - \Delta u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u(s)|^p \, ds & x \in \mathbb{R}^N, t > 0, \\ \\ u(0,x) = u_0(x), \; u_t(0,x) = u_1(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.4.1}$$

où $(u_0, u_1) \in H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N)$, $0 < \mu \leq N/2$, $N \geq 2$, $0 < \gamma < 1$ et $p > 1$. Γ est la fonction gamma d'Euler (Euler's gamma function) et $H^\mu(\mathbb{R}^N)$ est l'espace de Sobolev homogène d'ordre μ défini par

$$H^\mu(\mathbb{R}^N) = \left\{ u \in \mathcal{S}'; \; (-\Delta)^{\mu/2} u \in L^2(\mathbb{R}^N) \right\},$$

si $\mu \notin \mathbb{N}$, et par

$$H^\mu(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N); \; (-\Delta)^{\mu/2} u \in L^2(\mathbb{R}^N) \right\},$$

si $\mu \in \mathbb{N}$, and $(-\Delta)^{\mu/2}$ est le laplacien fractionnaire défini par

$$(-\Delta)^{\mu/2} u(x) := \mathcal{F}^{-1} \left(|\xi|^\mu \mathcal{F}(u)(\xi) \right)(x)$$

pour tout $u \in D((-\Delta)^{\mu/2}) = H^\mu(\mathbb{R}^N)$. $\mathcal{F}$ et $\mathcal{F}^{-1}$ sont la transformée de Fourier and son inverse respectivement.

Dans le cas de l'équation

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4.2}$$

une approche pour le non-existence de la solution globale a été étudiée pour la première fois par Keller [31], puis par John [26] et Kato [29]. Cette approche est basée sur l'intégration en espace de toute solution positive à support compact. Pour $p = p_c > 1$, où p_c est la racine positive de

$$(N-1)p^2 - (N+1)p - 2 = 0,$$

John a montré dans [26] que si $1 < p < p_c$, et $N = 3$ i.e. $1 < p < 1 + \sqrt{2}$, toute solution avec donnée initiale dans $C_c^\infty(\mathbb{R}^N)$ explose en temps fini. Le cas $p = 1 + \sqrt{2}$ a été étudié par Glassey [22] qui a démontré que toute solution, avec donnée initiale dans $C_c^\infty(\mathbb{R}^N)$ et ayant des moyennes strictement positives, explose en temps fini.

Dans cette partie, on étudie l'existence locale et l'explosion en temps fini de la solution pour une classe de non linéarité de type non-locale en temps. Notre résultat sur l'explosion est similaire à celui de Glassey.

Avant de présenter le premier résultat de cette partie, on a besoin de quelques notions, lemmes et théorèmes. On commence par définir les exposants $\sigma-$admissibles.

Définition 1.4.1 *On dit que la paire d'exposants (q, r) est $\sigma-$admissible si $q, r \geq 2$, $(q, r, \sigma) \neq (2, \infty, 1)$ et*

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}. \tag{1.4.3}$$

Si dans (1.4.3) on a une égalité, on dit que (q, r) est $\sigma-$admissible critique et on dit que (q, r) est $\sigma-$admissible non-critique dans le cas contraire. Particulièrement, si $\sigma > 1$, le point final

$$P = \left(2, \frac{2\sigma}{\sigma - 1}\right)$$

est $\sigma-$admissible critique.

Keel et Tao ont montré dans [30] une nouvelle version de l'estimation de Strichartz qui est l'idée de base pour démontrer notre résultat sur l'existence de solutions locales.

Théorème 1.4.2 (*Keel et Tao* [30])
*Supposons que $N \geq 2$ et (q, r) et $(\widetilde{q}, \widetilde{r})$ sont deux paires d'exposants admissibles pour l'équation des ondes, i.e. $\sigma = (N-1)/2$, avec $r, \widetilde{r} < \infty$. Si u est une solution faible du problème (**??**) dans $\mathbb{R}^N \times [0, T]$ pour un certain $0 < T < \infty$, alors*

$$\begin{aligned} \|u\|_{L^q([0,T];L^r_x)} &+ \|u\|_{C([0,T];H^\mu)} + \|\partial_t u\|_{C([0,T];H^{\mu-1})} \\ &\leq \overline{C}\left(\|u_0\|_{H^\mu} + \|u_1\|_{H^{\mu-1}} + \|f\|_{L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'}_x)}\right), \end{aligned} \tag{1.4.4}$$

sous la condition d'analyse dimensionnelles ("gap" condition) suivante :

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - \mu = \frac{1}{\widetilde{q}'} + \frac{N}{\widetilde{r}'} - 2 \tag{1.4.5}$$

où $\overline{C} > 0$ est une constante indépendante de T. Les exposants $\widetilde{r}', \widetilde{q}'$ sont les exposants conjugués de $\widetilde{r}, \widetilde{q}$ respectivement et $L^p_x := L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$.

Ici, on considère la solution douce définie par :

Définition 1.4.3 (*Solution douce*)
Soit $0 < T < \infty$ et $(u_0, u_1) \in H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N)$. On dit que u est une solution douce du problème (1.4.1) sur $[0, T]$ si et seulement si $(u, u_t) \in C([0, T]; H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N))$ et telle que u vérifie l'équation intégrale

$$u(t) = \dot{K}(t)u_0 + K(t)u_1 + (K * J^\alpha_{0|t}(|u|^p))(t), \quad t \in [0, T], \tag{1.4.6}$$

où

$$K(t)g := (-\Delta)^{-1/2}\sin((-\Delta)^{1/2}t) := \mathcal{F}^{-1}\left(\frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}(g)(\xi)\right)$$

$$\dot{K}(t)g := \cos((-\Delta)^{1/2}t) := \mathcal{F}^{-1}\left(\cos(t|\xi|)\mathcal{F}(g)(\xi)\right),$$

pour toute fonction g assez régulière.

Nous sommes maintenant en mesure de présenter notre premier résultat sur l'existence locale d'une solution.

Théorème 1.4.4 (*Existence locale*)
Supposons que $(u_0, u_1) \in H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N)$, $N \geq 2$, $0 < \mu \leq N/2$ *et* $p > 1$ *tels que*

$$\begin{cases} \dfrac{2}{\widetilde{r}'} + \dfrac{2\mu}{N} \leq p \leq \dfrac{2}{\widetilde{r}'}\dfrac{N}{N-2\mu} & si\ \mu < N/2, \\ 1 + \dfrac{1}{\widetilde{r}'} \leq p < \infty & si\ \mu = N/2, \end{cases} \tag{1.4.7}$$

où $\widetilde{r}$ *est défini dans* (5.3.4) *du chapitre* 5 *ci-dessous. Alors, il existe un temps* $T > 0$ *et une unique solution* u *du problème* (1.4.1) *tels que* $(u, u_t) \in C([0,T], H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N))$.

La preuve de ce théorème s'appuit sur le théorème de point fixe de Banach et le théorème 1.4.2. Pour étendre la solution sur un intervalle de temps maximal, on prend $\mu = 1$. D'où le

Corollaire 1.4.5 *Supposons que* $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, $N \geq 2$ *et* $p > 1$ *tels que*

$$\begin{cases} 1 + \dfrac{2}{N} \leq p \leq \dfrac{N}{N-2} & si\ N > 2, \\ \dfrac{3}{2} \leq p < \infty & si\ N = 2. \end{cases} \tag{1.4.8}$$

Alors, il existe un temps maximal $0 < T_{\max} \leq \infty$ *et une unique solution* u *du problème* (1.4.1) *tels que* $(u, u_t) \in C([0, T_{\max}), H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$. *De plus, si* $T_{\max} < \infty$ *on a* $(\|u\|_{C([0,t],H^1(\mathbb{R}^N))} + \|u_t\|_{C([0,t],L^2(\mathbb{R}^N))}) \to \infty$ *lorsque* $t \to T_{\max}$.

Enfin, on donne le résultat d'explosion de la solution.

Théorème 1.4.6 *Supposons que* $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} u_0 > 0$, $\int_{\mathbb{R}^N} u_1 > 0$, $N \geq 2$, *et* $p > 1$ *tels que*

$$\begin{cases} 1 + \dfrac{2}{N} \leq p \leq 1 + \dfrac{3-\gamma}{(N-2+\gamma)_+} := p* & si\ \ N > 2, \\ \dfrac{3}{2} \leq p \leq \dfrac{3}{\gamma} & si\ \ N = 2, \end{cases} \tag{1.4.9}$$

ou

$$\begin{cases} 1 + \dfrac{2}{N} \leq p < \dfrac{1}{\gamma} & si\ \ N > 2, \\ \dfrac{3}{2} \leq p < \dfrac{1}{\gamma} & si\ \ N = 2, \end{cases} \tag{1.4.10}$$

avec les conditions suivantes

$$\begin{cases} \dfrac{N-2}{N} \leq \gamma < 1 & if\ \ N > 2, \\ 0 < \gamma < 1 & if\ \ N = 2, \end{cases} \tag{1.4.11}$$

ou

$$\begin{cases} \dfrac{N-2}{N} \leq \gamma < \dfrac{N}{N+2} & \textit{if } N > 2, \\ \\ 0 < \gamma \leq \dfrac{2}{3} & \textit{if } N = 2, \end{cases} \tag{1.4.12}$$

respectivement. Alors la solution du problème (1.4.1) *explose en temps fini.*

Chapitre 2

Decay of mass for nonlinear equation with fractional Laplacian

Ahmad Z. FINO and Gregorz KARCH

Abstract

The large time behavior of non-negative solutions to the reaction-diffusion equation $\partial_t u = -(-\Delta)^{\alpha/2}u - u^p$, $(\alpha \in (0,2],\ p > 1)$ posed on $\mathbb{R}^N$ and supplemented with an integrable initial condition is studied. We show that the anomalous diffusion term determines the large time asymptotics for $p > 1 + \alpha/N$, while nonlinear effects win if $p \leq 1 + \alpha/N$.

Keywords : Large time behavior, fractional Laplacian, blow-up of solutions, critical exponent.
MSC : 35K55; 35B40; 60H99

2.1 Introduction

We study the behavior, as $t \to \infty$, of solutions to the following initial value problem for the reaction-diffusion equation with the anomalous diffusion

$$\partial_t u = -\Lambda^\alpha u + \lambda u^p, \qquad x \in \mathbb{R}^N, t > 0, \tag{2.1.1}$$

$$u(x,0) = u_0(x), \tag{2.1.2}$$

where the pseudo-differential operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ with $0 < \alpha \leq 2$ is defined by the Fourier transformation : $\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi)$. Moreover, we assume that $\lambda \in \{-1, 1\}$ and $p > 1$.

Nonlinear evolution problems involving fractional Laplacian describing *the anomalous diffusion* (or α-stable Lévy diffusion) have been extensively studied in the mathematical and physical literature (see [6, 14, 28] for references). One of possible ways to understand the interaction between the anomalous diffusion operator (given by Λ^α or, more generally, by the Lévy diffusion operator) and the nonlinearity in the equation under consideration is the study of the large time asymptotics of solutions to such equations. Our goal is to contribute to this theory and our results can be summarized as follows. For $\lambda = -1$ in Eq. 2.1.1, non-negative solutions to the Cauchy problem exist globally in time. Hence, we study the decay properties of the mass $M(t) = \int_{\mathbb{R}^N} u(x,t)\,dx$ of the solutions $u = u(x,t)$ to problem 2.1.1 and 2.1.2. We prove that $\lim_{t\to\infty} M(t) = M_\infty > 0$ for $p > 1 + \alpha/N$ (cf. Theorem 1, below), while $M(t)$ tends to zero as $t \to \infty$ if $1 < p \le 1 + \alpha/N$ (cf. Theorem 2). As a by-product of our analysis, we show the blow-up of all non-negative solutions to 2.1.1 and 2.1.2 with $\lambda = 1$ in the case of the critical nonlinearity exponent $p = 1 + \alpha/N$ (see Theorem 3, below).

The idea which allows to express the competition between diffusive and nonlinear terms in an evolution equation by studying the large time behavior of the space integral of a solution was already introduced by Ben-Artzi and Koch [4] who considered the viscous Hamilton-Jacobi equation $u_t = \Delta u - |\nabla u|^p$ (see also Pinsky [44]). An analogous result for the equation $u_t = \Delta u + |\nabla u|^p$ (with the growing-in-time mass of solutions) was proved by Laurençot and Souplet [39]. Such questions concerning the asymptotic behavior of solutions to the Hamilton-Jacobi equation with the Lévy diffusion operator were answered in [28].

In the case of the classical reaction-diffusion equation (i.e. Eq. 2.1.1 with $\alpha = 2$), for $p < 1 + 2/N$, Fujita [18] proved the nonexistence of non-negative global-in-time solution for any nontrivial initial condition. On other hand, if $p > 1 + 2/N$, global solutions do exist for any sufficiently small non-negative initial data. The proof of a blow-up of all non-negative solutions in the critical case $p = 1 + 2/N$ was completed in [23, 36, 51]. Analogous blow-up results for problem 2.1.1 and 2.1.2 with the fractional Laplacian (and with the critical exponent $p = 1 + \alpha/N$ for the existence/nonexistence of solutions) are contained, e.g. in [8, 20, 21, 51].

2.2 Statement of results

In all theorems below, we always assume that $u = u(x,t)$ is the non-negative (possibly weak) solution of problem 2.1.1 and 2.1.2 corresponding to the non-negative initial datum $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Let $u_0 \not\equiv 0$, for simplicity of the exposition. We refer the reader to [14] for several results on the existence, the uniqueness and the regularity of solutions to 2.1.1 and 2.1.2 as well as for the proof of the maximum principle (which assures that the solution is non-negative if the corresponding initial datum is so).

First, we deal with Eq. 2.1.1 containing the absorbing nonlinearity ($\lambda = -1$) and we study the decay of the "mass"

$$M(t) \equiv \int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx - \int_0^t \int_{\mathbb{R}^N} u^p(x,s)\,dxds. \tag{2.2.1}$$

Remark. In order to obtain equality 2.2.1, it suffices to integrate Eq. 2.1.1 with respect to x and t. Another method which leads to 2.2.1 and which requires weaker regularity assumptions on a solution consists in integrating with respect to x the integral formulation of problem 2.1.1 and 2.1.2 (see 2.3.8, below) and using the Fubini theorem.

Since we limit ourselves to non-negative solutions, the function $M(t)$ defined in 2.2.1 is non-negative and non-increasing. Hence, the limit $M_\infty = \lim_{t\to\infty} M(t)$ exists and we answer the question whether it is equal to zero or not.

In our first theorem, the diffusion phenomena determine the large time asymptotics of solutions to 2.1.1 and 2.1.2.

Theorem 2.2.1 *Assume that $u = u(x,t)$ is a non-negative nontrivial solution of 2.1.1 and 2.1.2 with $\lambda = -1$ and $p > 1 + \alpha/N$. Then $\lim_{t\to\infty} M(t) = M_\infty > 0$.*

Moreover, for all $q \in [1,\infty)$

$$t^{\frac{N}{\alpha}\left(1-\frac{1}{q}\right)} \|u(t) - M_\infty P_\alpha(t)\|_q \to 0 \quad \text{as} \ \ t \to \infty, \tag{2.2.2}$$

where the function $P_\alpha(x,t)$ denotes the fundamental solution of the linear equation $u_t + \Lambda^\alpha u = 0$ (cf. Eq. 2.3.3 below).

In the remaining range of p, the mass $M(t)$ converges to zero and this phenomena can be interpreted as the domination of nonlinear effects in the large time asymptotic of solutions to 2.1.1 and 2.1.2. Note here that the mass $M(t) = \int_{\mathbb{R}^N} u(x,t)\,dx$ of every solution to linear equation $u_t + \Lambda^\alpha u = 0$ is constant in time.

Theorem 2.2.2 *Assume that $u = u(x,t)$ is a non-negative solution of problem 2.1.1 and 2.1.2 with $\lambda = -1$ and $1 < p \le 1 + \alpha/N$. Then $\lim_{t\to\infty} M(t) = 0$.*

Let us emphasize that the proof of Theorem 2 is based on the so-called the rescaled test function method which was used by Mitidieri and Pokhozhaev (*cf. e.g.* [40, 41] and the references therein) to prove the nonexistence of solutions to nonlinear elliptic and parabolic equations.

As the by-product of our analysis, we can also contribute to the theory on the blow-up of solutions to 2.1.1 and 2.1.2 with $\lambda = +1$. Recall that the method of the rescaled test function (which we also apply here) was use in [20, 21] to show the blow-up of all positive solutions to 2.1.1 and 2.1.2 with $\lambda = 1$ and $p < 1 + \alpha/N$. Here, we complete that result by the simple proof of the blow-up in the critical case $p = 1 + \alpha/N$.

Theorem 2.2.3 *If $\lambda = 1$, $\alpha \in (0,2]$ and $p = 1 + \alpha/N$, then any non-negative nonzero solution of 2.1.1 and 2.1.2 blows up in a finite time.*

2.3 Proofs of Theorems 1, 2, and 3

Note first that any (sufficiently regular) non-negative solution to 2.1.1 and 2.1.2 satisfies

$$0 \le \int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx + \lambda \int_0^t \int_{\mathbb{R}^N} u^p(x,s)\,dx\,ds. \tag{2.3.1}$$

Hence, for $\lambda = -1$ and $u_0 \in L^1(\mathbb{R}^N)$, we immediately obtain

$$u \in L^\infty([0,\infty), L^1(\mathbb{R}^N)) \cap L^p(\mathbb{R}^N \times (0,\infty)). \tag{2.3.2}$$

Proof of Theorem 1. First, we recall that the fundamental solution $P_\alpha = P_\alpha(x,t)$ of the linear equation $\partial_t u + \Lambda^\alpha u = 0$ can be written via the Fourier transform as follows

$$P_\alpha(x,t) = t^{-N/\alpha} P_\alpha(xt^{-1/\alpha}, 1) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix.\xi - t|\xi|^\alpha}\, d\xi. \tag{2.3.3}$$

It is well-known that for each $\alpha \in (0,2]$, this function satisfies

$$P_\alpha(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad P_\alpha(x,t) \geq 0, \quad \int_{\mathbb{R}^N} P_\alpha(x,t)\, dx = 1, \tag{2.3.4}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, using the Young inequality for the convolution and the self-similar form of P_α, we have

$$\|P_\alpha(t) * u_0\|_p \leq Ct^{-N(1-1/p)/\alpha} \|u_0\|_1, \tag{2.3.5}$$
$$\|\nabla P_\alpha(t)\|_p = Ct^{-N(1-1/p)/\alpha - 1/\alpha}, \tag{2.3.6}$$
$$\|P_\alpha(t) * u_0\|_p \leq \|u_0\|_p, \tag{2.3.7}$$

for all $p \in [1, \infty]$ and $t > 0$.

In the next step, using the following well-known integral representation of solutions to 2.1.1 and 2.1.2

$$u(t) = P_\alpha(t) * u_0 - \int_0^t P_\alpha(t-s) * u^p(s)\, ds, \tag{2.3.8}$$

we immediately obtain the estimate $0 \leq u(x,t) \leq P_\alpha(x,t) * u_0(x)$. Hence, by 2.3.5 and 2.3.7 we get

$$\begin{aligned} \|u(t)\|_p^p &\leq \|P_\alpha(t) * u_0\|_p^p \\ &\leq \min\left\{Ct^{-N(p-1)/\alpha} \|u_0\|_1^p; \|u_0\|_p^p\right\} \equiv H(t,p,\alpha,u_0). \end{aligned} \tag{2.3.9}$$

Now, for fixed $\varepsilon \in (0,1]$, we consider the solution $u^\varepsilon = u^\varepsilon(x,t)$ of 2.1.1 and 2.1.2 with the initial condition $\varepsilon u_0(x)$. The comparison principle implies that $0 \leq u^\varepsilon(x,t) \leq u(x,t)$ for every $x \in \mathbb{R}^N$ and $t > 0$. Hence, it suffices to show that for small $\varepsilon > 0$, which will be determined later, we have

$$M_\infty^\varepsilon \equiv \lim_{t\to\infty} \int_{\mathbb{R}^N} u^\varepsilon(x,t)\, dx > 0.$$

Note first the using equality (2.3.1) in the case of the solution u^ε, we obtain

$$M_\infty^\varepsilon = \varepsilon \left\{ \int_{\mathbb{R}^N} u_0(x)\, dx - \frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x,t))^p\, dx\, dt \right\}. \tag{2.3.10}$$

Now, we apply (2.3.9) with u replaced by u^ε. Observe that the function H defined in 2.3.9 satisfies $H(t,p,\alpha,\varepsilon u_0) = \varepsilon^p H(t,p,\alpha,u_0)$. Hence

$$\frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x,t))^p\, dxdt \leq \frac{1}{\varepsilon} \int_0^\infty H(t,p,\alpha,\varepsilon u_0)\, dt$$

$$= \varepsilon^{p-1} \int_0^\infty H(t,p,\alpha,u_0)\,dt.$$

It is follows immediately from the definition of the function H that the integral on the right-hand side is convergent for $p > 1+\alpha/N$. Consequently,

$$\frac{1}{\varepsilon}\int_0^\infty \int_{\mathbb{R}^N} (u^\varepsilon(x,t))^p\,dx\,dt \to 0 \qquad \text{as} \quad \varepsilon \searrow 0,$$

and the constant M_∞^ε given by (2.3.10) is positive for sufficiently small $\varepsilon > 0$.

From now on, the proof of the asymptotic relation 2.2.2 is standard, hence, we shall be brief in details. First we recall that for every $u_0 \in L^1(\mathbb{R}^N)$ we have

$$\lim_{t\to\infty} \|P_\alpha(t) * u_0 - MP_\alpha(t)\|_1 = 0, \tag{2.3.11}$$

where $M = \int_{\mathbb{R}^N} u_0(x)\,dx$. This is the immediate consequence of the Taylor argument combined with an approximation argument. Details of this reasoning can be found in [6, Lemma 3.3].

Now, to complete the proof of Theorem 1, we adopt the reasoning from [39]. It follows from the integral equation (2.3.8) and inequality (2.3.5) with $p = 1$ that

$$\|u(t) - P_\alpha(t-t_0) * u(t_0)\|_1 \le \int_{t_0}^t \|u(s)\|_1\,ds \quad \text{for all} \quad t \ge t_0 \ge 0.$$

Hence, using the triangle inequality we infer

$$\begin{aligned} \|u(t) - M_\infty P_\alpha(t)\|_1 \le & \int_{t_0}^t \|u(s)\|_1\,ds \\ & + \|P_\alpha(t-t_0) * u(t_0) - M(t_0)P_\alpha(t-t_0)\|_1 \\ & + \|M(t_0)(P_\alpha(t-t_0) - P_\alpha(t))\|_1 \\ & + \|P_\alpha(t)\|_1\,|M(t_0) - M_\infty|\,. \end{aligned} \tag{2.3.12}$$

Applying first 2.3.4 and 2.3.11 with $u_0 = u(t_0)$, and next passing to the limit as $t \to \infty$ on the right-hand side of 2.3.12, we obtain

$$\limsup_{t\to\infty} \|u(t) - M_\infty P_\alpha(t)\|_1 \le \int_{t_0}^\infty \|u(s)\|_1\,ds + |M(t_0) - M_\infty|\,.$$

By letting t_0 go to $+\infty$ and using 2.3.2 we conclude that

$$\|u(t) - M_\infty P_\alpha(t)\|_1 \to 0 \qquad \text{as} \quad t \to \infty. \tag{2.3.13}$$

In order to obtain the asymptotic term for $p > 1$, observe that by the integral equation 2.3.8 and estimate 2.3.5, for each $m \in [1,\infty]$, we have

$$\|u(t)\|_m \le \|P_\alpha(t) * u_0\|_m \le Ct^{-N(1-1/m)/\alpha}\|u_0\|_1. \tag{2.3.14}$$

Hence, for every $q \in [1, m)$, using the Hölder inequality, we obtain

$$\begin{aligned}\|u(t) - M_\infty P_\alpha(t)\|_q &\le \|u(t) - M_\infty P_\alpha(t)\|_1^{1-\delta} \left(\|u(t)\|_m^\delta + \|M_\infty P_\alpha(t)\|_m^\delta\right) \\ &\le Ct^{-N(1-1/q)/\alpha} \|u(t) - M_\infty P_\alpha(t)\|_1^{1-\delta},\end{aligned}$$

with $\delta = (1 - 1/q)/(1 - 1/m)$. Finally, applying 2.3.13 we complete the proof of Theorem 1.

Proof of Theorem 2. Let us define the function $\varphi(x,t) = (\varphi_1(x))^\ell (\varphi_2(t))^\ell$ where

$$\ell = \frac{2p-1}{p-1}, \quad \varphi_1(x) = \psi\left(\frac{|x|}{BR}\right), \quad \varphi_2(t) = \psi\left(\frac{t}{R^\alpha}\right), \quad R > 0,$$

and ψ is a smooth non-increasing function on $[0, \infty)$ such that

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2. \end{cases}$$

The constant $B > 0$ in the definition of φ_1 is fixed and will be chosen later. In fact, it plays some role in the critical case $p = 1 + \alpha/N$ only while in the subcritical case $p < 1 + \alpha/N$ we simply put $B = 1$. In the following, we denote by Ω_1 and Ω_2 the supports of φ_1 and φ_2, respectively :

$$\Omega_1 = \left\{x \in \mathbb{R}^N \;:\; |x| \le 2BR\right\}, \qquad \Omega_2 = \{t \in [0, \infty) \;:\; t \le 2R^\alpha\}.$$

Now, we multiply Eq. 2.1.1 by $\varphi(x,t)$ and integrate with respect to x and t to obtain

$$\begin{aligned}&\int_{\Omega_1} u_0(x)\varphi(x,0)\,dx - \int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\varphi(x,t)\,dxdt \\ &= \int_{\Omega_2}\int_{\mathbb{R}^N} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x))^\ell\,dxdt \\ &\quad - \int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1^\ell(x)\partial_t\varphi_2^\ell(t)\,dxdt \\ &\le \ell\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\varphi_1^{\ell-1}(x)\Lambda^\alpha\varphi_1(x)\,dxdt \\ &\quad -\ell\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1^\ell(x)\varphi_2^{\ell-1}(t)\partial_t\varphi_2(t)\,dxdt.\end{aligned} \tag{2.3.15}$$

In 2.3.15, we have used the inequality $\Lambda^\alpha\varphi_1^\ell \le \ell\varphi_1^{\ell-1}\Lambda^\alpha\varphi_1$, (see [13, Prop. 2.3] and [27, Prop. 3.3] for its proof) which is valid for all $\alpha \in (0,2]$, $\ell \ge 1$, and any sufficiently regular, non-negative, decaying at infinity function φ_1.

Hence, by the ε-Young inequality $ab \le \varepsilon a^p + C(\varepsilon)b^{\ell-1}$ (note that $1/p + 1/(\ell-1) = 1$) with $\varepsilon > 0$, we deduce from 2.3.15

$$\begin{aligned}&\int_{\Omega_1} u_0(x)\varphi(x,0)\,dx - (1+2\ell\varepsilon)\int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\varphi(x,t)\,dx\,dt \\ &\le C(\varepsilon)\ell\left\{\int_{\Omega_2}\int_{\Omega_1} \varphi_1\varphi_2^\ell\left|\Lambda^\alpha\varphi_1\right|^{\ell-1}\,dxdt + \int_{\Omega_2}\int_{\Omega_1} \varphi_1^\ell\varphi_2\left|\partial_t\varphi_2\right|^{\ell-1}\,dxdt\right\}.\end{aligned} \tag{2.3.16}$$

Recall now that the functions φ_1 and φ_2 depend on $R > 0$. Hence changing the variables $\xi = R^{-1}x$ and $\tau = R^{-\alpha}t$, we easily obtain from 2.3.16 the following estimate

$$\int_{\Omega_1} u_0(x)\varphi(x,0)\,dx - (1+2\ell\varepsilon)\int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\varphi(x,t)\,dxdt \le CR^{N+\alpha-\alpha(\ell-1)}, \tag{2.3.17}$$

where the constant C on the right hand side of 2.3.17 is independent of R. Note that $N+\alpha-\alpha(\ell-1) \le 0$ if and only if $p \le 1+\alpha/N$. Now, we consider two cases.

For $p < 1+\alpha/N$, we have $N+\alpha-\alpha(\ell-1) < 0$. Hence, computing the limit $R \to \infty$ in 2.3.17 and using the Lebesgue dominated convergence theorem, we obtain

$$M_\infty = \int_{\mathbb{R}^N} u_0(x)\,dx - \int_0^\infty \int_{\mathbb{R}^N} u^p(x,t)\,dxdt \le 2\ell\varepsilon \int_0^\infty \int_{\mathbb{R}^N} u^p\,dxdt.$$

Since $u \in L^p(\mathbb{R}^N \times (0,\infty))$ (cf. 2.3.2) and since $\varepsilon > 0$ can be chosen arbitrary small, we immediately obtain that $M_\infty = 0$.

In the critical case $p = 1+\alpha/N$, we estimate first term on the right hand side of inequality 2.3.15 using again by the ε-Young inequality and the second term by the Hölder inequality (with $\bar{p} = p/(p-1) = \ell - 1$) as follows

$$\begin{aligned}
&\int_{\Omega_1} u_0(x)\varphi(x,0)\,dx - \int_{\Omega_2}\int_{\Omega_1} u^p\varphi(x,t)\,dxdt \\
&\le \ \ell\varepsilon \int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\,dxdt \\
&\quad + C(\varepsilon)\int_{\Omega_2}\int_{\Omega_1} \varphi_2^{\ell\bar{p}}(t)\varphi_1^{(\ell-1)\bar{p}}(x)\,|\Lambda^\alpha\varphi_1(x)|^{\bar{p}}\,dxdt \\
&\quad + \ell\left(\int_{\Omega_3}\int_{\Omega_1} u^p(x,t)\,dxdt\right)^{1/p} \\
&\quad\ \times \left(\int_{\Omega_2}\int_{\Omega_1} \varphi_1^{\ell\bar{p}}(x)\varphi_2^{(\ell-1)\bar{p}}(t)\,|\partial_t\varphi_2(t)|^{\bar{p}}\,dxdt\right)^{1/\bar{p}}.
\end{aligned} \tag{2.3.18}$$

Here, $\Omega_3 = \{t \in [0,\infty) \ : \ R^\alpha \le t \le 2R^\alpha\}$ is the support of $\partial_t\varphi_2$. Note that

$$\int_{\Omega_3}\int_{\Omega_1} u^p(x,t)\,dx\,dt \to 0 \quad \text{as} \quad R \to \infty,$$

because $u \in L^p(\mathbb{R}^N \times [0,\infty))$ (cf. 2.3.2).

Now, introducing the new variables $\xi = (BR)^{-1}x$, $\tau = R^{-\alpha}t$ and recalling that $p = 1+\alpha/N$, we rewrite 2.3.18 as follows

$$\begin{aligned}
&\int_{\Omega_1} u_0(x)\varphi(x,0)\,dx - \int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\varphi(x,t)\,dxdt - \varepsilon\ell\int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\,dxdt \\
&\qquad \le C_1 B^{N/\bar{p}}\left(\int_{\Omega_3}\int_{\Omega_1} u^p(x,t)\,dxdt\right)^{1/p} + C_2C(\varepsilon)B^{-\alpha},
\end{aligned} \tag{2.3.19}$$

where the constants C_1, C_2 are independent of R, B, and of ε. Passing in 2.3.19 to the limit as $R \to +\infty$ and using the Lebesgue dominated convergence theorem we get

$$\int_{\mathbb{R}^N} u_0(x)\,dx - \int_0^\infty \int_{\mathbb{R}^N} u^p(x,t)\,dxdt - \varepsilon\ell \int_0^\infty \int_{\mathbb{R}^N} u^p(x,t)\,dxdt \\ \leq C_2 C(\varepsilon) B^{-\alpha}. \tag{2.3.20}$$

Finally, computing the limit $B \to \infty$ in 2.3.20 we infer that $M_\infty = 0$ beacuse $\varepsilon > 0$ can be arbitrarily small. This complete the proof of Theorem 2.

Proof of Theorem 3. The proof proceeds by contradiction. Let u be a non-negative non-trivial solution of 2.1.1 and 2.1.2 with $\lambda = 1$. Take the test function φ the same as in the proof of Theorem 2. Repeating the estimations which lead to (2.3.19), we obtain

$$\int_{\Omega_1} u_0(x)\varphi(x,0)\,dx + \int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\varphi(x,t)\,dxdt \\ - \varepsilon\ell \int_{\Omega_2}\int_{\Omega_1} u^p(x,t)\,dxdt \\ \leq C_1 B^{N/\bar{p}} \left(\int_{\Omega_3}\int_{\Omega_1} u^p(x,t)\,dxdt \right)^{1/p} + C_2 C(\varepsilon) B^{-\alpha}.$$

Now, we chose $\varepsilon = 1/(2\ell)$ in 2.3.21 and we pass to the following limits : first $R \to \infty$, next $B \to \infty$. Using the Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^N} u_0(x)\,dx + \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^N} u^p(x,t)\,dxdt \leq 0.$$

Hence, $u(x,t) = 0$ which contradicts our assumption imposed on u.

Chapitre 3

On certain time-and space-fractional evolution equations

Ahmad Z. FINO and Mokhtar KIRANE

Abstract

In this article, we present first a new technique to prove, in a general case, the recent result of Cazenave, Dickstein and Weissler [10] on the blowing-up solutions to a temporally nonlocal nonlinear parabolic equation. Then, we study the blow-up rate, and the existence of global solutions. Furthermore, we establish necessary conditions for local or global existence.

Keywords : Parabolic equation, mild and weak solutions, local and global existence, blow-up, blow-up rate, maximal regularity, interior regularity, Schauder's estimates, critical exponent, fractional Laplacian, Riemann-Liouville fractional integrals and derivatives.
MSC : 35K55; 26A33; 35B44; 74G25; 35B33; 74G40

3.1 Introduction

In this paper, we investigate the nonlinear parabolic equation with a nonlocal in time nonlinearity

$$\left\{\begin{array}{ll} u_t+(-\Delta)^{\beta/2}u=\dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t(t-s)^{-\gamma}|u|^{p-1}u(s)\,ds & x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=u_0(x) & x\in\mathbb{R}^N, \end{array}\right. \tag{3.1.1}$$

where $u_0\in C_0(\mathbb{R}^N)$, $N\geq 1$, $0<\beta\leq 2$, $0<\gamma<1$, $p>1$ and the nonlocal operator $(-\Delta)^{\beta/2}$ is defined by

$$(-\Delta)^{\beta/2}v(x):=\mathcal{F}^{-1}\left(|\xi|^{\beta}\mathcal{F}(v)(\xi)\right)(x)$$

for every $v \in D((-\Delta)^{\beta/2}) = H^\beta(\mathbb{R}^N)$, where $H^\beta(\mathbb{R}^N)$ is the fractional Sobolev space of order β defined by

$$H^\beta(\mathbb{R}^N) = \left\{u \in \mathcal{S}';\ (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N)\right\},$$

if $\beta \notin \mathbb{N}$, and by

$$H^\beta(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N);\ (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N)\right\},$$

if $\beta \in \mathbb{N}$ where $\mathcal{S}'$ is the space of Schwartz distributions. $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for its inverse, Γ is the Euler gamma function and $C_0(\mathbb{R}^N)$ denotes the space of all continuous functions decaying to zero at infinity.
When Eq. (3.1.1) is considered with a nonlinearity of the form u^p, it reads

$$u_t + (-\Delta)^{\beta/2} u = u^p.$$

This equation has been considered by Nagasawa and Sirao [43], Kobayashi [35], Fino and Karch [16], Birkner, Lopez-Mimbela and Wakolbinger [8], Guedda and Kirane [20], and by Kirane and Qafsaoui [34].

The fractional Laplacian $(-\Delta)^{\beta/2}$ is related to Lévy flights in physics. Many observations and experiments related to Lévy flights (super-diffusion), e.g., collective slip diffusion on solid surfaces, quantum optics or Richardson turbulent diffusion, have been performed in recent years. The symmetric $\beta-$stable processes ($\beta \in (0, 2)$) are the basic characteristics for a class of jumping Lévy's processes. Compared with the continuous Brownian motion ($\beta = 2$), symmetric $\beta-$stable processes have infinite jumps in an arbitrary time interval. The large jumps of these processes make their variances and expectations infinite according to $\beta \in (0, 2)$ or $\beta \in (0, 1]$, respectively (see [32]). Let us also mention that when $\beta = 3/2$, the symmetric $\beta-$stable processes appear in the study of stellar dynamics (see [11]).

As a physical motivation, the problem (3.1.1) suggested to us the possibility of an interesting physical model in which a superdiffusive medium is coupled to a classically diffusive medium. The right-side of (3.1.1) might be interpreted as the effect of a classically diffusive medium that is nonlinearly linked to a superdiffusive medium. Such a link might come in the form of a porous material with reactive properties that is partially insulated by contact with a classically diffusive material. For more informations see the recent result of Roberts and Olmstead [45].
Our article is motivated Mathematically by the recent and very interesting paper by Cazenave, Dickstein and Weissler [10] which deals with the global existence and blow-up for the parabolic equation with nonlocal in time nonlinearity

$$u_t - \Delta u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s)\, ds \qquad x \in \mathbb{R}^N,\ t > 0, \tag{3.1.2}$$

where $0 \leq \gamma < 1$, $p > 1$ and $u_0 \in C_0(\mathbb{R}^N)$, which is a particular case of (3.1.1); it corresponds to $\beta = 2$.

If we set
$$p_\gamma = 1 + \frac{2(2-\gamma)}{(N-2+2\gamma)_+}$$
and
$$p_* = \max\left\{\frac{1}{\gamma}, p_\gamma\right\} \in (0,+\infty],$$
where $(\cdot)_+$ is the positive part, they proved that

(i) If $\gamma \neq 0$, $p \leq p_*$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.

(ii) If $\gamma \neq 0$, $p > p_*$, and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$ (where $q_{sc} = N(p-1)/(4-2\gamma)$) with $\|u_0\|_{L^{q_{sc}}}$ sufficiently small, then u exists globally.

If $\gamma = 0$ then all nontrivial positive solutions blow-up as proved by Souplet in [49]. The study of Cazenave, Dickstein and Weissler reveals the surprising fact that for equation (3.1.2) the critical exponent in Fujita's sense p_* is not the one predicted by scaling.
This can be explained by the fact that their equation can be formally converted into
$$D^\alpha_{0|t}u_t - D^\alpha_{0|t}\Delta u = |u|^{p-1}u, \tag{3.1.3}$$
where $D^\alpha_{0|t}$ is the left-sided Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ defined in (3.2.7) below (we have set in (3.1.3), $\alpha = 1-\gamma \in (0,1)$).
Eq. (3.1.3) is pseudo parabolic equation and as is well known scaling is efficient for detecting the Fujita exponent only for equations of parabolic type.
Needless to say that the equation considered by Cazenave, Dickstein and Weissler [10] is a genuine extension of the one considered by Fujita in his pioneering work [18].
In this article, concerning blowing-up solutions, we present a different proof from the one presented in [10], and for the more general equation (3.1.1). Our proof is more versatile and can be applied to more nonlinear equations (see the Remark 3.4.4).
Our analysis is based on the observation that the nonlinear differential equation (3.1.1) can be written in the form :
$$u_t + (-\Delta)^{\beta/2}u = J^\alpha_{0|t}\left(|u|^{p-1}u\right), \tag{3.1.4}$$
where $\alpha := 1-\gamma \in (0,1)$ and $J^\alpha_{0|t}$ is the Riemann-Liouville fractional integral defined in (3.2.9).
We will show that :

(**1**) For $u_0 \geq 0$, $u_0 \not\equiv 0$, and $u_0 \in C_0(\mathbb{R}^N)$, if
$$p \leq 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+} \quad \text{or} \quad p < \frac{1}{\gamma},$$
then all solutions of problem (3.1.1) blow-up in finite time.

(**2**) For $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, where $p_{sc} := N(p-1)/\beta(2-\gamma)$, if
$$p > \max\left\{1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+}; \frac{1}{\gamma}\right\},$$

and $\|u_0\|_{L^{p_{sc}}}$ is sufficiently small, then u exists globally.

The method used to prove the blow-up theorem is the test function method introduced by Mitidieri and Pohozaev [40, 41], then used by Kirane et al. [20, 33].

Furthermore, in the case $\beta = 2$, we derive the blow-up rate estimates for the parabolic equation (3.1.1). We shall prove that, if $u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and if u is the blowing-up solution of (3.1.1) at the finite time $T^* > 0$, then there are constants $c, C > 0$ such that $c(T^*-t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(\cdot\,, t) \leq C(T^*-t)^{-\alpha_1}$ for $1 < p \leq 1+2(2-\gamma)/(N-2+2\gamma)_+$ or $1 < p < 1/\gamma$ and all $t \in (0, T^*)$, where $\alpha_1 := (2-\gamma)/(p-1)$. We use a scaling argument to reduce the problems of blow-up rate to Fujita-type theorems (it is similar to blow-up analysis in elliptic problems to reduce the problems of a priori bounds to Liouville-type theorems). As far as we know, this method was first applied to parabolic problems by Hu [24], and then was used in various parabolic equations and systems (see [12, 15]). We notice that in the limiting case when $\gamma \to 0$, we obtain the constant rate $2/(p-1)$ found by P. Souplet [50].
For more informations, we refer the reader to the excellent paper of Andreucci and Tedeev [2] for the blow-up rate by an alternative method

The organization of this paper is as follows. In section 3.2, we present some definitions and properties. In Section 3.3, we derive the local existence of solutions for the parabolic equation (3.1.1). Section 3.4 contains the blow-up result of solutions for (3.1.1). Section 3.5 is dedicated to the blow-up rate of solutions. Global existence is studied in Section 3.6. Finally, a necessary condition for local and global existence is given in Section 3.7.

3.2 Preliminaries

In this section, we present some definitions and results concerning fractional Laplacians, fractional integrals and fractional derivatives that will be used hereafter.
First, if we take the usual linear fractional diffusion equation

$$u_t + (-\Delta)^{\beta/2} u = 0, \quad \beta \in (0,2],\ x \in \mathbb{R}^N,\ t > 0, \tag{3.2.1}$$

then, its fundamental solution S_β can be represented via the Fourier transform by

$$S_\beta(t)(x) := S_\beta(x,t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix.\xi - t|\xi|^\beta}\, d\xi. \tag{3.2.2}$$

It is well-known that this function satisfies

$$S_\beta(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad S_\beta(x,t) \geq 0, \quad \int_{\mathbb{R}^N} S_\beta(x,t)\, dx = 1, \tag{3.2.3}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, using Young's inequality for the convolution and the following self-similar form $S_\beta(x,t) = t^{-N/\beta} S_\beta(xt^{-1/\beta}, 1)$, we have

$$\|S_\beta(t) * v\|_q \;\leq\; Ct^{-\frac{N}{\beta}(\frac{1}{r}-\frac{1}{q})}\|v\|_r, \tag{3.2.4}$$

for all $v \in L^r(\mathbb{R}^N)$ and all $1 \leq r \leq q \leq \infty$, $t > 0$.
Moreover, as $(-\Delta)^{\beta/2}$ is a self-adjoint operator with $D\left((-\Delta)^{\beta/2}\right) = H^\beta(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^{\beta/2}v(x)\,dx = \int_{\mathbb{R}^N} v(x)(-\Delta)^{\beta/2}u(x)\,dx, \tag{3.2.5}$$

for all $u, v \in H^\beta(\mathbb{R}^N)$.
In an open bounded domain Ω, we denote by $\Delta_D^{\beta/2}$ the fractional Laplacian in Ω with Dirichlet condition. We have

Lemma 3.2.1 *Let λ_k $(k = 1, ..., +\infty)$ be the eigenvalues for the Laplacian operator in $L^2(\Omega)$ and let φ_k be the eigenfunction corresponding to λ_k. Then*

$$\Delta_D^{\beta/2}\varphi_k = \lambda_k^{\beta/2}\varphi_k$$

and

$$D(\Delta_D^{\beta/2}) = \left\{ u \in L^2(\Omega) \;\; \textit{such that} \;\; \|\Delta_D^{\beta/2}u\|_{L^2(\Omega)} := \sum_{k=1}^{k=+\infty} |\lambda_k^{\beta/2} < u, \varphi_k > |^2 < +\infty \right\}.$$

So, for $u \in D(\Delta_D^{\beta/2})$ we have

$$\Delta_D^{\beta/2}u = \sum_{k=1}^{k=+\infty} \lambda_k^{\beta/2} < u, \varphi_k > \varphi_k,$$

and then we get the following integration by parts

$$\int_\Omega u(x)\Delta_D^{\beta/2}v(x)\,dx = \int_\Omega v(x)\Delta_D^{\beta/2}u(x)\,dx, \tag{3.2.6}$$

for all $u, v \in D(\Delta_D^{\beta/2})$.

Next, if $AC[0,T]$ is the space of all functions which are absolutely continuous on $[0,T]$ with $0 < T < \infty$, then, for $f \in AC[0,T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D_{0|t}^\alpha f(t)$ and $D_{t|T}^\alpha f(t)$ of order $\alpha \in (0,1)$ are defined by (see [32])

$$D_{0|t}^\alpha f(t) \;:=\; DJ_{0|t}^{1-\alpha}f(t), \tag{3.2.7}$$

$$D_{t|T}^\alpha f(t) \;:=\; -\frac{1}{\Gamma(1-\alpha)}D\int_t^T (s-t)^{-\alpha}f(s)\,ds, \tag{3.2.8}$$

for all $t \in [0,T]$, where $D := d/dt$ is the usual derivative, and

$$J_{0|t}^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \tag{3.2.9}$$

is the Riemann-Liouville fractional integral defined in [32], for all $f \in L^q(0,T)$ $(1 \leq q \leq \infty)$.
Now, for every $f, g \in C([0,T])$, such that $D^\alpha_{0|t}f(t), D^\alpha_{t|T}g(t)$ exist and are continuous, for all $t \in [0,T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.64) p. 46 in [46])

$$\int_0^T \left(D^\alpha_{0|t}f\right)(t)g(t)\,dt \;=\; \int_0^T f(t)\left(D^\alpha_{t|T}g\right)(t)\,dt. \tag{3.2.10}$$

Note also that, for any $f \in AC^2[0,T]$, we have (see (2.2.30) in [32])

$$-D.D^\alpha_{t|T}f = D^{1+\alpha}_{t|T}f, \tag{3.2.11}$$

where

$$AC^2[0,T] := \{f : [0,T] \to \mathbb{R} \text{ such that } Df \in AC[0,T]\}.$$

Moreover, for all $1 \leq q \leq \infty$, the following equality (see [32, Lemma 2.4 p.74])

$$D^\alpha_{0|t}J^\alpha_{0|t} = Id_{L^q(0,T)} \tag{3.2.12}$$

holds almost everywhere on $[0,T]$.
Later on, we will use the following results.
• If $w_1(t) = (1 - t/T)^\sigma_+$, $t \geq 0$, $T > 0$, $\sigma \gg 1$, then

$$D^\alpha_{t|T}w_1(t) \;=\; \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}T^{-\alpha}\left(1-\frac{t}{T}\right)^{\sigma-\alpha}_+, \tag{3.2.13}$$

$$D^{\alpha+1}_{t|T}w_1(t) \;=\; \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}T^{-(\alpha+1)}\left(1-\frac{t}{T}\right)^{\sigma-\alpha-1}_+, \tag{3.2.14}$$

for all $\alpha \in (0,1)$; so

$$\left(D^\alpha_{t|T}w_1\right)(T) = 0 \quad ; \quad \left(D^\alpha_{t|T}w_1\right)(0) = C\,T^{-\alpha}, \tag{3.2.15}$$

where $C = (1-\alpha+\sigma)\Gamma(\sigma+1)/\Gamma(2-\alpha+\sigma)$; indeed, using the Euler change of variable $y = (s-t)/(T-t)$, we get

$$\begin{aligned}
D^\alpha_{t|T}w_1(t) &:= -\frac{1}{\Gamma(1-\alpha)}D\left[\int_t^T (s-t)^{-\alpha}\left(1-\frac{s}{T}\right)^\sigma ds\right]\\
&= -\frac{T^{-\sigma}}{\Gamma(1-\alpha)}D\left[(T-t)^{1-\alpha+\sigma}\int_0^1 (y)^{-\alpha}(1-y)^\sigma\,ds\right]\\
&= \frac{(1-\alpha+\sigma)B(1-\alpha;\sigma+1)}{\Gamma(1-\alpha)}T^{-\sigma}(T-t)^{\sigma-\alpha},
\end{aligned}$$

where $B(\cdot\,;\cdot\,)$ stands for the beta function. Then, (3.2.13) follows using the relation

$$B(1-\alpha;\sigma+1) = \frac{\Gamma(1-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}.$$

Moreover, using (3.2.11) and (3.2.13), we obtain (3.2.14).

• If $w_2(t) = (1 - t^2/T^2)_+^\ell$, $T > 0$, $\ell \gg 1$, then, using the change of variable $y = (s - t)/(T - t)$, we have

$$D^\alpha_{t|T} w_2(t) = \frac{T^{-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} C_1(\ell, k, \alpha) \left(1 - \frac{t}{T}\right)^{\ell+k-\alpha}, \tag{3.2.16}$$

$$D^{1+\alpha}_{t|T} w_2(t) = \frac{T^{-\alpha-1}}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} C_2(\ell, k, \alpha) \left(1 - \frac{t}{T}\right)^{\ell+k-\alpha-1}, \tag{3.2.17}$$

for all $-T \le t \le T$, $\alpha \in (0, 1)$, where

$$\begin{cases} C_1(\ell, k, \alpha) := c^k_\ell (1 - \alpha + \ell + k) 2^{\ell-k} (-1)^k \frac{\Gamma(k+\ell+1)\Gamma(1-\alpha)}{\Gamma(k+\ell+2-\alpha)}, \\ C_2(\ell, k, \alpha) := (\ell + k - \alpha) C_1(\ell, k, \alpha), \\ c^k_\ell := \frac{\ell!}{(\ell-k)!k!}; \end{cases}$$

so

$$\left(D^\alpha_{t|T} w_2\right)(T) = 0 \quad ; \quad \left(D^\alpha_{t|T} w_2\right)(-T) = C_3(\ell, k, \alpha)\, T^{-\alpha}, \tag{3.2.18}$$

where

$$C_3(\ell, k, \alpha) := \frac{2^{2\ell-\alpha}(-1)^\ell}{\Gamma(1-\alpha)} \sum_{k=0}^{\ell} c^k_\ell (1 - \alpha + \ell + k) \frac{\Gamma(k + \ell + 1)\Gamma(1 - \alpha)}{\Gamma(k + \ell + 2 - \alpha)}.$$

3.3 Local existence

This section is dedicated to proving the local existence and uniqueness of mild solutions to the problem (3.1.1). Let $T(t) := e^{-t(-\Delta)^{\beta/2}}$. As $(-\Delta)^{\beta/2}$ is a positive definite self-adjoint operator in $L^2(\mathbb{R}^N)$, $T(t)$ is a strongly continuous semigroup on $L^2(\mathbb{R}^N)$ generated by the fractional power $-(-\Delta)^{\beta/2}$ (see Yosida [53]). It holds $T(t)v = S_\beta(t) * v$ (see [42, *Prop*.3.3]), for all $v \in L^2(\mathbb{R}^N), t > 0$, where S_β is given by (3.2.2) and $u * v$ is the convolution of u and v. We start by giving the

Definition 3.3.1 (*Mild solution*)
Let $u_0 \in L^\infty(\mathbb{R}^N)$, $0 < \beta \le 2$, $p > 1$ and $T > 0$. We say that $u \in C([0, T], L^\infty(\mathbb{R}^N))$ is a mild solution of the problem (3.1.1) if u satisfies the following integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t - s) J^\alpha_{0|s} \left(|u|^{p-1}u\right) ds, \quad t \in [0, T]. \tag{3.3.1}$$

Theorem 3.3.2 (*Local existence*)
Given $u_0 \in C_0(\mathbb{R}^N)$ and $p > 1$, there exist a maximal time $T_{\max} > 0$ and a unique mild solution $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$ to the problem (3.1.1). Furthermore, either $T_{\max} = \infty$ or else $T_{\max} < \infty$ and $\|u\|_{L^\infty((0,t)\times\mathbb{R}^N)} \to \infty$ as $t \to T_{\max}$. In addition, if $u_0 \ge 0$, $u_0 \not\equiv 0$, then $u(t) > 0$ for all $0 < t < T_{\max}$. Moreover, if $u_0 \in L^r(\mathbb{R}^N)$, for $1 \le r < \infty$, then $u \in C([0, T_{\max}), L^r(\mathbb{R}^N))$.

Proof For arbitrary $T>0$, we define the Banach space

$$E_T := \left\{u \in L^\infty((0,T), C_0(\mathbb{R}^N));\ \|u\|_1 \leq 2\|u_0\|_{L^\infty}\right\},$$

where $\|\cdot\|_1 := \|\cdot\|_{L^\infty((0,T),L^\infty(\mathbb{R}^N))}$. Next, for every $u \in E_T$, we define

$$\Psi(u) := T(t)u_0 + \int_0^t T(t-s) J^\alpha_{0|s}\left(|u|^{p-1}u\right)\, ds.$$

As usual, we prove the local existence by the Banach fixed point theorem.

• $\mathbf{\Psi} : \mathbf{E_T} \to \mathbf{E_T}$ **:** Let $u \in E_T$, using (3.2.4), we obtain with $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^N)}$

$$\begin{aligned}
\|\Psi(u)\|_1 &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)}\|\int_0^t\int_0^s (s-\sigma)^{-\gamma}\|u(\sigma)\|_\infty^p\, d\sigma\, ds\|_{L^\infty(0,T)}\\
&= \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)}\|\int_0^t\int_\sigma^t (s-\sigma)^{-\gamma}\|u(\sigma)\|_\infty^p\, ds\, d\sigma\|_{L^\infty(0,T)}\\
&\leq \|u_0\|_\infty + \frac{T^{2-\gamma}}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)}\|u\|_1^p\\
&\leq \|u_0\|_\infty + \frac{T^{2-\gamma}2^p\|u_0\|_{L^\infty}^{p-1}}{\Gamma(3-\gamma)}\|u_0\|_\infty,
\end{aligned}$$

thanks to the following formula $\Gamma(x+1) = x\Gamma(x)$ for all $x>0$. Now, if we choose T small enough such that

$$\frac{T^{2-\gamma}2^p\|u_0\|_\infty^{p-1}}{\Gamma(3-\gamma)} \leq 1, \tag{3.3.2}$$

we conclude that $\|\Psi(u)\|_1 \leq 2\|u_0\|_\infty$, and then $\Psi(u) \in E_T$.

• Ψ **is a contracting map :** For $u, v \in E_T$, taking account of (3.2.4), we have

$$\begin{aligned}
\|\Psi(u)-\Psi(v)\|_1 &\leq \frac{1}{\Gamma(1-\gamma)}\|\int_0^t\int_0^s (s-\sigma)^{-\gamma}\||u|^{p-1}u(\sigma)-|v|^{p-1}v(\sigma)\|_\infty\, d\sigma\, ds\|_{L^\infty(0,T)}\\
&= \frac{1}{\Gamma(1-\gamma)}\|\int_0^t\int_\sigma^t (s-\sigma)^{-\gamma}\||u|^{p-1}u(\sigma)-|v|^{p-1}v(\sigma)\|_\infty\, ds\, d\sigma\|_{L^\infty(0,T)}\\
&\leq \frac{T^{2-\gamma}}{\Gamma(3-\gamma)}\||u|^{p-1}u-|v|^{p-1}v\|_1\\
&\leq \frac{C(p)2^p\|u_0\|_\infty^{p-1}T^{2-\gamma}}{\Gamma(3-\gamma)}\|u-v\|_1\\
&\leq \frac{1}{2}\|u-v\|_1,
\end{aligned}$$

thanks to the following inequality

$$|\,|u|^{p-1}u - |v|^{p-1}v| \leq C(p)|u-v|(|u|^{p-1}+|v|^{p-1}); \tag{3.3.3}$$

T is chosen such that

$$\frac{T^{2-\gamma}2^p\|u_0\|_\infty^{p-1}\max(2C(p),1)}{\Gamma(3-\gamma)} \leq 1. \tag{3.3.4}$$

Then, by the Banach fixed point theorem, there exists a unique mild solution $u \in \Pi_T$, where $\Pi_T := L^\infty((0,T), C_0(\mathbb{R}^N))$, to the problem (3.1.1).

• **Uniqueness of solution :** If u, v are two mild solutions in E_T for some $T > 0$, using (3.2.4) and (3.3.3), we obtain

$$\begin{aligned} \|u(t)-v(t)\|_\infty &\le \frac{C(p)2^p\|u_0\|_\infty^{p-1}}{\Gamma(1-\gamma)}\int_0^t\int_0^s (s-\sigma)^{-\gamma}\|u(\sigma)-v(\sigma)\|_\infty\,d\sigma\,ds \\ &= \frac{C(p)2^p\|u_0\|_\infty^{p-1}}{\Gamma(1-\gamma)}\int_0^t\int_\sigma^t (s-\sigma)^{-\gamma}\|u(\sigma)-v(\sigma)\|_\infty\,ds\,d\sigma \\ &= \frac{C(p)2^p\|u_0\|_\infty^{p-1}}{\Gamma(2-\gamma)}\int_0^t (t-\sigma)^{1-\gamma}\|u(\sigma)-v(\sigma)\|_\infty\,d\sigma. \end{aligned}$$

So the uniqueness follows from Gronwall's inequality.

Next, using the uniqueness of solutions, we conclude the existence of a solution on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup\{T > 0\,;\ \text{there exist a mild solution } u \in \Pi_T \text{ to (3.1.1)}\} \le +\infty.$$

Note that, using the continuity of the semigroup $T(t)$, we can easily conclude that

$$u \in C([0, T_{\max}), C_0(\mathbb{R}^N)).$$

Moreover, if $0 \le t \le t+\tau < T_{\max}$, using (3.3.1), we obtain

$$\begin{aligned} u(t+\tau) &= T(\tau)u(t) + \frac{1}{\Gamma(1-\gamma)}\int_0^\tau T(\tau-s)\int_0^s (s-\sigma)^{-\gamma}|u|^{p-1}u(t+\sigma)\,d\sigma\,ds \\ &+ \frac{1}{\Gamma(1-\gamma)}\int_0^\tau T(\tau-s)\int_0^t (t+s-\sigma)^{-\gamma}|u|^{p-1}u(\sigma)\,d\sigma\,ds. \end{aligned} \tag{3.3.5}$$

To prove that if $T_{\max} < \infty$, then $\|u\|_{L^\infty((0,t)\times\mathbb{R}^N)} \to \infty$ as $t \to T_{\max}$, we proceed by contradiction. Suppose that u is a solution of (3.3.1) on some interval $[0, T)$ with $\|u\|_{L^\infty((0,t)\times\mathbb{R}^N)} < \infty$ and $T_{\max} < \infty$. So, using the fact that the last term in (3.3.5) depends only on the values of u in the interval $(0, t)$ and using again a fixed-point argument, we conclude that u can be extended to a solution on some interval $[0, T')$ with $T' > T$. If we repeat this iteration, we obtain a contradiction with the fact that the maximal time $T_{\max}$ is finite.

• **Positivity of solutions :** If $u_0 \ge 0$ and $u_0 \not\equiv 0$, then we can construct a nonnegative solution on some interval $[0, T]$ by applying the fixed point argument in the set $E_T^+ = \{u \in E_T;\ u \ge 0\}$. In particular, it follows from (3.3.1) that $u(t) \ge T(t)u_0 > 0$ on $(0, T]$. It is not difficult by uniqueness to deduce that u stays positive on $(0, T_{\max})$.

• **Regularity of solutions :** If $u_0 \in L^r(\mathbb{R}^N)$, for $1 \le r < \infty$, then by repeating the fixed point argument in the space

$$E_{T,r} := \{u \in L^\infty((0,T), C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N));\ \|u\|_1 \le 2\|u_0\|_{L^\infty}, \|u\|_{\infty,r} \le 2\|u_0\|_{L^r}\},$$

instead of E_T, where $\|\cdot\|_{\infty,r} := \|\cdot\|_{L^\infty((0,T),L^r(\mathbb{R}^N))}$, and by estimating $\|u^p\|_{L^r(\mathbb{R}^N)}$ by $\|u\|^{p-1}_{L^\infty(\mathbb{R}^N)}\|u\|_{L^r(\mathbb{R}^N)}$ in the contraction mapping argument, using (3.2.4), we obtain a unique solution in $E_{T,r}$; we conclude then that

$$u \in C([0,T_{\max}), C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)).$$

□

We say that u is a global solution if $T_{\max} = \infty$; when $T_{\max} < \infty$, u is said to blow up in a finite time and in this case we have $\|u(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $t \to T_{\max}$.

3.4 Blow-up of solutions

Now, we want to derive a blow-up result for Eq. (3.1.1). To do this, we show for later use that a mild solution is a weak solution. Hereafter

$$\int_{Q_T} = \int_0^T \int_{\mathbb{R}^N} dx\, dt, \qquad \int_{\mathbb{R}^N} = \int_{\mathbb{R}^N} dx.$$

Definition 3.4.1 (*Weak solution*)
Let $u_0 \in L^\infty_{Loc}(\mathbb{R}^N)$, $0 < \beta \leq 2$ and $T > 0$. We say that u is a weak solution of the problem (3.1.1) if $u \in L^p((0,T), L^\infty_{Loc}(\mathbb{R}^N))$ and verifies the equation

$$\begin{aligned} \int_\Omega u_0(x)\varphi(x,0) + \int_0^T \int_\Omega J^\alpha_{0|t}(|u|^{p-1}u)(x,t)\varphi(x,t) &= \int_0^T \int_\Omega u(x,t)(-\Delta)^{\beta/2}\varphi(x,t) \\ &- \int_0^T \int_\Omega u(x,t)\varphi_t(x,t), \end{aligned} \tag{3.4.1}$$

for all $\varphi \in C^1([0,T], H^\beta(\mathbb{R}^N))$ such that $\Omega :=$suppφ is compact with $\varphi(\cdot\,,T) = 0$, where $\alpha := 1-\gamma \in (0,1)$.

Lemma 3.4.2
Consider $u_0 \in L^\infty(\mathbb{R}^N)$ and let $u \in C([0,T], L^\infty(\mathbb{R}^N))$ be a mild solution of (3.1.1), then u is a weak solution of (3.1.1), for all $0 < \beta \leq 2$ and all $T > 0$.

Proof Let $T > 0$, $0 < \beta \leq 2$, $u_0 \in L^\infty(\mathbb{R}^N)$ and let $u \in C([0,T], L^\infty(\mathbb{R}^N))$ be a solution of (3.3.1). Given $\varphi \in C^1([0,T], H^\beta(\mathbb{R}^N))$ such that supp$\varphi =: \Omega$ is compact with $\varphi(\cdot\,,T) = 0$. Then after multiplying (3.3.1) by φ and integrating over $\mathbb{R}^N$, we have

$$\int_\Omega u(x,t)\varphi(x,t) = \int_\Omega T(t)u_0(x)\varphi(x,t) + \int_\Omega \left(\int_0^t T(t-s)J^\alpha_{0|s}\left(|u|^{p-1}u\right)(x,t)\,ds\right)\varphi(x,t).$$

So after differentiating in time, we obtain

$$\frac{d}{dt}\int_\Omega u(x,t)\varphi(x,t) \;=\; \int_\Omega \frac{d}{dt}\left(T(t)u_0(x)\varphi(x,t)\right)$$

$$+ \quad \int_\Omega \frac{d}{dt}\int_0^t T(t-s)J^\alpha_{0|s}\left(|u|^{p-1}u\right)(x,s)\,ds\varphi(x,t). \tag{3.4.2}$$

Now, using (3.2.5) and a property of the semigroup $T(t)$ ([9, Chapter 3]), we have :

$$\begin{aligned}\int_\Omega \frac{d}{dt}\left(T(t)u_0(x)\varphi(x,t)\right) &= \int_\Omega A\left(T(t)u_0(x)\right)\varphi(x,t) + \int_\Omega T(t)u_0(x)\varphi_t(x,t)\\ &= \int_\Omega T(t)u_0(x)A\varphi(x,t) + \int_\Omega T(t)u_0(x)\varphi_t(x,t),\end{aligned} \tag{3.4.3}$$

and

$$\begin{aligned}\int_\Omega \frac{d}{dt}\int_0^t T(t-s)f(x,s)\,ds\varphi(x,t) &= \int_\Omega f(x,t)\varphi(x,t) + \int_\Omega\int_0^t A\left(T(t-s)f(x,s)\right)\,ds\varphi(x,t)\\ &+ \int_\Omega\int_0^t T(t-s)f(x,s)\,ds\varphi_t(x,t)\\ &= \int_\Omega f(x,t)\varphi(x,t) + \int_\Omega\int_0^t T(t-s)f(x,s)\,dsA\varphi(x,t)\\ &+ \int_\Omega\int_0^t T(t-s)f(x,s)\,ds\varphi_t(x,t),\end{aligned} \tag{3.4.4}$$

where $f := J^\alpha_{0|t}\left(|u|^{p-1}u\right) \in C([0,T];L^2(\Omega))$.
Thus, using (3.3.1), (3.4.3) and (3.4.4), we conclude that (3.4.2) imply

$$\frac{d}{dt}\int_\Omega u(x,t)\varphi(x,t) = \int_\Omega u(x,t)A\varphi(x,t) + \int_\Omega u(x,t)\varphi_t(x,t) + \int_\Omega f(x,t)\varphi(x,t).$$

Finally, we conclude by integrating in time over $[0,T]$ and use the fact that $\varphi(\cdot,T)=0$. □

Theorem 3.4.3 *Let $u_0 \in C_0(\mathbb{R}^N)$ be such that $u_0 \geq 0$ and $u_0 \not\equiv 0$. If*

$$p \leq 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+} := p^* \quad or \quad p < \frac{1}{\gamma}, \tag{3.4.5}$$

for all $\beta \in (0,2]$, then any mild solution to (3.1.1) blows-up in a finite time.
Note that in the case where $p = p^*$ and $\beta \in (0,2)$ we take $p > N/(N-\beta)$ with $N > \beta$.

Proof The proof is by contradiction. Suppose that u is a global mild solution to (3.1.1), then u is a solution of (3.1.1) in $C([0,T],C_0(\mathbb{R}^N))$ for all $T>0$ such that $u(t)>0$ for all $t\in[0,T]$.
Then, using Lemma 3.4.2, we have

$$\begin{aligned}\int_{\text{supp}\varphi} u_0(x)\varphi(x,0) + \int_0^T\int_{\text{supp}\varphi} J^\alpha_{0|t}(u^p)(x,t)\varphi(x,t) &= \int_0^T\int_{\text{supp}\varphi} u(x,t)(-\Delta)^{\beta/2}\varphi(x,t)\\ &- \int_0^T\int_{\text{supp}\varphi} u(x,t)\varphi_t(x,t),\end{aligned}$$

for all test function $\varphi \in C^1([0,T], H^\beta(\mathbb{R}^N))$ such that suppφ is compact with $\varphi(\cdot\,,T)=0$, where $\alpha := 1-\gamma \in (0,1)$.
Now we take $\varphi(x,t) = D^\alpha_{t|T}\left(\tilde{\varphi}(x,t)\right) := D^\alpha_{t|T}\left((\varphi_1(x))^\ell\,\varphi_2(t)\right)$ with $\varphi_1(x) := \Phi\left(|x|/T^{1/\beta}\right)$, $\varphi_2(t) := (1-t/T)^\eta_+$, where $\ell \geq p/(p-1)$, $\eta \geq \max\{(\alpha p+1)/(p-1); \alpha+1\}$ and Φ a smooth nonnegative non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$, $|\Phi'(r)| \leq C_1/r$, for all $r>0$. Using (3.2.15), we then obtain

$$\begin{aligned} &\int_\Omega u_0(x) D^\alpha_{t|T}\tilde{\varphi}(x,0) + \int_{\Omega_T} J^\alpha_{0|t}(u^p)(x,t) D^\alpha_{t|T}\tilde{\varphi}(x,t) \\ &= \int_{Q_T} u(x,t)(-\Delta)^{\beta/2} D^\alpha_{t|T}\tilde{\varphi}(x,t) - \int_{\Omega_T} u(x,t) D D^\alpha_{t|T}\tilde{\varphi}(x,t), \end{aligned} \tag{3.4.6}$$

where

$$\Omega_T := [0,T]\times\Omega \ \text{ for } \ \Omega = \left\{x \in \mathbb{R}^N \ ; \ |x| \leq 2T^{1/\beta}\right\}, \quad \int_\Omega = \int_\Omega dx \ \text{ and } \int_{\Omega_T} = \int_{\Omega_T} dx\,dt.$$

Furthermore, using (3.2.10) and (3.2.15) in the left hand side of (3.4.6), and (3.2.11) in the right hand side, we conclude that

$$\begin{aligned} &C\,T^{-\alpha}\int_\Omega u_0(x)\varphi_1^\ell(x) + \int_{\Omega_T} D^\alpha_{0|t} J^\alpha_{0|t}(u^p)(x,t)\tilde{\varphi}(x,t) \\ &= \int_{Q_T} u(x,t)(-\Delta)^{\beta/2} D^\alpha_{t|T}\tilde{\varphi}(x,t) + \int_{\Omega_T} u(x,t) D^{1+\alpha}_{t|T}\tilde{\varphi}(x,t). \end{aligned} \tag{3.4.7}$$

Moreover, using (3.2.12), we may write

$$\begin{aligned} &\int_{\Omega_T} u^p(x,t)\ \tilde{\varphi}(x,t) + C\,T^{-\alpha}\int_\Omega u_0(x)\varphi_1^\ell(x) \\ &= \int_{Q_T} u(x,t)(-\Delta)^{\beta/2}\varphi_1^\ell(x) D^\alpha_{t|T}\varphi_2(t) + \int_{\Omega_T} u(x,t) D^{1+\alpha}_{t|T}\tilde{\varphi}(x,t). \end{aligned} \tag{3.4.8}$$

So, Ju's inequality $(-\Delta)^{\beta/2}\left(\varphi_1^\ell\right) \leq \ell\varphi_1^{\ell-1}(-\Delta)^{\beta/2}\left(\varphi_1\right)$ (see the Appendix) allows us to write :

$$\begin{aligned} &\int_{\Omega_T} u^p(x,t)\ \tilde{\varphi}(x,t) + C\,T^{-\alpha}\int_\Omega u_0(x)\varphi_1^\ell(x) \\ &\leq C\int_{\Omega_T} u(x,t)\ \varphi_1^{\ell-1}(x)\left|(-\Delta)^{\beta/2}\varphi_1(x) D^\alpha_{t|T}\varphi_2(t)\right| \\ &+ \int_{\Omega_T} u(x,t)\ \varphi_1^\ell(x)\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right| \\ &= C\int_{\Omega_T} u(x,t)\ \tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\varphi_1^{\ell-1}(x)\left|(-\Delta)^{\beta/2}\varphi_1(x) D^\alpha_{t|T}\varphi_2(t)\right| \end{aligned}$$

$$+\int_{\Omega_T} u(x,t)\, \tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\varphi_1^{\ell}(x)\left|D_{t|T}^{1+\alpha}\varphi_2(t)\right| \tag{3.4.9}$$

Therefore, using Young's inequality

$$ab \leq \frac{1}{2p}a^{\,p} + \frac{2^{\widetilde{p}-1}}{\widetilde{p}}\, b^{\,\widetilde{p}} \qquad \text{where } \; p\widetilde{p} = p+\widetilde{p}, \;\; a>0, b>0, \;\; p>1, \widetilde{p}>1, \tag{3.4.10}$$

with

$$\begin{cases} a = u(x,t)\, \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p}\varphi_1^{\ell-1}(x)\left|(-\Delta)^{\beta/2}\varphi_1(x)D_{t|T}^{\alpha}\varphi_2(t)\right|, \end{cases}$$

in the first integral of the right hand side of (3.4.9), and with

$$\begin{cases} a = u(x,t)\, \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p}\varphi_1^{\ell}(x)\left|D_{t|T}^{1+\alpha}\varphi_2(t)\right|, \end{cases}$$

in the second integral of the right hand side of (3.4.9), we obtain

$$\begin{aligned} &(1-\frac{1}{p})\int_{\Omega_T} u^p(x,t)\, \tilde{\varphi}(x,t) \\ &+C\int_{\Omega_T} (\varphi_1(x))^{\ell-\widetilde{p}}\, (\varphi_2(t))^{-\frac{1}{p-1}}\left|(-\Delta_x)^{\beta/2}\varphi_1(x)D_{t|T}^{\alpha}\varphi_2(t)\right|^{\widetilde{p}} \\ &\leq C\int_{\Omega_T} (\varphi_1(x))^{\ell}\, (\varphi_2(t))^{-\frac{1}{p-1}}\left|D_{t|T}^{1+\alpha}\varphi_2(t)\right|^{\widetilde{p}}, \end{aligned} \tag{3.4.11}$$

as $u_0 \geq 0$. At this stage, we introduce the scaled variables : $\tau = T^{-1}t$, $\xi = T^{-1/\beta}x$, use formula (3.2.13) and (3.2.14) in the right hand-side of (3.4.11), to obtain :

$$\int_{\Omega_T} u^p(x,t)\, \tilde{\varphi}(x,t) \leq \; C\, T^{-\delta}, \tag{3.4.12}$$

where $\delta := (1+\alpha)\widetilde{p} - 1 - (N/\beta)$, $C = C(|\Omega_2|\;,\; |\Omega_3|)$, ($|\Omega_i|$ stands for the measure of Ω_i, with

$$\Omega_2 := \left\{\xi \in \mathbb{R}^N \;\; ; \;\; |\xi| \leq 2\right\} \quad , \quad \Omega_3 := \{\tau \geq 0 \;\; ; \;\; \tau \leq 1\}\,.$$

Now, noting that, as

$$p \leq p^* \quad \text{or} \quad p < \frac{1}{\gamma} \quad \Longleftrightarrow \quad \delta \geq 0 \quad \text{or} \quad p < \frac{1}{\gamma}, \tag{3.4.13}$$

we have to distinguish three cases :

• The case $p < p^*$ ($\delta > 0$) : we pass to the limit in (3.4.12), as T goes to ∞; we get

$$\lim_{T\to\infty}\int_0^T\int_{|x|\leq 2T^{1/\beta}} u^p(x,t)\, \tilde{\varphi}(x,t)\, dx\, dt = 0.$$

Using the Lebesgue dominated convergence theorem, the continuity in time and space of u and the fact that $\tilde{\varphi}(x,t) \to 1$ as $T \to \infty$, we infer that

$$\int_0^\infty \int_{\mathbb{R}^N} u^p(x,t)\,dx\,dt = 0 \qquad \Longrightarrow \qquad u \equiv 0.$$

Contradiction.

• The case $p = p^*$ $(\delta = 0)$: using inequality (3.4.12) with $T \to \infty$, taking into account the fact that $p = p^*$, we have on one hand

$$u \in L^p((0,\infty), L^p(\mathbb{R}^N)); \tag{3.4.14}$$

on the other hand, we repeat the same calculation as above by taking this time $\varphi_1(x) := \Phi\left(|x|/(B^{-1/\beta}T^{1/\beta})\right)$, where $1 \leq B < T$ is large enough such that when $T \to \infty$ we don't have $B \to \infty$ in the same time, so we arrive at

$$\int_{\Sigma_T} u^p(x,t)\;\tilde{\varphi}(x,t) \leq \; C\;B^{-N/\beta} \;+\; C\;B^{-N/\beta+\widetilde{p}}, \tag{3.4.15}$$

thanks to the following rescaling : $\tau = T^{-1}t$, $\xi = (T/B)^{-1/\beta}\,x$, where

$$\Sigma_T := [0,T] \times \left\{x \in \mathbb{R}^N \;\;;\;\; |x| \leq 2B^{-1/\beta}T^{1/\beta}\right\} \quad \text{and} \quad \int_{\Sigma_T} = \int_{\Sigma_T} dx\,dt.$$

Thus, using $p > N/(N-\beta)$ and taking the limit when $T \to \infty$ and then $B \to \infty$, we get :

$$\int_0^\infty \int_{\mathbb{R}^N} u^p(x,t)\,dx\,dt = 0 \qquad \Longrightarrow \qquad u \equiv 0,$$

which is a contradiction.

Note that, in the case $\beta = 2$ it is not necessary to take the condition $p > N/(N-\beta)$ with $N > \beta$. Indeed, from (3.4.9) with the new function φ_1, we may write

$$\begin{aligned}
&\int_{\Sigma_T} u^p(x,t)\;\tilde{\varphi}(x,t)\\
&\leq C\int_{\Sigma_T} u(x,t)\;\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\left(\varphi_1(x)\right)^{\ell}\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|\\
&+ C\int_{\Delta_B} u(x,t)\;\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\;\left(\varphi_1(x)\right)^{\ell-1}\left|\Delta_x\varphi_1(x)\;D^{\alpha}_{t|T}\varphi_2(t)\right|,
\end{aligned} \tag{3.4.16}$$

where

$$\Delta_B = [0,T] \times \left\{x \in \mathbb{R}^N \;\;;\;\; B^{-1/2}T^{1/2} \leq |x| \leq 2B^{-1/2}T^{1/2}\right\} \subset \Sigma_T \;\text{ and }\; \int_{\Delta_B} = \int_{\Delta_B} dx\,dt.$$

Moreover, using the following Young's inequality

$$ab \leq \;\frac{1}{p}a^{\,p} \;+\; \frac{1}{\widetilde{p}}\,b^{\,\widetilde{p}} \qquad \text{where}\;\; p\widetilde{p} = p + \widetilde{p},\;\; a > 0, b > 0,\;\; p > 1, \widetilde{p} > 1, \tag{3.4.17}$$

with

$$\begin{cases} a = u(x,t)\ \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p}\,(\varphi_1(x))^{\ell}\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|, \end{cases}$$

in the first integral of the right hand side of (3.4.16), and using Hölder's inequality

$$\int_{\Delta_B} ab \le \left(\int_{\Delta_B} a^{\,p}\right)^{1/p}\ \left(\int_{\Delta_B} b^{\,\widetilde{p}}\right)^{1/\widetilde{p}}, \qquad \text{where } p\widetilde{p} = p+\widetilde{p},\ \ a>0, b>0,\ \ p>1, \widetilde{p}>1,$$

with

$$\begin{cases} a = u(x,t)\ \tilde{\varphi}^{1/p}, \\ b = \tilde{\varphi}^{-1/p}\ \ (\varphi_1(x))^{\ell-1}\left|\Delta_x\varphi_1(x)\ D^{\alpha}_{t|T}\varphi_2(t)\right|, \end{cases}$$

in the second integral of the right hand side of (3.4.16), we obtain

$$\begin{aligned} &(1-\frac{1}{p})\int_{\Sigma_T} u^p(x,t)\ \tilde{\varphi}(x,t) \\ &\le C\int_{\Sigma_T} (\varphi_1(x))^{\ell}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}} \\ &+C\left(\int_{\Delta_B} u^p\ \tilde{\varphi}\right)^{1/p}\left(\int_{\Delta_B}(\varphi_1(x))^{\ell-\widetilde{p}}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|\Delta_x\varphi_1(x)D^{\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}}\right)^{1/\widetilde{p}}. \end{aligned} \tag{3.4.18}$$

Taking account of the scaled variables : $\tau = T^{-1}t$, $\xi = (T/B)^{-1/2}\,x$, and the fact that $\delta = 0$, we get

$$\int_{\Sigma_T} u^p(x,t)\ \tilde{\varphi}(x,t) \le\ C\ B^{-N/2}\ +\ C\ B^{-\frac{N}{2\widetilde{p}}+1}\left(\int_{\Delta_B} u^p\ \tilde{\varphi}\right)^{1/p}. \tag{3.4.19}$$

Now, as

$$\lim_{T\to\infty}\left(\int_{\Delta_B} u^p\ \tilde{\varphi}\right)^{1/p} = 0 \quad (\text{from } (3.4.14)),$$

then, passing to the limit in (3.4.19), as $T\to\infty$, we get

$$\int_0^{\infty}\int_{\mathbb{R^N}} u^p(x,t)\,dx\,dt \le\ C\ B^{-N/2}.$$

We conclude that $u\equiv 0$ by taking the limit when B goes to infinity, contradiction.

• For the case $p\ <(1/\gamma)$, we repeat the same argument as in the case $p<p^*$ by choosing the test function as follows : $\varphi(x,t) = D^{\alpha}_{t|T}\overline{\varphi}(x,t) := D^{\alpha}_{t|T}\left(\varphi_3^{\ell}(x)\varphi_4(t)\right)$ where $\varphi_3(x) = \Phi\left(|x|/R\right)$, $\varphi_4(t) = (1-t/T)_+^{\eta}$ and $R\in(0,T)$ large enough such that in the case when $T\to\infty$ we don't have $R\to\infty$ at the same time, with the same functions Φ as above. We then obtain

$$\begin{aligned} &\int_{\mathcal{C}_T} u^p(x,t)\ \overline{\varphi}(x,t)\ +\ C\,T^{-\alpha}\int_{\mathcal{C}}(\varphi_3(x))^{\ell}\,u_0(x) \\ &\le\ C\int_{\mathcal{C}_T} u(x,t)\ \overline{\varphi}^{1/p}\overline{\varphi}^{-1/p}\,(\varphi_3(x))^{\ell}\left|D^{1+\alpha}_{t|T}\varphi_4(t)\right| \end{aligned}$$

$$+ C\int_{\mathcal{C}_T} u(x,t)\,\overline{\varphi}^{1/p}\overline{\varphi}^{-1/p}(\varphi_3(x))^{\ell-1}\left|(-\Delta_x)^{\beta/2}\varphi_3(x)\; D^{\alpha}_{t|T}\varphi_4(t)\right|, \tag{3.4.20}$$

where

$$\mathcal{C}_T := [0,T]\times\mathcal{C} \quad \text{for} \quad \mathcal{C} := \left\{x\in\mathbb{R}^N \;\; ; \;\; |x|\leq 2R\right\}, \quad \int_{\mathcal{C}} = \int_{\mathcal{C}} dx \;\; \text{and} \;\; \int_{\mathcal{C}_T} = \int_{\mathcal{C}_T} dx\,dt.$$

Now, by Young's inequality (3.4.10), with

$$\begin{cases} a = u(x,t)\,\overline{\varphi}^{1/p}, \\ b = \overline{\varphi}^{-1/p}\,(\varphi_3(x))^{\ell}\left|D^{1+\alpha}_{t|T}\varphi_4(t)\right|, \end{cases}$$

in the first integral of the right hand side of (3.4.20) and with

$$\begin{cases} a = u(x,t)\,\overline{\varphi}^{1/p}, \\ b = \overline{\varphi}^{-1/p}(\varphi_3(x))^{\ell-1}\left|(-\Delta_x)^{\beta/2}\varphi_3(x)\; D^{\alpha}_{t|T}\varphi_4(t)\right|, \end{cases}$$

in the second integral of the right hand side of (3.4.20) and using the positivity of u_0, we get

$$\begin{aligned} (1-\frac{1}{p})\int_{\mathcal{C}_T} u^p(x,t)\,\overline{\varphi}(x,t) \;&\leq\; C\int_{\mathcal{C}_T}(\varphi_3(x))^{\ell}\,(\varphi_4(t))^{-\frac{1}{p-1}}\left|D^{1+\alpha}_{t|T}\varphi_4(t)\right|^{\widetilde{p}} \\ &+\; C\int_{\mathcal{C}_T}(\varphi_3(x))^{\ell-\widetilde{p}}\,(\varphi_4(t))^{-\frac{1}{p-1}}\left|(-\Delta_x)^{\beta/2}\varphi_3 D^{\alpha}_{t|T}\varphi_4\right|^{\widetilde{p}}. \end{aligned}$$

Then, the new variables $\xi = R^{-1}x$, $\tau = T^{-1}t$ and (3.2.13) – (3.2.14) allow us to write the estimate

$$\int_{\mathcal{C}_T} u^p(x,t)\,\overline{\varphi}(x,t)\,dx\,dt \leq\; C\,T^{1-(1+\alpha)\widetilde{p}}\,R^N \;+\; C\,T^{1-\alpha\widetilde{p}}\,R^{N-\beta\widetilde{p}}.$$

Taking the limit as $T\to\infty$, we infer, as $p < 1/\gamma \iff 1-\alpha\widetilde{p} < 0$, that

$$\int_0^{\infty}\int_{\mathcal{C}} u(x,t)^p\,(\varphi_3(x))^{\ell}\;dx\,dt = 0.$$

Finally, by taking $R\to\infty$, we get a contradiction as $u(x,t) > 0$ for all $x\in\mathbb{R}^N$, $t>0$. □

Remark 3.4.4
(**1**) *We can extend our analysis to the equation*

$$u_t = -(-\Delta)^{\beta/2}u + \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\psi(x,s)|u(s)|^{p-1}u(s)}{(t-s)^{\gamma}}\,ds, \qquad x\in\mathbb{R}^N, \tag{3.4.21}$$

where $p>1$, $\beta\in(0,2]$, $0<\gamma<1$ *and* $\psi\in L^1_{Loc}(\mathbb{R}^N\times(0,\infty))$, $\psi(\cdot,t)>0$ *for all* $t>0$,

$$\begin{cases} \psi(B^{-1/\beta}T^{1/\beta}\xi, T\tau) \geq C > 0 & \textit{if } p\leq p^* \\ \psi(R\xi, T\tau)\geq C>0 & \textit{if } p<1/\gamma, \end{cases}$$

for any $0 < R, B < T$, $\tau \in [0,1]$ *and* $\xi \in [0,2]$.

(**2**) *If we take* $\beta = 2$ *and* $v(x,t) = (\Gamma(1-\gamma))^{(1-\gamma)/(p-1)} u(\Gamma(1-\gamma)^{1/2}x, \Gamma(1-\gamma)t)$ *where* u *is a solution of* (3.1.1), *we recover the result in* [10] *as a particular case.*

(**3**) *We can take the nonlocal porous-medium spatio-fractional problem which is the real our motivation to extend the paper of* [10] :

$$\begin{cases} u_t + (-\Delta)^{\beta/2}|u|^{m-1}u = \dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t (t-s)^{-\gamma}|u|^{p-1}u(s)\,ds & x \in \mathbb{R}^N, t > 0, \\ \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N, \end{cases}$$

where $\beta \in (0,2]$, $0 < \gamma < 1$, $1 \leq m < p$, $u_0 \geq 0$ *and* $u_0 \not\equiv 0$.
The threshold on p *will be*

$$p \leq 1 + \frac{(2-\gamma)(N(m-1)+\beta)}{(N-\beta+\beta\gamma)_+} \qquad or \qquad p < \frac{m}{\gamma}.$$

3.5 Blow-up Rate

In this section, we present the blow-up rate for the blowing-up solutions to the parabolic problem (3.1.1) in the case $\beta = 2$.
We take the solution of (3.1.1) with an initial condition satisfying

$$u_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \quad u(\cdot\,,0) = u_0 \geq 0, \quad u_0 \not\equiv 0. \tag{3.5.1}$$

The following lemma will be used in the proof of Theorem 3.5.2 below.

Lemma 3.5.1 *Let* φ *be a nonnegative classical solution of*

$$\varphi_t = \Delta\varphi + J^{1-\gamma}_{-\infty|t}(\varphi^p) \qquad in\ \mathbb{R}^N \times \mathbb{R}, \tag{3.5.2}$$

where $\gamma \in (0,1)$, $p > 1$ *and*

$$J^{1-\gamma}_{-\infty|t}(\varphi^p)(t) := \frac{1}{\Gamma(1-\gamma)}\int_{-\infty}^t (t-s)^{-\gamma}\varphi^p(s)\,ds.$$

Then $\varphi \equiv 0$ *whenever*

$$p \leq p^* \quad or \quad p < \frac{1}{\gamma}. \tag{3.5.3}$$

Proof We repeat the same computations as in Theorem 3.4.3 with $(1 - t^2/T^2)_+^\eta$ instead of $(1-t/T)_+^\eta$ for $\eta \gg 1$, using (3.2.16) – (3.2.18) and taking account of the following inequality

$$J^{1-\gamma}_{-\infty|t}(\varphi^p) \geq J^{1-\gamma}_{-T|t}(\varphi^p).$$

Moreover, we take $\varphi_1^{\ell/p}\varphi_1^{-\ell/p}$ instead of $\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}$ (resp. $\overline{\varphi}^{1/p}\overline{\varphi}^{-1/p}$) in (3.4.9), (3.4.16) (resp. in (3.4.20)) for $\ell \gg 1$ to use the Young and Hölder's inequality.
Note that here, we use rather the $\varepsilon-$Young inequality

$$ab \leq \frac{\varepsilon}{2}a^p + C(\varepsilon)b^{\tilde{p}},$$

for $0 < \varepsilon < 1$. □

Theorem 3.5.2 *Let u_0 satisfies (3.5.1). For $p \leq p^*$ or $p < (1/\gamma)$, let $\alpha_1 := (2-\gamma)/(p-1)$ and let u be the blowing-up mild solution of (3.1.1) in a finite time $T_{\max} := T^*$. Then there exist two constants $c, C > 0$ such that*

$$c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(\cdot\,, t) \leq C(T^* - t)^{-\alpha_1}, \qquad t \in (0, T^*). \tag{3.5.4}$$

Proof The proof is split into two parts :

• The upper blow-up rate estimate. Let

$$M(t) := \sup_{\mathbb{R}^N \times (0,t]} u, \qquad t \in (0, T^*).$$

Clearly, M is positive, continuous and nondecreasing on $(0, T^*)$. As $\lim_{t \to T^*} M(t) = \infty$, then for all $t_0 \in (0, T^*)$, we can define

$$t_0^+ := t^+(t_0) := \max\{t \in (t_0, T^*) :\; M(t) = 2M(t_0)\}.$$

Choose $A \geq 1$ and let

$$\lambda(t_0) := \left(\frac{1}{2A}M(t_0)\right)^{-1/(2\alpha_1)}. \tag{3.5.5}$$

we claim that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \leq D, \qquad t_0 \in \left(\frac{T^*}{2}, T^*\right), \tag{3.5.6}$$

where $D > 0$ is a positive constant which does not depend on t_0.
We proceed by contradiction. If (3.5.6) were false, then there would exist a sequence $t_n \to T^*$ such that

$$\lambda_n^{-2}(t_n^+ - t_n) \longrightarrow \infty,$$

where $\lambda_n = \lambda(t_n)$ and $t_n^+ = t^+(t_n)$. For each t_n choose

$$(\widehat{x}_n, \widehat{t}_n) \in \mathbb{R}^N \times (0, t_n] \quad \text{such that} \quad u(\widehat{x}_n, \widehat{t}_n) \geq \frac{1}{2}M(t_n). \tag{3.5.7}$$

Obviously, $M(t_n) \to \infty$; hence, $\widehat{t}_n \to T^*$. Next, rescale the function u as

$$\varphi^{\lambda_n}(y, s) := \lambda_n^{2\alpha_1} u(\lambda_n y + \widehat{x}_n, \lambda_n^2 s + \widehat{t}_n), \qquad (y, s) \in \mathbb{R}^N \times I_n(T^*), \tag{3.5.8}$$

where $I_n(t) := (-\lambda_n^{-2}\widehat{t}_n, \lambda_n^{-2}(t-\widehat{t}_n))$ for all $t>0$. Then φ^{λ_n} is a mild solution of

$$\varphi_s = \Delta\varphi + J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|s}(\varphi^p) \qquad \text{in } \mathbb{R}^N \times I_n(T^*), \tag{3.5.9}$$

i.e., for $G(t) := G(x,t) := (4\pi t)^{-N/2} e^{-|x|^2/4t}$ and $*$ being the space convolution, we have

$$\varphi^{\lambda_n}(s) = G(s+\lambda_n^{-2}\widehat{t}_n) * \varphi^{\lambda_n}(-\lambda_n^{-2}\widehat{t}_n) + \int_{-\lambda_n^{-2}\widehat{t}_n}^{s} G(s-\sigma) * J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|\sigma}((\varphi^{\lambda_n})^p)\, d\sigma \tag{3.5.10}$$

in $\mathbb{R}^N \times I_n(T^*)$; whereupon, as $\varphi^{\lambda_n}(0,0) \geq A$,

$$0 \leq \varphi^{\lambda_n} \leq \lambda_n^{2\alpha_1} M(t_n^+) = \lambda_n^{2\alpha_1} 2M(t_n) = 4A \qquad \text{in } \mathbb{R}^N \times I_n(t_n^+),$$

thanks to (3.5.5) and the definition of t_n^+.
Moreover, as

$$\varphi^{\lambda_n} \in C([-\lambda_n^{-2}\widehat{t}_n, T], C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)) \quad \text{for all } T \in I_n(T^*),$$

so, as in Lemma 3.4.2, φ^{λ_n} is a weak solution of (3.5.9).
On the other hand, if we write φ^{λ_n} as $\varphi^{\lambda_n}(s) = v(s) + w(s)$ for all $s \in I_n(T^*)$, where

$$v(s) := G(s+\lambda_n^{-2}\widehat{t}_n) * \varphi^{\lambda_n}(-\lambda_n^{-2}\widehat{t}_n) \quad \text{and} \quad w(s) := \int_{-\lambda_n^{-2}\widehat{t}_n}^{s} G(s-\sigma) * J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|\sigma}((\varphi^{\lambda_n})^p)\, d\sigma,$$

we have, see [9, Chapter 3], for $T \in I_n(T^*)$

$$v \in C((-\lambda_n^{-2}\widehat{t}_n, T); H^2(\mathbb{R}^N)) \cap C^1((-\lambda_n^{-2}\widehat{t}_n, T); L^2(\mathbb{R}^N)) \subset L^2((-\lambda_n^{-2}\widehat{t}_n, T); H^1(\mathbb{R}^N))$$

and, using the fact that $f(s) := J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|s}((\varphi^{\lambda_n})^p) \in L^2((-\lambda_n^{-2}\widehat{t}_n, T); L^2(\mathbb{R}^N))$ and the maximal regularity theory, we have

$$w \in W^{1,2}((-\lambda_n^{-2}\widehat{t}_n, T); L^2(\mathbb{R}^N)) \cap L^2((-\lambda_n^{-2}\widehat{t}_n, T); H^2(\mathbb{R}^N)) \subset L^2((-\lambda_n^{-2}\widehat{t}_n, T); H^1(\mathbb{R}^N)).$$

It follows that

$$\varphi^{\lambda_n} \in C([-\lambda_n^{-2}\widehat{t}_n, T], L^2(\mathbb{R}^N)) \cap L^2((-\lambda_n^{-2}\widehat{t}_n, T), W^{1,2}(\mathbb{R}^N));$$

so from the parabolic interior regularity theory (cf. [38, Theorem 10.1 p. 204]) there is $\mu \in (0,1)$ such that the sequence φ^{λ_n} is bounded in the $C^{\mu,\mu/2}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$-norm by a constant that does not depend on n, where $C^{\mu,\mu/2}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$ is the locally Hölder space defined in [38]. Similar uniform estimates for $J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|s}(\varphi^p)$ follow if μ is sufficiently small. The parabolic interior Schauder's estimates (see [37, Th. 8.11.1 p. 130]), using the existence theorems in Hölder's space, imply now that the $C^{2+\mu,1+\mu/2}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$-norm of φ^{λ_n} is uniformly bounded. Hence, we obtain a subsequence converging in $C^{2+\mu,1+\mu/2}_{loc}(\mathbb{R}^N \times \mathbb{R})$ to a solution φ of

$$\varphi_s = \Delta\varphi + J^{\alpha}_{-\infty|s}(\varphi^p) \qquad \text{in } \mathbb{R}^N \times (-\infty, +\infty),$$

such that $\varphi(0,0) \geq A$ and $0 \leq \varphi \leq 4A$ in $\mathbb{R}^N \times \mathbb{R}$. Wherefrom, using Lemma 3.5.1, we infer that $\varphi \equiv 0$ in $\mathbb{R}^N \times (-\infty, +\infty)$. Contradiction with the fact that $\varphi(0,0) \geq A > 1$. This proves (3.5.6). Next we use an idea from Hu [24]. From (3.5.5) and (3.5.6) it follows that

$$(t_0^+ - t_0) \leq D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \qquad \text{for any } t_0 \in \left(\frac{T^*}{2}, T^*\right).$$

Fix $t_0 \in (T^*/2, T^*)$ and denote $t_1 = t_0^+, t_2 = t_1^+, t_3 = t_2^+, \ldots$. Then

$$\begin{aligned} t_{j+1} - t_j &\leq D(2A)^{1/\alpha_1} M(t_j)^{-1/\alpha_1}, \\ M(t_{j+1}) &= 2M(t_j), \end{aligned}$$

$j = 0, 1, 2, \ldots$. Consequently,

$$\begin{aligned} T^* - t_0 &= \sum_{j=0}^{\infty} (t_{j+1} - t_j) \leq D(2A)^{1/\alpha_1} \sum_{j=0}^{\infty} M(t_j)^{-1/\alpha_1} \\ &= D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \sum_{j=0}^{\infty} 2^{-j/\alpha_1}. \end{aligned}$$

Finally, we conclude that

$$u(x, t_0) \leq M(t_0) \leq C(T^* - t_0)^{-\alpha_1}, \qquad \forall\, t_0 \in (0, T^*)$$

where

$$C = 2A\left(D\sum_{j=0}^{\infty} 2^{-j/\alpha_1}\right)^{\alpha_1};$$

consequently

$$\sup_{\mathbb{R}^N} u(\cdot\,, t) \leq C(T^* - t)^{-\alpha_1}, \qquad \forall\, t \in (0, T^*).$$

• The lower blow-up rate estimate. If we repeat the proof of the local existence of Theorem 3.3.2, by taking $\|u\|_1 \leq \theta$ instead of $\|u\|_1 \leq 2\|u_0\|_\infty$ in the space E_T for all positive constant $\theta > 0$ and all $0 < t < T$, then the condition (3.3.2) of T will be :

$$\|u_0\|_\infty + CT^{2-\gamma}\theta^p \leq \theta, \tag{3.5.11}$$

and then, like before, we infer that $\|u(t)\|_\infty \leq \theta$ for (almost) all $0 < t < T$. Consequently, if $\|u_0\|_\infty + Ct^{2-\gamma}\theta^p \leq \theta$, then $\|u(t)\|_\infty \leq \theta$. Applying this to any point in the trajectory, we see that if $0 \leq s < t$ and

$$(t - s)^{2-\gamma} \leq \frac{\theta - \|u(s)\|_\infty}{C\theta^p}, \tag{3.5.12}$$

then $\|u(t)\|_\infty \leq \theta$, for all $0 < t < T$.
Moreover, if $0 \leq s < T^*$ and $\|u(s)\|_\infty < \theta$, then :

$$(T^* - s)^{2-\gamma} > \frac{\theta - \|u(s)\|_\infty}{C\theta^p}. \tag{3.5.13}$$

Indeed, arguing by contradiction and assume that for some $\theta > \|u(s)\|_\infty$ and all $t \in (s, T^*)$ we have

$$(t-s)^{2-\gamma} \leq \frac{\theta - \|u(s)\|_\infty}{C\theta^p},$$

then, using (3.5.12), we infer that $\|u(t)\|_\infty \leq \theta$ for all $t \in (s, T^*)$; this contradicts the fact that $\|u(t)\|_\infty \to \infty$ as $t \to T^*$.
Next, for example, by setting $\theta = 2\|u(s)\|_\infty$ in (3.5.13), we see that for $0 < s < T^*$ we have :

$$(T^* - s)^{2-\gamma} > C^{'} \|u(s)\|_\infty^{1-p},$$

and by the positivity and the continuity of u we get

$$c(T^* - s)^{-\alpha_1} < \sup_{x \in \mathbb{R}^N} u(x, s), \qquad \forall\, s \in (0, T^*). \tag{3.5.14}$$

□

3.6 Global existence

In this section, we prove the existence of global solutions of (3.1.1) with initial data small enough. We give a similar proof as that in [10] just for the seek of completeness. In the following, we use the notation $p_{sc} := N(p-1)/\beta(2-\gamma)$. As $p^* > 1+\beta(2-\gamma)/N$, we note that $p > p^* \Rightarrow p_{sc} > 1$.

Theorem 3.6.1 *Let $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$ and $0 < \beta \leq 2$. If*

$$p > \max\{\frac{1}{\gamma}; p^*\}, \tag{3.6.1}$$

and $\|u_0\|_{L^{p_{sc}}}$ is sufficiently small, then u exists globally.
Note that we can take $|u_0(x)| \leq C|x|^{-\beta(2-\gamma)/(p-1)}$ instead of $u_0 \in L^{p_{sc}}$.

Proof As $p > (1/\gamma)$, then we have the possibility to take a positive constant $q > 0$ so that :

$$\frac{2-\gamma}{p-1} - \frac{1}{p} < \frac{N}{\beta q} < \frac{1}{p-1}, \quad q \geq p. \tag{3.6.2}$$

It follows, using (3.6.1), that

$$q > \frac{N(p-1)}{\beta} > p_{sc} > 1. \tag{3.6.3}$$

Let

$$b := \frac{N}{\beta p_{sc}} - \frac{N}{\beta q} = \frac{2-\gamma}{p-1} - \frac{N}{\beta q}. \tag{3.6.4}$$

Then, using (3.6.2) − (3.6.4), we conclude that

$$b > \frac{1-\gamma}{p-1} > 0, \quad pb < 1, \quad \frac{N(p-1)}{\beta q} + (p-1)b + \gamma = 2. \tag{3.6.5}$$

As $u_0 \in L^{p_{sc}}$, using (3.2.4) and (3.6.4), we get, for all $t > 0$,

$$\sup_{t>0} t^b \|e^{-t(-\Delta)^{\beta/2}} u_0\|_{L^q} \leq C\|u_0\|_{L^p_{sc}} = \eta < \infty. \tag{3.6.6}$$

Set

$$\Xi := \left\{ u \in L^\infty((0,\infty), L^q(\mathbb{R}^N));\ \sup_{t>0} t^b \|u(t)\|_{L^q} \leq \delta \right\}, \tag{3.6.7}$$

where $\delta > 0$ is to be chosen sufficiently small. If we define

$$d_\Xi(u,v) := \sup_{t>0} t^b \|u(t) - v(t)\|_{L^q}, \qquad \forall u, v \in \Xi, \tag{3.6.8}$$

then (Ξ, d) is a complete metric space. Given $u \in \Xi$, let's set :

$$\Phi(u)(t) := e^{-t(-\Delta)^{\beta/2}} u_0 + \frac{1}{\Gamma(1-\gamma)} \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}} \int_0^s (s-\sigma)^{-\gamma} |u|^{p-1} u(\sigma)\, d\sigma\, ds, \tag{3.6.9}$$

for all $t \geq 0$. We have by (3.2.4), (3.6.6) and (3.6.7)

$$\begin{aligned} t^b \|\Phi(u)(t)\|_{L^q} &\leq \eta + Ct^b \int_0^t (t-s)^{-\frac{N}{\beta}(\frac{p}{q}-\frac{1}{q})} \int_0^s (s-\sigma)^{-\gamma} \|u^p(\sigma)\|_{L^{\frac{q}{p}}}\, d\sigma\, ds \\ &\leq \eta + C\delta^p t^b \int_0^t \int_0^s (t-s)^{-\frac{N(p-1)}{\beta q}} (s-\sigma)^{-\gamma} \sigma^{-bp}\, d\sigma\, ds. \end{aligned} \tag{3.6.10}$$

Next, using (3.6.2) and $pb < 1$, we get

$$\begin{aligned} \int_0^t \int_0^s \frac{(t-s)^{-\frac{N}{\beta q}(p-1)}}{(s-\sigma)^\gamma} \sigma^{-bp}\, d\sigma\, ds &= \left(\int_0^1 (1-\sigma)^{-\gamma} \sigma^{-bp}\, d\sigma \right) \int_0^t \frac{(t-s)^{-\frac{N(p-1)}{\beta q}}}{s^{bp+\gamma-1}}\, ds \\ &= Ct^{-\frac{N(p-1)}{\beta q} - bp - \gamma + 2} = Ct^{-b}, \end{aligned} \tag{3.6.11}$$

for all $t \geq 0$. So, we deduce from (3.6.10) – (3.6.11) that

$$t^b \|\Phi(u)(t)\|_{L^q} \leq \eta + C\delta^p. \tag{3.6.12}$$

Therefore, if η and δ are chosen small enough so that $\eta + C\delta^p \leq \delta$, we see that $\Phi : \Xi \to \Xi$. Similar calculations show that (assuming η and δ small enough) Φ is a strict contraction, so it has a unique fixed point $u \in \Xi$ which is a solution of (3.1.1). Now, we show that $u \in C([0,\infty), C_0(\mathbb{R}^N))$.

First, we show that $u \in C([0,T], C_0(\mathbb{R}^N))$ if $T > 0$ is sufficiently small. Indeed, note that the above argument shows uniqueness in Ξ_T, where, for any $T > 0$,

$$\Xi_T := \left\{ u \in L^\infty((0,T), L^q(\mathbb{R}^N));\ \sup_{0<t<T} t^b \|u(t)\|_{L^q} \leq \delta \right\}.$$

Let $\widetilde{u}$ be the local solution of (3.1.1) constructed in Theorem 3.3.2. Since $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, then, using the fact that $u_0 \in L^q(\mathbb{R}^N)$ and (3.6.3), we have $\widetilde{u} \in C([0,T_{\max}), L^q(\mathbb{R}^N))$ by Theorem 3.3.2. It follows that $\sup_{0<t<T} t^b \|\widetilde{u}(t)\|_{L^q} \leq \delta$ if $T > 0$ is sufficiently small. Therefore, by uniqueness, $u = \widetilde{u}$ on $[0,T]$, so that $u \in C([0,T], C_0(\mathbb{R}^N))$.

Next, we show that $u\in C([T,\infty),C_0(\mathbb{R}^N))$ by a bootstrap argument. Indeed, for $t>T$, we write

$$\begin{aligned} u(t)-e^{-t(-\Delta)^{\beta/2}}u_0 &= \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}}\int_0^T (s-\sigma)^{-\gamma}|u|^{p-1}u(\sigma)\,d\sigma\,ds\\ &+ \int_0^t e^{-(t-s)(-\Delta)^{\beta/2}}\int_T^s (s-\sigma)^{-\gamma}|u|^{p-1}u(\sigma)\,d\sigma\,ds\\ &\equiv I_1(t)+I_2(t). \end{aligned}$$

Since $u\in C([0,T],C_0(\mathbb{R}^N))$, it follows that $I_1\in C([T,\infty),C_0(\mathbb{R}^N))$. Also, by the calculations used to construct the fixed point, using the fact that $t^{-b}\le T^{-b}<\infty$ and $pq>q$, $I_1\in C([T,\infty),L^q(\mathbb{R}^N))$. Next, note that $N(p/q-1/q)/\beta<1$ by (3.6.3). Therefore, there exists $r\in(q,\infty]$ such that

$$\frac{N}{\beta}\Big(\frac{p}{q}-\frac{1}{r}\Big)<1. \tag{3.6.13}$$

Let $T<s<t$ (the case of $s\le T\le t$ is obvious). Since $u\in L^\infty((0,\infty),L^q(\mathbb{R}^N))$, we have $|u|^{p-1}u\in L^\infty((T,s),L^{q/p}(\mathbb{R}^N))$, and it easily follows, using (3.2.4) and (3.6.13), that $I_2\in C([T,\infty),L^r(\mathbb{R}^N))$. As the terms $e^{-\cdot(-\Delta)^{\beta/2}}u_0$ and I_1 both belong to $C([T,\infty),C_0(\mathbb{R}^N))\cap C([T,\infty),L^q(\mathbb{R}^N))$, we see that $u\in C([T,\infty),L^r(\mathbb{R}^N))$. Iterating this procedure a finite number of times, we deduce that $u\in C([T,\infty),C_0(\mathbb{R}^N))$. This completes the proof.

3.7 Necessary conditions for local and global existence

In this section, we establish necessary conditions for the existence of local or global solutions to the problem (3.1.1); these conditions depend on the behavior of the initial condition for large x.

Theorem 3.7.1 (*Necessary conditions for global existence*)
Let $u_0\in L^\infty(\mathbb{R}^N)$, $u_0\ge 0$, $0<\beta\le 2$ and $p>1$. If u is a global mild solution to problem (3.1.1), then there is a positive constant $C>0$ such that

$$\liminf_{|x|\to\infty}\big(u_0(x)|x|^{\frac{\beta(2-\gamma)}{p-1}}\big)\le C. \tag{3.7.1}$$

Proof Let u be a global mild solution to (3.1.1), then $u\in C([0,R^\beta],L^\infty(B_{2R}))$ for all $R>0$ sufficiently large, where B_{2R} stands for the closed ball of center 0 and radius $2R$. So, we repeat the same calculation as in the proof of Theorem 3.3.2 (here in bounded domain) by taking $\varphi(x,t):=D^\alpha_{t|T}\tilde\varphi(x,t):=D^\alpha_{t|T}\left(\varphi_1(x/R)\varphi_2(t)\right)$ instead of the one chosen in Theorem 3.3.2, where $0\le\varphi_1\in D(\Delta_D^{\beta/2})$ is the first eigenfunction of the fractional Laplacian operator $\Delta_D^{\beta/2}$ in B_2, with homogeneous Dirichlet boundary condition (see Lemma 3.2.1), associated to the first eigenvalue $\lambda:=\lambda_1^{\beta/2}$, and $\varphi_2(t):=\left(1-t/R^\beta\right)_+^\ell$ for $\ell\gg 1$ large enough.
Then, as for the estimate (3.4.11), we obtain, with $\Sigma:=[0,R^\beta]\times B_{2R}$,

$$\int_\Sigma u^p\,\tilde\varphi\,dx\,dt+C\,R^{-\alpha\beta}\int_{|x|\le 2R}u_0(x)\varphi_1(x/R)\,dx$$

$$\leq C\int_{\Sigma}\varphi_1(x/R)\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|D^{1+\alpha}_{t|R^\beta}\varphi_2(t)\right|^{\widetilde{p}}\,dx\,dt$$
$$+C\int_{\Sigma}(\varphi_1(x/R))^{-\frac{1}{p-1}}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|\Delta_D^{\beta/2}\varphi_1(x/R)D^{\alpha}_{t|R^\beta}\varphi_2(t)\right|^{\widetilde{p}}\,dx\,dt, \qquad (3.7.2)$$

where $\alpha := 1-\gamma$ and $\widetilde{p} := p/(p-1)$. If we take the scaled variables $\tau = t/R^\beta$, $\xi = x/R$ and use the fact that $\Delta_D^{\beta/2}\varphi_1(x/R) = R^{-\beta}\lambda\varphi_1(x/R)$ in the right-hand side of (3.7.2), take into account the positivity of u, we infer that

$$C\;R^{-\alpha\beta}\int_{|\xi|\leq 2}u_0(R\xi)\varphi_1(\xi)\,d\xi$$
$$\leq C(R)\int_{|\xi|\leq 2}\varphi_1(\xi)\,d\xi$$
$$= C(R)\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(\widetilde{p}-1)}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\varphi_1(\xi)\,d\xi$$
$$\leq C(R)(2R)^{\beta(1+\alpha)(\widetilde{p}-1)}\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\varphi_1(\xi)\,d\xi$$

where $\overline{C}(R) = R^{\beta-(1+\alpha)\beta\widetilde{p}}(C+C\lambda)$, and so

$$\int_{|\xi|\leq 2}u_0(R\xi)\varphi_1(\xi)\,d\xi \leq C\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\varphi_1(\xi)\,d\xi. \qquad (3.7.3)$$

Using the estimate

$$\inf_{|\xi|>1}\left(u_0(R\xi)|R\xi|^{\beta(1+\alpha)(\widetilde{p}-1)}\right)\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\varphi_1(\xi)\,d\xi \;\leq\; \int_{1<|\xi|\leq 2}u_0(R\xi)\varphi_1(\xi)\,d\xi \;\leq\; \int_{|\xi|\leq 2}u_0(R\xi)\varphi_1(\xi)\,d\xi$$

in the left-hand side of (3.7.3), we conclude, after dividing by $\int_{|\xi|\leq 2}|R\xi|^{\beta(1+\alpha)(1-\widetilde{p})}\varphi_1(\xi)\,d\xi$ that

$$\inf_{|\xi|>1}\left(u_0(R\xi)|R\xi|^{\beta(1+\alpha)(\widetilde{p}-1)}\right) \leq C. \qquad (3.7.4)$$

Passing to the limit in (3.7.4), as $R\to\infty$, we obtain

$$\liminf_{|x|\to\infty}\left(u_0(x)|x|^{\beta(1+\alpha)(\widetilde{p}-1)}\right) \leq C.$$

□

Corollary 3.7.2 (*sufficient conditions for the nonexistence of global solutions*)
Let $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, $0 < \beta \leq 2$ and $p > 1$. If

$$\liminf_{|x|\to\infty}\left(u_0(x)|x|^{\frac{\beta(2-\gamma)}{p-1}}\right) = +\infty,$$

then the problem (3.1.1) cannot admit a global solution. □

Next, we give a necessary condition for local existence where we obtain a similar estimate of T founded in the proof of Theorem 3.3.2, as $|x|$ goes to infinity.

Theorem 3.7.3 (*Necessary conditions for local existence*)
Let $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, $\beta \in (0,2]$ and $p > 1$. If u is a local mild solution to problem (3.1.1) on $[0,T]$ where $0 < T < +\infty$, then we have

$$\liminf_{|x|\to\infty} u_0(x) \leq C\, T^{-\frac{2-\gamma}{p-1}}, \tag{3.7.5}$$

for some positive constant $C > 0$.
Note that, if $A := \liminf_{|x|\to\infty} u_0(x)$, then we obtain a similar estimate as that found in (3.3.4),

$$\frac{T^{2-\gamma}A^{p-1}}{C^{p-1}} \leq 1.$$

Proof We take here, for $R > 0$ sufficiently large, $\varphi(x,t) := D^\alpha_{t|T}\tilde{\varphi}(x,t) := D^\alpha_{t|T}(\varphi_1(x/R)\varphi_2(t))$ where $\varphi_2(t) := (1-t/T)^\ell_+$ instead of the one chosen in Theorem 3.7.1. Then, as (3.7.2), we obtain

$$\begin{aligned}
&\int_{\Sigma_1} u^p\,\tilde{\varphi}\,dx\,dt + C\,T^{-\alpha}\int_{|x|\leq 2R} u_0(x)\varphi_1(x/R)\,dx \\
&\leq C\int_{\Sigma_1}\varphi_1(x/R)\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}}\,dx\,dt \\
&+C\int_{\Sigma_1}(\varphi_1(x/R))^{-\frac{2}{p-1}}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|\Delta^{\beta/2}_D\varphi_1(x/R)D^\alpha_{t|T}\varphi_2(t)\right|^{\widetilde{p}}\,dx\,dt,
\end{aligned}\tag{3.7.6}$$

where $\Sigma_1 := [0,T]\times\left\{x \in \mathbb{R}^N;\ |x| \leq 2R\right\}$, $\alpha := 1-\gamma$ and $\widetilde{p} := p/(p-1)$. Now, in the right-hand side of (3.7.6), we take the following scale of variables $\tau = T^{-1}t$, $\xi = R^{-1}x$ and we use the fact that $\Delta^{\beta/2}_D\varphi_1(x/R) = R^{-\beta}\lambda\varphi_1(x/R)$, while in the left-side we use the positivity of u, then we get

$$C\,T^{-\alpha}\int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi \leq \left(C\,T^{1-(1+\alpha)\widetilde{p}} + C\lambda\,T^{1-\alpha\widetilde{p}}R^{-\beta\widetilde{p}}\right)\int_{|\xi|\leq 2}\varphi_1(\xi)\,d\xi;$$

and so

$$\int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi \leq C(R,T)\int_{|\xi|\leq 2}\varphi_1(\xi)\,d\xi \tag{3.7.7}$$

where $C(R,T) = C\,T^{(1+\alpha)(1-\widetilde{p})} + C\,T^{1+\alpha(1-\widetilde{p})}R^{-\beta\widetilde{p}}$.
Using the estimate

$$\begin{aligned}
\inf_{|\xi|>1}(u_0(R\xi))\int_{|\xi|\leq 2}\varphi_1(\xi)\,d\xi &\leq \int_{1<|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi \\
&\leq \int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi
\end{aligned}$$

in the left-hand side of (3.7.7), we conclude, after dividing by $\int_{|\xi|\leq 2}\varphi_1(\xi)\,d\xi$, that

$$\inf_{|\xi|>1} u_0(R\xi) \leq C(R,T). \tag{3.7.8}$$

Passing to the limit in (3.7.8), as $R \to \infty$, we obtain

$$\liminf_{|x|\to\infty} u_0(x) \leq C\, T^{(1+\alpha)(1-\widetilde{p})} = C\, T^{-\frac{2-\gamma}{p-1}}.$$

□

Appendix

In this appendix, we give a proof of Ju's inequality (see proposition 3.3 in [27]), in dimension $N \geq 1$ where $\delta \in [0,2]$ and $q \geq 1$, for all nonnegative Schwartz function ψ (in the general case)

$$(-\Delta)^{\delta/2}\psi^q \leq q\psi^{q-1}(-\Delta)^{\delta/2}\psi. \tag{3.7.9}$$

The cases $\delta = 0$, $\delta = 1$ and $\delta = 2$ are obvious.

- If $\delta \in (0,1)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{\psi(x+z)-\psi(x)}{|z|^{N+\delta}}\,dz, \quad \forall\, x \in \mathbb{R}^N, \tag{3.7.10}$$

where $c_N(\delta) = \delta\Gamma((N+\delta)/2)/(2\pi^{N/2+\delta}\Gamma(1-\delta/2))$. Then

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{(\psi(x))^{q-1}\psi(x+z)-(\psi(x))^q}{|z|^{N+\delta}}\,dz.$$

The case $q = 1$ is clear. If $q > 1$, then by Young's inequality we have

$$(\psi(x))^{q-1}\psi(x+z) \leq \frac{q-1}{q}(\psi(x))^q + \frac{1}{q}(\psi(x+z))^q.$$

Therefore,

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) \geq \frac{-c_N(\delta)}{q}\int_{\mathbb{R}^N} \frac{(\psi(x+z))^q-(\psi(x))^q}{|z|^{N+\delta}}\,dz = \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.$$

- If $\delta \in (1,2)$, we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{\psi(x+z)-\psi(x)-\nabla\psi(x).z}{|z|^{N+\delta}}\,dz, \quad \forall\, x \in \mathbb{R}^N. \tag{3.7.11}$$

Then

$$\begin{aligned}
(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) &\geq -c_N(\delta)\int_{\mathbb{R}^N} \frac{\frac{1}{q}\left((\psi(x+z))^q-(\psi(x))^q\right)-(\nabla\psi(x).z)(\psi(x))^{q-1}}{|z|^{N+\delta}}\,dz \\
&= \frac{-c_N(\delta)}{q}\int_{\mathbb{R}^N} \frac{(\psi(x+z))^q-(\psi(x))^q-\nabla(\psi(x))^q.z}{|z|^{N+\delta}}\,dz \\
&= \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.
\end{aligned}$$

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Chapitre 4

On a certain time-fractional evolution system

Ahmad Z. FINO and Mokhtar KIRANE

Abstract

In this paper, we investigate the local existence and the finite-time blow-up of solutions of a semilinear parabolic system with nonlocal in time nonlinearities. Moreover, we investigate the blow-up rate of blowing-up solutions and the necessary conditions for local or global existence.

Keywords : Parabolic system, mild and weak solutions, local existence, blow-up, blow-up rate, maximal regularity, interior regularity, Schauder's estimate, Riemann-Liouville fractional integrals and derivatives.
MSC : 35K55; 26A33; 35B44; 74G40

4.1 Introduction

This article is concerned with the study of the following semilinear parabolic system with nonlocal in time nonlinearities

$$\begin{cases} u_t - \Delta u = \dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t (t-s)^{-\gamma}|v|^{p-1}v(s)\,ds & x \in \mathbb{R}^N, t>0, \\ v_t - \Delta v = \dfrac{1}{\Gamma(1-\delta)}\displaystyle\int_0^t (t-s)^{-\delta}|u|^{q-1}u(s)\,ds & x \in \mathbb{R}^N, t>0, \end{cases} \tag{4.1.1}$$

supplemented with the initial conditions

$$u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \;\; x\in\mathbb{R}^N, \tag{4.1.2}$$

where $u_0, v_0 \in C_0(\mathbb{R}^N)$, $\gamma,\delta\in(0,1)$ and $p,q>1$.
Here u_t stands for the derivative in time of u, $-\Delta$ for the Laplacian operator with domain $D(-\Delta) =$

$H^2(\mathbb{R}^N)$, where $H^2(\mathbb{R}^N)$ is the standard Sobolev space, Γ is the Euler gamma function. The space $C_0(\mathbb{R}^N)$ denotes the set of all continuous functions decaying to zero at infinity.
Our analysis is based on the observation that the nonlinear differential system (4.1.1) can be written in the form :

$$\begin{cases} u_t - \Delta u = J^{\alpha}_{0|t}\left(|v|^{p-1}v\right) & x \in \mathbb{R}^N, t > 0, \\ \\ v_t - \Delta v = J^{\beta}_{0|t}\left(|u|^{q-1}u\right) & x \in \mathbb{R}^N, t > 0, \end{cases} \tag{4.1.3}$$

where $\alpha := 1-\gamma \in (0,1)$, $\beta := 1-\delta \in (0,1)$, and $J^{\theta}_{0|t}$, is the Riemann-Liouville fractional integral defined in (4.2.8) for $\theta \in (0,1)$.

In the case of a single equation

$$\begin{cases} u_t - \Delta u = \dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t (t-s)^{-\gamma}|u|^{p-1}u(s)\,ds & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \tag{4.1.4}$$

where $u_0 \in C_0(\mathbb{R}^N)$, $\gamma \in (0,1)$, $p > 1$ and $u \in C([0,T], C_0(\mathbb{R}^N))$ for all $0 < T < \infty$, Cazenave, Deicktsein and Weissler [10] addressed the local existence, global existence and blow-up questions while in the recent work of Fino and Kirane [17], one can find the blow-up rate of blowing-up solutions and necessary conditions for the local or the global existence.
In the paper [10], it was shown that

(i) If $p \leq p_* := \max\{1/\gamma\ ;\ 1 + 2(2-\gamma)/(N-2+2\gamma)_+\}$ and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in a finite time.

(ii) If $p > p_*$ and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$ (where $q_{sc} = N(p-1)/(4-2\gamma)$) with $\|u_0\|_{L^{q_{sc}}}$ sufficiently small, then u exists globally.

It was shown later in [17] that in the case $p \leq 1 + 2(2-\gamma)/(N-2+2\gamma)_+$ or $p < (1/\gamma)$ and $u_0 \geq 0$, $u_0 \not\equiv 0$, two positive constants $c, C > 0$ exist such that

$$c(T^* - t)^{-\frac{2-\gamma}{p-1}} \leq \sup_{x \in \mathbb{R}^N} u(x,t) \leq C(T^* - t)^{-\frac{2-\gamma}{p-1}} \quad \text{for all } t \in (0, T^*),$$

where T^* denotes the maximal time of existence.

In this paper, we generalize the work of [10] and [17] to 2×2 systems. The main results of this article are :
If $u_0, v_0 \in C_0(\mathbb{R}^N)$ are such that $u_0, v_0 \geq 0$ and $u_0, v_0 \not\equiv 0$, and if

$$\frac{N}{2} \leq \max\left\{\frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}\ ;\ \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1}\right\}$$

or

$$p < \frac{1}{\delta} \qquad \text{and} \qquad q < \frac{1}{\gamma},$$

then any solution (u, v) to (4.1.1) – (4.1.2) blows-up in a finite time.
To understand the profile of (u, v) near the blow-up time, the first step usually consists in deriving a bound for the blow-up rate. So in the case of the blowing-up in a finite time $T_{\max} := T^*$ solutions, we have

$$\begin{cases} c_1(T^*-t)^{-\alpha_1} \le \sup_{\mathbb{R}^N} u(\cdot\,,t) \le C_1(T^*-t)^{-\alpha_1}, & t\in(0,T^*) \\ c_2(T^*-t)^{-\alpha_2} \le \sup_{\mathbb{R}^N} v(\cdot\,,t) \le C_2(T^*-t)^{-\alpha_2}, & t\in(0,T^*), \end{cases}$$

for four positives constants c_i, C_i $(i = 1, 2)$, where

$$\alpha_1 := \frac{(2-\gamma)+(2-\delta)p}{pq-1} \quad \text{and} \quad \alpha_2 := \frac{(2-\delta)+(2-\gamma)q}{pq-1},$$

under the condition that $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0, v_0 \ge 0$ and $u_0, v_0 \not\equiv 0$.

The organization of this paper is as follows : In section 4.2, some preliminaries are set. In Section 4.3, a local existence theorem of solutions for the parabolic system (4.1.1) – (4.1.2) is proved. Section 4.4 contains a blow-up result of solutions for (4.1.1) – (4.1.2). Section 4.5 is dedicated to the blow-up rate of blowing-up solutions. Finally, necessary conditions for local or global existence are presented in section 4.6.

4.2 Preliminaries

In this section, we present some definitions and results concerning fractional integrals and fractional derivatives needed to prove the main results.
First, the fundamental solution of the heat equation

$$u_t - \Delta u = 0, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{4.2.1}$$

is given by

$$G_t(x) := G(t,x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}. \tag{4.2.2}$$

It is well-known that this function satisfies

$$G_t \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad G_t(x) \ge 0, \quad \int_{\mathbb{R}^N} G_t(x)\,dx = 1, \tag{4.2.3}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, using the Young inequality for convolution, it holds

$$\|G_t * v\|_r \;\le\; \|v\|_r, \tag{4.2.4}$$

for any $v \in L^r(\mathbb{R}^N)$ and any $1 \le r \le \infty$, $t > 0$.
The semigroup on $L^2(\mathbb{R}^N)$ generated by the laplacian Δ is $e^{t\Delta}v := G_t * v$ for all $v \in L^2(\mathbb{R}^N)$, $t > 0$ where $u * v$ stands for the convolution of u and v. Moreover, as $(-\Delta)$ is a self-adjoint operator with $D(-\Delta) = H^2(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)v(x)\,dx = \int_{\mathbb{R}^N} v(x)(-\Delta)u(x)\,dx, \tag{4.2.5}$$

for all $u, v \in H^2(\mathbb{R}^N)$.
Next, if $AC[0,T]$ is the space of all functions which are absolutely continuous on $[0,T]$ with $0 < T < \infty$, then, for $f \in AC[0,T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D^\alpha_{0|t} f(t)$ and $D^\alpha_{t|T} f(t)$ of order $\alpha \in (0,1)$ are defined by (see [32])

$$D^\alpha_{0|t} f(t) := DJ^{1-\alpha}_{0|t} f(t), \tag{4.2.6}$$

$$D^\alpha_{t|T} f(t) := -\frac{1}{\Gamma(1-\alpha)} D \int_t^T (s-t)^{-\alpha} f(s)\, ds, \tag{4.2.7}$$

for all $t \in [0,T]$, where $D := d/(dt)$ is the usual derivative, and

$$J^\alpha_{0|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \tag{4.2.8}$$

is the Riemann-Liouville fractional integral defined in [32], for all $f \in L^q(0,T)$ $(1 \le q \le \infty)$.
Now, for every $f, g \in C([0,T])$, such that $D^\alpha_{0|t} f(t), D^\alpha_{t|T} g(t)$ exist and are continuous, for all $t \in [0,T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.64) p. 46 in [46])

$$\int_0^T \left(D^\alpha_{0|t} f\right)(s) g(s)\, ds = \int_0^T f(s) \left(D^\alpha_{t|T} g\right)(s)\, ds. \tag{4.2.9}$$

Note also that, for all $f \in AC^2[0,T]$, we have (see 2.2.30 in [32])

$$-D.D^\alpha_{t|T} f = D^{1+\alpha}_{t|T} f, \tag{4.2.10}$$

where

$$AC^2[0,T] := \{f : [0,T] \to \mathbb{R} \text{ and } Df \in AC[0,T]\}.$$

Moreover, for all $1 \le q \le \infty$, the following equality (see [Lemma 2.4 p.74][32])

$$D^\alpha_{0|t} J^\alpha_{0|t} = Id_{L^q(0,T)} \tag{4.2.11}$$

holds almost everywhere on $[0,T]$. Later on, we will use the following results.
• If $w_1(t) = (1 - t/T)^\sigma_+$, $t \ge 0$, $T > 0$, $\sigma \gg 1$, where $(\cdot)_+$ is the positive part, then

$$D^\alpha_{t|T} w_1(t) = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} T^{-\alpha} \left(1 - \frac{t}{T}\right)_+^{\sigma-\alpha}, \tag{4.2.12}$$

$$D^{\alpha+1}_{t|T} w_1(t) = \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} T^{-\alpha-1} \left(1 - \frac{t}{T}\right)_+^{\sigma-\alpha-1}, \tag{4.2.13}$$

for all $\alpha \in (0,1)$; so

$$\left(D^\alpha_{t|T} w_1\right)(T) = 0 \quad ; \quad \left(D^\alpha_{t|T} w_1\right)(0) = C\, T^{-\alpha}, \tag{4.2.14}$$

where $C = (1-\alpha+\sigma)\Gamma(\sigma+1)/\Gamma(2-\alpha+\sigma)$; indeed, using the Euler change of variable $y = (s-t)/(T-t)$, we get

$$D^\alpha_{t|T} w_1(t) := -\frac{1}{\Gamma(1-\alpha)} D \left[\int_t^T (s-t)^{-\alpha} \left(1 - \frac{s}{T}\right)^\sigma ds\right]$$

$$
\begin{aligned}
&= -\frac{T^{-\sigma}}{\Gamma(1-\alpha)} D\left[(T-t)^{1-\alpha+\sigma}\int_0^1 (y)^{-\alpha}(1-y)^{\sigma}\,ds\right]\\
&= +\frac{(1-\alpha+\sigma)B(1-\alpha;\sigma+1)}{\Gamma(1-\alpha)}T^{-\sigma}(T-t)^{\sigma-\alpha},
\end{aligned}
$$

where $B(\cdot\,;\cdot\,)$ stands for the beta function. Then, (4.2.12) follows using the relation

$$B(1-\alpha;\sigma+1)=\frac{\Gamma(1-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}.$$

Moreover, (4.2.13) follows from (4.2.10) applied to (4.2.12).

• If $w_2(t)=(1-t^2/T^2)_+^{\ell}$, $T>0$, $\ell\gg 1$, then, using the change of variable $y=(s-t)/(T-t)$, we have

$$D^{\alpha}_{t|T}w_2(t) = \frac{T^{-\alpha}}{\Gamma(1-\alpha)}\sum_{k=0}^{\ell} C_1(\ell,k,\alpha)\left(1-\frac{t}{T}\right)^{\ell+k-\alpha}, \tag{4.2.15}$$

$$D^{1+\alpha}_{t|T}w_2(t) = \frac{T^{-\alpha-1}}{\Gamma(1-\alpha)}\sum_{k=0}^{\ell} C_2(\ell,k,\alpha)\left(1-\frac{t}{T}\right)^{\ell+k-\alpha-1}, \tag{4.2.16}$$

for all $-T\le t\le T$, $\alpha\in(0,1)$, where

$$\begin{cases} C_1(\ell,k,\alpha) := c_\ell^k(1-\alpha+\ell+k)2^{\ell-k}(-1)^k\frac{\Gamma(k+\ell+1)\Gamma(1-\alpha)}{\Gamma(k+\ell+2-\alpha)},\\ C_2(\ell,k,\alpha) := (\ell+k-\alpha)C_1(\ell,k,\alpha),\\ c_\ell^k := \frac{\ell!}{(\ell-k)!k!};\end{cases}$$

so

$$\left(D^{\alpha}_{t|T}w_2\right)(T)=0 \quad ; \quad \left(D^{\alpha}_{t|T}w_2\right)(-T)=C_3(\ell,k,\alpha)\,T^{-\alpha}, \tag{4.2.17}$$

where

$$C_3(\ell,k,\alpha) := \frac{2^{2\ell-\alpha}}{\Gamma(1-\alpha)}\sum_{k=0}^{\ell} c_\ell^k(1-\alpha+\ell+k)2^{\ell-k}(-1)^k\frac{\Gamma(k+\ell+1)\Gamma(1-\alpha)}{\Gamma(k+\ell+2-\alpha)}.$$

4.3 Local existence

In this section, we derive the existence of a local mild solution for the system (4.1.1) – (4.1.2). First, we give the

Definition 4.3.1 (*Mild solution*)
Let $u_0, v_0\in L^\infty(\mathbb{R}^N)$ and $T>0$. We say that $(u,v)\in C([0,T],L^\infty(\mathbb{R}^N)\times L^\infty(\mathbb{R}^N))$ is a mild solution of (4.1.1) – (4.1.2) *if*

$$\begin{cases} u(t)=e^{t\Delta}u_0+\displaystyle\int_0^t e^{(t-s)\Delta}J^{\alpha}_{0|s}(|v|^{p-1}v)\,ds, & t\in[0,T],\\[2ex] v(t)=e^{t\Delta}v_0+\displaystyle\int_0^t e^{(t-s)\Delta}J^{\beta}_{0|s}(|u|^{q-1}u)\,ds, & t\in[0,T].\end{cases} \tag{4.3.1}$$

Theorem 4.3.2 (*Local existence of a mild solution*)
Given $u_0, v_0 \in C_0(\mathbb{R}^N)$ and $p, q > 1$, then there exist a maximal time $T_{\max} > 0$ and a unique mild solution $(u, v) \in C([0, T_{\max}), C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$ to the system (4.1.1) − (4.1.2).
Furthermore, either $T_{\max} = \infty$ or else $T_{\max} < \infty$ and $(\|u\|_{L^\infty((0,t)\times\mathbb{R}^N)} + \|v\|_{L^\infty((0,t)\times\mathbb{R}^N)}) \to \infty$ as $t \to T_{\max}$. If $u_0, v_0 \geq 0$, $u_0, v_0 \not\equiv 0$, then $u(t), v(t) > 0$ for all $0 < t < T_{\max}$. Moreover, if $u_0, v_0 \in L^r(\mathbb{R}^N)$ for $1 \leq r < \infty$, then $u, v \in C([0, T_{\max}), L^r(\mathbb{R}^N))$.

Proof For arbitrary $T > 0$, we define the Banach space

$$E_T = \left\{(u, v) \in L^\infty((0, T), C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N));\ \ |||(u, v)||| \leq 2\left(\|u_0\|_\infty + \|v_0\|_\infty\right)\right\}, \tag{4.3.2}$$

where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^N)}$ and $|||\cdot|||$ is the norm of E_T defined by :

$$|||(u, v)||| := \|u\|_1 + \|v\|_1 := \|u\|_{L^\infty((0,T)\times\mathbb{R}^N)} + \|v\|_{L^\infty((0,T)\times\mathbb{R}^N)}.$$

Next, for every $(u, v) \in E_T$, we define $\Psi(u, v) := (\Psi_1(u, v), \Psi_2(u, v))$, where

$$\Psi_1(u, v) := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} J^\alpha_{0|s}(|v|^{p-1}v)\, ds, \quad t \in (0, T)$$

and

$$\Psi_2(u, v) := e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} J^\beta_{0|s}(|u|^{q-1}u)\, ds, \quad t \in (0, T).$$

We will prove the local existence by the Banach fixed point theorem.

• $\mathbf{\Psi} : \mathbf{E_T} \to \mathbf{E_T}$: Let $(u, v) \in E_T$, using (4.2.4), we obtain

$$\begin{aligned}
|||\Psi(u, v)||| &\leq \|u_0\|_\infty + C_1\Big\|\int_0^t\int_0^s (s-\sigma)^{-\gamma}\|v(\sigma)\|^p_\infty\, d\sigma\, ds\Big\|_{L^\infty(0,T)}\\
&+ \|v_0\|_\infty + C_2\Big\|\int_0^t\int_0^s (s-\sigma)^{-\delta}\|u(\sigma)\|^q_\infty\, d\sigma\, ds\Big\|_{L^\infty(0,T)}\\
&= \|u_0\|_\infty + C_1\Big\|\int_0^t\int_\sigma^t (s-\sigma)^{-\gamma}\|v(\sigma)\|^p_\infty\, ds\, d\sigma\Big\|_{L^\infty(0,T)}\\
&+ \|v_0\|_\infty + C_2\Big\|\int_0^t\int_\sigma^t (s-\sigma)^{-\delta}\|u(\sigma)\|^q_\infty\, ds\, d\sigma\Big\|_{L^\infty(0,T)},
\end{aligned}$$

where

$$C_1 := \frac{1}{\Gamma(1-\gamma)}, \qquad C_2 := \frac{1}{\Gamma(1-\delta)}.$$

So, using the fact that $(u, v) \in E_T$, we get

$$\begin{aligned}
|||\Psi(u, v)||| &\leq \|u_0\|_\infty + C_3T^{2-\gamma}\|v\|^p_1\\
&+ \|v_0\|_\infty + C_4T^{2-\delta}\|u\|^q_1\\
&\leq (\|u_0\|_\infty + \|v_0\|_\infty) + \max\left\{C_3T^{2-\gamma}\|v\|^{p-1}_1; C_4T^{2-\delta}\|u\|^{q-1}_1\right\}(\|v\|_1 + \|u\|_1)\\
&\leq (\|u_0\|_\infty + \|v_0\|_\infty) + 2T(u_0, v_0)(\|u_0\|_\infty + \|v_0\|_\infty),
\end{aligned}$$

where

$$C_3 := \frac{C_1}{(1-\gamma)(2-\gamma)} = \frac{1}{\Gamma(3-\gamma)}, \qquad C_4 := \frac{C_2}{(1-\delta)(2-\delta)} = \frac{1}{\Gamma(3-\delta)}$$

and

$$T(u_0,v_0) := \max\left\{C_3T^{2-\gamma}2^{p-1}(\|u_0\|_\infty+\|v_0\|_\infty)^{p-1}; C_4T^{2-\delta}2^{q-1}(\|u_0\|_\infty+\|v_0\|_\infty)^{q-1}\right\}.$$

Now, if we choose T such that

$$2T(u_0,v_0)\leq 1, \tag{4.3.3}$$

we conclude that $\|\Psi(u,v)\|_1 \leq 2(\|u_0\|_\infty+\|v_0\|_\infty)$, and then $\Psi(u,v)\in E_T$.

• **Ψ is a Contraction map :** For $(u,v),(\widetilde{u},\widetilde{v})\in E_T$, we have

$$\begin{aligned} |||\Psi(u,v)-\Psi(\widetilde{u},\widetilde{v})||| &\leq C_1\Big\|\int_0^t\int_0^s(s-\sigma)^{-\gamma}\||v|^{p-1}v(\sigma)-|\widetilde{v}|^{p-1}\widetilde{v}(\sigma)\|_\infty\,d\sigma\,ds\Big\|_{L^\infty(0,T)}\\ &+ C_2\Big\|\int_0^t\int_0^s(s-\sigma)^{-\delta}\||u|^{q-1}u(\sigma)-|\widetilde{u}|^{q-1}\widetilde{u}(\sigma)\|_\infty\,d\sigma\,ds\Big\|_{L^\infty(0,T)}\\ &= C_1\Big\|\int_0^t\int_\sigma^t(s-\sigma)^{-\gamma}\||v|^{p-1}v(\sigma)-|\widetilde{v}|^{p-1}\widetilde{v}(\sigma)\|_\infty\,ds\,d\sigma\Big\|_{L^\infty(0,T)}\\ &+ C_2\Big\|\int_0^t\int_\sigma^t(s-\sigma)^{-\delta}\||u|^{q-1}u(\sigma)-|\widetilde{u}|^{q-1}\widetilde{u}(\sigma)\|_\infty\,ds\,d\sigma\Big\|_{L^\infty(0,T)}. \end{aligned}$$

Now, by the same computations as above, we obtain

$$\begin{aligned} |||\Psi(u,v)-\Psi(\widetilde{u},\widetilde{v})||| &\leq C_3T^{2-\gamma}\||v|^{p-1}v-|\widetilde{v}|^{p-1}\widetilde{v}\|_1+C_4T^{2-\delta}\||u|^{q-1}u-|\widetilde{u}|^{q-1}\widetilde{u}\|_1\\ &\leq C(p)C_3T^{2-\gamma}(\|v\|_1^{p-1}+\|\widetilde{v}\|_1^{p-1})\|v-\widetilde{v}\|_1\\ &+ C(q)C_4T^{2-\delta}(\|u\|_1^{q-1}+\|\widetilde{u}\|_1^{q-1})\|u-\widetilde{u}\|_1\\ &\leq 2C(p,q)T(u_0,v_0)|||(u,v)-(\widetilde{u},\widetilde{v})|||\\ &\leq \frac{1}{2}|||(u,v)-(\widetilde{u},\widetilde{v})|||, \end{aligned}$$

thanks to the standard estimate :

$$|\,|u|^{p-1}u-|v|^{p-1}v|\leq C(p)|u-v|(|u|^{p-1}+|v|^{p-1}) \quad \text{for every } u,\ v \text{ and all } p>1, \tag{4.3.4}$$

and the choice of T :

$$\max\{2C(p,q),1\}T(u_0,v_0)\leq\frac{1}{2}. \tag{4.3.5}$$

Hence, by the Banach fixed point theorem, the system (4.1.1) − (4.1.2) admits a unique mild solution $(u,v)\in E_T$.
Note, for later use, that it is sufficient to take the space

$$\left\{(u,v)\in L^\infty((0,T),C_0(\mathbb{R}^N)\times C_0(\mathbb{R}^N));\ \ \|u\|_1\leq 2\|u_0\|_\infty \text{ and } \|v\|_1\leq 2\|v_0\|_\infty\right\}$$

instead of E_T and to choose $T>0$ small enough so that

$$\begin{cases} C(p)2^pC_3T^{2-\gamma}\|v_0\|_\infty^p \leq \dfrac{1}{2^\eta}\|u_0\|_\infty, \\ \\ C(q)2^qC_4T^{2-\delta}\|u_0\|_\infty^q \leq \dfrac{1}{2^\eta}\|v_0\|_\infty, \end{cases} \tag{4.3.6}$$

where $\eta>0$ is a positive constant large enough so that

$$\max\left\{\frac{\Gamma(1-\delta)}{C(q)2^{pq+\eta(q+1)+q-p}}\ ,\ \frac{\Gamma(1-\gamma)}{C(p)2^{pq+\eta(p+1)+p-q}}\right\} \leq \frac{1}{2}.$$

Thus, the conditions (4.3.6) imply that

$$\begin{cases} 2^{pq+\eta q+\eta+q}(C(p)C_3)^qC(q)C_4T^{(2-\delta)+(2-\gamma)q}\|v_0\|_\infty^{pq-1} \leq 1, \\ \\ 2^{pq+\eta p+\eta+p}(C(q)C_4)^pC(p)C_3T^{(2-\gamma)+(2-\delta)p}\|u_0\|_\infty^{pq-1} \leq 1. \end{cases} \tag{4.3.7}$$

• **Uniqueness of solution :** Let $(u,v),(\widetilde{u},\widetilde{v}) \in E_T$ be two mild solutions in E_T, for $T>0$. So, using (4.2.4) and (4.3.4), we have for $t\in[0,T]$:

$$\begin{aligned} \|u(t)-\widetilde{u}(t)\|_\infty + \|v(t)-\widetilde{v}(t)\|_\infty &\leq C_1C(p)2^p\|v_0\|_\infty^{p-1}\int_0^t\int_0^s(s-\sigma)^{-\gamma}\|v(\sigma)-\widetilde{v}(\sigma)\|_\infty\,d\sigma\,ds \\ &+ C_2C(q)2^q\|u_0\|_\infty^{q-1}\int_0^t\int_0^s(s-\sigma)^{-\delta}\|u(\sigma)-\widetilde{u}(\sigma)\|_\infty\,d\sigma\,ds \\ &= C_1C(p)2^p\|v_0\|_\infty^{p-1}\int_0^t\int_\sigma^t(s-\sigma)^{-\gamma}\|v(\sigma)-\widetilde{v}(\sigma)\|_\infty\,ds\,d\sigma \\ &+ C_2C(q)2^q\|u_0\|_\infty^{q-1}\int_0^t\int_\sigma^t(s-\sigma)^{-\delta}\|u(\sigma)-\widetilde{u}(\sigma)\|_\infty\,ds\,d\sigma \\ &= \frac{C(p)2^p\|v_0\|_\infty^{p-1}}{\Gamma(2-\gamma)}\int_0^t(t-\sigma)^{1-\gamma}\|v(\sigma)-\widetilde{v}(\sigma)\|_\infty\,d\sigma \\ &+ \frac{C(q)2^q\|u_0\|_\infty^{q-1}}{\Gamma(2-\delta)}\int_0^t(t-\sigma)^{1-\delta}\|u(\sigma)-\widetilde{u}(\sigma)\|_\infty\,d\sigma. \end{aligned}$$

So, for

$$C' := \max\{\frac{C(p)2^p\|v_0\|_\infty^{p-1}}{\Gamma(2-\gamma)};\frac{C(q)2^q\|u_0\|_\infty^{q-1}}{\Gamma(2-\delta)}\}$$

and for

$$f(\gamma,\delta) := \begin{cases} \min\{\gamma,\delta\} & \text{if } (t-\sigma)>1, \\ \max\{\gamma,\delta\} & \text{if } (t-\sigma)<1, \end{cases}$$

we conclude that

$$\|u(t)-\widetilde{u}(t)\|_\infty + \|v(t)-\widetilde{v}(t)\|_\infty \leq C'\int_0^t(t-\sigma)^{1-f(\gamma,\delta)}[\|u(\sigma)-\widetilde{u}(\sigma)\|_\infty + \|v(\sigma)-\widetilde{v}(\sigma)\|_\infty\,d\sigma,$$

and by Gronwall's inequality (see [9, Lemme 8.1.1]) we obtain the uniqueness.

Now, using the fact that the solution is unique, we conclude the existence of a unique solution on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup\{T(u_0, v_0) > 0 : \ (u, v) \text{ is a solution to } (4.1.1) - (4.1.2) \text{ in } E_T\} \leq \infty.$$

Moreover, a simple calculation allows us to prove that $(u, v) \in C([0, T_{\max}), C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$. Furthermore, if $0 \leq t \leq t + \tau < T_{\max}$, using (4.3.1), we have

$$\begin{aligned} u(t+\tau) &= T(\tau)u(t) + \frac{1}{\Gamma(1-\gamma)}\int_0^\tau T(\tau - s)\int_0^s (s-\sigma)^{-\gamma}|v|^{p-1}v(t+\sigma)\,d\sigma\,ds \\ &+ \frac{1}{\Gamma(1-\gamma)}\int_0^\tau T(\tau - s)\int_0^t (t+s-\sigma)^{-\gamma}|v|^{p-1}v(\sigma)\,d\sigma\,ds, \end{aligned} \tag{4.3.8}$$

and

$$\begin{aligned} v(t+\tau) &= T(\tau)v(t) + \frac{1}{\Gamma(1-\delta)}\int_0^\tau T(\tau - s)\int_0^s (s-\sigma)^{-\delta}|u|^{q-1}u(t+\sigma)\,d\sigma\,ds \\ &+ \frac{1}{\Gamma(1-\delta)}\int_0^\tau T(\tau - s)\int_0^t (t+s-\sigma)^{-\delta}|u|^{q-1}u(\sigma)\,d\sigma\,ds. \end{aligned} \tag{4.3.9}$$

Note that the last terms in (4.3.8) and (4.3.9) depends only on the values of v and u respectively in the interval $(0, t)$. So, using again a fixed-point argument, we can easily deduce that if (u, v) is a solution of (4.3.1) on some interval $[0, T)$ and if $\|u\|_{L^\infty((0,t)\times\mathbb{R}^N)} < \infty$ and $\|v\|_{L^\infty((0,t)\times\mathbb{R}^N)} < \infty$, then (u, v) can be extended to a solution on some interval $[0, T')$ with $T' > T$. This shows that if $T_{\max} < \infty$, then

$$(\|u\|_{L^\infty((0,t)\times\mathbb{R}^N)} + \|v\|_{L^\infty((0,t)\times\mathbb{R}^N)}) \to \infty \quad \text{as} \quad t \to T_{\max}.$$

• **Positivity of solutions :** If $u_0, v_0 \geq 0$ and $u_0, v_0 \not\equiv 0$, then we can construct a non-negative solution on some interval $[0, T]$ by applying the fixed-point argument in the set $E_T^+ = \{(u, v) \in E_T;\ u, v \geq 0\}$. In particular, it follows from (4.3.1) that $u(t) \geq e^{t\Delta}u_0 > 0$ and $v(t) \geq e^{t\Delta}v_0 > 0$ on $(0, T]$. It is not difficult, by uniqueness and contradiction's principle, to deduce that (u, v) stays positive on $(0, T_{\max})$.

• **Regularity of solutions :** If $u_0, v_0 \in L^r(\mathbb{R}^N)$, for $1 \leq r < \infty$, then by repeating the fixed point argument in the space

$$\begin{aligned} E_{T,r} \ :=\ & \{(u,v) \in L^\infty((0,T), (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)) \times (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))) : \\ & |||(u,v)||| \leq 2(\|u_0\|_{L^\infty} + (\|v_0\|_{L^\infty}), |||(u,v)|||_r \leq 2(\|u_0\|_{L^r} + \|v_0\|_{L^r})\}, \end{aligned}$$

instead of E_T, where

$$|||(u,v)|||_r \ := \ \|u\|_{L^\infty((0,T),L^r(\mathbb{R}^N))} + \|v\|_{L^\infty((0,T),L^r(\mathbb{R}^N))},$$

and by estimating $\|u^p\|_{L^r(\mathbb{R}^N)}$ by $\|u\|^{p-1}_{L^\infty(\mathbb{R}^N)}\|u\|_{L^r(\mathbb{R}^N)}$, the same for v, in the contraction mapping argument, using (4.2.4), we obtain a unique solution in $E_{T,r}$. Thus, we conclude that

$$(u,v) \in C([0, T_{\max}), (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)) \times (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))).$$

□

We say that (u,v) is a global solution if $T_{\max}=\infty$, while in the case of $T_{\max}<\infty$, we say that (u,v) blows up in a finite time and in this case we have

$$\lim_{t\to T_{\max}}\left(\|u(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}+\|v(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}\right)=\infty.$$

4.4 Blowing-up solutions

Hereafter

$$\int_{Q_T}=\int_0^T\int_{\mathbb{R}^N}dx\,dt \quad \text{for all} \quad T>0; \qquad \int_{\mathbb{R}^N}=\int_{\mathbb{R}^N}dx.$$

In this section, we prove a blow-up result for system (4.1.1) – (4.1.2). First we give the

Definition 4.4.1 (*Weak solution*)
Let $T>0$ and $u_0,v_0\in L^\infty_{Loc}(\mathbb{R}^N)$. We say that

$$(u,v)\in L^q((0,T),L^\infty_{Loc}(\mathbb{R}^N))\times L^p((0,T),L^\infty_{Loc}(\mathbb{R}^N))$$

is a weak solution of the problem (4.1.1) – (4.1.2) if

$$\begin{aligned}\int_{\Omega_1}u_0(x)\varphi(x,0)+\int_{Q_T}J^\alpha_{0|t}(|v|^{p-1}v)(x,t)\varphi(x,t) &= -\int_0^T\int_{\Omega_1}u(x,t)\Delta\varphi(x,t)\\ &\quad -\int_0^T\int_{\Omega_1}u(x,t)\varphi_t(x,t),\end{aligned}\tag{4.4.1}$$

and

$$\begin{aligned}\int_{\Omega_2}v_0(x)\psi(x,0)+\int_0^T\int_{\Omega_2}J^\beta_{0|t}(|u|^{q-1}u)(x,t)\psi(x,t) &= -\int_0^T\int_{\Omega_2}v(x,t)\Delta\psi(x,t)\\ &\quad -\int_0^T\int_{\Omega_2}v(x,t)\psi_t(x,t),\end{aligned}\tag{4.4.2}$$

for all $\varphi,\psi\in C^1([0,T],H^2(\mathbb{R}^N))$ with $\Omega_1:=$suppφ, $\Omega_2:=$suppψ (compact support) and $\varphi(\cdot\,,T)=\psi(\cdot\,,T)=0$, where $\alpha=1-\gamma$ and $\beta=1-\delta$.

Lemma 4.4.2 (*Mild $\to$ Weak*)
Let $T>0$, $u_0,v_0\in L^\infty(\mathbb{R}^N)$ and $U:=(u,v)$, where $u,v\in C([0,T],L^\infty(\mathbb{R}^N))$. If U is a mild solution of (4.1.1) – (4.1.2), then U is a weak solution of (4.1.1) – (4.1.2).

Proof Let $T>0$, $u_0,v_0\in L^\infty(\mathbb{R}^N)$ and let $U:=(u,v)$ be a solution of (4.3.1) such that $u,v\in C([0,T],L^\infty(\mathbb{R}^N))$. For a compact support functions $\varphi,\psi\in C^1([0,T],H^2(\mathbb{R}^N))$ such that supp$\varphi=:\Omega_1$, supp$\psi=:\Omega_2$ and $\varphi(\cdot\,,T)=\psi(\cdot\,,T)=0$, we have, after multiplying the first (resp. second) equation in (4.3.1) by φ (resp. ψ) and integrating over $\mathbb{R}^N$,

$$\int_{\Omega_1}u(x,t)\varphi(x,t)=\int_{\Omega_1}T(t)u_0(x)\varphi(x,t)+\int_{\Omega_1}\int_0^tT(t-s)J^\alpha_{0|s}\left(|v|^{p-1}v\right)(x,t)\,ds\varphi(x,t),$$

and

$$\int_{\Omega_2} v(x,t)\psi(x,t) = \int_{\Omega_2} T(t)v_0(x)\psi(x,t) + \int_{\Omega_2}\int_0^t T(t-s)J^{\beta}_{0|s}\left(|u|^{q-1}u\right)(x,t)\,ds\psi(x,t),$$

where $\alpha = 1-\gamma$ and $\beta = 1-\delta$. So, after differentiation in time, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_1} u(x,t)\varphi(x,t) &= \int_{\Omega_1}\frac{d}{dt}\left(T(t)u_0(x)\varphi(x,t)\right)\\ &+ \int_{\Omega_1}\frac{d}{dt}\int_0^t T(t-s)J^{\alpha}_{0|s}\left(|v|^{p-1}v\right)(x,s)\,ds\varphi(x,t),\end{aligned}\tag{4.4.3}$$

and

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_2} v(x,t)\psi(x,t) &= \int_{\Omega_2}\frac{d}{dt}\left(T(t)v_0(x)\psi(x,t)\right)\\ &+ \int_{\Omega_2}\frac{d}{dt}\int_0^t T(t-s)J^{\beta}_{0|s}\left(|u|^{q-1}u\right)(x,s)\,ds\psi(x,t).\end{aligned}\tag{4.4.4}$$

Now, using (4.2.5) and a property of the semigroup $T(t)$ (see [9, Chapter 3]), we get :

$$\begin{aligned}\int_{\Omega_1}\frac{d}{dt}\left(T(t)u_0(x)\varphi(x,t)\right) &= \int_{\Omega_1}\Delta\left(T(t)u_0(x)\right)\varphi(x,t)\,dx + \int_{\Omega_1} T(t)u_0(x)\varphi_t(x,t)\\ &= \int_{\Omega_1} T(t)u_0(x)\Delta\varphi(x,t) + \int_{\Omega_1} T(t)u_0(x)\varphi_t(x,t),\end{aligned}\tag{4.4.5}$$

$$\begin{aligned}\int_{\Omega_2}\frac{d}{dt}\left(T(t)v_0(x)\psi(x,t)\right) &= \int_{\Omega_2}\Delta\left(T(t)v_0(x)\right)\psi(x,t)\,dx + \int_{\Omega_2} T(t)v_0(x)\psi_t(x,t)\\ &= \int_{\Omega_2} T(t)v_0(x)\Delta\psi(x,t) + \int_{\Omega_2} T(t)v_0(x)\psi_t(x,t),\end{aligned}\tag{4.4.6}$$

$$\begin{aligned}\int_{\Omega_1}\frac{d}{dt}\int_0^t T(t-s)f_1(x,s)\,ds\varphi(x,t) &= \int_{\Omega_1} f_1(x,t)\varphi(x,t) + \int_{\Omega_1}\int_0^t \Delta\left(T(t-s)f_1(x,s)\right)\,ds\varphi(x,t)\\ &+ \int_{\Omega_1}\int_0^t T(t-s)f_1(x,s)\,ds\varphi_t(x,t)\\ &= \int_{\Omega_1} f_1(x,t)\varphi(x,t) + \int_{\Omega_1}\int_0^t T(t-s)f_1(x,s)\,ds\Delta\varphi(x,t)\\ &+ \int_{\Omega_1}\int_0^t T(t-s)f_1(x,s)\,ds\varphi_t(x,t),\end{aligned}\tag{4.4.7}$$

and, similarly,

$$\int_{\Omega_2}\frac{d}{dt}\int_0^t T(t-s)f_2(x,s)\,ds\psi(x,t) = \int_{\Omega_2} f_2(x,t)\psi(x,t) + \int_{\Omega_2}\int_0^t T(t-s)f_2(x,s)\,ds\Delta\psi(x,t)$$

$$+ \int_{\Omega_2} \int_0^t T(t-s) f_2(x,s)\, ds \psi_t(x,t), \tag{4.4.8}$$

where

$$f_1 := J^{\alpha}_{0|t}\left(|v|^{p-1}v\right) \in C([0,T], L^2(\Omega_1)) \quad \text{and} \quad f_2 := J^{\beta}_{0|t}\left(|u|^{q-1}u\right) \in C([0,T], L^2(\Omega_2)).$$

Thus, using (4.3.1) and (4.4.5) – (4.4.8), we conclude that (4.4.3) and (4.4.4) imply

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_1} u(x,t)\varphi(x,t) &= \int_{\Omega_1} u(x,t)\Delta\varphi(x,t) + \int_{\Omega_1} u(x,t)\varphi_t(x,t) \\ &+ \int_{\Omega_1} f_1(x,t)\varphi(x,t), \end{aligned} \tag{4.4.9}$$

and

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_2} v(x,t)\psi(x,t) &= \int_{\Omega_2} v(x,t)\Delta\psi(x,t) + \int_{\Omega_2} v(x,t)\psi_t(x,t) \\ &+ \int_{\Omega_2} f_2(x,t)\psi(x,t). \end{aligned} \tag{4.4.10}$$

Finally, after integrating in time over $[0,T]$ and using the fact that $\varphi(\cdot\,,T) = \psi(\cdot\,,T) = 0$, we conclude the result. □

Theorem 4.4.3 *Let $u_0, v_0 \in C_0(\mathbb{R}^N)$ be such that $u_0, v_0 \geq 0$ and $u_0, v_0 \not\equiv 0$. If*

$$\frac{N}{2} \leq \max\left\{\frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}\ ;\ \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1}\right\} \tag{4.4.11}$$

or

$$p < \frac{1}{\delta} \qquad \textit{and} \qquad q < \frac{1}{\gamma} \tag{4.4.12}$$

then any mild solution (u,v) to (4.1.1) – (4.1.2), blows-up in a finite time.

Proof The proof is by contradiction. Suppose that (u,v) is a global mild solution to (4.1.1) – (4.1.2), then (u,v) is a solution of (4.1.1) – (4.1.2) where $u, v \in C([0,T], C_0(\mathbb{R}^N))$, for all $T > 0$, such that $u(t), v(t) > 0$ for all $t \in [0,T]$.
Thus, using Lemma 4.4.2, we conclude that u and v verify (4.4.1) and (4.4.2), respectively, for all compact support functions $\varphi, \psi \in C^1([0,T], H^2(\mathbb{R}^N))$ such that $\varphi(\cdot\,,T) = \psi(\cdot\,,T) = 0$.
Now, we take $\varphi(x,t) = D^{\alpha}_{t|T}\left(\tilde{\varphi}(x,t)\right) := D^{\alpha}_{t|T}\left((\varphi_1(x))^{\ell}\varphi_2(t)\right)$ and $\psi(x,t) = D^{\beta}_{t|T}\left(\tilde{\varphi}(x,t)\right)$, for $\alpha := 1-\gamma$ and $\beta := 1-\delta$, with $\varphi_1(x) := \Phi\left(|x|/(T^{1/2})\right)$, $\varphi_2(t) := (1-t/T)^{\eta}_{+}$, where $\ell \geq pq/((p-1)(q-1))$, $\eta \gg 1$, $1 \leq B < T$ large enough such as in the case of $T \to \infty$ we don't have $B \to \infty$ in the same time, and Φ is a smooth nonnegative non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$, $|\Phi'(r)| \leq C_1/r$, for all $r > 0$. So, using (4.2.14), we obtain

$$\begin{aligned}&\int_{\Omega} u_0(x) D^{\alpha}_{t|T}\tilde{\varphi}(x,0) + \int_{\Omega_T} J^{\alpha}_{0|t}(v^p)(x,t) D^{\alpha}_{t|T}\tilde{\varphi}(x,t)\\ &= -\int_{\Omega_T} u(x,t)\Delta D^{\alpha}_{t|T}\tilde{\varphi}(x,t) - \int_{\Omega_T} u(x,t) D D^{\alpha}_{t|T}\tilde{\varphi}(x,t),\end{aligned} \tag{4.4.13}$$

and

$$\begin{aligned}&\int_{\Omega} v_0(x) D^{\beta}_{t|T}\tilde{\varphi}(x,0) + \int_{\Omega_T} J^{\beta}_{0|t}(u^q)(x,t) D^{\alpha}_{t|T}\tilde{\varphi}(x,t)\\ &= -\int_{\Omega_T} v(x,t)\Delta D^{\beta}_{t|T}\tilde{\varphi}(x,t) - \int_{\Omega_T} v(x,t) D D^{\beta}_{t|T}\tilde{\varphi}(x,t),\end{aligned} \tag{4.4.14}$$

where

$$\Omega_T := [0,T]\times\Omega, \quad \text{for} \;\; \Omega = \left\{x \in \mathbb{R}^N \;\; ; \;\; |x| \leq 2T^{1/2}\right\}, \;\; \int_{\Omega_T} = \int_{\Omega_T} dx\, dt \;\; \text{and} \;\; \int_{\Omega} = \int_{\Omega} dx.$$

Furthermore, using (4.2.9) and (4.2.14) in the left hand sides of (4.4.13) and (4.4.14) while in the right hand sides we use (4.2.10), we conclude that

$$\begin{aligned}&C\,T^{-\alpha}\int_{\Omega} u_0(x)\varphi_1^{\ell}(x) + \int_{\Omega_T} D^{\alpha}_{0|t} J^{\alpha}_{0|t}(v^p)(x,t)\tilde{\varphi}(x,t)\\ &= -\int_{\Omega_T} u(x,t)\Delta D^{\alpha}_{t|T}\tilde{\varphi}(x,t) + \int_{\Omega_T} u(x,t) D^{1+\alpha}_{t|T}\tilde{\varphi}(x,t),\end{aligned} \tag{4.4.15}$$

and

$$\begin{aligned}&C\,T^{-\beta}\int_{\Omega} v_0(x)\varphi_1^{\ell}(x) + \int_{\Omega_T} D^{\beta}_{0|t} J^{\alpha}_{0|t}(u^q)(x,t)\tilde{\varphi}(x,t)\\ &= -\int_{\Omega_T} v(x,t)\Delta D^{\beta}_{t|T}\tilde{\varphi}(x,t) + \int_{\Omega_T} v(x,t) D^{1+\beta}_{t|T}\tilde{\varphi}(x,t).\end{aligned} \tag{4.4.16}$$

Moreover, from (4.2.11), we may write

$$\begin{aligned}&\int_{\Omega_T} v^p(x,t)\;\tilde{\varphi}(x,t) + C\,T^{-\alpha}\int_{\Omega} u_0(x)\varphi_1^{\ell}(x)\\ &= -\int_{\Omega_T} u(x,t)\Delta\varphi_1^{\ell}(x) D^{\alpha}_{t|T}\varphi_2(t) + \int_{\Omega_T} u(x,t) D^{1+\alpha}_{t|T}\tilde{\varphi}(x,t),\end{aligned} \tag{4.4.17}$$

and

$$\begin{aligned}&\int_{\Omega_T} u^q(x,t)\;\tilde{\varphi}(x,t) + C\,T^{-\beta}\int_{\Omega} v_0(x)\varphi_1^{\ell}(x)\\ &= -\int_{\Omega_T} v(x,t)\Delta\varphi_1^{\ell}(x) D^{\beta}_{t|T}\varphi_2(t) + \int_{\Omega_T} v(x,t) D^{1+\beta}_{t|T}\tilde{\varphi}(x,t).\end{aligned} \tag{4.4.18}$$

Then, the inequality $(-\Delta)\left(\varphi_1^\ell\right) \leq \ell\varphi_1^{\ell-1}(-\Delta)\varphi_1$ allows us to write :

$$\begin{aligned}
&\int_{\Omega_T} v^p(x,t)\ \tilde{\varphi}(x,t) + C\ T^{-\alpha}\int_\Omega u_0(x)\varphi_1^\ell(x)\\
&\leq C\int_{\Omega_T} u(x,t)\ \varphi_1^{\ell-1}(x)\left|(-\Delta)\varphi_1(x)D^\alpha_{t|T}\varphi_2(t)\right|\\
&+\int_{\Omega_T} u(x,t)\ \varphi_1^\ell(x)\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right|\\
&= C\int_{\Omega_T} u(x,t)\ \tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\varphi_1^{\ell-1}(x)\left|(-\Delta)\varphi_1(x)D^\alpha_{t|T}\varphi_2(t)\right|\\
&+\int_{\Omega_T} u(x,t)\ \tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\varphi_1^\ell(x)\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right| \qquad (4.4.19)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_T} u^q(x,t)\ \tilde{\varphi}(x,t)\,dx\,dt + C\ T^{-\beta}\int_\Omega v_0(x)\varphi_1^\ell(x)\\
&\leq C\int_{\Omega_T} v(x,t)\ \varphi_1^{\ell-1}(x)\left|(-\Delta)\varphi_1(x)D^\beta_{t|T}\varphi_2(t)\right|\\
&+\int_{\Omega_T} v(x,t)\ \varphi_1^\ell(x)\left|D^{1+\beta}_{t|T}\varphi_2(t)\right|\\
&= C\int_{\Omega_T} v(x,t)\ \tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\varphi_1^{\ell-1}(x)\left|(-\Delta)\varphi_1(x)D^\beta_{t|T}\varphi_2(t)\right|\\
&+\int_{\Omega_T} v(x,t)\ \tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\varphi_1^\ell(x)\left|D^{1+\beta}_{t|T}\varphi_2(t)\right| \qquad (4.4.20)
\end{aligned}$$

Therefore, as $u_0, v_0 \geq 0$, using Hölder's inequality, we obtain

$$\int_{\Omega_T} v^p(x,t)\ \tilde{\varphi}(x,t) \leq \left(\int_{\Omega_T} u^q(x,t)\ \tilde{\varphi}(x,t)\right)^{1/q}\mathcal{A}, \qquad (4.4.21)$$

$$\int_{\Omega_T} u^q(x,t)\ \tilde{\varphi}(x,t) \leq \left(\int_{\Omega_T} v^p(x,t)\ \tilde{\varphi}(x,t)\right)^{1/p}\mathcal{B}, \qquad (4.4.22)$$

where

$$\mathcal{A} := C\left(\int_{\Omega_T}\varphi_1^\ell\varphi_2^{-\frac{1}{q-1}}\left|D^{1+\alpha}_{t|T}\varphi_2\right|^{\widetilde{q}}\right)^{1/\widetilde{q}} + C\left(\int_{\Omega_T}\varphi_1^{\ell-\widetilde{q}}\varphi_2^{-\frac{1}{q-1}}\left|\Delta_x\varphi_1 D^\alpha_{t|T}\varphi_2\right|^{\widetilde{q}}\right)^{1/\widetilde{q}},$$

and

$$\mathcal{B} := C\left(\int_{\Omega_T}\varphi_1^\ell\varphi_2^{-\frac{1}{p-1}}\left|D^{1+\beta}_{t|T}\varphi_2\right|^{\widetilde{p}}\right)^{1/\widetilde{p}} + C\left(\int_{\Omega_T}\varphi_1^{\ell-\widetilde{p}}\varphi_2^{-\frac{1}{p-1}}\left|\Delta_x\varphi_1 D^\beta_{t|T}\varphi_2\right|^{\widetilde{p}}\right)^{1/\widetilde{p}},$$

with $\widetilde{p} := p/(p-1)$ and $\widetilde{q} := q/(q-1)$. Now, combining (4.4.21) and (4.4.22), we get

$$\left\{\begin{array}{l}
\left(\displaystyle\int_{\Omega_T} v^p(x,t)\ \tilde{\varphi}(x,t)\right)^{1-1/pq} \leq \mathcal{B}^{1/q}\mathcal{A},\\[2ex]
\left(\displaystyle\int_{\Omega_T} u^q(x,t)\ \tilde{\varphi}(x,t)\right)^{1-1/pq} \leq \mathcal{A}^{1/p}\mathcal{B}.
\end{array}\right. \qquad (4.4.23)$$

At this stage, we introduce the scaled variables : $\tau = T^{-1}t$, $\xi = T^{-1/2}x$; using formula (4.2.12) and (4.2.13) in the right hand-side of (4.4.23), we obtain the estimates

$$\begin{cases} \left(\int_{\Omega_T} v^p(x,t)\,\tilde{\varphi}(x,t)\right)^{1-1/pq} \leq CT^{\theta_1}, \\ \\ \left(\int_{\Omega_T} u^q(x,t)\,\tilde{\varphi}(x,t)\right)^{1-1/pq} \leq CT^{\theta_2}, \end{cases} \tag{4.4.24}$$

where

$$\theta_1 := \left(-(1+\alpha)\widetilde{q} + (1+\frac{N}{2})\right)\frac{1}{\widetilde{q}} + \left(-(1+\beta)\widetilde{p} + (1+\frac{N}{2})\right)\frac{1}{q\widetilde{p}}, \tag{4.4.25}$$

and

$$\theta_2 := \left(-(1+\beta)\widetilde{p} + (1+\frac{N}{2})\right)\frac{1}{\widetilde{p}} + \left(-(1+\alpha)\widetilde{q} + (1+\frac{N}{2})\right)\frac{1}{p\widetilde{q}}. \tag{4.4.26}$$

Note that inequality (4.4.11) is equivalent to $\theta_1 \leq 0$ or $\theta_2 \leq 0$. So, we have to distinguish three cases :

• The case $\theta_1 < 0$ (resp. $\theta_2 < 0$): we pass to the limit in the first equation (resp. second equation) in (4.4.24), as T goes to ∞; we get

$$\lim_{T\to\infty}\int_0^T\int_{|x|\leq 2T^{1/2}} v^p(x,t)\,\tilde{\varphi}(x,t)\,dx\,dt = 0.$$

(resp.

$$\lim_{T\to\infty}\int_0^T\int_{|x|\leq 2T^{1/2}} u^q(x,t)\,\tilde{\varphi}(x,t)\,dx\,dt = 0.)$$

Using the dominated convergence theorem and the continuity in time and space of v (resp. u), we infer that

$$\int_{Q_\infty} v^p(x,t)\,dx\,dt = 0 \qquad \Longrightarrow \qquad v \equiv 0.$$

(resp.

$$\int_{Q_\infty} u^q(x,t)\,dx\,dt = 0 \qquad \Longrightarrow \qquad u \equiv 0.)$$

Moreover, from (4.4.22) (resp. (4.4.21)), we infer that

$$\int_0^T\int_{|x|\leq 2T^{1/2}} u^q(x,t)\,\tilde{\varphi}(x,t)\,dx\,dt = 0,$$

(resp.

$$\int_0^T\int_{|x|\leq 2T^{1/2}} v^p(x,t)\,\tilde{\varphi}(x,t)\,dx\,dt = 0),$$

and by the same argument as above, we conclude that $u \equiv v \equiv 0$; contradiction.

• The case $\theta_1 = 0$ (resp. $\theta_2 = 0$): in this case, using (4.4.24) as $T \to \infty$, we conclude that

$$v \in L^p((0,\infty), L^p(\mathbb{R}^N)), \tag{4.4.27}$$

(resp.

$$u \in L^q((0,\infty), L^q(\mathbb{R}^N)). \tag{4.4.28}$$

Now, we take $\varphi_1(x) := \Phi\left(|x|/(B^{-1/2}T^{1/2})\right)$ instead of the one chosen above, where $1 \le B < T$ is large enough such that when $T \to \infty$ we don't have $B \to \infty$ in the same time, then if we repeat the same calculation as above and take account of the support of $\Delta\varphi_1$, we obtain (as in (4.4.19) – (4.4.20))

$$\begin{aligned} \int_{\Sigma_B} v^p\,\tilde{\varphi} &\le C\int_{\Sigma_B} u\,\tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\,(\varphi_1(x))^{\ell}\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right| \\ &+ C\int_{\Delta_B} u\,\tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\;(\varphi_1(x))^{\ell-1}\left|(-\Delta_x)\varphi_1(x)\;D^{\alpha}_{t|T}\varphi_2(t)\right|, \end{aligned} \tag{4.4.29}$$

and

$$\begin{aligned} \int_{\Sigma_B} u^q\,\tilde{\varphi} &\le C\int_{\Sigma_B} v\,\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\,(\varphi_1(x))^{\ell}\left|D^{1+\beta}_{t|T}\varphi_2(t)\right| \\ &+ C\int_{\Delta_B} v\,\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\;(\varphi_1(x))^{\ell-1}\left|(-\Delta_x)\varphi_1(x)\;D^{\beta}_{t|T}\varphi_2(t)\right|, \end{aligned} \tag{4.4.30}$$

where

$$\Sigma_B := [0,T]\times\left\{x\in\mathbb{R}^N\;\;;\;\;|x|\le 2B^{-1/2}T^{1/2}\right\},\qquad \int_{\Sigma_B} = \int_{\Sigma_B} dx\,dt,$$

and

$$\Delta_B := [0,T]\times\left\{x\in\mathbb{R}^N\;\;;\;\;B^{-1/2}T^{1/2}\le|x|\le 2B^{-1/2}T^{1/2}\right\},\qquad \int_{\Delta_B} = \int_{\Delta_B} dx\,dt.$$

On the other hand, as (u,v) is a global solution, then u (resp. v) verifies (4.4.1) (resp. (4.4.2)) locally and in particular on Δ_B. Thus, we obtain

$$\begin{aligned} \int_{\Delta_B} v^p\,\tilde{\varphi} &\le C\int_{\Delta_B} u\,\tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\,(\varphi_1(x))^{\ell}\left|D^{1+\alpha}_{t|T}\varphi_2(t)\right| \\ &+ C\int_{\Delta_B} u\,\tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\;(\varphi_1(x))^{\ell-1}\left|(-\Delta_x)\varphi_1(x)\;D^{\alpha}_{t|T}\varphi_2(t)\right|, \end{aligned} \tag{4.4.31}$$

and

$$\begin{aligned} \int_{\Delta_B} u^q\,\tilde{\varphi} &\le C\int_{\Delta_B} v\,\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\,(\varphi_1(x))^{\ell}\left|D^{1+\beta}_{t|T}\varphi_2(t)\right| \\ &+ C\int_{\Delta_B} v\,\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\;(\varphi_1(x))^{\ell-1}\left|(-\Delta_x)\varphi_1(x)\;D^{\beta}_{t|T}\varphi_2(t)\right|. \end{aligned} \tag{4.4.32}$$

At this stage, we set

$$\mathcal{U}_1 := \int_{\Sigma_B} u^q\;\tilde{\varphi}\,dx\,dt,\quad \mathcal{U}_2 := \int_{\Delta_B} u^q\;\tilde{\varphi}\,dx\,dt,$$

and

$$\mathcal{V}_1 := \int_{\Sigma_B} v^p \,\tilde{\varphi}\, dx\, dt, \quad \mathcal{V}_2 := \int_{\Delta_B} v^p \,\tilde{\varphi}\, dx\, dt.$$

Then by using the Hölder inequality in (4.4.29), (4.4.30), (4.4.31) and (4.4.32), we obtain

$$\begin{cases} \mathcal{V}_1 \leq \mathcal{U}_1^{1/q}\mathcal{A}_1 + \mathcal{U}_2^{1/q}\mathcal{C}_1, \\ \\ \mathcal{U}_1 \leq \mathcal{V}_1^{1/p}\mathcal{B}_1 + \mathcal{V}_2^{1/p}\mathcal{C}_2, \end{cases} \tag{4.4.33}$$

and

$$\begin{cases} \mathcal{V}_2 \leq \mathcal{U}_2^{1/q}\mathcal{A}_2 + \mathcal{U}_2^{1/q}\mathcal{C}_1, \\ \\ \mathcal{U}_2 \leq \mathcal{V}_2^{1/p}\mathcal{B}_2 + \mathcal{V}_2^{1/p}\mathcal{C}_2, \end{cases} \tag{4.4.34}$$

where

$$\mathcal{A}_1 := C\left(\int_{\Sigma_B} \varphi_1^{\ell}\varphi_2^{-\frac{1}{q-1}} \left|D_{t|T}^{1+\alpha}\varphi_2\right|^{\widetilde{q}}\right)^{1/\widetilde{q}},$$

$$\mathcal{A}_2 := C\left(\int_{\Delta_B} \varphi_1^{\ell}\varphi_2^{-\frac{1}{q-1}} \left|D_{t|T}^{1+\alpha}\varphi_2\right|^{\widetilde{q}}\right)^{1/\widetilde{q}} \leq \mathcal{A}_1,$$

$$\mathcal{B}_1 := C\left(\int_{\Sigma_B} \varphi_1^{\ell}\varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{1+\beta}\varphi_2\right|^{\widetilde{p}}\right)^{1/\widetilde{p}},$$

$$\mathcal{B}_2 := C\left(\int_{\Delta_B} \varphi_1^{\ell}\varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{1+\beta}\varphi_2\right|^{\widetilde{p}}\right)^{1/\widetilde{p}} \leq \mathcal{B}_1,$$

$$\mathcal{C}_1 := C\left(\int_{\Delta_B} \varphi_1^{\ell-\widetilde{q}}\varphi_2^{-\frac{1}{q-1}} \left|\Delta_x\varphi_1 D_{t|T}^{\alpha}\varphi_2\right|^{\widetilde{q}}\right)^{1/\widetilde{q}},$$

and

$$\mathcal{C}_2 := C\left(\int_{\Delta_B} \varphi_1^{\ell-\widetilde{p}}\varphi_2^{-\frac{1}{p-1}} \left|\Delta_x\varphi_1 D_{t|T}^{\beta}\varphi_2\right|^{\widetilde{p}}\right)^{1/\widetilde{p}}.$$

Combining (4.4.33) and (4.4.34), we obtain

$$\begin{aligned} \mathcal{V}_1 \;&\leq\; \mathcal{V}_1^{1/pq}\mathcal{B}_1^{1/q}\mathcal{A}_1 + \mathcal{V}_2^{1/pq}\mathcal{C}_2^{1/q}\mathcal{A}_1 \\ &+\; \mathcal{V}_2^{1/pq}\mathcal{B}_2^{1/q}\mathcal{C}_1 + \mathcal{V}_2^{1/pq}\mathcal{C}_2^{1/q}\mathcal{C}_1 \end{aligned} \tag{4.4.35}$$

and

$$\begin{aligned} \mathcal{U}_1 \;&\leq\; \mathcal{U}_1^{1/pq}\mathcal{A}_1^{1/p}\mathcal{B}_1 + \mathcal{U}_2^{1/pq}\mathcal{C}_1^{1/p}\mathcal{B}_1 \\ &+\; \mathcal{U}_2^{1/pq}\mathcal{A}_2^{1/p}\mathcal{C}_2 + \mathcal{U}_2^{1/pq}\mathcal{C}_1^{1/p}\mathcal{C}_2. \end{aligned} \tag{4.4.36}$$

To estimate the first term in the right-hand sides of (4.4.35) and (4.4.36), we apply Young's inequality

$$ab \leq \frac{1}{pq}a^{pq} + \frac{pq-1}{pq}b^{\frac{pq}{pq-1}} \quad p > 1, q > 1, \;\; a > 0, b > 0.$$

This yields

$$(1-\frac{1}{pq})\mathcal{V}_1 \leq \mathcal{B}_1^{\frac{p}{pq-1}}\mathcal{A}_1^{\frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq}\left[\mathcal{C}_2^{1/q}\mathcal{A}_1 + \mathcal{B}_2^{1/q}\mathcal{C}_1 + \mathcal{C}_2^{1/q}\mathcal{C}_1\right], \qquad (4.4.37)$$

and

$$(1-\frac{1}{pq})\mathcal{U}_1 \leq \mathcal{A}_1^{\frac{q}{pq-1}}\mathcal{B}_1^{\frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq}\left[\mathcal{C}_1^{1/p}\mathcal{B}_1 + \mathcal{A}_2^{1/p}\mathcal{C}_2 + \mathcal{C}_1^{1/p}\mathcal{C}_2\right].$$

Using the definition of φ and applying the following change of variables

$$\tau = T^{-1}t, \qquad \xi = T^{-1/2}B^{1/2}x,$$

in the integrals in $\mathcal{A}_i, \mathcal{B}_i$ and $\mathcal{C}_i$ for $i=1,2$, we get

$$\mathcal{V}_1 \leq CT^{\theta_1\frac{pq}{pq-1}}B^{\delta_1\frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq}\left[CT^{\theta_1}B^{\delta_2} + CT^{\theta_1}B^{\delta_3} + CT^{\theta_1}B^{\delta_4}\right], \qquad (4.4.38)$$

and

$$\mathcal{U}_1 \leq CT^{\theta_2\frac{pq}{pq-1}}B^{\eta_1\frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq}\left[CT^{\theta_2}B^{\eta_2} + CT^{\theta_2}B^{\eta_3} + CT^{\theta_2}B^{\eta_4}\right], \qquad (4.4.39)$$

where

$$\begin{cases} \delta_1 := -\frac{N}{2}(\frac{1}{q\widetilde{p}} + \frac{1}{\widetilde{q}}), & \delta_2 := \frac{1}{q} - \frac{N}{2}(\frac{1}{q\widetilde{p}} + \frac{1}{\widetilde{q}}), \\ \delta_3 := 1 - \frac{N}{2}(\frac{1}{q\widetilde{p}} + \frac{1}{\widetilde{q}}), & \delta_4 := 1 + \frac{1}{q} - \frac{N}{2}(\frac{1}{q\widetilde{p}} + \frac{1}{\widetilde{q}}), \end{cases}$$

and

$$\begin{cases} \eta_1 := -\frac{N}{2}(\frac{1}{p\widetilde{q}} + \frac{1}{\widetilde{p}}), & \eta_2 := \frac{1}{p} - \frac{N}{2}(\frac{1}{p\widetilde{q}} + \frac{1}{\widetilde{p}}), \\ \eta_3 := 1 - \frac{N}{2}(\frac{1}{p\widetilde{q}} + \frac{1}{\widetilde{p}}), & \eta_4 := 1 + \frac{1}{p} - \frac{N}{2}(\frac{1}{p\widetilde{q}} + \frac{1}{\widetilde{p}}). \end{cases}$$

Let us recall that $\theta_1 = 0$ (resp. $\theta_2 = 0$) imply that

$$\mathcal{V}_1 \leq CB^{\delta_1\frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq}\left[CB^{\delta_2} + CB^{\delta_3} + CB^{\delta_4}\right], \qquad (4.4.40)$$

(resp.

$$\mathcal{U}_1 \leq CB^{\eta_1\frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq}\left[CB^{\eta_2} + CB^{\eta_3} + CB^{\eta_4}\right],) \qquad (4.4.41)$$

Now, as $v \in L^p(0,\infty;L^p(\mathbb{R}^N))$, (resp. $u \in L^q(0,\infty;L^q(\mathbb{R}^N))$), we have

$$\lim_{T\to\infty}\mathcal{V}_2 = 0,$$

(resp.

$$\lim_{T\to\infty}\mathcal{U}_2 = 0,).$$

Taking the limit as $T \to \infty$ in (4.4.40), respectively in (4.4.41), and taking account of the dominated convergence theorem, we conclude that

$$\int_0^\infty\int_{\mathbb{R}^N} v^p(x,t)\,dx\,dt \leq CB^{\delta_1\frac{pq}{pq-1}}. \qquad (4.4.42)$$

(resp.

$$\int_0^\infty \int_{\mathbb{R}^N} u^q(x,t)\,dx\,dt \le C B^{\eta_1 \frac{pq}{pq-1}}.) \tag{4.4.43}$$

Finally, as $\delta_1 < 0$ and $\eta_1 < 0$, taking the limit as $B \to \infty$ in (4.4.42), respectively in (4.4.43), and using the continuity of u and v, we conclude that

$$v \equiv 0 \quad \text{or} \quad u \equiv 0,$$

and (4.4.33) implies that $u \equiv v \equiv 0$, which is a contradiction.

• The case $p < (1/\delta)$ and $q < (1/\gamma)$: we repeat the same procedure as in the case ($\theta_1 < 0$ and $\theta_2 < 0$) by choosing the following test function $\tilde{\varphi}(x,t) = (\varphi_1(x))^\ell \varphi_2(t)$ where $\varphi_1(x) = \Phi\left(|x|/R\right)$, $\varphi_2(t) = (1 - t/T)_+^r$, $r \gg 1$ and $R \in (0,T)$ large enough such that in the case of $T \to \infty$ we don't have $R \to \infty$ in the same time, and with the same functions Φ as above. Then, as in (4.4.23), we obtain

$$\left\{\begin{array}{l} \left(\displaystyle\int_{C_T} v^p\, \tilde{\varphi}(x,t)\right)^{1-1/pq} \le \mathcal{D}_2^{1/q}\mathcal{D}_1, \\ \\ \left(\displaystyle\int_{C_T} u^q\, \tilde{\varphi}(x,t)\right)^{1-1/pq} \le \mathcal{D}_1^{1/p}\mathcal{D}_2, \end{array}\right. \tag{4.4.44}$$

where

$$C_T := [0,T] \times \left\{x \in \mathbb{R}^N \;;\; |x| \le 2R\right\} \qquad \int_{C_T} = \int_{C_T} dx\,dt,$$

$$\mathcal{D}_1 := C\left(\int_{C_T} \varphi_1^\ell \varphi_2^{-\frac{1}{q-1}} \left|D_{t|T}^{1+\alpha}\varphi_2\right|^{\widetilde{q}}\right)^{1/\widetilde{q}} + C\left(\int_{C_T} \varphi_1^{\ell-\widetilde{q}} \varphi_2^{-\frac{1}{q-1}} \left|\Delta_x\varphi_1 D_{t|T}^{\alpha}\varphi_2\right|^{\widetilde{q}}\right)^{1/\widetilde{q}},$$

and

$$\mathcal{D}_2 := C\left(\int_{C_T} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{1+\beta}\varphi_2\right|^{\widetilde{p}}\right)^{1/\widetilde{p}} + C\left(\int_{C_T} \varphi_1^{\ell-\widetilde{p}} \varphi_2^{-\frac{1}{p-1}} \left|\Delta_x\varphi_1 D_{t|T}^{\beta}\varphi_2\right|^{\widetilde{p}}\right)^{1/\widetilde{p}}.$$

Then, the scaled variables $\xi = R^{-1}x$, $\tau = T^{-1}t$ and (4.2.12) – (4.2.13) allow us to write :

$$\left\{\begin{array}{l} \left(\displaystyle\int_{C_T} v^p\, \tilde{\varphi}(x,t)\right)^{1-1/pq} \le C_1(T,R), \\ \\ \left(\displaystyle\int_{C_T} u^q\, \tilde{\varphi}(x,t)\right)^{1-1/pq} \le C_2(T,R), \end{array}\right. \tag{4.4.45}$$

where

$$C_1(T,R) := CT^{\alpha_1}R^{\beta_1} + CT^{\alpha_2}R^{\beta_2} + CT^{\alpha_3}R^{\beta_3} + CT^{\alpha_4}R^{\beta_4},$$

$$C_2(T,R) := CT^{\gamma_1}R^{\sigma_1} + CT^{\gamma_2}R^{\sigma_2} + CT^{\gamma_3}R^{\sigma_3} + CT^{\gamma_4}R^{\sigma_4},$$

with

$$\begin{cases} \alpha_1 := \dfrac{1}{q}\left[\dfrac{1}{\widetilde{p}} - (1+\beta)\right] + \left[\dfrac{1}{\widetilde{q}} - (1+\alpha)\right], & \alpha_2 := \dfrac{1}{q}\left[\dfrac{1}{\widetilde{p}} - (1+\beta)\right] + \left[\dfrac{1}{\widetilde{q}} - \alpha\right], \\ \alpha_3 := \dfrac{1}{q}\left[\dfrac{1}{\widetilde{p}} - \beta\right] + \left[\dfrac{1}{\widetilde{q}} - (1+\alpha)\right], & \alpha_4 := \dfrac{1}{q}\left[\dfrac{1}{\widetilde{p}} - \beta\right] + \left[\dfrac{1}{\widetilde{q}} - \alpha\right], \end{cases}$$

$$\begin{cases} \gamma_1 := \dfrac{1}{p}\left[\dfrac{1}{\widetilde{q}} - (1+\alpha)\right] + \left[\dfrac{1}{\widetilde{p}} - (1+\beta)\right], & \gamma_2 := \dfrac{1}{p}\left[\dfrac{1}{\widetilde{q}} - (1+\alpha)\right] + \left[\dfrac{1}{\widetilde{p}} - \beta\right], \\ \gamma_3 := \dfrac{1}{p}\left[\dfrac{1}{\widetilde{q}} - \alpha\right] + \left[\dfrac{1}{\widetilde{p}} - (1+\beta)\right], & \gamma_4 := \dfrac{1}{p}\left[\dfrac{1}{\widetilde{q}} - \alpha\right] + \left[\dfrac{1}{\widetilde{p}} - \beta\right], \end{cases}$$

$$\begin{cases} \beta_1 := \dfrac{N}{\widetilde{p}q} + \dfrac{N}{\widetilde{q}}, & \beta_2 := \dfrac{N}{\widetilde{p}q} + \dfrac{N}{\widetilde{q}} - 2, \\ \beta_3 := \dfrac{1}{q}\left[\dfrac{N}{\widetilde{p}} - 2\right] + \dfrac{N}{\widetilde{q}}, & \beta_4 := \dfrac{1}{q}\left[\dfrac{N}{\widetilde{p}} - 2\right] + \dfrac{N}{\widetilde{q}} - 2, \end{cases}$$

and

$$\begin{cases} \beta_1 := \dfrac{N}{\widetilde{q}p} + \dfrac{N}{\widetilde{p}}, & \beta_2 := \dfrac{N}{\widetilde{q}p} + \dfrac{N}{\widetilde{p}} - 2, \\ \beta_3 := \dfrac{1}{p}\left[\dfrac{N}{\widetilde{q}} - 2\right] + \dfrac{N}{\widetilde{p}}, & \beta_4 := \dfrac{1}{p}\left[\dfrac{N}{\widetilde{q}} - 2\right] + \dfrac{N}{\widetilde{p}} - 2. \end{cases}$$

Taking the limit as $T \to \infty$, we infer, as

$$p < \frac{1}{\delta} \iff \frac{1}{\widetilde{p}} - \beta < 0 \quad \text{and} \quad q < \frac{1}{\gamma} \iff \frac{1}{\widetilde{q}} - \alpha < 0,$$

that

$$\begin{cases} \left(\displaystyle\int_0^\infty \int_{|x|\le 2R} v^p \, \tilde{\varphi}(x,t)\, dx\, dt\right)^{1-1/pq} = 0, \\ \left(\displaystyle\int_0^\infty \int_{|x|\le 2R} u^q \, \tilde{\varphi}(x,t)\, dx\, dt\right)^{1-1/pq} = 0. \end{cases}$$

Finally, by letting $R \to \infty$, we get a contradiction with the fact that $u(x,t) > 0$ and $v(x,t) > 0$ for all $x \in \mathbb{R}^N$, $t > 0$. □

Remark 4.4.4 *We can extend our analysis to the following system*

$$\begin{cases} u_t + (-\Delta)^{\alpha/2} u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t \dfrac{\psi_1(x,s)|v|^{p-1}v(s)}{(t-s)^\gamma}\, ds & x \in \mathbb{R}^N, t > 0, \\ v_t + (-\Delta)^{\alpha/2} v = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t \dfrac{\psi_2(x,s)|u|^{q-1}u(s)}{(t-s)^\delta}\, ds & x \in \mathbb{R}^N, t > 0, \end{cases}$$

where $\psi_1, \psi_2 \in L^1_{Loc}(\mathbb{R}^N \times (0,\infty))$, $\psi_1(\cdot\,,t), \psi_2(\cdot\,,t) \geq 0$ *for all* $t \geq 0$, $\psi \not\equiv 0$, *and for all* $0 < R, B < T$, $\tau \in [0,1]$, $\xi \in [0,2]$, $p, q > 1$, $0 < \alpha \leq 2$ *and* $0 < \gamma, \delta < 1$. *We have*

$$\begin{cases} \psi_1(B^{-1/\alpha}T^{1/\alpha}\xi, T\tau) \geq C > 0, \\ \\ \psi_2(B^{-1/\alpha}T^{1/\alpha}\xi, T\tau) \geq C > 0, \end{cases}$$

if (p,q) *verifies* (4.4.11), *while*

$$\begin{cases} \psi_1(R\xi, T\tau) \geq C > 0, \\ \\ \psi_2(R\xi, T\tau) \geq C > 0, \end{cases}$$

if (p,q) *verifies* (4.4.12).

4.5 Blow-up Rate

In this section, we study the blow-up rate of the blowing-up solutions of the parabolic system (4.1.1) − (4.1.2) requiring the following on the initial data :

$$u_0, v_0 \geq 0, \quad u_0, v_0 \not\equiv 0 \quad \text{and} \quad u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N). \tag{4.5.1}$$

First, we give a lemma which will play a crucial role in the sequel.

Lemma 4.5.1 *Let* (u,v) *be a nonnegative classical solution of*

$$\begin{cases} u_t = \Delta u + J^{1-\gamma}_{-\infty|t}(v^p) & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ v_t = \Delta v + J^{1-\delta}_{-\infty|t}(u^q) & \text{in } \mathbb{R}^N \times \mathbb{R}, \end{cases} \tag{4.5.2}$$

where $\gamma, \delta \in (0,1)$, $p, q > 1$ *and*

$$J^{\delta}_{-\infty|t} f(t) := \frac{1}{\Gamma(\delta)} \int_{-\infty}^{t} (t-s)^{\delta-1} f(s)\, ds, \quad \text{for all } f \in L^r(\mathbb{R}) \ (1 \leq r \leq \infty).$$

Then, for

$$\frac{N}{2} \leq \max\left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1} \,;\, \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\}, \tag{4.5.3}$$

or

$$p < \frac{1}{\delta} \qquad \text{and} \qquad q < \frac{1}{\gamma}, \tag{4.5.4}$$

we have $u \equiv v \equiv 0$.

Proof It is sufficient to observe that

$$J^{1-\gamma}_{-\infty|t}(v^p) \geq J^{1-\gamma}_{-T|t}(v^p) \quad \text{and} \quad J^{1-\delta}_{-\infty|t}(u^q) \geq J^{1-\delta}_{-T|t}(u^q).$$

Then, by repeating the same computations as in Theorem 4.3.2 with $\varphi_2(t) := (1-t^2/T^2)_+^\eta$, $\eta \gg 1$, taking into account (4.2.15) − (4.2.17), and by taking

$$(\varphi_3(t)\varphi_1^\ell(x))^{1/q}(\varphi_3(t)\varphi_1^\ell(x))^{-1/q} \quad \text{instead of} \quad \widetilde{\varphi}^{1/q}\widetilde{\varphi}^{-1/q}$$

in (4.4.19) − (4.4.29) − (4.4.31) and

$$(\varphi_3(t)\varphi_1^\ell(x))^{1/p}(\varphi_3(t)\varphi_1^\ell(x))^{-1/p} \quad \text{instead of} \quad \widetilde{\varphi}^{1/p}\widetilde{\varphi}^{-1/p}$$

in (4.4.20) − (4.4.30) − (4.4.32), where $\ell \gg 1$ and $\varphi_3(t) := (1-t/T)_+^\eta$, with a use of Hölder's inequality, we conclude the result.
We note that, here, we need also to use the fact that $\varphi_3(t)\varphi_1^\ell(x) \leq \varphi(x,t)$ before the combination as in (4.4.21) − (4.4.22) and (4.4.33) − (4.4.34). □

Theorem 4.5.2 *Let*

$$\alpha_1 := \frac{(2-\gamma)+(2-\delta)p}{pq-1} \quad \textit{and} \quad \alpha_2 := \frac{(2-\delta)+(2-\gamma)q}{pq-1}. \tag{4.5.5}$$

If

$$\frac{N}{2} \leq \max\left\{\frac{(2-\delta)p+(1-\gamma)pq+1}{pq-1}\ ;\ \frac{(2-\gamma)q+(1-\delta)pq+1}{pq-1}\right\},$$

or

$$p < \frac{1}{\delta} \qquad \textit{and} \qquad q < \frac{1}{\gamma},$$

and (u,v) is the blowing-up solution of (4.1.1) − (4.1.2) − (4.5.1) in a finite time $T_{\max} := T^*$, then two constants $c_i, C_i > 0$, $i = 1,2$, exist such that

$$\begin{cases} c_1(T^*-t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(\cdot\,,t) \leq C_1(T^*-t)^{-\alpha_1}, & t \in (0,T^*), \\ c_2(T^*-t)^{-\alpha_2} \leq \sup_{\mathbb{R}^N} v(\cdot\,,t) \leq C_2(T^*-t)^{-\alpha_2}, & t \in (0,T^*). \end{cases} \tag{4.5.6}$$

Proof It is split into two parts :

• **The upper blow-up rate estimate :** Let

$$M_1(t) := \sup_{\mathbb{R}^N\times(0,t]} u \quad \text{and} \quad M_2(t) := \sup_{\mathbb{R}^N\times(0,t]} v, \qquad t \in (0,T^*).$$

First, we show that there is $\eta \in (0,1)$ such that

$$\eta \leq M_1(t)^{-1/\alpha_1}M_2(t)^{1/\alpha_2} \leq \eta^{-1}, \qquad t \in \left(\frac{T^*}{2}, T^*\right). \tag{4.5.7}$$

Indeed, we proceed by contradiction, as in [15][Proof of (2.3)]. If (4.5.7) were false then there would exists a sequence $t_n \to T^*$ such that

$$M_1(t_n)^{-1/\alpha_1}M_2(t_n)^{1/\alpha_2} \to 0 \quad \text{or} \quad M_2(t_n)^{-1/\alpha_2}M_1(t_n)^{1/\alpha_1} \to 0. \tag{4.5.8}$$

We will study the first case in (4.5.8) while the second one can be treated by proceeding in the same way. Then, using the fact that the condition (4.5.1) and Theorem 4.3.2 imply $M_2 > 0$, we infer that M_1 diverges as $t_n \to T^*$. So, for each t_n, we choose

$$(\widehat{x}_n, \widehat{t}_n) \in \mathbb{R}^N \times (0, t_n] \quad \text{such that} \quad u(\widehat{x}_n, \widehat{t}_n) \geq \frac{1}{2} M_1(t_n). \tag{4.5.9}$$

Obviously, $M_1(t_n) \to \infty$ implies $\widehat{t}_n \to T^*$.
We rescale the solution (u, v) about the corresponding point $(\widehat{x}_n, \widehat{t}_n)$ with the scaling factor $\lambda_n := \lambda(t_n) := (1/(2A)M_1(t_n))^{-1/(2\alpha_1)}$, for $A \geq 1$, as follows :

$$\varphi^{\lambda_n}(y, s) := \lambda_n^{2\alpha_1} u(\lambda_n y + \widehat{x}_n, \lambda_n^2 s + \widehat{t}_n), \qquad (y, s) \in \mathbb{R}^N \times I_n(T^*), \tag{4.5.10}$$

$$\psi^{\lambda_n}(y, s) := \lambda_n^{2\alpha_2} v(\lambda_n y + \widehat{x}_n, \lambda_n^2 s + \widehat{t}_n), \qquad (y, s) \in \mathbb{R}^N \times I_n(T^*), \tag{4.5.11}$$

where $I_n(t) := (-\lambda_n^{-2}\widehat{t}_n, \lambda_n^{-2}(t - \widehat{t}_n))$ for all $t > 0$. Then $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ is a mild solution of the system

$$\varphi_s = \Delta\varphi + J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|s}(\psi^p) \qquad \text{in } \mathbb{R}^N \times I_n(T^*), \tag{4.5.12}$$

$$\psi_s = \Delta\psi + J^{\beta}_{-\lambda_n^{-2}\widehat{t}_n|s}(\varphi^q) \qquad \text{in } \mathbb{R}^N \times I_n(T^*), \tag{4.5.13}$$

i.e., for $G(t) := G(x, t) := (4\pi t)^{-N/2} e^{-|x|^2/4t}$, we have

$$\varphi^{\lambda_n}(s) = G(s + \lambda_n^{-2}\widehat{t}_n) * \varphi^{\lambda_n}(-\lambda_n^{-2}\widehat{t}_n) + \int_{-\lambda_n^{-2}\widehat{t}_n}^{s} G(s - \sigma) * J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|\sigma}((\psi^{\lambda_n})^p)\, d\sigma, \tag{4.5.14}$$

$$\psi^{\lambda_n}(s) = G(s + \lambda_n^{-2}\widehat{t}_n) * \psi^{\lambda_n}(-\lambda_n^{-2}\widehat{t}_n) + \int_{-\lambda_n^{-2}\widehat{t}_n}^{s} G(s - \sigma) * J^{\alpha}_{-\lambda_n^{-2}\widehat{t}_n|\sigma}((\varphi^{\lambda_n})^q)\, d\sigma, \tag{4.5.15}$$

in $\mathbb{R}^N \times I_n(T^*)$ such that $\varphi^{\lambda_n}(0, 0) \geq A$, and

$$\begin{cases} 0 \leq \varphi^{\lambda_n} \leq \lambda_n^{2\alpha_1} M_1(t_n) = 2A, \\ 0 \leq \psi^{\lambda_n} \leq \lambda_n^{2\alpha_2} M_2(t_n) = (2A)^{\alpha_2/\alpha_1} M_2(t_n) M_1(t_n)^{-\alpha_2/\alpha_1}, \end{cases}$$

in $\mathbb{R}^N \times (-\lambda_n^{-2}\widehat{t}_n, 0]$; here $*$ is the space convolution.
So, as in Lemma 4.4.2, $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ is a weak solution of (4.5.12) – (4.5.13).
Now, invoking the interior regularity (cf. [38, Theorem 10.1 p. 204]), there exists $\mu \in (0, 1)$ such that for any $K > 0$, the sequences $\varphi^{\lambda_n}, \psi^{\lambda_n}$ are bounded in the $C^{\mu,\mu/2}(\overline{S}_{2K})$-norm by a constant independent of n, where

$$S_K := \left\{(x, t) \in \mathbb{R}^N \times \mathbb{R};\ |x| < K,\ -K < t \leq 0\right\}.$$

We notice, here, that we have used the maximal regularity to obtain a sufficient regularity for $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ to apply the interior regularity (for more precise information, see [17][Proof of Th. 3].

Moreover, Schauder's estimates imply that the $C^{2+\mu,1+\mu/2}(\overline{S}_K)$-norm of $\varphi^{\lambda_n}, \psi^{\lambda_n}$ is uniformly bounded. Thus, there is a subsequence converging to a solution (φ,ψ) of the following system

$$\varphi_s = \Delta\varphi + J^{\alpha}_{-\infty|t}(\psi^p), \quad \psi_s = \Delta\psi + J^{\beta}_{-\infty|t}(\varphi^q), \quad \text{in } \mathbb{R}^N \times (-\infty, 0], \tag{4.5.16}$$

such that $\varphi(0,0) \geq A$, and

$$0 \leq \varphi \leq 2A, \qquad 0 \leq \psi \leq \lim_{n\to\infty} (2A)^{\alpha_2/\alpha_1} M_2(t_n) M_1(t_n)^{-\alpha_2/\alpha_1} = 0.$$

Combining the last inequality with the system (4.5.16), we obtain $\varphi \equiv 0$ on $\mathbb{R}^N \times (-\infty, 0]$. This contradicts $\varphi(0,0) \geq A$. Consequently, (4.5.7) holds.

On the other hand, M_1, M_2 are positive, continuous and nondecreasing on $(0,T^*)$. Moreover, as $\lim_{t\to T^*} M_1(t) = \infty$, this allow us, for all $t_0 \in (0,T^*)$, to define

$$t_0^+ := t^+(t_0) := \max\{t \in (t_0, T^*) :\ M_1(t) = 2M_1(t_0)\}. \tag{4.5.17}$$

Let

$$\lambda(t_0) := \left(\frac{1}{2A} M_1(t_0)\right)^{-1/(2\alpha_1)}. \tag{4.5.18}$$

We claim that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \leq D, \qquad t_0 \in \left(\frac{T^*}{2}, T^*\right), \tag{4.5.19}$$

where $0 < D < \infty$ is a positive constant which does not depend on t_0.
Indeed, if (4.5.19) were false, then there would exist a sequence $t_n \to T^*$ such that

$$\lambda_n^{-2}(t_n^+ - t_n) \longrightarrow \infty,$$

where $\lambda_n = \lambda(t_n)$ and $t_n^+ = t^+(t_n)$. For each t_n choose $(\widehat{x}_n, \widehat{t}_n)$ as in (4.5.9) and rescale (u,v) arround $(\widehat{x}_n, \widehat{t}_n)$ as in (4.5.10)-(4.5.11). Then $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ is a mild solution (so weak solution) of the systems (4.5.12)-(4.5.13), in $\mathbb{R}^N \times I_n(T^*)$ such that $\varphi(0,0) \geq A$, and in $\mathbb{R}^N \times I_n(t_n^+)$ we have from (4.5.7) and the definition of t_n^+ the following inequalities

$$\begin{aligned} 0 &\leq \varphi^{\lambda_n} \leq \lambda_n^{2\alpha_1} M_1(t_n^+) = \lambda_n^{2\alpha_1} 2M_1(t_n) := 4A, \\ 0 &\leq \psi^{\lambda_n} \leq \lambda_n^{2\alpha_2} M_2(t_n^+) \\ &\leq \lambda_n^{2\alpha_2} \eta^{-\alpha_2} (M_1(t_n^+))^{\alpha_2/\alpha_1} = (4A)^{\alpha_2/\alpha_1} \eta^{-\alpha_2}. \end{aligned}$$

Interior regularity and uniform Schauder's estimates for $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ yield a subsequence converging in $C^{2+\mu,1+\mu/2}_{loc}(\mathbb{R}^N \times \mathbb{R}) \times C^{2+\mu,1+\mu/2}_{loc}(\mathbb{R}^N \times \mathbb{R})$ to a solution (φ,ψ) of

$$(\varphi)_s = \Delta\varphi + J^{\alpha}_{-\infty|t}(\psi^p), \quad (\psi)_s = \Delta\psi + J^{\beta}_{-\infty|t}(\varphi^q), \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{4.5.20}$$

such that $\varphi(0,0) \geq A$ and

$$0 \leq \varphi \leq 4A, \qquad 0 \leq \psi \leq (4A)^{\alpha_2/\alpha_1} \eta^{-\alpha_2}.$$

It follows from Lemma 4.5.1 that $\varphi \equiv \psi \equiv 0$. This is a contradiction with $\varphi(0,0) \geq A \geq 1$. Now (4.5.19) is true.
Next we use an idea from Hu [24]. From (4.5.18) and (4.5.19) it follows that

$$(t_0^+ - t_0) \leq D(2A)^{1/\alpha_1} M_1(t_0)^{-1/\alpha_1} \qquad \text{for any } t_0 \in \left(\frac{T^*}{2}, T^*\right).$$

Fix $t_0 \in (T^*/2, T^*)$ and denote $t_1 = t_0^+, t_2 = t_1^+, t_3 = t_2^+, \ldots$. Then

$$\begin{aligned} t_{j+1} - t_j &\leq D(2A)^{1/\alpha_1} M_1(t_j)^{-1/\alpha_1}, \\ M_1(t_{j+1}) &= 2M_1(t_j), \end{aligned}$$

$j = 0, 1, 2, \ldots$. Consequently,

$$\begin{aligned} T^* - t_0 &= \sum_{j=0}^{\infty}(t_{j+1} - t_j) \leq D(2A)^{1/\alpha_1} \sum_{j=0}^{\infty} M_1(t_j)^{-1/\alpha_1} \\ &= D(2A)^{1/\alpha_1} M_1(t_0)^{-1/\alpha_1} \sum_{j=0}^{\infty} 2^{-j/\alpha_1}. \end{aligned}$$

We conclude that

$$u(x,t_0) \leq M_1(t_0) \leq C(T^* - t_0)^{-\alpha_1}, \qquad \forall\, t_0 \in (0,T^*)$$

where

$$C_1 = 2A\left(D\sum_{j=0}^{\infty} 2^{-j/\alpha_1}\right)^{\alpha_1},$$

and consequently

$$\sup_{\mathbb{R}^N} u(\cdot\,,t) \leq C_1(T^* - t)^{-\alpha_1}, \qquad \forall\, t \in (0,T^*).$$

Finally, (4.5.7) implies

$$v(x,t_0) \leq M_2(t_0) \leq C_2(T^* - t_0)^{-\alpha_2}, \qquad \forall\, t_0 \in (0,T^*)$$

where

$$C_2 = \eta^{-\alpha_2} C_1^{\alpha_2/\alpha_1};$$

whereupon

$$\sup_{\mathbb{R}^N} v(\cdot\,,t) \leq C_2(T^* - t)^{-\alpha_2}, \qquad \forall\, t \in (0,T^*).$$

• **The lower blow-up rate estimate :** If we repeat the same proof of the local existence in Theorem 4.3.2, by taking $\|u\|_1 \leq \theta_1$ and $\|v\|_1 \leq \theta_2$ instead of $|||(u,v)||| \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)$ in the space E_T in (4.3.2) for all positive constants $\theta_1, \theta_2 > 0$ and all $0 < t < T$, then the condition (4.3.3) on T will be :

$$\|u_0\|_\infty + C_3 T^{2-\gamma}\theta_2^p \leq \theta_1 \quad \text{and} \quad \|v_0\|_\infty + C_4 T^{2-\delta}\theta_1^q \leq \theta_2. \tag{4.5.21}$$

Then, by the same reasoning, we infer that $\|u(t)\|_\infty \leq \theta_1$ and $\|v(t)\|_\infty \leq \theta_2$ for all $0 < t < T$. Consequently, if

$$\|u_0\|_\infty + C_3 t^{2-\gamma}\theta_2^p \leq \theta_1 \quad \text{and} \quad \|v_0\|_\infty + C_4 t^{2-\delta}\theta_1^q \leq \theta_2,$$

then $\|u(t)\|_\infty \leq \theta_1$ and $\|v(t)\|_\infty \leq \theta_2$. Applying this to any point in the trajectories, we see that if $0 \leq s < t$ and

$$(t-s)^{2-\gamma} \leq \frac{\theta_1 - \|u(s)\|_\infty}{C_3\theta_2^p} \quad \text{and} \quad (t-s)^{2-\delta} \leq \frac{\theta_2 - \|v(s)\|_\infty}{C_4\theta_1^q}, \tag{4.5.22}$$

whereafter $\|u(t)\|_\infty \leq \theta_1$ and $\|v(t)\|_\infty \leq \theta_2$, for all $0 < t < T$.
Moreover, if $0 \leq s < T^*$, $\|u(s)\|_\infty < \theta_1$ and $\|v(s)\|_\infty < \theta_2$, then :

$$(T^*-s)^{2-\gamma} > \frac{\theta_1 - \|u(s)\|_\infty}{C_3\theta_2^p} \quad \text{and} \quad (T^*-s)^{2-\delta} > \frac{\theta_2 - \|v(s)\|_\infty}{C_4\theta_1^q}. \tag{4.5.23}$$

Indeed, suppose to the contrary that for some $\theta_1 > \|u(s)\|_\infty, \theta_2 > \|v(s)\|_\infty$ and all $t \in (s, T^*)$ we have

$$(t-s)^{2-\gamma} \leq \frac{\theta_1 - \|u(s)\|_\infty}{C_3\theta_2^p} \quad \text{or} \quad (t-s)^{2-\delta} \leq \frac{\theta_2 - \|v(s)\|_\infty}{C_4\theta_1^q};$$

then, using (4.5.22), we infer that $\|u(t)\|_\infty \leq \theta_1$ or $\|v(t)\|_\infty \leq \theta_2$ for all $t \in (s, T^*)$. Contradiction with the fact that $\|u(t)\|_\infty \to \infty$ and $\|v(t)\|_\infty \to \infty$ as $t \to T^*$.
Next, for example, letting $\theta_1 = 2\|u(s)\|_\infty$ and $\theta_2 = 2\|v(s)\|_\infty$ in (4.5.23), we see that for $0 < s < T^*$ we have :

$$(T^*-s)^{2-\gamma} > \frac{\|u(s)\|_\infty}{2^p C_3\|v(s)\|_\infty^p} \quad \text{and} \quad (T^*-s)^{2-\delta} > \frac{\|v(s)\|_\infty}{2^q C_4\|u(s)\|_\infty^q},$$

which, imply

$$c_1(T^*-t)^{-\alpha_1} < \|u(s)\|_\infty \quad \text{and} \quad c_2(T^*-t)^{-\alpha_2} < \|v(s)\|_\infty,$$

where

$$c_1 := \left(2^{p(1+q)}C_3C_4^p\right)^{-\frac{1}{pq-1}}, \qquad c_2 := \left(2^{q(1+p)}C_4C_3^q\right)^{-\frac{1}{pq-1}}.$$

By the continuity and the positivity of u and v, we get

$$c_1(T^*-s)^{-\alpha_1} < \sup_{x\in\mathbb{R}^N} u(x,s) \quad \text{and} \quad c_2(T^*-s)^{-\alpha_2} < \sup_{x\in\mathbb{R}^N} v(x,s), \quad \forall\, s \in (0,T^*). \tag{4.5.24}$$

□

4.6 Necessary conditions for local or global existence

A necessary condition for the existence of local or global solutions to the problem (4.1.1) − (4.1.2) are presented in this section. We obtain that these conditions depend on the behavior of the initial conditions at infinity.

Theorem 4.6.1 (*Necessary conditions for global existence*)
Let $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, $u_0, v_0 \geq 0$ and $p, q > 1$. If (u, v) is a mild global solution to problem (4.1.1) − (4.1.2), then there is a positive constant $C > 0$ such that

$$\liminf_{|x|\to\infty}(u_0(x)|x|^{2\alpha_1}) \leq C \quad \textit{and} \quad \liminf_{|x|\to\infty}(v_0(x)|x|^{2\alpha_2}) \leq C, \tag{4.6.1}$$

where α_1, α_2 are defined in (4.5.5) and $C > 0$ is a real positive number which may change from line to line.

Proof Let (u, v) be a global mild solution to (4.1.1) − (4.1.2), then $u \in C([0, R]; L^\infty(B_{2R}))$ for all $R \gg 1$, where B_{2R} stands for the closed ball of center 0 and radius $2R$. So we repeat the same computation as in the proof of Theorem 4.4.3 (here in bounded domain) by taking $\tilde{\varphi}(x, t) := \varphi_1(x/R)\,\varphi_2(t)$ where $\varphi_2(t) := (1 - t/(R^2))_+^\ell$ instead of the one chosen in Theorem 4.4.3, where $0 \leq \varphi_1 \in H^2(B_2)$ is the first eigenfunction for $-\Delta_x$ relative to the first eigenvalue $\lambda_1 := \inf\{\|u\|_{H^1};\ \|u\|_{L^2} = 1 \text{ and } u = 0 \text{ in } \partial B_2\}$. Then, as in (4.4.19) − (4.4.20), we have

$$\begin{aligned}
&\int_{\Sigma_1} v^p\, \tilde{\varphi}\, dx\, dt \;+\; C\, R^{-2\alpha} \int_{|x|\leq 2R} u_0(x)\varphi_1(x/R)\, dx \\
&\leq\; C \int_{\Sigma_1} u\, \tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\varphi_1(x/R)\left|D^{1+\alpha}_{t|R^2}\varphi_2(t)\right|\, dx\, dt \\
&+ C \int_{\Sigma_1} u\, \tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}\left|(-\Delta_x)\varphi_1(x/R)\; D^{\alpha}_{t|R^2}\varphi_2(t)\right|\, dx\, dt,
\end{aligned} \tag{4.6.2}$$

and

$$\begin{aligned}
&\int_{\Sigma_1} u^q\, \tilde{\varphi}\, dx\, dt \;+\; C\, R^{-2\beta} \int_{|x|\leq 2R} v_0(x)\varphi_1(x/R)\, dx \\
&\leq\; C \int_{\Sigma_1} v\, \tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\varphi_1(x/R)\left|D^{1+\beta}_{t|R^2}\varphi_2(t)\right|\, dx\, dt \\
&+ C \int_{\Sigma_1} v\, \tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}\left|(-\Delta_x)\varphi_1(x/R)\; D^{\beta}_{t|R^2}\varphi_2(t)\right|\, dx\, dt.
\end{aligned} \tag{4.6.3}$$

where $\alpha := 1 - \gamma$, $\beta := 1 - \delta$ and

$$\Sigma_1 := \left\{(x, t) \in \mathbb{R}^N \times [0, \infty);\ |x| \leq 2R,\ t \leq R^2\right\}.$$

Hence, using Hölder's inequality, we get

$$\mathcal{V}_3 + CR^{-2\alpha}\int_{|x|\leq 2R} u_0(x)\varphi_1(x/R)\, dx \leq \mathcal{U}_3^{1/q}\left[\mathcal{E}_1^{1/\tilde{q}} + \mathcal{F}_1^{1/\tilde{q}}\right], \tag{4.6.4}$$

and

$$\mathcal{U}_3 + CR^{-2\beta}\int_{|x|\leq 2R} v_0(x)\varphi_1(x/R)\, dx \leq \mathcal{V}_3^{1/p}\left[\mathcal{E}_2^{1/\tilde{p}} + \mathcal{F}_2^{1/\tilde{p}}\right], \tag{4.6.5}$$

where

$$\mathcal{U}_3 := \int_{\Sigma_1} u^q\, \tilde{\varphi}\, dx\, dt, \qquad \mathcal{V}_3 := \int_{\Sigma_1} v^p\, \tilde{\varphi}\, dx\, dt$$

$$\mathcal{E}_1 := C\left(\int_{\Sigma_1} \varphi_1\, \varphi_2^{-\frac{1}{q-1}} \left|D^{1+\alpha}_{t|R^2}\varphi_2\right|^{\widetilde{q}}\, dx\, dt\right)^{1/\widetilde{q}},$$

$$\mathcal{E}_2 := C\left(\int_{\Sigma_1} \varphi_1\, \varphi_2^{-\frac{1}{p-1}} \left|D^{1+\beta}_{t|R^2}\varphi_2\right|^{\widetilde{p}}\, dx\, dt\right)^{1/\widetilde{p}},$$

$$\mathcal{F}_1 := C\left(\int_{\Sigma_1} (\varphi_1\varphi_2)^{-\frac{1}{q-1}} \left|(-\Delta_x)\varphi_1 D^{\alpha}_{t|R^2}\varphi_2\right|^{\widetilde{q}}\, dx\, dt\right)^{1/\widetilde{q}},$$

and

$$\mathcal{F}_2 := C\left(\int_{\Sigma_1} (\varphi_1\varphi_2)^{-\frac{1}{p-1}} \left|(-\Delta_x)\varphi_1 D^{\beta}_{t|R^2}\varphi_2\right|^{\widetilde{p}}\, dx\, dt\right)^{1/\widetilde{p}}.$$

Combining (4.6.4) and (4.6.5) we obtain

$$\mathcal{V}_3 + CR^{-2\alpha}\int_{|x|\le 2R} u_0(x)\varphi_1(x/R)\, dx \le \mathcal{V}_3^{1/pq}\left[\left(\mathcal{E}_2^{1/q\widetilde{p}} + \mathcal{F}_2^{1/q\widetilde{p}}\right)\left(\mathcal{E}_1^{1/\widetilde{q}} + \mathcal{F}_1^{1/\widetilde{q}}\right)\right], \tag{4.6.6}$$

and

$$\mathcal{U}_3 + CR^{-2\beta}\int_{|x|\le 2R} v_0(x)\varphi_1(x/R)\, dx \le \mathcal{U}_3^{1/pq}\left[\left(\mathcal{E}_1^{1/p\widetilde{q}} + \mathcal{F}_1^{1/p\widetilde{q}}\right)\left(\mathcal{E}_2^{1/\widetilde{p}} + \mathcal{F}_2^{1/\widetilde{p}}\right)\right]. \tag{4.6.7}$$

Moreover, using Young's inequality in the right-hand side of (4.6.6) − (4.6.7), allows us to obtain

$$\mathcal{V}_3 + CR^{-2\alpha}\int_{|x|\le 2R} u_0(x)\varphi_1(x/R)\, dx \le \mathcal{V}_3 + \left[\left(\mathcal{E}_2^{1/q\widetilde{p}} + \mathcal{F}_2^{1/q\widetilde{p}}\right)\left(\mathcal{E}_1^{1/\widetilde{q}} + \mathcal{F}_1^{1/\widetilde{q}}\right)\right]^{\frac{pq}{pq-1}},$$

and

$$\mathcal{U}_3 + CR^{-2\beta}\int_{|x|\le 2R} v_0(x)\varphi_1(x/R)\, dx \le \mathcal{U}_3 + \left[\left(\mathcal{E}_1^{1/p\widetilde{q}} + \mathcal{F}_1^{1/p\widetilde{q}}\right)\left(\mathcal{E}_2^{1/\widetilde{p}} + \mathcal{F}_2^{1/\widetilde{p}}\right)\right]^{\frac{pq}{pq-1}}.$$

These imply

$$CR^{-2\alpha}\int_{|x|\le 2R} u_0(x)\varphi_1(x/R)\, dx \le \left[\left(\mathcal{E}_2^{\frac{p-1}{pq-1}} + \mathcal{F}_2^{\frac{p-1}{pq-1}}\right)\left(\mathcal{E}_1^{\frac{p(q-1)}{pq-1}} + \mathcal{F}_1^{\frac{p(q-1)}{pq-1}}\right)\right], \tag{4.6.8}$$

and

$$CR^{-2\beta}\int_{|x|\le 2R} v_0(x)\varphi_1(x/R)\, dx \le \left[\left(\mathcal{E}_1^{\frac{q-1}{pq-1}} + \mathcal{F}_1^{\frac{q-1}{pq-1}}\right)\left(\mathcal{E}_2^{\frac{q(p-1)}{pq-1}} + \mathcal{F}_2^{\frac{q(p-1)}{pq-1}}\right)\right]. \tag{4.6.9}$$

Now, if we take the scaled variables $\tau = R^{-2}t$, $\xi = R^{-1}x$ and use the fact that $(-\Delta_x)\varphi_1(x/R) = R^{-2}\lambda_1\varphi_1(x/R)$, we arrive at

$$CR^{-2\alpha}\int_{|\xi|\le 2} u_0(R\xi)\varphi_1(\xi)\, d\xi \le \overline{C}_1(R)\int_{|\xi\le 2} \varphi_1(\xi)\, d\xi, \tag{4.6.10}$$

and

$$CR^{-2\beta}\int_{|\xi|\le 2} v_0(R\xi)\varphi_1(\xi)\, d\xi \le \overline{C}_2(R)\int_{|\xi\le 2} \varphi_1(\xi)\, d\xi, \tag{4.6.11}$$

where

$$\overline{C}_1(R) := CR^{-2\frac{(1+\beta p)+p(1+\alpha q)}{pq-1}} \quad \text{and} \quad \overline{C}_2(R) := CR^{-2\frac{(1+\alpha q)+q(1+\beta p)}{pq-1}}.$$

Inequalities (4.6.10)-(4.6.11) can be arranged into

$$\begin{aligned} \int_{|\xi|\le 2} u_0(R\xi)\varphi_1(\xi) \le \overline{C}_3(R)\int_{|\xi\le 2}\varphi_1(\xi) &= \overline{C}_3(R)\int_{|\xi\le 2}|R\xi|^{2\alpha_1}|R\xi|^{-2\alpha_1}\varphi_1(\xi) \\ &\le C\int_{|\xi|\le 2}|R\xi|^{-2\alpha_1}\varphi_1(\xi), \end{aligned} \tag{4.6.12}$$

and

$$\begin{aligned} \int_{|\xi|\le 2} v_0(R\xi)\varphi_1(\xi) \le \overline{C}_4(R)\int_{|\xi\le 2}\varphi_1(\xi) &= \overline{C}_4(R)\int_{|\xi\le 2}|R\xi|^{2\alpha_2}|R\xi|^{-2\alpha_2}\varphi_1(\xi) \\ &\le C\int_{|\xi|\le 2}|R\xi|^{-2\alpha_2}\varphi_1(\xi), \end{aligned} \tag{4.6.13}$$

where $\overline{C}_3(R) := CR^{2\alpha}\overline{C}_1(R) = CR^{-2\alpha_1}$ and $\overline{C}_4(R) := CR^{2\beta}\overline{C}_2(R) = CR^{-2\alpha_2}$. Using the estimate

$$\begin{aligned} \inf_{|\xi|>1}(u_0(R\xi)|R\xi|^{2\alpha_1})\int_{|\xi|\le 2}|R\xi|^{-2\alpha_1}\varphi_1(\xi) &\le \int_{1<|\xi|\le 2} u_0(R\xi)\varphi_1(\xi) \\ &\le \int_{|\xi|\le 2} u_0(R\xi)\varphi_1(\xi) \end{aligned}$$

and

$$\begin{aligned} \inf_{|\xi|>1}(v_0(R\xi)|R\xi|^{2\alpha_2})\int_{|\xi|\le 2}|R\xi|^{-2\alpha_2}\varphi_1(\xi) &\le \int_{1<|\xi|\le 2} v_0(R\xi)\varphi_1(\xi) \\ &\le \int_{|\xi|\le 2} v_0(R\xi)\varphi_1(\xi) \end{aligned}$$

in the right-hand side of (4.6.12) and (4.6.13), we conclude, after dividing by $\int_{|\xi|\le 2}|R\xi|^{-2\alpha_1}\varphi_1(\xi)$ and $\int_{|\xi|\le 2}|R\xi|^{-2\alpha_2}\varphi_1(\xi)$, that

$$\inf_{|\xi|>1}(u_0(R\xi)|R\xi|^{2\alpha_1}) \le C, \tag{4.6.14}$$

and

$$\inf_{|\xi|>1}(v_0(R\xi)|R\xi|^{2\alpha_2}) \le C. \tag{4.6.15}$$

Passing to the limit in (4.6.14) − (4.6.15), as $R \to \infty$, we obtain

$$\liminf_{|x|\to\infty}(u_0(x)|x|^{2\alpha_1}) \le C,$$

and

$$\liminf_{|x|\to\infty}(v_0(x)|x|^{2\alpha_2}) \le C.$$

□

Corollary 4.6.2 (*sufficient conditions for the nonexistence of global solutions*)
Let $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, $u_0, v_0 \geq 0$ and $p, q > 1$. If

$$\liminf_{|x|\to\infty}(u_0(x)|x|^{2\alpha_1}) = \liminf_{|x|\to\infty}(v_0(x)|x|^{2\alpha_2}) = +\infty,$$

then the system (4.1.1) − (4.1.2) cannot admit a global solution. □

Finally, a necessary condition is given for local existence. We obtain a similar estimate of T founded in the proof of Theorem 4.3.2, as $|x|$ goes to infinity.

Theorem 4.6.3 (*Necessary conditions for local existence*)
Let $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, $u_0, v_0 \geq 0$ and $p, q > 1$. If (u, v) is a local solution to system (4.1.1) − (4.1.2) on $[0, T]$ where $0 < T < \infty$, then we have the estimates

$$\liminf_{|x|\to\infty} u_0(x) \leq C\, T^{-\alpha_1} \quad \textit{and} \quad \liminf_{|x|\to\infty} v_0(x) \leq C\, T^{-\alpha_2}, \tag{4.6.16}$$

for some positive constant $C > 0$. Note that, if $A := \liminf_{|x|\to\infty} u_0(x)$ and $B := \liminf_{|x|\to\infty} v_0(x)$ then we obtain, a similar estimate to the one founded in (4.3.7),

$$\begin{cases} C^{-(pq-1)}T^{(2-\gamma)+(2-\delta)p}A^{pq-1} \leq 1, \\ \\ C^{-(pq-1)}T^{(2-\delta)+(2-\gamma)q}B^{pq-1} \leq 1. \end{cases}$$

Proof Take here, for $R > 0$ sufficiently large, $\varphi(x,t) := \varphi_1(x/R)\varphi_2(t)$ where $\varphi_2(t) := (1 - t/T)_+^\ell$, instead of the one chosen in Theorem 4.6.1. Then, as for (4.6.8) − (4.6.9), we obtain, with $\Sigma_2 := \left\{(x,t) \in \mathbb{R}^N \times [0,\infty);\ |x| \leq 2R,\ t \leq T\right\}$,

$$CT^{-\alpha}\int_{|x|\leq 2R} u_0(x)\varphi_1(x/R)\,dx \leq \left[\left(\mathcal{G}_2^{\frac{p-1}{pq-1}} + \mathcal{H}_2^{\frac{p-1}{pq-1}}\right)\left(\mathcal{G}_1^{\frac{p(q-1)}{pq-1}} + \mathcal{H}_1^{\frac{p(q-1)}{pq-1}}\right)\right], \tag{4.6.17}$$

and

$$CT^{-\beta}\int_{|x|\leq 2R} v_0(x)\varphi_1(x/R)\,dx \leq \left[\left(\mathcal{G}_1^{\frac{q-1}{pq-1}} + \mathcal{H}_1^{\frac{q-1}{pq-1}}\right)\left(\mathcal{G}_2^{\frac{q(p-1)}{pq-1}} + \mathcal{H}_2^{\frac{q(p-1)}{pq-1}}\right)\right], \tag{4.6.18}$$

where

$$\mathcal{G}_1 := C\left(\int_{\Sigma_2} \varphi_1\, \varphi_2^{-\frac{1}{q-1}} \left|D_{t|T}^{1+\alpha}\varphi_2\right|^{\widetilde{q}}\,dx\,dt\right)^{1/\widetilde{q}},$$

$$\mathcal{G}_2 := C\left(\int_{\Sigma_2} \varphi_1\, \varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{1+\beta}\varphi_2\right|^{\widetilde{p}}\,dx\,dt\right)^{1/\widetilde{p}},$$

$$\mathcal{H}_1 := C\left(\int_{\Sigma_2} (\varphi_1\varphi_2)^{-\frac{1}{q-1}} \left|(-\Delta_x)\varphi_1 D_{t|T}^{\alpha}\varphi_2\right|^{\widetilde{q}}\,dx\,dt\right)^{1/\widetilde{q}},$$

and

$$\mathcal{H}_2 := C\left(\int_{\Sigma_2} (\varphi_1\varphi_2)^{-\frac{1}{p-1}} \left|(-\Delta_x)\varphi_1 D_{t|T}^{\beta}\varphi_2\right|^{\widetilde{p}}\,dx\,dt\right)^{1/\widetilde{p}},$$

with $\alpha := 1-\gamma$ $\beta := 1-\delta$, $\widetilde{p} := p/(p-1)$ and $\widetilde{q} := q/(q-1)$. If we take the scaled variables $\tau = T^{-1}t$, $\xi = R^{-1}x$, taking into account the fact that $(-\Delta_x)\varphi_1(x/R) = R^{-2}\lambda_1\varphi_1(x/R)$, then (4.6.17) – (4.6.18) implies

$$CT^{-\alpha}\int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi \leq \overline{C}_3(R,T)\int_{|\xi\leq 2}\varphi_1(\xi)\,d\xi, \tag{4.6.19}$$

and

$$CT^{-\beta}\int_{|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi)\,d\xi \leq \overline{C}_4(R,T)\int_{|\xi\leq 2}\varphi_1(\xi)\,d\xi, \tag{4.6.20}$$

where

$$\overline{C}_3(R,T) := C\left(T^{-\frac{1+\beta p}{pq-1}} + T^{-\frac{1-p(1-\beta)}{pq-1}}R^{-\frac{2p}{pq-1}}\right)\left(T^{-\frac{(1+\alpha q)p}{pq-1}} + T^{-\frac{(1-q(1-\alpha))p}{pq-1}}R^{-\frac{2pq}{pq-1}}\right),$$

and

$$\overline{C}_4(R,T) := C\left(T^{-\frac{1+\alpha q}{pq-1}} + T^{-\frac{1-q(1-\alpha)}{pq-1}}R^{-\frac{2q}{pq-1}}\right)\left(T^{-\frac{(1+\beta p)q}{pq-1}} + T^{-\frac{(1-p(1-\beta))q}{pq-1}}R^{-\frac{2pq}{pq-1}}\right).$$

Thus

$$\int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi \leq \overline{C}_5(R,T)\int_{|\xi\leq 2}\varphi_1(\xi)\,d\xi, \tag{4.6.21}$$

and

$$\int_{|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi)\,d\xi \leq \overline{C}_6(R,T)\int_{|\xi\leq 2}\varphi_1(\xi)\,d\xi, \tag{4.6.22}$$

where

$$\overline{C}_5(R,T) := CT^{\alpha}\overline{C}_3(R,T) \quad \text{and} \quad \overline{C}_6(R,T) := CT^{\beta}\overline{C}_4(R,T).$$

Using the estimate

$$\begin{aligned}\inf_{|\xi|>1}(u_0(R\xi))\int_{|\xi|\leq 2}\varphi_1(\xi)\,d\xi &\leq \int_{1<|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi\\ &\leq \int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi\end{aligned}$$

(resp.

$$\begin{aligned}\inf_{|\xi|>1}(v_0(R\xi))\int_{|\xi|\leq 2}\varphi_1(\xi)\,d\xi &\leq \int_{1<|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi)\,d\xi\\ &\leq \int_{|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi)\,d\xi)\end{aligned}$$

in the left-hand side of (4.6.21) (resp. (4.6.22)), we conclude, after dividing by the term

$$\int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi)\,d\xi, \quad \text{(resp. } \int_{|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi)\,d\xi),$$

that

$$\inf_{|\xi|>1} u_0(R\xi) \leq \overline{C}_5(R,T), \tag{4.6.23}$$

and

$$\inf_{|\xi|>1} v_0(R\xi) \leq \overline{C}_6(R,T). \tag{4.6.24}$$

Passing to the limit in (4.6.23) − (4.6.24), as $R \to \infty$, we arrive at

$$\liminf_{|x|\to\infty} u_0(x) \leq C\, T^{-\alpha_1},$$

and

$$\liminf_{|x|\to\infty} v_0(x) \leq C\, T^{-\alpha_2}.$$

□

Chapitre 5

Hyperbolic equation with a nonlocal nonlinearity

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Abstract

In this article, we study the existence of local solutions to a hyperbolic equation with a nonlocal in time nonlinearity. Moreover, we give a blow-up theorem under some conditions on the initial data and the exponents of the nonlinear forcing term.

Keywords : Hyperbolic equation, mild and weak solutions, local existence, Strichartz estimate, blow-up, Riemann-Liouville fractional integrals and derivatives.
MSC : 58J45; 26A33; 35B44

5.1 Introduction

We investigate the following nonlinear hyperbolic equation which contains a nonlocal in time nonlinearity

$$\left\{\begin{array}{ll} u_{tt}-\Delta u=\dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t (t-s)^{-\gamma}|u(s)|^p\,ds & x\in\mathbb{R}^N, t>0,\\ \\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x) & x\in\mathbb{R}^N,\end{array}\right. \tag{5.1.1}$$

where $(u_0,u_1)\in H^\mu(\mathbb{R}^N)\times H^{\mu-1}(\mathbb{R}^N)$, $0<\mu\leq N/2$, $N\geq 2$, $0<\gamma<1$, $p>1$, Δ is the usual laplacian and u_{tt} stands for the second in time derivative of u, Γ is the Euler gamma function and $H^\mu(\mathbb{R}^N)$ is the homogeneous Sobolev space of order μ defined by

$$H^\mu(\mathbb{R}^N)=\left\{u\in\mathcal{S}';\ (-\Delta)^{\mu/2}u\in L^2(\mathbb{R}^N)\right\},$$

if $\mu\notin\mathbb{N}$, and by

$$H^\mu(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N);\ (-\Delta)^{\mu/2}u\in L^2(\mathbb{R}^N)\right\},$$

if $\mu \in \mathbb{N}$, where $\mathcal{S}'$ is the space of Schwartz' distributions and $(-\Delta)^{\mu/2}$ is the fractional laplacian operator defined by

$$(-\Delta)^{\mu/2}u(x) := \mathcal{F}^{-1}\left(|\xi|^{\mu}\mathcal{F}(u)(\xi)\right)(x)$$

for every $u \in D((-\Delta)^{\mu/2}) = H^{\mu}(\mathbb{R}^N)$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, respectively.

The study of the non-existence (blow-up) of global solutions to semilinear wave equations has been initiated in the early sixties and intensively developed since then. An approach for the non-existence of global solutions has been initiated by Keller, then by John and Kato. It is based on an averaging method for positive solutions, usually with compact support. Much has been devoted to the case of the equation

$$u_{tt} - \Delta u = |u|^p, \quad p > 1. \tag{5.1.2}$$

For $p = p_c > 1$, where p_c is the positive root of

$$(N-1)p^2 - (N+1)p - 2 = 0,$$

John first proved in [26] that if $1 < p \leq p_c$, there exist smooth initial data arbitrarily small in $C_c^{\infty}(\mathbb{R}^N)$ such that no corresponding global solution exists. Actually, when $N = 3$, John's result states that when $1 < p < 1 + \sqrt{2}$ all solutions with initial data in $C_c^{\infty}(\mathbb{R}^N)$ blow-up in finite time. Later on, the initial value $1 + \sqrt{2}$ was included in Glassey's proof [22] under the additional assumption that u_0 and u_1 have both positive average. Glassey's technique is to derive differential inequalities which are satisfied by the average function $t \longmapsto \int_{\mathbb{R}^N} u(x,t)\,dx$.
Some other cases were considered by Sideris [48] and Schaeffer [47] who gave extensions of Glassey's result for different dimensions but with the same conditions on initial data as in Glassey's work (compact support and positive average).
A sharp result under a weaker assumptions was obtained by Kato [29] with an easier proof. In particular, Kato pointed out the role of the exponent $p_0 = (N+1)/(N-1)$ for $N \geq 2$ in order to have more general initial data, but still with compact support. The fact that the support of $u(.,t)$ is included in a cone $\{x;\ |x| \leq t+R\}$ plays a fundamental role in deriving the differential inequalities.

In this paper, we prove the local existence and the non-existence of global mild solutions of a new type of class of equations (4.1.1) with nonlocal in time nonlinearities. Note that our blow-up result is similar to that of Glassey.
Our analysis is based on the observation that Eq. (5.1.1) can be written in the following form :

$$u_{tt} - \Delta u = J^{\alpha}_{0|t}(|u|^p), \tag{5.1.3}$$

where the left-sided Riemann-Liouville fractional integral $J^{\alpha}_{0|t}$ of order $\alpha \in (0,1)$, defined in (5.2.16), plays a crucial role in the proof of the theorem on the blow-up ; we have set in (5.1.3) $\alpha = 1 - \gamma \in (0,1)$. We have three results :

– If $(u_0, u_1) \in H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N)$, $N \geq 2$, $0 < \mu \leq N/2$ and $p > 1$ verifies the condition (5.3.5) below, then there exist $T > 0$ and a unique mild solution u to the problem (5.1.1) such that $(u, u_t) \in C([0,T], H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N))$.

– If $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, $N \geq 2$ and $p > 1$ verifies condition (5.3.19) below, there exist a maximal time $T_{\max} > 0$ and a unique maximal mild solution u to the problem (5.1.1) such that $(u, u_t) \in C([0, T_{\max}), H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$.

– If $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ have a positive average, $N \geq 2$ and let $p > 1$ be such that

$$\begin{cases} 1 + \dfrac{2}{N} \leq p \leq 1 + \dfrac{3-\gamma}{(N-2+\gamma)_+} := p* & \text{if } N > 2 \\ \dfrac{3}{2} \leq p \leq \dfrac{3}{\gamma} & \text{if } N = 2, \end{cases} \tag{5.1.4}$$

or

$$\begin{cases} 1 + \dfrac{2}{N} \leq p < \dfrac{1}{\gamma} & \text{if } N > 2 \\ \dfrac{3}{2} \leq p < \dfrac{1}{\gamma} & \text{if } N = 2, \end{cases} \tag{5.1.5}$$

where $(\cdot)_+$ is the positive part and γ verifies the conditions (5.4.11) or (5.4.12) (respectively) below, then the solution of (5.1.1) blows up in finite time.
In the proof of the existence of the local mild solution, we use Strichartz' estimate founded in [30], while in the proof of the blow-up of solution we use the method of Pohozaev [40] based on a rescaling of the variables of the test function in the weak formulation.

The organization of this paper is as follows. In section 5.2, we give some properties, results and notations that will be used in the sequel. In Section 5.3, we present the local existence result of solutions for the hyperbolic equation (5.1.1). Section 5.4, contains the blow-up result of solutions for (5.1.1).

5.2 Preliminaries, notations

In this section, we present some definitions, notations and results concerning the wave operator, fractional integrals and fractional derivatives that will be used hereafter. For more information see [1], [19], [32] and [46].
Let us consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \mathbb{R}^N. \end{cases} \tag{5.2.1}$$

We define $K(t)$ and $\dot{K}(t)$ by $K(t) := \omega^{-1} \sin \omega t$ and $\dot{K}(t) := \cos \omega t$ where ω^{-1} is the inverse of the fractional laplacian operator $\omega := (-\Delta)^{1/2}$ of order $1/2$ defined above. The solution of the Cauchy problem (5.2.1) can be written, according to Duhamel's principle, as $u = v + w$, where v is the solution of the homogeneous equation

$$\begin{cases} v_{tt} - \Delta v = 0 & (x,t) \in \mathbb{R}^N \times (0,T), \\ v(x,0) = u_0(x),\ v_t(x,0) = u_1(x) & x \in \mathbb{R}^N, \end{cases} \tag{5.2.2}$$

namely

$$v(t) = \dot{K}(t)u_0 + K(t)u_1 \quad \left(v_t(t) = K(t)\Delta u_0 + \dot{K}(t)u_1\right),$$

which is denoted by $H(t)U_0 = (v(t), v_t(t))$ where $U_0 := (u_0, u_1)$; and w is the solution of the inhomogeneous equation with zero initial data

$$\begin{cases} w_{tt} - \Delta w = f, & (x,t) \in \mathbb{R}^N \times (0,T), \\ w(x,0) = 0,\ w_t(x,0) = 0, & x \in \mathbb{R}^N, \end{cases} \tag{5.2.3}$$

which can be written, for $t \geq 0$, as

$$w(t) = \int_0^t K(t-s)f(s)\,ds =: K * f(t) \quad \left(w_t(t) = \int_0^t \dot{K}(t-s)f(s)\,ds =: \dot{K} * f(t)\right).$$

The initial data $U_0 = (u_0, u_1)$ of the problem (5.2.1) will be taken in the space

$$Y^\mu := H^\mu(\mathbb{R}^N) \times H^{\mu-1}(\mathbb{R}^N). \tag{5.2.4}$$

In addition, for $(u,v) \in Y^\mu$, we define the norm of (u,v) by

$$\|(u,v)\|_{Y^\mu} := \|u\|_{H^\mu} + \|v\|_{H^{\mu-1}}.$$

The important case is the one of $\mu = 1$ which is encountered in physics'.

Next, we give the admissible version of the Strichartz estimates due to Keel and Tao [30]. Before we state the theorem of Strichartz estimate, we give the definition of $\sigma-$admissible pair for the wave equation for $\sigma = (N-1)/2$.

Definition 5.2.1 (*see Definition* 1.1 *in* [30]) *We say that the exponents pair (q,r) is $\sigma-$admissible if $q, r \geq 2$, $(q,r,\sigma) \neq (2,\infty,1)$ and*

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}. \tag{5.2.5}$$

If equality holds in (5.2.5), we say that (q,r) is sharp $\sigma-$admissible, otherwise we say that (q,r) is nonsharp $\sigma-$admissible. Note in particular that when $\sigma > 1$ the endpoint

$$P = \left(2, \frac{2\sigma}{\sigma - 1}\right)$$

is sharp $\sigma-$admissible. □

Theorem 5.2.2 *(see Corollary 1.3 in [30]) Suppose that $N \geq 2$ and (q,r) and $(\widetilde{q},\widetilde{r})$ are wave-admissible pairs (i.e. $\sigma = (N-1)/2$) with $r, \widetilde{r} < \infty$. If u is a (weak) solution to the problem (5.2.1) in $\mathbb{R}^N \times [0,T]$ for some data u_0, u_1, f and time $0 < T < \infty$, then*

$$\begin{aligned} \|u\|_{L^q([0,T];L^r_x)} &+ \|u\|_{C([0,T];H^\mu)} + \|\partial_t u\|_{C([0,T];H^{\mu-1})} \\ &\leq \overline{C}\left(\|u_0\|_{H^\mu} + \|u_1\|_{H^{\mu-1}} + \|f\|_{L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'}_x)}\right), \end{aligned} \tag{5.2.6}$$

under the assumption that the dimensional analysis (or "gap") condition

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - \mu = \frac{1}{\widetilde{q}'} + \frac{N}{\widetilde{r}'} - 2 \tag{5.2.7}$$

holds, where $\overline{C} > 0$ is a positive constant independent of T.
We denote by $\widetilde{r}', \widetilde{q}'$ the conjugate exponents of $\widetilde{r}, \widetilde{q}$ and by $L^p_x := L^p(\mathbb{R}^N)$ the standard Lebesgue x space for all $1 \leq p \leq \infty$.
Conversely, if (5.2.6) holds for all u_0, u_1, f, T, then (q,r) and $(\widetilde{q},\widetilde{r})$ must be wave-admissible and the gap condition must hold.
When $r = \infty$, the estimate (5.2.6) holds with the L^r_x norm replaced with the Besov norm $\dot{B}_{r,2}$, and similarly for $\widetilde{r} = \infty$. □

Corollary 5.2.3 *(Strichartz estimate for u_0)*
Suppose that $N \geq 2$ and (q,r) is a wave-admissible pair (i.e. $\sigma = (N-1)/2$) with $r < \infty$. If u is a (weak) solution to the problem (5.2.1) with $u_1 = f = 0$ in $\mathbb{R}^N \times [0,T]$ for $0 < T < \infty$, then

$$\|\dot{K}(t)u_0\|_{L^q([0,T];L^r_x)} + \|\dot{K}(t)u_0\|_{C([0,T];H^\mu)} + \|\Delta K(t)u_0\|_{C([0,T];H^{\mu-1})} \leq \overline{C}\|u_0\|_{H^\mu}, \tag{5.2.8}$$

under the assumption that the dimensional analysis (or "gap") condition

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - \mu \tag{5.2.9}$$

holds, where $u = \dot{K}(t)u_0$, $u_t = \Delta K(t)u_0$ and $\overline{C} > 0$ is the constant of (5.2.6). □

Corollary 5.2.4 *(Strichartz estimate for u_1)*
Suppose that $N \geq 2$ and (q,r) is a wave-admissible pair (i.e. $\sigma = (N-1)/2$) with $r < \infty$. If u is a (weak) solution to the problem (5.2.1) with $u_0 = f = 0$ in $\mathbb{R}^N \times [0,T]$ for $0 < T < \infty$, then

$$\|K(t)u_1\|_{L^q([0,T];L^r_x)} + \|K(t)u_1\|_{C([0,T];H^\mu)} + \|\dot{K}(t)u_1\|_{C([0,T];H^{\mu-1})} \leq \overline{C}\|u_1\|_{H^{\mu-1}}, \tag{5.2.10}$$

under the assumption that the dimensional analysis (or "gap") condition

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - \mu \tag{5.2.11}$$

holds, where $u = K(t)u_1$, $u_t = \dot{K}(t)u_1$.

□

Corollary 5.2.5 (*Strichartz estimate for f*)
Suppose that $N \geq 2$ and (q,r) and $(\widetilde{q},\widetilde{r})$ are wave-admissible pairs (i.e. $\sigma = (N-1)/2$) with $r,\widetilde{r} < \infty$. If u is a (weak) solution to the problem (5.2.1) with $u_0 = u_1 = 0$ in $\mathbb{R}^N \times [0,T]$ for $0 < T < \infty$, then

$$\begin{aligned}\|K * f(t)\|_{L^q([0,T];L^r_x)} + \|K * f(t)\|_{C([0,T];H^\mu)} &+ \|\dot{K} * f(t)\|_{C([0,T];H^{\mu-1})} \\ &\leq \overline{C}\|f\|_{L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'}_x)},\end{aligned} \tag{5.2.12}$$

under the assumption that the dimensional analysis (or "gap") condition

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - \mu = \frac{1}{\widetilde{q}'} + \frac{N}{\widetilde{r}'} - 2 \tag{5.2.13}$$

holds, where $u = K * f(t)$, $u_t = \dot{K} * f(t)$. □

On the other hand, if $AC[0,T]$ is the space of all functions which are absolutely continuous on $[0,T]$ with $0 < T < \infty$, then, for $f \in AC[0,T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D^\alpha_{0|t}f(t)$ and $D^\alpha_{t|T}f(t)$ of order $\alpha \in (0,1)$ are defined by (see [32])

$$D^\alpha_{0|t}f(t) := DJ^{1-\alpha}_{0|t}f(t), \tag{5.2.14}$$

$$D^\alpha_{t|T}f(t) := -\frac{1}{\Gamma(1-\alpha)}D\int_t^T (s-t)^{-\alpha}f(s)\,ds, \tag{5.2.15}$$

for all $t \in [0,T]$, where $D := \frac{d}{dt}$ and

$$J^\alpha_{0|t}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \tag{5.2.16}$$

is the Riemann-Liouville fractional integral (see [32]), for all $f \in L^q(0,T)$ $(1 \leq q \leq \infty)$.
Furthermore, for every $f,g \in C([0,T])$, such that $D^\alpha_{0|t}f(t), D^\alpha_{t|T}g(t)$ exist and are continuous, for all $t \in [0,T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.64) p. 46 in [46])

$$\int_0^T \left(D^\alpha_{0|t}f\right)(s)g(s)\,ds = \int_0^T f(s)\left(D^\alpha_{t|T}g\right)(s)\,ds. \tag{5.2.17}$$

Note also that, for all $f \in AC^{n+1}[0,T]$ and all integer $n \geq 0$, we have (see 2.2.30 in [32])

$$(-1)^n D^n.D^\alpha_{t|T}f = D^{n+\alpha}_{t|T}f, \tag{5.2.18}$$

where

$$AC^{n+1}[0,T] := \{f : [0,T] \to \mathbb{R} \text{ and } D^n f \in AC[0,T]\}$$

and D^n is the usual n times derivative.
Moreover, for all $1 \leq q \leq \infty$, the following equality (see [Lemma 2.4 p.74][32])

$$D^\alpha_{0|t}J^\alpha_{0|t} = Id_{L^q(0,T)} \tag{5.2.19}$$

holds almost everywhere on $[0,T]$.
Later on, we will use the following results.
If $w_1(t)=(1-t/T)_+^\sigma$, $t\geq 0$, $T>0$, $\sigma>>1$, then

$$D^\alpha_{t|T}w_1(t) = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}T^{-\sigma}(T-t)_+^{\sigma-\alpha}, \tag{5.2.20}$$

$$D^{\alpha+1}_{t|T}w_1(t) = \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}T^{-\sigma}(T-t)_+^{\sigma-\alpha-1}, \tag{5.2.21}$$

$$D^{\alpha+2}_{t|T}w_1(t) = \frac{(1-\alpha+\sigma)(\sigma-\alpha)(\sigma-\alpha-1)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}T^{-\sigma}(T-t)_+^{\sigma-\alpha-1}, \tag{5.2.22}$$

for all $\alpha\in(0,1)$; so

$$\left(D^\alpha_{t|T}w_1\right)(T)=0 \quad ; \quad \left(D^\alpha_{t|T}w_1\right)(0)=C\,T^{-\alpha}, \tag{5.2.23}$$

and

$$\left(D^{\alpha+1}_{t|T}w_1\right)(T)=0 \quad ; \quad \left(D^{\alpha+1}_{t|T}w_1\right)(0)=\widetilde{C}\,T^{-\alpha-1}, \tag{5.2.24}$$

where

$$C=\frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} \quad \text{and} \quad \widetilde{C}=\frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}.$$

Indeed, using the Euler change of variable $y=(s-t)/(T-t)$, we get

$$\begin{aligned} D^\alpha_{t|T}w_1(t) &:= -\frac{1}{\Gamma(1-\alpha)}D\left[\int_t^T(s-t)^{-\alpha}\left(1-\frac{s}{T}\right)^\sigma ds\right] \\ &= -\frac{T^{-\sigma}}{\Gamma(1-\alpha)}D\left[(T-t)^{1-\alpha+\sigma}\int_0^1(y)^{-\alpha}(1-y)^\sigma\,ds\right] \\ &= +\frac{(1-\alpha+\sigma)B(1-\alpha;\sigma+1)}{\Gamma(1-\alpha)}T^{-\sigma}(T-t)^{\sigma-\alpha}, \end{aligned}$$

where $B(\cdot\,;\cdot)$ stands for the beta function. Then, (5.2.20) follows using the relation

$$B(1-\alpha;\sigma+1)=\frac{\Gamma(1-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}.$$

Furthermore, (5.2.21) and (5.2.22) follows from the formula (5.2.18) applied to (5.2.20). □

Finally, note that

$$\int_{\mathbb{R}^N}v(x)(-\Delta)u(x)\,dx=\int_{\mathbb{R}^N}u(x)(-\Delta)v(x)\,dx, \tag{5.2.25}$$

for all $u,v\in H^2(\mathbb{R}^N)$.

5.3 Local existence

In order to prove the existence of a local mild solution for the problem (5.1.1), we convert it into

$$\left\{\begin{array}{ll} \partial_t U - AU = F(U) & x\in\mathbb{R}^N, t>0, \\ \\ U(x,0)=U_0(x)={}^t(u_0(x),u_1(x)) & x\in\mathbb{R}^N, \end{array}\right. \tag{5.3.1}$$

where $U = {}^t(u, u_t)$, $A = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}$ and $F(U) := \begin{bmatrix} 0 \\ J^{\alpha}_{0|t}(|u|^p) \end{bmatrix}$ with $\alpha := 1 - \gamma \in (0, 1)$.

Definition 5.3.1 (*Mild solution of* (5.1.1)) *We say that u is a mild solution of* (5.3.1) *if and only if $(u, u_t) \in C([0, T]; Y^\mu)$ and u verifies the integral equation*

$$u(t) = \dot{K}(t)u_0 + K(t)u_1 + (K * J^{\alpha}_{0|t}(|u|^p))(t), \quad t \in [0, T]. \tag{5.3.2}$$

Definition 5.3.2 (*Mild solution of* (5.3.1)) *We say that $U = (u, v)$ is a mild solution of* (5.3.1) *if and only if $U \in C([0, T]; Y^\mu)$ and verifies the integral equation*

$$U(t) = \begin{bmatrix} \dot{K}(t)u_0 + K(t)u_1 \\ \Delta K(t)u_0 + \dot{K}(t)u_1 \end{bmatrix} + \begin{bmatrix} (K * J^{\alpha}_{0|t}(|u|^p))(t) \\ (\dot{K} * J^{\alpha}_{0|t}(|u|^p))(t) \end{bmatrix} := H(t)U_0 + I(U)(t), \tag{5.3.3}$$

for all $t \in [0, T]$.

Lemma 5.3.3 *Let $U = (u, u_t) \in C([0, T]; Y^\mu)$. Then u is a mild solution of* (5.1.1) *if and only if U is a mild solution of* (5.3.1).

Now, for later use, let $\widetilde{r}, \widetilde{q}$ two constants such that

$$\frac{1}{\widetilde{q}} + \frac{N-1}{2\widetilde{r}} \leq \frac{N-1}{4} \quad \text{and} \quad \frac{N}{2} - \mu = \frac{1}{\widetilde{q}'} + \frac{N}{\widetilde{r}'} - 2, \tag{5.3.4}$$

with $2 \leq \widetilde{r} < \infty$, $\widetilde{q} \geq 2$ and $(\widetilde{q}, \widetilde{r}, (N-1)/2) \neq (2, \infty, 1)$.

Theorem 5.3.4 (*local existence of mild solution of* (5.1.1))
Given $(u_0, u_1) \in Y^\mu$, $N \geq 2$, $0 < \mu \leq N/2$ and let $p > 1$ be such that

$$\begin{cases} \dfrac{2}{\widetilde{r}'} + \dfrac{2\mu}{N} \leq p \leq \dfrac{2}{\widetilde{r}'} \dfrac{N}{N - 2\mu} & \text{if } \mu < N/2 \\ 1 + \dfrac{1}{\widetilde{r}'} \leq p < \infty & \text{if } \mu = N/2, \end{cases} \tag{5.3.5}$$

where $\widetilde{r}$ is defined in (5.3.4) *below. Then, there exist $T > 0$ and a unique solution u to the problem* (5.1.1) *such that $(u, u_t) \in C([0, T], Y^\mu)$.*

Proof To prove the local existence of solution of (5.1.1), its sufficient to prove the local existence of solution of (5.3.1). Let $T > 0$ and consider the following Banach space, with the same $\overline{C}$ as above,

$$E := \left\{ U = (u, v) \in C([0, T] : Y^\mu; \ \|U\|_E \leq 2\overline{C} \left(\|u_0\|_{H^\mu} + \|u_1\|_{H^{\mu-1}} \right) = 2\overline{C}\|U_0\|_{Y^\mu} \right\},$$

where $U_0 = (u_0, v_0)$ and

$$\|U\|_E = \|(u, v)\|_E := \|u\|_{C([0,T];H^\mu)} + \|v\|_{C([0,T];H^{\mu-1})}$$

and $d(U,V) := \|U-V\|_E$ is the distance on E, for all $U,V \in E$.
In order to use the Banach fixed point Theorem, we introduce the following map Φ on E defined by

$$\Phi(U)(t) := H(t)U_0 + I(U)(t).$$

It is sufficient to prove that Φ is an application from E to E and is a contraction on E.

- $\mathbf{\Phi} : \mathbf{E} \longrightarrow \mathbf{E}$: Let $U=(u,v) \in E$, using Strichartz's estimates (5.2.8), (5.2.10) and (5.2.12), we have

$$\|\Phi(U)\|_E \leq \overline{C}\left(\|u_0\|_{H^\mu} + \|u_1\|_{H^{\mu-1}} + \|J^\alpha_{0|t}(|u|^p)\|_{L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'}_x)}\right), \tag{5.3.6}$$

where $\widetilde{r}$ and $\widetilde{q}$ are given in (5.3.4). As

$$\|J^\alpha_{0|t}(|u|^p)\|_{L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'}_x)} \leq \|J^\alpha_{0|t}(\|u\|^p_{L^{p\widetilde{r}'}_x})\|_{L^{\widetilde{q}'}([0,T])},$$

and as the condition

$$\begin{cases} \dfrac{2}{\widetilde{r}'} < \dfrac{2}{\widetilde{r}'} + \dfrac{2\mu}{N} \leq p \leq \dfrac{2}{\widetilde{r}'}\dfrac{N}{N-2\mu} & \text{if } \mu < N/2 \\ \dfrac{2}{\widetilde{r}'} < 1 + \dfrac{1}{\widetilde{r}'} \leq p < \infty & \text{if } \mu = N/2, \end{cases} \tag{5.3.7}$$

imply the following Sobolev imbedding

$$H^\mu(\mathbb{R}^N) \hookrightarrow L^{p\widetilde{r}'}(\mathbb{R}^N),$$

we get

$$\|\Phi(U)\|_E \leq \overline{C}\left(\|U_0\|_{C([0,T];Y^\mu)} + C_1^p \|J^\alpha_{0|t}(\|u\|^p_{H^\mu(\mathbb{R}^N)})\|_{L^{\widetilde{q}'}([0,T])}\right), \tag{5.3.8}$$

where C_1 is the positive constant of the Sobolev imbedding. Using the fact that $U \in E$, we have

$$\|J^\alpha_{0|t}(\|u\|^p_{H^\mu(\mathbb{R}^N)})\|_{L^{\widetilde{q}'}([0,T])} \leq \frac{1}{(1+(1-\gamma)\widetilde{q}')^{1/\widetilde{q}'}\Gamma(2-\gamma)} T^{(1-\gamma)+1/\widetilde{q}'}\|u\|^p_{C([0,T];H^\mu)}.$$

So

$$\begin{aligned} \|\Phi(U)\|_E &\leq \overline{C}\left(\|U_0\|_{C([0,T];Y^\mu)} + C_2 T^{(1-\gamma)+1/\widetilde{q}'}\|u\|^p_{C([0,T];H^\mu)}\right) \\ &\leq \overline{C}\left(\|U_0\|_{C([0,T];Y^\mu)} + 2^p C_2 \overline{C}^p T^{(1-\gamma)+1/\widetilde{q}'}\|U_0\|^{p-1}_{C([0,T];Y^\mu)}\|U_0\|_{C([0,T];Y^\mu)}\right), \end{aligned}$$

where

$$C_2 = \frac{C_1^p}{(1+(1-\gamma)\widetilde{q}')^{1/\widetilde{q}'}\Gamma(2-\gamma)}.$$

Now, if we choose $T>0$ small enough such that

$$2^p C_2 \overline{C}^p T^{(1-\gamma)+1/\widetilde{q}'}\|U_0\|^{p-1}_{C([0,T];Y^\mu)} \leq 1,$$

we conclude that $\Phi(U) \in E$, and so $\Phi : E \longrightarrow E$.

• **Φ is a Contraction map :** For $U=(u,v), V=(\widetilde{u},\widetilde{v})\in E$, using the Strichartz estimate (5.2.12) for $f=J^{\alpha}_{0|t}(|u|^p-|\widetilde{u}|^p)$, we have

$$\begin{aligned}\|\Phi(U)-\Phi(V)\|_E &\leq \overline{C}\|J^{\alpha}_{0|t}(|u|^p-|\widetilde{u}|^p)\|_{L^{\widetilde{q}'}(0,T;L^{\widetilde{r}'}_x)}\\ &\leq C(p)\overline{C}\|J^{\alpha}_{0|t}\|(u-\widetilde{u})(|u|^{p-1}+|\widetilde{u}|^{p-1})\|_{L^{\widetilde{r}'}_x}\|_{L^{\widetilde{q}'}(0,T)},\end{aligned} \tag{5.3.9}$$

thanks to the inequality

$$|\,|u|^p-|\widetilde{u}|^p\,|\leq C(p)|\,|u|-|\widetilde{u}|\,|\left(|u|^{p-1}+|\widetilde{u}|^{p-1}\right)\leq C(p)|u-\widetilde{u}|\left(|u|^{p-1}+|\widetilde{u}|^{p-1}\right).$$

Now, we have to distinguish 2 cases : ($\mu=N/2$ or $\mu<N/2$).

− If $\mu=N/2$, then using the Hölder inequality

$$\|(u-\widetilde{u})\left(|u|^{p-1}+|\widetilde{u}|^{p-1}\right)\|_{L^{\widetilde{r}'}_x}\leq\|u-\widetilde{u}\|_{L^{2\widetilde{r}'}_x}\||u|^{p-1}+|\widetilde{u}|^{p-1}\|_{L^{2\widetilde{r}'}_x},$$

we obtain

$$\begin{aligned}\|\Phi(U)-\Phi(V)\|_E &\leq C(p)\overline{C}\left\|J^{\alpha}_{0|t}\|u-\widetilde{u}\|_{L^{2\widetilde{r}'}_x}\left(\||u|^{p-1}\|_{L^{2\widetilde{r}'}_x}+\||\widetilde{u}|^{p-1}\|_{L^{2\widetilde{r}'}_x}\right)\right\|_{L^{\widetilde{q}'}(0,T)}\\ &= C(p)\overline{C}\left\|J^{\alpha}_{0|t}\|u-\widetilde{u}\|_{L^{2\widetilde{r}'}_x}\left(\|u\|^{p-1}_{L^{2(p-1)\widetilde{r}'}_x}+\|\widetilde{u}\|^{p-1}_{L^{2(p-1)\widetilde{r}'}_x}\right)\right\|_{L^{\widetilde{q}'}(0,T)}.\end{aligned}$$

Moreover, as

$$2\widetilde{r}'\geq 2\quad\text{and}\quad p\geq 1+\frac{1}{\widetilde{r}'}\quad\Longleftrightarrow\quad 2\widetilde{r}'(p-1)\geq 2$$

imply the Sobolev imbeddings

$$H^{\mu}(\mathbb{R}^N)\hookrightarrow L^{2\widetilde{r}'}(\mathbb{R}^N)\quad\text{and}\quad H^{\mu}(\mathbb{R}^N)\hookrightarrow L^{2\widetilde{r}'(p-1)}(\mathbb{R}^N), \tag{5.3.10}$$

respectively ; we then conclude that

$$\|\Phi(U)-\Phi(V)\|_E\leq C_3C_4^{p-1}C(p)\overline{C}\left\|J^{\alpha}_{0|t}\|u-\widetilde{u}\|_{H^{\mu}}\left(\|u\|^{p-1}_{H^{\mu}}+\|\widetilde{u}\|^{p-1}_{H^{\mu}}\right)\right\|_{L^{\widetilde{q}'}(0,T)}, \tag{5.3.11}$$

where C_3 and C_4 are the constants of the two Sobolev's imbeddings in (5.3.10), respectively. So, using the fact that $U,V\in E$, we conclude that

$$\begin{aligned}\|\Phi(U)-\Phi(V)\|_E &\leq C_3C_4^{p-1}C(p)\overline{C}^p2^{p-1}\|U_0\|^{p-1}_{C([0,T];Y^{\mu})}\|U-V\|_{C([0,T];Y^{\mu})}\left\|J^{\alpha}_{0|t}1\right\|_{L^{\widetilde{q}'}(0,T)}\\ &\leq C_52^{p-1}\overline{C}^pT^{(1-\gamma)+1/\widetilde{q}'}\|U_0\|^{p-1}_{C([0,T];Y^{\mu})}\|U-V\|_{C([0,T];Y^{\mu})},\end{aligned} \tag{5.3.12}$$

for

$$C_5:=\frac{C_3C_4^{p-1}C(p)}{(1+(1-\gamma)\widetilde{q}')^{1/\widetilde{q}'}\Gamma(2-\gamma)}.$$

In this case, we choose $T>0$ such that

$$C_52^{p-1}\overline{C}^pT^{(1-\gamma)+1/\widetilde{q}'}\|U_0\|^{p-1}_{C([0,T];Y^{\mu})}\leq\frac{1}{2},$$

in order to obtain

$$\|\Phi(U)-\Phi(V)\|_E \le \frac{1}{2}\|U-V\|_{C([0,T];Y^\mu)}. \tag{5.3.13}$$

– If $\mu < N/2$, then using the Hölder inequality

$$\|(u-\widetilde{u})\left(|u|^{p-1}+|\widetilde{u}|^{p-1}\right)\|_{L_x^{\widetilde{r}'}} \le \|u-\widetilde{u}\|_{L_x^{\frac{2N}{N-2\mu}}} \|\,|u|^{p-1}+|\widetilde{u}|^{p-1}\|_{L_x^{\frac{2N\widetilde{r}'}{2N-(N-2\mu)\widetilde{r}'}}},$$

we arrive at

$$\|\Phi(U)-\Phi(V)\|_E \le C(p)\overline{C}\left\| J_{0|t}^{\alpha}\|u-\widetilde{u}\|_{L_x^{s_1}}\left(\|u\|_{L_x^{s_2}}^{p-1}+\|\widetilde{u}\|_{L_x^{s_2}}^{p-1}\right)\right\|_{L^{\widetilde{q}'}(0,T)} \tag{5.3.14}$$

where

$$s_1 := \frac{2N}{N-2\mu} \quad \text{and} \quad s_2 := \frac{2N\widetilde{r}'(p-1)}{2N-(N-2\mu)\widetilde{r}'}.$$

Moreover, the critical Sobolev exponent, and the condition on p

$$\frac{2}{\widetilde{r}'}+\frac{2\mu}{N} \le p \le \frac{2}{\widetilde{r}'}\frac{N}{N-2\mu} \iff 2 \le s_2 \le \frac{2N}{N-2\mu}$$

imply that

$$H^\mu(\mathbb{R}^N) \hookrightarrow L^{s_1}(\mathbb{R}^N) \quad \text{and} \quad H^\mu(\mathbb{R}^N) \hookrightarrow L^{s_2}(\mathbb{R}^N), \tag{5.3.15}$$

respectively ; we then conclude, as above, that

$$\begin{aligned}\|\Phi(U)-\Phi(V)\|_E &\le C_6 2^{p-1}\overline{C}^p T^{(1-\gamma)+1/\widetilde{q}'}\|U_0\|_{C([0,T];Y^\mu)}^{p-1}\|U-V\|_{C([0,T];Y^\mu)} &(5.3.16)\\ &\le \frac{1}{2}\|U-V\|_{C([0,T];Y^\mu)}, &(5.3.17)\end{aligned}$$

for

$$C_6 := \frac{C_7 C_8^{p-1} C(p)}{(1+(1-\gamma)\widetilde{q}')^{1/\widetilde{q}'}\Gamma(2-\gamma)},$$

where C_7 and C_4 are the constants of the two Sobolev imbeddings in (5.3.15), respectively ; Thanks to the choice of $T>0$:

$$C_6 2^{p-1}\overline{C}^p T^{(1-\gamma)+1/\widetilde{q}'}\|U_0\|_{C([0,T];Y^\mu)}^{p-1} \le \frac{1}{2}.$$

Hence the general choice of T is :

$$2^p \max\{C_2, C_5, C_6\}\overline{C}^p T^{(1-\gamma)+1/\widetilde{q}'}\|U_0\|_{C([0,T];Y^\mu)}^{p-1} \le 1. \tag{5.3.18}$$

So, by the Banach fixed point theorem, there exists a unique solution $U \in E$, i.e. $U \in C([0,T];Y^\mu)$ is a solution to (5.3.3) such that $\|U\|_E \le 2\overline{C}\|U_0\|_{C([0,T];Y^\mu)}$.
Finally, as $U=(u,v) \in C([0,T],Y^\mu)$ is a solution of (5.3.3), so it is easy to see that $u \in C([0,T];H^\mu(\mathbb{R}^N))$ verifies (5.3.2). Thus, using (5.2.2) and the fact that $K(0)=0$, $u_t \in C([0,T];H^{\mu-1}(\mathbb{R}^N))$ and verifies the second line in (5.3.3) and by the uniqueness of U in E we deduce that $(u,u_t) = U \in C([0,T];Y^\mu)$. So, using Lemma's 5.3.3, we deduce the result. □
Next, we give a local existence of the maximal mild solution for the problem (5.1.1) in the case $\mu = 1$.

Corollary 5.3.5 *(Case of $\mu = 1$)*
Given $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ (i.e. $\mu = 1$), $N \geq 2$ and let $p > 1$ be such that

$$\begin{cases} 1 + \dfrac{2}{N} \leq p \leq \dfrac{N}{N-2} & \text{if } N > 2 \\ \dfrac{3}{2} \leq p < \infty & \text{if } N = 2. \end{cases} \tag{5.3.19}$$

Then, there exist a maximal time $0 < T_{\max} \leq \infty$ and a unique maximal mild solution u to the problem (5.1.1) such that $(u, u_t) \in C([0, T_{\max}), H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$. Moreover, if $T_{\max} < \infty$, we have $(\|u\|_{C([0,t],H^1(\mathbb{R}^N))} + \|u_t\|_{C([0,t],L^2(\mathbb{R}^N))}) \to \infty$ when $t \to T_{\max}$.

Proof From theorem's 5.3.4 there exist $T > 0$, small enough, and a unique solution u to the problem (5.1.1) such that $(u, u_t) \in C([0, T], Y^1)$. To extend this solution to a maximal interval, we need a uniqueness result. Indeed, if u and v are 2 mild solutions of (5.1.1) such that $(u, u_t), (v, v_t) \in C([0, T'], Y^1)$ for arbitrary $T' > 0$, using the corollary 5.2.3, 5.2.4 and 5.2.5, we conclude that

$$\begin{aligned} \|u(t) - v(t)\|_{H^1(\mathbb{R}^N)} &\leq \overline{C}\|J^\alpha_{0|s}(|u|^p - |v|^p)\|_{L^{\widetilde{q}'}(0,t;L^{\widetilde{r}'}_x)} \\ &\leq C(p)\overline{C}\|J^\alpha_{0|s}\|(u-v)(|u|^{p-1} + |v|^{p-1})\|_{L^{\widetilde{r}'}_x}\|_{L^{\widetilde{q}'}(0,t)}, \end{aligned}$$

where

$$\frac{1}{\widetilde{q}} + \frac{N-1}{2\widetilde{r}} \leq \frac{N-1}{4} \quad \text{and} \quad \frac{N}{2} - \mu = \frac{1}{\widetilde{q}'} + \frac{N}{\widetilde{r}'} - 2,$$

with $2 \leq \widetilde{r} < \infty$, $\widetilde{q} \geq 2$ and $(\widetilde{q}, \widetilde{r}, (N-1)/2) \neq (2, \infty, 1)$.
As $\mu = 1$, thus by taking $\widetilde{r} = 2$ and $\widetilde{q} = \infty$, we conclude that

$$\|u(t) - v(t)\|_{H^1(\mathbb{R}^N)} \leq C(p)\overline{C}\|J^\alpha_{0|s}\|(u-v)(|u|^{p-1} + |v|^{p-1})\|_{L^2_x}\|_{L^1(0,t)}, \tag{5.3.20}$$

So, using (5.3.19) and the Sobolev imbedding (5.3.10) in the case $N = 2$, respectively (5.3.15) in the case $N > 2$, we get

$$\begin{aligned} \|u(t) - v(t)\|_{H^1(\mathbb{R}^N)} &\leq C\|J^\alpha_{0|s}\|u(s) - v(s)\|_{H^1}(\|u\|^{p-1}_{H^1} + \|v\|^{p-1}_{H^1})\|_{L^1(0,t)} \\ &\leq C'\int_0^t\int_0^s (s-\sigma)^{-\gamma}\|u(\sigma) - v(\sigma)\|_{H^1}\, d\sigma\, ds \\ &= C''\int_0^t (t-\sigma)^{1-\gamma}\|u(\sigma) - v(\sigma)\|_{H^1}\, d\sigma, \end{aligned}$$

for all $t \in [0, T']$, and by Gronwall's inequality we conclude the result.
By this uniqueness result, we can extend the solution to a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup\{T > 0\,;\ \text{there exist a solution } u \text{ to (5.1.1) with } (u, u_t) \in C([0, T]; Y^1)\}.$$

□

We say that u is a global solution of (5.1.1) if $T_{\max} = \infty$, while in the case of $T_{\max} < \infty$, we say that u blows up in finite time.

5.4 Blow-up of solutions

This section is devoted to the blow-up of solutions of the problem (5.1.1), with $\mu = 1$, under some conditions on the initial data (u_0, u_1) and the exponents γ and p of the nonlinear term. To do this, we have to introduce the definition of the weak solution of (5.1.1) and prove that mild solutions are weak solutions of (5.1.1). Hereafter

$$\int_{Q_T} = \int_0^T \int_{\mathbb{R}^N} dx\,dt \quad \text{for all} \quad T > 0, \qquad \int_{\mathbb{R}^N} = \int_{\mathbb{R}^N} dx.$$

Definition 5.4.1 (*Weak solution*) *Let $u_0, u_1 \in L^1_{Loc}(\mathbb{R}^N)$. We say that u is a weak solution of (5.1.1) if and only if $u \in\in L^p((0,T), L^p_{Loc}(\mathbb{R}^N))$ satisfying the following*

$$\begin{aligned} &\int_0^T \int_\Omega J^\alpha_{0|t}(|u|^p)(x,t)\varphi(x,t) + \int_\Omega u_1(x)\varphi(x,0) - \int_\Omega u_0(x)\varphi_t(x,0) \\ &= \int_0^T \int_\Omega u(x,t)\varphi_{tt}(x,t) - \int_0^T \int_\Omega u(x,t)\Delta\varphi(x,t) \end{aligned} \tag{5.4.1}$$

for all compact support function $\varphi \in C^2([0,T] \times \mathbb{R}^N)$ such that $\Omega :=$suppφ and $\varphi(\cdot\,,T) = 0$, where $\alpha := 1 - \gamma \in (0,1)$.

Lemma 5.4.2 (*Mild $\to$ Weak*)
Given $(u_0, u_1) \in Y^1$, $N \geq 2$ and $p > 1$ verifies (5.3.19). If u is the mild solution of (5.1.1) with $(u, u_t) \in C([0,T], Y^1)$, then u be a weak solution of (5.1.1), for all $T > 0$.

Proof Let $T > 0$, u be a mild solution of (5.1.1) and $\varphi \in C^2([0,T] \times \mathbb{R}^N)$ be a compact support function such that supp$\varphi =: \Omega$ and $\varphi(\cdot\,,T) = 0$. Then, using Lemma 5.3.3 and (5.3.3), we have

$$u(t) = \dot{K}(t)u_0 + \Delta K(t)u_1 + (K * J^\alpha_{0|t}(|u|^p))(t), \tag{5.4.2}$$

and

$$u_t(t) = \Delta K(t)u_0 + \dot{K}(t)u_1 + (\dot{K} * J^\alpha_{0|t}(|u|^p))(t). \tag{5.4.3}$$

So, after multiplying (5.4.3) by φ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned} \int_\Omega u_t(x,t)\varphi(x,t) &= \int_\Omega \Delta K(t)u_0(x)\varphi(x,t) + \int_\Omega \dot{K}(t)u_1(x)\varphi(x,t) \\ &+ \int_\Omega \int_0^t \dot{K}(t-s)J^\alpha_{0|s}(|u|^p)(x,s)\,ds\varphi(x,t). \end{aligned}$$

Then, after differentiating in time, we obtain

$$\frac{d}{dt}\int_\Omega u_t(x,t)\varphi(x,t) = \frac{d}{dt}\int_\Omega \Delta K(t)u_0(x)\varphi(x,t) + \frac{d}{dt}\int_\Omega \dot{K}(t)u_1(x)\varphi(x,t)$$

$$+ \int_\Omega \frac{d}{dt}\int_0^t \dot{K}(t-s)J^\alpha_{0|s}(|u|^p)(x,s)\,ds\varphi(x,t). \tag{5.4.4}$$

Now, using (5.2.2) and the fact that the Laplacian is a negative self-adjoint operator, we have :

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Delta K(t)u_0(x)\varphi(x,t) + \frac{d}{dt}\int_\Omega \dot{K}(t)u_1(x)\varphi(x,t)\\
&= \int_\Omega \Delta\left[\dot{K}(t)u_0(x)+K(t)u_1(x)\right]\varphi(x,t) + \int_\Omega \left[\Delta K(t)u_0(x)+\dot{K}(t)u_1(x)\right]\varphi_t(x,t)\\
&= \int_\Omega \left[\dot{K}(t)u_0(x)+K(t)u_1(x)\right]\Delta\varphi(x,t) + \int_\Omega \left[\Delta K(t)u_0(x)+\dot{K}(t)u_1(x)\right]\varphi_t(x,t),
\end{aligned} \tag{5.4.5}$$

and

$$\begin{aligned}
&\int_\Omega \frac{d}{dt}\int_0^t \dot{K}(t-s)f(x,s)\,ds\varphi(x,t)\\
&= \int_\Omega f(x,t)\varphi(x,t) + \int_\Omega\int_0^t \Delta\left(K(t-s)f(x,s)\right)\,ds\varphi(x,t)\\
&+ \int_\Omega\int_0^t \dot{K}(t-s)f(x,s)\,ds\varphi_t(x,t)\\
&= \int_\Omega f(x,t)\varphi(x,t) + \int_{\mathbb{R}^N}(K*f)(x,t)\Delta\varphi(x,t) + \int_\Omega(\dot{K}*f)(x,t)\varphi_t(x,t)
\end{aligned} \tag{5.4.6}$$

where $f := J^\alpha_{0|t}(|u|^p) \in C([0,T];L^2(\Omega))$.
Thus, using (5.4.2) – (5.4.3) and (5.4.5) – (5.4.6), we conclude that (5.4.4) implies that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_t(x,t)\varphi(x,t) &= \int_\Omega u(x,t)\Delta\varphi(x,t) + \int_\Omega u_t(x,t)\varphi_t(x,t)\\
&+ \int_\Omega f(x,t)\varphi(x,t).
\end{aligned} \tag{5.4.7}$$

Next, after integrating in time (5.4.7) over $[0,T]$ and using the fact that $\varphi(\cdot\,,T)=0$, we conclude that

$$\begin{aligned}
-\int_\Omega u_1(x)\varphi(x,0) &= \int_0^T\int_\Omega u(x,t)\Delta\varphi(x,t) - \int_0^T\int_\Omega u(x,t)\varphi_{tt}(x,t)\\
&- \int_\Omega u_0(x)\varphi_t(x,0) + \int_\Omega f(x,t)\varphi(x,t).
\end{aligned} \tag{5.4.8}$$

□

Theorem 5.4.3 *Let* $(u_0,u_1)\in Y^1$, $\int_{\mathbb{R}^N}u_0>0$, $\int_{\mathbb{R}^N}u_1>0$, $N\geq 2$, *and let* $p>1$ *be such that*

$$\begin{cases} 1+\dfrac{2}{N}\leq p\leq 1+\dfrac{3-\gamma}{(N-2+\gamma)_+} := p* & \text{if } N>2\\ \dfrac{3}{2}\leq p\leq \dfrac{3}{\gamma} & \text{if } N=2, \end{cases} \tag{5.4.9}$$

or

$$\begin{cases} 1+\dfrac{2}{N} \leq p < \dfrac{1}{\gamma} & \text{if } N>2 \\ \dfrac{3}{2} \leq p < \dfrac{1}{\gamma} & \text{if } N=2. \end{cases} \tag{5.4.10}$$

with the following conditions

$$\begin{cases} \dfrac{N-2}{N} \leq \gamma < 1 & \text{if } N>2 \\ \\ 0<\gamma<1 & \text{if } N=2, \end{cases} \tag{5.4.11}$$

or

$$\begin{cases} \dfrac{N-2}{N} \leq \gamma < \dfrac{N}{N+2} & \text{if } N>2 \\ \\ 0<\gamma\leq \dfrac{2}{3} & \text{if } N=2, \end{cases} \tag{5.4.12}$$

respectively. Then the solution of (5.1.1) blows up in finite time.

Proof The proof proceeds by contradiction. Let u be a global mild solution of the problem (5.1.1), then u is a solution of (5.1.1) in $C([0,T],H^1(\mathbb{R}^N))\cap C^1([0,T],L^2(\mathbb{R}^N))$ for all $T\gg 1$ large enough.
Then, using Lemma 5.4.2, we have

$$\begin{aligned} &\int_0^T\int_{\text{supp}\varphi} J^{\alpha}_{0|t}(|u|^p)(x,t)\varphi(x,t)+\int_{\text{supp}\varphi} u_1(x)\varphi(x,0)\,dx-\int_{\text{supp}\varphi} u_0(x)\varphi_t(x,0)\\ &=\int_0^T\int_{\text{supp}\varphi} u(x,t)\varphi_{tt}(x,t)-\int_0^T\int_{\text{supp}\varphi} u(x,t)\Delta\varphi(x,t) \end{aligned} \tag{5.4.13}$$

for all compact support function $\varphi\in C^2([0,T]\times\mathbb{R}^N)$ such that $\varphi(\cdot\,,T)=0$, where $\alpha:=1-\gamma\in(0,1)$.
For some reason that will be precised below, we have to distinguish three cases :

• <u>The case $p<p^*$</u> : Let $\varphi(x,t)=D^{\alpha}_{t|T}\left(\widetilde{\varphi}(x,t)\right):=D^{\alpha}_{t|T}\left((\varphi_1(x))^{\ell}\varphi_2(t)\right)$ with $\varphi_1(x):=\Phi\left(|x|/T\right)$, $\varphi_2(t):=(1-t/T)^{\eta}_+$, where $\ell,\eta\gg 1$ large enough and Φ be a smooth non-increasing function such that

$$\Phi(r)=\begin{cases} 1 & \text{if } 0\leq r\leq 1,\\ 0 & \text{if } r\geq 2,\end{cases}$$

$0\leq\Phi\leq 1$, $|\Phi^{'}(r)|\leq C_1/r$, for all $r>0$. Then, we have

$$\int_{\Omega_T} J^{\alpha}_{0|t}(|u|^p)(x,t)D^{\alpha}_{t|T}\widetilde{\varphi}(x,t)+\int_{\Omega} u_1(x)D^{\alpha}_{t|T}\widetilde{\varphi}(x,0)-\int_{\Omega} u_0(x)DD^{\alpha}_{t|T}\widetilde{\varphi}(x,0)$$

$$= \int_{\Omega_T} u(x,t) D^2 D^\alpha_{t|T}\widetilde{\varphi}(x,t) - \int_{\Omega_T} u(x,t)\Delta D^\alpha_{t|T}\widetilde{\varphi}(x,t), \tag{5.4.14}$$

where

$$\Omega_T = [0,T]\times\Omega \ \text{ for } \ \Omega := \left\{x\in\mathbb{R}^N \ ; \ |x|\leq 2T\right\}, \quad \int_{\Omega_T} = \int_{\Omega_T} dx\,dt, \quad \int_\Omega = \int_\Omega dx.$$

Moreover, from (5.2.17), (5.2.18), (5.2.23) and (5.2.24) we may write

$$\begin{aligned}
&\int_{\Omega_T} D^\alpha_{0|t}J^\alpha_{0|t}|u|^p\,\widetilde{\varphi} \ + \ C\,T^{-\alpha}\int_\Omega (\varphi_1(x))^\ell\,u_0(x) \ + \ \widetilde{C}\,T^{-\alpha-1}\int_\Omega (\varphi_1(x))^\ell\,u_1(x)\\
&= \int_{\Omega_T} u\,(\varphi_1(x))^\ell\,D^{2+\alpha}_{t|T}\varphi_2(t) \ + \ \int_{\Omega_T} u(-\Delta_x)\,(\varphi_1(x))^\ell\,D^\alpha_{t|T}\varphi_2(t).
\end{aligned} \tag{5.4.15}$$

So, (5.2.19) and the formula $\Delta\left(\varphi_1^\ell\right) = \ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2$ will allow us to write :

$$\begin{aligned}
&\int_{\Omega_T} |u|^p\,\widetilde{\varphi} \ + \ C\,T^{-\alpha}\int_\Omega (\varphi_1(x))^\ell\,u_0(x) \ + \ \widetilde{C}\,T^{-\alpha-1}\int_\Omega (\varphi_1(x))^\ell\,u_1(x)\\
&= \ C\int_{\Omega_T} u\,(\varphi_1(x))^\ell\,D^{2+\alpha}_{t|T}\varphi_2(t) \ - \ C\int_{\Omega_T} u\,(\varphi_1(x))^{\ell-1}\,\Delta_x\varphi_1(x)\,D^\alpha_{t|T}\varphi_2(t)\\
&- C\int_{\Omega_T} u\,(\varphi_1(x))^{\ell-2}\,|\nabla\varphi_1(x)|^2\,D^\alpha_{t|T}\varphi_2(t)\\
&\leq \ C\int_{\Omega_T} |u|\,(\varphi_1(x))^\ell\left|D^{2+\alpha}_{t|T}\varphi_2(t)\right| \ + \ C\int_{\Omega_T} |u|\,(\varphi_1(x))^{\ell-1}\left|\Delta_x\varphi_1(x)\,D^\alpha_{t|T}\varphi_2(t)\right|\\
&+ C\int_{\Omega_T} |u|\,(\varphi_1(x))^{\ell-2}\,|\nabla\varphi_1(x)|^2\,\left|D^\alpha_{t|T}\varphi_2(t)\right|
\end{aligned} \tag{5.4.16}$$

Therefore, as

$$\int_{\mathbb{R}^N} u_0 > 0, \int_{\mathbb{R}^N} u_1 > 0 \ \Longrightarrow \ \int_{\text{supp}\varphi_1} (\varphi_1(x))^\ell\,u_0(x) \geq 0, \ \int_{\text{supp}\varphi_1} (\varphi_1(x))^\ell\,u_1(x) \geq 0, \tag{5.4.17}$$

(here $\text{supp}\varphi_1 = \Omega$), we obtain

$$\begin{aligned}
\int_{\Omega_T} |u|^p\,\widetilde{\varphi} \ \leq \ & C\int_{\Omega_T} |u|\,\widetilde{\varphi}^{1/p}\widetilde{\varphi}^{-1/p}\,(\varphi_1(x))^\ell\left|D^{2+\alpha}_{t|T}\varphi_2(t)\right|\\
&+ C\int_{\Omega_T} |u|\,\widetilde{\varphi}^{1/p}\widetilde{\varphi}^{-1/p}\,(\varphi_1(x))^{\ell-1}\left|\Delta_x\varphi_1(x)\,D^\alpha_{t|T}\varphi_2(t)\right|\\
&+ C\int_{\Omega_T} |u|\,\widetilde{\varphi}^{1/p}\widetilde{\varphi}^{-1/p}\,(\varphi_1(x))^{\ell-2}\,|\nabla\varphi_1(x)|^2\,\left|D^\alpha_{t|T}\varphi_2(t)\right|,
\end{aligned} \tag{5.4.18}$$

So, using the Young inequality

$$ab \leq \frac{1}{2p}a^{\,p} \ + \ \frac{2^{\widetilde{p}-1}}{\widetilde{p}}\,b^{\,\widetilde{p}} \qquad \text{where } \ p\widetilde{p} = p+\widetilde{p}, \ \ p>1, \widetilde{p}>1, \ \ a>0, b>0, \tag{5.4.19}$$

with

$$\begin{cases} a = |u|\ \widetilde{\varphi}^{1/p}, \\ b = \widetilde{\varphi}^{-1/p}\,(\varphi_1(x))^{\ell}\left|D^{2+\alpha}_{t|T}\varphi_2(t)\right|, \end{cases}$$

in the first integral of the right hand side of (5.4.18),

$$\begin{cases} a = |u|\ \widetilde{\varphi}^{1/p}, \\ b = \widetilde{\varphi}^{-1/p}\ (\varphi_1(x))^{\ell-1}\left|\Delta_x\varphi_1(x)\ D^{\alpha}_{t|T}\varphi_2(t)\right|, \end{cases}$$

in the second integral of the right hand side of (5.4.18) and with

$$\begin{cases} a = |u|\ \widetilde{\varphi}^{1/p}, \\ b = \widetilde{\varphi}^{-1/p}\ (\varphi_1(x))^{\ell-2}\,|\nabla\varphi_1(x)|^2\left|D^{\alpha}_{t|T}\varphi_2(t)\right|, \end{cases}$$

in the third integral of the right hand side of (5.4.18), we obtain

$$\begin{aligned} \int_{\Omega_T}|u(x,t)|^p\ \widetilde{\varphi}(x,t) &\leq C\int_{\Omega_T}(\varphi_1(x))^{\ell}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|D^{2+\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}} \\ &+ C\int_{\Omega_T}(\varphi_1(x))^{\ell-\widetilde{p}}\,(\varphi_2(t))^{-\frac{1}{p-1}}\left|\Delta_x\varphi_1(x)D^{\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}} \\ &+ C\int_{\Omega_T}(\varphi_1(x))^{\ell-2\widetilde{p}}\,(\varphi_2(t))^{-\frac{1}{p-1}}\,|\nabla\varphi_1(x)|^{2\widetilde{p}}\left|D^{\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}}. \end{aligned} \quad (5.4.20)$$

At this stage, we introduce the scaled variables : $\tau = T^{-1}t$ and $\xi = (T)^{-1}x$; using formulas (5.2.20) and (5.2.22) in the right hand-side of (5.4.20), we obtain :

$$\int_{\Omega_T}|u(x,t)|^p\ \widetilde{\varphi}(x,t) \leq\ C\,T^{-\delta}, \quad (5.4.21)$$

where $\delta := (2+\alpha)\widetilde{p} - 1 - N$, $C = C(|\Omega_2|\ ,\ |\Omega_3|)$, ($|\Omega_i|$ stands for the measure of Ω_i, for $i = 2,3$), with

$$\Omega_2 := \left\{\xi \in \mathbb{R}^N\ ;\ |\xi| \leq 2\right\} \quad , \quad \Omega_3 := \{\tau \geq 0\ ;\ \tau \leq 1\}\,.$$

Passing to the limit in (5.4.21), as T goes to ∞, and taking into account the fact that $p < p*$ ($\Longleftrightarrow \delta > 0$), we conclude that

$$\lim_{T\to\infty}\int_0^T\int_{|x|\leq 2T}|u(x,t)|^p\ \widetilde{\varphi}(x,t)\,dx\,dt = 0.$$

Using the dominated convergence theorem, we infer that

$$\int_{Q_\infty}|u(x,t)|^p\,dx\,dt = 0 \qquad \Longrightarrow \qquad u = 0\ \text{ for all } t \text{ and a.e. } x.$$

This Contradicts with $\int_{\mathbb{R}^N} u_0 > 0$.

• The case $p = p^*$: In this case, we take $\widetilde{\varphi}(x,t) = (\varphi_1(x))^\ell \varphi_2(t)$ with $\varphi_1(x) := \Phi\left(|x|/B^{-1}T\right)$, $\varphi_2(t) := (1-t/T)_+^\eta$, instead of the one used in the last case, where $\ell, \eta \gg 1$ and $1 \leq B < T$ large enough such that when $T \to \infty$, we don't have $B \to \infty$ in the same time. Here Φ is the same function used above.
So, by repeating the same computations as in the case $p < p*$, we obtain

$$\begin{aligned}
&\int_{\Sigma_B} |u|^p\, \widetilde{\varphi} \;+\; C\, T^{-\alpha} \int_{\Omega_B} (\varphi_1(x))^\ell\, u_0(x) \;+\; \widetilde{C}\, T^{-\alpha-1} \int_{\Omega_B} (\varphi_1(x))^\ell\, u_1(x) \\
&\leq\; C \int_{\Sigma_B} |u|\, \widetilde{\varphi}^{1/p} \widetilde{\varphi}^{-1/p} (\varphi_1(x))^\ell \left| D^{2+\alpha}_{t|T} \varphi_2(t) \right| \\
&+ C \int_{\Delta_B} |u|\, \widetilde{\varphi}^{1/p} \widetilde{\varphi}^{-1/p}\; (\varphi_1(x))^{\ell-1} \left|\Delta_x \varphi_1(x)\; D^{\alpha}_{t|T} \varphi_2(t)\right| \\
&+ C \int_{\Delta_B} |u|\, \widetilde{\varphi}^{1/p} \widetilde{\varphi}^{-1/p}\; (\varphi_1(x))^{\ell-2} \left|\nabla \varphi_1(x)\right|^2 \left| D^{\alpha}_{t|T} \varphi_2(t)\right|,
\end{aligned} \tag{5.4.22}$$

where

$$\Sigma_B = [0,T] \times \Omega_B := [0,T] \times \left\{ x \in \mathbb{R}^N \;\; ; \;\; |x| \leq 2B^{-1}T \right\}, \quad \int_{\Sigma_B} = \int_{\Sigma_B} dx\, dt, \quad \int_{\Omega_B} = \int_{\Omega_B} dx$$

and

$$\Delta_B := [0,T] \times \left\{ x \in \mathbb{R}^N \;\; ; \;\; B^{-1}T \leq |x| \leq 2B^{-1}T \right\}, \quad \int_{\Delta_B} = \int_{\Delta_B} dx\, dt.$$

Moreover, using the Young inequality

$$ab \leq \frac{1}{p} a^{\,p} \;+\; \frac{1}{\widetilde{p}}\, b^{\,\widetilde{p}} \qquad \text{where } \; p\widetilde{p} = p + \widetilde{p}, \;\; p > 1, \widetilde{p} > 1, \;\; a > 0, b > 0, \tag{5.4.23}$$

with

$$\begin{cases} a = |u|\; \widetilde{\varphi}^{1/p}, \\ b = \widetilde{\varphi}^{-1/p} (\varphi_1(x))^\ell \left| D^{2+\alpha}_{t|T} \varphi_2(t) \right|, \end{cases}$$

in the first integral of the right hand side of (5.4.22), and using Hölder's inequality

$$\int_{\Delta_B} ab \leq \left(\int_{\Delta_B} a^{\,p} \right)^{1/p} \left(\int_{\Delta_B} b^{\,\widetilde{p}} \right)^{1/\widetilde{p}}, \;\; p > 1, \widetilde{p} > 1, \;\; a > 0, b > 0,$$

with

$$\begin{cases} a = |u|\; \widetilde{\varphi}^{1/p}, \\ b = \widetilde{\varphi}^{-1/p}\; (\varphi_1(x))^{\ell-1} \left| \Delta_x \varphi_1(x)\; D^{\alpha}_{t|T} \varphi_2(t) \right|, \end{cases}$$

in the second integral of the right hand side of (5.4.22) and with

$$\begin{cases} a = |u|\; \widetilde{\varphi}^{1/p}, \\ b = \widetilde{\varphi}^{-1/p}\; (\varphi_1(x))^{\ell-2} \left|\nabla \varphi_1(x)\right|^2 \left| D^{\alpha}_{t|T} \varphi_2(t) \right|, \end{cases}$$

in the third integral of the right hand side of (5.4.22), and taking account of (5.4.17), we obtain

$$\begin{aligned}
&\int_{\Sigma_B} |u(x,t)|^p\, \widetilde{\varphi}(x,t) \\
&\leq C\int_{\Sigma_B} (\varphi_1(x))^\ell\, (\varphi_2(t))^{-\frac{1}{p-1}} \left|D^{2+\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}} \\
&+C\left(\int_{\Delta_B} |u|^p\, \widetilde{\varphi}\right)^{1/p}\left(\int_{\Delta_B} (\varphi_1(x))^{\ell-\widetilde{p}}\, (\varphi_2(t))^{-\frac{1}{p-1}} \left|\Delta_x\varphi_1(x) D^{\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}}\right)^{1/\widetilde{p}} \\
&+C\left(\int_{\Delta_B} |u|^p\, \widetilde{\varphi}\right)^{1/p}\left(\int_{\Delta_B} (\varphi_1(x))^{\ell-2\widetilde{p}}\, (\varphi_2(t))^{-\frac{1}{p-1}} |\nabla\varphi_1(x)|^{2\widetilde{p}} \left|D^{\alpha}_{t|T}\varphi_2(t)\right|^{\widetilde{p}}\right)^{1/\widetilde{p}}. \quad (5.4.24)
\end{aligned}$$

Taking account of the scaled variables : $\tau = T^{-1}t$, $\xi = (T/B)^{-1}\, x$, the formulas (5.2.20), (5.2.22) and the fact that $p = p*$, we get

$$\int_{\Sigma_B} |u(x,t)|^p\, \widetilde{\varphi}(x,t) \leq\; C\, B^{-N} \;+\; C\, B^{2-\frac{N}{\widetilde{p}}} \left(\int_{\Delta_B} |u(x,t)|^p\, \widetilde{\varphi}(x,t)\right)^{1/p}. \quad (5.4.25)$$

Now, from (5.4.21) and the fact that ($p = p* \Longleftrightarrow \delta = 0$), we have the following implication

$$\lim_{T\to\infty}\int_{\Sigma_B} |u(x,t)|^p\, \widetilde{\varphi}(x,t) \leq\; C \;\;\Longrightarrow\;\; \int_{Q_\infty} |u(x,t)|^p \leq\; C,$$

and so

$$\lim_{T\to\infty}\left(\int_{\Delta_B} |u|^p\, \widetilde{\varphi}\right)^{1/p} = \left(\lim_{T\to\infty}\int_0^T\int_{|x|\leq 2B^{-1}T} |u|^p\, \widetilde{\varphi} - \lim_{T\to\infty}\int_0^T\int_{|x|\leq B^{-1}T} |u|^p\, \widetilde{\varphi}\right)^{1/p} = 0.$$

Thus, passing to the limit in (5.4.25), as $T\to\infty$, we get

$$\int_0^\infty\int_{\mathbb{R}^{\mathbb{N}}} |u(x,t)|^p\, dx\, dt \leq\; C\, B^{-N}.$$

Then, taking the limit when B goes to infinity, we obtain $u = 0$ for all t and for almost everywhere x; contradiction with the fact that $\int_{\mathbb{R}^N} u_0 > 0$.

• The case $p\; < (1/\gamma)$: We repeat the same argument as in the case $p < p^*$ by choosing the following function $\widetilde{\varphi}(x,t) = (\varphi_1(x))^\ell\, \varphi_2(t)$ where $\varphi_1(x) = \Phi\left(|x|/R\right)$, $\varphi_2(t) = (1-t/T)^\eta_+$, $\ell,\eta \gg 1$ and $R\in(0,T)$ large enough such that when $T\to\infty$ we don't have $R\to\infty$ in the same time, with the same function Φ as above. We then obtain

$$\begin{aligned}
&\int_{\mathcal{C}_T} |u|^p\, \widetilde{\varphi} \;+\; C\, T^{-\alpha}\int_{\mathcal{C}} (\varphi_1(x))^\ell\, u_0(x) \;+\; C\, T^{-\alpha-1}\int_{\mathcal{C}} (\varphi_1(x))^\ell\, u_1(x) \\
&\leq\; C\int_{\mathcal{C}_T} |u|\; \widetilde{\varphi}^{1/p}\widetilde{\varphi}^{-1/p}\, (\varphi_1(x))^\ell \left|D^{2+\alpha}_{t|T}\varphi_2(t)\right| \\
&+ C\int_{\mathcal{C}_T} |u|\; \widetilde{\varphi}^{1/p}\widetilde{\varphi}^{-1/p}(\varphi_1(x))^{\ell-1} \left|\Delta_x\varphi_1(x)\; D^{\alpha}_{t|T}\varphi_2(t)\right|
\end{aligned}$$

$$+ C \int_{\mathcal{C}_T} |u| \, \widetilde{\varphi}^{1/p} \widetilde{\varphi}^{-1/p} (\varphi_1(x))^{\ell-2} |\nabla \varphi_1(x)|^2 \left| D^{\alpha}_{t|T} \varphi_2(t) \right|, \tag{5.4.26}$$

where

$$\mathcal{C}_T := [0,T] \times \mathcal{C} := [0,T] \times \left\{ x \in \mathbb{R}^N \ ; \ |x| \le 2R \right\}, \quad \int_{\mathcal{C}_T} = \int_{\mathcal{C}_T} dx \, dt, \quad \int_{\mathcal{C}} = \int_{\mathcal{C}} dx.$$

Now, by Young's inequality (5.4.19), with the same a and b as above and using (5.4.17), we get

$$\begin{aligned} \int_{\mathcal{C}_T} |u|^p \, \widetilde{\varphi} \ &\le \ C \int_{\mathcal{C}_T} (\varphi_1(x))^{\ell} (\varphi_2(t))^{-\frac{1}{p-1}} \left| D^{2+\alpha}_{t|T} \varphi_2(t) \right|^{\widetilde{p}} \\ &+ \ C \int_{\mathcal{C}_T} (\varphi_1(x))^{\ell-\widetilde{p}} (\varphi_2(t))^{-\frac{1}{p-1}} \left| \Delta_x \varphi_1(x) D^{\alpha}_{t|T} \varphi_2(t) \right|^{\widetilde{p}} \\ &+ \ C \int_{\mathcal{C}_T} (\varphi_1(x))^{\ell-2\widetilde{p}} (\varphi_2(t))^{-\frac{1}{p-1}} |\nabla \varphi_1(x)|^2 \left| D^{\alpha}_{t|T} \varphi_2(t) \right|^{\widetilde{p}}. \end{aligned}$$

Then, the new variables $\xi = R^{-1}x$, $\tau = T^{-1}t$ and formulas (5.2.20) and (5.2.22) allow us to obtain

$$\int_{\mathcal{C}_T} |u(x,t)|^p \, \widetilde{\varphi}(x,t) \le \ C \, T^{1-(2+\alpha)\widetilde{p}} \, R^N \ + \ C \, T^{1-\alpha \widetilde{p}} \, R^{N-2\widetilde{p}}.$$

Taking the limit as $T \to \infty$, we infer, as $p < \frac{1}{\gamma}$ ($\iff \ 1 - \alpha\widetilde{p} < 0$), that

$$\int_0^{\infty} \int_{\mathcal{C}} |u(x,t)|^p \, (\varphi_1(x))^{\ell} \, dx \, dt = 0.$$

Finally, by taking $R \to \infty$, we get a contradiction. □

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