The theory of combinatorial maps and its use in the graph-topological computations.

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Abstract

In this work we investigate combinatorial maps, see [2, 11, 12, 18, 19, 22, 23], applying the geometrical idea of considering the corners between the edges in the embedding of the graph on the surface to be the elements on which the permutations act [58].

Combinatorial maps as well partial combinatorial maps are considered, theory of cycle covers, that give objects that correspond to the cycles in the graphs, are developed. Some formulas in permutations are found that calculate useful characteristics of maps. The aim of the work is to find useful applications: finding some characteristics computable in permutations that has graph-theoretical characteristics in correspondence.

The computer system is implemented that computes the formulas and algorithms obtained from the theory.

1 Preface

In this work we consider combinatorial maps that have the graphs on orientable surfaces in correspondance. We are working only with combinatorial maps in permutation technique because of the applications that we are aiming at and because of the reasons explained further. This work in its main outline is the result of three articles, i.e. [58, 60, 61].

We have discovered the combinatorial maps in the present outline independently from the other authors, and this fact makes our consideration distinct from the authors, which have this subject developed in the mathematical community. We base our geometrical interpretations only on the one particular, namely, elements in our permutations are the corners between the edges in the graphs embedding on the surface. Secondly, we are speaking about submaps and subgraphs but not about multimaps and multigraphs as other authors do, for example [23]. We speak about knots whereas some authors speak about zigzag walks[see [2]]and other objects [12].

In the chapters 3 we consider permutations in general and in 10 some technique is developed in the case when some partitioning for the set whereupon the permutations act is given. In the chapter 5 the general theory of combinatorial maps is developed, that is discovered by the author independently. Here the statement of the direct correspondence between permutations and maps, i.e. graphs is made. This statement we put in the category of some principle and this principal we use in some principal decisions about the use of the results of our theory.

In the chapter 6 we develop the theory of partial maps. The notion of the image and shortened image of the partial map is entered, that prove to be very useful in the calculations of different maps, accomplished in the chapter 11.

The theory of cycle covers both non-colored and colored are developed correspondingly in the chapters 8 and 9. Through the notion of the cycle from the cycle cover we get the access to the notion of the cycle in the graph on surface and so to the topology of the graph embedded in the surface. Here we try to use the principles: to find some simple objects in permutations that have graph-theoretical invariants in the correspondence.

In the chapter 12 we show our experience in the use of the results of the theory in the experimental system of the calculation of the permutations and the types of the maps and the relations between them obtained in the theory.

In the chapter 14 we show the history of the way of our evolution in developing effective graph-theoretical algorithms and how it led to the subject discussed here.

Contents

2 Introduction

Graph theory has its well known advantages in the possibility to use its invariants as simple objects where precise mathematical notions can be encapsulated in intuitively well understandable geometrical objects. Therefore the graph theory is increasingly used in different fields of mathematics , not only in theory but also in computer algorithms.

This last application of the graph theory brings forward new requirements for the graph theory itself. Simply speaking, we want more and more things in computer algorithms be expressed in some object oriented way of thinking, where one way of expressing these simple objects are just graphs and invariants in them. So we are tending to intertwine ourselves in the graphs more and more to use them further in algorithms.

The graph drawing, for example, is a field that is developing extremely rapidly. We are still far from being able to add something substantially to this subject in our theory, but nevertheless this motivates us to develop our theory.

The author has worked intensively on constructing effective graph-theoretical algorithms for some years and he has come to the conclusion that some rotational features in the graphs need to be more developed in order that they could be used in the algorithms. We get the simplest and most beautiful rotational scheme in graphs when we fix the order of the neighbors around the vertices and generate edges in the order as they appear one after another giving us the facets on some surface. It was proved by Edmonds [4], that they actually are faces of the graph embedded on an orientable surface, while the process itself was already well-known to Hefter [7].

2.1 Preliminary considerations on the rotational schemes

The earliest graph rotation scheme is the one found by Hefter [7], rediscovered by many other mathematicians, and rediscovered also by D. Zeps [47]. Particularly this scheme gives us what we call the rotation of a graph.

Let us imagine a the code of a graph be given as a list of vertices where each vertex has the fixed rotation of its neighboring vertices (or incident edges) around it in correspondence. These local rotations fix the global rotation of the whole graph that we obtain in the following way: let us chose an arbitrary vertex from the vertex list.

Next let us take the other end of some arbitrary chosen incident edge to this vertex and mark this edge as visited and the new vertex as the currant vertex, then let us chose the next vertex in the corresponding rotation after the vertex that was current before and continue in this way further. Since our graph is finite later or sooner we should come to the first chosen vertex. As a result, the cycles of the neighboring vertices generate the faces of this graph, whom we could image as beeing embedded on some orientable surface with just these faces.

Let us show this more formally as a pseudocode. Let $adj(v)$ be cyclical list of neighboring vertices of vertex v and $next(l, a)$ be next vertex in the list l after vertex a. Then

let $v \in V and w \in adj(v)$

$a := v; b := w; FLAG := true;$
while FLAG do
let $c = next(adj(b), a))$ in
$a := b; b = c; FLAG := b = v;$
end.

This loop should always stop. Why? Because for every vertex b this loop marks as c and a two of the members of $adj(b)$, except for the first vertex $b = v$. Thus, this loop will generate some closed sequence of vertices. When some other vertex is chosen, that was not visited by this first walk through the graph, then the visited vertices in the next walk should be distinct from already visited in the previous walk. And so on.

Usually dual graphs are defined both notionally and actualy only for the planar graphs. For an arbitrary graph it is reasonable to speak about the dual graph only when it has a fixed rotation. Thus, the rotation of the edges around the vertices and the generated global rotation fixes graphs dual graph as well. Further we will show that this holds also for subrotations, where by a subrotation we understand the rotation that is generated by the restriction on some subset of the neighbors.

When a graph is coded in the usual way as pair of the sets of vertices and edges, in the dual graph the vertices have the faces in correspondence and the edges have the edges in correspondence. Thus, the scheme of duality in this setting is not dual itself. Yet it is possible to change the vertices with some other objects that are dual with itself (similarly as the with the edges previously) in the duality scheme. These new objects are the corners between the two sequencing edges in the rotation. Solely, these corners do not suffice to determine the graph. We need to take also the dual rotation of the corners as well. In this way we obtain an additional symmetry by the fact that the dual object is not distinguishable from the object itself, what we denote as self-duality.

We can come to the notion of the corner as a useful topological object yet in a more simple way. Usually, speaking about the graphs on orientable surfaces it is assumed that the edges incident with the vertices are cyclically ordered, i.e. the rotation of such graphs are fixed. Yet as a matter of fact, the objects that actually are fixed, are the ordered pairs of the edges that follow one another in the cyclical order.

Now, we call such pairs of edges in the rotation *corners* due to their natural geometrical interpretation. Thinking right in the terms of corners we may observe, that the corners around vertices are ordered cyclically. Moreover, the cyclical sequences of the corners of different vertices do not overlap, hence they form together one common permutation on the set of all corners. This very fact is the most crucial in our theory.

Let us fix our attention on this permutation of the cyclical sequence of the corners.

We look on this permutation as the rotation of this graph. Further, if we "forget" the other information about the graph except this one permutation, then the graph cannot be restored uniquely, but if we take the corresponding permutations of the graph and its dual graph together, then the graph is restorable (up to the isomorphism), yet of course together with its rotation.

In general, two arbitrary permutations p_1 and p_2, define some graph with a fixed rotation (up to the isomorphism)whenever all orbits of $p_1^{-1} \cdot p_2$ are transpositions,. Here we are dealing just with such combinatorial objects; we called them *(combinatorial) maps*[23], which are pairs of permutations [11, 2].

Further, we define a binary operation on the maps that corresponds to the multiplication of the permutations. This operation is defined only if the product $p_1^{-1} \cdot p_2$ is equal for both of the maps and hence it is common for all the set of the maps that is closed under this operation. This set is as large as the set of all permutations (of some fixed degree).

Thus, in the frames on such closed class of the maps each map from this class is determined only by one permutation. Consequently, these facts give us possibility to establish one-one map between all maps (i.e. graphs on surfaces) and all permutations. We have arrived at the situation where one permutation corresponds just to one graph, and, if we consider this permutation as the rotation of this graph, then this graph is given when this rotation is given.

This insight enables us to speak about the graphs as the rotations instead of graphs with (fixed) rotations, whenever we consider them as the whole set, that is closed under one common operation, i.e. the multiplication of the rotations.

Combinatorial maps have been studied in many papers [17, 18, 19, 23, 22, 11, 2]. Particularly successful has been approach of so called 3-graphs [11, 2] that embrace both graphs and multigraphs on both orientable and unorientable surfaces. Yet in our corner approach we confine ourselves on graphs on orientable surfaces. The only generalization, that we intensively examine are partial maps, that are defined as arbitrary pairs of permutations. Behind the objects that are pairs of permutations without any constriction we are being predisposed just to see partial graphs and subgraphs rather than multigraphs [compare [23]].

Most of the results in this work are well know, but before appearing in sufficiently conceivable textbooks may be rather unfamiliar to those that are working in the graph theory and are less familiar with combinatorics. We aim to use this theory in the graph theoretical algorithms as much as possible. We aim to work with this theory in order to turn more graph theorists interest to these questions, to be in the role of some disseminator of the idea, that graph rotational schemes must not be overlooked in the traditional graph theory. Not completely the least motivation to consider these well known results in this work is that the author has discovered them independently.

2.2 Preliminary considerations on topology

In this work we are considering combinatorial maps, but with the aim to use them in the graph applications, thus all we do in the theory are done with the direct look in the graph-theoretical background and possible applications in it and in possibility to obtain new geometrical interpretations that would lead to more concise picture of the whole subject, where the usual concepts of graph-theoretic nature and the graph-topological notions and the notions of the graph-rotational schemes are intertwined between themselves.

Here the most useful notion is the partial maps, because together with the notions of the image of the partial map and its shorter image gives very effective tools for investigations. Almost all the best facts found in this work are connected with partial maps. Thus, partial maps such as cycle cover cycle submap and cut submap, as well simply cycle submap turn out to be very useful. These subgraphs are very easy to be calculated, but they hide behind then non trivial and difficalt objects, that, as it seems to us, could be easy overlooked.

Working practically with the partial maps we have come to the conclusion that because of the very simple feature of them that every pair of permutations makes some partial map, whenever anyone pair has some geometrical interpretation, it turns out to be very useful. In this way some very interesting partial maps had been found only by combining already well known maps in pairs. Here is good to use a computer programm to quickly check every such a pair that raises some interest.

Yet the most hopful seems to us the fact that following the very simple principle that maps (or simply permutations) have graphs on surfaces in correspondence has given some considerable results. Actually, finding some graph-topological invariant that is sizable in maps sufficiently simply it is expressable as a simple formula in permutations. That sometimes works [see section 9.2.3], and the permutational formulas that we have found in this way give us hope, that this way must be continued. The use of the computer programm that gives us possibility this work to make sufficiently fast promises some progress in this direction.

2.3 Preliminary considerations on computations

The main aim of our work is the applications of the theoretical results. For this reason we have built an environment where theoretical results can be as directly as possible applied in corresponding algorithms. Permutations because of their simplicity are generally recognized tool for mathematical investigations, but in the same time they are possible to calculate very effectively, i.e. most useful operations with permutations can be done in linear time and space. Thus the multiplication of permutations, that is the main operation, as well other operations that are built based on some restrictions on specified subsets are all mostly linear operations.

Whenever our theory gives us a good formula in permutations with these simplest operations, obviously we immediately have an efficient procedure that computes this formula. So naturally we aim to find such formulas as many as possible. This type of operations we call *1-type* operation in the computer program, .i.e. such operation that simply computes some formula in permutations using the simplest operations all of them being expressed in the language of permutations and subsets and restrictions on them.

Sometimes we have an operation, that can be calculated effectively with some simple algorithm, but not completely expressible in permutations and their restrictions. If the arguments and the result of this operation is expressible in permutations and it is near to that that some formula is expectable to be found that calculates it, then we call such operation *2-type* operation. In another case we call it *3-type* operation. Of course, no precise distinction could be made between the operations of the 2-type and 3-type. The distinction of the operations in this way shows more the present state of the development of the whole system.

The examples of 2-type and 3-type operations are given in the chapter 12. The one demonstrating this distinction is as follows: the computation of the greatest common submap we consider as the 2-type operation, because we have hope to find for it solution in permutations, i.e. to effect in the 1-type operation; but the finding of the maximal noncrossing subset of cross-edges we consider as the 3-type operation, because it essentially requires a graphical solution, i.e. the maximal independent subset must be found, followingly this is an NP-complete problem.

We have implemented the permutation calculus in the PASCAL program, that gives us possibility to reprove our ideas on different series of maps, that are entered either manually or generated randomly. In the environment in this way built, we both build the algorithms to do some computations and compute many things with the simplest formulas with permutations. At the present state the investigations on this sytem are intended to find sufficiently large set of permutational formulas with graph-topological operations in correspondence, to use it as an independent vehicle for topological calculus already without any algorithmical support.

2.4 Preliminary considerations on the use of the theory of combinatorial maps in computations

Before we saw that the coding the graph with fixed rotation, its edges appear two times in the lists. Doing the same for the corners, they do not repeat, i.e. the succession of corners is the substitution in the essence. We see here coincidence of two things: firstly, the corner is a definite and tangible object of the graph drawn on the surface, and secondly, at the same time corners are elements of some substitution. In this fact we are prone to see the main idea of our approach and all this work. We have an opportunity to build a combinatorial theory, where the abstract objects in the time are physical objects, or rather the code of them in the computer algorithms.

Generally thinking, may be it is a bad stile to build mathematical theory, that depends on some physical objects or some physical picture of the nature. General theories are those that are detached from their possibility beginnings of physical pictures. But in our case we believe, that our physical code is as simple as possible and also as general as possible, that it can not spoil the generality of the theory. As a result, our theory is as general as combinatorial theories are.

But still it may be dangerous to be bound too much to some physical nature of the object, because it can lead to the situation when, thinking seemingly abstractly, actually we can not find out whether we think really abstractly or we think in the terms of physical objects.

In our case the pay for our attempt is this peril that is always present of confusing real objects with their generalizations. But what we gain is this thing. Our permutational formulas are applicable direct to the code. It is very convenient in the situation that abstract objects are used in the programs. Let us assume, that we have some system already build that works with graphs, but without any connection with rotational schemes. Then, if only we might expect that in this system the corners between edges are computatible,we could apply our procedures directly to these objects.

3 Permutations

The main mathematical apparatus in this work is permutations and operations with them. Here we consider some notations that are used further in this work. We do not make difference between permutations and substitutions until we reach the chapter where we consider the coding of permutations in computer algorithms.

Speaking about the permutations, almost always we suggest that they are in cyclical representation,i.e. a permutation p is equal to a set of cycles c_i where $c \in [1..k]k > 0$. Then *fixed elements* in permutations are those that form cycles of the length one. These cycles may be left not written by suppression and solely suggested to be present. *Transpositions* are cycles of the length two.

With the Greek letters $\pi, \varrho, \sigma, \tau$, etc we denote permutations only with transpositions and fixed elements, being called *involutions*. Involutions are those permutations that multiplied by itself gives the identical permutation. Yet more, involution are the permutations that coinsides with its reverse permutation, i.e. that with elements written in opposite direction in its cyclical representation. Involutions without fixed elements we call *matchings*. Matchings partition the set on which they act into a set of pairs of elements that cover all this set.

Using permutations in the combinatorial scheme that has graphs behind it, our permutations act on a universal set C the elements of which we call *corners* (or sometimes simply *elements*). The number of corners are n.

For a permutation P and $c \in C$, c^P denotes that element of P to which c goes over. We are multiplying the permutations from left to right. For two permutations P and Q,

$$P^Q = Q^{-1} \cdot P \cdot Q$$

and this multiplication is a conjugate permutation to P (with respect to Q). When S is a cycle $(c_1c_2\ldots)$ in P, then S^Q is a cycle $(c_1^Q c_2^Q \ldots)$ in P^Q. (For arbitrary permutations P, Q, R it holds:

$$(P^Q)^R = P^{Q \cdot R}$$

and

$$(P \cdot Q)^R = P^R \cdot Q^R.$$

I denotes identical permutation, i.e. that with all elements being fixed.

One way to express a cycle $(c_1c_2\ldots c_k)(k > 0)$ of some permutation as the multiplication of transpositions is this:

$$(c_1c_2\ldots c_k) = (c_kc_{k-1})\ldots(c_kc_1).$$

In chapters further we more often use the name orbit for the name cycle, because cycles there should be used in other and, particularly, the graphical sense too.

We are revisiting the permutations in the chaper 10, where we consider permutations with the given partition of the set of corners into two parts and, particularly, into two equal parts.

4 Combinatorial maps. General approach

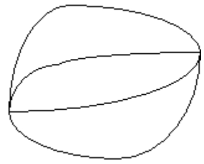
There are many ways to consider combinatorial maps. As is commonly accepted, the main is that considered in the work [22]. Very general way of developing the map theory is that of 3-graphs developed by Little [11] and now made manifest by the book [2].

5 Combinatorial maps

We now consider the simplest way of the combinatorial maps that have graphs on orientable surfaces behind them.

Let us have a set C with $2 \cdot m$ elements called corners. Permutations in this chapter act on this set C.

We say that two permutations P and Q of equal order n are*differing* whenever $i^P \neq i^Q$ for every $i \in [1\ldots n]$. An oriented pair of differing permutations (P, Q) we call combinatorial map whenever $P^{-1} \cdot Q$ $(= \pi)$ is involution. For every combinatorial map π is of course a matching. Truly, if we had some fixed element in π then permutations P and Q were not differing.



Picture 1: In the picture we see the corresponding graphs implementation

Example 1. *Let us consider an example. The natural number are used as denotations for corners in a graphs embedding in the combinatorial surface.*

$$P = (123)(45)(678)$$

$$Q = (164)(28)(357)$$

We can calculate the matching π *for this combinatorial map. It equals to* $\pi = (15)(26)(38)(47)$.

It is convenient to look upon some set of maps with the common matching π. In these cases, instead of (P, Q) we speak about a map P, keeping the matching π (and the second permutation Q equal to $P \cdot \pi$) in mind. Then we use to say that π is fixed and the map OP is considered with the respect to this π.

Let π be fixed and let it act on$2m$ corners of C dividing it into m pairs. Let (c_1, c_2) be such a pair, i.e. $c_1^\pi = c_2$. Then there exists such pair of corners (c, c') that $(c, c') \cdot P = P \cdot (c_1, c_2)$. These pairs are conjugate by P. We call the pair (c, c') *edge of the map* and (c_1, c_2) we call *next edge of the map.* We call matching π *next-edge-matching(n-matching)* of the map.

For a map (P, Q) there exists some other unique matching ϱ satisfying $\varrho \cdot P = P \cdot \pi$. We call the matching ϱ *edge-matching(e-matching) of the map.* We say that π *contains* next edges and ϱ contains edges of the map P , n-matching π being fixed.

For a map P (with fixed π) we denote $P \cdot \pi$ with P^*. They both have the same e-matching: from $\varrho^P = \pi$ follows $\varrho^{P \cdot \pi} = \pi^\pi = \pi$. We call (Q, P) (equal (P^*, Q^*))*dual map* to (P, Q) . From [22] we know that this duality coincides with the usual notion of the duality of maps, i.e. graphs on surfaces.

Example 2. *Example of a map:*

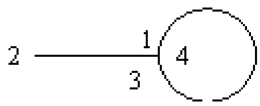
$$P = (143)$$

$$Q = (132)$$

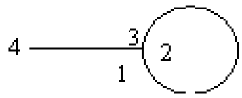
$$\pi = (12)(34)$$

$$\varrho = (14)(23)$$

This map has two vertices: (143) and (2) and two faces: (132) and (4). It has two edges: the edge (14) has the next edge (34) in correspondence, and the edge (23) has the next edge (34) in correspondence.



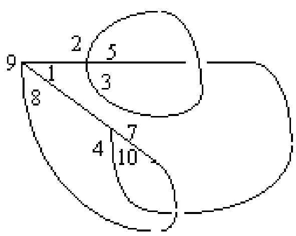
Picture 2: Drawing corresponding to this map



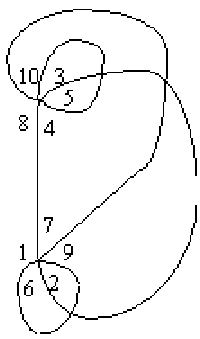
Picture 3: Drawing of the dual map (Q, P)

Example 3. *Example of a map:*

$$P = (189)(2536)(47\bar{0})$$
$$Q = (17926)(35480\bar{0})$$
$$\pi = (12)(34)(56)(78)(9\bar{0})$$
$$\varrho = (14)(23)(5\bar{0})(69)(78)$$



Picture 4: Drawing corresponding to the map in the example



Picture 5: Drawing of the dual map (Q, P)

5.1 Simple features of the combinatorial maps

Theorem 1. *Let $c \in C$ be a corner in (P, Q). Then (c^P, c^Q) is a next edge, and $(c^{P^{-1}}, c^{Q^{-1}})$ is an edge.*

Proof From the definition $P^{-1} \cdot Q$ is equal to π. Then

$$c^P = c^{Q \cdot \pi} = c_1^\pi,$$

where $c_1 = c^Q$, and the first pair in the theorems conditions equals to (c_1^π, c_1).

Further,

$$c^{Q^{-1}} = c^{P^{-1}\cdot\varrho} = c_2^{\varrho},$$

where $c_2 = c^{P^{-1}}$, and the second pair is (c_2, c_2^{ϱ}).

This theorem expresses what we observe when we look on the combinatorial map given in the cyclical form. The pair of elements that follows some element in P and Q is a next edge, and the pair before them is an edge.

Theorem 2. $\varrho^{P^{-1}} = \pi$, $\pi^P = \varrho$, $\varrho^{Q^{-1}} = \pi$, $\pi^Q = \varrho$.

It follows from the definition.

Theorem 3. *P [and Q] works as a map between the next edges and the edges in the sense that for a next edge (a,b) P [and Q] matches corresponding edge (c,d), i.e.*

$$(a,b)^P = (c,d), [(a,b)^Q = (c,d)].$$

It follows from the previous theorem.

5.2 Closed classes of combinatorial maps

We are going to convince ourselves that our choice to fix the matching π for a class of maps was very reasonable.

Permutations in cyclical representation has their orbits in the form where they do not overlap. But it isn't the only possible way to represent permutations. We can depict it in as many overlapping cycles as we like not changing the permutation as a whole. Then we must only agree ourselves in what direction we are reading the representation, i.e. multiply the permutations. We are reading from left to right. Consequently, we are multiplying both the cycles of permutation and the permutations in the whole from left to right too.

Let us define multiplication of two maps S and T (with fixed the same n-matching π) as usual multiplication of permutations, i.e. we put

$$(S_1, S_2) \cdot (T_1, T_2) = (S_1 \cdot T_1, S_1 \cdot T_2)$$

by definition, where on the left side of the expression the sign $'\cdot'$ stands for the multiplication of maps. In practice, writing $S \cdot T$ we mean both multiplication of permutations S and T and multiplication of corresponding maps with common n-matching π.

A multiplication for maps with different n-matchings is not defined. This agreement about the operation of multiplication gives us subsets of maps with a common n-matching, where the different members of any class is got by multiplying the members of this class.

It is easy to sea that it has the following consequence.

Theorem 4. *For any fixed n-matching (with m next edges) all permutations from S_{2m} with $2m$ elements define and form a class of maps that is closed against multiplication (of maps). For two differing permutations we have two different (not equal) maps in correspondence.*

Followingly, we have established a one-one map between the set of all permutations (with $2m$ elements) and the set of the maps (with m edges). So, now we may think in terms of one fixed n-matching π and the set of all permutations, under each permutation seeing a graph P that stands for the ordered pair $(P, P \cdot \pi)$.

Thus, keeping a particular π in the mind, for an arbitrary permutation P we have also a graph P (formally $(P, P \cdot \pi)$). Let us take the map (I, π) as identity map. Graphically interpreted it consists of m isolated edges [compare with [22]]. Its dual map (π, I) consists of m isolated loops.

P^{-1} being the reverse permutation of P, $(P^{-1}, Q^{-\pi})$ is called *reversed* map of (P, Q). It is easy to see that, so defined, both map and its reverse map belong to the same class of maps closed under the multiplication. Evidently as a consequence, because of the fact that all permutations with $2m$ elements form the group S_{2m} [1, 26], then, similarly, all maps with one fixed n-matching form a group that is isomorphic to S_{2m}.

If (a,b) is a transposition, $(a,b) \cdot P$ graphically can be interpreted as a union of two vertices at given corners in the graph corresponding to the map P.

Remembering that each permutation can be given as a sequence of multiplications of simple transpositions, each map can be generated by this sequence of the operations of union of two corners: if

$$P = (a_l b_l)(a_{l-1} b_{l-1})...(a_1 b_1),$$

then

$$P_0 = I,$$

$$\dots$$

$$P_k = (a_k, b_k) \cdot P_{k-1},$$

$0 < k \leq l$, $P = P_l$. Graphically interpreted, in this way l unions of vertices at given corners giving the graph P are done.

For entirely pragmatic reasons, it is convenient to choose one special class of maps with n-matching equal to

$$\pi = (12)(34) \dots (2i-1\ 2i) \dots (2m-1\ 2m).$$

The maps of this class we call *normal.* It is convenient for the practical work to be get used to some particularly chosen n-matchings and this one called *normal* is sufficiently natural for this pragmatic aim. Other applications may suggest other convenient fixed n-matchings.

Let us notice that the maps in the examples above were normal maps.

Further, it is convenient to choose one big natural number, say M, and consider it as the bound for possible numbers of edges in maps . Taking the number $2M$ as the order of all permutations. i.e. fixing the universal set of the corners $[1 \dots 2M]$, all maps should have M edges, but most of them should be isolated. Such a view is convenient also because it can be used directly in the computer implementation. [The only pay for this approach is that we loose possibility to have isolated edges, when we actually need them, from some graph theoretical point of view.] Then, following the graphical interpretation of the multiplication the map from the left with a transposition, we have in the beginning M isolated edges (as being initialized), and each currant graph, multiplied by the permutation from the left, gives some set of transpositions, that union and split vertices at distinct corners, giving in this way a sequence of new graphs.

5.3 Classes of maps with fixed edges

For a map P with n-matching π its e-matching ϱ is equal to $P \cdot \pi \cdot P^{-1}$ or $\pi^{P^{-1}}$. Truly, every permutation S with respect which the matchings π and ϱ are conjugate, defines a map with the same e-matching ϱ.

Let us define a set K_ϱ of maps as the class of all maps whose e-matching is ϱ. For different ϱ these classes K_ϱ are subclasses of maps of all the class of maps with fixed π, under which one this subclass is special, namely, K_π, that comprise the maps with e-matching equal to π. This subclass is not empty, because, for example, the maps (π, I) and (π, π) belong to it. Really, $\varrho_I = \pi^I = \pi$.

Theorem 5. *For two maps S, T with n-matching π e-matching of their multiplication $S \cdot T$ is equal to $\varrho_T^{S^{-1}}$, i.e. $\varrho_{S \cdot T}^S = \varrho_T$.*

Proof

$$\varrho_{S \cdot T} = S \cdot T \cdot \pi \cdot T^{-1} \cdot S^{-1} = S \cdot \varrho_T \cdot S^{-1} = \varrho_T^{S^{-1}}$$

.

Let us denote by $P \cdot K_\sigma$ the class of maps $\{P \cdot Q | Q \in K_\sigma\}$. From theorem 1 we know that $P \cdot K_\sigma$ goes into K_ϱ, where $\varrho = \sigma^{P^{-1}}$. We are going to prove the equality of these classes. First we start with the following.

Theorem 6. *K_π is a group.*

Proof If $\varrho_S = \varrho_T = \pi$ then

$$\varrho_{S \cdot T} = S \cdot T \cdot \pi \cdot T^{-1} \cdot S^{-1} = S \cdot \pi \cdot S^{-1} = \pi.$$

Further,

$$\varrho_{S^{-1}} = S^{-1} \cdot \pi \cdot S = \pi.$$

The identity map also belongs to this subclass , and, consequently, it is a subgroup.

Theorem 7. $P \cdot K_\sigma = K_{\sigma^{P^{-1}}}$. $(P \cdot K_{T^P} = K_T.)$

Proof If $\sigma = \pi$ then $P \cdot K_\sigma$ is a left coset of K_σ equal to $K_{\pi^{P^{-1}}}$ and the assertion of the theorem is right.

Let $\sigma \neq \pi$ and Q be such a map that $Q \cdot K_\pi = K_\sigma$, i.e. K_σ is a left coset of K_π and Q is some of its elements: $\varrho_Q = \pi^{Q^{-1}}$ and Q belongs to K_σ by theorem 1. If $P \cdot Q = R$ then we have

$$P \cdot K_\sigma = P \cdot Q \cdot K_\pi = R \cdot K_\pi = K_{\pi^{R^{-1}}} = K_{\sigma^{P^{-1}}}.$$

As a conclusion of this theorem we have that the class K_π and its left cosets are classes with fixed edges.

Corollary 1. *Left cosets of K_π are classes of maps with fixed edges.*

Thus arbitrary map P with e-matching $\varrho = \pi^{P^{-1}}$ belongs to $K_\varrho = P \cdot K_\pi$.

For two involutions σ and τ (i.e. $\sigma^2 = \tau^2 = I$) we write $\sigma \subseteq \tau$, if every transposition of σ is also a transposition in τ.

Lemma 1. $P^\pi = P$ *if and only if* $P \in K_\pi$.

Proof

$$\varrho_P = \pi \equiv \pi = P \cdot \pi \cdot P^{-1} \equiv P = \pi \cdot P \cdot \pi \equiv P = P^\pi.$$

Let c be a cycle of P. The cycle c^π is called *conjugate* cycle (with respect to π). If $c = c^\pi$ then it is called *selfconjugate* cycle. If every cycle in P has its conjugate cycle in P or is selfconjugate then P is called *selfconjugate map*. Lemma says that K_π is the class of selfconjugate maps with respect to π.

We reformulate this in a theorem.

Theorem 8. *K_π is the class of selfconjugate maps with respect to π.*

We can reveal the structure of these maps as follows.

Theorem 9. *K_π is isomorphic to $S_m \cdot S_2^m$.*

Proof Let $P \in K_\pi$ and P act on C and $C_1 \cup C_2$ be some partitioning of C induced by π. Then there is such an involution $\sigma \subseteq \pi$ that $P = Q \cdot \sigma$ and cycles of Q belong completely to C_1 or C_2. i.e. if a cycle c goes into C_1 then c^π goes into C_2 and reversally. Q can be expressed as $Q_1 \cdot Q_2$ where Q_1 has corners of C_1 and Q_2 corners of C_2.

Then Q_1 and Q_2 are both isomorphic between themselves and to some permutation in S_m and σ isomorphic to some permutation of S_2^m.

Theorem 10. $\mid K_\pi \mid = m! \times 2^m$.

Proof $\mid S_m \mid = m!$; $\mid S_2^m \mid = 2^m$.

Theorem 11. *K_π has (together with itself) $(2m-1)!!$ left cosets.*

Proof There are as many left cosets of K_π as many e-matchings can be generated. Their number is equal to $(2m-1)!!$.

One more way to see, that all left cosets of K_π have the same number of elements, i.e. $(2m-1)!!$, and that they do not really overlay each other, is from the equality

$$(2m-1)!! \times m! 2^m = (2m)!.$$

5.4 Combinatorial knot

Let (P, Q) be a map on the set of corners C and $\varrho = \pi^{P^{-1}}$. Let $C_1 \cup C_2$ be some partitioning of C such that it is induced both by π and ϱ, i.e. for every both edge and next edge one of its corner belongs to C_1 and other to C_2. In this case we say that C is *well* partitioned or *well colored in two colors* or we say that $C_1 \cup C_2$ is *well coloring* of C induced by the map (P, Q).

Theorem 12. *There always exists well coloring of C induced by an arbitrary map (P, Q).*

Proof Let $c_1c_2\dots c_{2k}$ be a cycle of corners such that $c_2 = c_1^\pi$ and $c_3 = c_2^\varrho$ and so on in an alternating way, i.e. $c_{2i} = c_{2i-1}^\pi$ and $c_{2i+1} = c_{2i}^\varrho$,for $i = 1, \dots k$, where $c_{2k}^\varrho = c_{2k+1} = c_1$. Let us, for an instant, suppose that $c_{2k-1}^\pi = c_{2k} = c_1$.

Then $c_{2k-1} = c_1^\pi = c_2$ and $c_{2k-2}^\varrho = c_2$ and $c_{2k-2} = c_2^\varrho = c_3$ and so on, until $c_{k+1} = c_k$, but it isn't possible. It follows, that these cycles may have only even number of elements.

The proof shows that this map does not uniquely define the coloring of C.

Now, following the routine in the proof of the theorem, we may put the odd elements of cycle in C_1 and the even elements in C_2 . If this cycle runs through all corners, then we have only one possibility to color the corners. In the other case, we chose arbitrary non colored corner taking it as c_1 and proceed as before. In the end we get a well partitioning $C_1 \cup C_2$ induced by the map.

Let us define the permutation μ as having the cycles as in the previous proof, i.e. if $C_1 \cup C_2$ is well coloring of the set C, then $c^\mu = c^\pi$ if $c \in C_1$ and $c^\mu = c^\varrho$ if $c \in C_2$. The permutation μ is called *combinatorial knot*. In place we use shorter name *knot* for μ. In the 'corner' interpretation μ really has to do with (alternating) knot of the graph [3], i.e. with some alternating knot, where the corresponding graph is isomorphic to this map.

Lemma 2. *If μ is a knot then if μ' is obtained from μ with some cycle of μ changed in the opposite direction then μ' is also a knot.*

Proof Starting a new cycle in the proof of theorem 12 we could choose a corner arbitrary . Consequently, a cycle in the knot can go in one or another direction. Thus, if μ is a knot then also μ^{-1} is a knot. It is easy to see that a knot μ depends only upon π and ϱ, thus it is common for all the class K_ϱ. The following theorem shows that, if we consider a map $(\mu, \mu \cdot \pi)$ then $\varrho_\mu = \varrho$.

Theorem 13. $\pi^\mu = \varrho$.

Proof Taking $(cc^\pi) \in \pi$, $(cc^\pi)^\mu$ equals to $(c^\pi c^{\pi\varrho})$ or $(c^\varrho c^{\pi\pi}) = (c^\varrho c)$, both cases giving that $(cc^\pi)^\mu \in \varrho$.

By $\mu(\pi, \varrho)$ we denote arbitrary knot of some map belonging to K_ϱ. Previous theorem and the theorem 7 gives the following result .

Theorem 14. $K_\varrho = \mu(\pi, \varrho) \times K_\pi$.

This gives right for what follows.

Corollary 2. *Every map can be expressed as a knot of this map multiplied by some selfconjugate map.*

Example 4. *Normal planar map corresponding to K_4:*

$$P = (19\bar{1})(4\bar{2}8)(236)(57\bar{0})$$

$$Q = (1\bar{0}6)(24\bar{1})(358)(79\bar{2})$$

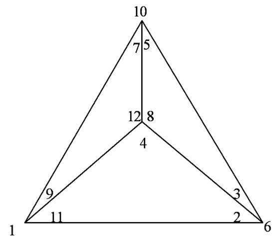
$$\varrho = (17)(28)(3\bar{0})(49)(5\bar{2})(6\bar{1})$$

A knot of this map is

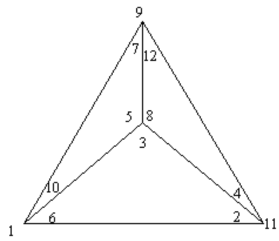
$$\mu = (1287)(349\bar{0})(56\bar{1}\bar{2})$$

The corresponding knotting [see 11] is:

$$\alpha = (1\bar{0}\bar{1}29\bar{2})(358)(467)$$



Picture 6: Drawing corresponding to the map in the example



Picture 7: Drawing of the dual map (Q, P)

Example 5. *Normal planar map corresponding to 'prism' graph*

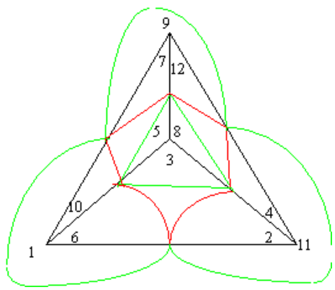
$$P = (1\bar{3}7)(2\bar{0}\bar{1})(38\bar{6})(4\bar{7}9)(5\bar{5}\bar{4})(6\bar{2}\bar{8})$$

$$Q = (1\bar{4}6\bar{1})(2937)(4\bar{8}5\bar{6})(8\bar{5}\bar{3})(\bar{0}\bar{2}\bar{7})$$

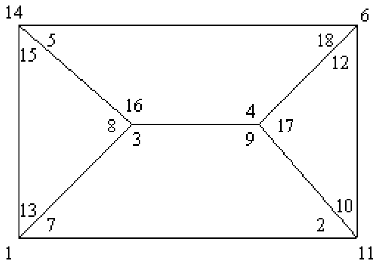
$$\varrho = (1\bar{5})(2\bar{7})(3\bar{3})(4\bar{2})(58)(6\bar{0})(7\bar{1})(9\bar{6})(\bar{4}\bar{8})$$

$$\mu = (12\bar{7}\bar{8}\bar{4}\bar{3}34\bar{2}\bar{1}78560\bar{0}9\bar{6}\bar{5})$$

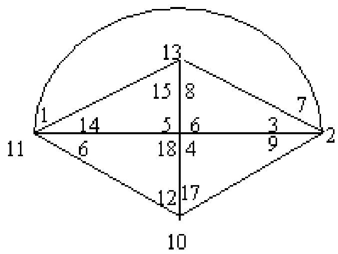
$$\alpha = (1\bar{4}6\bar{5}3723\bar{3}5\bar{4}8)(9\bar{1}\bar{8})(\bar{0}\bar{2}\bar{7})$$



Picture 8: The knot with alternatively green-red colored corners



Picture 9: Drawing corresponding to the map (Q, P)



Picture 10: Drawing of the dual map (Q, P)

Obviously the maps in these examples are normal maps.

5.5 Isomorphism of the combinatorial maps

Let us notice that $A \in K_\pi$ if and only if $\pi^A = \pi$, because $\pi^{A^{-1}} = \varrho_A$. In general, two maps $(P, P \cdot \pi_1)$ and $(Q, Q \cdot \pi_2)$ are isomorphic if and only if there exist such a permutation A that $\pi_2^A = \pi_1$ and $Q^A = P$. We write $P \simeq Q$, saying that P is *isomorphic* to Q if they are conjugate with respect to some A.

Theorem 15. *Let $(P, P \cdot \pi_1) \simeq (Q, Q \cdot \pi_2)$, i.e. they are conjugate with respect to A. Both maps belong to a common closed against multiplication class of maps if and only if $A \in K_\pi$ with $\pi = \pi_1$.*

Proof If $A \in K_{\pi_1}$ then π_1^A is equal to π_1 ,but because of conjugacy of π_1 and π_2 against A also equal to π_2. So $\pi_1 = \pi_2$ and P and Q are maps in one class with $\pi = \pi_1 = \pi_2$. Conversely, if $\pi_1 = \pi_2$, then $\pi_1^A = \pi_2 = \pi_1$ and $A \in K_\pi$.

5.6 Graphical isomorphism of combinatorial maps

We write $a^{P^*} = b$ for P acting on C, if there exists such k $(k \geq 0)$ that $a^{P^k} = b$. We say that b is P-reachable from a and reversely.

For two permutations P and Q acting on C let us write $Q < P$ if whenever $a^{Q^*} = b$ then also $a^{P^*} = b$. We say that Q is less then P. Thus $Q < P$ if whenever b is Q-reachable from b then also b is P-reachable from a.

Theorem 16. *For two P and Q permutations acting on C, if $Q < P$ then there holds*

$$P < P^Q < P.$$

Proof Let b be P-reachable from a. Then a^Q is Q-reachable from b^Q, because trivially a^Q is Q-reachable from a, and b^Q is Q-reachable from b. This means that $P < P^Q$. But Q-reachability is equivalent with the Q^{-1}-reachability, and the same consideration gives that $P^Q < P$.

The operation P^Q, i.e. potentiation with the lesser permutation, rearrange the elements of the orbits of P without changing the reachability relation between elements. This crucial feature is needed us in the definition of the graphical isomorphism.

Two maps T and S with the common next-edge rotation π are *graphically isomorphic* if there exist a pair of permutations (A, R) that $A \in K_\pi$ and $R < T$ that there holds

$$T^{A \cdot R} = S.$$

It is easy to see that the assertion of the theorem is right.

Theorem 17. *If T is graphically isomorphic to S then there exist A and R such that $A \in K_\pi$ and $R < T$ and*

$$T^{R \cdot A} = S.$$

Graphical isomorphism should be needed when we will study the possibility to reduce the genus of some embedding of the graph on the surface.

6 Partial combinatorial maps

As we have seen, combinatorial maps have very clear geometrical interpretation, i.e. graphs with rotation on surfaces, both orientable and, in more general case, nonorientable. Let us recall that, defining combinatorial maps as pairs of permutations (p, q), the necessary condition was that the multiplication $p \cdot q^{-1}$ is always an involution without fixed elements, i.e. a matching.

We could now ask. If we take off this last condition, have these objects convenient (or at all) geometrical interpretation similarly as in the case of combinatorial maps? Stahl [17, 18, 19]and other mathematicians [23, 22], who studied this subject, replied affirmatively and accordingly gave their interpretation. But we suggest a more natural way , as it seems to us, for this geometrical interpretation. It should be seen later, when we come to the submaps of c-maps.

Let us define the partial combinatorial map as an ordered pair of permutations. In the place of their full name — partial combinatorial map — we shall use a shorter name — p-map. Let us have two permutations P and Q, then the pair (P, Q) is a p-map. Having a p-map (P, Q) given, the cycles (or orbits) of permutation P we call the vertices or the vertex-rotation, the cycles of Q — the faces or the face-rotation, and the cycles of the product $Q \cdot P^{-1}$ — the edges or the edge-rotation.

Sometimes it would be more convenient to present p-map as the triple (P, Q, R) of permutations P, Q, R respectively with additional condition that R equals to $Q \cdot P^{-1}$. Then cycles of (P, Q, R) are vertex-, face- and edge-rotations respectively. Cycles of the product $P^{-1} \cdot Q$ we call next edges or mirror edges of the p-map (P, Q).

The lengths of the cycles of P, Q, R are degrees of vertices, faces and edges respectively. Thus, p-maps may have edges of arbitrary degree, not only of degree two as in the simple maps, i.e. c-maps. For this reason other investigators name these and analogous object — multigraphs, see [23]. We retain the name —partial because of the connection with subgraphs, as it should be seen later.

It is easy to see that combinatorial maps as defined before are also p-maps, but just such that have all edges of degree two. The condition in the definition of a combinatorial map is equivalent to an additional condition of p-map that all edges must be of degree two.

Example 6. *P-map* $((1234), (13)(24))$ *has one vertex* (1234)*, two faces* (13) *and* (24) *and one edge* (1234) *and corresponding next edge* (1234)*.*

Example 7. *P-map* $((1234)(567), (1537264))$ *has two vertices,* (1234) *and* (567) *respectively, one face* (1537264) *and four edges,* $(17), (25), (36), (4)$ *. The edge* (4) *is of degree one, others of degree two.*

6.1 The drawing of the p-maps

Let us first show how to draw a p-map, giving the proof of this procedure further.

For a given p-map let us draw a vertex $(c_1 \dots c_k)$ of degree $k > 0$ as a star graph with $2k$ halfedges, i.e. $2k$ corners. Further let us mark every second corner, taking them in clockwise direction, with elements $c_1, \dots, c_k$, leaving every second corner empty.

Having done this for vertices, ley us join halfedges looking on faces of p-map similarly as we did in the case of combinatorial maps: for some element c and d equal to c^Q let us join the halfedge that follows the corner marked by c with the halfedge that is followed by the corner marked by d at some (other or the same) vertex.

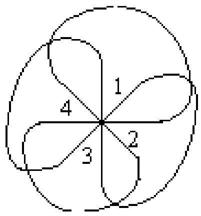
In this way we may join all pairs of halfedges determined by the succession of corners in the permutation Q, because the number of halfedges is even. Thenafter, going around the face that contains these elements c and d in the anticlockwise direction, we shall meet first c and then d as it is in the corresponding face-rotation of p-map.

Example 8. *P-map*

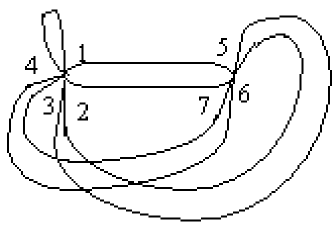
$$\left\{ \begin{array}{l} (1234) \\ \underline{(13)(24)} \\ (1234) \end{array} \right.$$

Example 9. *P-map*

$$\left\{ \begin{array}{l} (1234)(567) \\ \underline{(1537264)} \\ (17)(25)(36) \end{array} \right.$$



Picture 11: Drawing of the partial map in the example



Picture 12: Drawing of the partial map in the example

Let us observe, looking on these pictures, that going around faces of unmarked corners clockvise we have marked corners on-the-other-side-of-the-face in the sequence that of edges.

6.2 Image of the p-map.

In the procedure of the drawing of p- maps in previous chapter we got some map that has as many unmarked corners as the given number of all corners. Let us formalize this observation.

Let permutations P, Q, R act on the set of corners C. Let us have a map $u : C \to \bar{C}$ that bijectively maps C into some new set $\bar{C}$ such that $C \cap \bar{C} = \emptyset$. For $c \in C$ let $\bar{c}$ be equal to $u(c)$ and $\tilde{c}$ be $c\bar{c}$.

Applying this for a cycle $c = (c_1 \dots c_k)$, let $\bar{c}$ be $(\bar{c_1} \dots \bar{c_k})$ and $\tilde{c}$ be equal to $(c_1\bar{c_1} \dots c_k\bar{c_k})$. Elements of $\bar{P}$ are all from the set $\bar{C}$.

Now we are ready to show that the procedure of drawing of p-map always gives some c-map. This c-map is the 'drawing' of this p-map.

Theorem 18. *Let p-map (P, Q, R) be arbitrary taken. The p-map $(\tilde{P}, Q \cdot \bar{R}^{-1})$ is a c-map.*

Proof Let us denote this p-map that correspondence uniquely to the p-map (P, Q, R) as $(\mathcal{P}, \mathcal{Q})$. We must prove that all edges of it are of degree two. Let $c \in C$. Application of $Q\bar{R}^{-1}$ gives c^{Q}.

Further, application of $\tilde{P}^{-1}$ gives $\bar{c}^{\bar{Q}\cdot\bar{P}^{-1}}$. Followingly, first two elements of the edge are c and $\bar{c}^{\bar{Q}\cdot\bar{P}^{-1}}$). Further, applying $Q \cdot \bar{R}^{-1}$ once more we get $\bar{c}$ and applying $\tilde{P}$ we get c. Consequently, the edge is of degree two.

Let $a \in \bar{C}$ and $a = \bar{c}$, for some $c \in C$. Similarly, as before

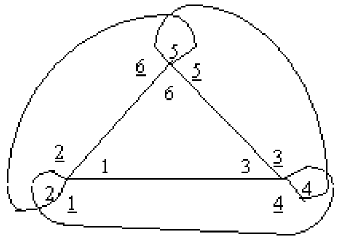
$$\bar{c} \to \bar{c}^{\bar{Q}\cdot\bar{P}^{-1}} \to c^{P\cdot Q^{-1}} \to c^{P} \to \bar{c}.$$

This edge also is of degree two equal to $(\bar{c}\ c^{R^{-1}})$.

The consequence of this theorem is that we may now connect with every p-map some combinatorial map in a unique way, when we have chosen a disjoined set $\bar{C}$ and a bijection u from C to $\bar{C}$. We call the c-map $(\mathcal{P}, \mathcal{Q})$ for (P, Q) its *image*. It is easy to convince oneself that we are drawing just this image when we follow the procedure given before.

Example 10. *P-map*

$$\left\{ \begin{array}{l} (12)(34)(56) \\ \underline{(135)(246)} \\ (145236) \end{array} \right.$$



Picture 13: Drawing of the image of the map

Image of this p-map

$$\left\{\begin{array}{l} (1\bar{1}2\bar{2})(3\bar{3}4\bar{4})(5\bar{5}) \\ \underline{(135)(246)(\bar{6}\bar{3}\bar{2}\bar{5}\bar{4}\bar{1})} \\ (1\bar{4})(2\bar{3})(3\bar{6})(4\bar{5})(5\bar{2})(6\bar{1}) \end{array}\right.$$

In the pictures the elements, that are overlined in examples, are underlined.

6.3 Gluing up the edges of the p-map

It is possible to give one more geometrical interpretation of the image of p-map.

Let us call the orbits of the face rotation of the image of p-map, that are not orbits of the p-map, *cut out*, or *scissored*, or *removed*, or *black* faces of the p-map. Let us call *half-edge -type* cut out face that which have an edge in correspondence with the degree equal to one.

Let us call *edge -type* cut out face that which have an edge in correspondence with the degree equal to one.

Let us call *essential* or *proper* cut out face that, which have an edge in correspondence with the degree greater that two.

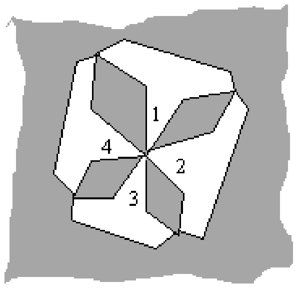
Let us imagine the p-map to be the combinatorial map with these faces being cut out, that correspond to the black or removed or scissored or cut out edges of the p-map. [We see that all these names of these faces are appropriately describing what is done with the p-map in this geometrical illustration.] Then computing the faces in the image of the p-map, that are not faces of the-map itself, we compute just these removed faces.

Then we can imagine the opposite operation, i.e. putting the removed faces back in the p-map and receiving the combinatorial map that is coinciding with the image of the p-map. Let us call this operation *gluing up* of cut faces.

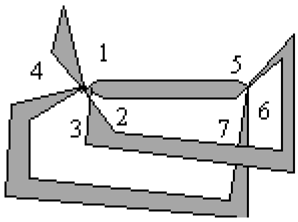
Then the computing of the image we can now imagine as follows. We compute the edges $(\bar{c}\ c^{R^{-1}})$ going round the cut out faces of the p-map. Then we add cut out faces or glue them up in the p-map, by this operation forming it in a c-map. Then we compute the vertex rotation by the simple formula $\mathcal{P} = \varrho \cdot \mathcal{Q}$, where ϱ is the edge rotation of the glued up c-map, i.e. of the image of p-map.

6.4 Shorter image of the p-map

In the chapter 18 we saw that the edges of the image of p-map have a form $(\bar{c}\ c^{R^{-1}})$. This fact may be used to build the image and also some modified forms of the image.



Picture 14: Drawing of the partial map in the example 8



Picture 15: Drawing of the partial map in the example 9

Suppose the p-map (p, q, r) is given. Let us behave in the following way. From the edge rotation r let us eliminate all orbits of lenght one and two, i.e. all half-edge-type and all edge-type cut out faces. From $\bar{C}$ let us eliminate all elements that corresponded to these orbits, assuming that the remaining set is C'. Let us take for this shortened image only these edges which fit into C'. This new image we call *shortened* image.

More formally we have restricted the image on the set from where are eliminated the corners of all non proper cut out faces. It must be proved that the shorter image is c-map.

Theorem 19. *The shorter image is c-map.*

Proof Eliminating from the [full] image of the p-map one edge for each non proper cut out face gives just a shortened image of the p-map.

6.5 Submaps

Geometrical interpretation of p-maps beside that of multigraphs [23] can be also connected with the notion of subgraph. Let us look this way.

Assume a permutation p acts on the set C and S be subset of C. We denote with $p\mid_S$ the restriction of this permutation on the subset S, i. e.the permutation that acts on S. Having p given in the cyclical representation we can get $p\mid_S$ simply by taking out of the orbits those elements that do not belong to S.

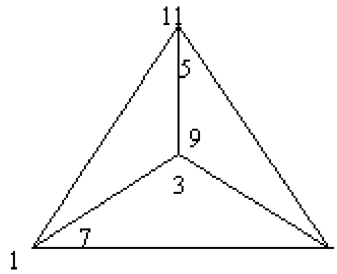
Then, having some p-map (P, Q) given, we can get a restriction on both permutations denoting this $(P, Q)\mid_S$. We say that (P_1, Q_2) equal to $(P, Q)\mid_S$ is a submap of the p-map (P, Q).

Immediately we get such a result.

Theorem 20. *Every p-map is a submap to arbitrary many combinatorial maps.*

Proof First of all, p-map is a submap of its image, when we restrict the image on the set C. So, it can be the first *over-map* we find. This image can be taken as a new p-map and its image is a new over-map differing from the first one. Proceeding in this way we can get arbitrary many c-maps, and for all of them p-map taken in the beginning is a submap .

Looking on simple examples, it is easy to see, that images are not the only c-maps, that contain a given p-map as its submap.



Picture 16: Example of a submap of map

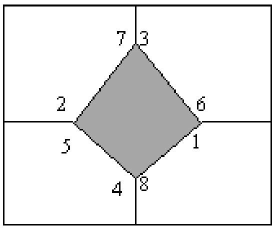
Example 11. *C-map:*

$$\left\{ \begin{array}{l} (1\bar{2}7)(246)(389)(5\bar{0}\bar{1}) \\ \underline{(1\bar{1}6)(237)(459)(8\bar{0}\bar{2})} \\ (1\bar{0})(29)(3\bar{2})(4\bar{1})(58)(67) \end{array} \right.$$

Submap:

$$\left\{ \begin{array}{l} (17)(39)(5\bar{1}) \\ \underline{(1\bar{1})((37)(59)} \\ (153)(79\bar{1}) \end{array} \right.$$

6.6 Examples of most famous p-maps



Picture 17: K_4 on torus with cut out planar face

6.7 Clases of p-maps close under multiplication from the left

Let us have a p-map (p, q) with $r = qp^{-1}$. Multiplying with some permutation t from the left side both p and q we get a new p-map, but its edge-rotation are conjugate to r: because

$$(tp, tq)r' = tq \cdot p^{-1} \cdot t^{-1} = r^{t^{-1}}.$$

Then followingly all p-maps with conjugate edge-rotation are closed under multiplication where the multiplication is defined as

$$(p_1, q_1) \times (p_2, q_2) = (p_1 p_2, p_1 q_2)$$

with the condition that $q_1 \cdot p_1^{-1} = q_2 \cdot p^{-1}$. Let us fix some permutaton s of the order k. Then the pair (e_k, s), where e_k is identical permutation on the same elements where s is acting, is a p-map with k corners and the edge-rotation s. Let (p, q) be p-map having conjugate edges r to s. Then there exists a permutation δ such that $(p, q) = (\delta, \delta \cdot s)$. Dividing δ into multi-transpositions $t_l, \dots, t_1$ $(l > 0)$ we can express (p, q) as $t_l t_{l-1} \dots t_2 t_1 \times (e_k, s) andr = s^{t_1 \dots t_l}$.

6.8 Multiplication by a transposition

Theorem 21. *Let some transposition $t = (a, b)$. Then $(a, b) \cdot (p, q)$ (with the edges ϱ) gives the p-map $((a, b) \cdot p, (a, b) \cdot q)$ and the edges $\varrho^{(a,b)}$ and with the image*

$$(ab \cdot \tilde{p}, ab \cdot q \cdot \bar{\varrho}^{-1})^{(\bar{a}, \bar{b})}.$$

Proof Let us assume that p has a cycle of the form $AaBb$. Then we get

$$(a, b)(\tilde{A} a \bar{a} \tilde{B} b \bar{b}, q\bar{\varrho}^{-1}) = (\bar{b} \tilde{A} a; \bar{a} \tilde{B} b, (a, b) \cdot q\bar{\varrho}^{-1}).$$

We have the image corresponding to the p-map unto which (a, b) multiplication from the left was applied, but the elements a,b are matched into $\bar{a}$ and $\bar{b}$ reversaly. Changing their names we get the image

$$(a\bar{a}\tilde{A}; b\bar{b}\tilde{B}, (a, b) \cdot q \cdot \bar{\varrho}^{(\bar{a}, \bar{b})}).$$

Then succeedingly, corresponding p-map is $(aA; bB, (a, b)q)$ with the edges

$$\varrho^{(a,b)}.$$

Then , in the case of a sequence of transpositions, if $\delta = t_l \dots t_1$ then $\delta \cdot (p, q)$ with r gives as a resulting p-map $(\delta p, \delta q)$ with the edges $r^{\delta^{-1}}$ and the image

$$(\delta \mathcal{P}, \mathcal{Q}^{\delta^{-1}}).$$

6.9 The class K_π

Let us find the class of p-maps that corresponds to K_π in the combinatorial map theory. Let a permutation s be fixed and

$$\Pi = \{a \mid a \cdot s = s \cdot a, a \in S_{2m}\},$$

i.e. Π is the set of all permutations commuting with s. Then for a p-map $(b, b \cdot s)$ the edges are $s^{-b} = b \cdot s \cdot b^{-1} = s$. Then the class

$$K_\Pi = \{(b, a \cdot s) \mid a \in \Pi\}.$$

Similarly as in the case of the combinatorial maps we get

Lemma 3. *Π is a group.*

This us shows what the set Π should be like. Let for this chosen permutation $s \in C$ C_{fixed} be the set of fixed points of p, C_{inv} — elements on which p acts as involution (without fixed points) and C_{rest} - the rest elements on which p acts, i.e. where the orbits of p are longer than two. Then, we can presume this same set's C characterization be kept for all members of Π, i.e.

1) arbitrary permutation without fixed points and transpositions acting on the set C_{rest}, multiplied by

2) arbitrary involution without fixed points acting on the set C_{inv}, multiplied by

3) identical permutation acting on the set C_{fixed}.

6.10 Combinatorial surfaces

In this part we are going to consider combinatorial surfaces examining partial maps, which should us give orientable surfaces in correspondence. Using more general clases of maps we would get in this way unorientable surfaces too.

As it is well know from Edmonds, graph with rotation defines an orientable surface into which the graph with its given rotation is embedded. Let us examine the situation with p-maps.

We define the Eulerian characteristic of p-maps similarly as in [18]:

$$\chi = \parallel P \parallel + \parallel Q \parallel + \parallel \varrho \parallel - n,$$

where n is the number of corners in p-map.

Theorem 22. *P-map and its image have the same Eulerian characteristic.*

Proof

$$\begin{aligned} \chi_{(\mathcal{P},\mathcal{Q})} &= \parallel \mathcal{P} \parallel + \parallel \mathcal{Q} \parallel + \parallel \varrho_{(\mathcal{P},\mathcal{Q})} \parallel - n_{(\mathcal{P},\mathcal{Q})} \\ &= \parallel P \parallel + \parallel Q \parallel + \parallel \varrho \parallel + n - 2n \\ &= \chi_{(P,Q)}. \end{aligned}$$

It affirms that $\chi_{(P,Q)}$ is a characteristic of the surface on which p-map's (P, Q) image can be embedded and of the surface too where the p-map itself can be embedded.

Thus we can imagine that this is the same surface where they both can be embedded, because their Eulerian characteristics are always the same. Rightly, thinking geometrically, embedding the image we also embed the p-map. Similarly as it is by a drawing, drawing the image of some p-map we draw also the p-map itself.

With the following theorem we convince ourselves that we can't go outside the orientable surfaces considering maps in this way as we do.

Theorem 23. *$\chi_{(P,Q)}$ is always even.*

Proof Without loose of generality we may think in the terms of classes of p-maps with fixed next edges π equal to $P^{-1} \cdot Q$. (P, Q) can be expressed as a sequence of transpositions' multiplication

$$t_1 t_2 \dots t_k \cdot (e, \varrho),$$

$l > 0, t_i$ is a transposition. For a p-map (e, ϱ) its Eulerian characteristic χ is equal to two.

Every multipication by t_i $i = 1, \dots t_k$ leaves χ the same or changes by 2, whereas ϱ remains with the same cycle structure. Thus, $\chi_{(P,Q)}$ remains even always.

Let us define genus of p-map as

$$\gamma = c \quad - \quad \frac{\chi}{2},$$

where c is a number of components of connectivity in p-map.

Further we are going to prove that the genus of submap is not greater than that of the p-map that contains this submap.

To prove this theorem, first we need an additional lemma to be proved.

Lemma 4. *Let $a \in C, b = a^{P^{-1}}$ and $R_1 = R \cdot (ab)$. Then submap $(P, Q) |_{/a}$ has edges $R_1 |_{/a}$.*

Proof P-map (P, Q, R) has the image $(\tilde{P}, Q\bar{R}^{-1})$. Let us divide the elimination of the corner a from p-map in two parts, performing it in a 'graphical' way. First, let us multiply the image with $(\bar{a}, \bar{b})$:

$$(\bar{a}\bar{b})(\tilde{P}, Q\bar{R}^{-1}) = ((P |_{/a}) \cdot (a\bar{a}), Q(\bar{R} \cdot (\bar{a}\bar{b}))^{-1}).$$

Second, let us eliminate vertex $(a\bar{a})$ from the resulting c-map. We get then

$$(\widetilde{P |_{/a}}, Q |_{/a} \cdot Q(\bar{R} \cdot (\bar{a}\bar{b}))^{-1} |_{/\bar{a}}).$$

This is the image of p-map $(P, Q) |_{/a}$ with edges $R \cdot (ab) |_{/a}$.

Now we may go to the theorem.

Theorem 24. *Let (P, Q) be some p-map and (P_1, Q_1) be some submap of it. Then genus of this submap is less or equal of the genus of (P, Q).*

Proof Let us eliminate an arbitrary corner $a \in C$ from (P, Q) and prove that genus by this operation didn't increase.

For $a \in C$, $p |_{/a}$ denotes p without a.

$$\begin{aligned}
\gamma_{(P,Q)|_{/a}} &= 1 - \frac{1}{2} \cdot [\| P |_{/a}\| + \| Q |_{/a}\| + \| R \cdot (a, b) |_{/a}\| - n + 1] \\
&\leq 1 - \frac{1}{2} \cdot [\| P \| + \| Q \| + \| R \| - 1 - n + 1] \\
&= \gamma_{(P,Q)},
\end{aligned}$$

when $\| P |_{/a}\| = \| P \|$ and $\| Q |_{/a}\| = \| Q \|$. When $\| P |_{/a}\| \leq \| P \|$, then must hold $b = a$. When $\| Q |_{/a}\| \leq \| Q \|$ then we apply the theorem to the dual p-map (Q, P) with the same genus γ.

If p-map has genus γ, then one should expect, that there exist some other p-maps that also have the same genus γ. Following theorem says something about such p-maps.

Theorem 25. *If the triple (P, Q, R) is a p-map with genus γ then p-maps are also (Q^{-1}, R^{-1}, P) and (R, P^{-1}, Q^{-1}) with the same genus γ.*

Proof There must hold $R = Q \cdot P^{-1}$ for the first p-map. Checking for second triple gives:

$$R^{-1} \cdot (Q^{-1})^{-1} = P \cdot Q^{-1} \cdot Q = P.$$

And the further checking gives:

$$P^{-1} \cdot R^{-1} = P^{-1} \cdot P \cdot Q^{-1} = Q^{-1}.$$

It is evident that these all p-maps have the same genus, because the permutations with precision to reverse are the same , but the reversing of the permutation doesn't change the number of cycles.

Theorem 24 admits the following transformation

$$(P, Q, R) \to (P\mid_{/a}, Q\mid_{/a}, R\cdot(aa^{P^{-1}}))\mid_{/a}),$$

where the arrow denots the transformation of the elimination the corner a from the partial map. Symmetrically, we can rewrite it as follows

$$(P, Q, R) \to (P, Q, R\cdot(aa^{P^{-1}}))\mid_{/a}.$$

Using the result of the last theorem as a rotational sheme, we now could write following three transformations:

$$(P, Q, R) \;\to\; (P, Q, R\cdot(aa^{P^{-1}}))\mid_{/a}, \tag{1}$$

$$(P, Q, R) \;\to\; (P\cdot(aa^{Q^{-1}}), Q, R)\mid_{/a} \tag{2}$$

$$(P, Q, R) \;\to\; (P, (aa^{R^{-1}})\cdot Q, R)\mid_{/a}. \tag{3}$$

On the right sides we get necessarely subgraphs of the graph in the beginning only in the case of transformation 1. In both other cases these transformations give subgraphs in some other sense.

In general, using these transforms we get from some legitimate p-maps some other legitimate p-maps. But, preserving number of cycles in permutation, the genus of the corresponding surface should be preserved.

Let us consider such eliminations of corners, that do not reduce genus of p-maps.

Theorem 26. *If in three possible transforms corners a and b belong to different edges (1st case), different vertices(2nd case) or different faces(3rd case) then the genus of the combinatorial surface should not be reduced.*

Proof Let us consider the first case and let corners belong to diferent edges. Then $R\cdot(ab)$ gives reduction of edges and together with the reduction of corners by one, the genus remains the same. Rotating this result we get this same for other two transformations.

6.11 The space of μ

In this section we consider some things that are discussed in the chapters lower.

Let π be fixed and $C_1 \cup C_2 = C$ be such that $C_1^\pi = C_2$. If arbitrary p be such that $C_1^p = C_1$ and $\bar{p} = p^\pi$, then $C_2^{\bar{p}} = C_2$.

Let us denote by $\tilde{p}^\pi$ a *twisting by* π what is defined as follows: if $(c_1c_2\dots c_k)$ be cycle of p then $(\bar{c}_1\bar{c}_2\dots\bar{c}_k)$ be the cycle in $\bar{p}$ and $(c_1\bar{c}_1c_2\bar{c}_2\dots c_k\bar{c}_k)$ the cycle of $\tilde{p}^\pi$, where $\bar{c}_i = c_i^\pi$. Then the knot μ is just such permutation of the form $\tilde{p}^\pi$ by the given partitioning $C_1 \cup C_2$. Each partitioning $C_1 \cup C_2$ with $C_1^\pi = C_2$ presents some subspace of μ where each $p \in S_n$ that $C_1^p = C_1$ gives a sample of μ: $\mu = \tilde{p}^\pi$.

Let us use the known formula $\mu = \zeta_2\pi\zeta_1^{-1}$ and the fact that μ is a twist $\tilde{p}^\pi$. We can conclude that firstly:

$$\mu = \tilde{p}^\pi = \begin{cases} C_1: & \pi \\ C_2: & \pi p \end{cases} = \pi p,$$

secondly:

$$\mu = \zeta_2\pi\zeta_1^{-1} = \begin{cases} C_1: & \pi \\ C_2: & \zeta_2\pi\zeta_1^{-1} \end{cases} = \zeta_2\pi\zeta_1^{-1}.$$

Then, taking both these together we have

$$\pi p = \zeta_2\pi\zeta^{-1}$$

and

$$p = \pi\zeta_2\pi\zeta_1^{-1} = \zeta_2^\pi\cdot\zeta_1^{-1}.$$

7 Nonorientable maps

Nonorientable maps could be called these that correspond to the graphs embedded on the nonorientable surface. Nonorientable maps can be considered both by the formalism of Tutte in [22] and Little using 3-graphs [11].

8 The theory of covers. Noncolored covers

Investigating graphs on surfaces, useful objects are cycles and cuts in graphs. What traditional topology does with closed lines on the surface, in the graph embeddings this same do cycles and, thinking in duality, the cuts of edges. Here and further also in our combinatorial world we find objects analogous to cycles and cuts in graphs. We use graph-theoretical notions in building the terminology in our map theory, to show correspondence between these notions that we are considering.

8.1 A choice operator

Firstly let us enter a useful notion of the choice operator.

Let us denote with ∇ the choice operator between two permutations and with $P\nabla Q$ a choice operation of two permutations P and Q, where $c^{P\nabla Q}$ is equal to c^P or c^Q. Then there exists a set of permutations $\{R \mid R = P\nabla Q\}$. Let us write $P\nabla Q$ in place of $\{R \mid R = P\nabla Q\}$, when there does not arise confusion. For arbitrary P and Q the set $P\nabla Q$ is not empty. For the pair of identical permutations this set contains one element, i.e. identical permutation. In the general, it contains at least P and Q.

8.2 Traces, paths and cycles

Let (P, Q) be a partial map. A sequence $a_1, \ldots, a_n$ $(n > 0)$ is a *trace* in (P, Q), if $a_{i+1} = a_i^{P\nabla Q}$ $(0 < i < n)$. A trace is closed when $a_1 = a_n^{P\nabla Q}$. A trace in general has elements with repetition. Let us call trace without repetitions - *path* and a closed path - *cycle*. It is easy to see, that the trace in general does not have any particular permutation behind it. Only, when a trace is a path, then $P\nabla Q$ contains some permutation τ such that $a_{i+1} = a_i^{\tau}$ for currant i modulo n.

Writing $a \in \tau$ we say, that the cycle a is an orbit of τ.

Theorem 27. *The set*

$$\bigcup_{\tau \in P\nabla Q} \bigcup_{a \in \tau} a$$

have all possible cycles of (P, Q) *as its elements.*

Proof By induction the proposition is right when $\mid C \mid = 1$, and let us assume that it is right when $\mid C \mid < 2m$. Let C be arbitrary cycle built using the choice operation $P\nabla Q$. Let us consider the restriction of P, Q on the set $C- \mid c \mid$. By the induction assumption the proposition of the theorem is right on it and we have it proved also on the whole set C, if only the chosen cycle c fits in the other part of C, but it is evidently, when the restricted part is an orbit.

8.3 Cycle covers

For an arbitrary partial map (P, Q) elements of $P\nabla Q$ we call *cycle covers over* (P, Q). We write (P, Q, τ) for the partial map (P, Q) with the fixed cycle cover τ.

The previous theorem said, that for an arbitrary cycle there exist always some cycle cover with this cycle in this cycle cover.

8.4 Cycle covers of combinatorial maps

Let us consider further cycle covers for combinatorial maps only, i.e. with edge rotation as involution without fixed points. It is possible to distinguish four types of edges in the combinatorial map with chosen cycle cover (P, Q, τ). Let us take arbitrary edge e with corners (c_1, c_2, c_3, c_4) such that $c_2^P = c_1$, $c_3^{\pi} = c_1$ and $c_4^{\varrho} = c_2$. It is easy to state what follows.

Theorem 28. *There are four possibilities:*
1) c_1 *and* c_2 *belong to one cycle and* c_3 *and* c_4 *to another cycle;*
2) or c_1 *and* c_4 *belong to one cycle and* c_2 *and* c_3 *to another cycle;*
3) or c_1 *is next element after* c_2 *and* c_3 *is next element after* c_4*;*
4) or c_1 *is next element after* c_4 *and* c_3 *is next element after* c_2.

It is easy to see that excluded are possibilities that corners of one edge belongs to more than two cycles of the cover, and that some cycle could go through c_1 and c_3 (c_2 and c_4)or escaping c_2 or c_4 (or c_1 or c_3) .

It is convenient to give names to each of this type of edges distinctly, separating into two parts of types of edges. An edge in (P, Q, τ) is called *inner*, if all of its corners go into one cycle of τ. In another case, i.e. when it goes into two cycles of τ it is called an *outer* edge. Then further, if κ is a cycle from τ, and $c_1 = c_2^\kappa$ (and $c_3 = c_4^\kappa$) then we call it *cross* edge.

Similarly, if $c_1 = c_4^\kappa$ (and $c_3 = c_2^\kappa$) then we call it *recurrence* edge.

If κ_1 and κ_2 are two distinct cycles from τ, and if $c_1 = c_2^{\kappa_1}$ (and $c_3 = c_4^{\kappa_2}$) then we call it *cut* edge.

Graphically inner edges belong to the cycles and outer edges do not. Cycle edges belong to two cycles, but cut edges join corners in different cycles. Recurrence edges are twice belonging to the same cycle, but cross edges join corners in the same cycle.

Similarly, if $c_1 = c_4^{\kappa_1}$ (and $c_3 = c_2^{\kappa_2}$) then we call it *cycle* edge.

Let us partition π for (P, Q, τ) into

$$\pi_{cycle} \cdot \pi_{cut} \cdot \pi_{cross} \cdot \pi_{recurr},$$

where π_{cycle} have next edges for all cycle edges of τ, and correspondingly, π_{cut} - for all cut edges, π_{recurr} - for all recurrence edges, and π_{cross} for all cross edges.

Similarly, we partition also the universal set C according these four edge types, i.e.

$$C = C_{cycle} \cup C_{cup} \cup C_{recurr} \cup C_{cross},$$

where C_{cycle} contain the corners of next edges of π_{cycle}, and other subsets of C similarly for other types of edges.

Sometimes we are going to use the following evident partitioning also , i.e. $\pi = \pi_{inner} \cdot \pi_{outer}$ and correspondingly, $C = C_{inner} \cup C_{outer}$.

Example 12. *The map from the example 3:*

$$P = (189)(2536)(47\bar{0})$$

$$Q = (17926)(3548\bar{0}).$$

One of possible cycle covers is:

$$(18\bar{0}479)(26)(35).$$

It has edges of all types:

$$(12)(34) - cut - edges[red(simple)],$$

$$(56) \quad - cycle - edge[green(bold)],$$

$$(78) \quad - cross - edge[yellow(pointed)],$$

$$(9\bar{0}) \quad - recurrence - edge[blue(dashed - pointed)].$$

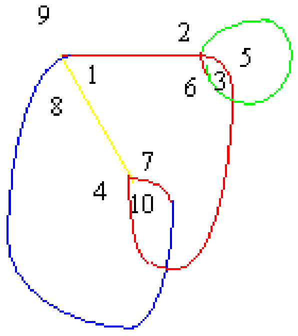
Example 13. *Normal planar map corresponding to prism graph, [see example 5]:*

$$P = (1\bar{3}7)(2\bar{0}\bar{1})(38\bar{6})(4\bar{7}9)(5\bar{5}\bar{4})(6\bar{2}\bar{8})$$

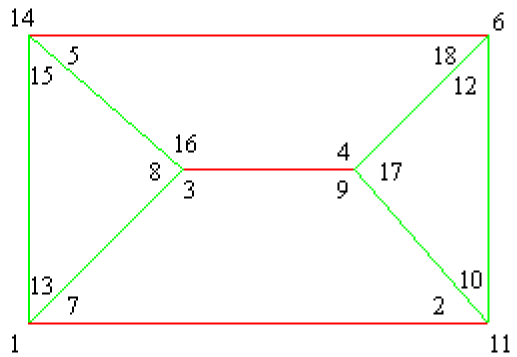
$$Q = (1\bar{4}6\bar{1})(2937)(4\bar{8}5\bar{6})(8\bar{5}\bar{3})(\bar{0}\bar{2}\bar{7}).$$

A cycle cover without inner edges [in this case the only possible]:

$$(1\bar{4}5\bar{6}37)(294\bar{8}6\bar{1})(8\bar{5}\bar{3})(\bar{0}\bar{2}\bar{7}).$$



Picture 18: Drawing corresponding to the map in the example



Picture 19: Drawing of the prism graph with marked with the cycle cover cycle and cut edges.

8.5 Duality in the edge types

Let us consider a cycle-covered map (P, Q, τ) and its dual map with the same cycle cover (Q, P, τ). This possibility follows from the symmetry of the choice operation.

It is very easy to see, that the following proposition is right.

Theorem 29. *The cycle edges in (P, Q, τ) have the cut edges in correspondence in the dual (Q, P, τ), and reversely; and the recurrence edges in (P, Q, τ) have cross edges in correspondence in the dual (Q, P, τ), and reversely.*

Theorem 30. *The inner edges of (P, Q, τ) are inner edges of the dual (Q, P, τ); and similarly, outer edges of (P, Q, τ) are outer edges of the dual (Q, P, τ).*

8.5.1 Cycles and cuts in duality

In graph-topological view cycles in a graph on the surface have cuts of edges in the dual graph in correspondence. It is necessary to distinct cycles and corresponding cuts in the combinatorial view also.

We try to do this in the following way.

Let for (P, Q, τ) and its dual (Q, P, τ) we chose cycle κ in τ. Then the orbit, the restriction of the cycle κ on cycle edges of the c-map , is

$$\kappa\,|_{C_{cycle}},$$

and it evidently have only these corners that correspond to cycle edges in (P, Q, τ). Let us use for this orbit denotation κ_{cycle}.

But in the dual object (Q, P, τ) this orbit κ_{cycle} represents the cut edges of the cycle κ.

Similarly we can find the orbit κ_{cut}, equal to

$$\kappa\,|_{C_{cut}},$$

that represents such a cut in the first map (P, Q, τ), that has a cycle in the map (Q, P, τ) in the correspondence.

Further we have to prove that these orbits κ_{cycle} and κ_{cut} graphically have to do with real cycles and cuts.

8.6 The cycle cover submap of the combinatorial map

Let we have (P, Q, τ) given, and let us consider the restriction of P to the subset $C_{cycle} \cup C_{recurr}$ and denote it $P_{cyclical}$ calling it the cyclical part of the vertex rotation of this combinatorial map with the given cycle cover. Similarly τ restricted on the cycle end recurrence edges we call cyclical par of the cover and denote $\tau_{cyclical}$. Let us consider the partial combinatorial map

$$(P_{cyclical}, \tau_{cyclical}),$$

calling it the *cycle cover submap* of the covered map (P, Q, τ).

It is one of the particular maps that we are going to use to explore the topology of maps.

Theorem 31. *The cycle-cover submap is a combinatorial map.*

Proof. We must prove that the cycle cover submap is the submap that remains from the combinatorial map, when from it is removed all cut edges and cross edges. Then we would be proved that the resulting map is combinatorial map. The proof follows from the fact that the edge rotation of the cycle cover submap is equal to $\pi_{cycle} \cdot \pi_{recurr}$.

Further let us call the genus of the cycle cover submap - the *genus of the cover*.

We explore cycle covered maps and their genera and the way to change the cycle cover to get change in the genus of the cover in one or another direction.

A special case of the cycle cover is useful, when all edges of the cycle cover are cyclical.

Theorem 32. *Let us suppose that for some cycle cover of (P, Q) all the edges of the map are cyclical, i.e. cycle and recurrence edges. Then the cycle cover is equal to Q.*

Proof. Let us think in the terms of the application of the choice operator $P\nabla Q$. For the cycle cover Q the choice operator gives always the second choice, i.e. the access to Q. But, if we for a single case retreat from this choice, then we must get immediately some non-cyclical edge. This suffices to prove what was stated.

As a consequence of this theorem, we can distinguish in our consideration such type of maps which cycle cover Q has only cycle edges, i.e. it has no recurrence edge in it. These maps correspond to the two triangle graphs in the graph-topological outlook.

Similarly, as we defined cycle cover submap, it is useful to define the subnotion of this. Let τ be cycle cover of some map, but σ is such, that the orbits of it are also orbits of τ. Let C_σ be the set of corners in σ. The permutation

$$(P\mid_{C_\sigma}, \sigma)$$

is called the *cycle submap* of σ. Similarly, we speak about the genus of the σ, considering it equal to the genus of the corresponding cycle submap.

When we say - cycle submap, we assume, that it is built on one or possibly several cycles from the cycle cover.

Theorem 33. *Let the map is such that it contains some orbit κ that κ has the recurrence edge e in its boundary. Then this edge is either isthmus in the map or there exist such sequence of orbits $\lambda_1, \ldots, \lambda_k, k > 0$, that $\lambda_1 = \kappa_1$ and $\lambda_k = \kappa_2$, where the orbits κ_1 and κ_2 are remaining parts of κ, when the edge e is eliminated, and for each pair $(\lambda_i, \lambda_{i+1})$ there exist depending cycle edge in Q $e_i, 0 < i < k$, such that the cycle subgraph of all orbits $\lambda_i, 0 < i < k+1$, has genus greater than zero, i.e. it is non-planar.*

Proof. It is easy to see that an isthmus is always the recurrence edge in the cycle cover Q.

If two orbits κ_1 and κ_2 are not joined with such sequence of orbits, as described in the theorem, then these orbits belong to parts of the graph that are disconnected or are connected only through an isthmus.

The previous theorem is possible to make simpler and stronger.

Theorem 34. *Let the map is such that it contains some orbit κ that κ has the recurrence edge e in its boundary. Then this edge is either isthmus in the map or the cycle subgraph built on κ has genus equal to one, i.e. it is on the torus.*

Proof. The cycle subgraph on κ is nonplanar K_4 with the planar face cut out.

Corollary 3. *If the map has genus equal to zero and the cycle cover Q have recurrence edges then these edges are isthmuses in the corresponding graph.*

An essential class of maps is that correspond to the two connected graphs embedded on the orientable surface so, that edges separate only distinct faces. These graphs have in correspondence combinatorial maps with no recurrence edges in the cycle cover Q.

8.7 The space of the cycle covers

Theorem 35. *Let τ belongs to $P\nabla Q$, and π_α is a submap of π. Then*

$$\tau \cdot \pi_\alpha$$

also belongs to $P\nabla Q$.

Proof. If for an edge belonging to π_α τ chooses P, then $\tau \cdot \pi_\alpha$ chooses Q, and, reversly, when τ chooses Q, then $\tau \cdot \pi_\alpha$ chooses P.

This result us shows that we can chose one particular cycle cover, say, τ, and it multiply with all possible submaps π_α of π, and in this way get all possible cycle covers in (P, Q). This set of all covers must be equal to $P\nabla Q$.

Theorem 36. *The cardinality of $P\nabla Q$ is equal to $2^{1/2|C|}$.*

Proof. In the theorem 27 was shown that all cycles are possible in the cycle covers. More over, when some path is chosen with the last element c, then always exist two possible continuations, namely, with c^P and c^Q. This essentially proves what was stated.

8.8 The computations of the cycle cover and its characteristics

Theorem 37. *For* (P, Q, τ)
$$P \cdot \tau^{-1} = \varrho_{cycle} \cdot \varrho_{recurr}.$$

Proof. It must be proved that
$$\varrho_{cyclical} \cdot P = \tau.$$
If the cycle cover cuts the edge, then the edge is not cyclical and P is applied simply in the choice operator, as it is by the formula, that must be proved.

If the cycle cover goes along the edge, then this edge is cyclical, and Q is applied in the choice operator, i.e. the same results , of course, with $\varrho \cdot P$.

Symmetrical result is possible.

Theorem 38. *For* (P, Q, τ)
$$P^{-1} \cdot \tau = \pi_{cycle} \cdot \pi_{recurr}.$$

Proof. We proved that
$$\varrho_{cyclical} \cdot P = \tau.$$
Using the formula $\varrho^P_{cyclical} = \pi_{cyclical}$, we get
$$\varrho_{cyclical} \cdot P = P \cdot \pi_{cyclical} \cdot P^{-1} \cdot P = P \cdot \pi_{cyclical} = \tau,$$
and followingly
$$\pi_{cyclical} = P^{-1} \cdot \tau.$$
Let us call the cycle cover $\tau \cdot \pi$ $(= \sigma)$ the *complementary* cycle cover of τ.

Then immediately follows, that for σ
$$P \cdot \sigma^{-1} = \varrho_{cut} \cdot \varrho_{cross},$$
where the edge types are with respect to the cycle cover τ.

Symmetrically, with the same assumption
$$P^{-1} \cdot \sigma = \pi_{cut} \cdot \pi_{cross}.$$

8.9 One more theorem for cycle covers

Theorem 39. *For a chosen cycle covered map* (P, Q, τ) *the map*
$$(P \cdot \pi_{cyclical}, P \cdot \pi_{non-cyclical})$$
is a combinatorial map, i.e. it has a graph in correspondence.

Proof.
$$(P, Q) \cdot pi_{cyclical} = (P \cdot \pi_{cyclical}, P \cdot \pi \cdot \pi_{cyclical} = (P \cdot \pi_{cyclical}, P \cdot \pi_{non-cyclical}).$$
This result is essentially used in the chapter 11.2.

8.10 Transformations of the cycle covers of combinatorial maps.

In this chapter we see how (P, Q, τ) changes when τ is multiplied by an arbitrary next edge (a, b).

We say that an edge depends upon the cycle κ from the cover τ, when all the corners of this edge, that is either cross edge or recurrence edge, belongs to this cycle.

We say that an edge depends upon the cycles $\kappa 1$ and $\kappa 2$ from the cover τ, when the corners of this edge, that is either cut edge or cycle edge, belongs to these both cycles.

Theorem 40. *Let (a,b) be cycle edge, i.e. $(a,b) \in \pi_{cycle}$ in τ and it depends in τ upon the cycles κ_1 and κ_2. Then in $\tau \cdot (a,b)$ it is a cross edge depending upon the new cycle κ, that is equal to*

$$\kappa = \kappa_1 \cdot \kappa_2 \cdot (a,b),$$

and all the cycle edges that were depending upon κ_1 and κ_2 now becomes recurrence edges, but all the cut edges that were depending upon κ_1 and κ_2, now become cross edges. Other edges, either depending upon any of those cycles, in the changed cycle cover do not change.

Proof. Multiplying τ by (a,b), only orbits κ_1 and κ_2 are changed in the cycle cover τ. Then the edges that are dependent upon these orbits change, as is stated by the theorem. But remains to persuade ourselves that the edges which corners also goes in these cycles, i.e. they depend from one or another of then, do not change their typeness. Really, when it was recurrence edge or cross edge, i.e. it was inner edge depending from one of the cycles, κ_1 or κ_2, then they remain such also in th new cycle κ as like. Non depending edges upon these cycles, do not change, of course.

As a corollary let us state, that changing cycle edge to cross edge other depending cycle edges change to recurrence edges, but depending cut edges change to cross edges. It means that some outer edges became inner edges, but all the inner edges remained to be inner.

Doing opposite operation, i.e. changing cross edge (a,b) multiplying τ by (a,b) , it becomes cycle edge. It is fixed in the next theorem.

Theorem 41. *Let (a,b) be cross edge, i.e. $(a,b) \in \pi_{cross}$ in τ and it depends in τ upon the cycle κ. Then in $\tau \cdot (a,b)$ it is a cycle edge depending upon the new cycle κ_1 and κ_2, that are equal to*

$$\kappa_1 \kappa_2 = \kappa \cdot (a,b),$$

and some of the cross edges that were depending upon κ now become cut edges, but some of them remain to be cross edges, but some of the recurrence edges that were depending upon κ, now become cross edges, but some of them remain to be recurrence edges as before the change. Other edges, depending upon this cycle, in the changed cycle cover do not change.

Proof. Looking this change as an opposite to the first, it is easy to see, which of the cross edges become cycle edges and which not, and similarly, which of the recurrence edges change to cycle edges and which remain to be recurrence edges. Dividing the orbit κ in τ into two orbits κ_1 and κ_2, some corners of the edges are now depending from the both new cycles and some are depending only from the one of them. The first group of edges change their type by this division of κ, but the other group of edges do not change their type.

Let us look the next pair of operations, when the cut edge changes to the recurrence edge.

Theorem 42. *Let (a,b) be cut edge, i.e. $(a,b) \in \pi_{cut}$ in τ and it depends in τ upon the cycles κ_1 and κ_2. Then in the changed cycle cover $\tau \cdot (a,b)$ it is a recurrence edge depending upon the new cycle κ, that is equal to*

$$\kappa = \kappa_1 \cdot \kappa_2 \cdot (a,b),$$

and all the cycle edges that were depending upon κ_1 and κ_2 now becomes recurrence edges, but all the cut edges that were depending upon κ_1 and κ_2, now become cross edges. Other edges, either depending upon any of those cycles or both, in the changed cycle cover do not change.

Proof. As before, multiplying τ by (a,b), only orbits κ_1 and κ_2 are changed in the cycle cover τ. Then the edges that are dependent upon these orbits change, as is stated by the theorem. Inner edges that were depending upon one of the cycles, κ_1 or κ_2, remain of the same type as before the change.

As a very similar corollary to the previous let us state, that changing the cut edge to the recurrence edge other depending cycle edges change to recurrence edges, but depending cut edges change to cross edges. It means that some outer edges became inner edges, but all the inner edges remained to be inner.

Opposite operation is symmetrical to the previous.

Theorem 43. *Let (a,b) be recurrence edge, i.e. $(a,b) \in \pi_{recurr}$ in τ and it depends in τ upon the cycle κ. Then in $\tau \cdot (a,b)$ it is a cut edge depending upon the new cycle κ_1 and κ_2, that are equal to*

$$\kappa_1 \kappa_2 = \kappa \cdot (a,b),$$

and some of the cross edges that were depending upon κ now become cut edges, but some of them remain to be cross edges, but some of the recurrence edges that were depending upon κ, now become cross edges, but some of them remain to be recurrence edges as before the change. Other edges, depending upon this cycle, in the changed cycle cover do not change.

Further we may find out the affected genus of the cycle cover by the multiplication of some next edge (a, b).

Theorem 44. *Let (a, b) be cycle edge depending in τ upon the cycles κ_1 and κ_2. Then in $\tau \cdot (a, b)$, when it becomes the cross edge, the genus of the changed cycle cover remains the same or is reduced by one.*

Proof. The cycle cover graph by the change of the cycle cover looses the edge (a, b). Then either the cycle was contractible and elimination of this edge reduces the genus of τ or the cycle is contractible and its elimination doesnt affect the genus of τ.

Theorem 45. *Let (a, b) be recurrence edge depending in τ upon the cycles κ_1 and κ_2. Then in $\tau \cdot (a, b)$, when it becomes the cut edge, the genus of the changed cycle cover remains the same or is reduced by one.*

Proof. The cycle cover graph by the change of the cycle cover looses the edge (a, b). Then either the cycle was contractible and elimination of this edge reduces the genus of τ or the cycle is contractible and its elimination doesnt affect the genus of τ.

Further we discuss some possible operations with sequences of edges multiplied to the cycle cover to change its genus in some desirable direction.

Theorem 46. *For arbitrary combinatorial map (P, Q) there exists such a cycle cover τ, that its genus γ_τ is equal to the genus of τ.*

Proof. Let us take arbitrary cycle cover τ.

Let us chose arbitrary cycle edge (a, b) and multiplying by it, change the cycle cover. Let us do this until no one cut edge is present in the cycle cover.

Let us further chose such cross edge that multiplication by it augments the genus. Change this cross edge to cycle edge. Repeat until no such cross edge is possible to find. If the genus of this equals to that of the map, then the theorem is proved.

Let the genus of the cycle cover is not yet equal to the genus of the map. Then we proceed as follows. Let us find two cross edges, (a_1, b_1) and (a_2, b_2), such that the orbit (a_1, a_2, b_1, b_2) is suborbit of some orbit of τ or its reverse τ^{-1}. We can this orbit (or its reverse) of the cycle cover express in the way

$$a_1 A_1 a_2 A_2 b_1 B_1 b_2 B_2,$$

denoting other sequences (possible empty) of elements of the orbit with big letters. Let us firstly multiply τ by the first cross edge (a_1, b_1). Then we would get as a result two orbits

$$a_1 A_1 a_2 A_2 and b_1 B_1 b_2 B_2,$$

where the edge (a_1, b_1) should become the cycles edge, but the edge (a_2, b_2) should become cut edge. Let us multiply the cycle cover with the second edge (a_2, b_2), that now is a cut edge. We get what follows

$$a_2 A_2 a_1 A_1 b_2 B_2 b_1 B_1,$$

and both edges become now recurrence edges. It is easy to see, that the genus of the cycle cover by this step is increased.

Let us repeat this routine until the genus of the cycle cover is equal with the genus of the combinatorial map.

Let us suppose the opposite, i.e. that we can not find any crossing cross edges and the genus of the cycle cover still is lesser that that of the map.

Then must be so, that all crossing edges are mutually non crossing. But then, multiplying by them the cycle cover, all of them turn into cycle edges, but do not increase the genus of the cycle cover. But then the cycle cover contain all the edges of the map, and in this case, it must be equal to face rotation of the map, and corresponding genus of this cover is evidently equal to that of the map. We have come to the contradiction. This says that the previous step of augmentation of the genus of cycle cover must be such that reaches the genus of the map.

It is possible to get rid of cut edges in this procedure in some more exact way, than how was suggested in the previous proof.

Theorem 47. *Let τ be cycle cover of the map (P, Q). Let $(a_1, b_1) \dots (a_k, b_k)$, $(k > 0)$ be the sequence (σ) of such next edges of the map that are cut edges of this cycle cover, that there are $2 \cdot k$ orbits in τ that each distinct orbit corresponds to one corner for each edge in the sequence. If this sequence is maximal, i.e. it can not been incremented, then the multiplication*

$$\tau \cdot \sigma$$

is without cut edges in it.

Proof. The fact, that the sequence σ can not be augmented, says, that there is not cut edges in the new cycle cover.

8.11 The computation

Let us discuss what from the considered things can be computed using multiplication and submap selection, i.e. restriction.

The most useful fact we got in the chapter 8.8.

Let us suppose that the combinatorial map (P, Q) is given. Let some process has found the cycle cover τ.

It is possible very easy to get cyclical edges separated from non cyclical by multiplication, i.e.

$$\varrho_{cycle} \cdot \varrho_{recurr} = P \cdot \tau^{-1}.$$

Symmetrically, for next edges

$$\pi_{cycle} \cdot \pi_{recurr} = P^{-1} \cdot \tau.$$

Non cyclical edges are

$$\varrho_{cut} \cdot \varrho_{cross} = Q \cdot \tau^{-1}.$$

And symmetrically,

$$\pi_{cut} \cdot \pi_{cross} = Q^{-1} \cdot \tau.$$

Let us denote by $\mathcal{G}[P, Q]$ the greatest common submap of P and Q.

Using the notion of the greatest common submap, we can separate inner edges from the outer in the following way:

$$\varrho_{recurr} = \mathcal{G}[\varrho_{cyclical}, \tau]$$

and

$$\pi_{recurr} = \mathcal{G}[\pi_{cyclical}, \tau]$$

Using the formula above, we can write

$$\varrho_{recurr} = \mathcal{G}[P \cdot \tau^{-1}, \tau]$$

and

$$\pi_{recurr} = \mathcal{G}[P^{-1} \cdot \tau, \tau]$$

Similarly for noncyclical edges:

$$\varrho_{cross} = \mathcal{G}[Q \cdot \tau^{-1}, \tau]$$

and

$$\pi_{cross} = \mathcal{G}[Q^{-1} \cdot \tau, \tau]$$

Next, more can be computed with local operations, i.e. such that uses as large computational space as is the updates space.

For example, in the process of eliminating of cut edges from the cycle cover:

Computation 1. *let* $\sigma = (a_1 b_1) \dots (a_k b_k)$
for $i := 1$ *to* k *do*
if $(a_i b_i)$ *is still* $cut - edge$ *then*
$\tau := \tau \cdot (a_i b_i)$.

The only non localized operation remains here checking, either (a, b) is a cut edge in the currant cycle cover or not.

9 Theory of the cycle covers of combinatorial maps. Colored cycle covers.

We assume the same definitions as in the case of non colored cycle covers.

For a combinatorial map (P, Q), a sequence of elements $c_1, \ldots, c_n$ $(n > 0)$ is *a trace*, if for c_i, $(0 < i < n)$, next element c_{i+1} in the sequence is equal to c_i^P or c_i^Q.

If this same holds also for c_n and c_1, i.e. c_1 equals to c_n^P or c_n^Q, then the path is closed, forming *a cyclical* trace.

Cycles are cyclical traces without repetition of elements. For a combinatorial map cycles are transitive permutations on some subsets of the set C with the appropriate choice of next elements.

A cycle cover of the combinatorial map (P, Q) is a permutation acting on the whole set C, where each of its orbits is a cycle in (P, Q). Then orbits in cycle covers are always cycles. Simplest cycle covers of (P, Q) are P and Q themselves.

Trivially enough, multiplying a cycle cover of some combinatorial map with some submap [58]of its π, i.e. next-edge-rotation, we get another (and possibly every) cycle cover of this combinatorial map.

We say, that two cycles ζ_1 and ζ_2 in (P, Q) *touch* each other when for same element e_1 of ζ_1 e_1^P or e_1^Q or e_1^{-P} or e_1^{-Q} is passed through by the cycle ζ_2.

The notion of the touching one cycle by another is most essential for colored cycle covers.

Let us suppose, that the elements of C for some combinatorial map (P, Q) with a fixed cycle cover are colored in such a way, that

1)elements of cycles of the cycle cover are colored in the same color, and

2) cycles with equal element coloring do not touch. Such a coloring of elements of combinatorial map we call *a cycle cover coloring.*

9.1 Two-colorable cycle covers in combinatorial maps

Further we explore only two-colorable cycle covers.

Firstly let us consider two-colorable cycle covers generally.

Theorem 48. *Let us suppose, that for a combinatorial map (P, Q) with cycle cover ζ elements are colored in two colors, so that this is also a cycle cover coloring. Let an arbitrary edge be with its (possibly not all distinct) corners c_1, c_2, c_3, c_4, such that $c_2 = c_1^{-P}$, $c_3 = c_1^{\pi}$ and $c_4 = c_1^{-Q}$.*
Then there are three possibilities:
1) c_1 and c_2 belong to the same color and c_3 and c_4 to the other color , [then we call such an edge cut-edge*];*
2) c_1 and c_4 belong to the same color and c_2 and c_3 to the other color, [then we call such an edge cycle-edge*];*
3) all corners of the edge are of the same color,[then we call such an edge inner edge*].*

Proof An edge is either a place where cycles of different color touch, or the same cycle meets itself (going either along the edge [like cycle-edge] or across the edge [like cut-edge]).

Let us notice, that , when the edge is not an inner edge, the pairs (c_1, c_3) and (c_2, c_4) contain corners of different coloring. So, edges of one color in π and ϱ are inner edges, but with different color corners - cycle-edges and cut-edges.

For further considerations most useful are cycle cover colorings without inner edges, because this case must show us, that all the results of noncolored cycle covers are applicable also in the case of colored cycle covers.

For two colorable cycle covers immediately is right what follows.

Theorem 49. *Let us suppose, that for a combinatorial map (P, Q) with cycle cover ζ elements are colored in two colors, green and red, so that this is also a cycle cover coloring and there are not inner edges. Then $\mid C_{green} \mid = \mid C_{red} \mid$, where $C_{green} \cup C_{red} = C$, and π and ϱ are one-one matches between C_{green} and C_{red}.*

Only cycle and cut edges are present in such covers, and every edge have two green corners and two red corners, then it is easy to see, that the cardinality of the sets C_{green} and C_{red} must be the same. Most essential is the fact, that involutions π and ϱ show themselves as bijections in more direct way, that they are one-one maps between C_{green} and C_{red}.

Further we speak about the way how to get two-colorable cycle covers without inner edges in combinatorial maps, using a knot of this map.

Combinatorially, knot is a zigzag walk cover in the combinatorial map [2] and [58].

Zigzag walk always has orbits of even degree, so it is possible to connect with a zigzag walk a coloring of elements of combinatorial map. More over, elements of one color form cycles of one color, resulting in the cycle cover with coinciding coloring.

Theorem 50. *Let for a combinatorial map (P, Q) is given coloring $C_{green} \cup C_{red}$, and a knot of this map by this same coloring is colored alternatively. Then there exists one unique cycle cover of this map with this same coloring as the cycle cover coloring without inner edges.*

Proof Let us suppose the opposite, and elements of one color do not form cycles. Then there must be an element c, say of green color, but the colors of both $c^P (= c_1)$ and $c^Q (= c_3)$ are red. But the pair (c_1, c_3) belongs to π and its elements must have different colors. We have come to a contradiction. And of course, there can not be inner edges.

Thereafter, inversely, each cycle cover without inner edges fixes some knot with precision to the reverse.

Theorem 51. *Let for a combinatorial map (P, Q) a knot of this map is found, and let us fix with this knot a coloring of corners of the map. Then there exists one unique cycle cover of this map with this same coloring as the cycle cover coloring without inner edges.*

Proof Fixing the coloring of the corners by the knot's coloring, each edge gets corners colored in such a way, that corners of π-transition are colored in different colors and similarly, corners of ϱ-transition are colored in different colors. But this condition suffices to define such coloring for the corners of the map, that also cycle cover is defined, that has this coloring as its coloring. Rightly, let us choose in cycle cover for a corner that corner that has the same color. Then this defines cycle cover.

Now we may persuade ourselves that two-colorable cycle covers define the same edge types as non-colored cycle covers.

Theorem 52. *Let for a combinatorial map (P, Q) some two colorable cycle cover is found. Then it is also a cycle cover in the sense as was discussed in the chapter 8. Further, edges, that are cycle edges, are cycle edges as defined in the chapter 8 and also edges, that are cut edges, are cut edges as defined in the chapter 8.*

Proof The statement of the theorem follows from the fact, that there touch each other only cycles with different color in colorable cycle covers, but this corresponds to the case, when two cycles depend upon some edges. Looking from the edges, when they have present two colors, then it is the same, as it depends upon two different cycles.

As a note can be stated, that this correspondence remains also in the case, when we allow in the colored cycle cover case cycles be such, that some cycles visit some edges repeatedly, i.e. that they have also recurrence and cross edges.

It is easy to get following features of colored cycle cover ζ without inner edges for (P, Q).

Theorem 53. *1) $\zeta \cdot \pi = \zeta_{altern}$, where ζ_{altern} is a cycle cover with alternating coloring of its elements;*
2) $\zeta^{-1} \cdot P = \zeta_{altern}^{-1} \cdot Q = \pi_{cycle}$, where π_{cycle} have all cycle-edges and only them;
3) and $\zeta_{altern}^{-1} \cdot P = \zeta^{-1} \cdot Q = \pi_{cut}$, where π_{cut} have all cut-edges and only them;
4) and π_{cycle} and π_{cut} are complementary involutions and thus $\pi_{cycle} \cdot \pi_{cut} = \pi$.

Proof Second and third points follows from the theorem 37.

9.2 Graphs on surfaces.

9.2.1 Introduction.

We have done preparations to do some explorations in maps with objects and functions that has in correspondence some topological objects and functions. We have the cycles and the paths and we can compute the genera of submaps, that all are graph-topological notions in essence.

We try to show the usefulness of our approach showing that we can investigate cycles and cycle structures in the graph on the surface. We can compute whether the cycle is contractible topologically to the point or not. We try to use it to find independent set of noncontractible cycles and to develop some apparatus to compute these things sufficiently fast and simply.

Here in this section we find out the 1-type operations, i.e. these computations, that are done only in permutations. In our computations we use also 2-type operations, these that are expressed in permutations, but themselves are to be done with hidden algorithm.

9.2.2 A theorem about genera of three maps.

Thus further we think more in terms of graphs and topology (than combinatorically), but corresponding manipulations do in maps, i.e. permutations.

Let us partition C into $C_{cycle} \cup C_{cut}$, where the first set contains elements of π_{cycle} and the second - of π_{cut}.

Theorem 54. *The partial map $(P, \zeta) \mid_{C_{cycle}}$ is a combinatorial map on C_{cycle}.*

Applying the theorem dually, we get, that also (Q, ζ) restricted on C_{cut} is a combinatorial map on this set.

Let us write ζ_{cycle} in the place of $\zeta \mid_{cycle}$, and so also ζ_{cut} in the place of $\zeta \mid_{cut}$.

We call $(P, \zeta) \mid_{C_{cycle}}$ and $(P, \zeta) \mid_{C_{cut}}$ correspondingly a *cycle graph* and a *cut graph.* The second one is in general a partial map.

Before we come to our theorem of this section we must notice what follows.

Theorem 55. *All cycles of the cycle cover for (P, Q) are also cycles in p-map (ζ, Q).*

Theorem 56.

$$\gamma_{(P,Q)} > \gamma_{(P,\zeta)} + \gamma_{(Q,\zeta)}.$$

Proof We must prove that

$$\gamma_{(P,Q)} > \gamma_{(P,\zeta)} + \gamma_{(Q,\zeta)},$$

using that

$$\gamma_{(P,Q)} = 1 - 1/2[\| P \| + \| Q \| + \| R \| - l],$$

and

$$\gamma_{(P,\zeta)} = c_{(P,\zeta)} - 1/2[\| P \| + \| \zeta \| + \| R_{(P,\zeta)} \| - l],$$

where $c_{(P,\zeta)}$ - number of components in (P, ζ), but $R_{(P,\zeta)}$ - the edge rotation, and

$$\gamma_{(Q,\zeta)} = c_{(Q,\zeta)} - 1/2[\| Q \| + \| \zeta \| + \| R_{(Q,\zeta)} \| - l],$$

where $c_{(Q,\zeta)}$ - number of components in (Q, ζ), but $R_{(Q,\zeta)}$ - the edge rotation.

Then there must hold the inequality

$$\| R \| < 2 \| C_\zeta \| + \| R_{(P,\zeta)} \| + \| R_{(Q,\zeta)} \| - l - 2c_{(P,\zeta)} - 2c_{(Q,\zeta)} + 2.$$

and

$$\| Q \cdot P^{-1} \| < 2 \| \zeta \| + \| \zeta \cdot P^{-1} \| + \| \zeta \cdot Q^{-1} \| - l - 2c_{(P,\zeta)} - 2c_{(Q,\zeta)} + 2$$

and

$$\| \zeta \cdot P^{-1} \| + \| \zeta \cdot Q^{-1} \| - \| Q \cdot P^{-1} \| + 2 \| \zeta \| > l - 2c_{(P,\zeta)} - 2c_{(Q,\zeta)} + 2,$$

and

$$\| \zeta \cdot P^{-1} \| + \| \zeta \cdot Q^{-1} \| + 2 \| \zeta \| > 3/2l - 2c_{(P,\zeta)} - 2c_{(Q,\zeta)} + 2,$$

and

$$\| \zeta \cdot P^{-1} \| + \| \zeta \cdot Q^{-1} \| + 2 \| \zeta \| > 3/2 \cdot l + 2c_{(P,\zeta)} + 2c_{(Q,\zeta)} - 2,$$

and

$$\| \pi_1 \| + \| \pi_2 \| + 2 \| \zeta \| > 3/2 \cdot l + 2c_{(P,\zeta)} + 2c_{(Q,\zeta)} - 2,$$

i.e. $\| \zeta \| > c_{(P,\zeta)} + c_{(Q,\zeta)} - 1$; where $c_{(P,\zeta)}$ and $c_{(Q,\zeta)}$ are numbers of components in the corresponding partial maps. But cycles of the cycle cover of (P, Q) are also orbits of both (ζ, P) and (ζ, Q), but being cut as a separate component only in the one of them. This proves what was stated.

Further we try to show that this rather simple fact about permutations causes less trivial consequence in graphic-topological view.

9.2.3 Cutting surface along cycles in the graph embeddings.

Now we show, that the expression of the permutations in the previous section has an equivalent in the topology of graphs on surfaces. Namely, we show that the p-maps (P,ζ) and (Q,ζ) have a geometrical interpretation and could be useful in applications.

From the graph topological point of view, when we cut a surface, in which a graph is embedded, along some cycle, then the surface is either cut to two parts or its genus is reduced depending whether the cut line along the cycle is contractible in the topological sense to point or not. Choosing some cycle in cycle cover and cutting orbits into orbits of one color in P along this cycle do the same thing in the combinatorial maps.

Let us denote multiplication $P \cdot P_{cycle}^{-1}$ by P' and consider it nearer, seeing behind the partial map (P',Q) one with cut embeddance surface along the cycles of cycle cover.

Theorem 57. *P', i.e. $P \cdot P_{cycle}^{-1}$ is equal to $\zeta \cdot \zeta_{cycle}^{-1}$ and the genus of (P',Q) is equal to the genus of (ζ,Q).*

Proof First we prove some lemmas.

Lemma 5. *Orbits of P_{cycle} have an alternating coloring, but orbits of P' have elements of one color.*

Lemma 6. $P_{cycle} \cdot \pi_{cycle} = \zeta_{cycle}$.

Proof Before it is shown that $P \cdot \pi_{cycle} = \zeta$. Restricting this expression onC_{cycle} we get $(P \cdot \pi_{cycle}) \mid_{cycle} = \zeta_{cycle}$, but the left side is equal to $P_{cycle} \cdot \pi_{cycle}$, what was to be proved.

Let us prove that $\zeta \cdot \zeta_{cycle}^{-1} = P'$. Really,

$$\zeta \cdot \zeta_{cycle}^{-1} = P \cdot \pi_{cycle} \cdot \pi_{cycle} \cdot P_{cycle}^{-1} = P \cdot P_{cycle}^{-1} = P'.$$

It remains to prove that $\gamma_{(P',Q)} = \gamma_{(\zeta,Q)}$. It follows from the fact, that multiplication of ζ by reverse ζ_{cycle} do not changes the genus of the partial map (ζ,Q), because the orbits of the reverse of ζ_{cycle} are subedges of multiedges of p-map (P',Q), where the last are the remnants of the cutting the surface along the cycles.

This completes the proof of the theorem.

Let us discuss what we did.

P is the vertex rotation of the c-map (P,Q) with the cut cycles of the cycle cover ζ. Let us notice how easy it can be calculated, when necessary, i.e.

$$P = P \cdot P_{cycle}^{-1}.$$

Moreover, this rotation can be calculated in the optional way, i.e. it is equal also to $\zeta \cdot \zeta_{cycle}^{-1}$.

Example 14. *The map:*

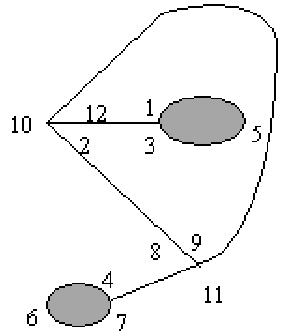
$$P = (163)(2\bar{0}\bar{2})(475)(89\bar{1})$$

$$Q = (15329\bar{2})(48\bar{0}\bar{1}76)$$

$$\mu = (1287)(34\bar{1}\bar{2})(56)(9\bar{0})$$

$$P_{cycle} = (36)(45)$$

$$P = P \cdot P_{cycle} = (13)(2\bar{0}\bar{2})(47)(89\bar{1})$$



Picture20 : Drawing of the p-map (P,Q) with the cut cycle marked

9.2.4 Conclusions of the comparing two facts.

Let us compare our main theorem and the topological theorem.

The main theorem above says that the difference between genera of (P, Q) and (ζ, Q) plus the difference between genera of (Q, P) and (ζ, P) is greater then genus of (P, Q).

Trying to translate this fact in the topological language, it would mean, that every cover of cycles ζ have enough cycles in the sense that, cutting along them surface both of (P, Q) and its dual (Q, P), reduces its genus completely. The uncontractible to the point cycles from ζ_{cycle} in the graph (ζ, Q) give the necessary uncontractible cycles, but in the graph (ζ, P) the genus-reducing-cuts of the edges.

Choosing another cycle cover ζ from the possible ones we could expect some change of the value of genus of (ζ, Q), that leads to greater or lesser difference from the genus of (P, Q), what would mean that we have greater or lesser set of uncontractible cycles in the cycle cover. Further in the chapter 13 we discuss the possibility to explore such features in an experimental system.

The results of this section gives a hope, that it is possible to find more useful operations, which would result in some permutational calculus with topological application.

10 Permutations revisited

In the next chapter we will develop some calculus of maps and their characteristics when the coloring of the universal set C is given. In this order we are going to elaborate special mechanisms for work with permutations when we have some extra partitioning of the set where on the permutations act. Here and further e denots an identical permutation.

Let us use special denotation for permutations when they act on partitioning

$$C_1 \bigcup C_2 \bigcup \ldots \bigcup C_k = C^{\Sigma} = C$$

writing

$$p = (C_1 : p_1, C_2 : p_2, \ldots, C_k : p_k)$$

where p_i for all i from 1 to k are injections from C_i's into C^{Σ}, such that the images of p_i comprise some partitioning of C possible different from C^{Σ}.

In special case, having a set C where upon some permutations act, let us have another disjoined set $\bar{C}$ of the same cardinality and bijection u from C to $\bar{C}$ and let us make act our permutations on extended domain $C \bigcup \bar{C}$. We are going to use following denotations. $\bar{e}$ stands for $u(e)$. For $p = (C_1 : p_1, C_2 : p_2)$ we write also $p = \left\{ \begin{array}{l} C_1 : p_1 \\ C_2 : p_2 \end{array} \right.$.

Let us have p defined on C. $\bar{p}$ we define as follows: for $c \in C$ and $c^p = d$ and $\bar{c} \in \bar{C}$ let $\bar{c}^{\bar{p}} = \bar{d}$.

It is easy to see that, if $p = (C : p, \bar{C} : e)$ then $\bar{p} = (C : e, \bar{C} : u \cdot p \cdot u) = u \cdot p \cdot u$.

Let us define $\tilde{p}$ as $(C : u, \bar{C} : \bar{p} \cdot u)$. Trivially, $\bar{p} \cdot u = u \cdot p$. Hence,

$$\tilde{p} = (C : u, \bar{C} : u \cdot p) = u \cdot p = \bar{p} \cdot u.$$

Let us prove a technical lemma, that helps us to deal with these special permutations.

Lemma 7. *Let $C^{p_1} = C$ and $C^{p_2} = C$. Then*

$$p = \left\{ \begin{array}{ll} C : & p_1 \cdot u \\ \bar{C} : & u \cdot p_2 \end{array} \right. = p_1 \cdot u \cdot p_2.$$

Immediate calculation gives the result.

11 The image of the partial map and other characteristics. Their calculation in permutations

. Using the notions of the previous chapter we are going to calculate in permutations the characteristics of combinatorial maps, that was considered in the upper chapters, i.e. images of p-maps, cycle covers, knots and so on.

In chapter 6.2 we introduced the notion of image of p-map and showed how to find it for a given p-map. Now, using the apparatus of chapter 10, we can calculate the image of p-map using some chosen bijection u for C and for corresponding set of 'empty 'corners that we denote also as $\bar{C}$.

Let us chose u as we did in the chapter 6.2. For an arbitrary corner c the corresponding corner $u(c)$ or $\bar{c}$ is the 'empty' corner that followed c in clockwise direction around the vertex where c goes in.

Then we may use formulas of previous chapter: for p such that $c^p = C$

$$p = \left\{ \begin{array}{ll} C : & p \\ \bar{C} : & e \end{array} \right. ; \quad \bar{p} = \left\{ \begin{array}{ll} C : & u \\ \bar{C} : & up \end{array} \right. = upu; \quad \tilde{p} = \left\{ \begin{array}{ll} C : & u \\ \bar{C} : & \bar{p}u \end{array} \right. = \left\{ \begin{array}{ll} C : & u \\ \bar{C} : & up \end{array} \right. = up = \bar{p}u.$$

Now we are ready to present the image of given p-map (P, Q) in a new view.

Theorem 58. *Image of p-map (P, Q) is equal to $A = (A_1, A_2) = (uP, QuPQ^{-1}u)$.*

Proof Direct calculation gives

$$A = (\tilde{P}, Q\bar{P}\bar{Q}^{-1}) = (uP, QuPuuQ^{-1}u) = (uP, QuPQ^{-1}u).$$

In chapter 6.2 we proved that the image is always a c-map. We can easy persuade ourselves in it using this new apparatus. Let us calculate π_A:

$$\pi_A = A_1^{-1} \cdot A_2 = P^{-1}uQuPQ^{-1}u = \begin{cases} C: & P^{-1}PQ^{-1}u \\ \bar{C}: & uQ \end{cases} = \begin{cases} C: & Q^{-1}u \\ \bar{C}: & uQ \end{cases} = Q^{-1}uQ.$$

From this it is easy to see that π_A is an involution without fixed points. Really, for $c \in C$ $c^{\pi_A} = \bar{c}^{\bar{Q}^{-1}} \neq c^{\pi_A}$. And $\pi_A^2 = Q^{-1}uQQ^{-1}uQ = e$.

Hence, we have

$$\pi_A = [(\bar{c}\ c^Q)|c \in C] = [(c\bar{c}^{\bar{Q}^{-1}})|c \in C].$$

Similarly, we may calculate edges also, i.e. ϱ_A:

$$\varrho_A = QuPQ^{-1}uP^{-1}u = \begin{cases} C: & QP^{-1}u \\ \bar{C}: & uPQ^{-1} \end{cases} = QP^{-1}uPQ^{-1} = \varrho u \varrho^{-1}.$$

Edges of image A can be expressed as

$$\varrho_A = [(\bar{c}c^{\varrho^{-1}})|c \in C] = [(c\bar{c}^{\bar{\varrho}})|c \in C].$$

We have reproved already known fact about the way how to find the edge rotation of the image of p-map. The method how to use this in the practical drawing of the image was discussed in the end of the section 6.3.

11.1 The knot of the image of p-map and the knotting α

A characteristic α of the map [see the corollary 2], when μ is chosen, is always determined. We call α a *knotting*. Partitioning of the corners of the image is a well coloring , because $C^\pi = \bar{C}$ and $C^\varrho = \bar{C}$. Consequently, using this well coloring we obtain also defined a knot of this c-map. Let us take μ_A equal to

$$\begin{cases} C: & \pi_A \\ \bar{C}: & \varrho_A \end{cases}.$$

Let us calculate this μ_A:

$$\mu_A = \begin{cases} C: & \pi_A \\ \bar{C}: & \varrho_A \end{cases} = \begin{cases} C: & Q^{-1}u \\ \bar{C}: & uPQ^{-1} \end{cases} = Q^{-1}uPQ^{-1}.$$

And its dual is equal to:

$$\mu_A \cdot \pi_A = Q^{-1}uPQ^{-1}Q^{-1}uQ = \begin{cases} C: & Q^{-1}Q \\ \bar{C}: & uPQ{-}2u \end{cases} = uPQ^{-2}u = \bar{P}\bar{Q}^{-2}.$$

Let us calculate the knotting α using expression $P = \mu \cdot \alpha$:

$$\alpha_A = \mu_A^{-1} \cdot A_1 = QP^{-1}uQuP = \begin{cases} C: & QP^{-1}P \\ \bar{C}: & uQu \end{cases} = \begin{cases} C: & Q \\ \bar{C}: & uQu \end{cases} = QuQu = Q \cdot \bar{Q}.$$

Its dual α is equal to:

$$\alpha_A \cdot \pi_A = QuQuQ^{-1}uQ = \begin{cases} C: & QQ^{-1}u \\ \bar{C}: & uQQ \end{cases} = \begin{cases} C: & u \\ \bar{C}: & uQ^2 \end{cases} = uQ^2 = \widetilde{Q^2}.$$

When μ or α is known, sometimes it is useful to know how to find the bijection u. Now it shall be some known function of p-map. We denote it by β. From $\pi_A = Q^{-1}uQ$ we can receive that β_A is equal to π_A^{-Q}. This funtion has an interesting geometrical interpretation.

11.2 Using of the image calculations to c-maps

Let us have a given c-map (P,Q). From the discussion before we know that the corner set C can be partitioned into $C_1 \cup C_2$ so that $C_1^\pi = C_2$ and $C_1^\varrho = C_2$. This we call a well coloring of corners and this defines also some knot μ. Then μ can be taken equal to $(C_1 : \pi, C_2 : \varrho)$.

Further, we can separate edges and correspondingly next edges in a way that it holds $C_1^{P\pi_1} = C_1$ and $C_1^{P\pi_2} = C_2$, where $\pi_1 \cdot \pi_2 = \pi$, i.e. π_1 comprise one part of edges with ϱ_1 in correspondence, and π_2 the second part of edges(with ϱ_2 in correspondence).

Then also holds $C_2^{P\pi_1} = C_2$ and $C_2^{P\pi_2} = C_1$. In this case, π_1 comprise those edges that has color $\chi(c^{P^{-1}}) = \chi(c^\pi)$. And, consequently, π_2 comprise those edges that has color $\chi(c^{Q^{-1}}) = \chi(c^\pi)$.

Theorem 59. *Let (P,Q) be arbitrary c-map. Then $(P\pi_1, P\pi_2)$ is a c-map and $P\pi_1$ can be expressed as a multiplication $\zeta_1\zeta_2$ or reversibly, $\zeta_2\zeta_1$ where ζ_1 acts in C_1 and ζ_2 acts in C_2.*

Proof By the definition of π_1 and π_2 cycles in $P \cdot \pi_1$ have corners of the same color. Cycles of the same color should give ζ_1 and ζ_2 respectively.

We now apply the previous theory of partial maps and their images. We are going to prove that the map $P\pi_1$ is image of some p-map, that can be calculated.

Theorem 60. *C-map $(P\pi_1, P\pi_2)$ is an image of p-map $(\zeta_1\zeta_2^\pi, \zeta_1)$ with the bijection u equal to $\zeta_1\pi\zeta_2^{-1}$.*

Proof Let us compute the image of some p-map $(p,q) = (\zeta_1\zeta_2^\pi, \zeta_1)$ using bijection u equal to $\zeta_1\pi\zeta_1^{-1}$. Then

$$A_1 = \tilde{p} = up = \zeta_1\pi\zeta_1^{-1}\zeta_1\pi\zeta_2\pi = \zeta_1\zeta_2\pi = P\pi_1.$$

And

$$A_2 = qupq^{-1}u = \zeta_1\zeta_1\pi\zeta_1^{-1}\zeta_1\pi\zeta_2\pi\zeta_1^{-1}\zeta_1\pi\zeta_1^{-1} = \zeta_1^2\zeta_2\zeta_1^{-1}.$$

Using equality $\zeta_1\zeta_2 = \zeta_2\zeta_1$ we get

$$A_2 = \zeta_1\zeta_2 = P\pi_2.$$

The proof in its necessary part is already done. We may reprove some more facts using this apparatus.

Let us count π_A and ϱ_A.

$$\pi_A = q^{-1}uq = \zeta_1^{-1}\zeta_1\pi\zeta_1^{-1}\zeta_1 = \pi.$$

And further,

$$\begin{aligned}
\varrho_A &= qp^{-1}upq^{-1} \\
&= \zeta_1\pi\zeta_2^{-1}\pi\zeta_1^{-1}\zeta_1\pi\zeta_1^{-1}\zeta_1\pi\zeta_2\pi\zeta_1^{-1} \\
&= \zeta_1\pi\zeta_2^{-1}\pi\zeta_2\pi\zeta_1^{-1} \\
&= \begin{cases} C_1 : & \zeta_1\pi\zeta_2^{-1} \\ C_2 : & \pi\pi\zeta_2\pi\zeta_1^{-1} \end{cases} \\
&= \zeta_1\zeta_2\pi\zeta_2^{-1}\zeta_1^{-1} \\
&= \varrho.
\end{aligned}$$

We may proceed in calculating of functions of image of p-map $(\zeta_1\zeta_2^{-1}, \zeta_1)$ with the function u equal to $\zeta_1\pi\zeta_1^{-1}$. Thus, the knot of this image is

$$\mu_A = q^{-1}upq^{-1} = \zeta_1^{-1}\zeta_1\pi\zeta_1^{-1}\zeta_1\pi\zeta_2\pi\zeta_1^{-1} = \zeta_2\pi\zeta_1^{-1}.$$

Moreover, we may express it as follows:

$$\mu = \zeta_2\zeta_1^{-1}u = \zeta_2\zeta_1^{-1}\beta.$$

Let us calculate a knotting also:

$$\begin{aligned}
\alpha &= q\bar{q} = \zeta_1\zeta_1^u = \zeta_1\zeta_1\pi\zeta_1^{-1}\zeta_1\zeta_1\pi\zeta_1^{-1} \\
&= \zeta_1^2\pi\zeta_1\pi\zeta_1^{-1} \\
&= \begin{cases} C_1: & \zeta_1^2\zeta_1^{-1} \\ C_2: & \pi\zeta_1\pi \end{cases} \\
&= \zeta_1\pi\zeta_1\pi \\
&= \zeta_1\zeta_1^{\pi}.
\end{aligned}$$

$$\bar{\alpha} = uq^2 = \zeta_1\pi\zeta_1^{-1}\zeta_1^2 = \zeta_1\pi\zeta_1.$$

We may check that

$$\mu\alpha = \zeta_2\pi\zeta_1^{-1}\zeta_1\pi\zeta_1\pi = \zeta_2\zeta_1\pi.$$

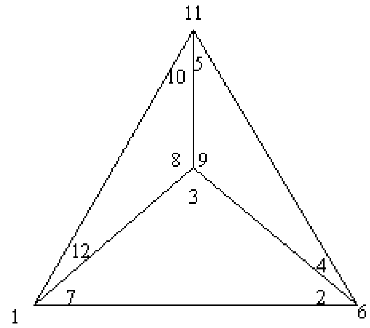
Example 15.

$$(P,Q) = \begin{cases} (1\bar{2}7)(246)(389)(5\bar{0}\bar{1}) \\ (1\bar{1}6)(237)(459)(8\bar{0}\bar{2}) \end{cases}$$

$$\begin{aligned}
\pi &= (12)(34)(56)(78)(9\bar{0})(\bar{1}\bar{2}) \\
\varrho &= (1\bar{0})(29)(3\bar{2})(4\bar{1})(58)(67) \\
C_1 &= \{1,3,5,7,9,\bar{1}\} \\
C_2 &= \{2,4,6,8,\bar{0},\bar{2}\} \\
\zeta_1 &= (1\bar{1}5937) \\
\zeta_2 &= (8\bar{0}\bar{2})(246) \\
\pi_1 &= (12)(34)(56) \\
\varrho_1 &= (67)(29)(4\bar{1}) \\
\pi_2 &= (78)(9\bar{0})(\bar{1}\bar{2}) \\
\varrho_2 &= (3\bar{2})(58)(1\bar{0})
\end{aligned}$$

$$\begin{aligned}
\mu &= \begin{cases} C_1: & (12)(34)(56)(78)(9\bar{0})(\bar{1}\bar{2}) \\ C_2: & (1\bar{0})(29)(3\bar{2})(4\bar{1})(58)(67) \end{cases} \\
&= \begin{cases} C_1: & [1\mapsto 2, 3\mapsto 4, 5\mapsto 6, 7\mapsto 8, 9\mapsto \bar{0}, \bar{1}\mapsto \bar{2}] \\ C_2: & [2\mapsto 9, 4\mapsto \bar{1}, 6\mapsto 7, 8\mapsto 5, \bar{0}\mapsto 1, \bar{2}\mapsto 3] \end{cases} \\
(P\pi_1, P\pi_2) &= \begin{cases} (1\bar{2}7238945\bar{0}\bar{1}6) \\ (1\bar{1}5937)(246)(8\bar{0}\bar{2}) \end{cases} \\
&= A_{(p,q)},
\end{aligned}$$

where $(p,q) = \begin{cases} (173951\bar{1}) \\ (1\bar{1}5937) \end{cases}$ *with* $u = (1\bar{2})(38)(5\bar{0})(72)(94)(\bar{1}6) = \beta.$



Picture 21: Drawing of the map (P, Q)

11.3 Some more expressions

The knot μ creats some set of bijections from one colored part of C to other: the main of them are $\pi, \varrho, \beta, \delta$. From previous chapter we got that $\mu = \zeta_2 \pi \zeta^{-1}$. Denoting the multiplication $\zeta_2 \zeta^{-1}$ with ϕ, we can get following expressions:

$$1) \quad \pi = \beta^{\zeta_1} = \delta^{\zeta_2};$$

$$2) \quad \beta = \delta^{\mu} = \delta^{\phi};$$

$$3) \quad \mu = \phi \cdot \beta = \delta \cdot \phi.$$

12 Exploration of the graph-topological operations

12.1 Finding of the knot of the combinatorial map

When some procedure has performed the coloring of the corners such that both the edge rotation and the next edge rotation are bijections from the one color colored corners (C_1) to the other (C_2), then the knot is already found in that sense that it can be calculated by permutational formula $\left\{ \begin{array}{cc} C_1 : & \pi \\ C_2 : & \varrho \end{array} \right.$, i.e. it is, of course, 1-type operation.

When the well coloring (in the two colors) of the corners is not given, then the finding of the knot is just this process of the coloring of the corners in the established way, i.e. when π andϱ become the bijections between the both color corners.

Finding the knot itself does not chose one particular coloring, because we must make a choise about what coloring from the possible colorings is to be used. As it is stated there are 2^k possible colorings when the knot has k links.

We consider the finding of the knot the 3-type operation, because essentially it is a walk through the map, and, similarly as in literature, we call this walk — zigzag walk.

12.2 Reduction of the genus of the graph embedding on the surface.

One of the main tasks to be solved in the graph-topology is to find such rotation of the graph that its genus be reduced, and so the embedding on the surface with lower genus is found. In general it is NP-complete problem, i.e. it can not be solved with some polinomial algorithm. But in different situations in the practice this task can be tried to solve more effectively than it is expectable in general. Here we investigate purely theoretical possibilities to solve the problems, and besides we turn more our attention how to do some operations effectively manually or in some dialog with interchanged manual and computer search operations.

First let us consider some special case what can be present rather often in large graph embeddings with very large genus.

Theorem 61. *Let us suppose that a map has a submap $((ac)(bd);(cd))$, where $(ab) \in \pi$. Then the edge (ab) can be reimplemented in the orbit with the suborbit (cd) so, that the genus of the imbedding is not made greater.*

The proof of this statement is trivial, but it is a very useful fact in applications.

Actualy there could be more than one such submaps around the edge (ab) when the multiedges are aloυded.

Let us show how the map should be changed when we want to do this change of the last theorem.

Theorem 62. *The edge rotation P must be changed in the way*

$$(a^{P^{-1}}ac) \times (b^{P^{-1}}bd) \times P$$

in order to get the edge reset in the orbit with elements a, b, c, d as in the conditions of the theorem 61.

This operation is easy implementable in the program and made run on the computer. This is done in the system descibed lower.

12.3 Finding of the multiedge submap

The mutiedge submap we call the map isomorphical to

$$(ab)(cd);(ac)(bd).$$

Immediately can be observed, that

Theorem 63. *If a map has the submap $(ab)(cd);(ac)(bd)$, then the corresponding graph has a pair of separation at the vertices corresponding to the vertex rotations orbits with the corners a and c.*

The corresponding change of the graph at the points of separation could proceed as follows. We disconnect the map at the corner pairs (a,b) and (c,d). The one disconnected part must be turned over. i.e. corresponding vertex rotation must be reversed. Then the parts must be connected back with the specified pairs of corners.

The operation of the finding of all the multiedge submaps in the map we consider as the 2-type operation, but the corresponding update of the map itself — the 3-type operation.

13 Exploration of the graph-topological features in the computer system *CombMap*

We have built a system *CombMap*and implemented it in the PASCAL program, which explores the maps and the graphs on the surfaces using the theoretical support of the combinatorial map theory.

The system gives possibility to check quickly ideas, that arise in the process of the work, to prove or (maybe sometimes) reprove experimentally, that mathematically are already proved or suggested facts, on the series of maps, to find the sequences of operations, that do particular manipulations with the graphs.

We enter the graphs manually or generate them randomly. Manually entered graphs are usually in the cyclical form and similarly are also displeyed on the monitor. Randomly graphs are firstly, where simply random permutations are generated or secondly, graphs with regular degree, for example — cubic, are generated. Planar graph series is possible to create too.

One of the first series of experiments was done cutting the noncontractable to the point cycles from the cycle cover that was found by the zigzag walk. Exploring the graphs in this way we came to the theory about the cycle cover cycle subgraphs and the possibility to calculate the partial map with the cut faces just corresponding to the cut cycles from the cycle cover with the permutations.

In this system has been explored possibilities to find such sequence of operations of transformation of the edges from the cycle cover from one type to some other type in order to find a cycle cover with the maximal possible genus, i.e. it that is equal to the genus of the whole map. These explorations still are in the process.

The system is aimed to be developed in that way that the well known macro-algorthms would be possible to be model with the operations present. For example, the planarization of the graph [33, 44, 45] following some chosen process, eliminating the edges or making them cross with new vertices introduction in the intersections, or else; or the division in the 3-connected components [34, 54, 48]. We do not consider this as an aim in the sense that in this way we are aiming to reach some progress, but rather through this facilies being possible to do these works in our system we would reach such level of development, that we could easy do wichever possibly series of operations interchanging them with the manual operations in the dialog.

In this direction we see the necessity to develop also technological system to do possible experiments with large graphs. Up to now the largest our graphs were several hundred corners up to the thousend maximally.

Another direction of the development we see the educational system to explore the graphs on the surface.

The system *CombMap* is implemented in the PASCAL program.

```
****************************************************
prism graph with two nonplanar edges [Genus=2]
P=(1 12 15 7)(2 4 16 6)(3 14 8 9)(5 10 11 13)
Q=(1 11 14 7 2 3 13 6)(4 15 8 10 12 16 5 9)
 [4:11 3:14 1:7 2:2 3:3 4:13 2:6 1:1] [1:15 3:8 4:10 1:12 2:16 4:5 3:9 2:4]
ro=(1 10)(2 9)(3 11)(4 12)(5 8)(6 7)(13 16)(14 15)
mi=(1 2 9 10)(3 4 12 11)(5 6 7 8)(13 14 15 16)
pi1=(1 2)(3 4)(5 6)(7 8)(11 12)(15 16)
pi2=(9 10)(13 14)
cycle map [genus=0]=(9 14)(10 13)
          =(9 13)(10 14)
cut map [genus=0]=(1 12 15 7)(2 4 16 6)(3 8)(5 11)
          =(1 11 6)(2 3 7)(4 15 8)(5 12 16)
cycle dmap[genus=1]=(1 11 7 2 3 6)(4 15 8 12 16 5)
          =(1 12 15 7)(2 4 16 6)(3 5)(8 11)
cut dmap[genus=0]=(9 10)(13 14)
          =
gamma=(1 12 15 7)(2 4 16 6)(3 13 5 9)(8 10 11 14)

NewP:(1 12 15 7)(2 16 6)(5 11)
 NewFiltered cycliCover:(1 11 6)(2 15 8 7)(5 12 16)
 NewGenus=0 New cycliCover:(1 11 13 6)(2 4 15 8 9 3 14 7)(5 10 12 16)
 Recurrence edges(7 8)
 Cycle edges(1 2)(5 6)(11 12)(15 16)
 Cross edges(3 4)
 Cut edges(9 10)(13 14)
 NewP:(1 12 7)(2 6)(5 11)
 NewFiltered cycliCover:(1 11 6)(2 5 12 8 7)
 NewGenus=0 New cycliCover:(1 11 13 6)(2 4 16 5 10 12 15 8 9 3 14 7)
 Recurrence edges(7 8)
 Cycle edges(1 2)(5 6)(11 12)
 Cross edges(3 4)(9 10)(15 16)
 Cut edges(13 14)
 NewP:(1 12 7)(2 6)(5 11 13)(8 14)
 NewFiltered cycliCover:(1 11 14 7 2 5 12 8 13 6)
 NewGenus=1 New cycliCover:(1 11 14 7 2 4 16 5 10 12 15 8 9 3 13 6)
 Recurrence edges(1 2)(5 6)(7 8)(11 12)(13 14)
 Cycle edges
 Cross edges(3 4)(9 10)(15 16)
 Cut edges
 NewP:(1 7)(2 6)(5 13)(8 14)
 NewFiltered cycliCover:(1 8 13 6)(2 5 14 7)
 NewGenus=0 New cycliCover:(1 12 15 8 9 3 13 6)(2 4 16 5 10 11 14 7)
 Recurrence edges
 Cycle edges(1 2)(5 6)(7 8)(13 14)
 Cross edges
 Cut edges(3 4)(9 10)(11 12)(15 16)
 NewP:(2 6)(5 13)
 NewFiltered cycliCover:(1 2 5 14 13 6)
 NewGenus=0 New cycliCover:(1 12 15 7 2 4 16 5 10 11 14 8 9 3 13 6)
 Recurrence edges(1 2)(5 6)(13 14)
 Cycle edges
 Cross edges(3 4)(7 8)(9 10)(11 12)(15 16)
 Cut edges
 NewP:(5 13)
 NewFiltered cycliCover:(5 14 13 6)
 NewGenus=0 New cycliCover:(1 12 15 7)(2 4 16 5 10 11 14 8 9 3 13 6)
```

Picture 22: A part of the protocol of the experiment with a graph

But the results were surprising. We found out that the program could do much more that it was intended in the beginning. It could be used to generate different classes of graphs, to maintain the graph and in the same time to have it as a base of knowledge that can give different answers to different questions about the status of the graph. These results were told in Leningrad [49] and Moscow [50] and in Novosibirsk in 1986 [52]where we were already thinking in the terms of the base of knowledge. But the problem about the complicacy of the program remained. The program was made public in [51].

The algorithmic solution of the dynamical partition of the graph into 3-connected components were described in the work [48]. This result seamed to us very essential. But we found out one strange as it seamed to us fact. The work [48] was very lucid and clear in that, that it gave a clear algorithm how to implement it, but in the same time it wasnt possible to join this work with the Tuttes work, and because of this fact that it didnt have any theoretical background it didnt contain any theorem. The only explanation for it, that we could find, that the theory of Tutte is essentially statistic, but our approach is essentially dynamic.

It gave us the impulse to try to find a dynamical theory in essence, that could solve the question. As it seemed to us, we did find such more or less appropriate theory [54, 55] that was more developed in the direction of dynamicy that previous theory. Working on these questions we came to some graph rotational schemes.

14 History

In this chapter the history of our evolution in creating graph-theoretical algorithms and the corresponding theoretical investigations is told.

In year 1977 we started to work with Paulis Ķikusts on the implementation of the algorithm of the partitioning the graph into the three-connected components using the theory and methods of Tarjan , Hopcroft etc. [32, 27]. We were extremely surprised to find out how fast these algorithms were running on the computer. In following years we proceeded in this direction and worked on finding the planar part of the graph and other algorithms [43].

Working in year 1983 on the algorithm how to realize intersections of edges to implement arbitrary graph in the plane we came to the thought that it would be good to have the graph maintained partitioned into three-connected components. In the year 1984 this question were solved [48]and a program was written that did the task. The theoretical solution was rather simple and easy to grasp. But the implementation of the computer program was very hard to accomplish. We came to the necessary to use the methods of specification of programs putting in service META IV language [28]. The program worked but was very large and hard to maintain.

But the results were surprising. We found out that the program could do much more that it was intended in the beginning. It could be used to generate different classes of graphs, to maintain the graph and in the same time to have it as a base of knowledge that can give different answers to different questions about the status of the graph. These results were told in Leningrad [49] and Moscow [50] and in Novosibirsk in 1986 [52]where we were already thinking in the terms of the base of knowledge. But the problem about the complicacy of the program remained. The program was made public in [51].

The algorithmic solution of the dynamical partition of the graph into 3-connected components were described in the work [48]. This result seamed to us very essential. But we found out one strange as it seamed to us fact. The work [48] was very lucid and clear in that, that it gave a clear algorithm how to implement it, but in the same time it wasnt possible to join this work with the Tuttes work, and because of this fact that it didnt have any theoretical background it didnt contain any theorem. The only explanation for it, that we could find, that the theory of Tutte is essentially statistic, but our approach is essentially dynamic.

It gave us the impulse to try to find a dynamical theory in essence, that could solve the question. As it seemed to us, we did find such more or less appropriate theory [54, 55] that was more developed in the direction of dynamicy that previous theory. Working on these questions we came to some graph rotational schemes.

Already in the year 1984 the author found independently the rotations in the graph, that was well known already since Hefter [7], being rediscovered by many mathematicans [4, 13]. Then in 1993 in [53]was found the combinatorial object that we call now combinatorial map. In 1995 we found that Tutte has developed this question in his book [22]. Only in 1995 we found the works of Stahl, Little, Vince, Bonnington and others.

Now we are working in the development of graph drawing algorithmical support. We hope that the results of the theory of combinatorial maps could be applied in the graph drawings also, where usually we are tended to think about graphs as geometrical and plane objects in usual sense.

15 Conclusions

Since the year 1978 we are concerning with the effective algorithms and tried to do our own contribution in the subject, building effective algorithms and developing the corresponding theoretical background [40, 33, 34, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 36, 39, 54, 55, 56]. For example the attempt to solve algorithmically the problem of dividing the graph into 3-connected components resulted in the works [48, 51, 54, 55].

Some years later this subject was attacked by a group of investigators [,for example in the works [29, 30, 31]] and the subject was solved more deeply as it was done by us. Evidently, it was our fault that we did not publish our results already in the 1984. when we had the first progress. But, as it is explaned in the previous chapter, we were kept by the fact, that there was not sufficiently good theoretical background in our disposal.

In the years after, the investigations have been developing in the direction of the topology and particularly combinatorics. The contribution of the last years are to be seen in the works [53, 35, 57, 58, 59, 60, 61, 62].

In the general, we see that the graph theory makes greater and greater influence on different fields of mathematics and mostly in applications. But in recently years some new directions in the graph theory extremely starts to develop. This as we see has occurred in the case of the combinatorial map theory, which is interesting to the graph theorists due to its possible applications in the graph theory field.

Coming independently to this field, we developed ourselves the theory of combinatorial maps and partial combinatorial maps. After we found the other autors that work in this field, we have chosen our own specific way of evolution, as it seems to us, we develop the theory and both have implemented our calculus in the computer program, that gives us possibility to prove and reprove many theoretical ideas which we come upon during the work.

We have persuaded ourselves that the rotational features are very hard to discover but whence discovered they show very easy ways to implement these found results in the computations.

The main result that we have reached is that we develop the combinatorial theory and directly implement the results in the system that is theoretically completely based on this theory in the development.

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