# Contrôlabilité et stabilisation frontière pour l'équation de Korteweg-de Vries.

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Thèse présentée par


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pour obtenir le grade de

**DOCTEUR EN SCIENCE**
**SPÉCIALITÉ MATHÉMATIQUES**

Sujet de la thèse :

# Contrôlabilité et stabilisation frontière pour l'équation de Korteweg-de Vries.

Soutenue le 5 juin 2008 devant le jury composé de :

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à Cathy

# Remerciements

Je voudrais commencer ces lignes en exprimant mes remerciements sincères à Jean-Michel Coron. Sa capacité à proposer des problèmes intéressants et fécondes, à éclairer avec des idées simples et efficaces, et ses encouragements constants, font de lui un directeur de recherche formidable. Par ailleurs, je voudrais dire que j'ai trouvé en lui, un être humain extraordinaire, un modèle à suivre.

J'adresse également ma vive reconnaissance à Jean-Pierre Puel et Bing-Yu Zhang pour avoir accepté d'être rapporteurs de cette thèse. Je remercie sincèrement Jean-Pierre Raymond, Jean-Claude Saut et Emmanuel Trélat de me faire l'honneur de participer à mon jury.

Je tiens à remercier Emmanuelle Crépeau pour sa disponibilité à travailler avec moi et sa gentillesse. Je remercie aussi Marianne Chapouly pour nos discussions enrichissantes et son amitié.

Je remercie le Laboratoire de Mathématiques de Orsay, qui m'a accueilli pour mes activités de recherche durant ces trois années. Notamment l'équipe d'Analyse Numérique et EDP dirigée par Patrick Gérard. Un chaleureux merci pour Valérie Lavigne et Catherine Poupon pour leur aide et leur sympathie.

Je remercie Carlos Conca et Axel Osses pour m'avoir encouragé à suivre mes études en France... ils n'avaient pas tort.

Cette thèse a pu être développée grâce à une bourse *Master 2 Investigación y Doctorado*, Embajada de Francia en Chile - CONICYT. Je remercie aussi le projet C-QUID, financé par l'Agence Nationale de la Recherche.

Un grand merci aux amis avec qui j'ai partagé mes joies et tristesses pendant ces années. Ils font une longue liste... mais j'y vais : Cristóbal, Claudia, Marianne, Fernando, Tamara, Mauricio, Ricardo, Jeanne, Mario, Alison, M. Isabel, Néstor, Draltan, Giorgio, Lucho, Miguel, Claudio, Jaliste, Kbzon, Joe, Guille et Pierre. Mention spéciale à Beto, avec qui on a commencé cette aventure parisienne.

Je remercie le soutien à la distance de ma famille au Chili : ma mère Patricia, mon père Eduardo, mes soeurs Pachy et Vero, mon neveu Vicho et ma belle-mère Fátima.

Toutes ces années n'auraient pas eu de sens sans la compagnie de Cathy et je voudrais donc profiter de ces dernières lignes pour lui exprimer tout mon amour et gratitude pour partager sa vie avec moi... sa présence m'est tout simplement indispensable.

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## 1.1 Introduction

L'équation de Korteweg-de Vries (KdV)

$$\partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \tag{1.1}$$

a été introduite par Korteweg et de Vries dans [31] pour modéliser la propagation d'une vague de petite amplitude se propageant à droite dans un canal uniforme peu profond. Plus généralement, cette équation sert à décrire l'évolution d'une onde unidimensionnelle dans certains modèles physiques où il faut considérer un effet non linéaire. C'est précisément la non linéarité présente dans l'équation qui rend possible l'existence de solutions particulières applées *solitons*. Ces solutions remarquables ont permis l'application de l'équation de KdV à divers phénomènes physiques. Le livre [53] de Whitham est une bonne référence pour comprendre la déduction et l'interprétation de cette équation à partir des lois de la physique.

D'un point de vue mathématique, cette équation a été étudiée à partir des années 60, lors de la découverte des *solitons* et de l'introduction de la méthode *scattering inverse transform* pour résoudre le problème de Cauchy associé à l'équation de KdV. La plupart des travaux ont été consacrés à l'étude sur la droite réelle $\mathbb{R}$ ou sur un domaine périodique $\mathbb{T}$ dans les espaces de Sobolev $H^s$. L'équation (1.1) est bien posée, lorsque l'on travaille sur la droite, dans $H^s(\mathbb{R})$ pour $s > -\frac{3}{4}$ (voir [26] et [9]) et lorsque l'on travaille sur un domaine périodique, dans $H^s(\mathbb{T})$ pour $s \geq -1$ (voir [25]).

Dans cette thèse, l'équation KdV sera considérée sur un domaine borné non périodique, et les résultats les plus adaptés à cette étude seront ceux portant sur le système suivant, posé sur l'intervalle $(0, L)$,

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \quad y(0,x) = \phi(x), \\ y(t,0) = h_1(t), \quad y(t,L) = h_2(t), \quad \partial_x y(t,L) = h_3(t), \end{cases} \tag{1.2}$$

où $\phi$ est la condition initiale et $h_j$ pour $j = 1, 2, 3$, sont les conditions aux bords. Dans le cas homogène, c'est-à-dire lorsque $h_j = 0$ pour $j = 1, 2, 3$, Zhang a démontré, dans [54], que (1.2) est bien posé dans $H^{3k+1}(0, L)$ avec $k \in \mathbb{N}$. Plus tard, dans [38], Perla Menzala, Vasconcellos et Zuazua ont démontré le même résultat dans $L^2(0, L)$. Dans le cas des conditions aux bords non homogènes, Bona, Sun and Zhang obtient, dans [3], que le système est bien posé dans $H^s(0, L)$ avec $s > 0$ pourvu que $h_j \in H^{(s+1)/3}_{loc}$ pour $j = 1, 2$, et $h_3 \in H^{s/3}_{loc}$. Dans [23], Holmer améliore ce dernier résultat en l'obtenant pour tout $s > -\frac{3}{4}$.

Dans cette thèse, nous allons donc considérer le système de contrôle suivant

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \quad y(0,x) = y_0(x), \\ y(t,0) = y(t,L) = 0, \quad \partial_x y(t,L) = u(t), \end{cases} \tag{1.3}$$

où l'état du système est $y(t,\cdot) : [0,L] \to \mathbb{R}$, le contrôle est $u(t) \in \mathbb{R}$ et la condition initiale est $y_0$. Comme on peut le voir, on impose des conditions Dirichlet homogènes et on a un contrôle sur la condition Neumann à droite de l'intervalle. D'après les résultats mentionnés précédemment, pour un temps $T > 0$ arbitraire, un contrôle $u \in L^2(0,T)$ et une condition initiale $y_0 \in L^2(0,L)$ suffisamment petits, on aura l'existence et l'unicité d'une solution de (1.3) dans l'espace $C([0,T];L^2(0,L))$.

Nous allons considérer deux types de problèmes qui sont étroitement liés. Tout d'abord la contrôlabilité, puis la stabilisation du système (1.3). On s'intéressera à l'existence de contrôles permettant d'amener notre système d'un état initial à un état final. Si l'on est capable de faire ceci en un temps fini, on parlera de contrôlabilité exacte. Par contre si l'on s'approche de l'état cible asymptotiquement en temps et si la commande au temps $t$ ne dépend que de l'état au temps $t$, on parlera de stabilisation.

## 1.2 Problème de contrôlabilité exacte

Soient $T, L > 0$. Etant donnés un contrôle $u \in L^2(0,T)$ et une condition initiale $y_0 \in L^2(0,L)$ dans (1.3), nous sommes dans le cadre de régularité nécessaire pour pouvoir parler de $y(T,\cdot)$, où $y$ est la solution de (1.3), comme d'un élément de l'espace d'états $L^2(0,L)$. Cela s'avère fondamental pour poser la question de la contrôlabilité exacte : étant donnés deux états $y_0$ et $y_T$ dans l'espace $L^2(0,L)$, existe-t-il une commande $u \in L^2(0,T)$ tel que la solution $y = y(t,x)$ de (1.3) satisfasse $y(T,\cdot) = y_T$ ?

Cette question a été étudiée pour le système de contrôle frontière (1.3) par Rosier dans [39]. Il a montré la contrôlabilité exacte locale pour des domaines $(0,L)$ dits non critiques, c'est-à-dire, le théorème suivant.

**Théorème 1.2.1** (voir [39]) *Soit $T > 0$ et $L > 0$ tel que*

$$L \notin N := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}; \, k,l \in \mathbb{N}^* \right\}. \tag{1.4}$$

*Alors, il existe $r > 0$ tel que pour tout couple $(y_0, y_T) \in L^2(0,L)^2$ avec $\|y_0\|_{L^2(0,L)} < r$ et $\|y_T\|_{L^2(0,L)} < r$, il existe $u \in L^2(0,T)$ et*

$$y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

*satisfaisant* (1.3) *et* $y(T,\cdot) = y_T$.

La première étape de la preuve consiste à montrer le résultat pour le système linéarisé autour de l'origine :

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, & y(0,\cdot) = y_0, \\ y(t,0) = y(t,L) = 0, & \partial_x y(t,L) = u(t), \end{cases} \tag{1.5}$$

à l'aide de la méthode des multiplicateurs et de la méthode HUM introduite par J.-L Lions (voir [34]). Rosier utilise ensuite un argument de point fixe pour revenir au problème non linéaire. Il montre de plus que lorsque $L \in N$, le système linéaire (1.5) n'est plus contrôlable et ainsi la première étape de la preuve échoue. En effet, dès que $L \in N$, il existe un sous espace de $L^2(0,L)$ de dimension finie, noté $M$, tel que pour tout état $\psi \in M$ non nul, pour tout contrôle $u \in L^2(0,T)$ et pour tout $y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ solution de (1.5) avec $y_0 = 0$, alors $y(T,\cdot) \neq \psi$. L'espace $M$ est l'espace d'états qui ne sont pas atteignables pour le système linéaire.

Ce problème des domaines critiques n'apparaît plus dès que l'on introduit d'autres contrôles dans notre système. Si l'on considère

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = u_1(t), \quad y(t,L) = u_2(t), \quad \partial_x y(t,L) = u_3(t), \end{cases} \tag{1.6}$$

où $u_j$ pour $j = 1,2,3$, sont des contrôles, on a d'après Zhang dans [55], la contrôlabilité exacte locale dans $H^s(0,L)$ pour $s \geq 0$ autour d'une solution régulière pour n'importe quelle valeur de $L > 0$. De plus, Rosier a remarqué dans [39] que dans le cas où on dispose seulement des deux contrôles $u_2$ et $u_3$ (on impose $u_1 = 0$), on a le même résultat dans $L^2(0,L)$ autour de l'origine.

Dans les chapitres 2 et 3 de cette thèse, nous nous intéressons précisément aux longueurs critiques. Notre but est de démontrer que malgré la perte de contrôlabilité du système linéaire, le terme non linéaire $y\partial_x y$ nous permet d'obtenir la contrôlabilité pour le système non linéaire. Pour faire ceci nous allons utiliser la méthode de développement en séries entières introduite dans le cadre de la dimension infinie par Coron et Crépeau dans [15] justement pour traiter le premier cas critique. Ils prouvent le Théorème 1.2.1 pour les longueurs critiques $L = 2k\pi$ avec $k \in \mathbb{N}^*$ tel que

$$\nexists (m,n) \in \mathbb{N}^* \times \mathbb{N}^* \text{ avec } m^2 + mn + n^2 = 3k^2 \text{ et } m \neq n. \tag{1.7}$$

Dans ces cas-là, le sous espace $M$ est unidimensionel : $M = \langle 1 - \cos(x) \rangle$. Leur méthode consiste à bouger le système le long de cette direction par des développements d'ordre supérieur à un (un développement de premier ordre donne le système linéaire qui n'est pas capable de bouger dans cette direction), et puis à appliquer un théorème de point fixe.

Nous allons expliquer d'abord le comportement du système linéaire, notamment la perte de contrôlabilité dans les cas critiques. Ensuite, nous expliquerons la méthode employée et les résultats obtenus.

### 1.2.1 Système de contrôle linéaire

Si l'on veut étudier la contrôlabilité du système de contrôle (1.5), il est bien connu que l'on est ramené à montrer une inegalité d'observabilité pour l'équation adjointe. Plus précisément il faut montrer qu'il existe une constante $C > 0$ telle que tout $\varphi_0 \in L^2(0, L)$, la solution de

$$\begin{cases} \partial_t\varphi + \partial_x\varphi + \partial_x^3\varphi = 0, \quad \varphi(0, \cdot) = \varphi_0, \\ \varphi(t, 0) = \varphi(t, L) = 0, \quad \partial_x\varphi(t, L) = 0, \end{cases} \tag{1.8}$$

satisfait

$$\|\varphi_0\|_{L^2(0,L)} \leq C\|\partial_x\varphi(t, 0)\|_{L^2(0,T)}. \tag{1.9}$$

Rosier a montré dans [39] que (1.9) est vraie si $L \notin N$. Par contre, cette inegalité n'est plus satisfaite si $L \in N$ et ainsi l'équation (1.5) n'est plus contrôlable, ce qui justifie le nom de *longueurs critiques* pour les éléments de l'ensemble $N$. Par exemple, si $L = 2\pi$, on peut prendre $\varphi = (1 - \cos(x))$ qui est la solution de (1.8) avec $\varphi_0 = (1 - \cos(x))$, mais l'inegalité d'observabilité n'est pas satisfaite car $\varphi_0 \neq 0$ et $\partial_x\varphi(\cdot, 0) = 0$. Comme on l'a dit auparavant, dans ce cas $M = \langle 1 - \cos(x)\rangle$. L'espace $M^\perp \subset L^2(0, L)$ est l'espace des conditions initiales $\varphi_0$ pour lesquelles l'inegalité (1.9) est satisfaite, ou de manière équivalente l'espace où le système est contrôlable.

Dans le cas critique général, le même phénomène apparaît . Soit $L \in N$. On a un nombre fini, $n$, de couples différents $\{(k_j, l_j)\}_{j=1}^n \subset \mathbb{N}^* \times \mathbb{N}^*$ avec $k_j \geq l_j$ tels que

$$L = 2\pi\sqrt{\frac{k_j^2 + k_jl_j + l_j^2}{3}}. \tag{1.10}$$

On introduit les notations

$$J^> := \{\, j \in \{1, \ldots, n\}\, ;\, k_j > l_j\},$$

$$J^= := \{\, j \in \{1, \dots, n\}\,;\, k_j = l_j\}, \quad n^> := |J^>|,$$

et on définit les nombres réels

$$\gamma_1^j := -\frac{1}{3}(2k_j + l_j)\frac{2\pi}{L}, \qquad \gamma_2^j := \gamma_1^j + k_j\frac{2\pi}{L}, \qquad \gamma_3^j := \gamma_2^j + l_j\frac{2\pi}{L} \tag{1.11}$$

qui nous permettent de définir les espaces suivants

– Pour $j \in J^>$, soit

$$M_j := \{\lambda_1\varphi_1^j + \lambda_2\varphi_2^j;\, \lambda_1, \lambda_2 \in \mathbb{R}\} = \langle \varphi_1^j, \varphi_2^j \rangle,$$

où les fonctions $\varphi_1^j, \varphi_2^j$ sont données par

$$\begin{aligned} \varphi_1^j(x) &:= C_j\left(\cos(\gamma_1^j x) - \tfrac{\gamma_1^j - \gamma_3^j}{\gamma_2^j - \gamma_3^j}\cos(\gamma_2^j x) + \tfrac{\gamma_1^j - \gamma_2^j}{\gamma_2^j - \gamma_3^j}\cos(\gamma_3^j x)\right), \\ \varphi_2^j(x) &:= C_j\left(\sin(\gamma_1^j x) - \tfrac{\gamma_1^j - \gamma_3^j}{\gamma_2^j - \gamma_3^j}\sin(\gamma_2^j x) + \tfrac{\gamma_1^j - \gamma_2^j}{\gamma_2^j - \gamma_3^j}\sin(\gamma_3^j x)\right), \end{aligned} \tag{1.12}$$

avec $C_j$ une constante choisie telle que $\|\varphi_1^j\|_{L^2(0,L)} = \|\varphi_2^j\|_{L^2(0,L)} = 1$.

– Pour $j \in J^=$, soit

$$M_j := \{\lambda(1 - \cos x);\, \lambda \in \mathbb{R}\} = \langle 1 - \cos(x) \rangle.$$

Finalement, on introduit les espaces

$$M := \bigoplus_{j=1}^{n} M_j \quad \text{et} \quad H := M^\perp.$$

Le sous espace $H$ contient les états atteignables pour le système linéaire. En effet, d'après [39], on a le résultat suivant.

**Théorème 1.2.2** *Pour tout couple $(y_0, y_T) \in H \times H$, il existe $u \in L^2(0,T)$ tel que la solution $y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ de (1.5) satisfait $y(T,\cdot) = y_T$. De plus, si $y_0 \in H$, pour tout $u \in L^2(0,T)$, la solution $y$ de (1.5) vérifie $y(t,\cdot) \in H$ pour tout temps $t$.*

Par contre, le sous espace $M$ contient toutes les conditions initiales pour lesquelles la solution de (1.8) ne satisfait pas l'inégalité d'observabilité (1.9).

### 1.2.2 Méthode et résultats

Pour montrer que la non linéarité nous permet d'obtenir la contrôlabilité, nous allons utiliser la méthode de développement en séries entières qui a été introduit dans [15].

Expliquons la méthode. Soit $y = y(t,x)$ la solution de (1.3) avec un contrôle $u = u(t)$. Considérons formellement le développement suivant

$$y = \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \epsilon^4 y_4 + \dots$$
$$u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \epsilon^4 y_4 + \dots$$

où $\epsilon$ est un paramètre petit. Le terme non linéaire s'écrit alors

$$y\partial_x y = \epsilon^2 y_1 \partial_x y_1 + \epsilon^3 y_1 \partial_x y_2 + \epsilon^3 y_2 \partial_x y_1 + o(\epsilon^3)$$

et les trois premiers ordres sont donnés par

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ \partial_x y_1(t,L) = u_1(t), \end{cases}$$

$$\begin{cases} \partial_t y_2 + \partial_x y_2 + \partial_x^3 y_2 = -y_1 \partial_x y_1, \\ y_2(t,0) = y_2(t,L) = 0, \\ \partial_x y_2(t,L) = u_2(t), \end{cases}$$

et

$$\begin{cases} \partial_t y_3 + \partial_x y_3 + \partial_x^3 y_3 = -\partial_x(y_1 y_2), \\ y_1(t,0) = y_1(t,L) = 0, \\ \partial_x y_1(t,L) = u_3(t). \end{cases}$$

Puisque la trajectoire $y_1$ ne peut pas *entrer* dans l'espace $M$ ($y_1$ est simplement la solution du système linéaire), l'idée est de considérer des trajectoires $y_1$ allant de zéro à zéro qui engendrent des trajectoires $y_2$ capables d'atteindre les états appartenant à $M$. Ensuite, en se servant du premier ordre pour contrôler dans $H$ et des ordres supérieurs pour contrôler dans $M$, on utilise un argument de point fixe pour obtenir le résultat cherché pour le système non linéaire.

Remarquons que pour *faire jouer* les ordres supérieurs, il faut absolument que $y_1(T,\cdot)$ soit nul. Sinon, "$\epsilon y_1(T) + \epsilon^2 y_2(T) + \epsilon^3 y_3(T) \approx \epsilon y_1(T)$", c'est-à-dire, ce serait le premier ordre qui l'emporterait.

Dans [15], Coron et Crépeau étudient le cas où l'espace $M$ est de dimension un. Ils montrent qu'un développement d'ordre deux n'est pas suffisant et ils doivent donc aller jusqu'au troisième ordre. Dans le chapitre 2 de cette thèse, on montre que dans le cas où $M$ est de dimension deux, un développement à l'ordre deux suffit pour obtenir la contrôlabilité exacte locale autour de l'origine. Comme dans [15], on voit que l'on peut entrer dans l'espace $M$ pour tout temps $T > 0$. Par contre, pour atteindre tous les états dans $M$ il nous faut, au moins avec notre métode, un certain temps. Ainsi, on voit apparaître une condition sur le temps de contrôle. Le cas critique général ($M$ de dimension supérieur à deux) fait l'objet du chapitre 3. Avec la même méthode, on arrive à montrer le théorème suivant

**Théorème 1.2.3** *Soit $L \in N$. Alors, il existe $T_L \geq 0$ tel que le système de contrôle (1.3) est localement exactement contrôlable en temps $T$ pourvu que $T > T_L$.*

## 1.3 Problème de stabilisation rapide

Le but de cette partie est de construire quelques lois de *feedback* pour le système (1.3) tel que le système en boucle fermée ait une décroissance exponentielle vers zéro, autrement dit, nous allons étudier la stabilisation exponentielle. Nous allons en plus, imposer à l'avance le taux de décroissance souhaité pour le système bouclé, raison pour laquelle on parle de *stabilisation rapide*.

La stabilisation exponentielle pour l'équation de KdV sur un domaine borné a d'abord été étudiée pour le système avec des conditions aux bords périodiques. Dans [30], Komornik, Russell et Zhang montrent qu'avec un contrôle distribué tout au long du domaine, on peut rendre exponentiellement stable le système. Plus tard, dans [44] et [46], les auteurs montrent la même proprieté mais avec un contrôle supporté dans un sous domaine ouvert. Pour la stabilisation frontière, Russell et Zhang ont étudié dans [45] le système

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, \quad y(0,\cdot) = y_0, \\ y(t,0) = y(t,\pi), \quad \partial_x^2 y(t,0) = \partial_x^2 y(t,\pi), \quad \partial_x y(t,L) = \alpha \partial_x y(t,0), \end{cases} \tag{1.13}$$

où $-1 < \alpha < 1$. Ils montrent la stabilité exponentielle pourvu que $\alpha \neq -\frac{1}{2}$. Le cas $\alpha = -\frac{1}{2}$ a été traité par Sun dans [50] en utilisant une autre méthode.

Pour notre système de contrôle, Zhang, dans [54], a considéré (1.3) dans le cas $L = 1$. Il propose le feedback

$$u(t) = \alpha \partial_x y(t,0) \tag{1.14}$$

avec $0 < |\alpha| < 1$. On voit qu'au moins formellement, les solutions du système bouclé satisfont

$$\frac{d}{dt}\int_0^1 |y(t,x)|^2 dx = -(1-\alpha)|\partial_x y(t,0)|^2$$

et une décroissance vers zéro est donc espérée. Zhang montre en fait que le système est bien posé dans $L^2(0,1)$ et qu'ils existent des constantes $\omega > 0$ et $C > 0$ telles que

$$\forall y_0 \in L^2(0,1), \forall t > 0, \quad \|y(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\omega t}\|y_0\|_{L^2(0,1)}, \tag{1.15}$$

où $y$ est la solution de

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = \alpha \partial_x y(t,0), \\ y(0,\cdot) = y_0. \end{cases} \tag{1.16}$$

Ainsi, le feedback proposé rend stable le système (1.3). Il faut quand même remarquer que d'après [39], la valeur de la longueur $L$ s'avère être très importante. En effet, comme on a déjà vu, Rosier montre qu'il existe un ensemble de *valeurs critiques* $N$ (voir la définition dans (1.4)), tel que si $L \in N$, ils existent des conditions initiales $y_0$ telles que la solution de (1.3) avec $u = 0$, a une norme $L^2$ constante (noter que $1 \notin N$). Ces solutions satisfont aussi $\partial_x y(t,0) = 0$, et un feedback comme (1.14) ne stabilise donc pas le système.

Dans les cas des longueurs non critiques, on pourrait introduire un feedback comme Zhang dans [54] pour montrer que le système est stabilisable, mais ce n'est pas nécessaire. En fait, dans [38], Perla Menzala, Vasconcellos, et Zuazua montrent que l'on a (1.15) avec tout simplement $u = 0$, c'est-à-dire, (1.3) est exponentiellement stable dans les cas non critiques.

Afin de traiter les cas critiques, dans [38] les auteurs ont introduit un terme de *damping* $b(x)y$ qui permet de montrer (1.15) pour les solutions de

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + b(x)y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = 0, \\ y(0,\cdot) = y_0, \end{cases} \tag{1.17}$$

pourvu que $b(x) > 0$ sur un sous domaine $I \subset (0,L)$ et que $I$ contienne un ensemble de la forme $(0,\delta) \cup (L-\delta, L)$ avec $\delta > 0$. Cette dernière hypothèse a été suprimée par Pazoto dans [37], et réduit à l'hypothèse que $I$ soit un ouvert non vide. Ces résultats s'appliquent aussi pour l'équation non linéaire de KdV et ils ont été étendus par Rosier et Zhang dans [42] et par Linares et Pazoto dans [33] pour l'équation de KdV généralisée.

On s'intéressera dans cette thèse au cas sans terme de damping. D'après les travaux mentionnés ci-dessus, nous savons que le cas non critique est stable et l'on veut donc montrer que pour n'importe quelle valeur de $\omega > 0$, on peut construire une loi de feedback telle que le système en boucle fermée a une décroissance exponentielle de taux au moins $\omega$. En fait, la méthode dont nous allons nous servir nous permet de montrer que le taux de décroissance est au moins égal à $2\omega$, assurant la stabilisation exponentielle rapide pour le système de contrôle étudié.

Dans la suite, nous allons présenter la méthode utilisée et les résultats obtenus.

### 1.3.1 Méthode pour la stabilisation rapide

Pour des systèmes de contrôle en dimension infinie, il y a peu de méthodes permettant d'atteindre un taux de décroissance exponentielle aussi grand que l'on veut. On peut citer celle de Slemrod dans [48] pour des opérateurs de contrôle bornés (le cas d'un contrôle interne) et celle de Komornik dans [28] pour des opérateurs de contrôle non bornés (le cas d'un contrôle frontière). Ces méthodes sont des extensions à la dimension infinie de la méthode introduite indépendamment par Kleinman dans [27] et par Luke dans [36] et qui est basée sur le Gramien de contrôlabilité. Dans le même esprit, Urquiza a généralisé à la dimension infinie, dans [52], une méthode appelée *Bass method* par Russell (voir [43, pages 117-118]). Pour ce faire, il s'est inspiré des résultats numériques obtenus par Briffaut (voir [4]) avec la méthode due à Komornik. Numériquement, le taux de décroissance effective était deux fois meilleur que celui prédit théoriquement. Pour arriver à déplacer de $2\omega$ la partie réelle des modes du système, Urquiza utilise le Gramien de controlabilité sur un horizon infini. Expliquons sa méthode sur le système de contrôle

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), \\ y(0) = y_0, \end{cases} \tag{1.18}$$

où l'état $y(t)$ appartient à un espace de Hilbert $Y$ et le contrôle $u(t)$ appartient à un espace de Hilbert $U$. On considère un opérateur $A$ à domaine dense et antiadjoint, c'est-à-dire, tel que $A^* = -A$ dans $Y$. L'opérateur $B$ est un opérateur non borné de $U$ à $Y$. Supposons que $A$ et $B$ satisfassent les hypothèses suivantes

(H1) L'opérateur $A$ génère un groupe fortement continu sur $Y$.

(H2) L'opérateur $B : U \to D(A)'$ est borné.

(H3) *Régularité.* Pout tout $0 < T < \infty$ il existe une constante $C_T > 0$ telle que

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \leq C_T \|y\|_Y^2, \quad \forall y \in D(A^*).$$

(H4) *Observabilité.* Il existe $T > 0$ et $c_T > 0$ tels que

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \geq c_T \|y\|_Y^2, \quad \forall y \in D(A^*).$$

Urquiza a montré dans [52] le théorème suivant dont la preuve utilise des résultats sur les équations de Ricatti algébriques associées au problème du régulateur linéaire quadratique.

**Théorème 1.3.1** *(voir [52, Theorem 2.1]) Considérons des opérateurs $A$ et $B$ satisfaisant* (H1)-(H4). *Alors, pour $\omega > 0$ quelconque,*

(*i*) *L'opérateur symétrique et positif $\Lambda_\omega$ défini par*

$$(\Lambda_\omega x, z)_Y = \int_0^\infty \left(B^* e^{-\tau(A+\omega I)^*} x, B^* e^{-\tau(A+\omega I)^*} z\right)_U d\tau, \quad \forall x, z \in Y,$$

*est coercif et un isomorphisme sur $Y$.*

(*ii*) *Soit $F_\omega := -B^* \Lambda_\omega^{-1}$. L'opérateur $A + BF_\omega$ avec $D(A + BF_\omega) = \Lambda_\omega(D(A^*))$ génère un semigroupe fortement continu sur $Y$ .*

(*iii*) *Le système en boucle fermée (le système (1.18) avec la loi de feedback $u = F_\omega(y)$) est exponentiellement stable avec un taux de décroissance au moins égal à $2\omega$, autrement dit,*

$$\exists C > 0, \forall y_0 \in Y, \quad \|e^{t(A+BF_\omega)} y_0\|_Y \leq C e^{-2\omega t} \|y_0\|_Y.$$

On peut voir que pour un paramètre $\omega > 0$ quelconque, un feedback est construit explicitement. Cette loi de rétroaction force les solutions du système en boucle fermée à avoir une décroissance exponentielle vers zéro avec une vitesse au moins égale à $2\omega$.

### 1.3.2 Résultats obtenus

Pour pouvoir appliquer le résultat ci-dessus, il faut d'abord mettre notre système de contrôle sous la forme (1.18) avec des opérateurs $A$ et $B$ satisfaisant (H1)-(H4). Pour satisfaire (H1) il faut le modifier afin de le rendre reversible en temps. Nous ferons donc le changement de fonction de contrôle suivant :

$$u(t) = \partial_x y(t, 0) + v(t),$$

où $v$ sera le nouveau contrôle. Ainsi, le système devient

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, \\ y(t, 0) = y(t, L) = 0, \\ \partial_x y(t, L) - \partial_x y(t, 0) = v(t). \end{cases} \tag{1.19}$$

Nous écrivons maintenant (1.19) sous la forme abstraite (1.18) en prenant $A$ et $B$ comme suit :

$$D(A) := \left\{ w \in H^3(0,L); w(0) = w(L) = 0, w'(0) = w'(L) \right\},$$

$$Aw := -w' - w''',$$

$$B : s \in \mathbb{R} \mapsto L_s \in D(A^*)',$$

$$L_s : z \in D(A^*) \mapsto sz'(L) \in \mathbb{R}.$$

L'opérateur $A$ satisfait ainsi $A^* = -A$ et

$$(Aw, w)_{L^2(0,L)} = 0, \quad \forall w \in D(A).$$

Il satisfait donc l'hypothèse (H1) d'après la théorie classique des semigroupes. De plus, il est à résolvante compacte, et son spectre contient donc seulement des valeurs propres complexes pures $(i\lambda_k)_{k\in\mathbb{Z}}$ et les modes propres $(\phi_k)_{k\in\mathbb{Z}}$ forment une base de l'espace $L^2(0,L)$. L'opérateur $B$ satisfait quant à lui l'hypothèse (H2). Pour montrer (H3) et (H4), nous utiliserons dans le chapitre 4 une étude asymptotique des valeurs et modes propres de l'opérateur $A$ et la technique des *inégalités d'Ingham*. Mais afin de pouvoir formuler les résultats obtenus, il faut introduire les espaces suivants.

**Définition 1.3.1** *Soit $Z$ l'ensemble de combinaisons finies des fonctions propres $(\phi_k)_{k\in\mathbb{Z}}$. Alors $Z$ est un sous espace dense dans $L^2(0,L)$. Pour $s \in \mathbb{R}$ quelconque, on définit l'espace $H_s$ comme étant la complétation de $Z$ par rapport à la norme définie par*

$$\Big\| \sum_{k\in\mathbb{Z}} c_k\phi_k \Big\|_s := \Big( \sum_{k\in\mathbb{Z}} (1+|\lambda_k|)^{\frac{2}{3}s} |c_k|^2 \Big)^{1/2}. \tag{1.20}$$

*De plus, dans chaque espace $H_s$, on a la base orthonormale $\big((1+|\lambda_k|)^{-\frac{s}{3}}\phi_k\big)_{k\in\mathbb{Z}}$.*

Nous savons que si $L \in N$, l'inégalité d'observabilité (H4) n'est pas satisfaite. Par contre, si $L \notin N$, nous allons montrer dans le chapitre 4 qu'il existe deux contantes $c_T$ et $C_T$ telles que

$$c_T \|z_0\|^2_{H_1} \le \int_0^T |z_x(t,L)|^2 dt \le C_T \|z_0\|^2_{H_1}, \quad \forall z_0 \in H_1, \tag{1.21}$$

où $z$ est la solution de l'équation homogène

$$\begin{cases} \partial_t z + \partial_x z + \partial_x^3 z = 0, \ z(0,.) = z_0, \\ z(t,0) = z(t,L) = 0, \ \partial_x z(t,L) - \partial_x z(t,0) = 0 \end{cases}$$

Cela signifie, que les hypothèses (H3) et (H4) sont satisfaites dans l'espace $H_1$, ce qui entraîne, par la dualité observabilité-contrôlabilité, un résultat de contrôlabilité dans l'espace $H_{-1}$ pour le système (1.19) avec des contrôles dans $L^2(0,T)$. De cette façon nous pouvons appliquer la méthode due à Urquiza à notre système de contrôle linéaire.

Afin de pouvoir définir la loi de feedback, nous introduisons d'abord la forme bilinéaire

$$a_\omega(q_0,\psi_0) := \int_0^\infty e^{-2\omega\tau} q_x(\tau,L)\psi_x(\tau,L)d\tau, \tag{1.22}$$

où $q$ et $\psi$ sont les solutions de

$$\begin{cases} \partial_\tau q + \partial_x q + \partial_x^3 q = 0,\ q(0,.) = q_0, \\ q(\tau,0) = q(\tau,L) = 0,\ \partial_x q(\tau,L) - \partial_x q(\tau,0) = 0 \end{cases} \tag{1.23}$$

et

$$\begin{cases} \partial_\tau \psi + \partial_x \psi + \partial_x^3 \psi = 0,\ \psi(0,.) = \psi_0, \\ \psi(\tau,0) = \psi(\tau,L) = 0,\ \partial_x \psi(\tau,L) - \partial_x \psi(\tau,0) = 0. \end{cases} \tag{1.24}$$

Nous définisson alors l'opérateur

$$\begin{aligned} F_\omega : H_1 &\longrightarrow \mathbb{R} \\ z &\longrightarrow F_\omega(z) := -q_0'(L), \end{aligned} \tag{1.25}$$

où $q_0$ est la solution du problème variationnel

$$a_\omega(q_0,\psi_0) =< z,\psi_0 >_{H_{-1},H_1}, \quad \forall \psi_0 \in H_1. \tag{1.26}$$

Finalement, nous obtenons la stabilisation rapide qui sera aussi illustrée par des simulations numériques.

**Théorème 1.3.2** *Soit $\omega > 0$. Le système en boucle fermée*

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0,\ y(0,.) = y_0, \\ y(t,0) = y(t,L) = 0,\ \partial_x y(t,L) - \partial_x y(t,0) = F_\omega(y(t)), \end{cases} \tag{1.27}$$

*est bien posé dans $H_1$. De plus, les solutions décroissent vers zéro avec un taux exponentiel au moins égal à $2\omega$, i.e.,*

$$\exists C > 0, \forall y_0 \in H_1, \quad \|y(t,\cdot)\|_{H_1} \le Ce^{-2\omega t}\|y_0\|_{H_1}.$$

# Chapitre 2

# Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain

This chapter is contained in [5].

## Sommaire

## 2.1 Introduction

Let $L > 0$ be fixed. Let us consider the following Korteweg-de Vries (KdV) control system with the Dirichlet boundary condition

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \end{cases} \tag{2.1}$$

where the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. This is a well known example of a nonlinear dispersive partial differential equation. This equation has been introduced by Korteweg and de Vries in [31] to describe approximately long waves in water of relatively shallow depth. A very good book to understand both physical motivation and deduction of the KdV equation, is the book by Whitham [53].

We are concerned with the exact controllability properties of (2.1). In [39] Rosier has proved that this control system is locally exactly controllable around the origin provided that the length of the spatial domain is not critical. This was done using multiplier techniques and the HUM method introduced by Lions (see [34]).

**Theorem 2.1.1** (see [39, Theorem 1.3]) *Let $T > 0$ and assume that*

$$L \notin N := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}};\ k,l \in \mathbb{N}^* \right\}. \tag{2.2}$$

*Then there exists $r > 0$ such that, for every $(y_0, y_T) \in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)} < r$ and $\|y_T\|_{L^2(0,L)} < r$, there exist $u \in L^2(0,T)$ and*

$$y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

*satisfying* (2.1)*, $y(0,\cdot) = y_0$ and $y(T,\cdot) = y_T$.*

Moreover, Rosier proved that the linearized control system of (2.1) around the origin, which is given by

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \end{cases} \tag{2.3}$$

is not controllable if $L \in N$. Indeed, there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $M$, which is unreachable for the linear system when starting from the origin. More precisely, for every non zero state $\psi \in M$, for every $u \in L^2(0,T)$ and for every $y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ satisfying (2.3) and $y(0,\cdot) = 0$, one has $y(T,\cdot) \neq \psi$.

**Remark 2.1.2** *If one is allowed to use more than one boundary control input, there is no critical spatial domain and the exact controllability holds for any $L > 0$. More precisely, let us consider the nonlinear control system*

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = u_1(t), \quad y(t,L) = u_2(t), \quad \partial_x y(t,L) = u_3(t), \end{cases} \tag{2.4}$$

*where the controls are $u_1(t), u_2(t)$ and $u_3(t)$. As it has been pointed out by Rosier in [39], for every $L > 0$ the system (2.4) with $u_1 \equiv 0$ is locally exactly controllable in $L^2(0,L)$ around the origin. Moreover, using all the three control inputs, Zhang proved in [55] that for every $L > 0$ the system (2.4) is exactly controllable in the space $H^s(0,L)$ for any $s \geq 0$ in a neighborhood of a given smooth solution of the KdV equation.*

Recently, Coron and Crépeau in [15] have proved Theorem 2.1.1 for the critical lengths $L = 2k\pi$ with $k \in \mathbb{N}^*$ satisfying

$$\nexists (m,n) \in \mathbb{N}^* \times \mathbb{N}^* \ \text{ with } \ m^2 + mn + n^2 = 3k^2 \ \text{ and } \ m \neq n. \tag{2.5}$$

For these values of $L$, the subspace $M$ of missed directions is one-dimensional and is generated by the function $f(x) = 1 - \cos(x)$. Their method consists, first, in moving along this direction by performing a power series expansion of the solution and then, in using a fixed point theorem.

**Remark 2.1.3** *The condition (2.5) has been communicated to the author by J.-M. Coron and E. Crépeau. They pointed out that if it is not satisfied, then the dimension of the missed directions subspace is higher than one and the proof given in [15] does not work anymore.*

In this chapter, we follow the method of Coron and Crépeau to investigate the case of critical lengths for which the subspace $M$ is two-dimensional. The set of lengths for which it holds is denoted by $N'$. We will see in section 2.2 that $N'$ contains an infinite number of lengths.

This chapter is organized as follows. First, in section 2.2, we study the linearized control system (2.3) and we provide a complete description of the space $M$ in terms of the length $L$ of the spatial domain $(0,L)$. Then, in section 2.3, we prove in the case $L \in N'$, that the nonlinear term $y\partial_x y$ allows us to reach all the missed directions provided that the time of control is large enough. We give an explicit expression of the minimal time required by our method. Finally, in section 2.4, we get the local exact controllability by means of a fixed point theorem, i.e. we prove our main result.

**Theorem 2.1.4** *Let $L \in N'$. There exists $T_M > 0$ such that for any $T > T_M$, there exist $C > 0$ and $r > 0$ such that for every $(y_0, y_T) \in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)} < r$ and $\|y_T\|_{L^2(0,L)} < r$, there exist $u \in L^2(0,T)$ with*

$$\|u\|_{L^2(0,T)} \leq C(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)})^{1/2} \tag{2.6}$$

*and*

$$y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

*satisfying* (2.1), *$y(0,\cdot) = y_0$ and $y(T,\cdot) = y_T$.*

**Remark 2.1.5** *The power $1/2$ in the estimate (2.6) comes, as we will see, from performing a power series expansion of second order to deal with the nonlinearity. The same estimate holds with power $1/3$ for the critical lengths studied in [15] (third order expansion) and with power $1$ for the noncritical lengths studied in [39] (first order expansion).*

**Remark 2.1.6** *In order to complete the study of the exact controllability of system (2.1), it is necessary to investigate the case where the dimension of the space $M$ is bigger than 2. An approach would be to use the exact controllability of the nonlinear equation around nontrivial stationary solutions proved by Crépeau (in [18] for the domains $(0, 2\pi k)$ and in [19] for any other domain $(0, L)$), and then to apply the method introduced in [12] (see also [1, 2]), that is the return method (see [10, 11]) together with quasi-static deformations (see also [16]). With such a method, one should obtain the exact controllability of (2.1) for a large time. However, it seems that the minimal time required with this approach is far from being optimal. In chapter 3 we propose a different approach relying again on power series expansion.*

**Remark 2.1.7** *In Theorem 2.1.4, we get the local controllability for (2.1) provided that the time of control is large enough. However, we may wonder if this condition on the time is really necessary. This is an interesting open problem since one knows that even if the speed of propagation of the Korteweg-de Vries equation is infinite, it may exist a minimal time of control. This is for example the case of a nonlinear control system for the Schrödinger equation studied by Beauchard and Coron in [2]. They proved the local controllability of this system along the ground state trajectory for a time large enough. And more recently, Coron proved in [13] and [14, Theorem 9.8] that this local controllability does not hold in small time, even if the Schrödinger equation has an infinite speed of propagation.*

**Remark 2.1.8** *In [1] and [2], there appear Schrödinger linear control systems which are not controllable. One could try to apply the method used in this chapter to prove the local controllability of the corresponding nonlinear control systems. The main difficulty is that in those cases, the subspace of missed directions for the linear system is not finite-dimensional.*

**Remark 2.1.9** *Concerning the stabilization of the KdV equation, some results in the case of periodic boundary conditions can be found in [30] (damping distributed all along the domain), [46] (damping distributed with localized support) and [45] (boundary damping). In the case of Dirichlet boundary condition, exponential decay of the solution has been obtained in [38] by adding a localized damping term (see also [42] for a generalization of this result). However, the decay rate is unknown. A natural open problem is to design for the control system (2.1) (or the linearized one (2.3)) stabilizing feedback laws which give us explicit decay rate. This kind of results, even with a prescribed arbitrarily large decay rate, has been obtained in [28] and [52] for a general class of second-order (in time) systems including the wave equation and plate-like systems. It uses the fact that these systems are time-reversible. This is not the case of the control system (2.1), but as we shall see in chapter 4, we can slightly modify it in order to be able to apply this approach. Thus, we will built some feedback laws forcing the linear system to have a exponential decay to zero with decay rate as large as desired.*

## 2.2 Linearized control system

We first recall some properties proved by Rosier in [39]. Let $L > 0$ and $T > 0$. In order to study the following linear KdV equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = f, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{2.7}$$

we define the space $\mathcal{B} := C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ endowed with the norm

$$\|y\|_{\mathcal{B}} = \max_{t\in[0,T]} \|y(t)\|_{L^2(0,L)} + \left(\int_0^T \|y(t)\|^2_{H^1(0,L)} dt\right)^{1/2}.$$

Let $A$ denote the operator $Aw = -w' - w'''$ on the domain $D(A) \subset L^2(0,L)$ defined by

$$D(A) := \left\{ w \in H^3(0,L);\ w(0) = w(L) = w'(L) = 0 \right\}.$$

One can see that both $A$ and its adjoint $A^*$ are closed and dissipative. Hence $A$ generates a strongly continuous semigroup of contractions. Using this fact and the multiplier method, Rosier proved the following existence and uniqueness result.

**Proposition 2.2.1** (see [39, Propositions 3.2 and 3.7]) *There exist unique continuous linear maps* $\Psi$ *and* $\delta$

$$\begin{array}{rcl} \Psi : L^2(0,L) \times L^2(0,T) \times L^1(0,T,L^2(0,L)) & \longrightarrow & \mathcal{B}, \\ (y_0,u,f) & \longmapsto & \Psi(y_0,u,f), \\ \delta : L^2(0,L) \times L^2(0,T) \times L^1(0,T,L^2(0,L)) & \longrightarrow & L^2(0,T), \\ (y_0,u,f) & \longmapsto & \delta(y_0,u,f), \end{array}$$

*such that, for* $y_0 \in D(A)$*,* $u \in C^2([0,T])$ *with* $u(0)=0$ *and* $f \in C^1([0,T],L^2(0,L))$*, then* $\Psi(y_0,u,f)$ *is the unique classical solution of* (2.7) *and*

$$\delta(y_0,u,f) = \partial_x \Psi(y_0,u,f)(\cdot,0).$$

The function $\Psi(y_0,u,f)$ is called the mild solution or simply the solution of (2.7) in the context of this thesis.

Now, we focus our attention on the domains of critical length. In particular, we describe the space $M$ of unreachable states for the linear control system (2.3). Let $L \in N$. There exist a finite number of pairs $\{(k_j,l_j)\}_{j=1}^n \subset \mathbb{N}^* \times \mathbb{N}^*$ with $k_j \geq l_j$ such that

$$L = 2\pi \sqrt{\frac{k_j^2 + k_j l_j + l_j^2}{3}}. \tag{2.8}$$

From the work of Rosier in [39], we know that for each $j \in \{1,\ldots,n\}$ there exist two non zero real-valued functions $\varphi_1^j = \varphi_1^j(x)$ and $\varphi_2^j = \varphi_2^j(x)$ such that $\varphi^j := \varphi_1^j + i\varphi_2^j$ is a solution of

$$\begin{cases} -ip(k_j,l_j)\varphi^j + \varphi^{j\prime} + \varphi^{j\prime\prime\prime} = 0, \\ \varphi^j(0) = \varphi^j(L) = 0, \\ \varphi^{j\prime}(0) = \varphi^{j\prime}(L) = 0, \end{cases} \tag{2.9}$$

where, for $(k,l) \in \mathbb{N}^* \times \mathbb{N}^*$, $p(k,l)$ is defined by

$$p(k,l) := \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}.$$

Easy computations lead to

$$\begin{aligned} \varphi_1^j &= C\left(\cos(\gamma_1^j x) - \tfrac{\gamma_1^j-\gamma_3^j}{\gamma_2^j-\gamma_3^j}\cos(\gamma_2^j x) + \tfrac{\gamma_1^j-\gamma_2^j}{\gamma_2^j-\gamma_3^j}\cos(\gamma_3^j x)\right), \\ \varphi_2^j &= C\left(\sin(\gamma_1^j x) - \tfrac{\gamma_1^j-\gamma_3^j}{\gamma_2^j-\gamma_3^j}\sin(\gamma_2^j x) + \tfrac{\gamma_1^j-\gamma_2^j}{\gamma_2^j-\gamma_3^j}\sin(\gamma_3^j x)\right), \end{aligned} \tag{2.10}$$

where $C$ is a constant and the numbers $\gamma_m^j$ with $m = 1, 2, 3$ are the three roots of $x^3 - x + p(k_j, l_j) = 0$. One can easily verify that these roots are given by

$$\gamma_1^j = -\frac{2\pi}{L}\left(\frac{2k_j + l_j}{3}\right), \quad \gamma_2^j = \gamma_1^j + \frac{2\pi k_j}{L}, \quad \gamma_3^j = \gamma_2^j + \frac{2\pi l_j}{L}. \tag{2.11}$$

Moreover, by choosing the constant $C$, we can assume that

$$\|\varphi_1^j\|_{L^2(0,L)} = \|\varphi_2^j\|_{L^2(0,L)} = 1.$$

Roughly speaking, the functions $\varphi_1^j$ and $\varphi_2^j$ for $j = 1, \dots, n$ are unreachable states for the linear KdV control system (2.3) since the following functions,

$$y_1(t,x) = Re(e^{-ip(k_j,l_j)t}\varphi^j(x)) \quad \text{and} \quad y_2(t,x) = Im(e^{-ip(k_j,l_j)t}\varphi^j(x)),$$

are solutions of (2.3) with $u(t) \equiv 0$ but they do not satisfy the next observability inequality leading to the exact controllability

$$\|y(0,x)\|_{L^2(0,L)} \le C\|\partial_x y(t,0)\|_{L^2(0,T)}.$$

Let us define the following subspaces of $L^2(0, L)$

$$M := \langle\{\varphi_1^1, \varphi_2^1, \dots, \varphi_1^n, \varphi_2^n\}\rangle \quad \text{and} \quad H := M^\perp.$$

**Remark 2.2.2** *If $p(k_j, l_j) = 0$ for some $j \in \{1, \dots, n\}$, then $\varphi_1^j = \varphi_2^j = 1 - \cos(x)$. It occurs when $k_j = l_j$, i.e. if $L = 2\pi k_j$. If $k_j$ satisfies the condition (2.5), then the space $M$ is one-dimensional. This is the case treated in [15]. It corresponds for example to the length $L = 2\pi$.*

**Remark 2.2.3** *If $p(k_j, l_j) \neq 0$, it is easy to see that $\varphi_1^j \perp \varphi_2^j$. Moreover, for distinct $j_1, j_2 \in \{1, \dots, n\}$, $\varphi_m^{j1} \perp \varphi_s^{j2}$ for $m, s = 1, 2$. Let us give some examples. The pair* $(2, 1)$ *defines a critical length for which the space $M$ is two-dimensional. The pair* $(11, 8)$ *defines a critical length for which the space $M$ is four-dimensional since the pairs* $(11, 8)$ *and* $(16, 1)$ *define the same critical length.*

At this point, we can state the following controllability result which follows directly from the work of Rosier in [39, Propositions 3.3 and 3.9].

**Theorem 2.2.4** *Let $T > 0$. For every $(y_0, y_T) \in H \times H$, there exist $u \in L^2(0,T)$ and $y \in \mathcal{B}$ satisfying* (2.3)*, $y(0,\cdot) = y_0$ and $y(T,\cdot) = y_T$.*

Now, let us define the set $N'$ by

$$N' := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}; (k,l) \in \mathbb{N}^* \times \mathbb{N}^* \text{ satisfying } k > l, \text{ and (2.13)} \right\}, \quad (2.12)$$

$$\forall m,n \in \mathbb{N}^*\backslash\{k\},\ k^2+kl+l^2 \neq m^2+mn+n^2. \quad (2.13)$$

It is easy to see that $N'$ is the set of critical lengths for which the space of unreachable states is two-dimensional. Indeed, let $L \in N'$, from (2.13) there exists a unique pair $(k_1, l_1) := (k, l)$ satisfying (2.8) and since $k_1 > l_1$, $p(k_1, l_1) > 0$ and therefore the functions $\varphi_1^1, \varphi_2^1$ are orthogonal.

Let us follow the proof of Proposition 8.3 in [14] in order to see that $N'$ contains an infinite number of elements. Let $q \geq 1$ be an integer satisfying

$$\forall m,n \in \mathbb{N}^*\backslash\{q\}, \quad m^2+mn+n^2 \neq 7q^2. \quad (2.14)$$

Let us consider the critical length $L_q$ defined by the pair $(2q, q)$, that is

$$L_q := 2\pi\sqrt{\frac{(2q)^2+2q^2+q^2}{3}} = 2\pi q\sqrt{\frac{7}{3}}.$$

From (2.14), it is easy to see that $L_q \in N'$. One can verify that (2.14) holds for $q = 1, 2, 3$ and therefore $L_1, L_2, L_3 \in N'$. Moreover, the following lemma says that the set $N'$ contains an infinite number of lengths $L_q$.

**Lemma 2.2.5** *There are infinitely many positive integers $q$ satisfying* (2.14)*.*

**Proof.** Let $q > 3$ be a prime integer which does not satisfy (2.14). That is, such that

$$\exists m,n \in \mathbb{N}^*\backslash\{q\}, \quad m^2+mn+n^2 = 7q^2. \quad (2.15)$$

From (2.15) one gets

$$-3mn = (m-n)^2 \pmod{q}, \quad mn = (m+n)^2 \pmod{q}. \quad (2.16)$$

It is easy to see that $m+n \neq 0 \pmod{q}$ and consequently from (2.16) we have

$$-3 = \left((m+n)^{-1}(m-n)\right)^2 \pmod{q}, \quad (2.17)$$

that is $-3$ is a square modulo $q$. Let us introduce the Legendre symbol, where $s$ is a prime and $x \in \mathbb{Z}$ is a integer not divisible by $s$

$$\left(\frac{x}{s}\right) := \begin{cases} 1 & \text{if } x \text{ is a square modulo } s, \\ -1 & \text{if } x \text{ is not a square modulo } s. \end{cases}$$

We have the quadratic reciprocity law due to Gauss for every prime integers $z > 2$, $s > 2$ (see [47, Chapter 3])

$$\left(\frac{s}{z}\right) = \left(\frac{z}{s}\right)(-1)^{\epsilon(z)\epsilon(s)}, \tag{2.18}$$

where

$$\epsilon(z) = \begin{cases} 0 & \text{if } z = 1 \pmod 4, \\ 1 & \text{if } z = -1 \pmod 4. \end{cases}$$

From [47, Chapter 3], we also have that for every $x, y$ coprime to $s$

$$\left(\frac{xy}{s}\right) = \left(\frac{x}{s}\right)\left(\frac{y}{s}\right) \tag{2.19}$$

and for every $s > 2$ prime integer

$$(-1)^{\epsilon(s)} = \left(\frac{-1}{s}\right). \tag{2.20}$$

Using (2.18), (2.20), (2.19), (2.17) with $s = q$, $z = 3$ and since $\epsilon(3) = 1$, one obtains

$$\left(\frac{q}{3}\right) = \left(\frac{3}{q}\right)(-1)^{\epsilon(q)} = \left(\frac{3}{q}\right)\left(\frac{-1}{q}\right) = \left(\frac{-3}{q}\right) = 1,$$

that is $q = 1 \pmod 3$.

Hence, if $q > 3$ is a prime integer such that $q = 2 \pmod 3$, then $q$ satisfies (2.14). As there are two possible non zero congruences modulo 3, the Dirichlet density theorem (see [47, Chapter 4]) says that (2.14) holds on a set of prime integers of density 1/2. In particular, there are infinitely many positive integers $q$ satisfying (2.14). ■

From now on and until the end of this chapter, we consider $L \in N'$. From (2.13), for each $L \in N'$ we can define a unique

$$p := \frac{(2k + l)(k - l)(2l + k)}{3\sqrt{3}(k^2 + kl + l^2)^{3/2}}$$

and the space $M$ is then defined by

$$M := \langle \varphi_1, \varphi_2 \rangle = \{\alpha\varphi_1 + \beta\varphi_2\,;\ \alpha, \beta \in \mathbb{R}\}$$

where $\varphi_1$ and $\varphi_2$ are given by (2.10) with $\gamma_m^j$ replaced by $\gamma_m$, where $\gamma_1, \gamma_2$ and $\gamma_3$ are the three roots of $x^3 - x + p = 0$. From (2.9) we also have that $\varphi_1$ and $\varphi_2$ satisfy

$$\begin{cases} \varphi_1' + \varphi_1''' = -p\,\varphi_2, \\ \varphi_1(0) = \varphi_1(L) = 0, \\ \varphi_1'(0) = \varphi_1'(L) = 0, \end{cases} \tag{2.21}$$

and

$$\begin{cases} \varphi_2' + \varphi_2''' = p\,\varphi_1, \\ \varphi_2(0) = \varphi_2(L) = 0, \\ \varphi_2'(0) = \varphi_2'(L) = 0. \end{cases} \tag{2.22}$$

Now, we investigate the evolution of the projection on the subspace $M$ of a solution of (2.3). Let us consider $(y, u) \in \mathcal{B} \times L^2(0, T)$ satisfying (2.3). Let us multiply (2.21) by $y$ and integrate on $[0, L]$. Using integrations by parts we get

$$\frac{d}{dt}\left(\int_0^L y(t,x)\varphi_1(x)dx\right) \;=\; -p\int_0^L y(t,x)\varphi_2(x)dx. \tag{2.23}$$

Similarly, multiplying (2.22) by $y$, we get

$$\frac{d}{dt}\left(\int_0^L y(t,x)\varphi_2(x)dx\right) \;=\; p\int_0^L y(t,x)\varphi_1(x)dx. \tag{2.24}$$

Hence, from (2.23) and (2.24), we obtain

$$\int_0^L y(t,x)\varphi_1(x)\,dx = \int_0^L y(0,x)(\cos(p\,t)\varphi_1(x) - \sin(p\,t)\varphi_2(x))\,dx, \tag{2.25}$$

$$\int_0^L y(t,x)\varphi_2(x)\,dx = \int_0^L y(0,x)(\sin(p\,t)\varphi_1(x) + \cos(p\,t)\varphi_2(x))\,dx. \tag{2.26}$$

From (2.25) and (2.26), we see that the projection on $M$ of $y(t,\cdot)$, denoted by $P_M(y(t,\cdot))$, only turns in this two-dimensional subspace and therefore conserves its $L^2(0, L)$-norm. The period of this rotation is $2\pi/p$. Furthermore, we see that if the initial condition $y(0,\cdot)$ lies in $H$, the solution too for every time $t$. Combining this rotation with Theorem 2.2.4, we obtain the following proposition.

**Proposition 2.2.6** *Let $y_0, y_1 \in L^2(0, L)$ be such that*

$$\|P_M(y_0)\|_{L^2(0,L)} = \|P_M(y_1)\|_{L^2(0,L)}.$$

*Then, there exists $t^* \leq \frac{2\pi}{p}$ and $u \in L^2(0, t^*)$ such that the solution $y = y(t, x)$ of (2.3) with $y(0,\cdot) = y_0$, satisfies $y(t^*,\cdot) = y_1$.*

**Proof.** Let $y_M = y_M(t,x)$ be the solution of (2.3) with $y_M(0,\cdot) = P_M(y_0)$ and without control ($u \equiv 0$). We know that there exists a time $0 < t^* \leq \frac{2\pi}{p}$ such that $y_M(t^*,\cdot) = P_M(y_1)$. On the other hand, from Theorem 2.2.4 there exists a control $u_H \in L^2(0,t^*)$ such that the corresponding solution $y_H = y_H(t,x)$ of (2.3) satisfies

$$y_H(0,\cdot) = P_H(y_0) \in H \quad \text{and} \quad y_H(t^*,\cdot) = P_H(y_1).$$

Then $y(t,x) := y_H(t,x) + y_M(t,x)$ satisfies (2.3) with $u = u_H$, $y(0,\cdot) = y_0$ and $y(t^*,\cdot) = y_1$, which ends the proof of this proposition. ■

## 2.3 Motion in the missed directions

Let us first explain the general idea of the method. Let $y = y(t,x)$ be a solution of (2.1) with control $u = u(t)$. We consider a power series expansion of $(y,u)$ with the same scaling on the state and on the control

$$\begin{aligned} y &= \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 \ldots \\ u &= \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 \ldots \end{aligned}$$

In this way, we see that the nonlinear term is given by

$$y\partial_x y = \epsilon^2 y_1 \partial_x y_1 + \epsilon^3 y_1 \partial_x y_2 + \epsilon^3 y_2 \partial_x y_1 + \text{(higher terms)}$$

and therefore, for a small $\epsilon$, we have the expansion of second order $y \approx \epsilon y_1 + \epsilon^2 y_2$, where $y_1$ and $y_2$ are given by

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ \partial_x y_1(t,L) = u_1(t), \end{cases}$$

and

$$\begin{cases} \partial_t y_2 + \partial_x y_2 + \partial_x^3 y_2 = -y_1 \partial_x y_1, \\ y_2(t,0) = y_2(t,L) = 0, \\ \partial_x y_2(t,L) = u_2(t). \end{cases}$$

The strategy consists first, in proving that the expansion to the second order of $y = y(t,x)$, i.e. $\epsilon y_1 + \epsilon^2 y_2$, can reach all the missed directions and then, in using a fixed point argument to prove that it is sufficient to get Theorem 2.1.4. This is a classical approach to study the local controllability of a finite-dimensional control system and it has been applied in [15] to prove the local exact controllability around the origin of the control system (2.1) for some critical domains.

Now, we see that we can "enter" into the subspace $M$. More precisely, the result we prove is the following one.

**Proposition 2.3.1** *Let $T > 0$. There exists $(u, v) \in L^2(0, T)^2$ such that if $\alpha = \alpha(t, x)$ and $\beta = \beta(t, x)$ are the solutions of*

$$\begin{cases} \partial_t \alpha + \partial_x \alpha + \partial_x^3 \alpha = 0, \\ \alpha(t, 0) = \alpha(t, L) = 0, \\ \partial_x \alpha(t, L) = u(t), \\ \alpha(0, \cdot) = 0, \end{cases} \tag{2.27}$$

*and*

$$\begin{cases} \partial_t \beta + \partial_x \beta + \partial_x^3 \beta = -\alpha \partial_x \alpha, \\ \beta(t, 0) = \beta(t, L) = 0, \\ \partial_x \beta(t, L) = v(t), \\ \beta(0, \cdot) = 0, \end{cases} \tag{2.28}$$

*then*

$$\alpha(T, \cdot) = 0 \quad \textit{and} \quad \beta(T, \cdot) \in M \backslash \{0\}.$$

**Proof.** In order to study the trajectory $\beta = \beta(t, x)$, we set $\beta = \beta^u + \beta^v$ where $\beta^u = \beta^u(t, x)$ and $\beta^v = \beta^v(t, x)$ are the solutions of

$$\begin{cases} \partial_t \beta^u + \partial_x \beta^u + \partial_x^3 \beta^u = -\alpha \partial_x \alpha, \\ \beta^u(t, 0) = \beta^u(t, L) = 0, \\ \partial_x \beta^u(t, L) = 0, \\ \beta^u(0, \cdot) = 0, \end{cases} \tag{2.29}$$

and

$$\begin{cases} \partial_t \beta^v + \partial_x \beta^v + \partial_x^3 \beta^v = 0, \\ \beta^v(t, 0) = \beta^v(t, L) = 0, \\ \partial_x \beta^v(t, L) = v(t), \\ \beta^v(0, \cdot) = 0. \end{cases} \tag{2.30}$$

If $u \in L^2(0,T)$ is given, by Theorem 2.2.4 one can find $v \in L^2(0,T)$ such that

$$\beta^v(T,\cdot) = -P_H(\beta^u(T,\cdot))$$

and thus $\beta(T,\cdot) = P_M(\beta^u(T,\cdot))$. From this fact, one sees that the proof of Proposition 2.3.1 can be reduced to prove

$$\exists u \in L^2(0,T) \quad \text{such that} \quad \alpha(T,\cdot) = 0 \quad \text{and} \quad P_M(\beta^u(T,\cdot)) \neq 0. \tag{2.31}$$

Let $u \in L^2(0,T)$. Let us multiply (2.29) by $\varphi_1$ and integrate the resulting equality on $[0,L]$. Then, using integration by parts, (2.21), boundary and initial conditions in (2.29), one gets

$$\frac{d}{dt}\left(\int_0^L \beta^u(t,x)\varphi_1(x)dx\right) = -p\int_0^L \beta^u(t,x)\varphi_2(x)dx + \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_1'(x)dx.$$

In a similar way, if we now multiply (2.29) by $\varphi_2$, we get

$$\frac{d}{dt}\left(\int_0^L \beta^u(t,x)\varphi_2(x)dx\right) = p\int_0^L \beta^u(t,x)\varphi_1(x)dx + \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_2'(x)dx.$$

If we call

$$\eta_k(t) := \int_0^L \beta^u(t,x)\varphi_k(x)dx \quad \text{for } k = 1,2,$$

we can write the system

$$\begin{cases} \begin{pmatrix} \dot\eta_1(t) \\ \dot\eta_2(t) \end{pmatrix} = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_1'(x)dx \\ \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_2'(x)dx \end{pmatrix} \\ \eta_1(0) = 0, \quad \eta_2(0) = 0. \end{cases} \tag{2.32}$$

The solution of (2.32) is given by

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} \cos(p\,t) & -\sin(p\,t) \\ \sin(p\,t) & \cos(p\,t) \end{pmatrix} \begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix}$$

where

$$I_1(t) := \frac{1}{2}\int_0^t\int_0^L \alpha^2(s,x)(\cos(ps)\varphi_1'(x) + \sin(ps)\varphi_2'(x))dx\,ds,$$
$$I_2(t) := \frac{1}{2}\int_0^t\int_0^L \alpha^2(s,x)(-\sin(ps)\varphi_1'(x) + \cos(ps)\varphi_2'(x))dx\,ds.$$

If we work with complex numbers calling $\varphi := \varphi_1 + i\varphi_2$, we get

$$\eta_1(t) + i\eta_2(t) = \frac{1}{2}e^{ip\,t}\int_0^t\int_0^L e^{-ips}\alpha^2(s,x)\varphi'(x)dx\,ds.$$

Now, let us assume that (2.31) fails to be true, i.e. let us suppose that

$$\forall\, u \in L^2(0,T), \quad \eta_1(T) = \eta_2(T) = 0 \quad \text{or} \quad \alpha(T,\cdot) \neq 0. \tag{2.33}$$

If we define

$$U_{ad} := \left\{u \in L^2(0,T)\,;\ \text{the solution } \alpha \text{ of (2.27) satisfies } \alpha(T,\cdot) = 0\right\},$$

then condition (2.33) implies that

$$\forall\, u \in U_{ad}, \quad \int_0^T \int_0^L e^{-ips} \alpha^2(s,x) \varphi'(x) dx\, ds = 0. \tag{2.34}$$

Let $\alpha_1 = \alpha_1(t,x)$ and $\alpha_2 = \alpha_2(t,x)$ be two solutions of (2.27) such that

$$\alpha_1(T,\cdot) = \alpha_2(T,\cdot) = 0.$$

Now, for $(\rho_1,\rho_2) \in \mathbb{R}^2$, let $\alpha := \rho_1\alpha_1 + \rho_2\alpha_2$ and $u := \alpha_x(\cdot, L)$. By linearity, we see that $\alpha = \alpha(t,x)$ is a solution of (2.27) and $u \in U_{ad}$. Consequently, (2.34) implies that, for every $(\rho_1,\rho_2) \in \mathbb{R}^2$,

$$\rho_1^2 \int_0^T \int_0^L e^{-ips} \alpha_1^2(s,x) \varphi'(x) dx\, ds + 2\rho_1\rho_2 \int_0^T \int_0^L e^{-ips} \alpha_1(s,x)\alpha_2(s)\varphi'(x) dx\, ds \\ + \rho_2^2 \int_0^T \int_0^L e^{-ips} \alpha_2^2(s,x) \varphi'(x) dx\, ds = 0.$$

Looking at the coefficient of $\rho_1\rho_2$, we get

$$\int_0^T \int_0^L e^{-ips} \alpha_1(s,x)\alpha_2(s,x)\varphi'(x) dx\, ds = 0. \tag{2.35}$$

Let $t_1, t_2$ be such that $0 < t_1 < t_2 < T$. We choose the trajectories $\alpha_1 = \alpha_1(t,x)$ and $\alpha_2 = \alpha_2(t,x)$ such that

$$\alpha_2 \text{ is not identically equal to } 0, \tag{2.36}$$

$$\alpha_2(t,x)|_{([0,t_1]\cup[t_2,T])\times[0,L]} = 0 \quad \text{and} \quad \alpha_1(t,x)|_{[t_1,t_2]\times[0,L]} = Re(e^{\lambda t} y_\lambda(x)), \tag{2.37}$$

where $\lambda \in \mathbb{C}\backslash\{\pm ip\}$ and $y_\lambda = y_\lambda(x)$ is a complex-valued function which satisfies

$$\begin{cases} \lambda y_\lambda + y_\lambda' + y_\lambda''' = 0, \\ y_\lambda(0) = y_\lambda(L) = 0. \end{cases} \tag{2.38}$$

If $\lambda \neq \pm ip$, one can see that $Re(y_\lambda), Im(y_\lambda) \in H$ and then by Theorem 2.2.4 there exists such a trajectory $\alpha_1 = \alpha_1(t,x)$.

Let us introduce the operator $\tilde{A}w = -w' - w'''$ on the domain $D(\tilde{A}) \subset L^2(0,L)$ defined by

$$D(\tilde{A}) := \left\{ w \in H^3(0,L);\, w(0) = w(L) = 0, w'(0) = w'(L) \right\}.$$

It is not difficult to see that $i\tilde{A}$ is a self-adjoint operator on $L^2(0,L)$ with compact resolvent. Hence, the spectrum $\sigma(\tilde{A})$ of $\tilde{A}$ consists only of eigenvalues. Furthermore, the spectrum is a discrete subset of $i\mathbb{R}$.

If we take $\lambda$ such that $(-ip+\lambda) \notin \sigma(\tilde{A})$, the operator $(\tilde{A} - (-ip+\lambda)I)$ is invertible, and thus, there exists a unique complex-valued function $\phi_\lambda = \phi_\lambda(x)$ solution of

$$\begin{cases} (-ip+\lambda)\phi_\lambda + \phi_\lambda' + \phi_\lambda''' = y_\lambda \varphi', \\ \phi_\lambda(0) = \phi_\lambda(L) = 0, \\ \phi_\lambda'(0) = \phi_\lambda'(L). \end{cases} \tag{2.39}$$

We multiply (2.39) by $\alpha_2(t,x)e^{(-ip+\lambda)t}$, integrate on $[0,L]$ and use integrations by parts together with (2.27), boundary and initial conditions in (2.39) to get

$$e^{-ip\,t}\int_0^L e^{\lambda t} y_\lambda \alpha_2(t,x)\varphi'(x)dx = \frac{d}{dt}\left(\int_0^L e^{(-ip+\lambda)t}\phi_\lambda(x)\alpha_2(t,x)dx\right) - e^{(-ip+\lambda)t}\phi_\lambda'(L)\partial_x\alpha_2(t,x)\Big|_{x=0}^{L}.$$

Then, let us integrate this equality on $[0,T]$ and use the fact that $\alpha_2(0,\cdot) = 0$ and $\alpha_2(T,\cdot) = 0$. We obtain

$$\int_0^T\int_0^L e^{-ip\,t}e^{\lambda t}y_\lambda\alpha_2(t,x)\varphi'(x)dx\,dt = -\phi_\lambda'(L)\int_0^T e^{(-ip+\lambda)t}\left(\partial_x\alpha_2(t,L) - \partial_x\alpha_2(t,0)\right)dt. \tag{2.40}$$

On the other hand, by (2.35) and (2.37), it follows that

$$\int_0^T\int_0^L e^{-ip\,t}Re(e^{\lambda t}y_\lambda)\alpha_2(t,x)\varphi'(x)dx\,dt = 0, \tag{2.41}$$

and, since one can also take a trajectory $\tilde{\alpha}_1 = \tilde{\alpha}_1(t,x)$ such that

$$\tilde{\alpha}_1(t,x)|_{[t_1,t_2]\times[0,L]} = Im(e^{\lambda t}y_\lambda(x)),$$

one deduces from (2.35) that

$$\int_0^T \int_0^L e^{-ipt} Im(e^{\lambda t} y_\lambda)\alpha_2(t,x)\varphi'(x)dx\,dt = 0. \tag{2.42}$$

Therefore, from (2.41) and (2.42), one gets

$$\int_0^T \int_0^L e^{-ipt} e^{\lambda t} y_\lambda \alpha_2(t,x)\varphi'(x)dx\,dt = 0$$

and consequently from (2.40), for every $\lambda \neq \pm ip$ such that $(-ip+\lambda) \notin \sigma(\tilde{A})$, one has

$$\phi'_\lambda(L) \int_0^T e^{(-ip+\lambda)t} \left(\partial_x \alpha_2(t,L) - \partial_x \alpha_2(t,0)\right) dt = 0. \tag{2.43}$$

Let $a \in \mathbb{R}\backslash[-1/\sqrt{3}, 1/\sqrt{3}]$. We take $\lambda = 2ai(4a^2-1)$. Let

$$y_\lambda(x) = Ce^{(-\sqrt{3a^2-1}-ai)x} + (1-C)e^{(\sqrt{3a^2-1}-ai)x} - e^{2aix}, \tag{2.44}$$

where

$$C = \frac{e^{2aiL} - e^{(\sqrt{3a^2-1}-ai)L}}{e^{(-\sqrt{3a^2-1}-ai)L} - e^{(\sqrt{3a^2-1}-ai)L}}.$$

One easily checks that such a $y_\lambda = y_\lambda(x)$ satisfies (2.38) and $y_\lambda \neq 0$. Let us define

$$\Sigma := \left\{ a \in \mathbb{R}\backslash[-1/\sqrt{3}, 1/\sqrt{3}]\,;\, \lambda \notin \sigma(\tilde{A}), (\lambda - ip) \notin \sigma(\tilde{A}) \right\},$$

where $\lambda = 2ai(4a^2-1)$. Then the function $S : \Sigma \to \mathbb{C}$, $S(a) = \phi'_\lambda(L)$ is continuous. Now we use the fact that $S$ is not identically equal to the function 0 (the proof of this statement will be given in Lemma 2.3.6 at the end of this section). Then, there exist $\widehat{a} \in \Sigma$ and $\epsilon > 0$ such that for every $a \in \Sigma$ with $|a - \widehat{a}| < \epsilon$, $S(a) \neq 0$. From (2.43) one gets

$$\forall\, a \in \Sigma, \quad |a - \widehat{a}| < \epsilon, \quad \int_0^T e^{(-p+2a(4a^2-1))i\,t} \left(\partial_x \alpha_2(t,L) - \partial_x \alpha_2(t,0)\right) dt = 0$$

and since the function $\beta \in \mathbb{C} \mapsto \int_0^T e^{\beta t} \left(\partial_x \alpha_2(t,L) - \partial_x \alpha_2(t,0)\right) dt \in \mathbb{C}$ is holomorphic, it follows that

$$\forall\, \beta \in \mathbb{C}, \quad \int_0^T e^{\beta t}\big(\partial_x \alpha_2(t,L) - \partial_x \alpha_2(t,0)\big)dt = 0,$$

which implies that $\partial_x \alpha_2(t,0) - \partial_x \alpha_2(t,L) = 0$ for every $t$. In summary, one has that $\alpha_2 = \alpha_2(t,x)$ satisfies

$$\begin{cases} \partial_t \alpha_2 + \partial_x \alpha_2 + \partial_x^3 \alpha_2 = 0, \\ \alpha_2(t,0) = \alpha_2(t,L) = 0, \\ \partial_x \alpha_2(t,0) = \partial_x \alpha_2(t,L), \\ \alpha_2(0,\cdot) = 0, \\ \alpha_2(T,\cdot) = 0. \end{cases} \tag{2.45}$$

If we multiply (2.45) by $\alpha_2$, integrate on $[0, L]$ and use integration by parts together with the boundary conditions, we obtain that

$$\frac{d}{dt}\int_0^L |\alpha_2(t,x)|^2 dx = 0,$$

which, together with $\alpha_2(0,\cdot) = 0$, implies that

$$\alpha_2(t,x) = 0 \quad \forall\, x \in [0,L],\, \forall\, t \in [0,T]. \tag{2.46}$$

But this is in contradiction with (2.36). Thus, we have proved (2.31) and therefore Proposition 2.3.1. ■

From now on, for each $T_c > 0$, we denote by $(u_c, v_c) \in L^2(0,T)^2$ the controls given by Proposition 2.3.1 and by $(\alpha_c, \beta_c)$ the corresponding trajectories. Let us define $\tilde{\varphi}_1 := \beta_c(T_c, \cdot)$. Let us notice that by scaling the controls, we can assume that $\|\tilde{\varphi}_1\|_{L^2(0,L)} = 1$. We will prove now that in any time $T > \pi/p$, we can reach all the states lying in $M$.

**Proposition 2.3.2** *Let $T > \pi/p$. Let $\psi \in M$. There exists $(u,v) \in L^2(0,T)^2$ such that if $\alpha = \alpha(t,x)$ and $\beta = \beta(t,x)$ are the solutions of* (2.27) *and* (2.28)*, then*

$$\alpha(T,\cdot) = 0 \quad \textit{and} \quad \beta(T,\cdot) = \psi.$$

**Proof.** Let $\hat{T} > 0$ be such that $T = (\pi/p) + \hat{T}$. Let $T_c$ be such that $0 < T_c < \hat{T}$. Let $T_a := T - T_c$. If we take in (2.27) and (2.28) the controls

$$(u^1, v^1)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a), \\ (u_c(t - T_a), v_c(t - T_a)) & \text{if } t \in (T_a, T), \end{cases}$$

we obtain that $\beta^1(T,\cdot) = \tilde{\varphi}_1$, where $\beta^1 = \beta^1(t,x)$ is the corresponding solution of (2.28). Now, we use the rotation showed in section 2.2 (see in particular (2.25) and (2.26)) in order to reach other states lying in $M$. Let us define $\tilde{\varphi}_2 := \beta^2(T,\cdot)$, where $\beta^2 = \beta^2(t,x)$ is defined by the controls

$$(u^2, v^2)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - \frac{\pi}{2p}), \\ (u_c(t - T_a + \frac{\pi}{2p}), v_c(t - T_a + \frac{\pi}{2p})) & \text{if } t \in (T_a - \frac{\pi}{2p}, T - \frac{\pi}{2p}), \\ (0,0) & \text{if } t \in (T - \frac{\pi}{2p}, T). \end{cases}$$

In a similar way, the controls

$$(u^3, v^3)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - \frac{\pi}{p}), \\ (u_c(t - T_a + \frac{\pi}{p}), v_c(t - T_a + \frac{\pi}{p})) & \text{if } t \in (T_a - \frac{\pi}{p}, T - \frac{\pi}{p}), \\ (0,0) & \text{if } t \in (T - \frac{\pi}{p}, T), \end{cases}$$

allow us to define $\tilde{\varphi}_3 := \beta^3(T, \cdot)$. Notice that $\tilde{\varphi}_3 = -\tilde{\varphi}_1$.

Let $T_\theta$ be such that $0 < T_\theta < \min\{\pi/(2p), \hat{T} - T_c\}$ and let $T_b := (\pi/p) + T_\theta$. Let us define $\tilde{\varphi}_4 := \beta^4(T, \cdot)$, where $\beta^4 = \beta^4(t, x)$ is the solution of (2.28) with

$$(u^4, v^4)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - T_b), \\ (u_c(t - T_a + T_b), v_c(t - T_a + T_b)) & \text{if } t \in (T_a - T_b, T - T_b), \\ (0,0) & \text{if } t \in (T - T_b, T). \end{cases}$$

We have thus proved that we can reach the missed directions $\{\tilde{\varphi}_k\}_{k=1}^4$. Let us now define the cones

$$\begin{aligned} M_1 &:= \{d_1\tilde{\varphi}_1 + d_2\tilde{\varphi}_2; d_1 > 0, d_2 \geq 0\}, \\ M_2 &:= \{d_1\tilde{\varphi}_2 + d_2\tilde{\varphi}_3; d_1 > 0, d_2 \geq 0\}, \\ M_3 &:= \{d_1\tilde{\varphi}_3 + d_2\tilde{\varphi}_4; d_1 > 0, d_2 \geq 0\}, \\ M_4 &:= \{d_1\tilde{\varphi}_4 + d_2\tilde{\varphi}_1; d_1 > 0, d_2 \geq 0\}. \end{aligned}$$

By construction of these cones, one has that $M = \cup_{k=1}^4 M_k$.

**Remark 2.3.3** *It is easy to see that if one chooses $T_c$, $T_\theta$ such that $T_c < T_\theta$, then the supports of the trajectories $\alpha^k = \alpha^k(t, x)$ for $k = 1, \ldots, 4$ are disjoint.*

For each $w = (w_1, w_2) \in \mathbb{R}^2$, let us define

$$\rho_w := \sqrt{w_1^2 + w_2^2} \quad \text{and} \quad z_w := (w_1\varphi_1 + w_2\varphi_2)/\rho_w \in M.$$

We have that $z_w \in M_i$ for some $i \in \{1, \ldots, 4\}$ and hence there exist $d_{1w} > 0$ and $d_{2w} \geq 0$ such that $z_w = d_{1w}\tilde{\varphi}_i + d_{2w}\tilde{\varphi}_{i+1}$. If we take the control

$$(u_w, v_w) = (d_{1w}^{1/2} u^i + d_{2w}^{1/2} u^{i+1}, d_{1w} v^i + d_{2w} v^{i+1})$$

and use the fact that the trajectories $\alpha^k$ for $k = 1, \ldots, 4$ are disjoints, then we see that the corresponding solution $\beta_w = \beta_w(t, x)$ of (2.28) satisfies $\beta_w(T, \cdot) = z_w$.

Finally, let $\psi \in M$. With $R := \|\psi\|_{L^2(0,L)}$ we can write $\psi = Rz_w$ for a $(w_1, w_2) \in \mathbb{R}^2$ such that $w_1^2 + w_2^2 = 1$. It is easy to see that the control $(u, v) = (R^{1/2}u_w, Rv_w)$ allows us to reach the state $\psi$ and so the proof of this proposition is ended. ■

**Remark 2.3.4** *The proof of Proposition 2.3.2 is the only part which needs a time large enough. Hence, Theorem 2.1.4 holds for $T_M := \pi/p$.*

**Remark 2.3.5** *In [15] an expansion to the second order is not sufficient and the authors must go to the third order to "enter" into the subspace of missed directions. Since in their case this subspace is one-dimensional and since they use an odd order expansion, one can reach all the missed states with a scaling argument. Our case is different. We can also "enter" into the subspace of missed directions in any time, but in order to reach all these states, our method needs a time large enough.*

It remains to prove the following lemma to complete the proof of Proposition 2.3.1.

**Lemma 2.3.6** *The function $S$ is not identically equal to 0.*

**Proof.** Let $a \in \Sigma$ and $\lambda = 2ai(4a^2 - 1)$. Let $\mu \in \mathbb{C}$ and let $y_\mu = y_\mu(x)$ be a solution of

$$\begin{cases} \mu y_\mu + y'_\mu + y'''_\mu = 0, \\ y_\mu(0) = y_\mu(L) = 0. \end{cases}$$

We multiply (2.39) by $y_\mu$ and integrate by parts on $[0, L]$. Thus, we get

$$(\lambda - ip + \mu) \int_0^L \phi_\lambda y_\mu dx - \phi'_\lambda(L)\big(y'_\mu(L) - y'_\mu(0)\big) = \int_0^L y_\lambda \varphi' y_\mu dx. \tag{2.47}$$

From now on, we set $\mu = \mu(a) := -\lambda + ip$. With this choice we obtain from (2.47)

$$-S(a)\big(y'_\mu(L) - y'_\mu(0)\big) = \int_0^L y_\lambda \varphi' y_\mu dx.$$

Therefore, if we prove that the function

$$a \in \Sigma \longrightarrow J(a) := \int_0^L y_\lambda \varphi' y_\mu dx \in \mathbb{C},$$

is not identically equal to 0, the proof of this lemma is ended. Let $b \in \mathbb{R}$ be such that $\mu = 2bi(4b^2 - 1)$. We take the function $y_\mu$ given by

$$y_\mu(x) = De^{(-\sqrt{3b^2-1}-bi)x} + (1-D)e^{(\sqrt{3b^2-1}-bi)x} - e^{2bix}, \tag{2.48}$$

where

$$D = \frac{e^{2biL} - e^{(\sqrt{3b^2-1}-bi)L}}{e^{(-\sqrt{3b^2-1}-bi)L} - e^{(\sqrt{3b^2-1}-bi)L}}.$$

In the next computations, we use the fact that $e^{i\gamma_1 L} = e^{i\gamma_2 L} = e^{i\gamma_3 L}$ (see (2.11)) and the following formula

$$\int_0^L e^{(v+iw)x} \varphi' = \frac{\big(1 + \gamma_1^2 - 2p/\gamma_1\big)\big(1 - e^{(v+iw+i\gamma_1)L}\big)(vi - w)}{(vi - w)^3 - (vi - w) + p} \tag{2.49}$$

which holds if $v + iw \neq -i\gamma_m$ for $m = 1, 2, 3$.

We want to show that as $a \to \infty$, the following expression diverges, which is in contradiction with the fact that $J(a) \equiv 0$

$$R(a) := \frac{(e^{(-\sqrt{3a^2-1}-ai)L} - e^{(\sqrt{3a^2-1}-ai)L})(e^{(-\sqrt{3b^2-1}-bi)L} - e^{(\sqrt{3b^2-1}-bi)L})}{1+\gamma_1^2 - 2p/\gamma_1} J(a).$$

In fact, by using (2.49), one computes explicitly $J(a)$ and thus one sees that as $a$ tends to infinity, the dominant term of $R(a)$ is given by

$$\begin{aligned}
Z(a) := e^{(\sqrt{3a^2-1}+\sqrt{3b^2-1})L} \Big\{ & \frac{(e^{(-ai-bi)L} - e^{(ai+bi+\gamma_1 i)L})(-2a-2b)}{(-2a-2b)^3 - (-2a-2b) + p} \\
& + \frac{e^{(-ai-bi)L}(-i\sqrt{3a^2-1} - i\sqrt{3b^2-1} + a + b)}{(-i\sqrt{3a^2-1} - i\sqrt{3b^2-1} + a + b)^3 - (-i\sqrt{3a^2-1} - i\sqrt{3b^2-1} + a + b) + p} \\
& - \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3a^2-1} + i\sqrt{3b^2-1} + a + b)}{(i\sqrt{3a^2-1} + i\sqrt{3b^2-1} + a + b)^3 - (i\sqrt{3a^2-1} + i\sqrt{3b^2-1} + a + b) + p} \\
& + \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3a^2-1} + a - 2b)}{(i\sqrt{3a^2-1} + a - 2b)^3 - (i\sqrt{3a^2-1} + a - 2b) + p} \\
& - \frac{e^{(-ai-bi)L}(-i\sqrt{3b^2-1} - 2a + b)}{(-i\sqrt{3b^2-1} - 2a + b)^3 - (-i\sqrt{3b^2-1} - 2a + b) + p} \\
& + \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3b^2-1} - 2a + b)}{(i\sqrt{3b^2-1} - 2a + b)^3 - (i\sqrt{3b^2-1} - 2a + b) + p} \\
& - \frac{e^{(-ai-bi)L}(-i\sqrt{3a^2-1} + a - 2b)}{(-i\sqrt{3a^2-1} + a - 2b)^3 - (-i\sqrt{3a^2-1} + a - 2b) + p} \Big\}
\end{aligned}$$

Using that as $a \to \infty$, $b \to -\infty$ and $a + b \sim -p/(24a^2)$, we obtain the following asymptotical expression for the right hand factor of $Z(a)$,

$$\frac{-(e^{\frac{p}{24a^2}iL} - e^{-\frac{p}{24a^2}iL + i\gamma_1 L})}{12a^2} \sim \begin{cases} -\frac{(1-e^{i\gamma_1 L})}{12a^2} & \text{if } e^{i\gamma_1 L} \neq 1, \\ -\frac{ipL}{144a^4} & \text{if } e^{i\gamma_1 L} = 1. \end{cases}$$

One can see that in both cases $Z(a)$ diverges as $a \to \infty$ and therefore $R(a)$ does, which implies that $J(a)$ is not identically equal to 0. It ends the proof of this lemma. ■

## 2.4 Proof of Theorem 2.1.4

### 2.4.1 Existence and uniqueness results

Let us recall the existence property proved by Coron and Crépeau in [15] for the following nonlinear KdV equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = f, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \\ y(0,\cdot) = y_0. \end{cases} \tag{2.50}$$

**Proposition 2.4.1** (see [15, Proposition 14]) *Let $L > 0$ and $T > 0$. There exist $\epsilon > 0$ and $C > 0$ such that, for every $f \in L^1(0,T,L^2(0,L))$, $u \in L^2(0,T)$ and $y_0 \in L^2(0,L)$ such that*

$$\|f\|_{L^1(0,T,L^2(0,L))} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)} \leq \epsilon,$$

*there exists at least one solution of* (2.50) *which satisfies*

$$\|y\|_{\mathcal{B}} \leq C(\|f\|_{L^1(0,T,L^2(0,L))} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}). \tag{2.51}$$

And for the uniqueness, one has

**Proposition 2.4.2** (see [15, Proposition 15]) *Let $T > 0$ and let $L > 0$. There exists $C > 0$ such that for every $(y_{01}, y_{02}) \in L^2(0,L)^2$, $(u_1,u_2) \in L^2(0,T)^2$ and $(f_1,f_2) \in L^1(0,T,L^2(0,L))^2$ for which there exist solutions $y_1 = y_1(t,x)$ and $y_2 = y_2(t,x)$ of* (2.50), *one has the following estimates :*

$$\int_0^T \int_0^L |\partial_x y_1(t,x) - \partial_x y_2(t,x)|^2 dxdt \leq e^{C(1+\|y_1\|^2_{L^2(0,T,H^1(0,L))}+\|y_2\|^2_{L^2(0,T,H^1(0,L))})} \cdot \left(\|u_1-u_2\|^2_{L^2(0,T)} + \|f_1-f_2\|^2_{L^1(0,T,L^2(0,L))} + \|y_{01}-y_{02}\|^2_{L^2(0,L)}\right),$$

$$\int_0^L |y_1(t,x) - y_2(t,x)|^2 dx \leq e^{C(1+\|y_1\|^2_{L^2(0,T,H^1(0,L))}+\|y_2\|^2_{L^2(0,T,H^1(0,L))})} \cdot \left(\|u_1-u_2\|^2_{L^2(0,T)} + \|f_1-f_2\|^2_{L^1(0,T,L^2(0,L))} + \|y_{01}-y_{02}\|^2_{L^2(0,L)}\right),$$

*for every $t \in [0,T]$.*

### 2.4.2 Settings and a technical lemma

Until the end of this chapter, we adopt some important notations. Let us denote, for $D > 0$ and $R > 0$,

$$B_R^D := \Big\{ \xi \in L^2(0,D)\,;\ \|\xi\|_{L^2(0,D)} \leq R \Big\},$$

and recall that for each $w = (w_1, w_2) \in \mathbb{R}^2$, we write $\rho_w := \sqrt{w_1^2 + w_2^2}$ and $z_w := (w_1\varphi_1 + w_2\varphi_2)/\rho_w$. We also write $(u_w, v_w) \in L^2(0,T)$ the controls defined in section 2.3 in order to drive the solutions $\beta_w = \beta_w(t,x)$ from zero at $t = 0$ to $z_w$ at $t = T$.

By the work of Rosier in [39], we know that for each $y_0 \in L^2(0,L)$ there exists a continuous linear affine map (it is a consequence of applying the HUM method to prove Theorem 2.2.4)

$$\Gamma_0 : h \in H \subset L^2(0,L) \longmapsto \Gamma_0(h) \in L^2(0,T),$$

such that the solution of

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = \Gamma_0(h), \\ y(0,\cdot) = P_H(y_0), \end{cases}$$

satisfies $y(T,\cdot) = h$. Moreover, there exist constants $D_1, D_2 > 0$ such that

$$\forall y_0 \in L^2(0,L),\ \forall h \in H \quad \|\Gamma_0(h)\|_{L^2(0,T)} \leq D_1(\|h\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}), \tag{2.52}$$

$$\forall y_0 \in L^2(0,L),\ \forall h, g \in H \quad \|\Gamma_0(h) - \Gamma_0(g)\|_{L^2(0,T)} \leq D_2\|h - g\|_{L^2(0,L)}. \tag{2.53}$$

Let $y_0 \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} < r$, $r > 0$ to be chosen later. Let us define the functions $G$ and $F$

$$\begin{aligned} G : L^2(0,L) &\longrightarrow\ L^2(0,T), \\ z = P_H(z) + w_1\varphi_1 + w_2\varphi_2 &\mapsto\ G(z) = \Gamma_0(P_H(z)) + \rho_w^{1/2} u_w + \rho_w v_w, \end{aligned}$$

$$\begin{aligned} F : B_{\epsilon_1}^T \cap L^2(0,T) &\longrightarrow\ L^2(0,L), \\ u &\longmapsto\ F(u) = y(T,\cdot), \end{aligned}$$

where $y = y(t,x)$ is the solution of

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{2.54}$$

and $\epsilon_1$ is small enough so that the function $F$ is well defined. It holds if $\epsilon_1 + r < \epsilon$ where $\epsilon$ is given by Proposition 2.4.1. The map $F$ is even continuous according to Proposition 2.4.2. Let $y_T \in L^2(0,L)$ be such that $\|y_T\| < r$. Let $\Lambda_{y_0,y_T}$ denote the map

$$\begin{aligned} \Lambda_{y_0,y_T} : B^L_{\epsilon_2} \cap L^2(0,L) &\longrightarrow \quad L^2(0,L), \\ z &\longmapsto \quad \Lambda_{y_0,y_T}(z) = z + y_T - F \circ G(z), \end{aligned}$$

where $\epsilon_2$ is small enough so that $\Lambda_{y_0,y_T}$ is well defined ($\epsilon_2$ exists according to Proposition 2.4.1 and to the continuity of $G$).

Let us notice that if we find a fixed point $\tilde{z} \in L^2(0,L)$ of the map $\Lambda_{y_0,y_T}$, then we will have $F \circ G(\tilde{z}) = y_T$ and this means that the control $u := G(\tilde{z}) \in L^2(0,T)$ drives the solution of (2.54) from $y_0$ at $t = 0$ to $y_T$ at $t = T$.

Let us assert the following technical result which will be needed to study the map $\Lambda_{y_0,y_T}$.

**Lemma 2.4.3** *There exist $\epsilon_3 > 0$ and $C_3 > 0$ such that, for every $z, y_0 \in B^L_{\epsilon_3}$, the following estimate holds.*

$$\|z - F(G(z))\|_{L^2(0,L)} \leq C_3(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}).$$

**Proof.** Let $z, y_0 \in L^2(0,L)$. Let $w = (w_1, w_2) \in \mathbb{R}^2$ be such that $z = P_H(z) + \rho_w z_w$. Let $y = y(t,x)$ be a solution of

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = G(z), \\ y(0,\cdot) = y_0. \end{cases} \tag{2.55}$$

From (2.52) and since $\rho_w \leq \|z\|_{L^2(0,L)}$, one deduces that if $\|z\|_{L^2(0,L)}$ is small enough, then there exists a constant $D_3$ such that

$$\|G(z)\|_{L^2(0,T)} \leq D_3(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}). \tag{2.56}$$

By using (2.51) and (2.56), one can find $\epsilon_2, C_2 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_2}$ the unique solution of (2.55) satisfies

$$\|y\|_{\mathcal{B}} \leq C_2(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}). \tag{2.57}$$

Let $\tilde{y} = \tilde{y}(t,x)$, $\alpha_w = \alpha_w(t,x)$, $\beta_w = \beta_w(t,x)$ and $\beta^0 = \beta^0(t,x)$ be the solutions of

$$\begin{cases} \partial_t \tilde{y} + \partial_x \tilde{y} + \partial_x^3 \tilde{y} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \partial_x \tilde{y}(t,L) = \Gamma_0(P_H(z)), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \partial_t \alpha_w + \partial_x \alpha_w + \partial_x^3 \alpha_w = 0, \\ \alpha_w(t,0) = \alpha_w(t,L) = 0, \\ \partial_x \alpha_w(t,L) = u_w(t), \\ \alpha_w(0,\cdot) = 0, \end{cases}$$

$$\begin{cases} \partial_t \beta_w + \partial_x \beta_w + \partial_x^3 \beta_w = -\alpha_w \partial_x \alpha_w, \\ \beta_w(t,0) = \beta_w(t,L) = 0, \\ \partial_x \beta_w(t,L) = v_w(t), \\ \beta_w(0,\cdot) = 0, \end{cases}$$

$$\begin{cases} \partial_t \beta^0 + \partial_x \beta^0 + \partial_x^3 \beta^0 = 0, \\ \beta^0(t,0) = \beta^0(t,L) = 0, \\ \partial_x \beta^0(t,L) = 0, \\ \beta^0(0,\cdot) = P_M(y_0). \end{cases}$$

Let us define

$$\phi := y - \tilde{y} - \rho_w^{1/2} \alpha_w - \rho_w \beta_w - \beta^0.$$

We have that $\phi = \phi(t,x)$ satisfies

$$\begin{cases} \partial_t \phi + \partial_x \phi + \partial_x^3 \phi + \phi \partial_x \phi = -\partial_x(\phi a) - b, \\ \phi(t,0) = \phi(t,L) = 0, \\ \partial_x \phi(t,L) = 0, \\ \phi(0) = 0, \end{cases}$$

where $$a := \tilde{y} + \rho_w^{1/2} \alpha_w + \rho_w \beta_w + \beta^0,$$

$$\begin{aligned} b := \; & \tilde{y} \partial_x \tilde{y} + \partial_x(\tilde{y}(\rho_w^{1/2} \alpha_w + \rho_w \beta_w + \beta^0)) + \rho_w^{3/2} \partial_x(\alpha_w \beta_w) \\ & + \rho_w^2 \beta_w \partial_x(\beta_w) + \rho_w^{1/2} \partial_x(\alpha_w \beta^0) + \rho_w \partial_x(\beta_w \beta^0) + \beta^0 \partial_x \beta^0. \end{aligned}$$

It is easy to see that there exists $C_4 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_2}$

$$\|\phi\|_{\mathcal{B}} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}), \tag{2.58}$$

$$\|a\|_{\mathcal{B}} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}), \tag{2.59}$$

$$\|b\|_{L^1(0,T,L^2(0,L))} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}). \tag{2.60}$$

One can also prove that there exists $C_5 > 0$ such that for every $f, g \in \mathcal{B}$

$$\|\partial_x(fg)\|_{L^1(0,T,L^2(0,L))} \leq C_5\|f\|_{\mathcal{B}}\|g\|_{\mathcal{B}}. \tag{2.61}$$

By Proposition 2.4.2, (2.60) and (2.61), there exists $C_6 > 0$ such that

$$\|\phi\|^2_{\mathcal{B}} \leq C_6(\|\phi\|^2_{\mathcal{B}}\|a\|^2_{\mathcal{B}} + \|y_0\|^2_{L^2(0,L)} + \|z\|^3_{L^2(0,L)}),$$

which, together with (2.58) and (2.59), implies the existence of $\epsilon_3$ and $C_7$ such that for every $z, y_0 \in B^L_{\epsilon_3}$

$$\|\phi\|_{\mathcal{B}} \leq C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}). \tag{2.62}$$

Finally, from (2.62) one obtains with $C_3 := C_7 + 1$

$$\begin{aligned}
\|z - F \circ G(z)\|_{L^2(0,L)} &\leq \|z - F \circ G(z) - \beta^0(T)\|_{L^2(0,L)} + \|\beta^0(T)\|_{L^2(0,L)} \\
&= \|\phi(T)\|_{L^2(0,L)} + \|\beta^0(T)\|_{L^2(0,L)} \\
&\leq C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}) + \|y_0\|_{L^2(0,L)} \\
&\leq C_3(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}),
\end{aligned}$$

which ends the proof of Lemma 2.4.3. ■

### 2.4.3 Fixed point in the subspace $H$

For $w = (w_1, w_2) \in \mathbb{R}^2$ fixed, let us study the map $\Pi := P_H \circ \Lambda_{y_0,y_T}(\cdot + \rho_w z_w)$ on the subspace $H$ (recall that $\rho_w z_w = w_1\varphi_1 + w_2\varphi_2$).

$$\begin{aligned}
\Pi : H &\longrightarrow H, \\
h &\longmapsto \Pi(h) = h + P_H(y_T) - P_H(F \circ G(h + \rho_w z_w)).
\end{aligned}$$

In order to prove the existence of a fixed point of the map $\Pi$, we will apply the Banach fixed point theorem to the restriction of $\Pi$ to the closed ball $B^L_R \cap H$ with $R > 0$ small enough. By using Lemma 2.4.3 we see that

$$\begin{aligned}
\|\Pi(h)\|_{L^2(0,L)} &\leq \|y_T\|_{L^2(0,L)} + \|h + \rho_w z_w - F \circ G(h + \rho_w z_w)\|_{L^2(0,L)} \\
&\leq \|y_T\|_{L^2(0,L)} + C_3(\|y_0\|_{L^2(0,L)} + \|h + \rho_w z_w\|^{3/2}_{L^2(0,L)}) \\
&\leq C_3'(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + \rho_w^{3/2}) + C_3\|h\|^{3/2}_{L^2(0,L)} \\
&\leq C_3'(2r + \rho_w^{3/2}) + C_3\|h\|^{3/2}_{L^2(0,L)},
\end{aligned}$$

where $C_3' := C_3 + 1$. Hence, if we choose $R$ such that $R^{3/2} \leq \frac{R}{2C_3}$ and $r, \rho_w$ such that

$$C_3'(2r + \rho_w^{3/2}) \leq \frac{R}{2},$$

then it follows that

$$\|\Pi(h)\|_{L^2(0,L)} \leq R \quad \text{and so} \quad \Pi(B_R^L \cap H) \subset (B_R^L \cap H).$$

It remains to prove that the map $\Pi$ is a contraction. Let $g, h \in B_R^L \cap H$. Let $y = y(t,x)$, $q = q(t,x)$, $\tilde{y} = \tilde{y}(t,x)$ and $\tilde{q} = \tilde{q}(t,x)$ be the solutions of the following problems

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = G(g + \rho_w z_w), \\ y(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} \partial_t q + \partial_x q + \partial_x^3 q + q\partial_x q = 0, \\ q(t,0) = q(t,L) = 0, \\ \partial_x q(t,L) = G(h + \rho_w z_w), \\ q(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} \partial_t \tilde{y} + \partial_x \tilde{y} + \partial_x^3 \tilde{y} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \partial_x \tilde{y}(t,L) = \Gamma_0(g), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \partial_t \tilde{q} + \partial_x \tilde{q} + \partial_x^3 \tilde{q} = 0, \\ \tilde{q}(t,0) = \tilde{q}(t,L) = 0, \\ \partial_x \tilde{q}(t,L) = \Gamma_0(h), \\ \tilde{q}(0,\cdot) = P_H(y_0). \end{cases}$$

Let us define $\phi := y - \tilde{y}$, $\psi := q - \tilde{q}$ and $\gamma := \phi - \psi$. One sees that $\gamma$ satisfies

$$\begin{cases} \partial_t \gamma + \partial_x \gamma + \partial_x^3 \gamma + \gamma\partial_x \gamma = -\partial_x(\gamma a) - b, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \partial_x \gamma(t,L) = 0, \\ \gamma(0) = 0, \end{cases} \tag{2.63}$$

where

$$a := \tilde{y} + \psi \quad \text{and} \quad b := \partial_x \left(q(\tilde{y} - \tilde{q})\right) + (\tilde{y} - \tilde{q})\partial_x(\tilde{y} - \tilde{q}).$$

It is easy to see that there exists a constant $C_8 > 0$ such that

$$\|b\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\tilde{y} - \tilde{q}\|_{\mathcal{B}}, \tag{2.64}$$

$$\|\partial_x(a\gamma)\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\gamma\|_{\mathcal{B}}. \tag{2.65}$$

By using Propositions 2.4.2, (2.64) and (2.65) we get the existence of $C_9 > 0$ such that

$$\|\gamma\|_{\mathcal{B}}^2 \leq C_9(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}})^2(\|\tilde{y} - \tilde{q}\|_{\mathcal{B}}^2 + \|\gamma\|_{\mathcal{B}}^2). \tag{2.66}$$

In addition, since $w := \tilde{y} - \tilde{q}$ satisfies the following linear equation

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w = 0, \\ w(t,0) = w(t,L) = 0, \\ \partial_x w(t,L) = \Gamma_0(g) - \Gamma_0(h), \\ w(0,\cdot) = 0, \end{cases}$$

there exists $C_{10} > 0$ such that

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10}\|\Gamma_0(g) - \Gamma_0(h)\|_{L^2(0,T)}$$

and so, from (2.53), one gets

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10}D_2\|g - h\|_{L^2(0,L)}. \tag{2.67}$$

Moreover, it is easy to see that there exists a constant $C_{11} > 0$ such that

$$\|q\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} \leq C_{11}(\|y_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|g\|_{L^2(0,L)} + \rho_w^{1/2}). \tag{2.68}$$

Thus, using (2.66), (2.67) and (2.68) we see that if $R, \rho_w, r$ are small enough, it follows that

$$\|\gamma\|_{\mathcal{B}} \leq \frac{1}{2}\|g - h\|_{L^2(0,L)}.$$

Therefore, we have

$$\begin{aligned} \|\Pi(g) - \Pi(h)\|_{L^2(0,L)} &\leq \|g - F \circ G(g + \rho_w z_w) - h + F \circ G(h + \rho_w z_w)\|_{L^2(0,L)} \\ &= \|\gamma(T)\|_{L^2(0,L)} \leq \|\gamma\|_{\mathcal{B}} \\ &\leq \tfrac{1}{2}\|g - h\|_{L^2(0,L)}, \end{aligned}$$

which implies the existence of a unique fixed point $h(y_0, y_T, w_1, w_2) \in B_R^L \cap H$ of the map $\Pi|_{B_R^L \cap H}$. Moreover, follows the more precise proposition.

**Proposition 2.4.4** *There exist $R_0 > 0$, $D > 1$, such that for every $0 < R < R_0$, for every $y_0, y_T \in B_{R/D}^L$, $(w_1, w_2) \in \mathbb{R}^2$ with $\rho_w < R/D$, there exists a unique $h(y_0, y_T, w_1, w_2) \in B_R^L \cap H$ fixed point of the map $\Pi|_{B_R^L \cap H}$.*

### 2.4.4 Fixed point in the subspace $M$

We now apply the Brouwer fixed point theorem to the restriction of the map

$$\begin{aligned} \tau : M &\longrightarrow M, \\ w_1\varphi_1 + w_2\varphi_2 &\rightarrow P_M(\rho_w z_w + y_T - F \circ G(\rho_w z_w + h(y_0, y_T, w_1, w_2))), \end{aligned}$$

to the closed ball $B^L_{\hat{R}} \cap M$ with $\hat{R}$ small enough. Using Lemma 2.4.3, the continuity (in a neighborhood of $0 \in (L^2(0,L))^2 \times \mathbb{R}^2$) of the map $(y_0, y_T, w_1, w_2) \longmapsto h(y_0, y_T, w_1, w_2)$ and choosing $r$ small enough, we get the existence of a radius $\hat{R} > 0$ such that $\tau(B^L_{\hat{R}} \cap M) \subset B^L_{\hat{R}} \cap M$. This inclusion and the continuity of the map $\tau$ allow us to apply the Brouwer fixed point theorem. Therefore, there exists $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{R}^2$ with $\tilde{w}_1^2 + \tilde{w}_2^2 \leq \hat{R}^2$ such that $\tilde{h} := h(y_0, y_T, \tilde{w}_1, \tilde{w}_2)$ satisfies

$$P_M(y_T - F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = 0. \tag{2.69}$$

Using the fact that

$$\Pi(\tilde{h}) = P_H(\tilde{h} + y_T - F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = \tilde{h},$$

we obtain

$$P_H(y_T - F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = 0,$$

which together with (2.69), implies that

$$y_T = F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2),$$

which ends the proof of Theorem 2.1.4. Let us remark that from our proof it follows that if $r$ is chosen small enough, one can take $\hat{R} := rD$ where $D > 0$ is given by Proposition 2.4.4. By using this proposition one obtains the estimate (2.6).

# Chapitre 3

# Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain

This chapter is contained in [6] which has been written in collaboration with Emmanuelle Crépeau.

## Sommaire

## 3.1 Introduction and main result

Let $L > 0$ be fixed. Let us consider the following Neumann boundary control system for the Korteweg-de Vries (KdV) equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = \kappa(t), \end{cases} \tag{3.1}$$

where the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $\kappa(t) \in \mathbb{R}$. This equation has been introduced by Korteweg and de Vries in [31] to describe the propagation of small amplitude long waves in a uniform channel. The KdV equation also appears in the study of various physical phenomena like long internal waves in a density-stratified ocean, ionic-acoustic waves in a plasma, etc.

In this chapter, we are concerned with the controllability of (3.1). More precisely, for a time $T > 0$, we want to prove the following local exact controllability property.

$\mathcal{P}(T)$ *There exists* $r > 0$ *such that, for every* $(y_0, y_T) \in L^2(0,L)^2$ *with* $\|y_0\|_{L^2(0,L)} < r$ *and* $\|y_T\|_{L^2(0,L)} < r$, *there exist* $\kappa \in L^2(0,T)$ *and*

$$y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

*satisfying* (3.1), $y(0,\cdot) = y_0$ *and* $y(T,\cdot) = y_T$.

In order to deal with the nonlinear term in (3.1), one can perform a power series expansion of $(y,\kappa)$ around 0. To find the different terms of this development, one can write, formally, for a parameter $\epsilon$ small,

$$\begin{aligned} y &= \epsilon\alpha + \epsilon^2\beta + \epsilon^3\gamma + \dots \\ \kappa &= \epsilon u + \epsilon^2 v + \epsilon^3 w + \dots \end{aligned}$$

thus, the nonlinear term can be written as

$$yy_x = \epsilon^2\alpha\alpha_x + \epsilon^3(\alpha\beta)_x + (\text{higher terms})$$

and therefore the three main orders are given by

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} = 0, \\ \alpha(t,0) = \alpha(t,L) = 0, \\ \alpha_x(t,L) = u(t), \end{cases} \tag{3.2}$$

$$\begin{cases} \beta_t + \beta_x + \beta_{xxx} = -\alpha\alpha_x, \\ \beta(t,0) = \beta(t,L) = 0, \\ \beta_x(t,L) = v(t), \end{cases} \tag{3.3}$$

and

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} = -(\alpha\beta)_x, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = w(t). \end{cases} \tag{3.4}$$

In [39] Rosier studies the control system (3.1) by using a first order expansion, i.e. he considers the linear control system (3.2) where the state is $\alpha(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. First, by using multiplier technique and the HUM method (see [34]), he proves that (3.2) is exactly controllable if and only if

$$L \notin N := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k,l \in \mathbb{N}^* \right\}, \tag{3.5}$$

and then, by means of a fixed point theorem, he gets the following result.

**Theorem 3.1.1** (see [39, Theorem 1.3]) *If $L \notin N$, then property $\mathcal{P}(T)$ holds for every $T > 0$.*

**Remark 3.1.2** *If one is allowed to use more than one boundary control input, there is no critical spatial domain and the exact controllability holds for any $L > 0$. More precisely, let us consider the nonlinear control system*

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = u_1(t), \quad y(t,L) = u_2(t), \quad y_x(t,L) = u_3(t), \end{cases} \tag{3.6}$$

*where the controls are $u_1(t), u_2(t)$ and $u_3(t)$. As it has been pointed out by Rosier in [39], for every $L > 0$ the system (3.6) with $u_1 \equiv 0$ is locally exactly controllable in $L^2(0,L)$ around the origin. Moreover, using all the three control inputs, Zhang proves in [55] that for every $L > 0$, the system (3.6) is exactly controllable in the space $H^s(0,L)$ for any $s \geq 0$, in a neighborhood of a given smooth solution of the KdV equation.*

If $L \in N$, one says that $L$ is a critical length since the linear control system (3.2) is no more controllable. Indeed, Rosier proves in [39] that there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $M = M(L)$, which is unreachable from 0 for the linear system. More precisely, for every non zero state $\psi \in M$, for every $u \in L^2(0,T)$ and for every $\alpha \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ satisfying (3.2) and $\alpha(0,\cdot) = 0$, one has $\alpha(T,\cdot) \neq \psi$.

Let us recall the description of $M$ given in chapter 2. Let $L \in N$. There exist $n$ distinct pairs $(k_j, l_j) \in \mathbb{N}^* \times \mathbb{N}^*$ with $k_j \geq l_j$ such that

$$\forall j \in \{1,\dots,n\}, \quad L = 2\pi\sqrt{\frac{k_j^2 + k_j l_j + l_j^2}{3}}. \tag{3.7}$$

$$\left(L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}},\ k \geq l,\ k,l \in \mathbb{N}^*\right) \implies (\exists j \in \{1,\dots,n\} \text{ s.t. } k = k_j \text{ and } l = l_j) \tag{3.8}$$

Let us introduce the notation

$$J^> := \{\, j \in \{1,\dots,n\}\,;\, k_j > l_j\}, \quad J^= := \{\, j \in \{1,\dots,n\}\,;\, k_j = l_j\}, \quad n^> := |J^>|. \tag{3.9}$$

For every $j \in \{1,\dots,n\}$, we define the real number

$$p_j := (2k_j + l_j)(k_j - l_j)(2l_j + k_j)\left(\frac{2\pi}{3L}\right)^3. \tag{3.10}$$

We have then, (see [39]),

$$\xi^3 - \xi + p_j = (\xi - \gamma_1^j)(\xi - \gamma_2^j)(\xi - \gamma_3^j)$$

with,

$$\begin{cases} \gamma_1^j = -\frac{1}{3}(2k_j + l_j)\frac{2\pi}{L}, \\ \gamma_2^j = \gamma_1^j + k_j\frac{2\pi}{L}, \\ \gamma_3^j = \gamma_2^j + l_j\frac{2\pi}{L} \end{cases} \tag{3.11}$$

**Lemma 3.1.3** *With the previous notations, we get*

1. *if* $j \in J^>$, $p_j \neq 0$,
2. *if* $j \in J^=$, $p_j = 0$,
3. *if* $i \neq j$, $p_i \neq p_j$.

**Proof.** Items 1. and 2. are obvious with (3.10). Let $i, j \in J$ such that $p_i = p_j$. Then, $\gamma_k^i = \gamma_k^j$ for $k = 1, 2, 3$. With the definitions of $\gamma_k^j$, (3.11) we obtain $k_i = k_j$, $l_i = l_j$ and hence $i = j$. ■

**Remark 3.1.4** *We can easily notice that* $|J^{=}| \leq 1$.

Thus we can reorganize the indexes such that

$$p_1 > p_2 > \cdots > p_n \geq 0.$$

With this notation, we define,

– for $j \in J^{>}$, the subspace of $L^2(0, L)$

$$M_j := \{\lambda_1 \varphi_1^j + \lambda_2 \varphi_2^j;\ \lambda_1, \lambda_2 \in \mathbb{R}\} = \langle \varphi_1^j, \varphi_2^j \rangle,$$

where the real-valued functions $\varphi_1^j, \varphi_2^j$ are given by

$$\begin{aligned} \varphi_1^j(x) &:= C_j \left( \cos(\gamma_1^j x) - \tfrac{\gamma_1^j - \gamma_3^j}{\gamma_2^j - \gamma_3^j} \cos(\gamma_2^j x) + \tfrac{\gamma_1^j - \gamma_2^j}{\gamma_2^j - \gamma_3^j} \cos(\gamma_3^j x) \right), \\ \varphi_2^j(x) &:= C_j \left( \sin(\gamma_1^j x) - \tfrac{\gamma_1^j - \gamma_3^j}{\gamma_2^j - \gamma_3^j} \sin(\gamma_2^j x) + \tfrac{\gamma_1^j - \gamma_2^j}{\gamma_2^j - \gamma_3^j} \sin(\gamma_3^j x) \right), \end{aligned} \tag{3.12}$$

where $C_j$ is a constant chosen so that $\|\varphi_1^j\|_{L^2(0,L)} = \|\varphi_2^j\|_{L^2(0,L)} = 1$

– for $j \in J^{=}$, the subspace of $L^2(0, L)$

$$M_j := \{\lambda(1 - \cos x);\ \lambda \in \mathbb{R}\} = \langle 1 - \cos(x) \rangle.$$

Then, one can define the following subspaces of $L^2(0, L)$

$$M := \bigoplus_{j=1}^{n} M_j \quad \text{and} \quad H := M^{\perp}.$$

Note that

$$\bigcup_{j=1}^{n^{>}} \{\varphi_1^j, \varphi_2^j\} \text{ (if } L \neq 2\pi k \text{ for any } k) \quad \text{or} \quad \{1 - \cos(x)\} \bigcup_{j=1}^{n^{>}} \{\varphi_1^j, \varphi_2^j\} \text{ (if } L = 2\pi k \text{ for some } k)$$

is an orthogonal basis from $M$.

The subspace $H$ is the space of reachable states for the linear control system. More precisely, from the work of Rosier one has the exact controllability in $H$ for the control system (3.2).

**Theorem 3.1.5** *Let $L > 0$ and $T > 0$. For every $(y_0, y_T) \in H \times H$, there exist $u \in L^2(0, T)$ and $\alpha \in C([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$ satisfying* (3.2), *$\alpha(0, \cdot) = y_0$ and $\alpha(T, \cdot) = y_T$.*

In [15], Coron and Crépeau study the first critical case :

$$n = 1 \quad \text{and} \quad k_1 = l_1 = k.$$

In this case the subspace M is one-dimensional. First, they prove that one can reach all the missed directions lying in $M$, i.e. $(1 - \cos(x))$ and $(\cos(x) - 1)$, with a third order power series expansion.

**Proposition 3.1.6** (see [15, Proposition 8]) *Let $L \in N$ be such that* $\dim M(L) = 1$. *Let $T > 0$. There exist $(u^\pm, v^\pm, w^\pm) \in L^2(0,T)^3$ such that if $\alpha^\pm, \beta^\pm, \gamma^\pm$ are the solutions of (3.2), (3.3) and (3.4) with initial conditions*

$$\alpha^\pm(0,\cdot) = 0, \quad \beta^\pm(0,\cdot) = 0, \quad \gamma^\pm(0,\cdot) = 0,$$

*then*

$$\alpha^\pm(T,\cdot) = 0, \quad \beta^\pm(T,\cdot) = 0, \quad \gamma^\pm(T,\cdot) = \pm(1 - \cos(x)).$$

Then, using Theorem 3.1.5 and a fixed point theorem, they prove that property $\mathcal{P}(T)$ holds for every $T > 0$ (see [15, Theorem 2]). They also prove that for this first critical case, a second order expansion is not sufficient to enter into the subspace $M$ (see [15, Corollary 19]).

**Remark 3.1.7** *The proof of $\mathcal{P}(T)$ given in [15] requires that the subspace $M$ is one-dimensional, but this is not implied by the the fact that $L = 2k\pi$ for some $k \in \mathbb{N}^*$. It is necessary to add a condition as the following one*

$$\left(m^2 + mn + n^2 = 3k^2, m \in \mathbb{N}^*, n \in \mathbb{N}^*\right) \Rightarrow (m = n = k)\,. \tag{3.13}$$

*This condition, not explicitly given in [15], appears in [14]. In this book is also proved that there are infinitely many positive integers $k$ satisfying (3.13) and therefore there are infinitely many lengths $L$ such that $M$ is one-dimensional.*

In chapter 2, the same approach is used to treat the second critical case :

$$n = 1 \quad \text{and} \quad k_1 > l_1.$$

In this case, the space $M$ is two-dimensional and a second order expansion allows to enter into the subspace $M$.

**Proposition 3.1.8** (see Proposition 2.3.1 in chapter 2) *Let* $L \in N$ *be such that* $\dim M(L) = 2$*. Let* $T > 0$*. There exist* $u, v \in L^2(0,T)$ *such that if* $\alpha, \beta$ *are the solutions of (3.2) and (3.3) with initial conditions*

$$\alpha(0, \cdot) = 0, \quad \beta(0, \cdot) = 0,$$

*then*

$$\alpha(T, \cdot) = 0, \quad \beta(T, \cdot) \in M \setminus \{0\}.$$

It is also proved that if the time of control is large enough, one can reach all the missed directions. Using this and a fixed point argument, one obtains property $\mathcal{P}(T)$ provided that the time of control $T$ is large enough (see Theorem 2.1.4 in chapter 2).

The aim of this chapter is to prove $\mathcal{P}(T)$ in the critical cases for which $n > 1$, i.e. when the dimension of the subspace $M$ is higher than 2. We use an expansion to the second order if $L \neq 2\pi k$ for any $k \in \mathbb{N}^*$ and an expansion to the third order if $L = 2\pi k$ for some $k \in \mathbb{N}^*$. Our main result is the following.

**Theorem 3.1.9** *Let* $L \in N$*. Then, there exists* $T_L \geq 0$ *such that* $\mathcal{P}(T)$ *holds provided that* $T > T_L$*.*

This chapter is organized as follows. First, in section 3.2, we recall the well-posedness results for both linear and nonlinear KdV control systems. Next, in section 3.3, we prove by using a second order power series expansion, that one can reach all the missed states in the subspaces $M_j$ for $j \in J^>$. Then, in section 3.4, we prove that if $L = 2\pi k$, one can reach the missed states $\pm(1 - \cos(x))$ with a third order expansion and finally, in section 3.5 we get Theorem 3.1.9 by using a fixed point argument.

**Remark 3.1.10** *From our proof of Theorem 3.1.9, it follows that there exists a constant* $C > 0$ *such that for every* $y_0, y_T \in L^2(0,L)$ *small enough, the solution* $y$ *and the control* $\kappa$ *given by property* $\mathcal{P}(T)$ *satisfy*

$$\|y\|_{C([0,T],L^2(0,L))} + \|y\|_{L^2(0,T,H^1(0,L))} + \|\kappa\|_{L^2(0,T)} \leq C \left(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)}\right)^{1/3}$$

*if* $L = 2k\pi$ *for some* $k \in \mathbb{N}^*$ *and*

$$\|y\|_{C([0,T],L^2(0,L))} + \|y\|_{L^2(0,T,H^1(0,L))} + \|\kappa\|_{L^2(0,T)} \leq C \left(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)}\right)^{1/2}$$

*if* $L \neq 2k\pi$ *for any* $k \in \mathbb{N}^*$*. The power* $1/3$ *and* $1/2$ *come from the order of the series expansion needed in each case.*

**Remark 3.1.11** *One can find other results on the controllability of KdV control systems in [22, 40, 41, 46, 55] and the references therein.*

**Remark 3.1.12** *The power series expansion method is a classical tool to study finite-dimensional control systems. It has been used for the first time in infinite dimension in [15]; see also [2] for a Schrödinger equation. This method and others such as quasi-static deformations (see [1, 16, 17] and [14, Chapter 7]) and the return method (see [1, 10, 11, 12] and [14, Chapter 6]) are very useful to deal with nonlinear systems and to get properties which are not a consequence of the linearized system behavior.*

## 3.2 Well-posedness results

The aim of this section is to precise what we mean by "a solution" of the KdV equations appearing in this thesis and to recall the existence and uniqueness results we will use.

Let us introduce the space $\mathcal{B} := C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ endowed with the norm

$$\|y\|_{\mathcal{B}} := \max_{t\in[0,T]} \|y(t)\|_{L^2(0,L)} + \left(\int_0^T \|y(t)\|^2_{H^1(0,L)} dt\right)^{1/2}.$$

Let us begin with the linear case.

**Definition 3.2.1** *Let $T > 0$, $f \in L^1(0,T,L^2(0,L))$, $y_0 \in L^2(0,L)$ and $\kappa \in L^2(0,T)$ be given. A solution of the Cauchy problem*

$$\begin{cases} y_t + y_x + y_{xxx} = f, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = \kappa(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{3.14}$$

*is a function $y \in \mathcal{B}$ such that, for every $\tau \in [0,T]$ and for every $\phi \in C^3([0,\tau]\times[0,L])$ satisfying*

$$\phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0,\ \forall t \in [0,\tau],$$

*one has*

$$-\int_0^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y dx dt - \int_0^\tau \kappa(t)\phi_x(t,L)dt + \int_0^L y(\tau,x)\phi(\tau,x)dx - \int_0^L y_0(x)\phi(0,x)dx = \int_0^\tau \int_0^L f\phi dx dt.$$

With this definition and from the work of Rosier in [39], we have the following result.

**Theorem 3.2.2** *Let $T > 0$, $f \in L^1(0, T, L^2(0, L))$, $y_0 \in L^2(0, L)$ and $\kappa \in L^2(0, T)$. Then, there exists one and only one solution of the Cauchy problem (3.14).*

Let us now give the definition of a solution for the nonlinear equation.

**Definition 3.2.3** *Let $T > 0$, $g \in L^1(0, T, L^2(0, L))$, $y_0 \in L^2(0, L)$ and $\kappa \in L^2(0, T)$ be given. A solution of the Cauchy problem*

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = g, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = \kappa(t), \\ y(0, \cdot) = y_0, \end{cases} \tag{3.15}$$

*is a function $y \in \mathcal{B}$ satisfying (3.14) with $f = g - yy_x$.*

**Remark 3.2.4** *Note that if $y \in \mathcal{B}$, then $yy_x \in L^1(0, T, L^2(0, L))$ and therefore $(g - yy_x)$ as well.*

**Theorem 3.2.5** [15, Appendix] *Let $T > 0$. Then there exists $\epsilon > 0$ such that, for every $g \in L^1(0, T, L^2(0, L))$, $y_0 \in L^2(0, L)$ and $\kappa \in L^2(0, T)$ satisfying*

$$\|g\|_{L^1(0,T,L^2(0,L))} + \|y_0\|_{L^2(0,L)} + \|\kappa\|_{L^2(0,T)} \leq \epsilon, \tag{3.16}$$

*the Cauchy problem (3.15) has one and only one solution. Furthermore, there exists a constant $C > 0$ such that this solution satisfies*

$$\|y\|_{\mathcal{B}} \leq C(\|g\|_{L^1(0,T,L^2(0,L))} + \|y_0\|_{L^2(0,L)} + \|\kappa\|_{L^2(0,T)}). \tag{3.17}$$

**Remark 3.2.6** *In [3] and [23], one can find some well-posedness results in the case where there are nonhomogeneous Dirichlet boundary conditions.*

**Remark 3.2.7** *Recently, in [8] the author proved Theorem 3.2.5 with $\epsilon = \infty$, that is, without a smallness condition on the data.*

## 3.3 Motion in the missed subspaces $M_j$, for $j \in J^>$

Here and in the sequel, we denote by $L$ a critical length such that $\dim M(L) > 2$ and by $P_A$ the orthogonal projection on a subspace $A$ in $L^2(0, L)$. We also adopt the notations introduced in section 3.1.

The first point is that for any $j \in J^>$, we can *enter* into the two-dimensional subspace $M_j$. The strategy is the same as in [15] and in chapter 2. We consider a power series expansion of $(y, \kappa)$ with the same scaling on the state $y$ and on the control $\kappa$. One has the following result that can be proved in the same way as Proposition 2.3.1 in chapter 2.

**Proposition 3.3.1** *Let $T > 0$. For every $i = 1, \ldots, n^>$, there exists $(u_i, v_i) \in L^2(0,T)^2$ such that if $\alpha_i = \alpha_i(t,x)$ and $\beta_i = \beta_i(t,x)$ are the solutions of*

$$\begin{cases} \alpha_{it} + \alpha_{ix} + \alpha_{ixxx} = 0, \\ \alpha_i(t,0) = \alpha_i(t,L) = 0, \\ \alpha_{ix}(t,L) = u_i(t), \\ \alpha_i(0,.) = 0, \end{cases} \tag{3.18}$$

*and*

$$\begin{cases} \beta_{it} + \beta_{ix} + \beta_{ixxx} = -\alpha_i\alpha_{ix}, \\ \beta_i(t,0) = \beta_i(t,L) = 0, \\ \beta_{ix}(t,L) = v_i(t), \\ \beta_i(0,.) = 0, \end{cases} \tag{3.19}$$

*then*

$$\alpha_i(T,.) = 0,\ P_H(\beta_i(T,.)) = 0 \text{ and } P_{M_i}(\beta_i(T,.)) \neq 0.$$

Let us denote, for $j = 1, \ldots, n^>$,

$$\phi_i^j := P_{M_j}(\beta_i(T,.)).$$

From Proposition 3.3.1, $\phi_i^i \neq 0$. Consequently, using scaling on the controls, we can assume that $\|\phi_i^i\|_{L^2(0,L)} = 1$. Notice that the previous proposition says nothing about $\phi_i^j$ for $j \neq i$.

Now, we shall prove that we can reach all the states lying in the subspace

$$M^> := \bigoplus_{i \in J^>} M_i,$$

in any time $T > T^>$, where

$$T^> := \pi \sum_{i=1}^{n^>} (n^> + 1 - i)\frac{1}{p_i}.$$

In order to do that, we will strongly use the fact (proved in chapter 2) that if there is no control (i.e. $\kappa = 0$) and if the initial condition lies in $M_j$ for $j \in J^>$ (i.e. $y_0 \in M_j$), then the solution $y$ of the linear KdV equation only turns in the two-dimensional subspace $M_j$ with an angular velocity equal to $p_j$ (defined in (3.10)) and conserves its $L^2$-norm. More precisely, we have the following result.

**Lemma 3.3.2** *Let $j \in J^>$. Let $y_0 \in M_j$. Let $\lambda \geq 0$ and $\delta \in [0, 2\pi)$ be such that*

$$y_0 = \lambda \cos(\delta)\varphi_1^j + \lambda \sin(\delta)\varphi_2^j. \tag{3.20}$$

*Then the solution of*

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,\cdot) = y_0 \end{cases} \tag{3.21}$$

*is given by*

$$y(t,x) = \lambda \cos(p_j t + \delta)\varphi_1^j + \lambda \sin(p_j t + \delta)\varphi_2^j. \tag{3.22}$$

For the sake of brevity we introduce, for $j \in J^>$, $\theta \in \mathbb{R}$ and $y_0 \in M_j$ reading as (3.20), the notation

$$R^j(y_0, \theta) := \lambda \cos(\theta + \delta)\varphi_1^j + \lambda \sin(\theta + \delta)\varphi_2^j, \tag{3.23}$$

i.e. $R^j(\cdot, \theta)$ represents a rotation of an angle $\theta$ in the subspace $M_j$. Thus, the solution of (3.21) can be written as

$$y(t,x) = R^j(y_0, p_j t).$$

**Proposition 3.3.3** *Let $T > T^>$. Let $\psi \in M^>$. There exists $(u_\psi, v_\psi) \in L^2(0,T)^2$ such that if $\alpha_\psi = \alpha_\psi(t,x)$ and $\beta_\psi = \beta_\psi(t,x)$ are the solutions of (3.18) and (3.19), then*

$$\alpha_\psi(T,.) = 0,\ \beta_\psi(T,.) = \psi.$$

**Proof.** First at all, let us notice that if $L = 2k\pi$ for some $k \in \mathbb{N}^*$, then $M_n = \langle 1 - \cos x \rangle$ and *a priori* $P_{M_n}(\beta_\psi(T,\cdot))$ may be non-null. However, we know from [15, Corollary 19] that a second order expansion is not sufficient to enter into the subspace $M_n$ and therefore $P_{M_n}\beta_\psi(T.\cdot) = 0$. That is the reason for which we do not care about the projection on $M_n$ of second-order trajectories.

The case $n^{>} = 1$ has already been studied in chapter 2. Let us consider the case $n^{>} = 2$, i.e. where we have 2 subspaces, $M_1$ and $M_2$ associated to $(k_1, l_1)$ and $(k_2, l_2)$ with $p_1 > p_2 > 0$ (for instance, $L = 2\pi\sqrt{91}$ leads to the couples $(k_1, l_1) = (16, 1)$ and $(k_2, l_2) = (11, 8)$).

Let $T > \dfrac{2\pi}{p_1} + \dfrac{\pi}{p_2}$. Let $T_1$ be such that

$$T_1 > \frac{\pi}{p_1} \quad \text{and} \quad T - T_1 > \frac{\pi}{p_1} + \frac{\pi}{p_2}.$$

Let $T_\theta > 0$ and $T_c > 0$ be such that

$$T_c < T_\theta, \quad T_c < \frac{\pi}{p_1},$$

$$T_c + T_\theta < \min\left(T - T_1 - \frac{\pi}{p_1} - \frac{\pi}{p_2}, \frac{\pi}{p_2} - \frac{\pi}{p_1}, T_1 - \frac{\pi}{p_1}\right).$$

Thanks to Proposition 3.3.1, there exist two pairs of controls, $(u_1, v_1)$ and $(u_2, v_2)$ in $L^2(0, T_c)$ such that the respective solutions of (3.18) and (3.19), $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$, satisfy $P_{M_1}(\beta_1(T_c, \cdot)) \neq 0$ and $P_{M_2}(\beta_2(T_c, \cdot)) \neq 0$. With the notations introduced before,

$$\begin{cases} (\phi_1^1, \phi_1^2) = (P_{M_1}(\beta_1(T, \cdot)), P_{M_2}(\beta_1(T, \cdot))), \\ (\phi_2^1, \phi_2^2) = (P_{M_1}(\beta_2(T, \cdot)), P_{M_2}(\beta_2(T, \cdot))). \end{cases}$$

We now use the rotation phenomena explained before and Proposition 3.3.1 to reach a basis for the missed directions lying in $M^{>}$. For the seek of clarity in our control strategy, we define for a time $t_1$,the following control in $L^2(0, T)$.

$$(U_{t_1}, V_{t_1})(t) := \begin{cases} (0, 0) & \text{if } t \in (0, t_1), \\ (u_1(t - t_1), v_1(t - t_1)) & \text{if } t \in (t_1, t_1 + T_c), \\ (0, 0) & \text{if } t \in (t_1 + T_c, T). \end{cases}$$

This control becomes active at time $t = t_1$, between $t = t_1$ and $t = t_2$, it drives the system to enter into the space $M_1$ and after $t = t_2$, it becomes inactive, producing a rotation in $M_1$.

Now, we define the controls

$$\begin{aligned} (u_1^1, v_1^1) &:= (U_{t_1}, V_{t_1}) && \text{with } t_1 = T - T_c, \\ (u_1^2, v_1^2) &:= (U_{t_1}, V_{t_1}) && \text{with } t_1 = T - T_c - \tfrac{\pi}{2p_1}, \\ (u_1^3, v_1^3) &:= (U_{t_1}, V_{t_1}) && \text{with } t_1 = T - T_c - \tfrac{\pi}{p_1}, \\ (u_1^4, v_1^4) &:= (U_{t_1}, V_{t_1}) && \text{with } t_1 = T - T_c - \tfrac{\pi}{p_1} - T_\theta. \end{aligned}$$

Let $\alpha_1^j, \beta_1^j \in B$ be the solutions of (3.18) and (3.19) with controls $u_1^j$ and $v_1^j$ for $j = 1, \ldots, 4$ and let us denote

$$\psi_1^j := P_{M_1}\beta_1^j(T,\cdot) \quad \text{and} \quad \tilde{\psi}_2^j := P_{M_2}\beta_1^j(T,\cdot)$$

It is easy to see that

$$\begin{array}{ll} \psi_1^1 = \phi_1^1, & \tilde{\psi}_2^1 = \phi_1^2 \\ \psi_1^2 = R^1(\phi_1^1, \frac{\pi}{2}), & \tilde{\psi}_2^2 = R^2(\phi_1^2, \frac{p_2\pi}{2p_1}) \\ \psi_1^3 = R^1(\phi_1^1, \pi) = -\phi_1^1, & \tilde{\psi}_2^3 = R^2(\phi_1^2, \frac{p_2\pi}{p_1}) \\ \psi_1^4 = R^1(-\phi_1^1, p_1 T_\theta), & \tilde{\psi}_2^4 = R^2(\phi_1^2, p_2(T_\theta + \frac{\pi}{p_1})) \end{array}$$

Thus, we have constructed some controls allowing to reach the missed states

$$\psi_1^1 + \tilde{\psi}_2^1, \quad \psi_1^2 + \tilde{\psi}_2^2, \quad \psi_1^3 + \tilde{\psi}_2^3, \quad \text{and} \quad \psi_1^4 + \tilde{\psi}_2^4.$$

Now, we define for a time $t_2$, the following control in $L^2(0,T)$

$$(U^{t_2}, V^{t_2})(t) := \begin{cases} (0,0) & \text{if } t \in (0, t_2), \\ (u_1(t-t_2), v_1(t-t_2)) & \text{if } t \in (t_2, t_2 + T_c), \\ (0,0) & \text{if } t \in (t_2 + T_c, t_2 + \frac{\pi}{p_1}), \\ (u_1(t - t_2 - \frac{\pi}{p_1}), v_1(t - t_2 - \frac{\pi}{p_1})) & \text{if } t \in (t_2 + \frac{\pi}{p_1}, t_2 + \frac{\pi}{p_1} + T_c), \\ (0,0) & \text{if } t \in (t_2 + \frac{\pi}{p_1} + T_c, T), \end{cases}$$

which is the superposition of two controls of type $(U_{t_1}, V_{t_1})$

$$(U^{t_2}, V^{t_2})(t) = (U_{t_2 + \frac{\pi}{p_1}}, V_{t_2 + \frac{\pi}{p_1}}) + (U_{t_2}, V_{t_2})$$

This fact means that the solution corresponding to the controls $(U^{t_2}, V^{t_2})$ is the addition of two trajectories which enter into $M$ and then turn during different times.

We define the following controls in $L^2(0,T)$.

$$\begin{array}{ll} (u_1^1, v_1^1) = (U^{t_2}, V^{t_2}) & \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - T_c, \\ (u_2^1, v_2^1) = (U^{t_2}, V^{t_2}) & \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - T_c - T_\theta, \\ (u_3^1, v_3^1) = (U^{t_2}, V^{t_2}) & \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - \frac{\pi}{p_2} - T_c, \\ (u_4^1, v_4^1) = (U^{t_2}, V^{t_2}) & \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - \frac{\pi}{p_2} - T_c - T_\theta. \end{array}$$

Let $\alpha_2^j, \beta_2^j \in B$ be the solutions of (3.18) and (3.19) with controls $u_2^j$ and $v_2^j$ for $j = 1, \ldots, 4$ and let us denote

$$\psi_2^j := P_{M_2}\beta_2^j(T,\cdot)$$

Here, it is very important to note that, by construction and since $p_1 > p_2$, one has

$$P_{M_1}\beta_2^1(T,\cdot) = 0 \quad \text{and} \quad \psi_2^1 = R^2(\phi_1^2, p_2T_1) + R^2(\phi_1^2, p_2(T_1 + \pi/p_1)) \neq 0$$

Thus, we have constructed some controls allowing to reach the following missed states

$$\psi_2^1, \quad \psi_2^2, \quad \psi_2^3, \quad \text{and} \quad \psi_2^4.$$

where

$$\begin{aligned}
\psi_2^2 &= R^2(\psi_2^1, p_2T_\theta)\\
\psi_2^3 &= R^2(\psi_2^1, \pi) = -\psi_2^1\\
\psi_2^4 &= R^2(-\psi_2^2, p_2T_\theta)
\end{aligned}$$

Furthermore, we have for $k = 1, 2$

$$M_k = \bigcup_{j=1}^{4} M_k^j \tag{3.24}$$

where

$$\begin{aligned}
M_k^1 &:= \{d_k^1\psi_k^1 + d_k^2\psi_k^2; d_k^1 > 0, d_k^2 \geq 0\},\\
M_k^2 &:= \{d_k^1\psi_k^2 + d_k^2\psi_k^3; d_k^1 > 0, d_k^2 \geq 0\},\\
M_k^3 &:= \{d_k^1\psi_k^3 + d_k^2\psi_k^4; d_k^1 > 0, d_k^2 \geq 0\},\\
M_k^4 &:= \{d_k^1\psi_k^4 + d_k^2\psi_k^1; d_k^1 > 0, d_k^2 \geq 0\}.
\end{aligned}$$

Let $\psi \in M^>$. From (3.24), we know that $P_{M_1}(\psi) \in M_1^i$ for some $i \in \{1, \ldots, 4\}$. Hence, there exist $d_1^1 > 0, d_1^2 \geq 0$, such that

$$\psi = d_1^1\psi_1^i + d_1^2\psi_1^{i+1} + P_{M_2}(\psi).$$

Let us write $\psi$ as follows

$$\psi = d_1^1\psi_1^i + d_1^2\psi_1^{i+1} + d_1^1\tilde{\psi}_2^i + d_1^2\tilde{\psi}_2^{i+1} + \left(P_{M_2}(\psi) - d_1^1\tilde{\psi}_2^i - d_1^2\tilde{\psi}_2^{i+1}\right).$$

Since the states $\tilde{\psi}_2^i, \tilde{\psi}_2^{i+1}$ lie in $M_2$, there exists $j \in \{1, \ldots, 4\}$ such that

$$P_{M_2}(\psi) - d_1^1\tilde{\psi}_2^i - d_1^2\tilde{\psi}_2^{i+1} \in M_2^j$$

and therefore there exist $d_2^1 > 0, d_2^2 \geq 0$ such that

$$\psi = d_1^1(\psi_1^i + \tilde{\psi}_2^i) + d_1^2(\psi_1^{i+1} + \tilde{\psi}_2^{i+1}) + d_2^1\psi_2^j + d_2^2\psi_2^{j+1}.$$

Thus, we have decomposed $\psi$ in terms of reachables directions for the second-order expansion. Now, we take the controls $u_\psi, v_\psi$ defined by

$$(u_\psi, v_\psi) = \left(\sqrt{d_1^1}u_1^i + \sqrt{d_1^2}u_1^{i+1} + \sqrt{d_2^1}u_2^j + \sqrt{d_2^2}u_2^{j+1}, d_1^1v_1^i + d_1^2v_1^{i+1} + d_2^1v_2^j + d_2^2v_2^{j+1}\right),$$

and $\alpha_\psi, \beta_\psi \in \mathcal{B}$ the corresponding solutions of (3.18) and (3.19) respectively. Here, it is important to note that, with the choices of $T, T_1, T_c$ and $T_\theta$, the supports of the trajectories $\alpha_k^j$ for $k=1,2$ and $j=1,\dots,4$ are disjoint and that all these trajectories go from 0 at $t=0$ to 0 at $t=T$, i.e.

$$\alpha_k^j(0,\cdot) = \alpha_k^j(T,\cdot) = 0.$$

Thus, it is not difficult to verify that

$$\alpha_\psi(T,\cdot) = 0 \quad \text{and} \quad \beta_\psi(T,\cdot) = \psi$$

which ends the proof in the case $n^> = 2$. The previous method can be easily adapted to the case where $n^> > 2$. In order to construct the controls needed in the general case, our method requires a time of control $T$ greater than $T^>$. ■

## 3.4 Motion in the missed directions $\pm(1-\cos x)$

We assume in this section that $L = 2k\pi$ for some $k \in \mathbb{N}^*$. Let us recall that in this case we have

$$M_n = \langle 1-\cos x\rangle \quad \text{and} \quad n^> = n-1. \tag{3.25}$$

Thanks to [15], we have the following result that one can prove in a similar way to [15, Proposition 8].

**Proposition 3.4.1** *Let $T_c > 0$. There exists $(u,v,w)$ in $L^2(0,T_c)^3$ such that, if $\alpha$, $\beta$, $\gamma$ are the mild solutions of*

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} = 0, \\ \alpha(t,0) = \alpha(t,L) = 0, \\ \alpha_x(t,L) = u(t), \\ \alpha(0,.) = 0, \end{cases} \tag{3.26}$$

$$\begin{cases} \beta_t + \beta_x + \beta_{xxx} = -\alpha\alpha_x, \\ \beta(t,0) = \beta(t,L) = 0, \\ \beta_x(t,L) = v(t), \\ \beta(0,.) = 0, \end{cases} \tag{3.27}$$

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} = -(\alpha\beta)_x, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = w(t), \\ \gamma(0,.) = 0, \end{cases} \tag{3.28}$$

*then*

$$\alpha(T_c,.) = 0,\ \beta(T_c,.) = 0 \text{ and } \gamma(T_c,.) = (1-\cos x) + \sum_{i=1}^{n^>} P_{M_i}(\gamma(T_c,.)).$$

The idea to vanish the projections of $\gamma(T_c,\cdot)$ on $M_i$, and thus to reach the direction $(1-\cos(x))$, is the same as before, that is, to use the rotation phenomena given in Lemma 3.3.2. In addition, we use the fact that the function $(1-\cos x)$ satisfies

$$\begin{cases} y_x + y_{xxx} = 0, \\ y(0) = y(2k\pi) = y_x(2k\pi) = 0. \end{cases}$$

The case $n = 1$ has already been considered in [15]. We deal with the case $n = 2$ (for example, $L = 14\pi$ leads to the couples $(k_1, l_1) = (11, 2)$ and $(k_2, l_2) = (7, 7)$).

Let us define the following control lying in $L^2(0,T)^3$, where $T > \pi/p_1$. (Here, we omit the time translation needed for the controls $u, v$ and $w$ which are defined in $(0, T_c)$)

$$(u_+, v_+, w_+)(t) = \begin{cases} (0,0,0) & \text{if } t \in (0, T - T_c - \frac{\pi}{p_1}), \\ (u,v,w) & \text{if } t \in (T - T_c - \frac{\pi}{p_1}, T - \frac{\pi}{p_1}), \\ (0,0,0) & \text{if } t \in (T - \frac{\pi}{p_1}, T - T_c), \\ (u,v,w) & \text{if } t \in (T - T_c, T). \end{cases}$$

By defining $\alpha_+, \beta_+, \gamma_+ \in \mathcal{B}$ as the solutions of (3.26) with control $u_+$, (3.27) with control $v_+$ and (3.28) with control $w_+$ respectively, it is not difficult to see that

$$\alpha_+(T,.) = 0,\ \beta_+(T,.) = 0,\ \gamma_+(T,.) = 2(1-\cos x). \tag{3.29}$$

Now, if we consider the control $(u_-, v_-, w_-) := (-u_+, v_+, -w_+)$ we get

$$\alpha_-(T,.) = 0,\ \beta_-(T,.) = 0,\ \gamma_-(T,.) = -2(1-\cos x), \tag{3.30}$$

where obviously $\alpha_-, \beta_-, \gamma_- \in \mathcal{B}$ are the solutions of (3.26), (3.27) and (3.28) with controls $u_-, v_-$ and $w_-$ respectively. Thus we can reach all directions in $M_2$ in a time $T > \dfrac{\pi}{p_1}$.

We can easily deduce the same result in the case $n > 2$. We just have to construct a control that vanishes the components in the other missed subspaces $M_j$, $j \in J^>$. In order to do that, a time of control $T$, with

$$T > T^n := \pi \sum_{i=1}^{n-1} \frac{1}{p_i}, \tag{3.31}$$

is sufficient.

## 3.5 Fixed point argument

If $L \neq 2k\pi$, then we can use the same proof as in chapter 2 and get property $\mathcal{P}(T)$ for every $T > T^{>}$. Thus the only interesting case we detail here is when $L = 2k\pi$ and $\dim M(L) > 2$.

### 3.5.1 Preliminaries

Recall that for $L \in N$, we have $n$ pairs $(k_j, l_j)$ such that (3.7) and (3.8) hold. We have introduced some important notations

$$J^{>} := \{j; k_j > l_j\}, \quad n^{>} := |J^{>}|, \quad M^{>} := \bigoplus_{j=1}^{n^{>}} M_j.$$

In this section, we consider the case where $n^{>} = (n-1)$ and consequently where $M_n = \langle 1 - \cos x \rangle$. Thus we can write any $z \in L^2(0, L)$ as

$$z = P_H(z) + \rho_z \psi_z + d_z(1 - \cos x), \tag{3.32}$$

where

$$\rho_z := \|P_{M^{>}}(z)\|_{L^2(0,L)}, \quad \rho_z \psi_z := P_{M^{>}}(z), \quad \text{and} \quad d_z(1 - \cos x) = P_{M_n}(z).$$

Let us also denote, for $D > 0$ and $R > 0$,

$$B_R^D := \Big\{\xi \in L^2(0, D)\,;\, \|\xi\|_{L^2(0,D)} \leq R\Big\}.$$

From the work of Rosier in [39], we know that for every $y_0 \in L^2(0, L)$ there exists a continuous linear affine map

$$\Gamma_0 : h \in H \subset L^2(0, L) \longmapsto \Gamma_0(h) \in L^2(0, T),$$

such that the solution of

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = \Gamma_0(h), \\ y(0, \cdot) = P_H(y_0), \end{cases}$$

satisfies $y(T, \cdot) = h$. Moreover, there exist two constants $D_1, D_2 > 0$ such that

$$\forall y_0 \in L^2(0, L), \forall h \in H, \quad \|\Gamma_0(h)\|_{L^2(0,T)} \leq D_1(\|h\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}), \tag{3.33}$$

$$\forall y_0 \in L^2(0, L), \forall h, g \in H, \quad \|\Gamma_0(h) - \Gamma_0(g)\|_{L^2(0,T)} \leq D_2\|h - g\|_{L^2(0,L)}. \tag{3.34}$$

From sections 3.3 and 3.4, we have the existence of the controls $u_\pm, v_\pm, w_\pm \in L^2(0,T^n)$ and for every $\psi \in M^>$, the controls $u_\psi, v_\psi \in L^2(0,T^>)$. As we shall see later, we need that the corresponding trajectories of first order $\alpha_\pm$ and $\alpha_\psi$ are disjoint and therefore for every $z \in L^2(0,L)$ written as (3.32), and for every $T$ satisfying

$$T > T_L := T^n + T^>,$$

we define the following controls lying in $L^2(0,T)$

$$(\tilde{u},\tilde{v},\tilde{w})(t) := \begin{cases} (0,0,0) & \text{if } t \in (0, T-T_L), \\ (u_{\mathrm{sign}(d_z)}, v_{\mathrm{sign}(d_z)}, w_{\mathrm{sign}(d_z)})|_{(t-T+T_L)} & \text{if } t \in (T-T_L, T-T^>), \\ (0,0,0) & \text{if } t \in (T-T^>, T) \end{cases}$$

and

$$(\hat{u},\hat{v})(t) := \begin{cases} (0,0) & \text{if } t \in (0, T-T^>), \\ (u_{\psi_z}, v_{\psi_z})|_{(t-T+T^>)} & \text{if } t \in (T-T^>, T), \end{cases}$$

where we use the notation

$$\mathrm{sign}(d_z) = \begin{cases} + & \text{if } d_z \geq 0, \\ - & \text{if } d_z < 0. \end{cases} \tag{3.35}$$

Let $y_0 \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} < r$, where $r > 0$ has to be chosen later. Using (3.32), we define the functions $G$ and $F$ by

$$\begin{aligned} G : L^2(0,L) &\longrightarrow L^2(0,T), \\ z &\mapsto G(z) := \Gamma_0(P_H(z)) + \rho_z^{1/2}\hat{u} + \rho_z\hat{v} + |d_z|^{1/3}\tilde{u} + |d_z|^{2/3}\tilde{v} + |d_z|\tilde{w}, \end{aligned}$$

$$\begin{aligned} F : B_{\epsilon_1}^T \cap L^2(0,T) &\longrightarrow L^2(0,L), \\ \kappa &\longmapsto F(\kappa) := y(T,\cdot), \end{aligned}$$

where $y = y(t,x)$ is the solution of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = \kappa(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{3.36}$$

and $\epsilon_1$ is small enough so that the function $F$ is well defined.

Let $y_T \in L^2(0,L)$ be such that $\|y_T\| < r$. Let $\Lambda_{y_0,y_T}$ denotes the map

$$\begin{aligned} \Lambda_{y_0,y_T} : B^L_{\epsilon_2} \cap L^2(0,L) &\longrightarrow L^2(0,L), \\ z &\longmapsto \Lambda_{y_0,y_T}(z) := z + y_T - F \circ G(z), \end{aligned}$$

where $\epsilon_2$ is small enough so that $\Lambda_{y_0,y_T}$ is well defined.

Let us remark that if we find a fixed point $\tilde{z} \in L^2(0,L)$ of the map $\Lambda_{y_0,y_T}$, then we will have

$$F \circ G(\tilde{z}) = y_T$$

which means that the control

$$\kappa := G(\tilde{z}) \in L^2(0,T)$$

drives the solution of (3.36) from $y_0$ at $t = 0$ to $y_T$ at $t = T$. In the following sections, we prove that such a fixed point does exist.

### 3.5.2 A technical lemma

Let us assert the following technical result which will be needed to study the map $\Lambda_{y_0,y_T}$.

**Lemma 3.5.1** *There exist $\epsilon_3 > 0$ and $C_1 > 0$ such that, for every $z, y_0 \in B^L_{\epsilon_3}$, the following estimate holds*

$$\|z - F \circ G(z)\|_{L^2(0,L)} \leq C_1(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}).$$

**Proof.** Let $z, y_0 \in L^2(0,L)$. Let $y = y(t,x)$ be the solution of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = G(z), \\ y(0,\cdot) = y_0. \end{cases} \tag{3.37}$$

From (3.33) and the fact that $\rho_z \leq \|z\|_{L^2(0,L)}$, one deduces that if $\|z\|_{L^2(0,L)}$ is smaller than 1 (and therefore $\|z\|_{L^2(0,L)} \leq \|z\|^{1/2}_{L^2(0,L)}$), then there exists a constant $C_2$ such that

$$\|G(z)\|_{L^2(0,T)} \leq C_2(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}). \tag{3.38}$$

Thus, one can find $\epsilon_4, C_3 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_4}$, the unique solution of (3.37) satisfies

$$\|y\|_{\mathcal{B}} \leq C_3(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}). \tag{3.39}$$

Let $\tilde{y}$, $\hat{\alpha}$, $\hat{\beta}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\hat{y}$ be the solutions of

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \tilde{y}_x(t,L) = \Gamma_0(P_H(z)), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases} \tag{3.40}$$

$$\begin{cases} \hat{\alpha}_t + \hat{\alpha}_x + \hat{\alpha}_{xxx} = 0, \\ \hat{\alpha}(t,0) = \hat{\alpha}(t,L) = 0, \\ \hat{\alpha}_x(t,L) = \hat{u}(t), \\ \hat{\alpha}(0,\cdot) = 0, \end{cases} \tag{3.41}$$

$$\begin{cases} \hat{\beta}_t + \hat{\beta}_x + \hat{\beta}_{xxx} = -\hat{\alpha}\hat{\alpha}_x, \\ \hat{\beta}(t,0) = \hat{\beta}(t,L) = 0, \\ \hat{\beta}_x(t,L) = \hat{v}(t), \\ \hat{\beta}(0,\cdot) = 0, \end{cases} \tag{3.42}$$

$$\begin{cases} \tilde{\alpha}_t + \tilde{\alpha}_x + \tilde{\alpha}_{xxx} = 0, \\ \tilde{\alpha}(t,0) = \tilde{\alpha}(t,L) = 0, \\ \tilde{\alpha}_x(t,L) = \tilde{u}(t), \\ \tilde{\alpha}(0,\cdot) = 0, \end{cases} \tag{3.43}$$

$$\begin{cases} \tilde{\beta}_t + \tilde{\beta}_x + \tilde{\beta}_{xxx} = -\tilde{\alpha}\tilde{\alpha}_x, \\ \tilde{\beta}(t,0) = \tilde{\beta}(t,L) = 0, \\ \tilde{\beta}_x(t,L) = \tilde{v}(t), \\ \tilde{\beta}(0,\cdot) = 0, \end{cases} \tag{3.44}$$

$$\begin{cases} \tilde{\gamma}_t + \tilde{\gamma}_x + \tilde{\gamma}_{xxx} = -(\tilde{\alpha}\tilde{\beta})_x, \\ \tilde{\gamma}(t,0) = \tilde{\gamma}(t,L) = 0, \\ \tilde{\gamma}_x(t,L) = \tilde{w}(t), \\ \tilde{\gamma}(0,\cdot) = 0, \end{cases} \tag{3.45}$$

$$\begin{cases} \hat{y}_t + \hat{y}_x + \hat{y}_{xxx} = 0, \\ \hat{y}(t,0) = \hat{y}(t,L) = 0, \\ \hat{y}_x(t,L) = 0, \\ \hat{y}(0,\cdot) = P_M(y_0). \end{cases} \tag{3.46}$$

Let us define

$$\phi := y - \tilde{y} - \rho_z^{1/2}\hat{\alpha} - \rho_z\hat{\beta} - |d_z|^{1/3}\tilde{\alpha} - |d_z|^{2/3}\tilde{\beta} - |d_z|\tilde{\gamma} - \hat{y}.$$

Then $\phi = \phi(t,x)$ satisfies

$$\begin{cases} \phi_t + \phi_x + \phi_{xxx} + \phi\phi_x = -(\phi a)_x - b, \\ \phi(t,0) = \phi(t,L) = 0, \\ \phi_x(t,L) = 0, \\ \phi(0,\cdot) = 0, \end{cases} \tag{3.47}$$

where $$a := y - \phi,$$

$$\begin{aligned} b := \; & \tilde{y}\tilde{y}_x + \hat{y}\hat{y}_x + \rho_z^2\hat{\beta}\hat{\beta}_x + \rho_z^{3/2}(\hat{\alpha}\hat{\beta})_x + |d_z|^{4/3}\tilde{\beta}\tilde{\beta}_x + |d_z|^{5/3}(\tilde{\beta}\tilde{\gamma})_x + |d_z|^{4/3}(\tilde{\alpha}\tilde{\gamma})_x + \\ & |d_z|^2\tilde{\gamma}\tilde{\gamma}_x + \Big(\tilde{y}(\rho_z^{1/2}\hat{\alpha} + \rho_z\hat{\beta} + |d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma} + \hat{y})\Big)_x + \\ & \Big((\rho_z^{1/2}\hat{\alpha} + \rho_z\hat{\beta})(|d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma} + \hat{y})\Big)_x + \\ & \Big(\hat{y}(|d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma})\Big)_x. \end{aligned}$$

Here, in order to use equation (3.47) we need some estimates on its right-hand side.

**Lemma 3.5.2** *There exists $C_4 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_4}$,*

$$\|\phi\|_{\mathcal{B}} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}), \tag{3.48}$$

$$\|a\|_{\mathcal{B}} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}), \tag{3.49}$$

$$\|b\|_{L^1(0,T,L^2(0,L))} \leq C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}). \tag{3.50}$$

**Proof of Lemma 3.5.2.** Let us prove (3.48). One has

$$\begin{aligned} \|\phi\|_{\mathcal{B}} \;\leq\; & \|y\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \rho_z^{1/2}\|\hat{\alpha}\|_{\mathcal{B}} + \rho_z\|\hat{\beta}\|_{\mathcal{B}} + |d_z|^{1/3}\|\tilde{\alpha}\|_{\mathcal{B}} + |d_z|^{2/3}\|\tilde{\beta}\|_{\mathcal{B}} + |d_z|\|\tilde{\gamma}\|_{\mathcal{B}} + \|\hat{y}\|_{\mathcal{B}} \\ \leq\; & C(\|G(z)\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}) + C(\|\Gamma_0(P_H(z))\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}) \\ & + C\rho_z^{1/2}\|\hat{u}\|_{L^2(0,T)} + C\rho_z(\|\hat{v}\|_{L^2(0,T)} + \|\hat{\alpha}\hat{\alpha}_x\|_{L^1(0,T,L^2(0,L))}) + C|d_z|^{1/3}\|\tilde{u}\|_{L^2(0,T)} \\ & + C|d_z|^{2/3}(\|\tilde{v}\|_{L^2(0,T)} + \|\tilde{\alpha}\tilde{\alpha}_x\|_{L^1(0,T,L^2(0,L))}) \\ & + C|d_z|(\|\tilde{w}\|_{L^2(0,T)} + \|(\tilde{\alpha}\tilde{\beta})_x\|_{L^1(0,T,L^2(0,L))}) + C\|P_M(y_0)\|_{L^2(0,L)}. \end{aligned}$$

One needs at this point the following trivial estimate.

$$\exists C_5 > 0, \forall f,g \in \mathcal{B}, \quad \|(fg)_x\|_{L^1(0,T,L^2(0,L))} \leq C_5\|f\|_{\mathcal{B}}\|g\|_{\mathcal{B}}. \tag{3.51}$$

By noticing that if $z = P_H(z) + \rho_z\psi_z + d_z(1-\cos(x))$, then

$$\|z\|^2_{L^2(0,L)} = \|P_H(z)\|^2_{L^2(0,L)} + \rho_z^2 + d_z^2\|1-\cos(x)\|^2_{L^2(0,L)},$$

and using (3.38) and (3.51), one gets (3.48). Estimate (3.49) follows from (3.48) and the definition of the function $a$. To prove (3.50), one uses (3.51) being very careful with the powers which appear. For instance, looking at the function $b$, one finds the term $(\rho_z^{1/2}\hat{\alpha}|d_z|^{1/3}\tilde{\alpha})$ which apparently is not bounded by $C_4\|z\|^{4/3}_{L^2(0,L)}$ for $z \in B_1^L$. This is the reason for which one takes the trajectories $\tilde{\alpha}$ and $\hat{\alpha}$ disjoint. ■

Thus, from (3.47) one obtains the existence of $C_6 > 0$ such that

$$\|\phi\|^2_{\mathcal{B}} \leq C_6(\|\phi\|^2_{\mathcal{B}}\|a\|^2_{\mathcal{B}} + \|y_0\|^2_{L^2(0,L)} + \|z\|^{8/3}_{L^2(0,L)}),$$

i.e. one has

$$\|\phi\|^2_{\mathcal{B}}(1 - C_6\|a\|^2_{\mathcal{B}}) \leq C_6(\|y_0\|^2_{L^2(0,L)} + \|z\|^{8/3}_{L^2(0,L)}),$$

which, together with (3.49), implies the existence of $\epsilon_5$ and $C_7$ such that for every $z, y_0 \in B^L_{\epsilon_5}$

$$\|\phi\|_{\mathcal{B}} \leq C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}). \tag{3.52}$$

Finally, from (3.52) one obtains

$$\begin{aligned}
\|z - F\circ G(z)\|_{L^2(0,L)} &\leq \|z - F\circ G(z) + \hat{y}(T,\cdot)\|_{L^2(0,L)} + \|-\hat{y}(T,\cdot)\|_{L^2(0,L)}\\
&= \|\phi(T,\cdot)\|_{L^2(0,L)} + \|\hat{y}(0,\cdot)\|_{L^2(0,L)}\\
&\leq \|\phi\|_{\mathcal{B}} + \|y_0\|_{L^2(0,L)}\\
&\leq C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}) + \|y_0\|_{L^2(0,L)}\\
&\leq (C_7+1)(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}),
\end{aligned}$$

which ends the proof of Lemma 3.5.1 with $C_1 := C_7 + 1$ and $\epsilon_3 := \epsilon_5$. ■

### 3.5.3 Fixed point in $H$

For $w = (w_1^1, w_1^2, \ldots, w_{n-1}^1, w_{n-1}^2, w_n) \in \mathbb{R}^{2n-1}$ fixed, let us denote

$$\Psi_w := w_n(1-\cos x) + \sum_{j=1}^{n-1}(w_j^1\varphi_j^1 + w_j^2\varphi_j^2), \tag{3.53}$$

where the functions $\varphi_j^i$ for $i = 1, 2$, $j = 1, \ldots, n-1$ are given in (3.12). Let us study the map

$$\Pi := P_H \circ \Lambda_{y_0,y_T}(\cdot + \Psi_w)$$

on the subspace $H$.

$$\begin{aligned}
\Pi : H &\longrightarrow H,\\
h &\longmapsto \Pi(h) = h + P_H(y_T) - P_H(F\circ G(h+\Psi_w)).
\end{aligned}$$

In order to prove the existence of a fixed point of the map $\Pi$, we will apply the Banach fixed point theorem to the restriction of $\Pi$ to the closed ball $B_R^L \cap H$ with $R > 0$ small enough. Using Lemma 3.5.1 we see that

$$\begin{aligned}
\|\Pi(h)\|_{L^2(0,L)} &\leq \|y_T\|_{L^2(0,L)} + \|h + \Psi_w - F \circ G(h + \Psi_w)\|_{L^2(0,L)} \\
&\leq \|y_T\|_{L^2(0,L)} + C_1(\|y_0\|_{L^2(0,L)} + \|h + \Psi_w\|_{L^2(0,L)}^{4/3}) \\
&\leq (C_1 + 1)(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + |w|^{4/3}) + C_1\|h\|_{L^2(0,L)}^{4/3} \\
&\leq (C_1 + 1)(2r + |w|^{4/3}) + C_1\|h\|_{L^2(0,L)}^{4/3}.
\end{aligned}$$

Hence, if we choose $R, r$ and $w$ such that

$$R^{4/3} \leq \frac{R}{2C_1} \quad \text{and} \quad (2r + |w|^{4/3}) \leq \frac{R}{2(C_1 + 1)},$$

then it follows that

$$\|\Pi(h)\|_{L^2(0,L)} \leq R \quad \text{and so} \quad \Pi(B_R^L \cap H) \subset (B_R^L \cap H).$$

It remains to prove that the map $\Pi$ is a contraction. Let $g, h \in B_R^L \cap H$. Let $y = y(t, x)$, $q = q(t, x)$, $\tilde{y} = \tilde{y}(t, x)$ and $\tilde{q} = \tilde{q}(t, x)$ be the solutions of the following problems

$$\begin{cases}
y_t + y_x + y_{xxx} + yy_x = 0, \\
y(t, 0) = y(t, L) = 0, \\
y_x(t, L) = G(g + \Psi_w), \\
y(0, \cdot) = y_0,
\end{cases}$$

$$\begin{cases}
q_t + q_x + q_{xxx} + qq_x = 0, \\
q(t, 0) = q(t, L) = 0, \\
q_x(t, L) = G(h + \Psi_w), \\
q(0, \cdot) = y_0,
\end{cases}$$

$$\begin{cases}
\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \\
\tilde{y}(t, 0) = \tilde{y}(t, L) = 0, \\
\tilde{y}_x(t, L) = \Gamma_0(g), \\
\tilde{y}(0, \cdot) = P_H(y_0),
\end{cases}$$

$$\begin{cases}
\tilde{q}_t + \tilde{q}_x + \tilde{q}_{xxx} = 0, \\
\tilde{q}(t, 0) = \tilde{q}(t, L) = 0, \\
\tilde{q}_x(t, L) = \Gamma_0(h), \\
\tilde{q}(0, \cdot) = P_H(y_0).
\end{cases}$$

Let us define $\phi := y - \tilde{y}$, $\psi := q - \tilde{q}$ and $\gamma := \phi - \psi$. One sees that $\gamma$ satisfies

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \gamma\gamma_x = -(\gamma a)_x - b, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = 0, \\ \gamma(0,\cdot) = 0, \end{cases} \tag{3.54}$$

where

$$a := \tilde{y} + \psi \quad \text{and} \quad b := \left(q(\tilde{y} - \tilde{q})\right)_x + (\tilde{y} - \tilde{q})(\tilde{y} - \tilde{q})_x.$$

It is easy to see that there exists a constant $C_8$ such that

$$\|b\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\tilde{y} - \tilde{q}\|_{\mathcal{B}}, \tag{3.55}$$

$$\|(a\gamma)_x\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\right) \|\gamma\|_{\mathcal{B}}. \tag{3.56}$$

Thus, we get the existence of $C_9 > 0$ such that

$$\|\gamma\|_{\mathcal{B}}^2 \leq C_9(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}})^2(\|\tilde{y} - \tilde{q}\|_{\mathcal{B}}^2 + \|\gamma\|_{\mathcal{B}}^2). \tag{3.57}$$

In addition, since $w := \tilde{y} - \tilde{q}$ satisfies the following linear equation

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \\ w(t,0) = w(t,L) = 0, \\ w_x(t,L) = \Gamma_0(g) - \Gamma_0(h), \\ w(0,\cdot) = 0, \end{cases}$$

there exists $C_{10} > 0$ such that

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10} \|\Gamma_0(g) - \Gamma_0(h)\|_{L^2(0,T)}$$

and so, from (3.34), one gets

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10} D_2 \|g - h\|_{L^2(0,L)}. \tag{3.58}$$

Moreover, it is easy to see that there exists a constant $C_{11} > 0$ such that

$$\|q\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} \leq C_{11}(\|y_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|g\|_{L^2(0,L)} + |w|^{1/3}). \tag{3.59}$$

Thus, using (3.57), (3.58) and (3.59) we see that if $R, |w|, r$ are small enough, it follows that

$$\|\gamma\|_{\mathcal{B}} \leq \frac{1}{2} \|g - h\|_{L^2(0,L)}.$$

Therefore, we have

$$\begin{aligned}\|\Pi(g)-\Pi(h)\|_{L^2(0,L)} &\leq \|g-F\circ G(g+\Psi_w)-h+F\circ G(h+\Psi_w)\|_{L^2(0,L)}\\ &= \|\gamma(T)\|_{L^2(0,L)} \leq \|\gamma\|_{\mathcal{B}}\\ &\leq \tfrac{1}{2}\|g-h\|_{L^2(0,L)},\end{aligned}$$

which implies the existence of a unique fixed point $h(y_0,y_T,w)\in B_R^L\cap H$ of the map $\Pi|_{B_R^L\cap H}$.

### 3.5.4 Fixed point in $M$

We now apply the Brouwer fixed point theorem to the restriction of the map

$$\begin{aligned}\tau: M &\longrightarrow M,\\ \Psi_w &\mapsto P_M(\Psi_w+y_T-F\circ G(\Psi_w+h(y_0,y_T,w))),\end{aligned}$$

to the closed ball $B_{\hat{R}}^L\cap M$ with $\hat{R}$ small enough.

In section 3.5.1, the controls $\hat{u},\hat{v},\tilde{u},\tilde{v}$ and $\tilde{w}$ were chosen in such a way so that the function $G$ is continuous. Thus, it is easy to see that the map $(y_0,y_T,w)\longmapsto h(y_0,y_T,w)$ is also continuous in a neighborhood of $0\in L^2(0,L)^2\times\mathbb{R}^{2n-1}$. Using this continuity, Lemma 3.5.1, and choosing $r$ small enough, we get the existence of a radius $\hat{R}>0$ such that $\tau(B_{\hat{R}}^L\cap M)\subset B_{\hat{R}}^L\cap M$. This inclusion and the continuity of the map $\tau$ allow us to apply the Brouwer fixed point theorem. Therefore, there exists $\tilde{w}\in\mathbb{R}^{2n-1}$ with $|\tilde{w}|\leq\hat{R}$ such that $\tilde{h}:=h(y_0,y_T,\tilde{w})$ satisfies

$$P_M(y_T-F\circ G(\tilde{h}+\Psi_{\tilde{w}}))=0. \tag{3.60}$$

Using the fact that

$$\Pi(\tilde{h})=P_H(\tilde{h}+y_T-F\circ G(\tilde{h}+\Psi_{\tilde{w}}))=\tilde{h},$$

we obtain

$$P_H(y_T-F\circ G(\tilde{h}+\Psi_{\tilde{w}}))=0,$$

which together with (3.60), implies that

$$y_T=F\circ G(\tilde{h}+\Psi_{\tilde{w}}),$$

which ends the proof of Theorem 3.1.9.

## 3.6 Conclusion

In this article, we have proved that in the last remaining critical cases, i.e. when $\dim M > 2$, the nonlinear KdV equation is controllable in a time large enough. First, we have performed a power series expansion of the solution and of the control. Next, we have constructed special controls allowing to reach a basis of missed directions and thus all the missed states. Then if $\dim M$ is even, the fixed point theorems used in chapter 2 are directly applicable. If $\dim M$ is odd, we prove the controllability using fixed point mixing proofs of [15] and chapter 2.

The following open problem arises naturally from the results of this work.

**Open Problem 1** *Let $L \in N$ such that the dimension of the subspace $M$ is higher than 1. Does $\mathcal{P}(T)$ holds for every $T > 0$ ?*

This is an interesting question since even if the speed of propagation of the KdV equation is infinite, it may exist a minimal time of control. For example, in [2] Beauchard and Coron proved, for a time large enough, the local exact controllability along the ground state trajectory of a Schrödinger equation and Coron proved in [13] and [14, Theorem 9.8] that this local controllability does not hold in small time, even if the Schrödinger equation has an infinite speed of propagation. Our guess, based in second order computations in some particular critical cases where the space $M$ is two-dimensional, is that there exists a minimal time of control, this means there exists a time $T_0$ such that for any time $T < T_0$, $\mathcal{P}(T)$ does not hold. Thus, the answer to Open Problem 1 should be negative.

We have seen that the nonlinearity gives us the controllability in the critical cases even if the linear system is not controllable. We may wonder if the nonlinearity gives us the stability.

**Open Problem 2** *Let $L \in N$. Let $y_0 \in L^2(0, L)$ and $y$ the solution of*

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = 0, \\ y(0,\cdot) = y_0. \end{cases} \tag{3.61}$$

*Does the solution $y$ decay to zero as $t$ goes to infinity ?*

In order to answer this question, a really nonlinear method is needed because with a first-order approximation one obtains the linear system which has some solution conserving its $L^2$-norm. On the other hand, it is not clear that our method applies. It strongly needs the controls to be able to use higher-order approximations.

# Chapitre 4

# Rapid exponential stabilization for a linear KdV equation

This chapter is contained in [7] which has been written in collaboration with Emmanuelle Crépeau.

## Sommaire

## 4.1 Introduction

In this chapter we address the boundary stabilizability problem for a linear Korteweg-de Vries (KdV) equation on a bounded domain. We consider a system with homogeneous Dirichlet boundary conditions where the control acts on the Neumann boundary condition at the right endpoint. This issue has been studied in the litterature firstly in the case of periodic boundary conditions, mainly by adding a damping term to the equation. For example, in [30] a damping term distributed all along the domain is considered ; in [46] the authors use a damping term distributed with localized support and in [45] the authors use a boundary damping term. In all these papers, an exponential decay of the solutions is proved. In the case of homogeneous Dirichlet boundary conditions with a localized damping term, the same stability property has been proved in [38] and in [42].

Here, we are interested in the case where there is no damping. In [54], Zhang considers a feedback law which allows him to prove that solutions decay exponentially to zero. In [38], Perla Menzala, Vasconcellos and Zuazua prove that the solutions decay exponentially to zero even in the case without control. It is done when the length of the domain does not belong to a countable set of *critical values* introduced by Rosier in [39]. In the case of critical domains, it is known that there exist some initial conditions such that the corresponding solutions conserve their $L^2$-norm.

Our main aim in this work is to prove that for any $\omega > 0$, one can build a feedback law such that the closed-loop system has an exponential decay rate $\omega$ at least. This is a big difference with the previous works, where one proves the exponential decay, but nothing is said about its rate. There exist a few results of this kind for control systems of partial differential equations. Among them, one can cite the works by Slemrod in [48] for some bounded control operators (the case of distributed controls) and by Komornik in [28] for some unbounded control operators (the case of point or boundary controls). Both methods use a Gramian approach and are inspired by the one introduced independently by Kleinman in [27] and by Luke in [36] in a finite-dimensional framework. Recently, Urquiza in [52] has generalized to infinite-dimensional control systems a method called the Bass method by Russell (see [43, pages 117-118]). Actually, Urquiza was inspired on numerical computations with the Komornik's approach performed by Briffaut (see [4]). These numerical results showed a decay rate twice better than the one predicted by Komornik's theorem. This faster decay is exactly the decay achieved by the Urquiza's approach for a controllable system where one has an unbounded control operator and where the free-control system is defined by a skew-operator which is the infinitesimal generator of a strongly continuous group. As we will see, the main task to do, in order to be able to apply this method to our control system, is to obtain its exact controllability. After proving that, we obtain the wanted result of stabilizability. We also perform some numerical computations in order to verify in practice the exponential decay predicted by our theoretical result.

**Remark 4.1.1** *In order to get rapid stabilization for some PDE control system, there exists an other method, called backstepping method, which use neither a Gramian approach nor operator Ricatti equations. We cite [49, 32] by Krstic and Smyshlyaev, [35] by Liu and the references therein.*

## 4.2 Statement of the problem and Urquiza's method

Let $L > 0$ be fixed. Let us consider the following linear control system for the KdV equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = u(t), \end{cases} \tag{4.1}$$

where the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. In [54], Zhang considers the following feedback law with $\alpha \in (0,1)$ and $L = 1$

$$u(t) = \alpha y_x(t,0). \tag{4.2}$$

One obtains, for the closed-loop system, that the energy satisfies

$$\frac{d}{dt}\int_0^1 |y(t,x)|^2 dx = -(1-\alpha)|y_x(t,0)|^2.$$

Thus, a decay to zero of the solutions is naturally expected. In fact, Zhang proves that the closed-loop system is well posed in $L^2(0,1)$ and that there exist $\omega > 0$ and $C > 0$ such that

$$\forall y_0 \in L^2(0,1), \forall t > 0, \quad \|y(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\omega t}\|y_0\|_{L^2(0,1)}, \tag{4.3}$$

where $y$ is the solution of (4.1)-(4.2) with initial data $y_0$ and $L = 1$. That means, the feedback law (4.2) stabilizes the control system (4.1) to the origin. Then, Rosier proves in [39] that for some values of $L$, called *critical values* (see the definition of the set $N$ in

(4.30)), there exist some initial conditions $y_0$ such that the corresponding solution of (4.1) with $u = 0$, conserves its $L^2$-norm. These solutions also satisfy

$$y_x(t, 0) = 0,$$

and therefore a feedback law as (4.2) does not stabilize the system. (Note that $1 \notin N$). Later, in [38], Perla, Vasconcellos and Zuazua prove that (4.3) actually holds for (4.1) with $u = 0$, provided that the lenght $L$ of the interval is not critical.

In this chapter, we are interested in the design of some feedback laws

$$u(t) = \Pi(y(t, \cdot)), \tag{4.4}$$

such that the closed-loop system (4.1) and (4.4) has an exponential decay rate (the constant $\omega$ in (4.3)) as large as desired. In order to get this stabilizability property we use a method due to Urquiza [52]. Let us explain his result on the following abstract control system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), \\ y(0) = y_0, \end{cases} \tag{4.5}$$

with state $y(t)$ in a Hilbert space $Y$ and control $u(t)$ in a Hilbert space $U$. Here, the initial condition $y_0 \in Y$, $A$ is a skew-adjoint operator (i.e. $A^* = -A$) in $Y$ whose domain is dense in $Y$, and $B$ is an unbounded operator from $U$ to $Y$. Let us assume that these operators satisfy the following hypothesis.

(H1) The skew-adjoint operator $A$ is the infinitesimal generator of a strongly continuous group on $Y$.

(H2) The operator $B : U \to D(A)'$ is linear continuous.

(H3) *Regularity property.* For every $0 < T < \infty$ there exists $C_T > 0$ such that

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \leq C_T \|y\|_Y^2, \quad y \in D(A^*).$$

(H4) *Controllability property.* There exist $T > 0$ and $c_T > 0$ such that

$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \geq c_T \|y\|_Y^2, \quad y \in D(A^*).$$

Then, one has the following result whose proof mainly relies on general results about the algebraic Riccati equation associated with the linear quadratic regulator problem (see [21]).

**Theorem 4.2.1** *(see [52, Theorem 2.1]) Consider operators $A$ and $B$ under assumptions* (H1)-(H4). *For any $\omega > 0$, we have*

*(i) The symmetric positive operator $\Lambda_\omega$ defined by*

$$(\Lambda_\omega x, z)_Y = \int_0^\infty \left(B^* e^{-\tau(A+\omega I)^*} x, B^* e^{-\tau(A+\omega I)^*} z\right)_U d\tau, \quad \forall x, z \in Y,$$

*is coercive and is an isomorphism on $Y$.*

*(ii) Let $F_\omega := -B^*\Lambda_\omega^{-1}$. The operator $A + BF_\omega$ with $D(A + BF_\omega) = \Lambda_\omega(D(A^*))$ is the infinitesimal generator of a strongly continuous semigroup on $Y$.*

*(iii) The closed-loop system (system (4.5) with the feedback law $u = F_\omega(y)$) is exponentially stable with a decay rate equals to $2\omega$, that is,*

$$\exists C > 0, \forall x \in Y, \quad \|e^{t(A+BF_\omega)} x\|_Y \leq Ce^{-2\omega t}\|x\|_Y.$$

As one can see, the feedback operator is built in an explicit way. This fact and the free choice of the parameter $\omega$ are the main advantages of this method. The first point to check in order to be able to apply this theorem to our linear KdV control system is (H1). As we easily see, hypothesis (H1) holds if we take as control, the function $v$ defined by

$$u(t) = y_x(t, 0) + v(t).$$

Hence our system becomes

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) - y_x(t, 0) = v(t). \end{cases} \tag{4.6}$$

We can rewrite (4.6) in the abstract form (4.5) by defining the operators $A$ and $B$ as follows

$$D(A) := \left\{w \in H^3(0, L); w(0) = w(L) = 0, w'(0) = w'(L)\right\},$$

$$Aw := -w' - w''',$$

$$B : s \in \mathbb{R} \mapsto L_s \in D(A^*)',$$

$$L_s : z \in D(A^*) \mapsto sz_x(L) \in \mathbb{R}.$$

It is not difficult to see that $A^* = -A$ and that

$$(Aw, w)_{L^2(0,L)} = 0, \quad \forall w \in D(A).$$

Hence, from classical semigroup results, one sees that the operator $A$ satisfies (H1). We also see that (H2) holds for the operator $B$. Hypothesis (H3) and (H4) are more delicate to show and will be proved in section 4.3. As our operator $B$ stands for a boundary control, we will see that assumption (H3) is a sharp trace regularity. Concerning (H4), it implies an exact controllability result that will be stated below. Then, in section 4.4, by applying Theorem 4.2.1, a feedback law for our control system is given in an explicit way and the rapid stabilizability is asserted in a precise way. Finally, in section 4.5 we check the performance of our feedback laws on some numerical simulations.

## 4.3 Proof of (H3) and (H4)

In this section we first study the asymptotic behavior of the eigenvalues of the operator $A$. Then, we apply a classical Ingham's inequality to prove that (H3) and (H4) hold for our control system. An exact boundary controllability result is also stated.

### 4.3.1 Spectral properties of the operator $A$

It is not difficult to see that the skew-adjoint operator $A$ has a compact resolvent. Hence the spectrum $\sigma(A)$ of $A$ consists only of eigenvalues. Furthermore the spectrum of $A$ is a discrete subset of $i\mathbb{R}$ and the eigenfunctions form an orthonormal basis of $L^2(0,L)$. For the work to be done here, we require very detailed informations about the asymptotic behavior of the eigen-elements of the operator $A$. Let us denote by $(i\lambda_k)_{k\in\mathbb{Z}}$ the eigenvalues of $A$ and by $(\phi_k)_{k\in\mathbb{Z}}$ its eigenfunctions.

**Proposition 4.3.1** *The real numbers $(\lambda_k)_{k\in\mathbb{Z}}$ have the asymptotic form*

$$\lambda_k = \frac{8\pi^3 k^3}{L^3} + O(k^2) \quad \text{as } k \to \pm\infty.$$

**Proof.** The eigenvalue problem to be considered is

$$\begin{cases} -\phi' - \phi''' = i\lambda\phi, \\ \phi(0) = \phi(L) = 0, \\ \phi'(0) = \phi'(L). \end{cases} \tag{4.7}$$

To each $\lambda$ corresponds at least a real $a$ such that $\lambda = 2a(4a^2-1)$. Thus, the three solutions of

$$z^3 + z + i\lambda = 0$$

read as

$$z_0 = \sqrt{|3a^2-1|} - ai, \quad z_1 = -\sqrt{|3a^2-1|} - ai, \quad z_2 = 2ai.$$

We distinguish 3 cases.

1. If $3a^2 - 1 < 0$.

   In this case, it is easy to see that the eigenfunction $\phi$ of $A$ associated to the eigenvalue $\lambda = 2a(4a^2-1)$ may be written

$$\phi(x) = e^{-iax}(\alpha\cos(\sqrt{1-3a^2}x) + \beta\sin(\sqrt{1-3a^2}x)) + \gamma e^{2iax}, \tag{4.8}$$

   where $\alpha, \beta$ and $\gamma$ are some constants such that $\phi(0) = \phi(L) = 0$ and $\phi'(0) = \phi'(L)$. That means, such that

$$\alpha + \gamma = 0, \tag{4.9}$$

$$e^{-iaL}(\alpha\cos(\sqrt{1-3a^2}L) + \beta\sin(\sqrt{1-3a^2}L)) - \alpha e^{2iaL} = 0, \tag{4.10}$$

$$\begin{aligned} -ia\alpha + \beta\sqrt{1-3a^2} + 2ia\gamma &= -iae^{-iaL}\big[\alpha\cos(\sqrt{1-3a^2}L) + \beta\sin(\sqrt{1-3a^2}L)\big] \\ &+ 2ia\gamma e^{2iaL} + e^{-iaL}\sqrt{1-3a^2}\big[-\alpha\sin(\sqrt{1-3a^2}L) + \beta\cos(\sqrt{1-3a^2}L)\big]. \end{aligned} \tag{4.11}$$

   From (4.10), one obtains

$$\beta = \alpha\frac{e^{3iaL} - \cos(\sqrt{1-3a^2}L)}{\sin(\sqrt{1-3a^2}L)}. \tag{4.12}$$

   Taking the real part of equation (4.11), one obtains that $a$ must satisfy

$$\begin{aligned} &\sqrt{1-3a^2}\cos(2aL) - 3a\sin(aL)\sin(\sqrt{1-3a^2}L) \\ &\qquad = \sqrt{1-3a^2}\cos(aL)\cos(\sqrt{1-3a^2}L). \end{aligned} \tag{4.13}$$

   The number of parameters $a$ satisfying (4.13) is finite and depends on $L$. As if $a$ satisfies (4.13), then $(-a)$ so, we find in this case $2N_L$ eigenvalues

$$\{\lambda_{-N_L}, ..., \lambda_{-1}, \lambda_1, ..., \lambda_{N_L}\}$$

2. If $3a^2 - 1 = 0$.

   We don't find any eigenfunction in this case. In fact, here

$$z_0 = z_1 = \frac{i\sqrt{3}}{3},\ z_2 = \frac{-2i\sqrt{3}}{3} \quad \text{or} \quad z_0 = z_1 = \frac{-i\sqrt{3}}{3},\ z_2 = \frac{2i\sqrt{3}}{3}$$

   and the candidate function to be an eigenfunction cannot satisfy the boundary conditions.

3. If $3a^2 - 1 > 0$.

   In this case, it is easy to see that the eigenfunction $\phi$ of $A$ associated to the eigenvalue $\lambda = 2a(4a^2 - 1)$ may be written

$$\phi(x) = e^{-iax}(\alpha \cosh(\sqrt{3a^2-1}x) + \beta \sinh(\sqrt{3a^2-1}x)) + \gamma e^{2iax} \tag{4.14}$$

   where $\alpha, \beta$ and $\gamma$ are some constants such that $\phi(0) = \phi(L) = 0$ and $\phi'(0) = \phi'(L)$. That means, such that

$$\alpha + \gamma = 0, \tag{4.15}$$

$$e^{-iaL}(\alpha \cosh(\sqrt{3a^2-1}L) + \beta \sinh(\sqrt{3a^2-1}L)) - \alpha e^{2iaL} = 0, \tag{4.16}$$

$$\begin{aligned} -ia\alpha + 2ia\gamma + \beta\sqrt{3a^2-1} &= -iae^{-iaL}\big[\alpha \cosh(\sqrt{3a^2-1}L) + \beta \sinh(\sqrt{3a^2-1}L)\big] \\ &+ 2ia\gamma e^{2iaL} + e^{-iaL}\sqrt{3a^2-1}\big[\alpha \sinh(\sqrt{3a^2-1}L) + \beta \cosh(\sqrt{3a^2-1}L)\big] \end{aligned} \tag{4.17}$$

   We deduce from (4.16)-(4.17)

$$\beta = \alpha \frac{e^{3iaL} - \cosh(\sqrt{3a^2-1}L)}{\sinh(\sqrt{3a^2-1}L)}, \tag{4.18}$$

$$\begin{aligned} -3a + \Im(\beta)\sqrt{3a^2-1} &= -3a\cos(2aL) \\ &+ \sqrt{3a^2-1}\sin(-aL)(\sinh(\sqrt{3a^2-1}L) + \Re(\beta)\cosh(\sqrt{3a^2-1}L)) \\ &+ \sqrt{3a^2-1}\cos(aL)\Im(\beta)\cosh(\sqrt{3a^2-1}L). \end{aligned} \tag{4.19}$$

   From these equations, one obtains that $a$ satisfies the following one

$$\begin{aligned} \sqrt{3a^2-1}\cos(2aL) - 3a\sin(aL)\sinh(\sqrt{3a^2-1}L) \\ = \sqrt{3a^2-1}\cos(aL)\cosh(\sqrt{3a^2-1}L). \end{aligned} \tag{4.20}$$

   If one neglects the terms $e^{-L\sqrt{3a^2-1}}$ as $a \to \pm\infty$, one gets

$$e^{L\sqrt{3a^2-1}} = \frac{\cos(2aL)}{2\cos(aL - \pi/3)} \tag{4.21}$$

   and hence there exists, for $k \in \mathbb{N}$ large enough, a unique solution $a_{k+N_L}$ (respectively $a_{-k-N_L}$) in the interval

$$\left[k\frac{\pi}{L}, (k+1)\frac{\pi}{L}\right], \quad \left(\text{respectively } \left[-(k+1)\frac{\pi}{L}, -k\frac{\pi}{L}\right]\right)$$

defined by equation (4.20) and given asymptotically by

$$a_k = \frac{5\pi}{6L} + \frac{k\pi}{L} + O(\frac{1}{k}) \quad \big(\text{respectively } a_{-k} = -\frac{5\pi}{6L} - \frac{k\pi}{L} + O(\frac{1}{k})\big). \tag{4.22}$$

The associated eigenfunction, $\phi_k$ is

$$\phi_k(x) = \alpha_k \left[ e^{-ia_k x} \left( \cosh(\sqrt{3a_k^2 - 1}x) + \frac{e^{3ia_k L} - \cosh(\sqrt{3a_k^2 - 1}L)}{\sinh(\sqrt{3a_k^2 - 1}L)} \sinh(\sqrt{3a_k^2 - 1}x) \right) - e^{2ia_k x} \right], \tag{4.23}$$

where $\alpha_k$ is chosen in such a way that $\|\phi_k\|_{L^2(0,L)} = 1$. Asymptotically, one sees that $(\alpha_k)$ converge to $1/\sqrt{L}$ as $k$ goes to $\infty$.

From (4.22), one deduces the asymptotic behavior of the eigenvalues and therefore the proof of this proposition is complete. ■

**Remark 4.3.2** *We easily deduce from equations (4.13) and (4.20) that*

$$\forall k \in \mathbb{Z}, \quad a_k = -a_{-k} \quad \textit{and} \quad \lambda_k = -\lambda_{-k}.$$

**Remark 4.3.3** *Similar asymptotical behaviors have been found out in [45] and [54].*

From the proof of the last proposition, we deduce the following lemma.

**Lemma 4.3.4** *There exists a constant $C$ such that*

$$\lim_{k \to +\infty} \frac{\phi_k'(L)}{k} = C \tag{4.24}$$

### 4.3.2 Ingham's inequality

Given the asymptotic behavior of the eigenvalues of $A$, we have to modify the choice of the state space $L^2(0, L)$ in order to prove (H3) and (H4). From the previous section, we know that $\{\phi_k\}_{k\in\mathbb{Z}}$ is a basis of $L^2(0, L)$. Thus, for any $f \in L^2(0, L)$ there exists a unique sequence $\{f_k\}_{k\in\mathbb{Z}}$ with $\sum_{k\in\mathbb{Z}} |f_k|^2 < \infty$ such that

$$f = \sum_{k\in\mathbb{Z}} f_k \phi_k \quad \text{and} \quad \|f\|_{L^2(0,L)} = \Big(\sum_{k\in\mathbb{Z}} |f_k|^2\Big)^{1/2}.$$

Let us now define some useful spaces.

**Definition 4.3.5** *Let us denote by $Z$ the linear hull of the basis functions $\{\phi_k\}_{k\in\mathbb{Z}}$. Then $Z$ is a dense subspace of $L^2(0,L)$. For any $s\in\mathbb{R}$ we define the space $H_s$ as the completion of $Z$ with respect to the norm defined by*

$$\Big\| \sum_{k\in\mathbb{Z}} c_k\phi_k \Big\|_s := \Big( \sum_{k\in\mathbb{Z}} (1+|\lambda_k|)^{\frac{2}{3}s} |c_k|^2 \Big)^{1/2}. \tag{4.25}$$

*In each space $H_s$, one has the orthonormal basis $\{(1+|\lambda_k|)^{-\frac{s}{3}}\phi_k\}_{k\in\mathbb{Z}}$.*

With this definition we can state the following well-posedness result whose proof is direct from the previous analysis.

**Proposition 4.3.6** *For any $z_0 = \sum_{k\in\mathbb{Z}} z_0^k \phi_k \in H_s$, there exists a unique solution of the homogeneous problem*

$$\begin{cases} z_t + z_x + z_{xxx} = 0, z(0,\cdot) = z_0, \\ z(t,0) = z(t,L) = 0, z_x(t,L) - z_x(t,0) = 0, \end{cases} \tag{4.26}$$

*which belongs to $C(\mathbb{R}, H_s)$ and is given by*

$$z(t,x) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k(x). \tag{4.27}$$

*Moreover, as $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, one has that*

$$\forall t\in\mathbb{R}, \quad \|z(t,\cdot)\|_s = \|z_0\|_s.$$

Now, we are interested in the regularity needed to obtain $z_x(\cdot,L)\in L^2(0,T)$ for any $T>0$. As one has at least formally,

$$z_x(t,L) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t} z_0^k \phi_k'(L),$$

one sees the importance of Lemma 4.3.4 which gives us the asymptotic behavior of $\phi_k'(L)$ as $k$ tends to $\pm\infty$. In order to find the regularity needed, we use the following classical result mainly due to Ingham (see [24] and [29]).

**Lemma 4.3.7 (Ingham's inequality)** *Let $T>0$. Let $\{\beta_k\}_{k\in\mathbb{Z}}\subset\mathbb{R}$ be a sequence of pairwise distinct real numbers such that*

$$\lim_{|k|\to+\infty} \beta_{k+1} - \beta_k = +\infty.$$

*Then there exist two strictly positive constants $C_1$ and $C_2$ such that for any sequence $\{\gamma_k\}_{k\in\mathbb{Z}}$ satisfying $\sum_{k\in\mathbb{Z}}\gamma_k^2 < \infty$, the series $f(t) = \sum_{k\in\mathbb{Z}}\gamma_k e^{i\beta_k t}$ converges in $L^2(0,T)$ and satisfies*

$$C_1\sum_{k\in\mathbb{Z}}\gamma_k^2 \le \int_0^T |f(t)|^2 dt \le C_2\sum_{k\in\mathbb{Z}}\gamma_k^2. \tag{4.28}$$

Let us apply this lemma. Let $z_0 = \sum_{k\in\mathbb{Z}} z_0^k\phi_k \in H_s$ for some $s \ge 0$. We want to take $\beta_k = \lambda_k$ and $\gamma_k = z_0^k\phi_k'(L)$. From the asymptotic behavior, we have that if $s \ge 1$, then

$$\sum_{k\in\mathbb{Z}} |z_0^k|^2|\phi_k'(L)|^2 < \infty.$$

This together with Proposition 4.3.1 allow us to apply the Ingham's inequality and get the existence of two constants $C_1$, $C_2 > 0$ such that

$$C_1\sum_{k\in\mathbb{Z}} |z_0^k|^2|\phi_k'(L)|^2 \le \int_0^T |z_x(t,L)|^2 dt \le C_2\sum_{k\in\mathbb{Z}} |z_0^k|^2|\phi_k'(L)|^2, \quad \forall z_0 \in H_s,\ s \ge 1. \tag{4.29}$$

We can estimate by above the right-hand side in terms of the $H_1$-norm of $z_0$, and consequently, in terms of any $H_s$-norm with $s \ge 1$. To get inequalities (H3) and (H4) in the space $H_1$, we need to estimate by below the left-hand side in terms of the $H_1$-norm of $z_0$. In order to do that we can not lose any coefficient $z_0^k$. Thus, the condition

$$\forall k \in \mathbb{Z}, \quad \phi_k'(L) \neq 0,$$

is required. From the work of Rosier in [39], we know that if $L$ satisfies

$$L \notin N := \left\{2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k,l \in \mathbb{N}^*\right\}, \tag{4.30}$$

then, there exist no $\mu \in \mathbb{C}$, $\varphi \in H^3(0,L)\backslash\{0\}$ satisfying

$$\begin{cases} \mu\varphi + \varphi' + \varphi''' = 0, \\ \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0. \end{cases} \tag{4.31}$$

In particular, this implies that $\phi_k'(L) \neq 0$ for any $k \in \mathbb{Z}$ and therefore from (4.29), we get the existence of positive constants $c_T$ and $C_T$ such that

$$c_T\|z_0\|_{H_1}^2 \le \int_0^T |z_x(t,L)|^2 dt \le C_T\|z_0\|_{H_1}^2, \quad \forall z_0 \in H_1. \tag{4.32}$$

The left-hand inequality in (4.32) is called an observability inequality and as we will see below it implies an exact controllability result. This ends the proof of (H3) and (H4).

### 4.3.3 Controllability

A direct consequence of (4.32) is the exact boundary controllability of our control system (4.6). Let us define what we mean by a solution of this system.

**Definition 4.3.8** *Let $T > 0$ be fixed. Let $y_0 \in H_{-1}$ and $v \in L^2(0,T)$. A solution of the Cauchy problem*

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ y(0,.) = y_0, \\ y(t,0) = y(t,L) = 0, \ y_x(t,L) - y_x(t,0) = v(t), \end{cases} \tag{4.33}$$

*is a function $y \in C([0,T], H_{-1})$ satisfying $y(0) = y_0$ and*

$$\forall \tau \in [0,T], \forall z_0 \in H_1, \quad \langle y(\tau), z(\tau)\rangle_{H_{-1},H_1} = \langle y_0, z_0\rangle_{H_{-1},H_1} + \int_0^\tau z_x(t,L)v(t)dt,$$

*where $z \in C([0,T], H_1)$ is the solution of the Cauchy problem*

$$\begin{cases} z_t + z_x + z_{xxx} = 0, z(0,\cdot) = z_0, \\ z(t,0) = z(t,L) = 0, z_x(t,L) - z_x(t,0) = 0. \end{cases} \tag{4.34}$$

With this definition, we obtain the following result whose proof is classical and hence omitted here (see for example [29, page 13]).

**Proposition 4.3.9** *Let $T > 0$. Let $y_0 \in H_{-1}$ and $v \in L^2(0,T)$. Then, the problem (4.33) has a unique solution.*

Let us now focus our attention on the controllability problem. It is a classical result, that the observability inequality previously proved in this chapter implies the following theorem.

**Theorem 4.3.10 (Exact controllability)** *Let $T > 0$ and $L > 0$ be such that $L \notin N$. Let $y_0, y_T \in H_{-1}$. Then there exists a control $v \in L^2(0,T)$ such that the solution of (4.33) satisfies $y(T,\cdot) = y_T$.*

**Remark 4.3.11** *Using the Hilbert Uniqueness Method (see [34]) one can choose a control $v \in L^2(0,T)$ of minimal $L^2$-norm among all the controls driving the system from $y_0$ at $t = 0$ to $y_T$ at $t = T$.*

**Remark 4.3.12** *In order to have a controllability result in more regular spaces, one may apply the method used in [51] and [20]. This method mainly consists in considering more regular controls allowing us to derive the equation. Applying that, one could get Theorem 4.3.10 in the space $H_2$ with controls in $H^1(0,T)$.*

## 4.4 Rapid stabilization

In this section, we apply Urquiza's method to our linear control system (4.6). Let us design the feedback laws allowing us to get the rapid stabilization result. We first define, for any $q_0$ and $\psi_0 \in H_1$, the bilinear form

$$a_\omega(q_0, \psi_0) := \int_0^\infty e^{-2\omega\tau} q_x(\tau, L)\psi_x(\tau, L)d\tau, \tag{4.35}$$

where $q$ and $\psi$ are the respective solutions of

$$\begin{cases} q_\tau + q_x + q_{xxx} = 0,\ q(0,.) = q_0, \\ q(\tau, 0) = q(\tau, L) = 0,\ q_x(\tau, L) - q_x(\tau, 0) = 0 \end{cases} \tag{4.36}$$

and

$$\begin{cases} \psi_\tau + \psi_x + \psi_{xxx} = 0,\ \psi(0,.) = \psi_0, \\ \psi(\tau, 0) = \psi(\tau, L) = 0,\ \psi_x(\tau, L) - \psi_x(\tau, 0) = 0. \end{cases} \tag{4.37}$$

We then define the operator $\Lambda_\omega : H_1 \longrightarrow H_{-1}$ assumed to satisfy

$$< \Lambda_\omega q_0, \psi_0 >_{H_{-1},H_1} = a_\omega(q_0, \psi_0), \quad \forall q_0, \psi_0 \in H_1. \tag{4.38}$$

Finally, we define the following operator

$$\begin{aligned} F_\omega : H_1 &\longrightarrow \mathbb{R} \\ z &\longrightarrow F_\omega(z) := -q_0'(L), \end{aligned} \tag{4.39}$$

where $q_0$ is the solution of the following Lax-Milgram problem

$$a_\omega(q_0, \psi_0) =< z, \psi_0 >_{H_{-1},H_1}, \quad \forall \psi_0 \in H_1. \tag{4.40}$$

From section 4.3 and Theorem 4.2.1 one easily gets the following result.

**Theorem 4.4.1** *Let $\omega > 0$. The closed-loop system*

$$\begin{cases} y_t + y_x + y_{xxx} = 0,\ y(0,.) = y_0, \\ y(t, 0) = y(t, L) = 0,\ y_x(t, L) - y_x(t, 0) = F_\omega(y(t)), \end{cases} \tag{4.41}$$

*is globally well posed in $H_1$. Moreover, the solutions decay to zero with an exponential rate of $2\omega$, i.e.,*

$$\exists C > 0, \forall y_0 \in H_1, \quad \|y(t, \cdot)\|_{H_1} \le Ce^{-2\omega t}\|y_0\|_{H_1}.$$

## 4.5 Numerical simulations

Let $\omega > 0$ be fixed. We use the Galerkin method and an approximation by modal superposition to decompose our solutions as in [4]. Let $(i\lambda_k, \phi_k)$ be the eigenmodes of

$$\begin{cases} -\phi_k' - \phi_k''' = i\lambda_k\phi_k, \\ \phi_k(0) = \phi_k(L) = 0, \\ \phi_k(L) - \phi_k(0) = 0. \end{cases}$$

Let us define

$$V_N = Span\{\phi_{-N}, ..., \phi_{-1}, \phi_1, ..., \phi_N\}, \quad \forall N \in \mathbb{N}^*.$$

For any $z_0 \in H_1$ let $q_N^0 \in V_N$ be the unique solution of the variational equation

$$a_\omega(q_N^0, \psi_N^0) = \int_0^\infty e^{-2\omega\tau} q_{N\,x}(\tau, L)\psi_{N\,x}(\tau, L)d\tau = \int_0^L z_0(x)\psi_N^0(x)dx, \qquad \forall \psi_N^0 \in V_N, \tag{4.42}$$

where $q_N$ and $\psi_N$ are the respective solutions of

$$\begin{cases} q_{N\,\tau} + q_{N\,x} + q_{N\,xxx} = 0, \\ q_N(\tau, 0) = q_N(\tau, L) = 0, \\ q_{N\,x}(\tau, L) - q_{N\,x}(\tau, 0) = 0, \\ q_N(0, .) = q_N^0 \end{cases} \tag{4.43}$$

and

$$\begin{cases} \psi_{N\,\tau} + \psi_{N\,x} + \psi_{N\,xxx} = 0, \\ \psi_N(\tau, 0) = \psi_N(\tau, L) = 0, \\ \psi_{N\,x}(\tau, L) - \psi_{N\,x}(\tau, 0) = 0, \\ \psi_N(0, .) = \psi_N^0. \end{cases} \tag{4.44}$$

We define the discrete operator,

$$L_N : z_0 \mapsto q_N^0.$$

As $(q_N^0, \psi_N^0) \in V_N \times V_N$ we can write $q_N^0 = \sum_{k=-N}^N \gamma_k^0\phi_k(x)$ and $\psi_N^0 = \sum_{k=-N}^N \hat{\gamma}_k^0\phi_k(x)$ and we easily deduce from (4.43) and (4.44) that

$$q_N(\tau, x) = \sum_{k=-N}^{N} e^{i\lambda_k\tau}\gamma_k^0\phi_k(x), \tag{4.45}$$

$$\psi_N(\tau, x) = \sum_{k=-N}^{N} e^{i\lambda_k\tau}\hat{\gamma}_k^0\phi_k(x), \tag{4.46}$$

Let us define $m_k = \phi_k'(L)$ for $k \in \mathbb{N}^*$. Then

$$a_\omega(q_N^0, \psi_N^0) = \sum_{k,j=-N}^{N} \frac{\gamma_k^0 \hat{\gamma}_j^0 m_k m_j}{2\omega - i\lambda_k - i\lambda_j}. \tag{4.47}$$

In order to solve the stabilization problem we write the problem in a weak form where the boundary term appears. We multiply (4.6) by $w \in D(A)$ and get by integration by parts

$$\int_0^L y_t w dx - \int_0^L y(w_x + w_{xxx}) dx = v(t) w_x(L). \tag{4.48}$$

We take as an approximation of the controlled solution, $y_N : [0, \infty] \to V_N$ solution of

$$\int_0^L y_{N\,t}(t,x) w(x) dx - \int_0^L y_N(t,x)(w_x + w_{xxx}) dx = v_N(t) w_x(L), \quad \forall w \in D(A), \tag{4.49}$$

where the stabilizing control is chosen as

$$v_N(t) = \frac{\partial}{\partial x} L_N(y_N(t,.))$$

with initial data, $y_N(0) = P_N(y_0)$ ($P_N$ is the orthogonal projection on $V_N$). Let $q_N^0(t) = L_N(y_N(t,.))$. It satisfies

$$\begin{aligned} \sum_{k,j=-N}^{N} \frac{\gamma_k^0 \hat{\gamma}_j^0 m_k m_j}{2\omega - i\lambda_k - i\lambda_j} &= \int_0^L \sum_{-N}^{N} y_k(t)\phi_k(x) \sum_{-N}^{N} \hat{\gamma}_l^0 \phi_l(x) \\ &= \sum_{k,l=-N}^{N} y_k(t) \gamma_l^0 \int_0^L \phi_k(x)\phi_l(x) \\ &= \sum_{k=-N}^{N} y_k(t)\gamma_k^0 \end{aligned} \tag{4.50}$$

Let $A_\omega$ be the matrix with coefficients

$$(A_\omega)_{l,j} = \frac{m_l m_j}{2\omega - i\lambda_l - i\lambda_j}.$$

Let $J$ be the one antidiagonal matrix, $\Gamma_N^0$ the vector of coefficients $\gamma_k^0$ and $Y_N(t)$ the vector of coefficients $y_k(t)$ . Then we get

$$\Gamma_N^0 = A_\omega^{-1} J Y_N(t) \tag{4.51}$$

and the control is

$$v_N(t) = \sum_{k=-N}^{N} [A_\omega^{-1} J Y_N(t)]_k m_k.$$

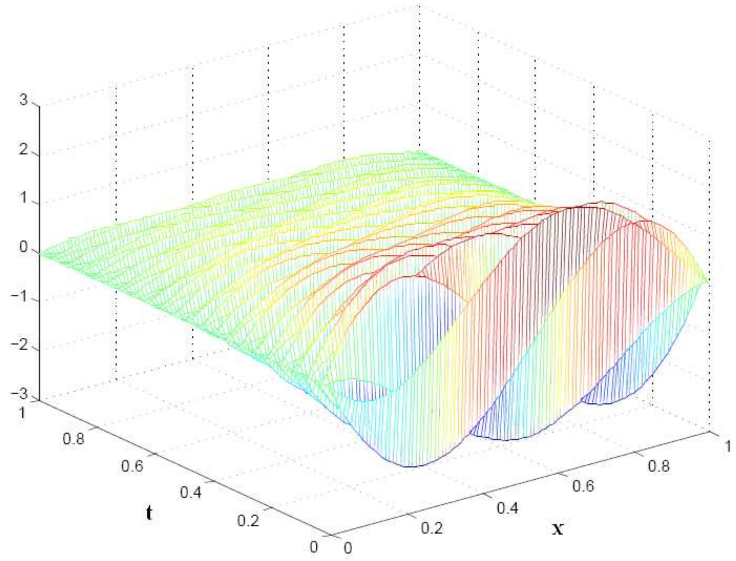


FIG. 4.1 – Evolution of the solution $y$ with $\omega = 2$.

So with (4.49), choosing $w = \phi_{-k}$ for $k = -N, ..., -1, 1, ..., N$,

$$Y_N'(t) + \begin{pmatrix} -i\lambda_{-N} & & \\ & \ddots & \\ & & -i\lambda_N \end{pmatrix} Y_N(t) = KY_N(t) \tag{4.52}$$

with $K$ the matrix with coefficients

$$(K)_{l,j} = m_{-l} \sum_{k=-N}^{N} [A_\omega^{-1} J]_{kj} m_k.$$

The matrix equation (4.52) can easily be solved. Results are drawn on some figures for $N = 10$ and the initial condition given by $Y_N^0(k) = 1$, for $k = -N, ..., -1, 1, ...N$. On Figure 4.1 and Figure 4.2, we show the evolution of the solution for $\omega = 2$ and $\omega = 3$ respectively. Note particularly, on Figure 4.3, the excellent agreement between theoretical and numerical results for the time-evolution of the $H_1$-norm.

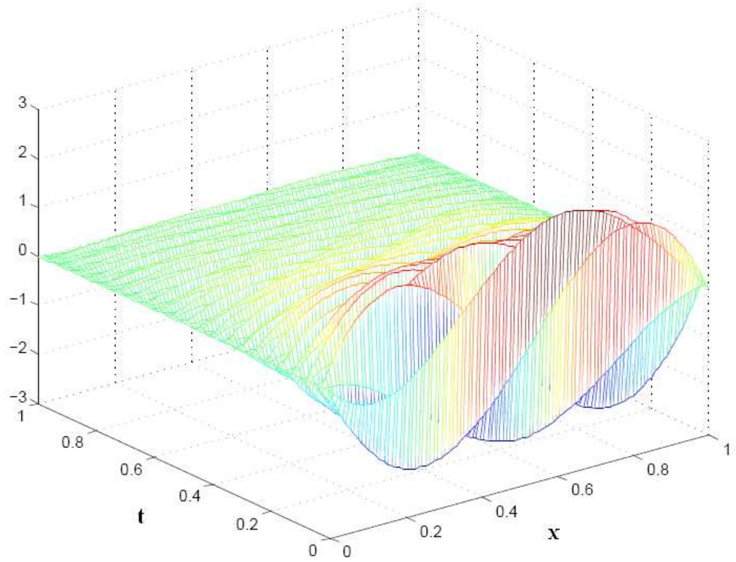


FIG. 4.2 – Evolution of the solution $y$ with $\omega = 3$.

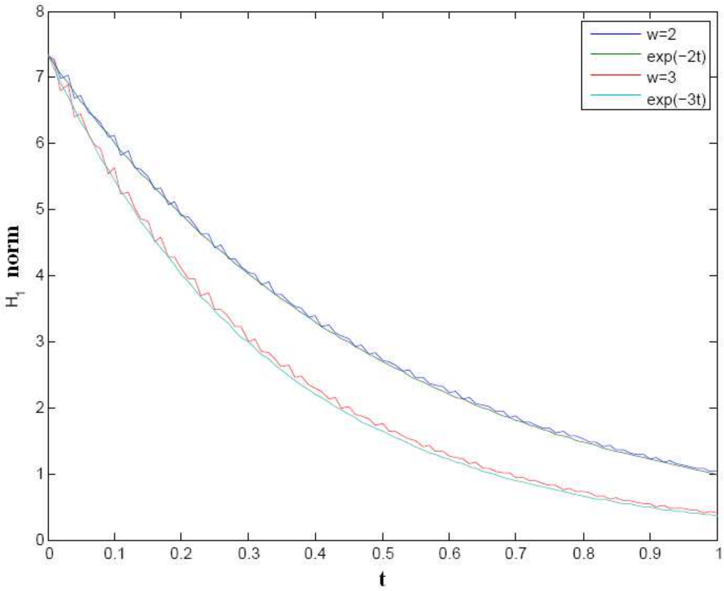


FIG. 4.3 – Time-evolution of the norm $\|y\|_{H_1}$ compared with $e^{-\omega t}\|y_0\|_{H_1}$ for $\omega = 2$ and $\omega = 3$.

# Chapitre 5

# Conclusion et perspectives

## Sommaire

## 5.1 Conclusion

Dans les chapitres 2 et 3 de cette thèse, nous avons étudié la contrôlabilité frontière de l'équation de KdV sur un domaine borné. Nous nous sommes intéressés à des domaines dont la longueur est dite critique car il y une perte de contrôlabilité exacte pour le système linéarisé autour de l'origine. Ceci est dû à l'existence d'un sous espace de dimension finie composé des directins dans lesquelles on ne peut pas amener le système de contrôle linéaire si l'on part de zéro. Nous avons montré que la non linéarité nous permet d'obtenir la contrôlabilité pour le système non linéaire dans tous les cas critiques. Ceci a été fait en utilisant, d'abord, un développement d'ordre supérieur à un pour la solution et le contrôle de notre système afin de mener ce dernier vers les directions manquantes. Ensuite, avec un argument de point fixe nous avons obtenu le résultat cherché pour le système non linéaire. Malheureusement, notre approche nous a amené à ajouter une condition sur le temps de contrôle : on montre la contrôlabilité uniquement pour des temps suffisamment grands.

Dans le chapitre 4 de ce manuscrit, nous avons abordé la question de la stabilisation par feedback pour l'équation linéaire de KdV sur un domaine borné. Nous avons montré que pour un paramètre positif $\omega$ quelconque il existe une loi de feedback telle que le système en boucle fermée ait une décroissance exponentielle vers zéro avec un taux au moins égal à $2\omega$. Pour construire de tels feedbacks, nous avon utilisé une méthode, due à Urquiza, qui garantit que, sous certaines hypothèses, le système en boucle fermée est bien posé et que l'on a la décroissance exponentielle cherchée. Nous avons aussi illustré ce résultat théorique par quelques simulations numériques.

## 5.2 Perspectives

A partir des travaux développés dans cette thèse, certaines questions apparaissent naturellement. Nous allons décrire donc dans cette section quelques problèmes auxquels nous pourrons nous intéresser par la suite.

### 5.2.1 Temps minimal de contrôle

A-t-on, pour un temps de contrôle quelconque, la contrôlabilité exacte montrée dans les chapitres 2 et 3 ? Nous avons montré qu'une condition sur le temps de contrôle est suffisante, mais est-elle nécessaire ? Il n'y a pas *a priori* de réponse évidente à cette question car même si la vitesse de propagation pour l'équation de KdV est infinie, il pourrait exister un temps minimal de contrôle. Par exemple dans [2], Beauchard et Coron montrent, pour des temps assez grands, la contrôlabilité exacte autour d'une trajectoire propre pour l'équation de Schrödinger et dans [13] et [14] Coron montre que ce résultat n'est plus vrai pour des temps petits, même si l'équation de Schrödinger a une vitesse de propagation infinie. En se basant sur des calculs d'ordre deux explicites développés, on conjecture, pour certaines longueurs critiques au moins, l'existence d'un temps de contrôle minimal tout comme pour Schrödinger.

### 5.2.2 Stabilité du système non linéaire dans les cas critiques

On a vu que dans l'étude de la contrôlabilité dans les cas critiques, la non linéarité de KdV joue un rôle très important. Le comportement du système linéaire est très différent de celui du système non linéaire. Ainsi, on peut se demander si la non linéarite joue aussi un rôle majeur dans l'étude de la stabilité, c'est-à-dire, lorsque on n'a pas de contrôle (on prend $u = 0$). Pour répondre à cette question, une méthode vraiment non linéaire est nécessaire, mais la nôtre ne semble pas être la plus adéquate car elle a besoin des contrôles pour faire jouer les ordres supérieurs.

### 5.2.3 Stabilisation rapide pour le système non linéaire

Dans le cas non critique, on a construit pour le système linéaire des feedbacks exponentiellement stabilisant avec un taux de décroissance arbitrairement grand. Une question naturelle est si ces mêmes lois de feedback forcent le système non linéaire à avoir une décroissance exponentielle vers l'origine au moins localement avec un taux de décroissance arbitrairement grand.

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