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Elementary Student Teachers Practising Mathematical Enquiry at their Level: Experience and Affect

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

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In light of the significant contribution of her work to mine, and though I never had the opportunity to meet her in person, I dedicate this thesis to Leone Burton (1936-2007).

Eva Knoll, December 9th, 2007.

Abstract

From the time of publication of Polya's "How to Solve It" (1954), many researchers and policy makers in mathematics education have advocated an integration of more problem solving activities into the mathematics classroom. In contemporary mathematics education, this development is sometimes taken further, through programmes involving students in mathematics research projects. The activities promoted by some of these programmes differ from more traditional classroom activities, particularly with regards to the pedagogic aim.

Several of the programmes which can claim to belong to this trend are designed to promote a less static view of the discipline of mathematics, and to encourage a stronger engagement in the community of practice that creates it. The question remains, however, about what such an experience can bring the students who engage in it, particularly given the de-emphasis on the acquisition of notional knowledge. In the study described in this thesis, I investigate possible experiential and affective outcomes of such a programme in the context of a mathematics course targeted at elementary student teachers.

The study is composed of three main parts. Firstly, the theoretical foundations of the teaching approach are laid down, with the expressed purpose of creating a module that would embody these foundations. The teaching approach is applied in an elementary teacher education context and the experience of the participating students, as well as its affective outcomes, are examined both from the point of view of authenticity with respect to the exemplar experience, and for the expected–and unexpected–affective outcomes. Both of these examinations are based on the establishment of a theoretical framework which emerges from an investigation of mathematicians' experience of their research work, as well as the literature on affective issues in mathematics education.

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Definitions and Abbreviations

General Terms

Module: the context of the intervention which is the subject of this study is a component of a modular programme of study in an American university. The module refers to the whole time that the students spend in class and out on work done in this component.

Study: the research project that is the subject of this thesis.

Theoretical Framework

Agency: control over the process of enquiry, including the choice of starting point, the directions of exploration, and the selection of a satisfactory goal-state (Grenier & Payan, 2003; Burton, 2004)

Conventional/observational; applicational; theorisational mathematical notions: the three levels of mathematical objects that form the basis of the epistemological view of mathematics used in this thesis

- *Conventional/observational:* the basic building blocks that are reasoned upon in the application of the remaining categories

- *Applicational:* the product or application of some form of mathematical reasoning upon the previous two categories, and can therefore be explained and traced back to this reasoning

- *Theorisational:* the notions that make possible the reasoning, which itself produces the applicational notions and promotes their adaptability (adapted from Hewitt, 1999 and Burton, 2004)

Engagement, passive; active; critico-creative: degree to which a participant involves her/himself in a task; the three terms describe a continuum, whereby the lowest level involves minimal attendance to the task, and the highest, critical and creative input (adapted from Passmore, 1967; Burton, 1984 and Mason, 1992)

Experience; experiential outcomes: distinction between the two terms

- *experience had* corresponds to the sum of an individual's apprehensions of a situation or collection of situations in which s/he is engaged, including affective, cognitive and psychomotor components (*Erlebnis*)

- *experience gained* corresponds to the outcomes of the experience (*Erfahrung*, adapted from Habermas, 1963)

Knowing that/how; why; when: the three levels of knowing that form the basis of the pluralistic epistemology of mathematics that is used in this thesis.

- *Knowing-that/how* is an unquestioning knowing rooted in the reliance on the word of an external figure of authority whom one implicitly trusts

- *knowing-why* corresponds to a familiarity with a mathematical notion that is known because its mathematical derivation is known

- *knowing-when* knowing which allows the knower to recognize, by analogy, generalization or specialization, the similitude of structure between applicational notions, thereby providing the powerful tools of rigour for the expansion of their applicability, and by extension, for the negotiation of their meaning (adapted from Burton, 2004, Schmalz, 1988)

Mathematical enquiry (ME); Mathematical Problem Solving in the Classroom (MPSC):

- *ME* is the practice that is ascribed to full participants of the community of research mathematicians; the exemplar upon which the teaching practice is based (adapted from Hadamard, 1945; Mason, 1992; Grenier & Payan, 2003 and Burton, 2004)

- *MPSC* is the practice of mathematical problem solving that is applied in many contemporary educational contexts and described in the mathematics education literature. (Mason, 1978; Burton, 1984; Brown, 1994 and Sowder, 1993)

Mathematical structures: the underlying invariants and generalisations inherent in the discipline of mathematics. The object of 'knowing-when' and the basis of 'knowing-why' (see above).

Peripheral; genuine full; stand-in full participants:

- *peripheral participants* who are learning to become full participants in the community of practice

- *genuine full participants* who are full participants within both the 'whole population' and a 'sample population' community of practice

- *stand-in full participant* who take on the role of full participant in a 'sample population' without being a full participant in the 'whole population' community of practice (adapted from Lave & Wenger, 1991)

Problems to be solved using mathematics; Problems which are mathematical in nature:

- *problem solved using mathematics* is a problem the solving of which requires mathematics but the solution itself is not mathematic in nature (adapted from Polya's problems to find, 1957)

- *problem which is mathematical in nature* is a problem the solving of which elicits the creation of a mathematical model, and the obstacle cannot be reduced to a lack of knowledge of mathematics (adapted from Polya's problems to prove; also Lesh & Harel, 2003). A good indication of this distinction is that the solution to the latter type is conducive to the formulation of new problems or questions, in a never-ending process.

Routine; non-routine; or critico-creative task:

- *routine task* is a task that "tests the [participant's] mastery of a narrowly focused technique, usually one that was recently 'covered'", as opposed to a "question that cannot be answered immediately ('exercise')

- *non-routine task* is a task for which the participant cannot at once decide what rule to apply or how it applies ('problem')

- *critico-creative* task is a non-routine task that involves not only the selection of the 'rule to apply', but also its creation and justification. It requires critico-creative engagement ('mathematical enquiry'; Polya, 1945; Passmore, 1967 and Zeitz, 1990)

'Sample population'; 'whole population' community of practice:

- *'sample population' community of practice* is a subset of the community of practice associated with a discipline, usually a microcosm within the community as a whole. A classroom or school is an example.

- *'whole population community of practice* corresponds to the community of practice , which is associated with a discipline, as a whole (adapted from Lave & Wenger, 1991)

Research Methodology

Action research: the overall methodological framework used in this study. Conventionally, it is constituted of the following stages: initial observation and exploration of a problem, plan, implementation and observation of the proposed solution, reflection on the proposed solution. (Mc Niff, 1988)

Components of the study: this study has three main components, the establishment of a solution to the problem observed, based on a theoretical analysis of literature, the verification of the authenticity of the experience provided by this solution, based on the intentions laid out by the theoretical analysis, and an evaluation of the outcomes of the intervention.

Intervention: the designed teaching approach and its implementation within the context of this study

Student- , teacher-; teaching assistant- and researcher-participant:

- *student-participants* are the participants of this study that take the role of students in the intervention

- teacher-participant and teaching assistant-participant are the participants that are responsible for the teaching of the module

- researcher-participant is responsible for the research component of the intervention

Surveys: the two data collection events that resulted in quantitative data provided by the student-participants

Teaching Approach

Design criteria for the teaching approach: the criteria, which emerged from the review of the pertinent literature, that formed the basis for the teaching approach

Teaching approach: the sum of the decisions regarding the activities of the teaching team, and the expectations from the student-participants

Teaching team: the teacher-participant and the teaching assistant-participant

Chapter 1: Introduction

Historical Context

From the time of publication of Polya's "How to Solve It" (1957), many researchers and policy makers in mathematics education have advocated an integration of more problem solving activities into the mathematics classroom. In several current curricula, this shift is manifested through changes in the language used, showing more emphasis on aspects of mathematics other than notional knowledge, including skills in problem solving, reasoning, and communication. In the United States, where the data for my study was collected, the National Council of Teachers of Mathematics promotes 'Process Standards', alongside the 'Content Standards'. This additional set of recommended outcomes includes problem solving, which is described as "engaging in a task for which the solution is not known in advance" (NCTM, 1989, 2000). Similarly, in England, the National Curriculum considers problem solving a "Key Skill" at all key stages (National Curriculum, n.d.).

In contemporary mathematics education, this development is sometimes taken further, through programmes involving students in mathematics research projects. Programmes of this type can target undergraduate students, as in the National Science Foundations' *Research Experience for Undergraduates* (REU); or school children, as in the Education Development Center's *Making Mathematics* (also sponsored by the NSF), or France's *MATh.en.JEANS*.

An analysis which I present in the literature review reveals that the activities promoted by some of these programmes differ in three main ways from those practised in response to the above mentioned curriculum policies:

The position or role of the various participants (students, teacher and perhaps researcher) is different, meaning that traditional relationships are altered, with respect, for example to individual power, authority, and responsibility.

The types of mathematical situation these programmes use are different. To understand this distinction, it is necessary to differentiate between a problem which requires the use of mathematics in order to be solved (as exemplified by traditional 'word problems', involving trains travelling in opposite directions, pies that need subdividing, bathtubs filling up, etc.), and a problem which is itself mathematical in nature, such as, for example, the search for patterns in Pascal's triangle (from the *Making Mathematics* programme). This distinction is related to that defined by Polya (1957) between "problems to find" and "problems to prove", although its implications are broader. In a problem of a mathematical nature, the practice elicits the creation of a mathematical model (Lesh & Harel, 2003), and the obstacle cannot be reduced to a lack of knowledge of mathematics. Instead, the barrier is overcome by the use of the mathematical thinking which can elicit such a mathematical model.

In some of the situations used in these programmes, the participant focuses on the posing and solving of problems which are by their very nature mathematical, rather than on the solving of problems *using* mathematics. Implications of this choice include the authority of the participants to individually formulate and reformulate the question(s), their independence from specific notional knowledge (since the question can be posed at a level appropriate to each participant), and the strong potential for extension to analogous or more general instances of the situation.

A shift in the focus of the discourse takes place during the programme, away from the acquisition of notional knowledge, to a more holistic experience of mathematics as a discipline. In *Making Mathematics* (1999), for example, the aim was to:

> introduce students, teachers, and parents to mathematics as a research discipline, rather than a body of facts to be memorized. Our mathematics research projects developed students' investigative skills and creativity, and they emphasized the habits of mind used by working mathematicians and scientists.

As Burton (2004) expressed it, the pedagogic aim of such programmes differs from more traditional teaching approaches:

> As Tony Ralston pointed out to me, understanding what he called the "mathematical enterprise" is more important than knowing quantities of facts or skills, that is, it is about how we engage students in the activity of mathematics, not about how much they learn (private communication). (p. 198)

Several of the programmes which can claim to belong to this trend are designed to promote a less static view of the discipline of mathematics, and to encourage a stronger

engagement in the community of practice that creates it. The question remains, however, about what such an experience can bring the students who engage in it, particularly given the de-emphasis on the acquisition of notional knowledge. In the study described in this thesis, I investigate possible experiential and affective outcomes of such a programme in the context of a mathematics course targeted at elementary student teachers. The choice of this particular population is particularly significant in that, as they take on their role in the education system, they will be the representatives of best practices in mathematics in the eyes of their pupils.

Theoretical Background

According to van Bendegem (1993):

> Most mathematicians would agree with the following statements: (1) there is something like a mathematical universe, (2) this universe is unique, and (3) in it all mathematical problems are settled. The mathematician's task is to discover and chart this universe, with the knowledge that a complete map is impossible. (p. 23)

The language used in this statement, implying as it does that mathematicians are cartographers of the "mathematical universe", suggests that the activities of practising mathematics researchers are an important source of what our society accepts as mathematics, and that the product of these activities could be contained in the discourses found in textbooks and classroom instruction, journals and conference lectures, a mapping of sorts. This would lead to the conclusion that all mathematical knowledge can be recorded into such static forms.

In contrast to this interpretation, Davis (1993), in the same collection of chapters about the philosophy of mathematics and mathematics education, states the following:

> A more recent view, connected perhaps with the names of Kuhn and Lakatos, is that knowledge is socially justified belief. In this view, *knowledge is not located in the written word or in symbols of whatever kind. It is located in the community of practitioners*. We do not create this knowledge as individuals, but we do it as part of a belief community. Ordinary individuals gain knowledge by making contact with the community experts. (p. 188, my italics)

Davis suggests that there is more to (mathematical) knowledge than that which can be recorded in written form. This view is connected to recent contributions in education (Schön, 1990; Lave & Wenger, 1991) which posit that there is a more subtle component of the knowledge of a discipline, which is acquired through personal engagement in the specific practice of the discipline, and can therefore not be captured through mere

retelling. This theoretical perspective suggests that direct transmission of the product of a practice does not do justice to the discipline it represents in a way comparable to the direct experience of its practice. This perspective contrasts with those of 'transmission of knowledge', which conceptualises learning as assimilation, and 'social constructivism', which conceptualises it as social construction. Lave and Wenger (1991) call this direct experience *legitimate peripheral participation (LPP) in the community of practice* of the discipline. The conceptual language of this phrase is deliberately specific as it is used to describe a particular occurrence. According to Wenger (1998), to start, a community of practice is defined by the following three criteria:

- What it is about – its joint enterprise as understood and continually renegotiated by its members.
- How it functions – mutual *engagement* that bind members together into a social entity.
- What capability it has produced – the shared repertoire of communal resources (routines, sensibilities, artefacts, vocabulary, styles, etc.) that members have developed over time. (Wenger, 1998, my italics)

In effect, the expression 'community of practice' refers to a social entity comprising individuals who, while possibly disagreeing about significant issues, are nevertheless engaging, in the larger sense, in the pursuit of a common goal, and are presumably operating within a common practice and therefore having a comparable experience. This assumption that the practice and consequently the experience of students in the mathematics classroom context are indeed comparable to that of full participants is what this study is questioning.

In this respect, the peripherality cited by Lave and Wenger connects to the level of competency of the non-expert participant. In their view, a competent participant in the community of practice operates at the centre of the community, while a non-expert operates closer to the periphery of the community, where the tasks are simpler and require a less complete contribution on her/his part, though, the authors claim, s/he can nevertheless have a comparable experience. Legitimate participation would therefore mean an engagement and experience that are authentic with respect to that of full participants of the community of practice. In relation to classrooms, this authenticity of the experience, however, is questionable as it is dependent on the authenticity of the practice the participants engage in and the goals they pursue as well as on the social structure within which they take place. In contrast to the practice taking place at the 'centre of the community of practice', in the classroom this social structure is largely dictated by the interaction between the teacher and the students. In socio-cultural

theories of learning, for example, the concept of scaffolding is presented as a teaching tool:

> In order to qualify as scaffolding, [Mercer and Fisher (1993, in Wells, 1999)] propose, a teaching and learning event should: a) enable the learners to carry out the task which they would not have been able to manage on their own; b) be intended to bring the learner to a state of competence which will enable them eventually to complete such a task on their own; and c) be followed by evidence of the learners having achieved some greater level of *independent competence* as a result of the scaffolding experience (Wells, 1999, p. 221). The emphasis of their definition is on the collaboration between the teacher and the learner in constructing knowledge and skill in the former. (Verenikina, 2003)

This description clearly gives the teacher, as 'sanctioned' authority, an active role in the practice, and therefore in the experience of the student, thus impacting its authenticity with respect to that of full participants. Socio-cultural theorists defend the view that the role of the teacher is to manipulate the situation so that the leap of understanding that the student is required to make fits into the 'zone of proximal development', which is defined as:

> the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers (Vygotsky, 1978, p. 86)

In doing so, they propose a model which is dissimilar to the experiences of full participants, who often need to build the very structure that will allow them to reach their goals. The effectiveness of this model is dependent on the type of learning sought. It could be argued, for example, that developing the skills to build one's own scaffold is itself a worthy learning goal. In addition, if a participant never experiences this last practice, it is difficult to justify the notion that his/her overall experience will give him/her a complete appreciation of mathematics as a discipline.

Lave and Wenger's proposed framework is effective for the project because it does not simply suggest a return to the source of knowledge, or the intimate involvement of the full participant in the specific tasks and activities of the peripheral participant. Instead, it presents a case for strong personal engagement, on the part of the *peripheral participant*, with the practice that produces this knowledge as well as its context, thus justifying the shift of emphasis on experience rather than acquisition of an objectified set of knowledge outcomes, and consequently promoting a more holistic view of learning.

In the case of mathematics, the practice of full participants in the community of practice is reflected by the full range of experiences of researchers as well as end-users of mathematics, including creation and theorisation as well as application. To experience an authentic participation in this community of practice therefore means to research and develop as well as to use mathematics. Implementing a programme providing experience of the research component entails an examination of the experience of the full participants in this component. In order to complete this examination, however, a clarification is to be made regarding the word experience, as it can have several meanings. To define my preferred use of the term, I turn to Habermas (1973), who, writing in German, had two distinct words at his disposal for the single English term.

The first meaning of the word experience, corresponding to Habermas' 'Erlebnis' (1973), refers to the sum of an individual's apprehensions of a situation or collection of situations in which s/he is engaged, including affective, cognitive and psychomotor components. In other words, it refers to how the individual feels, both in a sensory and emotional sense and what s/he thinks while s/he is experiencing the situation(s); the experience s/he *has* while the situation is in place. This definition is distinct from that of 'Erfahrung', referring to the outcomes of the experience. That is the experience s/he *gained*.

The phrases in the previous paragraph, which use the term 'experience' with the different meanings, reflect a difference in views of education between a transfer-of-knowledge paradigm (experience gained) to knowledge-through-participation paradigms (experience had). For this reason, and because of the theoretical perspective I choose, I focus on experience had, 'Erlebnis', in this project. In particular, I compare the participants' experience with that which researchers have, when they are involved in their own practice. From this point on, I use the word experience in that sense, in contrast with *experiential outcomes*, which refers to the 'experience gained'.

As the study is concerned with the experience of authentic mathematics research practice, an examination of this practice and the nature of the experience of its full participants is essential. The emphasis on the participant's experience, rather than on the knowledge s/he may gain also shifts the emphasis of the teaching programme, accordingly, from a push for more sophisticated mathematical content to a focus on the participants' process, necessitating, in turn, the use of topics possessing specific attributes, including open-endedness, and both accessibility and newness from the

perspective of the students. If these topics are pitched at the appropriate level, the ad hoc provision, by the teacher, of scaffolds is de-emphasised, providing a more authentic experience of full participation in mathematics research, where there are peers but no absolute authority figure. The key to this condition is that a mathematical problem need not be complex or unsolved by the community of practice as a whole to form the basis of an experience which is authentic, with respect to that of full participants, for the peripheral participant. On the other hand, the situation should not be so accessible that the 'goal state' can be easily envisioned, with or without external help.

The literature on the philosophy of mathematics informs my perspective of the epistemological foundations of mathematics. In particular, this will contribute to the design of the teaching approach with respect to the mathematical truth-authority, the position of the participants in terms of who has the final answer, and the responsibility and power in the interaction that is being experienced.

The literature on the experience of practising mathematicians will help in the evaluation of the authenticity of the participating students' experience, in comparison with that of full participants. In particular, Burton's (2004) report on the interviews she conducted with 70 contemporary mathematicians provides significant insights into the nature of this experience as well as into its possible applicability in the classroom.

I also consider the conditions of the transposition of the experience from the context of the professional life of trained researchers, to that of the classroom. This transposition cannot be simple, as the two contexts differ in significant ways. Designing this programme therefore entails the establishment of criteria of value regarding aspects of the original experience, which, in combination with an evaluation of the classroom context, determine the parameters of the teaching approach, and rates their relative importance. For example, the context of the intervention dictates time and scheduling constraints in the form of class times and reasonable expectations of non-class time spent on course work. In contrast, the professional practice of mathematics researchers entails a temporal flexibility, in the sense that inspiration can strike in many forms and at various times, and the participant needs the flexibility to strike when the iron is hot (Hadamard, 1945).

I resolve the theoretical issues of this portion of the project through an investigation of philosophical questions in mathematics education, particularly with respect to the place

and meaning of problem solving in mathematics teaching. This last component is essential for the purpose of characterising of the activities in the planned teaching approach, particularly in comparison with those in more traditional versions of problem solving teaching.

Finally, I develop tools to evaluate the suggested programme in terms of its goals. To recall, the intention is two-fold: first, I plan to provide the participants with an opportunity to engage strongly in an authentic experience of peripheral participation in a community of practice, and second, I expect them to develop insights from the experience, mainly but not exclusively in the form of changes in views, attitudes and beliefs about mathematics and themselves in relation to the discipline. These two aspects are evaluated separately, using distinct tools. As I mentioned earlier, the literature on the experience of mathematicians forms the basis of the evaluation of the programme as the participants' experienced it. The fact that this part of the evaluation serves as a verification of the authenticity of the participants' experience, peripheral though it may be, of the practices of the community, makes reports about mathematicians' experience the appropriate source for the comparison. In particular, the roles of the various participants are interdependent and connected to issues of truth-authority and problem ownership.

The portion of the evaluation which focuses on the insights which the participants developed, on the other hand, is limited to affective issues and based on a review of theories of affect, including emotions and attitudes as well as beliefs. As mathematical problem solving is the classroom activity which most closely resembles that which takes place in this programme, the literature I consult focuses on affect with respect to problem solving in mathematics (Schoenfeld, 1985; Lester, Garofalo & Kroll, 1989; Mandler, 1989; Silver & Metzger, 1989; Goldin, 2000).

Personal Context and Motivation

The motivation for this study emerged from my own experience with mathematical enquiry. This experience mainly took place outside my regular schooling, through personal pursuits in the interdisciplinary study of mathematics and fine art, design and the decorative arts. My mathematical journeys led me to explore topics such as tessellations and transformational geometry, 2- and 3-D Euclidean geometry, polyhedra and space-filling, non-Euclidean geometries, and number theory.

To show the relevance to the project of my past experience with mathematical enquiry, I discuss three categories of parameters, relate them back to some of the issues discussed in the background section and illustrate each through a personal mathematical enquiry culminating in a publication. I begin by considering the nature of the starting points of my investigations. In a second instance, I debate some characteristics of the position of researcher which I took in my investigations. I consider this role particularly in relation to other members of the community of practice (Lave and Wenger, 1991) within which I operated, notably the full-participating 'experts'. Finally, I review the nature of my experience of enquiry, focusing on the experiential (Hadamard, 1945) and heuristic (Polya, 1957; Schoenfeld, 1992; etc.) cycles through which I passed, and my relationship with the community of practice.

The Nature of the Starting Points

The proceedings of the 2002 conference *Bridges: Mathematical Connections between Art, Music and Science* contain the report of one of my mathematical enquiries, titled "From a Subdivided Tetrahedron to the Dodecahedron: Exploring Regular Colorings" (Knoll, 2002). This report is the culmination of a project which began with the study of the subdivision of regular polyhedra into congruent parts, and ended with the discovery of an interesting relationship between two Platonic Solids, the regular tetrahedron and dodecahedron. Consistently with the philosophical underpinnings of this thesis, I do not focus here on the mathematical content of the enquiry. Rather, I use the project to illustrate what I consider important from the point of view of the nature of the starting point. Interested readers can find the full paper in Appendix 1.

In order to accomplish this task, I begin by differentiating between what I mean by a question or problem, and a research situation (Grenier & Payan, 2003), either of which can serve as the starting point of a mathematical enquiry. In essence, a problem or question is well-defined and implies, in its formulation, the form that the solution, if it exists, will take. If the solution does not exist, the answer will take the form of a justification for this finding. In the example under discussion, part of the enquiry consisted in the posing, and answering, of the following question: "What is the simplest non-adjacent regular colouring of a tetrahedron whose faces have been subdivided into sets of three kites?" (see figure 1 in Appendix 1). An important property of this question is that it can be definitely answered, most simply in this case by a diagram or a physical model. In effect, there are a 'given state' and a 'goal state' (Mayer, 1985). In contrast, a

research situation presents the enquirer with a mathematical situation (a given state), without a specific direction of enquiry, or even an implied goalpost. In the case of the enquiry reported in the publication, a question, indeed several questions were formulated as well as answered during the process, but the enquiry as a whole did not begin with these questions. Rather, I began with a mathematical situation which showed some potential for exploration, and the exploration consisted, among other stages, in the posing of specific questions and the selection and answering of some of them. The distinction, therefore, lies in the openness of the starting point of the task. This open nature of the starting point manifests itself at three distinct levels (Lock, 1990): not only is the solution or result open, but so are the method and even the initial focus of the problem(s) or question(s), the formulation of which then becomes an integral part of the process. This aspect has significant consequences with respect to my role as enquirer and the resulting experience, as I discuss later.

In addition, a research situation, according to Grenier and Payan (2003), is accessible to the enquirer(s), without recourse to sophisticated notional knowledge, including manipulation algorithms, or symbolic notations. The starting point is easy to grasp and inviting, and barriers of notional knowledge are minimised. In the example in point, I was already somewhat familiar with the regular polyhedra, at least intuitively, and had investigated non-adjacent regular colourings and the symmetries they illustrate, though mainly in 2-dimensional grids. More importantly, however, the situation was accessible largely because it dealt with objects which, though mathematical, can relatively easily be visualised, manipulated and experimented on using models made of paper or other materials. Though this last characteristic is shared by most of my work of mathematical enquiry, in a more general context, accessibility depends on an individual's existing knowledge, and the selection of the starting point is therefore critical to the nature of the experience.

In their characterisation, Grenier and Payan (2003) also require that the research situation be unsolved by the parties engaged in the enquiry. In their case, and because they involve professional researchers in their activities, this means that the situation has to be unresolved even within the community as a whole. The situations they use in their work are in fact taken from current research in mathematics. Their hypothesis is that this will impact the way in which the students will interact with the situation. Overall, the condition can be expressed as follows: the starting point is established by one of the

participants, and it is essential that none of the participants know the solution, or even a sure way to proceed. As I was not too concerned with the societal acknowledgement of my results, and I was effectively the only participant the situation needed only be new to and initiated by me.

As I mention earlier, an important distinction of the starting point in the current example is that it presents a situation which is mathematical by nature, rather than only by the method of its resolution. This aspect of the situation has an important repercussion for the activities, and therefore for the experience; as Grenier and Payan (2003) explain: "a resolved question very often suggests a new question. The situation has no definite end. There are only local 'end' criteria."[1] A problem which, according to this criterion, is not itself mathematical in nature is solved once the mathematical result is re-interpreted into a non-mathematical context. In the example in point, there were in fact several questions asked during the enquiry, each leading to at least one other, and the report even ends with another question, suggesting the possibility for further enquiry.

In this section, I suggested that the nature of the starting point of a mathematical investigation can impact on the nature of a participant's engagement with it, and by extension on the nature of the experience it provides. I further explained that to design this starting point to follow criteria inspired from the practice of full participants in the mathematics research community can contribute to providing an experience that is authentic with respect to this practice. These criteria are as follows: (1) the starting point should consist of an open situation, without an implied 'goal state' (Mayer, 1985; Grenier & Payan, 2003), nor even specified method of resolution (Lock, 1990; Grenier & Payan, 2003); additionally, (2) the situation needs to be accessible, without recourse to complex mathematical knowledge; and (3) the situation should be unsolved for all the participants (Grenier & Payan, 2003).

My Position as Mathematical Enquirer

In 2000, I presented "Decomposing Deltahedra" at the International Society of the Arts, Mathematics, and Architecture conference. Deltahedra are polyhedra which have a special property: all their faces are congruent equilateral triangles. The paper consisted of the report of an investigation into the possibility of a classification and the development of a method for generating deltahedra which belonged to a specific class,

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[1] My translation

contingent on the classification. In this example, again, I do not focus on the results, or indeed even on the methods. I instead focus on the role that I gave myself in the enquiry. The paper can be found in Appendix 2.

As in the previous example, the enquiry was self-initiated, largely in reaction to experimentation I had done with Origami using circular paper (Knoll, 1999, 2001, 2000b). My work in this area had led me to consider polyhedra which, with slight modification, could be transformed into deltahedra, as this made them buildable using the Origami method I had devised. I had found these in references on the subject (Cundy & Rollet, 1961), but was dissatisfied with what I had uncovered in the literature. In particular, as reported in the paper, the traditional way of generating polyhedra from each other, that is through truncation (slicing off parts) and stellation (extending faces until they meet others, usually further away from the centre), appeared to leave out certain possibilities (notably the snub icosahedron), which nevertheless had common properties with simpler polyhedra, including certain symmetries. In essence, I considered the canon to be incomplete, showing a gap in the mapping of this specific area of mathematics. Rather than pursuing a deeper literature search, I began a mathematical enquiry of the situation using the Origami method I had devised.

Interaction with a professional researcher yielded some ideas for approaches. Most notably, a discussion of the Origami method was conducive to an investigation of a theorem of differential geometry, which, when considered in the discrete case of polyhedral surfaces, was quite accessible to me, particularly since my Origami method appeared to be its embodiment! The nature of the researcher's contribution is an important aspect of the experience: I was not handed the solution to the problem or even a definite hint as to the 'correct', or 'best' direction to follow. Instead, the contribution amounted to a way of expressing the phenomenon observed, and a general idea of its behaviour. With this new tool in hand, I was able to investigate my class of polyhedra and develop a generating rule for them.

Through this short description of the enquiry, I can now bring up several points concerning my role in it. Firstly, as it was self-initiated and indeed self-motivated, I retained agency (Burton, 1999, 2004), at all times, with respect to the direction of enquiry. In other words, having formulated the question, I owned it, and this gave me the power to change the question, refine it in any way I found relevant, or abandon it altogether. In addition, this gave me ownership of the method of enquiry as, even

though I relied on the canon in certain respects, mine was the choice of what mathematics to use. Indeed, even the contribution made by the researcher could be rejected if I did not see a way of integrating it into my process. I therefore remained relatively self-directed, within the community of practice, at least during the process, thus experiencing a practice that is authentic with respect to that of genuine full participants. This last aspect manifests itself through what one might call the responsibility of the enquirer: It is the enquirer who is responsible for the direction which the process will take, and consequently, for its ultimate result. This responsibility, in turn, rests on a control of the criteria of sanction of the results and an accountability of their justification. Though these criteria and their justification are taken from the canon, their application is at the discretion of the enquirer. In effect, I decided what constituted a satisfactory result. This final point is an essential one for the nature of the experience, which I discuss next: having the power to accept or reject a result based on criteria I had chosen meant that *I* was the first 'gatekeeper of mathematical knowledge', in this context. It was only at a later stage, during the peer-reviewed process that the power was passed on. Burton (2004) discusses this aspect in her report of the practice of mathematics researchers. In the section focusing on writing for publication, she describes the process of sharing findings with the community of practice at large. A finding that surprised her was that

> although many of the mathematicians felt that the paper should be correct, they did not see it as the job of the referee to check whether or not it was. Few reported doing the mathematical work to ensure correctness in others' papers and not even always in their own. (p. 147)

Epistemologically speaking, as a member of the community of practice, I could not think of mathematics as something created exclusively by entities outside myself, be they 'experts' (research mathematicians), 'nature' or God, who were then in charge of evaluating/verifying the 'correctness' of my results: I had to convince myself, before I could try to convince others.

In this section, I developed the idea that the position of enquirer which is taken in a research situation is key to the experience it provides. In effect, if the participant is largely self-directed, if s/he owns the questions s/he poses to the degree that s/he can change, or reformulate, or abandon them altogether, and finally, if s/he relies, at least initially, on her own judgement regarding the reliability of the results, her/his

experience will come that much closer to that of full participants in mathematics research.

The Nature of the Experience

Finally, in the context of my masters' study (1995-1997), I pursued a mathematical enquiry into the feasibility of the transfer of two tiling design methods from 2- into 3-dimensional space.

I had the privilege of carrying out this programme in the context of a research laboratory where a mathematical tool was being developed, which operated within an axiomatic (mathematical) framework similar to my own. The closeness of the relationship to the community of practice which this circumstance provided produced a distinctive experience with regards both to the practice of mathematical enquiry and the experience I had while doing it. In particular, this is significant in terms of the level of interaction (Burton, 2004) which this context made possible. The mathematical tool designed in the laboratory became relevant as a support for the enquiry both from the point of view of the modelling of the behaviour of the mathematical objects and the point of view of the theoretical framework it provided. Indeed, both the tool and my study operated within the axiomatic context of projective geometry, where the rules are different from those of ordinary Euclidean geometry.

This project marked the start of my more substantial interaction with the pertinent community of practice, beginning with my discovery of the work of a Constructivist painter, Hans Hinterreiter, whose work paralleled mine to some degree, through work with my supervisory team and the personnel in the research laboratory, and followed by my continued participation in conferences and meetings about art and mathematics. The nature of this interaction is relevant for this discussion, and connects back to the position of researcher which I adopted, as described above: In my interactions with fellow members of the community, I communicated my results through formal interactions such as presentations and publications, I discussed other member's achievements as manifested by the same and learned from them, and most importantly for this discussion, I consulted with my peers, informally, regarding my own investigations. Importantly, however, in none of the situations did the relationship develop into a dependence on an 'expert' for decisions regarding directions of research. The formal nature of the first two types of interaction prevents this since the results are

presented at a relatively advanced stage of the enquiry. In the third and only informal type of interaction, the interaction was often simple advice or encouragement, which I was free to use or ignore.

In particular, in the course of my research, I consulted with an eminent mathematician regarding a specific element of my project. His advice was pitched at a level that made it useful without undermining my ownership of the enquiry. This experience is typical of interactions with members of this community of practice in that they preserve the position of the enquirer with respect to the characteristics discussed in the previous section: the interaction is reciprocal in that communication occurs in both directions, but the individual researchers or teams can remain self-directed, and are free to take the input into account or not.

Two more themes emerge from my reflection on the nature of my experience in mathematical enquiry. Firstly, it consisted of the practice of a variety of methods of mathematical heuristics, including the developing of conjectures, their refutation, and the formulation of proofs (Lakatos, 1976) as well as, for example, "creating, testing, falsifying and validating, the setting of boundary conditions that influence whether conjectures are, or are not, valid, the constructing and challenging of argument and the deliberate use of reflecting" (Burton, 2004, p. 199). This component of the experience is important in the way it replicates the practice of full participants in the community of practice, as described in the literature, validating the authenticity of my experience as one of legitimate peripheral participation in the community of practice that creates mathematical knowledge.

Finally, the experiential cycle and the timeline of my mathematical enquiries present a picture not unlike that described by Hadamard (1945) in his discussion of the invention[2] process in mathematics (and further discussed in Burton's 2004 report). As Hadamard (1945) portrays it, the cycle of enquiry proceeds through stages which are not entirely under the control of the researcher. He presents the four stages as preparation (initiation), incubation, illumination and verification (Liljedahl, 2004), each of which involves different activities and frames of mind, making them identifiable in their own right. My experience in mathematical enquiry outside regular schooling seems to

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[2] The question has been discussed in various contexts as to whether mathematics is 'invented', or 'discovered'. Though an important philosophical issue, I do not debate it here, as it is only marginally relevant to the current topic. From now on I use the more neutral "mathematical enquiry".

emulate that of professional researchers, as described by Hadamard and others, more closely than it does classroom activities. This suggests the possibility of a link between my experience of enquiry and my attitudes and practices with mathematics in general. If this is the case, can an intervention be designed that will allow this experience be replicated for other learners, and will they gain comparable insights and benefits from it? I discuss the complexity revealed by this two-pronged question in the following section.

Aims and Objectives

The aim of the intervention, from the point of view of the participants, is to complete their overall experience of mathematics by providing them with a component which is analogous to that of practicing research mathematicians engaged in mathematical enquiry, thus giving them the opportunity to gain a deeper understanding of the practice of doing mathematics. The implication behind this statement is that the students' existing experience, as provided by their regular schooling, endows them with only a partial appreciation of mathematics as a discipline, and that the experience provided by the intervention is meant to add to, rather than replace this existing experience, particularly in the case of student teachers.

Based on my theoretical outlook, I claim that enriching the participants' mathematical experience in this way can contribute not only to this appreciation but to their self-confidence, their knowledge and indeed their capability in mathematics. It would make the knowledge they do have more readily available by making them aware of its value, thus transforming it from dormant knowledge (available only through direct invocation by the context) to readily accessible (even without deliberate invocation). In particular, this is the case for their grasp of mathematical heuristics such as those I mentioned in the previous section. This in turn would give them more intellectual agency and, potentially, a higher level of motivation.

In summary, the approach I advocate focuses on making an acknowledged learning environment (the classroom) the context for legitimate peripheral participation in the community of practice of *mathematical researchers*. I accomplish this by engaging the participants with mathematical enquiry, emulating, as closely as possible, the experience of full participants. This purpose is facilitated by the selection of a teaching team whose members all have genuine experience of mathematical enquiry. Finally, the

criteria for this emulation are those I disclosed in the discussion of my own experience and which I analyse as part of the literature review.

The uniqueness of the approach lies not only in the modification of a context which is traditionally heavily laden with presuppositions and assumption about knowledge and its epistemology, but in a reduction of emphasis on notional knowledge criteria for evaluation. In effect, the focus of the classroom interaction is on the process which the participants experience, and particularly the heuristics they practice. More importantly, this process is not only one they experience directly, but one which they *develop* themselves. This shift in the functioning of the classroom is achieved through the careful designing of the didactical contract (Brousseau, 1997) between the teaching team and the participating students. In other words, consideration is given to the interaction between the students' and the teachers' expectations of each other, and of the setting, and how this will, in turn, affect their practice and by extension their experience. As Borasi (1991) declared:

> The mathematical experiences provided by prior schooling are likely to make students react with disbelief at the very suggestion of their ability to engage in mathematical enquiry. In order to overcome this response and the many difficulties intrinsic to engaging in open-ended explorations, mathematics educators will need to create an environment supportive of student enquiry in the mathematics classroom. (p.189)

In the present study, I use existing instruments, guided by my own experience, to design a teaching approach which provides the participating students with the opportunity to strongly engage in an authentic experience of mathematics research practice, at their own level.

Research Objectives

The central hypothesis of this study is that it is possible to provide, in a classroom context, an experience of mathematics research practice which is analogous to that of 'expert' practitioners, and that engaging in this experience can have a positive impact on affective responses of the participants, with regards, in particular, to their view of mathematics as a discipline and of themselves as mathematical individuals. This hypothesis can be tested by answering the following questions:

- What could be the design criteria of a teaching approach which aims to provide an experience of practice analogous to that of research mathematicians? Which of these criteria are feasible in the given context?

- Can the experience of engagement with the resulting teaching approach successfully simulate that of engagement in the practice on which it is based, according to the design criteria?
- What are the affective outcomes, and are they as anticipated?

In order to answer the first question, I characterise the experience of mathematical enquiry which I use as exemplar for that with which I intend for the participants to engage. The characterisation of mathematical enquiry, as applicable in the teaching approach, is based on an analysis, informed by my own experience, of writings about the practice and experiences of mathematics researchers and of the literature on problem solving and investigational work in the classroom. In particular, it focuses on the processes that are practised in either situations, and on the experience that results from them. The purpose of this analysis aims at the establishment of design parameters for the teaching approach, and also provides a theoretical basis for the second component of the study.

In response to the second question, I evaluate the authenticity of the participants' experience. To accomplish this, I develop tools and criteria based on this characterisation and focus my analysis on the participants' report of their experience of the classroom practice and their thoughts and reactions about it, including the match with their expectations. The tools I use in this part of the analysis are taken from interpretive methodology and remain fundamentally open-ended in order to capture the richness of the participants' responses.

To respond to the third question, and based on a review of the literature on affective outcomes in mathematical problem solving in the classroom, I adapt and develop tools which I then use to evaluate the participants' construction of their affective interaction with mathematics, particularly in terms of the change between their affective responses before and after. Specifically, the framework I implement focuses on the students' views and beliefs about mathematics as a discipline, including its epistemology, and about themselves as mathematical agents.

An additional question which could be asked but is beyond the scope of this project is the following: Are the affective outcomes meaningful as well as valuable as an instructional achievement, making this practice a useful one, in the training of teachers or in education in general?

The study as a whole is described as follows: based on my own experience with and a survey of literature about processes of mathematical enquiry, I develop a teaching approach which is intended to emulate the experience of professional mathematicians. This teaching approach is then evaluated in terms of two distinct sets of criteria. Firstly, I investigate the authenticity of the participants' experience with respect to that of professional mathematicians, and secondly, I assess whether this teaching approach has any impact on the participants' affective responses to mathematics.

Chapter 2: Literature Review

According to Davis' (1993) description of a recent philosophy of mathematics,

> knowledge is not located in the written word or in symbols of whatever kind. It is located in the community of practitioners. We do not create this knowledge as individuals, but we do it as part of a belief community. Ordinary individuals gain knowledge by making contact with the community experts. The teacher is a representative of the belief community. (p. 188)

Subscribing to this view can have different implications. To begin with, it can be interpreted to mean that mathematical learning cannot be based solely on perusal of the reified 'written word or symbols' produced by the 'experts'. Instead, to be authentic, learning needs to be derived from engagement in an experience (Habermas, 1963) of legitimate peripheral participation in the community of practice (Lave & Wenger, 1991) that creates and uses mathematical knowledge within the larger society.

In this light, I contend that 'mathematical problem solving in the classroom' as it is described in the literature and generally practised in schools does not do justice to the experience of legitimate (either full or peripheral) participation in communities of practice of mathematical enquiry (where new knowledge is created) and can therefore not be considered authentic in that respect. This has an impact on the outcomes of the experience the participants in the classroom practice do have, creating a discrepancy between the outlooks of the two communities. This difference may include but not limit itself to the participants' perception of mathematics as a discipline or field of enquiry and of themselves as mathematical individuals. If this is the case, and expected as they are to become the teacher-authority, or in Davis' words, the representatives of the belief-community, student teachers in particular would benefit from having authentic experiences upon which to draw.

A Few Definitions

The argument, as presented above, requires the clarification of a few terms, which I briefly define here. Firstly, the term engagement, as it is used in the argument, refers to the nature and degree to which an individual participates in a situation. In this respect, it

is influenced by the extent to which the situation is accessible (Lave & Wenger, 1991) and the extent to which the participant can function autonomously within it. This autonomy depends on the power dynamics within the situation (Bourdieu, 1977). Based on these influences, the engagement can be characterised in terms of a positioning on a continuum ranging from non-existent to passive, to active, to critico-creative (Passmore, 1967), the last referring to the fullest possible engagement, as exemplified by full-participants. To illustrate, a non-participating individual is unengaged, though present in the situation, an 'innocent bystander'. A passive participant does what s/he is told, step-by-step, without contributing to any decisions. An active participant shows initiative and does contribute to the decision-making process, without engaging in critical reflection about the situation. Finally, a critico-creative participant engages in critique and develops new options for the decisions through the use of "mental processes that lead to solutions, ideas, conceptualizations, artistic forms, theories or products that are unique and novel" (Johnson-Laird, 1988, p. 203, as cited in Kay, 1994, p. 117). In particular, Johnson-Laird's categorisations of creativity contain the following types, from the lowest to the highest:

- "expressive creativity" or the development of a unique idea with no concern about quality;
- "technical creativity," or proficiency in creating products with consummate skill, as in shaping a Stradivarius violin, without much evidence of expressive spontaneity;
- "inventive creativity," or the ingenious use of materials to develop new use of old parts and new ways of seeing old things, possibly through novel plots or cartoons or the inventions of an Edison, a Bell, and a Marconi whose products are novel and appropriate but do not represent contributions of new basic ideas;
- "innovative creativity," or the ability to penetrate basic foundational principles or established schools of thought and formulate innovative departures, as in the case of Jung and Adler building their theories on Freudian psychology or a Copernicus extending and reinterpreting Ptolemaic astronomy; and
- "emergentive creativity," a rarely attained quality of excellence since it incorporates "the most abstract ideological principles or assumptions underlying a body of art or science" (Taylor, 1975, p. 307), as in the work of an Einstein and a Freud in science and a Picasso and a Wright in art. (pp. 267-268) (Kay, 1994, p. 117)

The list suggests that there is a more powerful degree of engagement possible than simply 'active' participation. This higher degree of participation is a critical ingredient of mathematical enquiry as practised by mathematical researchers and defined below, in contrast with the active engagement required in mathematical problem solving in the classroom, which I define further in a later section.

The second term, experience, is defined in chapter 1. Briefly, I use the meaning of the German word 'Erlebnis', which refers to the sum of an individual's apprehensions of a situation or collection of situations that he/she engaged with, in contrast with 'Erfahrung', which refers to the outcomes of the experience, and which are often reified as 'educational objectives' (Bloom et al., 1956). From this point on, therefore, experience will be used in opposition to outcomes, which represent the resultant transformations of the individual by her/his experience. This definition connects back to the engagement of the participant in that the experience of a participant is directly related to the positioning of her/his engagement on the continuum described previously.

In this light, the modifier 'authentic', as it is used in the above argument, refers to the extent to which an experience emulates the exemplar it is based upon. In the present case, the authenticity of the peripheral participation known as mathematical problem solving is compared to that of full participation in mathematical enquiry as practised by professional researchers. In Hanks' (1991) words, legitimate peripheral participation is:

> an interactive process in which the apprentice engages by simultaneously performing in several roles – status subordinate, learning practitioner, sole responsible agent in minor parts of the performance, aspiring expert and so forth – each implying a different sort of responsibility, a different set of role relations, and different interactive involvement (p. 23)

This multiplicity of the roles of the peripheral participant is important to consider in that it creates the possibility of leaving out some of them, thus giving the peripheral participant an incomplete experience of the practice as a whole, which in turn will impact her/his affective outlook on the discipline. The criteria of this comparison, therefore, emerge from a review of what I define as mathematical enquiry, based on the literature on the practice of research mathematicians (Burton, 2004, Hadamard, 1945), against what is meant by mathematical problem solving as applied in the classroom context. The use of 'authentic', in addition to Lave and Wenger's 'legitimate' helps to reinforce the need to verify the match between the two experiences.

Finally, I use the term practice in the sense given to Practicum as used by Schön (1990), and integrated into Lave and Wenger's 'communities of practice'. This meaning contrasts with that used in statements such as 'practising the violin' in that it incorporates the attitudes and perspectives as well as the acts and habits associated with the activities to which it refers. In this case the practice is that of creating mathematical knowledge.

The discussion is presented in the following sequence: To begin with, I lay the foundations of the argument through an examination of the educational theory presented by Lave and Wenger's model of situated learning, comparing it briefly to models presented by Social Constructivism and Socio-Cultural Theory. This section also presents an epistemological and a practical perspective on mathematics, which lead into the following discussion, on the difference between the classroom practice of mathematical problem solving and the professional one of mathematical enquiry. This part serves as the theoretical basis for both the development of the design criteria for the intervention in student teacher experience that this thesis explores, and the examination of this intervention for authenticity, with respect to the exemplar practice and experience.

Following this section, I discuss the theoretical framework which supports the examination of the affective outcomes of the experience, and finally, I discuss the pertinence of the intervention in teacher education.

Educational Foundations of the Theoretical Framework

In the introduction, I briefly discussed the importance of preserving an experiential connection between the practice of mathematics researchers and that of the mathematics classroom. This perspective aligns itself with the framework developed by Lave & Wenger (1991), according to which learning occurs as a result of a deepening process of participation in a community of practice, suggesting that it consists largely in a cycle of progressively more complete emulation of the practice of a selected group of 'experts' who embody the full, exemplary practice. In the case of mathematics, this practice has its roots in an engagement that is positioned, at least some of the time, in the critico-creative category, as mathematicians are creative:

> Mathematics is not a contemplative, but a creative subject; no one can draw much conclusion from it when he has lost the power or the desire to create. (Hardy, 1967, p. 143)

Decomposing the above description of learning into terms that connect with teaching practice can be illuminating: Teaching, in this framework, pertains to the activities that facilitate increased participation in the community of practice, from peripheral to full. Burton (2004), investigated the practice of contemporary research mathematicians to this aim. Her findings showed evidence suggesting the need for a pedagogical shift which:

> makes dialogue a feature integral to mathematics learning. Such dialogue involves talking with and about, not being talked 'at', talking to learn through the negotiation of meaning, not accepting the meaning of others. Used as a style, it emphasises for the learner that their identity in the classroom has shifted from dependency upon the teacher or text to their agency as a member of a supportive community. (Burton, 2004, p. 185)

This idea of agency, which she suggests is central to a more authentic experience, is a theme which is stitched throughout the coming discussion. The intention of providing learners with agency, however, does not imply an attitude of 'anything goes'. Indeed, in mathematical enquiry, critico-creative thinking is only part of the practice. Furthermore, in those instances, the need for 'creative' thinking is counter-balanced by a requirement for the scientific rigour that supports the 'critical' aspect of the practice, the criteria for which are culturally determined by the community of practice (Burton, 2004, p. 143). As for the 'experts', they are acknowledged as being the participants who are the most highly functional within the specific community of practice, and in the case in point, they are called mathematicians. A problem arises if the local picture of the specific classroom never reflects the global picture of the community as a whole. If we accept that the classroom represents a 'sample' community of practice, in contrast to the 'whole population' community of practice of mathematics, then, in the classroom, the teacher plays the role of full participant, and therefore represents the global full participants. If s/he has not experienced this global full participation her/himself, however, s/he can, at best, claim to be a stand-in full participant, which is significantly different from a genuine full participant. This distinction has deep repercussions in the development of pupils as demonstrated by research investigating the correlation between teacher and pupils' knowledge and attitude (Bell et Al., 1983; Carré & Ernest, 1993; Phillipou & Christou, 1998)

Alternate Educational Perspectives

In contrast, the prevailing perspective in current educational theory, Social Constructivism, is based on two main concepts, the metaphor of carpentry or architecture (Spivey, 1995), and the idea that knowledge, including mathematical knowledge, is socially derived (Ernest, 1995; Gergen, 1995; Jaworski, 1998). The construction metaphor is represented by the three tenets of Constructivism, as posited by von Glasersfeld:

- Knowledge is not passively received through the senses or by way of communication. Knowledge is actively built up by the cognizing subject.
- The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability.

- Cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality. (von Glasersfeld, 1990)

Developed in contrast with the older paradigm, often referred-to as 'transfer-of-knowledge', this perspective acknowledges that there is a distinction between what the teacher is attempting to convey to the student and what the latter learns. Using the construction metaphor, every learner's building is different. This theory ascribes a higher responsibility for learning to the student; and therefore requires active engagement on her/his part.

In the context of mathematics teaching, the social dimension of this perspective concerns both a characterisation of social derivation of mathematics as object of the instruction, and a suggestion of the mechanism of learning. In other words, the social entity to which the learner belongs includes members, which are called experts, who determine what is true (mathematically), and what is to be learned by these apprentice-members. This perspective, however, can create an unforeseen conflict:

> Although socially derived, this [mathematical] knowledge takes on perceived absolutist properties. Those interacting with it, teachers and students, may come to regard it as objective and external to human endeavor. (Jaworski, 1998, pp. 113-14)

… or certainly external to their own endeavour. In other words, there is a benchmark, against which the learner's acquired knowledge, rather than her/his mathematical rigour and creativity, is consistently measured, and the source of this knowledge necessarily remains external to her/him, despite the fact that s/he is purported to be a member, however peripheral, of the community which determines it. The learner has no voice in the negotiation of meaning and need not, therefore, develop much of a rigorous, critical eye.

In this framework, teaching could be expressed as pertaining to the activities that facilitate the construction, on the part of the learner, of a model of mathematical meaning. This model of mathematical meaning correlates with correspondingly static external expectations. It does not, however, provide her/him with agency, as these expectations are framed both by the teacher in his role as gatekeeper, and by social norm, as established by the 'experts', including mathematicians, curriculum and textbook writers and educational policy setters. In this situation, where the focus of assessment is on this replication of static knowledge instead of on the development of critico-creative mathematical thinking, the less critico-creative learner is given leave to interpret the interactions of the classroom as a guessing game, where the prize is in the

formal mark and the test in the replication of the teachers' discourse. This is expressed eloquently in what Brousseau (1997) termed the Topaze Effect. In the eponymous play, a school teacher gives a dictation to a student and, seeing the difficulty he is having, progressively gives more clues until the answer is given entirely and the exercise moot, to humorous effect.

In contrast, at the edge of the canon, though the researcher can make use of socially derived knowledge, s/he needs also to practice critico-creative thinking in order to create new knowledge. In the context of school, it is easy to forego this practice as all the knowledge needed is available. This being the case, if the purpose of learning is simply its acquisition, then this strategy is sufficient. If, however, the purpose is extended to include the development of critico-creative thinking, the exclusive reliance on socially derived meaning is unproductive as it shuts down critico-creative enquiry.

This is well illustrated in the application of this perspective known as scaffolding, when it pertains to the development of notional knowledge. In this pedagogical method, the teacher establishes a scaffold by designing her/his interaction with the learner so as to bridge the 'zone of proximal development', which corresponds to

> the distance between problem-solving abilities exhibited by the learner working alone and that learner's problem-solving abilities when assisted by or collaborating with more-experienced people (Lave & Wenger, 1991, p. 48).

As the description suggests, the scaffold created by the teacher forms an external construction against which the learner can lean on her/his way up (presumably) to a higher state of knowing. As the learner develops, the scaffold can be reduced, until it can be removed altogether. This idea can be applied at different levels, with correspondingly different implications for learning. Firstly, it can be understood at the level of curriculum design, where ideas about the hierarchical nature of mathematical knowledge can dictate the order in which notions can best be taught. Secondly, it can be understood as the necessity

> for the teacher to take control only when needed and to hand over the responsibility to the students whenever they are ready. Through interactions with the supportive teacher, the students are guided to perform at an increasingly challenging level. In response, the teacher gradually fades into the background and acts as a sympathetic coach, leaving the students to handle their own learning. The teacher is always monitoring the discussions, however, and is ready to take control again when understanding fails (Brown et al., p. 141).

In this case, the scaffolding can be considered ad hoc in the sense that it is introduced in response to a learner's difficulties with a specific situation. The teacher can of course prepare for this eventuality, but the help is meant to be only introduced when the need arises. The reactive nature of this type of response can make it difficult to gauge and, if misused, can consequently produce a dependence of the learner on the teacher, particularly if the latter intervenes too readily and thoroughly, causing the learner to further relinquish any sense of agency, since s/he can always rely on the 'expert' to take her/him over the hump. As Mason (1978) says:

> [...] The widespread use of Hints is particularly unfortunate. It indicates that the originator is concealing a solution which the student is to find. The student's problem turns into guessing what is in the originator's mind. A much more neutral word is suggestion, and it is best if mutually contradictory suggestions are made, indicating that there are several ways to proceed. (p. 47)

Mason brings up an important point: it is not sufficient to react, in an ad hoc manner, to the learner's attempts. It is important to pitch the response at the appropriate level, so that the learner retains agency in the process of doing mathematics.

The level of the response can be seen to belong to two possible categories: scaffolding for meaning or for rigour. Scaffolding for meaning focuses the response on the contribution of notional knowledge missing to the process of resolution. Scaffolding for rigour constitutes a response that addresses the direction of thought of the student in a way that does not close off avenues by providing missing steps, but rather opens opportunities for critical reflection.

In a context of activities in which the development of critico-creative mathematical thinking is emphasised, therefore, the application of scaffolding changes. In this situation, scaffolding only makes sense if it is directed at the development of rigour in the mathematical thinking, in which case agency, on the part of the peripheral learner is preserved. Indeed, as the focus of the scaffolding is on rigour, the student retains agency on the development of meaning, and the scaffold is not specific to the meaning of the activity. The scaffold, determined by the socially derived norms is given as a tool for the negotiation of meaning, and, if applied consistently, contributes to the development of fuller participation on the part of the peripheral participant.

In the wider context of learning, if one scaffold for meaning is removed when it is deemed superfluous, only to be replaced by another, the learner never experiences the working out of problems on his own, thereby acquiring a 'learned helplessness' (Mc

Leod, 1989) that can be detrimental to the development of learning as Bruner expressed it:

> Learning should not only take us somewhere; it should allow us later to go further more easily. (Bruner, 1960, p. 17)

The idea of a potentially detrimental effect to such a systematic use of scaffolding for meaning suggests that introducing learning experiences that break this pattern of dependence might have a beneficial impact.

The language used in the Social Constructivist descriptions retains the metaphor of learning as internalisation of an externally, though socially, established 'truth'. As Lave and Wenger (1991) describe:

> Conventional explanations view learning as a process by which a learner internalises knowledge, whether 'discovered', 'transmitted' from others, or 'experienced in interaction' with others. […] It establishes a sharp dichotomy between inside and outside … (p. 47)

In this respect, Social Constructivism preserves the distinction between the learner and the outside, knowing, 'expert' community, whose roles include the establishment, preservation and control of the meaning of mathematical notions, and of truth-claims about them, and into which the learner is inducted through a process referred to as learning. Indeed, these meanings and truth-claims serve as the building blocks of the scaffolds. Furthermore, the scaffold and its originator serve to mediate between the learner and the object of learning, resulting in an application of the Vygotskian theory of mediated activity (Kozulin, 1998, p. 62), and thereby perpetuating the pattern.

In the theory of Situated Cognition, every member of the community of practice has a voice in the negotiation of meaning, though not an equal one, and the social nature of mathematical truth is not external to the learner. Instead, s/he is an integral part of the community and therefore, even if the teacher enacts an ad hoc intervention aimed at scaffolding, the learner, as agent, has the power to critique the intervention for its applicability to their own context at that time. If this response is not only allowed but fostered, the guidance is benevolent in that it preserves the agency of the learner. The issue, then, becomes one of pitching the intervention at the right level so as to preserve this context, as illustrated by Burton's (2004) "dialogue involve[ing] talking with and about, not being talked 'at', talking to learn through the negotiation of meaning, not accepting the meaning of others". I discuss this aim in more detail in the next chapter, on the teaching approach.

A model of the application of the two perspectives described here, of teacher as mediator versus teacher as guide, emerge from the differentiation I develop, juxtaposing mathematical enquiry (ME) as practised by research mathematicians and as a teaching approach, with mathematical problem solving in the classroom (MPSC), as it is currently described in the education literature and practised in the classroom. As Burton (2004) argues in her description of mathematical enquiry as practised by full participants:

> The world of mathematical [enquiry], […] is very different from that to be found in the mathematical literature where the results and the techniques are reified and the participatory practices of the mathematicians, as recorded here, are expunged. If, however, we think of these mathematicians as learners, their enquiry processes provide an excellent model to use for less sophisticated learners. Viewed as an educational process, then, and when based on participation as, in this case, a collaborative practice, Etienne Wenger pointed out that such processes:
>
>> are effective in fostering learning not just because they are better pedagogical ideas, but more fundamentally because they are 'epistemologically correct', so to speak. There is a match between knowing and learning, between the nature of competence and the process by which it is acquired, shared, and extended. (1998: 101/102) (Burton, 2004, p. 133)

In effect, for learning to be 'epistemologically correct', it needs to take place in a social context where the negotiation of meaning and of truth-claims is practised by the learners as well as the 'experts'. Brown (1997) expresses this as:

> a shift in emphasis from the learner focussing on mathematics as an externally created body of knowledge to be learned, to this learner engaging in mathematical activity taking place over time. Such a shift locates the learner within any account of learning that he offers, thus softening any notion of a human subject confronting an independent object. (p. 49)

The criteria for a social context conducive to this shift are the goal of the present discussion. In order to better define the differentiation that allows these criteria to emerge, I begin with a discussion of epistemological views in mathematics. Later, I use the results of this discussion in the characterisation of the two practices.

Epistemological Discussion of Mathematics

According to a definition put forth by the Mathematics Sections of the Association of Teachers in Colleges and Departments of Education,

> A one-sentence description of mathematics is that it is the study of relationships. […] Traditionally mathematics includes the topics of number and the relational aspects of space (as distinct from the aesthetic aspects); it is also fairly well agreed that the classification and ordering of any material are activities of a mathematical nature. Further, we may examine relations such as that between a statement and its negation (1967, p. 8).

From the epistemological point of view, this can be interpreted very generally as a view that mathematical knowledge is about mathematical objects and of relationships between them. In addition, these relationships can themselves be regarded as objects of higher-level relationships, as in the example of algebra, which studies and manipulates relationships between variables, or mathematical logic, which uses relational statements as objects. At any given moment in a mathematical situation, therefore, mathematical statements are treated either as statements of relationship, or the object of relationships with one, several or a whole class of other mathematical objects. For this reason, and to lighten the flow of the text, I use the term (mathematical) notions to include both (mathematical) objects and (mathematical) relationships, unless the distinction is significant. For example, $1+1=2$ can be seen as a statement about the relationship between the numbers 1 and 2, or as an object in a statement relating it to, for example, $2-1=1$. Depending on this focus, the statement is used to define '2', for example, or to define the operation of addition.

A Historical Perspective on Epistemology

Epistemologically, this does not establish whether this 'mathematical reality' is 'real' in itself, in the way that the ground under our feet is real, that is, whether it exists independently of our awareness of it, or whether it is merely real in the minds of the humans who think of it, i.e. it is simply a model or a set of models that is used to study perceptible reality. Throughout the history of the development of mathematics, views on this topic have changed. In the case of the earlier example, $1+1=2$, for example, though the statement might appear empirically self-evident, more recent views of the nature of the discipline have demanded a more rigorous, rational justification.

The beginnings of mathematics are traditionally traced back to the accounting activities of scribes. In such activities, empirical evidence that mathematical statements, as typified by the example, are correct would have been sufficient. Mathematics, then, was a way to describe invariants in perceived reality. By the time of classic Greek thought, however, the sophistication of mathematical findings had evolved to a point where this was no longer deemed to be the case. As Davis and Hersh (1981, p. 147) said, the "first proof in the history of mathematics is said to have been given by Thales of Miletus (600 B.C.)". They explain further that "the genius of the act was to understand that a proof is possible and necessary". This last statement is significant in that it expresses the manifestation of the cultural shift from a mainly empirical science to one where there is

a need to justify the mathematical validity of a statement, beyond direct observation, through rigorous (deductive) reasoning.

According to the Platonist philosophy, mathematics is largely an empirical science, the results of which stem from observation of a mathematical reality that "exists independently of human beings. It is 'out there somewhere'" (Davis & Hersh, 1981, p. 68-69). The difference with previous, completely empirical philosophies is that the observable reality on which mathematical findings are based is a concrete representation of an idealised world that contains objects which are

> not physical or material. They exist outside the space and time of physical existence. They are immutable—they are not created and they will not change or disappear. (Davis & Hersh, p. 69)

Though Plato's mathematical objects are idealised, they behave like the objects which we can interact with in such a way as to become observable 'by proxy'. In this manner, a connection can be made between purely empirical, direct observation, and rational reasoning about 'generalised', ideal objects. Mathematics, in short, describes "relations that do not change between objects that do not change. […] It reflects the reality of the Forms." (Restivo, 1993, p. 6)

A rationalist view of mathematics, in contrast, emphasises the deductive reasoning component of mathematical thought, requiring formal proof. This is well exemplified by Descartes' method (1637), in which he declared that:

> if we accepted none as true that was not so in fact, and kept to the right order in deducing one from the other, there was nothing so remote that it could not be reached, nothing so hidden that it could not be discovered.

He sought to minimise the influence of observation, or empiricism, while expanding the dominion of reasoning in scientific (and mathematical) thought. He was not, however, introducing any new components to mathematical method, but merely requiring a further shift of balance between the two existing, observation and reasoning. Mathematics was still described as explaining the patterns manifest in perceived reality, by using both direct observation and deductive reasoning, only reasoning was given preference over 'mere' observation, though these two means of constructing reality were still seen as inseparable aspects of a single discipline, which had its source of data in an 'objective' reality.

Problems, however were to come as a result of this emphasis on minimised observation and maximised reasoning, in the form of "the appearance on the mathematical scene a century and a half ago of non-Euclidean geometries" (Davis & Hersh, 1981), among other unexpected findings. In this particular case, the problem arose as a result of the efforts of mathematicians to formalise their discipline, particularly geometry, based on the contents of Euclid's Elements, which had long been considered to be the standard for mathematical descriptions of reality.

The basic structure of the Elements is key to this collapse. In them, the geometric properties of space are presented as derived from a minimal set of statements, called axioms or postulates, which are to be accepted without derivation, as the basis of all further geometrical reasoning. In the original text, these number five, and are demonstrated to be sufficient to derive the propositions that follow, that is, to describe geometric reality.

The crisis began when these five axioms were examined more closely, in order to determine whether they were all needed to achieve this aim. Unfortunately, it was found that by using only the first four, different geometries could indeed be derived that would be equally internally consistent, and, moreover, that at least one of these also fit our observations of reality, without requiring the fifth axiom (Kline, 1980, pp. 81-88). As Fang (1970) put it:

> The discovery of the independence of the parallel axiom revealed, once and for all, the folly in the extensive reliance on spatial intuition. When the visual dust settled, as it were, the Euclidian geometry turned out to be a parabolic geometry in the company of elliptic, hyperbolic and spheric geometries. The myth of the absolute truth of mathematics was gone for the time being, if not for ever […]. (p. 80)

Another development in mathematics added energy to the debate. In the nineteenth century, "the development of analysis […] overtook geometrical intuition, as in the discovery of space-filling curves and nowhere-differential curves" (Davis & Hersh, 1981, p. 330-31). This development negated the universality of geometric intuition as a foundation for mathematics, further reducing the importance of empirical intuition in mathematical discovery.

This turn of events is unprecedented in that mathematics had up to that point been built up from what were thought to be unassailable foundations. It had been thought to be limited in its evolution only by human imagination and effort. From this point on, however, the very foundations of mathematical knowledge had to be re-examined, and

mathematical 'truths' that had long been accepted as unimpeachable were now placed back under an (albeit different) microscope. This opened the way for foundationalist studies in mathematics, whose aim it was to establish a new set of unassailable foundations on which a new mathematical edifice could be (re-)built.

The first attempts to correct this problem involved the reduction of all mathematics to statement of logic or set theory (which were then considered to be effectively equivalent, Davis & Hersh, 1981, p. 331). Logicists such as Russell and Whitehead proceeded to develop a rigid logical system that would form the 'rules' according to which mathematical 'truths' could be derived from each other. In their view, mathematical statements became the objects of the argument, rather than the arguments themselves, and the focus of research became rigorous deduction for its own sake. In other words, the rationalist half of the classical mathematics dyad was to become the whole of mathematics, and statements derived from observations, such as the idea of the stability of the results of arithmetic operations, became mere objects of the logical argument. In this system, the epistemological connection of mathematics to reality fell away, and the discipline became an attempt to construct a complicated, yet indubitable tautology. To make this system account for all the mathematics then accepted, the concept of infinity in all its different forms had to be accounted for, including not only the idea of an infinitely large number, but also of the infinitely many numbers between 0 and 1, and the idea of an infinite set. Unfortunately, the logicists programme too contained the seed of its own failure as a foundation for mathematics:

> By the time [it] had been patched up to exclude the paradoxes, it was a complicated structure which one could hardly identify with logic in the philosophical sense of "the rules for correct reasoning". (Davis & Hersh, 1981, p. 332)

In contrast, Intuitionists preached a return to a more empirical source. According to Brouwer, "the natural numbers are given to us by a fundamental intuition, which is the starting point for all mathematics" (Davis & Hersh, 1981, pp. 333-34). Anything else in mathematics can be developed by constructing it from these 'raw materials'. His followers further contended that, unlike the Platonists, they did "not attribute an existence independent of our thoughts, i.e., a transcendental existence, to [even] the integers or to any other mathematical objects" (Heyting, 1964, p. 53), thereby further acknowledging the empirical nature of their view. Unfortunately for its supporters, this theory did not account for a large part of what was already accepted as 'true', rejected much of it and was therefore never widely accepted.

Hilbert, the figurehead of the Formalists, responded to the issues surrounding Brouwer's solution by introducing a programme that attempted to reclaim all of classical mathematics by proving, at least, its internal consistency. He designed this programme carefully, using three main components. Firstly, he borrowed the Logicists' formal system of logical rules; secondly, he developed a "theory of the combinatorial properties of this formal language, regarded as a finite set of symbols subject to permutations and rearrangements as provided by the rules of inference" and thirdly, he used this programme to prove "by purely finite arguments that a contradiction […] cannot be derived within this system" (Davis & Hersh, 1981, p. 335-36). The main difference between the Formalist programme and the Logicist's was therefore that Hilbert rejected the concept of an infinite set, thus avoiding some of Russell and Whitehead's failures. Hilbert, however, as Russell and Whitehead before him, had detached his mathematical system from observable reality and turned it into "a meaningless game" (Ibid.).

To summarise, the philosophical problem began with the attempt to shift the balance, within the discipline, between empirical and rational activities in the pursuit of mathematical results. The push for more rationalism, in turn, uncovered inconsistencies in what had until then been accepted as mathematically true. In response, Brouwer's Intuitionists prescribed a 'return to nature' that would swing the pendulum back towards a more empiricist outlook, which in turn rejected many of the results already accepted. In contrast, Russell and Whitehead's Logicists and Hilbert's Formalists attempted to restructure the rational side of the discipline, succeeding only in uncovering further irregularities. All this activity was not, however, in vain. As Kline (1980) explains:

> There is no question that the axiomatic movement of the late 19th century was helpful in shoring up the foundations of mathematics, even though it did not prove to be the last word in settling foundational problems. (p. 284)

Despite this, the field remained open for a methodology that would prove generally reliable. It is within this atmosphere of uncertainty that fallibilism emerged as a possible framework. This framework is particularly apt as it capitalises on the very uncertainty that had made earlier ones fail.

In Lakatos' account, this framework is represented by a process that develops in three stages, beginning with the informal mathematical finding of a pattern (largely empirical), through a formalised proof that incorporates definition-building as well as

the framing of the domain of the result (largely rational), to the development of a programme of research that refines the area. The process of this Lakatosian cycle is described as follows, beginning with what he called:

> […] the stage of 'naive trial and error'. […] At some stage the naive conjecture is subjected to a sophisticated attempted refutation; analysis and synthesis starts: this is the second stage of discovery which [he] called 'proof-procedure'. This proof-procedure generates first the brand-new proof-generated theorem and then a rich research programme. The naive conjecture disappears, the proof-generated theorems become ever more complex and the centre of the stage is occupied by the newly invented lemmas, first as hidden (enthymemes), and later as increasingly well articulated auxiliary assumptions. It is these hidden lemmas which, finally, become the hard core of the programme. (Lakatos, 1978, p. 96)

The final stage, according to Lakatos, consists of the development of a research programme based on the activities of the second stage, that is, of a systematic verification of the findings. The interesting aspect of Lakatos' framework is that it integrates the opposition between empiricism and rationalism in a way consistent with the historical development described thus far: Empirical evidence is collected and interpreted first, and the rational, deductive justification is constructed over time, involving activities of concept formation and definition building as well as theorem proving. In effect, both thought currents are incorporated into a framework that presents a macroscopic view of validation of mathematical statements, in the sense that these are not so much considered 'true until proven false' (through counter-examples), but more that this is how they are constructed at all, during the second of his three stages, by the very defining of the concepts and their domain of validity, and through cycles of refutation and re-definition, which he called the 'method of lemma-incorporation' or the 'method of proof and refutations'.

The process as a whole combines two modes of activity, which Lakatos referred to as 'informal' and 'formal' mathematics. The first he likens to Popperian science, in that:

> it grows by a process of successive criticism and refinement of theories and the advancement of new and competing theories *(not* by the deductive pattern of formalized mathematics). (Davis & Hersh, 1981, p. 349)

Burn refers to this distinction as the 'divergence between formal mathematical structures and their genesis' (2002, p. 21), and cites Freudenthal: "No mathematical idea has ever been published in the way it was discovered" (Ibid).

As it has in the epistemological and historical development of mathematics, Lakatos holds that the 'formal' component of mathematical development occurs later, in the

'proof-procedure' stage (1978, p. 96). In separating the two stages in time, Lakatos implies that they are not so much mutually exclusive, but rather, that they work together towards a common end. A mathematical finding is pushed through the stages and progressively refined. Lakatos therefore incorporates both empiricist and rationalist views into a theory according to which the process of mathematical discovery traces a convergent asymptotic trajectory:

> We start with a naive conjecture and we have to invent the lemmas, and even perhaps the conceptual framework in which the lemmas can be framed. Moreover we find that in a heuristically fruitful analysis most of the hidden lemmas will be found on examination to be false, and even known to be false at the time of their conception. […] In my conception the problem is not to prove a proposition from lemmas or axioms but to discover a particularly severe, imaginative 'test-thought experiment' which creates the tools for a 'proof-thought experiment', which, however, instead of proving the conjecture improves it. The synthesis is an 'improof', not a 'proof', and may serve as a launching pad for a research programme. (Lakatos, 1978, p. 96)

Once the second stage begins to wind up, a systematic research programme can be developed, that will tie loose ends and examine cases that had been excluded in order to give a more complete picture of the domain. Findings at this stage can also form the beginning of a new Lakatosian cycle, in a new direction.

In descriptions of such a fallibilist view of mathematics, an important metaphor seems to emerge, of research as convergence towards 'truth' as likened to a collective 'discourse', where the various voices contribute to a body of work that is refined through the development and incorporation of exceptions and counter-examples, formalisations and definitions. According to this perspective, mathematical knowledge can be defined as 'socially justified belief' (Davis, 1993, p. 188), which, therefore, is not only constantly revised and refined through the work of a community of practice which establishes, collectively, what it accepts as true; it lives within this community, and is made visible through its activities, including publications, conferences, and applications.

An Integrated View of Epistemology in Mathematics

Based on the above historical perspective, and considering that the development of distinct components of mathematics, or what I call notions, occurred under the aegis, as it were, of divergent epistemological perspectives, I propose that, rather than a single overall epistemology of mathematics, a pluralistic epistemological perspective involving a categorisation of individual notions could prove more pragmatic.

In the first case, which I call conventional, the notions in question are not the result of empirical observation or logical derivations from more basic or fundamental notions. They have been chosen by the experts or imposed by simple enculturation (Pimm, 1995) as convenient for the task, are accepted socially, remain unquestioned, and are treated as monolithic. An example of this in the school context is the adoption of the standard number notation, which integrates place value: It makes column addition and long division possible, but is not the only way to represent numbers. Hewitt (1999) referred to these as 'arbitrary' and explained that "all students will need to be informed of the arbitrary" (p. 4), that is, the only way for an individual to know these notions is to be told by another, presumably fuller, member of the community of practice within which s/he is being inducted by this teaching. Cockcroft (1982) integrated this first category into the components of school mathematics as follows:

- **Facts** are items of information which are essentially unconnected or arbitrary. They include notional conventions–for example that 34 means three tens plus four and not four tens plus three–conversion factors such as that '2.54 centimeters equal 1 inch' and the names allotted to particular concepts, for example trigonometric ratios. The so-called 'number facts', for example 4+6=10, do not fit into this category since they are not unconnected or arbitrary but follow logically from an understanding of the number system.
- **Skills** include not only the use of the number facts and the standard computational procedures of arithmetic and algebra, but also of any well established procedures which it is possible to carry out by the use of a routine. (p. 71)

In the first of these descriptions, a point is made to distinguish 'facts' from other mathematical notions which "follow logically from an understanding of the number system". 'Skills', on the other hand, are described as "well established procedures" and involving "the use of a routine". Though they imply a somewhat dynamic quality in that they generate a transformation, essentially they are also treated as static as well as monolithic in that no part can be changed without the whole notion being put into question: they can metaphorically be thought of as a machine whose mechanism can remain unknown, but which we are trained to use.

In the second case, observational notions are the result of basic empirical observation, without explanatory content. An example of this category is the stability of the results of arithmetic operations; in lower elementary grades, the use of a variety of counters helps develop in the learner the idea that however these are arranged, their total number is stable. The epistemological source of these notions lies in personal sensory experience with a phenomenon. Together with the conventional notions, they are the basic building

blocks that are reasoned upon in the application of the remaining categories, and can therefore be grouped together as the conventional/observational category.

In contrast, applicational notions are the product or application of some form of mathematical reasoning upon the previous two categories, and can therefore be explained and traced back to this reasoning. An example of this category is illustrated in the necessity of a common denominator in fraction addition, which can be derived, through reasoning, from the meaning of the fraction notation. They are the result of a combination of mathematics and its structures with conventional and observational notions (in this case the notation for fractions, and its corresponding meaning). Cockcroft's (1982) corresponding categories in school mathematics are described as follows:

- **Conceptual Structures** are richly inter-connected bodies of knowledge, including the routines required for the exercise of skills. […] They underpin the performance of skills and their presence is shown by the ability to remedy a memory failure or to adapt a procedure to a new situation. (p. 71)
- **General Strategies** are procedures which guide the choice of which skill to use or what knowledge to draw upon at each stage in the course of solving a problem or carrying out an investigation.

In these descriptions, in contrast to the previous, the language used implies a more flexible knowing; notions of this class can be transformed to adapt to a broader range of situations, or re-created to remedy a memory lapse. In addition, applicational notions can be seen as the first step in the organising of conventional/observational notions. This class mirrors the previous one in that there is a static category, Conceptual Structures, parallel to the previous 'Facts' and a dynamic category, General Strategies, parallel to the previous 'Skills'. In addition, both problem solving and investigations (which I discuss in a later section) are cited as the context for the use of 'General Strategies', suggesting that knowing them leads to better performance in these activities.

There is a fourth class which is necessary to a complete picture of the discipline of mathematics: the notions that make possible the reasoning, which itself produces the applicational notions and promotes their adaptability. Adding this extra element completes the picture both with regards to mathematics, and to the practice of mathematical enquiry. It also connects back to the highest of the three levels of engagement I described earlier: Meaningful critico-creative engagement requires the use of this class of mathematical notions, which corresponds to the basic mathematical structures that determine the properties of the applicational class, and the rules of

engagement in the process of development of mathematics, theorisation[3], in which these notions are used to construct models of the underlying mathematical structures. In a sense, these notions allow the knower to work in the reverse direction, theorising instead of applying, and I therefore refer to this class as theorisational notions. An example of such a notion is the awareness that it makes no sense to consider the parity[4] of a real, non-whole number. In Cockcroft's (1982) categorisation, they can, at best, be found in:

- **Appreciation** involves awareness of the nature of mathematics and attitudes towards it.

The language of this last category, manifested by the use of words such as 'awareness', implies a knowing which is more intuitive and can therefore not be as easily reified (Wenger, 1998), or indeed communicated explicitly in the discourses of textbooks, classroom instruction, journals, etc.: The awareness of the nature of mathematics, if it is 'true', encompasses a recognition of the underlying structure of mathematics, the theorisational notions, which helps the knower build more mathematical knowledge. In this respect, it corresponds to the knowing which is necessary to the application of rigour in mathematical reasoning, and is therefore essential to a true participation in the negotiation of meaning which I described as an essential component of 'epistemologically correct' learning (Wenger, 1998).

The four categories of mathematical notions are summarised in Table 1, below:

Class	Conventional/ Observational notions	Applicational notions	Theorisational notions
Perceived as	unquestioned 'fact' that needs to be provided by a fuller participant or are directly observable by the individual	result of logical derivations	underlying structure of mathematics, rules of logical derivations
Examples	symbolic notations, including place value, fractions, variables in algebra, etc. procedures such as long division and column addition	why the procedures in the previous category work, or why $(x-y)(x+y)=x^2-y^2$	problem posing, classification into cases, abstraction, generalisation, conjecture building, proofs (Polya, 1957, Lakatos, 1976)

Table 1: Four levels of mathematical notions, grouped into three classes

Each successive epistemological level corresponds to a deeper, more fundamental stratum of mathematics. At the shallowest level, both conventional and observational

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[3] Theorisation is not to be confused with formalisation, which involves rigorous, even ritualized socially accepted norms of expression, including the use of standardized symbolic notations.

[4] whether a whole number is odd or even

notions are simply fixed monolithic elements, which can be interpreted as absolute and determined by an external, god-like authority, without loss of understanding. They are not applications of reasoning by the participant her/himself, but external to her/him; they are observed but can remain unquestioned. At the next level, applicational notions are more reason-bound, operate at a deeper level and serve as organisational elements for the conventional and observational notions. The theorisation on which they are based, however, is unquestioned. Finally, the theorisational notions form the matrix from which applicational notions emerge. In addition, they are surpassing conventional and observational notions and in fact are the element that allows their critical examination.

As Bishop explains it, each of these levels needs to be acknowledged:

> Educating people mathematically consists of much more than just teaching them some mathematics. […] It requires a fundamental awareness of the values which underlie mathematics […] It is not enough merely to teach them mathematics, we need also to educate them about mathematics, to educate them through mathematics, and to educate them with mathematics. (Bishop, 1988, p. 3)

The practical consequences of considering these distinctions are connected to the perception of the nature of mathematics as a discipline. Indeed, different participants in the community of practice can perceive a specific mathematical notion as conventional/observational, applicational, or theorisational, depending on their own experience with it, and consequently, their perception of mathematics as a whole will be constructed from the same. Mathematics research is often concerned with a deeper understanding of specific mathematical notions, transferring them from one class to another. For example, the 'discovery' of non-Euclidean geometry emerged from a questioning of what had hitherto remained unquestioned (largely conventional): the fundamental nature of the parallel postulate in Euclid's geometry (Davis & Hersh, 1981). In the following section, I discuss the idea that many mathematical notions can be viewed as conventional/observational, applicational or theorisational, *depending on the situation.*

Practical Discussion of the Epistemology of Mathematics

In parallel with the tripartite classification of mathematical notions discussed above, three levels of knowing appear to be applied in practice. The reason this differentiation is replicated is that a participant's 'knowing' of a mathematical notion can be interpreted as conventional/observational, applicational or theorisational somewhat

independently of the notion's epistemological nature, and indeed differently according to the situation.

At the lowest level, as discussed earlier, mathematical notions can be perceived to be merely conventional/observational. This knowing corresponds to what Skemp reported Mellin-Olsen as calling instrumental understanding:

> It is what I have in the past described as 'rules without reasons', without realising that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by 'understanding'. (Skemp, 1976).

These mathematical 'rules' are assumed to have been determined by an external, trustworthy authority, and the participant feels no need to understand the reason behind them. Some mathematical notions really are conventions, although there are often reasons why specific conventions are adopted, and that can make them easier to understand and remember but does not make them inevitable. In practice, the conventional/observational knowing, which I call knowing-that/how, is an unquestioning knowing rooted in the reliance on the word of an external figure of authority whom one implicitly trusts. The participant in the community of practice repeats what s/he has been told verbatim, and knows how to apply it to specific, obviously relevant cases. Even if *s/he* has derived it her/himself in the past, s/he treats the notion in the same way, as a monolithic object. In terms of communities of practice, the knowing is strictly replicative, where the peripheral participant emulates the behaviour of full(er) participant without questioning.

At the middle level, a mathematical notion is known because its mathematical derivation is known. This level, which I call knowing-why corresponds to Skemp's relational understanding: "knowing both what to do and why" (1976). An example of this at secondary school could be the factorisation of $x^2 + 2xy + y^2$ as $(x+y)^2$. A participant could know why because s/he can re-derive the former from the latter by simply multiplying it out, then collecting like terms. These kinds of statements of mathematical relationship can be derived logically, but are soon treated as conventional/observational after continued use in examples and exercises. Indeed, proponents of drill exercises favour this outlook. Such a level of knowing often remains implicit, which makes it less likely to be developed by more peripheral participants. When achieved, however, it allows the knowing participant to adapt the element, perhaps to situations where one or both of the terms are different, as in $4a^2 + 12a + 9$

being equal to $(2a+3)^2$. In terms of communities of practice, this type of knowing cannot be reified as it is inherently a practice, that of adapting knowing-that/how to wider domains. It can be developed if the community endorses the autonomous construction, by the peripheral participant, of an understanding of the notion. To be effective, this endorsement takes the form not only of an allowance for the development of understanding, but also of a valuing of the resultant understanding.

At the higher level, the participant's knowing allows her/him to recognize, by analogy, generalization or specialization, the similitude of structure between applicational notions, thereby providing the powerful tools of rigour for the expansion of their applicability, and by extension, for the negotiation of their meaning. I call this level knowing-when, because it allows the knowing participant to frame and re-frame the context of relevance (when) of an applicational notion. For example, in Knoll (2000), several of the developments of ideas were due to the awareness of an analogy of structure between branches of mathematics such as group theory as applied to polyhedral geometry, vector calculus, and number theory, allowing the transfer from one representation of this structure to another and back again (see Appendix 2). In terms of communities of practice, again, this type of knowing is part of genuine full participation, in the form of its practice, and cannot therefore be reified, though it needs to be endorsed, if it is to be engaged with.

To summarise, the three levels of mathematical knowing are described in Table 2:

Level	Knowing-what/how	Knowing-why	Knowing-when
Register	Low level	Mid-level	High level
Manifestation	Recalling a fact or performing a process	Monitoring a process	Abstracting from a process
Engagement	Passive	Active	Critico-Creative
Properties	Replicative	Transferable	Constructive
Perception	Determined by 'external authority' To be memorised	Instance of reasoning behind a notion	Rigour in the structure of mathematics. Can be used to (re-)construct notions

Table 2: Three levels of mathematical knowing and their properties

In the table, the lowest level of mathematical knowing, knowing-what/how is described as being manifested through 'recalling a fact or performing a process'. This is reminiscent of what is often expected in traditional testing, where the participant is asked to perform a replication of a process or recall verbatim a fact that he has been told. In such situations, the participant need not have much depth of insight into the reasoning behind the performance. Her/his engagement is passive in nature, in that s/he

simply replicates what s/he has been shown, on command, and her/his knowing is replicative. The perception is that the notions have been determined by some nebulous external entity, nature, God, or simply 'the experts', or indeed remains unaddressed.

In contrast, in the problem-solving literature, the monitoring of processes is often required: choices have to be made and some form of understanding is necessary. The participant is asked not only to perform previously seen tasks, but also to make decisions about which to use, thereby demonstrating deeper understanding and a more sophisticated knowing. S/he therefore needs to be able to monitor her/his activities by comparing them to an intended result, and indeed to transfer her/his understanding from similar situations. This kind of knowing is established as being developed through experience in problem solving (Polya, 1945, p. 130), and belongs to the mid-level register: knowing-why. It is transferable in that the applicability of the knowledge is wider than for the previous level, and the knowing participant has sufficient understanding of the notion that s/he can begin to adapt it to more diverse situations. It is also often connected to metacognition (Carr & Biddlecomb, 1998; Garofalo & Lester, 1985; Schoenfeld, 1987, 1992).

The third category of knowing, as the table indicates, involves a monitoring and a critical awareness of the monitoring activities inherent in problem solving, what Cockcroft referred to as mathematical 'appreciation' (1982): an added intuitive understanding of the mathematical structure of which the concepts applied and the choices made are a manifestation. This involves a higher level of abstraction than the other two registers. It consists of the rigour that mathematicians use when they are being critico-creative and involves creative manipulation and critical examination of abstract mathematical concepts. It entails an awareness of similarities and differences which in turn is conducive to abstraction and therefore to problem posing, generalisation, classification into cases, expansion and contraction of validity, etc.: knowing-when a mathematical structure does or does not apply.

Schmalz (1988) delineates an analogous tri-partite categorisation when she speaks of the goals of teaching mathematics. First, she describes the aim of teaching knowing-that/how:

> At the lowest level the goal is to impart some applicable facts and to have these facts applied in simple situations... (p. 42, as cited in Burton, 2004, p. 190)

Then, she describes the aim of knowing-why:

> On a higher level, it is to teach deductive reasoning and proof construction. (ibid)

And finally, of knowing-when:

> At an even higher level, it is to discover connections between facts that are later verified in an accurate proof, or to solve a problem by creative use of some applicable facts. Thus, the goal is not simply to pass on some tools useful in applications; it is not simply to pass on a set of useful problem-solving techniques. It is also to create situations where students will discover the power of their intuition. (ibid, my italics)

As I suggest in the beginning of this section, a mathematical notion can be known on at least these three levels, and mathematicians often operate at all three in concert, jumping back and forth and indeed, re-positioning each notion in the category most useful at that moment. When doing research in abstract analysis, for example, a researcher can implement certain often-used algorithms without trying to re-derive them each time, thereby treating them as s/he would a conventional/observational notion. S/he does, however, possess an awareness, however tacit, of the underlying theorisation that produced it, which in turn gives her/him not only the power to use it at will but to modify and adapt it, thereby treating it as an applicational notion. S/he can even re-construct the concepts underlying it in order to apply them elsewhere, thereby treating it as a theorisational notion. Mason (1992) expresses this idea of the changeability of the epistemological level of a mathematical notion when he discusses the integration of new findings into practice:

> When research findings are translated into practice, they turn from observation into rules, from heuristics into content. Attempts to pass on insights become attempts to teach patterns of thought. Once the "patterns of thought", the heuristics, become content to be learned, instruction in problem-solving takes over and thinking tends to come to a halt. (p. 18)

In other words, as Mason's 'heuristics' (theorisational notions) become familiar through experience or formalisation, they can become the object of simply knowing-why, or even knowing-that/how, thereby being treated as a form of applicational, or conventional/observational knowledge, respectively. As he says, (higher level) thinking then tends to come to a halt. The ability to shift a notion between levels of knowing is certainly a skill in itself, without which the participant cannot use the notion to its highest potential.

If the knowing is developed at a more shallow level of awareness, however, its flexibility and adaptability are lost, and the notion is calcified into a monolithic,

conventional/observational element which then cannot effectively be used in critico-creative enquiry. This phenomenon is particularly likely to occur in situations where the use of scaffolding, as presented in the earlier discussion on alternate educational perspectives, is aimed at the development of a shallower level of knowing such as knowing-that/how or -why at the expense, or disregard, of knowing-when. In this instance, knowing-when is devalued to the degree that the learner can remain unaware of this component of mathematical knowledge, operating only at the shallower levels of knowing-that/how and -why.

In the case of an authentic form of mathematical enquiry, knowing-when becomes an essential component of the practice and therefore of the experience, since it provides the participant with the tools of rigour needed to sustain productive critico-creative agency. The epistemological framework developed here can therefore serve as the foundation of the coming discussion of the nature of mathematical enquiry and allows distinctions to be made between it and the field of mathematical problem solving in the classroom.

Mathematical Enquiry (ME) as Distinguished from Mathematical Problem Solving in the Classroom (MPSC)

To develop a framework which distinguishes mathematical enquiry (ME) as practised by research mathematicians from mathematical problem solving in the classroom (MPSC), I examine definitions and perspectives from the literature which characterise the latter. In each case, I then position the former with respect to the specific perspective, in order to justify a design criterion for the teaching approach. In addition, I examine the literature on 'investigational work' (Cockcroft, 1982; Mason, 1978; Wells, 1987; Ernest, 1991; Driver, 1988), which presents a teaching approach that, the authors claim, emulates more closely that of ME. The theoretical perspective on epistemology which I developed in the previous section serves as the basis of a refutation of this argument.

The definitions and perspectives on which this positioning is based can be described through a characterisation of mathematical tasks and the impact of the characteristics of the preferred tasks on the participants' experience of them. This characterisation takes two forms: First I discuss what makes a mathematical question a candidate for 'problem solving', and second, I discuss what makes a problem genuinely mathematical in nature. The overall idea is that for a mathematical situation to lead to genuine mathematical

enquiry, it needs to be both critico-creative and genuinely mathematical in nature, as opposed to simply being an application of mathematics.

A Classification of Mathematical Tasks

In general terms, a problem, according to Passmore (1967) presents a "situation where the [participant] cannot at once decide what rule to apply or how it applies" (and therefore requires knowing-why), in contrast with an exercise, which presents "a situation in which this is at once obvious" (p. 206), and which can therefore be solved strictly with knowing-that/how[5]. In the case of an exercise, the method required can be inherently obvious, or it can be made so by being specified in the formulation of the initial problem.

Zeitz, in parallel, indicates that "an exercise is a question that tests the [participant's] mastery of a narrowly focused technique, usually one that was recently 'covered'" (knowing-that/how), as opposed to a "question that cannot be answered immediately" (1990, p. ix). In this respect, both Passmore and Zeitz align themselves with Polya (1957) who distinguished routine from non-routine problems:

> In general, a problem is a "routine problem" if it can be solved either by substituting special data into a formerly solved general problem, or by following step by step, without any trace of originality, some well worn conspicuous example', and non-routine problems where these conditions are not present. (p. 171)

In Polya and many of his followers' language, what Passmore and Zeitz call an exercise is referred to as a problem, though it is often qualified as 'routine'. This semantic ambiguity reflects an issue which Polya had already mentioned and which Goldin (1982), in his analysis of obstacles to problem solving, brings to the fore: part of the distinction of a routine problem (i.e. an exercise), is determined by the participant's previous experience relative to the problem. He cites for this the following definitions taken from the psychology literature:

- A problem arises when a living creature has a goal but does not know how this goal can be reached (Duncker, 1945, p. 1)
- […] (A problem is) any situation in which the end result cannot be reached immediately (Radford and Burton, 1974, p. 39),
- […] A person is confronted with a problem when he wants something and does not know immediately what series of actions he can perform to get it (Newell and Simon, 1972, p. 72), etc. (cited by Golding, 1982, p. 87)

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[5] … keeping in mind that a specific notion can be either known-that/how or known-why.

In each of these definitions, there is an oblique mention of the relationship between the solver's 'existing state' and the requirements for the resolution of the problem. This aspect is significant, and indeed, it is partially addressed in Polya's (1957) distinction, above: he implies that in the case of routine problems the resolution is simply a matter of substituting variables and executing steps, which presumably the participant has available. This availability lies in the participant's knowing-that/how. This is the case even when the method of execution is specified, as success still depends on the participant's understanding of the specification.

In contrast, a non-routine problem requires the participant to select the appropriate response, based on rationales established by her/his understanding of the situation (knowing-why). A problem, in essence, is neither routine, nor non-routine per se. It is really the interaction between problem and learner that characterises a problem as routine in a particular context, depending on the knowledge available to guide the participant.

In the earlier epistemological discussion, I pointed out that a specific mathematical notion can be known in different ways. This accords itself with the framework I describe here in that a given problem might for one person be routine, requiring only knowing-that/how, and for another, non-routine, requiring knowing-why. In effect, the same mathematical notion may be familiar enough to one participant that s/he recognises it at once as the appropriate one for a given problem (knowing-that/how), and it might for another participant be more difficult to assess as appropriate and requiring knowing-why.

Though Polya's description presents a clear dichotomy, which is echoed by the epistemological distinctions I made earlier, the characterisation is better described as a continuum, depending on the relative familiarity of the problem to the participant. In Goldin's (1982) categorisation of problem solving situations, based on the solver's starting point and 'givens', he proposed the following possible categories:

1. The subject 'knows the answer' or is already at the goal when the task is posed. Operationally, the outcome measures […] do not detect any steps, processes, or significant time lag between the posing of the task and the correct response.
2. The subject does not 'know the answer,' but 'possesses a correct procedure' for arriving at the answer (operationally, arrives through correct processes at the correct answer or goal), and furthermore 'knows' (can correctly state) that he or she possesses the procedure, and furthermore is able to describe the procedure verbally before carrying it out. The procedure may be a standard algorithm taught

as part of the mathematics curriculum, or it may be a non-routine procedure which the subject possesses by virtue of prior learning or problem-solving experience.

3. Same as 2, but the subject is unable to describe the procedure in advance of carrying it out.
4. Same as 3, but the subject 'does not know for sure' (cannot state with certainty) that he or she possesses the procedure until after the problem has been attempted.
5. The subject does not possess a procedure for arriving at the answer (operationally, does not arrive through correct processes at the answer or goal until additional information or assistance is provided). (p. 95-96)

Goldin's five-fold categorisation differentiates more subtly than Polya's dichotomous framework because it separates the cases by the degree or 'magnitude' of the obstacles to the execution of the task. The interesting feature of Goldin's categorisation is that the obstacles are clearly in the nature of the solver, not in that of the problem. This is illustrated by the language used: the subject 'knows' or 'does not know' the answer, s/he 'possesses' the correct procedure, etc.

Other articles of theoretical perspective as well as research reports define problem solving within their framework, often through fairly succinct statements. Brown (1997), for example, discusses "students' ability to make use of a wide array of inductive and deductive skills as they operate on incomplete knowledge" (p. 36). When he specifies that the students operate on incomplete knowledge, the implication is that the problems are non-routine, according to Polya's definition. The issue with the use of 'problem solving' in learning contexts is then that the person setting the problem cannot always completely determine its positioning within the dimensions of Goldin's model, as s/he does not know each solver's individual experiential context. Consequences of this situation are that the level of difficulty and associated affective responses can vary greatly.

In 1992, Mason advanced a different definition:

> I take the word problem to refer to a person's state of being in question, and problem solving to refer to seeking to resolve or reformulate unstructured questions for which no specific technique comes readily to mind. (p. 17, footnote)

In his statement, Mason clearly shifts the focus onto the solver's state of mind, describing it as 'in question'. He further suggests a description of the actions involved when the participant engages her/himself in this state: s/he then 'resolves or reformulates […] questions'. There is an obstacle, which means the situation does not fall under Goldin's category 1, or Polya's routine problem, but the nature of the obstacle is not specified at all: the technique could 'not come to mind' for any number of

reasons, up to and including affective ones. Furthermore, the participant must, despite this obstacle, become sufficiently engaged to be in question.

By definition, then, according to Mason, an individual is solving a problem if s/he is engaged with it, as long as the solution has not yet been found and regardless of the reasons why this is the case. The key to Mason's statement is the student's engagement with the problem. This condition to problem solving is neither a feature of the problem, nor indeed of the solver, at the start of the solving process. Instead, it is the first 'act' required of the solver: s/he has to *become engaged* with the problem, at a level that s/he deems appropriate. Mason takes the situation back, therefore, to an initial condition which involves the nature of the participant's engagement. This characteristic relates to the level of knowing the participant requires and has available, along with her/his attitudes, views, etc. If the problem is routine, the participant can rely on knowing-that/how to execute it and never really becomes actively engaged, in order to do so. If the problem is non-routine, s/he needs to be actively engaged and apply knowing-why.

Because of the importance of Mason's 'engagement' component in the characterisation of a problem solving situation, and to pre-empt any ambiguity between exercise, routine problem, problem and non-routine problem in this discussion, I choose from here on to refer to the general case as a 'task', and to characterise it based on the level of engagement required. Correspondingly, a task that, for the specific individual, requires simple execution without obstacle, that is, where knowing-that/how is sufficient, shall be referred to as a routine task, where minimal engagement is sufficient. A task that presents an obstacle and therefore requires knowing-why shall be referred to as a non-routine task, requiring active engagement that involves conscious, though largely uncritical, decision-making'. The case of mathematical enquiry (ME), which has yet to be examined, corresponds to a critico-creative task, requiring knowing-when, and therefore critical and creative engagement.

In summary, therefore, a non-routine task can be described as a situation, in which a participant is actively engaged (Mason, 1992) and the resolution of which presents an obstacle (Goldin, 1982) of unspecified nature, though it is a property of the task in relation to the previous experience of the participant (Goldin, 1982, Passmore, 1967, Polya, 1957, Zeitz, 1990), that is, it is a function of how much it is 'new to them'. This distinction of a given task as 'new to' the participants is useful both for what constitutes a non-routine task, and as a description of a critico-creative task such as ME. In the

latter case, this is in fact a firm requirement of research: it must expand existing knowledge of the community of practice in mathematics as a whole, and cannot be routine for it or, therefore, for any and all of its members. In other words, the extent to which the task is new for the participants is a condition to its adequacy as a critico-creative task of mathematical enquiry.

In the case of classroom activities, however, an issue lies in the determination, for a specific participant or group of participants, as to how routine a given task is, since it depends on their previous experience as well as, according to Mason, on their level of engagement. This is the first factor that will contribute to the authenticity of a classroom experience relative to ME.

'Investigational Work' in Britain

Some theoretical frameworks, particularly in Britain, describe 'investigational work', alongside problem solving, as possible higher-end classroom activities. These descriptions are often vague and unhelpful for differentiating the relevant activities from problem solving. Driver (1988), for example, claims that "an investigation must be a step into the unknown" (p. 2), suggesting that the task should be 'new to' the student. Cockcroft (1982), in his characterisation of the approach, is less assertive. He states that:

> Even practice in routine skills can sometimes, with benefit, be carried out in investigational form; for example, 'make up three subtraction sums which have 473 as their answer'. The successful completion of a task of this kind may well assist understanding the fact that subtraction can be checked by means of addition. (§ 251)

In effect, the nature of the task itself as routine or non-routine seems not to be an indicator of whether it can be counted as 'investigational work'. Overall, Cockcroft's description is not very helpful, as it tries to cover too much ground. Writers have commented on this; notably, Wells (1987) presents a discussion of the distinction between problem-solving and investigational work which expresses this issue:

> So far, so confused […] The confusion of the Cockcroft Report suggests more than a lack of clarity on the part of its authors. It suggests confusion among educators in general. […] So why the confusion? […] It can be no surprise that other documents show similar confusion. (p. 2-3)

Wells interrogates other documents (ATM, 1984; DES, 1985), and finds them similarly confused[6]. The point which he claims does emerge from the texts he reviews is that investigational work is divergent, where problem-solving is convergent. He further cites that the two documents he reviewed are in agreement as to the following:

> Clear distinctions do not exist between problem solving and investigational work. Nevertheless, in broad terms it is useful to think of problem solving as being a convergent activity where pupils have to reach a solution to a defined problem, whereas investigative work should be seen as a more divergent activity. (ATM, 1984, as cited in Wells, 1987, p. 3)

The report goes on to explain that even in the process of solving a problem, a pupil may engage in investigational work, and conversely, etc. What does this divergent property, which seems to be the only one which can be held onto, provide in the analysis? A second look at Cockcroft's (1982) contribution, dictated perhaps by my intention in the current context, reveals a possible criterion for work to be investigational in nature: the requirement for agency on the part of the learners. Cockcroft (1982) expresses this as follows:

> At the most fundamental level, and perhaps most frequently they should start in response to *pupils' questions*,
> […] the teacher must be willing to pursue the matter *when a pupil asks* 'could we have done the same thing with three other numbers?' or 'what would happen if...?'
> […] sometimes it may be appropriate to suggest that the pupil or a group of pupils, or even the whole class should try to *find the answer for themselves*;
> […] find time on another occasion to *discuss the matter*.
> […] There should be willingness on the part of the teacher to *follow some false trails*
> […] It is necessary to realise that much of the value of an investigation can be lost unless the outcome of the investigation is *discussed*. (§ 250-52, my italics)

In the fragments cited above, emphasis can be placed on the agency of the learner in that activities are derived from questions which the pupil themselves posed, avenues of exploration they themselves presumably suggested (and which might lead nowhere), solutions they developed, or discussions in which they have a negotiating role. This emphasis aligns itself with the Situated Cognition perspective in that the members of a community of practice all have some degree of agency, and therefore their practice and by extension their experiences are comparable to full participants'. Furthermore, the conditions that are necessary (though not necessarily sufficient) for the learner to retain agency as described here include, still, that cited earlier: that the situation (problem, question, starting point…) be 'new to them'. If both the conditions of divergence and

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[6] 'lack of clarity' or some form of the word 'confusion' are used 8 times, and there are 19 question marks in a 2½-page typewritten text

'new to them' are present, the task potentially falls into the critico-creative category, requiring knowing-when and qualifying as mathematical enquiry. In the adverse case, if the situation is not new to them, then the meaning has already been negotiated, and the learner has no agency.

Grenier & Payan's Framework

Grenier and Payan (2003) present a framework for the development of situations which support mathematical enquiry activities in which the conditions of divergence and 'new-ness' to the participants, which I defined previously, are required. In their characterisation of 'research situations for the classroom' as a starting point distinct from 'problem solving for the classroom', Grenier and Payan (2003) explain that:

> En situation de recherche, le chercheur peut, et doit, pour faire évoluer sa question, choisir lui-même le cadre de résolution, modifier les règles ou en changer, s'autoriser à redéfinir les objets ou à modifier la question posée. Il peut momentanément s'attaquer à une autre question si cela lui semble nécessaire.[7]

This practice, according to them, can be implemented in the classroom through five criteria, which I examine here.

Criterion 1: a novel starting point

To begin with, they explain that:

> Une SRC s'inscrit dans une problématique de recherche professionnelle. Elle doit être proche de questions non résolues. Nous faisons l'hypothèse que cette proximité à des questions non résolues - non seulement pour les élèves, pour l'ensemble de la classe, mais aussi pour l'enseignant, les chercheurs - va être déterminante pour le rapport que vont avoir les élèves avec la situation. (p. 189)[8]

According to their framework, in order to preserve learner agency, none of the participants, in the community of practice of mathematics as a whole, are to know the solution. They further hypothesise that this circumstance impacts the affective responses of the pupils, and consequently, their experience, as discussed above. It also connects back to the 'new to them' characteristic described earlier as corresponding to both non-

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[7] In a research situation, for the problem to evolve, the researcher can and should determine the domain of applicability of his questions, modify or replace the rules under which s/he operates, allow her/himself to redefine the objects of the problem or indeed the problem itself, focus temporarily on a different question if it seems necessary. (My translation)

[8] An RSC [Research Situation for the Classroom] is framed by a professional research question. It must be connected to problems which are unsolved in the canon. We make the hypothesis that the fact that the problem is unsolved, not only for the pupils, but for the instructors and for the participating professionals, is key to the rapport which the pupils will develop with the situation. (ibid)

routine and critico-creative tasks. In addition, it stops the task from becoming uniquely one of literary search, since this would be futile, as the solution does not yet exist in the canon, and by extension, in the literature. This also prevents the participants from thinking that engaging creatively in the task would amount to 're-inventing the wheel', a futile exercise. In addition, the participant can then feel that s/he can engage in the practice even without notional knowledge ('knowing-that/how or knowing-why) specific to the task.

An issue presents itself, however, in the implementation of this condition: given a proposed 'research situation' (as they call them), unless the full participants in the sample community are genuine full participants in the whole community, with specific knowledge of the relevant area of mathematics, how can they know the situation is unsolved in the canon? An obvious way to resolve this issue is of course to involve such a genuine full participant, by implicating her/him in the selection and presentation of the initial enquiry, as Grenier and Payan did.

This is not always possible, and at any rate, the important part of the condition is that the task be new to the sample community. A different way for this condition to be fulfilled is to leave the choice and the initial formulation of the task to the peripheral participants in the sample community themselves, thereby reinforcing the learner's agency. This, in fact, emulates the experience of many genuine full participants who select their tasks themselves from what they find "on email", "in someone else's work", "reading the journals", "at conferences", etc. (Burton, 2004, p. 128). The issue, then, is to lead the participants into an affective and experiential state where they feel that they can choose and formulate a task that requires the appropriate level of engagement, which in this case is critico-creative. This condition is important in that it allows the learner the added agency to be able to re-formulate the task, based on their discoveries, as described by Grenier & Payan, above. Burton refers to this condition when she explains that:

> Solvers must have the feeling that the problem 'belongs' to them. To generate this feeling, choice is most important. Observations with pupils confirm that choosing a problem introduces no additional burdens and it does affect attitudes to problem solving positively. (Burton, 1984, p. 19)

Mason (1978) concurs:

> The question by itself cannot replace the process leading to its articulation, so the student is not in the same state as the originator. (p. 45)

In the case of this study, I choose to implement this last way to fulfil the condition: the first criterion for the design of the teaching approach is to create a social context within which the peripheral participants can take on the role of originator, by choosing and formulating the starting point of their own task. I describe the implications of this and the following conditions in more detail in Chapter 4: The Teaching Approach.

Criterion 2: an open-ended process

Grenier and Payan's (2003) next condition focuses on the openness of the process which the participants are expected to undergo, suggesting the necessity of divergence to agency:

> Plusieurs stratégies d'avancée dans la recherche et plusieurs développements sont possibles, aussi bien du point de vue de l'activité (construction, preuve, calcul) que du point de vue des notions mathématiques (p. 189).[9]

In effect, they imply that decisions regarding the choice of the process involved in the resolution should be the pupil's. This condition connects to a point made earlier, in the discussion of non-routine tasks, where no process of resolution is specified (Zeitz, 1999), and to Cockcroft's (1982) expression of learners' agency through their power to suggest possible avenues of exploration. In Grenier and Payan's words, several possible strategies and developments are possible, and, by implication, the choice is left open, which means that the task cannot require simply knowing-that/how. Instead, knowing-why and knowing-when can help provide these avenues. The condition in point can be expressed as a two-component design criterion as follows: the process of resolution is to be unspecified and open to the peripheral participant's choice, and it is to be the focus of the experience.

Criterion 3: an open-ended goal-state

The last condition of Grenier and Payan's (2003) framework, concerning the eventual solution, also connects to the necessity of divergence to agency:

> Une question résolue renvoie très souvent une nouvelle question. La situation n'a pas de « fin ». Il n'y a que des critères de fin locaux. (p. 189).[10]

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[9] Several investigation approaches and developments are possible, both from the point of view of the activity (construction, proof, calculation), as from the point of view of the mathematical knowledge required (my translation).

[10] An answered question often leads to a new question. The situation has no 'goal-state'. There are only criteria of local resolution.

In Grenier and Payan's framework, the situation is presented in such a way that even if a result is achieved, it can easily lead to a new question, requiring further work. The question, then, lies in the way to control for this condition, and can be answered through the development of a distinction which I briefly mentioned in the introduction, between problems which are mathematical in nature, and problems which simply use mathematics in order to be solved. This distinction is fundamental to a characterisation of mathematical enquiry in that it establishes a set of criteria that help differentiate it from many mathematical problem solving tasks in which students engage in the classroom. As expected, these criteria are reflected mainly by the nature of the solution sought, which is in turn reflected by the nature of the starting point. In essence, a problem which is mathematical in nature presents a situation where a mathematical structure, either applicational or theorisational, is the intended solution or product of the process of resolution. In contrast, a 'problem requiring mathematics to be solved' simply necessitates the application of mathematical notions, whether conventional/observational, applicational or theorisational. The possible exception to this distinction is the case of a problem which requires mathematics to be solved, but which appears to have no answer, in which case a justification for this finding is required.

Polya (1957) proposed a dichotomy, between 'problems to find' and 'problems to prove', which provides a suggestion of this distinction. In his view,

> The aim of a "problem to find" is to find a certain object, the unknown of the problem. […] We may seek all sorts of unknowns; we may try to find, to obtain, to acquire, to produce, or to construct all imaginable kinds of objects. […] The principal parts of a "problem to find" are the unknown, the data and the condition. (p. 154-55)

In his definition, Polya implies that the solution manifests itself as the classic 'single right answer', the 'unknown', which can be 'found', 'obtained', 'acquired', 'produced', or 'constructed'. Polya's discussion of such problems includes examples from mathematics as well as problems outside it. Although the former are solved using mathematics, they do not belong to the category of 'problems which are mathematical in nature' since they do not produce a mathematical structure, but *use* or apply one (or several), in order to produce a solution. Though this solution may take the form of a number, or geometric figure, or equation, etc, it is not a mathematical structure as much as the result or manifestation of one (or several).

For example, Polya uses the problem of the construction of a triangle whose sides a, b and c have given lengths (1957, p. 155). In his words, the 'unknown' is the constructed triangle, the 'data' are the lengths and the 'condition' is that the sides of the triangle satisfy the length requirement. Though the solution of the problem is a geometric figure, it is not a mathematical structure so much as the manifestation of one (i.e. that fixing the lengths of the sides of a triangle is sufficient to produce a figure which is unique under symmetry transformations, in Euclidean geometry). Relating back to the epistemological framework of Table 2, the mathematical knowledge used is treated as knowing-that/how and/or knowing-why, exclusively, since there is no need to use tools which help to either develop or uncover mathematical structures.

In contrast to this first type of problem, Polya discusses 'problems to prove':

> The aim of a 'problem to prove' is to show conclusively that a certain clearly stated assertion is true, or else to show that it is false. We have to answer the question: Is this assertion true or false? And we have to answer conclusively, either by proving the assertion true, or by proving it false. (1957, p. 154)

The goal of a mathematical 'problem to prove'[11], in Polya's vernacular, is the presentation of a mathematical argument which supports or denies the hypothesis by showing this conclusively. In his example, the proof of the Pythagorean Theorem consists in the application of mathematical structures pertaining to measurement, lengths and angles in Euclidean geometry, to the justification of a theorem relating these in the case of right-angled triangles. The solution is therefore a mathematical structure: the relationship between the lengths of the sides of a triangle, as applied to the domain of right-angled triangles in Euclidean geometry. In fact, in this example, the end goal, i.e. the truth of the Pythagorean Theorem, is already provided, and it is the mathematical structure that supports its assertion which is required.

Though I presented the distinction between problems requiring mathematics to be solved and problems which are mathematical in nature as being clear-cut, it is possible to contend that it implies a false distinction. After all, an argument is also something which is 'found', 'obtained', 'acquired', 'produced', or 'constructed'. Indeed, the justification of a mathematical statement is also, on some level, the result or manifestation of a mathematical structure. The difference, however, lies in the fact that

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[11] Polya also given examples of non-mathematical 'problems to prove', but I shall concentrate on the mathematical ones.

the solution itself is also a mathematical structure. To support this claim, I show that the distinction is one of focus, as demonstrated by the initial formulation of the problem.

In the example I used for this topic in the introduction, one of the questions which was asked and answered was: "What is the simplest non-adjacent regular colouring of a tetrahedron whose faces have been subdivided into sets of three kites?" (Knoll, 2002, see Appendix 1). If we decompose this question according to Polya's structure of a 'problem to find', we find that the data take the form of a regularly subdivided tetrahedron, the unknown is its 'colouring' (that is the association of each 'face' with a unique colour), under specific conditions involving simplicity (the least possible number of colours), non-adjacency (no two 'faces' that share an edge will have the same colour) and regularity (interchanging two colour attributions will not change the overall symmetry). Each of these conditions ties the solution of the problem to the mathematical structure pertaining to it, but it is not the structure itself.

These structures can remain implicit, as long as a simultaneous manifestation of all of them, the solution, is found. To change this problem into one which is mathematical in nature would entail the requirement for the explicit formulation of the mathematical structures that determine both the solution and its existence and uniqueness, in the form of a conclusive argument. This would transform the problem into one 'to prove' (Polya, 1957), as it would focus on the why of the solution. In the concluding statement of the paper, such a structure is alluded to as the 'spatial relationship between two Platonic solids that are not each other's duals'.

In summary, there are two types of problems which can be posed in a mathematical context: (1) problems which are mathematical by their very nature, that is, the solution of which requires the development of a mathematical structure (Polya's 'problems to prove', 1957), and (2) problems which simply require mathematics to be solved, and the solution of which is merely a manifestation of one or several mathematical structures (Polya's 'problems to find', 1957). In addition, the initial formulation of the task is an indicator of the type, as demonstrated by the focus of the formulation. This focus is itself connected to the level of engagement required of the participant, and by extension, to the level of knowing required. An exception to this connection is important to note: it is of course always possible, in the case of a known mathematical structure, to memorise a proof, or indeed any result to a question, whether it is mathematical in nature, or simply requires mathematics for its resolution. In this case, the task of

reproducing this structure is one of routine, and is performed using knowing-that/how. The distinction between problems which are mathematical in nature and problems which require mathematics to be solved is therefore only relevant for the distinction between non-routine tasks and critico-creative tasks.

In Grenier and Payan's framework, the situation is presented in such a way that even if a result is achieved, it can easily lead to a new question, requiring further work. This connects to the characterisation of problems which are mathematical in nature: if a mathematical structure is found or developed, for a particular range of mathematical situations, it is possible to question whether this structure can be applied to other cases. In Polya's example of the proof of the Pythagorean Theorem, it is possible to examine the case of triangles with no right angle, or polygons with more sides, or perhaps investigate spherical triangles, etc. This condition can be expressed as a design criterion as follows: there should be no goal-state (Mayer, 1985), implicit or explicit, in the initial presentation of the task, so that there is no implied end.

I have so far established three requirements for a social context in which learners have agency with respect to the meaning of mathematical notions and truth-claims associated with them. These requirements correspond to the three parts of the task: (1) The learners need to be able to choose and formulate the starting point of their own task; (2) the process of resolution is to be open to the peripheral participant's choice, and it is to be the focus of the experience; and (3) there should be no implied 'goal-state'. This last condition connects back to both the others in that no implied end is necessary if both the starting point and the process are to be open to re-negotiation on the part of the learner.

Criterion 4: an atmosphere of security

Grenier & Payan's (2003) framework presents an additional characteristic, reflected by the second and third conditions, which they impose on the design of an RSC, concerning the accessibility of the initial situation to all the participants:

> La question initiale est facile d'accès : la question est « facile » à comprendre. Pour que la question soit facilement identifiable par l'élève, le problème doit se situer hors des mathématiques formalisées et c'est la situation elle-même qui doit « amener » l'élève à l'intérieur des mathématiques.

> Des stratégies initiales existent, sans que soient indispensables des prérequis spécifiques. De préférence, les connaissances scolaires nécessaires sont les plus élémentaires et les plus réduites possibles.[12] (p. 189)

These conditions directly connect to the earlier discussion of what constitutes a problem: according to Mason (1992), a participant becomes engaged in problem solving if s/he allows her/himself to be 'in question'. This initial act on the part of a participant is voluntary and takes place, in the classroom context, under specific conditions which are mostly derived from the didactical contract. According to Brousseau (1997), the didactical contract is the relationship that:

> determines—explicitly to some extent, but mainly implicitly—what each partner, the teacher and the students, will have the responsibility for managing, and in some way or other, be responsible to the other person for. (p. 31)

In the classroom context, a participant will engage in a task if s/he feels that it is accessible under the didactical contract, and that the level of engagement required is feasible, based on the knowledge available to her/him. Mason (1989) describes this phenomenon as a tension which arises from this 'contract':

> This tension arises between pupils and teachers in the following way. The pupils know that the teacher is looking to them to behave in a particular way. The teacher wishes the pupils to behave in a particular way as a result of, or even a manifestation of, their understanding of the concepts or the topic. The more explicit the teacher is about the specific behaviour being sought, the more readily the pupils can provide that sought after behaviour, but simply by producing the behaviour and not as a manifestation of their understanding. Tension arises because the pupils are seeking the behaviour and expect the teacher to be explicit about that behaviour, whereas the teacher is in the bind that the more explicit he is, the less effective the teaching. (1989, p. 155)

In the case of a critico-creative task, where the outcome is uncertain, a context needs to be provided where this tension is not problematic. In other words, the participant needs to feel both motivated and sufficiently in control to take the necessary risks. The last criterion, therefore, requires the implementation of an atmosphere of security which promotes and encourages the taking of critico-creative risks. Given that other criteria of design for the teaching approach preclude the extensive use of scaffolding, an alternate strategy needs to be put in place to provide this sense of security. This can be accomplished, for example, by shifting the focus of the formal assessment onto the process of development of the result, rather than on the result, which is uncertain. Given

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[12] The initial question is easily accessible: the question is "easy" to understand. For the question to be easy to identify, the problem must be situated outside of formalised mathematics, and must pull in the pupil.
Initial strategies exist, without requiring specific pre-requisite knowledge. Preferably, this required knowledge is made minimal.

this shift, the use of the modifier 'didactical' for the tacit contract between teacher and student is no longer pertinent, as the deliberate orientation towards a specific didactical goal has been subverted. Instead, and because the relationship still does stand in some form, I use the modifier 'social' [contract], from now on.

In summary, the design criteria, with respect to the nature of the mathematical task, are as follows: the teaching approach needs to (1) create a context within which the peripheral participants take on the role of choosing and formulating the starting point of their task; (2) leave the process of resolution unspecified and open to the peripheral participant's choice; (3) imply no goal-state (Mayer, 1985), implicit or explicit, in the initial presentation of the task, so that there is no implied end; (4) present an atmosphere of security that motivates the taking of creative risks by shifting the focus on the process of development of the result, rather than on the result, which is uncertain. With respect to types of classroom activities described in the mathematics education literature, these criteria connect more strongly to 'investigative work' than to 'problem-solving' (Cockcroft, 1982; Wells, 1987; Driver; 1988; Ernest, 1991), particularly in terms of the divergent or open ended nature of the task, and the agency of the participant, at each stage of the activity. In the next section, I discuss the implications of these conditions for the practice and the experience it provides, with specific focus on the heuristic and affective cycles invoked as well as on the encapsulated epistemological model.

The Nature and Experience of the Practice: Criterion 5

Besides formal definitions, mathematics educators have also attempted to frame what mathematical problem solving in the classroom might be by describing its components, or stages. In Table 3, below, I summarise some of the heuristic cycles of mathematical problem solving which have been proposed by mathematics educators. Though it is acknowledged that a solver does not necessarily follow the order of these stages linearly and only once, still, these schemes are represented in this way.

The scheme by Mason (1978) is designed to describe both problem solving (if it is exploratory in nature) and investigational work. In the earlier discussion on the latter, I asserted that it is characterized by the manifestation of learner agency in some form at some stage. Reflecting this perspective, Mason emphasises the process by which this engagement develops, as demonstrated by his 'energy states' 1 through 4. The actual

resolution stage is condensed into states 5 and 6, and the looking back step is included as state 7.

Burton's (1984) description of the problem solving process is more general, consisting as it does of fewer stages. She describes the activity as a whole as comprising a starting point (Entry), a process (Attack), and after the solution is found, two more stages, one of verification of the correctness of the process (Checking) and one of finding possible extensions (Looking back).

Basing himself on the work of Polya, Brown (1994) defined the problem solving process through five stages. His description of the stages of problem solving, not so different from Burton's, focuses on the choice and elaboration of the strategy of resolution, suggesting an emphasis on the control of the process as a whole, reminiscent of metacognition, on the part of the solver. In this way, he presents a similar scheme to Sowder's (1993), also based on Polya, though the latter only includes four stages. Where Burton (1984), Brown (1992) and Sowder (1993) emphasise the nature of the learner's behaviour using action descriptors, Mason (1978) discusses the learner's engagement using his 'energy states'.

Mason (1978)	Burton (1984)	Brown (1994)	Sowder (1993)
1. Getting started (recognising and accepting the problem)	1. Entry (trying to understand the problem and clarifying what must be done)	1. Gaining an awareness or understanding of the problem.	1. Understanding the problem
2. Getting involved (accepting ownership)			
3. Mulling (taking control)	2. Attack (finding a resolution and working on coming 'unstuck')	2. Considering possible strategies to solve it	2. Devising a plan
4. Keep going (committing to the process)		3. Choosing a strategy	
5. Insight (getting an idea)			
?		4. Carrying out the strategy	3. Carrying out the plan
The solution having been 'found'…			
6. Checking (verifying the correctness of the idea)	3. Review (examining the resolution)	5. Verify the solution	4. Looking back
7. Looking back (looking for understanding)	4. Extension (finding the seeds of a further problem or re-examining errors)	?	

Table 3: Comparing problem-solving and investigation schemes

Table 3 presents the four schemes, aligning the stages as feasible. The four schemes all agree on the agency of the solver during the process, as demonstrated by the language

used: recognising, accepting (Mason); trying to understand, finding a resolution (Burton); considering, choosing (Brown) and understanding, devising (Sowder). Mason, however, does not differentiate between getting an idea and executing it, since his 'Insight' is his last stage before the solution is found. After the solution is assumed to have been found, some of the schemes, again, differ. Where Brown describes the 'looking back' step to mean monitoring the solution, that is, verifying if there were any procedural or 'spelling' errors, the three other schemes go further in order to include what might be termed a generalisation attempt: Mason is looking for understanding, Burton is suggesting a quest for extensions, that is, similar situations for which this solution would be applicable, and Sowder considers that the interesting part is only beginning:

> Much teaching effort properly goes into developing several heuristics that may help in the devising-a-plan step. What may be neglected, however, is the last step, looking back.
> […] The solution of a problem is quite often followed only by "How about the next one?" Even such important questions as the following may be neglected: "Have we answered the question?" or "Is our answer reasonable?" We may too rarely ask, "How did we solve it?" to give emphasis and explicit attention to particular heuristics, or, "Is there another way?" to emphasize that problems may have more than one solution.
> […] But looking back has even more to offer than just the possibility of finding a more elegant or simpler solution. Looking back can give our students a glimpse at an exciting part of mathematics, the creation of conjectures. Looking back can give our students a small taste of mathematics in the making, as opposed to the consumption of polished mathematics. Looking back can develop the outlook that how one gets answers is more important than the answers. (Sowder, 1993, p. 235-36)

In this view, looking back and establishing understanding are ways to enhance the power of mathematical problem solving as a classroom activity through the application of and focus on metacognitive strategies that help develop the learning that "allow[s] us later to go further more easily" (Bruner, 1960, p. 17): if the process of problem solving is followed by an attempt to understand, extend and, where possible, generalise the results as well as the process, then the task comes to its full potential in educational terms. This conceptual perspective on problem solving corresponds to the criterion of an open goal-state which I discussed previously and therefore suggests a connection with the development of knowing-when as a purpose for the practice of mathematical problem solving in the classroom, and by extension, to the authentic practice of mathematical enquiry, as established in the previous section. In this respect, Mason (1978), Burton (1984) and Sowder's (1993) schemes most explicitly emphasise this potential, thereby making room for learner agency at the concluding stage of the task.

The problem with this view is that the last step it prescribes can be left out altogether from tasks that can be and often still are termed 'mathematical problem solving in the classroom', thereby justifying the introduction of a new term, mathematical enquiry, which necessarily does include this last stage.

In an article citing his problem solving heuristics shown above, Mason describes investigations as manifesting, under certain conditions, the undercurrent of a pursuit of understanding not only in the 'looking back' stage, but also in the initial approach, that is, when the problem is posed. In the 1990s, Brown and Walters opened a discussion on the importance of the problem posing stage of the problem-solving process. By doing this, they intended to take a closer look at the actions taking place at that early stage, and in the process, introduced the concept of agency into this earlier stage of the process of enquiry, the starting point, connecting back also to Mason's (1978) comment on the issues arising from a separation between the originator and the learner[13]. Opening up this earlier stage to learner agency completes the circle, connecting the first three criteria I derived from Grenier and Payan's (2003) framework: Agency in the choice of the starting point, the development of the process and the sanction of the result as successful goal-state. Without all three of these criteria, the task can therefore not be thought of as authentic mathematical enquiry as practised by full participants.

The fourth criterion that emerges from the analysis of Grenier & Payan (2003) framework is connected to the affective experience of mathematical enquiry. In this respect, it links to Mason's (1978) 'energy levels' or levels of engagement. To recall, a problem solver (1) gets started by recognising and accepting the problem, (2) gets involved by accepting ownership, (3) mulls, that is, takes control, (4) keeps going by committing to the process, (5) gains insight, that is, gets an idea, (6) checks, or verifies the correctness of the idea, and finally (7) looks back for understanding.

Mason's description certainly integrates the idea of agency into the process and end stages of the task, and potentially into the starting point. In addition, although his scheme suggests that the move from one stage to another is seamless, he discusses, alone and with Burton and Stacey (1982), the state of 'being stuck', that is, of being unable to move to the next energy state. They qualify this state as 'honourable' and 'an essential part of improving thinking' (p. ix), and devote two of the nine chapters in

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[13] See in section on alternate educational perspectives

Thinking Mathematically to it. As for Burton (2004), she cites 55 of her 70 participants as mentioning this state (p. 55). She further reports that

> […] the mathematicians usually had many problems on which they were working simultaneously, […] that they regarded errors as normal, that they frequently became stuck and, when they did, moved from what might be causing this state, to something different in order to unblock their thinking. (p. 194)

The strategies suggested in Burton's description rely on the same metacognitive awareness that is presumably developed in the 'looking back' step described above.

Mason's (1978) work represents a description of the experience of problem solving, through a focus on the engagement of the participant. This perspective evokes the experiential cycle of mathematical enquiry described by Hadamard (1945) according to whose scheme the mathematical enquirer experiences the following stages: initiation (or preparation), incubation, illumination and verification. The interesting aspect of Hadamard's scheme is his emphasis on the experience, that is the subjective 'apprehensions of a situation or collection of situations that he/she engaged with', as I defined it at the beginning of this chapter. The experience of each stage can have associated affective responses. In the initiation stage, for example, the participant needs to become engaged, which must be voluntary and therefore depends on a favourable affective context, including responses such as curiosity, feelings of security, etc., alluded to in the fourth design criterion, and which I discuss in the next section. Given the importance of positive affective responses at this stage, this stage of the teaching approach is particularly sensitive.

During incubation, in contrast, a loss of control on the part of the participant is likely, thereby evoking frustration and potentially, depending on the context, anxiety. Illumination, also known as the Aha! moment evokes pleasure, excitement, possibly relief. Hadamard's (1945) description integrates 'being stuck' (Mason et al., 1982) as its own stage, in contrast with many authors cited earlier. This distinction is significant in that incubation, in research, is seen as a productive time, even though it may not seem so, because it leads to illumination. Hadamard's illustration of this phenomenon is the story told by Poincaré who:

> was not working when he boarded the omnibus of Coutances: he was chatting with a companion; the idea [for the solution of his problem] passed through his mind for less than one second, just the time to put his foot on the step and enter the omnibus. (p. 36)

This discussion of the incubation stage and its subsequent release in illumination suggests that room should be made in the design of the intervention for this transition. Unfortunately, incubation largely escapes control on the part of the participant. In fact, it consists precisely in the hopefully temporary loss of control of the process. To make allowance for this phenomenon, Burton (2004) explains that:

> The strategy of a student working on more than one problem at a time, almost unheard of in mathematics classrooms, and of having time and space to retreat, reflect, research, is not only appropriate to the unsolved problems of research mathematicians. Students undertaking a mathematical challenge also need to have room to manoeuvre, to work together, to consult people or books, to think. Most of all, instead of being overwhelmed by frustration when stuck, students would benefit from knowing that mathematicians find it "an honourable and positive state, from which much can be learned" (Mason et al., 1982: 49) and more than that: mistakes and errors are part of mathematics. You cannot live without errors in mathematics. (p. 194-95)

Time, then is required, not only to experience mathematical enquiry in an authentic way, but also to move between the stages of creation and critical evaluation of the task and the findings. The final design criterion of the teaching approach, therefore, is time for the authentic experience to unfold.

In summary, the design criteria for a teaching approach that provides an authentic experience of mathematical enquiry as a practice are as follows: the teaching approach needs to

1. create a context within which the peripheral participants take on the role of choosing and formulating the starting point of their task;

2. leave the process of resolution unspecified and open to the peripheral participant's choice;

3. imply no goal-state (Mayer, 1985), implicit or explicit, in the initial presentation of the task, so that there is no implied end;

4. present an atmosphere of security which promotes and encourages the taking of creative risks by shifting the focus on the process of development of the result, rather than on the result, which is uncertain;

5. allow for enough time so that the experiential cycle(s) can be experienced in full.

I discuss the implementation of these criteria in Chapter 4: The Teaching approach.

Assessing the Potential Effect: Affective Outcomes

One of the goals of the present study is to examine the teaching approach, which is designed to respond to the criteria I have outlined, for its potential impact on the participants. In particular, I focus on this potential impact on affect, which, according to Bloom et al. (1964) is distinct from the cognitive and psychomotor domains. In order to establish the criteria of this examination, I explore the literature on affect in education, paying particular attention to affective issues in mathematics education, though not to the exclusion of the cognitive domain. In addition, as this is an extremely wide area of research in its own rights, with many perspectives, I focus on theoretical frameworks that are applicable to the present study, by selecting the theoretical perspectives that help support the main hypothesis, according to which an individual's experiences of a practice are a significant constituent of what forms their affective responses. In this section, I combine a historical perspective from wider to narrower with a search for a framework useful to this study.

Bloom et al.'s Taxonomy of Educational Objectives

Based on Bloom et al.'s taxonomy (1964), educational outcomes of an affective nature can be broken down into the following categories:

- Receiving (attending): awareness that a learner is conscious of something
- Responding: being sufficiently motivated so as not just be willing to attend, but actively attending
- Valuing: assigning something value sufficiently consistently to be described as a belief or attitude
- Organisation: follows internalization of values and applies to situations where several values are relevant
- Characterisation: acting consistently in accordance with internalized values

In the earlier epistemological discussion of mathematics, I cited Cockcroft's definition of appreciation as "involv[ing] awareness of the nature of mathematics and attitudes towards it" (1982, § 240). Several of the above categories, it seems, would fit into this very general description. For example, 'valuing', which entails a value assignation to something, could be expressed, in the case of mathematics, as an attitude towards it. As I demonstrate later, this observation is consistent with the usage of later writing, which includes emotions, attitudes, views and beliefs under affective outcomes. It poses a conundrum, however, in that it makes the distinction from cognitive objectives unclear. In Bloom et al.'s taxonomy (1964), cognitive outcomes are broken down into the following categories:

- Knowledge: behaviours and situations that emphasize remembering
- Comprehension: understanding a literal message without necessarily relating it to other material
- Application: ability to use previously learned material in situations which are new or contain new elements
- Analysis: breakdown of material into its constituent parts and detection of the relationship of the parts
- Synthesis: the putting together of elements and parts so as to form a whole
- Evaluation: the making of judgments about the value of ideas, works, etc.

The first five categories in this list seem clearly cognitive. The last one, however, presents a semantic similarity with "valuing: assigning something value sufficiently consistently to be described as a belief or attitude", taken from the list of categories of affective outcomes (see above). Though the distinction between the two categories can be interpreted as being between the rating of the quality of the object of the response against a standard or using certain criteria (cognitive) and seeing the importance of the object (affective), this rapprochement between categories in otherwise separate domains mirrors the problematic nature of the distinction between belief, as associated with affective issues, and knowledge, which is seen as cognitive, a distinction which is still an issue today. As Lester, Garofalo & Kroll (1989) put it,

> The distinction between beliefs and objective knowledge is […] unclear. The difference between the two rests with the notion that an individual's beliefs may or may not be logically true and may or may not be externally justifiable, whereas knowledge must have both characteristics in addition to being believed by the individual (cf. Kitcher, 1984). (p. 77)

The difference between the domains seems to be one of criteria for the response. In the case of cognitive responses, the expectation is that there are 'rational' justifications that can be verified externally, whereas beliefs lack this condition[14]. Once again, the apparent simplicity and clarity of the described framework is negated by a closer look highlighting overlap between categories, not only within the affective domain, but also between domains. This ambiguity between 'valuing' and 'evaluating', contrasts with the very unequivocal nature of the framework as implied in its clear structure, which suggests that a given 'objective' or desired behaviour and/or view only find itself placed in one cell of the table.

The criteria for possible distinction between 'valuing' and 'evaluating' suggested by Lester et al. do not necessarily resolve the conundrum from a subjective, internal stand point, since logical truth and external justification are not always a part of an

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[14] There is continuous philosophical debate in process about the validity of knowledge as 'justified true belief', with counter-examples and definitional adjustments in true Lakatosian manner. (Gettier, 1963)

individual's conscious awareness of her/his belief. And indeed, this does not separate these beliefs from the judgements referred to in 'evaluation'. Though this discussion shows that the characterisation of an objective as cognitive versus affective is not always clear, this does not necessarily make the framework unworthy as a whole; in any case, the two domains can be seen as mutually constitutive since our knowledge can inform our affective responses and vice-versa. As I am focusing on the affective slice of the framework, a solution is simply to include all the categories that can be seen as affective, even if they can also be interpreted as cognitive.

Affect and Participant Engagement

Some of the categories of Bloom et al.'s affective domain show a connection with the earlier discussion of the nature and degree of the engagement of a participant with mathematics, from passive (receiving), to active (responding) to critico-creative (valuing, organisation, then characterisation), suggesting a link between a participant's affect and the nature of her/his engagement. This link may or may not be causal, and, were this causality to exist, it could run in either direction, or be reciprocal, where the engagement impacts the affective responses and vice-versa. In addition, the connection to engagement suggested by the language used in Bloom et al.'s affective categories can help interpret participant's responses to the teaching approach in terms of their engagement with it.

Affect and Epistemological Perspectives

Bloom et al.'s categorisation also presents a connection with the epistemological perceptions of a participant, who can 'know' or indeed believe that a mathematical notion is true by convention/observation, true by application of logical reasoning or true as a manifestation of the underlying mathematical structure, albeit socially derived. Indeed, if a learner's epistemological perception is that a mathematical notion is 'true by convention', s/he will do no more that 'receive' it, whereas if s/he perceives a notion as true by observation, then s/he applies more attention in order to compare the notion to her/his own experience, in order to "actively attend" to it. Finally, in order for 'valuing', 'organisation' and 'characterisation' to take place, the participant needs to perceive it as justifiable through (critico-creative) reasoning. Bloom's categories are structured into an ordered list, where, parallel to the epistemological framework described in the beginning of the chapter, each successive category is richer and deeper than the previous.

Skemp's Model of Affect

In the 1970's, Skemp (1979) developed a model of affective responses in the mathematics education context which was based on the idea that an individual's behaviour is most often goal-directed (p. 2). Furthermore, the goal-states for which an individual is aiming, according to Skemp, are connected to her/his survival. In this context, the dimensions of his model relate to the success or failure of the individual to achieve this goal-state[15], which either s/he or someone else in authority has set for her/him, whilst simultaneously avoiding what Skemp terms the anti-goal-state. He then focuses on each of the combinations that can occur: the individual can be moving towards her/his goal-state, thus feeling pleasure, or s/he can move away from the goal-state, in which case s/he feels 'unpleasure' (p. 13). Conversely, s/he can be moving towards an anti-goal-state, generating fear, or away from it, generating relief. These four situations can be presented in a matrix form, as Table 4:

Movement	Towards	Away from
Goal-State	Pleasure	Unpleasure
Anti-Goal-State	Fear	Relief

Table 4: Skemp's responses to movement between goal- and anti-goal-states

In addition, to these four categories of emotional responses, Skemp introduces the idea that the individual may or may not feel an ability to direct the movement between states. This adds another dimension, and consequently, four more categories of 'emotions', as shown in Table 5, below. The focus on a goal-state which is foundational to Skemp's (1979) framework is a key element in the development of Criterion 4 for the design of the teaching approach.

Directing movement…	Feeling of Ability	Feeling of Inability
Towards Goal-State	Confidence	Frustration
Away from Anti-Goal-State	Security	Anxiety

Table 5: Skemp's knowledge of ability to respond to movement between goal- and anti-goal-states

The final variable which Skemp discusses relates to the novelty of the actions or activities that the individual engages in. As these become more routine, he says, the level of consciousness required diminishes, and so does the intensity of the emotional responses. In his framework, he describes the novelty of a situation as critical to the emotional response. A situation with no novelty generates a low level of consciousness;

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[15] Though they are related, this goal-state is not to be confused with the one I discuss in Criterion 3.

if there is some novelty, consciousness is raised. In his third category, there is so much novelty that the individual feels threatened in her/his ability to survive (p. 15). This of course connects back to the earlier discussion of routine versus non-routine mathematical tasks, where a routine task, which involves little or no novelty, requires a less active engagement than a non-routine task. The question then poses itself, does a 'critico-creative task of mathematical enquiry', as I described the third type, fall into Skemp's threatening category? To answer this, it is necessary to connect back to the discussion on risk-taking, and the social/didactic context within which this can be made acceptable to a participant, thus reducing its threatening nature, and placing it back within the tolerable realm of Skemp's framework (See Criterion 4: an atmosphere of security).

Skemp's framework is interesting in that it attempts to account for the internal response by classifying the external context as well as the individual's perception of their ability to cope with it. In addition, he incorporates a time dimension in that he considers prolonged or repeated exposure to a given situation or type of situation as diminishing the intensity of the response as a consequence of a lowering of the level of consciousness required by the given situation. He incorporates these four dimensions into his model with the underlying assumption that each consists of a dichotomy, or certainly a one-dimensional continuum. This structure of the framework can be useful in that it suggests a clarity of interpretation for participants' responses, but this clarity can be deceiving in that an individual's responses are rarely that simple. Though Skemp does take this into account in his description through the acknowledgement that the emotions can occur simultaneously due to different components of a given situation, this does not account for the complexity of human emotional experience. For example, the repeated experience of a type of situation does not simply dampen the affective response to it. Instead, as demonstrated by research into, for example, mathematics anxiety, the response can become so ingrained in the individual, that s/he can react in anticipation, or at the mere mention of such a situation. In other words, the emotional response of an individual to a situation is influenced by their past experience not only in the degree of routine-ness of the situation, but also by the emotional responses s/he had on those previous occurrences of the situation.

Later Developments

In attempting to create a coherent model of affective responses, later researchers used distinctions and connections of a temporal and causal nature between its components, while preserving Skemp's point that affective responses are reactions to the success or failure of the individual to achieve the goal-state. Mandler (1989), for example, introduced the notion that emotions have two characteristics. They "express some aspect of value" (suggesting a link to Bloom et al.'s earlier framework) and they are "hot, implying a gut reaction or a visceral response" (p. 6). The hot, immediate nature which

Mandler ascribes to emotions implies that there can also be cool, more stable affective responses, often referred to as attitudes, which "may result from the automatizing of a repeated emotional reaction" (McLeod, 1992, p. 581). In other words, repeated exposure to a class of situations contributes to the development of an attitude towards situations which the individual perceives as belonging to this class. In Ajzen's (2001) summary of theoretical perspectives in the psychological study of affect, attitudes are described as follows:

> There is general agreement that attitude represents a summary evaluation of a psychological object captured in such attribute dimensions as good-bad, harmful-beneficial, pleasant-unpleasant, and likable-dislikable (Ajzen & Fishbein 2000, Eagly & Chaiken 1993, Petty et al 1997; an in-depth discussion of issues related to evaluation can be found in Tesser & Martin 1996; see also Brendl & Higgins 1996). (p. 28)

Attitudes, therefore, have valence, that is, a position along a continuum between two opposite 'values', as do emotions, and both result from an evaluation of the specific situation in relation to previous exposure to what the individual perceives as comparable. This comparison is in fact what triggers the individual's response[16]. The value ascription implied by the specific response therefore suggests a mechanism of comparison to situations in the individual's previous experience which s/he perceives as connected. As McLeod (1987) expresses it, these values are dependent on the individual's "knowledge, beliefs and previous experience" (p. 135), which in turn impacts the selection of the class of previous experiences to which the current situation is being compared. In other words, the nature of the situation, including the interruption of plans, juxtaposed with the nature of the individual's past experience with analogous

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[16] In this respect, the very novelty of a situation can be the trait which focuses the comparison, and, in a seemingly paradoxical twist, a familiarity with novelty itself can make a novel situation less so.

situations, causes an affective response comprising a value component. As Ajzen (2001) expresses it:

> According to the most popular conceptualization of attitude, the expectancy-value model (see Feather 1982, Fishbein 1963, Fishbein & Ajzen 1975), this evaluative meaning arises spontaneously and inevitably as we form beliefs about the object (see Ajzen & Fishbein 2000). Each belief associates the object with a certain attribute, and a person's overall attitude toward an object is determined by the subjective values of the object's attributes in interaction with the strength of the associations. (p. 30)

In other words attitudes are at least partly derived from beliefs about the object of the attitude. When an individual forms several, possibly conflicting, beliefs about an object, Ajzen (2001) says that:

> Although people can form many different beliefs about an object, it is assumed that only beliefs that are readily accessible in memory influence attitude at any given moment. A belief's chronic accessibility tends to increase as a function of the frequency with which the expectancy is activated and the recency of its activation, as well as the belief's importance (Higgins 1996, Olson et al 1996). (p. 30)

According to recent results, therefore, though an attitude can be derived from the individual's beliefs about the object of the attitude, s/he can hold several conflicting beliefs at a given time, and the belief that determines the attitude at that time is selected according to criteria external to itself.

In Lester et al.'s (1989) summary of affect in the specific context of mathematical problem solving, affective responses are subdivided into three categories: (1) emotions, defined as "subjective reactions to specific situations", and which could therefore, in the current context, be used as markers of Hadamard's (1945) stages or Mason's (1978) energy states; (2) preferences and attitudes, which are "generally accepted [as being] traits, albeit perhaps transient ones, of the individual"; and (3) beliefs, which "constitute the individual's subjective knowledge about self, mathematics, […]" (p. 77). In this framework, emotions are reactions to a current situation, while attitudes and beliefs are formed over time, in a kind of accumulated effect.

In his summary, Ajzen (2001) mentions a few researchers that investigated the relative weight of beliefs and feelings (here referred to as emotions) on the evaluations that produce attitudes. Specifically, he explains that the relative influence of the two components differs between individuals:

> Haddock and Zanna (2000) summarized the results of several studies that provide support for the joint effects of beliefs and feelings on evaluations. Of more interest, they also show that individuals differ in their tendency to base their attitudes on

cognition or affect. […] Regression analyses showed the expected results: The attitudes of individuals identified as "thinkers" were predicted by their beliefs about the attitude objects, but not by their feelings, and the reverse was true for individuals identified as "feelers." (Ajzen, 2001, p. 34)

In other words, an individual's tendency to trust their feelings versus their intellect influences the power of either to determine their attitudes.

In Skemp's (1979) framework, the routine-ness of a situation diminished the intensity of the affective response while preserving its nature. What the later framework supplements is the notion that this less intense response, conversely, is stable. In the present framework, emotions are immediate responses to a situation, while attitudes and beliefs are responses that have been programmed by repeated exposure to a class of situations, such as, for example, mathematical problem solving, or by repeated social programming. In this respect, both attitudes and beliefs are general in nature in that they are associated with a collection of situations that have common traits. Emotions are impacted by the attitudes and beliefs formed previously, but they are associated with the immediate situation, whereas attitudes and beliefs are responses associated, by the individual, with categories of situations.

If we put these two frameworks together, we get beliefs, which influence attitudes, and emotions and attitudes which influence each other, with the relative importance of the influence on attitudes of emotions versus beliefs dictated by the individual's personal traits.

Lester et al. (1989) also include beliefs, which they define as subjective knowledge, into the affective domain. These are derived by a combination of experience and reflection thereon. This reflective component, which can be seen as cognitive, is what makes it problematic to define them as purely affective. Conversely, their causal relationship with attitudes, which is established by Ajzen (2001), together with the fact that they can be based on subjective and/or non-rational justifications, makes their clear characterisation as purely cognitive problematic. In the end, beliefs need to be included into either picture, since, as McLeod explains: "The role of beliefs is central in the development of attitudinal and emotional responses to mathematics" (1987, p. 579). As the present study focuses on all affective issues in mathematics education, I therefore include beliefs in this framework.

In the perspective developed so far, affective responses are characterised as being derived from a connection of the situation to what is perceived by the individual as a comparable class in their existing experience. This response, if it is immediate (hot) and focused on the specific moment, will be referred to here as emotional; if it is focused on the class of situations to which the current one belongs and expresses a value judgement, it will be referred to as an attitude, and if it presents itself as subjective knowledge (which has an associated truth-value judgement; Goldin, 2002), it will be deemed a belief. More specifically, and based on the literature, examples of emotional responses to mathematics include curiosity, puzzlement, bewilderment, frustration, anxiety, fear and despair, encouragement and pleasure, elation and satisfaction (Goldin, 2000), which are associated with discrete moments in time and therefore can be connected with the stages of the heuristic cycles discussed in the section on the nature and experience of the practice of authentic mathematical enquiry. Examples of attitudes in the mathematics education context could include interest, like/dislike of mathematics, the perceived importance and difficulty/ease of the subject by the individual, and attitude to learning and (in the case of teachers or student teachers) teaching it. Finally, beliefs concern truth-values which the individual perceives as objective and assigns to aspects of mathematics, themselves in relation to it, and its education. The clarity of the framework as defined thus far conceals the existence of cases which are ambiguous and can therefore be expressed as either a belief or an attitude. Several of these cases are highlighted throughout the following discussion, which I centre on beliefs.

Beliefs: a Categorisation

According to McLeod (1987), "there are a variety of ways to organize research on beliefs" (p. 579). In the context of mathematics education, they can be categorised using a range of models, depending on the focus of the research. Törner (2002) goes so far as to state that:

> there is still no consensus on a unique definition of the term belief […] [Some authors] even speak of 'definitional confusion among researchers'. Many authors seem to be aware of this deficiency and thus establish their own terms… (p. 75).

In his review of studies on beliefs in mathematics education, McLeod (1987) breaks these down into four sub-categories: beliefs (1) about mathematics (the discipline), (2) about self (the individual's view of her/himself as learner, user or creator of mathematics), (3) about mathematics teaching, and (4) about the social context (for example, the type of classroom environment conducive to learning mathematics, or the

overall social view of mathematics and its teaching). In terms of beliefs about education, McLeod's categories 3 and 4 could, I think, be melded together into a single category, about mathematics education. In addition, mathematical knowledge, if expressed as 'beliefs in the content of mathematics', could also be understood as belonging in this category, as for example I could say I believe (or know) that the value of π (that is, the ratio between the circumference and diameter of a circle) is an irrational number between 3 and 4, even though this was not always acknowledged. In summary, therefore, a categorisation could entail the following components:

- Beliefs about the discipline of mathematics (epistemological and practical)

- Beliefs about self as a mathematical being (or not)

- Beliefs about mathematics education in general

- Beliefs in the content of mathematics (which in some instances includes mathematical knowledge, as discussed in the earlier section on Bloom's Taxonomy)

As the practice of mathematical enquiry and its impact on the participants' affective responses is the subject of this study, I centre the framework on the view of mathematics as a discipline, without leaving out the two next beliefs. In any case, these can be seen as connected to the first: an individual's view of her-/himself as a mathematical being is informed by her/his view of what mathematics is, etc. As for the 'beliefs in the content of mathematics', it refers to the acceptance of the notions (as defined in the epistemological discussion), and, though they are experiential outcomes, will not contribute to the assessment of the intervention. I therefore do not discuss them further.

Beliefs about Mathematics

An individual's beliefs about the nature of mathematics as a discipline are informed by the practice of the community to which s/he belongs. In this respect, a characterisation can be expressed in terms of the activities involved in the practice by the specific community. In this section, I examine the characterisation that can be found in the literature, by comparing it with the epistemological framework I develop above, and the views expressed by active members of the mathematics research community.

For the case of mathematics, Ernest (1989) categorised beliefs into a hierarchy. At the shallowest level, he places Instrumentalism, which presents the view that "mathematics is an accumulation of facts, rules, and skills to be used in the pursuance of some

external end" (p. 250). In practice, this view is connected with the previously discussed 'conventional/observational' epistemology, according to which mathematics is constituted of "mathematical facts, rules and methods" (ibid), which might as well be ordained from above and are taught and learned because they 'work', without consideration for reasoning behind it. As Stipek et al. explain it:

> This conceptualization of mathematics is similar to what Skemp (1978) refers to as an instrumental concept of mathematics—a set of fixed plans for performing mathematical tasks involving step-by step procedures. It is also similar to what Kuhs and Ball (1986) refer to as a content focus, in which students' mastery of mathematical rules and procedures are emphasized. (2001, p. 214)

In this view, the source of mathematical truth is considered external to the individual, and left unquestioned by her/him, seen as residing instead in textbooks or the mind(s) of the authority figure(s). This perspective is only sustainable if s/he has surrendered curiosity, and any form of critical thinking. Goldin (2000) expresses the consequences of this position in problem solving situations as follows:

> The least unhealthy response available to the solver experiencing extremely negative feelings may now be the acceptance of authority-based problem solving, in which the solver tries to relieve the anxiety by *complying*. The problem solver may try to guess a response thought to be desired. Any rote procedure is now welcome. If helped at this point, the student's resolve may be to *imitate* the indicated procedure, regardless of "understanding the mathematics." (p. 216)

McLeod (1992) reports on researchers who:

> noted how students view mathematics as a skills-oriented subject, and how such limited views of the discipline lead to anxiety about mathematics (Greenwood, 1984) and more generally interfere with higher-order thinking in mathematics (Garofalo, 1989). (p. 580)

This position thus appears incompatible with the development of more sophisticated mathematical knowledge, which can only be achieved with great difficulty in the absence of higher-order thinking. Mathematics is seen as a subject that serves other areas of knowledge, such as physics, economics, accounting, cooking.

At the next level, Ernest (1989) places Platonism which, according to his analysis, views mathematics as "a static but unified body of certain knowledge" (p. 250). The use of the term 'certain' suggests an underlying assumption of this view that is the objectivity of mathematical knowledge. As Hardy (1967) expressed it:

> For me, and I suppose for most mathematicians, there is another reality, which I call a 'mathematical reality'; and there is no sort of agreement about the nature of mathematical reality among either mathematicians or philosophers. Some hold that

it is 'mental' and that in some sense we construct it, others that it is outside and independent of us. (p. 123)

Either way, according to Hardy, mathematics is an absolute in that the truth of a mathematical statement can be determined with complete certainty. As Davis and Hersh (1980) expressed it:

> the activity of mathematical research forces a recognition of the objectivity of mathematical truth. The 'Platonism' of the working mathematician is not really a belief in Plato's myth; it is just an awareness of the refractory nature, the stubbornness of mathematical facts. They are what they are, not what we wish them to be. (p. 362)

Hardy illustrates this Platonist perception in his discussion of 'pure mathematics' which:

> seems to [him] a rock on which idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but because it is so, because mathematical reality is built that way. (p. 130)

The Platonist view is metaphorically compatible with the image of the mathematician as cartographer of this unique mathematical 'reality'. In this respect, it is a more arrogant view, implying as it does that (human) mathematicians can create an accurate, complete map of mathematical reality as a whole, given enough time. In these terms, it fits well with the 'applicational' epistemology, in which mathematical applications are the product of a mathematical version of the scientific method.

Recent activity in the philosophy of mathematics, as well as in research mathematics, however, has suggested that this view is not completely accurate. In response to this state of affairs, Ernest (1989) concludes his hierarchy with the 'problem solving' view according to which mathematics is "a dynamic, continuously expanding field of human creation and invention, a cultural product" (p. 250). Ollerton & Watson (2001) describe this view as follows:

> Currently it is more usual to see [mathematics] as having been created by mathematicians in response to a variety of needs and interests, and consisting of conventional symbols and relations which have commonly accepted meanings. (p. 7)

This view is also consistent with Lakatos' (1976) fallibilist perspective according to which mathematics:

> like the natural sciences, is fallible, not indubitable; it too grows by the criticism and correction of theories which are never entirely free of ambiguity or the possibility of error or oversight. Starting from a problem or a conjecture, there is a simultaneous

> search for proofs and counterexamples. New proofs explain old counterexamples, new counterexamples undermine old proofs. (Davis & Hersh, 1981, p. 347)

In this perspective, therefore, the notion of truth is provisional at best, contingent as it is on the discovery of a 'new counterexample' that could later refute it, and mathematics, as a discipline, cannot be separated from the human context within which it was created. Indeed:

> To Lakatos, 'proof' in this context of informal mathematics does not mean a mechanical procedure which carries truth in an unbreakable chain from assumptions to conclusions. Rather, it means explanations, justifications, elaborations which make the conjecture more plausible, more convincing, while it is being made more detailed and accurate under the pressure of counterexamples. (Davis & Hersh, 1981, p. 347)

These explanations, justifications and elaborations provide the mathematical community with 'working theories' about mathematical notions, which theories are human creations, and can therefore be revised as new developments are made.

The problem with calling this view 'problem solving', lies in the implication it makes, that new mathematics is developed specifically in response to problems. As I discussed previously, problem solving, at least in the educational context, implies the existence of a specific starting point, and a solution, a 'goal-state'. As Grenier & Payan explain it, however, in an authentic mathematics research context, the researcher "can and should determine the domain of applicability of her/his questions, modify or replace the rules under which s/he operates, allow her/himself to redefine the objects of the problem or indeed the problem itself, focus temporarily on a different question if it seems necessary" (see my translation, in Grenier and Payan's Framework). In mathematical enquiry, it should therefore, according to them, be permissible to shift focus or direction, temporarily or permanently, in the course of one's research. How is this possible? To view the discipline of mathematics in such a way, a further step needs to be taken, away from a purely utilitarian, mathematics-as-servant-of-other-disciplines outlook, to what might be seen as a mathematics-for-mathematics'-sake standpoint. Davis & Hersh (1981) associate this distinction with that between 'pure' and 'applied' mathematics. Although, as Burton (2004) states it, the line between the two sub-disciplines is blurry at best, Davis and Hersh's (1981) make reference to Hardy's Mathematician's Apology in their discussion of this outlook is significant. To understand this reference clearly (I will cite it in a minute!), it is useful to know that Hardy (1967) likened the work of mathematicians to that of artist:

> A Mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. […] The mathematician's patterns, like the painter's or the poet's, must be beautiful… (pp. 84-85)

In this light, Hardy's proud contention that, despite the success of his career as a research mathematician, he "has never done anything 'useful'" supports what Davis & Hersh call:

> the dominant ethos of twentieth-century mathematics—that the highest aspiration in mathematics is the aspiration to achieve a lasting work of art. If, on occasion, a beautiful piece of pure mathematics turns out to be useful, so much the better. But utility as a goal is inferior to elegance and profundity. (1981, p. 86)

In this, Davis and Hersh's citation of Hardy, the reality of the researcher in pure mathematics is exposed: their purpose is not to work on solving problems presented to them by external users; they explore mathematics, the patterns it describes and the structures it possesses, without necessarily tracking the immediate or potential applicability of the results. Following this idea, and to distinguish it from 'problem solving' (Ernest, 1989), I call this belief category 'pattern analysis' after Schoenfeld's (1994) definition:

> Mathematics is the science of patterns. (p. 54)

The question remains, however, whether the belief in a mathematics-for-mathematics'-sake, is acceptable, or even useful, as an educational outcome. The answer lies in an examination of the epistemological framework I developed at the start of the chapter. If mathematics is the study of relationships or the science of patterns, the highest level of thinking in mathematics (described as knowing-when) is fundamental to the development of a rigorous model of mathematics which serves to study these relationships and patterns. As for this highest level of thinking, it occurs in the critico-creative realm where even the starting point, and not only the process or the goal-state, are open-ended. This perspective, or at least an experience of the connected practice, is then the goal for student teachers, particularly in light of their future roles as genuine full participant of the global community of practice, in the classroom.

In summary, beliefs about the nature of mathematics as a discipline can be categorized as follows:

- Instrumentalism: mathematics as an accumulation of facts, rules, and skills to be used in the pursuance of some external end (Ernest, 1989)

- Platonism: mathematics as a static but unified body of certain knowledge (Ernest, 1989)

- Problem Solving: mathematics as a dynamic, continuously expanding field of human creation and invention, a cultural product (Ernest, 1989), and

- Pattern Analysis: mathematics as the science of patterns (following Schoenfeld, 1994)

As I discussed throughout this section, an analogy exists between the hierarchy of beliefs about the discipline of mathematics and the structure I proposed for the epistemological nature of mathematics. In addition, just as a knowledgeable individual can treat a mathematical notion as conventional/observational, applicational or theorisational based on the requirement of the situation, an individual can choose to treat mathematics as a whole in an Instrumentalist, Platonist, Problem-solving or Pattern Analysis way, as the situation warrants, *provided s/he has experienced it as such*. As the aim of the teaching intervention is to provide an experience to complete this overall experience, this dimension is the primary one I investigate.

Derived Beliefs

In the earlier discussion on the sub-categories of beliefs, I described beliefs about self in relation to mathematics and beliefs about mathematics education as deriving from beliefs about mathematics as a discipline. Op T'Eynde, De Corte and Verschaffel (2002) collected the following subsets of sub-categories of beliefs from various sources:

1. Beliefs about mathematics education

   - Beliefs about mathematics as a subject
   - Beliefs about mathematical learning and problem solving
   - Beliefs about mathematical teaching in general

2. Beliefs about the self [in relation to mathematics]

   - Self-efficacy beliefs
   - Control beliefs
   - Task-value beliefs
   - Goal-orientation beliefs

3. Beliefs about the social context [of mathematics instruction]

   - Beliefs about social norms in their own class
   - The role and the functioning of the teacher
   - The role and the functioning of the students
   - Beliefs about socio-mathematical norms in their own classrooms (p. 28)

In the previous section, I discussed only beliefs about the nature of mathematics as a subject, characterising these in terms of the epistemological framework I developed earlier, using Ernest's (1989) hierarchy, and connecting them to the practices of contemporary research mathematicians. This does not give a complete picture of even 'Beliefs of mathematics as a subject', as it does not include the perceived utility of the

subject, or its difficulty. Altogether, therefore, the category of beliefs about the subject of mathematics could include (1) beliefs about the nature of the subject, (2) beliefs about its objective importance, and (3) beliefs about its objective difficulty.

In Beliefs: a Categorisation, I cited another three categories: beliefs about the self in relation to the discipline, beliefs about educational matters in relation to mathematics, and beliefs in the content of the discipline, that is, mathematical knowledge. I also explained that the last category would not be the subject of this study. It remains therefore to examine the two categories relating mathematics to self and to education.

Beliefs about the Self in Relation to Mathematics

If I re-order Op T'Eynde et al.'s subcategories, I can preserve the grouping they made of their subcategories focusing on the self. Later, I can collect all their categories about mathematics education in general together, including about mathematical learning and problem solving, about mathematical teaching in general, about the role and the functioning of the teacher and of the students and about socio-mathematical norms.

Sub-category	Illustrative example
Self-efficacy beliefs	I am confident I can understand the most difficult material presented in the readings of this mathematical course
Control beliefs	if I study in appropriate ways, then I will be able to learn the material for this course
Task-value beliefs	it is important for me to learn the course material in this mathematics class
Goal-orientation beliefs	the most satisfying thing for me in this mathematics course is trying to understand the content as thoroughly as possible

Table 6: Op T'Eynde et al.'s (2002) subcategories of beliefs about the self in relation to mathematics, with illustrative examples

In the case of beliefs about self, Op T'Eynde et al. (2002) consider four sub-categories: self-efficacy, control, task-value and goal-orientation beliefs. They illustrate each of these sub-categories with the examples shown in Table 6, below.

The authors use examples which focus on beliefs regarding a specific mathematics course. Generalising the concepts, it is, however, possible to expand each sub-category to reflect a belief about the self in relation to mathematics in general, without doing away with the distinction between the sub-categories.

In the case of self-efficacy, the issue lies in the participant's belief in her/his own potential as a mathematical being. The second component, 'control' focuses on the individual's belief of whether it is in her/him that lies the capability to be successful in mathematics. This of course connects to the first component, concerning self-efficacy. In addition, it suggests that the power to control the success or failure of an engagement

with mathematics can be internal to the individual, or external in the form of the social context (such as for example, the teaching ability of the instructor). It could be argued, of course that these first two sub-categories fall under attitude, since they incorporate a value judgement, as self-efficacy, for example, could be expressed as a value judgement of one's own capabilities. I keep these in the belief-about-self category, however, because they can subjectively be seen, by the individual, as an objective trait of their own, or of the context, not a preference.

The third sub-category represents the belief in the importance of the subject. In Op T'Eynde et al.'s (2002) illustrative example, the sub-category expresses a belief of the importance of the content of the course to the individual. A generalization of the concept presents an evaluation of the importance of the subject. This could again be thought of as a value judgement, but one that can be subjectively believed to be objectively true: mathematics is/is not important as a subject. Generalising this statement, however, has removed its connection with the self, placing it instead into the category of beliefs about the subject, and thereby connecting it back to the earlier discussion on this topic. If the importance of the subject is evaluated in terms of the individual's personal preference, on the other hand, it becomes an attitude.

Finally, under the last of Op T'Eynde et al.'s (2002) sub-categories focusing on the self in relation to mathematics and based on the illustrative example, goal-orientation beliefs appear to focus on the individual's feelings of satisfaction with their engagement in mathematical tasks. This presents the sub-category as referring to attitudinal traits or preferences, suggesting its incorporation into the attitude category.

In summary, in the category of beliefs about the self in relation to mathematics, I include Op T'Eynde et al.'s (2002) self-efficacy and control beliefs with regards to mathematical tasks, while I reposition the perceived importance of the subject as a sub-category of beliefs about mathematics, and goal-orientation beliefs as an attitude. The apparent mis-placement of this last category as a belief instead of an attitude may stem from the context of publication of Op T'Eynde et al.'s (2002) article: it belongs to a collection of papers entitled "Beliefs: a Hidden Variable in Mathematics Education", and it is therefore likely that the conceptual frameworks developed throughout the text were done in inclusive ways, whilst leaving out what the authors considered unambiguously to be an attitude.

Beliefs about Mathematics Education

In Op T'Eynde et al.'s (2002) framework, the subcategories focusing on beliefs about education in mathematics include the following: (1) beliefs about mathematical learning and problem solving, (2) beliefs about mathematical teaching in general, (3) beliefs about social norms in their own class, including the role and the functioning of the teacher and of the students (4) beliefs about socio-mathematical norms in their own classrooms.

Overall, these beliefs focus on the experiences associated with the learning process. As such they involve beliefs about the nature of learning and the roles of the participants in this process, and by extension, they connect to the individual's perceptions of the nature of the subject of learning, as explained by Lindgren (1996):

> The basis for the birth of the beliefs about teaching mathematics can be seen to be the individuals' conception of the nature of mathematics. (p. 113)

I alluded to this connection earlier in this chapter, in the section on educational perspectives, where I contrast the transfer-of-knowledge, the Social Constructivist and the Situated Cognition views of learning. I also connect the beliefs about the nature of mathematics to the epistemological hierarchy I develop, associating 'conventional /observational knowledge' to Instrumentalism, etc. Thompson, in her framework for conceptualising beliefs about mathematics instruction, uses a similar structure. As reported by Lindgren (1996), Thompson (1991) assigns the level identifiers 0 to 2 to the ordered characterisations. At Level 0, where the instructional practice focuses on facts, rules, and procedures:

> the role of the teacher is perceived as a demonstrator, and the students' role is to imitate the demonstrated procedures and to practice them diligently. Obtaining accurate answers is viewed as the goal of mathematics instruction. Problem solving is viewed as getting right answers - usually using prescribed techniques - to "story problems". (p. 113)

Based on the description of the roles of the participants (both the teacher and the students), this level can be associated with the transfer-of-knowledge perspective. At Level 1, in contrast, Thompson broadens the concept of mathematics to incorporate some "understanding of principles behind the rules", though, as Lindgren says,

> rules still play a basic part of mathematics. The teaching of mathematics is characterised by an awareness of the use of instructional representations and manipulatives. However, the use of manipulatives is often regarded as useful for promoting the view that "math is fun". The role of the teacher IS perceived

> somewhat as in Level O. Views of the role of the students include some
> understanding. (p. 113-14)

Analogously to the case of Level 0, this association with understanding the rules can be linked back to the earlier description of 'applicational knowledge' and to Platonism. In this view, the student's role is to incorporate understanding of the reasons behind the rules, into her/his knowledge of mathematics, wherever possible.

At Level 2, finally, Thompson's framework connects with Ernest's (1989) 'problem solving' view. At this level,

> the conception of how mathematics should be taught is characterised by a view that the student himself must engage in mathematical investigation. The view of teaching for understanding grows out of engagement in the very processes of doing mathematics. The role of the teacher is to steer students' thinking in mathematically productive ways. The students are given opportunities to express their ideas and the teacher listens and assesses their reasoning (Thompson, 1991 as cited in Lindgren, 1996, p. 114).

This level connects back to the earlier discussion of the use of scaffolding in the Social Constructivist perspective, where the learner develops her/his own understanding of the topics under investigation.

In the earlier discussion on the hierarchy of beliefs about the nature of mathematics, I included an additional view, which I termed 'pattern analysis', to distinguish from 'problem solving', which I perceived as too goal-state-focused. To supplement Thompson's levels so as to incorporate this last view, a fourth level, Level 3 could be defined. In Lindgren's (1996) outline each level is described with respect to the roles of the participants. In my characterisation of 'pattern analysis', I specified the necessity for agency on the part of the learner, at each of the stages of mathematical work. Level 3 could therefore be characterised thus: in this view, the role of the teacher is to facilitate interaction between the learner and the object of learning, without stepping between them and taking over the process. In the description of Level 2, above, Lindgren cites the teacher as "steer[ing] students' thinking", thereby suggesting that s/he retains control of the process. That view reflects a position which approves of the use of the hints which Mason (1978) warned against (as I discuss in Alternate Educational Perspectives). In contrast, the view I define as Level 3 assigns value to the learning that relies on the agency of the learner.

In the present perspective, beliefs about mathematics education, parallel to beliefs about the nature of the subject can be classified into four levels: (0) as mathematics is a set of

(conventional/observational) 'facts, rules, and procedures', the role of the teacher is to demonstrate those, and the student is required to replicate them; (1) as the discipline of mathematics is also made of the understanding behind those rules, the teacher's role is the same, except that the understanding is also to be communicated, and the role of the students is also to develop some understanding; (2) as mathematics is about the processes of doing mathematics, teaching consists of steering the students along in this process, while learning consists of developing one's own sense of how mathematics can be done, in collaboration with the authority/expert; finally (3) as mathematics is the science of patterns, teaching consists of supporting students in the development of their own understanding of these patterns, including a development of their own sense of the structures underlying these patterns.

Affective Outcomes of Experience in Mathematics Enquiry, a Summary

Throughout this section of the literature review, I examined writings on affective issues linked to mathematics education. In a first stage, I introduced (1) emotions, which are considered to be hot, immediate, and linked to the situation at hand, and (2) attitudes which are cooler, and "may result from the automatizing of a repeated emotional reaction" (McLeod, 1987), suggesting that they develop through repeated exposure to a class of situations which the individual perceives as similar. In addition, following Skemp (1979), I characterised emotions as diminishing in intensity after prolonged exposure, on the part of the individual, to the type of situation. In the discussion about emotions, I cited Skemp as associating emotions with the movements between goal- and anti-goal-states. This perspective can incorporate emotional responses such as pleasure, unpleasure, fear, relief, confidence frustration, security, anxiety (Skemp, 1979), in addition to curiosity, puzzlement, bewilderment, despair, encouragement, elation and satisfaction (Goldin, 2000). McLeod's (1989) description therefore distinguishes emotions and attitudes by the fact that the former are associated with the immediate situation the individual finds her/himself in, and attitudes as the affective response to the class of situations which s/he perceives as similar. Responses such as confidence, anxiety or curiosity, therefore, could fall into either category, depending on the circumstances.

Both of these categories express an individual's personal value judgement about an aspect of mathematics, its relation to her/him, and its instruction. In contrast, beliefs were defined as subjective views of an objectifiable truth-value of statements about

mathematics, and its instruction. For example, the truth-value of a statement such as "mathematics is difficult" can be negotiated and agreed-on by a community of practice, whereas "I find mathematics difficult" is the expression of an individual's value judgement.

Within this category of beliefs, the central component regards the nature of mathematics itself, while other components of beliefs, including beliefs about its importance and difficulty, beliefs about the self in relation to mathematics and beliefs about its instruction, are derived from it. These beliefs are furthermore associated with the epistemological perspectives I delineate at the beginning of the chapter. To wit, an Instrumentalist view of mathematics suggests an epistemological perspective limited to conventional/observational knowledge and a transfer-of-knowledge view of instruction, assigning the student a role of memorisation and replication, and therefore promoting a self-image incorporating memorisation skill as the measure of confidence in mathematics.

Teacher Education as a Context for this Study

In an earlier section of this chapter, I stated that, based on the Situated Cognition perspective, the function of classroom teacher could be seen as incorporating the role of full participant in the community of practice, relative to the peripheral role of the learner. In addition, I qualified this role as genuine if the teacher brought, to the sample community that the classroom is, an authentic experience of the practice, in the absence of which the teacher could only be said to be a stand-in for the full participants. In this light, the aim of the study is therefore to provide, for the potential full participants in the sample community that student teachers are, the experience that would complete their existing experience in those terms, thus informs their affective response, including their view of mathematics as a discipline, as well as other factors.

Diamond (1991) classified teacher education programmes into four categories based on their goals: competency-based, personalistic, focused on language and learning and based on perspective transformation. The above description suggests an association with the last category, as the aim is to refine the participants' perspective with regards to mathematics and its methods of enquiry and instruction. In addition, Klein (2001) argues for an enabling as well as engaging experience:

> Enablement is important, because it is constitutive of one's developing subjectivity and it is this 'knowing about oneself' as potential pedagogue that will significantly influence future practice, rather than constructed mathematical knowledge alone. (p. 265)

The enablement she is evoking emerges from experiences that allow the development of a subjectivity, that is, a cluster of affective responses, which serves as a support for the 'genuine full participant persona' necessary to effective teaching. In Ernest's (1989) words:

> Mathematics teachers' beliefs have a powerful impact on the practice of teaching. [...] The autonomy of the mathematics teacher depends on all three factors: beliefs, social context, and level of thought. [...] Only by considering all three factors can we begin to do justice to the complex notion of the autonomous mathematics teacher. (p. 253)

The autonomy of the mathematics teacher to which Ernest refers supports a 'genuine full participant persona', which I have cited as necessary to effective teaching. In particular, for this study, the teacher's beliefs and perhaps her/his level of thought could be transformed by the experience I am proposing, although, as Nesbitt, Vacc and Bright (1999) explained:

> Preservice teachers' general beliefs about teaching are tenacious (Holt-Reynolds, 1992) as are their beliefs about teaching and learning mathematics (Ball, 1989; McDiarmid, 1990). Learning new theories and concepts may have little effect in changing preservice teachers' general beliefs about teaching practices (Calderhead & Robson, 1991; Kagan, 1992). Instead, preservice teachers' beliefs seem to be drawn from previous vivid episodes or events in their lives (Pajares, 1992); their beliefs about teaching and learning appear to be generalizations derived from their own experiences as students (Holt-Reynolds, 1992; Knowles & Holt-Reynolds, 1991). (p. 91-92)

If student teachers' beliefs about teaching are informed by their experiences as learners of mathematics, and not as learners of educational theory, albeit specifically about mathematics, then an experience such as the one I am proposing, which is meant to provide 'vivid episodes' focused on mathematical enquiry, could benefit them more than instruction about such experiences. In addition, it connects well with the proposition, made earlier, that beliefs about mathematics education are derived from beliefs about the discipline of mathematics itself, specifically about its nature. Finally, if, as Burton explained, an epistemologically correct (Wenger, 1998) experience of mathematics requires the practice on which it is based to incorporate agency on the part of the learners and the opportunity to actively, indeed critico-creatively take part in the negotiation of meaning, it follows that an experience such as the one I am proposing

could benefit the participant with respect to her/his future role as genuine full participant in the sample community.

An issue that presents itself for a successful perspective transformation based on an experience provided in the context of a teacher education programme, and indeed any experience that could be regarded as relevant to a teaching practice is what Ernest (1989) called the 'espoused-enacted' distinction. He sees this distinction as necessary,

> because case studies have shown that there can be a great disparity between a teacher's espoused and enacted models of teaching and learning mathematics (for example Cooney, 1985). (p. 252)

In effect, even if a teacher or student teacher experiences a transformation in perspective, as intended by the approach, this does not necessarily bring on a transformation in practice as the latter also derives from the social context and level of thought (Ernest, 1989, p. 253). In particular, according to Ernest (1989), 'the teacher's level of consciousness of his or her own beliefs, and the extent to which the teacher reflects on his or her practice of teaching mathematics' also play a role. These other two factors of the autonomy of the teacher, as Ernest calls them, are related to the teachers' practice in the field, whereas the present study focuses on their beliefs during their education prior to that.

Conclusion

In summary, the theoretical foundations on which this study is built consist of the following components:

1. The education perspective developed by Lave & Wenger (1991) according to which learning develops through a progressive shift from peripheral to full participation in the practice to which an individual is exposed;

2. an epistemological perspective on mathematics that encapsulates three levels of knowing, including the traditional ones, referred to here as 'knowing-that/how' and 'knowing-why', and the additional, higher level one referred to as 'knowing-when', which focuses on the structure underlying mathematics and which helps develop the why;

3. the existence of a fundamental difference between mathematical enquiry as practised by research mathematicians and mathematical problem solving as practised in the mathematics classroom, which manifests itself through (a) the roles ascribed to the

various participants, (b) the focus of the social contract that connects them and (c) the nature and experience of the two practices;

4. the possibility of designing a teaching approach which can provide an experience emulating that of research mathematicians, in a teacher education context;

5. a theoretical perspective on affective responses to mathematics that hypothesizes a connection between experiences of mathematics, beliefs about its nature, and attitudinal responses to it; and

6. the idea that the change in beliefs and attitudes that can emerge from such an additional experience can influence the practice of future (and possibly current) teachers.

Chapter 3: Research Methodology

The central hypothesis of this study is that experiencing and reflecting on a practice of mathematical enquiry that is authentic with respect to that of the community of researchers in mathematics can transform the affective responses of a participant, giving her/him a broader and therefore more meaningful view of mathematics as a discipline and of her/himself as a mathematical individual, ultimately transforming her/his own practice as a teacher of mathematics. The aim of my project is therefore to assess a teacher education practice, based on an investigation of practice in mathematical enquiry, for its potential affective outcomes. Decomposing this statement reveals a complex undertaking. To begin, there are three principal components of the assessment:

- What could be the design criteria of a teaching approach which aims to provide an experience of practice analogous to that of research mathematicians? Which of these criteria are feasible in the given context?

- Can the experience of the resulting teaching approach successfully simulate that of the practice on which it is based, according to the design criteria?

- What are the affective outcomes, and are they as anticipated? (See Chapter 1)

Responses to the first item constitute the first two steps of the study, the establishment of the design criteria (see Mathematical Enquiry (ME) as distinguished from Mathematical Problem Solving in the Classroom (MPSC) in Chapter 2) and the design and implementation of the teaching approach based on these criteria and others imposed by the wider context (see Chapter 4: The Teaching Approach). The next two questions essentially form an assessment of the resulting teaching approach in terms of the design criteria and of the affective framework established in Chapter 2, with the aim of adjusting any future practice of the teaching approach.

Though there are three components, the project as a whole remains the assessment of a practice in terms of its purpose. In this respect and because the assessment can provide new insights into future practice, the project places itself within a cycle of refinement of a practice. In addition, the intention of the teaching practice includes not only a

broadening of vision on the part of the participants, but connected to it, an emancipation of their mathematical selves, developed through access to an alternate experience of mathematics as a discipline. Whether this can be achieved through the designed teaching approach is in fact a central question of this study.

The potential change of perspective and/or of practice on the part of the researcher-, teacher-, and teaching assistant-participant suggests that this project places itself in the action research tradition. Defined as…

> a form of self-reflective enquiry undertaken by participants […] in social (including educational) situations in order to improve […] their understanding of these practices […] (Carr and Kemmis, 1986),

… action research very much depends on active engagement in self-reflection on the part of the participants whose practice is being investigated. In this form of research, the participants whose practice is under scrutiny are expected to self-reflect, that is, to rethink their actions and the context of these actions. Action research, therefore, prescribes reflection at these levels of participation. In addition, the purpose of this self-reflection, in the context of the action research, is to transform (future practice), and connects, therefore, to Bruner's comment, that:

> Learning should not only take us somewhere; it should allow us later to go further more easily. (Bruner, 1960, p. 17)

In the wider context of the research, the researcher's and the teacher's self-reflection on practice contribute to the assessment of the reported experience of the participants (compared to the intended experience), and of the predicted affective outcomes and their value, and by extension, to the re-design of subsequent practice.

Action research is often described as cyclical or spiral-shaped, and constituted of the following stages:

> Planning a change, acting and observing the process and consequences of the change, reflecting on these processes and consequences, and then replanning, acting and observing, reflecting and so on (Kemmis & McTaggart, 2000, p. 595)

In the broader context of my research, this study constitutes a first cycle and thus includes the planning, acting and observing, and reflecting stages, and begins with the initial observation of an educational practice. The stages of this cycle are reflected in the three questions the study is attempting to answer and these are distinct not only in the stage of action research they address, but also in their focus, and consequently in their

methods. As a result, the overall research methodology is also constituted of three components.

In the context of this study, each of the four stages described by Kemmis and McTaggart can be connected with a specific stage of the process described in this thesis, and with the specific questions it attempts to answer. In Figure 1, below, I show these stages, including the component that each addresses, and the section or chapter that describes them in detail. After this first cycle, which includes an initial observation/exploration of the problem (see below), the research programme can continue to cycle through the latter three stages, as shown by the yellow curved arrows.

Initial Observation/Exploration of Problem
Discrepancy between the affective responses to mathematics of professional researchers and that which research has found to be prevalent in the general public. Hypothesis: this is caused by the difference in experiences of the practice of mathematics of the two groups. (Chapter 1, part of Chapter 2)

Plan: Question 1
What could be the design criteria of a teaching approach which aims to provide an experience of practice analogous to that of research mathematicians? Which of these criteria are feasible in the given context? (Chapter 2, part of Chapter 4)

Implementation/Observation: Question 2
Can the experience of the resulting teaching approach successfully simulate that of the practice on which it is based, according to the design criteria? (Part of Chapters 4, Chapter 6)

Reflection: Question 3
What are the affective outcomes, and are they as anticipated? How could they be further improved? (Chapters 7, 8, 9)

Figure 1   The action research cycle, as applied to the present study

The starting point of the study is an interrogation about the apparent discrepancy between widely held affective responses to mathematics and mine (and those of other mathematics researchers with whom I've interacted). This discrepancy elicits a questioning as to the origin of these views (see Chapter 1), and constitutes the first stage: the initial observation/exploration of the problem. As Carr & Kemmis put it:

> […] Action research involves relating practices and understandings and situations to one another. It involves discovering correspondences and non-correspondences

> between understandings and practices […], between practices and situations […], and between understandings and situations. (Carr & Kemmis, 1983, p. 182)

In this case, the 'understandings' that are being examined concern the concept of mathematics as a discipline, its practice, and the experience of this practice as well as the way in which these are perceived in the educations context, which corresponds to the 'situations' under scrutiny. The 'practices' are those associated with classroom teaching, particularly with respect to problem solving and those associated with mathematical enquiry (see Chapter 2).

The implementation of this stage differs from the conventional starting point of educational action research in which:

> The objects of action research—the things that action researchers research and that they aim to improve—are their own educational practices, their understanding of these practices, and the situations in which they practice. (Carr & Kemmis, 1983, p. 180)

In the present case, my role in the initially observed practice is that of 'product' of what is deemed a successful 'action'. Instead of a more traditional situation where I would critically observe my existing action as teacher, in order to refine it, I investigate the theoretical foundation of an existing, wide-spread practice, and analyse it for what it does not provide, by comparing it to what I consider a successful one, with the ultimate goal of developing an improved teaching practice, according to my criteria.

Hypothesising that an individual's overall affective responses to mathematics are significantly influenced by her/his personal experience with the subject, I investigate the difference between the experience that I and many professional researchers in mathematics have, and that generally provided by the education system. This investigation yields a theoretical framework (described in the first part of Chapter 2), which focuses on epistemological questions and leads to the next step, which is encapsulated by Question 1:

- What could be the design criteria of a teaching approach which aims to provide an experience of practice analogous to that of research mathematicians? Which of these criteria are feasible in the given context?

To answer this question, I examine the literature about mathematics as the subject of school activities such as problem solving, and mathematics as the subject of the practice of professional researchers. For this purpose, I position myself in the community of professional researchers as I associate my own practice to theirs. The comparison,

focusing on various aspects of the practice, reveals that a significant difference between the two activities seems to lie in the social contexts within which they take place. These social contexts are particularly distinct in the roles they ascribe to the participants, together with considerations of the epistemology of mathematics and the typology of the activities in either practice and this opens the way for a framing of the design criteria that constitute the answer to question 1, and concludes the first step of the planning stage of the action research approach (this is articulated in the second part of Chapter 2). At this stage, my role changes from product of action to originator of action. I am now taking active part in the practice, by articulating the intentions of the practice into a set of criteria for the design of the intervention.

The next step is to plan an implementation of these design criteria in the form of a teaching module, with the purpose of assessing it in terms relating to the hypothesis. This plan is generated through collaboration and negotiation with the participant who is implementing it. The interaction can be termed collaborative because the contributions of the two participants are equitable though different: I contribute the knowledge of the theoretical framework, he manages the practical considerations (see description in Chapter 4) and together we refine the approach itself. The interaction gains:

> meaning and significance […] by virtue of its being understood […] by actors themselves as social agents, by the people with whom they interact, or by scientific observers. (Carr & Kemmis, 1983, p. 181)

In this case, I play the role of 'scientific observer' through the development of the theoretical underpinnings, and of 'actor as social agent' through my contribution to the implementation (for example through the journal feedback), while my collaborator acts as 'actor as social agent'. In this respect, and because our collaboration has at this stage reached a degree of maturity (we have worked together since 1999), the collaboration amounts to the pursuit of the common goal of a worthwhile teaching practice, which is ensured by a continued, intense interaction during the intervention. From this point on, I refer to my collaborator as the teacher-participant.

The implementation/observation stage of the project follows, during which the planned teaching strategies are put into practice and adjusted as the need arises. This is also the stage during which data is collected both in terms of the correspondence of the experience with the exemplar, which has been established in the planning phase, and in terms of the impact of said experience. The first of these terms is necessary in order to

verify that the participants' experience corresponds to the intended one. The second term focuses on an answer to the hypothesis, according to which this experience can have an effect on the participants, particularly their affective responses. This step, again, differs from the conventional one in that I do not directly execute the plan. Rather, due to circumstance surrounding the selection of participants, the teaching approach is implemented by my collaborator, who is in agreement with the intentions and plan for the intervention. My role during this stage is one of scientific observer, though, following the action research tradition, we also continue to renegotiate the practice throughout the implementation.

The reflection stage focuses on an examination of the results of the two components of the study and includes an assessment of these results with respect to the aims of the intervention. This stage prepares the way for a new planning cycle, which leads to a new implementation, which is then assessed, etc. During this stage, my role within the action research approach is the central one: I assess the practice in terms of the theoretical framework and examine the results in terms of the intentions, preparing for additional cycles of action research.

Methodologies

At the initial observation and planning stage of the study, the methodology focuses on a combination of theoretical findings, which emerge both from the relevant literature and reflection on my own experience, and from practical considerations such as those imposed by the social context within which the study takes place. The theoretical foundations of the planning stage are developed through a comparative review of two bodies of literature, that focusing on problem solving on the mathematics classroom and that focusing on mathematicians' professional practice. This review is informed by my collaborator's and my own experiences with both practices, and by considerations in the philosophy of mathematics, and its results are presented in the relevant sections of Chapter 2.

The practical foundations of the planning stage also include ethical considerations regarding the intervention such as the necessity of preserving the integrity of the course module with respect to the wider degree programme to which it belonged. These considerations and their bearing on the implementation are described in Chapter 4.

Within the action and observation stage of the study, the focus of the methodology is on both the second and third question. To answer these questions, two separate assessment processes take place: (1) a qualitative assessment of the experience of the participating students, through the analysis of participants' reflections recorded during the intervention, and (2) the measure of the participants' change of affective perspective (or lack thereof), through a comparative analysis of responses to pre- and post-module questionnaires. This last assessment is directed at the initial hypothesis, that the experience provided by the intervention can transform affective responses. Despite the lack of control group, the validity of the findings of this particular component is supported by several contextual factors. Firstly, the module is given in an academic context within which the participants follow distinct schedules and do not therefore share extensive experiences outside this module. In addition, the methods implemented to answer the second question focus on the participants' experience in this specific module and the responses can therefore be used to verify the connection between the findings of the questionnaire analysis and the intervention itself. Finally, the post-module questionnaire itself includes items that address explicitly the participants' experiences *in* the intervention.

In terms of their methods, these components of the study are of course interconnected, particularly with respect to the data collection since I do not collect data for them sequentially, but concurrently. They are also connected to the implementation of the teaching approach. Consequently the responses to each of the instruments are likely to be impacted by exposure to other instruments. For this reason, though I describe the respective methodologies separately, I describe all the data collection events chronologically rather than by research component. In the case of the analysis, in contrast, I discuss each part of the study in turn, because of the divergence of their nature and the questions they answer.

The reflection stage, finally, consists of an evaluation of the outcomes in terms of the intentions of the teaching practice. In particular, the affective outcomes are measured in order to evaluate the worth of the approach in terms of its goals. In addition, the experience that the participants report to have had is compared to the exemplar and recommendations are made to refine the approach for future cycles of action research.

Assessing the Teaching Approach for Authenticity

On the basis of my own history in mathematical enquiry and the literature on the work of mathematics researchers, I focus the assessment of the authenticity of the teaching approach on the students' reports of their experience[17] in the class. This decision stems from the theoretical perspective, delineated in the literature review, which suggests that it is the authenticity of an experience, with respect to that of full participants in the community of practice, which contributes to the appropriateness of the affective outcomes. In essence, I use the journals to capture the transitory affective responses of the students, and then analyse them to assess the reported experience in terms of the model of full participant engagement established in the literature review.

As this experience is defined as that had while the situation is in place, the data sought needs to be generated during the experience, on a continuous basis and with as much immediacy as possible, suggesting an autonomous form of data generation. This is further supported by the theoretical perspective which I use to ascertain this authenticity, that is, the findings about Hadamard's (1945) stages of mathematical discovery and Mason's (1978) energy states, which are by definition transient. These can be used to interpret the students' self-reported affective states during the experience. As such, they are mainly represented by an individual's emotional states, described by Lester et al. (1989) as "generally accepted [as being] traits, albeit perhaps transient ones, of the individual" (See Later Developments in Chapter 2). In addition, in order both to help reify the experience of the participants, and to enhance its anticipated effect, a methodology promoting self-reflection is advisable.

In this part of the study, therefore, the choice of methodology is made based on the following criteria: the form of the data is continuous yet immediate in nature, and its production is individualised and autonomous though a tool is built into the process that encourages continued application. These criteria suggest the use of journals on the part of the participants, with a periodical mechanism of response to the entries to both clarify the observations and motivate continued reflection. This last condition is linked to the aim of the intervention in that, from a pedagogical perspective, the journals are also expected to provide, for the participants, a platform for self-reflection regarding their experience in a mathematics classroom context, from the learner's point of view. In

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[17] 'Refers to how the individual feels, both in a sensory and emotional sense and what s/he thinks *while* s/he is experiencing the situation(s); the experience s/he *has* during the situation. (see Chapter 1)

effect, the students are encouraged to think of these journals as opportunities to 'speak to their future teaching selves'. The journals are therefore designed to serve a cathartic as well as a data generation purpose (Lincoln & Guba, 1985).

As this component of the study is comparative in nature, I disconnect the analysis from the interpretation, only bringing in the comparative aspect and the nature of the exemplar at the interpretation stage. In the analysis, the codification emerges from the data. The focus of the analysis, however, remains on the student-participants' affective responses, including particularly their emotions (Lester et al., 1989). In addition, the phase of the teaching approach, as well as the object of the affective response, is considered as they are a critical component of the response.

The interpretation of the data, on the other hand, takes the form of a qualitative comparison of the results of the analysis against the exemplar emerging from the literature. Specifically, it focuses on two aspects: (1) a parallel is drawn between the emotional content of the descriptions given in their journals by the responding student teachers, and the emotional states associated with Hadamard's (1945) stages and Mason's (1978) 'energy states', allowing for a reconstruction of the student teachers' process in terms of the exemplar stages; and (2) evidence of the responding student teachers' agency (Burton, 2004) in the practice of mathematical enquiry is sought across the range of their responses. These two aspects are combined to provide an indication of the authenticity of their overall experience, from an affective perspective.

In addition to the data generated in the participants' journals, a few questions in the post-module questionnaire also focus on the experience they had during the intervention (see Items 2-4, 6-9 and 40-44, in Appendix 4: Post-Module Questionnaire. These include a comparative item rating the three principal phases of the module, as well as a few open-ended questions and a yes/no item.

Assessing the Affective Outcomes

In assessing the outcomes of an intervention, it is useful to compare measurements taken before and after the event, in order to measure outcomes against 'baseline scores' (AllPsychOnline, n.d.). In this case, the focus is on the affective responses to mathematics of the student-participants. In the literature review, I discussed three types of affective outcomes: emotions, which are immediate, hot, connected to specific situations, and have a valence; attitudes, which also have valence but are cooler and

more general; and beliefs, which "constitute the individual's subjective knowledge about self, mathematics, […]" (Lester et al., 1989, p. 77). The fact that emotions are connected to immediacy precludes them from being used effectively to measure a change due to the intervention, unless they are measured during a situation that would elicit them, which is not the case here. Attitudes and beliefs do not have such a limitation, and the questionnaires used to assess the outcomes therefore focus on these last two components.

The questionnaires are designed using a combination of items using different formats. Multiple choice items are used mostly in the sections measuring demographics, which can then be analysed as interval or nominal data. Short answer items are used to investigate the participants' responses to their experience both in the past and during the intervention in a way that is open-ended enough for the variety of their perspectives to come through. I create the items belonging to these first two categories based on the aspects of the experience which are relevant to the research questions and context, including a sense of the sample of population, both in general terms and in terms of their experience with mathematics, and the participants' perception of the intervention.

In addition, I call descriptive items the four items in each questionnaire (they are the same both times), in which the participants are asked to choose three descriptors, out of a list, corresponding to a criterion given in the question. These items are adapted from an article entitled 'Teachers' Definition of Math: Creating and Implementing an Instrument" (Pachnowski, 1987), and can be used to compare participants' views before and after the intervention.

Finally, a series of 25 Likert items, focusing on the views and attitudes of the participants, is composed of items taken from a list which has been compiled and provided by my thesis supervisor of the time, in addition to a few more, which I added. These items are, again, the same in both questionnaires, and correspond to six subscales emerging from the literature review, two measuring attitudes, and the remaining measuring views of mathematics.

The pre-module questionnaire is piloted on a comparable test population, as I describe in a later section, and the resulting, modified version is used, with a few modifications, before and after the intervention. The modifications mainly entail the removal of

demographic items (which do not change) and their replacement with items focusing on the participants' apprehension of their experience during the intervention.

The participant sample, described in detail in Chapter 5, is taken by opportunity in that it is constituted of the students registered in the module given by the teacher-participant who is collaborating with me. The sample is taken from a population of elementary student teachers, and this is taken into account in the theoretical framework, but it cannot be considered representative of a general population of such students due to the sampling method. The results I find with regards to their affective outcomes are therefore used to understand the possibility of a change in view, rather than to affirm a transferable, generalisable result. In addition, the use of a control group, which is impractical given the circumstances in any case, is not especially useful for the same reason. Instead, the post-module questionnaire includes several open-ended items that explicitly refer to the impact of the intervention, asking the participant to describe it in their own words, thereby giving it credibility. Reliability for the results is also verifiable through the comparison of the results of the questionnaire component with those of the journal analysis: in addition to answering the second research question, it can also provide a sense of the authenticity of the responses to the questionnaires. I discuss this issue of reliability in more detail in Chapter 8.

The operational hypothesis for this intervention is that it does have an effect on the participants' views and attitudes, though this can be mitigated by properties of the measured outcome, as I discuss in the appropriate section of the literature review.

Ethics Considerations in the Overall Design of the Methodology

The design of the intervention as well as the research that I conduct on it involves several considerations that can be discussed under the umbrella of ethics. Given that a significant part of the data collected concerns the participants' affective responses to the intervention, an ethics issue exists concerning the possible conflict of interest of the participants who are awarding the mark for the module, if they have access to the sensitive data. In order to pre-empt this issue, the data collected during the course of the intervention that is not used for the mark is withheld from the teaching team.

This aspect of the data collection is also explained to all the participants, not only because of ethics considerations but also in order to ensure the reliability of the data collected. This is especially relevant in the case of the journals, which are meant to

record the feelings and attitudes of the participants during the interventions, and which should be censured as little as possible. In summary, the teacher- and teaching assistant-participants only have access to the data collected which is connected to the marking scheme (see Appendix 6). As researcher-participant, I have access to all data. In order to ensure this is the case, the sensitive data that is collected is kept in a special locked box.

In order to ensure anonymity of the participants, in addition, I use only pseudonyms in all writing about the study, with the exception of my own name. The real names are used in the questionnaire data collection, in order to collate the data (see Appendices 3 and 4), though the questionnaire does stipulate that anonymity is to be preserved. This includes the teacher (to whom I refer as Dr. Zachary) and the teaching assistant-participant (to whom I refer as Alan). Though there is a risk of people closest to the research context seeing through the pseudonyms, this is not a significant problem as the intervention dates back several years, there were several groups of students taking the module, only one of which participated in the study, and the participants have since been dispersed.

The risk to the participating students can be considered minimal as most of the official content of the course normally constitutes a review of previously seen material and permission by the department(s) has been granted. In addition, there is a portion of the semester for 'catching up with the curriculum' (see Chapter 4). Participants that considered this to be an issue are informed that the researcher is available for discussion. In addition, the students can opt out of participation.

Time taken away from the teaching approach for the purpose of data collection is kept to a minimum, and takes place mostly in the first and last sessions.

Methods

Both the planning and the action and observation stages of the action research cycle represented in this study require the application of research methods beyond reviews of literature and introspection on the part of the researcher. In addition to the literature review and teaching approach design, the planning stage of the study also includes a small pilot study focused on the content of the questionnaires that I use to assess the change in affective responses. As part of the action and observation stage of the study,

data is collected, analysed and interpreted towards the formulation of an answer to each of the research questions. Each of these components is described here in turn.

Pilot Study

Due to circumstances surrounding the project such as the time constraints, the format of the doctorate programme within which this study takes place, and the design of the study itself, only a small portion of the research design is piloted, specifically, the pre-module questionnaire (see Appendix 3), leaving most of the teaching and research methods to be tested in the main study. This pilot study is performed with the participation of an opportunity sample taken from a comparable population, the elementary student teachers at the institution where I am postulating for a doctorate degree. Reasons for this choice include accessibility and comparability to the anticipated population of the main study: both groups are pursuing qualifications for teaching positions in elementary education, which includes the subject of mathematics. In one of the sessions near the end of its programme, the pilot sample is kind enough to fill in the planned questionnaire and in particular, to answer the following question:

> Please use this section to comment on the questionnaire in general, or any question in particular. This will help to fine tune the questionnaire.

The responses to the questionnaire as a whole and the last question (above) in particular allow a revision, though not a substantial shortening of the questionnaire. Section 1, which focuses on the demographic data, is expanded to provide a more complete picture of the sample, including the year of higher education they are currently pursuing (this is necessary as the modular nature of the degree programme means that the participants can be at different stages).

Section 2, which focuses on the participants' past experience with 'open-ended, investigative lessons', is shortened from six to three general items differentiated by schooling stages, and the wording is altered. This change is due to a response indicating the difficulty of remembering these experiences in that level of detail.

In Section 3, items 11 and 13 involve choosing five items from a list of twenty-three that described mathematics. In the final questionnaire, the questions are changed to a choice of three terms from a list of fourteen, thereby making the response to these items easier.

In the case of the Likert Scale items of Section 4, an analysis of the items reveals that the section cannot be reduced from the original 25 without losing information about the students' attitudes and beliefs. This analysis mainly focuses on the component of the theoretical framework that is relevant to each item and allows for the breakdown of Table 7, below:

Item type	Example	Direction	Items
Like / dislike	I find solving mathematics problems to be dull and boring	Negative	2
Self-confidence	Mathematics is a subject I find easy	Positive	2
Anxiety	I am anxious about teaching math	Negative	0
Perceived utility	Mathematics is as important as any other subject (Philippou & Constantinos, 1998, p. 197)	Positive	0
Instrumentalist view	Someone who is good at mathematics never makes a mistake	Positive	11
Platonist	Mathematics consists of a set of fixed, everlasting truths	Positive	5
Problem-solving view	Only gifted professional mathematicians can be creative in mathematics	Positive	4
Pattern Analysis	Investigating a puzzle can lead to significant new mathematics	Positive	8

Table 7: Examples of Likert items for some affective categories about mathematics

The first four Likert items focus on attitudes of the participants to mathematics, and correspond to two separate variables: like/dislike, and self-confidence. Having two items addressing the same 'variable' can serve as an internal reliability test for the responses given.

The remaining Likert items (19-39 in the updated questionnaire) focus on views of the nature of mathematics, as discussed in the literature review. Breaking these down into the view(s) that each addresses shown the breakdown of table 7, above. The fact that the numbers of items for each category do not add up to 21 is because some items address more than one view. For example, "Exploring number patterns is not real mathematics" is both a positive statement for Instrumentalism and a negative one for Pattern Analysis. This breakdown further shows that only a small number of items corresponds to each category, the highest being Instrumentalist, with 11. It is important to keep a reasonable number in, so as to allow for the verification of reliability of the responses.

A question also arises as to the reliability of responses to items expressed as negative statements, with the response choices being: 'YES! / yes / ?? / no / NO!'. An example of this issue is item 36: 'A person should not mind risking a mistake when trying to solve mathematics problem'. In the end, the wording of all the items is retained, as the response scale is clearly indicated at the top of the section, as follows:

- YES! STRONGLY AGREE
- yes   AGREE
- ??   NEUTRAL or UNDECIDED
- no   DISAGREE
- NO! STRONGLY DISAGREE

Overall, the terminology is modified in order to adapt to the language of the responding participants, as the pilot study was done in the United Kingdom, and the main study took place in North America. Expressions such as 'doing your sums' come to mind. The pilot study mainly contributes to the refinement of the questionnaire, and is consequently not as helpful in developing the research design as could be the case, particularly as it focuses on only one aspect of the study.

Sample Selection

In the context of the main study, the participants are selected by opportunity, as the intervention is designed in cooperation with the teacher, who is a member of the teaching body in the host institution. The modular nature of the degree programme, and indeed of the institution as a whole, however, provides a random element to the selection of the sample. Although the module is intended for students aiming for a certification in elementary education, via the completion of a Bachelor of Education[18], it is open to all students, from those who have not yet 'selected a major', to graduate students pursuing their teaching certification through the available Masters' degree. In addition, the modular nature of the degree programme leaves the choice open to the students of when they take this module, both in terms of the weekly schedule, and in terms of the year of study. These circumstances suggest that a variety of profiles are built into the sample, justifying the need for a more extensive collection of demographic data on the sample (see Chapter 5 for a detailed description of the sample).

Data Collection

As I discuss in an earlier section, though the various data collection methods correspond to different aspects of the study, they are interconnected, both because the students are aware of the collection, and because the data address connected topics. Consequently, the different elements of the study are combined in a way to maximise both their impact on the aim of the intervention, and their reliability. The main criterion for organising the various components, aside from their purpose for the study, is therefore their chronological position, in relation to the progression of the intervention. For example,

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[18] B.S.Ed. (Foundations), which I describe briefly in Chapter 4.

the pre-module survey, by definition, is done as close as possible to the beginning of the intervention, making it an event occurring at a discrete point early in the time line. Similarly, the post-module survey takes place as close as possible to the end of the intervention. The student journals, on the other hand, are continuous and immediate, with regular interaction with me in the form of collection and written feedback. They are therefore spread out over a continuous stretch of the time line.

The module as a whole is subdivided into four phases (see Chapter 4). The first phase, which takes place from the beginning of classes on September 3rd until October6th, is designed to initiate the participants into the working mode of mathematical enquiry. In the second phase, lasting from October 8th to 29th the participants are meant to engage in authentic mathematical enquiry. The third phase is used to address the course content as it is normally taught, and lasts until November twenty-sixth. Finally, the assessment phase takes place in the first two weeks of December. Based on these criteria, the various methods can be organised both chronologically and logically (see Figure 2, below).

Phase	Mini-projects											Main project							Regular teaching								Assess.			
Dates: (month)	09	09	09	09	09	09	09	09	09	10	10	10	10	10	10	10	10	10	11	11	11	11	11	11	11	11	12	12	12	12
(day)	01	03	08	10	15	17	22	24	29	01	06	08	13	15	20	22	27	29	03	05	10	12	17	19	24	26	01	03	08	10
Pre- and post-module surveys	✘																										✘			
Whole class discussion	✘																													
Journal distribution		✘																												
Journal collection			✘	✘	✘	✘						✘		✘					✘	✘	✘	✘								✘
Recorded presentations																												✘	✘	
Portfolio collection																														✘

(Thanksgiving class: optional, at 11/26)

Figure 2   Timeline of module with student data collection events[19]

In the first class, I collect data through the pre-module survey and a recorded discussion which I have with the whole class about what they think a mathematician does. This conversation takes place after the questionnaires have been returned so that the responses to the latter are individual, and is conducted mainly in order to set the tone

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[19] see Chapter 4 for a description of the phases.

and kick-start the student's reflections about their affective responses to mathematics and its practice. It also takes place without the teacher-participant, in order to clearly separate the two purposes, as I discuss in the section on ethics considerations, above.

In the second class, I distribute the journals. As I discuss in the section on methodologies, the purpose of the journals is to record the participants' experience of the practice with which they engage during the intervention. This choice of method fits well with the intentions of this component of the study in several ways. In addition to being a more continuous and immediate instrument than either interviews or questionnaires, which serves the purpose of the study, the journals provide the participants with a platform for an internal dialog about mathematics which can:

> uniquely contribute to the learning process because of a combination of attributes: writing can engage all students actively in the deliberate structuring of meaning; it allows learners to go at their own pace; and it provides unique feedback, since writers can immediately read the product of their own thinking on paper (Borasi & Rose, 1989, p. 348)

In this respect, the careful design of the journal writing guidelines can steer the participants towards a more purposeful use of the strategy:

> Overall, I found that giving prompts for the journal entries acquired a more positive response from the students than the open-ended reflections did in the first semester... (Liebars, 1997, p. 2)

The guidelines in this case, which can be found in Appendix 5, are designed so as to leave the field open to the participants about their reflections, as Liebars explains, without being so vague as to discourage engagement in the writing.

During the term, I collect a selection of these at more or less regular intervals, recording the entries and responding to them. In addition to the self-engendered feedback described by Borasi and Rose (1989), my collecting and responding to the journals provides an additional means of communication, with an external agent, through regular interaction, thereby giving acknowledgement and value to their reflections. As Liebars describes it, in this kind of strategy:

> Some students [can find] the journals to be an excellent source of communication between them and me. I did write comments and answer questions where appropriate and I think the students appreciated this and found it a way to get questions answered that perhaps they felt intimidated to ask in class. (Liebars, 1997, p. 5)

In the second to last week of class, I collect the students' responses to the post-module survey, and in the following two classes, I record the students' project presentations,

using video. Finally, in the last class, the students submit their journals and their portfolios, which in some cases include re-worked homework. These are all recorded electronically. I also take digital pictures of all submitted work, both in written form and constructed models.

The above data collection methods are all student-centred. In addition, I also collect data from the other participants, for the purpose of confirmation. These include observations I make in almost every session that does not involve my direct interaction with the students as well as recall notes in the alternate situation. Both members of the teaching team also keep their own journal, which I collect at the end. After most sessions, I interview the teacher-participant, and, a few times, the teaching assistant-participant. I find it important to record both parties' recollections of their interactions with the groups during the project phase, since this is a delicate matter with regards to participant agency and a central part of the teaching approach.

Finally, in one case, an external observer, in the form of a lecturer in mathematics education, sits in on the class, after which I interview him. Though initially the plan is for the external observer to attend a session and be interviewed in each phase of the module, circumstances mean that only the first observation was accompanied by a recorded follow-up interview.

It is important here to recall that the main part of this study centres on the students' experience, based on data that originates from them. All the other forms of data collection contribute more towards the contextualisation and triangulation of the intervention. In addition, the circumstances of the study mean that I only have access to the participants, with the exception of the teacher-participant, for the actual duration of the term, precluding the use of cycles of feedback beyond the intervention.

The Instruments

The first and almost last methods of data collection that I use for this study are the pre- and post-module surveys (see Appendices 3 and 4). The first one of these is administered, with a few exceptions, in the first session, that is, on September 3rd. The exceptions are due to the switching around between sections of a module which are de rigueur in the first weeks of a term in an American university, due to the modular and interchangeable nature of the modules. Of the six students who do not participate in the first session, four submit the questionnaire in the two weeks following, and the

remaining two do not participate in this aspect of the study. With the exception of three students who do so two days later, all students fill in the post-module questionnaire on the 1st of December.

These instruments themselves are modified versions of the one piloted in the United Kingdom (see the earlier section on the Pilot Study). The questionnaires are both subdivided into five sections. Distinctions between these sections fall into two main categories: the type of data collected, demographic data (section 1) versus data related to the study questions (sections 2-5); and the type of items, with section 2: multiple choice or short answers, section 3: selection of 3 best words or expressions, section 4: Likert items and section 5: short answers.

The questions of section 1 are used mainly to develop a profile of the sample, without particular intentions of comparison between types such as gender, age group, etc, as the sample (37) is too small. In the post-module questionnaire, section 1, which focuses on demographic data is reduced to the statement of their name for the purpose of collating the two data sets. Starting with the data transcription, in all subsequent stages of this study, these names are replaced by pseudonyms taken from a list generated in advance.

Section two, again, contains items that are different in the two cases. In the pre-module questionnaire, the items relate to the participants' past experience in the mathematics classroom, whereas in the post-module questionnaire, the section contains short answer or ranking items asking the participants to describe their overall experience in the intervention. The items from this section are developed based on the theoretical framework, according to which experience with a specific practice plays into the participants' views of mathematics. In consequence, in the pre-module questionnaires, these items focus on the participants' previous experiences similar to that planned for the study in order to establish a basis of comparison. In the post-module questionnaire, this section focuses on the participants' overall perception of their experience in the module.

Section three is the same in both questionnaires. Borrowed from "Teachers' Definition of Math: Creating and Implementing an Instrument" (Pachnowski, 1997) and modified based on the results of the pilot study, the items in this section require the participants to choose from lists the terms they feel describe mathematics the best, and in another question the least. The responses to these items form part of the comparative component

of the study, in which the group's responses before and after the intervention are compared.

In section four, the students are asked to respond to a series of Likert items constituted of a statement accompanied by five possible responses: YES! (strongly agree), yes (agree), ?? (neutral or undecided), no (disagree), or NO! (disagree strongly). The items in this section of both questionnaires are the same and the two sets of responses are compared.

In the final section, the students are required to give short answers to different open-ended questions in each questionnaire. Examples of these include:

- If you had to explain to one of your future pupils what a mathematician does, what would you say?" (pre-module)
- Please explain how you think your project fits into the context of your view of mathematics" (post-module).

In addition, in the post-module questionnaire, the last item relates back to the discussion I have with the whole class about mathematicians' work. In the first class, after the students have filled in the pre-module questionnaire, we spend about twenty minutes on a whole-class discussion regarding what the students think that a mathematician does. The debate begins with my addressing the group as a whole. As I do not yet know the students, and they have not formed any relationship with me (though perhaps with each other through other modules in the degree programme), the beginning is a little bit slow. Soon, however, responses such as "They solve problems using numbers" are given. Throughout the dialogue, I record their comments on the blackboard. This turns out to be very useful as the audio recording is of poor quality, and mostly my own voice is heard. As I usually repeat what the student says while I write it on the board, it is nevertheless recorded, and the digital photograph of the board is also preserved. Though this only provides a partial record of the discussion, it is enough to produce an additional item in the post-module questionnaire. In Item 44, indeed, I ask the students to circle the part of the diagram (reproduced from the discussion data) that they think corresponds most closely to the scope of their mathematical enquiry project (see Appendix 4). As I feel it to be a definite risk, for the Post-Module questionnaire, I ask the students not to try to remember their choices in the Pre-Module questionnaire, but to think about their current way of thinking.

Originally, the plan is that I would have discussions with smaller groups during the course of the term in order to gauge the students' experience in further detail and in a more social context than in their journals. This proves impractical particularly with regards to the consequent invasiveness of the research component, which I feel is already reaching uncomfortable levels.

In the second class, I distribute blank notebooks containing journal writing guidelines These are intended to serve several purposes, including as a mechanism to collect data about the participants' immediate affective responses to their experience, but also as a means of dialogue with me about it and as an opportunity to talk to their 'future teaching selves' from the learner's point of view. Following Liebars' (1997) remark cited earlier, I design the journal guidelines as a series of unfinished sentences, including for example:

- I've changed my mind about...

- I still think that...

- I am not sure about... (see Appendix 5: Journal Keeping Guidelines).

Regular interaction through feedback in the form of journal responses can encourage continued participation in the exercise, if only by reminding the participants of it every time I request some journals to be handed in. To the same effect, when I collect the journals, I not only read them, but also comment on some of their writing, although only occasionally. Deciding when and what to include is an exercise in restraint that owes a lot to interviewing techniques in the intention to minimise the impact of my viewpoint, thereby preserving the participants' agency, and therefore reducing the intrusiveness of my input and increasing the trustworthiness of the data. For example, to a quotation such as the following:

> I feel like they are always changing the [mark]ing policies and what we need to do. It is very difficult to follow. I just don't like feeling confused all the time. (Barbara [pseudonym], 19/09/03)

I respond:

> Please elaborate on this '[mark]ing policy issue'. This sounds important... (Eva, in Barbara's journal)

In this particular case, the student did not take the opportunity to respond or reflect further on the topic by replying to my comment. In other cases, such as the following:

> In class today we worked more with the circles. We folded another, new circle the same way, and discussed the ideas of an equilateral triangle and ways to prove it. There were four main ideas that we discussed in groups. The first idea (simply measuring the sides) seemed straight forward, but once the group began to think of things in more of a mathematical way, new ideas developed and our thinking expanded. (Isabel, 09/09/03)

I then ask:

> "More of a mathematical way..."? Please explain. (Eva, in Isabel's journal)

And she replies:

> We began to expand the way we thought about the problem. When we took in more of a mathematical perspective it didn't seem as straight forward. (Isabel, n.d.)

All in all, the students' journal entries are varied and often show serious personal consideration of the issues at hand. To preserve the anonymity of the journals during the interaction and to facilitate the organisation of the data, since it is collected repeatedly, I use a numbering stamp to identify each 2-page spread of each journal as follows: each journal is assigned a random three-digit number, for example 173. Each spread then contains a six-digit number composed of the journal identifier and the page identifier, for example 173004. This identifier helps both to preserve the order of the entries and to reconnect previously stored data with later versions of the same entries. The example cited above comes from page 231002, that is, from the second spread of journal 231. A first data collection yields only the initial entry and my feedback, while a later iteration contributes the participant's response. To preserve the link between these stages, spreads where the interaction has more than one cycle are recorded at subsequent collections and collated at a later stage. The journal numbers are correlated to the name of the student on a separate list, and the journals remain anonymous except for personal writing style. The journal identification numbers are more or less randomly spaced between 001 and 999 and so each time I collect journals, I request, for example, journals whose numbers begin with 0, 2 or 4, and later, 1, 3, 8 or 9, and finally, 5, 6 or 7. The data is recorded using a digital camera, and the images are transcribed and later analysed.

During the whole of the 'mini-enquiry projects' and 'regular teaching' phases (see Chapter 4: The Teaching Approach), I take field notes on my observations. These notes are not intended as a main source of data for analysis, but rather as a means of

contextualising the students' writing and as an aide-mémoire for the description of the teaching approach (see Chapter 4). In addition, and since I am actively participating in the interaction during the 'project' phase, I take notes of my recollections of these interactions. The journals written by the two members of the instructional team are also collected.

In each phase of the module, the students submit written work, and in some cases, small models. All these artefacts, which the student-participants produce in response to the teaching approach, are digitally recorded after they are assessed, for completeness and later reference. In addition, some of the students take advantage of the homework resubmission policy and these re-submissions are also recorded. At the end of term, the students also submit portfolios of the work completed during the term, which contain a complete report of their project work. Based on the provided guidelines (see Appendix 6), these reports focus largely on the students' process during the project phase, and are individual. As such, they can provide further data concerning their experience. Finally, at the end of term, the students give oral presentations of their projects, which are also recorded.

Though the above data collection methods provide me with a significant amount to wade through, many of the data sets are intended more for confirmation of findings than as the main focus of the analysis.

Finally, in the formal assessment phase, the teaching team, while assessing the project reports, respond to a questionnaire rating the projects in terms of their views of mathematical enquiry (see Appendix 4).

Description, Analysis and Interpretation

The second and third questions that this study is designed to answer are addressed using qualitative and quantitative methods, respectively. Examining the experience of the participants for authenticity is done by analysing their journal writing during the experience, while its impact on affective outcomes is investigated using a pre-/post-intervention questionnaire comparison. Unlike in the previous section regarding data collection, the two components of the analysis are treated separately.

Qualitative Component: Examining the Experience Had

This component of the study is mainly focused on the participants' individual experience during the intervention, as reported in their journals. In addition, the analysis is comparative in nature, with the exemplar being described in terms of participant agency (Burton, 2004, Grenier & Payan, 2003), of the stages of mathematical enquiry (Hadamard, 1945) and of the perceptions of the social contract (Brousseau, 1997, Mason, 1978).

As I discuss earlier and describe in detail in Chapter 4, the module is subdivided into four phases. The most important of these, with regards to the study, is the second, where the participants engage in what I intend to be authentic mathematical enquiry. Figure 3, below, charts the number of journal entries in each phase, for each participant.
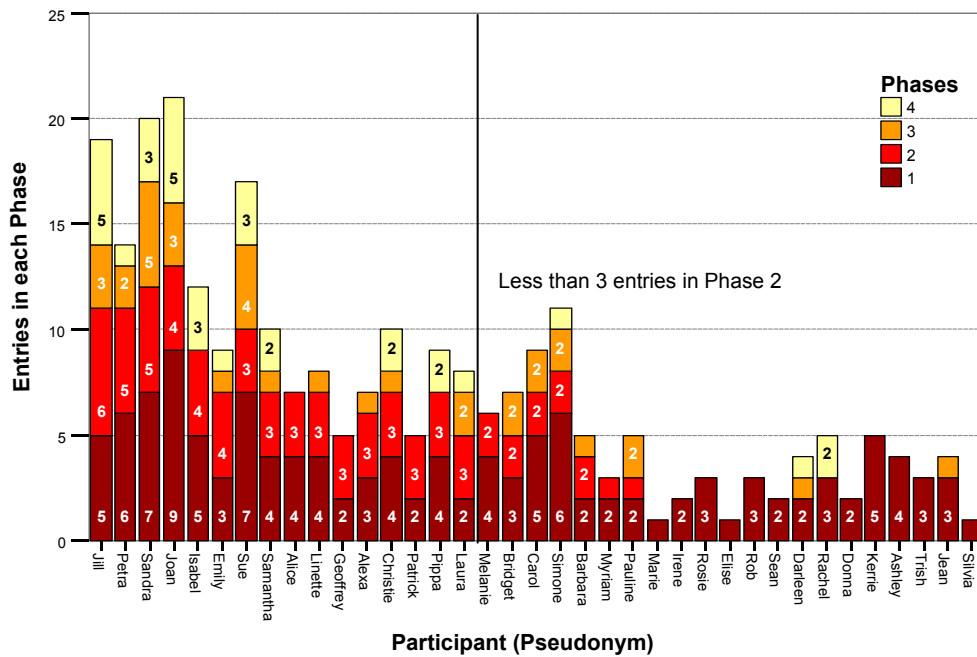
Figure 3   **Journal entries per participant (pseudonym), for each phase of the intervention**

The chart shows the 37 participants, ordered by the number of entries they make during the phase most representative of the intended experience, Phase 2 (in red). These numbers range from 6 to 0, whereas the total number of entries per participant ranges from 21 to 1, with a median and mode of 5 for both. As the purpose of the analysis is to examine the process that the participants experience during Phase 2, the experience reported by participants who submit less than 3 entries during this phase (to the right of the vertical line) is not likely to give a clear picture. Consequently, I do not consider these participants' journals in the analysis, focusing instead on the remaining 16. This,

of course, does not mean that their experience is not authentic. It simply means that they have not generated enough evidence as to whether it is. Conversely, the self-reflection provided by greater participation in the journal exercise cannot be discounted as a potentially cathartic element in the experience.

Design decisions regarding the nature of the data collected and the insights expected from it play a key role in the selection of strategies for the analysis stage. As Pirie puts it:

> When are we working with existing theory, and when are we hoping to build new theory? When can we predefine the coding of our data for analysis, and when do we prefer to allow a taxonomy to emerge from the data as they are gathered? (Pirie, 1998, p. 21)

The strength of the theoretical framework and the clearness of the question that this component is aiming to answer suggest the implementation of an analysis strategy that comprises a mechanism that excludes data not relevant to the question. This purposive method can facilitate the filtering and reduction of data to help frame the results. However, as the data generated in the journals is solicited in a very open-ended way (see Appendix 5 for the journal writing guidelines), using a coding set exclusively derived from the theoretical framework potentially yields too small a crop of quotations to get a good sense of the participants' experience. Making use of coding techniques taken from 'grounded theory' can help alleviate this problem. As Charmaz (2000) explains it:

> [...] grounded theorists develop analytic interpretations of their data to focus further data collection, which they use in turn to inform and refine their developing theoretical frameworks. (p. 509)

In the grounded theory perspective, data is coded in a 'snowball' process where,

> Through coding, we start to define and categorise our data. In grounded theory coding, we create codes as we study our data. [...] We should interact with our data and pose questions to them... (Charmaz, 2000, p. 515)

The data, in other words, generates the codes, and it is through an iterative process that the quotations are assigned relevant codes. The codes that emerge from this approach, together with some that are focused more specifically on the theoretical framework provide a rich foundation for the development of an interpretation of the data. At the interpretation stage, again, the findings can lead back to the data with new requirements for clarification, supported by new codes. At this stage, the use of auto-coding, made possible by CAQDAS (Computer-Assisted Qualitative Data Analysis) (Seale, 2000, p.

155) can be helpful as the terminology has been established by previous examination of the data. This multilayered set of methods provides the most useful approach for this components of the study as it combines the strengths of open-ended data generation with the those of the theoretical framework.

A preliminary analysis across some 21 journals shows the complexity of the data, even considering only the quotations that express affective responses. This suggests the use of a modular approach comprising a grammar-like hierarchy of codes, which can be applied in Boolean concatenation to make a complex analysis possible. The 'emerging codes' generated in this way fall into the following families:

0. Scope of the quotation

1. Object of the quotation

2. Verb, i.e. the nature of the affective response

3. Topic of the quotation, i.e. the specific aspect of the course

4. Orientation, i.e. the modifier for the 'verb'

8. Context of quotation, i.e. during which phase it is written

The necessity of adding the Context category (8) in addition to the Scope (0) is justified by the existence of situations in which a comment is made 'out of sequence' about a phase that is not concurrent with the quotation. For example Laura writes this comparative comment during Phase 3, which is associated with 'direct teaching':

> Working in the book is a lot different from working on the project. The project was more working on our own and with the book it's more of Dr Zachary teaching us and learning that way. (Laura, 18/11/03)

The quotation evokes not only the concurrent activities, but also those engaged in during the previous phase, which justifies a double tagging.

In addition to these coding families, I include more purposive codes, including 'Meaningful remarks' and 'Feedback and responses' for the collection of quotations that stand out and quotations that are generated by my feedback during the module, respectively. In the latter case, out of the 16 journals included in the final analysis, 15 contain a combined total of 43 feedback entries, ranging from 1 to 6, with a median of 3

per journal. 7 of the 15 participants then respond to 22 of these entries, ranging from 1 to 6 per participant, with a median of 3. In one particular case, the participant monitors her development through repeated responses to my feedback, as follows:

> -At times I find myself searching for ways to explain what I want to say, but not knowing what words to use.
>
> -[EK] Do you feel your communication has improved a bit now?
>
> -Um... yes
>
> - Yes definitely - still not perfect... (Joan, between 10/09 and 03/12)

Quotations tagged using either 'Meaningful remarks' or 'Feedback and responses' are not precluded from being tagged otherwise, of course, which is why these families are not part of the concatenation structure.

In addition, I include a family derived from the literature review. These other 'purposive codes' consist of direct evidence of the application of participant agency (Burton, 2004) at each stage of enquiry (according to design criteria 1-3, in Grenier & Payan's 2003 Framework), the stages of mathematical enquiry as described by Hadamard (1945) and which I used to establish criterion 5, and what I defined in the literature review as Mason's (1989) tension, after Brousseau's (1997) didactical contract (see criterion 4). The following code families are therefore added:

> 5. Meaningful remarks
>
> 7. Feedback and responses
>
> 9. Theoretical framework (Burton, 2004; Grenier & Payan, 2003; Hadamard, 1945; Mason, 1989)

The codes derived from the theoretical framework are used in the analysis to select the quotations which directly express the participants' awareness of the corresponding experience. For example, agency in the choice of starting point (criterion 1) is evidenced in the following quotation:

> [...] this project has been interesting in the way it allowed us to have freedom to choose what we liked/wanted to do. Math classes do not normally give this much freedom with assignments, so this has been a new experience for me. (Alexa, 15/10/03)

The final analysis is then done on the 16 relevant journals and takes place in steps, broken down first by the phases of the intervention during which the entries are written, and then into manageable chunks of 12 to 18 entries, as shown in Table 8, below. The 16 selected journals are written by participants from 12 of the 19 groups that worked together, as illustrated in the first column of the table.

Group	Participant	Ph 1		Ph 2		Ph 3		Ph 4	
J	Joan	9	C (14)	4	G (15)	3	J (12)	5	M (14)
J	Sandra	7	B (15)	5		5		3	
G	Sue	7	A (14)	3	F (15)	4		3	
M	Petra	6	D (14)	5		2	K (12)	1	
P	Jill	5	C (14)	6	G (15)	3		5	L (14)
I	Isabel	5	A (14)	4	F(15)	0		3	
F	Christie	4	E (14)	3		1		2	
P	Linette	4	B (15)	3	H (12)	1		0	
L	Samantha	4		3		1		2	
A	Alice	4	D (14)	3		0		0	
R	Pippa	4		3		0		2	
S	Emily	3	E (14)	4	I (16)	1		1	M (14)
D	Alexa	3		3		1		0	
I	Laura	2	A (14)	3		2		1	
A	Geoffrey	2	E (14)	3		0		0	
N	Patrick	2		3		0		0	
12	Total	71	6	58	4	24	2	28	2

Table 8: Analysis schedule for journal entries (each stage A to M of the analysis is accompanied by the corresponding number of entries)

During this process, issues arise about the integrity of the codes used, particularly as to the possible duplication of concepts across two or more codes. The coding hierarchy is therefore further refined, then finalised. A complete inventory of the codes sorted into families is shown in Figure 3, below (see Appendix 7 for a list containing the comments attached to each code). The figure also describes a possible concatenation between members of families that can be used for the deeper analysis.

In the interpretation, the emerged codes are used in Boolean concatenation to extract quotations which might indirectly express indications of the experiences had. For example, a quotation might express the participant's [4.0. negative] > [2.0. ease/difficulty] > in connection with the specific > [1.0. instruction content] > within the > [0.1. mini-projects]. S/he might have made that remark during [8.0. Stage 1]. This particular concatenation of tags shows quotations regarding the lack of accessibility of the mathematical content of the mini-project, thereby critiquing their (meaning) scaffolding (Criterion 4):

> Working with 3-D shapes has been difficult because you have to have a visual concept of these shapes, and that is something I don't really have. It is hard to spatially see where the lines of reflection and rotation occur. (Alexa, 17/09/03)

Topic
- 3.0. creativity in maths
- 3.0. independence of work
- 3.0. learning by doing
- 3.0. mathematicians' work
- 3.0. nature of maths
- 3.0. openness of instruction
- 3.0. text book as reference
- 3.0. the time it takes (critn 5)
- 3.0. portfolios
- 3.0. mathematical Heuristics

Miscellaneous
- 5. meaningful remarks
- 7. feedback and responses
- 7.1. with response

Modifier
- 4.0. negative
- 4.0. negative change in
- 4.0. positive
- 4.0. positive change in
- 4.0. neutral
- 4.0. slight negative
- 4.0. slight positive
- 4.0. unsure (positive/negative)
- 4.1. externalisation
- 4.1. internalisation
- 4.2. comparison of
- 4.2. reflection on

Scope
- 0.0. course as a whole
- 0.0. maths/maths education as a whole
- 0.0. other sections
- 0.0. previous maths classroom experience
- 0.1. 'direct instruction' (Nov)
- 0.1. mini-enquiry projects (Sept)
- 0.1. work on project (Oct)

Verb (response)
- 2.0. anxiety/worry/intimidation about
- 2.0. ease/difficulty with
- 2.0. excited/enjoying/bored with
- 2.0. found useful
- 2.0. frustration/feeling challenged about
- 2.0. interest in
- 2.0. learning about
- 2.0. like/dislike of/personal preference
- 2.0. self confidence about
- 2.0. understanding/confusion about
- 2.1. comparison of
- 2.1. reflection on
- 2.1. relating project to mini-projects
- 2.1. relating to 'real life'
- 2.1. relating to the elementary classroom
- 2.1. student's expectations/hopes about

Object
- 1.0. instruction content
- 1.0. instruction style
- 1.1. hands-on work
- 1.1. student's work (Hwk, etc.)
- 1.1. visualisation
- 1.1. whole class discussion
- 1.1. work in small groups
- 1.1. work with textbook
- 1.2. additional help
- 1.2. assessment requirements
- 1.3. maths - the subject
- 1.3. maths learning
- 1.3. maths teaching

Context
- 8.0. stage 1: September mini-projects
- 8.0. stage 2: October project
- 8.0. stage 3: November regular classes
- 8.0. stage 4: December assessment

Theoretical framework
- 9.1. B1 agency of starting point (criterion 1)
- 9.1. B2 agency of process (criterion 2)
- 9.1. B3 agency of end point (criterion 3)
- 9.2. H1 initiation/Planning (criterion 5)
- 9.2. H2 incubation (criterion 5)
- 9.2. H3 illumination (criterion 5)
- 9.2. H4 verification (criterion 5)
- 9.2. H5 overall process
- 9.3. M1 security /Mason's tension (criterion 4)

Autocoding
- 6.0 'book'
- 6.0 'concrete'
- 6.0 'elementary', 'children'
- 6.0 'journal'
- 6.0 'want', 'decide', 'freedom'

Figure 4   **Finalised codes used in the analysis of 16 journals, sorted into families**

During the analysis of the data using both the emerging and purposive codes, similar concatenations evolve that correspond to the concepts of the theoretical framework, and can therefore be used in the interpretation. Examples of subtleties in the coding also arise. For instance, the quotation "So far we have found it is difficult for most colorings to have two colors only" (Sue, 07/10/03) does not represent an affective response 'found difficult' to the activity. Rather, it expresses the idea that a mathematical phenomenon is rare.

The interpretation of the qualitative data, which is used to compare the participants' experience to the exemplar, is organised according to the criteria of the teaching approach design, as established in the literature review. In each section, the relevant data is then examined for themes emerging from the responses and for evidence of the success or failure of the intentions behind the design criteria and their corresponding teaching strategies. For example, the 'ramping up' strategy implemented in September is evaluated for its usefulness.

While analysing the data against the theoretical framework, in addition, the data selected through the extraction of quotations corresponding to relevant codes can itself suggest the further selection of quotations, for instance, of adjacent text, or of specific terms. For example, the discussion of the accessibility of the starting points in Phase 1 (see Chapter 4) yields several quotations using the opposition pair 'concrete' and 'abstract'. A quick automatic coding using those terms harvests a few additional comments that help clarify the discussion.

Each of the design criteria developed in Chapter 2 is examined in this way, and so are the additional constraints described in Chapter 4, including the presence of research artefacts and the constraints imposed by the overall context of the study.

Quantitative component: Probing for Potential Change

The third question that this study is designed to answer focuses on the potential effect of the experience on the participants' affective responses to mathematics. As this answer concerns a change in response, the measure takes the form of a comparison of responses before and after the intervention (Items 4 and 11-39, See Appendices 3 and 4). In addition, I include 9 items in the post-module questionnaire (2, 3, 6-9 and 40-44) that focus on the overall experience, without having a counterpart in the pre-module questionnaire. Though these items do not elicit responses that allow a comparison to be made, they can nevertheless help to frame the views of the students.

Items 6 to 9 of the post-module questionnaire are designed to elicit a ranking of the three main phases of the teaching approach, the 'ramping up', the main mathematical enquiry and the 'regular teaching'. In each case, the participant is asked to rank the three phases by giving them a number between 1 and 3 to rate them according to a comparative descriptor. Some of the students, rather than using the numbers, simply enter an X in the box corresponding to the phase they feel most relevant to the

statement. Because of this, rather than comparing the ranking of each phase for each item, the preferred phase is chosen and data regarding second or third ranking by the remaining participants is discarded. The results are then tabulated and illustrated in a bar graph showing the relative ranking of each of the three main phases.

The short responses to Items 2, 3, and 40 to 44 are categorised and counted by matching them with a classification taken either from the theoretical framework if possible, from the codes which emerged in the qualitative analysis of the journals, or from the responses themselves, otherwise.

Items 4 and 5 are essentially multiple-choice and responses are therefore presented simply as a count for each choice. In the case of item 4, the count is compared to that taken from the pre-module questionnaire.

The descriptive items of the two questionnaires (Items 11-14) are the same and each consists of the selection of three terms, taken from a given list, that best correspond to the participants' views. There are two lists, one containing nouns and nominative phrases (Items 11 and 13), and the second containing adjectives and modifying phrases (Items 12 and 14).

In the case of Items 11 and 12, the participants are asked to select the terms that they feel best describe mathematics, in general, as they see it. Conversely, Items 13 and 14 ask which terms describe mathematics the least. For each term of each list, the response count is added together and the responses to Items 13 and 14 (the "least" descriptors) are subtracted from the responses to Items 11 and 12 (the best descriptors). Though the items require the participant to rank her/his three choices, some questionnaires show evidence that this is not done and the three choices are therefore rated equally in the analysis, each term given the same value of 1 if chosen as best, 0 if not chosen, and -1 if chosen as worst descriptor. The responses are presented in a table that contains columns for the score of each response in each item. In addition, columns present the net score for both questionnaires, (best minus worst) and the net change between the pre- and post-module questionnaire scores.

The null hypothesis for each case is of course that the numbers for each question are evenly distributed (three times 35 responses over 14 or 12 options, respectively). Calculation is as follows: if the students all picked nouns at random, the probability of each noun being picked as a, say, first choice would be $35/14 = 2.5$ or $35/12 \approx 2.9$,

respectively. The same can be said for the second or third choice. As for being picked at least once, the probability is 3 x 2.5 = 7.5 or 3 x 2.9 = 8.7, respectively.

I discount the two participants that only provide post-module data in order to maintain the totals in both questionnaires. In addition, in Item 13 of the pre-module questionnaire, two participants give faulty responses, so that the totals do not add up. One only chose a single option instead of three, and the other duplicated a choice, which I then only count once.

In addition to the table, I present the net count before and after in a mock-log graph with the extreme value regions compressed to de-emphasise them (see Chapter 7). In the case of the results of Items 12 and 14, the most significant result is the change in orientation of the value found for 'concrete'. Coupled with the findings from the qualitative study, this prompts a closer look at the scores, and the specific distribution of both 'abstract' and 'concrete' in the results of these items. The combined selections are presented in two tables, one per questionnaire, and the results are discussed.

In the case of items 12 and 14, in addition, the data can be further examined in terms of the type of responses the participants choose. According to the theoretical framework (see Chapter 2), individuals can be categorised into those who tend to base their attitudes on emotional responses and those who tend to base them on beliefs. The descriptors can be sorted into three supra-categories corresponding to emotions, attitudes and beliefs. In the list of 12 descriptors, four belong to each supra-category. I compare the counts for each of these before and after and find a change. Considering that attitude can be based on either an emotion- or belief-based response, I assign it a neutral value, with a positive for belief choices and a negative for emotion choices. The net value for the change in the balance between the three supra-categories is examined and suggests a further investigation into the actual responses.

Section 4 contains 25 Likert items (15-39) focusing on the participants' like/dislike of mathematics (2 items), their self-confidence (2 items) and their views of the nature of mathematics (21 items).

The items are categorised, based on the theoretical framework, in terms of the affective responses they elicit, as shown in Table 9, below. The four views that the last 21 items are measuring are not necessarily distinct from each other, or indeed mutually exclusive.

For this reason, some of the items can be used for more than one view, possibly in reverse, notably, item 23.

	Measure	Direction
15. I find solving mathematics problems to be dull and boring	Like/Dislike	Negative
16. I like mathematics better than most other subjects	Like/Dislike	Positive
17. Mathematics is a subject I find easy	Self-Confidence	Positive
18. I have never been confident in mathematics	Self-Confidence	Negative
19. Someone who is good at mathematics never makes a mistake	Instrumentalist	Positive
20. Mathematics consists of a set of fixed, everlasting truths	Platonist	Positive
21. Mathematics is about the study of all possible patterns	Pattern Analysis	Positive
22. Mathematics is basically doing calculations	Instrumentalist	Positive
23. Only gifted professional mathematicians can be creative in mathematics	Instrumentalist	Positive
	Pattern Analysis	Negative
24. There are many ways of solving any problem in mathematics	Problem-solving	Positive
	Instrumentalist	Negative
25. The discoveries of mathematics are permanent	Platonist	Positive
26. Exploring number patterns is not real mathematics	Instrumentalist	Positive
	Pattern Analysis	Negative
27. In mathematics there is always a right answer	Platonist	Positive
28. Puzzles and investigations are not genuine mathematics	Problem-solving	Negative
	Pattern Analysis	Negative
29. There are many problems in mathematics which have never been solved	Problem-solving	Positive
	Pattern Analysis	Positive
30. Basic number skills are more important than creativity in mathematics	Instrumentalist	Positive
31. Mathematics is always changing and growing	Platonist	Negative
32. The procedures and methods in mathematics guarantee right answers	Instrumentalist	Positive
33. Some mathematics problems have many answers, some have none	Pattern Analysis	Positive
34. Mathematics is exact and certain	Platonist	Positive
35. There is only one correct way of solving any mathematics problem	Instrumentalist	Positive
36. A person should not mind risking a mistake when trying to solve a mathematics problem	Instrumentalist	Negative
	Problem-solving	Positive
37. Investigating a puzzle can lead to significant new mathematics	Pattern Analysis	Positive
38. Knowing how to solve a problem is more important than the right answer	Instrumentalist	Negative
39. I think that creativity and mathematics are related	Instrumentalist	Negative
	Pattern Analysis	Positive

Table 9: Likert items, what they measure, and their direction (note that some items measure more than one value)

Items 15 to 18. The first four Likert items measure affective responses of an attitudinal nature, specifically like/dislike of the subject, and self-confidence. Two subscales are created using the appropriate items, and statistical methods are used to analyse the resultant data, including the interpretation of bar graphs showing the situation before and after and the changes as well as the *p*-value of a paired-sample t-test (indicating

statistical significance) and Cohen's *d* (indicating the practical significance). I explain these two tests in more detail below.

Items 19-39. In the case of the remaining items, the fact that the theoretical, face validities of the items can overlap creates a more complex situation. In addition, the interpretability of the views established by the theoretical framework means that the assignment of items to views can be questioned. To remedy this situation, I perform a repeated examination of the correlations of the pre-module data against the subscales proposed in Table 9, above (see Appendix 10), reassigning items to subscales to which they correlate until the subscale definitions stabilise. The resulting subscale assignments are listed in the following table:

	Instrumentalism	Platonism	Problem Solving	Pattern Analysis
19. Someone who is good at mathematics never makes a mistake				
20. Mathematics consists of a set of fixed, everlasting truths		+		
21. Mathematics is about the study of all possible patterns			-	
22. Mathematics is basically doing calculations				
23. Only gifted professional mathematicians can be creative in mathematics	+			-
24. There are many ways of solving any problem in mathematics			+	
25. The discoveries of mathematics are permanent		+		
26. Exploring number patterns is not real mathematics	+			
27. In mathematics there is always a right answer		+		-
28. Puzzles and investigations are not genuine mathematics	+			-
29. There are many problems in mathematics which have never been solved	-			+
30. Basic number skills are more important than creativity in mathematics	+			-
31. Mathematics is always changing and growing	-			+
32. The procedures and methods in mathematics guarantee right answers				
33. Some mathematics problems have many answers, some have none				+
34. Mathematics is exact and certain	+	+	-	
35. There is only one correct way of solving any mathematics problem	+		-	
36. A person should not mind risking a mistake when trying to solve a mathematics problem	-		+	
37. Investigating a puzzle can lead to significant new mathematics	-	-		+
38. Knowing how to solve a problem is more important than the right answer				
39. I think that creativity and mathematics are related	-			+

Table 10: **New subscale definitions (note: the items marked in yellow are not used)**

These new subscales are then used to examine the pre- and post-module data for change. The examination focuses on the statistical significance of the results, using a paired-sample t-test and on the effect size (practical significance) using a Cohen effect size scale. This last value is computed as follows:

$$d = \frac{\overline{x}_2 - \overline{x}_1}{\sqrt{\dfrac{SD_2{}^2 + SD_1{}^2}{2}}} \text{, where } \overline{x} \text{ is the mean and } SD \text{ the standard deviation.}$$

The purpose of this statistical test is to examine the size of the effect of the intervention, regardless of the N value (in this case 35). Finally, the results are examined in terms of the theoretical framework, which proposed the existence of four distinct views, including one that is not yet in the literature on views of mathematics in mathematics education. In addition, each of the subscales is tested for internal consistency using Cronbach's α (see Appendix 11). The results of this test show a clear internal consistency (> .7) for the Like/Dislike (.721), Self-Confidence (.887), Instrumentalist (.750) and Pattern Analysis (.773) subscales, and slightly lower values for Platonist (.645) and Problem-solving (.654) subscales. The lower values for the last two subscales can be seen as demonstrating the fact that either the items used or the corresponding categories themselves need refining. Conversely, the value of N (35 in the pre-module questionnaire) makes this test less meaningful.

Conclusion

This study makes use of a great number of methodologies and methods because the questions it is answering address very different aspects of the intervention it examines. The three questions are as follows:

- What could be the design criteria of a teaching approach which aims to provide an experience of practice analogous to that of research mathematicians? Which of these criteria are feasible in the given context?

- Can the experience of engagement with the resulting teaching approach successfully simulate that of engagement in the practice on which it is based, according to the design criteria?

- What are the affective outcomes, and are they as anticipated?

The first question is answered using a combination of literature review and reflection on my personal experience, added to contextual factors to produce first a set of design

criteria, then a teaching approach, which is then implemented by a collaborator and described using a digest of the ethnographical notes that I take during the intervention combined with selected extracts from the journals written by the various participants. The results of this component of the study are described in Chapter 4.

The second question is answered by an analysis of a selection of the journals written during the intervention by the student-participants. These journals contain responses to open-ended prompts about the student-participants' experience during the intervention and the purpose of the analysis is to verify the authenticity of this experience with respect to the exemplar derived in the answer to the first question. The methods used in this part of the study need to be both connected to the theory because of this comparison aspect, yet participant-centred because of the open-ended method of data solicitation. I therefore combine a grounded theory approach for the analysis stage with a more closed approach in the interpretation stage. The results of this component are in Chapter 6.

The third question corresponds to a pre-experimental design in that it measures the effect of the intervention on a single group, without control group, using a pre-test/post-test format. The analysis focuses on the participants' responses that measure change in affective responses to mathematics, including particularly attitudes and beliefs. The characteristics of the sample are described in Chapter 5 and the results of the comparative analysis are in Chapter 7.

In each of these components, the methods chosen reflect characteristics of the question that is being answered. In the first component, the focus is on theoretical underpinnings of teaching approaches and the methodology therefore concentrates on the existing literature, though the investigation involves a novel juxtaposition that is dictated by the initial observation/examination of the problem. The resulting teaching approach is also under examination in this component in the sense that, following the action research framework, a continuous, reflective process is layered over its application, which incorporates its constant re-evaluation and adjustment. This is done through frequent discussions between the researcher-participant and the teaching team, particularly the teacher-participant.

In the second, the focus is on the experience of the participants and the methodology therefore relies on methods that reflect the continuity of this experience. Possible methods that are not used include single or group interviews and recorded whole class

discussions, which, with the exception of the one carried out in the first session, are considered to be too intrusive and time consuming for the purpose at hand.

In the third component, the focus is on the effect of the intervention, that is, on the change in affective responses of the student-participants. The method of choice for pre-experimental designs of this kind is the questionnaire. Possibility exists of timing the post-module at different points during the intervention. For example, as this is the first of two modules that are connected, the option existed of doing the survey at the end of the second part, or conversely, at the end of the key phase of the teaching approach (see Chapter 4), at the end of October, after the participants complete their enquiry projects. The decision to use the end of the autumn term stems from two considerations: the idea that the module as a whole is under examination, including the phase that is closer to a traditional teaching approach means that the end of the enquiry phase is too early, and the modular characteristic of the programme of study (see Chapter 4) means that the constitution of the class during part 2 of the module cannot be guaranteed to be the same as in part 1. This would present a difficulty for the acquiring of data about the original sample.

Several additional data collection methods are used, mainly for triangulation and for reference, should the already mentioned methods prove deficient. These include regular interviews with the teacher-participant, pictures taken during the sessions (including some that are used by the student-participants in their written submissions), a third-party observation by a mathematics educator, the video recording of the presentations of the projects, and the recording of all work submitted for a mark.

One notable method that is discarded both for practical reasons and because it is not necessary for the study as it is designed, is any form of follow-up with the participants after further course work, or even after some teaching experience. As it is designed, this study is not investigating the longer-term effect of the intervention.

Several data analysis methods have also been discarded, including the correlation between findings in the journals and the responses to the questionnaires, and the construction of case studies examining the data for a single participant across all data source, mainly for practical reasons of timing.

Trustworthiness of the findings

Given that each question answered by this study is answered using a different methodology, the trustworthiness of the results for each component relies on different factors. For the first question, distinct literary sources, i.e. the literature on problem solving in mathematics, and that about the philosophy of mathematics and the practice of its research discipline, are analysed and contrasted using a lens that is considered current. This lens, combined with that of my own experience with mathematics research provide the first part of a triangulation mechanism for the verification of the results to the first question, and the second and third questions are in fact further verification mechanisms for these results.

The authenticity of the results of the second component of this study, which concerns the experience of the student-participants, is ensured by three mechanisms. First, the reliability of the raw data is ensured by the separation of teacher- and teaching assistant-participant which I describe earlier: the student-participants are assured that the participants involved in marking do not see the journal contents until after the marks have been reported. Second, the analysis method is verified by a colleague of the researcher, who uses the given codes to analyse a small subset of the data for comparison. Finally, in addition to an examination of the findings in terms of the theoretical framework, the analysis includes a component where, in line with action research methodologies, I purposefully look for results outside this framework that can help develop the teaching approach for future iterations of the module.

The verifiability of the results of the third component of the study, is provided by the literature in the field on affective issues in education. This literature predicts the stability and instability of categories of affective outcomes, which are mirrored in the results.

As the study as a whole takes place within the framework of action research, the overall trustworthiness of the study is not a question of transferability to any similar sample, but one of the development of a practice, based on the experience gained and the interpreted findings.

Chapter 4: Design and Implementation of the Teaching Approach

The teaching approach, as implemented in this intervention, is largely based on the design criteria established in Chapter 2. In addition, it is informed by other factors, including the integration of constraints imposed by both the educational study itself and external factors emerging from the social context of the intervention. I use the term teaching approach in a wide sense, encompassing any classroom activities, independent studies components, assessment and other events connected to the intervention. It also incorporates the teachers' background, attitudes and beliefs, insofar as they inform the approach, and the social contract as it develops during the teaching interaction. In a first instance, I discuss the form taken by the application of each of these constraints on the approach design, and later, I describe, chronologically, how the intervention takes place, integrating the changes that are determined by small-scale action research cycles within the time span of the intervention.

The Design of the Teaching Approach

At the design stage of the intervention, each of the design constraints is transposed into the selection of corresponding teaching strategies, and these are later applied to the creation of a plan for the overall intervention. The design constraints can be classified into three main groups. Firstly, I consider the design constraints focusing on the intended experience. These emerged both from the criteria defined in the literature review and from the need for a focused emphasis on the process. Secondly, constraints derive from the integration of the educational research component of the intervention, particularly the data collection events. Lastly, consideration is given to the wider context within which the intervention takes place, that is, the syllabus of the specific module and the degree programme to which it belongs.

The Intended Experience

In the part of the literature review delineating design criteria for the teaching approach, I establish the requirement of open-ended-ness for the starting point, process and goal state of the practice (criteria 1-3), after which I ascertain the importance of presenting an atmosphere of security that promotes and encourages the taking of creative risks (criterion 4). In the implementation of these criteria into the selection and refinement of planning and teaching strategies, I proceed in chronological order, that is, I review the criteria in the order in which they become relevant throughout the implementation of the intervention.

Criterion 4: an atmosphere of security

Before confronting the participating students with the potentially anxiety-inducing critico-creative component of the practice to which I intend to expose them, it is important to develop of an atmosphere of security that promotes and encourages the taking of creative risks, thereby securing sustained engagement. Three strategies are available to achieve this aim. Firstly, following Grenier and Payan's (2003) model, the mathematical content can be carefully selected to require little specific pre-requisite knowledge for the successful engagement in the activities, largely doing away with the need for ad hoc scaffolds for the peripheral participants. In Grenier & Payan's (2003) work, this condition is mainly fulfilled through control of the presentation of the initial 'research situation', and is therefore primarily under the aegis of the teachers (the 'full participants'), implying that agency regarding the choice of the starting point is theirs. As established in the literature review, however, I choose to shift this element of participant agency entirely into the peripheral participants' sphere of influence. Consequently, a different strategy is required, one that secures the student's agency: I design the approach so that they choose, and articulate their own starting points.

Secondly, a period of preliminary 'training' in the intended practice can be integrated into the approach, allowing a habituation of the participants in relation to the creativity of the required thinking mode. This initial 'ramping up' phase of classroom activities is comprised of several 'mini-enquiry' projects chosen for their accessibility as manifested by the nature of their mathematical content. This strategy can also resolve the issue of participant agency regarding the starting point in that the peripheral participants can, in the later, 'main enquiry' phase, propose their own starting point.

Thirdly, the focus of the social contract can be shifted away from more traditional expectations, directing itself instead on the participants' engagement, thereby promoting a more student-centred character for the intervention as a whole. This focus can be achieved through two means: the nature of the classroom interaction between the full and peripheral participants, which I discuss in the later section about the implementation of criterion 2, and characteristics of the assessment strategy. Rather than requiring the participants to demonstrate success either in the acquisition of mathematical content knowledge or in the resolution of the problem, this assessment strategy is designed to bring the participants' engagement in the proposed practice to the fore. In addition, as discussed in the section on criterion 5, an important aspect of the practice of mathematical enquiry is the process that the participant experiences. Arter et al. (1995) cite the "tracking of growth over time" as one of three common assessment uses of portfolios, which makes them an appropriate choice for this intervention. Defined as "collections of authentic tasks gathered across time and across contexts" (Zollmann & Jones, 1994, p. 5), portfolios constitute one of the main assessment strategies of the module, worth 65% of the final mark (see appendix 10). Additional components of the formal assessment strategy include an oral presentation of the projects, class participation, and the results of short quizzes on the more content-oriented part of the module (see constraints imposed by the Programmatic Context, below).

Advantages of portfolios as a main component of the assessment strategy include their compatibility with the time frame devoted to the mathematical enquiry. In addition, all portfolio parts that focus on the mathematical enquiry component of the intervention, take the form of written reports, focusing on the process. As Bagley and Gallenberger (1992) explain:

> Writing is more than just a means of expressing what we think; it is a means of knowing what we think—a means of shaping, clarifying, and discovering our ideas. (Bagley & Gallenberger, 1992, p. 660)

The use of collections of process-based writing therefore reinforces, again, the importance of the process.

In summary, to develop an atmosphere of security for the students, I use three distinct strategies: the module starts with a 'ramping up' phase in which the approach consists of 'mini-enquiry' projects, which give the participants a sense of the expected level of enquiry. In addition, these early projects are chosen for their accessibility, thus limiting

the necessity for ad hoc scaffolds. Finally, the social contract and particularly the assessment strategy are selected and refined to reinforce the focus on the process.

Criterion 1: a novel starting point

By following up the ramping up stage with the activities constituting the key experience of the intervention, the design provides a benefit with respect to criterion 2, the novelty of the starting point for the participant. As I discuss in the relevant part of the literature review, this property of the starting point, for the peripheral participant, is connected to her/his agency through what Mason (1978) called its 'articulation', which, he says, cannot simply be replaced by the question. If the participant is given agency in the selection and articulation of her/his own starting point for the main enquiry, a higher level of ownership and by extension, motivation can be attained, and, given the right affective and experiential state, the novelty of the starting point for the specific participant can be achieved. In addition, transferring the articulation of the starting point to the peripheral participant has implications for the next criterion, of an open-ended process. In effect, as the articulation of the starting point is left in the hands of the peripheral participant, s/he exerts control over it making it more difficult for the full participants to 'take over'.

Criterion 2: an open-ended process

In order to further preserve the peripheral participants' agency in the mathematical enquiry process after the initial articulation of the starting point, the interaction between them and the full participants involved in the intervention needs to be carefully managed. This can be done through the development of a teaching objective, but the views and past experiences of the full participants are also very important. Indeed, if this objective is incompatible with the views of the full participants, a conflict can arise, thereby subverting the intended practice. In the case in point, the teaching team is constituted of two members, both of which have had exposure to mathematical enquiry. The teacher-participant is a recent PhD graduate in pure mathematics (whose pseudonym throughout is Dr. Zachary), and the teaching assistant is a doctoral student about to move from course work to the supervised research stage, also in pure mathematics (referred to as Alan). The teaching objective is to suppress as much as possible and ideally eliminate the affective outcomes of ad hoc scaffolding (as described in the literature review), in order to simulate the exemplar practice. However, rather

than attempting to reify this objective into an articulated written document, I discuss it with the instruction team. Examples of practical suggestions include:

- Do not lead

- Refer back to the initial problem posing

- Do not give them answers

- Avoid giving hints (Mason, 1978)

- Act as a resource only, only rarely redirecting their focus

- Resolve student questions by suggesting how they can resolve them for themselves. Good research supervisors do not know the outcome in advance.

- Give indirect guidance when they make errors

Because of the experiential background in mathematics research of each member of the teaching team as well as my own, we are all able to integrate this idea into the practice. The teacher-participant of the module, who participates in the design of the intervention from the beginning, is central to the development of the intentions of the intervention. In his own words, the teaching assistant-participant expressed this as follows:

> The goal of the class (as I understand it) is not to instil a large body of technical knowledge into the students, since that would not be at all useful to them.
>
> So I agree with practices such as "don't answer any questions". Answering a question directly very efficiently instils knowledge into the student, but it doesn't give the student any experience at the mathematical process of figuring it out for yourself. Telling them the answer produces a light-bulb moment for the student, but a somewhat dim and unsatisfying one. It also doesn't prepare you at all for being in front of a class and being asked a question that you don't quite know the answer to. (Alan [pseudonym], personal journal, 10/09/03)

Throughout the intervention, the team itself experiences the teaching style and refines its sense of the responses, as I show in the chronology of the intervention, below. Connected to this need to hold back easy answers and thereby preserving the agency of the peripheral participant regarding the process of enquiry is the necessity to preserve agency regarding the satisfactory nature of a resolution, as I discuss in the next section.

Criterion 3: an open-ended goal-state

As discussed in the literature review, in the context of authentic mathematical enquiry, there is no implied end to the enquiry, since, as Grenier and Payan (2003) put it, "an

answered question often leads to a new question", and therefore, an evaluative decision has to be made regarding a "criterion of local resolution" (ibid). In the spirit of the intervention, this decision is left to the peripheral participant, although, as I discuss in the next section, it is impacted by an otherwise unconnected constraint, the time frame.

Criterion 5: a practical time frame

In the literature review, I define criterion 5 as the requirement for time for the authentic experience to unfold. This is important for two reasons. Firstly, in order for the experience to be complete and therefore authentic, the participants need to progress through all the stages of Hadamard's (1945) scheme, including the move from incubation to illumination. In particular, the participants need to persist beyond the experience of the lack of immediate results, to illumination, so that they can realise that mulling over and committing to the process (Mason, 1978) even when it is not straight forward, can produce results. This aim is expressed as follows, in the module planning journal:

> We may need to take them to a boredom/frustration point so they get beyond it. Otherwise they remain spoon fed. (Dr Zachary, personal journal, p. 1-99, 01/08/03)

This requirement connects to the second reason for this criterion, the necessity of allowing the participants to shift their focus away from the resolution to reflect instead on the process the experience of mathematical enquiry. To this end, the class time and independent work of the first two months of the term are entirely devoted to the mini-enquiry projects and main project.

Conclusion

As the intervention embodies an unequivocal departure from the traditional classroom context[20], this change needs to be accentuated wherever possible in order to ensure the participants' appropriate level of engagement, particularly with respect to the social contract. This intention is realised through the following means:

- The part of the course devoted to the experience of mathematical enquiry is scheduled to take place first, to prevent the establishment of a social contract based on more 'traditional' instruction and therefore potentially counter-productive to the expected outcomes of the experience. It is given two whole months to unfold.

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[20] I discuss the negotiation for the implementation of the intervention in the degree programme in the second-to-next section: Constraints Imposed by the Programmatic context.

- The first phase of the intervention serves as a ramping up of the practice, through the introduction of mini-enquiry projects that are chosen for the accessibility of their starting points.

- The articulation of the starting point, the direction of the process and the evaluation of the goal state are left to the peripheral participants, through the development of teaching guidelines that are reactive and held back.

- The formal assessment mechanism is designed to focus on the process. This is achieved through the use of portfolios and through the articulation of the writing guidelines.

Overall, these strategies all aim at reinforcing both the authenticity of the experience of mathematical enquiry as defined in the literature review, and its impact on the participants, through increased reflection on said experience. In addition, some of the strategies integrated into the intervention for the purpose of the educational research study or the integration into the wider context of the degree programme also help to reinforce these aims. For example, the mathematical topics that are the subject of the module lend themselves to the articulation of accessible mini-enquiry starting points. Conversely, the reflective components of the data gathering, particularly the journal writing but also the questionnaires, help to direct the participants' reflections regarding their practice.

Constraints of the Educational Study: Data Collection

The intervention's aim of transforming the participating students' perceptions is reinforced through the deeper reflexivity required by participation in the research study components of the intervention. This aspect is important to consider because the participants' experience in the module is altered by the presence of artefacts of the educational study. These artefacts include the pre- and post-module questionnaires, an initial group discussion about what the participants think that mathematicians do, and the feedback provided through the reflective journals. In addition, and since I sit in on every session, I choose to interact with the students in class, providing me with additional 'ethnographical' data in the form of researcher's field notes, and adding another member to the instructional team in an in-class feedback capacity. I describe these interactions in more detail in the data collection section of Chapter 3: Methodology. In particular, the journals provide the participants with an important opportunity:

> Writing forces a slow down of one's thought processes, thereby allowing one to reflect and clarify their own thinking. […] Journals can only enhance the learning of mathematics for a wide range of students, including […] pre-service educators. (Liebars, 1997, p. 5)

The journals therefore provide a reinforcement mechanism in that it gives the participant an opportunity to articulate their experience of the practice. The other research instruments can have an analogous impact, and need therefore be considered an integral part of the teaching approach, as are the strategies that emerged from the overall context of the intervention, which I discuss below.

Constraints Imposed by the Programmatic Context

The context of the module within which the intervention takes place dictates some additional design constraints, particularly with respect to the mathematical content. Official documentation produced by the institution describes the module as follows:

> [Parts I and II of this course] are required of all prospective elementary school teachers in the undergraduate program. […] This sequence is unusual compared to what is being offered at other institutions. […]
> The key to success is the method of presentation. These courses are taught in small classes of size 30. The students work most of the time in groups of 3 or 4. Their learning activities are guided by a faculty member and a teaching assistant who are both present during all of the class meetings. This labor-intensive approach to instruction makes it possible for us to considerably broaden the students' mathematical perspective, thereby increasing the likelihood that they will pass on a positive message about mathematics to the school children that they will teach. (n.d.)

Though the module is based on a specific textbook, there is flexibility in the selection of topics that are addressed in each part. In addition, the description given of the teaching approach is compatible with that which I plan for the intervention. As a consequence, the constraints imposed by the wider context of the module are light, and centre mainly on the mathematical topics to be covered. In order to conform to these requirements, the part of the module that is not devoted to the practice of authentic mathematical enquiry and its preparation is used to fill in the gaps in the syllabus.

This is a complex consideration. On the one hand, the module is part one of two which can be taken individually, in different sections. Consequently, though the various topics in the double module can be worked on in a different order, there can be issues of overlap with the other module, particularly for students changing sections between parts 1 and 2[21]. However, this is less of an issue because most of the mathematical content areas have been addressed previously in the participants' education. The third phase of

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[21] The modular system used in the institution allows the students to choose the schedule of the modules that are taught by more than one instructor and this flexibility causes groups of students to change 'section' from one term to another.

the module and the choice of the topics for the mini-enquiry projects are therefore the areas that are most impacted by the context of the module.

In preparation for the study, the teacher-participant discusses this intervention with the mathematics department representative in charge of the degree programme. The course itself has originally been created in response to a document, *A Call for Change* (1991), which is described as "a set of recommendations for the mathematical preparation of teachers from the Mathematical Association of America". In response to this document, the course is designed to present a:

> labor-intensive approach to instruction [which] makes it possible for us to considerably broaden the students' mathematical perspective, thereby increasing the likelihood that they will pass on a positive message about mathematics to the school children that they will teach.

As the intervention, though innovative, is within the spirit of this endeavour, the administrator agrees to the intervention with no conditions beyond the 'covering' of the appropriate material.

At the onset of the intervention, the module plan contained the following components:

- The module starts with a series of data collection events, including a pre-module questionnaire, a class discussion about what mathematicians do, and the distribution of the journals and their guidelines (see appendices for details of each).

- The first phase of teaching consists of a 'ramping up' of the participants' practice of mathematical enquiry, in the form of a series of mini-enquiry projects that are chosen for the accessibility of their starting points.

- The second phase consists of a series of sessions completely devoted to the peripheral participants' pursuit of full-scale mathematical enquiry projects based on self-articulated starting points. During this phase, the instruction team and researcher interact with the peripheral participants in specifically determined way so as to preserve the agency of the latter.

- The third phase focuses on the topics prescribed by the module description that are not addressed by the previous phases. The teaching approach at this point returns to a more conventional style, though the class still works in small groups and through class discussions.

- The fourth phase encapsulates most of the formal assessment, including the presentations of the participants' enquiry projects, the submission of a portfolio containing both assignments and project writings of various stages, and a quiz addressing the material of phase 3.

Though each of the above phases and their characteristics are carefully designed to address one or several of the design criteria discussed in the literature review, in the tradition of action research, adaptations are made during the intervention, to fine tune

the intended experience of the practice. In the following section, I describe the intervention chronologically and in more detail.

Chronology of the Intervention in Practice

As a methodological approach, action research integrates continual adjustment of the practice it is investigating. In addition, the implementation of the teaching approach is an essential part of the response to the first research question. Consequently, it is important to consider the teaching approach not only as it is planned, but also how it is carried out, as it took place, with emphasis on changes and their rationale. This section presents a chronological narrative of the intervention. For the sake of lightness of reading, I refer to the peripheral participants as students, and the full participants using the role they are assigned, i.e. the teacher-participant, the teaching assistant-participant, and the researcher-participant.

Phase 0: Data Collection and Module Introduction

The presence of research components in an intervention can have an impact on the perception that the participants have of the teaching approach being assessed. Making use of this impact helps design the way in which the artefacts connected more closely to it are implemented.

In the first session, before the teaching begins, data is collected in two forms. The students fill out the pre-module questionnaire, then, to further emphasise the difference in teaching approach, part of the session is devoted to a recorded, moderated whole-class discussion of what the students think a mathematician does, both within society and in their day-to-day activities. On the one hand, this discussion constitutes an act of data collecting, but, most importantly, from the students' point of view, it can help set the tone for the module by bringing up issues of a more reflective nature about what it is about. This intention is supported by some of the items in the questionnaire as well as, and more significantly, by another data collection instrument: the personal journals. These are introduced in the second class, and the guidelines for them contain such statements as "[the journal] is also intended as an aid for you to reflect on the experience of learning mathematics, in particular in terms of your future pupils." and "Focus on your attitude and feelings about the content and the way you are learning, your process of understanding, and your overall experience" (See Journal Keeping Guidelines in Appendix 5).

The last two research artefacts that can, by their presence in the teaching approach, have an impact on the students' views are the video recording of the students' oral presentations at the end of the term, and the post-module questionnaire collection.

In addition to the research artefacts, the first session also includes the distribution of the brief course syllabus (see Appendix 8). This syllabus includes a mark breakdown, and a short description of the mathematical enquiry project. As the main project needs to be given its rightful weight, it is assigned a significant share of the mark: the handed-in write-up is worth 35% of the mark and in addition, the oral project presentation (at the end of the module), together with general classroom participation, are worth an additional 15% of the final mark. Finally, part of the homework encompassed by an additional 30% is a result of the activities in the first phase, the mini-enquiry projects of September. This description does not mention the connection to full participant, professional practice, but discloses the focus on the process, and the open-ended nature of the task, citing that the planned "explorations" should be:

> "a step into the unknown. [...] The principal hope for an investigation should be that totally unexpected things turn up, that different kinds of approaches to problems should appear as different pupils tackle different aspects of the problem in different ways." (Cited from Driver: 'Investigative Mathematics in School' *Mathematics in School* volume 17 number 1, 1988, p. 2).

Discussion of this project component of the module is deliberately kept brief so that the focus can remain on the tasks at hand. The decision to withhold the connection with professional practice stems from the teacher-participant's impression of the group, that this might have an adverse effect on their engagement.

Phase 1: Mini-projects

The purpose of the first phase of the teaching approach proper is two-fold. Firstly, it is designed to establish a new kind of social contract between the students and the teacher and teaching assistant, by presenting the former with situations within which they are required to engage in mathematical enquiry, thereby developing their sense of the practice that is expected from them. Secondly, the mathematical topics are selected, not for any curriculum areas that they might address, but for the accessibility of their content.

Typically, each session is structured around a mathematical topic introduced through a situation that is discussed briefly with the whole class, after which the students break

into smaller groups to explore one aspect or another in more detail, either solving a specific problem or investigating the mathematical structure of the topic, for instance by generating examples. This is done in alternation with whole-class discussions of the results of the small-group work. Occasionally, the teacher-participant asks students to present their results in front of the whole class.

The first lesson, which is seen as the one setting the tone for the module, centres on the concept of proper colouring[22] as applied to the regular triangular grid. As such, the lesson is very accessible since there is at first glance little mathematical content. This impression is deceiving, of course, since the number of allowed colours, for example, has an essential part to play in the number of possible solutions. Rather quickly, the students realise that with two colours, the result is determined by the colour choice for the first tile. This deduction is discussed, and the students compare it to the case of three colours. The new situation is much more interesting, presenting the additional complexity that colour choices are not entirely determined at each stage, nor are they entirely open. Figure 5, below, presents an example of response to this topic.
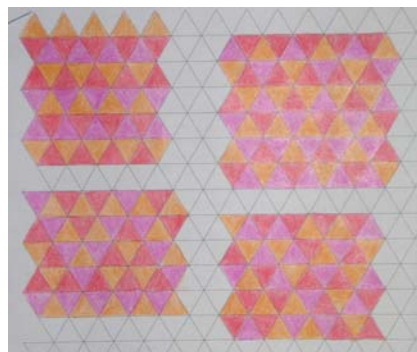
Figure 5   **Four solutions to the proper 3-colouring of a regular triangular grid (Alexa, 10/09/03)**

During this phase the class makes a brief foray into circular Origami (Knoll, 2001), investigates polyhedra and tilings, their symmetries and the Euler characteristic[23], colours the polyhedra and considers the symmetries of the result, considers the Platonic solids and why there are only five, and finally investigates some connected graphs and their relationship to polyhedra.

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[22] In this case, a proper colouring is defined as a way of colouring a region of connected tiles (no gaps and no overlaps) so that no two tiles with a common edge (side) have the same colour.

[23] The Euler characteristic describes the relationship between the number of faces, edges and vertices of geometric solids.

When the teacher interacts with the group as a whole, he focuses on the solicitation of ideas from the students, emphasising modes of thinking that rely on the structural forms of mathematics[24]. A typical observation from these classes recorded in the field notes is as follows:

> He explains what to do and then says "does that make sense?" [...]
>
> In the student presentation phase he doesn't say much and lets them present without comments. [...]
>
> After all the presentations [of the students' results], he does a recapitulation asking them what was done/said. (Researcher's field notes, p. 53, 08/09/03)

And an example of something to be found on a blackboard:

> Once you have got a pattern, check [that] it works for different examples, then ask [yourself] why it may always be true. Do the simple cases very thoroughly. (Blackboard, 24/09/03)

Adjustments to the teaching style are made during this initial phase. For example, the following reflection, noted in the teacher-participant's journal:

> Many good ideas mentioned in class by student presenters were not always written up [in the homework]. They need to be. [A] further ideas section is needed. (Dr. Zachary, personal journal, p. 2-12, 15/09/03)

… prompts an adjustment to the instruction. The plan is modified to include:

> a few minutes group time at the end of each concept to think of ideas to explore next [that] might help them develop problem posing skills. (Dr. Zachary, personal journal, p. 2-14, 16/09/03)

In addition, from this point on, the homework includes a bonus item:

> New: Interesting thing to look at next (Blackboard 2, 15/09/03)

This additional work had various purposes; besides helping the students understand the connectedness of the topics:

> Asking the students to guess the next concept is good. They can see why we define things certain ways. (Dr. Zachary, personal journal, p. 9, 25/03/03)

… the bonus questions makes them think a little more deeply about the topics and gives them a source from which to choose their project topics.

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[24] 'knowing when', see literature review.

An emphasis is also placed on forms of communication and correct vocabulary. For example, the teacher-participant exclaims:

> "do you see how powerful it is to have these labels, [when it comes time] to write down the instructions? Now I can just…" (Researcher's field notes, p. 1-55, 10/09/03)

Much important vocabulary is introduced during this phase as well, mainly through direct experience: words such as vertices, faces and edges, coplanar, collinear, geometry and topology, conjecture, counter-example and proof, and local and global properties of mathematical objects. Typically, he introduces the last in this list, and then asks:

> "How do you think I want to use these here?" (Researcher's field notes, p. 2-62, 17/09/30)

This kind of discourse expresses the intention of the teacher-participant to develop agency in the students. This is also in evidence in censuses he conducts about the students' findings, or through the language, for example when he says:

> "I want each group to *decide*[25] on the symmetries of the pentagonal dipyramid" (Researcher's field notes, p. 1-58, 10/09/03)

Though the connection to full participation is never entirely explicitly made, the suggestion is there in the discourse:

> "You're learning what it's like for the people who [discovered the mathematics] in the first place" (Researcher's field notes, p. 1-69, 29/09/03)

Another issue that presents itself during this phase of teaching, and which is connected to the agency of the students concerns the distinction between the modelling of mathematical enquiry and the creation of a space within which the students could experience it firsthand:

> Eva warns: do not model how to do the mini[-enquiry] projects so much that they see it rather than do it. (Dr. Zachary, personal journal, p. 2-14, 15/09/03)

This remark reinforces the need to give the students opportunities to direct the enquiry themselves.

In addition to the managing of whole-class discussions, the instruction team also interacts with the students through discussions with individuals or small groups,

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[25] My italics.

typically at their desks. Both the teacher- and teaching assistant-participant and the researcher-participant engage in this component of the approach, and it rapidly becomes the context for the reinforcement of agency in the process of enquiry. Alan, the teaching assistant-participant, remarks on the challenge of this mode of interaction:

> I've noticed in class recently that I'm answering questions a little bit more than I really should. I found that […] many groups gave up quickly […]. I found it hard to encourage them to keep looking without explicitly saying that there were more […] for them to find. (19/09/03)
>
> […] I did better today with asking more general, less leading questions.
>
> I thought the students did well […]. They very nearly happened upon the […] argument […]. It would have been nice to have a few more minutes of class to see if they could make the final connection. (Alan, personal journal, 22/09/03)

He expresses also a bird's eye view of the students' progress in that he may have a keener awareness of where they are headed. This awareness makes it all the more difficult to resist the temptation of leading them, thereby taking away their agency. To prevent this from occurring, two devices are used. Firstly, the mini-enquiry projects are never closed, that is, a final, 'right' answer is not given. Secondly, the homework that is assigned is carefully formulated, as described in the section on the design of the teaching approach, to focus on the reporting of the process of enquiry. This aims at encouraging a more epistemologically correct practice.

At the end of every second session (the group meets twice a week), the teacher-participant delineates the requirements for the homework which is due a week later. Generally, the content of the homework constitutes a continuation of the discussed findings for that class, a generalisation, an application or a transfer of context. This device therefore provides an additional opportunity for the students to engage in enquiry, this time more independently.

In the session between the assigning and handing in of the homework, some time is also spent answering queries about it and generally discussing it, giving a sense that the work is a process that could be revised and is not set in stone in a single attempt.

In her journal, Kerrie gives an interesting account of her experience with the first assignment, continuing on from the described enquiry about proper colourings of the triangular grid:

> When I heard our first assignment was to color patterns, I thought this was a "no-brainer". [...] I also assumed the writing we had to do about our patterns was probably the more significant part of the assignment.
>
> When I actually sat down to color my patterns, I soon realized more thought was required than what I had originally anticipated. [...] Careful planning was required to design and create a pattern that was able to stay within the guidelines prescribed and continue on in all directions on the page. A process of trial and error and trying something else is what I used to come up with a pattern [...] because I did not realize at first without seeing it on the paper that certain types [...] would not work. (Kerrie, personal journal, 16/09/03)

Interestingly, already at that stage, and probably due to the open-ended nature of the assignment, the students submit work that has a wide range of mathematical thinking registers. Some are simply trying to generate coloured regions which satisfy the chosen conditions of proper colouring and describing their selection process at each stage. Others try to extract what is happening on a more analytical thinking register, experimenting with systems of rules, as illustrated in Rachel's report:

> [...] however, the free placement did not last long. It was not long until my [...] color choice was affecting other [...] placement. I had to consciously watch and think ahead about color placement. I realize now that the forced color choice came through order. It depended when the diamond was filled in and where it lay on the plane. (Rachel, homework 1, 10/09/03)

Response to the homework focuses on reinforcing this kind of thinking. In addition, the mini-enquiry projects and their corresponding homework topics are used as the basis for the students' articulation of the main projects, as to both form and content:

> Part of the reflective process for the students is to critically assess whether any part of the course warrants further investigation in the form of a project. (Dr. Zachary, personal journal, p. 1-7, 23/03/03)

Overall, the purpose of this phase is fulfilled through the selection of the topics for the enquiries, and the fine-tuning of the interactions, preparing the way for phase 2, during which the students focus on their own enquiry.

Phase 2: Projects

Though it is the most substantial and certainly the most important, phase 2 requires the least amount of planning. On October First, the homework is set as follows:

> For Monday 6th: Write down which activities either in class or for homework you enjoyed the most or might be interested in exploring further.

> If you already have an idea you want to discuss for your project, write about it and say what you have found so far, even if it is not working properly. (Blackboard, 01/10/03)

The project has been little discussed previously, outside of discussions of the marking scheme, so that the students remain focused on the mini-enquiry projects and are not distracted by the idea of the big project.

When the project proposals are returned to the students, they contain only very general comments, a response that becomes typical of the interaction during this phase, that is, during the sessions of most of October in which the students work on their projects in class:

> The project was [kept] very hands-off. [The students] owned them. We were not trying to impress (with complex cases like the torus, for example). We were guiding at best, sometimes only with general strategies. We didn't close any doors. (Researcher's field notes, 18/12/03)

For seven sessions, the students sit with their chosen groups, and explore their topics, while the teacher, teaching assistant and I 'visit' with the groups, asking them what they are up to and, as the case may be, suggesting possible avenues. As the teacher describes it:

> Dr. Z: "This is where creativity begins" (Researcher's field notes, 01/10/03)

> Dr. Z: "the process of refining/changing the problem posed might in effect be the most educational part of the experience for them" (Researcher's field notes, 27/11/03)

Every week, each group of students is required to submit a 'progress report' which is then examined, though not formally assessed, and used for more formal yet still comparatively hands-off feedback, for example, in the case of a student investigating colourings of semi-regular tilings:

> Can you come up with comparisons of the different colorings of the same tessellations? (Dr. Zachary in Silvia's progress report, 14/10/03)

The purpose of this mode of interaction is to supplement the more informal classroom discussions in order to keep the activities rolling and give the students a sense of what is expected. The format of these reports is left open, resulting in a range of responses:

> I am looking at the project [progress] reports. […] Several only wrote half a page, no pictures and no explanations. Two groups, I think. Most groups wrote a lot. (Dr Zachary, personal journal, p. 40, 19/10/03)

During this phase, the teaching consists almost exclusively of individualised interactions with the groups. The intention is the same as in the previous phase, and examples of the interactions show both the difficulty of resisting the draw to scaffold:

> I had to stop myself:
>
> - telling Rob and Rachel[26] to look at tori
>
> - telling the tessellation people to look at Archimedean tilings (Dr Zachary, personal journal, p. 40, 19/10/03)

… and the fact that the students have a sense of what the teaching team is doing:

> Rob asked about his 'number of manipulations'. 'Is this one or two?' he asked. I [wanted to ask] him to define it his way, but he completed the sentence [for me]. (Dr Zachary, personal journal, p. 38, 13/10/03)

Though the teacher and I have prepared 'fall-back projects' in case a student or group of students should not have a topic for the project, this is not an issue as each group has a question that they articulate based on the work done in phase 1. Indeed, one student is so intrigued by one of the topics in phase 1 that he starts working on his project then, weeks before the start of phase 2. As the teaching assistant summarises:

> This project time is allowing some students to solidify older concepts that still aren't quite clear to them, and allowing other students to continue on and extend the concepts we've already studied or put them together in novel ways.
>
> It's been slightly surprising how well the project time has gone. I might have expected to be running around and constantly answering questions and dealing with frustrated students—but overall, the students seem to be doing extremely well on their own. Much of the class time passes with students working quietly among themselves. (Alan, personal journal, 16/10/03)

At the end of phase 2, each student is asked to submit a typed draft of their individual report. These reports are meant to reflect the spirit of the following quote (see Appendix 6: Project Report Guideline):

> We often hear that mathematics consists mainly in "proving theorems". Is a writer's job mainly that of "writing sentences"? A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure our minds are not playing tricks. (Rota, 1981, p. xviii)

In addition, the students are expected to outline four aspects of their project:

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[26] In two different groups…

- the initial problem posed (taken from their proposal),

- the process that the group went through, including dead ends encountered and changes of direction,

- the results,

- finally, possible new directions to follow.

The second section is expected to be the most substantial. After all:

> The process in September gives them a chance to look for patterns and rules and in October they can explore them more systematically. [The] goal for September-October is process. [They should be] assessed on process. (Researcher's field notes, 27/07/03)

As for the third part, describing the results, it is expected to contain key examples and/or counter-examples and a precise statement of the student's claims and reasons why. The students are also reminded that reasons why something does not work are acceptable mathematical results.

Initially, on the 8th of October, the due date for these first drafts is set for the 29th of October. On the 20th, this reveals itself to be a problem and the date is shifted to the 3rd of November. Though the project is not concluded because there is an additional cycle of feedback and resubmission planned, phase 2 is completed and the remainder of the sessions are devoted to phases 3 and 4. The drafts are examined and handed back, and the final version is due on December 10th, while the oral project presentations take place on December 3rd and 8th.

Phase 3: 'Regular Teaching'

On November 3rd, a more standard pace of classroom activities is begun. The focus now is on mathematical content and homework is taken from the assigned textbook. The syllabus driving the class includes sequences and series, combinatorics and probability. The teaching style, however, does not completely move to a traditional 'direct instruction' mode: Activities include whole-class and small group discussions and class interrogation similar to those in phase 1 as well as demonstrative periods and the use of prepared worksheets. An attempt is also made to connect some of the topics to other domains of mathematics, notably in the use of geometric representations of sums of series. Based on class observation notes, I can see that the students' minds often wander away from the topics at hand, possibly due to the new pace or to the fact that the topics

are at least partly known to them. Peppered throughout my observation notes are remarks such as:

> You can hear the hum of the [overhead] lights.[...]
>
> He is not really asking them to do any thinking. [...]
>
> He is leading to find [the] formula. He has to spend 10 minutes to illustrate the division—class is really quiet—[learning from experience] is forgotten. (Researcher's field notes, 17/11/03)

In other instances, the teaching style contains remnants of the previous phase:

> "This is really important because we're choosing a notation." (Researcher's field notes, 19/11/03)

This part of the module also contains two quizzes, the second of which, though it took place in the session set aside for the final exam, counts for the same part of the mark. In addition, due to the American holiday of Thanksgiving, the November 24th session is declared optional as many students travel home for the Thursday celebration. In the end, there are four students in that class, and the time is spent discussing their projects and examining the contents of a CD presenting material bringing together art and mathematics (Emmer & Schattschneider, 2002).

A notable strategy that is carried through this phase is the inclusion of the bonus 'Interesting thing to look at next' question. This is done in order to create an experiential link between the enquiry-oriented phases and the content-oriented phase, since, as the teacher-participant puts it, thinking of what comes next is:

> a huge part of professional practice in mathematics (Interview, 19/10/03)

This whole content-oriented phase is characterised by a distinct change in pace, as described by the teaching assistant:

> I've noticed that student interest in the class has plummeted. Attendance is lacklustre, students just up and leave at the break, some students sit in class and read things for other classes (or the Daily, or things not for any class). When asked to work in groups, some students barely even turn to work with their group members. It's a little disheartening. (Alan, personal journal, 17/11/03)

The shift to a focus on the mathematical content—which many if not all of the students have seen before—appears to be dissatisfying.

Phase 4: Summative Assessments and Final Data Collection

The last two weeks of class, from December 1st to 10th are devoted to most of the module assessment: December 1st, the students fill in the post-module questionnaire, then spend class preparing for the presentation, December 3rd and 8th are devoted to the class presentations and December 10th, the students take the final quiz. They are also given the opportunity to revise their project reports and everything, including homework re-submissions, is handed in, in the form of a portfolio, at the end of term (December 10th).

Overall the module is rich in teaching approaches and the strategies are combined with the selection of content focus to spotlight the potentially indeterminate nature of mathematics. The interactions between the teaching team and researcher, and the students occur at various levels, including the whole class interactions, which provide qualitative guidance regarding the teacher's expectations, individual group interactions, which fine-tune the scaffolding for rigour in the practice, and the written interactions, both through homework feedback providing more scaffolding for rigour, and through journal responses providing feedback on the participants' reflections about their practice and experience.

Chapter 5: The Participating Students

In the introduction and in the literature review, I briefly discuss the overall context of the study. Though the module is designed for elementary student teachers (EST) in an integrated, 4-year undergraduate programme, due to the modular system of university education in the United States only a portion of the participating students are effectively registered as such. One of the consequences of this factor is that sample uniformity cannot be assumed, and that demographic characteristics therefore need to be explored. In addition, other attributes of the group can have a significant role in the initial characterisation of the sample. I explore these attributes here through an examination of the group's responses to the pre-module questionnaire, beginning with demographic data, and following with responses specific to mathematics.

The group is composed of 33 women and 4 men. Moreover, at the time of the survey (only 35 students responded due to module section changes, as I explain in Chapter 4), their ages range from less than twenty-one (16 students), to between twenty-six and thirty (3 students). The class records show that of the 37 students registered, 24 are pursuing the predicted undergraduate degree in education, though not necessarily in elementary teaching, 3 are pursuing other degrees and 6 have not yet declared a 'major'. In addition, 4 students are working on a course-based master's degree in education. This option is offered to students who are pursuing elementary teacher certification, but who already have an undergraduate degree. Of the students registered as undergraduates, 4 are in their second year (of four), 15 in their third and 12 in their final year.

As I discuss in the description of the programmatic context (Chapter 4), the module relevant to the study is one of two parts, which together represent two of four mathematics modules that EST are required to take. At the time of the survey, I have data for 23 students pursuing an undergraduate teaching degree. 22 of these student have also completed 'College Algebra and Probability', 7: 'Statistics', 3: 'Calculus, and 1 each 'Logic' and 'Functions, Statistics and Trigonometry' (FST). Of the remaining 12 undergraduate students (non-EST), only one student has not taken 'College Algebra'. A

majority of the students registered to an undergraduate degree have therefore comparatively recently been exposed to higher-education mathematics. Of the 35 responding students, 2 rate themselves as having 'excellent' mathematical ability, 15 as 'competent', 14 'average', 2 'weak' and 1 'poor'. Moreover, 25 choose algebra as their stronger topic, against 4 for geometry and 6 for arithmetic. Overall, therefore, the participants' experience with college-level mathematical content is comparatively recent and significantly higher than that which they are required to teach in their chosen career, at least for the EST.

A closer look at their experience of the discipline of mathematics is revealed by their responses to items 8-10, and 40-41 (see Appendix 3). When asked whether, at different stages in their schooling, they had been taught using open-ended, investigative activities where the teacher gives class time to explore mathematical topics, the students respond as illustrated in Figure 6, below.
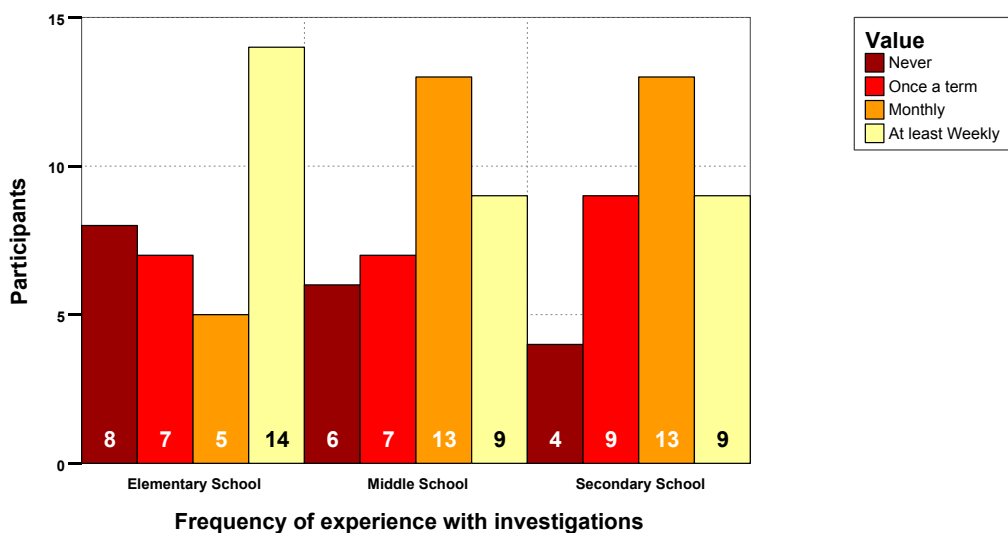
Figure 6   Experience with investigation in the mathematics classroom

This chart tells us that in most school levels, at most about one quarter of the responding participants have not done investigative work (8 of 35 in Elementary, 6 in Middle and 4 in Secondary School, in maroon in the chart), and at least half had done so at least monthly if not weekly (5+14=19 of 35 in Elementary, 22 in Middle and 22 in Secondary School, orange plus yellow), though, as discussed in the literature review, these investigative approaches can include a wide ranging spectrum of activities, which are not specified in the question.

Correlating the data between the three items reveals that of the 35 responding participants, a total of 14 remember doing investigative work at least monthly

throughout their schooling, 6 of which did so at least weekly throughout. Conversely, 5 participants remember experiencing investigative work at most once a term, in at least one stage, including 2 participants not remembering ever doing it at all.

Item 40 and 41 report on the participants' first stumbling block in their overall mathematical learning process, as illustrated in Table 11, below:

Stage	Elementary	Secondary	College	Word Prob's
Respondents	9	17	6	3

Table 11:   **Categorised responses to item 40 of the pre-module questionnaire**

The format of the item is open-ended, and the responses are therefore categorised by the schooling stage to which the topic is associated, with the exception of word problems, which are encountered throughout. 'Elementary', therefore, includes division, long division, fractions and generally arithmetic, 'Middle' is generally known in the US to be a time when the material from elementary is reviewed and deepened, which is why it was left out; 'Secondary' includes algebra, geometry, trigonometry; and 'College' topics (even if taken in secondary school) included calculus and logarithms. The significant result of this item is that about half of the respondents revealed geometry to be their first big stumbling block. This result is particularly relevant as the topic is a significant part of the teaching approach.

In summary, the group is composed of the following ratios:

Variable	Ratio
Gender	33 (women) : 4 (man)
Age	16 (<21) : 16 (21-25) : 3 (>25 years old)
Degree Programme	24 (B.S.Ed.) : 4 (Masters') : 9 (other/undecided)
Year of study	4 (2nd) : 15 (3rd) : 12 (last) : 4 (Masters)
Self assessment	3 (excellent) : 15 (competent) : 14 (average) : 3 (weak/poor)
Investigative experience	2 (never at all) : 24 (variedly) : 9 (often)
Stage of stumble	9 (Elementary) : 17 (Secondary) : 6 (College) : 3 (word problems)

Table 12:   **Demographic information summary**

In addition to the participants' general demographic information and specific mathematics experience, I collect data regarding their views of the role of the professional research mathematician through item 42:

> If you had to explain to one of your future pupils what a mathematician does, what would you say?

Though this question seems directed specifically at students who are pursuing the education certification degree, who only constitute about 4/5 of the class, the nature of

the module means that the students expect it to relate to the teaching of school children. The responses to this open-ended question are parsed into categories that emerged from the data in light of the literature review on views of mathematics as a subject (Chapter 2). The participants' views of the role of mathematicians (as opposed to the views of the subject) are defined as follows:

- Application of mathematics (analogous to Instrumentalist view): The mathematician knows facts, has skills, etc and uses this to an external end, applying it to other areas such as physics, accounting, economics, etc.

- Understanding of mathematical 'reality' (analogous to Platonist view): The mathematician develops an understanding of a static but unified external 'reality' called mathematics. The problems he solves originate in this mathematical 'reality'.

- Creation of new mathematics (analogous to Problem-Solving view): The mathematician is an integral part of a community that develops a dynamic, expanding creation with a cultural attribute.

- Education: The mathematician is responsible for communicating mathematical knowledge to lay-people.

- Unspecified: These answers cannot be placed into one of the other categories.

The results are illustrated in Table 13, below.

Categories	Example of response	Count
Application	Uses the field of math to explore, research and solve problems in all different areas of life. (Bridget)	13
Application/ Understanding	It's a way to practice solving problems and understanding/learning how to make complicated situations into solvable smaller situations. (Christie)	4
Understanding	Mathematicians research, study, and work with mathematics to find better explanations and solutions to complex and everyday mathematics. (Isabel)	9
Understanding/ Creation	A mathematician explores mathematical concepts and tries to find new and innovative ways to solve problems. They seek understanding, and they try to do this by coming up with new processes. (Alexa)	4
Creation	-	0
Education	Makes math problems easier for us to understand. (Elise)	1
Education/ Understanding	A mathematician works on solving new mathematical problems and explains them to other people. (Irene)	2
Unspecified	A mathematician is someone who is always dealing with numbers. (Jean)	2
	It's a necessity whether people like it or not it needs to be taught and learned. (Geoffrey)	
Total	-	35

Table 13: **Categorised responses to Item 42 of the pre-module questionnaire**

As the table shows, the majority of respondents (26/35) express the view that mathematicians apply mathematical facts and skills and/or understand an external mathematical reality. This view is not really surprising if one considers that their

experience with mathematics is most likely limited to the classroom, with its assessment and instruction practicalities.

Overall, the sample presents a profile that is consistent with the experience of mathematics provided by contemporary classrooms, with its emphasis on understanding as well as skill, a focus on 'real life' applications, but few opportunities for experience of or reflection on the practice of full participants in mathematical enquiry. In Chapter 7, I examine the participants' affective responses to mathematics more thoroughly, with an emphasis on the change, or lack of change in these affective responses across the intervention.

Chapter 6: The Experience of the Participants

The qualitative component of this study consists of a comparative analysis of the participants' experience of the intervention, as reported in their journals, against an exemplar developed both from my own experience and from an examination of the literature about mathematician's full professional practice. This comparison focuses on the 5 criteria developed in the literature review and applied to the design of the teaching approach:

1. A novel starting point (in this case, of their own choosing),

2. An open-ended process,

3. An open-ended goal-state,

4. An atmosphere of security, and

5. An overall experience of the practice that is similar to that of full-participants in mathematics research.

Across the 16 journals that I retain for the final analysis (see Chapter 3 for the journal selection criteria), the 181 entries yield 536 quotations. I break down the results in terms of each component of the theoretical framework, beginning with Criterion 4, followed by Criteria 1-3, and finally by Criterion 5. Within each component, I examine the data in terms of the four teaching phases (see Chapter 4) and of the experience's success with respect to the aim of the intervention. In the case of criteria 1-3 and 5, I focus on a small number of individual participants by developing a narrative of their responses to the experience throughout the intervention.

Overall, the quotations manifesting valent (positive or negative) affective responses (433 of 536) show an 8:5 proportion of positive to negative orientation. This bias is not unsurprising if one considers the need to please the reader (myself) after a personal

interaction, and given the participants' awareness[27] of my stake in the intervention. Conversely, it is possible to consider that the participants who are dissatisfied with the experience simply choose to opt out of the journaling component by reducing their entries (17 participants wrote less than 5 entries), thereby excluding themselves from selection for this analysis. This proportion is therefore important to consider in the interpretation. Indeed, absence of data relating to an expected experience does not mean that the experience itself is absent. The open-ended nature of the instrument can also mean that a given experience is simply left unreported, for any number of reasons, including the perception that it is not unusual, or even that it is a non-event (see for example, Hadamard's (1945) incubation, in the description of criterion 5 in Chapter 2).

In addition, the presence of a small number of participants pursuing their teaching qualification through a Masters' degree should be noted. Of the 16 selected participants, three are in this situation, Jill[28] (19 entries), Linette (8 entries) and Patrick (5 entries). This ratio ($3{:}16{\approx}19\%$) is higher than for the group as a whole ($4{:}37{\approx}10\%$). Their status is emphasised, from this point on, with an attached asterisk.

Pseudonym	Team (members)	Degree pursued	Year in Programme	Age Group	Gender	Self-determined ability – before
Geoffrey	A (2)	B.S Ed	Junior (3)	< 21	Male	Average
Alice		B.S Ed	Senior (4)	21-25	Female	Competent
Alexa	D (1)	B.S Ed	Junior	< 21	Female	Competent
Christie	F (1)	Undecided	Junior	26-30	Female	Competent
Sue	G (3)[29]	B.S Ed	Junior	< 21	Female	Average
Laura	I (2)	B.S Ed	Senior	21-25	Female	Competent
Isabel		B.S Ed	Junior	< 21	Female	Competent
Sandra	J (3)	B.S Ed	Junior	< 21	Female	Competent
Joan		B.S Ed	Junior	< 21	Female	Competent
Samantha	L (2)	B.S Ed	Senior	21-25	Female	Average
Petra	M (4)	B.S Ed	Senior	21-25	Female	Average
Patrick*	N (1)	Masters	N/A	26-30	Male	Competent
Jill*	P (3)	Masters	N/A	21-25	Female	Weak
Linette*		Masters	N/A	21-25	Female	Competent
Pippa	R (2)	B.S Ed	Senior	21-25	Female	Average
Emily	S (2)	B.S Ed	Junior	< 21	Female	Average

Table 14: **Demographic data for participants selected for the journal analysis**

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[27] In this chapter, I use the term awareness as I define it when I discuss mathematics appreciation (Cockcroft, 1982): to characterise responses as "a knowing which is more intuitive and can therefore not be as easily reified".

[28] These names are all pseudonyms. Mine is the only real name that is used throughout.

[29] The journals of Sue's two partners are not among the 16 I analyse.

In Chapter 2, the theoretical foundation of this intervention is presented as a departure from what is traditionally thought of as 'problem solving in the mathematics classroom' to a practice which differs by the implementation of criteria for the teaching approach that I cite above. In this chapter, I examine the way in which the participants' experience corresponds to this intended practice. Keeping in mind that the experience of the practice is different between individuals, particularly as their previous experiences with mathematics are heterogeneous, I examine the data for evidence of the impact of the design criteria.

Criterion 4: An Atmosphere of Security

In the description of Criterion 4, I discuss the conditions that allow the participant to feel safe to engage in the proposed critico-creative tasks. These conditions include a component focusing on the nature of the mathematical content, and one focusing on the didactical approach implemented by the teaching team, including the way it is manifested in the marking scheme. Each of these components is discussed in the analysed journals, as shown below.

Accessibility of the Starting Point

In the 'ramping up' phase (Phase 1, see Chapter 4), students are of two minds about the mathematical content. Some are frustrated that so much time and effort is spent on topics that they consider simple:

> So far this class seems too easy and basic. I think that a lot of us are getting bored very quickly. I like the fact that we get to work in groups but it doesn't really seem necessary since we all know the answers already. (Laura, 10/09/03)

Others are glad that the content is accessible:

> Up to this point in the class I feel very comfortable with the concepts and ideas […]. Some concepts were very easy for me to understand and some were difficult. (Geoffrey, 10/09/03)

And if they find it difficult, they rationalize it as being a result of their personal attributes:

> Working with 3-D shapes has been difficult because you have to have a visual concept of these shapes, and that is something I don't really have. It is hard to spatially see where the lines of reflection and rotation occur. (Alexa, 17/09/03)

… or their past experience:

> In order to complete last week's homework I visited office hours. […] It became apparent that some of the old geometric concepts are not as clear as they seemed. (Patrick*, 20/09/03)

In some cases, the comment reflects a view of mathematics as a thinking discipline:

> It's difficult to [go] back to geometry and to try to prove that something is true. It is easier to think in terms of disproving an idea. (Jill*, 10/09/03)

Later, during the main enquiry phase (2), a small number of students comment on the connection between the two phases, which suggests that the 'ramping up' is a useful device for the participants to develop their own sense of the level at which they are expected to work:

> A lot of this goes back to the project that we did in small groups, combining triangles and determining a pattern, but what is interesting is the fact that when 2 different types of faces are used, the equation still holds true. (Jill*, 19/10/03)

> What really excites me is that all the earlier concepts have mostly come into play in my work. (Patrick*, 27/10/03)

Four of the 16 participants express their responses to the mathematical content through the use of the opposition pair 'concrete' and 'abstract'. This choice of terminology is interesting in that it seems to parallel the theoretical framework regarding the participants' view of mathematics as a discipline. Jill*, for example, associates mathematics with concreteness because "There is always a right answer" (Jill*, 17/09/03), and feels relieved in November because:

> It is always nice to have a concrete formula to follow, and to just plug in numbers and solve for a certain variable. (Jill*, 12/11/03)

Petra associates concreteness in the mathematics classroom context with an increased amount of teacher direction:

> We are working on prisms and answering, how many closed up shapes have every face the same, and every vertex the same? This is making me really not like math. I need more direction, my mind does not work in a mathematical way and I find no interest in learning more and trying to discover. I need more *concrete* things to go by. (Petra, 17/09/03)

Alexa contrasts 'concrete' with 'conceptual':

> All of the work we do in this class is so conceptual that I have a hard time understanding. It's like there's nothing *concrete*. [...] I do know that I am looking forward to beginning work in the workbook, because maybe the concepts in there are a little more *concrete*.
>
> [EK: What do you mean by concrete? Please give examples...] *Concrete* to me means: formulas, applications of formulas, actual problems in a workbook with answers. *Not concrete* to me means: guesswork and ideas. I know that the more abstract ideas are needed in order to understand the more concrete stuff like formulas and the application of formulas, but there comes a point where there is just too much guesswork. (Alexa, 28/09/03 and later)

These definitions contrast with that suggested by Hadamard, as cited and interpreted by Pimm:

> [T]he mathematician Jacques Hadamard has offered the maxim: "The concrete is the abstract made familiar by time". He is suggesting that concreteness is relative to our past experiences, rather than being an attribute of certain things in themselves. (Pimm, 1995, p. 27)

For Jill*, Petra and Alexa, mathematics is concrete when it corresponds to the Instrumentalist view, and becomes abstract or conceptual when it leaves it behind. Linette*'s use of concrete perhaps approaches Hadamard's perspective more closely. Her responses give a range of meanings. In September, she explains that:

> I enjoy algebra much more than geometry because it seems much more *concrete* to me.
>
> [EK: Please elaborate on what you mean here by *'concrete'*...] By *concrete* I mean something that I can put in words or envision through the process. (Linette*, 06/10/03 and later)

This description suggests, as Pimm does, that concreteness is an attribute of the activity in relation to the participants' past experience, just as the routine-ness of a problem is defined, according to Polya and others, in terms of its familiarity to the participant. This parallel suggests that the concreteness that the participants find lacking in the tasks is indeed a manifestation of the shift from routine, to non-routine and (perhaps even) critico-creative tasks that the intervention is promoting. In November, during 'regular teaching', she relates concreteness both to connections to 'real life examples' and to working with ready-made formulas, suggesting a more routine approach:

> Now that we've been working with formulas I feel a little more comfortable. I feel like this is more *concrete*, meaning that I can see where everything is coming from and I can envision a conclusion. (Linette*, 17/11/03)

In the 'mathematical enquiry' phase (2), the participants make no comment on the accessibility of the mathematical content, perhaps because their agency in the choice of topic allows them to select one that is well suited to their level in this respect. Instead, many comments focus on the process they are undergoing, as I discuss later, in the section on criteria.

In the 'regular teaching' phase (3), implemented in November, the teaching approach is still largely about conceptual understanding rather than about skill reproduction. The responses to the content are rare and there are more comments focusing on the instruction, reflecting this emphasis:

> The sequence stuff was very easy for me in college algebra, but now it is not so easy. He explains things weird or different from how I learn. It seems like simple concepts are being turned into complex concepts because he is explaining things awkwardly. Who knows. (Samantha, 11/11/03)

> I appreciate Dr. Zachary's method of teaching. Although we are duplicating the same processes in arithmetic and geometric sequences, he presents it in a way that is more of a search than just plug in your numbers here. (Jill*, 17/11/03)

> I really like how Dr. Zachary explains why formulas are certain ways and has us critically think before we hear the answers. (Joan, 24/11/03)

Overall, the participants' responses to the mathematical content reflect the intention of the teaching approach with respect to the atmosphere of security required for their engagement. Though the responses vary, the ramping up phase seems to accomplish its aim in this respect, and the absence of comments in October can be seen as manifesting a good comfort level on the part of the students. The presence of comments in Phase 3 suggests that the topic is one that the participants do consider, further justifying the interpretation of its absence in phase 2.

In the theoretical framework regarding affective responses, the model proposed by Skemp (1979) is relevant to this aspect of the participants' experience: the teaching approach integrated issues of threats to comfort, which Skemp described as generating fear (in the case of movement towards an anti goal-state) or unpleasure (in the case of movement away from a goal-state), by modifying the social contract in the classroom to one in which the goal-state is not focused on the development of a successful, certain mathematical result. In many cases, the responses described in this section illustrate the difficulty of this change of perspective.

The Social Contract

In the section of the literature review focusing on Criterion 4, I introduce Mason's (1989) notion of the tension that 'arises between pupils and teachers', regarding the behaviour each party expects of the other. In the theoretical framework, this is presented as an effect of the social contract, and can be an obstacle to the participants' full, critico-creative engagement in the experience promoted by the intervention, particularly with respect to the expectations the teacher has of the students.

In the design of the teaching approach, this issue is dealt with not only through a careful monitoring of the complexity of the mathematical content used, but also through the development of appropriate teaching guidelines and the design and implementation of the formal assessment mechanism (marking). The obstacle presents an opposition between the necessarily open-ended nature of the task and the need for security, on the part of the participants. This issue is particularly acute with respect to the module assessment criteria. The expectation is that the content of the responses evolves during the course of the intervention, and the results are therefore presented chronologically.

Overall, the journal responses focusing on this issue reflect the social context at each phase of the intervention. For example, a leitmotiv of concern about the teacher's expectations is carried through Phase 1, with phrases like 'unclear what we were supposed to tell the class' (08/09/03), 'minimal explanation in class' (14/09/03), 'assignments … so abstract' (Jill*, 17/09/03), 'don't know what he expects or wants' (Jill*, 23/09/03).

Many responses to the teaching in Phase 1 manifest this tension through personal preference:

> I like more structure so this assignment was a little harder for me to do. I never know if I was doing it correctly. (Petra, 09/09/03)

In some cases, this response is compared to past experience, where the teacher is not the only resource:

> What frustrates me most about the homework is that when I am stuck, I don't know what to do. We have no textbook to refer to for any sort of help, so when I can't understand why something is the way it is, I feel helpless. (Alexa, 28/09/03)

In other instances, it expresses an enquiring attitude:

> I am also a little curious where we are going with all of this and how we will be assessed. (Joan, 10/09/03)

> How is he going to [mark] our first assignment? (Sandra, 09/09/03)

… or anxiety about the formal assessment:

> I hope that I can get my brain to work well enough to get a good [mark] in this class. It would be nice to have a syllabus for this class so I know what to expect and when to expect it. (Pippa, 29/09/03)

Interestingly, Christie interprets this situation as an opportunity for self-reflection:

> I enjoy doing group work because I am able to compare my comprehension of the class to others'. This helps me monitor whether I am behind (need help), ahead (boring me) or on track (just like everyone else). I am on track because my classmates have similar questions, doubts, and observations to mine. (Christie, 22/09/03)

… and Alice, for furthering her mathematical understanding:

> After receiving my homework back I decided to redo it. I was apparently unclear on the guidelines and had a lot marked wrong. After reading the instructor's comments, I have a better understanding of the Platonic solids. (Alice, 06/10/03)

In Phase 2, the tone changes. Though some participants still express tension regarding the teachers' expectations:

> I wish we had some guidelines or a rubric for the project. I have no idea about what they are looking for. There need to be [mark]ing expectations. This class seems to have no structure which is frustrating because I always feel lost. (Samantha, 26/10/03)

> I wish I had a list of requirements so I knew exactly what's expected of me. (Pippa, 13/10/03)

… in some cases, it is combined with more self-confident comments:

> I don't really understand the project or what we're supposed to do for it, but hopefully I figure it out. (Isabel, 15/10/03)

> The project is going well so far, but I'm still confused on how to do the write up. (Isabel, 22/10/03).

In addition, many responses use language that expresses more ownership, including the use of 'decide' (Petra, 09/10/03, 15/10/03, 26/10/03), 'want to work with' (Joan, 08/10/03; Petra, 09/10/03), 'freedom' (Joan, 11/10/03; Petra, 09/10/03), suggesting a

shift in the feeling of agency on the part of the participants. I discuss this phenomenon more extensively in the report of the results regarding Criteria 1-3, below.

Several students still mention the need for feedback, but there seems to be a loosening of the interaction, suggesting a decrease in importance given to the canon and the full participants of the sample community:

> I still don't think [my project] wrapped up completely, so I'm eager to get my draft back to see how I can improve on the project. (Jill*, 03/11/03)

> We got feedback on our assignment and it seems as though we are on the right track. I feel confident and that is a good thing. […] I really feel really confident! I like my group and all of the helpful feedback we have been receiving. (Joan, 15/10/03)

> It would be interesting to get feedback on our first draft and see if our conclusions are really ideas you can get from our work. (Emily, 27/10/03)

> I just finished my rough draft. It wasn't as difficult as I thought it would be, now let's see when I get it back. (Joan, 02/11/03)

Isabel, in her description of this issue with respect to the project conveys a strong reaction:

> Today I stumped the professor. He wanted to find a glitch in our data so that we'd have something else to work on, but my partner and I covered a lot of ground. It was reassuring that he could not come up with anything, because it means we did a good job with our project. (Isabel, 27/10/03)

The shift in the nature of the interaction is strong enough that she sees the teacher's role as reduced to a reaction to the work she has done with her team, rather than guidance towards a goal that he already sees.

Emily's journal shows her clear awareness of the purpose of the approach with respect to the issue of engagement. Already during Phase 1, she writes:

> I think that he might just be helping us become independent and group learners. (Emily, 15/09/03)

In Phase 2, she continues with:

> I am becoming a much better independent learner. […] I understand that this class is not so much about right or wrong answers, but about how you came to think the way you did; hence, why we explain in words everything we do in our homework. […] Also, it helps the professor know how you analyzed the problem even if your answer was wrong. (Emily, 20/10/03)

Her expectations seem to have essentially lined themselves up with the intention of the intervention, at least with respect to the atmosphere of security.

In Phases 3 and 4, the responses demonstrating students' expectations show an awareness of the return to a more 'normal' teaching approach:

> It was a relief to begin working out of the book. It is always nice to have a concrete formula to follow, and to just plug in numbers and solve for a certain variable. I feel like there is an end to my process, and that it is wrapped up, rather than left hanging as it was with the project. (Jill*, 12/10/03)

Emily comments that the project is more difficult for her than the teaching that she is currently experiencing. She interprets this difference as stemming from the 'lack of structure':

> It is nice to be done with [the project] for a while because I was getting kinda "bored" with thinking only about the project every class period. The lack of structure was making me not work as much as I could have been. It is just all about getting myself motivated to think critically and analyze thoroughly. (Emily, 02/11/03)

In the discussion of Criterion 4, I establish the necessity for an alternative to 'normal' scaffolding, to promote thinking about the structure underlying mathematics and thereby developing 'knowing-when'. This takes the form of the re-conceptualisation of the social contract (see Chapter 4). Despite this, participants such as Emily still find the approach in Phase 2 unsettling. The fact that the social structure that she alludes to is not provided as expected remains uncomfortable and a sense of relief is expressed by several others when it is returned. In contrast, Christie is happier with mathematical enquiry:

> The new information is a little complex and I much prefer doing mathematical research than mathematical theory!! It is much less interesting and is limiting to one right answer. (Christie, 05/11/03)

… Petra finds the return to a more directed approach difficult as well:

> We are back to working in the textbook, and I seem to always get frustrated when the teacher teaches on the board. I get it at the beginning and then I slowly lose the teacher as they go on because I feel like math is information overload. I wish it was split up more so we had chances to look closer. (Petra, 05/11/03)

… though others, like Joan, find it reassuring:

> I really like how Dr. Zachary explains why formulas are certain ways and has us critically think before we hear the answers. (Joan, 24/11/03)

In contrast, others still find this return to be less than they hoped:

> It is so frustrating because it is a formula. You just put in the numbers and it should work. But I can't get it to! I worked with two others and they can't get it either! I'm giving up now. (Sue, 18/11/03)

Overall, the responses made in Phases 3 and 4, with respect to the tension described by Mason (1989), generally depict a need for understanding which can be achieved through further interaction:

> I had a lot of trouble with this homework. I can't get how it has anything to do with what we did in class. It is very frustrating because I want to understand it, but I can't. (Sue, 04/12/03)

> My group and I just met with Alan. The assignment was very misleading, and some of the questions seem impossible. [Alan] worked through them with us, and they make more sense now, but I'm not sure I would have made it to the end of the problem without some guidance. (Jill*, 05/12/03)

> I went to get help from Alan again tonight. Once again everything makes so much more sense. Hopefully I can retain it for the quiz. (Sue, 08/12/03)

> It really helped to be able to work on it in class and get immediate feedback when needed. (Samantha, 09/12/03)

According to the theoretical framework, the nature of the social interactions between peripheral and full participants, with respect both to the accessibility of the starting point and to the social contract, forms the context within which the peripheral participants can develop their sense of agency with respect to the main project and the direction of the explorations. In the next section, I discuss this resulting sense of agency, as reported in the 16 analysed journals.

Criteria 1 to 3: Agency in the Overall Process

Excluding Laura, who mainly focused her responses during Phase 2 on a comparison to the other section of the module, the 15 other participants included in the analysis discuss agency in their experience of mathematical enquiry during the pertinent phase. Emily, whose responses on the topic of the social contract showed an awareness of the intention behind the intervention, makes several remarks about agency. In Phase 1, where she mentions the idea of becoming more independent learners, she also says:

> The ideas presented about how to prove […] have made me question what is an accurate way. Before the discussion today, I just accepted the ideas were true when we wrote them on Monday. I never questioned why they would be right or wrong. Although all of the ideas still aren't perfectly clear, my understanding has increased and I know I will continue to understand more each day. (Emily, 10/09/03)

> To back track a little, I really enjoyed using the humongous triangles out in the hall. It not only taught me math, but also how to develop ideas that are both my own and my group member's. (Emily, 29/09/03)

During Phase 2, she discusses agency in the starting point:

> I was skeptical at first about whether I would even know where to start, (Emily, 13/10/03)

… in the process:

> Our original purpose of the project varies from the purpose we were asked to describe in our findings. I know that the new ideas given to us are probably related but trying to bring in a new concept at the end is hard since we thought we [had] figured out the problem that was at hand. (Emily, 27/10/03)

> Since these are our projects, asking the professor questions does not make sense, so me and my partner decide on the answer ourselves. (Emily, 20/10/03)

… and in the goal-state:

> Forming an overall conclusion about our project is frustrating. (Emily, 27/10/03)

In this last comment, she displays awareness that the goal-state, in mathematical enquiry, can be elusive, as I discuss in the section of Chapter 2 describing the nature of the goal-state.

This level of awareness of agency throughout the enquiry is comparatively high, and Emily seems overall to have reflected seriously on her experience. Other participants' responses display less awareness. This difference can be seen as lying in either the awareness or the experience. Pippa, for example only discusses the topic as follows:

> I think the project I've picked will be really interesting. I'm not quite sure where to go with it. (Pippa, 13/10/03)

After which she focuses on the learning she achieves, and how it relates to what the other section is doing.

Geoffrey's responses suggest that he attempted to develop agency in the process, but relied heavily on feedback:

> Everything I have tried so far has not worked. I am at a dead end unsure of where to go next. (Geoffrey, 22/10/03)

> After talking with the professor at the end of last week I feel a little more confident about where my project is headed. They gave me good advice to take another step in my project. Even though it's almost a completely new task it still correlates to our original topic. (Geoffrey, 30/10/03)

His responses suggest a low level of agency in the process, but the project report contains instances of choices for direction of enquiry, suggesting that the journal simply does not report accurately the experience of agency. This inaccuracy has two aspects. It means that the journals alone do not convey the experience the participants have, as I explain earlier, since an unreported experience does not mean an experience which has not been had. Conversely, the potential lack of awareness of an experience can mean a lack of reflection about the experience and therefore hinder any change in view despite the fact that the participant has an experience. I address this topic further in the discussion chapter.

Patrick*, in comparison, is aware of his low level of agency, but likes to attribute it to the choice of topic:

> I do feel like I need more feedback from Dr Zachary or Alan to make sure I am on the right track, but that could just be the nature of what I am working on. (Patrick*, 27/10/03)

His situation is noteworthy, because though he shows a high level of engagement, he also chooses to interact a lot with the teaching team and researcher, as reported by the latter in their journals and field notes. Patrick*'s example shows an instance of the distinction between agency overall and engagement. Though highly engaged in the mathematical enquiry, he chooses to keep his agency at a low level. Other examples include Joan, who gives a prominent place to feedback:

> We got feedback on our assignment and it seems as though we are on the right track. […] We've also taken out a few of our original ideas (triangles) because it just seemed too difficult for this amount of time. Maybe later. (Joan, 15/10/03)

As I report in the section on the social contract, the language in October demonstrates a higher level of agency, with words like 'decide', 'want to' and 'freedom'. In addition, comments focus on the experience of this agency, as shown by Alex's comment:

> Exploring ideas I choose is very different than exploring ideas the teacher/professor chooses. (Alexa, 15/10/03)

And Alice's responses:

> Some of our original hypotheses were incorrect, but these ideas led to new explorations and ideas. (Alice, 15/10/03)

> Even though we came across many road blocks along the way, we were able to come across new ideas and things to explore. (Alice, 02/11/03)

Other participants, though they are engaging with self-agency at the expected level, find it difficult to sustain. Linette*, for example writes:

> I understand the idea of exploring and coming up with ideas on our own, however it is difficult to get anywhere without much of a starting point. (Linette*, 17/09/03)

> The project is starting out to be very frustrating. My group and I find ourselves starting out with one idea and continually coming up with another idea. […] It seems like we're going to keep coming up with other things to do without ever getting anywhere. (Linette*, 18/10/03)

… though her response changes later:

> We seem to be getting somewhere with this project. At least we have narrowed down our focus to "stellation" and how it relates to Euler's formula and why it works, so that's the point in saying "so it works with stellation too." It doesn't seem to be a very exciting discovery. I don't know how it will come together. (Linette*, 27/10/03)

In the end, her view of the work her group accomplished is mixed:

> I feel like we've done a lot of work and not really come up with anything stunning. It is interesting, however, to look at the project as a whole. I do like that we came up with something that we had no idea about at the beginning […]. It was a good exploration, but somewhat abstract. [….] It would be good to be able to correlate this to something more concrete in my mind. (Linette*, 03/11/03)

This response can suggest a possible rethinking of her view of productivity in mathematical enquiry, away from that expected in more traditional mathematics teaching, where she considers herself competent (see Table 14, above).

In contrast, Sue, who rates of herself as average in mathematics, manifests her satisfaction with her results:

> My partners and I collated our information and I think we developed some great stuff! We came up with reasons why we saw the results we did. (Sue, 29/10/03)

Overall, the comments concerning the quality of the goal-state vary greatly. Linette*, who feels unimpressed with her results, already shows concern about the plausibility of a goal-state at an earlier stage:

> I don't see where this project is going to end. I'm not sure that we'll ever get to a conclusion. We started out looking at polyhedra, looking at different combinations of regular polygon faces. Now we're taking only combinations of triangles and squares. (Linette*, 18/10/03)

Jill* expresses a similar concern and, with some prompting, reflects on this issue with respect to its application in her future profession:

> … but I feel like we've reached a dead end, and we're having difficulty moving past it.
>
> [EK: Did you get past the 'dead-end'? How did you manage it? This might be a good tip for your future students...] In order to move on from the project we had [to] accept that our findings weren't profound, never thought of ideas, rather, we were able to apply this formula throughout the project and be satisfied that it continued to work. This is important for students to understand also... although things might not seem conclusive, it doesn't mean you haven't discovered something or strengthened your math skills. (Jill*, 22/10/03 and later)

Several comments are made on the presentations of other participants' results, notably expressing surprise as to the variety of directions that could be followed from the same starting ideas:

> I originally thought that many people would have the same projects. However, this isn't the case. Even in using similar ideas: proper colorings, faces, vertices and edges, and symmetries, all the groups have taken a slightly different approach to their project. (Jill*, 03/12/03)
>
> I didn't honestly realize a person could do so much with proper coloring, tesselations, or both. Many groups had really good ideas, (Isabel, 03/12/03)
>
> The other projects presented today were interesting. It's amazing the ideas people came up with and the things they are interested in when given the chance to explore. (Isabel, 08/12/03)

Overall, the selected journals reflect the students' responses to the open-ended nature of the task with comments about agency at all three levels, starting point, process and goal-state, with varying degrees of detail, reflectivity and frequency, though mainly during Phase 2 and when the presentations are given. The contrast with more traditional teaching is emphasised by the phases of the intervention, which are taught by the same team, and this is made explicit in Jill*'s comment:

> Now that we are in the book I appreciate the project even more. Having an idea and just going with it. I want to encourage this type of learning in my classroom. […] This type of learning could be exciting for children because then the math becomes their own, rather than it being information that they 'have' to learn. (Jill*, 17/11/03)

Criterion 5: The Nature and Experience of the Practice

In the section of the literature review where I establish the design criteria for the teaching approach, I include a fifth criterion focusing on the overall experience of mathematical enquiry and the time needed to do so to the full extent of the experience. In addition, as I discuss in an earlier section, mathematical enquiry requires the development and application of a variety of types of knowing, including at least knowing-why, and possibly knowing-when. Though this requirement is not explicitly framed as a distinct design criterion, it is encapsulated in Criterion 5. In this section, I therefore examine three aspects of the criterion: Hadamard's scheme, the issue of the time allotment, and the level of knowing, within the hierarchy established in the literature review, that is addressed.

Hadamard's (1945) Scheme

In the description of criterion 5, I focused on Hadamard's (1945) exposition of the cycle that researchers, notably in mathematics, go through in their practice. In his view, research evolves through initiation, incubation, illumination and verification.

Polya's framework describing problem solving, which forms the basis of much work on the subject, parallels this description in that it includes the initiation, illumination and verification stages. The first has been called 'getting started' (Mason, 1978), 'entry' (Burton, 1984), 'gaining an awareness' (Brown, 1992), 'understanding the problem' (Sowder, 1993), among others, and it generally signals the beginning of the task. The second corresponds to the resolution phase, and is referred to as 'carrying out the plan' (Polya, 1957), for example. The third corresponds to what Polya and his followers

emphasised as the need to verify the validity of the solution, or, in some cases, the search for understanding (Mason, 1978) or for a new question emerging from the result (Burton, 1984).

Both incubation and this broader meaning of verification are often de-emphasised in descriptions of problem solving but are emphasised in the teaching approach under study. As a consequence, I examine the data focusing on criterion 5 by addressing first the overall process that the participants experienced relative to Hadamard's scheme, and later, the incidence of responses manifesting incubation and verification, separately.

Initiation, for example, where the question is framed and approached, is a significant stage for Linette*:

> The project is starting out to be very frustrating. My group and I find ourselves starting out with one idea and continually coming up with another idea. […] We started out looking at polyhedra, looking at different combinations of regular polygon faces. Now we're taking only combinations of triangles and squares. It seems like we're going to keep coming up with other things to do without ever getting anywhere. (Linette*, 18/10/03)

Sandra, on the other hand, finds herself alternating between illumination and incubation, experiencing the uncertain nature of the experience:

> Every time I think I figure something out, I feel more confused and have more questions. (Sandra, 20/09/03)

Geoffrey, in his brief description of Phase 2 explains that:

> Everything I have tried so far has not worked. I am at a dead end unsure of where to go next. (Geoffrey, 22/10/03)

After a discussion with the teacher-participant, who makes suggestions for a redirection of his enquiry, he declares:

> After talking with the professor at the end of last week I feel a little more confident about where my project is headed. They gave me good advice to take another step in my project. Even though it's almost a completely new task it still correlates to our original topic. (Geoffrey, 30/10/03)

Geoffrey presents an interesting example: in the early stages of his enquiry, he articulates a problem which is at once too broad and too narrow: he is

> using 3-dimensional and 2-dimensional shapes to determine the number of colors needed for proper coloring. To do this [his partner and he] are looking at

the number of edges, [which they] believe will determine the minimal number of colors for proper coloring". (Geoffrey, Progress Report, 22/10/03)

This question is too broad in that the domain of resolution includes all tilings and all polyhedra, but at the same time too narrow in that the variables that they consider are, as luck would have it, not the ones that express the relationship, which does exist, between the structure of a tiling or polyhedron, and its least, proper colouring.

In terms of Hadamard's scheme, Geoffrey and his partner need to return to stage 1, initiation, though their experience is different from Linette*'s whose team, according to her report, is confronted with an excess of starting points.

The clearest sense of a narrative is given by Jill*'s entries. This is not surprising as she contributed 6 entries in Phase 2 alone. Jill*'s description is a good example in that it not only shows evidence of Hadamard's scheme, but also the idea that though it is described as a linear process, the experience is in fact recursive and non-linear, as is that of Polya's problem solving process. Throughout her enquiry, she encounters initiation:

> I think we have a good start to our projects, […] Right now, the project is overwhelming, but taking it a step at a time will hopefully bring us to some sort of conclusion. (Jill*, 13/10/03)

… illumination and a return to initiation:

> This [result] is interesting. […] Is this true for any solid? I want to try the equation with another combination, squares and pentagons, to see if it continues to hold true. (Jill*, 19/10/03)

… incubation:

> … but I feel like we've reached a dead end, and we're having difficulty moving past it. (Jill*, 22/10/03)

… and again illumination:

> Although all the ideas that my group discussed regarding our project seemed as though we wasted our time, it seems as though they all ended up tying together. At first when we started the project it seemed as though what we were doing was useless, but ideas continued to extend from these thoughts, and this continuous branching developed our final project. So, even if something seems like a 'bad', invalid idea it has the potential to spark a good idea. I hope another good one is sparked from our last discovery. (Jill*, 24/10/03)

She demonstrates an awareness of the importance of the last component of Hadamard's scheme, verification, though she does not consider that her enquiry successfully achieved it, in the end:

> I am relieved to get rid of my project, but I still don't think it wrapped up completely […]. (Jill*, 03/11/03)

It is interesting to note that this comment regarding the last phase of Hadamard's scheme made after the enquiry is concluded, contrasts with one she made earlier, in response to some feedback:

> Is this true for any solid? I want to try the equation with another combination, squares and pentagons, to see if it continues to hold true. […]
>
> [EK: So did it hold true?] For every solid except a sphere and a cylinder Euler's formula holds true. It is interesting because […] this formula exists in everything that surrounds [us] that are made of angles. It exists in tables, chairs, boxes... Anything where we can count edges, faces, and vertices. (Jill*, 19/10/03 and later)

Within Jill*'s practice of enquiry, she establishes a result, conceptualises and accepts a generalisation of it, then moves on, yet when she reflects on the overall experience, she considers it unfinished. This suggests an experience of the never-ending nature of mathematical enquiry which promotes a more critico-creative approach.

Other participants struggled with the last stage. Emily, for example, tries to resolve the apparent incompatibility between her group's starting point, the achieved goal-state, and the drive to generalise their result, which is a part of verification:

> Forming an overall conclusion about our project is frustrating. Our original purpose of the project varies from the purpose we were asked to describe in our findings. I know that the new ideas given to us are probably related but trying to bring in a new concept at the end is hard since we thought we figured out the problem that was at hand. Trying to apply our findings to general circumstances is harder than the actual specific cases, which you wouldn't think would happen. (Emily, 27/10/03)

Alice, in contrast, is undisturbed by this incompatibility, since, as she sees it, it is simply part of the process:

> Some of our original hypotheses were incorrect, but these ideas led to new explorations and ideas. (Alice, 15/10/03)

The verification phase of Hadamard's scheme is explicitly discussed by about half the participants selected for this analysis. At the same time, the written report that the

participants are required to write includes a component, titled 'Results', which is described as follows:

> This section should be a summary of your findings [...], referring to key examples and/or counter-examples. A precise statement of your claims and reasons why will be required. Note that reasons why something did not work are an acceptable mathematical result. [...] (Guidelines for the Project Report Guidelines, see Appendix 6)

This section requires the participants to formulate a goal-state and is followed by one called 'New ideas for further study', reinforcing the idea, formulated by Grenier & Payan (2003), that:

> An answered question often leads to a new question. The situation has no "goal state". There are only criteria of local resolution (p. 189, my translation, see Chapter 2).

The two sections, together, compel the participants to reflect on their enquiry in terms of Hadamard's 'verification' stage because they require the formulation of both a goal-state, and a possible extension. The fact that many participants do not mention this stage explicitly in the journal as well can stem from various reasons, and, as I discuss earlier, does not mean that this aspect is left un-experienced.

It is interesting to see, aside from references to 'verification', that 'incubation', even though it can be thought of as a non-event, (since nothing is 'happening' during that stage) is expressed by several participants, particularly as it pertains with their emotional responses, notably of frustration at being stuck. The significance of this phenomenon is that the participants are compelled to get un-stuck in order to continue their enquiry. This emotional response is manifested and interpreted by the participants in different ways. Alexa, for example, says:

> So far, this project has been very frustrating for me. I feel like I have no direction, and I feel like I am not really exploring anything. (Alexa, 15/10/03)

She interprets the state of being stuck as the lack of direction or of productivity. Other participants simply stated the fact:

> I also felt frustrated and overwhelmed looking at the project for so long. I am going to let everything sink in and work on it at home. (Sandra, 19/10/03)

> I am stuck for new ideas for my project. (Samantha, 26/09/03)

> I am still lost for new ideas. I think once I am able to start typing charts up, things will start to fall together. (Samantha, 28/10/03)

In Geoffrey's case, as I discuss earlier, the way out is through feedback from the teachers, and Jill* perceives a re-evaluation of the results to be a useful tool for coming out of the stage:

> … I feel like we've reached a dead end, and we're having difficulty moving past it.
>
> [EK: Did you get past the 'dead-end'? How did you manage it? This might be a good tip for your future students…] In order to move on from the project we had [to] accept that our findings weren't profound, never thought of ideas, rather, we were able to apply this formula throughout the project and be satisfied that it continued to work. This is important for students to understand also… although things might not seem conclusive, it doesn't mean you haven't discovered something or strengthened your math skills. (Jill*, 22/10/03)

The four stages described in Hadamard's (1945) theoretical perspective are all represented in the 16 analysed journals, albeit to various degrees and not always in order. In particular, the two stages that are often left implicit in descriptions of Mathematical Problem Solving in the Classroom, incubation and verification (sometimes called 'looking back'), do emerge to some extent from the selected data. In addition, the latter is ostensibly required in the last two parts of the final report, and therefore, though it is not discussed explicitly by every participant whose journal is selected for analysis, it is experienced, at least to some degree.

The Time it Takes

An important component of design criterion 5 (see Chapter 2) is the provision of enough time for the practice of mathematical enquiry to devolve through the stages presented by Hadamard (1945). In addition to evidence of the stages themselves, I therefore focus on the participants' statements focusing on this time issue. Several participants, for example, make observations about the extent of time used for what they sometimes consider unproductive tasks, particularly during Phase 1:

> I felt as if we were over analysing our drawings or just trying to make something up to say. (Christie, 08/09/03)
>
> So far this class seems too easy and basic. I think that a lot of us are getting bored very quickly. […] but it doesn't really seem necessary since we all know the answers already. (Laura, 10/09/03)
>
> I feel very frustrated with this class right now. It just seems like we keep doing the same thing over and over. (Laura, 22/09/03)

> I feel like we are focusing on simple subjects way too in depth. I barely listen
> in class and the directions never seem clear on what we are supposed to be
> discussing. I try to pay attention so I know what is going on, but it is becoming
> impossible. (Samantha, 28/09/03)

Others find this slower pace useful because it gives them time to really immerse themselves in the thinking:

> I find that discussion is proving helpful in this exercise because the more I
> stare at my Origami triangle with my own ideas in mind, the more questions I
> develop and the more I second guess my answers and ideas. Discussion helps me
> focus on a couple ideas at a time and concentrate on the topic at hand, i.e.
> proof of the equilateral triangle. (Linette*, 10/09/03)

In Phase 2, the comments focusing on the time allotted change. Some participants discuss the usefulness of being given class time to work on the enquiry. The responses are both positive:

> I liked the fact that we got class time to work on it. It's hard to get together
> with group members outside of class. Everyone is so busy that you usually end
> up doing most of the work on your own. (Laura, 29/10/03)

> It really helped to be able to work on it in class and get immediate feedback
> when needed. (Samantha, 09/12/03)

… and negative:

> We worked more on the project. It is hard to concentrate on this project for
> the whole class. I think we should devote half to the project and half to a
> lesson. […] I also felt frustrated and overwhelmed looking at the project for so
> long. I am going to let everything sink in and work on it at home. (Sandra,
> 19/10/03)

In Alexa's case, the observation elicits a reflection on the teaching approach itself and the time issue associated with it:

> I feel that we have been working on these projects for years. I also feel like
> we were given a little too much time. […]It must be very difficult to choose a
> due date because everyone is at such different points in their projects, and
> because everyone has such different projects. Choosing an amount of time for
> something in the classroom is often challenging because some children finish
> remarkably quickly, and some finish much later. Finding a balance is the trick.
> This math class has allowed me to realise that. (Alexa, 20/10/03)

And Emily's perspective regarding the time issue evolved during the experience:

> At first I didn't think that we would have enough to do for the month, but now I have realized how many questions and problems arise as I figure out my original problem. It takes time to sort everything out. (Emily, 20/10/03)

In several cases, the amount of time allotted to the experience is critical to the participants' engagement level. For example, Emily, though she acknowledges the importance of taking the time to 'sort things out', also demonstrates an awareness that the sustained attention required for critico-creative thinking can be difficult to muster:

> It is nice to be done with it for a while because I was getting kinda "bored" with thinking only about the project every class period. The lack of structure was making me not work as much as I could have been. It is just all about getting my self motivated to think critically and analyze thoroughly. (Emily, 02/11/03)

In contrast, both Linette* and Jill* demonstrate an awareness that time is a critical condition for the success of the enquiry. Linette*, for example, shows an awareness of the progressive nature of the enquiry when she says, towards the end of Phase 2:

> We seem to be getting somewhere with this project. At least we have narrowed down our focus to "stellation" and how it relates to Euler's formula and why it works, so that's the point in saying "so it works with stellation too." (Linette*, 27/10/03)

… whereas Jill* feels that the time spent, even on what seems a disconnected or negative result, can be justified when the ideas are pulled together later:

> Although all the ideas that my group discussed regarding our project seemed as though we wasted our time, it seems as though they all ended up tying together. At first when we started the project it seemed as though what we were doing was useless, but ideas continued to extend from these thoughts, and this continuous branching developed our final project. So, even if something seems like a 'bad', invalid idea it has the potential to spark a good idea. (Jill*, 24/10/03)

Several participants expressed their concerns about the time spent on the project by bringing up the other module sections, which are not involved in the intervention and are taught using a more traditional teaching approach:

> I'm a little worried though because after talking to people in the other class they said they aren't doing a project at all. I'm afraid that we're going to be behind for the second part of the math classes. (Laura, 15/10/03)

> Other classes seem to be covering so much more material. (Pippa, 24/10/03)

In Laura's case, the comment immediately preceding the one under discussion implies that her group's project is well under way at the time of writing (which is about half-way through the phase). As Alexa's remark suggests, the different groups need different amounts of time for their project, and Laura's may be an example of a shorter project, and she may be ready to move on at this stage. Two weeks later, however, she is still discussing the process of the project, suggesting that she and her group have found more to do:

> Our project is coming along. We've made really good progress so far. (Laura, 29/10/03)

Laura's description is compatible with Grenier & Payan's (2003) claim that there are only local goal-states and that results presents opportunities for new questions.

In Phase 4, few comments are made about the time frame, suggesting that the participants' observations of the intervention run along the lines of their expectations. Samantha, for example, remarks:

> I think we need more time to ask questions during class and get more feedback. (Samantha, 11/11/03)

Overall, in the 16 journals, the issue of time is treated differently across the four Phases. In Phase 1, an impatience with the time spent on topics is communicated, manifested by the participants' impression of a discrepancy between that warranted by the proposed activity, and expected by them, and the time actually spent.

In Phase 2, the issue of time is expressed as the frustration of having to invest sustained effort, over a significant portion of the module, into a task with uncertain results. Finally, the comment in Phase 4 is one that could have been made in any classroom where the teaching approach focuses on notional knowledge.

Epistemological Engagement

In the description of Criterion 5, one of the purposes of the intervention is said to ensure that the participants have an opportunity to develop 'knowing-when', the higher level of knowing. This is indeed one of the distinguishing features I ascribe to Mathematical Enquiry, as opposed to Mathematical Problem solving in the Classroom. Though this is not discussed with the students explicitly, evidence can be found in several journals that this intention does come through. During the ramping up of Phase 1, for example, Isabel says:

[…] once the group began to think of things in more of a mathematical way, new ideas developed and our thinking expanded.

[EK: "More of a mathematical way…"? Please explain.] We began to expand the way we thought about the problem. When we took in more of a mathematical perspective it didn't seem as straight forward.

[…] Thinking about the main ideas can be trying but it is a good way to get in the "mathematical mode".

[…] Today required a lot more analytical thinking.

[EK: What do you call 'analytical thinking'?] Good question. It's when you analyze, pick things apart, or breaking things down to look at things a different way and make connections.] (Isabel, 09/09/03 and later)

…and later:

I wish we had better explanations in the class. I have never been this confused in my life. [EK: Are you figuring it out now?] The class is starting to come together more so now that it is the end of the year. I was more confused in the beginning, but I realize that was the intention of the class. (Isabel, 18/09/03 and later)

Joan also observed a shift in thinking style:

I am just not used to learning this way. [EK: Please elaborate on 'learning this way'… sounds interesting.] Reasons behind why we do things the way we do— How the formulas work instead of just plugging things into a formula. (Joan, 28/09/03 and later)

… as did Linette*:

It is interesting to use discussions for mathematical concepts because I am used to hearing "this is true" and "that is true", but not why. (Linette*, 10/09/03)

I understand the idea of exploring and coming up with ideas on our own, however it is difficult to get anywhere without much of a starting point. I am often wondering where we are going with each session. (Linette*, 17/09/03)

I am starting to see the point of starting ideas in such a vague manner because it's a good way to use the rationale of every step leading to the big picture. It is still frustrating, however, to take home work and attempt to explain something that I don't completely understand. (Linette*, 29/09/03)

Pippa:

I find the projects confusing, and for that reason, frustrating. […] I've been learning math one way all through school and to switch to projects that are as unstructured as the ones we're doing is difficult. I'm not quite sure what I'm

supposed to be doing and not having a textbook with examples to consult is difficult.

[EK: What do you think about the approach you were used to before, as compared to this class, at this point?] The other approach was nice because I knew exactly what was expected of me. Also, having the formulas given to me and then having to apply them seemed more straight forward, and so more understandable. It doesn't sound great, but math is not something that interests me enough to be driven to understand the process behind the formulas. I like seeing how the formulas work, but it's not a priority in what I want to be studying. (Pippa, 17/09/03 and later)

… and Jill*

This is a new level of math for me where there might be a right answer, but the professor isn't sharing it with us, and I come up with a new answer every time I look at it.

[EK: Do these answers relate to each other at all?] The answers always tend to relate to each other, however, I get frustrated because I didn't figure it out the first time and sometimes it seems as though I wasted time. I know that I do learn a lot this way, and that then things build after each. (Jill*, 17/09/03)

During Phase 2, references to a higher level of knowing often discuss ways of getting unstuck, or ways of changing the domain of the question posed. Alice, for example explains:

Some of our original hypotheses were incorrect, but these ideas led to new explorations and ideas. (Alice, 15/10/03)

Christie navigated through a variety of domains and their relationship throughout her process. She first looks for patterns she can find in an open domain:

However, I am finding patterns in my research so I do feel like I am getting somewhere. (Christie, 13/10/03)

Then tries to narrow down the domain for the pattern she finds:

Once I find one type of shape I try to find other objects that are similar to put categories to them. (Christie, 20/10/03)

And later finds a new lens with which to observe her phenomenon, giving her a new outlook on it:

Some results I have identified about my project are that objects on a global level are much easier to categorize than objects observed on a local level. Objects become more complex and the little details make them unique and un-comparable to any other objects. (Christie, 31/10/03)

Jill*'s group works on a project that integrates an exploration of the outcomes of rules they have developed, including the number of possible results, and a reason behind that number:

> We are doing our project on combining two of the regular polygons to make 3D shapes, and we're trying to determine how many they are, and justify this according to our rules. However, there may be more or less depending upon which rules are in place. (Jill*, 13/10/03)

Linette*'s group, after an initial struggle to find a starting point:

> It seems like we're going to keep coming up with other things to do without ever getting anywhere. (Linette*, 18/10/03)

… settles on a question of generalisation of a result to a wider domain:

> We seem to be getting somewhere with this project. At least we have narrowed down our focus to "stellation" and how it relates to Euler's formula and why it works, so that's the point in saying "so it works with stellation too." (Linette*, 27/10/03)

Many students describe the thinking that takes place in the mathematical enquiry in language that suggests a search for the mathematical structure underlying the phenomenon they are observing. In the last example, Linette*'s group, for instance, is examining the consequences to the Euler characteristic[30] of replacing a face by a pyramid that fits on it. They are working on what amounts to an inductive proof of the invariance of the characteristic.

In Phases 3 and 4, the language of the entries changes again, and the focus is on whether the teaching approach helps to develop understanding, as the participants know it, that is, at the 'knowing-why' level.

> I also like the way he introduced [the new topic]. Dr Zachary tried to show us what was happening, and visualising the sequences as areas is helpful. Although, it can get rather confusing when we are supposed to derive the formula. (Alexa, 03/11/03)

Some of the comments do still suggest an awareness of the higher level, however:

> I really like how Dr. Zachary explains why formulas are certain ways and has us critically think before we hear the answers. (Joan, 24/11/03)

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[30] The Euler characteristic relates the number of edges, vertices and faces of a polyhedron in a unique way.

> I appreciate Dr. Zachary's method of teaching. Although we are duplicating the same processes in arithmetic and geometric sequences, he presents it in a way that is more of a search than just plug in your numbers here. (Jill*, 17/11/03)

And some participants are less comfortable with this approach:

> [This] was very easy for me in college algebra, but now it is not so easy. He explains things weird or different from how I learn. It seems like simple concepts are being turned into complex concepts because he is explaining things awkwardly. (Samantha, 11/11/03)

… than others:

> Now that we've been working with formulas I feel a little more comfortable. I feel like this is more concrete, meaning that I can see where everything is coming from and I can envision a conclusion. (Linette*, 17/11/03)

Most of the selected journals demonstrate some awareness of the higher level of knowing described in the theoretical framework. An exception is Geoffrey, who focuses, in the five entries he wrote, on his feelings, and does not discuss the practice itself, and therefore does not comment at all on the understanding he uses or develops. In the following section, I collect comments made by some of the 16 selected participants, regarding the learning they see themselves as having achieved in the module.

Other Significant Aspects of the Experience

Learning Derived from the Experience

As described in Chapter 4, the teaching approach is designed to de-emphasise traditional notion of achievement in both learning and successful problem resolution, through a re-formulation of the assessment strategy, focusing instead on the process that the participants go through. The question of the learning achieved by the participants, however, remains on their minds. Isabel, for example, questions this early on:

> I'm still confused about what we're doing, learning, and how we can apply this to our future teaching careers. (Isabel, 22/09/03)

Though she later acknowledges having achieved learning, without specifying what kind:

> I did […] learn things I didn't know and I began to think about things in new ways. (Isabel, 09/12/03)

Petra is more specific in that she evaluates the learning she derived from the experience in terms of notional knowledge:

> I […] feel that I have learned a lot through discovering on my own. I have learned a lot about tesselations and I think that I have learned more than I would have learning from a textbook. (Petra, 31/10/03)

Alexa and Jill* reflect on changes they have made in their attitudes to the learning of mathematics. Alexa has grown ambivalent, showing an awareness of the levels of knowing:

> I am torn between the need to understand why things are the way they are and the desire to simply be given the formula. The benefits of simply being given the formula is that it wastes less time in class, because we wouldn't have to go through every little step to finally reach the formula. On the reverse side, we wouldn't understand why we were using the formula, which wouldn't be very helpful. (Alexa, 03/11/03)

She also remarks:

> [EK: How do you relate this to your concept of mathematics as a subject?] My concept of mathematics as a subject has changed with this class. I always thought that mathematics was a subject more focused on right or wrong answers. After taking this class (and remembering back to Calculus, I can also see it), I realized that mathematics is more about the process used to get the correct answer. With that new concept of math, I will be better equipped to teach an elementary school classroom. (Alexa, after 03/11/03)

And Jill* has noticed a change in her behaviour when confronted with what she perceives as a non-routine problem:

> After finishing the quiz I finally understand some of the concepts in class that I wasn't connecting earlier. The last math class I took was […] four years ago. Then, if I would have come to a problem that I didn't understand, I probably would have given up, rather than working through it. On this quiz, at first I didn't feel like I could confidently answer one question, but as I thought about the problems, and what they were asking for, I was able to work through them, and feel fairly confident about my answers. I took the long way on some of them (coefficient problem), but once I was done, I understood [more]. (Jill*, 10/12/02)

Pippa reflects on the experience with respect to her views of the discipline:

> I have a new appreciation for the work mathematicians do. I also don't think their work is as boring as I once did. (Pippa, 02/12/03)

… and several students relate their experience to the elementary classroom, both positively:

> I want to encourage this type of learning in my classroom. I want the kids to feel free to explore an idea and present their findings. This type of learning could be exciting for children because then the math becomes their own, rather than it being information that they 'have' to learn. (Jill*, 17/11/03)

… and negatively:

> Class was very interesting, all the talk on how to do the worksheet was very unpredictable for me, but it seemed useful. I don't think that elementary kids would get the concept, but it sure got me thinking. (Joan, 29/09/03)

These few remarks about the learning that the participants feel to have achieved show a continued need to self-evaluate in terms of learning. At the same time, the variety of levels at which this achievement is seen to be made demonstrates an awareness, by some of the participants, that learning is not necessarily limited to notional knowledge.

Research Instruments

In the description of the teaching approach and of the research methodology I acknowledged the data collection instruments as being an explicit, integral part of the experience of the intervention, beyond the targeted practice. In the selected journals, very few references are made to these artefacts of the study. Two participants mention the journals. Early on, Geoffrey (10/09/03) displays uncertainty as to its purpose, and in her last entry, Isabel (09/12/03), addressing me directly, apologises for not writing more, even though her journal ranks sixths in the number of entries. The questionnaires and other sources of data are not mentioned at all. In Chapter 7, I discuss the possible correlation between participation in the journal writing and change in affective responses.

Overall context

Another set of constraints impacting the design of the teaching approach derives from the overall context within which it takes place, including the degree programme within which the module is taught. The participants' comments focusing on this aspect fall into two categories: the use of the required textbook and the difference between the participating group and the parallel section of the module.

Discussions focusing on the use of the textbook present a varied image, between participants who associate the use of the textbook with higher levels of comfort:

> We have no textbook to refer to for any sort of help, so when I can't understand why something is the way it is, I feel helpless. (Alexa, 28/09/03)

> I'm hoping that since we started the book, the whole class will make more sense to me now. (Laura, 03/11/03)

… participants who find its use dissatisfying:

> I do not like this textbook because it gives you no instruction whatsoever. (Petra, 15/11/03)

… and participants who experience difficulty when the book and the teacher do not match:

> When he gave us homework out of the book, the terminology and symbols were not the same and then it gets confusing. (Alexa, 05/11/03)

Discussions of the parallel module section also bring up the textbook, which is used there throughout:

> Now we are starting on the book. I am a little nervous! Since everyone in other sections says that it is terrible. (Joan, 02/11/03)

… and the need to 'keep up':

> I'm a little worried though because after talking to people in the other class they said they aren't doing a project at all. I'm afraid that we're going to be behind […] . We haven't even started the book yet so I don't really understand how this project is going to help us. (Laura, 15/10/03)

This last comment, together with other similar ones, manifests the view that learning is still to be measured in notional terms, though it is not shared by all the participants, as discussed earlier. Chapter 7, with its description of the results of the pre- and post-module questionnaires, examines this topic in more detail.

Conclusion

In Chapter 3, I describe the journals as a source of data that is continuous yet immediate. In consequence, the data that they provide reflect the responses at or as close as possible to the time of the experience. These responses themselves, however, are often already filtered by the participants' own reflective selves. This filtering occurs in different ways and at various levels: The participants are likely to focus on what stands out for them, while not necessarily discussing what seems ordinary; they may express their observations in terms which they think I want to see; though the writing guidelines are designed to be open-ended and as un-biased as feasible, the participants may interpret them otherwise. Despite these conditions, evidence of the effects of each of the

design criteria can be found in at least some of the analysed journals. The open-ended nature of the practice is commented on regarding the starting point (Criterion 1, for example by Emily), the process itself (Criterion 2) and the goal-state (Criterion 3). The idea of a shift in the social contract is taken up by the participants as well (for example by Samantha), though it might not have had the expected effect. The nature of the experience itself (Criterion 5) is also discussed at its various levels, including the parallel with Hadamard's scheme, the time issue, and the epistemological level of engagement. Finally, the additional, practical constraints are also evidenced in the participants' writing, including the presence of artefacts of the research study itself and the relevance of the module to the overall degree programme. A more thorough synthesis of these findings can be found in Chapter 8, where I discuss their pertinence for teacher education.

Chapter 7: The Effects of the Intervention

One of the most important questions that this study is designed to answer concerns the potential affective effect of a mathematics teaching approach that is innovative in its focus and strategies. Chapter 4 contains a description of the approach as it is applied in this study; in Chapter 2, the section titled "Mathematical Enquiry (ME) as Distinguished from Mathematical Problem Solving in the Classroom (MPSC)" describes the theoretical framework on which this approach is based; and chapter 6 shows the comparison between the effective and intended experience of the participants.

In this chapter, I describe the effect of the intervention on the student-participants. The data used for this assessment is taken from the post-module questionnaire (see Appendix 4), and is in some instances compared to the responses to the pre-module questionnaire, in order to measure the change in affective responses. A variety of item types are used in order to triangulate the results, including descriptive, where the participant selects preferred responses from a list, Likert, short answer, rating and multiple choice items.

Descriptive Items

The descriptive items of Section 3 are the same in the two questionnaires and can therefore be examined for the change in the responses.

Items 11 and 13

Items 11 and 13 consist of the selection of 3 items out of the same list, and the choice is of the best and least adequate descriptor of mathematics, respectively. The totals are calculated for each term and represented in Table 15, below. The count for worst descriptor is subtracted from that for best descriptor, giving a net positive total for terms more often selected as best than as worst and a net negative total for terms more often selected as worst. Several of the terms move very little between before and after.

For example, 'problem-solving' goes from 29 to 24, meaning that it is selected by a net count of 29 participants before, and only 24 after. This is not as significant as 'an exploration', which moves from -8 to 24, for a net increase of 32. Other significant changes are the counts for 'numbers and operations' (from 20 to 0, net), 'patterns and relations' (from 3 to 22, net), or 'measurements' (from -2 to -14, net).

	Before			After			Net Change
	Best	Worst	Net	Best	Worst	Net	
an exploration	0	8	-8	24	0	24	32
patterns and relations	4	1	3	22	0	22	19
an art	0	31	-31	1	23	-22	9
exercise for the mind	7	4	3	7	0	7	4
a history	1	25	-24	0	20	-20	4
rules	4	5	-1	2	4	-2	-1
a language	3	11	-8	2	14	-12	-4
problem-solving	29	0	29	24	0	24	-5
formulas	14	3	11	8	3	5	-6
logic	10	2	8	6	4	2	-6
a tool	7	3	4	3	5	-2	-6
a science	4	5	-1	3	10	-7	-6
measurements	2	4	-2	0	14	-14	-12
numbers and operations	20	0	20	6	6	0	-20

Table 15:   Frequencies of selection for terms from list 1

The counts that change not only in value but also in orientation, such as those of 'numbers and operations', 'a tool' and 'an exploration' are significant as they suggest a net reversal of view. The figure below uses a mock-log representation which compresses the extremities progressively, thereby reducing the visual effect of less significant changes occurring at extreme values and emphasising the changes near the tipping point (0). The terms highlighted in the table show the most significant changes, and all the values can be examined in terms of the theoretical framework. 'An exploration', for example, which fits the 'Pattern Analysis' view (see page 93), changes from a net count of -8 to one of 24, showing the single most significant change. This suggests that the open-ended nature of the experience provided by the teaching approach has been integrated into some of the participants' beliefs and is compatible with those participants' awareness of mathematics as a discipline at the time of the survey.
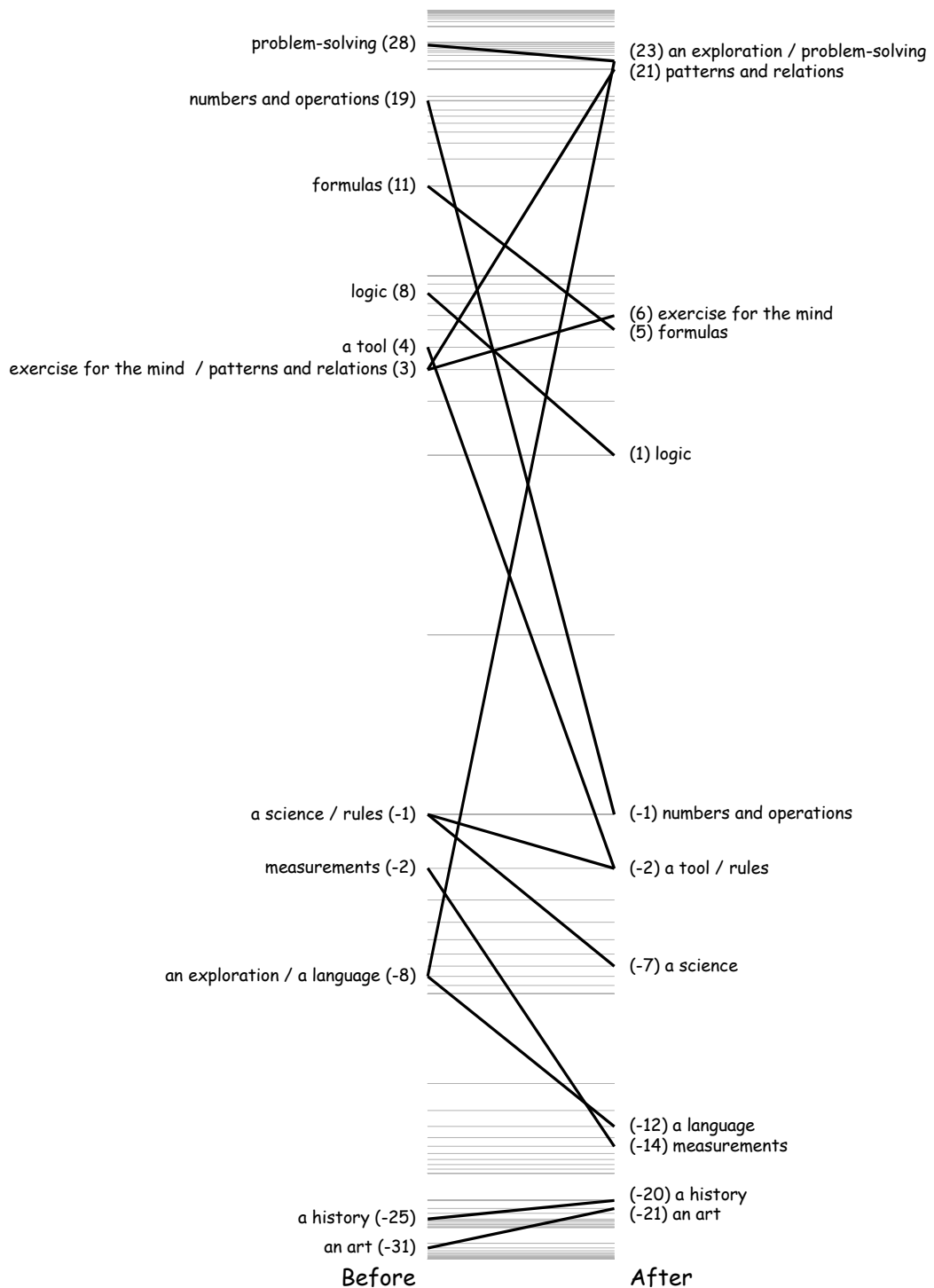
Figure 7   **Net frequency of selection of terms from list 1**

'Patterns and relations' and 'numbers and operations' have almost exactly changed places in terms of their net values. Whereas the former gained 19 counts through the intervention, going from 3 to 22, net, the latter lost 20 point, going from 20 to 0, net. This suggests a shift away from an Instrumentalist view focusing more on the application of facts and skills and towards a perspective more compatible with the Pattern Analysis view, which conceptualises the nature of mathematics by the phenomena it examines. In confirmation with this finding, the changes in the frequency

of selection of 'a tool' and of 'measurements' suggest that the view that mathematics is a service subject for other areas of knowledge has indeed lost ground, in favour of a more Pattern Analysis-like view.

Items 12 and 14

Items 12 and 14 require similar responses, taken from a list containing adjectives and other modifiers, as shown in the table below. Again, the count for worst descriptor is subtracted from that for best, both before and after the intervention. The last column denotes the change in these scores.

	Before			After			Net Change
	Best	Worst	Net	Best	Worst	Net	
gives meaning (View)	6	4	2	13	3	10	8
fun (Emotion +)	4	15	-11	4	7	-3	8
abstract (View +?[31])	15	4	11	19	2	17	6
frustrating (Emotion -)	14	0	14	18	1	17	3
useless (Attitude -)	2	27	-25	2	25	-23	2
empowering (Attitude +)	3	8	-5	3	7	-4	1
valuable (Attitude +)	24	0	24	21	0	21	-3
frightening (Emotion -)	3	14	-11	1	15	-14	-3
torture (Emotion -)	3	21	-18	1	22	-21	-3
practical (Attitude +)	17	1	16	14	2	12	-4
rigid (View -)	2	6	-4	2	11	-9	-5
concrete (View -?)	12	5	7	7	10	-3	-10

Table 16: Frequencies of selection for terms from list 2

In the figure below, the counts are again represented in a mock-log graph, which emphasises the changes near the tipping point (0) more than at the extremities. Only one option changes its orientation as well as its value: 'concrete'. It is also the one that changes the most in value (by -10, from 7 to -3). In the previous chapter, the opposing pair 'abstract' and 'concrete' is shown to be a significant semantic tool to describe the participants' experience. As I discuss there (starting page 170), the various participants use the term in different ways, and in fact, they are not always seen as mutually exclusive opposites.

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[31] The arithmetic signs denote the valence or, in the case of the views, the desirability of the view. In the case of abstract/concrete, this is open for debate, and discussed below.
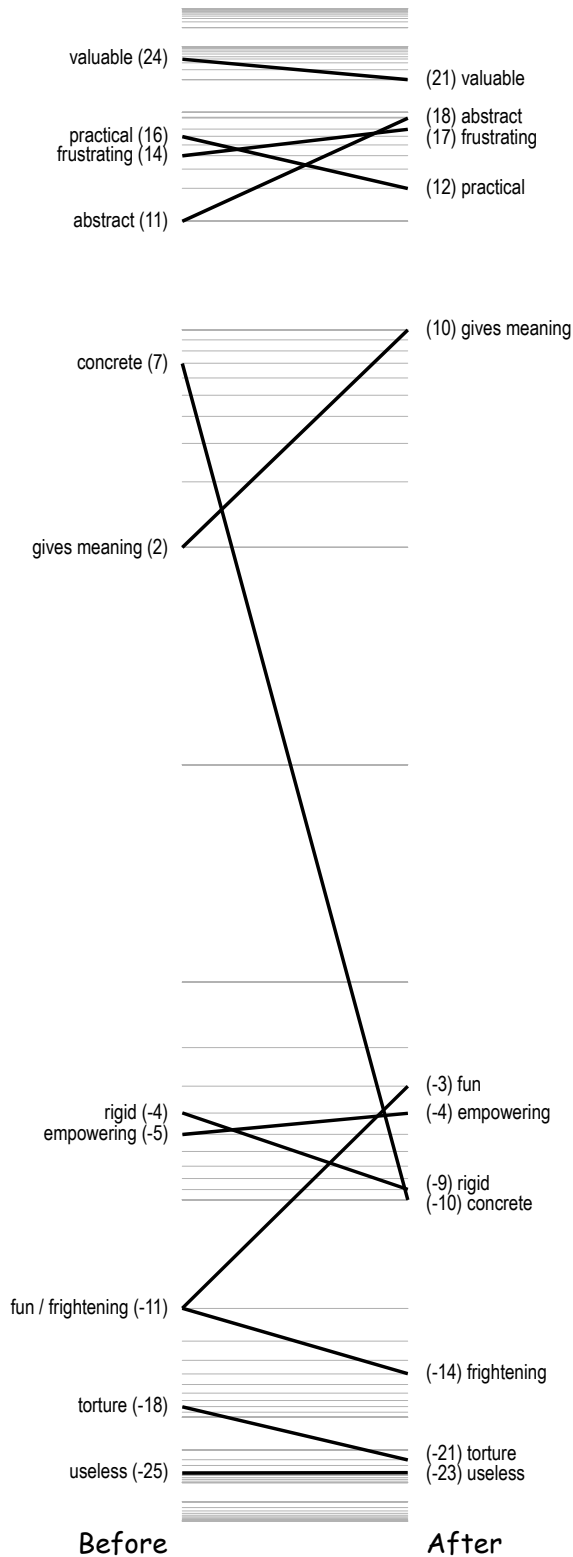
Figure 8 Net frequency of selection of terms from list 2

In the pre-module questionnaire, of the 15 participants that select 'abstract' as one of the best terms, two select 'concrete' as one of the best at the same time, suggesting that, for them, the two terms are not incompatible. In addition, the fact that no-one selects abstract both as best and worst suggests that the items are addressed as honestly as possible. The same is true for concrete. On the other hand, of the 12 participants that

select 'concrete' as their best term, besides the two who select 'abstract' alongside it, 3 select 'abstract' as one of the worst. In the post-module questionnaire, again, two participants select both abstract and concrete as best at the same time. Interestingly, the participants that choose both descriptors as best at the end (Geoffrey and Isabel) are different from the ones who do so initially (Melanie and Simone).

In the pre-module context, the two participants both select valuable as their third descriptor, and in the post-module context, the two participants select practical and valuable, respectively. This result takes us back to the discussion in Chapter 6 about the use of 'abstract' and 'concrete' in the participants journal entries, though the participants who use this language in their journals are, on the whole, not the same as the ones I mention here.

The meanings of the terms in this second list can bring further insight into the participants' affective responses. By their semantic nature, some of the terms are more easily associated with the attitudes or emotions sub-categories of the affective domain, as defined in the theoretical framework, while others relate to views of the nature of mathematics. The mix of terms presents an interesting situation: the participants can choose to focus their (positive or negative) responses on their beliefs, attitudes or emotions, depending which they find the most significant. As the items ask for both the best and worst descriptor, a participant may, depending on which is most relevant to them, respond with either a term denoting heat and valence (an emotion): frightening, frustrating, fun, or torture; a cool term denoting valence (an attitude): useless, empowering, valuable, or practical; or a beliefs about the nature of mathematics: abstract, rigid, concrete, or gives meaning. Responses to this set of items therefore connect to the component of the theoretical framework according to which some individuals base their attitudes on feelings (emotions) and others on beliefs (Haddock and Zanna, 2000, as cited by Ajzen, 2001). In the responses to this item, the frequencies of the choices therefore not only differentiate between the participants with various attitudes to or beliefs about mathematics, but can also serve to categorise the participants as 'feelers' or 'thinkers'.

The results show an interesting shift: whereas the balance between emotion-, attitude- and belief-based responses is 74:82:54 before the intervention, it is 69:74:67, after. In effect, many participants seem more inclined to select belief-based descriptors after the intervention. This suggests that they have a stronger basis on which to base beliefs,

because they have had an experience and reflection on which they can draw. In the group as a whole, 19 of the 35 participants that answered both surveys have shifted to a more belief-based response, 7 have maintained the same balance, and 9 have shifted to a more emotion-based response. I tabulate this shift in Appendix 9.

Overall, responses to these items show a less significant shift in the overall scores of each descriptor than the items using the previous list, with the exception of concrete, but the responses do tell us that the participants find it easier to form a belief-based response after the intervention.

Likert Items

The Likert items of Section 4 provide responses in both the pre- and post-module questionnaires and these can therefore be compared for change. The items themselves can be classified by the category of affective responses they elicit. Items 15 and 16 focus on the participants' like/dislike of mathematics, Items 17 and 18, on their self-confidence with the subject, and the remaining items focus on beliefs about mathematics. In each case, the participants are asked to select a response from the following five options:

- YES!   STRONGLY AGREE
- yes   AGREE
- ??   NEUTRAL or UNDECIDED
- no   DISAGREE
- NO!   STRONGLY DISAGREE

Figure 9, below, shows the total, absolute value of the change of views of all the 35 student-participants who participate in both surveys. The range is 3 to 24, the mode and median are 14. The highlighted bars are explained in Chapter 8.
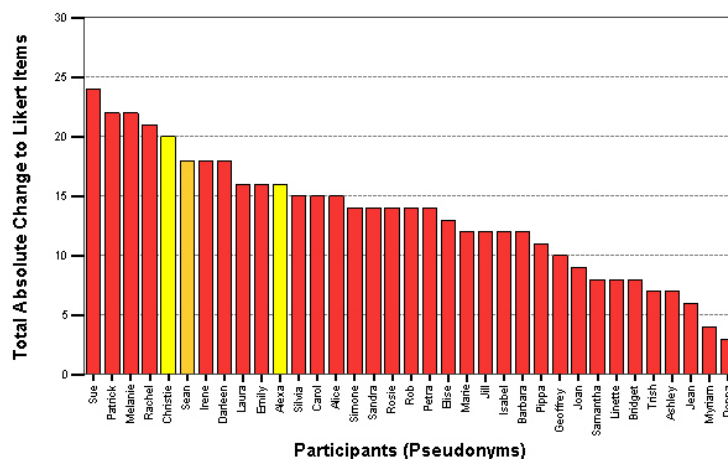

Figure 9   **Total absolute change to participants' responses to Likert items**

In Chapter 3, I group the items into subscales representing the component of the theoretical framework which they address. Items 15 and 16 measure like/dislike, and are formulated to do so using opposing scales; their subscale is expressed as value (Item 16) – value (Item 15). Items 17 and 18 measure self-confidence with regards to mathematics, again using opposing scales; their subscale is expressed as value (Item 17) – value (Item 18). The remaining items measure the participants' views in terms of Instrumentalism, Platonism, Problem-solving and Pattern Analysis (see chapter 2). The subscales representing each view are as follows:

	Positively loaded	Negatively loaded	Items
Instrumentalism	23, 26, 28, 30, 34, 35	29, 31, 36, 37, 39	11
Platonism	20, 25, 27, 34	37	5
Problem-solving	24, 36	21, 34, 35	5
Pattern Analysis	29, 31, 33, 37, 39	23, 27, 28, 30	9
Not used	19, 22, 32, 38		4

Table 17: The four subscales for views of mathematics

For each subscale in this section, I show a bar graph for the responses before and after the intervention, separated by a graph showing the movements using the values above.

Like/Dislike of Mathematics

Items 15 and 16 interrogate the participants regarding their like/dislike attitudes to mathematics and are oriented in opposite directions, as shown in the table below.

Items	Direction
15. I find solving mathematics problems to be dull and boring	Negative
16. I like mathematics better than most other subjects	Positive

Table 18: The 2 Likert items constituting the like/dislike subscale

The range of the subscale is from 2 to 10 and the results are illustrated in the figure below. The yellow chart on the left, illustrates the responses at the start of the course, and shows a bell curve with a slant towards the positive: the participants, at the start, express a tendency towards liking mathematics. In the maroon chart on the right, the post-module responses still form a bell curve slanted towards the positive, though the frequencies have changed. The chart in the middle shows the movement of the responses between the two surveys. Theses movements are concentrated in the middle, as are all of those measured by the Likert items, and the charts have therefore been truncated by half around the zero.
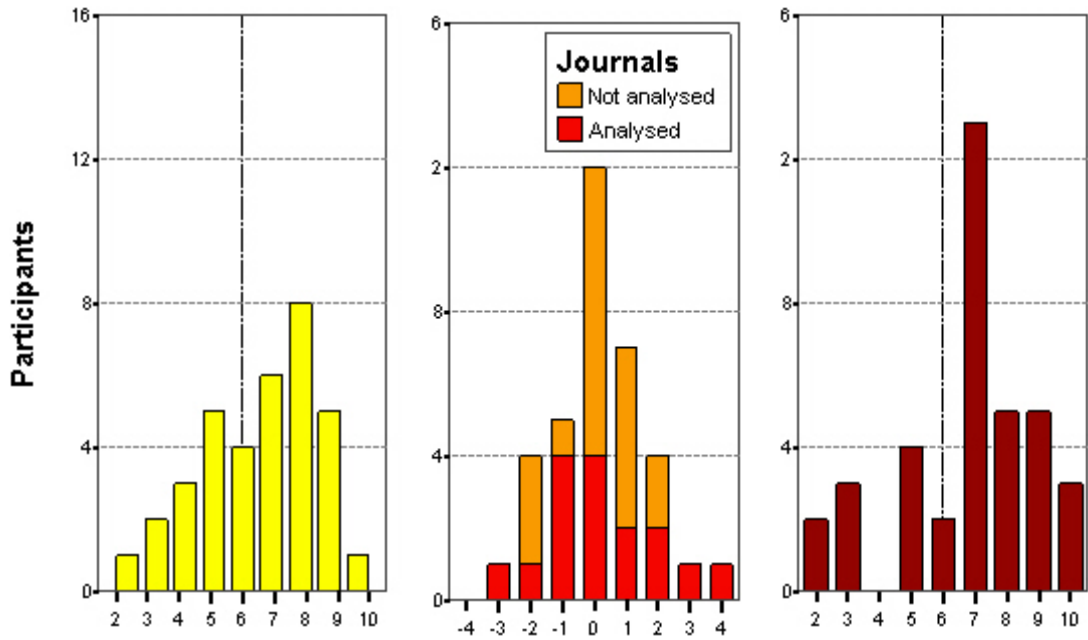
Figure 10   **Results for the like/dislike subscale: before / change / after**

As predicted by the theoretical framework, which states that attitudinal responses are stable over time, the changes are insignificant, both statistically (with a Paired-sample p-value of .505) and practically (with a Cohen's d of .083, where values between .200 and .500 denote a small effect).

Self-confidence in Mathematics

Items 17 and 18 interrogate the participants regarding their self-confidence in mathematics and are oriented in opposite directions, as shown in Table 19, and the range of the subscale is again between 2 and 10 and the results are illustrated in the figure 11, below.

Items	Direction
17. Mathematics is a subject I find easy	Positive
18. I have never been confident in mathematics	Negative

Table 19:   **The 2 Likert items constituting the self-confidence subscale**

The yellow chart on the left, illustrates the responses at the start of the course, and shows a bell curve with a slant towards the positive: the participants, at the start, express a tendency towards self-confidence in mathematics. In the maroon chart on the right, the post-module responses still form a bell curve slanted towards the positive, though the frequencies have changed. The chart in the middle shows the movement of the responses between the two surveys. As predicted by the theoretical framework, again, the changes are insignificant, both statistically (with a Paired-sample p-value of .768)

and practically (with a Cohen's *d* of .024, where values between .200 and .500 denote a small effect).
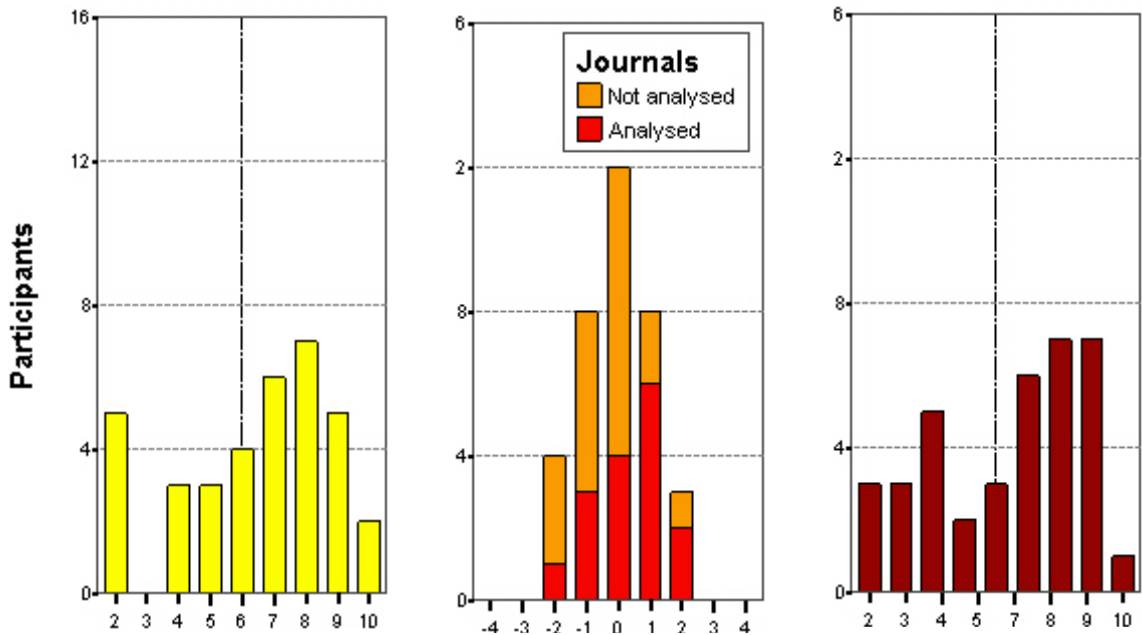

Figure 11   Results for the self-confidence subscale: before / change / after

The remaining Likert items of Section 4 focus on the participants views of the nature of mathematics. The subscales are defined by successive refining of the correlations with the Items (see Chapter 3 for the method and Appendix 10 for the values).
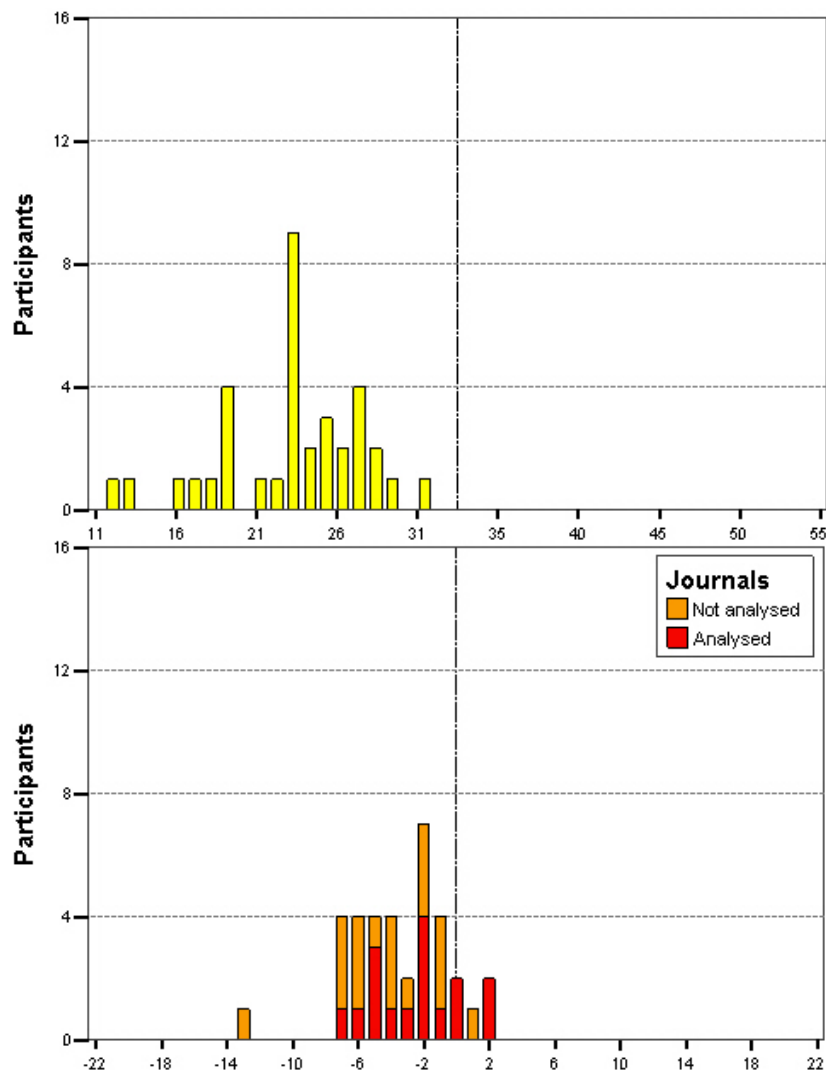
Instrumentalist View of Mathematics

The Instrumentalist view is represented by the responses to the items shown in Table 20:

Items	Direction
23. Only gifted professional mathematicians can be creative in mathematics	Positive
26. Exploring number patterns is not real mathematics	Positive
28. Puzzles and investigations are not genuine mathematics	Positive
29. There are many problems in mathematics which have never been solved	Negative
30. Basic number skills are more important than creativity in mathematics	Positive
31. Mathematics is always changing and growing	Negative
34. Mathematics is exact and certain	Positive
35. There is only one correct way of solving any mathematics problem	Positive
36. A person should not mind risking a mistake when trying to solve a mathematics problem	Negative
37. Investigating a puzzle can lead to significant new mathematics	Negative
39. I think that creativity and mathematics are related	Negative

Table 20:   The 11 Likert items constituting the Instrumentalist subscale

Together, these items create a subscale that ranges from 11 to 55 in value and the results are illustrated in the figure on the next page. For the Instrumentalist subscale, the first chart in yellow illustrates the participants' tendency at the start of the module and the maroon one, after. The situation prior to the intervention already presents results suggesting that Instrumentalism is not a view of choice (the median and mode are both at 23, with the theoretical value at 33). This tendency becomes more acute through the experience of the intervention, as shown by the new values for median and mode of 19 and 16 respectively. This is further reflected by the statistical significance (p-value of .000) and the practical significance (Cohen's d of -.815).
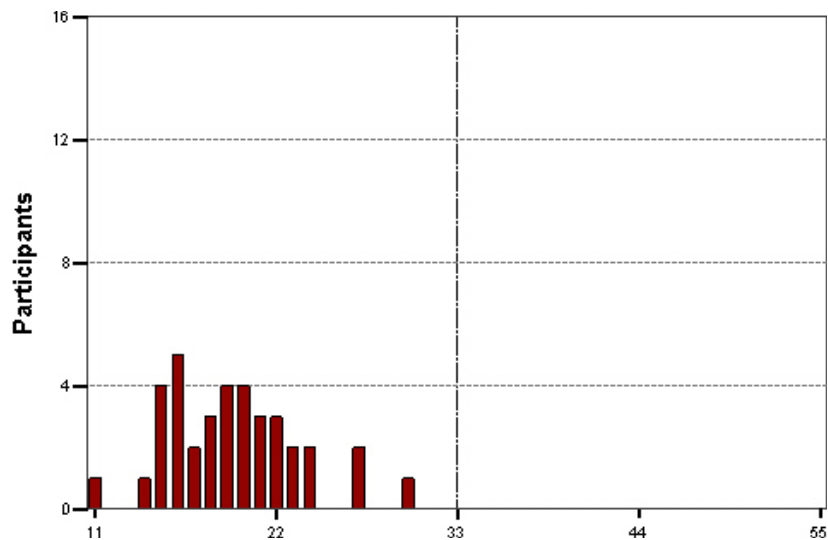
Figure 12   Results for the Instrumentalist subscale: before / change / after

Platonist View of Mathematics

The Platonist view is represented by the responses to the items shown in Table 21:

Items	Direction
20. Mathematics consists of a set of fixed, everlasting truths	Positive
25. The discoveries of mathematics are permanent	Positive
27. In mathematics there is always a right answer	Positive
34. Mathematics is exact and certain	Positive
37. Investigating a puzzle can lead to significant new mathematics	Negative

Table 21:   The 5 Likert items constituting the Platonist subscale

Together, these items create a subscale that ranges from 11 to 55 in value and the results are illustrated beginning below. For the Platonist subscale, the first chart in yellow illustrates the participants' tendency at the start of the module and the maroon one, after. For this view, again, the situation prior to the intervention shows that Platonism is already not a view of choice (the median and mode values are 11 and 10, respectively, with the theoretical value at 15), and, though this is less flagrant than in the previous subscale, the post-module values for the same two are both 10, once again well below the theoretical value of 15. The chart in the middle demonstrates that this tendency has become more acute through the experience of the intervention. This is reflected by the statistical significance (*p*-value of .000) and the practical significance (Cohen's *d* of -.767).
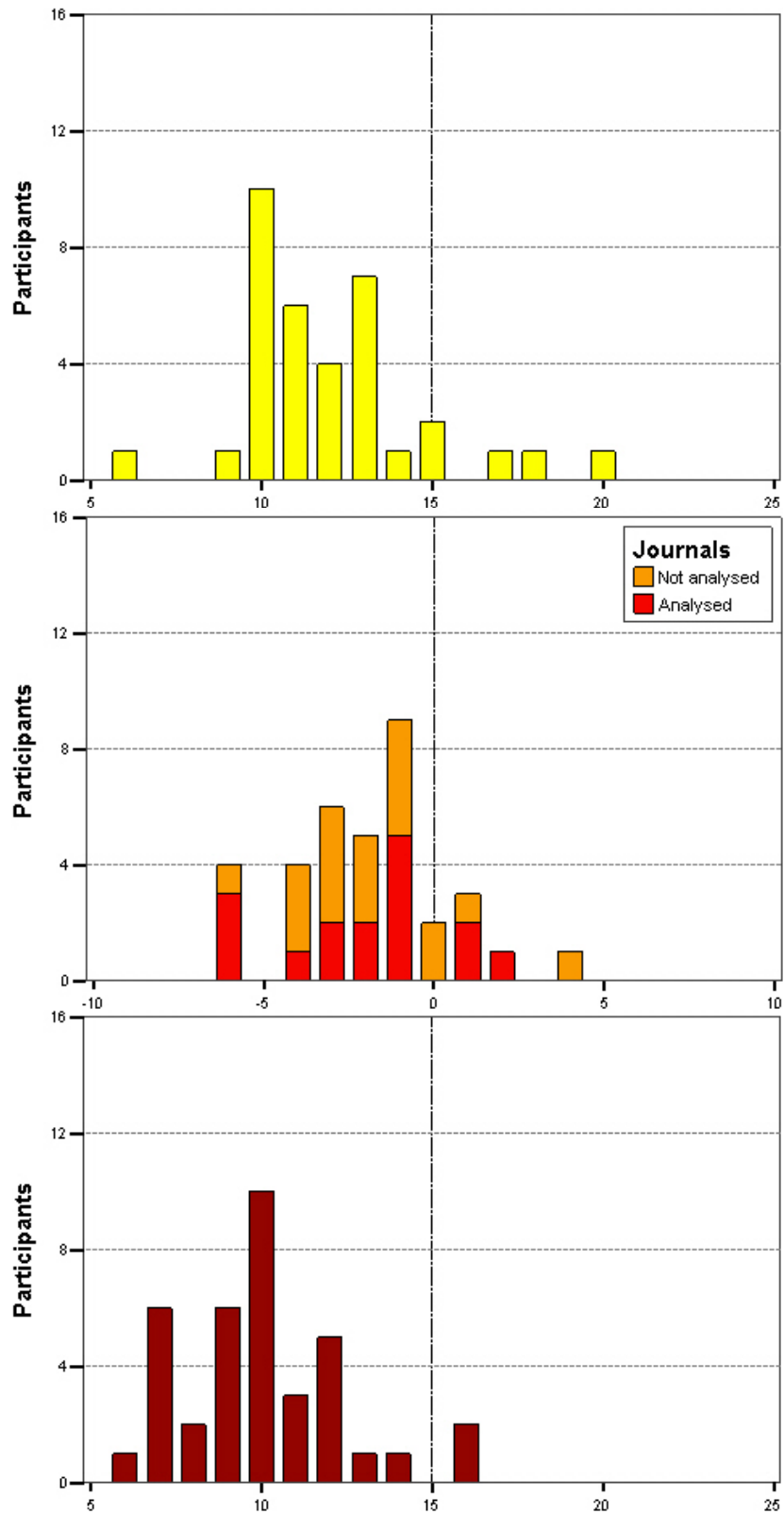
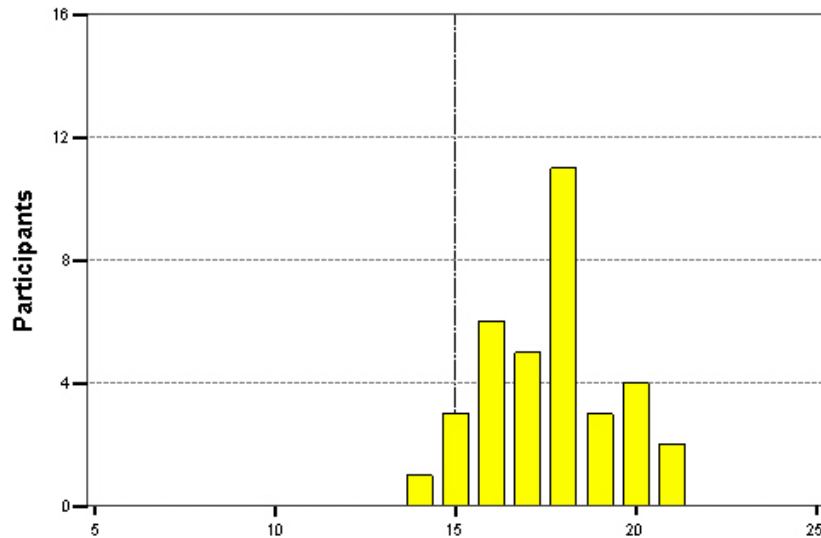**Figure 13   Results for the Platonist subscale: before / change / after**

Problem-Solving view of Mathematics

The Problem-solving view is represented by the responses to the items shown in Table 22:

Items	Direction
21. Mathematics is about the study of all possible patterns	Negative
24. There are many ways of solving any problem in mathematics	Positive
34. Mathematics is exact and certain	Negative
35. There is only one correct way of solving any mathematics problem	Negative
36. A person should not mind risking a mistake when trying to solve a mathematics problem	Positive

Table 22: **The 5 Likert items constituting the Problem-solving subscale**

Together, these items create a subscale that ranges from 5 to 25 in value and the results are illustrated in the figure on the following page. For the Problem-solving subscale, the first chart in yellow illustrates the participants' tendency at the start of the module and the maroon one, after. For this view, the situation prior to the intervention shows that Problem-solving is a view of choice (the median and mode values are both 18, with a theoretical value of 15). The post-module values for the same two increase to 19, further away from the theoretical value of 15); the effect is there, as demonstrated by the chart in the middle which shows the changes. This is also reflected by the statistical significance (p-value of .000) and the practical significance (Cohen's d of .919).
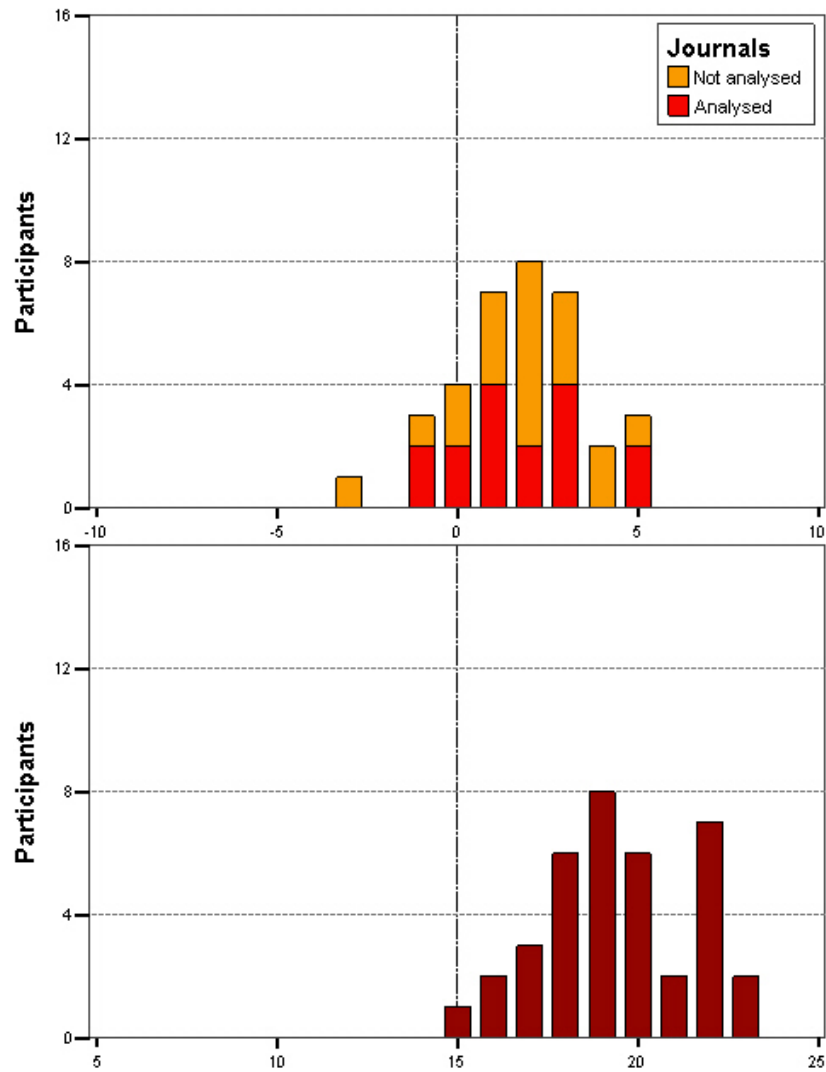
**Figure 14   Results for the Problem-solving subscale: before / change / after**

Pattern Analysis View of Mathematics

The Pattern Analysis view is represented by the responses to the items shown in Table 23:

Items	Direction
23. Only gifted professional mathematicians can be creative in mathematics	Negative
27. In mathematics there is always a right answer	Negative
28. Puzzles and investigations are not genuine mathematics	Negative
29. There are many problems in mathematics which have never been solved	Positive
30. Basic number skills are more important than creativity in mathematics	Negative
31. Mathematics is always changing and growing	Positive
33. Some mathematics problems have many answers, some have none	Positive
37. Investigating a puzzle can lead to significant new mathematics	Positive
39. I think that creativity and mathematics are related	Positive

Table 23:   The 9 Likert items constituting the Pattern Analysis subscale

Together, these items create a subscale that ranges from 9 to 45 in value and the results are illustrated in the figure starting below. For the Pattern Analysis subscale, the first chart in yellow illustrates the participants' tendency at the start of the module and the maroon one, after. For this view, again, the situation prior to the intervention shows that Pattern Analysis is a view of choice (the median and mode values are both 31, with a theoretical value of 27), and, though this is less flagrant than in the previous subscale (the post-module values to 34 and 33 respectively, further away from the theoretical value of 15), the effect is visible, as demonstrated by the chart in the middle. This is also reflected by the statistical significance (*p*-value of .000) and the practical significance (Cohen's *d* of .831).
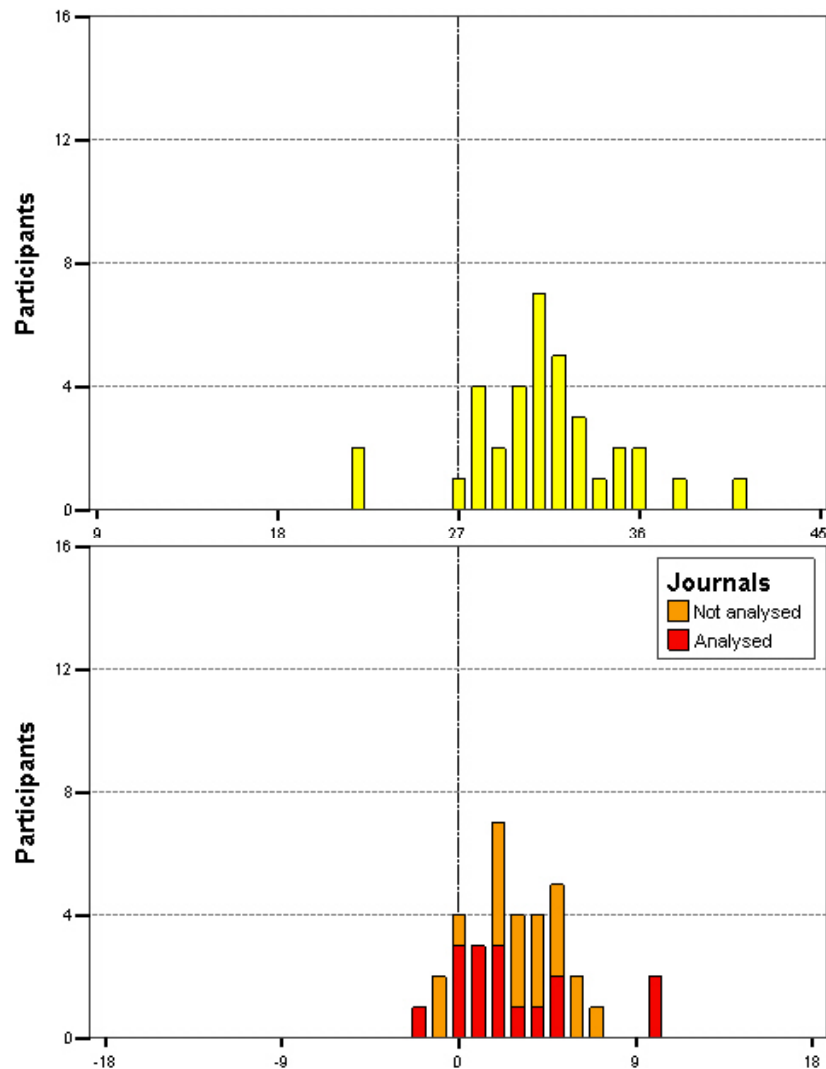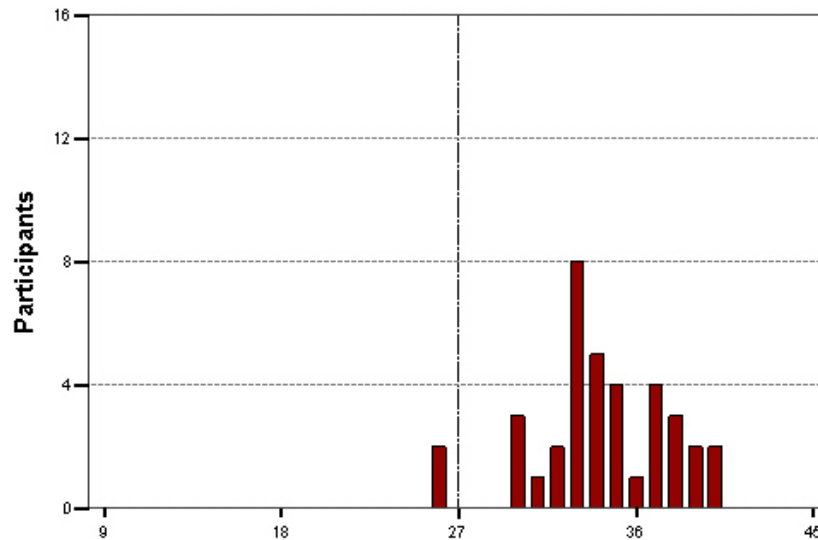
Figure 15   Results for the Pattern Analysis subscale: before / change / after

In the section of Chapter 2 that focuses on the potential effect of the intervention, I cited Ajzen as explaining that "only beliefs that are readily accessible in memory influence attitude at any given moment." (2001, p. 30). In addition, the theoretical framework also states that attitudes are more stable than either emotions or beliefs. In the results of the Likert subscales I examine here, this last point is clearly seen. In Table 24, below, I show a summary of the significance values:

		Items	Sign. (2-t'd)	Cohen's d
Attitudes	**Like/Dislike Subscale - oriented**	2	.505	0.083
	Self-Confidence Subscale - oriented	2	.768	-0.024
Beliefs	**Instrumentalist Subscale - oriented**	11	.000	-0.815
	Platonist Subscale - oriented	5	.000	-0.767
	Problem-solving Subscale - oriented	5	.000	0.919
	Pattern Analysis Subscale - oriented	9	.000	0.831

Table 24:   Summary of significance of results for the 6 Likert subscales

As predicted, the statistical significance values for the attitudes are too high for the result to be significant (with a value <.05 normally considered significant), and the effect size values too low (a value >.200 is necessary, for even a small effect size). In contrast, the values for the beliefs are well within the significant range.

In the theoretical framework, I make the point of including a view of mathematics that is not normally discussed in the literature about mathematics education, but emerged from the review of the philosophy of mathematics. I called it Pattern Analysis, to emphasise the lack of the expected goal-state implied in Problem-solving. In the analysis of the Likert items, I create a corresponding, fourth subscale. After a refining of the four subscales, I calculate the significance of the change in views of the participants.

To conclude, I examine the separate nature of this fourth view. In Table 17 I list the items that are constitutive of each subscale. As it is, Pattern Analysis has no items in common with Problem-solving, suggesting that there is a clear distinction between the two. In addition, Pattern Analysis has 2 items in common with Platonism (though in reverse directions), and 7 with Instrumentalism (again, in reverse direction). Though this last number is fairly high, this is partly due to the fact that the two subscales have as many as 9 and 11 items. To show to what extent the subscales are distinct, Table 25, below, presents correlations between the pre-module results in all four subscale. Note that Instrumentalism and Platonism correlate highly, even though they only share Items 34 and 37. Moreover, neither Instrumentalism nor Platonism correlate with Problem-solving, though they share 3 and 1 Item, respectively, with it.

		Instrumentalism Subscale - before	Platonistm Subscale - before	Problem-Solving Subscale - before	Pattern Analysis Subscale - before
Instrumentalism Subscale – before	Pearson Correlation	1	.432**[32]	-.287	-.795**
	Sig. (2-tailed)		.010	.094	.000
Platonism Subscale – before	Pearson Correlation	.432**	1	.148	-.430**
	Sig. (2-tailed)	.010		.396	.010
Problem Solving Subscale – before	Pearson Correlation	-.287	.148	1	-.046
	Sig. (2-tailed)	.094	.396		.791
Pattern Analysis Subscale – before	Pearson Correlation	-.795**	-.430**	-.046	1
	Sig. (2-tailed)	.000	.010	.791	

Table 25: Correlating the pre-module results in the four subscales

Though Pattern Analysis has a very high negative correlation with Instrumentalism, Tables 26 and 27, below, show that the two subscales are indeed distinct. As indicated in yellow in Table 26, the 'missing items', taken from the Pattern Analysis subscale, do not correlate highly with the Instrumentalism subscale.

		Instrumentalist Subscale – oriented – before
Item 27 – before	Pearson Correlation	0.218
	Sig. (2-tailed)	0.209
Item 33 – before	Pearson Correlation	-.176
	Sig. (2-tailed)	.312

Table 26: Correlating Items 27 and 33 to the Instrumentalist subscale

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[32] **Correlation is significant at the 0.01 level (2-tailed).

Table 27 shows that the equivalent converse is the case: the 'missing items' taken from the Instrumentalism subscale correlate poorly with the Pattern Analysis subscale. The two subscale are therefore not simply opposite poles of the same dimension. Though the subscales overlap by as many as 7 items, they can be considered distinct enough to measure a different attribute of the sample.

		Pattern Analysis Subscale - oriented - before
Item 26 – before	Pearson Correlation	-.099
	Sig. (2-tailed)	.571
Item 34 – before	Pearson Correlation	-.211
	Sig. (2-tailed)	.224
Item 35 – before	Pearson Correlation	-.042
	Sig. (2-tailed)	.810
Item 36 – before	Pearson Correlation	.118
	Sig. (2-tailed)	.499

Table 27:   **Correlating Items 26, 34, 35 and 36 to the Pattern Analysis subscale**

Short Answer Items

Items 2 and 3 of Section 2 and 40-44 of Section 5 of the post-module questionnaire require the participants to provide short answers.

Items 2 and 43: Discussing the Course

Items 2 and 43 required the participants to reflect on the course as a whole. Item 2 asked the participants if, given that they are taking part 2 of the double-module with the same teacher, they would be interested in working on a mathematical enquiry again. Of the 37 respondents, five say that they are not taking the module with the same teacher, two of which are expressing dismay at this, including Marie:

> I am unable to take his class next spring due to scheduling problems. However, in the fall I really hope he teaches 3118 and I would really enjoy doing a project like we did. I found it to be the most enjoyable part of class. (Marie, Post-module Questionnaire)

Of the remaining 33 participants, 27 expressed interest in more mathematical enquiry experience, with responses ranging from keen interest:

> Yes I am planning on taking this class in the spring. I would like to do a project because I felt it opened up our minds to be able to think of math in a whole new light. (Sue, Post-module Questionnaire)

… to a cooler response:

> Yes, I'm taking the class. I would work on a project if it were more structured and pertained more to the class and helping me teach in the future. (Isabel, Post-module Questionnaire)

The six participants who are responding negatively, do so mildly. Though they are taking part 2 with the same group, they explain that:

> … it is time consuming and took time away from my main focuses (Ashley, Post-module Questionnaire)

> I don't think I would be interested because I would not be a very good participant. I barely had enough time to write in my journal this semester. (Pauline, Post-module Questionnaire)

> … it helps us become experts in one subject but then we tend to forget about the other topics. (Laura, Post-module Questionnaire)

Most of the participants who answered positively included suggestions to improve the approach, which, combined with the responses to Item 43, provide a source for adapting the teaching approach. The responses to this item can be categorised using a combination of the theoretical framework (Chapter 2), the codes developed in the qualitative analysis (Chapter 3), and the responses themselves. These responses fall into the following classes:

Category	Example of Response	Count
Time Constraint	Sometimes I felt like you gave too much time when we were working in groups while you were lecturing. (Sandra)	6
Instruction Content	Make less 3-D shapes. (Rob)	6
Instruction Style	Integrate the book work into the course earlier. Maybe spend one day a week on projects and the other day on bookwork. (Silvia)	11
	Maybe inform us a little more about exactly what we were going to be doing over the semester. (Jean)	
Mason's Tension	At the beginning, I'd say what the agenda is to the class, everyday in September seemed to go through different unrelated topics which was confusing. (Sean)	6
No Project	I wouldn't do a project. I would tie material together, and I would explain more about teaching in a classroom. (Isabel)	1
No Comment	I cannot think of anything. (Darleen)	9

Table 28: **Categories and counts for responses to Item 43**

A small number of these responses manifest the continued misapprehension about the purpose of the course which I mention earlier.

Items 40 to 42: Discussing the Process of Enquiry

Responses to Items 40-42 focus on the process of enquiry which is presented in the intervention (see Appendix 4):

40. Did you find the bonus question about an 'interesting thing to look at next' useful? How?

41. Please explain how you decided on your project topic

42. Please explain how you think your project fits into the context of your view of mathematics

The first item focuses on the feature of the ramping-up phase that reminded the participants that there is no fixed goal-state in this kind of practice. Their responses to the item ranged from a terse 'no', to positive responses with varied justifications. For example, 16 participants interpreted the usefulness of the bonus question in a philosophical way, that is, their responses reveal a reflective standpoint that can be connected to their views of their engagement with mathematics. In contrast, 3 participants gave responses that focused on very pragmatic perspectives that have more to do with the realities of the classroom, regardless of the discipline, as illustrated by Samantha's response (4th row).

Category	Example of Response	Count
Yes (Philosophical)	Yes, because it gets me to critically think about what I want to learn and how to connect what I learned to other things. (Joan)	16
	Yes, it makes me stop and think about what I just worked on and why I worked on it. Also, it made me form relationships between different ideas. (Emily)	
Yes (Pragmatic)	It was useful for deciding on a project topic. (Silvia)	3
	Yes, it was an easy question that gave "cushion" points for things you got wrong. (Samantha)	
Ambivalent	Sometimes. It was difficult to know what to look at next and we never really covered what I thought we could look at next. It made me think, but it was never elaborated on. (Isabel)	14
	Not useful but a way to make you think critical about the homework but I never used it with other things/activities. (Christie)	
No	No, because I know I'll never explore what I suggest further, plus, it usually takes me longer to come up with that than to complete the actual homework.(Myriam)	4
	No because you never really did anything with it. (Petra)	

Table 29: Categories and counts for responses to Item 40

Responses to Item 41 are almost exclusively attitudinal, as shown in Table 30, below:

Category	Example of Response	Count
Interest	We started with a topic we were interested in (patterns and colorings) and then used trial and error to expand our ideas. (Alice)	16
	Took two topics that interested me and synthesised them. (Sue)	
Like	I really like colors, designs and patterns so I decided to do tesselations with the proper colorings. (Darleen)	8
	I really like the idea of color versus symmetry and how the two could relate. (Elise)	
Different	We wanted to look at something different that no one else was doing. We didn't come up with it right away. We did some experimenting first. (Sandra)	5

	I decided to do tesselations because it was a totally new idea to me. I wanted to learn about something I did not know already. (Emily)	
Group choice	Joined a group, took their basic topic and approached it or looked at it in a way I wanted to. (Kerrie)	2
Others	We thought it would be fun to explore closed-up shapes using only 2 different [polygons] and see if there was a formula. As we started our project it grew into different explorations of these shapes. (Simone)	7
	I accidentally went more in depth on an assignment than needed, and that is what we ended up working on in my group. (Trish)	
	I developed a hypothesis, a theory through homework but found that it did not work. (Geoffrey)	
	I chose the topic that I understood[33] the most, wouldn't have a difficult time putting together into a project and would be easy to discuss with the class. (Myriam)	

Table 30:   **Categories and counts for responses to Item 41[34]**

In this table, the responses are organised according to categories that emerged from the data, and in certain cases, these categories emulate those found in the literature on affect in mathematics education, as is the case with the first two: interest and like. The third category expresses the participants' aspiration to choose an original topic. Together the first three categories suggest that the condition stipulated by Criterion 4 (an atmosphere of security) is successfully implemented, at least for these participants.

Finally, responses to Item 42 fall into four distinct categories: comments on the participants' views of mathematics as a subject, views of the content of mathematics, the process of doing mathematics, and the experience associated with it. The distribution is as shown in Table 31, below:

Category	Example of Response	Count
View	It opened my views of math to a more broad idea where relations and patterns etc. fit in to the whole deal. (Trish)	10
	My project is somewhat complicated and I think I view mathematics also as complex. (Pauline)	
Content	Shapes are manipulated all the time to create other ones. Also, it shows changing one aspect changes more. (Sandra)	3
	It doesn't really fit. It was just one piece of mathematics and not anywhere close to the whole picture. (Isabel)	
Process	My project allowed me to understand the process of developing a mathematical idea. (Sue)	17
	This project taught me how to look for multiple ways to explore the same topic and how to analyse the results we found. (Silvia)	
Experience	It made me realise how creative and innovative math can be, not just numbers and problems and formulas. (Barbara)	8
	It was frustrating to try and break down the concepts that you already know are true. It also made a common theory uniquely mine. (Rosie)	

Table 31:   **Categories and counts for responses to Item 42**

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[33] My Italics.

[34] One participant expressed both interest and liking, giving the table a total of 38.

In the case of Item 42, some results manifest a change in views. Carol, in fact, is quite emphatic; she says: "My view has obviously changed from the beginning of semester". Overall, the language used in the responses to these three items shows creative engagement, through the use of positive attitudes for the selection of the starting point (terms like 'liking', 'interest' and 'fun' come up), and critical reflection regarding the essence of mathematics, its practice and experience.

Item 3: Awareness of Change in View of Mathematics

In Item 3 of the post-module questionnaire, the participants expressed their awareness of the change, or lack of change, in their view of mathematics. As this item requires a short answer (four lines are provided), the responses are general in nature and do not reflect the complexity shown by the Likert subscales. In addition, the responses made at this point reflect the participants' awareness of change. The counts, however, do mirror the findings of the Likert subscales: the most frequent category is found in the changed beliefs (2+22+3=27), while only 3 responses suggest a change in attitude.

Example of Response	Changed Attitude	Changed View	Count
I think of math now as something that can be discovered. Also as something interesting. (Sue)	yes	yes	2
Yes. Mathematics is more complicated to teach than I imagined. It is more difficult to explain your answers and make things clear to other people. (Pauline)	yes	-	1
Yes, I am more analytical about the "whys" of mathematics. Like trying to determine why an answer or solution is right and not just determining if it is right. (Alice)	-	yes	22
Yes this course has taught me about the wider implications of math. Before I saw math as primarily problems that you would solve from a book. This class showed me a range of new topics like applications. (Silvia)			
Somewhat. I spent more time thinking about how it can help me in my teaching rather than disliking it. (Elise)	no	yes	3
Not really. I view this class as frustrating, which leads to frustration with the math that we learned. However, it didn't change my view of math as a whole. (Isabel)	no	no	4
I've disliked the whole 'figure it out for yourself' method. I hate doing my homework or answering questions and thinking I understand the material only to be told later that I'm completely wrong. (Myriam)	no	-	2
No, not really. I have always enjoyed mathematics. This class was fun!			

| Not really. It showed me a different angle to do the same stuff. It was really cool and helpful but the answer is still the same. (Sean) | - | no | 2 |

Table 32: Categories and counts for responses to Item 3

Phase Rating Items

In the post-module questionnaire, Items 6 to 9 required the participants to rate the three main phases of the teaching approach in terms of four criteria. The question itself suggests the use of the ranks 1, 2 and 3, in order, for the ranking. Several participants responded simply by putting an X in the box corresponding to their best choice, leaving the other two blank. Others used the three numbers. Occasionally, a student marked more than one phase as best choice. For consistency of analysis, only the first ranking is used in this analysis. In figure 16, below, the participants' responses rate the phases for most interesting and most like the work they think mathematicians do.
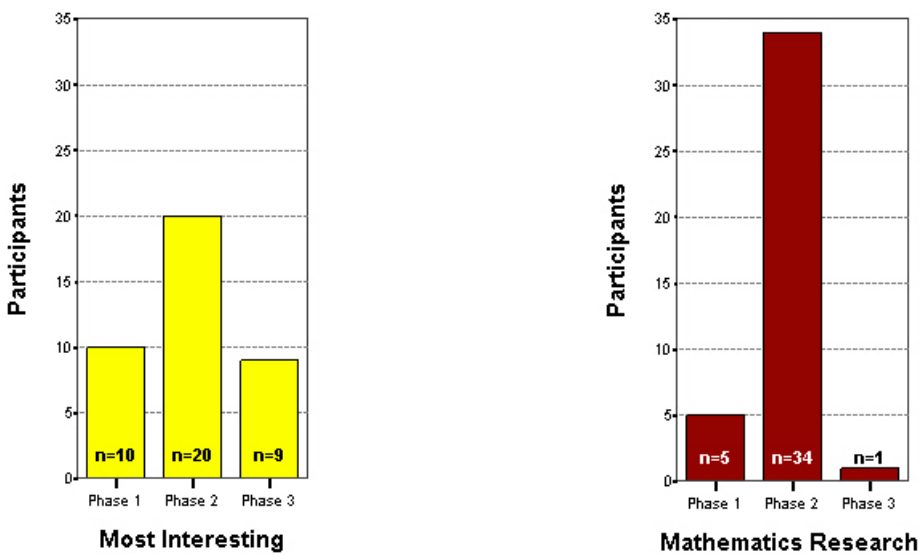

Figure 16 Rating the phase of the teaching approach for most interesting and most like the work of mathematics researchers

The responses in yellow, regarding the participants' interest shows that though phase two has interested many of them, presumably because they were able to choose their topics, the other two phases are almost head-to-head, and together almost match the count for phase 2. The responses for Item 9 are shown in maroon and correspond to the phase during which the participants think their "work was closest to what I think research mathematicians do". The chart clearly shows the participants' awareness that the work of phase 2 most closely emulated the exemplar.

In Items 7 and 8, the participants responded to inquiries regarding the phase during which they feel that they learned the most about teaching (see left side of Figure 17) and about mathematics (right side of figure 17).

Though the participants show a preference for phase 2 for learning about teaching (20 participants choose that option, versus 11 for phase 1), they prefer phase 3, which consisted of direct teaching, for the learning about mathematics. This result suggests that the phrase 'learning about mathematics' may be equivalent, at least in some cases, with 'learning mathematics', i.e. 'learning mathematical content'.
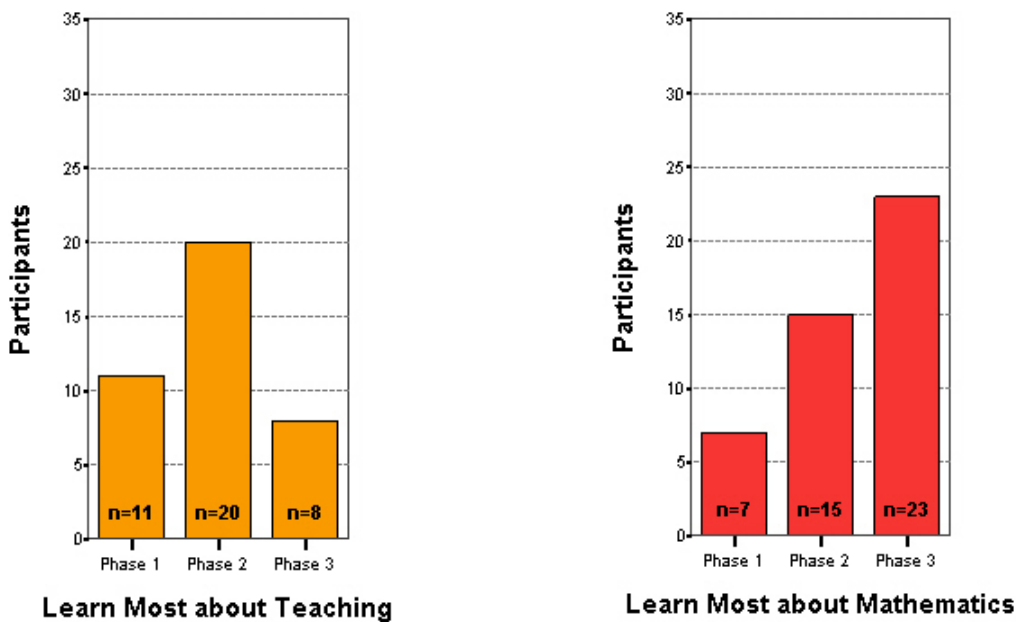

Figure 17   **Rating the phase of the teaching approach for most learning about teaching and about mathematics**

Multiple-Choice Items

Aside from the 'descriptive items' analysed previously, there are 2 items in the post-module questionnaire that can be considered multiple-choice. The first one of these, Item 4, requires the participants to comment on the impact that they expect the experience provided will have on their future practice. The results show a 32:4 bias towards the positive, with one response missing. Unfortunately, the item did not require a qualified response, and so this does not allow for much interpretation: the change could be in either direction.

The second multiple-choice item, numbered 5, concerns the participants' self-perceived ability in mathematics. Data for this item is collected in both surveys and the results can therefore be compared. Figure 18, below, shows the change in self-determined ability in mathematics. The chart focuses only on the participants who responded to both questionnaires.

As predicted by the theoretical framework, the changes are slight: the count for 'competent' goes down by 1, and that for 'weak' goes up. The change is so small that it can be considered insignificant. Once again, an attitudinal response is stable.
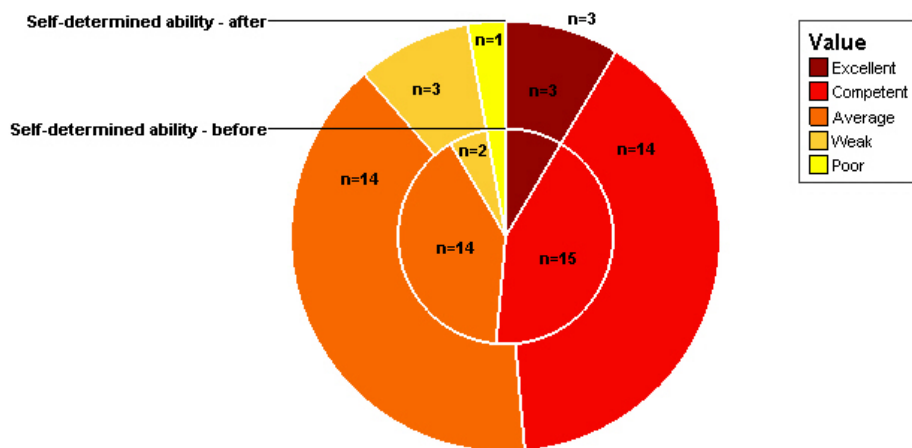

Figure 18   Self-determined ability in mathematics before and after

Overall, the responses to the post-questionnaire demonstrate what is predicted by the theoretical framework: the participants' view of mathematics shifts away from Instrumentalism and Platonism to a view more in line with current thought. Conversely, the attitudes remain essentially the same, thereby demonstrating their stability.

Chapter 8: Discussion

In the first chapter, I introduced the three questions that this study is designed to answer, to wit:

- What could be the design criteria of a teaching approach which aims to provide an experience of practice analogous to that of research mathematicians? Which of these criteria are feasible in the given context?

- Can the experience of engagement with the resulting teaching approach successfully simulate that of engagement in the practice on which it is based, according to the design criteria?

- What are the affective outcomes, and are they as anticipated?

Each of these questions is answered using the appropriate methods, and in the case of the first one, the results are used as the basis for the later two. Because of this structure, I discuss the results to each component of the study separately, after which I discuss some of their interactions.

Question 1: Designing the Teaching Approach

The formulation of the first question elicits a result that takes two forms. Firstly, a set of design criteria for a teaching approach is derived through a review of the pertinent literature, and secondly, the results are integrated with the realities of the classroom context, in order to create a feasible teaching approach.

The literature that provides the basis for the design criteria combines writings on the experience of research mathematicians with an examination of the mathematics teaching practice most comparable to it: mathematical problem solving as practised in the classroom. This result can be found in the last part of *Mathematical Enquiry (ME) as Distinguished from Mathematical Problem Solving in the Classroom (MPSC)*. It essentially consists of 5 criteria, according to which the teaching approach needs to:

1. create a context within which the peripheral participants take on the role of choosing and formulating the starting point of their task;

2. leave the process of resolution unspecified and open to the peripheral participant's choice;

3. imply no goal-state (Mayer, 1985), implicit or explicit, in the initial presentation of the task, so that there is no implied end;

4. present an atmosphere of security which promotes and encourages the taking of creative risks by shifting the focus on the process of development of the result, rather than on the result, which is uncertain;

5. allow for enough time so that the experiential cycle(s) can be experienced in full.

These criteria largely centre on the participating student's engagement, practice and experience in a module taught using this approach, leaving the mathematical content open. In particular, the element that promotes the engagement, on the part of the student-participants, needs to be carefully calibrated. This engagement, which I define, in the beginning of the literature review, as critico-creative, involves, by its very nature, a component of risk-taking, which is mediated by the design criteria. For example, criteria 1-3 shift the responsibility for decision making onto the student-participant, thereby providing her/him with agency, as I discuss in the literature review. This places the risk on the student-participant's shoulders. Conversely, criterion 4 mitigates this risk by de-emphasising the need for achieving an objectively normed result, focusing instead on the required level of engagement. Criterion 5, finally, is designed to increase the similarity of the student-participants' experience to that of research mathematicians, who wind down their enquiry when it demands it, rather than when 'time is up', or when they lose momentum. This different criterion for ending an enquiry circles back, again, to the enquirer's agency, which, in the case of research mathematicians, is more pronounced than in traditional mathematical problem solving instruction.

Promotion of the right level of engagement according to these criteria, together with other elements of the teaching approach such as the assessment scheme, is intended in turn to provide the appropriate context for a practice and experience similar to the exemplary to take place. In particular, the similarity of this experience to the exemplar is examined in the results of Question 2, below.

In chapter 4, I describe the implementation of the approach resulting from the integration of these criteria with constraints imposed by the specific context of the intervention. This implementation, in the spirit of action research, incorporates

continuous adjustments, due to the immediate circumstances of the specific context. For example, a concerted effort has to be made, on the part of the teaching participants, to focus their responses on scaffolding for rigour, rather than for meaning (see the description of this distinction in Educational Foundations of the Theoretical Framework). This strategy is unusual for a mathematics classroom and its purpose stems from the different objective of the teaching approach and the need to communicate this difference. It demands a higher level of vigilance on the part of the teaching participants and therefore also a changed attitude on their part. In addition, it impacts other components of the approach, such as the social contract that is established, however implicitly, between them and the student-participants. In effect, this change in interaction can only be acceptable to the student-participants if it is counter-balanced by shifts in the teacher's expectations of the students, as perceived by the latter. Removing the scaffold for meaning is acceptable because the rules of the interaction have changed otherwise.

In addition to adjustments that are made, during the implementation, in response to immediate circumstances, the results of the second research question can provide additional input into answering the first question: the analysis of the student's journals can help refine the teaching approach towards a future implementation. For example, the writings of several student-participants suggest a set of expectations on their part that does not accord with the purpose of the module, in its normal form or within the intervention: they expect the purpose of the module to be an examination of effective teaching methods for elementary school mathematics. Finally, some of the items of the post-module questionnaire can provide additional input for the future implementation of the approach.

Question 2: The Experience of the Participants

Question 2 of this study focuses on a comparison between the experience which student-participants have of the designed teaching approach and the exemplar on which it is based. This comparison is itself largely structured around the design criteria, though other results emerging from the data itself are also examined. In addition, only 16 of the 37 journals are examined, as I put aside any that show an insufficient engagement in the journaling exercise.

For each of the original design criteria, I find some responses suggesting the authenticity of student-participants' experience with respect to the exemplar. In some cases, the response is positive, that is, the participant expresses not only an awareness of having it, but also an appreciation of the experience. In other cases, the student-participant expresses a negative affective response to the experience, but this still can be counted as 'having the experience'. Some criteria are not alluded to by some participants. This can stem from three different possible scenarios: the student-participant does not have the specific experience, s/he is not aware of having it, or s/he does not express it in writing. The question that this component is used to answer is whether the experience can be simulated, and the evidence does suggest that it does, for each of these five criteria.

Criterion 4: The Sense of Security

In the case of criterion 4, two aspects emerge. Firstly, student-participants' comments on the accessibility of the mathematical content show a significant pattern: during the 'Ramping up' and 'Regular Teaching' phases the comments present a normal picture in that the views are distributed between those expressing too much ease and too much difficulty. In contrast, during the 'Main Enquiry' phase, the comments focusing on the content mainly consist of connections made with the topics of the previous phase, without evaluation of difficulty. This is not unexpected, of course, since the topics that the student-participants investigate during this phase are pitched, by them, at a level with which they are comfortable. This suggests conformity of the experience with the requirement of criterion 4, regarding the atmosphere of security.

Secondly, criterion 4 can be interpreted to be built on the social contract that binds the teaching participants and the student-participants, particularly with respect to the tension described by Mason (1989). In this context, the vehicle for this aspect of the teaching approach is the module syllabus, and particularly the assessment requirements. In the design of the teaching approach, this aspect is translated into the shift from the traditional focus of assessment in mathematics: the results of the enquiry, to a focus on the process undergone. In addition, the teacher-participant chooses to withhold the details of the assessment criteria, creating an uncomfortable situation for the student-participants. The student-participants' responses that discuss this aspect are varied, though many express this discomfort.

On the whole, responses to this topic express more dismay at the lack of communication than at the change of focus, suggesting that this issue can be remedied without fundamentally changing the design of the teaching approach.

Examining each phase separately also highlights a difference in response, with respect to the expectations of the participants and manifests the difference in the social context. In Phase 1, the participants are encouraged to develop a sense of their own power of evaluation of their work. The responses show a resistance to this pressure, possibly because of the view that the teachers still retain the last say as to the 'rightness' of the answer. In the second phase, the responses show an increased awareness of the peripheral participants' independence from the full participant, somewhat defusing Mason's (1989) tension. This is exemplified in the comment, made by Emily, that:

> Since these are our projects, asking the professor questions does not make sense, so me and my partner decide on the answer ourselves. (Emily, 20/10/03)

In contrast, other student-participants keep a close watch on the teacher-participants' feedback:

> After talking with the professor at the end of last week I feel a little more confident about where my project is headed. They gave me good advice to take another step in my project. Even though it's almost a completely new task it still correlates to our original topic. (Geoffrey, 30/10/03)

… suggesting a certain agency, on the part of the student-participants' about the level of independence they choose to take, which, in turn, implies that they can provide for their own sense of security, in a kind of meta-agency.

In the 'Regular Teaching' phase, the comments regarding the sense of security demonstrate a return to what is perhaps not so much a comfortable level, but certainly one with which the student-participants are more familiar.

Criteria 1 to 3: Agency in the Enquiry

Criteria 1-3 essentially encapsulate the push for the student-participants' taking of ownership of the enquiry process. In keeping with this intention, three components of the enquiry process are left to their choice: the selection of the starting point, the steering of process, and the criteria of acceptance of a result. Overall, a greater level of agency is manifested in the responses discussing the 'Main Enquiry' phase, as anticipated by the theoretical framework. This agency is mainly manifested through

comments regarding the experience of choosing or changing the direction of the enquiry in which they are engaging, and the difficulty of choosing a satisfactory goal-state. Recognising this difficulty can lead to insights about the quality of question(s) raised during an enquiry, both in terms of whether the question(s) can be answered, and whether they imply a goal-state (which is sometimes a desirable characteristic). The comments focusing on this criterion are generally also connected with those of Criterion 4, since, in essence, they are two sides of the same coin.

In the discussion of this topic that can be found in the literature review, I make the point that, without all three of these criteria, the task can not be thought of as authentic mathematical enquiry as practised by full participants. This condition is quite exclusive, and suggests the importance of agency at each of the three stages. The open-ended nature of the data collection instrument, however, does not allow for a correspondingly specific verification of authenticity. In addition, this authenticity is largely dependent on the engagement of the student-participants, which can only be coaxed, not imposed. It is therefore not possible to conclusively ascertain the unqualified fulfilment of this condition, or indeed the others, even were it the case.

Criterion 5: The Nature and Experience of the Practice

In the theoretical framework, I discuss two aspects of the experience of mathematical enquiry that I consider essential to the exemplar: the stages of the process, as described by Hadamard (1945), and the extended time frame that distinguishes it from the 'investigations' of the traditional mathematics classroom. In addition, the examination of the journal data yielded an additional component to this criterion: the epistemological engagement of the student-participants during the enquiry.

In the case of Hadamard's scheme, data show evidence of student-participants experiencing each the stages, including Initiation (or Preparation), Incubation, Illumination and Verification. It is not possible to ascertain, however, whether each student-participant experiences each stage. The Initiation stage becomes apparent in the case of student-participants who express the experience of returning to re-formulate their starting point, thereby both acknowledging agency at that level, and demonstrating awareness of the stage.

Incubation, in a sense, is a non-event, that is, the lack of something happening, and student-participants may therefore not be aware of its coming to pass, and even if they

are, they may not comment on it. There are several instances of this awareness, however, and this is often expressed as frustration about the process, or more specifically about the time spent on it, suggesting that Incubation is felt, albeit indirectly. I discuss this awareness of time spent more extensively below.

Verification, an important aspect that distinguishes Hadamard's scheme from some problem solving cycles, is not explicitly mentioned in every analysed journal in its obvious form of justification of the results. This form is exemplified by comments such as: "Is this true for any solid? I want to try the equation with another combination, squares and pentagons, to see if it continues to hold true" (Jill*, 19/10/03). Using this meaning of verification as a form of rigour, verification can then be an indicator of the participants' awareness of the need to connect findings to the underlying mathematical structures, suggesting a degree of knowing-when that is associated with fuller participation.

This stage is, however, manifested in its other form, which Sowder (1993) describes, in the case of problem solving, as entailing a search of possible extensions. As such, it connects to the earlier criterion concerning the acceptability of a goal-state, and, as such, is manifested more frequently. In addition, it manifested itself in the need, for some student-participants, to explore these possible new directions, because of another criterion: the time allotment. Indeed, some groups, having achieved a goal-state that is satisfactory to them, have enough time to continue the enquiry.

Hadamard's (1945) scheme, which encapsulates four modes of working that constitute mathematical enquiry, is discernible in the journal data, though each stage is, again, not explicit in each journal.

The student-participants comment on another aspect of the nature of their experience that I emphasise in the theoretical framework: the time that needs to be allotted for an authentic process of mathematical enquiry to unfold. This aspect is connected with the already mentioned one of Hadamard's (1945) Incubation and Verification stages. In the former case, because Incubation is a real presence on the time line of mathematical enquiry, and in the latter, because more time means the possibility of pursuing the possible extensions generated by Verification. Comments are also made reflecting the idea that the time needed for a mathematical enquiry, though it varies both between

individuals and between directions of enquiry, is a significant aspect of the experience of enquiry. For example, Emily says:

> At first I didn't think that we would have enough to do for the month, but now I have realized how many questions and problems arise as I figure out my original problem. It takes time to sort everything out. (Emily, 20/10/03)

The comment shows that Emily has developed an awareness of the importance of the questions asked during an enquiry, and of their quality. This awareness can, in turn, contribute to the development of a richer view of the discipline.

A final component of the nature of the experience, which emerged from the data, focuses on the level of epistemological engagement of the participants, and relates to the part of the theoretical framework regarding the epistemological foundations of mathematics. In the analysis, several instances of comments connecting to 'knowing-when' emerged, suggesting a successful shift of the focus of the teaching approach to this level of mathematical knowing. Some comments express reluctance to engage at this level, or concerns stemming from negative affective responses. Others show willingness and a sense of wonder that this is possible.

Several events occur during the intervention, only some of which are reported in the journals, that are also quite telling of the heterogeneity of the student-participants experience, some more relevant to the theoretical framework than others. For example, the teacher-participant remarks on Kerrie (who wrote 5 journal entries) who, he says, did not engage at the anticipated critico-creative level, based on the quality of the write-up she submits. The content of the submitted work is clear but does not communicate an investigative, risk-taking practice. She joined the class late enough in the term to miss participation in both the pre-module survey and the initial class discussion, and is a member of the only group of four student-participants that works together. It is possible that these circumstances contributed to this non-engagement. As she does not participate in the pre-module, it is not possible to measure the affective outcome of her experience.

Sean (2 entries), presents a completely different picture: half-way through the 'Ramping Up' phase, he sets himself aside from the rest of the class and begins work on his enquiry project, thus spending half again as much time on it as the others. Based on his questionnaire results, particularly pertaining to the Likert items, this experience has

transformed his views: his total change score for the Likert items is 18, with the class range of 3 to 24 and the mode and median of 14 (see Figure 9).

Alexa, a strong student with a more extensive journal (only 5 entries, but spanning 8 pages) and a high level of critico-creative engagement, uses the homework submission to communicate some of her affective responses to the experience. In one of her better structured and thought out submissions, she writes: "Help me I'm lost", which suggests that her valuing of her work is at issue. Her total Likert change score is also above the median (16).

Christie (10 entries) presents an analogous situation: in her discussion with the teaching participants, she questions the extent of the mathematical nature of her topic, which suggests a need for a broader understanding of the nature of the discipline. The change in her total absolute score in the Likert items is of 20, the fifth highest.

A note made by the teacher-participant in his journal is also illuminating. Darleen (4 entries) shows that she caught on to the requirements for the engagement level in this anecdote:

> I saw [Darleen] sitting doing nothing and asked her if she had done [the assigned task]. She said, Yes, it works, and the reason it works is because all the angles add up to 360°.
>
> I said: Ah! You even said why. She said Yes, I knew you were going to ask why, that is why I told you. (Dr. Zachary, personal journal, p. 2-14, 19/10/03)

The variety of backgrounds of the students (see Chapter 5), together with the highly open-ended nature of the approach means a high level of heterogeneity of experiences for the student-participants. Despite this, certain commonalities exist. For example, all the groups are able to select and refine their own starting points, without needing suggestions from the teaching team. With the possible exception of Kerrie, whom I mention above, all the student-participants engage at a level that satisfies the teacher-participant.

Given this highly heterogeneous result, one aspect of this intervention is difficult to ascertain: in the theoretical framework, I claim exclusive conditions of authenticity for the practice and its experience, formulated as the five design criteria. The evidence shows that each aspect of each criterion is present in some of the student-participants' experience, but this does not mean that each student-participant experiences each

aspect. The open-ended nature of the data collection instrument, however, means that the possibility does exist of a complete experience for all, though this may not be the case. In addition, a single experience, even successful, does not guarantee a similar response in a future, analogous context, particularly if one considers, as Ajzen (2001) does, the possible co-existence of conflicting attitudes:

> Although people can form many different beliefs about an object, it is assumed that only beliefs that are readily accessible in memory influence attitude at any given moment. A belief's chronic accessibility tends to increase as a function of the frequency with which the expectancy is activated and the recency of its activation, as well as the belief's importance (Higgins 1996, Olson et al 1996). (p. 30)

If this is the case, the replicability of the engagement, even by a participant with a successful participation, is uncertain, though possible.

Question 3: The Affective Outcome of the Intervention

The last question this study is attempting to answer involves the measurement of an affective change during the intervention. This potential change can be measured within the 'attitudes' category and the 'beliefs' category, and is exposed through the comparison of pre- and post-module data. In each of these categories, the results of the Likert and descriptive items are discussed below. In addition, some of the results of Chapter 7 are discussed in general, without particular emphasis on whether they pertain to attitudes or beliefs.

Attitudes

In the case of the attitudes, the results of the Likert items show the insignificant change that is predicted by the theoretical framework. Indeed, the literature on affect in education suggests that attitudes are too stable for a three months intervention to have much impact. Though this is counter to the intention of the intervention, it has a positive aspect: if the results of the 'belief' component are stronger, as is also suggested by the theoretical framework, the insignificance of the attitudinal change can be used as a justification of validity of the overall results, since the difference between the two sets of results is predicted by the framework.

In the descriptive items numbered 11-14, only the two using adjectives or other modifiers pertain to attitudes. To recall, the results of these two items are combined into a net score before and after the intervention, which denotes the number of times a term is chosen as best descriptor, minus the times it is selected as worst descriptor. This score

for before the intervention is then subtracted from that after the intervention, to show the change. Again, the attitudes scores are closer to the null-hypothesis value of 0 than the belief scores (see Table 16), further supporting the validity of the results. In effect, this part of the quantitative results mainly serves a validating purpose by its accordance with the theoretical framework.

Views of Mathematics as a Subject

Changes in the views of mathematics that are held by the participants are more significant that the changes in attitudes, as predicted. In particular, the Likert items that relate to the views of mathematics show changes that are both statistically and practically significant. In addition, within each subscale, the change is in the anticipated direction: Instrumentalist views, which are not strongly held by the participants even before the intervention, decrease in a significant way: the *p*-value is .000, that is, very highly significant, and Cohen's *d* (which denotes the practical significance, that is, whether the intervention is useful regardless of the N) has a value of -.815, which is rated as a large effect size. The negative connotes the fact that the intervention decreased Instrumentalism. In the case of mathematical Platonism, the situation is analogous: the *p*-value is .000 again and Cohen's *d* is -.767, approaching a large effect size.

The Problem-solving and Pattern Analysis subscales present similar situations, though in the reverse direction: their *p*-values are both .000 and their Cohen effect sizes are .919 and .831 respectively. In essence, though the participants' views already begin largely on the 'right' side of the continuum for each subscale (see charts in Chapter 7), the change increases this orientation in a way that can be considered significant enough to warrant the implementation of such a practice, at least for student teachers who are already on the 'right' side.

In Item 3 of the post-module questionnaire, the participants are asked to express their awareness of the change, or lack of change, in their view to mathematics. Besides the fact that several discuss their attitudes as well as their views, the participants show a high count of affirmative answers to the item. In Table 32, the total number of answers that suggests a change in views is 27 out of the 37 students who participated in the post-module survey, just under ¾, confirming the findings of the Likert items.

The descriptive items also show significant changes: in the case of Items 11 and 13, the choices made, before and after the intervention, amount to scores that vary by as much as 32 (for an exploration) and -20 (for numbers and operations). In most cases, the scores reflect the direction of the change that is also shown by the Likert items: the fact that 'an exploration', 'patterns and relations', and 'an art' increase and that 'measurements' and 'numbers and operations' decrease is consistent with the other findings as well as the hypothesis.

In the case of Items 12 and 14, and despite the apparent confusion with 'abstract' and 'concrete', all four items denoting views are within the significant range (in yellow in table 16). And certainly the two options that do not lead to confusion change in the appropriate direction: 'gives meaning' increases by 8 and 'rigid' decreases by 5.

These results, which I describe both in relation to the participants' attitudes and views, also connect with those described in Chapter 6, under the heading 'Learning Derived from the Experience'. In this section, I report on participants describing, in their journals, having noticed a change in their world view, particularly pertaining to mathematics. Alexa, for example, explains:

> After taking this class (and remembering back to Calculus, I can also see it), I realized that mathematics is more about the process used to get the correct answer. (after 03/11/03)

In another interesting example, Jill* notes a new behaviour with what she perceives as a non-routine problem:

> [In the last math class I took], if I would have come to a problem that I didn't understand, I probably would have given up, […]. On this quiz, at first I didn't feel like I could confidently answer one question, but as I thought about the problems, and what they were asking for, I was able to work through them, and feel fairly confident about my answers. (Jill*, 10/12/02)

These examples show possibilities of the impact of the intervention, both in the views and in the behaviours they associate with the practice of mathematics, and are positive for the practice of the teaching approach.

An interesting effect of this analysis also concerns the examination of the theoretical framework, which is set up in such a way that it adds to that developed from the existing literature on views of mathematics. In the literature review, I discuss the addition of a fourth view, Pattern Analysis, which is distinct from Problem-solving. It could be

hypothesised that the former is simply a subscale of the latter, or perhaps it corresponds to the opposite direction from one of the other two views, Instrumentalism or Platonism. However the analysis and correlation of the subscales, which I make in Chapter 7, shows that this is unlikely. Firstly, the subscales for Pattern Analysis and for Problem-solving have *no* common items. The two subscales and their associated dimensions are not equivalent. In the case of Instrumentalism and Platonism, the situation is less clear: the overlap with Pattern Analysis is of 7 out of 11 and 2 out of 5 items, respectively. In the case of Instrumentalism, in particular, only 4 items from Instrumentalism are not in the Pattern Analysis subscale, but an additional 2 items are in the Pattern Analysis subscale without being in the Instrumentalism one. I represent the situation as a whole in Figure 19, below. In effect, the data gathered from this group of participants suggests the existence of a view of mathematics that is distinct from those already described by the existing literature on views of mathematics.
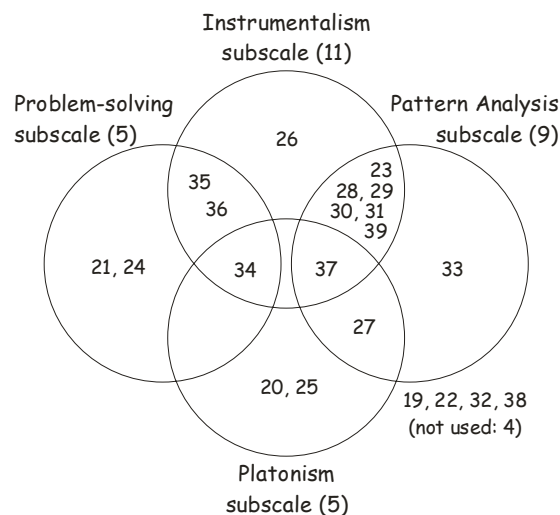

Figure 19   Venn diagram summarising the relationships between the subscales

Finally, a hypothesis is implied in the discussion of the design of the module as a whole, concerning the impact which the degree of engagement in the journaling exercise might have on the effect of the overall experience. In the scatter plot below, I chart the number of entries in the journals (not considering their individual word-count) against the total absolute change in the Likert items (the scores are added together regardless of their orientation, maximising the 'count'). The plot shows no sign of a tendency, suggesting that the hypothesis could be false. There are, however, two possible arguments against this. Participants can have experienced a strong change in affective outcomes resulting from intense reflection, without having recorded these reflections in the provided

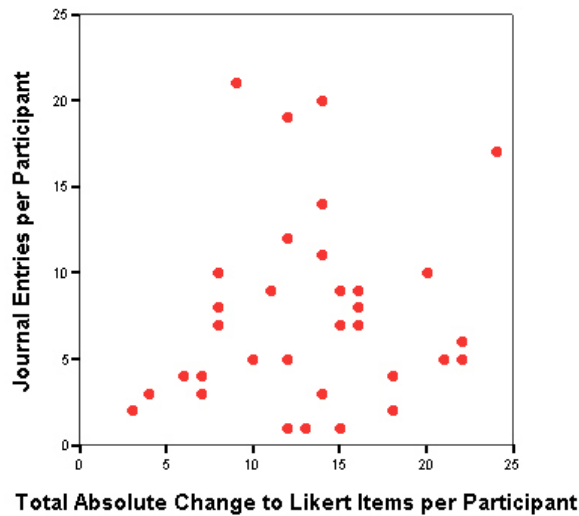journals. If this is the case, the dots are lower, within the plot, than they could have been.

Figure 20   Comparison of engagement in journaling versus absolute change in responses to Likert items

Conversely, participants can have reflected extensively on their experience, without changing their affective responses significantly, potentially because the experience already largely corresponded to their views. In this case, some of the dots are more to the left than they could have been.

Considerations for Future Implementation

Several items in the post-module questionnaire focus on the logistics of the teaching approach implementation. As such, they give suggestions for the refinement of the practice that is an inherent part of action research. In particular, the responses to Item 43 contain suggestions that revealed a need to adjust for Mason's tension (1989):

> Maybe inform us a little more about exactly what we were going to be doing over the semester. (Jean)

> I would have the professor better outline his expectations. (Pippa)

> A more solid outline of what is to be expected in the course in the beginning of the semester. (Melanie)

… suggestions that connect to the expectations of a mathematics module:

> Do book work and projects together throughout the class. (Rosie)

> I would change the amount of time spent on the project because I felt it took away some time from other important concepts in the book and it dragged on a little at the end. (Alexa)

… and suggestions revealing a belief that the module is about mathematics teaching methods:

> I'm not sure if there is a way to fit more ideas from the book just to get an idea of more ways to teach different things in elementary school. (Linette)

These last two categories of suggestions can be seen as an indication of the need for a more explicit disclosure, early on, of the pedagogical purpose of the module. They also indicate a very pragmatic, shallow view of education, which Dewey himself has discussed:

> Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular things he is studying at the time. Collateral learning… may be and often is much more important than the actual lesson. (Dewey, as cited in Mason, 1992, p. 18)

This kind of 'collateral learning', which seems to remain invisible to many of the participants, is precisely what this approach is trying to encourage, and these responses suggest that this needs to be clarified further.

Other participants discuss the time spent on the enquiry process, both in the short items of the questionnaire and in their journals. These responses mainly suggest that the participants see a benefit in curtailing the enquiry component, which contradicts the purpose of the practice and can therefore only be taken into limited consideration. The integration of this teaching practice is therefore problematic and requires a rigorous justification towards which this study is aiming.

Trustworthiness and Reliability of the Results

The results discussed above emerged through a variety of methodological processes. As such, their trustworthiness and reliability are verified through correspondingly different means. In several cases, the trustworthiness or reliability is provided by a triangulation with the results of other components of the study, and in other cases, by the expectations based on the literature review.

The proposed answers to the first question, which emerge largely from the literature, are reinforced by some of the results of the other two components. The subscale analysis of chapter 7 gives increased reliability to the development of a fourth category of views of mathematics, since the analysis showed a distinction between that and each of the other three, existing categories. In addition, many of the responses of the participants, in the

journals, demonstrate an awareness of the criteria of design, even though these are not discussed explicitly. This reinforces the reliability of the design criteria developed in answer to Question 1.

The credibility of the results pertaining to the second question is connected to the findings in the third component: the results of the quantitative analysis accord themselves with those in the journals. An unexpected example of this is found in the responses of the students who discussed the abstract/concrete pair. The semantic confusion revealed by the data appears in both the journals and the questionnaire responses, suggesting that the two data sets emerge from the same reflections. More broadly, the responses to the course evaluating items (6-9) and the journals and of the short answers and the journals show consistency.

The reliability of the results pertaining to the third question is connected to test-retest issues. For example, the idea that the participants can have undergone a maturation process is a relevant one. However, the fact that the data was collected at the beginning and the end of the intervention and in the same physical location suggests that the experience itself evokes the responses. In addition, the modular nature of the degree programme that the participants are following, in addition to the heterogeneity of the group (see Chapter 5), suggests that the individual participants have otherwise very different experiences, which would provide them with different directions and degrees of maturation. The mortality of the group is not an issue either as only two out of 37 participants did not participate in both surveys, and their responses are not considered in the contexts where the pre- and post-module data is compared. The regression-to-the-mean phenomenon can be excluded as well as a threat to validity as the group was self-selected, and contained regular and mature students, education 'majors' and others, and men and women (though the proportion is not balanced).

In the earlier section discussing the creation of an additional category in the views of mathematics framework, the results of the analysis of the questionnaire data are used to confirm the results of the literature analysis. Similarly, the question can be posed as to the creation of a category of mathematical practice (Mathematical Enquiry), distinct from that already practised in classrooms (Mathematical Problem-Solving in the Classroom). In the corresponding section of the literature review, I cite the necessity of each criterion for an authentic experience. The heterogeneity of the participants, and of their individual (reported and actual) experience throughout the intervention makes it

difficult to confirm the distinction which I emphasise. The evidence described in Chapter 6, however, shows that each aspect is experienced by at least some of the participants.

Another aspect of the teaching approach is expressed as essential to the experience: in Chapter 4, I make a strong point of the necessity of the 'Ramping up' phase, to ease the participants into the practice that they are expected to engage in. There is no explicit mention on the part of the participants about it. They do, however, discuss the process of selection of their enquiry topic and, in particular, Emily, talks about the mini-projects helping her and her colleagues to 'becoming independent learners' (15/09/03), suggesting that the ramping up served its purpose.

In summary, the first question is answered by the five criteria laid out for the design of the teaching approach. These criteria, together with a few additional components are used to answer the second question, and measure the authenticity of the participants' experience, with respect to the exemplar: professional mathematicians' enquiry practice. When I develop the theoretical framework, I hypothesise that each of the criteria are a condition of authenticity. The analysis shows that globally, each criterion is present, but it does not allow for the certainty that each participant experiences each criterion. The third question, which focuses on the affective outcomes of the experience, is answered as predicted by the theoretical framework: the participants' attitudinal responses do not change in a significant way, but their beliefs about mathematics do. Given that beliefs are one of the influences of attitudes, this is still a positive result, though further investigation is warranted. Finally, in addition to these result, the findings of the analysis of the Likert items justifies a part of the theoretical framework which I had added to the existing theory: the Pattern Analysis view of mathematics is shown to be distinct from the three categories that are already discussed in the literature. Finally, the feedback provided in both the journals and the questionnaires can be used to refine the teaching approach for future implementation, particularly with respect to classroom management issues and the selection of the mathematical situations used.

Chapter 9: Conclusion

> Traditional disciplines should be taught in such a way as to make
> their methods of enquiry visible. (Schön, 1990, p. 322)

A leitmotiv throughout this study is to provide the participant with a sense of agency higher than that provided in more traditional education contexts, thereby emulating more authentically the full-participation in the mathematical community of practice known as mathematical enquiry. The intention behind this move stems from a hypothesis according to which the experience that such a participation can provide would give the participant a richer understanding of the nature of mathematics, and by extension, a more positive attitude and motivation. In the previous chapter, I discuss the results of an intervention based on this hypothesis. In the following pages, in turn, I discuss the implications, at various levels, of these findings.

To begin, I address the possibility of future implementations of the teaching approach that formed the locus of the intervention, both in terms of an adjustment of the practice, and of its applicability to the teacher training context, within which the current implementation already takes place. In addition, I discuss the implications of the findings for my personal action research process, of which this is the first cycle. This section extends into a discussion of the theoretical framework upon which the research is based, and the implications of the findings for the said framework. Finally, I expose possible further research both within the context of continued application of the teaching approach, and beyond it, in connection with other aspects of the study.

Future Implementations of the Teaching Approach

In Chapter 7, I describe the participants' change in beliefs about the nature of mathematics as highly significant both statistically and practically. This being the case, the implementation of this teaching approach to other groups of participants seems a promising proposition. Should the possibility arise for such an undertaking, however, several points need to be taken into consideration that could, based on the responses to this implementation, improve the practice.

The first of these points concerns the strategies that the teacher-participant(s) practice in this teaching approach, and the ones that they need to avoid. In particular the notion of 'scaffolding for rigour' rather than 'scaffolding for meaning' needs to be articulated more thoroughly. A difficulty that presents itself for this aspect is the fact that most educational contexts require the latter and little attention is paid to the former. This means that most experiences of mathematical education situations illustrate the latter, eschewing the former, even (or perhaps particularly) for trained educators. Even for the teacher-participants of this intervention,

> … it is difficult to find a balance between correcting them and thereby removing the project from their ownership for the sake of "correctness", and leaving them to it. I notice this now in reading Rachel's report. [...] Actually, to leave errors in the process description of the project write-up is necessary, or it doesn't describe the process!! (Eva, research notebook, 01/02/04)

It is also difficult, when examining the work submitted by the student-participants, not to 'read results into their work', that is, mentally inserting the missing steps, or possibly even the missing results into their submissions. The danger of this practice is that the teacher-participants then assumes that the student-participants have reached a richer understanding of the mathematical objects they are investigating than they actually have.

So far, the teaching approach described in this study has been implemented by the same participants who helped develop it. If such a strong shift is to occur, the teaching-participant need not only espouse the intentions of the practice, but perhaps requires training as well. If the approach is to be implemented by other educators, therefore, a preceding step needs to be integrated into the practice, whereby the *teacher*-participant is inducted into the practice, perhaps her/himself through authentic experience, following the situated cognition model.

The teacher-participant and I, who designed the approach together, have discussed this throughout the intervention and the time following it, and several suggestions have emerged from this:

- PhD student and postdoctoral fellows in mathematics, who have experienced this practice as peripheral participants, could be a good source for teacher-participants. In particular, individuals who show an interest in learning to supervise research, would be appropriate. Indeed, the teaching assistant-participant that worked in this implementation, who was concurrently finishing his competency exams and beginning his research proved to be well up to the task.

- The teaching approach could, perhaps, be integrated with REU activities (see Chapter 1), or with a 'Research Supervision Seminar', which would help coordinate and guide the teacher-participants, and could count towards their degrees.

- Interested, and experienced, mathematics research supervisors could act in an advisory capacity, since their purpose in the supervision they provide, is more geared towards 'scaffolding for rigour'.

Some of the results of the present implementation suggested adaptations of the overall teaching approach. For example, the fact that the journals (and the project write-ups) contain few statements about higher level knowing suggests that this focus of the theoretical foundations of the approach need to be made more explicit to the student-participants. Indeed, what for the mathematically more able individual consists of the tacit understanding I described as knowing-when, may not exist as far as the mathematically less able are concerned. If this is explicitly made the focus of the experience, the effect of the experience might be enhanced[35]. Overall, a more thorough disclosure of the theoretical foundations underlying the approach could well enhance the emancipation that the experience is designed to promote.

The teaching approach, as it is designed and implemented in the current study comprises a few other limitations, some of which can be eliminated through viable adjustments. The first one of these, which is difficult to remove, concerns the amount of time that is needed for a proper implementation. This aspect of the approach is so fundamental to the practice that it is constitutive of one of the five design criteria of the approach. As such, it changes from a characteristic of the practice to a condition for its authenticity, and therefore the question becomes not one of adjustment, but of the worthwhile nature of the practice as a whole. Though many of the student-participants' suggestions focused on this issue, therefore, this limitation cannot be 'designed out' of the approach without threatening the integrity of the practice.

Another limitation of the approach as it is described in this study concerns the possibility of a student-participant 'opting out' of the required engagement by choosing a topic that is not 'new to' her/him. This might seem safer for the individual, who then needs to 'fake it' in the written reports. Though this prospect might seem more demanding to many, this is always a possibility, and, indeed, in the teacher-participant's view, this was the case for one student-participant.

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[35] In the second implementation of this approach, which takes place in the summer of 2007, with in-service teachers, this modification is implemented.

Finally, the implementation of an assessment strategy that does not focus on cognitive achievement presents a difficulty relating to the tension that Mason (1992) describes and which is much discussed by the student-participants both in their journals and their questionnaire responses. This difficulty reveals a disjoint between available, and expected, assessment models and the intentions of the intervention, and this weakness of the teaching approach is revealed by the implementation. This limitation of the design remains an open question, and can be added to the list of potential research directions emerging from this study which I describe in a later section.

Incorporation into Teacher Education

In the introduction, I describe the aim of the intervention, from the perspective of the student-participants, as one of completing their experience with mathematics, thereby giving them the opportunity to gain greater understanding of the discipline. An implication of this more complete experience, for the student teacher specifically, is hypothesised to be, in addition to a richer view of the nature of mathematics, a greater independence of mathematical thinking. In the late 1980's, when problem-solving instruction was heralded as the new, more efficient approach, Ernest (1989) already claimed that:

> A shift [in teaching] approach […] depends fundamentally on the teacher's system
> of beliefs, and in particular on the teacher's conception of the nature of mathematics
> and mental models of teaching and learning mathematics. (Ernest, 1989, p. 249)

At both the level of the researcher-participant and that of the student-participants, reflective action is assumed to be a conduit to transformed action. In order to bring about this transformation, however, it seems necessary not only to change standards of policy regarding practice, but also to transform the practitioners', that is, the teachers' beliefs regarding teaching practice. And for this transformation to occur, it seems reasonable to think that beliefs about the practice of the discipline being taught may need to be transformed. In general terms, I designed the teaching approach in order to effect such a transformation in the student-participants' views about and attitudes to mathematics. In this section, I explain some of the implications that, if successful, this change might signify in the wider social and (micro-)political context.

Contrasting the Social Context of the Classroom and of the Mathematics Research Community

One of the most important aspects of research, in any subject, is that the researcher is involved with questions that remain until then unanswered or unsatisfactorily answered

to anyone in his/her community. This is also true in mathematics research. In contrast, in the mathematics classroom, the questions with which the student is confronted, generally, are assumed to have a specific answer, which is itself assumed to be known by the teacher. This situation can create the tension that Mason (1989) describes, and which I discuss throughout the thesis: the relationship between the teacher and the pupil can be distorted into one in which the pupil focuses on satisfying the teacher's expectations rather than actually accomplishing the hoped-for learning, thus participating, effectively, in a social game, the prize of which is a positive assessment. In the research context, in contrast, there is no-one in the role of the teacher: There isn't a party with the 'answers' in his/her head, with whom the researcher can play this game.

According to Schoenfeld, "there is a social dimension to what is accepted as mathematical 'truth'" (1994, p. 60). Although this is difficult to deny, this social dimension is very different from that of the classroom as it is described by Mason (1989). In the research situation, the community of researchers, through conferences, peer-reviewed publications, etc., engage in a species of Habermasian discourse (Carr & Kemmis, 1986, beginning p. 141), in which truth by consensus is achieved, and where all members have (notionally) an equal voice. Indeed, pure mathematics research may be the closest to what Habermas refers to as an 'ideal speech situation', a "democratic form of [communal] discussion which allows for an uncoerced flow of ideas and arguments and for participants to be free from any threat of domination, manipulation or control" (Carr & Kemmis, p. 142). In this context, the rules for the consensus to be achieved hinges on the selfsame rigour for which scaffolds are built in the authentic practice. In contrast, in the classroom, there is a very strong power structure, reinforcing the authority of the teacher through the necessities of assessment, and scaffolding for meaning is a manifestation of this power structure.

Reconciling the Two Social Contexts

As the reality of education today is strongly oriented towards assessment, the implication is that making social approval (in the form of teachers' assessment) the measure of a pupil's success can potentially stifle their curiosity, their creativity, and the self-reliance of their thinking: It can make them rely on the signals of a never-ending line of 'higher authorities' and can prevent them from developing a healthy, independent self-confidence.

This focus on social, external approval is so deeply embedded in the education consciousness that Arter, (1995), when she emphasises that instructions for a portfolio assessment should be as detailed as possible, asks: "How can students become skilled self-assessors if they don't know the target at which they are aiming?" (§5)

Her inquiry implies that self-assessment is, or ought to be based on, or at least dependent on external assessment. This leads right back to Mason's (1989) tension, in the sense that the target is still given by the 'higher authority', and so, the game is still on. Klein, on the other hand, experimented with less rigid instructions, with these results:

> There might not have been a right or wrong way of doing something, but as marks/ grades show, there are better ways and worse ways of doing things. The university classroom is not the place to take risks where marks are concerned. [...] We know that there is something in the teacher's head, and that it determines how right or wrong our answer is, depending on how far our answer/folio correlates with, or deviates from the teacher's expectations. [...] Constructivism, was for me, disempowering. [student quote/data] (Klein, 2001, p. 263)

Again, the tension described by Mason comes through: Though she is aware of a changed atmosphere designed to encourage risk-taking, she still perceives the context as precluding an independent knowledge development process. In fact, the situation is worse in that the student's dilemma is between wanting to find the way to please the teacher, and being expected to think with self-reliance.

Jaworski (1998) suggests a different way out of this dilemma:

> From asking questions and the resulting investigation, students gain ownership of the mathematics they generate, which provides an experiential grounding for synthesis of particular mathematical ideas. (p.120)

The issue of ownership of knowledge, which Jaworski emphasises, and which is advocated in 'reform' education, forms an attempt at correcting this situation within the existing educational framework. The idea that the students can feel ownership of the mathematical ideas, and therefore that their affective responses to working in mathematics will be improved is, in this instance, the hoped-for outcome.

The school context, even in 'reform' teaching, does not allow the transfer of ownership to be carried out to the point where the learner has power over and responsibility for his own learning; the bottom line of assessment is still focused on whether the learner can perform the tasks related to the knowledge, rather than whether the learner has reached an understanding of the subject. This state not only relates to the disjoint I mention

above, between the intention of this intervention and the available assessment models, but it connect back to Dewey's (1916) distinction between educational and miseducational activities:

> Those activities were educational which led to further 'growth'. A miseducational activity or experience was something which blocked growth. However important, objectively speaking, a lesson or curriculum might be judged, if it turned pupils off or closed minds to further thought or dulled the sensibility. then it was not educational. (Pring, 2000, p. 12)

In other words, activities can be defined as educational if they lead to further growth, that is, if they produce something that can foster and sustain an attitude conducive to the self-directed development of more knowledge. In effect, the knowledge produced by educational activities ideally creates in the individual a knowing which possesses a life of its own, without the need for constant stimulation in the form of 'scaffolding for meaning'. This knowing would be of a dynamic rather than of a static nature, what Whitehead called 'inert' knowledge (Burton, 2004, p. 2): Rather than a metaphorical filing cabinet, however large and interconnected, it would resemble an ever-changing, growing organism with potential for self-healing and self-replicability. This self-replicability would be sustained by the component of learning addressing 'knowing-when', and this component therefore needs to be emphasised to a greater extent. Because of the realities of assessment and tradition in the classroom, however, the filing cabinet metaphor is probably closer to the realities of mathematics education, than that of the growing organism, showing that a pragmatic, Instrumentalist view is still manifested by the power structures and the implemented pedagogies.

In mathematics research, the researcher communicates directly, metaphorically speaking, with the mathematical entities, or objects, which he is studying. There is no third party that has such authority that it comes between the mathematical object and the observer, and imposes a lens through which the objects are to be viewed. Certainly, there is such a lens, that which is imposed by the enquirer's own context; but that lens is hers/his, and even in instances of collaborations, the researchers are in control of the process and it is their thoughts, ideas and processes, and their grasp of 'truth' which guide the development of the mathematics and their learning.

In the classroom, traditionally, the teacher takes on a role of intermediate between the learner and the mathematics. S/he therefore controls the interaction, the flow, the pace, and the focus. The contrast between these two situations is important from a social

perspective, because the relationship between the players has a great impact on the learning that takes place.

Giving Future Practitioners a Glimpse at the Reconciled Practice

The relationship between these two contexts is a central theme of this study because, in the classroom, the social context is altered by the existence of an inequitable power structure. Brousseau, (1997) explains that the students and teachers engage in a didactical contract, which determines the terms of their relationship with regards to the learning taking place in the classroom. This contract is so central to the goings on in the classroom that it impacts the interplay between the students' perception of their own knowledge (internalised reconstruction) and their perception of socially accepted (or imposed), so-called 'objective' truth: In the event of a conflict between the two, the socially constructed version would come up on top, because it is the reference for the assessment model. In a context where knowing-when is a central focus of the didactic contract and scaffolding for rigour the favoured form of communication, the self-replication metaphor becomes relevant to the learning process.

Though this scenario does reflect the realities of learning in most social contexts, it is not completely appropriate to the research context, where this reference is non-existent, and the rules of rigour (however tacit they may be), remain the only reference. It is the task of the researchers to construct the 'socially accepted' version of truth, based on their own subjective constructions, via a collective discourse with formal rules of rigour that are themselves socially imposed, though they are continuously debated and critically evaluated by the community of researchers. In the case of 'real', cutting-edge research, the method of evaluation of the knowledge is imposed socially, but the outcome is open. The collective discourse that researchers engage in is best described through Habermas' (Carr & Kemmis, from p. 134) idea of truth by consensus, which can be reached through a collective discourse:

> Any consensus arrived at within the framework of the appropriate discourse can, therefore, be regarded as a true consensus. (Carr & Kemmis, 1986, p. 141)

At first glance, it may seem that this model of a coming to truth-validity should be reserved for what is already socially accepted as a socially derived science, such as are the 'social sciences' (sic). Schoenfeld (1994), however, reminds us that (even) mathematics is socially derived. Given this epistemological perspective, even

mathematics can be subjected to Habermas' validation through collective discourse, where:

> In attempting to come to a 'rational' decision […], we must suppose that the outcome of our discussion will be the result simply of the force of the better argument and not of accidental or systematic constraints on discussion. Habermas' thesis is that the structure (of communication) is free from constraints only when for all participants there is a symmetrical distribution of chances to select and employ speech acts, when there is an effective equality of chances to assume dialogue roles. […] In other words, the conditions of the ideal speech situation must ensure discussion which is free from all constraints of domination. (McCarthy, 1975, as cited in Carr & Kemmis, 1986, p. 143)

This is far from the realities of the traditional mathematics classroom, with Brousseau's didactical contract, Mason's tension, and an environment where it is proposed that:

> the teacher should steer children away from nonproductive solutions, but not steer children towards productive solutions. The latter presupposes that the teacher is in possession of "the truth," rather than someone aware of the conventional nature of knowledge. […] In such a circumstance, there is a perception that there are "right answers" towards which to steer children (Ernest 1991a, as cited in Ernest, 1995, p. 464)

In contrast, in the teaching approach that is investigated in this study, I intend to give the students an opportunity to experience participation in a situation approximating a Habermasian discourse to a greater extent than a traditional classroom situation can allow, that is, to expose them to a consensual validation where they had:

> "the same chance to initiate and perpetuate discourse, to put forward, call into question, and give reasons for or against statements, explanations, interpretations, and justifications". (McCarthy, 1975, as cited in Carr & Kemmis, 1986, p. 143)

The purpose of the intervention is furthermore to increase the students' self-understanding, that is, the awareness of (a) their beliefs and belief systems regarding mathematics and themselves with regards to it, (b) possible causes of these beliefs and belief systems, and (c) possible consequences of these beliefs.

From a different perspective, the intention is to emancipate the student-participants from the teacher-authority, giving them a more direct experiential conduit to the research practice, by way of having power over and responsibility for the outcome of their work. This is especially pertinent for future teachers, who, as I discuss in Chapter 2, are seen as representing full partipation in mathematical practice.

The idea, therefore, is for the participants to develop an awareness of the epistemological foundations of mathematics, and consequently, to refine their teaching approach to include these insights. This desired course of events, however, can be met

with a strong obstacle, which Ernest (1991) terms the "difference between the teacher's espoused theories of teaching and learning, and the enacted versions of these theories." (p. 285). In effect, even were a teacher convinced of the appropriateness of a teaching theory, or a method, or, in this case a perspective on the nature of mathematics, the context in which the teaching is to take place, and the pressures of this context, may very well subvert her/his convictions.

From a political standpoint, however, there is another important issue. Indeed, what are the implications, should this intervention be successful, and assuming that the participants' future practice were impacted as anticipated? Beyond the ideals of freedom, democracy and justice, the issue here is one of power. In simple terms, the intention of this intervention is to emancipate the participating students, in the context of the mathematics classroom, from the teacher-authority. The students are expected to develop their own, independent thinking space, with validation mechanisms that do not rely on an entity external to themselves, but rather on a community of which they are, at least nominally, a member. This is particularly crucial in this case, where the participants are student teachers, that is, individuals who, in the foreseeable future, will be responsible for the knowledge and understanding of others. It is connected, therefore, with the position of the teacher in our social order. A further consideration, for which the scope of this thesis is insufficient, therefore lies in the repercussions of such a development, were it possible to effect it. If this intervention effects emancipation of thinking, without knowledge, it can be a destructive force. And can emancipation of thinking, without power of action be considered real?

Action Research: Practice of this Teaching Approach

I have suggested, throughout this document, that this study forms the first part of what I expect to be an extended cycle of action research, as defined by Carr and Kemmis (1986):

> a form of self-reflective enquiry undertaken by participants (teachers, students or principals for example) in social (including educational) situations in order to improve the rationality and justice of (a) their own social or educational practices (b) their understanding of these practices and (c) the situations and institutions in which these practices are carried out. (Carr and Kemmis, as cited in Wellington, 2000, p. 24)

I therefore expect the experience to influence my further research as well as my teaching practice. According to Mc Niff (1988), action research cycles through the following stages:

- The statement of problems
- The imagination of a solution
- The implementation of a solution
- The evaluation of the solution
- The modification of practice in the light of the evaluation (p. 38)

More specifically, I see myself as being at the stage of evaluating a proposed 'solution', using the results of the intervention and my reflection on them, with the intention of informing my own future practice both in research and in teaching, as well as (hopefully) the domain of mathematics education as a whole. In the case of my own cycle of action research, some of the implications of my findings concern the design of the research component of the study as well as the aspects I describe in other sections of this chapter.

For example, I find that some of the data collection methods that I use to answer the research question prove to be a poor match for the hypothesis they are designed to evaluate. The journals do not provide a conclusive measure of the authenticity of the participants' experience with respect to the design criteria. I probably do not write as many responses, in the journals, as could help with this issue. It might also be useful to emphasise that the journal entries should be more about the process of learning, and the associated experience than about specific content.

Theoretical Framework

Several implications of this study, which are not directly connected to the research questions, concern the theoretical framework that underlies it. For example, enriching the theoretical discourse surrounding mathematics education by the addition of mathematical enquiry as an activity separate from problem solving may produce an awareness, in the mathematics education community, of the amalgamation that has been made until now, of these two types of activities. The consequence of this change in perspective can lead to changes in practice, policy and theoretical work. The practice component of this shift is already under way in disparate locations, as I discuss in the

introduction, but much of the theoretical underpinnings of this practice need still be investigated[36], and policy changes are still pending

In Chapter 2, I justify my selection of 'Situated Cognition' as the educational perspective that underlies the theoretical framework of this study. This choice suggests an implication of the findings of this study: if, as proponents of Situated Cognition suggest:

> […] we learn the working practices of the setting in which we operate. […] working practices do not transfer from one culture to another. (Hughes et al., 2000, p. 16-17)

… couldn't we then develop 'transfer between contexts' as a practice? As Whitehead explained:

> The certainty of mathematics depends on its complete abstract generality. (cited in Hardy, 1967, p. 106)

Perhaps, then, mathematical thinking is the root of transferability of thinking, and as such, could be leveraged into a tool, not only of quantification and "the relational aspects of space" (Mathematics Sections of the Association of Teachers in Colleges and Departments of Education, 1967, p. 8), but also of the analogy of patterns (Hardy, 1967, Schoenfeld, 1994) in situations of all ilk.

Further Research with the Teaching Approach

During the course of both the intervention and the subsequent analysis, several new research questions emerged from the reflection engendered by the study:

1. A significant consequence of the methodological design is that it prevented a satisfactory answer to the second question, regarding verification of the authenticity of the experience for each student-participant. In addition to the fact that several student-participants did not sufficiently engage in the journaling, thereby not providing enough evidence for the ascertaining of said authenticity, the open-ended nature of the journaling activity itself prevented me from concluding anything about an absence of experience. Indeed, if a student-participant fails to mention an experience in her/his journal, this cannot be interpreted as her/him not having the experience. I can therefore not ascertain the full authenticity for each individual. Ironically, this flaw is a consequence of the agency that the overall design of the study is used to promote, at the meta-level of the overall experience with the intervention. The theoretical framework

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[36] Some of this work is being carried out in France, with the investigation of the mathematical thinking involved in mathematical enquiry (MATh.en.JEANS, n.d.).

suggests that each criterion is necessary to the authentic experience, but is it necessary to the research, that each student-participant report experiencing each aspect?

2. The writings of Burton (2004) are an important component of the theoretical framework regarding the authentic experience of mathematical enquiry as practised by full participants in the mathematics community. In addition to the criteria used in the present study, she suggests an additional component to the experience: In her report, she explains that communication and cooperation or collaboration are an essential part of the experience of full participation in the mathematics research community. An analysis of the journal responses in those terms could therefore be an additional investigation of the authenticity of the experience. This is in fact a topic that does come up in the journals, as exemplified by Joan's remark:

> It is nice that we can work in groups - it kind of takes away the frustration. (Joan, 03/10/03)

3. In the intervention as implemented within the context of this study, the third phase, of 'Regular Teaching' takes up a substantial proportion of the time. The question is not investigated, as to whether this is significant to the student-participants' experience, in retrospect perhaps, of the main mathematical enquiry practice.

4. The mathematical content of the submitted projects themselves can be investigated in terms of the theoretical framework describing mathematics. Examples of such considerations include the levels of knowing applied, or developed in the enquiry (see Practical Discussion of the Epistemology of Mathematics), or the idea that there may be a natural sense of 'goal-state' in mathematical enquiry situations (see Grenier & Payan's Framework).

5. It might be interesting to examine the changes in views, which are shown to be so strong, by filtering the sample according to various relevant criteria, for example by the initial view. More specifically, the students that tend towards an Instrumentalist-Platonist view might show a different change from those who are already on the side of Problem-solving and Pattern Analysis, even within the given group.

6. The closer examination of a few selected participants using a case study approach could give some insight into their specific experience.

7. A longitudinal study could investigate the possible consequence to both the participants' motivation to integrate the experience into their practice and the implications of this intention.

I envisage engaging in several of these possible new directions, even within the action research centred around the teaching practice. In addition, I describe, in the next section, other directions that are related but need not be applied to the teaching practice directly.

Further Research beyond the Teaching Approach

Several theoretical interrogations emerged from this study, which relate to the theoretical framework that is applied but do not depend on the implementation of the teaching approach:

1. The data generated to answer both Questions 2 and 3 revealed a variety of interpretations of the terms 'abstract' and 'concrete' in relation to mathematics. In particular, in the descriptive items of the questionnaires, the two terms are not always used in a mutually exclusive way. In the journals, the terms are often used in opposition, but they are defined in different ways by different participants. As descriptors of mathematics, they are useful to investigate individual views of the subject, and I can see this as the starting point of an interesting investigation, perhaps using personal construct theory (Kelly, 1955).

2. In connection with Burton (2004), a re-examination of the basis for the evaluation of the authenticity of the experience could include a question that perhaps the distinction between Mathematical Problem-Solving in the Classroom and Mathematical Enquiry is more a matter of degree than of exclusive conditions.

3. I propose earlier, in the discussion of future implementations of the teaching approach, that "what for the mathematically more able individual consists of the tacit understanding I described as knowing-when, may not exist as far as the mathematically less able are concerned". This statement is worth investigating further.

4. The discussion of critico-creative engagement elicited the following interrogation:

   > How much 'previous knowledge' do you need to be able to be creative [in mathematics]? (Eva, research notebook, 29/09/04)

5. It would be interesting to investigate assessment models that are both practical and relevant to the theoretical underpinnings of the teaching approach. Indeed, individualised interviewing can be helpful in determining the understanding reached by individuals, but it is neither practical, nor resistant to cheating, once the answers are known.

6. A more thorough investigation of the distinction between scaffolding for meaning or for rigour could be a fertile undertaking, especially within a larger project of characterisation of the concept.

The action research cycle that I am now engaged in promises not only to help refine my teaching, and research, practice, but to lead to many complex new investigations, which will hopefully contribute to knowledge about knowing, doing and teaching mathematics.

Appendices

Appendix 1   Exemplar 1

From a Subdivided Tetrahedron to the Dodecahedron: Exploring Regular Colorings

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Abstract

The following paper recounts the stages of a stroll through symmetry relationships between the regular tetrahedron whose faces were subdivided into symmetrical kites and the regular dodecahedron. I will use transformations such as stretching edges and faces and splitting vertices. The *simplest non-adjacent regular coloring*[1], which illustrates inherent symmetry properties of regular solids, will help to keep track of the transformations and reveal underlying relationships between the polyhedra. In the conclusion, we will make observations about the handedness of the various stages, and discuss the possibility of applying the process to other regular polyhedra.

Introduction

The symmetry relationships between the Platonic solids are well known, as are their simplest non-adjacent regular colorings. These properties can in fact be used as basis for many activities, including their construction and observations about their properties. The present example began with a particular coloring of the tetrahedron illustrating some of its more sutle properties of regularity, and ended up somewhere completely unexpected. The entire exploration is presented here in a narrative because the order of the events adds a significant clue as to its richness. Derived from activities in polyhedral Origami and mathematics education, the exploration gives a good example of discovery through exploration.

The experiment

2.1 The Premise. The entire experience began with the following question:

What is the simplest non-adjacent regular coloring of a tetrahedron whose faces have been subdivided into sets of three kites (see figure 1)?
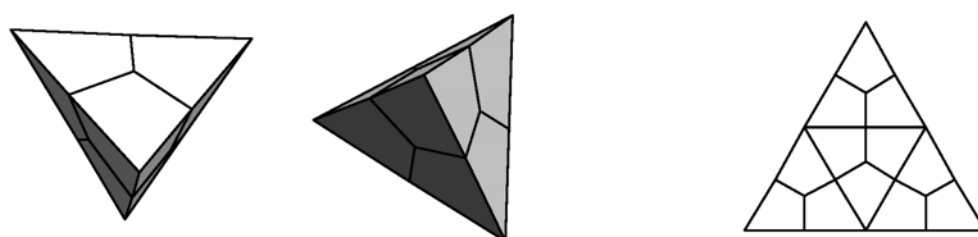

Figure 1: Two views of the tiled tetrahedron and one of its possible nets

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[1] In this paper, the simplest non-adjacent regular coloring is defined as the coloring using the least number of distinct colors where no two same-colored regions are adjacent and each set of regions of the same color is isometric to the others on the surface of the polyhedron.

2.2 The Simplest Non-adjacent Regular Coloring. This tiling of the tetrahedron conserves all the symmetries of the tetrahedron that supports it. Additionally, it contains twelve 'faces', twenty-four 'edges', six occurences of 4-vertices (at the mid-point of each edge of the supporting tetrahedron) and eight occurences of 3-vertices. The latter are subdivided into two groups: four at the vertices of the supporting tetrahedron and four at the centers of its faces. The simplest coloring, therefore, must contain at least three colors (because of the 3-vertices), but needs no more than four, even if we require that all the 'faces' remain equivalent (see figure 2). This is a very important condition, leading us to the next step.
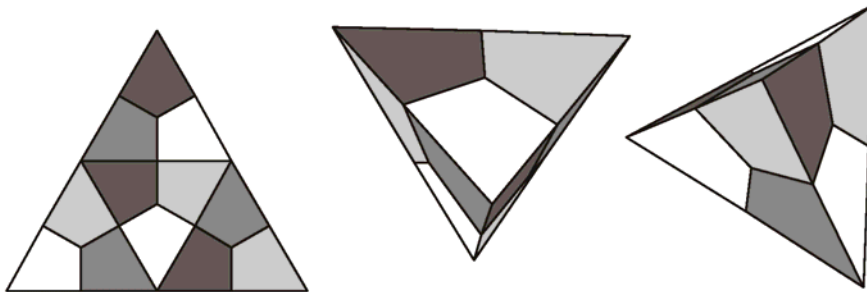

Figure 2: The simplest non-adjacent regular coloring of the tiled tetrahedron

2.3 Coloring the dodecahedron. If there are four colors and twelve faces, we can say that:

a) Each color is used three times.

b) The number of colors and their frequency are the same as in the regularly colored dodecahedron.

This is interesting, and worthy of further exploration: Consider the simplest non-adjacent coloring of our shape (figure 2) and the simplest non-adjacent coloring of the regular dodecahedron, which also contains four colors each used three times. If there is a symmetry link between the two colored polyhedra, it should be possible to create an animated sequence moving one to the other without changing the colors.

Let us further compare the tiled tetrahedron and the dodecahedron, counting each kite as a 'face':

Definition	'Faces'	'Edges'	Vertices	3-vertices	4-vertices
Tiled tetrahedron	12	24	14	8	6
Dodecahedron	12	30	20	20	0

2.4 Adjusting the Properties. Looking at the number of 'faces', things appear fine, but we seem to have a discrepancy in the number of 'edges', and once we reach the vertices, all seems definitely lost. Not only do the numbers not match, we don't even seem to have the right *type* of vertex! If we could somehow transform the six 4-vertices into twelve 3-vertices, and if the number of edges were then also adjusted, we would have something to work with.

One way to transform 4-vertices into 3-vertices is to stretch the vertex by introducing an edge between two pairs of incoming edges, as in figure 3.
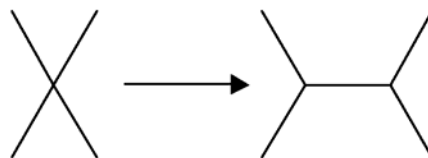

Figure 3: Transforming a 4-vertex into two 3-vertices

In figure 4, we can see the triangular net of figure 1 modified accordingly. The new edges are emphasized in the diagram on the right. Since the figure is meant to closed, the new edges on the perimeter of the net are in part the same, and therefore must be counted only once.
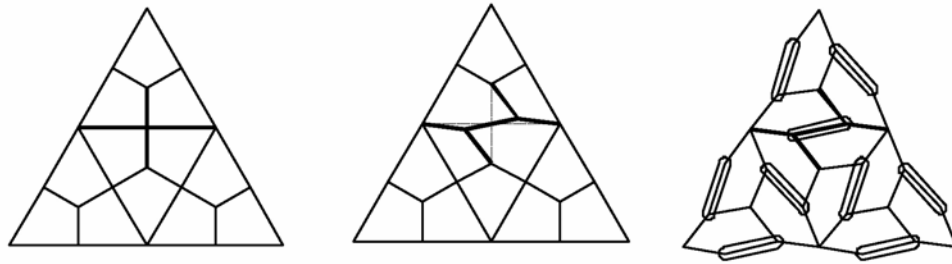

Figure 4: *Transforming the net*

We can now add a new line to the table as follows:

Definition	'Faces'	'Edges'	Vertices	3-vertices	4-vertices
Tiled tetrahedron	12	24	14	8	6
Stretched tiled tetrahedron	12	30	20	20	0
Dodecahedron	12	30	20	20	0

Topologically speaking, we now have the correct combination of 'faces', edges and vertices. There are of course many possible positions for these new vertices. Let us have a look at some significant ones:

a) The new points can be placed somewhere along the edge of the supporting tetrahedron. These solutions are interesting because they conserve the existing coplanarities among the 'faces', preserving the supporting tetrahedron.

b) The new points can be placed in such a way that all the edges in the net are of the same length.

As it turns out, there are two solutions that conform to both of these conditions simultaneously. The two solutions are mirror images of each other, but if they are combined with the coloring, which already possesses handedness, they become distinct. Figure 5 shows both the nets and the closed shape (including the coloring) for the two solutions. The net in the middle corresponds to the original subdivision, and the ones on the right and left sides show the new nets with the 4-vertices transformed into 3-vertices, all 'edges' being of equal length. Note that the 'faces' of each color retain their relative positions.
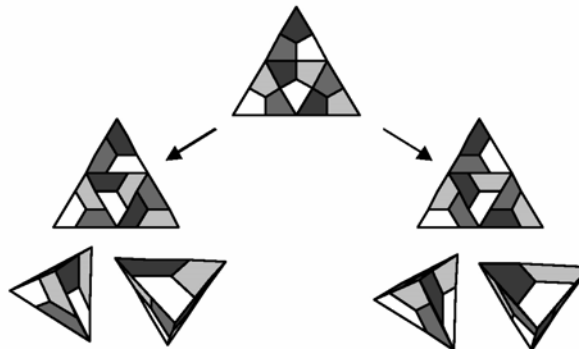

Figure 5: *Transforming the tiled tetrahedron*

2.5 The Transformation. Once the tetrahedron is tiled with the new faces, it is time to look at the transformation leading to the dodecahedron. Since all the edges are already of equal length, only the angles remain to be adjusted. Figure 6 shows the entire sequence from the kite-tiled tetrahedron to the stretched tiled tetrahedron, to the regular dodecahedron. Throughout the whole sequence, the four vertices of the supporting tetrahedron do not move, and in the second part, the vertices on the faces and edges pull out to become equivalent vertices on the dodecahedron.
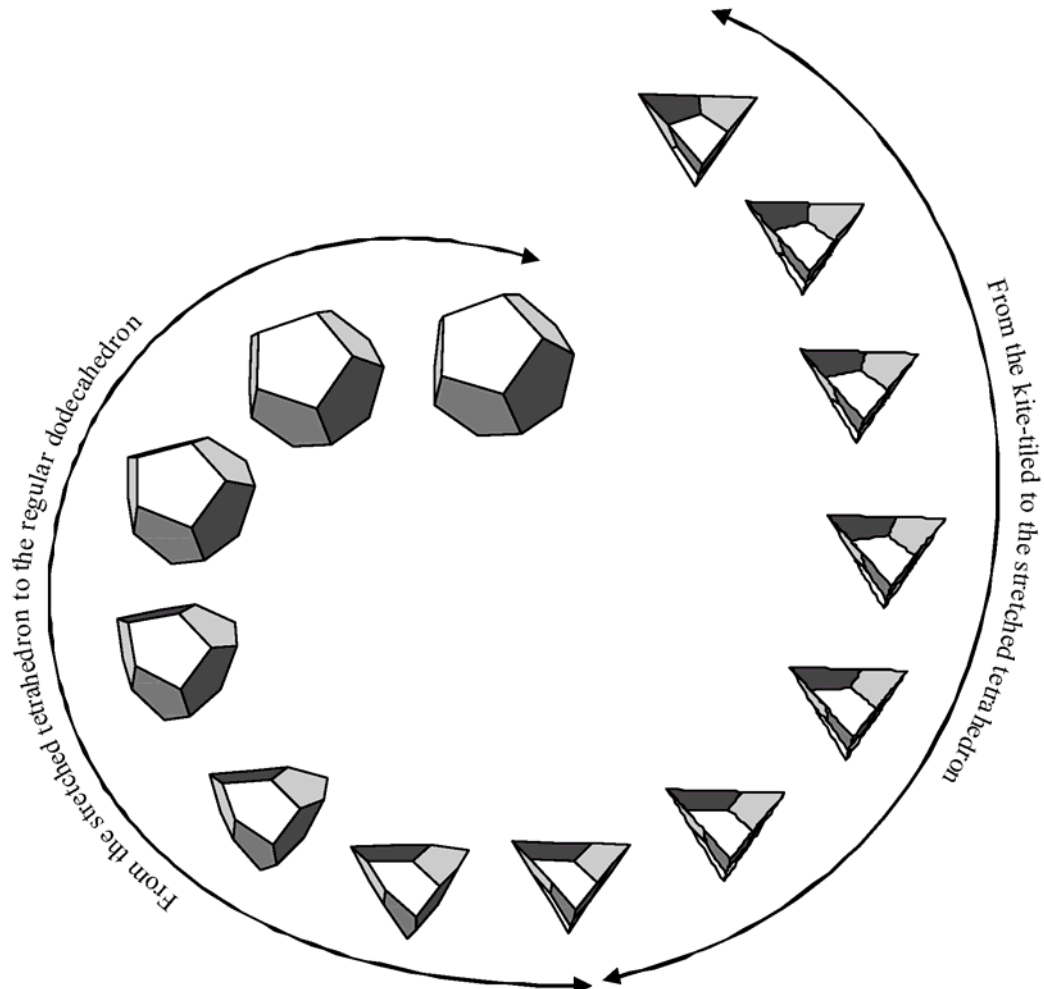

Figure 6: transformation from the kite-tiled tetrahedron to the stretched tiled tetrahedron, to the dodecahedron

Conclusion

3.1 Handedness. The series of procedures described above began as a stroll without prescribed destination. It lead us through interesting developments, each of which showed something about the Platonic solids and some of their derivative polyhedra. In several of these steps, the transformation introduced new factors, new symmetries, and new variations. In particular, at several stages the

transformations introduced handedness, meaning that there were really two separate solutions that were each other's mirror image. In most cases, this double solution can be counted as only one, but if handedness is introduced at more than one stage of the transformation, this may not be the case. In the present exercise, the handedness was introduced by the coloring, first, and then by the vertex stretching of figures 3 and 4. The handedness introduced in the first case gives two solutions that are in fact each other's mirror images. In the second assymmetric transformation, the stretching of the 4-vertex into two 3-vertices, the handedness that is introduced can be applied in the same direction or in opposition to the first one. Because of this, there are then four solutions, which we will call the left-handed/left-handed, left-handed/right-handed, right-handed/left-handed and right-handed/right-handed solutions. If we discounts the mirror image solutions in the present situation, there are only two solutions, one with same handedness in both transformations, and one with opposite handedness.

3.2 The Other Platonic Solids. Although the transformation from the kite-tiled tetrahedron to the dodecahedron was developed as a byproduct of the coloring problem, it demonstrates beautifully the spacial relationship between two Platonic solids that are not each other's duals. This brings up interesting possibilities for further investigation: Can there be similar coloring-conserving transformation between some other pairs of Platonic solids? The structural relationships between the five polyhedra, as well as their intrinsic properties, are well known, so the problem is not impossible to solve, but the result will not necessarily be as aesthetically pleasing as the tetrahedron-dodecahedron relationship.

Decomposing Deltahedra

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Abstract

Deltahedra are polyhedra with all equilateral triangular faces of the same size. We consider a class of we will call 'regular' deltahedra which possess the icosahedral rotational symmetry group and have either six or five triangles meeting at each vertex. Some, but not all of this class can be generated using operations of subdivision, stellation and truncation on the platonic solids. We develop a method of generating and classifying all deltahedra in this class using the idea of a generating vector on a triangular grid that is made into the net of the deltahedron.

We observed and proved a geometric property of the length of these generating vectors and the surface area of the corresponding deltahedra. A consequence of this is that all deltahedra in our class have an integer multiple of 20 faces, starting with the icosahedron which has the minimum of 20 faces.

Introduction

The Japanese art of paper folding traditionally uses square or sometimes rectangular paper. The geometric styles such as modular Origami [4] reflect that paradigm in that the folds are determined by the geometry of the paper (the diagonals and bisectors of existing angles and lines). Using circular paper creates a completely different design structure. The fact that chords of radial length subdivide the circumference exactly 6 times allows the use of a 60 degree grid system [5]. This makes circular Origami a great tool to experiment with deltahedra (Deltahedra are polyhedra bound by equilateral triangles exclusively [3], [8]).

After the barn-raising of an endo-pentakis icosi-dodecahedron (an 80 faced regular deltahedron) [Knoll & Morgan, 1999], an investigation of related deltahedra ensued. Although there are infinitely many deltahedra, beginning with the 8 convex ones [8][9], the scope of this paper is restricted to a specific class. First of all, the shapes under consideration have exclusively vertices with 5 or 6 triangles meeting. Since all the triangles are equilateral, the vertices all have $360°$ or $300°$ total flat angle. This means that there are always twelve 5-vertices, since the sum of the angle deficit ($360°$-flat angle) at all the vertices of a genus 0 polyhedron always equals $720°$ [6]). Second, the deltahedra have to be regular in the sense that each 5-vertex has a local 5-fold rotational symmetry that extends to the whole shape. We know from the symmetries of the icosahedron [7: p101] and [2], that the twelve 5-vertices are evenly spaced.

The intuitive route

Figure 1 shows examples of simple 3-D transformations applied to the icosahedron and the dodecahedron (the 2 platonic solids possessing 5-fold rotational symmetry). Using these transformations of truncation and dimpling (see figure), we found 4 deltahedra satisfying the above requirements:
- The icosahedron (1A) is one of the platonic solids.
- The endo-pentakis dodecahedron (1B) is transformed from the dodecahedron by 'dimpling' all the pentagons (endo-pentakis) .

- The endo-pentakis icosi-dodecahedron (1C) is transformed, either from the icosahedron or the dodecahedron, through first a truncation, then a 'dimpling'.
- Finally, the hexakis endo-pentakis truncated icosahedron (1D) is transformed from the icosahedron by first truncating it at the 1/3 point nearest to the vertex along the edge, then 'dimpling' the resulting pentagons[1].

But how does the endo-pentakis snub dodecahedron (1E) fit in? Can we find a sequence of simple transformations that will generate it from a platonic solid?

The platonic solids have both reflective and rotational symmetry. The simple transformations we used in figure 1 all preserve these symmetries. Due to its handedness, the endo-pentakis snub dodecahedron has only rotational symmetry. Therefore, this type of simple symmetry-preserving transformation cannot generate it from a platonic solid.

This defect prompted the search for a more systematic approach. In order to develop this approach, we need to find key structural properties of the deltahedra that can help us classify them. An exhaustive classification will help us develop a reliable recipe to generate the deltahedra of this class.
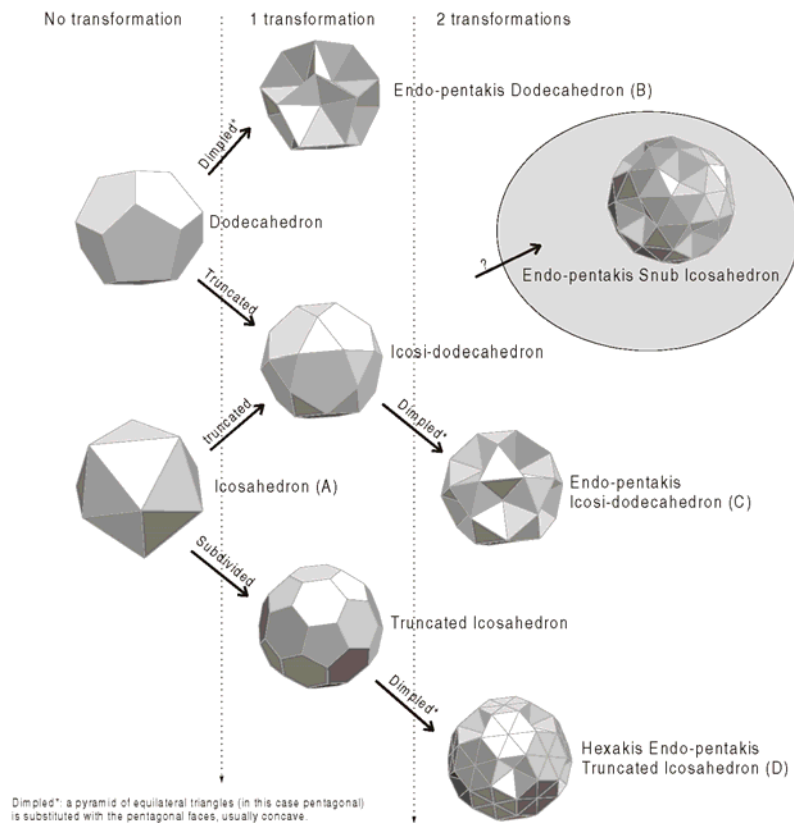

Figure 1: Deltahedra generated by simple 3-D transformations

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[1] Note that the hexagons have also been subdivided into triangles to retain the deltahedral identity of the polygon, but that the ensuing triangles remain coplanar.

The Systematic Approach

From 3d deltahedron to 2d vector

Based on the snowflake-net construction method used in the barn-raising [6: 133-135], we focussed on the shortest surface path between two nearest 5-vertices on the surface of the deltahedra. A first look showed a similarity of structure between the three first deltahedra: the shortest path simulates, in each case, a specific vector between 2 points of an equilateral triangular grid on a flat surface (figure 2). On the icosahedron, the shortest path (which we will call the GPS for geodesic path segment) runs along the edge connecting the two marked vertices. This GPS and the two triangles touching it were lifted off the surface of the icosahedron and flattened above, indicating the vector. In the endo-pentakis dodecahedron, the GPS runs through the midpoints of the faces between the two marked vertices, which in the flat translates into a line segment at 30° to the grid. In the endo-pentakis icosi-dodecahedron, the GPS runs along two edges, tracing the diagonal of the hexagon drawn. The vectors (2-D) in figure 2 can each be seen as representating a specific deltahedron (3-D) in the triangular grid.
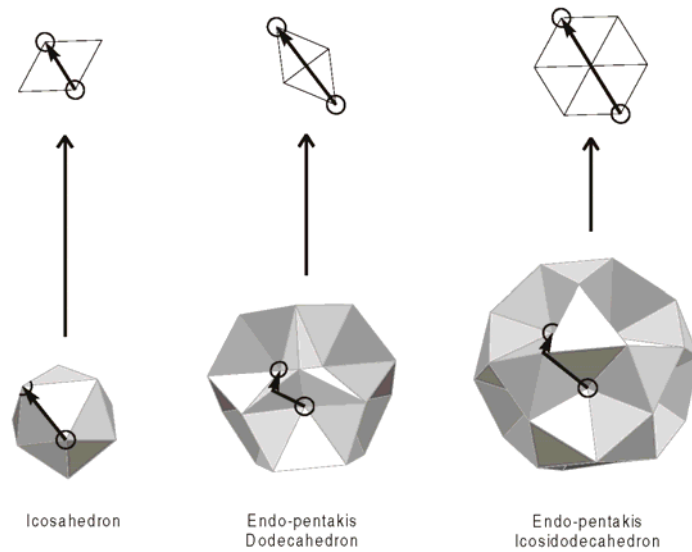

Icosahedron   Endo-pentakis Dodecahedron   Endo-pentakis Icosidodecahedron
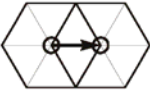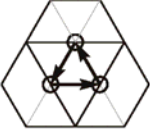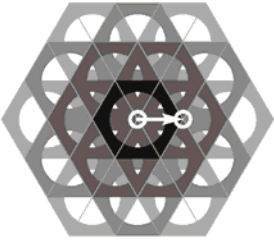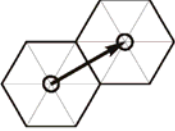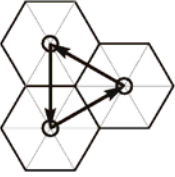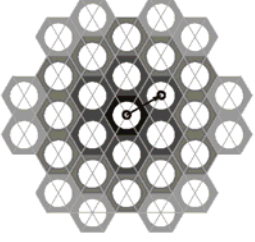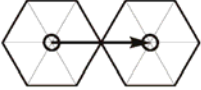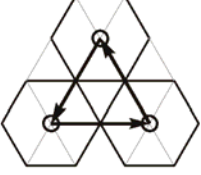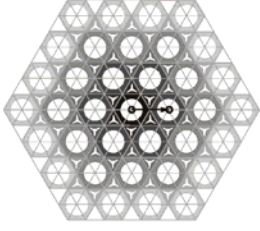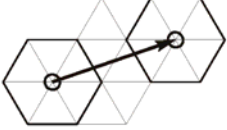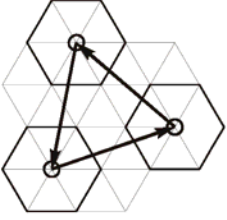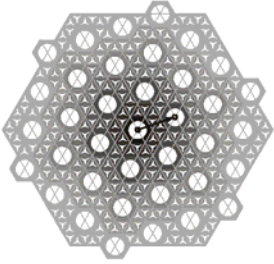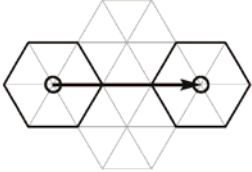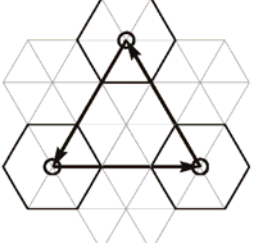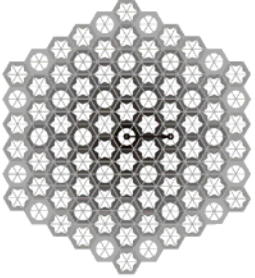
Figure 2: From deltahedra to vectors

From 2d vector to 3d deltahedron via a concentric net

Figure 3 shows all 5 deltahedra of figure 1 along with their corresponding vectors. In column A, the vector is depicted on part of the flat triangular grid together with a hexagon centered at each extremity.

A - Vector	B - Vector Triangulation	C - Grid

Figure 3: From vectors to deltahedra

○ Simple circular motif on hexagon to become pentagonal
△ Wavy motif on triangles between
○ Wavy motif on hexagons between
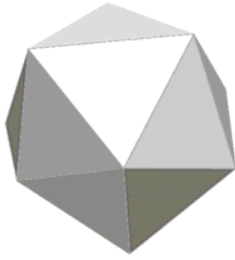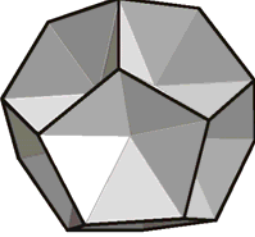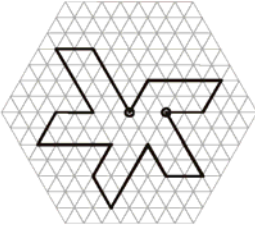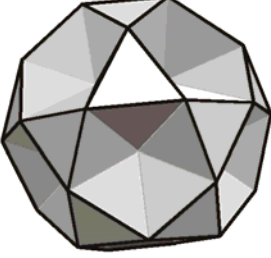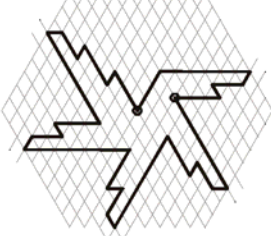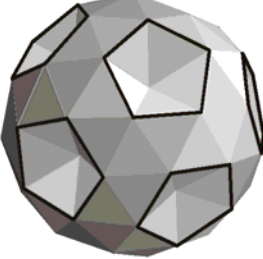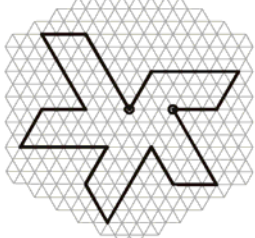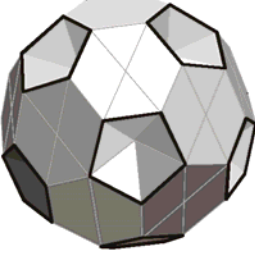
D - Concentric Net	E - Deltahedron	F - Properties II
		Reference 1-A n = 2, m = 0 square distance = 1 Icosahedron 20 faces, 30 edges, 12 5-vertices icosahedral symmetry group: reflections and rotations △ 0 ○ 12 (with 3-fold overlap) ○ 0 12x5 /3 = 20 triangles total
		Reference 2-B n = 3, m = 1 square distance = 3 Endo-pentakis dodecahedron 60 faces, 90 edges, 12 5-vertices, 20 6-vertices icosahedral symmetry group: reflections and rotations △ 0 ○ 12 ○ 0 12x5 = 60 triangles total
		Reference 2-A n = 4, m = 0 square distance = 4 Endo-pentakis icosi-dodecahedron 80 faces, 120 edges, 12 5-vertices, 30 6-vertices icosahedral symmetry group: reflections and rotations △ 20 ○ 12 ○ 0 20 + 12x5 = 80 triangles total
		Reference 3-B n = 5, m = 1 square distance = 7 Endo-pentakis snub dodecahedron 140 faces, 210 edges, 12 5-vertices, 60 6-vertices partial icosahedral symmetry group: rotations only △ 80 ○ 12 ○ 0 80 + 12x5 = 140 triangles total
		Reference 3-A n = 6, m = 0 square distance = 9 Endo-pentakis truncated icosahedron 180 faces, 90 edges, 12 5-vertices, 20 6-vertices icosahedral symmetry group: reflections and rotations △ 0 ○ 12 ○ 20 12x5 + 20x6 = 180 triangles total

Figure 3: From vectors to deltahedra (cont'd)

Each of these hexagons will collapse, by losing a triangle, into a pentagonal pyramid. They will then become a 5 triangle-vertex on the deltahedron (see column E). Thus, these two hexagons yield a closest pair of 5-vertices in the deltahedron.

In column C, the same vector is embedded in the triangular grid.gUsing the 6-fold rotational symmetry of the grid we generate the vector system[2] that determines the relative positions of the grid vertices that can become 5-vertices. The rendering, based on the vector system, emphasizes the different elements in the resulting deltahedron. The hexagons with simple circular motif become pentagonal pyramids with a central 5-vertex. The triangles and hexagons with the wavy motif remain triangles and hexagons on the surface of the deltahedra. The grids on this column show the 6-fold rotational symmetry of the vector systems corresponding to each deltahedron.

Building the concentric net for the icosahedron (column D), we use the wedge technique [6: 133]. Starting at the centre of the grid and removing a 60° wedge, we convert it into a 5-vertex when the net is closed. Moving out radially to the next ring of points to become 5-vertices (using the same vector system), we remove 5 new 60° wedges and so on as shown until we have enough triangles to construct the deltahedron without overlaps. The last 5-vertex (antipodal to the first) only needs one triangle at the end of each of the 5 branches of the net (there are only 5 branches since the 5-fold symmetry of the deltahedronmeant the removal of one branch with the first wedge cut). The vertices where the wedges are removed correspond to the centers of the simple circular motif in column C. In the case of the icosahedron, the endo-pentakis-icosi-dodecahedron and the endo-pentakis truncated icosahedron, the wedges start at each vertex of the net perimeter of column D, except for the last 5-vertex. The net is concentric because it has a central vertex around which the triangles are positioned in concentric rings corresponding to concentric rings on the deltahedron. The net gives a flattened view of the deltahedron as seen from the first 5-vertex.

It is worth noting here that the same net can produce different regular deltahedra. We could join up the net for the endo-pentakis-icosi-dodecahedron to make a large size icosahedron by having the 5-vertices point out instead of dimpling in. However, both shapes have the same number of triangles with the same edge connections. In one case the vertices point out and in another they point in. We can therefore say that a net gives a unique deltahedron, providing we assume the concavity of the 5-vertices where there is a choice[3] and we consider mirror images to be the same deltahedron[4].

Choosing one of six equivalent vectors
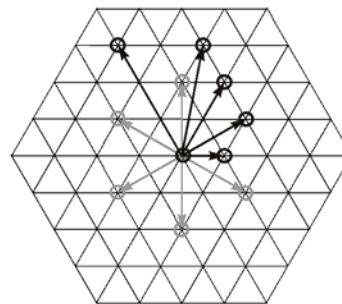
Figure 4: Representative vectors

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[2] The vector system is defined as the vector, its images under 60° rotation and combinations of these under vector addition.

[3] In the case of the icosahedron, the 5-vertices are necessarily convex because of their proximity.

[4] This applies to the endo-pentakis snub icosahedron, which we will discuss lower.

Figure 4 shows the representative vectors for the deltahedra of figure 1 on a triangular grid. Now that we have a method of constructing deltahedra from 2-D vectors we can take any vector on the grid and see what deltahedron it represents. When constructing the net, we use six rotated images of the vector. These are shown in gray in figure 4 for one of the vectors. We then see that each of the six rotated vector images represent the same deltahedron. Instead of drawing the whole grid and getting six copies of everything, we can simply draw one sixth of each set to get all the representative vectors without duplicates. In fact in figure 5 we have reduced the grid further to only a wedge of angle 30 degrees. This eliminates another kind of duplicate: mirror images of snub, or handed, deltahedra. The vector of a snub deltahedron if reflected in the horizontal base of the wedge will generate the mirror image of the snub deltahedron. Another way of seeing this duplication is: if the net in figure 3 column D is closed up with the facing surface on the outside yields a left handed snub deltahedron, then if it is closed up with the facing surface on the inside, the equivalent right handed snub deltahedron is built. Are there any more duplications? The answer here is no, because any two distinct vectors in the remaining wedge will either have different lengths or different sizes of angle from the horizontal in the grid, or both. Therefore according to this method, they will make different deltahedra.
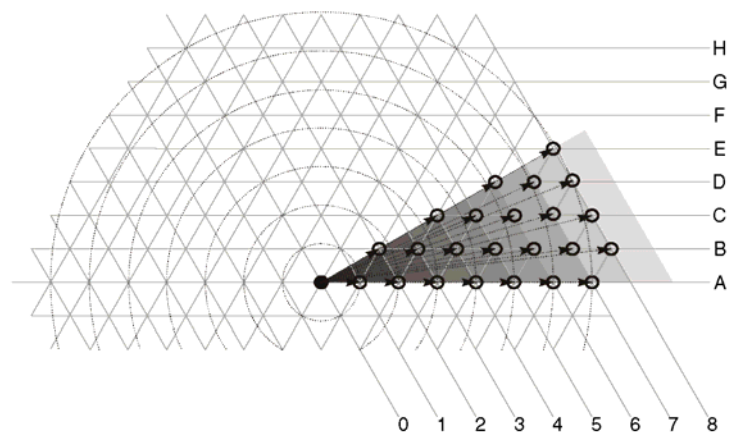
Figure 5: Wedge of vectors

From the wedge of vectors to the list of all regular deltahedra

We have now shown by construction that all regular deltahedra have a unique representative vector in our wedge. We have also shown that each vector can be used to construct a unique deltahedron. Therefore our vectors can act as a list of identifications for deltahedra. Using this identification, is it possible to create a systematic list of the regular deltahedra belonging to this family (as defined in the introduction)? Using the length of the vectors would be a great idea if it worked because simpler deltahedra having fewer triangles will have shorter vectors and come before more complex deltahedra in the list. This works only until the vector length equals 7 (the unit length corresponds to the edge length of a triangle). At a distance of 7 units from the origin, there are 2 points of the grid (7A and 8D in figure 5). Using instead the coordinate system of figure 5, we can use the rows and columns to list the vectors (and the deltahedra). In this particular case, because the rows are of infinite length, we will go 'column' by 'columns'.

In table 1, we list the vectors shown in the wedge of figure 5. To facilitate the use of the pythagorian theorem, we define n as follows: $n/2$ = the length of the horizontal component of the vector. This garantees that n is an integer since all points are horizontally at a multiple of ½ from the

origin. Similarly, we define m as follows: $m(\sqrt{3})/2 =$ the length of the vertical component of the vector. The third column lists the square value of the vector length, and in the last column, we have shown the names of the known deltahedra corresponding to their vectors in the table.

Vector	n	m	L^2	Deltahedron	Vector	n	m	L^2	Deltahedron
1-A	2	0	1	Icosahedron	6-D	9	3	27	...
2-B	3	1	3	E-P Dodecahed.	6-C	10	2	28	...
2-A	4	0	4	E-P Icosidodecah.	6-B	11	1	31	...
3-B	5	1	7	E-P Snub Icosah.	6-A	12	0	36	...
3-A	6	0	9	E-P Truncated Icosah.	7-D	11	3	37	...
4-C	6	2	12	...	7-C	12	2	39	...
4-B	7	1	13	...	7-B	13	1	43	...
4-A	8	0	16	...	7-A	14	0	49	...
5-C	8	2	19	...	8-E	12	4	48	...
5-B	9	1	21	...	8-D	13	3	49	...
5-A	10	0	25	...	8-C	14	2	52	...

Table 1: Squared distance table for Vectors and their Deltahedra

According to pythagorus, the squared length L^2 of a vector is given by:

$$L^2 = \left(\frac{n}{2}\right)^2 + \left(\frac{\sqrt{3}\times m}{2}\right)^2 ,$$

where n and m are positive integers.

In the example of figure 6 (vector 2-B), $n = 3$ and $m = 1$, giving a vector length of 3:

$$L^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}\times 1}{2}\right)^2 = \left(\frac{9}{4}\right) + \left(\frac{3}{4}\right) = \frac{12}{4} = 3 ,$$

Notice that vectors 7-A and 8-D are both of squared length 49 as shown in figure 5 and discussed above. We also see in the table that L^2 always seems to be an integer and that n and m always seem to be both odd or both even.
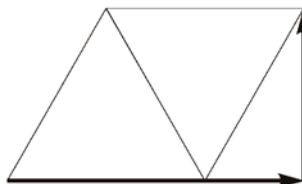
Figure 6: Vector 1B

The relative parity of N & M determines the integral value of L^2

Some interesting numerical properties of this system can be proven. First, the geometry of the wedge determines that if n is even, then so is m and conversely, if n is odd, so is m[5]. Symbolically,

---

[5] If we study the grid points in the wedge of figure 5, by rows, the pattern is clear: even 'numbered' rows have even values for n and m and odd 'numbered' rows have odd values for n & m.

$$\frac{(n+m)}{2} = p, \text{ a positive integer, where } p > m, n.$$

Substituting for n, we get:

$$n = 2p - m,$$

and in the Pythagoras equation,

$$L^2 = \left(\frac{2p-m}{2}\right)^2 + \left(\frac{\sqrt{3} \times m}{2}\right)^2$$

$$= \frac{4p^2}{4} - \frac{4mp}{4} + \frac{m^2}{4} + \frac{3m^2}{4}$$

$$= p^2 - mp + m^2$$

where m and p are positive integers and because $p > m$, we know that the last line is positive.

Relating the length of the vector to the surface area of the deltahedron

Returning to the cells of column B, figure 3, we take a closer look at a simple vector triangle from the vector system. The area of this vector triangle is calculated from the length of the vector as being L^2. As we have seen previously, that value is always an integer. Illustrating this using figure 3, in the first row the vector is of unit length and the triangle in column B is a single triangle (1 tile) of the grid. For the vector of length 2, as in the third row, the triangle is made up of 4 grid triangles. Therefore, the vector of squared length $2^2 = 4$ defines a vector triangle of area 4. From all this, we can make the following statement:

The vector triangle defined in each vector system shown in this paper has a surface area equal to the square of the length of the vector. Furthermore, this value is always an integer.

Comparing the value of L^2 to the number of triangles on the surface of the equivalent deltahedron (see column F) we observe that the total number of triangles is always 20 times the squared vector length and the area of the vector triangle, L^2. Does that mean that it always takes 20 vector triangles to make the whole polyhedron? We can see this is true in the case of rows 1, 3 and 5 of figure 3 (see column D). The 3 nets only differ in the relative scale of the grid and the outline. In fact, comparing the area of a single triangle in the net of row 1, the equivalent section of the net in row 3 has an area of 4 (L^2, where L=2), and the equivalent section of the net in row 5 has an area of 9 (L^2, where L=3). In the other rows, the vector triangle of column B allow us to deduce the same relationship (in row 2, the area is 3 and in row 4, 7). If we draw the vector systems directly on the deltahedra, with a vector connecting each neighbouring pair of 5-vertices as we started in the lower part of figure 2, we see that there are indeed always 20 vector triangles on each deltahedron of this class. This shows that the surface area is always $20 \times L^2$, where L is the vector length, and L^2 is an integer. In other words,

The deltahedra of this class are always composed of an integer multiple of 20 triangles. That multiple is in fact the squared length of the defining vector.

Note that the minimum number of 20 faces is achieved by the icosahedron.

Conclusion

In the course of this paper, we have shown that the classical operations of stellation, truncation and subdivision do not enable us to generate all the deltahedra of a specific class. More accurately, asymmetric or handed deltahedra cannot be generated from a platonic solid by these processes. Examining the relationship between nearest 5-vertices on our regular deltahedra allowed us to develop a 'vector system' method that can be used to list and generate all deltahedra of a class. We then applied this method to the class specified in the introduction.

Later, by examining the length properties of admissible grid vectors we were able to prove that for all deltahedra with the specified regularity property the number of faces is divisible by 20.

We have not however proven that every admissible grid vector gives rise to a buildable deltahedron. We do not know if all the triangles can always fit together in three-dimensional space with no gaps or overlaps. It is therefore possible that there may be some impossible deltahedra in our list. Evidence collected to this point seems to support the assumption that there are no such impossible deltahedra and we are optimistic that a proof can be produced by considering the way in which the vector triangles from figure 3, column B can be made to fit together in three space.

Recent developments in organic chemistry [1] have led to carbon molecules that take on shapes such as the truncated icosahedron, the soccer ball. The geometric building blocks of these molecules include regular hexagons and pentagons formed by rings of carbon atoms with fixed bond length. The equivalent deltahedron would have six flat triangles making a hexagon, and five triangles coming together to form a pyramid with a pentagonal base. There is therefore a precise geometric correspondence between carbon molecules and deltahedra. This is enough to provide additional motivation for studying the geometry of deltahedra, the polyhedra with equilateral triangular faces.

A further exercise for the reader would be to consider how to apply the work in this paper to the class of deltahedra with 4-vertices and 6-vertices and the octahedral symmetry group. Do we get the result that there are always an integer multiple of 8 faces in that class of deltahedra?

A different kind of situation arises if we combine 6-vertices and 7-vertices. The result is a hyperbolic saddle surface which never closes up and has an infinite number of faces.

Finally, future work could consider the classification of families of deltahedra with certain symmetry groups for given combinations of vertices. One such examples is the class of deltahedra with 3-vertices, 6-vertices and 8-vertices (which includes the stella-octangula possessing the octahedral symmetry group and 24 faces). Does every deltahedron in one of these families have an integer multiple of a number of faces, and does there exist a deltahedron that has the minimum possible number of faces?

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Appendix 3   Pre-Module Questionnaire about Mathematics

This survey is part of a research project on the teaching of math to future elementary teachers. By taking part, you are adding to our knowledge in this area, and helping to improve teacher education courses.

The researcher affirms that she will handle the data, including the identity of the participant, with strict confidentiality. When and if she has the opportunity to present findings based on this test, she will not refer to any specific participants, except anonymously, so as to eliminate possibilities of identification.

Thank you for your cooperation!

Section 1

1. Please indicate your name: _____

2. Please indicate which required math courses for your degree you've already taken:
    _____

Please circle the appropriate letter:

3. What is your gender?
(a) female   (b) male

4. What age group are you a part of?
(a) less than 21   (c) 26 – 30   (e) 36 – 40
(b) 21 – 25   (d) 31 – 35   (f) over 40

5. How would you describe your mathematical ability?
(a) excellent   (c) average   (e) poor
(b) competent   (d) weak

6. Are you better at:
(a) Geometry   (b) Arithmetic   (c) Algebra

7. Are you a:
(a) Freshman   (c) Junior   (e) Graduate Student
(a) Sophomore   (d) Senior

Section 2

Please circle the appropriate answer on the right side.

Were you taught mathematics through *open-ended, investigative lessons* where the instructor gave you class time to explore mathematical topics:

8. In Primary/Elementary School   Never / Once a term / Monthly / At least weekly

9. In Middle /Junior High School   Never / Once a term / Monthly / At least weekly

10. In Secondary /High School   Never / Once a term / Monthly / At least weekly

Section 3

11. Please select the five terms from the lists below that you think **best** describe mathematics in general, working down from the best describer (feel free to cross out choices to make your selection easier):

1. _____

2. _____

3. _____

(a) an art	(f) a language	(k) patterns and relations
(b) exercise for the mind	(g) logic	(l) rules
(c) an exploration	(h) measurements	(m) a science
(d) formulas	(i) numbers and operations	(n) a tool
(e) a history	(j) problem-solving	

12. Please select the three terms from the lists below that you think **best** describe mathematics in general, working down from the best describer, (feel free to cross out choices to make your selection easier):

7. _____

8. _____

9. _____

(a) abstract	(e) frustrating	(i) rigid
(b) concrete	(f) fun	(j) torture
(c) empowering	(g) gives meaning	(k) useless
(d) frightening	(h) practical	(l) valuable

13. Please select the five terms from the lists below that you think least describe mathematics in general, working up from the worst describer, (feel free to cross out choices to make your selection easier):

1. _____

2. _____

3. _____

(a) an art	(f) a language	(k) patterns and relations
(b) exercise for the mind	(g) logic	(l) rules
(c) an exploration	(h) measurements	(m) a science
(d) formulas	(i) numbers and operations	(n) a tool
(e) a history	(j) problem-solving	

14. Please select the three terms from the lists below that you think **least** describe mathematics in general, working up from the worst describer, (feel free to cross out choices to make your selection easier):

1 _____

2 _____

3 _____

(a) abstract	(e) frustrating	(i) rigid
(b) concrete	(f) fun	(j) torture
(c) empowering	(g) gives meaning	(k) useless
(d) frightening	(h) practical	(l) valuable

Section 4

Each statement in this section expresses an opinion of mathematics in general. Please show how much you agree with the statement by **circling** your choice of:

- YES! .... STRONGLY AGREE
- yes .... AGREE
- ?? .... NEUTRAL or UNDECIDED
- no .... DISAGREE
- NO! .... STRONGLY DISAGREE

15. I find solving mathematics problems to be dull and boring   YES! yes ?? no NO!

16. I like mathematics better than most other subjects   YES! yes ?? no NO!

17. Mathematics is a subject I find easy   YES! yes ?? no NO!

18. I have never been confident in mathematics   YES! yes ?? no NO!

19. Someone who is good at mathematics never makes a mistake   YES! yes ?? no NO!

20. Mathematics consists of a set of fixed, everlasting truths   YES! yes ?? no NO!

21. Mathematics is about the study of all possible patterns   YES! yes ?? no NO!

22. Mathematics is basically doing calculations   YES! yes ?? no NO!

23. Only gifted professional mathematicians can be creative in mathematics   YES! yes ?? no NO!

24. There are many ways of solving any problem in mathematics   YES! yes ?? no NO!

25. The discoveries of mathematics are permanent   YES! yes ?? no NO!

26. Exploring number patterns is not real mathematics   YES! yes ?? no NO!

27. In mathematics there is always a right answer   YES! yes ?? no NO!

28. Puzzles and investigations are not genuine mathematics   YES! yes ?? no NO!

29. There are many problems in mathematics which have never been solved   YES! yes ?? no NO!

30. Basic number skills are more important than creativity in mathematics   YES! yes ?? no NO!

31. Mathematics is always changing and growing   YES! yes ?? no NO!

32. The procedures and methods in mathematics guarantee right answers   YES! yes ?? no NO!

33. Some mathematics problems have many answers, some have none   YES! yes ?? no NO!

34. Mathematics is exact and certain   YES! yes ?? no NO!

35. There is only one correct way of solving any mathematics problem   YES! yes ?? no NO!

36. A person should not mind risking a mistake when trying to solve a mathematics problem

    YES! yes ?? no NO!

37. Investigating a puzzle can lead to significant new mathematics   YES! yes ?? no NO!

38. Knowing how to solve a problem is more important than the right answer   YES! yes ?? no NO!

39. I think that creativity and mathematics are related   YES! yes ?? no NO!

Section 5

Please answer the following questions as best you can. Feel free to use the back of the sheet for more space. If you do, clearly indicate which question you are answering.

40. Describe the topic that was your first big stumbling block when learning mathematics.

_____

_____

_____

_____And I was in grade/year _____

41. Described briefly what your strategy was to overcome this block.

_____

_____

_____

_____

42. If you had to explain to one of your future pupils what a mathematician does, what would you say?

_____

_____

_____

_____

Appendix 4   Post-Module Questionnaire
about Mathematics

This survey is part of a research project on the teaching of math to future elementary teachers. By taking part, you are adding to our knowledge in this area, and helping to improve teacher education courses.

The researcher affirms that she will handle the data, including the identity of the participant, with strict confidentiality. When and if she has the opportunity to present findings based on this test, she will not refer to any specific participants, except anonymously, so as to eliminate possibilities of identification.

Thank you for your cooperation!

Section 1

1. Please indicate your name: _____

Section 2

2. Are you planning to take Prof. Morgan's section of 3118 in the spring? If so, would you be interested in working on a project like this past October? Why / why not?

3. Do you view mathematics differently from the way you did before you took this course? How?

_____

_____

_____

_____

4. Do you think having taken this class will have any effect on the way you will teach?   Yes   No

5. How would you describe your mathematical ability?

   (a) excellent   (c) average   (e) poor
   (b) competent   (d) weak

Please rate the part of the course that best corresponds to the following statements by using 1 for most true, 3 for least true:

	Sept.	Oct.	Nov.
6. I found the course most interesting in:			
7. I learned the most about teaching in:			
8. I learned the most about mathematics in:			
9. My work was closest to what I think research mathematicians do in:			

Section 3

11. Please select the five terms from the lists below that you think **best** describe mathematics in general, working down from the best describer (feel free to cross out choices to make your selection easier):

1. _____

2. _____

3. _____

(a) an art	(f) a language	(k) patterns and relations
(b) exercise for the mind	(g) logic	(l) rules
(c) an exploration	(h) measurements	(m) a science
(d) formulas	(i) numbers and operations	(n) a tool
(e) a history	(j) problem-solving	

12. Please select the three terms from the lists below that you think **best** describe mathematics in general, working down from the best describer, (feel free to cross out choices to make your selection easier):

1. _____

2. _____

3. _____

(a) abstract	(e) frustrating	(i) rigid
(b) concrete	(f) fun	(j) torture
(c) empowering	(g) gives meaning	(k) useless
(d) frightening	(h) practical	(l) valuable

13. Please select the five terms from the lists below that you think least describe mathematics in general, working up from the worst describer, (feel free to cross out choices to make your selection easier):

1 _____

2 _____

3 _____

(a) an art	(f) a language	(k) patterns and relations
(b) exercise for the mind	(g) logic	(l) rules
(c) an exploration	(h) measurements	(m) a science
(d) formulas	(i) numbers and operations	(n) a tool
(e) a history	(j) problem-solving	

14. Please select the three terms from the lists below that you think **least** describe mathematics in general, working up from the worst describer, (feel free to cross out choices to make your selection easier):

1 _____

2 _____

3 _____

(a) an art	(f) a language	(k) patterns and relations
(b) exercise for the mind	(g) logic	(l) rules
(c) an exploration	(h) measurements	(m) a science
(d) formulas	(i) numbers and operations	(n) a tool
(e) a history	(j) problem-solving	

Section 4

Each statement in this section expresses an opinion of mathematics in general. Please show how much you agree with the statement by **circling** your choice of:

- YES! .... STRONGLY AGREE
- yes .... AGREE
- ?? .... NEUTRAL or UNDECIDED
- no .... DISAGREE
- NO! .... STRONGLY DISAGREE

15. I find solving mathematics problems to be dull and boring   YES! yes ?? no NO!

16. I like mathematics better than most other subjects   YES! yes ?? no NO!

17. Mathematics is a subject I find easy   YES! yes ?? no NO!

18. I have never been confident in mathematics   YES! yes ?? no NO!

19. Someone who is good at mathematics never makes a mistake   YES! yes ?? no NO!

20. Mathematics consists of a set of fixed, everlasting truths   YES! yes ?? no NO!

21. Mathematics is about the study of all possible patterns   YES! yes ?? no NO!

22. Mathematics is basically doing calculations   YES! yes ?? no NO!

23. Only gifted professional mathematicians can be creative in mathematics   YES! yes ?? no NO!

24. There are many ways of solving any problem in mathematics   YES! yes ?? no NO!

25. The discoveries of mathematics are permanent   YES! yes ?? no NO!

26. Exploring number patterns is not real mathematics   YES! yes ?? no NO!

27. In mathematics there is always a right answer   YES! yes ?? no NO!

28. Puzzles and investigations are not genuine mathematics   YES! yes ?? no NO!

29. There are many problems in mathematics which have never been solved   YES! yes ?? no NO!

30. Basic number skills are more important than creativity in mathematics   YES! yes ?? no NO!

31. Mathematics is always changing and growing   YES! yes ?? no NO!

32. The procedures and methods in mathematics guarantee right answers   YES! yes ?? no NO!

33. Some mathematics problems have many answers, some have none   YES! yes ?? no NO!

34. Mathematics is exact and certain   YES! yes ?? no NO!

35. There is only one correct way of solving any mathematics problem   YES! yes ?? no NO!

36. A person should not mind risking a mistake when trying to solve a mathematics problem
    YES! yes ?? no NO!

37. Investigating a puzzle can lead to significant new mathematics   YES! yes ?? no NO!

38. Knowing how to solve a problem is more important than the right answer   YES! yes ?? no NO!

39. I think that creativity and mathematics are related   YES! yes ?? no NO!

Section 5

Please answer the following questions as best you can. Feel free to use the back of the sheet for more space. If you do, clearly indicate which question you are answering.

40. Did you find the bonus question about an 'interesting thing to look at next' useful? How?

41. Please explain how you decided on your project topic:

42. Please explain how you think your project fits into the context of your view of mathematics:

43. If you had a chance to redesign this course, what would you change?

44. On the next page, you will find the results of our discussion of September 3rd. Please circle the part of the diagram you consider best describes your project. Feel free to add to it.

Blackboard 1

- Solve problems using numbers

  - geometry/…

- unsolved problems

- come up with new processes

- use formulas and symbols

- answering questions by proving/disproving

- understanding phenomena using numbers, patterns and relations

Helping people (rewarding)

Proving Wrong! (rewarding)

Blackboard 2

- Exploring relationships

  - (Speed) relationship

- Teaching

- Writing

- Trial and error—checking the answer

- Following the rules

  - Testing the rules

Physics

Accounting

It's new! (rewarding)

Proof

Find the limit — bend the limit

Experience

Appendix 5 Journal Keeping Guidelines

Researcher and class mentor, Eva Knoll, name@server.edu

This section of Mathematics for Elementary Teachers of the fall semester of 2003-2004 will be the object of a research project in mathematics education, and will include the involvement of a mathematics education student from the University of Exeter (UK), Eva Knoll, who will also serve as class mentor.

Data for the research will be collected throughout the course, in the form of handed-in, assessed work, interviews and classroom discussions, and the keeping of a journal by all participants, including the instructional team and the researcher.

I, Eva Knoll, declare that I will handle the data, including the identity of the participant, with strict confidentiality. When and if I have the opportunity to present findings based on the data, I will change the participant's name and details so as to eliminate the possibility of identification.

Journal keeping

The journal keeping is only partly for the purpose of the research project. It is also intended as an aid for you to reflect on the experience of learning mathematics, in particular in terms of your future pupils. The following are a few guidelines to help you in your journal writing.

- Describe the date, time and context of your entries (e.g. after doing project work, or a discussion with the mentor, or with a friend, etc) so you'll remember when you wrote them. Feel free to describe your mathematical work to contextualise your observations.

- Focus on your attitude and feelings about the content of the course and the way you are learning, your process of understanding, your overall experience. Don't forget that the journal is also an opportunity for a dialogue with me to help you with the course.

- Write only on the right hand page so you can add later reflections on those notes on the left.

- Write regularly and don't worry about running out of space. Remember: this will be a useful reference when you teach to see how it feels to be on the 'other side'.

- If you are not sure where to begin or have a writers' block, speak to the researcher or choose one of the following starting points:

  1. My experience in the last class was...

  2. Taking this course makes me realise...

  3. I've changed my mind about...

  4. I still think that...

  5. I am not sure about...

  6. I was surprised/shocked that...

  7. My mind has/hasn't been stretched because...

  8. This course has made me feel that I am more/less...

9. Writing this journal has made me realise that...

10. I think using portfolios in the mathematics classroom is...

11. I prefer working alone/in a team in this context because...

12. I feel what we are doing in class does/doesn't relate to 'real mathematics' because...

13. I think/don't think language affects understanding in the mathematics classroom

14. How do I feel about my knowledge of mathematics?

15. What tools and skills did I use to explain, record or understand the mathematical ideas in this course?

16. Did I experience either knowing or understanding or both in this course?

17. I think mathematicians must spend their days doing...

18. Try to compare or oppose your experience in this course with some of the following words: art, calculations, counting, creativity, exercise for the mind, explorations, a language, laws, logic, measurements, numbers, operations, patterns, problem-solving, a puzzle, relations, rules, a tool

Appendix 6   Guidelines for the Project Write-up

In the introduction to *The Mathematical Experience*[37]*,* Gian-Carlo Rota wrote:

> "We often hear that mathematics consists mainly in "proving theorems". Is a writer's job mainly that of "writing sentences"? A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure our minds are not playing tricks."

This class's project write-up should reflect the spirit of this quote.

These are the sections your project write-up should contain:

Introduction

This section will contain your motivation for choosing your topic and a short description of your original proposal (you can use your homework assignment 5 originally due October 6th).

Process

In this section, you should describe the project as it evolved, showing your work and thinking in full detail for each stage. Include any tables, diagrams, examples and counter-examples. Make sure you discuss your changes in focus or tactics and the cause of these changes. This section also includes ideas you decided not to follow up, for reasons of time constraints or others. Be careful to include all your ideas and observations even if you don't see their value. We can help you see and express the mathematical depth of your work.

Results

This section should be a summary of your findings from section 2, referring to key examples and/or counter-examples. A precise statement of your claims and reasons why will be required. Note that reasons why something did *not* work are an acceptable mathematical result. We will help you with this section, particularly regarding the formalisation of your arguments into precise mathematical language.

New ideas for further study

As in the September homework, write about new interesting things to look at that resulted from your project.

The write-up should also contain a full collection of examples and pictures illustrating your thinking. If you need photographs of your work, see Eva during class or email her at name@server.edu.

The draft and final version of the write-up should be type written and double-spaced. The first draft is due November 3rd.

---

[37] Davis, Philip J., Hersh, Reuben, *The Mathematical Experience*, Birkhäuser, Boston, 1981, p. xviii.

Project [mark]ing scheme

The [mark]ing scheme is included below to show the emphases you need to make in your work.

40%   What you've tried and found, clearly explained, including what didn't work

40%   Reasons why. This includes why something you tried did or didn't work, and if you make a mathematical claim, why *it* is true.

20%   What could you look at next. This will not only include what you would look at next at the end of the project, but also ideas that came up during the exploration that you may or may not have had time to work on.

Course [mark]ing scheme (this has not changed)

The [mark] percentage breakdown for the course will be as follows:

35%   Project portfolio

15%   Main project presentation and in class participation

30%   Homework

20%   Chapter quizzes

Appendix 7   Codes for Qualitative Analysis with Comments

Code Family: 0 Scope

- Code: 0.0. Course as a whole
"This is for comments about the course overall."

- Code: 0.0. mathematics/mathematics education as a whole

- Code: 0.0. Other sections
"For comparison. Participants discuss the difference between sections of the course"

- Code: 0.0. previous mathematics classroom experience
"Often for comparison"

- Code: 0.1. December assessment

- Code: 0.1. November regular instruction

- Code: 0.1. October project

- Code: 0.1. September mini-projects

Code Family: 1 Object

- Code: 1.0. Instruction content
"This refers to comments on the mathematical content of the class. About geometry, algebra, probabilities, etc., but also in the sense of the NCTM's categories 'problem solving', 'reasoning and proof', 'communication', 'connections' and 'representations'."

- Code: 1.0. instruction style
"This pertains to comments on HOW it is taught, and only if it is an overall comment not specific to other 1.1, 1.2 or 1.3 tags"

- Code: 1.1. hands-on work
"Includes worksheets and manipulatives"

- Code: 1.1. students' work (Homework, etc.)
"homework, quizzes, portfolio, presentations both formal and informal, etc, their content and so on, NOT as they relate to assessment (for that, use 'assessment requirements')"

- Code: 1.1. Visualisation

- Code: 1.1. whole class discussion

- Code: 1.1. Work in small groups
"NOT whole group discussion."

- Code: 1.1. Work with textbook
"This refers to work in 'Stage 3: November' as well as to comparisons of other class work to using a textbook."

- Code: 1.2. additional help
"office hours, feedback, etc. also other students"

- Code: 1.2. Assessment requirements
"... as a philosophical issue. (e.g. I feel like they are always changing the marking policies and what we need to do.) for comments on the work itself see 'student work'"

- Code: 1.3. about mathematics - the subject

- Code: 1.3. about mathematics learning

- Code: 1.3. about mathematics teaching

Code Family: 2 Verb/affective response

- Code: 2.0. Anxiety/worry/intimidation about

- Code: 2.0. Ease/difficulty with

- Code: 2.0. Excited/enjoying/bored with
  "also motivation"

- Code: 2.0. Found useful
  "Can also be 'found important' or 'helpful'"

- Code: 2.0. Frustration/feeling challenged about
  "not to be confused with 'bored' when the class is not challenging the participant! Even if s/he uses the word frustrated, s/he might mean bored, or not finding the work useful. Frustrated should relate to the idea of 'being stuck' in Mason et al."

- Code: 2.0. Interest in

- Code: 2.0. Learning about
  "This one is on a cognitive level. The student felt she learned something about... For the affective, see 'understanding/confusion about'"

- Code: 2.0. Like/dislike of/personal preference

- Code: 2.0. Self confidence about

- Code: 2.0. Understanding/confusion about

- Code: 2.1. Comparison of
  "This is tagged positive '+ve' if it is in favour of the evaluated course design."

- Code: 2.1. Reflection on
  "This is a cognitive tag. As opposed to 'learning about', there is not necessarily a 'lesson' to be learned. It is simply a comment on their experience."

- Code: 2.1. Relating project to mini-projects

- Code: 2.1. Relating to 'real life'

- Code: 2.1. Relating to the primary classroom

- Code: 2.1. Student's expectations/hopes about

Code Family: 3 Topic

- Code: 3. Creativity in mathematics

- Code: 3. Independence of work

- Code: 3. Learning by doing
  "as opposed to being told"

- Code: 3. Mathematicians' work
  "'Relating something to the work of the mathematician (reflective?) and also mentions of searching out why something is true/false"

- Code: 3. Openness of instruction

- Code: 3. Text book as reference
  "This is specifically about the 'authority' of the textbook as provider of answers."

- Code: 3. The time it takes
  "Any issue of timeline, including wasted time, or not enough spent on something"

- Code: 3.0 Portfolio
- Code: 3.0. Mathematical Heuristics

Code Family: 4 Modifier

- Code: 4.0. Negative
- Code: 4.0. Negative change in
- Code: 4.0. Positive
- Code: 4.0. Positive change in
- Code: 4.0. Neutral
  "Neutral assessment (e.g. 'the class seems to go pretty fast' is not labeled as 'good' or 'bad')."
- Code: 4.0. Slight negative
  "use of words like 'a little', 'sort of', 'kind of', etc also for 'occasional', 'at times'"
- Code: 4.0. Slight positive
  "use of words like 'a little', 'sort of', 'kind of', etc also for 'occasional', 'at times'"
- Code: 4.0. Unsure (positive/negative)
  "This is for when the student is unsure but also for statements that are ambiguous. If a statement contains both positive & negative bits, use both positive and negative tags."
- Code: 4.1. Externalization
  "Putting the blame in an entity external to the student (e.g. previous experience in mathematics class)."
- Code: 4.1. Internalization
  "Putting the 'blame' in the student her/himself."

Code Family: 5 Meaningful remarks

- Code: 5. Meaningful remarks

Code Family: 6 Auto-coded

- Code: 6.0 'Book'
- Code: 6.0 'Concrete'
- Code: 6.0 'Elementary', 'children'
- Code: 6.0 'Journal'
- Code: 6.0 'Want*', 'decide*', 'freedom'

Code Family: 7 Responses

- Code: 7.0. Feedback and responses
- Code: 7.1. With response

Code Family: 8 Context

- Code: 8.0. Stage 1: September mini-projects
- Code: 8.0. Stage 2: October project
- Code: 8.0. Stage 3: November regular classes
- Code: 8.0. Stage 4: December assessment

Code Family: 9 Theoretical Framework

- Code: 9.1 B1 Agency of starting point

- Code: 9.1 B2 Agency of process

- Code: 9.1 B3 Agency of end point

- Code: 9.2. H1 Initiation/Planning

- Code: 9.2. H2 Incubation

- Code: 9.2. H3 Illumination

- Code: 9.2. H4 Verification
  "As Sowder and others put it, this also includes asking new questions"

- Code: 9.2. H5 Overall process

- Code: 9.3. M1 Security/Mason's tension
  "... a tension arising from what Brousseau (1984) calls the didactic contract. This tension arises between pupils and teachers in the following way. The pupils know that the teacher is looking to them to behave in a particular way. The teacher wishes the pupils to behave in a particular way as a result of, or even a manifestation of, their understanding of the concepts or the topic. The more explicit the teacher is about the specific behaviour being sought, the more readily the pupils can provide that sought after behaviour, but simply by producing the behaviour and not as a manifestation of their understanding. Tension arises because the pupils are seeking the behaviour and expect the teacher to be explicit about that behaviour, whereas the teacher is in the bind that the more explicit he is, the less effective the teaching. (Mason, 1989, p. 155)"

Appendix 8   Course Syllabus
Mathematics for Elementary Teachers

Fall 03 - Time - Location

Instructor: Dr. Zachary, name@server.edu, office number
TA: Alan, name@server.edu, office number
Text: -

This section of Mathematics for Elementary Teachers includes a substantial component of project work. In addition, key areas of the curriculum will be covered using the course text, and assessed through homework assignments and quizzes. Students who successfully complete this section of part 1 will be ready for any section of part 2 to be taken in the spring semester of 2004.

Projects

There will be a series of small exploration projects in September, aimed at developing skills in problem *posing* and in the communication of mathematical concepts and ideas. A main exploration project for each student will develop from this work, culminating in a presentation at the end of the semester. Work for both the small and main exploration projects, including explorations in directions that did not work out, will go into a project portfolio for each student.

The project portfolio [mark] will be based on the process, communication and outcomes of the explorations. This is because exploration projects, by their nature, are "a step into the unknown. [...] The principal hope for an investigation should be that totally unexpected things turn up, that different kinds of approaches to problems should appear as different pupils tackle different aspects of the problem in different ways." (Driver: 'Investigative Mathematics in School' *Mathematics in School* volume 17 number 1).

Textbook chapter work

Chapters of the text will be covered in class group work, homework and quizzes. The chapter quizzes will last 1 hour and be presented at the end of each chapter covered in the text and will only examine the content of that chapter. The final chapter quiz will probably be held at the time scheduled for this class in the final exam period, but will still only examine the final chapter covered in the course. All chapter quizzes, including the final one will be given equal credit weighting.

[Mark]ing scheme

The [mark] percentage breakdown for the course will be as follows:

35%   Project portfolio

15%   Main project presentation and in class participation

30%   Homework

20%   Chapter quizzes

Appendix 9
Ratio between Emotion-, Attitude- and Belief-based Responses

	Beliefs		Attitudes		Emotions		Increase in Belief	Increase in Attitude	Increase in Emotion	Change Belief - Emotion
	Before	After	Before	After	Before	After				
Geoffrey	0	2	3	2	3	2	2	-1	-1	3
Joan	1	3	3	2	2	1	2	-1	-1	3
Barbara	3	4	0	1	3	1	1	1	-2	3
Jill	2	3	2	3	2	0	1	1	-2	3
Melanie	2	4	2	0	2	2	2	-2	0	2
Bridget	0	1	3	3	3	2	1	0	-1	2
Jean	1	2	3	3	2	1	1	0	-1	2
Linette	1	2	1	1	4	3	1	0	-1	2
Rachel	1	1	1	3	4	2	0	2	-2	2
Alexa	2	3	3	2	1	1	1	-1	0	1
Ashley	0	1	3	2	3	3	1	-1	0	1
Carol	2	3	3	2	1	1	1	-1	0	1
Darleen	2	3	2	1	2	2	1	-1	0	1
Donna	1	2	3	2	2	2	1	-1	0	1
Elise	1	2	4	3	1	1	1	-1	0	1
Emily	2	3	2	1	2	2	1	-1	0	1
Isabel	1	2	3	2	2	2	1	-1	0	1
Sue	0	1	4	3	2	2	1	-1	0	1
Alice	1	1	2	3	3	2	0	1	-1	1
Trish	2	3	3	1	1	2	1	-2	1	0
Myriam	2	2	2	2	2	2	0	0	0	0
Petra	2	2	1	1	3	3	0	0	0	0
Rob	0	0	3	3	3	3	0	0	0	0
Sean	0	0	3	3	3	3	0	0	0	0
Simone	3	3	2	2	1	1	0	0	0	0
Pippa	2	1	1	3	3	2	-1	2	-1	0
Irene	2	2	4	3	0	1	0	-1	1	-1
Rosie	2	2	3	2	1	2	0	-1	1	-1
Samantha	2	2	3	2	1	2	0	-1	1	-1
Sandra	2	2	2	1	2	3	0	-1	1	-1
Christie	2	1	2	3	2	2	-1	1	0	-1
Patrick	3	2	1	2	2	2	-1	1	0	-1
Laura	3	2	2	2	1	2	-1	0	1	-2
Marie	2	0	1	2	3	4	-2	1	1	-3
Silvia	2	0	2	3	2	3	-2	1	1	-3
Totals	54	67	82	74	74	69	13	-8	-5	18

Appendix 10   Correlations of Items 19 to 39 to the Four Belief Subscales

Correlations to Instrumentalist Subscale - oriented

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
Subscale 0: 11 Items	+			+	+	-		+		+		+				+	+	-	-	+	-
Pearson Cor.	0.151	-0.043	0.245	0.218	0.673	-0.320	0.154	0.428	0.199	0.416	-0.362	0.673	-0.229	0.324	-0.126	0.388	0.379	-0.540	-0.367	0.165	-0.414
Sig. (2-tailed)	0.386	0.806	0.157	0.207	0.000	0.061	0.378	0.010	0.251	0.013	0.033	0.000	0.186	0.058	0.469	0.021	0.025	0.001	0.030	0.344	0.013
Subscale 1: 10 Items					+			+		+	-	+	-			+	+	-	-		-
Pearson Cor.	0.223	-0.060	0.188	0.108	0.677	-0.144	0.304	0.437	0.207	0.576	-0.463	0.677	-0.353	0.143	-0.109	0.494	0.396	-0.559	-0.637	-0.181	-0.562
Sig. (2-tailed)	0.198	0.732	0.280	0.538	0.000	0.408	0.076	0.009	0.234	0.000	0.005	0.000	0.038	0.411	0.535	0.003	0.019	0.000	0.000	0.297	0.000
Subscale 2: 11 Items					+			+		+	-	+	-			+	+	-	-		-
Pearson Cor.	0.281	-0.105	0.166	0.094	0.659	-0.134	0.309	0.417	0.218	0.541	-0.515	0.637	-0.493	0.140	-0.176	0.462	0.359	-0.545	-0.642	-0.218	-0.627
Sig. (2-tailed)	0.102	0.548	0.342	0.592	0.000	0.442	0.071	0.013	0.209	0.001	0.002	0.000	0.003	0.424	0.312	0.005	0.034	0.001	0.000	0.208	0.000
Subscale 3: 11 Items					+			+		+	-	+	-			+	+	-	-		-

No change: done   **Positive items: 23, 26, 28, 30, 34, 35; Negative items: 29, 31, 36, 37, 39.**

Correlations to Platonist Subscale - oriented

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
Subscale 0: 5 Items		+					+		+				-			+			-		
Pearson Correlation	0.210	0.629	0.196	0.125	0.272	0.000	0.734	-0.001	0.637	0.097	-0.212	0.209	-0.290	0.233	-0.042	0.650	0.017	-0.108	-0.388	-0.108	-0.231
Sig. (2-tailed)	0.227	0.000	0.258	0.474	0.114	1.000	0.000	0.995	0.000	0.579	0.221	0.228	0.091	0.177	0.809	0.000	0.924	0.537	0.021	0.536	0.183
Subscale 1: 5 Items		+					+		+							+			-		
Pearson Correlation	0.140	0.686	0.194	0.109	0.316	-0.014	0.722	-0.019	0.614	0.172	-0.144	0.282	-0.078	0.240	0.041	0.665	0.083	-0.110	-0.529	-0.092	-0.130
Sig. (2-tailed)	0.424	0.000	0.263	0.534	0.064	0.938	0.000	0.915	0.000	0.324	0.409	0.101	0.656	0.165	0.816	0.000	0.635	0.529	0.001	0.600	0.458
Subscale 2: 5 Items		+					+		+							+			-		

No change: done   **Positive items: 20, 25, 27, 34; Negative item: 37.**

Correlations to Problem Solving Subscale - oriented

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
Subscale 0: 4 Items						+				-	+							+			
Pearson Correlation	-0.224	-0.148	-0.241	0.295	-0.297	0.781	0.035	-0.194	0.285	0.046	-0.107	-0.067	0.101	0.097	-0.215	-0.326	-0.493	0.824	0.166	0.142	0.250
Sig. (2-tailed)	0.195	0.395	0.163	0.085	0.083	0.000	0.841	0.265	0.097	0.793	0.541	0.701	0.565	0.580	0.216	0.056	0.003	0.000	0.341	0.414	0.148
Subscale 1: 3 Items						+										-	-	+			
Pearson Correlation	-0.205	-0.125	-0.270	0.317	-0.272	0.776	0.035	-0.233	0.286	-0.036	-0.098	-0.088	0.067	0.072	-0.199	-0.359	-0.698	0.778	0.200	0.123	0.194
Sig. (2-tailed)	0.237	0.474	0.117	0.064	0.114	0.000	0.843	0.178	0.096	0.836	0.576	0.614	0.701	0.683	0.252	0.034	0.000	0.000	0.250	0.482	0.263
Subscale 2: 4 Items			-			+										-	-	+			
Pearson Correlation	-0.131	-0.246	-0.368	0.215	-0.320	0.701	-0.122	-0.199	0.164	-0.022	-0.042	-0.136	0.059	-0.046	-0.199	-0.675	-0.673	0.731	0.255	0.175	0.241
Sig. (2-tailed)	0.453	0.154	0.030	0.215	0.061	0.000	0.484	0.253	0.346	0.899	0.812	0.437	0.737	0.794	0.253	0.000	0.000	0.000	0.139	0.316	0.163
Subscale 3: 5 Items			-			+										-	-	+			
Pearson Correlation	-0.078	-0.246	-0.601	0.137	-0.321	0.678	-0.153	-0.166	0.152	0.002	-0.037	-0.164	0.035	-0.062	-0.181	-0.693	-0.648	0.665	0.215	0.178	0.208
Sig. (2-tailed)	0.655	0.155	0.000	0.431	0.060	0.000	0.381	0.341	0.382	0.993	0.831	0.346	0.841	0.725	0.297	0.000	0.000	0.000	0.215	0.308	0.231
Subscale 4: 5 Items			-			+										-	-	+			

No change: done   **Positive items: 24, 36; Negative items: 21, 34, 35.**

Correlations to Pattern Analysis Subscale - oriented

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
Subscale 0: 8 Items			+		-			-		-	+	-			+				+		+
Pearson Correlation	-0.299	0.327	0.238	-0.094	-0.584	-0.102	-0.125	-0.308	-0.373	-0.559	0.658	-0.492	0.576	-0.200	0.489	-0.084	-0.077	0.192	0.633	0.191	0.658
Sig. (2-tailed)	0.081	0.056	0.168	0.592	0.000	0.561	0.473	0.072	0.027	0.000	0.000	0.003	0.000	0.249	0.003	0.630	0.660	0.269	0.000	0.272	0.000
Subscale 1: 10 Items		+			-				-	-	+	-	+		+				+		+
Pearson Correlation	-0.256	0.206	-0.026	-0.207	-0.632	-0.078	-0.213	-0.126	-0.593	-0.456	0.623	-0.641	0.592	-0.249	0.569	-0.197	-0.038	0.114	0.634	0.194	0.674
Sig. (2-tailed)	0.138	0.235	0.881	0.233	0.000	0.655	0.219	0.472	0.000	0.006	0.000	0.000	0.000	0.149	0.000	0.256	0.830	0.516	0.000	0.264	0.000
Subscale 2: 9 Items					-				-	-	+	-	+		+				+		+
Pearson Correlation	-0.256	0.206	-0.026	-0.207	-0.632	-0.078	-0.213	-0.126	-0.593	-0.456	0.623	-0.641	0.592	-0.249	0.569	-0.197	-0.038	0.114	0.634	0.194	0.674
Sig. (2-tailed)	0.138	0.235	0.881	0.233	0.000	0.655	0.219	0.472	0.000	0.006	0.000	0.000	0.000	0.149	0.000	0.256	0.830	0.516	0.000	0.264	0.000
Subscale 3: 9 Items					-				-	-	+	-	+		+				+		+

No change: done   **Positive items: 29, 31, 33, 37, 39; Negative items: 23, 27, 28, 30**

Appendix 11   Paired-Sample Statistics, t-test Results, Cohen's *d* and Cronbach α for the Six Subscales

	Items	Time of Survey	Theoretical Range	Mean	Median	Mode	N	SD	Std. Error Mean	$(SD)^2$	Sig. (2-t'd)	Cohen's *d*	Cronbach α
Like/dislike Subscale – oriented	2	Before	2-10	6.57	7	8	35	2.019	.341	4.076	.505	.083	.721
		After	2-10	6.74	7	7	35	2.119	.358	4.490			-
Confidence Subscale - oriented	2	Before	2-10	6.4	7	8	35	2.428	.410	5.894	.768	-.024	.887
		After	2-10	6.34	7	8	35	2.351	.397	5.526			-
Instrumentalism Subscale – oriented	11	Before	11-55	22.9	23	23	35	4.357	.737	18.987	.000	-.815	.750
		After	11-55	19.5	19	16	35	4.054	.685	16.432			-
Platonism Subscale - oriented	5	Before	5-25	12	11	10	35	2.684	.454	7.205	.000	-.767	.645
		After	5-25	10	10	10	35	2.370	.401	5.617			-
Problem-solving Subscale - oriented	5	Before	5-25	17.6	18	18	35	1.75039	.296	3.064	.000	.919	.654
		After	5-25	19.4	19	19	35	2.030441	.343	4.123			-
Pattern Analysis Subscale - oriented	9	Before	9-45	31.2	31	31	35	3.763603	.636	14.165	.000	.831	.773
		After	9-45	34.2	34	33	35	3.445372	.582	11.870			-

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