Approche algébrique des modèles de chaînes de spin et d’autres systèmes exactement solubles en physique quantique

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THÈSE

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*Approche algébrique des modèles de chaînes de spin
et d'autres systèmes exactement solubles en physique quantique*

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Résumé

Cette thèse est consacrée à l'étude de la théorie mathématique qui sous-tend la construction et la résolution d'une classe particulière de systèmes quantiques exactement solubles: son objectif est d'utiliser les superalgèbres de Lie comme un outil pour construire et résoudre des chaînes de spins intégrables.
Nous développons une approche générale et systématique permettant de construire et traiter simultanément une large classe de systèmes intégrables partageant la même super--symétrie, allant du cas bien connu où tous les sites portent la représentation fondamentale (comme par exemple dans le cas du modèle t-J) à des situations plus complexes d'intérêt physique comprennent chaînes de spins alternée, avec impuretés, etc…

Les deux premiers chapitres sont consacrés à un examen des résultats connus concernant le Yangien de la superalgèbre de Lie gl(m|n), nécessaire pour introduire la version graduée de la méthode de diffusion inverse quantique. Nous appliquons notre approche dans le chapitre 3 aux chaînes fermées et dans le chapitre 4 aux chaînes ouvertes. Dans ce chapitre sont étudiés les homologues super--symétriques de l'algèbre de réflexion et du Yangien twisté, qui sont les structures algébriques permettant d'imposer des conditions aux bords qui préservent l'intégrabilité. Dans le dernier chapitre, la méthode dite de fusion est traitée en détail pour des chaînes de spins avec supersymétrie sl(1|2).

La méthode de résolution que nous utilisons, tant dans le cas fermé que dans le cas ouvert, est la généralisation au cas supersymétrique de l'Ansatz de Bethe analytique, pour lequel les équations de Bethe paramétrant les nombres quantiques du système sont obtenus comme conditions d'analyticité pour les valeurs propres des Hamiltoniens.

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Introduction

The theory of integrable systems enlightens deep relationships between formal algebraic structures and mathematical properties of exactly solvable physical models, both at the classical and the quantum level. If the search for symmetries (in particular the ones described by Lie algebras) has always been an essential tool for the study of any physical theory or model, in the context of integrable systems it becomes something even more relevant: most of the developments of this branch of mathematical physics are in some sense related to generalizations of the very same concept of symmetry. Among these generalizations, the theory of Lie superalgebras stands out as a particularly powerful and fertile one.

The aim of this work is to use Lie superalgebras as an algebraic tool to build and solve integrable quantum *spin chains*. The appearance of these one–dimensional models dates back almost 80 years, and their prototype is the XXX chain, first introduced by Heisenberg in 1928 [1] as an attempt to elaborate a model for the ferromagnetic transition. This celebrated model can be described as a linear array of spin $1/2$ particles with uniform exchange interactions between nearest neighbours. In 1931, H. Bethe [2] presented a method for obtaining its exact spectrum. In his solution, the eigenvectors of the chain are built in terms of quasi–particle whose rapidities satisfy a set of algebraic equations (Bethe equations). A relevant feature of the Bethe approach is that the eigenstates are naturally characterized by a set of quantum numbers that can be used to classify them according to specific physical properties.

This approach, known today as Bethe Ansatz, has been used later as the main tool for the construction of the spectrum of innumerable one–dimensional quantum integrable systems, that are known to be solvable by means of some generalization of the Bethe Ansatz (coordinate, algebraic, functional, nested, etc.). The method has thus been expanded far beyond the ad hoc calculation tool it was in its first appearance, and integrable spin chains appear today in a huge variety of domains, from condensed matter physics to quantum optics, particle physics and string theory. Integrable vertex models, reaction–diffusion models and $1+1$ dimensional field theories can be related to spin chains by means of suitable transformations or limit procedures. A striking mathematical richness corresponds to this variety of physical applications: many algebraic tools used to build and solve integrable spin chains are today widely used in connection with knots theory, non–commutative geometry or quantum groups (whose introduction has been partially motivated by the study of spin chains).

From the integrability point of view these models share a common feature: the existence of a generating function (the transfer matrix $t(u)$), depending on a complex parameter, from which the Hamiltonians describing the interaction can be derived as elements of a set of commuting operators (thus simultaneously diagonalizable). This is essentially due to the fact that transfer matrices commute at different values of the spectral parameter u:

$$[t(u)\,,t(v)] = 0\,.$$

As a consequence, the coefficients of the u–series expansion of $t(u)$ are commuting operators and can be used to build the hamiltonian. In practice, the diagonalization of $t(u)$ allows the simultaneous calculation of the spectrum of all these operators.

The particular Bethe Ansatz approach we deal with in this work is the so–called *analytical Bethe Ansatz*. This variant of the Bethe Ansatz was developed in [3, 4, 5, 6] for closed spin chains and in [7, 8] for open chains, and it originates from the observation that the Bethe equations for the XXX Heisenberg chain are *analyticity conditions* for the eigenvalues of the transfer matrix $t(u)$.

The next relevant observation for our purposes is that if the generators of a symmetry algebra commute with $t(u)$, all the spin chains generated by the transfer matrix will share the same symmetry.

In an attempt to generalize the algebro–analytic approach devoleped in [9], we have fixed our attention on *supersymmetric* spin chains, i.e. such that their transfer matrices commute with all the generators of a superalgebra of the $gl(m|n)$ series. This is indeed the most general possible choice: it simultaneously deals with bosonic and fermionic degrees of freedom, and other finite–dimensional algebras and superalgebras can be obtained as sub–(super)algebras of $gl(m|n)$ (in a similar way the B_n, C_n and D_n algebras can be obtained as A_n subalgebras). On the other hand, even the first non–trivial case – $gl(1|2)$ – is the symmetry algebra of a surprisingly rich integrable model as the supersymmetric $t-J$ model, whose well–known diagonalization by means of the Bethe Ansatz we expect to recover as a subcase of our approach.

Now then, the main goals of our research program can be summarized as follows:

1. *is it possible to write the Bethe equations for all the spin chains with $gl(m|n)$ symmetry as analiticity conditions for the transfer matrices that generate them?*

2. *how should these Bethe equations be modified when dealing with supersymmetric open spin chains?*

The key point allowing a global treatment of all the $gl(m|n)$ supersymmetric spin chains – independent from the representation chosen for the spin variables – is that their integrability only relies on the algebraic structure underlying the construction of the transfer matrix (the so called Yangian of $gl(m|n)$), that closely resembles its non supersymmetric counterpart. The present introduction is correspondigly organized in two parts: in the next section, the main ideas of this algebraic treatment of integrable spin chains and of the (non supersymmetric) quantum inverse scattering method are very briefly summarized in the next section. We then recall a few fundamental definitions and results about the theory of Lie superalgebras we shall use throughout this work. The contents and results of this thesis will then be described in the last section of the introduction.

1.1 Yangians and the quantum inverse scattering method

In the theory of quantum integrable systems, Lie algebras and superalgebras naturally appear as the ones entailing the physical operators of the problem. Starting e.g. from a family of commuting operators $\{H_k\}$, one can embed the commuting algebra spanned by these observables into a bigger Lie algebra, in such a way that the Hilbert space of the system states becomes a representation of this algebra. It is then possible to find the eigenvalues of the $\{H_k\}$ by purely algebraic means.

A relevant feature of the quantum inverse scattering method is that the involved algebras are not finite–dimensional algebras, and are not even, generally speaking, Lie algebras. In their place, one considers the algebras spanned by the generators $T_{ij}(u)$, with $1 \leq i, j \leq N$, u being a complex parameter called the *spectral parameter*. Gathering the generators in an $N \times N$ matrix, their commutation relations are defined by the equation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)\,, \tag{1.1}$$

with the entries of the numerical matrix $R(u)$ play the role of a set of generalized constant structures. As a compatibility condition, they should satisfy the Yang–Baxter equation :

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)\,.$$

Any solution of the above equation defines the quadratic algebra $\mathcal{T}_R$ through equation(1.1). The representations of $\mathcal{T}_R$ can then be interpreted as a quantum integrable system whose Hilbert space is the module of the $\mathcal{T}_R$ representation, and whose commuting integrals of motion are the elements of the maximal commutative subalgebra $t(u) \subset \mathcal{T}_R$. The main aim of the quantum inverse scattering method is then to find the spectrum and the eigenvectors of this family. According to Sklyanin [10], one can summarize the steps of the quantum inverse scattering method as follows:

1. find the R matrix by solving the Yang–Baxter equation;
2. identify a representation of $\mathcal{T}_R$;
3. calculate the spectrum of $t(u)$;
4. calculate the correlation functions and other quantities of physical interest.

The first solutions to the Yang–Baxter equation have been found by empirical methods, until, with the work of Drinfeld [11], an axiomatic approach to the QISM based on Hopf algebras was introduced, whose main tool is a class of quasi–triangular Hopf algebras $Y(\mathfrak{g})$, parametrized by the simple Lie algebras $\mathfrak{g}$, and called *Yangians* by Drinfeld himself in honor of C. N. Yang who found a particular solution to the Yang–Baxter equation [12].

The Yangians, that we shall introduce in chapter 2, form a remarkable family of quantum groups related to rational solutions of the quantum Yang–Baxter equation . For each Lie algebra $\mathfrak{g}$, the corresponding Yangian $Y(\mathfrak{g})$ is defined as a deformation of the universal enveloping algebra $U(\mathfrak{g}[u])$ for the loop algebra $\mathfrak{g}[u]$.

In this work, we are going to focus our attention on the third step of the above summary (finding the spectrum of $t(u)$), using supersymmetric generalizations of the Yangians as our main algebraic tool.

1.2 Lie superalgebras

In the exposition and the notation of this section, summarizing some results about classical Lie superalgebras, we essentially follow [13], [14] and the comprehensive review [15].

Definition 1.1 ($\mathbb{Z}_2$ graded vector space) *A vector space V is called $\mathbb{Z}_2$–graded if it can be decomposed into the direct sum of two subspaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$. On the homogeneous elements $x \in V$ (i.e. those having zero projection onto one of these subspaces) one can define a degree function $[x]$ (also called grading or gradation) with values in $\mathbb{Z}_2$:*

$$\begin{aligned} [x] &= 0\,, \quad if\; x \in V_{\overline{0}}\,, \\ [x] &= 1\,, \quad if\; x \in V_{\overline{1}}\,. \end{aligned}$$

Elements of 0 (resp. 1) degree are called even (resp. odd) elements.

Definition 1.2 (Superalgebra) *Let $\mathcal{A}$ be an algebra over a field $\mathbb{K}$ of characteristic 0 (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) with internal composition laws $+$ and $\cdot$, and set $\mathbb{Z}_2 = \mathbb{Z}/(2\mathbb{Z}) = \{\overline{0}, \overline{1}\}$. $\mathcal{A}$ is called a superalgebra or $\mathbb{Z}_2$ graded algebra if it can be written as the direct sum of two spaces*

$$\mathcal{A} = \mathcal{A}_{\overline{0}} \oplus \mathcal{A}_{\overline{1}}$$

such that

$$\mathcal{A}_{\overline{0}} \cdot \mathcal{A}_{\overline{0}} \subset A_{\overline{0}}\,, \qquad \mathcal{A}_{\overline{0}} \cdot \mathcal{A}_{\overline{1}} \subset A_{\overline{1}}\,, \qquad \mathcal{A}_{\overline{1}} \cdot \mathcal{A}_{\overline{1}} \subset A_{\overline{0}}\,.$$

The elements $a \in \mathcal{A}_{\overline{0}}$ are called even, or of degree $[a] = 0$, while the elements $a \in \mathcal{A}_{\overline{1}}$ are called odd, or of degree $[a] = 1$.

Definition 1.3 (Lie superalgebra) *A Lie superalgebra is a superalgebra $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ endowed with a product $[\,,]$ satisfying the following axioms:*

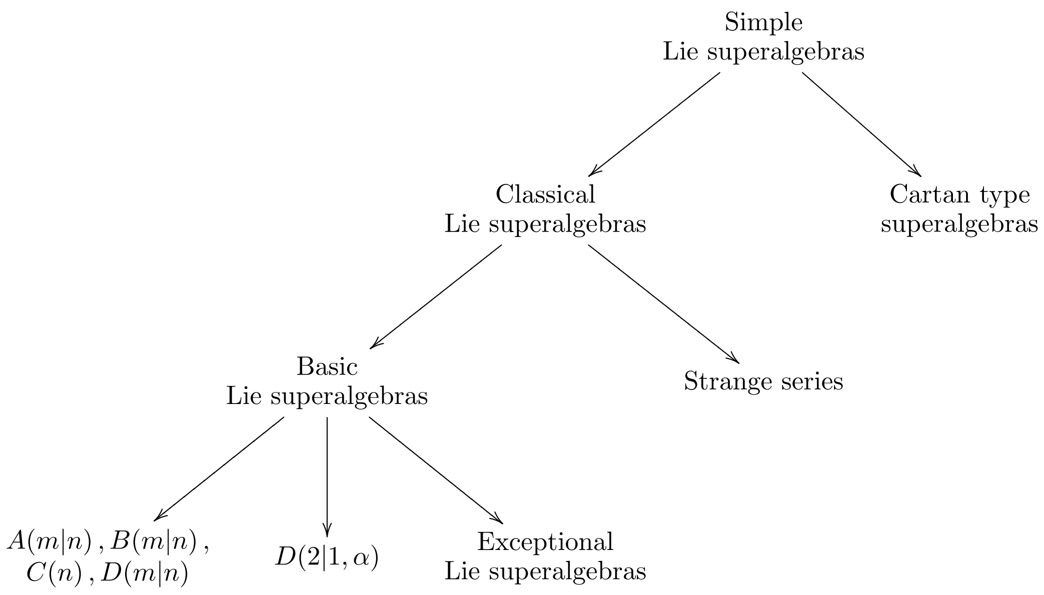
1. *$\mathbb{Z}_2$–gradation:*
$$[\mathcal{A}_i\,, \mathcal{A}_j] \subset \mathcal{A}_{i+j}$$
2. *graded–antisymmetry:*
$$[a_i\,, a_j] = -(-1)^{[a_i][a_j]}\,[a_j\,, a_i]$$
3. *generalized Jacobi identity:*
$$[a_i\,, [a_j\,, a_k]] = [[a_i\,, a_j]\,, a_k] + (-1)^{[a_i][a_j]}\,[a_j\,, [a_i\,, a_k]]$$

for $i\,, j \in \mathbb{Z}_2 = \left\{\bar{0}, \bar{1}\right\}$ and $a_i \in \mathcal{A}_i$.

Notice that $\mathcal{A}_{\bar{0}}$ – called the bosonic or even part of $\mathcal{A}$ – is a Lie algebra whose Lie bracket coincides with the restriction of $[\,,]$ to its elements, and that $\mathcal{A}_{\bar{1}}$, while not a Lie algebra in itself, is an $\mathcal{A}_{\bar{0}}$ module thanks to the $\mathbb{Z}_2$ gradation of $[\,,]$. The even subalgebra of any Lie superalgebra can therefore be represented on its odd part. It follows that each simple Lie superalgebra falls into one of two general families: for the *classical* Lie superalgebras the representation of the even subalgebra on the odd part is completely reducible, while for the *Cartan type* superalgebras such a property does not hold. Among the classical superalgebras, one naturally separates the basic series from the strange ones, obtaining the following classification:

1. four so called *basic series*, denoted $A(m|n)$, $B(m|n)$, $C(n)$, $D(m|n)$, that are in many ways like the A_n–D_n series of Lie algebras;
2. two exceptional Lie superalgebras $F(4)$ and $G(3)$, respectively 40–dimensional and 31–dimensional;
3. a one–parameter family of 17–dimensional superalgebras $D(2|1, \alpha)$.
They are α–deformations of $D(2|1)$;
4. two infinite families (the *strange* series) respectively denoted $P(n)$ and $Q(n)$.

The following diagram resumes the classification.



In the present work we shall only deal with the $A(m|n)$–$D(m|n)$ series, whose main definitions and properties we will now briefly recall.

I. The $gl(m|n)$ superalgebra

In this section we define the Lie superalgebras of the basic type as *matrix* superalgebras, while more formal equivalent definitions will be given later, together with the needed supercommutation relations. We thus start introducing the set $End\, V$ of linear transformations of a $\mathbb{Z}_2$ graded vector space (definition 1.1) $V = V_{\bar{0}} \oplus V_{\bar{1}}$ of dimension $m+n$, $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$. $End\, V$ being an associative superalgebra with respect to the composition of maps $\circ$, it is natural to define the bracket $[\,,]$ by the graded commutator

$$[a\,,b] = a \circ b - (-1)^{[a][b]} b \circ a\,.$$

One defines $gl(m|n)$ as the Lie superalgebra obtained endowing $End\, V$ with the above bracket. The role played by $gl(m|n)$ in Lie superalgebras theory is in many ways the same as that of $gl(n)$ in Lie algebras theory. Assume now that $e_1, \ldots, e_m$, $e_{m+1}, \ldots, e_{m+n}$ is a basis of V that is the union of bases of $V_{\bar{0}}$ and $V_{\bar{1}}$, i.e. $[e_i] = 0$ for $i = 1, \ldots, m$ and $[e_i] = 1$ for $i = m+1, \ldots, m+n$. It is then useful to attach the grading to the indices i, i.e.

$$[i] = \begin{cases} 0\,, & 1 \le i \le m \\ 1\,, & m+1 \le i \le m+n\,. \end{cases} \tag{1.2}$$

Such a basis is called homogeneous. In this basis an operator $X \in gl(m|n)$ can be represented in the form of block matrices with complex elements:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.3}$$

where $A \in gl(m)$ and $D \in gl(n)$, while B and C are respectively $m \times n$ and $n \times m$ complex matrices. All even elements are of the form

$$X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad [X] = 0\,,$$

and all odd elements read:

$$X = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \qquad [X] = 1\,. \tag{1.4}$$

The even part of $gl(m|n)$ is thus seen to be isomorphic to the direct sum of $gl(m)$ and $gl(n)$, while the $gl(m|n)$–module $gl(m|n)_{\bar{1}}$, whose elements are the matrices of the form (1.4), is isomorphic to $gl(m) \oplus gl(n)$. We can define a basis for $gl(m|n)$ taking as generators the elements

$$\mathcal{E}_{ij}\,, \qquad 1 \le i, j \le m+n\,,$$

with grading defined through (1.2)

$$[\mathcal{E}_{ij}] = [i] + [j]\,,$$

and satisfying the following supercommutation relations:

$$[\mathcal{E}_{ij}\,, \mathcal{E}_{kl}] = \delta_{jk}\mathcal{E}_{il} - (-1)^{([i]+[j])([k]+[l])}\delta_{il}\mathcal{E}_{kj}\,. \tag{1.5}$$

Finally, one can show that the value of the *supertrace*, defined as the functional in $gl(m|n)$:

$$str\, X = tr\, A - tr\, D = \sum_{k=1}^{m+n} (-1)^{[k]} X_{kk}\,,$$

does not depend on the choice of the basis.

II. Lie superalgebras $A(m|n)$

The property of the supertrace

$$str\,([a\,,b]) = 0$$

implies that the subalgebra

$$sl(m|n) = \{X \in gl(m|n) \mid str\, X = 0\}$$

is a 1–codimensional ideal of $gl(m|n)$, whose $\mathbb{Z}_2$ grading induces the same grading on $sl(m|n)$. The Lie superalgebra $sl(n|n)$ contains a 1–dimensional ideal consisting of scalar matrices $\lambda\, 1_{2n}$, $\lambda \in \mathbb{C}$. The *unitary* superalgebra $A(m|n)$ is thus defined as

$$\begin{aligned} A(m|n) &= sl(m+1|n+1)\,, \qquad m,n \geq 0\,, \\ A(m|m) &= sl(m+1|m+1)/\lambda\, 1_{2m+2}\,, \qquad m > 0\,. \end{aligned}$$

$A(m|n)$ can be realized as a matrix superalgebra by taking matrices of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

satisfying the supertracelessness condition:

$$str\, X = tr\, A - tr\, D = \sum_{k=1}^{m+n} (-1)^{[k]} X_{kk} = 0\,.$$

Thus $A(m|n)$ has dimension $(m+n)^2 - 1$ and rank $m+n-1$ for $m \neq n$, and its even part is $sl(m) \oplus sl(n) \oplus u(1)$.

III. Lie superalgebras $B(m|n)$, $C(n)$, $D(m|n)$

The Lie superalgebras containing the orthogonal and symplectic algebras are defined as the ones preserving the bilinear forms on V, $\dim V = m + 2n$, whose block diagonal matrix read

$$J_{m,n} = \left(\begin{array}{c|cc} 1_m & & \\ \hline & 0 & 1_n \\ & -1_n & 0 \end{array}\right).$$

Here, 1_m and 1_n denote identity matrices of the corresponding dimension. Thus, the basic series $B(m|n)$, $C(n)$ and $D(m|n)$ can be defined as the $gl(m|n)$ sub–superalgebras whose elements, written in the form (1.3), satisfy the following constraints:

$$A^t = -A\,, \qquad D^t J = -JD\,, \qquad B = C^t J\,,$$

where ^{t} denotes the usual transposition and the matrix J is given by

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} .$$

These so called *orthosymplectic* Lie superalgebras $osp(m|2n)$ fall into three series

$$\begin{aligned} B(m|n) &= osp(2m+1|2n)\,, \qquad m \geq 0\,,\ n > 0\,; \\ D(m|n) &= osp(2m|2n)\,, \qquad m \geq 2\,,\ n > 0\,; \\ C(n+1) &= osp(2|2n)\,, \qquad n \geq 1\,. \end{aligned}$$

The even part of $B(m|n)$ is given by $so(2m+1) \oplus sp(2n)$, while its odd part is the $(2m+1, 2n)$–dimensional representation of the even part; it has rank $m+n$ and dimension $2(m+n)^2+m+3n$. The superalgebra $C(n+1)$ has as even part the $so(2) \oplus sp(2n)$ Lie algebra and as odd part two copies of the fundamental representation of $sp(2n)$. Its rank is $n+1$, and its dimension $2n^2+5n+1$. Finally, the even part of $D(m|n)$ is $so(2m) \oplus sp(2n)$, whose $(2m, 2n)$–dimensional representation gives the odd part; it has rank $m+n$ and dimension $2(m+n)^2 - m + n$.

1.3 Outline of the thesis

The present thesis consists of a short introduction and five chapters. In the following we briefly summarize their contents. Each chapter is preceded by a more detalied synopsis, including the main bibliographical references.

- **Chapter 2. The Yangian of $gl(m|n)$.** This chapter is entirely devoted to a self–contained presentation of the graded Yangian of the general Lie superalgebra $gl(m|n)$. All of the algebraic structures (reflection algebra, twisted Yangians, etc.) from which we derive and solve integrable spin chains are later introduced as subalgebras of $Y(m|n)$.

  We begin the chapter establishing our notation and recalling the construction of the fundamental solutions of the graded Yang–Baxter equation. Then, the main definitions and properties of $Y(m|n)$ and its highest weight representations are summarized. The class of irreducible representations known as *evaluation representations* will be described.

  While almost all the results of the second chapter are well known, some new calculations are contained in its sections 4 and 5: namely, we study the action of $T^{-1}(u)$ on highest weight representations – a key point in order to gain valuable information about the values assumed by the central element of $Y(m|n)$ (the so–called *quantum Berezinian*) on evaluation representations.

- **Chapter 3. Closed spin chains.** The third chapter connects the algebraic setting described in chapter 2 with the graded version of the quantum inverse scattering method, describing how monodromy and transfer matrices leading to periodic spin chains can be sistematically built, exploiting the Hopf structure, by means of super Yangian representations. The symmetry superalgebra characterizing these transfer matrices (and all the hamiltonians they generate) will also be described. At the same time, the basic ideas of the analytical Bethe Ansatz are discussed. These can be summarized as follows:

  1. the representations theory of $Y(m|n)$ supplies us with a natural pseudovacuum eigenvector v^+ of the transfer matrix. The action of the monodromy matrix $\mathcal{T}(u)$ on v^+, and the corresponding transfer matrix eigenvalue $\Lambda_0(u)$, can be explicitly calculated.

2. It is then assumed that *all the eigenvalues* of the transfer matrix can be obtained by properly "dressing" the pseudovacuum eigenvalue with rational functions of the spectral parameter.
3. From the properties satisfied by the eigenvalues, one deduces a number of constraints that the form of the dressing functions should satisfy. In doing so, a set of quantum numbers labelling the eigenvalues is introduced. These are to be related to the Bethe roots in the last step of the analytical Bethe Ansatz.
4. The Bethe equations are obtained as *analyticity conditions* on the eigenvalues.

In the remaining part of the chapter, we sistematically apply this approach to the obtained transfer matrices. Several specific examples are discussed. Thanks to the purely algebraic nature of our approach, our results are valid for any Dynkin diagram and for any highest weight representation of $gl(m|n)$. The novelty of the content of chapter 3 consists in the full generality of the approach, but the resulting Bethe equations were actually already known, or at least conjectured, in several cases (see the synopsis preceding the chapter for bibliographical details).

- **Chapter** 4. **Open spin chains.** In the fourth chapter, the analytical Bethe Ansatz approach is extended to the case of supersymmetric open spin chains. In this kind of models, the first and last sites of the spin chain do not interact, and the corresponding term in the hamiltonian is replaced by some non–trivial boundary condition, described by numerical matrices K. As a consequence, the symmetry of the resulting models is reduced to some sub–superalgebra of the corresponding periodic chain symmetry. In order to preserve the integrability, one should redefine the monodromy matrix of the spin chain as follows:

$$B(u) = T^{+}(u)K(u)T^{-}(u)\,,$$

where, roughly speaking, $T^{+}(u)$ represents a spin–wave propagating towards the boundary, and $T^{-}(u)$ represents the reflected wave. The numerical matrix $K(u)$ entails the effect of the boundary on the reflection, and should satisfy consistency conditions (the so–called *reflection equations*) of the generic abstract form $RKRK = KRKR$.

Two main classes of integrable boundary conditions are considered, corresponding to different kinds of reflection equations. For historical reasons they are known in literature as *soliton preserving* (SP) and *soliton non-preserving* (SNP) boundary conditions. The integrability of these boundary conditions relies on the properties of two different subalgebras of the graded Yangian:

1. the so–called graded *reflection algebra* for the SP case;
2. the *twisted super Yangian* for the SNP case.

A discussion of the algebraic properties of the graded reflection algebras and of the twisted super Yangian is followed by the application of the analytical Bethe Ansatz to the corresponding transfer matrices. As in chapter 3, the general form of the Bethe equations is obtained as analyticity condition for the eigenvalues of the transfer matrix. Most of the results presented here are original. The results concerning the SP boundary conditions are already published in [16], while the dressing hypothesis and the Bethe equations for the SNP case are not.

- **Chapter** 5. **Fused** $sl(1|2)$ **models.** A particular feature of the representation theory of Lie superalgebras of the $sl(m|n)$ series is the existence of families of irreducible *typical* representations, labelled by a complex parameter b, that cannot be obtained as components of tensor products of lower dimensional representations. The usual fusion method used to solve spin chains corresponding to higher–dimensional representations of ordinary Lie algebras such as $sl(n)$ cannot be applied to these cases.

Nevertheless, it is still possible to obtain the Bethe equations corresponding to typical representations as analyticity conditions for the eigenvalues of the transfer matrix. This suggests the possibility of extending the analytical Bethe Ansatz approach to deal with these cases. This chapter is less general and systematic than the previous ones, and represents a first step in this direction. It is devoted to the study of the fusion procedure for the rank 2 superalgebra $sl(1|2)$. We discuss the advantages and drawbacks of our approach, and recover some known results about $sl(1|2)$ closed fused models. A short section is also devoted to the fusion procedure for open $sl(1|2)$ spin chains.

- **Chapter 6. Integrability from coalgebra symmetry: an $osp(1|2)$ chain.** The last chapter describes, through a single detailed example, the coalgebraic approach to the problem of the integrability of quantum spin chains. This set of techniques was originally formulated for classical integrable systems, and has been later generalized to the quantum case. In this concluding chapter we show how to apply it to supersymmetric models whose coalgebra symmetry can be described in terms of rank one superalgebras. We discuss the case of $osp(1|2)$ and of its q–deformation, building a set of commuting Gaudin–like hamiltonians related to the Casimir of this Lie superalgebra, and explicitly finding their spectrum and eigenvectors.

  Although not directly related to the mainstream of the quantum inverse scattering method, the integration algorithm we describe shows some similarity with the algebraic Bethe Ansatz: in particular the eigenvectors of the hamiltonians are obtained by repeated application of raising operators to a suitable reference state, and this procedure introduces a set of quantum numbers satisfying quantization rules related to the representation theory of $osp(1|2)$.

  Despite these similarities, we shall show that the class of models that are solvable through the coalgebraic approach are very different (and quite less general) from the ones we obtained in the previous part of the thesis. We thus conclude the chapter with an explicit comparison of the two approaches, analizing their advantages and drawbacks, and emphasizing their differences. The results collected in this chapter are already published in [17].

2

The Yangian of $gl(m|n)$

In this chapter we present the algebraic notions that we shall use throughout this work. In the first section, the main results about the well–known generalization of the Yang–Baxter equation to the graded case (see for instance [18, 19, 20],) will be briefly summarized, and the notation fixed. The solutions to the graded Yang–Baxter equation that we shall use in our approach will be discussed. These will provide a natural starting point for the presentation of the graded Yangian $Y(m|n)$, that will be the object of the remaining part of the chapter.

Excellent and comprehensive surveys of Yangians and twisted Yangians (to be introduced in chapter 4) from the algebraic point of view are, for instance, [21] and [22].

The defining relations of the graded Yangian [23] can be written in form of a single *ternary* (or RTT) relation satisfied by the matrix of the generators, exactly as in the non–graded case. As already mentioned in the introduction, this relation originates from the quantum inverse scattering theory (see [24, 25, 26]), and has a rich and extensive background.

The Yangians were primarily regarded as a tool to build rational solutions of the Yang–Baxter equation. Conversely, the ternary relation was used in [27] for studying quantum groups. Moreover, the Hopf structure of the Yangian can also be described in matrix form.

The graded Yangian, first introduced in [23], generalizes these structures to the supersymmetric case, providing a natural mathematical formulation for the graded quantum inverse scattering theory and for all Bethe Ansatz approaches to supersymmetric integrable models.

We shall describe the supercommutation relations, the Hopf structure, and some relevant morphism of the graded Yangian in section 2.2, devoting section 2.3 to summarize few facts, first established in [28], about the highest weight and evaluation representations of $Y(m|n)$, that we shall use to build integrable systems.

In sections 2.4 and 2.5 a brief but self–contained description of the center of the graded Yangian will be given, focusing on the more relevant features from the quantum inverse scattering point of view. To this end, we shall also remind some facts about quantum determinants and the center of the non–graded Yangian $Y(n)$.

A specific example of some features of the graded Yangian in the case of $Y(1|2)$ concludes the chapter.

As a general rule, most of this chapter results will be given without proof (but with bibliographical references pointing to the original articles). Detailed proofs will be presented only when new results or calculations are concerned.

2.1 Yang–Baxter equation

The present section is devoted to fix the notation and to review some preliminaries on the R–matrix formalism and the Yang–Baxter equation in the case of graded vector spaces. In this framework, one deals with multiple tensor products of the form

$$\mathbb{C}^{m|n} \otimes \cdots \otimes \mathbb{C}^{m|n} , \tag{2.1}$$

each factor being the graded vector space whose even (resp. odd) subspace is $\mathbb{C}^m$ (resp. $\mathbb{C}^n$):

$$(\mathbb{C}^{m|n})_{\bar{0}} = \mathbb{C}^m \,,$$
$$(\mathbb{C}^{m|n})_{\bar{1}} = \mathbb{C}^n \,.$$

We can act on each factor of the tensor product (2.1) with a copy of the Lie superalgebra $End\,\mathbb{C}^{m|n}$, hereafter called the auxiliary space. We will denote with e_{ij} the elementary $End(\mathbb{C}^{m|n})$ matrices, which have 1 in position (i,j), and with e_i the $\mathbb{C}^{m|n}$ vectors which have 1 in position i. Their $\mathbb{Z}_2$ gradation is defined through (1.2) as:

$$[e_{ij}] = [i] + [j] \quad \text{and} \quad [e_i] = [i] \,.$$

We shall use the so–called Leningrad notation, allowing us to distinguish among the operators acting on the different copies of $\mathbb{C}^{m|n}$: for an operator $A \in End\,\mathbb{C}^{m|n}$ and an integer number $k = 1, 2, \dots$ we set

$$A_l = 1^{\otimes(l-1)} \otimes A \otimes 1^{\otimes(k-l)} \in (End\,\mathbb{C}^{m|n})^{\otimes k} \,, \qquad 1 \le l \le k \,. \tag{2.2}$$

If $A \in (End\,\mathbb{C}^{m|n})^{\otimes 2}$, then for any i, j such that $1 \le i\,, j \le k$ and $i \ne j$, we denote by A_{ij} the operator in $(End\,\mathbb{C}^{m|n})^{\otimes k}$ which acts as A on the i–th and j–th copies, and as 1 on all other copies. That is, writing A in the basis spanned by the e_{ab},

$$A = \sum_{a,b,c,d} A_{abcd}\, e_{ab} \otimes e_{cd} \,, \qquad A_{abcd} \in \mathbb{C} \qquad \Rightarrow \qquad A_{ij} = \sum_{a,b,c,d} A_{abcd}\, (e_{ab})_i (e_{cd})_j \,,$$

where, according to (2.2),

$$(e_{ab})_i = 1^{\otimes(i-1)} \otimes e_{ab} \otimes 1^{\otimes(k-i)} \,.$$

Remark 2.1 *When the copies of the auxiliary space on which an operator acts are specified by indices as in the example above, we will often omit the tensor product sign, writing, e.g.,* $A_1 B_3$ *instead of* $A \otimes 1 \otimes B$.

If we write the components of two operators A and B in the e_{ij} basis as a_{ij} and b_{ij}, i.e.

$$A = \sum_{i,j} a_{ij}\, e_{ij} \,, \qquad B = \sum_{i,j} b_{ij}\, e_{ij} \,, \tag{2.3}$$

the components of their tensor product in the $e_{ij} \otimes e_{kl}$ basis are given by $a_{ij} b_{kl}$, i.e.

$$A \otimes B = \sum_{i,j,k,l} a_{ij} b_{kl}\, e_{ij} \otimes e_{kl} \,.$$

We shall also use the following notation for the components of a tensor product:

$$(A \otimes B)_{ij,kl} = a_{ij}\, b_{kl} \,.$$

Both $\mathbb{C}^{m|n}$ and $End\,\mathbb{C}^{m|n}$ are $\mathbb{Z}_2$-graded spaces, with $\mathbb{Z}_2$-grade

$$[\;] : \left\{ \begin{array}{lll} \mathbb{N}_{m+n} & \to & \{0,1\} \\ j & \mapsto & [j] \end{array} \right. \tag{2.4}$$

where $\mathbb{N}_{m+n} = \{1, 2, ..., m+n\}$, so that the tensor product $\otimes$ will always be graded, according to the following definition.

Definition 2.2 *Let $\mathcal{A}$ and $\mathcal{A}'$ be two superalgebras with multiplications $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{A}'}$. The tensor product between their elements is called graded if the induced $\mathcal{A}\otimes\mathcal{A}'$ multiplication*

$$\mu : (\mathcal{A}\otimes\mathcal{A}' \, , \mathcal{A}\otimes\mathcal{A}') \rightarrow \mathcal{A}\otimes\mathcal{A}'$$

is given by:

$$\mu(X\otimes X' \, , Y\otimes Y') = (-1)^{[X'][Y]}\mu_{\mathcal{A}}(X\, , Y)\otimes\mu_{\mathcal{A}'}(X'\, , Y')\, ,$$

for X,$Y\in\mathcal{A}$, X',$Y'\in\mathcal{A}'$, X' and Y being homogeneous elements.

Accordingly we have, for the elementary matrices e_{ij}:

$$(e_{ia}\otimes e_{kb})(e_{aj}\otimes e_{bl}) = (-1)^{([b]+[k])([a]+[j])}e_{ij}\otimes e_{kl}\, .$$

Thus, if A, B, C, $D\in End\,\mathbb{C}^{m|n}$, the following formula for the multiplication of their tensor products holds:

$$[(A\otimes B)(C\otimes D)]_{ij,kl} = \sum_{a,b} A_{ia}C_{aj}B_{kb}D_{bl}(-1)^{([b]+[k])([a]+[j])}\, . \tag{2.5}$$

The action of the operators on vectors is also graded: acting with an operator $A\otimes B$ of the form (2.3) on a vector

$$\mathbb{C}^{m|n}\otimes\mathbb{C}^{m|n}\ni\xi = \sum_{r,s}\xi_{rs}\, e_r\otimes e_s\, ,$$

the result is:

$$(A\otimes B)\,\xi = \sum_{i,j,k,l}(-1)^{[j]([k]+[l])}\xi_{jl}\, a_{ij}b_{kl}\,(e_i\otimes e_k)\, .$$

Remark 2.3 *The form of the right hand side of the above equation suggests a possible redefinition of the k-fold tensor product, widely used in literature (e.g. in [28, 29]), and given by*

$$\begin{aligned} A = &\quad a_{i_1j_1,i_2j_2,\ldots,i_kj_k}\,(e_{i_1j_1}\otimes e_{i_2j_2}\otimes\cdots\otimes e_{i_kj_k})\mapsto \\ \tilde{A} = &\quad (-1)^{\sum_{1\le l<p\le k}[j_l]([i_p]+[j_p])}a_{i_1j_1,i_2j_2,\ldots,i_kj_k}\,(e_{i_1j_1}\otimes e_{i_2j_2}\otimes\cdots\otimes e_{i_kj_k})\, . \end{aligned}$$

This convention automatically takes care of the gradation of the tensor product when multiplying several operators like in eq.(2.5). For this reason it is very useful in computer–assisted calculations, since matrix products reduce, through this mapping, to the usual (non–graded) ones. Nevertheless, it is much more heavy in analytical calculations, so that we shall avoid throughout this work the redefinition $A\mapsto\tilde{A}$.

Having established the notation, we will now give the basic definitions and properties connected with the graded Yang–Baxter equation. As in the non–graded case, this equation involves tensor products of three so–called *auxiliary spaces* V_1, V_2, V_3. We shall begin for simplicity with the case of three identical auxiliary spaces $V_1 = V_2 = V_3$, all coinciding with $End\,\mathbb{C}^{m|n}$: they can be considered as three copies of the fundamental representation of $gl(m|n)$. We will then briefly examine the slightly different case of two identical auxiliary spaces $V_1 = V_2 = End\,\mathbb{C}^{m|n}$, and an arbitrary representation of $gl(m|n)$ in V_3. More complicated cases, obtained through the so–called fusion procedure, will be introduced later.

Definition 2.4 (Graded YBE) *By graded Yang–Baxter on equation is meant the following equation*

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{2.6}$$

for the linear operator $R(u)\in(End\,\mathbb{C}^{m|n})^{\otimes 2}$.

The graded Yang–Baxter equation appears formally identical to its non–graded counterpart, the only difference being the gradation of the tensor product, since both sides of eq.(2.6) are elements of $(End\,\mathbb{C}^{m|n})^{\otimes 3}$. By projecting it on the matrix element $e_{ij}\otimes e_{kl}\otimes e_{mn}$ we obtain:

$$\begin{aligned} & (-1)^{([k]+[k_1])([i_1]+[j])+([r]+[r_1])([k_1]+[l])} R_{ii_1,kk_1}(u-v)R_{i_1j,rr_1}(u)R_{k_1l,r_1s}(v) = \\ = \; & (-1)^{[i_1]([r_1]+[s])+([j_1]+[k]+[l])([r]+[s])+[i]([r]+[r_1])+([i]+[j])([k]+[k_1])} \times \\ & \times R_{kk_1,rr_1}(v)R_{ii_1,r_1s}(u)R_{i_1j,k_1l}(u-v)\,, \end{aligned} \tag{2.7}$$

where the sum over repeated indices i_1, k_1, r_1 from 1 to $m+n$ is implied. Written in this form, eq.(2.6) can be considered a functional equation for the set of functions $R_{ij,kl}(u)$ of a complex parameter u, depending on four indices i,j,k,l running from 1 to $m+n$.

A solution $R(u)$ to the graded Yang–Baxter equation will be called an R–matrix, and the complex variable u will be called the spectral parameter. The R–matrices we shall be interested in throughout this work will always obey the following properties:

1. they will be *even* matrices, i.e. all non–zero elements of $R(u)$ shall be of 0 degree:

$$R_{ij,kl}(u)\neq 0 \quad \Rightarrow \quad [i]+[j]+[k]+[l]=0\,;$$

2. they will be *invariant relative to the supergroup $GL(m|n)$*, i.e. such that

$$[A_1A_2\,,R_{12}(u)]=0\,, \qquad \forall\, A\in GL(m|n)\,. \tag{2.8}$$

The restriction to even solutions of the Yang–Baxter equation leads to the following simplification of eq.(2.7):

$$\begin{aligned} & (-1)^{[i_1]([k]+[l])+[j]([k_1]+[l])} R_{ii_1,kk_1}(u-v)R_{i_1j,rr_1}(u)R_{k_1l,r_1s}(v) = \\ = \; & (-1)^{[i]([k]+[k_1])} R_{kk_1,rr_1}(v)R_{ii_1,r_1s}(u)R_{i_1j,k_1l}(u-v)\,, \end{aligned}$$

while the invariance condition allows to easily find a solution to eq.(2.6). There are a total of two operators in $End\,\mathbb{C}^{m|n}\otimes End\,\mathbb{C}^{m|n}$ that are invariant in the sense of (2.8) [19]: the identity operator $1\otimes 1$ and the graded permutation operator

$$P_{12}=\sum_{i,j=1}^{m+n}(-1)^{[j]}e_{ij}\otimes e_{ji} \tag{2.9}$$

whose action on the basis vectors and matrices reads:

$$\begin{aligned} & P_{12}(e_i\otimes e_j)=(-1)^{[i][j]}\,e_j\otimes e_i\,, \\ & P_{12}(e_{ij}\otimes e_{kl})P_{12}=(-1)^{([i]+[j])([k]+[l])}\,e_{kl}\otimes e_{ij}\,. \end{aligned}$$

Remark 2.5 *The permutation operator obeys the relation $P_{12}^2=1\otimes 1$, so that it is symmetric:*

$$P_{21}=P_{12}\,P_{12}\,P_{12}=P_{12}\,.$$

The simplest solution to the graded Yang–Baxter equation is then obtained [19] as a proper linear combination of these operators:

$$R_{ab}(u)=1-\frac{\hbar}{u}P_{ab}\,, \tag{2.10}$$

as can be checked by direct calculation. The following transformations obviously leave (2.6) invariant:

1. multiplication of $R(u)$ by an arbitrary scalar function $f(u)$;
2. similiarity transformations through a nondegenerate operator $A \in End\,\mathbb{C}^{m|n}$:

$$R'(u) = (A \otimes A)R(u)(A \otimes A)^{-1}\,.$$

In particular, multiplication by scalar functions allows one to build solutions to the graded Yang–Baxter equation with different normalizations, while, for solutions with the property (2.8), the latter transformation is a symmetry of the R–matrix itself.

Definition 2.6 (Regularity) *A graded Yang–Baxter equation solution will be called regular if there exists $u_0 \in \mathbb{C}$ such that $R(u_0)$ is proportional to the permutation operator:*

$$R(u_0) \propto P\,.$$

Remark 2.7 *Let us note that there also exists another system of notation where, instead of the operator R, the operator $\check{R}$ is used, differing from R by a multiplication by the permutation operator P:*

$$\check{R}_{ab}(u) = P_{ab}R_{ab}(u)\,.$$

In this case, the graded Yang–Baxter equation reads

$$(1 \otimes \check{R}(u-v))(\check{R}(u) \otimes 1)(1 \otimes \check{R}(v)) = (\check{R}(v) \otimes 1)(1 \otimes \check{R}(u))(\check{R}(u-v) \otimes 1)\,.$$

In this notation, the property of regularity writes

$$\check{R}(u_0) \propto 1\,.$$

Let us now summarize in a proposition the most important properties of the fundamental solution (2.10) of the Yang–Baxter equation.

Proposition 2.8 *The R–matrix (2.10) satisfies the following properties:*

1. *Unitarity:*
$$R_{ab}(u)R_{ba}(-u) = \zeta(u)\mathbb{I}_{ab}\,,$$
where
$$\zeta(u) = \left(1 - \frac{\hbar}{u}\right)\left(1 + \frac{\hbar}{u}\right)\,.$$
Equivalently, we can write $R^{-1}_{ab}(u) = \frac{1}{\zeta(u)}R_{ab}(-u)$;
2. *Symmetry:*
$$R_{ba}(u) = P_{ab}R_{ab}(u)P_{ab} = R^{t_a t_b}_{ab}(u) = R_{ab}(u)\,;$$
3. *Crossing unitarity :*
$$\left(R^{-1}_{ab}(u)\right)^{t_a} = \frac{1}{\zeta(u)}\left(R^{t_a}(u + \hbar(m-n))\right)^{-1}\,,$$
4. *$GL(m|n)$ invariance:*
$$[A_a A_b\,, R_{ab}(u)] = 0\,, \qquad A \in GL(m|n)\,.$$

Finally, one can obtain a regular solution $\tilde{R}(u)$ to the Yang–Baxter equation from the fundamental one (2.10), by multiplying it by $-u$:

$$\tilde{R}_{ab}(u) = -uR_{ab}(u) \quad \Rightarrow \quad \tilde{R}_{ab}(0) = \hbar P_{ab}\,. \tag{2.11}$$

As in the case of $gl(n)$, it is not difficult to write down a solution to the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{2.12}$$

in the case of $V_1 = V_2 = End\,\mathbb{C}^{m|n}$ and V_3 an arbitrary representation π of $gl(m|n)$:

$$\pi : \begin{cases} gl(m|n) \to V_3\,, \\ \mathcal{E}_{ij} \mapsto \pi(\mathcal{E}_{ij})\,. \end{cases}$$

Proposition 2.9 *Given any representation π of $gl(m|n)$, the R–matrix*

$$R_{a3}(u) = 1 - \frac{\hbar}{u}\sum_{k,l=1}^{m+n}(-1)^{[k]}(e_{kl})_a(\pi(\mathcal{E}_{lk}))_3\,, \qquad a = 1,2\,,$$

solves the Yang–Baxter equation (2.12) with $R_{12}(u)$ coinciding with the fundamental solution (2.10).

Proof: Multiplying both sides of (2.12) by $uv(u-v)$ and expanding in u and v, cubic, quadratic and zero degree terms in the spectral parameters trivially simplifies. The linear terms lead to the equation

$$\sum_{i,j,k,l}(-1)^{([l]+[j])}e_{ij}\otimes e_{kl}\otimes\left\{\pi(\mathcal{E}_{ji})\pi(\mathcal{E}_{lk}) - (-1)^{([i]+[j])([k]+[l])}\pi(\mathcal{E}_{lk})\pi(\mathcal{E}_{ji})\right\} =$$
$$= \sum_{i,j}(-1)^{[j]}1\otimes e_{ij}\otimes\pi(\mathcal{E}_{ji})\,P_{12} - P_{12}\sum_{i,j}(-1)^{[j]}1\otimes e_{ij}\otimes\pi(\mathcal{E}_{ji})\,.$$

Picking up the matrix elements in the auxiliary spaces V_1 and V_2 and using the fact that π is a representation, we see that the resulting condition on V_3 reads

$$\pi\left([\mathcal{E}_{ji}\,,\mathcal{E}_{kl}]\right) = \delta_{ik}\pi(\mathcal{E}_{jl}) - (-1)^{([i]+[j])([k]+[l])}\delta_{jl}\pi(\mathcal{E}_{ki})\,,$$

coinciding with the supercommutation relations of $gl(m|n)$. Thus, no new condition arises on the representations π. ■

2.2 Yangian of $gl(m|n)$

Definition 2.10 *The Yangian of $gl(m|n)$, hereafter denoted with $\mathcal{Y}_\hbar(m|n)$, is the graded associative algebra with unity $1_\mathcal{Y}$, and $\mathbb{Z}_2$–graded generators $T_{ab}^{(k)}$, $k>0$, $1\leq a,b\leq m+n$ satisfying the following supercommutation rules:*

$$\left[T_{ab}^{(k)}\,,T_{cd}^{(l)}\right] = (-1)^{[a][b]+[a][c]+[b][c]}\sum_{p=0}^{min(k,l)-1}\left(T_{cb}^{(p)}\,T_{ad}^{(k+l-1-p)} - T_{cb}^{(k+l-1-p)}\,T_{ad}^{(p)}\right)\,, \tag{2.13}$$

where $T_{ab}^{(0)} = \delta_{ab}1_\mathcal{Y}$. The gradation of the generators is given by

$$[T_{ab}^{(k)}] = [a]+[b]\,, \qquad \forall\, a\,,b\,.$$

We will mainly work in the so–called *distinguished* $\mathbb{Z}_2$-grade defined by

$$[i] = \begin{cases} 0\,, & 1 \leq i \leq m\,, \\ 1\,, & m+1 \leq i \leq m+n\,. \end{cases} \tag{2.14}$$

However, our results will be valid (unless explicitly specified) for different grading too, such as the *symmetric* $\mathbb{Z}_2$-grade, defined for even n:

$$[i] = \begin{cases} 0\,, & 1 \leq i \leq n/2 \quad \text{and} \quad m+n/2+1 \leq i \leq m+n\,, \\ 1\,, & n/2+1 \leq i \leq m+n/2\,. \end{cases} \tag{2.15}$$

The name of these grading refers to the $sl(m|n)$ Dynkin diagram (and simple roots) they are associated to, see below.

We will now gather the elements of $\mathcal{Y}_\hbar(m|n)$ in an $(m+n)$–dimensional square matrix, depending on a formal parameter $u \in \mathbb{C}$, to be identified later with the spectral parameter appearing in the $gl(m|n)$ invariant R–matrix. The resulting element of $\mathcal{Y}_\hbar(m|n)[u^{-1}] \otimes End\,\mathbb{C}^{m|n}$ will be denoted with $T(u)$. Following the strategy and the terminology of the quantum inverse scattering method, the (copies of the) Yangian $\mathcal{Y}_\hbar(m|n)$ will then be referred to as the *quantum space(s)*, while $End\,\mathbb{C}^{m|n}$ will be identified with the *auxiliary space*.

Let $u \in \mathbb{C}$ be a formal variable. We define

$$T(u) = \sum_{a,b=1}^{m+n} \sum_{k \geq 0} \frac{\hbar^k}{u^k}\, T^{(k)}_{ab}\, e_{ab} = \sum_{k\geq 0} \frac{\hbar^k}{u^k}\, T^{(k)} = \sum_{a,b}^{m+n} T_{ab}(u)\, e_{ab}\,.$$

The above matrix is an even degree element of $\mathcal{Y}_\hbar(m|n)[u^{-1}] \otimes End\,\mathbb{C}^{m|n}$. For any positive integer k we shall work on algebras of the form

$$\mathcal{Y}_\hbar(m|n)[u^{-1}] \otimes End\,\mathbb{C}^{m|n} \otimes \cdots End\,\mathbb{C}^{m|n}\,, \tag{2.16}$$

with k copies of $End\,\mathbb{C}^{m|n}$. For any $a \in \{1,\ldots,k\}$ we denote by $T_a(u)$ the matrix $T(u)$ which acts on the a–th copy of $End\,\mathbb{C}^{m|n}$. That is, $T(u)$ is an element of the algebra (2.16) of the form

$$T_a(u) = \sum_{i,j=1}^{m+n} T_{ij}(u)\, 1 \otimes \cdots \otimes 1 \otimes e_{ij} \otimes 1 \otimes \cdots \otimes 1\,.$$

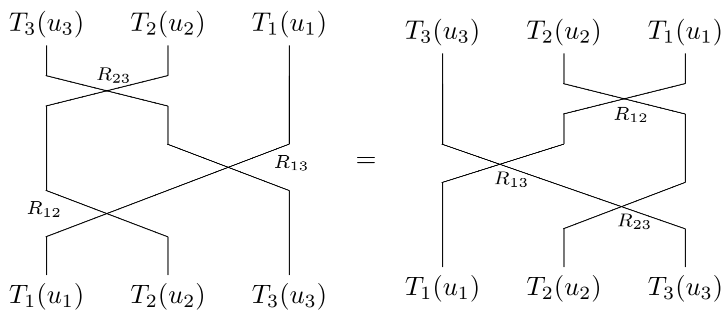
The following proposition establishes the relation between the quantum inverse scattering method and the Yangian.

Proposition 2.11 *The supercommutation relations (2.13) defining the Yangian $Y_\hbar(m|n)$ can be equivalently written:*

$$R_{12}(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R_{12}(u-v)\,, \tag{2.17}$$

where $R_{12}(u)$ is the R–matrix (2.10).

By repeated application of the fundamental exchange relation (2.17), the product $T_1(u_1)T_2(u_2)T_3(u_3)$ can be transformed into $T_3(u_3)T_2(u_2)T_1(u_1)$ in two different ways: either along the scheme $(123) \to (213) \to (231) \to (321)$ or $(123) \to (132) \to (312) \to (321)$, the former corresponding to left multiplication by $R_{12}R_{13}R_{23}$, and the latter to left multiplication by $R_{23}R_{13}R_{12}$, according to the following graphical interpretation:



where each line corresponds to an auxiliary space and R–matrices act at each crossing, intertwining two T matrices. Within the graded version of the quantum inverse scattering method, the graded Yang Baxter equation arises as the condition on the R–matrix (2.10) under the assumption that these two alternative paths give the same results.

Projecting the equivalent defining relations (2.17), called RTT relations, on the matrix element $e_{ij} \otimes e_{kl}$ gives us the supercommutation relations for the elements of the $T(u)$ matrix:

$$[T_{ab}(u)\,, T_{cd}(v)] = \frac{(-1)^{[a][b]+[a][c]+[b][c]}\,\hbar}{u-v}\Big(T_{cb}(u)\,T_{ad}(v) - T_{cb}(v)\,T_{ad}(u)\Big)\,.$$

The above supercommutation relations show that there exist various subalgebras of $\mathcal{Y}_\hbar(m|n)$. In particular, the following ones will play an important role in what follows:

1. the Yangian $\mathcal{Y}_\hbar(m)$ is the even subalgebra generated by $\{T_{ab}(u)\,|\,1 \le a, b \le m\}$;
2. the Yangian $\mathcal{Y}_{-\hbar}(n)$ is the even subalgebra with generators $\{T_{ab}(u)\,|\,m+1 \le a, b \le n\}$. The usual Yangian $\mathcal{Y}_\hbar(n)$ is obtained through the replacement $\hbar \mapsto -\hbar$ in equation (2.17). Notice that altough both $\mathcal{Y}_\hbar(m)$ and $\mathcal{Y}_{-\hbar}(n)$ are even subalgebras, together they do not form a subalgebra of $\mathcal{Y}_\hbar(m|n)$.
3. the Yangian $\mathcal{Y}_\hbar(1|1)$ is the subsuperalgebra with generators $\{T_{m,m+1}(u)\,, T_{m+1,m}(u)\,, T_{m,m}(u)\,, T_{m+1,m+1}(u)\}$.
4. the $gl(m|n)$ superalgebra is generated by $\Big\{(-1)^{[a]}\,T^{(1)}_{ab}\,|a, b = 1\,, \ldots, m+n\Big\}$. This can be seen by taking $k = l = 1$ in the supercommutation relations (2.13). The existence of a $gl(m|n)$ subsuperalgebra will be a key point when reconstructing the symmetry algebra of the integrable models we will build using the Yangian structure.

The map we introduce in the next definition is related to the last point above.

Definition 2.12 (Evaluation map) *The evaluation map*

$$ev : \begin{cases} \mathcal{Y}(m|n) \to \mathcal{U}(gl(m|n))\,, \\ \\ T_{ij}(u) \mapsto \delta_{ij} - (-1)^{[j]}\,\frac{\hbar}{u}\mathcal{E}_{ji}\,, \end{cases} \tag{2.18}$$

defines an algebra homomorphism from the Yangian to the universal enveloping algebra of $gl(m|n)$.

Remark 2.13 *The graded algebras $\mathcal{Y}_\hbar(m|n)$, $\hbar \in \mathbb{C} - \{0\}$, are all isomorphic: an isomorphism $\mathcal{Y}_\hbar(m|n) \mapsto \mathcal{Y}_{\hbar'}(m|n)$ can be defined by*

$$T_{ij}(u) \mapsto T_{ij}(\frac{\hbar'}{\hbar}u)\,.$$

The algebra $\mathcal{Y}_0(m|n)$ is a graded commutative one.

According to the above remark, the deformation parameter $\hbar$ is irrelevant, provided it is not zero, hence it is in general set to 1 for algebraic studies. However, in the context of spin chain models, it is set to $-i$, so that we keep it free to encompass these two choices. At the same time, we will simplify the notation $\mathcal{Y}_\hbar(m|n)$ to $\mathcal{Y}(m|n)$, dropping the $\hbar$, except when different choices of its value are needed in the same calculation.

Straightforward checks show that the following proposition holds.

Proposition 2.14 *The Yangian $\mathcal{Y}_{m|n}$ is a $\mathbb{Z}_2$ graded Hopf algebra. The coalgebraic structures are given by the following maps:*

- *Coproduct*

$$\Delta : \begin{cases} \mathcal{Y}(m|n) & \rightarrow \quad \mathcal{Y}(m|n) \otimes \mathcal{Y}(m|n) \\ \\ T_{ij}(u) & \mapsto \quad \Delta\left(T_{ij}(u)\right) = \sum_{k=1}^{m+n} T_{ik}(u) \otimes T_{kj}(u) \end{cases}$$

- *Counit*

$$\epsilon : \begin{cases} \mathcal{Y}(m|n) & \rightarrow \quad \mathbb{C} \\ \\ T_{ij}(u) & \mapsto \quad \delta_{ij} \end{cases}$$

- *Antipode*

$$S : \begin{cases} \mathcal{Y}(m|n) & \rightarrow \quad \mathcal{Y}(m|n) \\ \\ T_{ij}(u) & \mapsto \quad T_{ij}^{-1}(u)\,. \end{cases}$$

The most relevant object for the construction of integrable spin chains from Yangian is the coproduct Δ. Gathering the generators into matrices, it rewrites

$$\Delta\left(T(u)\right) = T(u)\dot{\otimes}T(u) \in \mathcal{Y}(m|n) \otimes \mathcal{Y}(m|n) \otimes End(\mathbb{C}^{m|n})\,.$$

It is then easy to check that Δ is a graded algebra morphism of $\mathcal{Y}(m|n)$ to $\mathcal{Y}(m|n) \otimes \mathcal{Y}(m|n)$, by simply showing that $\Delta(T(u))$ satisfies the fundamental relation (2.17). In doing so, one uses the fact that elements belonging to different copies of the Yangian (i.e. different quantum spaces) supercommute among themselves. As we will see in the following section, this is related to the so called *ultralocality* of the quantum integrable systems we will build using coproducts of $T(u)$ as monodromy matrices[1]. It is important to notice that Δ is a coassociative map:

$$\Delta^{(N)} = (\Delta^{(N-1)} \otimes \mathrm{id})\,\Delta = (\mathrm{id} \otimes \Delta^{(N-1)})\,\Delta\,,$$

whose N–th iteration $\Delta^{(N)}$ on the (i,j)–th matrix element are explicitly given by the following expression:

$$\Delta^{(N)}(T_{ij}(u)) = \sum_{k_1,\dots,k_{N-1}=1}^{m+n} T_{ik_1}(u) \otimes T_{k_1k_2}(u) \otimes \cdots \otimes T_{k_{N-2}k_{N-1}}(u) \otimes T_{k_{N-1}j}(u)\,.$$

Remark 2.15 *The coproduct Δ is not cocommutative.*

Let us also notice that the following map

$$\text{sign }: T(u) \mapsto T(-u)$$

defines an involutive antiautomorphisms of $\mathcal{Y}(m|n)$. The next proposition lists a few $\mathcal{Y}(m|n)$ automorphisms that will be used in what follows.

[1] An example of a supersymmetric model whose integrability entirely relies on the coproduct structure is given in appendix .

Proposition 2.16 *The following mappings define automorphisms of the graded algebra $\mathcal{Y}(m|n)$:*

1. *The shift in u:*
$$\sigma_a : T(u) \mapsto T(u+a)\,, \qquad a \in \mathbb{C}\,.$$
2. *The multiplication by a formal power series:*
$$\mu_f : T(u) \mapsto f(u)T(u)\,,$$
where
$$f(u) = 1 + f_1 u^{-1} + f_1 u^{-2} + \ldots \in \mathbb{C}[u^{-1}]\,.$$
More explicitly,
$$T_{ij}^{(1)} \mapsto T_{ij}^{(1)} + f_1 \delta_{ij}\,, \quad T_{ij}^{(2)} \mapsto T_{ij}^{(2)} + f_1 T_{ij}^{(1)} + f_2 \delta_{ij}\,, \quad etc.$$
3. *Composition of the inversion with the* sign *map:*
$$\mathrm{inv_s} : T(u) \mapsto T^{-1}(-u)\,.$$
4. *$*$–morphism:*
$$T(u) \mapsto T^*(u) = (T^{-1}(u))^t = \sum_{a,b=1}^{m+n} T_{ab}^*(u)\, e_{ab}\,,$$
where the graded transposition ^{t} is defined as
$$A^t = \sum_{i,j=1}^{m+n} (-1)^{[i][j]+[j]}\, A_{ji}\, e_{ij} = \sum_{i,j=1}^{m+n} \left(A^t\right)_{ij}\, e_{ij}\,, \tag{2.19}$$
that is $\left(A^t\right)_{ij} = (-1)^{[i][j]+[j]}\, A_{ji}\,.$

Proof: We only prove the last point, due to its relevance in our approach. Multiplying both sides of the fundamental exchange relation (2.17) by $T_2^{-1}(v)$, and transposing it in the second auxiliary space, we first get

$$R_{12}^{t_2}(u-v)T_2^*(v)T_1(u) = T_1(u)T_2^*(v)R_{12}^{t_2}(u-v)\,. \tag{2.20}$$

Repeating the same steps for $T_1(u)$ and the first auxiliary space, we get

$$R_{12}^{t_1t_2}(u-v)T_1^*(u)T_2^*(v) = T_2^*(v)T_1^*(u)R_{12}^{t_1t_2}(u-v)\,.$$

The proof ends noticing that the R–matrix is symmetric under graded transposition

$$R_{12}^{t_1t_2}(u) = R_{12}(u)\,.$$

∎

The exchange relations between $T(u)$ and $T^*(v)$ can be read off from equation (2.20), and are given by

$$\begin{aligned}\left[T_{ij}^*(u)\,, T_{kl}(v)\right] \;=\; & \frac{\hbar\,(-1)^{[k][j]}}{u-v}\Big(\delta_{jl}(-1)^{[j]+[k][i]} \sum_{a=1}^{m+n} (-1)^{[a][i]+[a]} T_{ka}(v)T_{ia}^*(u) \\ & -\delta_{ik}(-1)^{[i]} \sum_{a=1}^{m+n} (-1)^{[a][j]} T_{aj}^*(u)T_{al}(v)\Big)\,.\end{aligned}$$

The following facts are easy corollaries of proposition 2.16:

- For τ: $End\,\mathbb{C}^{m|n} \to End\,\mathbb{C}^{m|n}$ an arbitrary antiautomorphism of the graded algebra $End\,\mathbb{C}^{m|n}$, the composed map

$$\tau \circ \mathrm{sign} : T(u) \mapsto T^{\tau}(-u)$$

defines an automorphism of $\mathcal{Y}(m|n)$. In what follows, two different kinds of τ antiautomorphisms will be used: the supertransposition defined above and, when dealing with the twisted super–Yangian, the generalized transposition (see chapter 4);

- the antipode S (see proposition 2.14), acting on the $T(u)$ matrix as an inversion, can be written in the following way:

$$S = \mathrm{inv} \circ \mathrm{sign} : T(u) \mapsto T^{-1}(u)\,.$$

- we can endow $\mathcal{Y}(m|n)$ with different coproduct structures by simply composing Δ with the shift automorphism: defining

$$\Delta_{(a_1,a_2)} = (\sigma_{a_1} \otimes \sigma_{a_2}) \circ \Delta\,,$$

and

$$\Delta^{(N)}_{(a_1,\ldots,a_N)} = (\mathrm{id}^{\otimes N-1} \otimes \sigma_{a_N}) \circ (\Delta^{(N-1)}_{(a_1,\ldots,a_{N-1})} \otimes \mathrm{id}) \circ \Delta\,,$$

we obtain

$$\Delta^{(N)}_{\underline{a}} : \begin{cases} \mathcal{Y}(m|n) & \to \quad \mathcal{Y}(m|n)^{\otimes N} \\ \\ T(u) & \mapsto \quad T(u-a_1)\dot{\otimes}T(u-a_2)\dot{\otimes}\ldots\dot{\otimes}T(u-a_N)\,, \end{cases}$$

where $\underline{a} = (a_1, a_2, \ldots, a_N) \in \mathbb{C}^N$.

Before discussing some well known facts about the representation theory of the super Yangian, let us remind here the validity of the Poincaré–Birkhoff–Witt theorem for $\mathcal{Y}(m|n)$, first established in [28]

Proposition 2.17 *For any given ordering of the generators $T_{ij}^{(k)}$, $1 \le i,j \le m+n$, $k > 0$, the ordered products of the $T_{ij}^{(k)}$ containing no second and higher order powers of the odd generators form a basis of $\mathcal{Y}(m|n)$.*

2.3 Representations

Definition 2.18 (Highest weights) *A $\mathcal{Y}(m|n)$ module V is said to be a highest weight module if there exists $v^+ \in V$ such that*

$$\begin{cases} T_{aa}(u)\,v^+ = \lambda_a(u)\,v\,, \quad \lambda_a(u) \in \mathbb{C}[u^{-1}]\,, & \forall\, a = 1, ..., m+n \\ T_{ab}(u)\,v^+ = 0\,, & 1 \le b < a \le m+n \end{cases} \tag{2.21}$$

The vector $\lambda(u) \doteq (\lambda_1(u), ..., \lambda_{m+n}(u))$ is the highest weight of V, and v^+ a highest weight vector.

The following results about representations of $\mathcal{Y}(m|n)$ have been proved in [28].

Proposition 2.19 *Any finite–dimensional irreducible representation of $\mathcal{Y}(m|n)$ admits a unique highest weight vector (up to normalization).*

Proposition 2.20 *An irreducible representation with highest weight $\lambda(u)$ is finite–dimensional if and only if*

$$\frac{\lambda_a(u)}{\lambda_{a+1}(u)} = \frac{P_a(u+\hbar)}{P_a(u)}\,, \quad 1 \le a \le m+n \quad \textit{and} \quad a \neq m\,, \qquad \frac{\lambda_m(u)}{\lambda_{m+1}(u)} = \frac{P_m(u)}{P_{m+n}(u)}\,, \tag{2.22}$$

where all $P_a(u)$, called Drinfel'd polynomials, are monic polynomials.

Among the finite-dimensional highest weight representations, there is a class of particular interest, constructed from the evaluation map (2.18). This class of representations allows to extend any $gl(m|n)$ module to the superalgebra $\mathcal{Y}(m|n)$, and it will provide us with the main tool for building integrable models with $gl(m|n)$ symmetry.

Definition 2.21 (Evaluation representation) *An evaluation representation ev_{π_μ} is a morphism from the super-Yangian $Y(m|n)$ to a highest weight irreducible representation π_μ of $gl(m|n)$, obtained as the composition of the evaluation map (2.18) with π_μ:*

$$ev_{\pi_\mu} = \pi_\mu \circ ev\,.$$

Its action on the Yangian generators reads

$$ev_{\pi_\mu} : \begin{cases} \mathcal{Y}(m|n) \to \pi_\mu \\ T_{ij}(u) \mapsto ev_{\pi_\mu}(T_{ij}(u)) = \delta_{ij} - (-1)^{[j]}\frac{\hbar}{u}\pi_\mu(\mathcal{E}_{ji})\,, \qquad 1 \le i,j \le m+n\,. \end{cases}$$

The highest weight $\mu(u) = (\mu_1(u), ..., \mu_{m+n}(u))$ of the representation ev_{π_μ} can be immediately read from the definition, and it is given by:

$$\mu_i(u) = 1 - (-1)^{[i]}\,\mu_i\frac{\hbar}{u}\,, \qquad 1 \le i \le m+n\,,$$

where $\mu = (\mu_1, ..., \mu_{m+n})$ is the highest weight of the representation π_μ. Expanding $ev_{\pi_\mu}(T_{ij}(u))$ in powers of $\hbar/u$, one can see that

$$\begin{aligned} ev_{\pi_\mu}(T_{ij}^{(1)}) &= (-1)^{[i]}\,\pi_\mu(\mathcal{E}_{ji})\,, \\ ev_{\pi_\mu}(T_{ij}^{(r)}) &= 0 \quad \text{for} \quad r > 1\,. \end{aligned}$$

By composing it with the shift automorphism of the Yangian we can always make an evaluation representation depend on a parameter $a \in \mathbb{C}$, thus obtaining a one–parameter family of representations:

$$ev_{\pi_\mu}^a(T_{ij}(u)) = \delta_{ij} - (-1)^{[i]}\,\pi_\mu(\mathcal{E}_{ji})\,\frac{\hbar}{u-a}\,, \qquad 1 \le i \le m+n\,. \tag{2.23}$$

Remark 2.22 *When taking an evaluation representation of the fundamental exchange relation (2.17), we recover the Yang Baxter equation (2.12) for the case $End\,\mathbb{C}^{m|n} = V_1 = V_2 \neq V_3$. This means that*

$$R^{(m+n\,,\pi_\mu)}(u) = ev_{\pi_\mu}(T(u))$$

always supplies a solution to the Yang–Baxter equation . For example, the Yang–Baxter equation (2.6) and its solution (2.10) are obtained choosing π_μ to be the fundamental representation, whose highest weight is $\mu = (1, 0, \ldots, 0)$.

2.4 Relations for $T^{-1}(u)$

A necessary step in our approach to the construction of integrable spin chains is to build the supercommutation relations for the elements of $T^{-1}(u)$, as well as the eigenvalues on the highest weight vector v^+ of its diagonal elements $T^{-1}_{kk}(u)$. This is essentially due to the following circumstances:

- In the case of open spin chains, the monodromy matrix explicitly contains $T^{-1}(u)$.
- The definition of the central element of $\mathcal{Y}(m|n)$, the so called *quantum Berezinian*, involves, as we will see, the elements of $T^{-1}(u)$. Even in the case of closed spin chains, part of the information needed to find the spectrum of our models will rely on the properties of the quantum Berezinian. We will need to know its action (and therefore the action of $T^{-1}(u)$) on irreducible representations.

In this section we will first prove that v^+ is an highest weight vector for $T^{-1}(u)$. Working for simplicity in the distinguished Dynkin diagram case, we will then obtain a formula for the eigenvalues of its diagonal elements on v^+.

Let us first remark that the construction of $T^{-1}(u)$ involves taking inverses in the quantum as well as in the auxiliary space, i.e. we can say that $T^{-1}(u)$ is *defined* by the following relation

$$T(u)T^{-1}(u) = 1_{\mathcal{Y}}\,\mathbb{I}\,, \tag{2.24}$$

where $1_{\mathcal{Y}}$ is the unity of $\mathcal{Y}(m|n)$, and $\mathbb{I} = \sum_k e_{kk} \in End\,\mathbb{C}^{m|n}$ is the identity in the auxiliary space. For this reason, it is important to clearly distuinguish between the inverse of an element of $T(u)$ (i.e. the result, if well defined, of an inversion in the quantum space only), and an element of the inverse of $T(u)$. We will therefore adopt the following notation:

$$T'_{ij}(u) \doteq \left(T^{-1}(u)\right)_{ij}\,.$$

Equation (2.24) is understood as a power series in u^{-1}, so that we can reconstruct the generators $T'^{(k)}_{ab}$ from the generators $T^{(k)}_{ab}$, according to the following formula:

$$T'^{(k)}_{ab} = -T^{(k)}_{ab} - \sum_{c=1}^{m+n}\sum_{p=1}^{k-1} T'^{(k-p)}_{ac}T^{(p)}_{cb}\,. \tag{2.25}$$

Proposition 2.23 *The elements of $T(u)$ and $T^{-1}(u)$ satisfy the following supercommutation relations:*

$$\left[T'_{ij}(u)\,,T_{kl}(v)\right] = \frac{\hbar\,(-1)^{[k][j]}}{u-v}\sum_{a=1}^{m+n}\Big(\delta_{il}(-1)^{[k][i]+[i][j]}\,T_{ka}(v)\,T'_{aj}(u) - \delta_{jk}\,T'_{ia}(u)\,T_{al}(v)\Big)\,. \tag{2.26}$$

$$\left[T'_{ij}(u)\,,T'_{kl}(v)\right] = -\frac{(-1)^{[i][j]+[i][k]+[k][j]}\,\hbar}{u-v}\Big(T'_{kj}(u)\,T'_{il}(v) - T'_{kj}(v)\,T'_{il}(u)\Big)\,. \tag{2.27}$$

Proof: Starting from the RTT relation (2.17), and using the antiautomorphism property of S, we easily get:

$$T_2^{-1}(v)\,R_{12}(u-v)\,T_1(u) = T_1(u)\,R_{12}(u-v)\,T_2^{-1}(v)\,,$$
$$R_{12}(v-u)\,T_1^{-1}(u)\,T_2^{-1}(v) = T_2^{-1}(v)\,T_1^{-1}(u)\,R_{12}(v-u)\,.$$

By projecting the above relations on the matrix element $e_{ij}\otimes e_{kl}$, we get equations (2.26) and (2.27). ■

Expanding eq.(2.26) in powers of u^{-1} and v^{-1}, one gets the supercommutation relations between the $T'^{(k)}_{ab}$:

$$\left[T'^{(p+1)}_{ij}, T^{(s)}_{kl}\right] = (-1)^{[k][j]} \sum_{r=0}^{p} \sum_{a=1}^{m+n} \left(\delta_{il}(-1)^{[k][i]+[i][j]} T^{(s+r)}_{ka} T'^{(p-r)}_{aj} - \delta_{jk} T'^{(p-r)}_{ia} T^{(s+r)}_{al}\right) . \tag{2.28}$$

Proposition 2.24 *Let v^+ be a highest weight vector of the super-Yangian. Then, v^+ is also a highest weight vector for $T^{-1}(u)$:*

$$\begin{cases} T'^{(k)}_{ij}\, v^+ = 0\,, \quad i > j\,, \quad k > 0 \\ T'_{ij}(u)\, v^+ = 0\,, \quad i > j\,, \end{cases} \tag{2.29}$$

and

$$\begin{cases} T'^{(k)}_{ii}\, v^+ = \lambda'^{(k)}_i\, v^+\,, \quad k > 0 \\ T'_{ii}(u)\, v^+ = \lambda'_i(u)\, v^+\,, \end{cases} \tag{2.30}$$

where $i, j = 1, \ldots, m+n$, $\lambda'^{(k)}_i \in \mathbb{C}$, and, setting $\lambda'^{(0)}_i = 1$ for all i, we can write

$$\lambda'_i(u) = \sum_{k \geq 0} \frac{\hbar^k}{u^k} \lambda'^{(k)}_i \in \mathbb{C}[u^{-1}]\,.$$

Proof: To prove the proposition, we make a recursion on k. By means of eq.(2.25) it is easy to see that (2.29) and (2.30) are true for $k = 1$. Suppose now that we have for a given $s > 0$ and some scalars $\lambda'^{(k)}_i$

$$\begin{array}{l} T'^{(k)}_{il}\, v^+ = 0 \quad \text{for} \quad i > l\,,\ 0 < k < s \\ T'^{(k)}_{ii}\, v^+ = \lambda'^{(k)}_i\, v^+ \quad \text{for} \quad 0 < k < s\,, \end{array} \tag{2.31}$$

Applying (2.25) for $k = s$ and $i > l$ on v^+, one gets

$$\begin{aligned} T'^{(s)}_{il}\, v^+ &= -\sum_{c=1}^{l} \sum_{p=1}^{s-1} T'^{(s-p)}_{ic} T^{(p)}_{cl}\, v^+ = -\sum_{c=1}^{l} \sum_{p=1}^{s-1} \left[T'^{(s-p)}_{ic}, T^{(p)}_{cl}\right] v^+ = \\ &= \sum_{a=1}^{l} (-1)^{[a]} \sum_{p=1}^{s-2} p \sum_{c=1}^{l} \left[T'^{(s-p-1)}_{ic}, T^{(p)}_{cl}\right] v^+\,, \end{aligned} \tag{2.32}$$

where to get the last equality, we have used (2.28). Iterating r times (with $2 \leq r \leq s-1$) this calculation we are led to :

$$T'^{(s)}_{il}\, v^+ = A_{l,r} \sum_{p=1}^{s-r-1} B_{s,r,p} \sum_{c=1}^{l} \left[T'^{(s-p-r)}_{ic}, T^{(p)}_{cl}\right] v^+\,.$$

where $A_{l,r}$ and $B_{s,r,p}$ are some resummation numbers. Taking $r = s-1$ gives (2.29) for $n = s$, which is thus proven for all n. Finally, applying (2.25) for $n = s$ and $i = l$ on v^+, we have:

$$\begin{aligned} T_{ii}^{\prime(s)}\, v^+ \;=\; & -\lambda_i^{(s)}\, v^+ - \sum_{p=1}^{s-1} \lambda_i^{\prime(s-p)} \lambda_i^{(p)}\, v^+ + \\ & + \sum_{c=1}^{i-1} (-1)^{[c]} \sum_{p=1}^{s-1} p \left(\lambda_i^{\prime(s-p-1)} \lambda_i^{(p)} - \lambda_c^{\prime(s-p-1)} \lambda_c^{(p)} \right) v^+ + \\ & + \sum_{c=1}^{i-1} (-1)^{[c]} \sum_{p=1}^{s-2} p \left(\sum_{a=1}^{i-1} \left[T_{ia}^{\prime(s-p-1)} , T_{ai}^{(p)} \right] - \sum_{a=1}^{i-1} \left[T_{ca}^{(p)} , T_{ac}^{\prime(s-p-1)} \right] \right) v^+ \,. \end{aligned}$$

Again, iterating as in eq. (2.32), we see that only scalar terms acting on v^+ will survive in the right hand side. This proves the property. ■

After having proved that v^+ is the highest weight vector for the inverse of the $T(u)$ matrix, we need to compute the eigenvalues of $T'_{ii}(u)$ on it. Our objective is to find an expression for the $\lambda'_i(u)$ in terms of the $\lambda_i(u)$ only. An application of the evaluation map will then allow us to express everything in terms of the Drinfel'd polynomials in an arbitrary representation. However, before writing such an expression for the $\lambda'(u)$, it is necessary to introduce the graded analogue of the quantum contraction, and to resume few well-known facts about the center of $\mathcal{Y}(m|n)$.

2.5 Quantum contraction

The starting point is to define the following operator

$$Q_{12} = P_{12}^{t_2}\,,$$

where ^{t_a} denotes the supertransposition in the a–th auxiliary space. Since

$$Q_{12}^2 = (m-n) Q_{12}\,,$$

one can see that Q_{12} is proportional, in the $m \neq n$ case, to a projector, while it squares to zero in the $m = n$ case. Let us define, in the $m \neq n$ case, the normalized projector

$$\hat{Q}_{12} = \frac{1}{m-n} Q_{12}\,.$$

By applying it to the basis vectors $v_{kl} = e_k \otimes e_l$, one can see that $\hat{Q}_{12}$ projects $\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n}$ onto the one–dimensional subspace spanned by the vector $\xi = \sum_i (-1)^{[i]} e_i \otimes e_i$:

$$\hat{Q}_{12}\, v_{kl} = \delta_{k,l}\, \xi\,.$$

We can write Q_{12} and $\hat{Q}_{12}$ in terms of the regular R–matrix $\tilde{R}$ defined in equation (2.11) as follows:

$$\tilde{R}_{12}^{t_2}(0) = \hbar\, Q_{12} = \hbar\, P_{12}^{t_2} = \hbar \sum_{i,j=1}^{m+n} (-1)^{[j]+[i]+[i][j]} e_{ij} \otimes e_{ij}\,.$$

Remark 2.25 *Due to the properties of the supertransposition, and unlike the permutation operator, the $\hat{Q}_{12}$ projector is not symmetric; instead, one has*

$$\hat{Q}_{21} = P_{12}\,\hat{Q}_{12}\,P_{12} = \frac{1}{m-n}P_{12}^{t_1} = \frac{1}{m-n}\sum_{i,j=1}^{m+n}(-1)^{[i][j]}e_{ij}\otimes e_{ij} \neq \hat{Q}_{12} = \frac{1}{m-n}P_{12}^{t_2}\,.$$

The proof of the following proposition was first done in [23].

Proposition 2.26 *There exists an element $Z(u) \in \mathcal{Y}(m|n)$, called the quantum contraction of $T(u)$, such that*

$$T_2^*(u)\,T_1(u+\hbar(m-n))\,Q_{12} = Q_{12}\,T_1(u+\hbar(m-n))\,T_2^*(u) = Z(u)\,Q_{12}\,. \tag{2.33}$$

$Z(u)$ is a central element of $\mathcal{Y}(m|n)$, i.e.

$$Z(u)\,T_{ij}(u) = T_{ij}(u)\,Z(u)\,, \qquad 1 \le i,j \le m+n\,.$$

Using the above proposition, we can now extend the crossing relation for the R–matrix (2.10) to the Yangian level, and use the result in our quest for the eigenvalues of $T'(u)$ on the highest weight vector.

Proposition 2.27 *The $T(u)$ matrix satisfies the following crossing relation:*

$$T^*(u) = Z(u)\,T^t(u+\hbar(m-n))^{-1}\,. \tag{2.34}$$

<u>Proof:</u> Multiplying both sides of the second equality in relation (2.33) with $T_2^*(u)^{-1}$, we get

$$Q_{12}\,T_1(u+\hbar(m-n)) = Z(u)\,Q_{12}\,T_2^*(u)^{-1}\,.$$

Transposing in the auxiliary space 2 and multiplying both sides by P_{12}, we get

$$\left(T^*(u)^{-1}\right)^t = \frac{1}{Z(u)}\,T(u+\hbar(m-n))\,, \tag{2.35}$$

which is equivalent to 2.34. ■

Remark 2.28 *In the fundamental representation, where the matrix $T(u)$ reduces to the fundamental solution 2.10 of the Yang–Baxter equation, relation (2.34) reduces to the crossing unitarity relation satisfied by the R matrix, see proposition 2.8, showing that in the fundamental representation*

$$Z(u) = \frac{1}{\zeta(u)}\,.$$

The above results generalize to the graded case the analogous formulas holding in $\mathcal{Y}(n)$. In particular, for the even subalgebras $\mathcal{Y}_\hbar(m)$ and $\mathcal{Y}_{-\hbar}(n)$ of $\mathcal{Y}(m|n)$ we have:

$$\left(T_{-\hbar}^{(n)}(u)^t\right)^{-1} = z_{-\hbar}^{(n)}(u)\,T_{-\hbar}^{(n)*}(u+\hbar n)\,, \tag{2.36}$$

$$\left(T^{(m)}(u)^t\right)^{-1} = z^{(m)}(u)\,T^{(m)*}(u-\hbar m) \tag{2.37}$$

for some scalar functions $z^{(m)}(u)$ and $z_{-\hbar}^{(n)}(u)$. Remarkably, they are related to the quantum determinants of $\mathcal{Y}(m)$ and $\mathcal{Y}(n)$ through the following formulas, whose graded counterparts we will introduce later:

$$z^{(m)}(u) = \frac{\mathrm{qdet}\,T^{(m)}(u-\hbar)}{\mathrm{qdet}\,T^{(m)}(u)}\,, \tag{2.38}$$

$$z_{-\hbar}^{(n)}(u) = \frac{\operatorname{qdet} T^{(n)}(u+\hbar)}{\operatorname{qdet} T^{(n)}(u)} \,. \tag{2.39}$$

Let us remind here that $\operatorname{qdet} T(u)$ is the central element of $\mathcal{Y}(n)$:

Definition 2.29 (Quantum determinant) *The quantum determinant of the matrix $T(u)$ generating $\mathcal{Y}(n)$ is the formal series*

$$\operatorname{qdet} T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \cdots \in \mathcal{Y}(n)[u^{-1}]\,,$$

such that

$$A_n T_1(u) \cdots T_n(u - \hbar(n-1)) = T_n(u-\hbar(n-1)) \cdots T_1(u) A_n = \operatorname{qdet} T(u) A_n\,,$$

where A_n is the antisymmetrizer of $(End\, \mathbb{C}^n)^{\otimes n}$, i.e. the one dimensional projector projecting $(End\, \mathbb{C}^n)^{\otimes n}$ onto the subspace spanned by

$$\xi = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots e_{\sigma(n)}\,.$$

The definition of the quantum determinant in the case $n = 2$ first appeared in [31]. The basic ideas and formulas associated with the quantum determinant for an arbitrary n are contained in the survey paper [25].

Proposition 2.30 $\operatorname{qdet} T(u)$ *lies in the center of $\mathcal{Y}(n)$. That is, all of its coefficients are central elements.*

The values of this central element on highest weight representations can be computed applying the following formulas on v^+:

$$\begin{aligned} \operatorname{qdet} T(u) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, T_{\sigma(1)1}(u) \cdots T_{\sigma(n)n}(u - \hbar(n-1)) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, T_{1\sigma(1)}(u - \hbar(n-1)) \cdots T_{n\sigma(n)}(u)\,. \end{aligned} \tag{2.40}$$

The result for the subalgebras we are interested in are then

$$\begin{aligned} \operatorname{qdet} T^{(m)}(u) &= \lambda_1(u - \hbar(m-1)) \cdots \lambda_m(u)\,, \\ \operatorname{qdet} T^{(n)}_{-\hbar}(u) &= \lambda_1^{(n)}(u + \hbar(n-1)) \cdots \lambda_n^{(n)}(u)\,, \end{aligned}$$

where $\lambda_j^{(n)}(u) = \lambda_{m+j}(u)$, $j = 1, \ldots, n$. Substitution of the above formulas into (2.38) and (2.39) allows us to find the values of the scalar functions $z^{(m)}(u)$ and $z_{-\hbar}^{(n)}$ on highest weight representations:

$$z^{(m)}(u) = \frac{\lambda_1(u - \hbar m) \cdots \lambda_m(u - \hbar)}{\lambda_1(u - \hbar(m-1)) \cdots \lambda_m(u)}\,, \tag{2.41}$$

$$z_{-\hbar}^{(n)}(u) = \frac{\lambda_1^{(n)}(u + \hbar n) \cdots \lambda_n^{(n)}(u + \hbar)}{\lambda_1^{(n)}(u + \hbar(n-1)) \cdots \lambda_n^{(n)}(u)}\,. \tag{2.42}$$

We are now in position to find an expression for the eigenvalues of the diagonal elements of $T'(u)$. This is done in the following

Proposition 2.31 *Let $\lambda'_k(u)$ be the eigenvalue of $T'_{kk}(u)$ on v^+:*

$$T'_{kk}(u)\, v^+ = \lambda'_k(u)\, v^+ \,, \qquad , k = 1, \dots, m+n \,.$$

We have

$$\lambda'_k(u) = \begin{cases} \frac{\lambda_1(u+\hbar)\cdots\lambda_{k-1}(u+\hbar(k-1))}{\lambda_1(u)\cdots\lambda_k(u+\hbar(k-1))} \,, & k = 1, \dots, m \,, \\ \\ Z(u) \frac{\lambda_{k+1}(u+\hbar(2m-k))\cdots\lambda_{m+n}(u+\hbar(m-n+1))}{\lambda_k(u+\hbar(2m-k))\cdots\lambda_{m+n}(u+\hbar(m-n))} \,, & k = m+1, \dots, m+n \,. \end{cases} \tag{2.43}$$

Proof: In order to find the first m diagonal entries of $T'(u)$, we start writing

$$\sum_{j \le k} T_{ij}(u) T'_{jk}(u)\, v^+ = \delta_{ik}\, v^+ \,.$$

Taking $i, k \le m$ we can rewrite the above equation, in the distinguished grade, as follows

$$\sum_{j \le k} T^{(m)}_{ij}(u) T'_{jk}(u)\, v^+ = \delta_{ik}\, v^+ \qquad i, k \le m \,,$$

where $T^{(m)}(u)$ represents as usual the matrix collecting the generators of the even $\mathcal{Y}(m)$ subalgebra of $\mathcal{Y}(m|n)$. We can then consider this as an identity in $\mathcal{Y}(m|n)[u^{-1}] \otimes End(\mathbb{C}^m)$, and we can act on the left with $T'^{(m)}(u) = (T^{(m)}(u))^{-1}$, obtaining

$$T'_{kj}(u)\, v^+ = T'^{(m)}_{kj}(u)\, v^+ \,, \qquad k, j = 1, ..., m \,. \tag{2.44}$$

Let us stress that in (2.44), $T'_{kj}(u)$ is the entry (k, j) of the inverse of the $(m+n) \times (m+n)$ matrix $T(u)$, while $T'^{(m)}_{kj}(u)$ is the entry (k, j) of the inverse of the $m \times m$ matrix $T^{(m)}(u)$. In particular, we get the relation

$$T'_{kk}(u)\, v^+ = \lambda'^{(m)}_k(u)\, v^+ \,, \qquad k = 1, ..., m$$

where the $\lambda'^{(m)}_k(u)$ are the eigenvalues on v^+ of $T'^{(m)}_{kk}(u)$. It has been shown in [21, 53, 9] that these eigenvalues can be written as

$$\lambda'^{(m)}_k(u) = \frac{\lambda^{(m)}_1(u+\hbar) \cdots \lambda^{(m)}_{k-1}(u+\hbar(k-1))}{\lambda^{(m)}_1(u) \cdots \lambda^{(m)}_k(u+\hbar(k-1))} \,, \tag{2.45}$$

which leads to the first line of eq. (2.43).

For the last n diagonal entries of $T'(u)$ we start writing in block form the relation

$$T^t(u) \left(T^t(u)\right)^{-1} v^+ = v^+ \,,$$

setting

$$T^t(u) = \begin{pmatrix} T^{(m)}(u)^t & F(u) \\ G(u) & T^{(n)}_{-\hbar}(u)^t \end{pmatrix} , \qquad T^t(u)^{-1}\, v^+ = \begin{pmatrix} A(u) & 0 \\ * & D(u) \end{pmatrix} v^+ \,,$$

where $*$ denotes a complicated matrix with elements in the Yangian whose exact expression is not relevant in the proof. We then read from the lower right block

$$D(u)\, v^+ = \left(T^{(n)}_{-\hbar}(u)^t\right)^{-1} v^+ \,. \tag{2.46}$$

The l.h.s. of this equation is computed via the crossing relation (2.34) which implies, for $k > m$,

$$D_{k-m,k-m}(u)\, v^+ = \left(T^t(u)\right)^{-1}_{kk}\, v^+ = \frac{1}{Z(u-\hbar(m-n))}\, T'_{kk}(u-\hbar(m-n))\, v^+ \,.$$

The right hand side of the above equation is computed via eq. (2.36). Comparing the left and right hand sides leads to

$$\lambda'_k(u) = z^{(n)}_{-\hbar}(u+\hbar(m-n))\, Z(u)\, \lambda'^{(n)}_{k-m}(u+\hbar m) \qquad k = m+1, ..., m+n\,, \tag{2.47}$$

where the $\lambda'^{(n)}_k(u)$ are the eigenvalues on v^+ of diagonal elements of the $T^{(n)}_{-\hbar}(u)$ matrix. Applying eq. (2.45) to the $\mathcal{Y}_{-\hbar}(n)$ subalgebra, we can write these eigenvalues as

$$\lambda'^{(n)}_l(u) = \frac{\lambda^{(n)}_1(u-\hbar)\cdots\lambda^{(n)}_{l-1}(u-\hbar(l-1))}{\lambda^{(n)}_1(u)\cdots\lambda^{(n)}_l(u-\hbar(l-1))}\,,\ l = 1, ..., n\,.$$

Inserting the value (2.42) of $z^{(n)}_{-\hbar}$ in eq. (2.47) we find the second line of eq. (2.43). ■

In a finite dimensional irreducible representation, where relations (2.22) hold, we can rewrite eq. (2.43) in the following form:

$$\lambda'_k(u) = \begin{cases} \frac{1}{\lambda_1(u)}\prod_{l=1}^{k-1}\frac{P_l(u+\hbar(l+1))}{P_l(u+\hbar l)}\,, & k = 1,\ldots,m\,, \\[2ex] \frac{Z(u)}{\lambda_{m+n}(u+\hbar(m-n))}\prod_{l=k}^{m+n-1}\frac{P_l(u+\hbar(2m-l))}{P_l(u+\hbar(2m-l+1))}\,, & k = m+1,\ldots,m+n\,. \end{cases}$$

Remark 2.32 *Since the diagonal elements of $T^{-1}(u)$ are all even elements, the graded transposition does not affect them. As a consequence, formulas (2.43) also give the eigenvalue of $T^*(u)$ on the highest weight vector v^+.*

We will now define the quantum Berezinian, and review a few of its properties that we shall use in this work. We will first make use of it in order to express the quantum contraction $Z(u)$ in terms of the $\lambda_k(u)$, thus improving our expression (2.43). The quantum Berezinian, defined by Nazarov [23], plays a similar role in the study of the graded Yangian $\mathcal{Y}(m|n)$ as the quantum determinant does for the case of the Yangian $\mathcal{Y}(n)$.

Definition 2.33 *The quantum Berezinian is the following power series with coefficients in the Yangian:*

$$\begin{aligned} Ber(u) &= \sum_{\sigma\in S_m} \mathrm{sgn}(\sigma)\, T_{\sigma(1)1}(u+\hbar(m-n-1))\cdots T_{\sigma(m)m}(u-\hbar n) \\ &\quad\times \sum_{\tau\in S_n} \mathrm{sgn}(\tau)\, T^*_{n+\tau(1)\,,m+1}(u-\hbar n)\cdots T^*_{m+\tau(n)\,,m+n}(u-\hbar)\,. \end{aligned} \tag{2.48}$$

Referring to eq.(2.40), one can immediately recognize that

$$Ber(u) = \mathrm{qdet}\, T^{(m)}(u+\hbar(m-n-1))\, \mathrm{qdet}\, T^{*(n)}(u-\hbar n)\,. \tag{2.49}$$

The following proposition has been proven in [23]. A completely different proof, based on a triangular decomposition of the graded Yangian, has been given in [32].

Proposition 2.34 *The coefficients of the quantum Berezinian (2.48) are central in $\mathcal{Y}(m|n)$, i.e.:*

$$T_{ij}(u)Ber(u) = Ber(u)T_{ij}(u)\,, \qquad 1 \le i,j \le m+n\,.$$

They are related to the Liouville contraction through the identity

$$Ber(u)\, Z(u) = Ber(u+\hbar)\,. \tag{2.50}$$

Remark 2.35 *In [23], it was also conjectured that the coefficients of the quantum Berezinian generate the center of $Y(m|n)$. This conjecture has eventually been proven in a recent paper [33]. However relevant from the algebraic point of view, we shall not make use of this supplementary information in what follows.*

Eq.(2.50), relating the central elements of $\mathcal{Y}(m|n)$, can be considered as the graded counterpart of equation (2.42). Thanks to the above proposition, we can deduce the value of the quantum Berezinian on any representation of highest weight $\lambda(u)$ by simply acting with it on the highest weight vector v^+, and applying formula (2.48). Taking into account that

$$T^*_{ij}(u)\, v^+ = 0$$

for $i \geq j$, we get

$$Ber(u) = \prod_{l=1}^{m} \lambda_l(u - \hbar n + \hbar(l-1)) \prod_{l=m+1}^{m+n} \lambda'_l(u - \hbar(m+n-l+1))\,, \tag{2.51}$$

where the $\lambda'_l(u)$, $l = m+1, ..., m+n$ are given in eq. (2.43). Substitution of this expression in the identity (2.50) yields the following expression for $Z(u)$:

$$Z(u) = \frac{Ber(u+\hbar)}{Ber(u)} = \prod_{k=1}^{m} \frac{\lambda_k(u+\hbar k)}{\lambda_k(u+\hbar(k-1))} \prod_{l=m+1}^{m+n} \frac{\lambda_l(u+\hbar(2m-l))}{\lambda_l(u+\hbar(2m-l+1))}\,. \tag{2.52}$$

Inserting now this expression into eq. (2.43), one obtains:

Corollary 2.36 *The eigenvalues of the diagonal elements of $T^{-1}(u)$ on v^+ are given by*

$$\lambda'_k(u) = \frac{\prod_{l=1}^{k-1} \lambda_l(u+\hbar c_l)}{\prod_{l=1}^{k} \lambda_l(u+\hbar c_{l-1})}\,, \qquad k = 1, \ldots, m+n\,. \tag{2.53}$$

where we set $c_l = \sum_{k=1}^{l} (-1)^{[k]}$, $l = 1, \ldots, m+n$, and $c_0 = 0$.

Remark 2.37 *Using formulas (2.51) and (2.53), one can eliminate the $\lambda'_k(u)$ from the value of the quantum Berezinian, obtaining the more symmetric expression*

$$Ber(u) = \prod_{k=1}^{m} \lambda_k(u+\hbar c_{k-1}) \prod_{l=m+1}^{m+n} \frac{1}{\lambda_l(u+\hbar c_l)}\,. \tag{2.54}$$

The value of the quantum contraction (2.52) reads

$$Z(u) = \frac{\prod_{l=1}^{m+n} \lambda_l(u+\hbar c_l)}{\prod_{l=1}^{m+n} \lambda_l(u+\hbar c_{l-1})}\,.$$

In what follows, we shall need a different expression for $Ber(u)$, also proved in [23]:

Proposition 2.38

$$\begin{aligned} Ber^{-1}(u) \quad &= \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma)\, T^*_{\sigma(1)1}(u+\hbar(m-1)) \cdots T^*_{\sigma(m)m}(u) \times \\ &\times \sum_{\tau \in S_n} \mathrm{sgn}(\tau)\, T_{m+\tau(1),m+1}(u+\hbar(m-n)) \cdots T_{m+\tau(n),m+n}(u+\hbar(m-1))\,. \end{aligned} \tag{2.55}$$

Remark 2.39 *Applying eq.(2.55) to the highest weight vector v^+ we obtain*

$$Ber(u) = \prod_{k=1}^{m} \frac{1}{\lambda_k(u+\hbar c_{k-1})} \prod_{l=m+1}^{m+n} \lambda_l(u+\hbar c_l)\,,$$

in agreement with (2.54).

Applying to both factors of expression (2.49) for the quantum Berezinian the known identity (holding in $\mathcal{Y}_\hbar(n)$)

$$\mathrm{qdet}\, T(u) A_n = T_n(u-\hbar(n-1)) \cdots T_1(u) A_n\,, \tag{2.56}$$

where A_n is the normalized antisymmetrizer in the tensor space $End(\mathbb{C}^n)^{\otimes n}$, we can write

$$Ber(u) A_n A_m = T^{(m)}_m(u-\hbar n)\cdots T^{(m)}_1(u+\hbar(m-n-1)) T^{*(n)}_{m+n}(u+\hbar')\cdots T^{*(n)}_{m+1}(u+\hbar' n) A_m A_n\,,$$

where we have set $\hbar' = -\hbar$ in the second quantum determinant. The A_m and A_n antisymmetrizers are both one–dimensional projectors respectively acting on the tensor product of m and n copies of the auxiliary space, and can be written in terms of the R matrices defining $\mathcal{Y}(m)$ and $\mathcal{Y}_{\hbar'}(n)$:

$$A_n = (R_{12}\cdots R_{1m})\cdots R_{n-1,m}\,, \qquad R_{ij} = R^{(m)}_{ij}(u_i-u_j)\,, \quad u_i - u_{i+1} = \hbar\,,$$

while

$$A_n = \left(R'_{m+1,m+2}\cdots R'_{m+1,m+n}\right)\cdots R'_{m+n-1,m+n}\,, \qquad R'_{ij} = R^{(n),\hbar'}_{ij}(u_i-u_j)\,, \quad u_i - u_{i+1} = \hbar'\,.$$

For any given integer N, A_N projects the N–fold tensor product of auxiliary spaces $\mathbb{C}^N$ onto the one–dimensional subspace generated by the vector

$$\xi = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(N)}\,.$$

Writing now $T^{(m)}(u) = \mathbb{I}^{(m)} T(u) \mathbb{I}^{(m)}$ and $T^{*(n)}(u) = \mathbb{I}^{(n)} T^*(u) \mathbb{I}^{(n)}$, where

$$\mathbb{I}^{(m)} = \sum_{i,[i]=0} e_{ii}\,,$$
$$\mathbb{I}^{(n)} = \sum_{i,[i]=1} e_{ii}\,,$$

and setting $\Pi_{m|n} = (\mathbb{I}^m)^{\otimes m} \otimes (\mathbb{I}^n)^{\otimes n}$, we get

$$\begin{aligned} Ber(u) A_m A_n &= \Pi_{m|n} T_m(u-\hbar n)\cdots T_1(u+\hbar(m-n-1))\,\times \\ &\quad \times T^*_{m+n}(u-\hbar)\cdots T^*_{m+1}(u-\hbar n) A_m A_n\,. \end{aligned}$$

The same steps applied to eq. (2.55) lead to the following equation.

$$\begin{aligned} Ber^{-1}(u) A_m A_n &= \Pi_{m|n} T'_m(u+\hbar(m-1))\cdots T'_1(u)\,\times \\ &\quad \times T_{m+n}(u+\hbar(m-1))\cdots T_{m+1}(u+\hbar(m-n)) A_m A_n\,. \end{aligned}$$

The above expressions can be considered as the graded counterparts of eq. (2.56): both relations act on a number of copies of the auxiliary space equal to the dimension of the Yangian and relate a $(m+n)$–fold tensor product of T matrices to a central element by means of suitable one-dimensional projectors. We will apply this kind of projection in the *generalized fusion* procedure, a key step in finding the spectrum of our integrable models.

Before moving on, let us give an illustrative example in which the results obtained in this section are applied to the simple case of $\mathcal{Y}(1|2)$, and discussed in full detail.

Example 2.40 *Let us take $m = 1$, $n = 2$ with the distinguished grading*

$$[i] = \begin{cases} 0\,, & i = 1\,, \\ 1\,, & i = 2\,,3\,, \end{cases}$$

and let us consider two different representations π_f and π_4 of $gl(1|2)$: the fundamental three–dimensional one, simply given by

$$\pi_f(\mathcal{E}_{ij}) = e_{ij} \in End\,\mathbb{C}^{1|2}\,, \qquad 1 \le i,j \le 3\,,$$

and the following four–dimensional representation:

$$\pi_4(\mathcal{E}_{ij}) = E_{ij} \in End\,\mathbb{C}^{2|2}\,, \qquad 1 \le i,j \le 3\,,$$

where

$$E_{11} = \begin{pmatrix} 3 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} \quad E_{22} = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} \quad E_{12} = \begin{pmatrix} 0 & \sqrt{2} & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix}$$

$$E_{13} = \begin{pmatrix} & & \sqrt{2} & 0 \\ & & 0 & -1 \\ & & & \\ & & & \end{pmatrix} \quad E_{23} = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix} \quad E_{21} = \begin{pmatrix} 0 & 0 & & \\ \sqrt{2} & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}$$

$$E_{31} = \begin{pmatrix} & & & \\ & & & \\ \sqrt{2} & 0 & & \\ 0 & -1 & & \end{pmatrix} \quad E_{32} = \begin{pmatrix} 0 & & & \\ & 0 & 0 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} \quad E_{33} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

The module of the representation π_4 is the graded vector space $\mathbb{C}^{2|2}$, with grading $[1] = [4] = 0$, $[2] = [3] = 1$. We build an evaluation representation for the Yangian $\mathcal{Y}(1|2)$ using the representation π_4:

$$T^{\pi_4}_{ij}(u) = \delta_{ij} - (-1)^{[j]}\frac{\hbar}{u}E_{ji}\,, \qquad 1 \le i,j \le 3\,.$$

As already noticed, the evaluation of $T(u)$ supplies us with a representation of $\mathcal{Y}(1|2)$ as well as with a new solution of the Yang–Baxter equation for the case of two different auxiliary space. Explicitly, it reads

$$T(u) \doteq T^{\pi_4}(u) = 1 - \frac{\hbar}{u}\left(\begin{array}{cccc|cccc|cccc} 3 & & & & & & & & & & & \\ & 2 & & & -\sqrt{2} & & & & & & & \\ & & 2 & & & & & & -\sqrt{2} & & & \\ & & & 1 & & & -1 & & & 1 & & \\ \hline & \sqrt{2} & & & 1 & & & & & & & \\ & & & & & 0 & & & & & & \\ & & & 1 & & & 1 & & & -1 & & \\ & & & & & & & 0 & & & & \\ \hline & & \sqrt{2} & & & & & & 1 & & & \\ & & & -1 & & & -1 & & & 1 & & \\ & & & & & & & & & & 0 & \\ & & & & & & & & & & & 0 \end{array}\right),$$

where we only wrote the diagonal and the non–zero off–diagonal entries. Thus, we have a three–dimensional auxiliary space, coinciding with the fundamental representation of $gl(1|2)$, *while the quantum space has to be identified with the four–dimensional representation* π_4. *The highest weight vector* v^+ *for this representation of* $\mathcal{Y}(1|2)$ *is given by*

$$v^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^{2|2} \,,$$

whose highest weight is $\lambda(u) = u^{-1}(u-3\hbar, u-\hbar, u-\hbar)$, *on which* $T(u)$ *act as a lower–triangular matrix. In order to build the quantum Berezinian in this representation, and to show that it is indeed a central element for* $\mathcal{Y}(1|2)$ *(i.e. proportional to the identity) we shall first construct* $T^*(u)$. *This can be easily accomplished by noticing that* $T^{(k)} = 0$ *for* $k > 1$ *and that*

$$\left(T^{(1)}\right)^2 = -3\,T^{(1)} \,,$$

and using formula (2.25):

$$T'_{ij}(u) = \delta_{ij} + \sum_{k>0} (-1)^k \left(\frac{\hbar}{u}\right)^k \left(T^{(1)}_{ij}\right)^k = \delta_{ij} - \frac{\hbar}{u-3\hbar}\,T^{(1)}_{ij} = \delta_{ij} + (-1)^{[j]}\frac{\hbar}{u-3\hbar}E_{ji} \,.$$

We then see, as stated in proposition 2.24, that v^+ *is a highest weight vector for* $T^{-1}(u)$. *A straightforward check shows that the eigenvalues of* $T^{-1}(u)$ *are in agreement with (2.53):*

$$\begin{aligned} \lambda'_1(u) &= \frac{1}{\lambda_1(u)} = \frac{u}{u-3\hbar} \,, \\ \lambda'_2(u) &= \frac{\lambda_1(u+\hbar)}{\lambda_1(u)\lambda_2(u+\hbar)} = \frac{u-2\hbar}{u-3\hbar} \,, \\ \lambda'_3(u) &= \frac{\lambda_1(u+\hbar)\lambda_2(u)}{\lambda_1(u)\lambda_2(u+\hbar)\lambda_3(u)} = \frac{u-2\hbar}{u-3\hbar} \,. \end{aligned}$$

We are now able to deduce an expression for $T^*(u)$, *by supertransposing* $T^{-1}(u)$ *in the auxiliary space:*

$$T^*(u) = \sum_{i,j=1}^{3} (-1)^{[i][j]+[i]} e_{ij} \otimes T'_{ji}(u) \,,$$

i.e.

$$T^*_{ij}(u) = \delta_{ij} + (-1)^{[i][j]}\frac{\hbar}{u-3\hbar}E_{ij} \,.$$

The quantum Berezinian of $\mathcal{Y}(1|2)$ *is given by*

$$Ber(u) = T_{11}(u-2\hbar)\Big[T^*_{22}(u-2\hbar)T^*_{33}(u-\hbar) - T^*_{32}(u-2\hbar)T^*_{23}(u-\hbar)\Big] \,.$$

Both factors in the above expression are diagonal elements of the quantum space, the term in square brackets being equal to

$$T^*_{22}(u-2\hbar)T^*_{33}(u-\hbar) - T^*_{32}(u-2\hbar)T^*_{23}(u-\hbar) = \begin{pmatrix} \frac{u-3\hbar}{u-5\hbar} & & & \\ & \frac{u-3\hbar}{u-4\hbar} & & \\ & & \frac{u-3\hbar}{u-4\hbar} & \\ & & & 1 \end{pmatrix} \,.$$

Notice that this element of the quantum space commutes with the even subspace $\mathcal{Y}(2)$: as can be seen from the above expression, it is in fact the quantum determinant of $\mathcal{Y}(2)$ written in the direct sum of two one–dimensional and one two–dimensional representation of $\mathcal{Y}(2)$. Multiplication by $T_{11}(u-2\hbar)$ gives

$$Ber(u) = \left(\frac{u-3\hbar}{u-2\hbar}\right) 1_{\mathcal{Y}(1|2)}\,,$$

as in formula (2.51). An analogous calculation starting from eq.(2.38) leads to

$$Ber^{-1}(u) = T^*_{11}(u)\Big[T_{22}(u-\hbar)T_{33}(u) - T_{32}(u-\hbar)T_{23}(u)\Big] = \left(\frac{u-2\hbar}{u-3\hbar}\right) 1_{\mathcal{Y}(1|2)}\,.$$

The value of the quantum contraction $Z(u)$ in this representation can be read off from relation (2.52):

$$Z(u) = \frac{(u-2\hbar)^2}{(u-\hbar)(u-3\hbar)}\, 1_{\mathcal{Y}(1|2)}\,.$$

A straightforward calculation allows to check the crossing relation:

$$Z(u+\hbar)T^t(u)^{-1} = T^*(u+\hbar)\,.$$

3

Closed spin chains

The Hopf structure of the graded Yangian allows one to build a periodic N–site monodromy matrix satisfying the same exchange relations as the matrix of the generators of $Y(m|n)$, and the subsequent derivation of quantum commuting hamiltonians. At the same time, the properties of $Y(m|n)$ provide also relevant information about the symmetry of the resulting integrable models.

In this chapter, we will study the spectrum and Bethe Ansatz equations associated to these periodic chains, presenting the analytical Bethe Ansatz solution to the problem of finding the spectrum of the related transfer matrices. Our treatment of the problem will be global, i.e. the results will be valid for all m and n and for any Dynkin diagram, as well as independent on the chosen $Y(m|n)$ representation. Our aim is to generalize to the superalgebra case the algebraic approach developed in [9].

Sections 3.1, 3.2 and 3.3 deal with the construction of spin chains from the graded Yangian structures, and their aim is to prove their integrability and to describe their symmetry properties. The analytical Bethe Ansatz will be fully discussed in section 3.4, while the main results of the chapter (the Bethe equations) are presented in section 3.5 for the distinguished gradation and in section 3.6 for the remaining Dynkin diagrams, with special attention payed to the symmetric one. Six examples are worked out in section 3.7, including new integrable systems together with well–known ones (e.g., the celebrated supersymmetric t–J model [36, 37]) whose solution we recover in our approach.

Closed spin chains based on $gl(m|n)$ superalgebras in the distinguished Dynkin diagram were studied in [19] and [56], while in [34] the Bethe equations for closed chains in the fundamental representation, but for any Dynkin diagram, were deduced using the Baxter Q–operator. In a recent paper [35], an elegant and quite general approach based on Hirota equation was proposed. Some results of this chapter are thus already known from different approaches. Nevertheless, we present detailed proofs of all relevant steps to illustrate our method, that represents a new and systematic treatment of the problem. We will extend it to open chains with general boundary conditions in the next chapter.

3.1 Monodromy and transfer matrices

The starting point for the construction of supersymmetric integrable spin chains consists, exactly as in the non–graded case, in defining suitable monodromy and transfer matrices. These objects can be naturally defined exploiting the algebraic structures of $\mathcal{Y}(m|n)$ we described in the previous chapter. This will not only ensure integrability by prescribing fully general recipes for the construction of commuting transfer matrices, but also allow us to classify the integrable models generated by these transfer matrices by means of their symmetry properties.

For any given positive integer N (which shall correspond to the number of sites in the chain), the monodromy matrix is defined as

$$\mathcal{T}(u) = \Delta^{(N)}\left(T(u)\right) = T(u)\dot{\otimes}T(u)\dot{\otimes}\cdots\dot{\otimes}T(u) \ \in \mathrm{End}(\mathbb{C}^{(m|n)}) \otimes \mathcal{Y}(m|n)^{\otimes N}\,, \tag{3.1}$$

where the coproduct $\Delta^{(N)}$ can assume one of the different forms defined in the previous chapter. The application of an evaluation map on each term of this tensor product provides the 'usual' monodromy matrix: the different sites correspond to the terms in the tensor product (quantum spaces), and the evaluation map defines the 'spin' (the representation) carried by the site. Taking different representations of the super-Yangian allows to construct various type of closed super-spin chain models.

Remark 3.1 *We will sometimes need to refer to the individual copies of the quantum space: to this end we shall write*

$$T^{[k]}(u)\,, \qquad 1 \leq k \leq N\,,$$

for the k–th copy of the Yangian $\mathcal{Y}(m|n)$ in eq.(3.1). Lower indices shall always refer to the auxiliary space End $\mathbb{C}^{m|n}$. Using this notation, the elements of the monodromy matrix (3.1) would read:

$$\mathcal{T}_{ij}(u) = \sum_{k_1=1}^{m+n} \cdots \sum_{k_{N-1}=1}^{m+n} T^{[1]}_{ik_1}(u)\, T^{[2]}_{k_1 k_2}(u) \cdots T^{[N]}_{k_{N-1} j}(u)\,, \quad 1 \leq i, j \leq m+n\,. \tag{3.2}$$

Remark 3.2 *The coproduct $\Delta^{(N)}$ being a graded algebra homomorphism*

$$\Delta^{(N)} : \mathcal{Y}(m|n) \to \mathcal{Y}(m|n)^{\otimes N}\,,$$

the monodromy matrix (3.1) satisfies the RTT relation (2.17) with the same fundamental R–matrix (2.10):

$$R_{12}(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)R_{12}(u-v)\,. \tag{3.3}$$

The validity of the above equation relies on the fact that the different copies of the Yangian appearing in the monodromy matrix supercommute:

$$\left[T^{[i]}(u)\,, T^{[j]}(v)\right] = 0\,, \qquad i \neq j.$$

The meaning of the above condition, sometimes referred to as the ultralocality condition, is that local operators belonging to different sites of the chain always correspond to compatible observables.

As a consequence of relation (3.3) several properties of $T(u)$ also apply to the monodromy matrix. In particular, the first order of the expansion of $\mathcal{T}(u)$ in powers of u^{-1}

$$\mathcal{T}(u) = \sum_{k \geq 0} \left(\frac{\hbar}{u}\right)^k \mathcal{T}^{(k)}$$

generate a *global* $gl(m|n)$ superalgebra, whose generators are the sum of the corresponding *local* $gl(m|n)$ superalgebras acting on the individual copies of $\mathcal{Y}(m|n)$:

$$\mathcal{T}^{(1)}_{ij} = \sum_{l=1}^{N} \left(T^{[l]}_{ij}\right)^{(1)}\,, \qquad 1 \leq i, j \leq m+n\,,$$

or

$$\mathcal{T}^{(1)}_{ij} = T^{(1)}_{ij} \otimes 1^{\otimes N-1} + 1 \otimes T^{(1)}_{ij} \otimes 1^{\otimes N-2} + \cdots + 1^{\otimes N-1} \otimes T^{(1)}_{ij}\,.$$

A straightforward calculation shows that the set $\{(-1)^{[i]}\mathcal{T}^{(1)}_{ij}\,|1 \leq i, j \leq m+n\}$ satisfy the supercommutation relations of $gl(m|n)$.

Remark 3.3 *To avoid confusion with the single site generators we shall refer to this global $gl(m|n)$ as $gl^{(N)}(m|n)$, even tough they are obviously isomorphic superalgebras.*

In principle, the transfer matrices for our spin chains could be defined in terms of the trace:

$$t(u) = \sum_{i=1}^{m+n} \mathcal{T}_{ii}(u) \tag{3.4}$$

as well as the supertrace

$$st(u) = \sum_{i=1}^{m+n} (-1)^{[i]} \mathcal{T}_{ii}(u) \tag{3.5}$$

of the monodromy matrix. This is due to the following proposition.

Proposition 3.4 *Both the trace and the supertrace of the monodromy matrix generate commutative families of operators, i.e.:*

$$[t(u)\,, t(v)] = 0 \qquad and \qquad [st(u)\,, st(v)] = 0\,.$$

Proof: As a consequence of the RTT relation (3.3), we get

$$\mathcal{T}_1(u)\mathcal{T}_2(v) = \frac{1}{\zeta(u-v)} R_{12}(v-u)\mathcal{T}_2(v)\mathcal{T}_1(u)R_{12}(u-v)\,.$$

We can now take the trace or the supertrace in both auxiliary spaces of the above relation, and the fact that $R_{12}(u)$ is an even numerical matrix allows to use ciclicity in both cases, thus ending the proof. ■

It is important to remark that the two transfer matrices $t(u)$ and $st(u)$ do not commute with each other, so that the observable families they generate will differ. Further, they will have different symmetry algebras. The next proposition describes the symmetry of the transfer matrices (3.4) and (3.5).

Proposition 3.5 *For all m and n the following relations hold:*

$$[X\,, st(u)] = 0\,, \qquad X \in gl^{(N)}(m|n)\,, \tag{3.6}$$

$$[X\,, t(u)] = 0\,, \qquad X \in gl^{(N)}(m) \oplus gl^{(N)}(n)\,. \tag{3.7}$$

Proof: Expanding eq.(3.3) in powers of v^{-1} and taking the v^0 term, one obtains

$$\left[\mathcal{T}_{ij}(u)\,, \mathcal{T}^{(1)}_{kl}\right] = -(-1)^{[i][j]+[i][k]+[j][k]} \left(\mathcal{T}_{kj}(u)\delta_{il} - \mathcal{T}_{il}(u)\delta_{kj}\right)\,.$$

Taking the supertrace in the first auxiliary space, one immediately gets (3.6), while taking the trace results in

$$\left[\mathcal{T}_{ij}(u)\,, \mathcal{T}^{(1)}_{kl}\right] = ((-1)^{[k]} - (-1)^{[l]})\mathcal{T}_{kl}(u)\,.$$

The above expression vanishes if and only if k and l are both even or odd indices, so that eq.(3.7) holds. ■

According to this result, the supertrace enjoys a full global $gl(m|n)$ invariance, while the trace is only $gl(m) \oplus gl(n)$ invariant: it is then reasonable to think that the models associated to $st(u)$ are more relevant than the ones associated to $t(u)$ for the construction of super-spin chain models. We will nevertheless present the Bethe Anstaz for both transfer matrices. Note however that the construction of open spin chain models is possible for the supertrace only, emphasizing the difference between $t(u)$ and $st(u)$.

Remark 3.6 *It is possible to slightly generalize the form of the transfer matrices 3.4 and 3.5, multiplying the monodromy matrix by an invertible numerical K^+ matrix before taking the (super)trace:*

$$\tilde{st}(u) = \sum_{k,i=1}^{m+n} (-1)^{[i]} K^+_{ik} \mathcal{T}_{ki}(u)\,,$$

Proposition 3.4 will still hold, thanks to the $GL(m|n)$ invariance of the R–matrix:

$$\left[K_1^+ K_2^+\,, R(u)\right] = 0\,.$$

The transfer matrices $\tilde{st}(u)$ will still commute at different values of the spectral parameter:

$$\left[\tilde{st}(u)\,, \tilde{st}(v)\right] = 0\,,$$

but the symmetry algebra of the resulting family of commuting observables will be reduced with respect to the one described in proposition 3.5. Although the generalization of our results to this case is straightforward, we shall not make use of these so–called quasiperiodical boundary conditions in our study of closed spin chains, and fully general boundary conditions will be discussed in the case of open spin chains only.

3.2 Pseudovacuum

As in the case of the algebraic Bethe Ansatz, for the study of each different (either closed or open) spin chain it is necessary to find an eigenvector of the transfer matrix, the so–called pseudovacuum, such that the monodromy matrix act on it in a simple way (usually, one tries to satisfy upper or lower triangularity conditions). The representation theory of $\mathcal{Y}(m|n)$ allows one to easily construct a pseudovacuum starting from a $\mathcal{Y}(m|n)$ highest weight vector.

Proposition 3.7 *Suppose $V_1, \ldots, V_N$ are highest weight modules for $\mathcal{Y}(m|n)$, with highest weight vectors $v_1, \ldots, v_N$ and weights $\lambda^{[1]}(u), \ldots, \lambda^{[N]}(u)$. Then the vector*

$$v^+ = v_1 \otimes \cdots \otimes v_N \tag{3.8}$$

is a highest weight vector for the representation of the N–sites monodromy matrix (3.1) on the tensor product $V_1 \otimes \cdots \otimes V_N$:

$$\begin{aligned} &\mathcal{T}_{ij}(u)\, v^+ = 0\,, \quad 1 \le j < i \le m+n\,, \\ &\mathcal{T}_{kk}(u)\, v^+ = \left(\prod_{l=1}^{N} \lambda_k^{[l]}(u)\right) v^+ \doteq \lambda_k(u)\, v^+\,, \qquad 1 \le k \le m+n\,. \end{aligned} \tag{3.9}$$

<u>Proof:</u> Acting on v^+ with the monodromy matrix (3.2) we see that the only surviving terms are those with

$$i \le k_l \le k_{l+1} \le j\,, \qquad 1 \le l \le N-2\,.$$

Taking the equalities in the above relation, one gets (3.9). ∎

Since the action of the monodromy matrix on the pseudovacuum is upper triangular

$$\mathcal{T}(u)\, v^+ = \begin{pmatrix} \lambda_1(u) & * & \cdots & * \\ 0 & \lambda_2(u) & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & & \lambda_{m+n}(u) \end{pmatrix} v^+\,,$$

v^+ is an eigenvector of the transfer matrices (3.4) and (3.5), with the following eigenvalues:

1. for the trace case:

$$t(u)\, v^+ = \hat{\Lambda}_0(u)\, v^+ \,,$$
$$\hat{\Lambda}_0(u) \doteq \sum_{k=1}^{m+n} \lambda_k(u) = \sum_{k=1}^{m+n} \prod_{l=1}^{N} \lambda_k^{[l]}(u)\,;$$

2. for the supertrace:

$$st(u)\, v^+ = \Lambda_0(u)\, v^+ \,,$$
$$\Lambda_0(u) \doteq \sum_{k=1}^{m+n} (-1)^{[k]} \lambda_k(u) = \sum_{k=1}^{m+n} (-1)^{[k]} \prod_{l=1}^{N} \lambda_k^{[l]}(u)\,.$$

We will work with representations of the monodromy matrix that are obtained as tensor products of evaluation representations for the individual sites of the chain: using evaluation representations ev_{π_i} (2.23) for $1 \leq i \leq N$,

$$ev_{\vec{\pi}}\,(\mathcal{T}(u)) = (ev_{\pi_1} \otimes \cdots \otimes ev_{\pi_N})\,(\mathcal{T}(u)) = ev_{\pi_1}(T(u)) \otimes \cdots \otimes ev_{\pi_N}(T(u))\,. \tag{3.10}$$

Starting from the above expressions, and if the highest weights of the ev_{π_i} are

$$\lambda_k^{[i]}(u) = 1 - (-1)^{[k]} \frac{\hbar}{u - a_i} \mu_k^{[i]}\,, \quad 1 \leq i \leq N\,,$$

we easily get the highest weight of the representation of $\mathcal{T}(u)$:

$$ev_{\vec{\pi}}\,(\mathcal{T}_{kk}(u))\; v^+ = \prod_{i=1}^{N} \left(1 - (-1)^{[k]} \frac{\hbar}{u - a_i} \mu_k^{[i]}\right) v^+\,, \qquad k = 1, \ldots, m+n\,,$$

while for the transfer matrices:

$$ev_{\vec{\pi}}\,(st(u))\; v^+ = \sum_{k=1}^{m+n} (-1)^{[k]} \prod_{i=1}^{N} \left(1 + (-1)^{[k]} \frac{\hbar}{u - a_i} \mu_k^{[i]}\right) v^+\,,$$
$$ev_{\vec{\pi}}\,(t(u))\; v^+ = \sum_{k=1}^{m+n} \prod_{i=1}^{N} \left(1 + (-1)^{[k]} \frac{\hbar}{u - a_i} \mu_k^{[i]}\right) v^+\,.$$

3.3 Normalization

Since our objective for this chapter is to write down the Bethe equations for $gl(m|n)$ invariant spin chains as analyticity conditions for the eigenvalues of the transfer matrix, it is important, before moving on, to choose a suitable normalization for the monodromy matrix. Noticing that in an evaluation representation ev_π^a the entries of the matrix $(u-a)T(u)$ are analytical in the spectral parameter, we will use for the local and monodromy matrices the following normalizations:

$$T_k^{[i]}(u) \mapsto (u - a_i)\, T_k^{[i]}(u)\,, \qquad \text{and} \quad \mathcal{T}(u) \mapsto \prod_{i=1}^{N} (u - a_i) \mathcal{T}(u)\,, \tag{3.11}$$

that ensure analyticity of their entries. The transfer matrix will be accordingly normalized. With the normalization (3.11) the highest weights in the ev_{π_i} representations read:

$$\lambda_k^{[i]}(u) = u - a_i - (-1)^{[k]}\, \hbar\, \mu_k^{[i]} \quad \text{and} \quad \lambda_k(u) = \prod_{i=1}^{N} \left(u - a_i - (-1)^{[k]}\, \hbar\, \mu_k^{[i]}\right)\,. \tag{3.12}$$

Nevertheless, let us stress that the above calculation only relies on the existence of a highest weight vector, and thus remains valid for infinite dimensional (highest weight) representations. When the representations are finite dimensional, it is possible to rewrite $\Lambda_0(u)$ in terms of Drinfel'd polynomials. In this case, we will see that the Bethe Ansatz equations depend on the representation only through the Drinfel'd polynomials.

3.4 Dressing hypothesis

Having determined the form of the pseudovacuum eigenvalue we assume now the following form for the general transfer matrix eigenvalues:

$$\hat{\Lambda}(u) = \sum_{k=1}^{m+n} \lambda_k(u)\, \hat{A}_{k-1}(u)\,, \tag{3.13}$$

$$\Lambda(u) = \sum_{k=1}^{m+n} (-1)^{[k]} \lambda_k(u)\, A_{k-1}(u)\,. \tag{3.14}$$

The functions $A_i(u)$ and $\hat{A}_i(u)$, $0 \le i \le m+n-1$, are called *dressing function.*

Remark 3.8 *Taking $A_i(u) = 1$ and $\hat{A}_i(u) = 1$ for all i, we recover the eigenvalue of the transfer matrices on the pseudovacuum.*

This so called dressing hypothesis relies on several known results:

- in the $gl(1|2)$ and $gl(2|2)$ cases, the algebraic Bethe Ansatz leads to (3.13) and (3.14) [41, 65];
- it is a well–established hypothesis for the non graded cases [9], that can be considered the $n = 0$ subcase in our approach;
- it is supported by partial results on $gl(m|n)$ spin chains eigenvectors, as well as numerical evidence for the case of small N;
- a posteriori, the dressing hypothesis leads to the correct Bethe Ansatz equations, known in some cases from other approaches [35].

The aim of the remaining part of this section is to find and implement, by means of algebraic methods, enough constraints upon the spectrum of our transfer matrices, thus determining the form of the dressing functions. The outline of our approach goes as follows:

1. the R matrix and monodromy matrix (as well as the supercommutation relations of $\mathcal{Y}(m|n)$) being written in terms of rational functions of the spectral parameter u, one assumes that the $A_l(u)$ are also rational functions for all l;
2. analyticity requirements imposed on the spectrum imply that, whenever a dressing function $A_l(u)$ has a pole, there must be one and only one dressing function $A_{l'}(u)$, with $l \neq l'$, with a pole at the same position. We further assume that $A_l(u)$ (resp. $\widehat{A}_l(u)$) has common poles with $A_{l\pm 1}(u)$ (resp. $\widehat{A}_{l\pm 1}(u)$) only;
3. the poles of the dressing functions will be assumed simple: the relation between $A_l(u)$ and $A_{l+1}(u)$ poles is then the simplest one which ensures the analyticity of the eigenvalues;
4. information about the number of factors in the aforementioned rational functions will be extracted by comparing two different expressions for the asymptotic expansion of the transfer matrix ;

5. the generalized fusion provides relations among the dressing functions.

Requirements 1. and 2. above fix the following form for the dressing functions:

$$A_l(u) = \prod_{j=1}^{M^{(l)}} \frac{u-\alpha_j^{(l)}}{u-w_j^{(l)}} \prod_{j=1}^{M^{(l+1)}} \frac{u-\beta_j^{(l+1)}}{u-v_j^{(l+1)}} \,.$$

The following equivalent form, in which the dressing functions attached to the even and odd pseudavacuum eigenvalues acquire different denominators, will turn out to be more convenient:

$$A_l(u) = \begin{cases} \displaystyle\prod_{j=1}^{M^{(l)}} \frac{u-\alpha_j^{(l)}}{u-u_j^{(l)}-\hbar\frac{l}{2}} \prod_{j=1}^{M^{(l+1)}} \frac{u-\beta_j^{(l+1)}}{u-u_j^{(l+1)}-\hbar\frac{l+1}{2}}\,, & 0\le l<m\,, \\ \displaystyle\prod_{j=1}^{M^{(l)}} \frac{u-\alpha_j^{(l)}}{u-u_j^{(l)}-\hbar m+\hbar\frac{l}{2}} \prod_{j=1}^{M^{(l+1)}} \frac{u-\beta_j^{(l+1)}}{u-u_j^{(l+1)}-\hbar m+\hbar\frac{l+1}{2}}\,, & m\le l<m+n\,; \end{cases} \tag{3.15}$$

the shifts in the denominators can always be eliminated through a redefinition of the $u_j^{(l)}$. In the above expressions for the dressing functions it is assumed that $M^{(0)} = M^{(m+n)} = 0$, while the values of the integers $M^{(l)}$, $l = 1, \ldots, m+n-1$ are to be determined by means of asymptotic expansion (point 4. above) and are related, from the physical point of view, to the conserved $gl^{(N)}(m|n)$ charges of the chain as we will show below. In particular, they must be kept free in order that the spectrum of the transfer matrix can be generated by varying their values. The next step consists in finding constraints to determine $\alpha_j^{(l)}$ and $\beta_j^{(l)}$ in terms of $u_j^{(l)}$. This is achieved by means of a generalized fusion procedure, requiring the introduction of a transfer matrix built from the $T^*(u)$ generators. The fact that *,as defined in proposition 2.16 is an isomporphism of $\mathcal{Y}(m|n)$ allows to define another transfer matrix

$$st^*(u) = str\, T^*(u)$$

obeying

$$[st^*(u)\,, st^*(v)] = 0\,.$$

The supercommutation relations for $T(u)$ and $T^*(v)$ show that it commutes with $st(u)$:

$$[st(u)\,, st^*(v)] = 0\,,$$

so that we can consider the dressing of $st^*(u)$ simultaneously with the one of $st(u)$:

$$\Lambda^*(u) = \sum_{k=1}^{m+n} (-1)^{[k]} \lambda_k^*(u) A_k^*(u)\,, \tag{3.16}$$

where $T_{kk}^*(u)\,v^+ = \lambda_k^*(u)\,v^+$. The following form for the $T^*(u)$ dressing functions will be assumed:

$$A_l^*(u) = \begin{cases} \displaystyle\prod_{j=1}^{M^{(l)}} \frac{u-\alpha_j^{*(l)}}{u-u_j^{*(l)}-\hbar\left(m-\frac{l}{2}\right)} \prod_{j=1}^{M^{(l+1)}} \frac{u-\beta_j^{*(l+1)}}{u-u_j^{*(l+1)}-\hbar\left(m-\frac{l+1}{2}\right)}\,, & 0\le l<m\,, \\ \displaystyle\prod_{j=1}^{M^{(l)}} \frac{u-\alpha_j^{*(l)}}{u-u_j^{*(l)}-\hbar\frac{l}{2}} \prod_{j=1}^{M^{(l+1)}} \frac{u-\beta_j^{*(l+1)}}{u-u_j^{*(l+1)}-\hbar\frac{l+1}{2}}\,, & m\le l<m+n\,. \end{cases}$$

The next proposition will allow us to find the needed relations between the $u_j^{(k)}$ and the free parameters in the numerators of the dressing functions.

Proposition 3.9 *The dressing functions $A_i(u)$ and $A_i^*(u)$, $0 \leq i \leq m+n-1$, satisfy the following constraints:*

$$\prod_{k=0}^{m-1} A_k(u+\hbar\, c_k) \prod_{k=m}^{m+n-1} A_k^*(u-\hbar c_k+\hbar m) = 1\,, \tag{3.17}$$

$$\prod_{k=0}^{m-1} A_k^*(u+\hbar\, c_k) \prod_{k=m}^{m+n-1} A_k(u-\hbar c_k+\hbar m) = 1\,, \tag{3.18}$$

where the c_k integers, $k=1,\ldots,m+n$, are defined as in corollary 2.36, and $c_0=0$.

Proof: Let A_m, A_n and $\Pi_{m|n}$ be the one–dimensional graded projectors defined in the previous chapter, acting on $m+n$ auxiliary spaces $V_1,\ldots,V_{m+n}$ and denote

$$\mathcal{T}\mathcal{T}^* = \mathcal{T}_m(u-\hbar n)\cdots\mathcal{T}_1(u+\hbar(m-n-1))\mathcal{T}^*_{m+n}(u-\hbar)\cdots\mathcal{T}^*_{m+1}(u-\hbar n)\,.$$

Then, from the following relation

$$\mathcal{T}\mathcal{T}^* = Ber(u)A_mA_n + (1-\Pi_{m|n})\mathcal{T}\mathcal{T}^*A_mA_n + \mathcal{T}\mathcal{T}^*(1-A_mA_n)\,, \tag{3.19}$$

we deduce, by taking the supertrace in the spaces $1,\ldots,m+n$, that

$$st(u-\hbar n)\cdots st(u+\hbar(m-n-1))st^*(u-\hbar)\cdots st^*(u-\hbar n) = (-1)^n Ber(u) + st_{\mathfrak{f}}^{(1)}(u)\,,$$

where the so called fused transfer matrix $st_{\mathfrak{f}}^{(1)}(u)$ is given by

$$st_{\mathfrak{f}}^{(1)}(u) = str_{1\ldots m+n}\left[(1-\Pi_{m|n})\mathcal{T}\mathcal{T}^*A_mA_n + \mathcal{T}\mathcal{T}^*(1-A_mA_n)\right]\,.$$

Then, acting with relation (3.19) on any ($st(u)$ and $st^*(u)$) eigenvector v with eigenvalues $\Lambda(u)$, $\Lambda^*(u)$, one obtains

$$\begin{aligned} & \Lambda(u-\hbar n)\cdots\Lambda(u+\hbar(m-n-1))\Lambda^*(u-\hbar)\cdots\Lambda^*(u-\hbar n) = \\ = \; & (-1)^n \prod_{k=1}^{m} \lambda_k(u-\hbar(n-k+1)) \prod_{l=m+1}^{m+n} \lambda_l'(u+\hbar(m+n-l+1)) + \Lambda_{\mathfrak{f}}^{(1)}(u)\,, \end{aligned} \tag{3.20}$$

where $\Lambda_{\mathfrak{f}}^{(1)}(u)\,v = st_{\mathfrak{f}}^{(1)}(u)\,v$ and we have used eq. (2.51). Let us remark that this relation shows that v is also an eigenvector of $st_{\mathfrak{f}}^{(1)}(u)$. Using the postulated expression (3.14) for the eigenvalues and picking the term proportional to

$$\prod_{k=1}^{m} \lambda_k(u-\hbar(n-k+1)) \prod_{l=m+1}^{m+n} \lambda_l'(u+\hbar(m+n-l+1))$$

in eq. (3.20), we deduce the first constraint between the dressing functions, namely

$$A_0(u-\hbar n)\cdots A_{m-1}(u+\hbar(m-n-1))A_m^*(u-\hbar n)\cdots A^*_{m+n-1}(u-\hbar) = 1\,. \tag{3.21}$$

By shifting u in the above relation we get (3.17). An analogous calculation proves the second constraint: we start setting

$$\mathcal{T}'\mathcal{T} = \mathcal{T}_m'(u+\hbar(m-1))\cdots\mathcal{T}_1'(u)\mathcal{T}_{m+n}(u+\hbar(m-1))\cdots\mathcal{T}_{m+1}(u+\hbar(m-n))$$

and supertracing in all auxiliary spaces the identity

$$\mathcal{T}'\mathcal{T} = Ber^{-1}(u)A_mA_n + (1-\Pi_{m|n})\mathcal{T}'\mathcal{T}A_mA_n + \mathcal{T}'\mathcal{T}(1-A_mA_n)\,, \tag{3.22}$$

we get

$$st^*(u+\hbar(m-1))\cdots st^*(u)st(u+\hbar(m-1))\cdots st(u+\hbar(m-n)) = (-1)^n Ber^{-1}(u) + st_{\mathfrak{f}}^{(2)}(u)\,,$$

where $st_{\mathfrak{f}}^{(2)}(u) = str_{1\ldots m+n}\left[(1-\Pi_{m|n})\mathcal{T}'\mathcal{T}A_mA_n + \mathcal{T}'\mathcal{T}(1-A_mA_n)\right]$. Acting again with the above equation on v, one obtains

$$\begin{aligned}
&\Lambda^*(u+\hbar(m-1))\cdots\Lambda^*(u)\Lambda(u+\hbar(m-1))\cdots\Lambda(u+\hbar(m-n)) = \\
=\;& (-1)^n\prod_{l=1}^{m}\lambda'_l(u+\hbar(m-l))\prod_{l=m+1}^{m+n}\lambda_l(u+\hbar(2m-l)) + \Lambda_{\mathfrak{f}}^{(2)}(u)\,,
\end{aligned} \tag{3.23}$$

where $\Lambda_{\mathfrak{f}}^{(2)}(u)\,v = t_{\mathfrak{f}}^{(2)}(u)\,v$ and eq. (2.54) has been used. Picking up the term proportional to $\lambda'_l(u+\hbar(m-l))\prod_{l=m+1}^{m+n}\lambda_l(u+\hbar(2m-l))$, we get the second constraint on the dressing functions:

$$A_0^*(u+\hbar(m-1))\cdots A_{m-1}^*(u)A_m(u+\hbar(m-1))\cdots A_{m+n-1}(u+\hbar(m-n)) = 1\,,$$

i.e., after the shift $u\to u-\hbar(m-1)$, eq.(3.18). ■

In order to satisfy our constraints on the dressing function eq.(3.17) and eq.(3.18), we adopt the simplest non–trivial choices for the parameters $\alpha_j^{(k)}$, $\beta_j^{(k)}$, $\alpha_j^{*(k)}$, $\beta_j^{*(k)}$, that leaves the values of the $M^{(k)}$ parameters free. In the case of the constraint (3.17), the simplest choice corresponds to set:

$$\begin{aligned}
\alpha_j^{(k)} &= u_j^{(k)} + \frac{\hbar}{2}(k+2)\,,\\
\beta_j^{(k+1)} &= u_j^{(k+1)} + \frac{\hbar}{2}(k-1)\,,
\end{aligned}$$

for all j and $k=0,\ldots,m-1$, and

$$\begin{aligned}
\alpha_j^{*(k)} &= u_j^{*(k)} + \frac{\hbar}{2}(k+2)\,,\\
\beta_j^{*(k+1)} &= u_j^{*(k+1)} + \frac{\hbar}{2}(k-1)\,,
\end{aligned}$$

for all j and $k=m,\ldots m+n-1$, together with

$$u_j^{*(m)} = u_j^{(m)} - \hbar m\,,$$

in such a way that the dressing functions acquire the form

$$\begin{aligned}
A_k(u) &= \prod_{j=1}^{M^{(k)}}\frac{u-u_j^{(k)}-\hbar\frac{k+2}{2}}{u-u_j^{(k)}-\hbar\frac{k}{2}}\prod_{j=1}^{M^{(k+1)}}\frac{u-u_j^{(k+1)}-\hbar\frac{k-1}{2}}{u-u_j^{(k+1)}-\hbar\frac{k+1}{2}}\,, \qquad k=0,\ldots,m-1\,,\\
A_k^*(u) &= \prod_{j=1}^{M^{(k)}}\frac{u-u_j^{*(k)}-\hbar\frac{k+2}{2}}{u-u_j^{*(k)}-\hbar\frac{k}{2}}\prod_{j=1}^{M^{(k+1)}}\frac{u-u_j^{*(k+1)}-\hbar\frac{k-1}{2}}{u-u_j^{*(k+1)}-\hbar\frac{k+1}{2}}\,, \qquad k=m,\ldots,m+n-1\,,
\end{aligned}$$

and cancelations occur between dressing functions labeled by consecutive indices in expression (3.21). To satisfy this second constraint we set

$$\alpha_j^{(k)} = u_j^{(k)} + \hbar(m - \frac{k}{2} - 1)\,,$$
$$\beta_j^{(k+1)} = u_j^{(k+1)} + \hbar(m - \frac{k-1}{2})\,,$$

for $k = m, \ldots, m+n-1$, and

$$\alpha_j^{*(k)} = u_j^{*(k)} + \hbar(m - \frac{k}{2} - 1)\,,$$
$$\beta_j^{*(k+1)} = u_j^{*(k+1)} + \hbar(m - \frac{k-1}{2})\,,$$

for $k = 0, \ldots, m-1$, so that

$$A_k(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{(k)} - \hbar\left(m - \frac{k}{2} - 1\right)}{u - u_j^{(k)} - \hbar\left(m - \frac{k}{2}\right)} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{(k+1)} - \hbar\left(m - \frac{k-1}{2}\right)}{u - u_j^{(k+1)} - \hbar\left(m - \frac{k+1}{2}\right)}\,, \quad m \le k < m+n\,,$$

$$A_k^*(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{*(k)} - \hbar\left(m - \frac{k}{2} - 1\right)}{u - u_j^{*(k)} - \hbar\left(m - \frac{k}{2}\right)} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{*(k+1)} - \hbar\left(m - \frac{k-1}{2}\right)}{u - u_j^{*(k+1)} - \hbar\left(m - \frac{k+1}{2}\right)}\,, \qquad 0 \le k < m\,.$$

Again, it is seen that

$$u_j^{*(m)} = u_j^{(m)} - \hbar m\,.$$

Remark 3.10 *Relations (3.17) and (3.18) also hold when the $A_l(u)$, $A_l^*(u)$ functions are replaced with $\hat{A}_l(u)$, $\hat{A}_l^*(u)$, thus leading to the same form for the dressing functions appearing in the eigenvalues (3.13) and (3.14).*

Remark 3.11 *Using the c_k integers introduced in proposition 4.7, one can write a single expression for the dressing functions:*

$$A_k(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{(k)} - \frac{\hbar}{2}\left(c_{k+1} + (-1)^{[k+1]}\right)}{u - u_j^{(k)} - \frac{\hbar}{2} c_k} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{(k+1)} - \frac{\hbar}{2}\left(c_k - (-1)^{[k+1]}\right)}{u - u_j^{(k+1)} - \frac{\hbar}{2} c_{k+1}}\,,$$

$$A_k^*(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{*(k)} - \frac{\hbar}{2}(2m - c_{k+1} - (-1)^{[k+1]})}{u - u_j^{*(k)} - \frac{\hbar}{2}(2m - c_k)} \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{*(k+1)} - \frac{\hbar}{2}(2m - c_{k-1})}{u - u_j^{*(k+1)} - \frac{\hbar}{2}(2m - c_{k+1})}\,,$$

for $0 \le k \le m+n-1$. We will see that these expressions generalize to the case of Dynkin diagrams different from the distinguished one.

Remark 3.12 *As we have seen in section 3.1, the generators of the global finite–dimensional $gl^{(N)}(m|n)$ superalgebra commute with the transfer matrix $st(u)$. It is now possible to relate the integers $M^{(k)}$, $k = 1, \ldots, m+n-1$ appearing in the $\Lambda(u)$ dressing to the eigenvalues of the Cartan generators of $gl^{(N)}(m|n)$. This can be done by in the following way. Taking first the asymptotic expansions $u \to \infty$ in the expression* (3.14) *for $\Lambda(u)$ for an N sites chain (and taking for simplicity an evaluation representation with eigenvalues (3.12) where all $a_n = 0$), one gets*

$$\Lambda(u) \sim u^N(m-n) + u^{N-1} \sum_{k=1}^{m+n} \hbar\left((-1)^{[k]}\lambda_k^{(1)} - M^{(k-1)} + M^{(k)}\right)\,,$$

where we set $\lambda_k(u) = u^N + \hbar\lambda_k^{(1)} u^{N-1} + O\left(u^{N-2}\right)$, *i.e.*

$$\lambda_k^{(1)} = \sum_{l=1}^{N} \mu_k^{[l]} .$$

On the other hand, the same expansion performed on the transfer matrix $st(u)$ *leads to*

$$st(u) \sim u^N (m-n) + u^{N-1} \sum_{k=1}^{m+n} \hbar \left(\sum_{l=1}^{N} \mathcal{E}_k^{[n]} \right) ,$$

where $\sum_{l=1}^{N} \mathcal{E}_k^{[l]} = \sum_{l=1}^{N} (-1)^{[k]} \mathcal{T}_{kk}^{(1)[n]}$ *is the k-th diagonal generator of the global* $gl^{(N)}(m|n)$ *symmetry algebra of the chain. Starting then from a transfer matrix eigenvector with eigenvalue (3.14), it is reasonable to assume that*

$$h_k = (-1)^{[k]} \lambda_k^{(1)} - M^{(k-1)} + M^{(k)} ,$$

where h_k *is the eigenvalue of the diagonal generator* $\sum_{l=1}^{N} \mathcal{E}_k^{[l]}$. *Writing the Cartan generators of* $gl(m|n)$ *as* $s_k = (-1)^{[k]} \mathcal{E}_k - (-1)^{[k+1]} \mathcal{E}_{k+1}$, *one gets*

$$s_k \, v = \left(2M^{(k)} - M^{(k-1)} - M^{(k+1)} + (-1)^{[k]} \lambda_k^{(1)} - (-1)^{[k+1]} \lambda_{k+1}^{(1)} \right) v .$$

As in the non graded case, the values of the $M^{(k)}$ *are then related to the conserved charges of the global symmetry of the chain.*

3.5 Bethe equations

We have found in the previous section that $A_l(u) = \hat{A}_l(u)$, and that they have the form

$$\begin{aligned}
A_l(u) &= \prod_{k=1}^{M^{(l)}} \frac{u - u_k^{(l)} - \hbar \frac{l+2}{2}}{u - u_k^{(l)} - \hbar \frac{l}{2}} \prod_{k=1}^{M^{(l+1)}} \frac{u - u_k^{(l+1)} - \hbar \frac{l-1}{2}}{u - u_k^{(l+1)} - \hbar \frac{l+1}{2}} , \qquad 0 \le l < m , \\
A_l(u) &= \prod_{k=1}^{M^{(l)}} \frac{u - u_k^{(l)} - \hbar \left(m - \frac{l}{2} - 1\right)}{u - u_k^{(l)} - \hbar \left(m - \frac{l}{2}\right)} \prod_{k=1}^{M^{(l+1)}} \frac{u - u_k^{(l+1)} - \hbar \left(m - \frac{l-1}{2}\right)}{u - u_k^{(l+1)} - \hbar \left(m - \frac{l+1}{2}\right)} , \quad m \le l < m+n ,
\end{aligned}$$

with the convention $M^{(0)} = M^{(m+n)} = 0$. Substitution of the above equations into (3.14) and (3.14) gives the eigenvalues of the transfer matrices $st(u)$ and $t(u)$ in terms of the set of *Bethe roots* $u_k^{(l)}$, $l = 1, \ldots, m+n$, $k = 1, \ldots, M^{(k)}$. The Bethe equations will be now obtained as analyticity conditions for the eigenvalues $\Lambda(u)$ and $\hat{\Lambda}(u)$: one imposes that the their residues at $u = u_j^{(k)} + \hbar \frac{k}{2}$ for $1 \le j \le M^{(k)}$, $0 < k < m$, and at $u = u_j^{(k)} + \hbar (m - \frac{k}{2})$ for $1 \le j \le M^{(k)}$, $m \le k \le m+n-1$, all vanish. It is useful to introduce the following rational function:

$$\mathfrak{e}_k(u) \doteq \frac{u - \hbar \frac{k}{2}}{u + \hbar \frac{k}{2}} . \tag{3.24}$$

The Bethe equations naturally fall into three different sets, two of them corresponding to cancelation of poles inside the $gl(m)$ and $gl(n)$ terms of the transfer matrix, and one to the cancelation of the poles at $u = u_j^{(m)} + \hbar \frac{m}{2}$, leading to a $gl(1|1)$–like set of Bethe equations connecting the two even subalgebras. Let us then impose $\operatorname{Res} \Lambda(u)|_{u = u_j^{(k)} + \frac{\hbar}{2} c_k} = 0$, with j running from 1 to $M^{(k)}$, in the three subcases:

1. $1 \leq k < m$:

$$\prod_{l=1}^{M^{(k-1)}} \mathfrak{e}_{-1}(u_j^{(k)} - u_l^{(k-1)}) \prod_{l\neq j}^{M^{(k)}} \mathfrak{e}_{2}(u_j^{(k)} - u_l^{(k)}) \prod_{l=1}^{M^{(k+1)}} \mathfrak{e}_{-1}(u_j^{(k)} - u_l^{(k+1)}) = \frac{\lambda_k(u_j^{(k)} + \hbar\frac{k}{2})}{\lambda_{k+1}(u_j^{(k)} + \hbar\frac{k}{2})}\,;$$

2. $m < k < m+n$:

$$\prod_{l=1}^{M^{(k-1)}} \mathfrak{e}_{1}(u_j^{(k)} - u_l^{(k-1)}) \prod_{l\neq j}^{M^{(k)}} \mathfrak{e}_{-2}(u_j^{(k)} - u_l^{(k)}) \prod_{l=1}^{M^{(k+1)}} \mathfrak{e}_{1}(u_j^{(k)} - u_l^{(k+1)}) = \frac{\lambda_k(u_j^{(k)} + \hbar(m-\frac{k}{2}))}{\lambda_{k+1}(u_j^{(k)} + \hbar(m-\frac{k}{2}))}\,;$$

3. $k = m$:

$$\prod_{k=1}^{M^{(m-1)}} \mathfrak{e}_{-1}(u_j^{(m)} - u_k^{(m-1)}) \prod_{k=1}^{M^{(m+1)}} \mathfrak{e}_{1}(u_j^{(m)} - u_k^{(m+1)}) = \pm\,\frac{\lambda_{m+1}(u_j^{(m)} + \hbar\frac{m}{2})}{\lambda_{m}(u_j^{(m)} + \hbar\frac{m}{2})}\,.$$

In the last equation, the $+$ sign (resp. $-$ sign) corresponds to the $\Lambda(u)$ Bethe Ansatz equations (resp. $\hat{\Lambda}(u)$ Bethe Ansatz equations).

Remark 3.13 *One recognizes, in the indices of the $\mathfrak{e}$ functions appearing in the left hand side of the Bethe equations, the entries of the $sl(m|n)$ Cartan matrix. In the next section we shall see how this feature is preserved for different choices of the Dynkin diagram.*

Remark 3.14 *The left hand sides of the Bethe equations only depend on the chosen superalgebra, while the Yangian representations spanned by the spin chain only play a role in the right hand sides. When these representations are finite dimensional, the right-hand side can be re-expressed in terms of Drinfel'd polynomials. For instance, for the first set of BAEs, one gets*

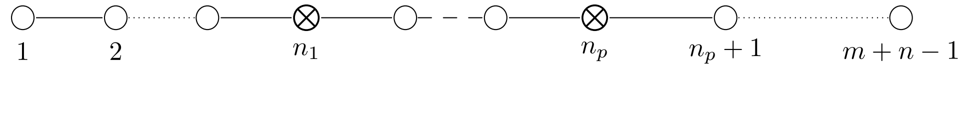
$$\frac{\lambda_i(u_j^{(i)} + \hbar\frac{i}{2})}{\lambda_{i+1}(u_j^{(i)} + \hbar\frac{i}{2})} = \frac{P_i(u_j^{(i)} + \hbar\frac{i}{2})}{P_i(u_j^{(i)} + \hbar\frac{i-1}{2})} \quad \textit{where} \quad P_i(u) = \prod_{k=1}^{N} P_i^{[k]}(u)\,, \tag{3.25}$$

$P_i^{[k]}(u)$ being the Drinfel'd polynomials for each site.

3.6 Bethe equations for arbitrary Dynkin diagrams

As already mentioned, up to now we have worked with the distinguished Dynkin diagram and its associated gradation. However, several Dynkin diagrams can be used to describe the same superalgebra, leading to inequivalent Dynkin diagram, and thus to different presentations of the Bethe equations. For each of the grading (i.e. for each inequivalent Dynkin diagram), one can apply the above procedure to determine the form of the dressing functions. This has been noticed in [54] for open super-spin chains in the fundamental representation of $sl(m|n)$. We generalize it for arbitrary super-spin chains. The dressing functions keep essentially the same structure, with the following rules.

The inequivalent Dynkin diagrams of the $sl(m|n)$ superalgebras contain only bosonic root of same square length (”white dots”), normalized to 2, and isotropic fermionic roots (”grey dots”), which square to zero. A given diagram is completely characterized by the p-uple of integers $0 < n_1 < \ldots < n_p < m+n$ labelling the positions of the grey dots of the diagram:



where the total number of (grey and white) dots is $m+n-1$. Formally, we define $n_0 = 0$ and $n_{p+1} = m+n$ although there is actually no root at these positions. Such a diagram defined by the p-uple $(n_i)_{i=1\dots p}$ corresponds to the superalgebra $sl(m|n)$ with

$$m = \sum_{\substack{i \text{ odd} \\ i \le p+1}} n_i - \sum_{\substack{i \text{ even} \\ i < p+1}} n_i \qquad \text{and} \qquad n = \sum_{\substack{i \text{ even} \\ i \le p+1}} n_i - \sum_{\substack{i \text{ odd} \\ i < p+1}} n_i \,. \tag{3.26}$$

Accordingly, the $\mathbb{Z}_2$-grading is defined by

$$[j] = \frac{1-(-1)^k}{2}\,, \quad \text{i.e.} \quad (-1)^{[j]} = (-1)^k\,, \quad \text{for} \quad n_k + 1 \le j \le n_{k+1}\,, \quad 0 \le k \le m+n\,. \tag{3.27}$$

$$\begin{array}{rcll}
[j] & = & 0 & \text{for} \quad 1 \le j \le n_1 \\
[j] & = & 1 & \text{for} \quad n_1 + 1 \le j \le n_2 \\
[j] & = & 0 & \text{for} \quad n_2 + 1 \le j \le n_3 \\
\vdots & & & \\
[j] & = & \frac{1-(-1)^p}{2} & \text{for} \quad n_p + 1 \le j \le m+n
\end{array}$$

For each of these gradings, one can compute a new value for the parameters

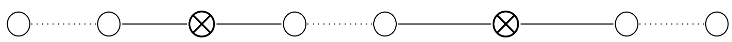
$$c_k = \sum_{j=1}^{k} (-1)^{[j]}\,, \qquad k = 1, \dots, m+n\,.$$

Then, the dressing functions will keep the form given in remark 3.11, but with now the above value for the parameters c_k. Computing the residues for $\Lambda(u)$ with these new dressing functions, leads to the following general recipe for the Bethe equations

$$\begin{aligned}
&(1-(-1)^{[l]}\langle\alpha_\ell,\alpha_\ell\rangle) \prod_{k=1}^{m+n-1} \prod_{j=1}^{M^{(k)}} \mathfrak{e}_{\langle\alpha_\ell,\alpha_k\rangle}(u_i^{(\ell)} - u_j^{(k)}) = \frac{\lambda_\ell(u_i^{(\ell)} + \frac{\hbar}{2}\, c_\ell)}{\lambda_{\ell+1}(u_i^{(\ell)} + \frac{\hbar}{2}\, c_\ell)}\,, \\
&i = 1, \dots, M^{(\ell)}\,, \quad 1 \le \ell < m+n-1\,.
\end{aligned} \tag{3.28}$$

where $\langle\alpha_\ell,\alpha_k\rangle$ is the scalar product of the simple roots, numbered *as they are ordered by the chosen Dynkin diagram*. Explicitly, in $sl(m|n)$, denoting α_j the simple roots, that we label according to their position $j = 1, \dots, m+n$ in the Dynkin diagram, their norm is given by $\langle\alpha_j,\alpha_j\rangle = (-1)^{[j]}\,2$ for the bosonic 'white' roots and by $\langle\alpha_j,\alpha_j\rangle = 0$ for the fermionic 'grey' roots. Moreover, the scalar products between different simple roots are all zero *but* for the simple roots which are linked in the Dynkin diagram. Linked roots have scalar product $\langle\alpha_j,\alpha_{j+1}\rangle = -(-1)^{[j+1]}$. For more informations on the construction of simple roots and Dynkin diagrams for superagebras, see e.g. [15]. The above formula generalizes the Bethe equations given in the previous section to any Dynkin diagram: the indices of the $\mathfrak{e}$ rational functions can still be identified as the corresponding Cartan matrix entries, as specified in remark 3.13. It should be clear that, since the different presentations (i.e. Dynkin diagrams) describe the same superalgebra and the same representations on the chain, the spectrum will be identical, although the dressing functions and the BAE look different. As an important example, let us fully discuss the case of the so–called *symmetric* Dynkin diagram.

Example 3.15 *In the case of $sl(m|2n)$, there exists a symmetric Dynkin diagram with two isotropic fermionic simple roots in positions n and $m+n$:*



n $\qquad\qquad$ $m+n$

We give here the explicit expression for the dressing functions and Bethe Ansatz equations for this diagram. The corresponding gradation of the indices is

$$[i] = \begin{cases} 0\,, & 1 \le i \le n \quad \text{and} \quad m+n+1 \le i \le m+2n\,, \\ 1\,, & n+1 \le i \le m+n\,, \end{cases}$$

implying the following values of the c_k parameters:

$$c_k = \begin{cases} k\,, & k \le n\,, \\ 2n-k\,, & n < k \le m+n\,, \\ k-2m\,, & m+n < k \le m+2n\,. \end{cases} \tag{3.29}$$

This choice of the grading implies that the elements of $T^{(m)}(u)$ (resp. $T^{(2n)}(u)$) generate now a $\mathcal{Y}_{-\hbar}(m)$ (resp. $\mathcal{Y}_{\hbar}(2n)$) bosonic subalgebra. As a consequence, the expressions for the quantum Berezinian and its inverse are modified as follows:

$$Ber(u) = \operatorname{qdet} T^{(2n)}(u - \hbar(m-2n+1)) \operatorname{qdet} T^{*(m)}(u - \hbar m)\,, \tag{3.30}$$

$$Ber^{-1}(u) = \operatorname{qdet} T^{*(2n)}(u + \hbar(2n-1)) \operatorname{qdet} T^{(m)}(u - \hbar(m-2n))\,. \tag{3.31}$$

To determine its value on an highest weight vector v we rewrite the quantum Berezinian for the symmetric Dynkin diagram case as

$$\begin{aligned} Ber(u) &= \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) T_{\sigma(1),1}(u - \hbar(m-2n+1)) \cdots T_{\sigma(n),n}(u - \hbar(m-n)) \times \\ &\times \; T_{m+\sigma(n+1),m+n+1}(u - \hbar(m-n+1)) \cdots T_{m+\sigma(2n),m+2n}(u - \hbar m) \times \\ &\times \sum_{\tau \in S_m} \operatorname{sgn}(\tau) T^*_{n+\tau(1),n+1}(u - \hbar m) \cdots T^*_{n+\tau(m),n+m}(u - \hbar)\,, \end{aligned}$$

obtaining:

$$Ber(u)\, v^+ = \prod_{l=1}^{n} \lambda_l(u - \hbar(m-l+1)) \prod_{l=n+1}^{m+n} \lambda^*_l(u - \hbar(m-l+n+1)) \prod_{l=m+n+1}^{m+2n} \lambda_l(u - \hbar(2m-l+1))\, v$$

In the same way we can compute the constant value of $Ber^{-1}(u)$ on the v module. Since

$$\begin{aligned} Ber^{-1}(u) &= \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) T^*_{\sigma(1),1}(u + \hbar(2n-1)) \cdots T^*_{\sigma(n),n}(u + \hbar n) \times \\ &\times \; T^*_{m+\sigma(n+1),m+n+1}(u + \hbar(n-1)) \cdots T^*_{m+\sigma(2n),m+2n}(u) \times \\ &\times \sum_{\tau \in S_m} \operatorname{sgn}(\tau) T_{n+\tau(1),n+1}(u - \hbar(m-2n)) \cdots T_{n+\tau(m),n+m}(u + \hbar(2n-1))\,, \end{aligned}$$

we get

$$Ber^{-1}(u)\, v^+ = \prod_{l=1}^{n} \lambda^*_l(u + \hbar(2n-l)) \prod_{l=n+1}^{m+n} \lambda_l(u + \hbar(2n-l+n)) \prod_{l=m+n+1}^{m+2n} \lambda^*_l(u + \hbar(m+2n-l))\, v^+\,.$$

The steps leading to the constraints on dressing functions can now be repeated as in the distuinguished Dynkin diagram case, taking into account the different form of the value of the quantum Berezinian: in particular, one can show that the constraints (3.17) and (3.18) are to be replaced with:

$$A_0(u) \cdots A_{n-1}(u + \hbar(n-1))\, A^*_n(u) \cdots A^*_{m+n-1}(u + \hbar(m-1)) \times$$

$$\times A_{m+n}(u+\hbar n)\cdots A_{m+2n-1}(u+\hbar(2n-1))=1\,, \tag{3.32}$$

and

$$A_0^*(u+\hbar(2n-1))\cdots A_{n-1}^*(u+\hbar n)\,A_n(u+\hbar(2n-1))\cdots A_{m+n-1}(u+\hbar(2n-m))\times$$
$$\times A_{m+n}^*(u+\hbar(n-1))\cdots A_{m+2n-1}^*(u)=1\,. \tag{3.33}$$

Both these constraints are satisfied by the dressing functions of remark 3.11. As a general rule, at each fermionic root two dressing functions A and A^* meet, and the $u_j^{(k)}$ parameters must satisfy an additional relation[1] *of the form $u_j^{*(k)}=u_j^{(k)}-\hbar m$. We are now in position to write the Bethe Ansatz equations for the symmetric Dynkin diagram, requiring the transfer matrix eigenvalue*

$$\Lambda(u)=\sum_{k=1}^{m+2n}(-1)^{[k]}A_{k-1}(u)\lambda_k(u)$$

to have vanishing residues at $u=u_j^{(l)}+\frac{\hbar}{2}c_l$ for $l=1,\ldots,m+2n-1$ and $j=1,\ldots,M^{(l)}$. The Bethe Ansatz equations take the form:

1. $1\le k<n$ *and* $m+n+1<k<m+2n$*:*

$$\prod_{l=1}^{M^{(k-1)}}\mathfrak{e}_{-1}(u_j^{(k)}-u_l^{(k-1)})\prod_{l\neq j}^{M^{(k)}}\mathfrak{e}_2(u_j^{(k)}-u_l^{(k)})\prod_{l=1}^{M^{(k+1)}}\mathfrak{e}_{-1}(u_j^{(k)}-u_l^{(k+1)})=\frac{\lambda_{k+1}(u_j^{(k)}+\frac{\hbar}{2}c_k)}{\lambda_k(u_j^{(k)}+\frac{\hbar}{2}c_k)}\,;$$

2. $k=n$*:*

$$\prod_{l=1}^{M^{(n-1)}}\mathfrak{e}_{-1}(u_j^{(n)}-u_l^{(n-1)})\prod_{l=1}^{M^{(n+1)}}\mathfrak{e}_1(u_j^{(n)}-u_l^{(n+1)})=\frac{\lambda_{n+1}(u_j^{(n)}+\frac{\hbar}{2}n)}{\lambda_n(u_j^{(n)}+\frac{\hbar}{2}n)}\,;$$

3. $n<k<m+n$*:*

$$\prod_{l=1}^{M^{(k-1)}}\mathfrak{e}_1(u_j^{(k)}-u_l^{(k-1)})\prod_{l\neq j}^{M^{(k)}}\mathfrak{e}_{-2}(u_j^{(k)}-u_l^{(k)})\prod_{l=1}^{M^{(k+1)}}\mathfrak{e}_1(u_j^{(k)}-u_l^{(k+1)})=\frac{\lambda_{k+1}(u_j^{(k)}+\frac{\hbar}{2}c_k)}{\lambda_k(u_j^{(l)}+\frac{\hbar}{2}c_k)}\,;$$

4. $k=m+n$*:*

$$\prod_{l=1}^{M^{(m+n-1)}}\mathfrak{e}_1(u_j^{(m+n)}-u_l^{(m+n-1)})\prod_{l=1}^{M^{(m+n+1)}}\mathfrak{e}_{-1}(u_j^{(m+n)}-u_l^{(m+n+1)})=\frac{\lambda_{m+n+1}(u_j^{(m+n)}+\frac{\hbar}{2}(n-m))}{\lambda_{m+n}(u_j^{(m+n)}+\frac{\hbar}{2}(n-m))}\,,$$

for j running from 1 to $M^{(k)}$ in each case.

[1] In the distinguished Dynkin diagram case there is only one fermionic root, corresponding to the $u_j^{*(m)}=u_j^{(m)}-\hbar m$ relation obtained in the previous section.

3.7 Examples

In this section we discuss the application of our approach to few examples. We will replace the $\hbar$ parameter with the imaginary unit $-i$, as it is customary in dealing with spin chains. Let us stress that, altough in the examples the energies will look identical (up to additive constants), the spectrum and the Hamiltonians are indeed different. In fact, the energies are function of the Bethe roots, which obey *different* Bethe Ansatz equations, specified by the representations entering the spin chain. Our first examples will deal with general situations to which our approach applies. They are to be considered universal, in the sense that the values of m and n will not be specified (except sometimes for the condition $m \neq n$): the properties of the resulting spectra and Hamiltonians will only rely on the general form of the Bethe equations. The last, more detailed, example will be related to the particular choice of the symmetry superalgebra $gl(1|2)$.

Example 3.16 (Fundamental representation of $gl(m|n)$) *Our first example deals with the basic situation in which the local operators acting on each site of a N–spin chain are written in the fundamental representation, which we shall denote π_f. This leads to the usual and widely studied super–spin chains. In our approach, the monodromy matrix with normalization (3.11) for this case is obtained by means of the simplest evaluation representation:*

$$ev_{\pi_f} \otimes \cdots \otimes ev_{\pi_f} \left(\mathcal{T}_{ij}(u)\right) ,$$

where

$$ev_{\pi_f} \left(T_{ij}(u)\right) = u\delta_{ij}1 - \hbar(-1)^{[j]} e_{ji} \,.$$

The highest weight of the fundamental representation is $\mu_f = (1, 0, \ldots, 0)$, and the eigenvalues of $\mathcal{T}_{kk}(u)$ read:

$$\lambda_k(u) = \begin{cases} (u+i)^N & k = 1\,, \\ u^N & k \neq 1\,. \end{cases} \tag{3.34}$$

Notice that the free complex parameters appearing in the evaluation representation have all been set equal to zero. Hence, up to the normalization, the local $T^{[k]}(u)$ coincide with the fundamental solution to the Yang–Baxter equation, so that we shall simply denote them with $R(u)$:

$$T(u) = u - \hbar P = R(u)\,.$$

The monodromy and transfer matrices read:

$$\mathcal{T}_a(u) = R_{a1}(u)R_{a2}(u)\cdots R_{aN}(u)\,, \tag{3.35}$$

$$st(u) = str_a \mathcal{T}_a(u) = \sum_{k=1}^{m+n} (-1)^{[k]} \mathcal{T}_{kk}(u)\,. \tag{3.36}$$

the index a referring to the auxiliary space. $R(u)$ being a regular solution of the Yang–Baxter equation :

$$R_{ab}(0) = -\hbar P_{ab}\,,$$

we can build a spin chain in the usual way. Since

$$\left.\frac{d}{du} R_{ab}(u)\right|_{u=0} = 1_{ab}\,,$$

the well–known formula

$$H = \frac{d}{du}\left(ln\ st(u)\right)\Big|_{u=0}\,. \tag{3.37}$$

leads to

$$H=\sum_{k=1}^{N}P_{k-1\,,k} \tag{3.38}$$

with the periodical boundary condition

$$P_{01}=P_{N1}$$

arising as a natural consequence of the ciclicity of the supertrace. The operator P_{ab} *permutes the* $m+n$ *configurations between the sites* a *and* b*, picking up a minus sign when both of the permuted configurations are fermionic. The Bethe equations establishing the analiticity of the supertrace eigenvalue*

$$\Lambda(u)=(u+i)^N\prod_{\ell=1}^{M^{(1)}}\frac{u-u_\ell^{(1)}-\frac{i}{2}}{u-u_\ell^{(1)}+\frac{i}{2}}+\cdots+u^N\prod_{\ell=1}^{M^{(m+n-1)}}\frac{u-u_\ell^{(m+n-1)}+i\left(\frac{m-n-1}{2}\right)}{u-u_\ell^{(m+n-1)}+i\left(\frac{m-n+1}{2}\right)}$$

are obtained plugging the eigenvalues (3.34) into the general Bethe equations of section 3.5. The results are:

$$\prod_{\ell\neq j}^{M^{(1)}}\mathfrak{e}_2(u_j^{(1)}-u_\ell^{(1)})\prod_{\ell=1}^{M^{(2)}}\mathfrak{e}_{-1}(u_j^{(1)}-u_\ell^{(2)})=\left(\frac{u_j^{(1)}+\frac{i}{2}}{u_j^{(1)}-\frac{i}{2}}\right)^N\,,\qquad j\leq M^{(1)}\,,$$

$$\prod_{\ell=1}^{M^{(k-1)}}\mathfrak{e}_{-1}(u_j^{(k)}-u_\ell^{(k-1)})\prod_{\ell\neq j}^{M^{(k)}}\mathfrak{e}_2(u_j^{(k)}-u_\ell^{(k)})\prod_{\ell=1}^{M^{(k+1)}}\mathfrak{e}_{-1}(u_j^{(k)}-u_\ell^{(k+1)})=1\,,\quad j\leq M^{(k)}\,,$$

$$\prod_{\ell=1}^{M^{(m-1)}}\mathfrak{e}_{-1}(u_j^{(m)}-u_\ell^{(m-1)})\prod_{\ell=1}^{M^{(m+1)}}\mathfrak{e}_1(u_j^{(m)}-u_\ell^{(m+1)})=1\,,\qquad 1\leq j\leq M^{(m)}\,,$$

where the second set of Bethe equations holds for $1<k<m+n$*, with* $k\neq m$*. The energies corresponding to the Hamiltonian (3.38) can be calculated by taking the logarithmic derivative of* $\Lambda(u)$ *and evaluating it at* $u=0$*, and are given by:*

$$E=N-\sum_{\ell=1}^{M^{(1)}}\frac{1}{(u_\ell^{(1)})^2+\frac{1}{4}}\,,$$

where $u_\ell^{(1)}$ *are the Bethe roots satisfying the above Bethe equations.*

Remark 3.17 *In the above example, the Bethe equations are written assuming that* $m\neq 1$*, otherwise the first and third set of equations would be in conflict: in that case, the first set of equations has to be eliminated, and the right hand side of the third set should be replaced with*

$$\left(\frac{u_j^{(1)}+\frac{i}{2}}{u_j^{(1)}-\frac{i}{2}}\right)^N\,.$$

We shall return on this case later, when dealing in full detail with the case of $gl(1|2)$*.*

Example 3.18 (Fundamental representation, one inhomogeneity) *A slightly more general case is obtained from the previous one by taking* $a_p=a\neq 0$ *in the evaluation representation correpsonding to the site* p*, leaving* $a_k=0$ *for* $k\neq p$*. The effect of this inhomogeneity is to* *modify the hamiltonian (3.38) (which is still obtained as a logarithmic derivative of $st(u)$) as follows:*

$$H = \sum_{\substack{k=1 \\ k\neq p,p+1}}^{N} P_{k,k-1} + \frac{1}{a^2+1}\left(a^2 P_{p-1,p+1} + P_{p+1,p} - iaP_{p+1,p-1}P_{p,p-1} + iaP_{p,p-1}P_{p+1,p-1}\right) .$$

The translational symmetry of the chain is broken, and next nearest neighbour terms appear around the inhomogeneity. The second and third sets of Bethe equations are as in example 3.16, while the right hand side of the first set gets modified as follows:

$$\prod_{\ell\neq j}^{M^{(1)}} \mathfrak{e}_2(u_j^{(1)} - u_\ell^{(1)}) \prod_{\ell=1}^{M^{(2)}} \mathfrak{e}_{-1}(u_j^{(1)} - u_\ell^{(2)}) = \frac{u_j^{(1)} + \frac{i}{2} + a}{u_j^{(1)} - \frac{i}{2} + a}\left(\frac{u_j^{(1)} + \frac{i}{2}}{u_j^{(1)} - \frac{i}{2}}\right)^{N-1} , \qquad j \leq M^{(1)} .$$

while the energies become

$$E = N - \frac{a}{a+i} - \sum_{\ell=1}^{M^{(1)}} \frac{1}{(u_\ell^{(1)})^2 + \frac14} .$$

Example 3.19 (Impurity) *Another case to which our formalism easily applies is the super–spin chain with one site (the so–called impurity) in a representation different from the others. The easiest case is again the spin chain where all sites are in the fundamental representation except for the p–th, associated to the highest weight $\mu_k^{[p]}$, $k = 1, \ldots, m+n$. The transfer for the N–site spin chain with one impurity can be written as*

$$st(u) = st_a\left(R_{a,N}(u) \cdots R_{a,p-1}(u)\, T_{ap}(u)\, R_{a,p+1}(u) \cdots R_{a,1}(u)\right) ,$$

$T_{ap}(u)$ being the local matrix in the auxiliary space a acting on the p–th quantum space. Its associated hamiltonian is

$$H = \sum_{\substack{k=1 \\ k\neq p,p-1}}^{N} P_{k,k+1} + i\, T^{-1}_{p+1,p}(0) + P_{p-1\,,p+1} T^{-1}_{p,p-1}(0) T_{p+1,p}(0) .$$

It is worth noticing that in this situation all but the p–th quantum spaces are isomorphic to the auxiliary space (the fundamental representation): hence, the local matrices $T_{kp}(u)$, $k \neq p$, and their inverses, are well–defined, and coincide with $T_{ap}(u)$ and $T_{ap}^{-1}(u)$. The spectrum of the hamiltonian (3.19) is given by

$$E = (N-1) + i\,\frac{\mu_1'(0)}{\mu_1(0)} - \sum_{\ell=1}^{M^{(1)}} \frac{1}{(u_\ell^{(1)})^2 + \frac14} . \tag{3.39}$$

The eigenvalues of the diagonal entries of the monodromy matrix appearing in the Bethe equa-

tions are as follows:

$$\begin{aligned}
\frac{\lambda_1(u_j^{(1)}-\frac{i}{2})}{\lambda_2(u_j^{(1)}-\frac{i}{2})} &= \frac{u_j^{(1)}-\frac{i}{2}-i\mu_1^{[p]}}{u_j^{(1)}-\frac{i}{2}-i\mu_2^{[p]}}\left(\frac{u_j^{(1)}+\frac{i}{2}}{u_j^{(1)}-\frac{i}{2}}\right)^{N-1}, \\
\frac{\lambda_k(u_j^{(k)}-i\frac{k}{2})}{\lambda_{k+1}(u_j^{(k)}-i\frac{k}{2})} &= \frac{u_j^{(k)}-i\frac{k}{2}-i\mu_k^{[p]}}{u_j^{(k)}-i\frac{k}{2}-i\mu_{k+1}^{[p]}}, \qquad 1<k<m, \\
\frac{\lambda_{m+1}(u_j^{(m)}-i\frac{m}{2})}{\lambda_m(u_j^{(m)}-i\frac{m}{2})} &= \frac{u_j^{(m)}-i\frac{m}{2}+i\mu_{m+1}^{[p]}}{u_j^{(m)}-i\frac{m}{2}-i\mu_m^{[p]}}, \\
\frac{\lambda_k(u_j^{(k)}-im+i\frac{k}{2})}{\lambda_{k+1}(u_j^{(m)}-im+i\frac{k}{2})} &= \frac{u_j^{(k)}-im+i\frac{k}{2}+i\mu_k^{[p]}}{u_j^{(k)}-im+i\frac{k}{2}-i\mu_{k+1}^{[p]}}, \qquad m+1\le k<m+n,
\end{aligned}$$

and the Bethe roots $u_\ell^{(1)}$, $\ell \le M^{(1)}$ *in eq.(3.39) satisfy the Bethe equations of section 3.5, with right hand sides given by the above ratios.*

Example 3.20 (Alternating spin chain) *In alternating spin chains (see e.g. [55] for a non–graded example), the spins belong alternatively to two different representations. As a particular example, one can take an even number of sites* N *for the chain, and let the spins in the even sites be in the fundamental representation, while the spins in the odd sites are in a different one. The transfer matrix for such a chain will then be given by*

$$st(u) = str_a\ (T_{a,1}(u)\,R_{a,2}(u)\cdots T_{a,N-1}(u)\,R_{a,N}(u))\ ; \tag{3.40}$$

here the auxiliary space a *is* $m+n$ *dimensional, and the matrices acting on the even sites correspond to fundamental representations (thus coinciding with the* R *matrix), while the ones acting on the odd sites are in a non–fundamental representation and are denoted by* $T(u)$. *From the transfer matrix (3.40) one gets a local hamiltonian:*

$$H = \sum_{k=1}^{N/2} T^{-1}_{2k-2,2k-1}(0)\Big(i + P_{2k-2,2k}T_{2k-2,2k-1}(0)\Big)\,.$$

Denoting with μ_k, $k=1,\dots,m+n$ *the weights of the representation on the odd sites, and* $\mu_k(u) = u + i(-1)^{[k]}\mu_k$, *one gets for the eigenvalues:*

$$\lambda_k(u) = \begin{cases} (u+i)^{N/2}\mu_1(u)^{N/2}\,, & k=1 \\ u^{N/2}\mu_k(u)^{N/2}\,, & 1<k\le m+n\,. \end{cases},$$

where we set $a_k = 0$ *for all* k. *This leads to the spectrum*

$$E = \frac{N}{2}\left(1 - \frac{\mu_1'(0)}{\mu_1(0)}\right) - \sum_{\ell=1}^{M^{(1)}} \frac{1}{(u_\ell^{(1)})^2 + \frac14}\,.$$

Choosing e.g. the adjoint representation for the odd sites, i.e. building a chain with highest weight $\mu_i^{[k]} = \delta_{i1}$ *for even* k, *and* $\mu_i^{[k]} = \delta_{i1} + \delta_{i,m+n}$ *for odd* k, *one gets the following form for the eigenvalues*

$$\lambda_k(u) = \begin{cases} \big(u+i\big)^N & k=1\,, \\ u^N & 1<k<m+n\,, \\ \big(u-i\big)^{N/2}u^{N/2}\,, & k=m+n\,, \end{cases}$$

where we set $a_i = 0$ for all i. The Bethe equations for $1 \le k \le m$ remain as in the fundamental representation case, while the equations for $m < k \le m+n-1$ are modified as follows:

$$\prod_{\ell=1}^{M^{(k-1)}} \mathfrak{e}_1(u_j^{(k)} - u_\ell^{(k-1)}) \prod_{\ell \neq j}^{M^{(k)}} \mathfrak{e}_{-2}(u_j^{(k)} - u_\ell^{(k)}) \prod_{\ell=1}^{M^{(k+1)}} \mathfrak{e}_1(u_j^{(k)} - u_\ell^{(k+1)}) =$$
$$= \begin{cases} 1 & , \; m < k < m+n-1 \,, \\ \left(\mathfrak{e}_1(u_j^{(k)} - i\frac{m-n}{2})\right)^{N/2} & , \; k = m+n-1 \,, \end{cases} \tag{3.41}$$

with $1 \le j \le M^{(k)}$.

Example 3.21 (Supersymmetric t-J model) *The most basic supersymmetric spin chain that has been proposed as a model for high–temperature superconductivity is the celebrated t-J model [36, 37, 38]*

$$H_{t-J} = -t \sum_{j=1}^{N} \mathcal{P} \sum_{\sigma=\uparrow,\downarrow} \left(c_{j,\sigma}^{\dagger} c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger} c_{j,\sigma}\right) \mathcal{P} + J \sum_{j=1}^{N} \left(\mathbf{S}_j \cdot \mathbf{S}_{j,j+1} - \frac{1}{4} n_j n_{j+1}\right) , \tag{3.42}$$

where $\mathcal{P}$ is the projector on the subspace of non–doubly occupied states. In our approach we recover its hamiltonian and spectrum at the supersymmetric point, as the fundamental representation case of $gl(1|2)$ (see example 3.16). It is indeed well known [39, 40] that, for $J = 2t = 2$, the hamiltonian (3.42) becomes globally $gl(1|2)$ invariant, and it is possible to rewrite it in terms of the graded permutation operator (up to an irrelevant shift):

$$H_{t-J}^{s} = N - 2N_e - \sum_{j=1}^{N} P_{j,j+1} \,, \tag{3.43}$$

where N_e is the number of fermions in the chain (a conserved charge[2], since $[H\,, N_e] = 0$). Hence, since there are three possible configurations for each site:

$$\phi_1 = |0\rangle \;, \qquad \phi_2 = |\uparrow\rangle = c_{\uparrow}^{\dagger}\,|0\rangle \;, \qquad \phi_3 = |\downarrow\rangle = c_{\downarrow}^{\dagger}\,|0\rangle \;,$$

with

$$\begin{aligned} &[\phi_1] = 0 \,, \\ &[\phi_2] = [\phi_3] = 1 \,, \end{aligned} \tag{3.44}$$

we can identify the Hamiltonian of the supersymmetric t-J model (3.43) with the fundamental graded chain of example 3.16 in the case of $m = 1$, $n = 2$, i.e. for the following normalized transfer matrix:

$$st(u) = str_a \mathcal{T}_a(u) = str_a \left(T_a^{[1]}(u) \cdots T_a^{[N]}(u)\right) ,$$

where

$$T_a^{[k]}(u) = R_{ak}(u) = \begin{pmatrix} u - \hbar e_{11}^{[k]} & \hbar e_{21}^{[k]} & \hbar e_{31}^{[k]} \\ -\hbar e_{12}^{[k]} & u + \hbar e_{22}^{[k]} & \hbar e_{32}^{[k]} \\ -\hbar e_{13}^{[k]} & -\hbar e_{23}^{[k]} & u + \hbar e_{33}^{[k]} \end{pmatrix} . \tag{3.45}$$

[2]The conserved $gl(1|2)$ charges of the t-J model can be identified, e.g., with the number of fermions and the third component of the total spin S^3.

Realizing the $gl(1|2)$ *generators in terms of canonical Fermi operators satisfying the anticommutation relations*

$$\{c^{\dagger}_{i,\sigma}\,, c_{j,\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'}\,,$$

and of the number operator $n_{i,\sigma} = c^{\dagger}_{i,\sigma}c_{i,\sigma}$ *for electrons with spin* σ *on site* i*, the local matrix (3.45) writes:*

$$T^{[k]}_a(u) = u - \hbar \begin{pmatrix} 1 - n_{k,\uparrow} - n_{k,\downarrow} & -c^{\dagger}_{\uparrow}(1-n_{k,\downarrow}) & -c^{\dagger}_{k,\downarrow}(1-n_{k,\uparrow}) \\ (1-n_{k,\downarrow})c_{k,\uparrow} & -n_{k,\uparrow} & c^{\dagger}_{k,\downarrow}c_{k,\uparrow} \\ (1-n_{k,\uparrow})c_{k,\downarrow} & c^{\dagger}_{k,\uparrow}c_{k,\downarrow} & -n_{k,\downarrow} \end{pmatrix}.$$

Since the case $m = 1$ *was not included in example 3.16 (see remark 3.17), we explicitly write down the Bethe equations for this case:*

$$\prod_{j=1}^{M^{(2)}} \frac{u^{(1)}_j - u^{(2)}_k + \frac{i}{2}}{u^{(1)}_j - u^{(2)}_k - \frac{i}{2}} = \left(\frac{u^{(1)}_k + \frac{i}{2}}{u^{(1)}_k - \frac{i}{2}}\right)^N, \qquad k \le M^{(1)}\,,$$
$$\prod_{j=1}^{M^{(1)}} \frac{u^{(2)}_j - u^{(1)}_k + \frac{i}{2}}{u^{(2)}_j - u^{(1)}_k - \frac{i}{2}} \prod_{j=1}^{M^{(2)}} \frac{u^{(2)}_j - u^{(2)}_k - i}{u^{(2)}_j - u^{(2)}_k + i} = 1\,, \qquad k \le M^{(2)}\,.$$

Due to our choice of the basis vectors and gradation (3.44), the pseudovacuum is the following eigenstate of the hamiltonian:

$$|\omega\rangle = \bigotimes_{i=1}^{N} |0\rangle_i\,.$$

The eigenvalues of the Cartan generators of the global $gl^{(N)}(1|2)$ *symmetry generators on the pseudovacuum can be read off from the asymptotic expansion* $u \sim \infty$ *of the diagonal entries of the* $\mathcal{T}(u)$ *matrix, and are given by:*

$$e^{(N)}_{11}\,|\omega\rangle = \sum_{k=1}^{N} e^{[k]}_{11}\,|\omega\rangle = N\,|\omega\rangle\,,$$
$$e^{(N)}_{22}\,|\omega\rangle = \sum_{k=1}^{N} e^{[k]}_{22}\,|\omega\rangle = 0\,,$$
$$e^{(N)}_{33}\,|\omega\rangle = \sum_{k=1}^{N} e^{[k]}_{33}\,|\omega\rangle = 0\,.$$

We can now identify the values of $M^{(1)}$ *and* $M^{(2)}$ *with the number of excitations of the chain with respect to the pseudovacuum, i.e. with the values of the conserved charges of the hamiltonian (3.43) on an eigenvector* $|v\rangle$*. Having in mind remark 3.12, we see that*

$$e^{(N)}_{11}\,|v\rangle = (N - M^{(1)})\,|v\rangle\,,$$
$$e^{(N)}_{22}\,|v\rangle = (M^{(1)} - M^{(2)})\,|v\rangle\,,$$
$$e^{(N)}_{33}\,|v\rangle = M^{(2)}\,|v\rangle\,,$$

with the condition

$$M^{(2)} \le M^{(1)}\,.$$

It is then easy to realize that $M^{(1)}$ has to be identified with the number of fermions N_e in the eigenstate $|v\rangle$, and $M^{(2)}$ with the total number of down spins $N_\downarrow$, so that the value of the third component of the total spin is simpliy

$$S^3 = \frac{M^{(1)}}{2} - M^{(2)}\,.$$

After rewriting the Bethe equations as

$$\prod_{j=1}^{N_\downarrow} \frac{u_j^{(1)} - u_k^{(2)} + \frac{i}{2}}{u_j^{(1)} - u_k^{(2)} - \frac{i}{2}} = \left(\frac{u_k^{(1)} + \frac{i}{2}}{u_k^{(1)} - \frac{i}{2}}\right)^N, \qquad k \le N_e\,,$$
$$\prod_{j=1}^{N_e} \frac{u_j^{(2)} - u_k^{(1)} + \frac{i}{2}}{u_j^{(2)} - u_k^{(1)} - \frac{i}{2}} = \prod_{j=1}^{N_\downarrow} \frac{u_j^{(2)} - u_k^{(2)} + i}{u_j^{(2)} - u_k^{(2)} - i}, \qquad k \le N_\downarrow\,,$$

we recognize the "generic Bethe equations" of Essler and Korepin. In their paper [41], they show they are equivalent to the original Schlottman and Lai formulation of the Bethe equations for the t-J model [37, 38]. The energy reads:

$$E = \sum_{k=1}^{N_e} \frac{1}{(u_k^{(1)})^2 + \frac{1}{4}} - N\,,$$

usually rewritten as

$$E = -2\sum_{\ell=1}^{N_e} \cos k_\ell + 2N_e - N\,,$$

through the parametrization $u_\ell^{(1)} = \frac{1}{2}\cot k_\ell$.

Example 3.22 (Fundamental–adjoint alternating spin chain: the $sl(1|2)$ case) *As a second specific example, we specialize the situation of example 3.20 to the case of $sl(1|2)$.*

We choose the fundamental representation for the even sites of the chain, and the adjoint for the odd ones, i.e. following expressions for the local matrices

$$R_{a,2j}(u) = u1_{a,2j} + i\,(\mathbf{e}_a \cdot \mathbf{e}_{2j})\,, \qquad j = 1, \ldots, \frac{N}{2}\,,$$
$$T_{a,2j+1}(u) = u1_{a,2j+1} + i\,(\mathbf{e}_a \cdot \mathcal{E}_{2j+1})\,, \qquad j = 0, \ldots, \frac{N}{2} - 1\,,$$

where $\mathbf{e}$ and $\mathcal{E}$ respectively denote the $sl(1|2)$ generators in the fundamental and adjoint representations. We choose the following basis for the fundamental representation

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad e_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

with grading

$$[e_1] = [e_2] = [e_5] = [e_8] = 0\,,$$
$$[e_3] = [e_4] = [e_6] = [e_7] = 1\,.$$

The matrix elements of the generators in the adjoint representation are given by:

$$(\mathcal{E}_i)_{jk} = C_{jik}\,,$$

C_{ijk} *being the structure constants of* $sl(1|2)$*. The inner product* $\cdot$ *is defined, as usual, in terms of the invariant, nondegenerate bilinear form* $K_{\alpha\beta}$ *on* $sl(1|2)$*:*

$$\mathbf{A}\cdot\mathbf{B} = \sum_{\alpha,\beta}(K^{-1})^{\alpha\beta}A_\alpha A_\beta\,,$$

where

$$K_{\alpha\beta} = str\ (e_\alpha e_\beta) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the $sl(1|2)$ *inner product reads*

$$\mathbf{e}\cdot\mathcal{E} = 2\,\mathbf{e}_1\,\mathcal{E}_1 - \frac{1}{2}\mathbf{e}_2\,\mathcal{E}_2 + \mathbf{e}_3\,\mathcal{E}_6 - \mathbf{e}_6\,\mathcal{E}_3 + \mathbf{e}_4\,\mathcal{E}_7 - \mathbf{e}_7\,\mathcal{E}_4 - \mathbf{e}_5\,\mathcal{E}_8 - \mathbf{e}_8\,\mathcal{E}_5\,.$$

Insertion of the above expressions into eq.(3.40) yields the transfer matrix for this model. The Hamiltonian involves nearest–neighbour and next–nearest–neighbour interaction terms and reads:

$$H = \sum_{j=1,j\ even}^{N/2} H^{(1)}_{j,j+1} + \sum_{j=1,j\ odd}^{N/2} H^{(2)}_{j-1,j,j+1}\,, \tag{3.46}$$

where

$$H^{(1)}_{j,j+1} = -\mathbf{e}_j\cdot\mathcal{E}_{j+1} - (\mathbf{e}_j\cdot\mathcal{E}_{j+1})^2\,, \tag{3.47}$$

$$H^{(2)}_{j-1,j,j+1} = -\left(\mathbf{e}_{j-1}\cdot\mathcal{E}_j\right)\left\{(\mathbf{e}_{j-1}\cdot\mathcal{E}_j) - 1\right\}\left(\mathbf{e}_{j-1}\cdot\mathbf{e}_{j+1}\right)\left(\mathbf{e}_{j-1}\cdot\mathcal{E}_j\right)\,. \tag{3.48}$$

4

Open spin chains

A relevant developement of the quantum inverse scattering theory was the introduction and solution of integrable systems on the finite interval with non–periodical boundary conditions on each end. Many efforts have been devoted to this issue, based on the pioneering approach of Sklyanin [58], and in the present chapter our aim is to build exactly solvable supersymmetric spin chains with non–trivial boundary conditions, and to sistematically apply to them the analytical Bethe Ansatz machinery presented in the previous chapter.

The spin chains we will be considering here are characterized by their *reflection matrices* K. These are numerical matrices obeying quadratic consistency equations with the R–matrix, with the generic abstract form $RKRK = KRKR$. These so–called quaternary relations, or reflection equations, first appeared in [57] and [58] in connection with integrable systems, while the algebraic structures related to them were discussed for instance in [53, 22]. Here we shall introduce the graded counterpart to these so called *reflection algebras* as subalgebras of $Y(m|n)$, as done in [53] for the $Y(n)$ case. This approach will allow us to apply several results of the previous chapter to the new situation of non periodical boundary conditions.

In [53], deep analogies were noted relating the (non graded) reflection algebras to the *twisted Yangians* introduced in [48] (see also [21] for a detailed exposition). These are, roughly speaking, subalgebras of the Yangian related to $o(n)$ and $sp(2n)$ ($osp(m|2n)$ in the graded case) in the same way as $Y(n)$ is related with $gl(n)$ ($gl(m|n)$ for the supersymmetric case): one writes the defining relations of the twisted Yangian in matrix form as a quaternary relation depending on a spectral parameter, and the first order expansion in $\hbar/u$ of the generators will turn out to be isomorphic to $osp(m|2n)$. We shall show how the reflection equations defining the twisted super Yangian lead, for non–trivial choices of the reflection matrix K, to a class of integrable boundary conditions different from the ones obtained from the reflection superalgebra.

The chapter is correspondingly divided in two main parts: in the first one, from section 4.1 to section 4.8, the reflection superalgebra is introduced as a subalgebra of $Y(m|n)$, and the corresponding monodromy and transfer matrices are described. As we shall see, the symmetry of the resulting integrable models will not be, however, the full reflection superalgebra but some subalgebra of it, unless K is chosen to be proportional to the identity matrix.

In the remaining part of the chapter, the twisted super Yangian $Y^{\theta_0,\varepsilon}(m|n)$ and the related commuting quantum transfer matrices are described. The Bethe equations for any representation and for any Dynkin diagram are derived through analytical Bethe Ansatz in the reflection algebra case, generalizing the results obtained in [54] to the case of arbitrary representations. The results obtained for the twisted super Yangian are less general, since we shall restrict ourselves to the most natural presentation of $Y^{\theta_0,\varepsilon}(m|n)$, i.e. the symmetric Dynkin diagram, and to diagonal boundary matrices.

Most of the results of this chapter are original. Open spin chains based on $gl(1|2)$ have been studied in detail in e.g. [68] and [64], while the $sl(m|n)$ case with diagonal K matrix, and with spins in the fundamental representation (but for any type of Dynkin diagram) has been done in [54]. The representation theory of the twisted super Yangian (with $K = 1$) is studied in [42], and non–graded integrable spin chains built from $Y^{\theta_0,\varepsilon}(n)$ are considered in [43]

4.1 The reflection algebra

Definition 4.1 (Reflection superalgebra) *The reflection superalgebra $\mathfrak{B}(m|n)$ is the graded associative algebra with unity $1_{\mathfrak{B}}$, and $\mathbb{Z}_2$–graded generators $\tilde{B}^{(k)}_{ab}$, $k \geq 0$, $1 \leq a, b \leq m+n$. We set $\tilde{B}^{(0)}_{ab} = \delta_{ab}\theta_a 1_{\mathfrak{B}}$, where*

$$\theta_a = \begin{cases} +1\,, & \ell_1 \leq a \leq \ell_2\,, \\ -1\,, & 1 \leq a < \ell_1\,,\ \ell_2 < a \leq m+n\,, \end{cases}$$

ℓ_1 and ℓ_2 being integer numbers characterizing the superalgebra. Introducing the even element of $\mathfrak{B}(m|n)[u^{-1}] \otimes End\,\mathbb{C}^{m|n}$:

$$\tilde{B}(u) = \sum_{a,b=1}^{m+n} \sum_{k\geq 0} \frac{\hbar^k}{u^k}\, \tilde{B}^{(k)}_{ab}\, e_{ab} = \sum_{k\geq 0} \frac{\hbar^k}{u^k}\, \tilde{B}^{(k)} = \sum_{a,b}^{m+n} \tilde{B}_{ab}(u)\, e_{ab}\,,$$

where $u \in \mathbb{C}$ is a formal variable, the defining relations of $\mathfrak{B}(m|n)$ are given by

$$R_{12}(u-v)\ \tilde{B}_1(u)\ R_{21}(u+v)\ \tilde{B}_2(v) = \tilde{B}_2(v)\ R_{12}(u+v)\ \tilde{B}_1(u)\ R_{21}(u-v)\,, \tag{4.1}$$

where $R(u)$ is the fundamental solution (2.10) to the graded Yang–Baxter equation, together with

$$\tilde{B}(u)\tilde{B}(-u) = 1_{\mathfrak{B}}\mathbb{I}\,. \tag{4.2}$$

Remark 4.2 *According to the above definition of the graded reflection algebra, which follows the non–graded one given in [53], the notation $\mathfrak{B}(m|n)$ should also contain the labels ℓ_1 and ℓ_2: we omit them for simplicity.*

Relation (4.1) corresponds to the original definition of the non–graded reflection algebra given by Sklyanin [58]. The unitarity relation (4.2), although absent from Sklyanin's formulation, allows one to show that $\mathfrak{B}(m|n)$ is a subalgebra of the graded Yangian $Y(m|n)$, and admits a simple interpretation in terms of unitarity conditions on the reflection matrices as we shall see below. Projecting the defining relation (4.1) on the auxiliary space matrix element $e_{ij} \otimes e_{kl}$, the supercommutation relations among the $\tilde{B}_{ij}(u)$ generators are obtained:

$$\begin{aligned}
\Big[\tilde{B}_{ij}(u)\,, \tilde{B}_{kl}(v)\Big] &= \frac{(-1)^{\eta(i,j,k)}\hbar}{u-v}\Big(\tilde{B}_{kj}(u)\tilde{B}_{il}(v) - \tilde{B}_{kj}(v)\tilde{B}_{il}(u)\Big) \\
&+ \frac{\hbar}{u+v}\Big((-1)^{[j]}\delta_{jk} \sum_{a=1}^{m+n} \tilde{B}_{ia}(u)\tilde{B}_{al}(v) - (-1)^{\eta(i,j,k)}\delta_{il} \sum_{a=1}^{m+n} \tilde{B}_{ka}(v)\tilde{B}_{aj}(u)\Big) \\
&- \frac{\hbar^2}{u^2-v^2}\delta_{ij}\Big(\sum_{a=1}^{m+n} \tilde{B}_{ka}(u)\tilde{B}_{al}(v) - \sum_{a=1}^{m+n} \tilde{B}_{ka}(v)\tilde{B}_{al}(u)\Big)\,,
\end{aligned} \tag{4.3}$$

where

$$\eta(i,j,k) = [i][j] + [i][k] + [j][k]\,.$$

In the following chapters we shall deal with a class of particular realizations of $\mathfrak{B}(m|n)$, defined by means of so–called reflection matrices $K(u)$. These are numerical matrices that will describe, as we shall see, the boundary conditions for open spin chains. They are solutions to the so–called *reflection equation*:

$$R_{12}(u-v)\ K_1(u)\ R_{21}(u+v)\ K_2(u) = K_2(v)\ R_{12}(u+v)\ K_1(u)\ R_{21}(u-v)\,. \tag{4.4}$$

Proposition 4.3 *The matrix*

$$B(u) = T(u)K(u)T^{-1}(-u)\,, \tag{4.5}$$

where $T(u)$ generates the Yangian $\mathcal{Y}(m|n)$, and $K(u)$ satisfies the reflection equation (4.4), obeys the defining relation of $\mathfrak{B}(m|n)$.

Proof: By definition of $B(u)$, we have

$$\begin{aligned} &R_{12}(u-v)B_1(u)R_{21}(u+v)B_2(v) = \\ &= R_{12}(u-v)T_1(u)K_1(u)T_1^{-1}(-u)R_{21}(u+v)T_2(v)K_2(v)T_2^{-1}(-v)\,. \end{aligned} \tag{4.6}$$

From the exchange relations (2.17) of $\mathcal{Y}(m|n)$ we find

$$T_1^{-1}(-u)R_{21}(u+v)T_2(v) = T_2(v)R_{21}(u+v)T_1^{-1}(-u)\,, \tag{4.7}$$

and, since the fact that K–matrices are numerical ones implies

$$[T_i(u)\,, K_j(v)]\;, \qquad i \neq j\,,$$

we can rewrite (4.6) as follows:

$$R_{12}(u-v)T_1(u)T_2(v)K_1(u)R_{12}(u+v)K_2(v)T_1^{-1}(-u)T_2^{-1}(-v)\,.$$

Using now the exchange relations together with the reflection equation (4.4), we bring the above expression to the form

$$T_2(v)T_1(u)K_2(v)R_{12}(u+v)K_1(u)T_2^{-1}(-v)T_1^{-1}(-u)R_{12}(u-v)\,,$$

that, thanks to eq.(4.7), coincides with

$$B_2(v)R_{21}(u+v)B_1(u)R_{12}(u-v)R_{21}(u+v)\,.$$

■

In [53], it is shown that the mapping

$$\varphi : \tilde{B}(u) \mapsto B(u)$$

is indeed an embedding of $\mathfrak{B}(m|n)$ into $\mathcal{Y}(m|n)$. Hereafter we will simply write the generators of $\mathfrak{B}(m|n)$ as $B_{ij}(u)$, identifying the reflection superalgebra with its realization (4.5). By rewriting the suppercommutation relations (4.3) in terms of the matrix elements of $T(u)$ and $K(u)$, one can see that $\mathfrak{B}(m|n)$ is a subalgebra of $\mathcal{Y}(m|n)$. Furhter, the following property holds:

Proposition 4.4 *The reflection algebra generated by $B(u)$ is a Hopf coideal of $\mathcal{Y}(m|n)$, i.e.*

$$\Delta\,(\mathfrak{B}) \subseteq \mathcal{Y}(m|n) \otimes \mathfrak{B}\,.$$

Proof: The coproduct Δ is a superalgebra homomorphism, so that

$$\Delta\,(B(u)) = \Delta\,(T(u))\,K(u)\Delta\left(T^{-1}(-u)\right)\,.$$

Writing in components the above relation, we see that

$$\Delta\;(B_{ij}(u)) = \sum_{l,k=1}^{m+n} (-1)^{([k]+[j])([k]+[l])} T_{il}(u)T'_{kj}(-u) \otimes B_{lk}(u)\,,$$

proving the proposition. ■

4.2 Solutions to the reflection equation

As we have said, in the framework of open spin chains, the matrix $K(u)$ will be related to the boundary conditions and to the symmetry of the model. Hence, a relevant task in this context is to classify the solution to the reflection equation (4.4). As far as the super Yangian of $gl(m|n)$ is concerned, this classification has been completed in [54]. In the following proposition we summarize the results, referring to the original work for more details.

Proposition 4.5 *Any invertible solution of the soliton preserving reflection equation (4.4) takes the form* $K(u) = U\left(\mathbb{E} + \frac{\xi}{u}\mathbb{I}\right)U^{-1}$ *where either*

1. $\mathbb{E}$ *is diagonal and* $\mathbb{E}^2 = \mathbb{I}$ *(diagonalizable solutions)*
2. $\mathbb{E}$ *is strictly triangular and* $\mathbb{E}^2 = 0$ *(non–diagonalizable solutions)*

The matrix U is an element of the group $GL(m) \times GL(n)$; ξ is a free parameter, and the classification is done up to multiplication by a function of the spectral parameter.

In this work, we will restrict to the case of diagonalizable solutions. The possible matrices $\mathbb{E}$ are then labeled by the integers ℓ_1 and ℓ_2, $0 \leq \ell_1 \leq \ell_2 \leq m+n$, which count the number of -1 on the diagonal of $\mathbb{E}$:

$$\mathbb{E} = \mathrm{diag}\,(\underbrace{-1,\ldots,-1}_{\ell_1},\underbrace{1,\ldots,1}_{\ell_2-\ell_1},\underbrace{-1,\ldots,-1}_{m+n-\ell_2}) \equiv \mathrm{diag}\,(\theta_1,\ldots,\theta_{m+n})\,.$$

In the following, we will choose the normalization of the resulting reflection matrix in such a way that its entries are analytical:

$$K(u) = \mathrm{diag}\,(\underbrace{\xi-u,\ldots,\xi-u}_{\ell_1\text{ terms}},\underbrace{u+\xi,\ldots,u+\xi}_{\ell_2-\ell_1\text{terms}},\xi-u,\ldots,\xi-u)\,. \tag{4.8}$$

Let us stress that the diagonalization matrix U being constant, it is sufficient to consider diagonal $K(u)$ matrices: the other cases are recovered by a conjugation $T(u) \to U^{-1}\,T(u)\,U$ on each site of the chain, which does not affect the reflection algebra, nor the transfer matrix. However, the algebraic structure of $\mathfrak{B}$ does depend on the choice for $K(u)$, through the labels ℓ_1 and ℓ_2. A particular consequence, relevant for the study of the symmetry of spin chains, of this feature is discussed in the following proposition:

Proposition 4.6 *For (i,j) such that $\theta_i = \theta_j$, the generators $(-1)^{[i]}\frac{1}{\theta_i+\theta_j}B^{(1)}_{ij}$ span a sub-superalgebra of $\mathfrak{B}(m|n)$ isomorphic to one of the following finite–dimensional superalgebras:*

1. $gl(m-\ell_1|\ell_2-m) \oplus gl(\ell_1|m+n-\ell_2)$, *for* $\ell_1 \leq m \leq \ell_2$
2. $gl(\ell_2-\ell_1) \oplus gl(m|n-\ell_2+\ell_1)$, *for* $m \leq \ell_1 \leq \ell_2$
3. $gl(\ell_2-\ell_1) \oplus gl(m-\ell_2+\ell_1|n)$, *for* $\ell_1 \leq \ell_2 \leq m$.

Proof: Expanding relation (4.5) in powers of u^{-1}, and picking up the term corresponding to $B^{(1)}_{ij}$, we find

$$B^{(1)}_{ij} = -\frac{\xi}{\hbar}\delta_{ij} + (\theta_i+\theta_j)T^{(1)}_{ij}\,.$$

Taking into account that the set $\{(-1)^{[i]}T^{(1)}_{ij}\,|\,1 \leq i,j \leq m+n\}$ generates a $gl(m|n)$ sub-superalgebra of the Yangian, the proof ends with an enumeration of the different possible choices for ℓ_1 and ℓ_2, and the $gl(m|n)$ subalgebras they lead to. ■

We present now a construction of highest weight representations of the reflection superalgebras based on the super-Yangian, and sharing the same highest weight vector. This construction will be used later on to build open spin chains. However, a complete classification, similar to the one done in [53] for reflection algebras based on the Yangian $\mathcal{Y}(n)$, remains to be done.

Proposition 4.7 *The vector v is a highest weight vector for the representations of the reflection algebra obtained from the representation (2.21) of $\mathcal{Y}(m|n)$. In particular, one has:*

$$B_{kl}(u)\, v = 0\,, \qquad 1 \le l < k \le m+n\,, \tag{4.9}$$

$$B_{kk}(u)\, v = \frac{2u}{2u-\hbar c_{k-1}}\, g_k(u)\, \lambda_k(u)\, \lambda'_k(-u)\, v - \sum_{j=1}^{k-1} g_j(u)\, a_j(u)\, v\,,\ 1 \le k \le m+n\,, \tag{4.10}$$

where $c_k = \sum_{a=1}^{k} (-1)^{[a]}$ and

$$g_k(u) = \begin{cases} (\xi - u)\,, & \text{if} \quad 1 \le k \le \ell_1 \\ (\xi + u - \hbar c_{\ell_1})\,, & \text{if} \quad \ell_1 < k \le \ell_2\,, \\ (\xi - u - \hbar(c_{\ell_1} - c_{\ell_2}))\,, & \text{if} \quad \ell_2 < k \le m+n\,, \end{cases} \tag{4.11}$$

$$a_k(u) = (-1)^{[k]}\, \hbar\, \frac{2u\lambda_k(u)\lambda'_k(-u)}{(2u-\hbar c_k)(2u-\hbar c_{k-1})}\,. \tag{4.12}$$

<u>Proof:</u> We start writing, for $k > l$,

$$B_{kl}(u)\, v = \sum_{j=1}^{l} T_{kj}(u)\, K_{jj}(u)\, T'_{jl}(-u)\, v = \sum_{j=1}^{l} K_{jj}(u)\, \left[T_{kj}(u)\,, T'_{jl}(-u)\right]\, v\,. \tag{4.13}$$

From the commutation relations, we find for $a \le l < k$

$$[T_{ka}(u)\,, T'_{al}(-u)]\ v = (-1)^{[a]}\frac{\hbar}{2u}\sum_{b=1}^{l} T_{kb}(u)\, T'_{bl}(-u)\, v\,, \tag{4.14}$$

Considering the case $a = l$, we see that the l.h.s. of (4.14) vanishes, so that

$$\sum_{b=1}^{l} T_{kb}(u)\, T'_{bl}(-u)\, v = 0\,.$$

Hence the right hand side of eq. (4.13) also vanishes, proving (4.9).
We now turn to the case $l = k$, i.e. to the eigenvalues of $B_{kk}(u)$ on v. We start defining

$$f_a(u) \doteq \sum_{k=1}^{a} T'_{ak}(-u)\, T_{ka}(u)\, v \quad \text{and} \quad \Psi_i(u) \doteq \sum_{k=1}^{i} T_{ik}(u)\, T'_{ki}(-u)\, v\,.$$

The supercommutation relations applied to these definitions imply

$$\begin{cases} f_a(u) = \dfrac{1}{2u-\hbar\, c_{a-1}}\left(2u\lambda_a(u)\lambda'_a(-u)\, v - \hbar\displaystyle\sum_{k=1}^{a-1}(-1)^{[k]}\Psi_k(u)\right) \\ \Psi_a(u) = \dfrac{1}{2u-\hbar\, c_{a-1}}\left(2u\lambda_a(u)\lambda'_a(-u)\, v - \hbar\displaystyle\sum_{k=1}^{a-1}(-1)^{[k]} f_k(u)\right)\,, \end{cases} \tag{4.15}$$

for $a = 1, \dots, m+n$. Since $f_1(u) = \Psi_1(u) = \lambda_1(u)\lambda'_1(-u)\, v$, the system (4.15) has a unique solution $f_a(u) = \Psi_a(u)$, so we can rewrite the expression of $f_a(u)$ as

$$\left(1 - \frac{\hbar}{2u}\, c_{a-1}\right) f_a(u) = \lambda_a(u)\lambda'_a(-u)\, v - \frac{\hbar}{2u} \sum_{k=1}^{a-1} (-1)^{[k]} f_k(u)\,. \tag{4.16}$$

Eq.(4.16) is a triangular linear system in the unknowns $f_a(u)$ whose unique solution can be written as:

$$f_j(u) = \frac{\lambda_j(u)\lambda'_j(-u)}{1 - \frac{\hbar}{2u}\, c_{j-1}}\, v - \sum_{l=1}^{j-1} \frac{(-1)^{[l]}\hbar\lambda_l(u)\lambda'_l(-u)}{2u\left(1 - \frac{\hbar}{2u}\, c_l\right)\left(1 - \frac{\hbar}{2u}\, c_{l-1}\right)}\, v = \frac{\lambda_j(u)\lambda'_j(-u)}{1 - \frac{\hbar}{2u}\, c_{j-1}}\, v - \sum_{l=1}^{j-1} a_l(u)\, v\,. \tag{4.17}$$

Using this expression it is now clear that for $j \le \ell_1$ we can write:

$$B_{jj}(u)\, v = (\xi - u) f_j(u) = \left(\frac{2u(\xi - u)\lambda_j(u)\lambda'_j(-u)}{2u - \hbar c_{j-1}} - (\xi - u) \sum_{k=1}^{j-1} a_k(u) \right) v\,.$$

For $\ell_1 < j \le \ell_2$ we have

$$\begin{aligned} B_{jj}(u)\, v &= (\xi + u) f_j(u) - 2u \sum_{k=1}^{\ell_1} T_{jk}(u) T'_{kj}(-u)\, v \\ &= (\xi + u - \hbar c_{\ell_1}) f_j(u) + \hbar \sum_{k=1}^{\ell_1} (-1)^{[k]} f_k(u)\,, \end{aligned} \tag{4.18}$$

where to get the last equality we have used supercommutation relations on $T_{jk}(u)T'_{kj}(-u)$. Using now eq. (4.17), we get

$$\hbar \sum_{k=1}^{\ell_1} (-1)^{[k]} f_k(u) = (2u - \hbar c_{\ell_1}) \sum_{k=1}^{\ell_1} a_k(u)\, v\,.$$

Substituting the above equation in eq. (4.18), we get the required result.
An analogous calculation for the $j > \ell_2$ case leads to (4.11). ■

4.3 Monodromy and transfer matrices

We shall now build the monodromy matrix for open spin chain models, exploiting the algebraic properties of the reflection algebra. The monodromy matrix on the auxiliary space a for an N–site open chain with boundary matrix $K(u)$ is defined as follows:

$$\mathcal{B}_a(u) = \Delta^{(N)}(B_a(u)) = \Delta^{(N)}(T_a(u))\, K_a(u)\, \Delta^{(N)}(T_a^{-1}(-u))\,, \tag{4.19}$$

i.e.

$$\mathcal{B}(u) = (T(u) \otimes \cdots \otimes T(u))\, K(u) \left(T^{-1}(-u) \otimes \cdots \otimes T^{-1}(-u)\right)\,. \tag{4.20}$$

The monodromy matrix for a specific model is again obtained taking evaluation representations on each factor of the tensor product (4.20). Using the notation introduced in remark 3.1, the matrix elements of $\mathcal{B}(u)$ read:

$$\mathcal{B}_{ij}(u) = \sum_{k_1,\dots,k_{(N-1)}} \sum_{j_1,\dots,j_{(N-1)}} \sum_{k,l} T^{[1]}_{ik_1}(u) \cdots T^{[N]}_{k_{(N-1)}k}(u)\, K_{kl}(u)\, T'^{[N]}_{lj_1}(-u) \cdots T'^{[1]}_{j_{(N-1)}j}(-u)\,,$$

for $1 \leq i, j \leq m+n$. Thanks to the ultralocality

$$\left[B^{[k]}(u)\,, B^{[l]}(u)\right] = 0\,, \qquad k \neq l$$

and to the homomorphism property of the coproduct, it obeys

$$R_{ab}(u_a - u_b)\mathcal{B}_a(u_a)R_{ba}(u_a + u_b)\mathcal{B}_b(u_b) = \mathcal{B}_b(u_b)R_{ab}(u_a + u_b)\mathcal{B}_a(u_a)R_{ba}(u_a - u_b)\,. \quad (4.21)$$

The transfer matrix associated with the monodromy (4.19) is defined as:

$$b(u) = str\left(K^+(u)\,\mathcal{B}(u)\right) = \sum_{k,l=1}^{m+n} (-1)^{[k]} K^+_{kl}(u)\,\mathcal{B}_{lk}(u)\,, \quad (4.22)$$

where the left boundary matrix $K^+(u)$ has been introduced. In order to preserve integrability, a sufficient condition is that $K^+(u)$ satisfy a *dual* reflection equation, as we will show in the next proposition.

Proposition 4.8 *If $K^+(u)$ satisfies the* dual *reflection equation:*

$$\begin{aligned} R_{12}(-u+v)\ K^+_1(u)^t\ R_{21}(-u-v+\hbar(m-n))\ K^+_2(v)^t = \\ K^+_2(v)^t\ R_{12}(-u-v+\hbar(m-n))\ K^+_1(u)^t\ R_{21}(-u+v)\,, \end{aligned} \quad (4.23)$$

the transfer matrix (4.22) generates a family of commuting observables

$$[b(u)\,, b(v)] = 0\,,$$

Proof: The proof is an application to graded case of the original Sklyanin's argument ([58]): one starts writing

$$\begin{aligned} b(u)b(v) &= str_1\left\{K^+_1(u)\mathcal{B}_1(u)\right\}\, str_2\left\{K^+_2(v)\mathcal{B}_2(v)\right\} = \\ &= str_{1,2}\left\{K^+_1(u)^{t_1}K^+_2(v)\mathcal{B}^{t_1}_1(u)\mathcal{B}_2(v)\right\}\,. \end{aligned}$$

We now insert the identity

$$\frac{1}{\rho(u+v)} R^{t_1}_{12}(-u-v+\hbar(m-n))\, R^{t_1}(u+v) = 1_{12}\,,$$

where $\rho(u) = -u(u+\hbar(m-n))$, in the expression for $b(u)b(v)$, bringing it to the form

$$\frac{1}{\rho(u+v)} str_{1,2}\left\{\left[K^+_1(u)^{t_1}R_{12}(-u-v+\hbar(m-n))K^+_2(v)^{t_2}\right]^{t_2}\left[\mathcal{B}_1(u)R_{12}(u+v)\mathcal{B}_2(v)\right]^{t_1}\right\} =$$

$$= \frac{1}{\rho(u+v)} str_{1,2}\left\{\left[K^+_1(u)^{t_1}R_{12}(-u-v+\hbar(m-n))K^+_2(v)^{t_2}\right]^{t_1 t_2}\mathcal{B}_1(u)R_{12}(u+v)\mathcal{B}_2(v)\right\}\,,$$

where to get the right hand side the ciclicity property of the supertrace has been used. Inserting now $R_{12}(v-u)R_{12}(u-v) = \zeta(u-v)1_{12}$ after the term in square brackets, and using the shorthand notation

$$\begin{aligned} u^+ &= u+v\,, \\ u^- &= u-v\,, \end{aligned}$$

we obtain

$$\begin{aligned} b(u)b(v) &\propto str_{1,2}\left\{\left[R_{12}(-u^-)K_1^+(u)^{t_1}R_{12}(-u^+ + \hbar(m-n))K_2^+(v)^{t_2}\right]^{t_1t_2}\right. \\ &\times \left. R_{12}(u^-)\mathcal{B}_1(u)R_{12}(u^+)\mathcal{B}_2(v)\right\}\,. \end{aligned}$$

It remains to apply eq.(4.21) and the dual reflection equation satisfied by $K^+(u)$ to reverse the order of factors in the above expression and repeat the whole chain of transformations in reverse order. In the end one arrives to $b(v)b(u)$, showing that $[b(u)\,, b(v)] = 0$. ■

The classification of the left boundary matrices $K^+(u)$ can now be deduced from proposition 4.5, by simply observing that the solutions of eq.(4.4) are in one to one correspondence with the ones of the dual reflection equation. Indeed, if $K(u)$ obeys the one, then $K(-u + \frac{\hbar}{2}(m-n))^t$ will satisfy the other. As a consequence, following the notation of proposition 4.5, we shall write

$$K^+(u) = U'\left(\mathbb{E}' + \frac{\xi'}{u}\mathbb{I}\right)U'^{-1}\,,$$

for some new complex parameter ξ', and matrices U' and $\mathbb{E}'$. We further assume that the matrix $K^+(u)$ commutes with $K^-(v)$. Then, all the matrices $K^\pm(u)$ are diagonalizable by conjugation with the same matrix U independent on the spectral parameter. Thus, we can assume that $K^+(u)$ is also diagonal and analytic:

$$K^+(u) = \text{diag}\,(\underbrace{\xi' - u, \dots, \xi' - u}_{\ell_1'}, \underbrace{u + \xi', \dots, u + \xi'}_{\ell_2' - \ell_1'}, \xi' - u, \dots, \xi' - u)\,. \tag{4.24}$$

In order to get analytical entries for the transfer matrix, thus allowing the analytical Bethe Ansatz treatment, we adopt the representation (3.10) and the normalization (3.11) for $T(u)$ and $\mathcal{T}(u)$, and define:

$$\vartheta(u) = \prod_{k=1}^{m+n}\prod_{i=1}^{N}\left(u + a_i - \hbar(c_k + (-1)^{[k]}\mu_k^{[i]})\right)\,, \tag{4.25}$$

and

$$\widehat{\mathcal{B}}(u) = \vartheta(u)\mathcal{B}(u)\,. \tag{4.26}$$

As can be checked by means of eq.(2.43), the normalized transfer matrix

$$\widehat{b}(u) = str\left(K^+(u)\widehat{\mathcal{B}}(u)\right)$$

in such a representation is analytical in u; in particular, as we will see in section 4.5 its eigenvalues on the pseudo–vacuum v^+ are analytical.

4.4 Symmetry of the transfer matrix

As we did in the previous chapter for the closed chain case, we now turn to determine the symmetry of the model whose transfer matrix is given by (4.22). For simplicity, we assume in what follows that $\ell_1 < m$ and $\ell_2 > m$.

Proposition 4.9 *We consider the transfer matrix $b(u)$ describing open spin chain models with boundary conditions given by $K(u)$ and $K^+(u)$, see eq. (4.8) and (4.24), with $\ell_1, \ell_1' < m$ and $\ell_2, \ell_2' > m$. Let*

$$\mathfrak{m}_j = \min(\ell_j, \ell_j') \quad \textit{and} \quad \mathfrak{M}_j = \max(\ell_j, \ell_j')\,, \qquad j = 1, 2\,.$$

Then, $b(u)$ admits a $gl(\mathfrak{m}_1|m+n-\mathfrak{m}_2)\oplus\mathcal{G}\oplus gl(m-\mathfrak{M}_1|\mathfrak{M}_2-m)$ symmetry, where

$$\mathcal{G}=\begin{cases} gl(\mathfrak{M}_1-\mathfrak{m}_1)\oplus gl(\mathfrak{M}_2-\mathfrak{m}_2)\,, & \textit{if } (\mathfrak{m}_1\,,\mathfrak{m}_2)=(\ell_1\,,\ell_2) \textit{ or } (\mathfrak{m}_1\,,\mathfrak{m}_2)=(\ell'_1\,,\ell'_2)\,,\\[2ex] gl(\mathfrak{M}_1-\mathfrak{m}_1|\mathfrak{M}_2-\mathfrak{m}_2) & \textit{otherwise}\,.\end{cases}$$

Proof: Supertracing in the first auxiliary space the supercommutation relations (4.3), and expanding them in u and v, one reads, from the v^1 order term

$$\Big[b(u)\,,B_{ij}^{(1)}\Big\}=-B_{ij}(u)(K_{ii}^+(u)-K_{jj}^+(u))(\theta_i+\theta_j)\,. \tag{4.27}$$

Using the expression of $B_{ij}^{(1)}$ computed in proposition 4.6, it is seen that $T_{ij}^{(1)}$ commutes with $b(u)$ when $\theta_i=\theta_j$ (otherwise the left hand side of eq.(4.27)identically vanishes) and, *at the same time*, $K_{ii}^+(u)=K_{jj}^+(u)$ (i.e. $\theta'_i=\theta'_j$). The proof ends with an enumeration of the cases satisfying the above conditions. ■

4.5 Pseudovacuum

The pseudovacuum vector is built in the same way as in the closed chain case, by taking $N-$fold tensor product of the highest weight vectors $v_1,\dots v_N$ for the local $T^{[1]}(u),\dots,T^{[N]}(u)$ appearing in the monodromy matrix:

$$v^+=v_1\otimes v_2\otimes\cdots\otimes v_N\,.$$

This naturally leads to an highest weight vector for the normalized monodromy matrix $\widehat{\mathcal{B}}(u)$, thanks to the homomorphism property of $\Delta^{(N)}$:

$$\begin{aligned}&\widehat{\mathcal{B}}_{ij}(u)\,v^+=0\,,\quad 1\le j<i\le m+n\,,\\ &\widehat{\mathcal{B}}_{kk}(u)\,v^+=\eta_k(u)\,v^+\,,\qquad 1\le k\le m+n\,,\end{aligned} \tag{4.28}$$

for some scalar functions $\eta_k(u)$, $k=1,\dots,m+n$, whose values are calculated in the next proposition.

Proposition 4.10 *The eigenvalues of the diagonal entries of $\widehat{\mathcal{B}}(u)$ on the pseudovacuum v^+ are given, for $1\le k\le m+n$, by*

$$\widehat{\mathcal{B}}_{kk}(u)\,v^+=\frac{2u}{2u-\hbar c_{k-1}}\,g_k(u)\,\beta_k(u)\,v^+-\sum_{j=1}^{k-1}g_j(u)\,a_j(u)\,v^+\,, \tag{4.29}$$

where

$$a_k(u)=(-1)^{[k]}\,\hbar\,\frac{2u\,\beta_k(u)}{(2u-\hbar c_k)(2u-\hbar c_{k-1})}\,,$$

the $g_k(u)$'s are the boundary–dependent functions defined in proposition 4.7 and

$$\beta_k(u)=\left(\prod_{l=1}^{k-1}\lambda_l(-u+\hbar c_l)\right)\lambda_k(u)\left(\prod_{l=k+1}^{m+n}\lambda_l(-u+\hbar c_{l-1})\right)\,. \tag{4.30}$$

Proof: A straightforward calculation, based on eq.2.43, gives the expression (4.30) for the normalized products $\lambda_k(u)\lambda'_k(-u)$. The proof is then identical to the one of proposition 4.7. ■

Remark 4.11 *The functions $\beta_k(u)$ are determined by the representations on the chain, while the $g_k(u)$ only depend on the boundary matrix $K^-(u)$.*

Remark 4.12 *As in the closed spin chain case, the global $\lambda_k(u)$'s in expression (4.30) are the products of the corresponding local eigenvalues:*

$$\lambda_k(u) = \prod_{i=1}^{N} \lambda_k^{[i]}(u)\,, \quad k = 1, \ldots, m+n\,.$$

As an easy corollary, we obtain the eigenvalue of $b(u)$ on the pseudovacuum, that will be used as the starting point for our dressing hypothesis.

Corollary 4.13 *The highest weight vector v^+ is an eigenvector of $\widehat{b}(u)$:*

$$\widehat{b}(u)\, v^+ = \sum_{j=1}^{m+n} (-1)^{[j]} K^+_{jj}(u)\, \widehat{\mathcal{B}}_{jj}(u)\, v^+ = \Lambda_0(u)\, v^+\,,$$

with eigenvalue

$$\Lambda_0(u) = \sum_{j=1}^{m+n} (-1)^{[j]}\, \gamma_j(u)\, \beta_j(u)\,.$$

The functions $\gamma_j(u)$, $j = 1, \ldots, m+n$, depend only on the boundary matrix:

$$\gamma_j(u) = \frac{2u(2u - \hbar(m-n))}{(2u - \hbar c_{j-1})(2u - \hbar c_j)} K_{jj}(u)\, K^+_{jj}(u)\,. \tag{4.31}$$

Remark 4.14 *It is important to notice that, despite the presence of poles in the $\gamma_j(u)$ at the values $u = \hbar\, c_l/2$, $l = 1, \ldots, m+n-1$, the pseudovacuum eigenvalue $\Lambda_0(u)$ is analytical in the spectral parameter. Explicitly, the residues of $\Lambda_0(u)$ read:*

$$\mathrm{Res}\, \Lambda_0(u)|_{u=\hbar\frac{c_l}{2}} = \hbar\, c_l(c_l - m + n) \left[\beta_l\left(\frac{\hbar}{2} c_l\right) - \beta_{l+1}\left(\frac{\hbar}{2} c_l\right)\right]\,,$$

but the term in square brackets vanishes, because eq.(4.30) implies

$$\beta_l\left(\frac{\hbar}{2} c_l\right) = \beta_{l+1}\left(\frac{\hbar}{2} c_l\right)\,, \qquad l = 1, \ldots, m+n-1\,.$$

4.6 Dressing functions for the open chain

The starting hypothesis of the analytical Bethe Ansatz is that all the eigenvalues of $b(u)$ can be written

$$\Lambda(u) = \sum_{k=1}^{m+n} (-1)^{[k]}\, \gamma_k(u)\, \beta_k(u)\, A_{k-1}(u)\,, \tag{4.32}$$

with $\gamma_k(u)$ and $\beta_k(u)$ given by (4.72) and (4.30) respectively, and dressing functions $A_k(u)$ to be determined. Guided by the results of the previous chapter, taking into account that exchange relations for the reflection algebra generators always involve both $R(u-v)$ and $R(u+v)$, and having in mind the algebraic Bethe Ansatz construction for the transfer matrix eigenvectors, we conclude that that the corresponding eigenvalues consist of terms of the form $f(u-u_j)f(u+u_j)$. Hence, we assume the following form for the dressing functions in (4.32):

$$\begin{aligned} A_k(u) &= \prod_{j=1}^{M^{(k)}} \frac{u-v_j^{(k)}}{u-u_j^{(k)}-\frac{\hbar}{2}\,c_k}\;\frac{u-w_j^{(k)}}{u+u_j^{(k)}-\frac{\hbar}{2}\,c_k} \\ &\times \prod_{j=1}^{M^{(k+1)}} \frac{u-\alpha_j^{(k+1)}}{u-u_j^{(k+1)}-\frac{\hbar}{2}\,c_{k+1}}\;\frac{u-\beta_j^{(k+1)}}{u+u_j^{(k+1)}-\frac{\hbar}{2}\,c_{k+1}}\,, \end{aligned}$$

where, as usual, $M^{(0)}=M^{(m+n)}=1$ by convention. The parameters $v_j^{(k)}$, $w_j^{(k)}$, $\alpha_j^{(k)}$ and $\beta_j^{(k)}$ can be determined taking into account that the vanishing of the residues of $\Lambda(u)$ at $u=\frac{\hbar}{2}c_k$ implies

$$A_{k-1}(\frac{\hbar}{2}c_k)=A_k(\frac{\hbar}{2}c_k)\,,\qquad \text{for}\quad 1\le k\le m-n-1\,. \tag{4.33}$$

The simplest non–trivial way to satisfy the constraint (4.33) is to set

$$\begin{aligned} v_j^{(k)} &= u_j^{(k)}+\frac{\hbar}{2}\left(c_{k+1}+(-1)^{[k+1]}\right)\,, \\ w_j^{(k)} &= -u_j^{(k)}+\frac{\hbar}{2}\left(c_{k+1}+(-1)^{[k+1]}\right)\,, \end{aligned}$$

and

$$\begin{aligned} \alpha_j^{(k)} &= u_j^{(k)}+\frac{\hbar}{2}\left(c_{k-1}-(-1)^{[k]}\right)\,, \\ \beta_j^{(k)} &= -u_j^{(k)}+\frac{\hbar}{2}\left(c_{k-1}-(-1)^{[k]}\right)\,, \end{aligned}$$

for $k=1,\ldots,m+n$ and $j\le M^{(k)}$. The resulting dressing functions read:

$$\begin{aligned} A_k(u) &= \prod_{j=1}^{M^{(k)}} \frac{u-u_j^{(k)}-\frac{\hbar}{2}\left(c_{k+1}+(-1)^{[k+1]}\right)}{u-u_j^{(k)}-\frac{\hbar}{2}\,c_k}\;\frac{u+u_j^{(k)}-\frac{\hbar}{2}\left(c_{k+1}+(-1)^{[k+1]}\right)}{u+u_j^{(k)}-\frac{\hbar}{2}\,c_k} \\ &\times \prod_{j=1}^{M^{(k+1)}} \frac{u-u_j^{(k+1)}-\frac{\hbar}{2}\left(c_k-(-1)^{[k+1]}\right)}{u-u_j^{(k+1)}-\frac{\hbar}{2}\,c_{k+1}}\;\frac{u+u_j^{(k+1)}-\frac{\hbar}{2}\left(c_k-(-1)^{[k+1]}\right)}{u+u_j^{(k+1)}-\frac{\hbar}{2}\,c_{k+1}}\,. \end{aligned}$$

4.7 Bethe equations

The Bethe equations for open spin chains are again obtained as analyticity conditions for the transfer matrix eigenvalue $\Lambda(u)$. As we have done for the closed spin chain case, let us impose that the residues of $\Lambda(u)$ at

$$u=u_j^{(k)}+\frac{\hbar}{2}c_k\,,\qquad k\le m+n-1\,,\quad j\le M^{(k)}\,,$$

all vanish. The dressing functions are chosen in such a way that cancelations of poles only happen between consecutive dressing functions: let us assume, for the present moment, that we are in the distinguished Dynkin diagram case, and separately treat the three sets of Bethe equations as in the previous chapter.

1. $1 \le k < m$:

$$\prod_{l\neq j}^{M^{(k)}} \mathfrak{e}_2(u_j^{(k)} - u_l^{(k)}) \prod_{j=1}^{M^{(k)}} \mathfrak{e}_2(u_j^{(k)} + u_l^{(k)}) \prod_{\tau=\pm1} \prod_{l=1}^{M^{(k+\tau)}} \mathfrak{e}_{-1}(u_j^{(k)} - u_l^{(k+\tau)})\, \mathfrak{e}_{-1}(u_j^{(k)} + u_l^{(k+\tau)}) =$$
$$= \frac{\beta_k(u_j^{(k)} + \frac{\hbar}{2}k)}{\beta_{k+1}(u_j^{(k)} + \frac{\hbar}{2}k)} \frac{\gamma_k(u_j^{(k)} + \frac{\hbar}{2}k)}{\gamma_{k+1}(u_j^{(k)} + \frac{\hbar}{2}k)};$$

2. $k = m$:

$$\prod_{l=1}^{M^{(m+1)}} \mathfrak{e}_1(u_j^{(m)} - u_l^{(m+1)})\mathfrak{e}_1(u_j^{(m)} + u_l^{(m+1)}) \prod_{l=1}^{M^{(m-1)}} \mathfrak{e}_{-1}(u_j^{(m)} - u_l^{(m-1)})\, \mathfrak{e}_{-1}(u_j^{(m)} + u_l^{(m-1)}) =$$
$$= \frac{\beta_m(u_j^{(m)} + \frac{\hbar}{2}m)}{\beta_{m+1}(u_j^{(m)} + \frac{\hbar}{2}m)} \frac{\gamma_m(u_j^{(m)} + \frac{\hbar}{2}m)}{\gamma_{m+1}(u_j^{(m)} + \frac{\hbar}{2}m)};$$

3. $m \le k \le m+n-1$:

$$\prod_{l\neq n}^{M^{(k)}} \mathfrak{e}_{-2}(u_j^{(k)} - u_l^{(k)}) \prod_{j=1}^{M^{(k)}} \mathfrak{e}_{-2}(u_j^{(k)} + u_j^{(k)}) \prod_{\tau=\pm1} \prod_{j=1}^{M^{(k+\tau)}} \mathfrak{e}_1(u_j^{(k)} - u_l^{(k+\tau)})\, \mathfrak{e}_1(u_j^{(k)} + u_l^{(l+\tau)}) =$$
$$= \frac{\beta_k(u_j^{(k)} + \hbar m - \frac{\hbar}{2}k)}{\beta_{k+1}(u_j^{(k)} + \hbar m - \frac{\hbar}{2}k)} \frac{\gamma_k(u_j^{(k)} + \hbar m - \frac{\hbar}{2}k)}{\gamma_{k+1}(u_j^{(k)} + \hbar m - \frac{\hbar}{2}k)}. \tag{4.34}$$

for j running from 1 to $M^{(k)}$ in each case.

Remark 4.15 *As in the closed case, the left hand side of the Bethe equations only depends on the choice of the algebra, while the right hand side explicitly depends on the choice of the representation (through the $\beta_k(u)$'s functions, eq. (4.30)) and on the reflection matrix (through the $\gamma_l(u)$'s functions, eq. (4.72)).*

4.8 Bethe equations for arbitrary Dynkin diagrams

We turn now to the calculation of the spectrum and Bethe equations of open super-spin chains for other Dynkin diagrams. The rules will be the same as the ones given for the closed case (see section 3.6). The functions $\gamma_k(u)$ have a form similar to (4.72), with a change of increasing or decreasing behaviour of the poles each time a grey (fermionic) root is met, due to the change in the definition of the $\mathbb{Z}_2$-grading, and thus in the parameters c_k.

The Bethe Ansatz equations read, for $\ell = 1, \ldots, m+n-1$ and $i = 1, \ldots, M^{(\ell)}$

$$\epsilon_\ell \prod_{k=1}^{m+n-1} \prod_{j=1}^{M^{(k)}} \mathfrak{e}_{\langle\alpha_\ell,\alpha_k\rangle}(u_i^{(\ell)} - u_j^{(k)})\; \mathfrak{e}_{\langle\alpha_\ell,\alpha_k\rangle}(u_i^{(\ell)} + u_j^{(k)}) \;=\; \frac{\beta_\ell(u_i^{(\ell)} + \frac{\hbar}{2}\,c_\ell)}{\beta_{\ell+1}(u_i^{(\ell)} + \frac{\hbar}{2}\,c_\ell)} \frac{\gamma_l(u_n^{(l)} + \frac{\hbar}{2}c_\ell)}{\gamma_{\ell+1}(u_i^{(l)} + \frac{\hbar}{2}\gamma_\ell)},$$

where $\epsilon_\ell = (1-(-1)^{[l]}\langle\alpha_\ell,\alpha_\ell\rangle)$, as in the closed spin chain case. As an example, we specialize the above formulas to the symmetric Dynkin diagram case.

Example 4.16 *In order to treat the symmetric Dynkin diagram case, we redefine $n \mapsto 2n$, ordering the indices as customary:*

$$[i] = \begin{cases} 0\,, & 1 \le i \le n \quad \textit{and} \quad m+n+1 \le i \le m+2n\,, \\ 1\,, & n+1 \le i \le m+n\,. \end{cases},$$

i.e.

$$c_k = \begin{cases} k\,, & k \le n\,, \\ 2n-k\,, & n<k\le m+n\,, \\ k-2m\,, & m+n<k\le m+2n\,. \end{cases}$$

The γ functions are in this case:

$$\gamma_l(u) = \frac{u(u+\frac{\hbar(m-2n)}{2})}{(u+\frac{\hbar(l-1)}{2})(u+\frac{\hbar l}{2})}\,, \quad l=1,\ldots,n\,,$$

$$\gamma_l(u) = \frac{u(u+\frac{\hbar(m-2n)}{2})}{(u+\frac{\hbar(2n-l+1)}{2})(u+\frac{\hbar(2n-l)}{2})}\,, \quad l=n+1,\ldots,m+n\,,$$

$$\gamma_l(u) = \frac{u(u+\frac{\hbar(m-2n)}{2})}{(u+\frac{\hbar(l-2m-1)}{2})(u+\frac{\hbar(l-2m)}{2})}\,, \quad l=m+n+1,\ldots,m+2n\,,$$

and the Bethe equations, obtained imposing the analiticity of $\Lambda(u)$ at the points $u=u_k^{(l)}+\hbar c_l/2$, for $1\le k\le M^{(l)}$ and $l=1,\ldots,m+2n-1$ are:

1. $1\le l<n$ *and* $m+n<l<m+n$:

$$\prod_{j\neq k}^{M^{(l)}} \mathfrak{e}_2(u_k^{(l)}-u_j^{(l)}) \prod_{j=1}^{M^{(l)}} \mathfrak{e}_2(u_k^{(l)}+u_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} \mathfrak{e}_{-1}(u_k^{(l)}-u_j^{(l+\tau)})\,\mathfrak{e}_{-1}(u_k^{(l)}+u_j^{(l+\tau)}) =$$
$$= \frac{\beta_l(u_k^{(l)}+\frac{\hbar}{2}\,c_l)}{\beta_{l+1}(u_k^{(l)}+\frac{\hbar}{2}\,c_l)}\,\frac{\gamma_l(u_n^{(l)}+\frac{\hbar}{2}c_l)}{\gamma_{l+1}(u_k^{(l)}+\frac{\hbar}{2}c_l)}\,;$$

2. $l=n$:

$$\prod_{j=1}^{M^{(n-1)}} \mathfrak{e}_{-1}(u_k^{(n)}-u_j^{(n-1)})\mathfrak{e}_{-1}(u_k^{(n)}+u_j^{(n-1)}) \prod_{j=1}^{M^{(n+1)}} \mathfrak{e}_1(u_k^{(n)}-u_j^{(n+1)})\mathfrak{e}_1(u_k^{(n)}+u_j^{(n+1)}) =$$
$$= \frac{\beta_n(u_k^{(n)}+\frac{\hbar}{2}n)}{\beta_{n+1}(u_k^{(n)}+\frac{\hbar}{2}n)}\,\frac{\gamma_n(u_k^{(n)}+\frac{\hbar}{2}n)}{\gamma_{n+1}(u_k^{(n)}+\frac{\hbar}{2}n)}\,;$$

3. $n<l<m+n$:

$$\prod_{j\neq k}^{M^{(l)}} \mathfrak{e}_{-2}(u_k^{(l)}-u_j^{(l)}) \prod_{j=1}^{M^{(l)}} \mathfrak{e}_{-2}(u_k^{(l)}+u_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} \mathfrak{e}_1(u_k^{(l)}-u_j^{(l+\tau)})\,\mathfrak{e}_1(u_k^{(l)}+u_j^{(l+\tau)}) =$$
$$= \frac{\beta_l(u_k^{(l)}+\hbar n-\frac{\hbar}{2}l)}{\beta_{l+1}(u_k^{(l)}+\hbar n-\frac{\hbar}{2}l)}\,\frac{\gamma_l(u_n^{(l)}+\hbar n-\frac{\hbar}{2}l)}{\gamma_{l+1}(u_k^{(l)}+\hbar n-\frac{\hbar}{2}l)}\,;$$

4. $l=m+n$:

$$\prod_{j=1}^{M^{(l-1)}} \mathfrak{e}_1(u_k^{(l)}-u_j^{(l-1)})\mathfrak{e}_1(u_k^{(l)}+u_j^{(l-1)}) \prod_{j=1}^{M^{(l+1)}} \mathfrak{e}_{-1}(u_k^{(l)}-u_j^{(l+1)})\mathfrak{e}_{-1}(u_k^{(l)}+u_j^{(l+1)}) =$$
$$= \frac{\beta_l(u_k^{(l)}+\frac{\hbar}{2}(n-m))}{\beta_{l+1}(u_k^{(l)}+\frac{\hbar}{2}(n-m))}\,\frac{\gamma_l(u_k^{(l)}+\frac{\hbar}{2}(n-m))}{\gamma_{l+1}(u_k^{(l)}+\frac{\hbar}{2}(n-m))}\,.$$

Example 4.17 (Open chain, fundamental representation) *As a first example of how our approach applies to open spin chains, let us discuss in some detail the well-known case of the fundamental representation, illustrating the main differences with the closed chain case. In this example, and in the next one, we will suppose that the grading is such that $c_k \neq 0$ for $k \neq 0$. In the case of distuinguished Dynkin diagram, this amounts to choose $m > n$. We start taking an evaluation representation*

$$ev_{\pi_f}\left(T^{[k]}(u)\right) = R(u)\,, \qquad k = 1, \ldots, N$$

on N quantum spaces. Having in mind that $R^{-1}(u) \propto R(-u)$, we build the monodromy matrix on the auxiliary space a as

$$\mathcal{B}_a(u) = R_{a1}(u) \cdots R_{aN}(u) K_a(u) R_{aN}(-u) \cdots R_{a1}(-u)\,,$$

with the usual (diagonal) right–boundary matrix. The transfer matrix is given by the supertrace of $\mathcal{B}(u)$, dressed with a diagonal left–boundary matrix $K^+(u)$:

$$b(u) = str_a\left(K_a^+(u)\mathcal{B}_a(u)\right)\,,$$

and it commutes with the following local hamiltonian:

$$H = \sum_{k=1}^{N} P_{k,k-1} + \frac{i}{2\xi} K_N'(0) - \frac{i}{2\rho\xi'} str_a \left(\frac{dK_a^+(u)}{du}\right)\Bigg|_{u=0}\,, \tag{4.35}$$

where $\rho = m - n$. With respect to the closed chain case, the interaction term between the first and the N–th site disappeared, being replaced by an interaction of the N–th site with the boundary matrix $K(u)$. The hamiltonian (4.35) is proportional to the derivative of the transfer matrix at $u = 0$:

$$H = \frac{1}{2i\xi\xi'\rho}\left(\frac{db(u)}{du}\right)\Bigg|_{u=0}\,.$$

The energy spectrum is given by:

$$E = \beta\,\frac{N}{2} - i\frac{\beta}{2}\sum_{k=2}^{m+n}\frac{\lambda_k'(-ic_{m-1})}{\lambda_k(-ic_{m-2})} + i\frac{\xi+\xi'}{2\,\xi\xi'} + \frac{1-\rho}{\rho} - \beta\sum_{\ell=1}^{M^{(1)}}\frac{1}{(u_\ell^{(1)})^2 + \frac{1}{4}}\,,$$

where we used the shorthand notation $\beta = \beta_1(0)$. Introducing the boundary dependent function:

$$Q_\ell(u) = \frac{K_\ell^+(u)K_\ell(u)}{K_{\ell+1}^+(u)K_{\ell+1}(u)}\,, \qquad 1 \le k < m+n\,,$$

and the notation

$$\hat{\mathfrak{e}}_\ell(u\,,v) = \mathfrak{e}_\ell(u-v)\mathfrak{e}_\ell(u+v)\,,$$

we can write the Bethe equations for the open chain in the fundamental representation in the following compact form:

$$\prod_{\ell=1}^{M^{(1)}} \hat{\mathfrak{e}}_2(u_j^{(1)}, u_\ell^{(1)}) \prod_{\ell=1}^{M^{(2)}} \hat{\mathfrak{e}}_{-1}(u_j^{(1)}, u_\ell^{(2)}) = \left(\frac{u_j^{(1)} + \frac{i}{2}}{u^{(1)} + \frac{i}{2}}\right)^{2N} Q_1(u_j^{(1)} + \frac{i}{2})\,, \qquad j \le M^{(1)}\,,$$

$$\prod_{\ell=1}^{M^{(m-1)}} \hat{\mathfrak{e}}_{-1}(u_j^{(m)}, u_\ell^{(m-1)}) \prod_{\ell=1}^{M^{(m+1)}} \hat{\mathfrak{e}}_1(u_j^{(m)}, u_\ell^{(m+1)}) = Q_m(u_j^{(m)} + i\frac{m}{2})\,, \qquad j \le M^{(m)}\,,$$

$$\prod_{\ell=1}^{M^{(k-1)}} \hat{\mathfrak{e}}_1(u_j^{(k)}, u_\ell^{(k-1)}) \prod_{\ell=1}^{M^{(k)}} \hat{\mathfrak{e}}_{-2}(u_j^{(k)}, u_\ell^{(k)}) \prod_{\ell=1}^{M^{(k+1)}} \hat{\mathfrak{e}}_1(u_j^{(k)}, u_\ell^{(k+1)}) = Q_k(u_j^{(k)} + \frac{i}{2}c_k)\,,\ j \le M^{(k)}\,,$$

the last equation holding for $1 < k \le m+n-1$, $k \neq m$.

Example 4.18 (The open alternating spin chain) *We define the transfer matrix for a $2N$–site open alternating chain as:*

$$b(u) = str_a\big[K^+(u)\,T_{a,1}(u)R_{a,2}(u)\cdots T_{a,2N-1}(u)R_{a,2N}(u)\,K(u)$$
$$R^{-1}_{a,2N}(-u)T^{-1}_{a,2N-1}(-u)\cdots R^{-1}_{a,2}(-u)T^{-1}_{a,1}(-u)\big]\,.$$

Here the matrices acting on the even sites are in the fundamental representation, coinciding again with $R(u)$, and the ones for the non–fundamental are denoted with $T(u)$ and act on the odd sites of the chain. The boundary matrix $K(u)$ is as in eq. (4.8). A local Hamiltonian can be obtained by taking the derivative of $b(u)$:

$$H = \frac{1}{\xi\xi'\rho}\frac{d}{du}b(u)\bigg|_{u=0}\,,$$

where we wrote $\rho = m-n$, while ξ and ξ' carachterize the left and right boundary matrices $K(u)$ and $K^+(u)$ respectively, as in the previous example. One shows that

$$\begin{aligned} H &= \frac{1}{\xi}K'_{2l}(0) + \frac{1}{\xi'\rho}str_a\left(\frac{d\,K^+_a(u)}{du}\right)\bigg|_{u=0} + \frac{2}{\rho}\,st_a\left\{(i\mathbb{I} + T_{a,1}(0)P_{a2})\,T^{-1}_{a,1}(0)\right\} \\ &+ 2\sum_{k=2}^{l}(i\mathbb{I} + T_{2k-2,2k-1}(0)P_{2k-2,2k})\,T^{-1}_{2k-2,2k-1}(0)\,. \end{aligned}$$

The energy spectrum is then given by:

$$\begin{aligned} E &= \beta N\left(1 + \sum_{l=1}^{m+n-1}\frac{1}{c_l} - i\sum_{l=1}^{m+n}\frac{\lambda'_l(-ic_{l-1})}{\lambda_l(-ic_{l-1})}\right) - \beta\sum_{j=1}^{M^{(1)}}\frac{1}{(u_j^{(1)})^2+\frac{1}{4}} \\ &+ i\beta\frac{\xi+\xi'}{\xi\xi'} + 2\beta\frac{1-\rho}{\rho}\,, \end{aligned}$$

where $\beta = \beta_1(0)$. For the distuinguished Dynkin diagram case, and choosing the adjoint representation for the odd sites, the Bethe equations read:

$$\prod_{j=1}^{M^{(1)}}\tilde{\mathfrak{e}}_2(u_k^{(1)},u_j^{(1)})\prod_{j=1}^{M^{(2)}}\tilde{\mathfrak{e}}_{-1}(u_k^{(1)},u_j^{(2)}) =$$
$$-\Big(\mathfrak{e}_{-1}(u_k^{(1)}-i)\,\mathfrak{e}_{-3}(u_k^{(1)}+i)\Big)^N\mathfrak{e}_1(u_k^{(1)})\,Q_1(u_k^{(1)}-\frac{i}{2})\,,\qquad k\le M^{(1)}\,,$$
$$\prod_{j=1}^{M^{(\ell)}}\tilde{\mathfrak{e}}_2(u_k^{(\ell)},u_j^{(\ell)})\prod_{\tau=\pm1}\prod_{j=1}^{M^{(\ell+\tau)}}\mathfrak{e}_{-1}(u_k^{(\ell)},u_j^{(\ell+\tau)}) = -\mathfrak{e}_1(u_k^{(\ell)})\,Q_\ell(u_k^{(\ell)}-\frac{i}{2}\ell)\,,\qquad k\le M^{(\ell)}\,,$$

for $1<\ell<m$;

$$\prod_{j=1}^{M^{(m+1)}}\tilde{\mathfrak{e}}_1(u_k^{(m)},u_j^{(m+1)})\prod_{j=1}^{M^{(m-1)}}\tilde{\mathfrak{e}}_{-1}(u_k^{(m)},u_j^{(m-1)}) = Q_m(u_k^{(m)}-\frac{i}{2}m)\,,\qquad k\le M^{(m)}\,,$$
$$\prod_{j=1}^{M^{(\ell)}}\tilde{\mathfrak{e}}_{-2}(u_k^{(\ell)},u_j^{(\ell)})\prod_{\tau=\pm1}\prod_{j=1}^{M^{(\ell+\tau)}}\tilde{\mathfrak{e}}_1(u_k^{(\ell)},u_j^{(\ell+\tau)}) = -\mathfrak{e}_{-1}(u_k^{(\ell)})\,Q_\ell(u_k^{(\ell)}-\frac{i}{2}(2m-\ell))\,,\qquad k\le M^{(\ell)}\,,$$

for $m < \ell < m+n-1$, and, finally

$$\prod_{j=1}^{M^{(\ell)}} \tilde{\mathfrak{e}}_{-2}(u_k^{(\ell)}, u_j^{(\ell)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(\ell+\tau)}} \tilde{\mathfrak{e}}_1(u_k^{(\ell)}, u_j^{(\ell+\tau)}) =$$

$$= \quad -\Big(\mathfrak{e}_{-1}(u_k^{(\ell)} - \frac{i}{2}\rho)\,\mathfrak{e}_{-1}(u_k^{(\ell)} + \frac{i}{2}\rho)\Big)^N \mathfrak{e}_1(u_k^{(\ell)})\, Q_{m+n-1}(u_k^{(\ell)} - \frac{i}{2}(\rho-1))\,,$$

In the above equations, we set

$$Q_\ell(u) = \frac{K_\ell^+(u) K_\ell(u)}{K_{\ell+1}^+(u) K_{\ell+1}(u)}\,, \qquad 1 \le k < m+n\,,$$

according to the chosen boundary matrix.

4.9 The twisted super Yangian

We shall now introduce another subalgebra of the super Yangian, that, unlike the reflection algebra, can only be defined in the $Y(m|2n)$ case. This so called twisted super Yangian $Y^{\theta_0,\varepsilon}(m|2n)$ will allow us to treat open spin chains with soliton non preserving boundary conditions. As we shall see, from the symmetry point of view this corresponds to passing from open chains whose superalgebra symmetries belong to the $A(m|n)$ series (see proposition 4.9), to models with $osp(m|2n)$ symmetry. We shall restrict ourselves, throughout the whole section, to the case of symmetric Dynkin diagram, and therefore suppose n is an even integer. We begin defining an antiautomorphism of $End\,\mathbb{C}^{(m|n)}$, the so called generalized transposition.

Definition 4.19 (Generalized transposition) *Given a square matrix $A \in End\,\mathbb{C}^{m|n}$, its generalized transposed matrix A^t is given by:*

$$A^t_{ab} = (-1)^{[a]([b]+[1])}\theta_a\theta_b E_{\bar{b}\bar{a}} \qquad \text{with} \qquad \theta_a = \pm 1\,,$$
$$\bar{a} = m+n+1-a \qquad \text{for} \qquad a = 1\,,\ldots\,, m+n\,. \tag{4.36}$$

The generalized transposition satisfies

$$(AB)^t = B^t A^t\,, \tag{4.37}$$

and the $\bar{a}$ are such that $[\bar{a}] = [a]$ for the symmetric Dynkin diagram. If we demand the generalized transposition to be of order 2 [1], i.e. $(A^t)^t = A$, we get the constraint:

$$(-1)^{[a]}\theta_a\theta_{\bar{a}} = \theta_0 = \pm 1\,.$$

Introducing the matrix

$$V_{ab} = \delta_{a\bar{b}}\theta_b\,, \qquad a\,, b = 1\,,\ldots m+n\,,$$

one can write the action of t as the usual supertransposition T followed by a conjugation with V:

$$A^t = V^{-1} A^T V\,.$$

The action of the generalized transposition on the R matrix reads:

$$R_{12}^{t_1}(u) \doteq \overline{R}_{12}(u) = R_{12}^{t_2}(u) = 1 - \frac{\hbar}{u} Q_{12} \doteq 1 - \frac{\hbar}{u} P_{12}^{t_1}\,,$$

and the following relation holds

$$P_{12} Q_{12} P_{12} = \theta_0 Q_{12}\,. \tag{4.38}$$

[1] Notice that the usual super transposition T is only of order 2 up to the grade of its elements. One has $(A^T)^T_{ij} = (-1)^{[i]+[j]} A_{ij}$.

Remark 4.20 *Notice that the Q_{12} defined above, while still proportional to a one dimensional projector, is not equivalent to the one used in chapter 2, due to the different definitions of ^{t}.*

We then define on $\mathcal{Y}(m|n)$:

$$\overline{T}(u) = \sum_{a,b} \overline{T}_{ab}(u) E_{ab} = \sum_{a,b} T_{ab}(-u - \hbar\rho) E^t_{ab}\,, \tag{4.39}$$

leading to the following morphism for the super Yangian generators:

$$\begin{cases} \tau: \quad \mathcal{Y}(m|n) \to \mathcal{Y}(m|n) \\ T_{ab}(u) \mapsto \tau(T_{ab}(u)) = \overline{T}_{ab}(u) = (-1)^{[a]([b]+1)} \theta_a \theta_b T_{\overline{b}\overline{a}}(-u - \hbar\rho)\,. \end{cases} \tag{4.40}$$

Proposition 4.21 *The τ morphism defined by (4.40) is a superalgebra automorphism for $\mathcal{Y}(m|n)$.*

Proof: We first apply the transposition t_1 and the operation $(u,v) \to (-u-\hbar\rho, -v-\hbar\rho)$ on the spectral parameters in eq.(2.17) to get

$$\tau\left[T_1(u)\right] R^{t_1}_{12}(-u+v) T_2(-v-\hbar\rho) = T_2(-v-\hbar\rho) R^{t_1}(-u+v) \tau\left[T_1(u)\right]\,.$$

Then we exchange the parameters $u \leftrightarrow v$, and apply the permutation $P(\cdot)P$ on both sides of the above equation (taking into account that $PQP = \theta_0 Q$, see eq.(4.38)):

$$\tau\left[T_2(v)\right] R^{t_1}_{12}(u-v) T_1(-u-\hbar\rho) = T_1(-u-\hbar\rho) R^{t_1}_{12}(u-v) \tau\left[T_2(v)\right]\,.$$

A second application of t_1 concludes the proof:

$$R_{12}(u-v) \tau\left[T_1(u)\right] \tau\left[T_2(v)\right] = \tau\left[T_2(v)\right] \tau\left[T_1(u)\right] R_{12}(u-v)\,.$$

■

The next step is the definition of numerical matrices, which we shall call K matrices in analogy with the reflection algebra case, which are solution to the soliton non–preserving reflection equation:

$$R_{12}(u-v) K_1(u) \overline{R}_{12}(u+v) K_2(v) = K_2(v) \overline{R}_{12}(u+v) K_1(u) R_{12}(u-v)\,. \tag{4.41}$$

Its solution has been classified in [54], according to the following

Proposition 4.22 *Any invertible solution of the soliton non–preserving reflection equation (4.41) is (up to a multiplication by a scalar function) a constant matrix such that $K^t = \varepsilon K$, where $\varepsilon = \pm 1$.*

Proof: Writing everything in terms of P_{12} and Q_{12}, and taking the part of eq.(4.41) which is symmetric in the exchange $u \leftrightarrow v$, one gets:

$$K_1(u) Q_{12} K_2(v) + K_2(u) Q_{12} K_1(v) = K_2(v) Q_{12} K_1(u) + K_1(v) Q_{12} K_2(u)\,.$$

Transposing the above equation in the first auxiliary space, and eliminating P_{12}, we have

$$K^t_1(u) K_2(v) + K_1(u) K^t_2(v) = K_1(v) K^t_2(u) + K^t_1(v) K_2(u)\,,$$

showing that the dependance of the K matrices on the spectral parameter is multiplicative only:

$$K(u) = f(u) K\,, \tag{4.42}$$

where K is now a numerical matrix. Inserting eq.(4.42) into eq.(4.41), we get the result. ∎

In what follows, we will restrict ourselves to the case of diagonal K matrices

$$K = diag(\zeta_1, \dots, \zeta_{m+n}), \qquad \text{with } \zeta_{\overline{k}} = \varepsilon\zeta_k,$$

and $\zeta_k \neq 0$ for $k = 1, \dots, m+n$ in order to ensure the invertibility of K. Notice that when m is odd, this condition forces $\varepsilon = +1$, because

$$\zeta_{\frac{m+n+1}{2}} = \varepsilon\zeta_{\frac{m+n+1}{2}}.$$

We now define

$$S(u) = T(u)K(u)\tau\left[T(u)\right] = \sum_{a,b=1}^{m+n} S_{ab}(u)E_{ab} = 1 + \sum_{a,b=1}^{m+n}\sum_{n>0}\frac{\hbar^n}{u^n}S_{ab}^{(n)},$$

where $K(u)$ is a solution to the soliton non–preserving reflection equation (4.41). The entries of $S(u)$ read:

$$S_{ab}(u) = \sum_{c=1}^{m+n} T_{ac}(u)\zeta_j T_{\overline{b}\overline{c}}(-u-\hbar\rho)\theta_b\theta_c(-1)^{[c]([b]+1)}. \tag{4.43}$$

Proposition 4.23 *S(u) obeys the following relations:*

$$R_{12}(u-v)S_1(u)R_{12}^{t_1}(-u-v-\hbar\rho)S_2(v) = S_2(v)R_{12}^{t_1}(-u-v-\hbar\rho)S_1(u)R_{12}(u-v), \tag{4.44}$$

$$\tau\left[S(u)\right] = \varepsilon S(u) - \theta_0\frac{\hbar}{2u+\hbar\rho}\left(S(-u-\hbar\rho) - S(u)\right). \tag{4.45}$$

Proof: Eq.(4.44) is a direct consequence of eq.(4.41), together with the fact that τ is an automorphism of $\mathcal{Y}(m|n)$. The second relation can be checked by direct calculation: inserting eq.(4.43) in

$$S_{ab}^t(-u-\hbar\rho) = S_{\overline{b}\overline{a}}(-u-\hbar\rho)\theta_a\theta_b(-1)^{[a]([b]+1)},$$

one gets

$$S_{ab}^t(-u-\hbar\rho) = \sum_{c=1}^{m+n} T_{\overline{b}\overline{c}}(-u-\hbar\rho)\,\zeta_{\overline{c}}\,T_{ac}(u)\theta_{\overline{a}}\theta_{\overline{c}}\theta_a\theta_b(-1)^{[c]([a]+1)+[a]([b]+1)}.$$

Using now $\zeta_{\overline{a}} = \varepsilon\zeta_a$ together with $(-1)^{[a]}\theta_a\theta_{\overline{a}} = \theta_0$ and exchanging the order of the T factors in the above equation, one has:

$$S_{ab}^t(-u-\hbar\rho) = \varepsilon\theta_0\sum_{c=1}^{m+n}\zeta_c\,\theta_b\theta_{\overline{c}}\Big(T_{ac}(u)T_{\overline{b}\overline{c}}(-u-\hbar\rho)(-1)^{[c][b]}$$

$$-\left[T_{\overline{b}\overline{c}}(-u-\hbar\rho), T_{ac}(u)\right](-1)^{[a][b]+[a][c]+[c]}\Big),$$

and eq.(4.45) follows using the supercommutation relations. ∎

Equations (4.44) and (4.45) *uniquely* define, for even values of n, the twisted super Yangian as the subalgebra $Y^{\theta_0,\varepsilon}(m|n) \subset Y(m|n)$ whose generators are the matrix elements of $S_{ij}(u)$.

Remark 4.24 *The supercommutation relations satisfied by the $S_{ij}(u)$ generators can be read off from eq.(4.44), and are given by:*

$$\begin{aligned}
[S_1(u)\,,S_2(v)] &= \frac{\hbar}{u-v}\big(P_{12}S_1(u)S_2(v) - S_2(v)S_1(u)P_{12}\big)\\
&+ \frac{\hbar}{u+v+\hbar\rho}\big(S_1(u)Q_{12}S_2(v) - S_2(v)Q_{12}S_1(u)\big) \qquad (4.46)\\
&+ \frac{\hbar^2}{(u-v)(u+v+\hbar\rho)}\big(P_{12}S_1(u)Q_{12}S_2(v) - S_2(v)Q_{12}S_1(u)P_{12}\big)\,,
\end{aligned}$$

or equivalently, after projection on the matrix element $e_{ab}\otimes e_{cd}$:

$$\begin{aligned}
[S_{ab}(u)\,,S_{cd}(v)] &= \frac{\hbar(-1)^{\eta([a],[b],[c])}}{u-v}\Big(S_{cb}(u)S_{ad}(v) - S_{cb}(v)S_{ad}(u)\Big)\\
&- \frac{\hbar(-1)^{[a][c]+[b][c]}}{u+v+\hbar\rho}\Big((-1)^{[a][c]}\theta_b\theta_{\bar c}S_{a\bar c}(u)S_{\bar b d}(v) - (-1)^{[b][d]}\theta_d\theta_{\bar a}S_{c\bar a}(v)S_{\bar d b}(u)\Big)\\
&+ \frac{\hbar^2(-1)^{[a]+[a][c]+[b][c]}}{(u-v)(u+v+\hbar\rho)}\theta_a\theta_b\Big(S_{c\bar a}(u)S_{\bar b d}(v) - S_{c\bar a}(v)S_{\bar b d}(u)\Big)\,. \qquad (4.47)
\end{aligned}$$

Proposition 4.25 *When $K = 1$ (and in that case $\varepsilon = +1$), the twisted super Yangian $Y^{\theta_0,+}(m|n)$ contains $osp(m|n)$ as Lie subsuperalgebra. It is generated by*

$$S^{(1)}_{ab} = T^{(1)}_{ab} - (-1)^{[a]([b]+1)}\theta_a\theta_b T^{(1)}_{\bar b\bar a}\,, \qquad 1\le a,b\le m+n\,.$$

Proof: Multiplying the above supercommutation relations by $(u-v)(u+v+\hbar\rho)$, and picking up the term proportional to u/v in their expansion, one gets

$$\Big[S^{(1)}_1\,,S^{(2)}_2\Big] = \Big[S^{(1)}_2\,,P_{12}-Q_{12}\Big]\,. \qquad (4.48)$$

Performing the same expansion on the symmetry relation 4.45, one has

$$S^{(1)^t} = -S^{(1)}\,. \qquad (4.49)$$

Equations (4.48) and (4.49) are just the defining relations of the $osp(m|n)$ superalgebra. ■

We shall now briefly present some algebraic results that will be useful in the following sections, and then proceed to define the monodromy matrix associated with the twisted super Yangian.

4.10 Quantum contraction and Sklyanin determinant for $Y^{\theta_0,\varepsilon}(m|n)$

Following the same steps that lead to the quantum contraction of $Y(m|n)$, one can define a quantum contraction $z(u)$ for the twisted super Yangian case.

Proposition 4.26 *There exists an element $z(v)\in Y^{\theta_0,\varepsilon}(m|n)$, called the quantum contraction of $S(u)$, such that*

$$\begin{aligned}
&Q_{12}S_1^{-1}(-v-\hbar\rho+\hbar(m-n))R_{12}(2\,v+\hbar\rho-\hbar(m-n))S_2(v) =\\
&= S_2(v)R_{12}(2\,v+\hbar\rho-\hbar(m-n))S_1^{-1}(-v-\hbar\rho+\hbar(m-n))Q_{12} = z(v)\,Q_{12}\,. \qquad (4.50)
\end{aligned}$$

Proof: Taking into account that

$$R_{12}^{-1}(u-v) \sim R(v-u)\,,$$
$$\left(R_{12}^{t_1}(u)\right)^{-1} = R_{12}^{t_1}(-u-\hbar(m-n))\,,$$

the defining relation (4.44) can be rewritten as follows:

$$S_2(v)R_{12}(v-u)S_1^{-1}(u)R_{12}^{t_1}(u+v+\hbar\rho-\hbar(m-n)) =$$
$$= R_{12}^{t_1}(u+v+\hbar\rho-\hbar(m-n))S_1^{-1}(u)R_{12}(v-u)S_2(v)\,. \qquad (4.51)$$

Equation (4.50) is then obtained as the residue of (4.51) at $u=-v-\hbar\rho+\hbar(m-n)$. ■

Generalizing the proof given in [51] for the non graded case, it is possible to show that $z(u)$ is a central element of the twisted super Yangian, i.e.

$$z(u)S_{ij}(v) = S_{ij}(v)z(u)\,, \qquad 1 \le i,j \le m+n\,.$$

Observe now that from the crossing relation

$$T^t(u)^{-1} = \frac{1}{Z(u+\hbar(m-n))}T^{-1}(u+\hbar(m-n))^t$$

it follows that

$$S_{kk}^{-1}(u) = \frac{1}{Z(-u-\hbar\rho+\hbar(m-n))}\sum_{c=1}^{m+n}(-1)^{[k]([c]+1)}\theta_c\theta_k\frac{1}{\zeta_c}T_{\bar{c}\bar{k}}^{-1}(-u-\hbar\rho+\hbar(m-n))T_{ck}^{-1}(u)\,.$$

Starting from the above relation, and using the shorthand notation $-u-\hbar\rho+\hbar(m-n)=u^*$, one can build the expression of $S_{\bar{k}\bar{k}}^{-1}(u)$:

$$S_{\bar{k}\bar{k}}^{-1}(u) = \frac{1}{Z(u^*)}\sum_{c=1}^{m+n}(-1)^{[k]([c]+1)}\theta_c\theta_{\bar{k}}\frac{1}{\zeta_c}T_{\bar{c}k}^{-1}(u^*)T_{c\bar{k}}^{-1}(u)\,.$$

After commuting the T matrices in the above equation, one can recognise that, on the highest weight vector v^+ of $T(u)$, the following useful relation holds:

$$S_{\bar{k}\bar{k}}^{-1}(u)\,v^+ = \frac{Z(u)}{Z(u^*)}\varepsilon\left(1+\frac{\varepsilon\theta_0\hbar}{2\,u+\hbar\rho-\hbar(m-n)}\right)S_{kk}^{-1}(u^*)\,v^+$$
$$-\frac{\theta_0\hbar}{2\,u+\hbar\rho-\hbar(m-n)}S_{kk}^{-1}(u)\,v^+\,. \qquad (4.52)$$

We now remind the definition and properties of the central element of the non–graded twisted Yangian $Y^{\theta_0,\epsilon}(N)$. By means of this so–called *Sklyanin determinant* we shall build a central element of $Y^{\theta_0,\varepsilon}(m|n)$ and use it in a generalized fusion procedure similar to the one adopted, starting from the quantum Berezinian, in chapter 3.

Definition 4.27 (Sklyanin determinant) *The Sklyanin determinant of the matrix $S(u)$ generating $Y^{\theta_0,\varepsilon}(N)$ is the formal series*

$$\mathrm{sdet}\,S(u) = 1 + c_1u^{-1} + c_2u^{-2} + \cdots \in Y^{\theta_0,\epsilon}(N)[u^{-1}]$$

such that

$$S_{\langle a_N\cdots a_1\rangle}(u)A_N = A_NS_{\langle a_N\cdots a_1\rangle}(u)A_N = \mathrm{sdet}\,S(u)\,A_N\,,$$

where A_N is the antisymmetrizer of $\left(\mathbb{C}^N\right)^{\otimes N}$,

$$S_{\langle a_N \cdots a_1\rangle}(u) = \left(\prod_{2\le k\le N}^{\leftarrow} S_{a_k}(u_k) R^{t_{a_k}}_{a_k a_{k-1}}(-u_k - u_{k-1} - \hbar\rho) \cdots R^{t_{a_k}}_{a_k a_1}(-u_k - u_1 - \hbar\rho)\right) S_{a_1}(u_1)\,,$$

and $u_k = u - \hbar(k-1)$.

Proposition 4.28 sdet $S(u)$ *lies in the center of $Y^{\theta_0,\epsilon}(N)$. That is, all its coefficients are central elements. Moreover, it satisfies the following relation*

$$\operatorname{sdet} S(u) = \operatorname{sdet} K(u)\, \operatorname{qdet} T(u)\, \operatorname{qdet} T(-u - \hbar(\rho - N + 1))\,,$$

with

$$\operatorname{sdet} K(u) = \zeta_1\zeta_2\cdots\zeta_N \frac{2u - (\theta_0\varepsilon + 1)[N/2]\hbar + \hbar + \hbar\rho}{2u - 2[N/2]\hbar + \hbar + \hbar\rho}\,.$$

Here $[N/2]$ denotes the integer part of $N/2$.

In order to write down an expression for a non–trivial central element of $Y^{\theta_0,\varepsilon}(m|n)$, we start introducing the matrix

$$S^*(u) = T^*(u)\, K^*\, T^*(-u - \hbar\rho')^t\,,$$

where $T^*(u) = T^{-1}(u)^t$, and $\rho' = \rho - \hbar(m-n)$.

Proposition 4.29 *$S^*(u)$ obeys the following relations:*

$$R_{12}(u-v) S_1^*(u) R_{12}^{t_1}(-u-v-\hbar\rho') S_2^*(v) = S_2^*(v) R_{12}^{t_1}(-u-v-\hbar\rho') S_1^*(u) R_{12}(u-v)\,, \quad (4.53)$$

$$S^*(-u-\hbar\rho')^t = \varepsilon S^*(u) - \frac{\hbar\theta_0\varepsilon}{2u+\hbar\rho'}\left(S^*(-u-\hbar\rho') - S^*(u)\right)\,. \quad (4.54)$$

Proof: To prove eq.(4.53) and (4.54), one proceeds as in the proof of proposition 4.23, taking into account that $K^* = \varepsilon K^{-1}$ statisfies the same soliton non preserving reflection equation as K. ■

Remark 4.30 *The crossing relation for the T matrix shows that $S^*(u)$ is proportional to $S^{-1}(-u-\hbar\rho')$:*

$$\begin{aligned} S^*(u) &= \frac{1}{\varepsilon} Z(u) T^t(u + \hbar(m-n))^{-1} K^{-1} T^{-1}(-u - \hbar\rho + \hbar(m-n)) = \\ &= \frac{1}{\varepsilon} Z(u) S^{-1}(-u - \hbar\rho + \hbar(m-n))\,. \end{aligned}$$

Introducing now the usual restriction of $S^*(u)$ and $S(u)$ to their even subalgebras:

$$S^{*(m)}(u) = \mathbb{I}^{(m)} S^*(u) \mathbb{I}^{(m)}\,,$$
$$S^{(n)}(u) = \mathbb{I}^{(n)} S(u) \mathbb{I}^{(n)}\,,$$

we can prove the following

Proposition 4.31 *The products of Sklyanin determinants*

$$\operatorname{sdet} S^{*(m)}(u - \hbar n + \hbar)\, \operatorname{sdet} S^{(n)}(u) \in Y^{\theta_0,\varepsilon}(m|n) \quad (4.55)$$

and

$$\operatorname{sdet} S^{(m)}(u - \hbar(m-n))\, \operatorname{sdet} S^{*(n)}(u + \hbar(n-1)) \in Y^{\theta_0,\varepsilon}(m|n) \quad (4.56)$$

are central elements of $Y^{\theta_0,\varepsilon}(m|n)$.

Proof: We shall prove the proposition by showing that (4.55) can be rewritten in terms of the quantum Berezinian. Using proposition 4.28, and remembering that, in the symmetric Dynkin diagram case, $T^{*(m)}(u)$ generates a $Y_{-\hbar}(m)$ subalgebra, we can write

$$\begin{aligned}&\operatorname{sdet} S^{*(m)}(u-\hbar n+\hbar)\operatorname{sdet} S^{(n)}(u)=\\&=\xi^{(m)}(u)\operatorname{qdet} T^{*(m)}(u-\hbar n+\hbar)\operatorname{qdet} {T^{*(m)}}^{t}(-u+\hbar n-\hbar-\hbar\rho')\times\\&\times\xi^{(n)}(u)\operatorname{qdet} T^{(n)}(u)\operatorname{qdet} {T^{(n)}}^{t}(-u-\hbar\rho)\,,\end{aligned}\tag{4.57}$$

where

$$\begin{aligned}\xi^{(m)}(u)&=\frac{1}{\zeta_{\frac{n}{2}+1}\cdots\zeta_{m+\frac{n}{2}}}\,\frac{2(u-\hbar n+\hbar)+(\theta_0\varepsilon+1)[m/2]\hbar-\hbar+\hbar\rho'}{2(u-\hbar n+\hbar)+2\hbar[m/2]-\hbar+\hbar\rho'}\,,\\ \xi^{(n)}(u)&=\zeta_1\cdots\zeta_{\frac{n}{2}}\zeta_{m+\frac{n}{2}+1}\cdots\zeta_{m+n}\,\frac{2u-(\theta_0\varepsilon+1)\frac{n}{2}\hbar+\hbar+\hbar\rho}{2u-\hbar n+\hbar+\hbar\rho'}\,.\end{aligned}$$

Referring to the definition (3.30) of the quantum Berezinian in the symmetric Dynkin diagram case, and taking into account the relation, holding in $Y(N)$,

$$\operatorname{qdet} T^{t}(-u)=\operatorname{qdet} T(-u-\hbar N+\hbar)\,,$$

we can identify the right hand side of eq.(4.57) with

$$\xi^{(m)}(u)\,\xi^{(n)}(u)\,Ber(u+\hbar(m-n+1))\,Ber(-u-\hbar\rho+\hbar m)\,.$$

An analogous calculation starting from (4.56) leads to

$$\begin{aligned}&\operatorname{sdet} S^{(m)}(u-\hbar(m-n))\operatorname{sdet} S^{*(n)}(u+\hbar(n-1))=\\&=\xi^{(m)}(u)\,\xi^{(n)}(u)\,Ber^{-1}(u)\,Ber^{-1}(u-\hbar\rho'-\hbar(n-1))\,,\end{aligned}$$

proving the proposition. ■

4.11 Monodromy and transfer matrices

The monodromy matrix used to construct an N–site open spin chain with soliton non–preserving boundary conditions takes the following form:

$$\mathcal{S}(u)=\Delta^{(N)}\left(S(u)\right)=\left(T(u)\otimes\cdots\otimes T(u)\right)K(u)(\overline{T}(u)\otimes\cdots\otimes\overline{T}(u))\,,\tag{4.58}$$

which is an element of $\mathrm{End}(\mathbb{C}^{m|n})\otimes(\mathcal{Y}(m|n))^{\otimes N}$, obeying

$$R_{12}(u-v)\mathcal{S}_1(u)\overline{R}_{12}(u+v)\mathcal{S}_2(v)=\mathcal{S}_2(v)\overline{R}_{12}(u+v)\mathcal{S}_1(u)R_{12}(u-v)\,,\tag{4.59}$$

and whose $a\,,b$ element reads

$$\mathcal{S}_{ab}(u)=\sum_{l}\sum_{k_1,\dots,k_{(N-1)}}\sum_{j_1,\dots,j_{(N-1)}}T_{ak_1}(u)\cdots T_{k_{N-1}l}(u)\,\zeta_l\,\overline{T}_{lj_1}(u)\cdots\overline{T}_{j_{N-1}b}(u)\,.$$

In the soliton non–preserving case the tansfer matrix becomes

$$s(u)=str\ (\mathcal{S}(u))=\sum_{k=1}^{m+n}(-1)^{[k]}\mathcal{S}_{kk}(u)\,,\tag{4.60}$$

satisfying a crossing relation deduced from eq.(4.45):

$$s(u) = \frac{2\,u\varepsilon + \hbar(\rho\,\varepsilon - \theta_0)}{2\,u + \hbar(\rho - \theta_0)}\, s(-u-\hbar)\,. \tag{4.61}$$

The normalization of $s(u)$ that leads to the analyticity of its entries in an evaluation representation is obtained defining

$$\eta(u) = \prod_{n=1}^{N} (u - a_n)(-u + \hbar\rho - a_n)$$

and

$$\hat{\mathcal{S}}(u) = \eta(u)\mathcal{S}(u)\,, \qquad \hat{s}(u) = str\left(\hat{\mathcal{S}}(u)\right)\,.$$

The analyticity of the eigenvalues of this normalized transfer matrix on the pseudovacuum vector will be shown in the next section. More general transfer matrices for the soliton non preserving spin chains can be built introducing non–trivial left boundary matrices $K^+(u)$:

$$s(u) = str_a K_a^+(u) S_a(u)\,. \tag{4.62}$$

The following proposition, establishing which form the $K^+(u)$ matrix may have in order to preserve the integrability, can be proven by direct calculation:

Proposition 4.32 *The transfer matrices with non–trivial left boundary conditions (4.62) commute for different values of the spectral parameters*

$$[s(u)\,, s(v)] = 0\,, \tag{4.63}$$

if the following reflection equation is satisfied by the left boundary matrix:

$$R_{12}(v-u)K_1^+(u)^t R_{12}^t(u+v+\hbar\rho)K_2^+(v)^t = K_2^+(v)^t R_{12}^t(u+v+\hbar\rho)K_1^+(u)^t R_{12}(v-u)\,. \tag{4.64}$$

Observe that $K^+(u) = 1$, leading to the transfer matrix (4.60), $K^+(u) = K^+ = K^t$, and $K^+ = K^{-1}$ are solutions to the above reflection equation.

4.12 Symmetry of the transfer matrix

In this section we show that, when special boundary conditions are considered, the generators $S_{ab}^{(1)}$ commute with the transfer matrix of the system. To this end, let us start noting that the supercommutation relations (4.46) imply

$$\left[S_1^{(1)}\,, S_2(v)\right] = P_{12}\Big(S_1(v)K_2 - K_1 S_2(v)\Big) + K_1 Q_{12} S_2(v) - S_2(v) Q_{12} K_1\,,$$

i.e., projecting on the matrix element $e_{ab} \otimes e_{cd}$,

$$\left[S_{ab}^{(1)}\,, S_{cd}(v)\right] = (-1)^{\eta([a],[b],[c])}\big(\zeta_c \delta_{cb} S_{ad}(v) - \zeta_a \delta_{ad} S_{cb}(v)\big)$$

$$-(-1)^{[a][c]+[b][c]}\big((-1)^{[a][c]}\zeta_a\theta_b\theta_{\bar{c}}\delta_{a\bar{c}}S_{\bar{b}d}(v) - (-1)^{[b][d]}\zeta_b\theta_d\theta_{\bar{a}}\delta_{d\bar{b}}S_{c\bar{a}}(v)\big)\,.$$

By adding the terms corresponding to the supertrace, we obtain

$$\left[S_{ab}^{(1)}\,, s(v)\right] = \sum_{c=1}^{m+n}(-1)^{[c]}\left[S_{ab}^{(1)}\,, S_{cc}(v)\right] =$$

$$= \big(\zeta_b - \zeta_a\big)\big(S_{ab}(v) - (-1)^{[a][b]+[a]}\theta_a\theta_b S_{\overline{b}\overline{a}}(v)\big) = \big(\zeta_b - \zeta_a\big)\big(S_{ab}(v) - S^t_{ab}(v)\big)\,. \qquad (4.65)$$

From the above relation, it is seen that for a generic K matrix, the elements $S^{(1)}_{ab}$ do not commute with the transfer matrix. However, there are special choices of the right boundary matrix K^+ such that commutativity holds, but the right and left boundary matrices have to be appropriately tuned. For instance, when $K^+ = K^{-1}$, it is easy to show that

$$\Big[S^{(1)}_{ab}\,, s(v)\Big] = 0\,, \qquad 1 \leq a, b \leq m+n\,.$$

For a given choice of the boundary matrices, eq.(4.65) entails the necessary information about the symmetry of the chain. The situation for the corresponding non–graded chains is discussed in [43, 94].

4.13 Pseudovacuum

As in the previous cases, one has to firstly find a particular eigenvector for the transfer matrix $\hat{s}(u)$. It is easy to check that the pseudovacuum vector v^+ can be built, in the usual way, by taking tensor products of the highest weight vectors v_1,...,v_N of the representations appearing in the monodromy matrix:

$$v^+ = v_1 \otimes v_2 \otimes \cdots \otimes v_N\,. \qquad (4.66)$$

The following proposition describes the triangular action of the monodromy matrix on the pseudovacuum.

Proposition 4.33 *The vector (4.66) satisfies the relations:*

$$\mathcal{S}_{kl}(u)\,v^+ = 0\,, \qquad\qquad 1 \leq l < k \leq m+n\,, \qquad (4.67)$$

$$\mathcal{S}_{kk}(u)\,v^+ = \begin{cases} \zeta_k\sigma_k(u)\,v^+\,, & k \leq \frac{m+n+1}{2}\,, \\[2ex] \frac{\zeta_k}{2u+\hbar\rho}\Big(\sigma_k(u)(2u+\hbar\rho-\varepsilon\hbar\theta_0) + \sigma_{\overline{k}}(u)\varepsilon\hbar\theta_0\Big)\,v^+\,, & k > \frac{m+n+1}{2}\,, \end{cases} \qquad (4.68)$$

where

$$\sigma_k(u) = \lambda_k(u)\lambda_{\overline{k}}(-u-\hbar\rho)\,, \qquad k = 1,\ldots,m+n. \qquad (4.69)$$

Proof: From the definition of $\mathcal{S}(u)$, and since $k > l \Rightarrow \bar{l} > \bar{k}$, one immediately gets (4.67). The equalities (4.68) are then obtained using (4.67) and applying once the supercommutation relations to $\mathcal{S}(u)$. ■

Notice that eq.(4.69) implies

$$\sigma_k(u) = \sigma_{\overline{k}}(-u-\hbar\rho)\,. \qquad (4.70)$$

The eigenvalue of $\hat{s}(u)$ on the pseudovacuum, the starting point of the dressing hypothesis, is obtained as a corollary of the above proposition.

Corollary 4.34 *The pseudovacuum v^+ is an eigenvector of the normalized transfer matrix $\hat{s}(u)$:*

$$\hat{s}(u)\,v^+ = \sum_{k=1}^{m+n}(-1)^{[k]}\hat{\mathcal{S}}_{kk}(u)\,v^+ = \Lambda_0(u)\,v^+\,,$$

where

$$\Lambda_0(u) = \sum_{k=1}^{m+n} (-1)^{[k]} g_k(u) \sigma_k(u) \,.$$

Here the $g_k(u)$, $k = 1, \ldots, m+n$ are the following boundary dependent functions:

$$g_k(u) = \begin{cases} \zeta_k \, \frac{2u+\hbar(\rho+\theta_0)}{2u+\hbar\rho} \,, & k < \frac{m+n+1}{2} \,, \\ \zeta_k \,, \quad \text{for odd } m \text{ and} & k = \frac{m+n+1}{2} \,, \\ \zeta_k \, \frac{2u+\hbar(\rho-\theta_0\varepsilon)}{2u+\hbar\rho} \,, & k > \frac{m+n+1}{2} \,. \end{cases} \tag{4.71}$$

while the $\sigma_k(u)$ functions (4.69) depend on the chosen representation and are given, in the normalized evaluation representation, by

$$\sigma_k(u) = \prod_{l=1}^{N} (u - a_l - (-1)^{[k]} \hbar \mu_k^{[l]})(-u - a_l + \hbar\rho - (-1)^{[k]} \hbar \mu_{\overline{k}}^{[l]}) \,.$$

Let us stress that in obtaining eq.(4.71) the symmetric grading has been used.

Remark 4.35 *Although each $g_k(u)$, $k \neq (m+n+1)/2$, has a pole at $u = -\hbar\rho/2$, $\Lambda_0(u)$ is still analytical at that value, thanks to eq.(4.70) which implies $\sigma_k(-\hbar\rho/2) = \sigma_{\overline{k}}(-\hbar\rho/2)$.*

Remark 4.36 *The functions $g_k(u)$ satisfy the following crossing relation:*

$$g_k(u) = \frac{2u\varepsilon + \hbar(\rho\varepsilon - \theta_0)}{2u + \hbar(\rho - \theta_0)} g_{\overline{k}}(-u - \hbar\rho) \,. \tag{4.72}$$

Let us now introduce the analogous formulas for the monodromy matrix associated to $S^{-1}(u)$. In order to deal with analytical entries, we first choose the following normalization for $\mathcal{S}^{-1}(u)$:

$$\tilde{\mathcal{S}}(u) = \prod_{k=1}^{m+n} \sigma_k(u + \hbar c_{k-1}) \, \mathcal{S}^{-1}(u) \,.$$

One can then prove the following

Proposition 4.37 *The eigenvalues of the diagonal entries of $\tilde{\mathcal{S}}(u)$ on the highest weight vector v^+ are given by:*

$$\tilde{\mathcal{S}}_{kk}(u) \, v^+ = \begin{cases} \frac{1}{\zeta_k} \, \tilde{\sigma}_k(u) \, v^+ \,, & k \leq \frac{m+n+1}{2} \,, \\ \left(\frac{2\,u+\hbar\rho-\hbar(m-n)-\varepsilon\hbar\theta_0}{2\,u+\hbar\rho-\hbar(m-n)} \frac{1}{\zeta_k} \, \tilde{\sigma}_k(u) + \frac{\varepsilon\hbar\theta_0}{2\,u+\hbar\rho-\hbar(m-n)} \frac{1}{\zeta_k} \, \tilde{\sigma}_{\overline{k}}(u) \right) v^+ \,, & k > \frac{m+n+1}{2} \,, \end{cases}$$

where

$$\tilde{\sigma}_k(u) = \prod_{l=1}^{k-1} \sigma_l(u + \hbar c_l) \prod_{l=k+1}^{m+n} \sigma_l(u + \hbar c_{l-1}) \,.$$

<u>Proof:</u> A direct calculation, using the supercommutation relations and taking into account eq.(4.52), and the following useful relations:

$$\frac{Z(u^*)}{Z(u)} = \prod_{k=1}^{m+n} \frac{\sigma_k(v + \hbar c_{k-1})}{\sigma_k(v + \hbar c_k)} \,,$$

$$\prod_{k=1}^{m+n} \sigma_k(u^* + \hbar c_k) = \prod_{k=1}^{m+n} \sigma_k(u + \hbar c_{k-1})\,. \tag{4.73}$$

■

A useful consequence of proposition 4.37 is that the eigenvalue of $\tilde{s}(u)$ on the highest weight vector can be explicitly calculated:

$$\tilde{s}(u)\, v^+ = \sum_{k=1}^{m+n} (-1)^{[k]} \tilde{S}_{kk}(u)\, v^+ = \sum_{k=1}^{m+n} (-1)^{[k]} \tilde{g}_k(u) \tilde{\sigma}_k(u)\, v^+\,, \tag{4.74}$$

where

$$\tilde{g}_k(u) = \frac{1}{\zeta_k^2}\, g_k(u - \frac{\hbar}{2}(m-n))\,,$$

and the $g_k(u)$ functions are as in eq.(4.71). Again, one can generalize the structure of the transfer matrix to include non trivial left boundary matrices:

$$\tilde{s}(u) = str_a \tilde{K}^+_a(u) \tilde{S}_a(u)\,. \tag{4.75}$$

The following generalization of proposition 4.32 establishes the compatibility of $s(u)$ and $\tilde{s}(u)$ as observables:

Proposition 4.38 *The transfer matrices with non–trivial left boundary conditions (4.62), (4.75) all commute for different values of the spectral parameters*

$$[s(u)\,, s(v)] = [\tilde{s}(u)\,, s(v)] = [\tilde{s}(u)\,, \tilde{s}(v)] = 0\,, \tag{4.76}$$

if the following reflection equations are satisfied by the boundary matrices:

$$\begin{aligned}
R_{12}(v-u) K_1^+(u)^t R_{12}^t(u+v+\hbar\rho) K_2^+(v)^t &= K_2^+(v)^t R_{12}^t(u+v+\hbar\rho) K_1^+(u)^t R_{12}(v-u)\,,\\
R_{12}(u-v) \tilde{K}_1^+(u)^t R_{12}^t(u^*-v) \tilde{K}_2^+(v)^t &= \tilde{K}_2^+(v)^t R_{12}^t(u^*-v) \tilde{K}_1^+(u)^t R_{12}(u-v)\,,\\
R_{12}^t(u-v^*) \tilde{K}_1^+(u)^t R_{12}(u-v-\hbar\mathcal{K}) K_2^+(v)^t &= K_2^+(v)^t R_{12}(u-v-\hbar\mathcal{K}) \tilde{K}_1^+(u)^t R_{12}^t(u-v^*)\,,
\end{aligned}$$

where $v^* = -v - \hbar\rho + \hbar(m-n)$ *and* $\mathcal{K} = m-n$.

In analogy with eq.(4.74), one can show that v^+ is also an eigenvector for the transfer matrix $s^*(u)$ built from $S^*(u)$:

$$s^*(u)\, v^+ = \sum_{k=1}^{m+n} (-1)^{[k]} S^*_{kk}(u)\, v^+ = \sum_{k=1}^{m+n} (-1)^{[k]} g_k^*(u) \sigma_k^*(u)\,,$$

where

$$\sigma_k^*(u) = \lambda_k^*(u) \lambda_k^*(-u - \hbar\rho + \hbar(m-n))\,,$$

and

$$g_k^*(u) = \begin{cases} \frac{\varepsilon}{\zeta_k} \frac{2u+\hbar(\rho'+\varepsilon\theta_0)}{2u+\hbar\rho'}\,, & k < \frac{m+n+1}{2}\,,\\[2ex] \frac{\varepsilon}{\zeta_k}\,, & k = \frac{m+n+1}{2}\,,\\[2ex] \frac{\varepsilon}{\zeta_k} \frac{2u+\hbar(\rho'-\theta_0)}{2u+\hbar\rho'}\,, & k > \frac{m+n+1}{2}\,. \end{cases}$$

Finally, the scalar value of the products of quantum Berezinians appearing in proposition 4.31 can be calculated by acting with them on the pseudovacuum vector v^+. This leads to the following expressions:

$$Ber(u + \hbar(m - n + 1))\, Ber(-u - \hbar\rho + \hbar m) = \tag{4.77}$$
$$= \prod_{l=1}^{n/2} \sigma_l(u - \hbar n + \hbar l) \prod_{l=n/2+1}^{m+n/2} \sigma_l^*(u - \hbar n + \hbar(l - n/2)) \prod_{l=m+n/2+1}^{m+n} \sigma_l(u - \hbar(m - n) + \hbar l)\,,$$
$$Ber^{-1}(u)\, Ber^{-1}(-u - \hbar\rho' - \hbar(n - 1)) = \tag{4.78}$$
$$= \prod_{l=1}^{n/2} \sigma_l^*(u + \hbar n - \hbar l) \prod_{l=n/2+1}^{m+n/2} \sigma_l(u + \hbar n - \hbar(l - n/2)) \prod_{l=m+n/2+1}^{m+n} \sigma_l^*(u + \hbar(m + n - l))\,.$$

4.14 Dressing functions

In analogy with the closed chains and with the open chains based on the reflection algebra, the starting hypothesis is that all the eigenvalues of $\hat{s}(u)$ can be written as

$$\Lambda(u) = \sum_{k=1}^{m+n} (-1)^{[k]} g_k(u) \sigma_k(u) A_{k-1}(u)\,, \tag{4.79}$$

$g_k(u)$ and $\sigma_k(u)$ being respectively given by (4.71) and (4.69) and dressing functions $A_k(u)$ to be determined. As in the reflection algebra case, we assume the dressing functions to be rational functions of the form:

$$\begin{aligned} A_k(u) \quad &= \quad \prod_{j=1}^{M^{(k)}} \frac{u - v_j^{(k)}}{u - u_j^{(k)} - \frac{\hbar}{2}\, c_k} \; \frac{u - w_j^{(k)}}{u + u_j^{(k)} - \frac{\hbar}{2}\, c_k} \\ &\times \prod_{j=1}^{M^{(k+1)}} \frac{u - \alpha_j^{(k+1)}}{u - u_j^{(k+1)} - \frac{\hbar}{2}\, c_{k+1}} \; \frac{u - \beta_j^{(k+1)}}{u + u_j^{(k+1)} - \frac{\hbar}{2}\, c_{k+1}}\,, \end{aligned}$$

for $1 \le k \le m + n$, with $M^{(0)} = M^{(m+n)} = 0$. We now look for algebraic constraints on the eigenvalue (4.79), in order to fix the relation between the parameters $v_j^{(k)}$, $w_j^{(k)}$, $\alpha_j^{(k)}$ and $\beta_j^{(k)}$ and the $u_j^{(k)}$. To this end (and unlike the reflection algebra case), it is not enough to impose the cancelation of the residues of $\Lambda(u)$ at the pole of the boundary functions $g_k(u)$, $u = -\frac{\hbar}{2}\rho$. A first constraint comes from the crossing relations (4.61) satisfied by the transfer matrix. Taking into account eq.(4.70) and eq.(4.72), one sees that

$$A_{k-1}(u) = A_{\overline{k}-1}(-u - \hbar\rho)\,, \tag{4.80}$$

and this latter relation is sufficient to prove that the residue of $\Lambda(u)$ at $u = -\frac{\hbar}{2}\rho$ vanishes. The constraint (4.80) shows that the integers number $M^{(k)}$, $k = 1, \ldots, m + n - 1$, should satisfy the *folding* relation

$$M^{(k)} = M^{(m+n-k)}\,.$$

The algebraic origin of this condition can be retraced in the fact that only half of the Cartan generators of $sl(m|n)$ survive after imposing soliton non preserving boundary conditions through equation (4.44). For instance, in the case of K proportional to the identity, these are exactly the generators of the $osp(m|n)$ superalgebra. In order to further constrain the dressing functions, we use a generalization of the fusion procedure adopted in chapter 3 for the closed spin chain case. Proposition 4.38 and remark 4.30 show that

$$[s^*(u)\,,s(v)]=0\,,$$

so that we can consider the dressing of $s^*(u)$ together with the one of $s(u)$:

$$\Lambda^*(u)=\sum_{k=1}^{m+n}(-1)^{[k]}g^*_k(u)\sigma^*_k(u)A^*_{k-1}(u)\,. \tag{4.81}$$

For the $A^*_k(u)$ dressing functions, the following form will be assumed:

$$\begin{aligned} A^*_k(u) \quad &= \prod_{j=1}^{M^{(k)}}\frac{u-v^{*(k)}_j}{u-u^{*(k)}_j-\frac{\hbar}{2}\left(2m-c_k\right)}\;\frac{u-w^{*(k)}_j}{u+u^{*(k)}_j-\frac{\hbar}{2}\left(2m-c_k\right)}\\ &\times\prod_{j=1}^{M^{(k+1)}}\frac{u-\alpha^{*(k+1)}_j}{u-u^{*(k+1)}_j-\frac{\hbar}{2}\left(2m-c_{k+1}\right)}\;\frac{u-\beta^{*(k+1)}_j}{u+u^{*(k+1)}_j-\frac{\hbar}{2}\left(2m-c_{k+1}\right)}\,, \end{aligned}$$

for $1\le k\le m{+}n$. We shall now show that the above dressing functions satisfy two constraints analogous to the ones obtained in the closed chain case for the symmetric Dynkin diagram. Let A_m, A_n and $\Pi_{m|n}$ be the usual projectors acting on $m+n$ auxiliary spaces $V_1,\ldots,V_{m+n}$, and denote

$$\mathcal{S}\mathcal{S}^*=S_{\langle m+n,\ldots,m+1\rangle}(u)S^*_{\langle m\ldots1\rangle}(u-\hbar n+\hbar)\,,$$

and

$$\mathcal{R}(u)=\prod_{j=2}^{m}\prod_{i=1}^{j-1}R^{t_j}_{ji}(u^*_j+u^*_i+\hbar\rho')\prod_{k=m+2}^{m+n}\;\prod_{l=m+1}^{k-1}R^{t_k}_{kl}(u_k+u_l+\hbar\rho)\,,$$

where

$$\begin{aligned} u_k&=u+\hbar m-\hbar(k-1)\,,\\ u^*_k&=u-\hbar n+\hbar k\,. \end{aligned}$$

Starting from the following decomposition

$$\begin{aligned} &\mathcal{R}(u)\,\mathcal{S}\mathcal{S}^*=\Pi_{m|n}\mathcal{R}(u)\,\mathcal{S}\mathcal{S}^*A_mA_n\;+\\ +\quad&(1-\Pi_{m|n})\mathcal{R}(u)\mathcal{S}\mathcal{S}^*A_mA_n+\mathcal{R}(u)\,\mathcal{S}\mathcal{S}^*(1-A_mA_n)\,, \end{aligned} \tag{4.82}$$

and taking the supertrace of both sides over all the auxiliary spaces $V_1,\ldots,V_{m+n}$, we get, after some calculation

$$\begin{aligned} &\gamma(u)\,s(u-\hbar n+\hbar)\cdots s(u)\,s^*(u-\hbar n+\hbar m)\cdots s^*(u-\hbar n+\hbar)=\\ &=\kappa(u)\,\xi^{(m)}(u)\xi^{(n)}(u)\,Ber(u+\hbar(m-n+1))\,Ber(-u-\hbar\rho+\hbar m)+s_{\mathfrak{f}}(u)\,, \end{aligned} \tag{4.83}$$

where the following notations have been introduced:

$$\begin{aligned} \gamma(u)=&\prod_{\ell=1}^{n-1}\frac{2u+\hbar\rho-2\hbar\ell+2\hbar}{2u+\hbar\rho-2\hbar\ell+\hbar}\prod_{\ell=n+1}^{m+n-1}\frac{2u-2\hbar n+\hbar\rho'+2\hbar(\ell-n)+2\hbar}{2u-2\hbar n+\hbar\rho'+2\hbar(\ell-n)+\hbar}\times\\ &\times\left(\frac{2u-2\hbar n+\hbar\rho+2\hbar}{2u-2\hbar n+\hbar\rho+\hbar}\right)\left(\frac{2u-2\hbar n+\hbar\rho'+2\hbar}{2u-2\hbar n+\hbar m+\hbar\rho'+\hbar}\right)\,, \end{aligned}$$

and
$$\kappa(u)=\left(\frac{2u+\hbar\rho-\hbar(3-\theta_0)\frac{n}{2}+\hbar}{2u+\hbar\rho-\hbar(n-1)}\right)\left(\frac{2u-2\hbar n+\hbar\rho'+\hbar(3-\theta_0)[\frac{m}{2}]-\hbar}{2u-2\hbar n+\hbar\rho'+\hbar m-\hbar}\right).$$
The fused transfer matrix $s_{\mathfrak{f}}(u)$ is given by
$$s_{\mathfrak{f}}(u)=str\,\left[(1-\Pi_{m|n})\mathcal{R}(u)\mathcal{S}\mathcal{S}^*A_mA_n+\mathcal{R}(u)\,\mathcal{S}\mathcal{S}^*(1-A_mA_n)\right],$$
and the matrix $\mathcal{R}(u)$ has been introduced in the process to ensure that the left hand side of relation (4.82) is a function of the transfer matrices $s(u)$ and $s^*(u)$. Acting now with relation (4.83) on any $s(u)$ and $s^*(u)$ eigenvector v with eigenvalues $\Lambda(u)$ and $\Lambda^*(u)$, and taking into account eq.(4.77), one obtains:
$$\begin{aligned}&\frac{\gamma(u)}{\kappa(u)\xi^{(m)}(u)\xi^{(n)}(u)}\Lambda(u-\hbar n+\hbar)\cdots\Lambda(u)\,\Lambda^*(u-\hbar n+\hbar m)\cdots\Lambda^*(u-\hbar n+\hbar)\,v=\\ =\;&\prod_{l=1}^{n/2}\sigma_l(u-\hbar n+\hbar l)\prod_{l=n/2+1}^{m+n/2}\sigma_l^*(u-\hbar n+\hbar(l-n/2))\prod_{l=m+n/2+1}^{m+n}\sigma_l(u-\hbar(m-n)+\hbar l)\,v+\\ +\;&\Lambda_{\mathfrak{f}}(u)\,v\,.\end{aligned}\tag{4.84}$$
Let us remark that this equation shows that v is also an eigenvector of $s_{\mathfrak{f}}(u)$. Using the postulated expressions for the transfer matrix eigenvalues (4.79) and (4.81), and picking up the term proportional to the product of quantum Berezinians in eq.(4.84), we deduce the following constraint on the dressing functions:
$$\begin{aligned}A_0(u)\cdots A_{\frac{n}{2}-1}(u+\hbar(\frac{n}{2}-1))\,A^*_{\frac{n}{2}}(u)\cdots A^*_{m+\frac{n}{2}-1}(u+\hbar(m-1))\times\\ \times A_{m+\frac{n}{2}}(u+\hbar\frac{n}{2})\cdots A_{m+n-1}(u+\hbar(n-1))=1\,.\end{aligned}\tag{4.85}$$
An analogous calculation starting from the decomposition of
$$\mathcal{S}^*\mathcal{S}=S^*_{\langle m+n,\dots,m+1\rangle}(u+\hbar(n-1))S_{\langle m\dots1\rangle}(u-\hbar(m-n))$$
provides us with a second constraint:
$$\begin{aligned}A_0^*(u+\hbar(n-1))\cdots A^*_{\frac{n}{2}-1}(u+\hbar\frac{n}{2})\,A_{\frac{n}{2}}(u+\hbar(n-1))\cdots A_{m+\frac{n}{2}-1}(u+\hbar(n-m))\times\\ \times A^*_{m+\frac{n}{2}}(u+\hbar(\frac{n}{2}-1))\cdots A^*_{m+n-1}(u)=1\,.\end{aligned}\tag{4.86}$$
Imposing the constraints (4.85) and (4.86), the dressing functions become, for $k<\left[\frac{m+n}{2}\right]-1$:
$$\begin{aligned}A_k(u)=&\prod_{j=1}^{M^{(k)}}\frac{u-u_j^{(k)}-\frac{\hbar}{2}c_{k+1}-(-1)^{[k+1]}\frac{\hbar}{2}}{u-u_j^{(k)}-\frac{\hbar}{2}c_k}\;\frac{u+u_j^{(k)}-\frac{\hbar}{2}c_{k+1}-(-1)^{[k+1]}\frac{\hbar}{2}}{u+u_j^{(k)}-\frac{\hbar}{2}c_k}\\ \times&\prod_{j=1}^{M^{(k+1)}}\frac{u-u_j^{(k+1)}-\frac{\hbar}{2}c_k+(-1)^{[k+1]}\frac{\hbar}{2}}{u-u_j^{(k+1)}-\frac{\hbar}{2}c_{k+1}}\;\frac{u+u_j^{(k+1)}-\frac{\hbar}{2}c_k+(-1)^{[k+1]}\frac{\hbar}{2}}{u+u_j^{(k+1)}-\frac{\hbar}{2}c_{k+1}}\,,\end{aligned}$$

where in evaluating $c_k = \sum_{l>k}(-1)^{[l]}$ the symmetric gradation has to be used. Let us now set $n = 2\nu$. When $m = 2\mu$, one has the following particular form for $A_{\mu+\nu-1}(u)$:

$$\begin{aligned} A_{\mu+\nu-1}(u) &= \prod_{j=1}^{M^{(\mu+\nu-1)}} \frac{u - u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu-1)}{2}}{u - u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu+1)}{2}} \frac{u + u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu-1)}{2}}{u + u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu+1)}{2}} \\ &\times \prod_{j=1}^{M^{(\mu+\nu)}} \frac{u - u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu+2)}{2}}{u - u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu)}{2}} \frac{u + u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu+2)}{2}}{u + u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu)}{2}} \\ &\times \frac{u - u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu-2+2\rho)}{2}}{u - u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu+2\rho)}{2}} \frac{u + u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu-2+2\rho)}{2}}{u + u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu+2\rho)}{2}}, \end{aligned} \tag{4.87}$$

while, when $m = 2\mu + 1$, we have the following particular form for $A_{\mu+\nu}(u)$:

$$\begin{aligned} A_{\mu+\nu}(u) &= \prod_{j=1}^{M^{(\mu+\nu)}} \frac{u - u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu-2)}{2}}{u - u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu)}{2}} \frac{u + u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu-2)}{2}}{u + u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu)}{2}} \\ &\times \frac{u - u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu-2+2\rho)}{2}}{u - u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu+2\rho)}{2}} \frac{u + u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu-2+2\rho)}{2}}{u + u_j^{(\mu+\nu)} + \frac{\hbar(\nu-\mu+2\rho)}{2}}. \end{aligned}$$

The remaining dressing functions are determined by the condition 4.80.

Remark 4.39 *Notice that when $m = 2\mu$, and $\rho = \mu - \nu$, one has to modify the form of (4.87). Indeed, in that case, the second productory in (4.87) is a square, which has to be omitted. This leads to the simpler form*

$$\begin{aligned} A_{\mu+\nu-1}(u) &= \prod_{j=1}^{M^{(\mu+\nu-1)}} \frac{u - u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu-1)}{2}}{u - u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu+1)}{2}} \frac{u + u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu-1)}{2}}{u + u_j^{(\mu+\nu-1)} - \frac{\hbar(\nu-\mu+1)}{2}} \\ &\times \prod_{j=1}^{M^{(\mu+\nu)}} \frac{u - u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu+2)}{2}}{u - u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu)}{2}} \frac{u + u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu+2)}{2}}{u + u_j^{(\mu+\nu)} - \frac{\hbar(\nu-\mu)}{2}} \end{aligned}$$

4.15 Bethe Equations

In what follows we will simultaneously treat the cases of even and odd m, writing $n = 2\nu$, and denoting with μ the integer part of $\frac{m}{2}$. We shall also write $Q_l(u) = K_l^+(u)K_l^-(u)\sigma_l(u)$ for $l = 1, \ldots, m+n$, and

$$\hat{\mathfrak{e}}_n(u,v) = \mathfrak{e}_n(u-v)\,\mathfrak{e}_n(u+v)\,.$$

Imposing $\Lambda(u)$ to be analytical at $u = u_l^{(1)} + \frac{\hbar}{2}$, $l \le M^{(1)}$, one gets the first set of Bethe equations for the soliton non preserving open spin chains:

$$\prod_{j=1}^{M^{(1)}} \hat{\mathfrak{e}}_{-2}(u_l^{(1)}, u_j^{(1)}) \prod_{j=1}^{M^{(2)}} \hat{\mathfrak{e}}_1(u_l^{(1)}, u_j^{(2)}) = -\frac{Q_2(u_l^{(1)} + \frac{\hbar}{2})}{Q_1(u_l^{(1)} + \frac{\hbar}{2})}\,.$$

Analyticity at $u = u_l^{(k)} + \frac{\hbar}{2}k$, where $k = 2, \ldots, \nu - 1$, implies:

$$\prod_{j=1}^{M^{(k)}} \hat{\mathfrak{e}}_{-2}(u_l^{(k)}, u_j^{(k)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(k+\tau)}} \hat{\mathfrak{e}}_1(u_l^{(k)}, u_j^{(k+\tau)}) =$$

$$= -\frac{Q_{k+1}\left(u_l^{(k)} + \frac{\hbar}{2}k\right)}{Q_k\left(u_l^{(k)} + \frac{\hbar}{2}k\right)}, \qquad l = 1, \ldots, M^{(k)},$$

while for $u = u_l^{(\nu)} + \frac{\hbar}{2}\nu$ one gets

$$\prod_{j=1}^{M^{(\nu+1)}} \hat{\mathfrak{e}}_{-1}(u_l^{(\nu)}, u_j^{(\nu+1)}) \prod_{j=1}^{M^{(\nu-1)}} \hat{\mathfrak{e}}_1(u_l^{(\nu)}, u_j^{(\nu-1)}) =$$

$$= \frac{Q_{\nu+1}\left(u_l^{(\nu)} + \frac{\hbar}{2}\nu\right)}{Q_\nu\left(u_l^{(\nu)} + \frac{\hbar}{2}\nu\right)}, \qquad l = 1, \ldots, M^{(\nu)}.$$

Analyticity at $u = u_l^{(k)} + \hbar\nu - \frac{\hbar}{2}k$, where $k = \nu+1, \ldots, \mu+\nu-1$, implies:

$$\prod_{j=1}^{M^{(k)}} \hat{\mathfrak{e}}_2(u_l^{(k)}, u_j^{(k)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(k+\tau)}} \hat{\mathfrak{e}}_{-1}(u_l^{(k)}, u_j^{(k+\tau)}) =$$

$$= -\frac{Q_{k+1}\left(u_l^{(k)} + \hbar\nu - \frac{\hbar}{2}k\right)}{Q_k\left(u_l^{(k)} + \hbar\nu - \frac{\hbar}{2}k\right)}, \qquad l = 1, \ldots, M^{(k)}.$$

Finally, to impose the vanishing of the residues of $\Lambda(u)$ at $u = u_l^{(\mu+\nu)} + \frac{\hbar}{2}(\nu-\mu)$, we have to separately treat the even and odd m cases, and, when $m = 2\mu$, the $\rho = \mu - \nu$ subcase. For $m = 2\mu$, $\rho \neq \mu - \nu$, one has

$$\prod_{j=1}^{M^{(\mu+\nu-1)}} \hat{\mathfrak{e}}_1(u_l^{(\mu+\nu)}, u_j^{(\mu+\nu-1)}) \prod_{j \neq l}^{M^{(\mu+\nu)}} \hat{\mathfrak{e}}_{-2}(u_l^{(\mu+\nu)}, u_j^{(\mu+\nu)})$$
$$\times \prod_{j=1}^{M^{(\mu+\nu-1)}} \hat{\mathfrak{e}}_1(u_l^{(\mu+\nu)} + \hbar(\nu-\mu) + \hbar\rho, u_j^{(\mu+\nu-1)}) \prod_{j=1}^{M^{(\mu+\nu)}} \hat{\mathfrak{e}}_{-2}(u_l^{(\mu+\nu)} + \hbar(\nu-\mu) + \hbar\rho, u_j^{(\mu+\nu)}) =$$

$$= -\frac{Q_{\mu+\nu+1}(u_l^{(\mu+\nu)} + \frac{\hbar}{2}(\nu-\mu))}{Q_{\mu+\nu}(u_l^{(\mu+\nu)} + \frac{\hbar}{2}(\nu-\mu))}.$$

When $m = 2\mu$, $\rho = \mu - \nu$, the above equation reduces to

$$\prod_{j=1}^{M^{(\mu+\nu-1)}} \left[\hat{\mathfrak{e}}_1(u_l^{(\mu+\nu)}, u_j^{(\mu+\nu-1)})\right]^2 \prod_{j=1}^{M^{(\mu+\nu)}} \hat{\mathfrak{e}}_{-2}(u_l^{(\mu+\nu)}, u_j^{(\mu+\nu)}) = -\frac{Q_{\mu+\nu+1}(u_l^{(\mu+\nu)} + \frac{\hbar}{2}(\nu-\mu))}{Q_{\mu+\nu}(u_l^{(\mu+\nu)} + \frac{\hbar}{2}(\nu-\mu))}.$$

For $m = 2\mu + 1$, one has:

$$\prod_{j=1}^{M^{(\mu+\nu-1)}} \hat{\mathfrak{e}}_1(u_l^{(\mu+\nu)}, u_j^{(\mu+\nu-1)}) \prod_{j=1}^{M^{(\mu+\nu)}} \hat{\mathfrak{e}}_{-2}(u_l^{(\mu+\nu)}, u_j^{(\mu+\nu)})\, \hat{\mathfrak{e}}_1(u_l^{(\mu+\nu)} + \hbar(\nu-\mu) + \hbar\rho, u_j^{(\mu+\nu)}) =$$

$$= -\frac{Q_{\mu+\nu+1}(u_l^{(\mu+\nu)} + \frac{\hbar}{2}(\nu-\mu))}{Q_{\mu+\nu}(u_l^{(\mu+\nu)} + \frac{\hbar}{2}(\nu-\mu))}.$$

5

Fused $sl(1|2)$ models

The generality of our approach to the analytical Bethe Ansatz resides in the fact that the construction of the pseudovacuum with its eigenvalue and of the dressing functions is independent on the choice of the superalgebra and of the Dynkin diagram. As we have seen, the monodromy matrices are built as representations of the Yangian $\mathcal{Y}(m|n)$, so that any representation can be used for the quantum space, thus allowing a simultaneous treatment of integrable transfer matrices with $gl(m|n)$ symmetry and with arbitrary spins.
The main drawback is that the auxiliary spaces we use always have the same dimension for a given superalgebra. In other words, the local matrices entering in the monodromy are always solution to the Yang Baxter equation on tensor products of an arbitrary representation of $gl(m|n)$ *with the fundamental one.* Since the supertrace of the monodromy matrix is taken on the auxiliary space, the transfer matrix acts on the quantum space only, and carries the arbitrary representation. However, most of the models solvable by algebraic Bethe Ansatz stem from *regular* solution of the Yang–Baxter equation: in our approach this situation is only allowed for the fundamental $gl(m|n)$ representation case, because the auxiliary and quantum spaces are generally of *different* dimension. Thus, several known results about the construction of commuting observables from transfer matrices cannot be directly applied to our transfer matrices. In particular, the commonly used recipe

$$H \propto \frac{d}{du} \ln st(u)$$

will generally not lead to local and nearest–neighbour interaction hamiltonians. It is therefore natural to investigate whether the analytical Bethe Ansatz approach can be extended to tackle the case of different auxiliary spaces. In this chapter we will present some possible generalizations of our approach to supersymmetric integrable models arising from more general solutions of the Yang–Baxter equation. We shall however restrict our task to the simplest non–trivial case of $sl(1|2)$ invariant models. In the first section we briefly recall the fundamental elements of the representation theory of $sl(1|2)$ that we shall use in our approach,

5.1 Definitions and notations

In order to work with a *simple* superalgebra, we first restrict ourselves from $gl(1|2)$ to $sl(1|2)$, by taking the quotient of $gl(1|2)$ with its central element $\mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33}$. We can then define $sl(1|2)$ as the superalgebra of 3×3 matrices with zero supertrace. Following the notation of [30], we shall write the $sl(1|2)$ generators as $\{B, S^3, S^+, S^-, V^+, V^-, W^+, W^-\}$, obeying the following supercommutation relations:

$$\begin{aligned}
&\left[B\,,S^3\right] = \left[B\,,S^\pm\right] = 0\,,\\
&\left[S^3\,,S^\pm\right] = \pm S^\pm\,, \qquad \left[S^+\,,S^-\right] = 2\,S^3\,,\\
&\left[S^\pm\,,V^\pm\right] = \left[S^\pm\,,W^\pm\right] = 0\,,\\
&\left[S^\pm\,,V^\mp\right] = V^\pm\,, \qquad \left[S^\pm\,,W^\mp\right] = W^\pm
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&\left[B\,,V^\pm\right] = \frac{1}{2}V^\pm\,, \qquad \left[B\,,W^\pm\right] = -\frac{1}{2}W^\pm\,,\\
&\left[S^3\,,V^\pm\right] = \pm\frac{1}{2}V^\pm\,, \qquad \left[S^3\,,W^\pm\right] = \pm\frac{1}{2}W^\pm\,,\\
&\left\{V^+\,,V^-\right\} = \left\{W^+\,,W^-\right\} = 0\,,\\
&\left\{V^\pm\,,W^\pm\right\} = \pm S^\pm\,, \qquad \left\{V^\pm\,,W^\mp\right\} = -S^3 \pm B\,.
\end{aligned} \tag{5.2}$$

The even subalgebra of $sl(1|2)$ is a $sl(2)\oplus u(1)$ Lie algebra, with S^3, $S^\pm$ generating the $sl(2)$ isospin and B generating the $u(1)$ hypercharge. The odd part of $sl(1|2)$ contains two isospin $\frac{1}{2}$ tensors of $su(2)$: $V^\pm$, with hypercharge $\frac{1}{2}$, and $W^\pm$ with hypercharge $-\frac{1}{2}$. $sl(1|2)$ has a quadratic Casimir:

$$C_2 = B^2 - S^2 - \frac{1}{2}\left(V^-W^+ + W^-V^+ - W^+V^- - V^+W^-\right)\,,$$

where S^2 is the $sl(2)$ Casimir, and a cubic one, C_3, whose expression we shall not need. There are two types of representations for most superalgebras. The *typical* ones are irreducible, and are similar to the usual representations of Lie algebras. The values of the Casimirs, for a given typical representation, are unique to the representation, so that they can be classified according to the Casimir eigenvalues. The *atypical* representations, on the other hand, have no counterpart in the ordinary Lie algebra representations, and the Casimir for two different atypical representations can take the same values.

The representation theory of $sl(1|2)$ has been studied in [30]: they are carachterized by the pair of labels $[b,j]$, where j is a non–negative integer or half–integer, and b an arbitrary complex number. The representations $[b,j]$ with $b\neq\pm j$ are typical and their dimension is $8j$, while the representations $[\pm j,j]$, sometimes referred to as $[j]_+$ and $[j]_-$ are atypical and their dimension is $4j+1$. In the typical representations $[b,j]$ the Casimir operators have eigenvalues $C_2 = j^2-b^2$, $C_3 = b\,(j^2-b^2)$, while they are identically zero in the atypical representations.

The typical representation $[b,j]$ decomposes, under the even part $sl(2)\oplus u(1)$, as

$$[b,j] = D_j(b)\oplus D_{j-1/2}(b-1/2)\oplus D_{j-1/2}(b+1/2)\oplus D_{j-1}(b)\,, \quad j\geq 1\,,$$

where $D_j(b)$ stands for the representation of $sl(2)\otimes u(1)$ with isospin j and hypercharge b. The case $j=1/2$ reduces to

$$[b,1/2] = D_{1/2}(b)\oplus D_0(b-1/2)\oplus D_0(b+1/2)\,.$$

The irreducible atypical representations $[\pm j,j]$ decompose under the even part as

$$\begin{aligned}
&[+j,j] = D_j(j)\oplus D_{j-1/2}(j+1/2)\,,\\
&[-j,j] = D_j(-j)\oplus D_{j-1/2}(-j-1/2)\,.
\end{aligned}$$

The fundamental 3–dimensional representation of $sl(1|2)$ is the atypical representation $[1/2, 1/2]$. We shall write it as follows:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad S^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \qquad V^- = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad W^+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad S^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

i.e. with the $j = 1/2$, $b = 1/2$ isospin multiplet in the lower right block, and the $j = 0$, $b = 1$ singlet in the upper left entry[1]. This amounts to choose the basis vectors as

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \phi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5.3}$$

with fermionic ϕ_2 and ϕ_3, i.e. $[\phi_1] = 0$, $[\phi_2] = [\phi_3] = 1$.
We shall write down the explicit form of other representations when needed in the following sections. For the moment, let us recall here the decomposition formula for tensor product of atypical $[+j, j]$ representations:

$$[j_1, j_1] \otimes [j_2, j_2] = [J, J] \bigoplus_{k=0}^{2j-1} [J + 1/2, J - 1/2 - k], \tag{5.4}$$

where $J = j_1 + j_2$ and $j = \min(j_1, j_2)$.

5.2 Fused R matrices

We begin writing the fundamental analytical solution to the graded Yang–Baxter equation in the case of $sl(1|2)$ as

$$R(u) = u - \hbar P, \tag{5.5}$$

with

$$P\,(\phi_i \otimes \phi_j) = (-1)^{[\phi_i][\phi_j]} (\phi_j \otimes \phi_i).$$

It is easy to show that $R(u)$ can be written in a graded symmetric way in terms of tensor products of the representations of the generators:

$$\begin{aligned} R(u) \;=\; & (u+\hbar)\, 1 \otimes 1 - 2\,\hbar\, B \otimes B + 2\,\hbar\, \mathbf{S} \dot{\otimes} \mathbf{S} \\ & -\hbar \left(V^- \otimes W^+ - V^+ \otimes W^- - W^+ \otimes V^- + W^- \otimes V^+\right), \end{aligned} \tag{5.6}$$

[1] Notice that the conventions of [30] and [65] differ by a sign in the definition of V^-. The diagonal elements of $gl(1|2)$ in the canonical basis can be reobtained as

$$\begin{aligned} e_{11} &= 2B - 1, \\ e_{22} &= 1 - B + S^3, \\ e_{33} &= 1 - B - S^3. \end{aligned}$$

or

$$R_{ij}(u) = (u+\hbar)\mathbf{1}_{ij} - 2\,\hbar\, B_i\, B_j + 2\,\hbar\,\mathbf{S}_i\cdot\mathbf{S}_j - \hbar\left(V_i^-\, W_j^+ + V_j^- W_i^+ + W_j^- V_i^+ + W_i^- V_j^+\right)\,,$$

where

$$\mathbf{S}_i\cdot\mathbf{S}_j = S_i^3 S_j^3 + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)$$

is the $sl(2)$ scalar product. We shall now consider the so called fused solution to the Yang–Baxter equation , i.e. solutions obtained through projections onto invariant subspaces of multiple tensor products of the fundamental solution (5.6) with itself. We shall first use this procedure to fuse the quantum spaces, recovering the Yang–Baxter equation solutions with fundamental auxiliary space introduced in proposition 2.9. These R matrices will then be fused again, this time in the auxiliary space, obtaining *regular* solutions of the Yang–Baxter equation with isomorphic auxiliary and quantum space in non–fundamental representations. The first step in order to build the R matrix corresponding to different representation is to notice that $R(u)$ becomes proportional to a projector at the special values of the spectral parameter $u = \pm\hbar$:

$$\frac{1}{2}\,R(\pm\hbar) = \mp\frac{\hbar}{2}(1 \pm P) = \mp\hbar\,\mathcal{P}^{\pm}\,,$$

where

$$\left(\mathcal{P}^{\pm}\right)^2 = \mathcal{P}^{\pm}\,.$$

The projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ are orthogonal and form a complete set:

$$\mathcal{P}^+\mathcal{P}^- = 0\,, \qquad \mathcal{P}^+ + \mathcal{P}^- = \mathbf{1}\,.$$

This is a fully general feature of any fundamental $gl(m|n)$ solution, and corresponds to the decomposition of the tensor product of two fundamental representations into the direct sum of the graded symmetric and antisymmetric representations. In the case of $sl(1|2)$, $\mathcal{P}^+$ and $\mathcal{P}^-$ respectively project on the 4–dimensional and the 5–dimensional invariant subspaces of $[1/2, 1/2] \otimes [1/2, 1/2]$, according to the decomposition of the tensor product of two atypical representations (5.4):

$$[\frac{1}{2}\,,\frac{1}{2}] \otimes [\frac{1}{2}\,,\frac{1}{2}] = [1\,,1] \oplus [\frac{3}{2}\,,\frac{1}{2}]\,.$$

We can identify the resulting $sl(1|2)$ representations by means of their basis vectors, that are obtained acting with $\mathcal{P}^{\pm}$ on the basis vectors of $[1/2, 1/2] \otimes [1/2, 1/2]$. One gets:

$$\left[\frac{3}{2}\,,\frac{1}{2}\right] = \begin{cases} \chi_1 = \phi_1 \otimes \phi_1 \\ \chi_2 = \frac{1}{\sqrt{2}}(\phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1) \\ \chi_3 = \frac{1}{\sqrt{2}}(\phi_1 \otimes \phi_3 + \phi_3 \otimes \phi_1) \\ \chi_4 = \frac{1}{\sqrt{2}}(\phi_2 \otimes \phi_3 - \phi_3 \otimes \phi_2)\,, \end{cases} \tag{5.7}$$

which is a graded symmetric representation, i.e.

$$P\chi_i = \chi_i\,, \qquad i = 1\,,\ldots\,,4\,.$$

Notice that the gradation of the χ vectors is derived from the one chosen for the ϕ_i, and is given by

$$[\chi_1] = [\chi_4] = 0\,, \qquad [\chi_2] = [\chi_3] = 1\,. \tag{5.8}$$

The other invariant subspace has the following basis:

$$[1\,,1]=\begin{cases}\psi_1=\frac{1}{\sqrt{2}}(\phi_1\otimes\phi_2-\phi_2\otimes\phi_1)\\[2mm] \psi_2=\frac{1}{\sqrt{2}}(\phi_1\otimes\phi_3-\phi_3\otimes\phi_1)\\[2mm] \psi_3=\phi_2\otimes\phi_2\\[2mm] \psi_4=\frac{1}{\sqrt{2}}(\phi_2\otimes\phi_3+\phi_3\otimes\phi_2)\\[2mm] \psi_5=\phi_3\otimes\phi_3\,.\end{cases}\tag{5.9}$$

which is a graded antisymmetric atypical representation:

$$P\psi_i=-\psi_i\,,\qquad i=1\,,\dots\,,5\,.$$

Let us write the generators of $sl(1|2)$ in the $[\frac{3}{2}\,,\frac{1}{2}]$ representation, using (5.7) as basis vectors:

$$B=\begin{pmatrix}2&&&\\&\frac{3}{2}&&\\&&\frac{3}{2}&\\&&&1\end{pmatrix}\qquad S^3=\begin{pmatrix}0&&&\\&\frac{1}{2}&&\\&&-\frac{1}{2}&\\&&&0\end{pmatrix}\qquad V^-=\begin{pmatrix}0&-\sqrt{2}&&\\0&0&&\\&&0&-1\\&&0&0\end{pmatrix}$$

$$V^+=\begin{pmatrix}&&\sqrt{2}&0\\&&0&-1\\&&&\\&&&\end{pmatrix}\qquad S^+=\begin{pmatrix}0&&&\\&0&1&\\&0&0&\\&&&0\end{pmatrix}\qquad W^+=\begin{pmatrix}0&0&&\\\sqrt{2}&0&&\\&&0&0\\&&1&0\end{pmatrix}$$

$$W^-=\begin{pmatrix}&&&\\&&&\\\sqrt{2}&0&&\\0&-1&&\end{pmatrix}\qquad S^-=\begin{pmatrix}0&&&\\&0&0&\\&1&0&\\&&&0\end{pmatrix}.$$

For the five–dimensional (atypical) representation $[1\,,1]$, they read:

$$B=\left(\begin{array}{cc|ccc}\frac{3}{2}&&&&\\&\frac{3}{2}&&&\\\hline&&1&&\\&&&1&\\&&&&1\end{array}\right)\qquad S^3=\left(\begin{array}{cc|ccc}\frac{1}{2}&&&&\\&-\frac{1}{2}&&&\\\hline&&1&&\\&&&0&\\&&&&-1\end{array}\right)\qquad V^-=\left(\begin{array}{cc|ccc}&&-\sqrt{2}&0&0\\&&0&-1&0\\\hline&&&&\\&&&&\\&&&&\end{array}\right)$$

$$V^+=\left(\begin{array}{cc|ccc}&&0&1&0\\&&0&0&\sqrt{2}\\\hline&&&&\\&&&&\\&&&&\end{array}\right)\qquad S^+=\left(\begin{array}{cc|ccc}0&1&&&\\0&0&&&\\\hline&&0&\sqrt{2}&0\\&&0&0&\sqrt{2}\\&&0&0&0\end{array}\right)\qquad W^-=\left(\begin{array}{cc|ccc}&&&&\\&&&&\\\hline\sqrt{2}&0&&&\\0&1&&&\\0&0&&&\end{array}\right)$$

$$W^-=\left(\begin{array}{cc|ccc}&&&&\\&&&&\\\hline0&0&&&\\1&0&&&\\0&\sqrt{2}&&&\end{array}\right)\qquad S^-=\left(\begin{array}{cc|ccc}0&0&&&\\1&0&&&\\\hline&&0&0&0\\&&\sqrt{2}&0&0\\&&0&\sqrt{2}&0\end{array}\right),$$

where the horizontal and vertical lines separate the isospin $1/2$, hypercharge $3/2$ and isospin 1, hypercharge 1 representations of the even sector.

5.2.1 R matrix for $[1/2, 1/2] \otimes [3/2, 1/2]$

We already know from proposition 2.9 how to write down a solution of the Yang–Baxter equation having $[1/2, 1/2]$ as auxiliary space and $[3/2, 1/2]$ as quantum space. Let us now show how the same solution can be obtained through fusion.
According to eq.(5.4), the tensor product of the second and third quantum spaces in the expression $R_{ab}(u+u_0)R_{ac}(u-u_0)$, where $R_{ab}(u)$ is the R–matrix (5.6), will carry the representations $[3/2, 1/2]$ and $[1, 1]$. The latter representation can be projected out by means of $\mathcal{P}^+_{bc}$. We then seek a solution of the form

$$R_{a(bc)}(u) = R_{ab}(u + u_0)R_{ac}(u - u_0)\mathcal{P}^+_{bc}\,, \tag{5.10}$$

to the fused Yang–Baxter equation :

$$R_{12}(u - v)R_{1(34)}(u)R_{2(34)}(v) = R_{2(34)}(v)R_{1(34)}(u)R_{12}(u - v)\,, \tag{5.11}$$

where the notation (34) stands for the tensor product of the spaces 3 and 4, considered as a single quantum space. Using the fact that $\mathcal{P}^+ \propto R(-\hbar)$, it is easy to show that a solution is obtained taking $u_0 = \hbar/2$. Eq.(5.11) is then a plain consequence of the relation:

$$\mathcal{P}^-_{bc}\, R_{ab}(u + \hbar/2)R_{ac}(u - \hbar/2)\, \mathcal{P}^+_{bc} = 0\,, \tag{5.12}$$

sometimes referred to as 'triangularity condition': its algebraic meaning is that the representations $[3/2, 1/2]$ and $[1, 1]$ are not mixed in the fused space. Thus, choosing for the 9–dimensional fused space (23) the union of the bases (5.7) and (5.9), the matrix

$$R_{a(bc)}(u) = R_{ab}(u + \hbar/2)R_{ac}(u - \hbar/2)\mathcal{P}^+_{bc}$$

in block form is upper triangular:

$$R_{a(bc)}(u) = \left(\begin{array}{c|c} R_1 & * \\ \hline 0 & R_2 \end{array}\right).$$

Our R–matrix with fundamental auxiliary space and $[3/2, 1/2]$ quantum space is then simply the restriction of $R_{a(bc)}(u)$ to its invariant $[3/2, 1/2]$ subspace. Explicitly, it reads:

$$R^{\left[\frac{1}{2},\frac{1}{2}\right],\left[\frac{3}{2},\frac{1}{2}\right]}(u) = \left(u - \frac{\hbar}{2}\right)\mathcal{P}^+_{23}\left[u + \frac{\hbar}{2} - \hbar\,(P_{12} + P_{13})\right]\mathcal{P}^+_{23}\,,$$

where the identity $P_{ab}P_{ac}\mathcal{P}^+_{bc} = P_{ac}\mathcal{P}^+_{bc}$ has been used.

5.2.2 R matrix for $[1/2, 1/2] \otimes [1, 1]$

The R matrix intertwining the fundamental with the $[1, 1]$ atypical representation can be built following the same steps as in the previous subsection: one seeks a solution to the fused Yang–Baxter equation (5.11) of the form:

$$R_{a(bc)}(u) = R_{ab}(u + u_0)R_{ac}(u - u_0)\mathcal{P}^-_{bc}\,,$$

and the triangularity condition complementary to eq.(5.12):

$$\mathcal{P}^+_{bc}\, R_{ab}(u + u_0)R_{ac}(u - u_0)\, \mathcal{P}^-_{bc} = 0$$

yields $u_0 = -\hbar/2$. The resulting R matrix is then

$$R^{\left[\frac{1}{2},\frac{1}{2}\right],[1,1]}(u) = \left(u + \frac{\hbar}{2}\right)\mathcal{P}^-_{23}\left[u - \frac{\hbar}{2} - \hbar\,(P_{12} + P_{13})\right]\mathcal{P}^-_{23}\,.$$

5.2.3 Fusion in the auxiliary space

Formally, one can use a different fused Yang–Baxter equation in which the spaces to be fused are interpreted as the auxiliary ones:

$$R_{(12)3}(u-v)R_{(12)4}(u)R_{34}(v) = R_{34}(v)R_{(12)4}(u)R_{(12)3}(u-v)\,. \tag{5.13}$$

Following the same steps as in the case of fused quantum space, we can build two solutions to the above equation, respectively acting on $[3/2,1/2]\otimes[1/2,1/2]$ and $[1,1]\otimes[1/2,1/2]$.

Proposition 5.1 *The fused R matrices*

$$R^{+}_{(12)3}(u) = \mathcal{P}^{+}_{12}\,R_{13}(u-\hbar/2)R_{23}(u+\hbar/2)\mathcal{P}^{+}_{12}\,, \tag{5.14}$$

$$R^{-}_{(12)3}(u) = \mathcal{P}^{-}_{12}\,R_{13}(u+\hbar/2)R_{23}(u-\hbar/2)\mathcal{P}^{-}_{12}\,, \tag{5.15}$$

solve eq.(5.13).

<u>Proof:</u> Let us first check the proposition for the solution 5.14. Noticing that the Yang–Baxter equation, together with the fact that $(\mathcal{P}^+)^2 = (\mathcal{P}^+)$, imply

$$R^{+}_{(12)3}(u) = R_{23}(u+\hbar/2)R_{13}(u-\hbar/2)\mathcal{P}^{+}_{12} = \mathcal{P}^{+}_{12}R_{13}(u-\hbar/2)R_{23}(u+\hbar/2)\,,$$

we can rewrite the left hand side of eq.(5.13) as

$$\mathcal{P}^{+}_{12}R_{13}(u-v-\hbar/2)R_{14}(u-\hbar/2)R_{23}(u-v+\hbar/2)R_{24}(u+\hbar/2)R_{34}(v)\mathcal{P}^{+}_{12}\,.$$

Applying twice the Yang–Baxter equation to bring $R_{34}(v)$ on the right and inserting again the projectors where needed, we get the right hand side of (5.13). The proof for the R matrix (5.15) goes the same. ∎

The solutions described in the above proposition coincide with their counterparts of the previous subsection, but the roles of quantum and auxiliary spaces are exchanged. For future reference, and to understand how these solutions act on the tensor product of representations, let us write them explicitly.

- The solution with auxiliary space $[3/2,1/2]$ is of dimension 12 and reads:

$$R^{+}_{(12)3}(u) = \big(u-\frac{\hbar}{2}\big)\big[u+\frac{\hbar}{2}-\hbar(P_{13}+P_{23})\big]\mathcal{P}^{+}_{12} =$$

$$= \big(u^2-\frac{\hbar^2}{4}\big)-\hbar\big(u-\frac{\hbar}{2}\big)\left(\begin{array}{ccc|ccc|ccc|ccc}
2 & & & & & & & & & & & \\
 & 0 & & \sqrt{2} & & & & & & & & \\
 & & 0 & & & & \sqrt{2} & & & & & \\
\hline
 & \sqrt{2} & & 1 & & & & & & & & \\
 & & & & -1 & & & -1 & & & & \\
 & & & & & 0 & & & & 1 & & \\
\hline
 & & \sqrt{2} & & & & 1 & & & & & \\
 & & & & -1 & & & 0 & & -1 & & \\
 & & & & & & & & -1 & & & \\
\hline
 & & & & & 1 & & -1 & & 0 & & \\
 & & & & & & & & & & -1 & \\
 & & & & & & & & & & & -1
\end{array}\right),$$

the blocks being elements of the fundamental representation. Writing it in terms of the generators acting on spaces i and j, we have

$$R^+_{ij}(u) = \big(u^2 - \frac{\hbar^2}{4}\big)\, 1_{ij} + 2\,\hbar\big(u - \frac{\hbar}{2}\big)\, r_{ij}\,,$$

where

$$r_{ij} = 1_{ij} + B_i\, B_j - \mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{2}\left(V_i^-\, W_j^+ + V_j^- W_i^+ + W_j^- V_i^+ + W_i^- V_j^+\right)\,.$$

- The solution with auxiliary space $[1,1]$ is of dimension 15 and reads:

$$R^-_{(12)3}(u) = \big(u + \frac{\hbar}{2}\big)\big[u - \frac{\hbar}{2} - \hbar(P_{13} + P_{23})\big]\mathcal{P}^+_{12} = \big(u^2 - \frac{\hbar^2}{4}\big) - \hbar\big(u + \frac{\hbar}{2}\big)\times$$

$$\times\left(\begin{array}{ccc|ccc|ccc|ccc|ccc}
1 & & & & & & & & & & & & & & \\
 & -1 & & & & & -\sqrt{2} & & & & & & & & \\
 & & 0 & & -1 & & & & & -1 & & & & & \\
\hline
 & & & 1 & & & & & & & & & & & \\
 & & -1 & & 0 & & & & & -1 & & & & & \\
 & & & & & -1 & & & & & & & -\sqrt{2} & & \\
\hline
 & -\sqrt{2} & & & & & 0 & & & & & & & & \\
 & & & & & & & -2 & & & & & & & \\
 & & & & & & & & 0 & & -\sqrt{2} & & & & \\
\hline
 & & -1 & & -1 & & & & & 0 & & & & & \\
 & & & & & & & & -1 & & -1 & & & & \\
 & & & & & & & & & & & -1 & & -\sqrt{2} & \\
\hline
 & & & & & -\sqrt{2} & & & & & & & 0 & & \\
 & & & & & & & & & & & -\sqrt{2} & & 0 & \\
 & & & & & & & & & & & & & & -2
\end{array}\right).$$

Writing again everything in terms of the $sl(1|2)$ generators on spaces i and j, we get

$$R^-_{ij}(u) = \big(u^2 - \frac{\hbar^2}{4}\big)\, 1_{ij} + 2\,\hbar\big(u + \frac{\hbar}{2}\big)\, r_{ij}\,.$$

Remark 5.2 *Being second degree polynomials in the spectral parameter, all the solutions obtained up to now through fusion coincide with the ones obtained in proposition 2.9 only up to multiplication by a scalar function (and a shift in u). We shall later sistematically factorize these functions to get solutions that are linear in the spectral parameter.*

5.2.4 *R matrix for* $[3/2, 1/2] \otimes [3/2, 1/2]$

Iterating the fusion procedure we can now build the R–matrix intertwining two $[\frac{3}{2}\,, \frac{1}{2}]$ representations. This R matrix will be a regular one, and we shall use it in the following section to build a transfer matrix to which we shall apply a generalization of our analytical Bethe Ansatz approach. It is obtained multiplying two R matrices of the $[\frac{1}{2}, \frac{1}{2}] \otimes [\frac{3}{2}\,, \frac{1}{2}]$ type, and projecting the result of this 'scattering' on the $[\frac{3}{2}\,, \frac{1}{2}]$ representation by means of $\mathcal{P}^+$:

$$R_{(12)(34)}(u) = \mathcal{P}^+_{12} R_{1(34)}(u - \hbar/2) R_{2(34)}(u + \hbar/2) \mathcal{P}^+_{12}\,. \tag{5.16}$$

A straightforward calculation based on eq.(5.13) shows that the R–matrix thus obtained satisfies a Yang–Baxter equation of the following form:

$$R_{(12)(34)}(u - v) R_{(12)(56)}(u) R_{(34)(56)}(v) = R_{(34)(56)}(v) R_{(12)(56)}(u) R_{(12)(34)}(u - v)\,. \tag{5.17}$$

Remark 5.3 *One could define the fused R matrix as a product of* $[\frac{3}{2}, \frac{1}{2}] \otimes [\frac{1}{2}, \frac{1}{2}]$ *matrices:*

$$\tilde{R}_{(12)(34)}(u) = \mathcal{P}^+_{34} R_{(12)3}(u + \hbar/2) R_{(12)4}(u - \hbar/2) \mathcal{P}^+_{34} \,.$$

A straightforward calculation shows that $R_{(12)(34)}(u) = \tilde{R}_{(12)(34)}(u)$*.*

In order to get an expression of $R_{(12)(34)}(u)$ in terms of the generators in the $[\frac{3}{2}, \frac{1}{2}]$ representation we start writing it explicitly as:

$$R_{(12)(34)}(u) = u(u-\hbar)\big[u(u+\hbar) - \hbar u(P_{13} + P_{14} + P_{23} + P_{24}) + \hbar^2(P_{14}P_{23} + P_{13}P_{24})\big]\mathcal{P}^+_{12}\mathcal{P}^+_{34} \,.$$

Remark 5.4 *The above expression implies that the following identities hold:*

$$R_{(34)(12)}(u) = R_{(21)(43)}(u) = R_{(12)(34)}(u) \,,$$

showing that the R matrix is symmetric under the exchange of fused spaces as well as under the permutation of indices in each fused space. These symmetry relations will be useful in the next section, when dealing with the fused reflection equation.

Let us first consider the $P_{13} + P_{14} + P_{23} + P_{24}$ term in the expression above. By acting with it on the $\chi_i \otimes \chi_j$ basis vectors, it is not difficult to see that the following relations holds on the $[\frac{3}{2}, \frac{1}{2}] \otimes [\frac{3}{2}, \frac{1}{2}]$ representation:

$$P_{13} + P_{14} + P_{23} + P_{24} = C - 4 \,, \tag{5.18}$$

where

$$C = 2\,B \otimes B - 2\,\mathbf{S} \dot{\otimes} \mathbf{S} - V^+ \otimes W^- + V^- \otimes W^+ - W^+ \otimes V^- + W^- \otimes V^+ \,. \tag{5.19}$$

Using relation (5.18) we now seek to write the term $P_{14}P_{23} + P_{13}P_{24}$ in terms of C. To this end, we start writing

$$\begin{aligned}(C-4)^2 &= (P_{13} + P_{14} + P_{23} + P_{24})^2 = \\ &= 4 + 2(P_{14}P_{23} + P_{13}P_{24}) + P_{13}P_{14} + P_{14}P_{13} + P_{23}P_{24} + P_{24}P_{23} + \\ &\quad + P_{13}P_{23} + P_{23}P_{13} + P_{14}P_{24} + P_{24}P_{14} \,.\end{aligned}$$

Observe now that, on the $[\frac{3}{2}, \frac{1}{2}] \otimes [\frac{3}{2}, \frac{1}{2}]$ representation, one can rewrite the above expression as

$$(C-4)^2 = 4 + 2(P_{14}P_{23} + P_{13}P_{24}) + 2(P_{13} + P_{14} + P_{23} + P_{24}) \,,$$

since

$$P_{ab}P_{ad}\mathcal{P}^+_{ac}\mathcal{P}^+_{bd} = P_{ad}\mathcal{P}^+_{ac}\mathcal{P}^+_{bd} \,.$$

We thus arrive to write:

$$P_{14}P_{23} + P_{13}P_{24} = \frac{1}{2}C^2 - 5C + 10 \,, \tag{5.20}$$

and we can express the R matrix in terms of the generators as a polynomial in the product (5.19):

$$R_{(12)(34)}(u) = u(u-\hbar)\left[(u^2 + 5\hbar u + 10\hbar^2) - \hbar(u + 5\hbar)C + \frac{\hbar^2}{2}C^2\right] . \tag{5.21}$$

5.2.5 R matrix for $[1,1]\otimes[1,1]$

Following the same lines as in the previous section, we can build the R–matrix for the $[1,1]\otimes[1,1]$ representation starting with

$$R_{(12)(34)}(u)=\mathcal{P}^-_{12}R_{1(34)}(u+\hbar/2)R_{2(34)}(u-\hbar/2)\mathcal{P}^-_{12}\,.$$

The result is

$$R_{(12)(34)}(u)=u(u{+}\hbar)\left[u(u-\hbar)-\hbar u(P_{13}+P_{14}+P_{23}+P_{24})+\hbar^2(P_{13}P_{24}+P_{23}P_{14})\right]\mathcal{P}^-_{12}\mathcal{P}^-_{34}\,,$$

or, after expressing everything in terms of (5.19):

$$R_{(12)(34)}(u)=u(u+\hbar)\left[(u^2+3\hbar u+2\hbar^2)-(u+3\hbar)C+\frac{\hbar^2}{2}C^2\right]\,. \qquad (5.22)$$

5.3 Models from fusion

Thanks to the fused Yang–Baxter equation (5.16), it is possible to build commuting transfer matrices with using the set of fused R matrices just obtained. A key point is that, since the auxiliary and quantum spaces coincide in (5.21) and (5.22), they can be rewritten as regular solutions. We shall discuss the resulting models in the following two subsections.

5.3.1 Four dimensional representation

Let us first consider the $[3/2,1/2]\otimes[3/2,1/2]$ case: since the term $P_{14}P_{23}+P_{13}P_{24}$ acts on the basis vectors as a graded permutation

$$\frac{1}{2}(P_{14}P_{23}+P_{13}P_{24})(\chi_i\otimes\chi_j)=(-1)^{[i][j]}\chi_j\otimes\chi_i\,,$$

we get a regular Yang–Baxter solution by removing a global factor $2\,u(u{-}\hbar)$ from the R matrix (5.22):

$$R^{[\frac{3}{2},\frac{1}{2}]^{\otimes 2}}(u)=\frac{1}{2}u^2+\frac{5}{2}\hbar\,u-\frac{\hbar}{2}u\,C+\hbar^2P\,.$$

The N–sites monodromy and transfer matrices are built as usual:

$$\mathcal{T}_a(u)=R_{a1}(u)R_{a2}(u)\cdots R_{aN}(u)\,,$$

$$st(u)=str_a\mathcal{T}_a(u)\,,$$

and thanks to the regularity, we know that the hamiltonian

$$H=\frac{d}{du}\ln t(u)\bigg|_{u=0}$$

can be written as

$$H=\sum_{j=1}^{N}P_{j,j+1}R'_{j,j+1}(0)\,,$$

subject to the periodical boundary condition $N+1\equiv 1$. Substituting in the Hamiltonian the expression of $R'(u)$, we get

$$H=\sum_{j=1}^{N}H_{j,j+1}\,,\qquad H_{j,j+1}=P_{j,j+1}(5-C_{j,j+1})\,.$$

We thus see that the local Hamiltonian $H_{j,j+1}$ does not reduce to a graded permutation. Thanks to the fact that eq.(5.20) implies:

$$PC = 4\,P + C - 4\,,$$

it is possible to rewrite the Hamiltonian, up to an irrelevant shift, as

$$H_{j,j+1} = P_{j,j+1} - C_{j,j+1}\,. \tag{5.23}$$

In order to write down a correlated electrons Hamiltonian starting from the above interaction term, let us now choose the following interpretation for our basis vectors: at a given chain site j there are four possible electronic states, described by canonical Fermi operators $c_{j,\sigma}$ and $c^\dagger_{j,\sigma}$ satisfying the anticommutation relations given by $\{c^\dagger_{i,\sigma}\,,c_{j,\tau}\} = \delta_{ij}\,\delta_{\sigma\tau}$, where $1 \le i,j \le N$ and $\sigma,\tau =\uparrow,\downarrow$:

$$|0\rangle\;,\quad |\uparrow\rangle_j = c^\dagger_{j,\uparrow}\,|0\rangle\;,\quad |\downarrow\rangle_j = c^\dagger_{j,\downarrow}\,|0\rangle\;,\quad |\uparrow\downarrow\rangle_j = c^\dagger_{j,\downarrow}c^\dagger_{j,\uparrow}\,|0\rangle\;\;.$$

We choose to identify them according to the gradation (5.8):

$$|0\rangle\, = \chi_1\,,\qquad |\uparrow\rangle\, = \chi_2\,,\qquad |\downarrow\rangle\, = \chi_3\,,\qquad |\uparrow\downarrow\rangle\, = \chi_4\,. \tag{5.24}$$

We can now realize the generators of the $sl(1|2)$ superalgebra in the $[\frac{3}{2}\,,\frac{1}{2}]$ representation as follows: we denote by $n_{j,\sigma} = c^\dagger_{j,\sigma}c_{j,\sigma}$ the number operator for electrons with spin σ on site j, and we write $n_j = n_{j,\uparrow} + n_{j,\downarrow}$. The bosonic operators B, S^3, S^+, S^- read

$$B = 2 - \frac{1}{2}n\,,\qquad S^+ = c^\dagger_\uparrow c_\downarrow\,,\qquad S^- = c^\dagger_\downarrow c_\uparrow\,,\qquad S^3 = \frac{1}{2}(n_\uparrow - n_\downarrow)\,,$$

while the fermionic operators are given by

$$\begin{aligned} V^- &= -n_\downarrow c_\uparrow - \sqrt{2}(1-n_\downarrow)c_\uparrow\,, &\qquad W^+ &= n_\downarrow c^\dagger_\uparrow + \sqrt{2}(1-n_\downarrow)c^\dagger_\uparrow\,,\\ V^+ &= -n_\uparrow c_\downarrow + \sqrt{2}(1-n_\uparrow)c_\downarrow\,, &\qquad W^- &= -n_\uparrow c^\dagger_\downarrow + \sqrt{2}(1-n_\uparrow)c^\dagger_\downarrow\,. \end{aligned}$$

By means of these realization of the generators in terms of fermionic operators, the graded permutation can be written as:

$$\begin{aligned} P_{ij} &= -2\,\mathbf{S}_i\cdot\mathbf{S}_j + \frac{1}{2}(n_i-1)(n_j-1) - \sum_\sigma c^\dagger_{i,\sigma}c^\dagger_{i,-\sigma}c_{\sigma,j}c_{-\sigma,j}\\ &+ \sum_\sigma (n_{\sigma,i}-\frac{1}{2})(n_{-\sigma,i}-\frac{1}{2}) + \sum_\sigma (n_{\sigma,j}-\frac{1}{2})(n_{-\sigma,j}-\frac{1}{2})\\ &+ \sum_\sigma (1-n_{\sigma,i}-n_{\sigma,j})\big(c^\dagger_{i,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{i,\sigma}\big)\,, \end{aligned}$$

while the C_{ij} term becomes:

$$\begin{aligned} C_{ij} &= 8 + n_i + n_j + \frac{1}{2}n_in_j - 2\,\mathbf{S}_i\cdot\mathbf{S}_j\\ &+ \sum_\sigma \big(2 - (2+\sqrt{2})(n_{\sigma,i}+n_{\sigma,j}) + (3+2\sqrt{2})n_{\sigma,i}n_{\sigma,j}\big)\big(c^\dagger_{i,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{i,\sigma}\big)\,. \end{aligned}$$

By putting the above expressions into eq.(5.23), and removing the constant terms, we arrive at the following expression for our Hamiltonian:

$$\begin{aligned} H &= -\sum_{\sigma,\langle i,j\rangle} \left(c^\dagger_{i,\sigma} c_{j,\sigma} + c^\dagger_{j,\sigma} c_{i,\sigma}\right) - \sum_{\sigma,\langle i,j\rangle} c^\dagger_{i,\sigma} c^\dagger_{i,-\sigma} c_{\sigma,j} c_{-\sigma,j} \\ &+ (1+\sqrt{2}) \sum_{\sigma,\langle i,j\rangle} \left(n_{-\sigma,i} + n_{-\sigma,j}\right)\left(c^\dagger_{i,\sigma} c_{j,\sigma} + c^\dagger_{j,\sigma} c_{i,\sigma}\right) \\ &- (1+\sqrt{2})^2 \sum_{\sigma,\langle i,j\rangle} n_{-\sigma,i}\, n_{-\sigma,j}\left(c^\dagger_{i,\sigma} c_{j,\sigma} + c^\dagger_{j,\sigma} c_{i,\sigma}\right) \\ &+ \sum_{\sigma,i} (n_{\sigma,i} - \frac{1}{2})(n_{-\sigma,i} - \frac{1}{2}) + \frac{3}{2} \sum_{\langle i,j\rangle} (n_i + n_j)\,, \end{aligned}$$

where $\langle i,j\rangle$ denotes the sum over nearest–neighbour sites. The hamiltonian (5.23) can thus be viewed as an extended Hubbard model with aditional nearest–neighbour interaction terms: the second one is a pair–hopping term, while the third and fourth are so–called bond–charge interaction terms. We shall write the Bethe equations and the eigenvalues for a generalization of this model in the following section.

5.4 Typical representations with $b \neq \frac{3}{2}$

In the previous section we built all possible R matrices intertwining the components of the tensor product of the fundamental representation of $sl(1|2)$ with itself. A very natural generalization of the integrable models we obtained from these fused R matrices would be to substitute the typical $[3/2, 1/2]$ with the $[b, 1/2]$ representation as single–site space of states in the hamiltonian (5.23). This would amount to add a free parameter in the hamiltonian, endowing it with an internal degree of freedom that preserves integrability. It turns out, however, that the fusion procedure does not allow to build the R matrix intertwining $[b, 1/2]$ and $[b', 1/2]$ representations: in other words, $[3/2, 1/2]$ is the only four dimensional representations that can be obtained from tensor product of the fundamental representation. Although it is possible to build the R matrix we need projecting the universal R matrix on $[b, 1/2]$ (see e.g. [65],[67]), in this section we shall take an euristic approach, and try to directly solve the graded Yang–Baxter equation. We firstly guess the following form for our R–matrix

$$R_{ab}(u) = (\alpha_1 u^2 + \alpha_2 \hbar u + \hbar^2 \alpha_3) + (\beta_1 \hbar u + \hbar^2 \beta_2) C_{ab} + \hbar^2 \gamma P_{ab}\,,$$

where P is the usual graded permutation of 4–dimensional auxiliary spaces, and C is given by the value of

$$C = 2\,B \otimes B - 2\,\mathbf{S} \dot{\otimes} \mathbf{S} - V^+ \otimes W^- + V^- \otimes W^+ - W^+ \otimes V^- + W^- \otimes V^+\,,$$

where both factors of the tensor product are in the $[b, \frac{1}{2}]$ representation of $sl(1|2)$, i.e. the $sl(2)$ generators are as in the $[3/2, 1/2]$ case, while the $u(1)$ generator and the fermions read:

$$B = \begin{pmatrix} b+\frac{1}{2} & & & \\ & b & & \\ & & b & \\ & & & b-\frac{1}{2} \end{pmatrix} \qquad V^- = \begin{pmatrix} 0 & -\sqrt{b+\frac{1}{2}} & & \\ 0 & 0 & & \\ & & 0 & -\sqrt{b-\frac{1}{2}} \\ & & 0 & 0 \end{pmatrix}$$

$$V^+ = \begin{pmatrix} & & \sqrt{b+\frac{1}{2}} & 0 \\ & & 0 & -\sqrt{b-\frac{1}{2}} \\ & & & \\ & & & \end{pmatrix} \qquad W^+ = \begin{pmatrix} 0 & 0 & & \\ \sqrt{b+\frac{1}{2}} & 0 & & \\ & & 0 & 0 \\ & & \sqrt{b-\frac{1}{2}} & 0 \end{pmatrix}$$

$$W^- = \begin{pmatrix} & & & \\ & & & \\ \sqrt{b+\frac{1}{2}} & 0 & & \\ 0 & -\sqrt{b-\frac{1}{2}} & & \end{pmatrix}.$$

We then seek for the values of the parameters $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma\}$ such that Yang–Baxter equation (5.17) is satisfied. In order to get a local Hamiltonian, we need a regular solution of Yang–Baxter equation, and we choose $R(0) = \hbar^2 P$. Up to a rescaling of the spectral parameter u, there is no loss of generalization in setting $\beta_1 = 1$. We thus get:

$$R_{ab}(u) = (\alpha_1 u + \alpha_2 \hbar)u + \hbar u\, C_{ab} + \hbar^2 P_{ab}\,.$$

Inserting the above expression in the Yang–Baxter equation, we get after some calculation the following constraints on the parameters:

$$\begin{aligned} \alpha_1 &= (b+\frac{1}{2})(b-\frac{1}{2}), \\ \alpha_2 &= -2\,(b+\frac{1}{2})(b-\frac{1}{2}) - 1, \end{aligned}$$

leading to the following regular solution of the Yang–Baxter equation:

$$R(u) = [(b+\frac{1}{2})(b-\frac{1}{2})(u-2\hbar) - \hbar]u + \hbar u C + \hbar^2 P\,. \tag{5.25}$$

Remark 5.5 *When $b = 3/2$, the R matrix (5.25) can be mapped in the known solution (5.21) by means of the spectral parameter redefinition $u \mapsto -u/2$.*

The realization of the $sl(1|2)$ generators in terms of Fermi operators changes according to the chosen value of b; the bosonic operators read:

$$B = b + \frac{1}{2} - \frac{1}{2}n\,, \qquad S^+ = c_\uparrow^\dagger c_\downarrow\,, \qquad S^- = c_\downarrow^\dagger c_\uparrow\,, \qquad S^3 = \frac{1}{2}(n_\uparrow - n_\downarrow)\,,$$

with unmodified $sl(2)$ sector, while the fermionic operators are given by

$$\begin{aligned} V^- &= -\sqrt{b-\frac{1}{2}}\,n_\downarrow c_\uparrow - \sqrt{b+\frac{1}{2}}\,(1-n_\downarrow)c_\uparrow\,, \quad W^+ = \sqrt{b-\frac{1}{2}}\,n_\downarrow c_\uparrow^\dagger + \sqrt{b+\frac{1}{2}}\,(1-n_\downarrow)c_\uparrow^\dagger\,, \\ V^+ &= -\sqrt{b-\frac{1}{2}}\,n_\uparrow c_\downarrow + \sqrt{b+\frac{1}{2}}\,(1-n_\uparrow)c_\downarrow\,, \quad W^- = -\sqrt{b-\frac{1}{2}}\,n_\uparrow c_\downarrow^\dagger + \sqrt{b+\frac{1}{2}}\,(1-n_\uparrow)c_\downarrow^\dagger\,. \end{aligned}$$

The Hamiltonian is modified as follows:

$$H=\sum_{j=1}^{N}H_{j,j+1}\,,\qquad H_{j,j+1}=P_{j,j+1}\left[2\left(b^2-\frac{1}{4}\right)+1-C_{j,j+1}\right]\,.$$

In order to write it as a linear combination of the Casimir and of the permutation, we use the following formulas, holding for all values of $b\neq\pm\frac{1}{2}$:

$$P=2\left(b^2-\frac{1}{4}\right)+1-\frac{2\left(b^2-\frac{1}{4}\right)+1}{\left(b^2-\frac{1}{4}\right)}\,C+\frac{1}{2\left(b^2-\frac{1}{4}\right)}\,C^2\,,$$

and

$$PC=C+2\left(b^2-\frac{1}{4}\right)P-2\left(b^2-\frac{1}{4}\right)\,.$$

As for the $b=\frac{3}{2}$ case, we get

$$H_{j,j+1}=P_{j,j+1}-C_{j,j+1}\,,\tag{5.26}$$

but it has to be stressed that the expression of the Casimir in terms of the Fermi operators will now contain an additional free parameter related to the choice of b, so that, after removing all constant terms we get a generalized version of our correlated electrons hamiltonian (5.23):

$$\begin{aligned}H\;=\;&-t\sum_{\sigma,\langle i,j\rangle}\left(c^\dagger_{i,\sigma}c_{j,\sigma}+c^\dagger_{j,\sigma}c_{i,\sigma}\right)-\sum_{\sigma,\langle i,j\rangle}c^\dagger_{i,\sigma}c^\dagger_{i,-\sigma}c_{\sigma,j}c_{-\sigma,j}\\&+\;(t+\sqrt{t(t+1)})\sum_{\sigma,\langle i,j\rangle}\left(n_{-\sigma,i}+n_{-\sigma,j}\right)\left(c^\dagger_{i,\sigma}c_{j,\sigma}+c^\dagger_{j,\sigma}c_{i,\sigma}\right)\\&-\;\frac{1}{t}(t+\sqrt{t(t+1)})^2\sum_{\sigma,\langle i,j\rangle}n_{-\sigma,i}\,n_{-\sigma,j}\left(c^\dagger_{i,\sigma}c_{j,\sigma}+c^\dagger_{j,\sigma}c_{i,\sigma}\right)\\&+\;\sum_{\sigma,i}(n_{\sigma,i}-\frac{1}{2})(n_{-\sigma,i}-\frac{1}{2})+\left(t+\frac{1}{2}\right)\sum_{\langle i,j\rangle}(n_i+n_j)\,,\end{aligned}$$

where we wrote $t=b-\frac{1}{2}$.

5.5 Bethe equations and eigenvalues

In this section we seek to find the Bethe equations and the spectrum of the models obtained from fused transfer matrices. This requires a generalization of our analytical Bethe Ansatz approach to the case of non fundamental auxiliary space. Let us state again the problem for the $[b,1/2]$ representation: with our solutions to the fused Yang–Baxter equations, it is possible to write two different transfer matrices:

$$t^{[b\,,\frac{1}{2}]}(u)=str_a\left\{R_{a1}^{[\frac{1}{2},\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)R_{a2}^{[\frac{1}{2},\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)\cdots R_{aN}^{[\frac{1}{2},\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)\right\}\,,\tag{5.27}$$

and

$$\tilde{t}^{[b\,,\frac{1}{2}]}(u)=str_a\left\{R_{a1}^{[b\,,\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)R_{a2}^{[b\,,\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)\cdots R_{aN}^{[b\,,\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)\right\}\,.\tag{5.28}$$

Both t and $\tilde{t}$ act on $[b\,,\frac{1}{2}]$ as a quantum space, but the auxiliary space in eq.(5.27) is the fundamental representation, while it is $[b\,,\frac{1}{2}]$ for the transfer matrix (5.28). The advantage of having the quantum space coincide with the auxiliary is that one can easily get a local Hamiltonian (as we have done in the previous section) thanks to the fact that the matrix

$R^{[b\,,\frac{1}{2}]\,,[b\,,\frac{1}{2}]}$ is a regular solution of Yang–Baxter equation. On the other hand, the fact that in eq.(5.27) the auxiliary space is the fundamental representation, allows our tratment through generalized fusion and dressing functions. The key point is now that, thanks to the following Yang–Baxter equation:

$$R_{12}^{[\frac{1}{2},\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u-v)R_{13}^{[\frac{1}{2},\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)R_{23}^{[b\,,\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(v) = R_{23}^{[b\,,\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(v)R_{13}^{[\frac{1}{2},\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u)R_{12}^{[\frac{1}{2},\frac{1}{2}]\,,[b\,,\frac{1}{2}]}(u-v)\,,$$

we are ensured that

$$\left[t(u)\,,\tilde{t}(v)\right] = 0\,. \tag{5.29}$$

Since it is clear that $[t(u)\,,t(v)] = 0$, we can build our Hamiltonian from $\tilde{t}(u)$ (so that it is local) and exctrat the commuting observables from $t(u)$ diagonalizing it in the usual way, and getting the Bethe Ansatz equations through analytical Bethe Ansatz. From the previous chapters, we know that result is the following set of Bethe equations:

$$\prod_{j=1}^{M^{(2)}} \frac{u_j^{(1)} - u_k^{(2)} - \frac{\hbar}{2}}{u_j^{(1)} - u_k^{(2)} + \frac{\hbar}{2}} = \frac{\lambda_1(u_k^{(1)} + \frac{\hbar}{2})}{\lambda_2(u_k^{(1)} + \frac{\hbar}{2})}\,, \qquad k \leq M^{(1)}\,, \tag{5.30}$$

$$\prod_{j=1}^{M^{(1)}} \frac{u_j^{(2)} - u_k^{(1)} - \frac{\hbar}{2}}{u_j^{(2)} - u_k^{(1)} + \frac{\hbar}{2}} \prod_{j=1}^{M^{(2)}} \frac{u_j^{(2)} - u_k^{(2)} + \hbar}{u_j^{(2)} - u_k^{(2)} - \hbar} = \frac{\lambda_2(u_k^{(2)})}{\lambda_3(u_k^{(2)})}\,, \qquad k \leq M^{(2)}\,, \tag{5.31}$$

holding for any value of b appearing in the right hand side. The only problem left is that the eigenvalues of H *cannot* be exctracted from $t(u)$, so that some other procedure is needed. We shall proceed in two steps:

1. we first solve the case $b = \frac{3}{2}$, when the transfer matrices are built from fusion, and satisfy so called *fusion relations*; the knowledge of the Bethe equations (5.30) and (5.31) will allow us to identify the eigenvalue of the transfer matrix;

2. we then tackle the generic b case, that will be solved by means of a minimal generalization.

We start obtaining a well-known functional relation satisfied by the $[3/2, 1/2]$ and $[1, 1]$ fused transfer matrices. Starting from the R matrix for the tensor product of the fundamental with the $[3/2, 3/2]$ representation (subsection 5.2.1) we can build the R matrices acting on $[3/2, 3/2] \otimes [3/2, 3/2]$ and $[1, 1] \otimes [3/2, 3/2]$ through fusion in the auxiliary space:

$$R_{1(34)}(u - \frac{\hbar}{2})R_{2(34)}(u + \frac{\hbar}{2})\mathcal{P}_{12}^{+} = R^{[\frac{3}{2},\frac{1}{2}]}(u)\,,$$
$$R_{1(34)}(u + \frac{\hbar}{2})R_{2(34)}(u - \frac{\hbar}{2})\mathcal{P}_{12}^{-} = R^{[1,1]}(u)\,.$$

In the above equations, the representations specified in the right hand sides refer to the auxiliary space while the quantum space is the graded symmetric representation $\left[\frac{3}{2}, \frac{1}{2}\right]$. They obviously imply the analogous conditions for the monodromy matrices for spin chains with N sites:

$$\mathcal{T}_1(u - \frac{\hbar}{2})\mathcal{T}_2(u + \frac{\hbar}{2})\mathcal{P}_{12}^{+} = \mathcal{T}^{[\frac{3}{2},\frac{1}{2}]}(u)\,,$$
$$\mathcal{T}_1(u + \frac{\hbar}{2})\mathcal{T}_2(u - \frac{\hbar}{2})\mathcal{P}_{12}^{-} = \mathcal{T}^{[1,1]}(u)\,.$$

Setting $a_k = 0$ for all sites, the eigenvalues on the pseudovacuum of the diagonal entries of the monodromy matrices appearing in the right hand side can be read from the expression of the corresponding fused R matrices and are given by

$$\begin{aligned}
&\mathcal{T}_{11}^{[\frac{3}{2},\frac{1}{2}]}(u) = (u-2\hbar)^N(u-\hbar)^N && \mathcal{T}_{11}^{[1,1]}(u) = (u+\hbar)^N(u-2\hbar)^N\\
&\mathcal{T}_{22}^{[\frac{3}{2},\frac{1}{2}]}(u) = u^N(u-\hbar)^N && \mathcal{T}_{22}^{[1,1]}(u) = (u+\hbar)^N(u-2\hbar)^N\\
&\mathcal{T}_{33}^{[\frac{3}{2},\frac{1}{2}]}(u) = u^N(u-\hbar)^N && \mathcal{T}_{33}^{[1,1]}(u) = u^N(u+\hbar)^N\\
&\mathcal{T}_{44}^{[\frac{3}{2},\frac{1}{2}]}(u) = u^N(u+\hbar)^N && \mathcal{T}_{44}^{[1,1]}(u) = u^N(u+\hbar)^N\\
& && \mathcal{T}_{55}^{[1,1]}(u) = u^N(u+\hbar)^N\,.
\end{aligned}$$

The relation between the eigenvalues of the fused transfer matrices can now be read taking the supertrace of both the equations above on a common eigenvector v and adding the results together, taking into account the completeness relation satisfied by the $\mathcal{P}^{\pm}$ projectors:

$$\Lambda\big(u-\frac{\hbar}{2}\big)\Lambda\big(u+\frac{\hbar}{2}\big) = \Lambda^{[\frac{3}{2},\frac{1}{2}]}(u) + \Lambda^{[1,1]}(u)\,. \tag{5.32}$$

From the dressing hypothesis, we can write the left hand side of eq.(5.32) for a N–site chain as:

$$\begin{aligned}
&\Bigg\{\big(u-2\hbar\big)^N \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}}{u-u_j^{(1)}-\hbar} - u^N \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}}{u-u_j^{(1)}-\hbar} \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}-\frac{3}{2}\hbar}{u-u_j^{(2)}-\frac{\hbar}{2}}\\
&\qquad -u^N \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}+\frac{\hbar}{2}}{u-u_j^{(2)}-\frac{\hbar}{2}}\Bigg\} \times \Bigg\{\big(u-\hbar\big)^N \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}}\\
&- \big(u+\hbar\big)^N \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}} \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}-\frac{\hbar}{2}}{u-u_j^{(2)}+\frac{\hbar}{2}} - \big(u+\hbar\big)^N \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}+\frac{3}{2}\hbar}{u-u_j^{(2)}+\frac{\hbar}{2}}\Bigg\}\,.
\end{aligned}$$

We can collect the eigenvalues on the pseudovacuum, rewriting the left hand side of (5.32) as:

$$u^N\big(u+\hbar\big)^N B(u) + \big(u-2\hbar\big)^N\big(u+\hbar\big)^N C(u) + u^N\big(u-\hbar\big)^N D(u) + \big(u-2\hbar\big)^N\big(u-\hbar\big)^N F(u)\,,$$

where, in particular,

$$\begin{aligned}
B(u) = &\prod_j^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}} + \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}}{u-u_j^{(1)}-\hbar} \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}-\frac{3}{2}\hbar}{u-u_j^{(2)}-\frac{\hbar}{2}}\, \frac{u-u_j^{(2)}+\frac{3}{2}\hbar}{u-u_j^{(2)}+\frac{\hbar}{2}}\\
&+ \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}-\hbar} \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}-\frac{3}{2}\hbar}{u-u_j^{(2)}+\frac{\hbar}{2}} + \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}+\frac{3}{2}\hbar}{u-u_j^{(2)}-\frac{\hbar}{2}}\,,
\end{aligned}$$

$$D(u) = -\prod_j^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}-\hbar} \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}-\frac{3}{2}\hbar}{u-u_j^{(2)}-\frac{\hbar}{2}} - \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}} \prod_j^{M^{(2)}} \frac{u-u_j^{(2)}+\frac{\hbar}{2}}{u-u_j^{(2)}-\frac{\hbar}{2}}\,,$$

and

$$F(u) = \prod_j^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}-\hbar}\,.$$

By inspection of the expression of the right hand side of eq.(5.32) when acting on the pseudovacuum we can see that the $D(u)$ and $F(u)$ terms should belong to the $\Lambda^{[\frac{3}{2},\frac{1}{2}]}(u)$ eigenvalue,

while $C(u)$ should belong to $\Lambda^{[1,1]}(u)$. The $B(u)$ term, on the other hand, can be in principle part of both eigenvalues. To pick up the two contributions, we note that $D(u)$ and $F(u)$ (i.e. the part of (5.32) already identified with $\Lambda^{[\frac{3}{2},\frac{1}{2}]}(u)$) have poles at $u = u_\ell^{(1)}$, $u_\ell^{(1)} + \hbar$ for $\ell \leq M^{(1)}$ and $u_\ell^{(2)} + \frac{\hbar}{2}$ for $\ell \leq M^{(2)}$, and that the Bethe equations (5.30) and (5.31) already lead to vanishing residues at the last two of these. If the same Bethe equations must lead to analyticity of $\Lambda^{[\frac{3}{2},\frac{1}{2}]}(u)$ at $u = u_\ell^{(1)}$, the first term of $B(u)$ must be included.
Collecting the various terms, we can express the eigenvalue of the transfer matrix with fused auxiliary space as:

$$\begin{aligned}\Lambda^{[\frac{3}{2},\frac{1}{2}]}(u) \quad = \quad & u^N \big(u+\hbar\big)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}} + \big(u-2\hbar\big)^N \big(u-\hbar\big)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}-\hbar} \\ & -u^N (u-\hbar)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}-\hbar} \prod_{j=1}^{M^{(2)}} \frac{u-u_j^{(2)}-\frac{3}{2}\hbar}{u-u_j^{(2)}-\frac{\hbar}{2}} \\ & -u^N (u-\hbar)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\hbar}{u-u_j^{(1)}} \prod_{j=1}^{M^{(2)}} \frac{u-u_j^{(2)}+\frac{\hbar}{2}}{u-u_j^{(2)}-\frac{\hbar}{2}} \, . \end{aligned} \tag{5.33}$$

We now generalize this formula to the $b \neq \frac{3}{2}$ case, assuming that $\Lambda^{[b,\frac{1}{2}]}(u)$ keep the same structure as $\Lambda^{[\frac{3}{2},\frac{1}{2}]}(u)$, but with the pseudovacuum eigenvalues replaced with the ones obtained from the R matrix (5.25):

$$\begin{aligned}\lambda_1^{[b,\frac{1}{2}]}(u) &= \frac{2}{b^2-\frac{1}{4}} \big[u-\big(b-\frac{1}{2}\big)\hbar\big] \big[u-\hbar\big(b+\frac{1}{2}\big)\big] \, , \\ \lambda_2^{[b,\frac{1}{2}]}(u) &= \frac{2}{b^2-\frac{1}{4}} u\big[u-\hbar\big(b-\frac{1}{2}\big)\big] \, , \\ \lambda_3^{[b,\frac{1}{2}]}(u) &= \frac{2}{b^2-\frac{1}{4}} u\big[u-\hbar\big(b-\frac{1}{2}\big)\big] \, , \\ \lambda_4^{[b,\frac{1}{2}]}(u) &= \frac{2}{b^2-\frac{1}{4}} u(u+\hbar) \, ,\end{aligned}$$

and with shifted dressing functions:

$$\begin{aligned}\Lambda^{[b,\frac{1}{2}]}(u) \quad = \quad & \big(\lambda_1^{[b,\frac{1}{2}]}(u)\big)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\alpha_1}{u-u_j^{(1)}+\beta_1} + \big(\lambda_4^{[b,\frac{1}{2}]}(u)\big)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\alpha_1}{u-u_j^{(1)}+\beta_2} \\ & -\big(\lambda_2^{[b,\frac{1}{2}]}(u)\big)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\alpha_1}{u-u_j^{(1)}+\beta_1} \prod_{j=1}^{M^{(2)}} \frac{u-u_j^{(2)}+\gamma_1}{u-u_j^{(2)}+\delta_1} \\ & -\big(\lambda_3^{[b,\frac{1}{2}]}(u)\big)^N \prod_{j=1}^{M^{(1)}} \frac{u-u_j^{(1)}+\alpha_1}{u-u_j^{(1)}+\beta_2} \prod_{j=1}^{M^{(2)}} \frac{u-u_j^{(2)}+\gamma_2}{u-u_j^{(2)}+\delta_1} \, . \end{aligned} \tag{5.34}$$

The shift parameters β_i, γ_i and δ_i can now be determined imposing that the analyticity conditions for $\Lambda^{[b,\frac{1}{2}]}(u)$ are again the Bethe equations (5.30) and (5.31), with right hand sides evaluated in a quantum space carrying the typical representation $(b, \frac{1}{2})$.

The results are:

$$\begin{aligned}&\beta_1 = -3\hbar + \hbar(b+\frac{1}{2}) \, , \qquad \beta_2 = -2\hbar + \hbar(b+\frac{1}{2}) \, , \\ &\gamma_1 = -3\hbar + \hbar\, b \, , \qquad \gamma_2 = -\hbar + \hbar\, b \, , \qquad \delta_1 = -2\hbar + \hbar\, b \, ,\end{aligned}$$

correctly reproducing the eigenvalue (5.33) in the $b \to 3/2$ limit. The value of α_1 cannot be determined by analyticity requests (except that it should be such that $\alpha_1 \to \hbar$ when $b \to 3/2$), since it appears in a global factor.

5.6 Fusion for open chains

The aim of this section is to define integrability conditions for the open counterpart of the fused model of sections 5.3.1. Since the auxiliary space does not coincide with the fundamental representation, the left and right boundary matrices $K^-(u)$ and $K^+(u)$ shall satisfy fused reflection equations in order to preserve integrability. We will show how the fusion procedure we presented in the previous sections can be applied to the boundary matrices of chapter 4, naturally leading to solutions of the fused reflection equations. The results we obtained are fully general, but they will mainly be applied to the typical four dimensional representation of $sl(1|2)$.

The first step is again to evaluate the reflection equation at his degeneration points (i.e. to choose the spectral parameters in such a way that the R matrix becomes a projector). Setting $v = u \pm \hbar$ in (4.4), one gets

$$\mathcal{P}^+_{12}\, K_1(u) R_{12}(2u+\hbar) K_2(u+\hbar) = K_2(u+\hbar) R_{12}(2u+\hbar) K_1(u)\, \mathcal{P}^+_{12}\,,$$
$$\mathcal{P}^-_{12}\, K_1(u) R_{12}(2u-\hbar) K_2(u-\hbar) = K_2(u-\hbar) R_{12}(2u-\hbar) K_1(u)\, \mathcal{P}^-_{12}\,,$$

impliying the triangularity conditions

$$\mathcal{P}^+_{12}\, K_1(u) R_{12}(2u+\hbar) K_2(u+\hbar) \mathcal{P}^-_{12} = 0\,,$$
$$\mathcal{P}^-_{12}\, K_1(u) R_{12}(2u-\hbar) K_2(u-\hbar) \mathcal{P}^+_{12} = 0\,.$$

To each solution $K(u)$ of the reflection equation, we can thus associate the fused boundary matrix

$$K_{(12)}(u) = \mathcal{P}^+_{12}\, K_1(u - \frac{\hbar}{2}) R_{12}(2u) K_2(u + \frac{\hbar}{2})\, \mathcal{P}^+_{12}\,, \tag{5.35}$$

that will satisfy the following generalized reflection equation:

$$R_{1(23)}(u-v) K_1(u) R_{(23)1}(u+v) K_{(23)}(v) = K_{(23)}(v) R_{(23)1}(u+v) K_1(u) R_{1(23)}(u-v)\,. \tag{5.36}$$

Using the above exchange relation, it is not difficult to prove the following

Proposition 5.6 *The boundary matrix (5.35) satisfies the following fused reflection equations:*

$$R_{(12)(34)}(u-v) K_{(12)}(u) R_{(34)(12)}(u+v) K_{(34)}(v) =$$
$$= K_{(34)}(v) R_{(34)(12)}(u+v) K_{(12)}(u) R_{(12)(34)}(u-v)\,, \tag{5.37}$$

with the fused matrix $R_{(12)(34)}(u)$ defined as in section 5.2.

Proof: For the sake of brevity, we omit the arguments of the R and K matrices as well as the projectors $\mathcal{P}^+$ acting on both sides of the equation. After trivial transformations, the left hand side of eq.(5.37) reads:

$$R_{1(34)} K_1 R_{2(34)} R_{12} R_{(34)1} K_2 R_{(34)2} K_{(34)}\,.$$

Owing to eq.(5.36) and to the fused Yang–Baxter equation we can write down the following chain of transformations, leading to the right hand side of eq.(5.37) and proving the proposition:

$$
\begin{aligned}
& R_{1(34)}K_1R_{(34)1}R_{12}R_{2(34)}K_2R_{(34)2}K_{(34)} = \\
&= R_{1(34)}K_1R_{(34)1}R_{12}K_{(34)}R_{(34)2}K_2R_{2(34)} = \\
&= R_{1(34)}K_1R_{(34)1}K_{(34)}R_{12}R_{(34)2}K_2R_{2(34)} = \\
&= K_{(34)}R_{(34)1}K_1R_{1(34)}R_{12}R_{(34)2}K_2R_{2(34)} = \\
&= K_{(34)}R_{(34)1}K_1R_{(34)2}R_{12}R_{1(34)}K_2R_{2(34)} = \\
&= K_{(34)}R_{(34)1}R_{(34)2}K_{(12)}R_{1(34)}R_{2(34)} \,.
\end{aligned}
$$

■

The monodromy matrices corresponding to open spin chains also depend on the inverse of the local T matrices. We thus need an expression for the inverse[2] of the fused matrix $R_{(12)(34)}(u)$. A straightforward calculation shows that

$$R_{1(23)}(u)R_{1(23)}(-u) = \big(u^2 - \frac{\hbar^2}{4}\big)\big(u^2 - \frac{9}{4}\hbar^2\big)\,\mathcal{P}^+_{23}\,,$$

and that

$$R_{(12)(34)}(u)R_{(12)(34)}(-u) \propto \mathcal{P}^+_{12}\mathcal{P}^+_{34}\,.$$

The above unitarity relations for the fused R matrices allow to define a monodromy matrix $\mathcal{B}_{(ab)}$ with fused auxiliary space (ab) for a N–site chain in the usual way:

$$\mathcal{B}_{(ab)}(u) = \mathcal{T}_{(ab)}(u)K_{(ab)}(u)\mathcal{T}^{-1}_{(ab)}(-u)\,, \tag{5.38}$$

where

$$\mathcal{T}_{ab}(u) = R_{(ab)(c_1d_1)}(u)R_{(ab)(c_2d_2)}(u)\cdots R_{(ab)(c_Nd_N)}(u)\,.$$

The quantum space is then identified with the tensor product of the spaces denoted with (c_kd_k), $k = 1,\ldots,N$. The following corollary of proposition 5.6 will allow us to build integrable systems with fused boundary matrices.

Corollary 5.7 *The monodromy matrix (5.38) satisfies the defining relation of the reflection algebra on fused spaces. As a consequence, the transfer matrix*

$$b(u) = str_{(ab)}\mathcal{B}_{(ab)}(u)$$

generates a family of commuting observables:

$$[b(u)\,,b(v)] = 0\,.$$

[2]By inverse of a fused R matrix, we mean the inverse of its restriction to the invariant subspace of the projectors $\mathcal{P}^{\pm}$.

6

Integrability from coalgebra symmetry: an $osp(1|2)$ spin chain

In this chapter, exposing the results of our work [17], we briefly discuss a different algebraic approach to integrable spin chains, based on a general procedure to construct and solve long–range spin chains with coalgebra symmetry in the case of rank 1 algebras or superalgebras. As a particular example, we consider a Gaudin model related to the q–deformed superalgebra $\mathcal{U}_q(osp(1|2))$, and present an exact solution to that system diagonalizing a complete set of commuting observables.

The aim is to emphasize the advantages and the drawbacks of the analytical Bethe Ansatz approach proposed in the previous chapters by comparing it to another general construction of integrable systems. We shall first describe the model and its solution, and then discuss some analogies of the proposed solution with the Bethe Ansatz. At the same time we shall compare the general features of the models solvable through the coalgebraic approach with the ones built from Yangians.

6.1 Integrability from coalgebras

The Gaudin model, introduced by M. Gaudin in 1976, is a quantum mechanical system involving long–range spin interaction [76, 77].

In [78] it was solved in the framework of the algebraic Bethe Ansatz. It was also shown there that the model is governed by a Yang–Baxter algebra, called the Gaudin algebra, with commutation relations linear in the generators and determined by a classical r-matrix. It is to be stressed that this features are present in the model despite its quantum mechanical nature. In fact the Gaudin model is one of a large class of models, with such an algebraic nature, so that its study becomes an important issue.

Let us recall that the superalgebra extension of the Gaudin algebra, and of the related $r-$matrix structure, has been worked out in some remarkable papers (see for instance [79, 80]) where the Gaudin model related to orthosymplectic Lie superalgebra $osp(1|2)$ has been constructed and solved through a brilliant generalization of the Bethe-Ansatz.

It is known that this algebraic richness and robustness allows one to use it as a testing ground for many ideas such as the Bethe Ansatz and the general procedure of separation of variables.

Among these approaches, the coalgebraic one was introduced in a series of papers [81, 82, 83, 84]. A general and constructive connection between coalgebras and integrability can be stated as follows: given *any* coalgebra $(g\,,\Delta)$ with Casimir element C, each of its representations gives rise to a family of completely integrable Hamiltonians $H^{(m)}$, $m = 1, ..., N$ with an arbitrary number N of degrees of freedom.

Endowing this coalgebra with a suitable additional structure (either a Poisson bracket or a non–commutative product on g), both classical and quantum mechanical systems can be obtained from the same (g, Δ). It is important to emphasize that the validity of this general procedure by no means depends on the explicit form of Δ (i.e., on whether the coalgebra (g, Δ) is deformed or not).

In this framework a particular class of coalgebras that can be used to construct systematically integrable systems are the so–called *q-algebras* [85]. The feature of such systems will be that they are integrable deformations of the ones obtained applying the same method to the corresponding non-deformed coalgebra.

We briefly recall here a general construction of completely integrable quantum systems associated with Lie (rank-1) superalgebras based on a coalgebraic approach [86]. Applying the method to higher ranks (super)algebras it is still possible to obtain commuting observables but completeness is by no means guaranteed [87].

Let us consider a Lie superalgebra g with Casimir $C \in \mathcal{U}(g)$, and a co-associative linear mapping $\Delta : \mathcal{U}(g) \mapsto \mathcal{U}(g) \otimes \mathcal{U}(g)$ (denoted as coproduct) such that Δ is a Lie homomorphism:

$$[\Delta(a), \Delta(b)] = \Delta([a, b]), \qquad \forall\, a, b \in \mathcal{U}(g)\,.$$

The coassociativity property allows one to construct from Δ in an unambiguous way subsequent homomorphisms

$$\Delta^{(2)} \doteq \Delta, \qquad \Delta^{(3)} : \mathcal{U}(g) \,\mapsto\, \mathcal{U}(g)^{\otimes 3}, \quad \ldots \quad , \Delta^{(N)} : \mathcal{U}(g) \,\mapsto\, \mathcal{U}(g)^{\otimes N}.$$

Thus, we can associate to our superalgebra, (or better *co-superalgebra*) a quantum integrable system with N degrees of freedom, whose Hamiltonian is an arbitrary function of the N-th coproduct of the generators and the remaining $N-1$ integrals of motion are provided by $\Delta^{(m)}(C)$, $m = 2, \ldots, N$.

In [88, 89] it has been shown how to associate to a Lie–Hopf superalgebra a quantum integrable system and how to extend this procedure to q-superalgebras. In fact, q-superalgebras are obtained by Lie–Hopf superalgebras through a process of deformation that preserve their Lie–Hopf structure. It is therefore possible to associate to q-superalgebras integrable systems that are deformed version of the ones associated to the original superalgebra.

We will consider an integrable q-deformation of the Lie superalgebra $osp(1|2)$ in order to obtain a Gaudin model with $\mathcal{U}_q(osp(1|2))$ symmetry.

6.2 A q-deformation of $\mathcal{U}(osp(1|2))$

The quantum superalgebra $\mathcal{U}_q(osp(1|2))$ [90, 91] as a deformation of the universal enveloping algebra of the Lie superalgebra $osp(1|2)$ is generated by three elements E, F, H, with gradation H=0 and $[E] = [F] = 1$. The q-deformed commutation relations between the generators are:

$$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}, \qquad [H, E] = E, \qquad [H, F] = -F \tag{6.1}$$

In the following we will also need the operators F^2 and E^2 fulfilling the commutation relations

$$\begin{aligned}
\left[F^2, E\right] &= \kappa \left(q^{H+1/2} + q^{-H-1/2}\right) F,\\
\left[E^2, F\right] &= -\kappa \left(q^{H-1/2} + q^{-H+1/2}\right) E,\\
\left[E^2, F^2\right] &= \kappa^2 - \kappa \frac{q^{2H+1/2} - q^{-2H-1/2}}{q - q^{-1}} + (q^H - q^{-H}) E\, F,\\
[H, E^2] &= 2\, E^2,\\
[H, F^2] &= -2\, F^2,
\end{aligned}$$

where

$$\kappa \doteq \frac{1}{q^{1/2} + q^{-1/2}}.$$

The center of $\mathcal{U}_q(osp(1|2))$ is spanned by the q-deformed Casimir element (provided that q is not a root of the unity [92]):

$$C(q) = \left(\frac{q^{H-1/2} + q^{-H+1/2}}{q - q^{-1}}\right)^2 - \kappa^2 E^2 F^2 - \left(q^{H-1} - q^{-H+1}\right) E\, F. \tag{6.2}$$

We now endow $\mathcal{U}_q(osp(1|2))$ with a coalgebra structure. This can be done assigning the following q-deformed coproduct:

$$\begin{aligned}
\Delta_q(H) &= H \otimes \mathbb{1} + \mathbb{1} \otimes H,\\
\Delta_q(E) &= E \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes E,\\
\Delta_q(F) &= F \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes F.
\end{aligned} \tag{6.3}$$

which establish a superalgebra homomorphism:

$$\begin{aligned}
[\Delta_q(E), \Delta_q(F)] &= \frac{\Delta_q(q^H) - \Delta_q(q^{-H})}{q - q^{-1}},\\
[\Delta_q(H), \Delta_q(E)] &= \Delta_q(E),\\
[\Delta_q(H), \Delta_q(F)] &= -\Delta_q(F).
\end{aligned}$$

For the sake of completeness we give the corresponding antipode and counit,

$$\epsilon(H) = \epsilon(E) = \epsilon(F) = 0, \qquad \epsilon(q^{\pm H}) = 1,$$

$$\sigma(E) = -q\, E, \qquad \sigma(F) = -q^{-1} F, \qquad \sigma(H) = -H, \qquad \sigma(q^{\pm H}) = q^{\mp H},$$

obtaining a Hopf superalgebra.

The coproducts (6.3) can be extended to the N-th order by means of the coassociativity property as in the non deformed case, taking into account that

$$\Delta_q(q^H) = q^H \otimes q^H.$$

Explicitly,

$$\begin{aligned}
\Delta_q^{(N)}(H) &= \sum_{i=1}^{N} H_i,\\
\Delta_q^{(N)}(E) &= \sum_{i=1}^{N} E_i\, q^{\frac{1}{2}\sum_{j=1}^{N} \mathrm{sgn}(i-j) H_j},\\
\Delta_q^{(N)}(F) &= \sum_{i=1}^{N} F_i\, q^{\frac{1}{2}\sum_{j=1}^{N} \mathrm{sgn}(i-j) H_j}.
\end{aligned}$$

Remark 6.1 *In the limit $q \to 1$ the above deformed supercommutation relations obviously reduce to well-known supercommutation relations of the Lie superalgebra $osp(1|2)$. Let us recall that $osp(1|2)$ has dimension five and rank one; the supercommutation relations between its generators are*

$$[E, F] = H, \qquad [H, E] = E, \qquad [H, F] = F.$$

$$[E, E] = 2E^2, \qquad [F, F] = 2F^2,$$

$$[E^2, F] = -E, \qquad [F^2, E] = F,$$

$$[H, E^2] = 2E^2, \qquad [H, F^2] = -2F^2, \qquad [F^2, E^2] = H.$$

We see that the operators H, E^2, F^2 generate the Lie algebra $\mathfrak{sl}(2)$. The above supercommutation relations define H, E^2, F^2 as the bosonic generators, and E, F as the fermionic ones, i.e. $[H] = [E^2] = [F^2] = 0$ and $[F] = [E] = 1$. This gradation can naturally be extended to the deformed enveloping superalgebra $\mathcal{U}_q(osp(1|2))$, since

$$[a\,b] = [a] + [b], \qquad \bmod 2 \qquad \forall\, a, b \in \mathcal{U}_q(osp(1|2)), \tag{6.4}$$

and $[q^H] = 0$.

In the same limit $q \to 1$, definitions (6.3) also reduce to the non–deformed coproducts for the superalgebra $osp(1|2)$, which we will denote with Δ.

6.3 Exact solution of a $\mathcal{U}_q(osp(1|2))$ Gaudin model

Now we have all we need to construct a Gaudin model with $\mathcal{U}_q(osp(1|2))$ symmetry in the coalgebra setting.

We consider the N commuting observables $\{C^{(n)}(q)\}_{n=1}^N$:

$$\left[C^{(m)}(q), C^{(n)}(q)\right] = 0, \quad \forall\, m, n = 1, ..., N,$$

where

$$\begin{aligned} C^{(m)}(q) &= \Delta_q^{(m)}\left[C(q)\right] = \\ &= \left[\frac{\Delta_q^{(m)}\left(q^{H-1/2}\right) + \Delta_q^{(m)}\left(q^{-H+1/2}\right)}{q - q^{-1}}\right]^2 - \kappa^2 \Delta_q^{(m)}(E^2)\, \Delta_q^{(m)}(F^2) + \\ &\quad - \left[\Delta_q^{(m)}\left(q^{H-1}\right) - \Delta_q^{(m)}\left(q^{-H+1}\right)\right] \Delta_q^{(m)}(E)\, \Delta_q^{(m)}(F). \end{aligned}$$

Hereafter we parametrize the deformation parameter with $z \doteq \ln q$.

A "physical" Gaudin Hamiltonian for the N-bodies system can be choosen as the N-th order deformed coproduct of the Casimir $\Delta_q^{(N)}\left[C(z)\right]$, namely

$$\begin{aligned} \mathcal{H}_q = &\frac{\sinh^2\left[z\left(\Delta_q^{(N)}(H) - \frac{1}{2}\right)\right]}{\sinh^2 z} - \kappa^2 \Delta_q^{(N)}(E^2)\, \Delta_q^{(N)}(F^2) + \\ &-2\cosh\left[z\left(\Delta_q^{(N)}(H) - 1\right)\right] \Delta_q^{(N)}(E)\, \Delta_q^{(N)}(F). \end{aligned} \tag{6.5}$$

This Hamiltonian can be written in any representation of the deformed superalgebra $\mathcal{U}_q(osp(1|2))$. While it's always possible to choose a particular one, we will work in the general case of spin j representation (with integer or half-integer j). Further generalization can be obtained by allowing site–dependent representations $(j_1, ..., j_N)$. However, for the sake of simplicity, we will not deal with this more general case in this chapter.

A complete set of independent commuting observables is provided by

$$\left\{\Delta_q^{(N)}(H), C^{(2)}(z), \ldots, C^{(N)}(z)\right\}. \tag{6.6}$$

We can write the Hamiltonian (6.5) in the following form:

$$\mathcal{H} = \frac{\sinh^2\left[z\left(\sum_{i=1}^N H_i - \frac{1}{2}\right)\right]}{\sinh^2 z} - \kappa^2 \sum_{i,j,k,l=1}^N \eta_i\, \eta_j\, \phi_k\, \phi_l - 2\cosh\left[z\left(\sum_{i=1}^N H_i - 1\right)\right] \sum_{i,j=1}^N \eta_i\, \phi_j,$$

where

$$\eta_i \doteq E_i\, q^{\frac{1}{2}\sum_{j=1}^N \mathrm{sgn}(i-j)H_j}, \qquad \phi_i \doteq F_i\, q^{\frac{1}{2}\sum_{j=1}^N \mathrm{sgn}(i-j)H_j}.$$

Notice that the interaction involves more than two sites: this non–local feature is a peculiar property of q–deformed models.

We will show that the common eigenstates of the family of observables (6.6) take the form

$$\varphi_z(k, m_l, s_{m_l}; \ldots, 0, 0) = \left[\Delta_q^{(N)}(E)\right]^{k-m_l} \psi_z(m_l, s_{m_l}; \ldots, 0, 0), \tag{6.7}$$

where $\psi_z(m_l, s_{m_l}; \ldots; 0, 0)$ is an element of the basis spanning the kernel of the lowering operator $\Delta_q^{(s_{m_l})}(F)$. These elements can be obtained through the recursive formula:

$$\psi_z(m_l, s_{m_l}; \ldots; 0, 0) = \sum_{i=0}^{\delta m} \alpha_i(z) \left[\Delta_q^{(s_{m_l}-1)}(E)\right]^{\delta m - i} (E_{s_{m_l}})^i\, \psi_z(m_{l-1}, s_{m_{l-1}}; \ldots; 0, 0), \tag{6.8}$$

where $\delta m \doteq m_l - m_{l-1}$ and $\{\alpha_i(z)\}_{i=1}^{\delta m}$ denotes a set of suitable coefficients. Since in each representation of $\mathcal{U}_q(osp(1|2))$ we have $E^{4j+1} = 0$, j being the spin of the choosen representation, the sum in formula (6.8) will have at most $4j+1$ terms, so that $\delta m \leq 4j$. If we consider the pseudo–vacuum state

$$\psi_z(0,0) = |\downarrow \cdots \downarrow\rangle \in \mathrm{Ker}\left(\Delta_q^{(k)}(F)\right), \qquad \forall\, k = 1, \ldots, N, \tag{6.9}$$

we recognize that m_l stands for the total number of excitations with respect to $\psi_z(0,0)$ and s_{m_l} indicates the number of the last excited site (counting from the left).

Proposition 6.2 *The states (6.8) are annihilated by $\Delta_q^{(s_{m_l})}(F)$ if and only if*

$$\frac{\alpha_{i+1}(z)}{\alpha_i(z)} = (-1)^{i+1} e^{\frac{z}{2}(\tau+\delta m-2)} \frac{(-1)^{\delta m - i}\sinh\left[z\left(\tau+\delta m - i - \frac{1}{2}\right)\right] - \sinh\left[z\left(\tau - \frac{1}{2}\right)\right]}{(-1)^{2j}\sinh\left[z\left(j+\frac{1}{2}\right)\right] + \sinh\left[z\left(i-j+\frac{1}{2}\right)\right]}, \tag{6.10}$$

$i = 0, \ldots, \delta m - 1$, *where τ is the eigenvalue of $\Delta_q^{(s_{m_l})}(H)$.*

Proof: A straightforward computation. Notice that it may be useful the following expression

$$F\,E^k + (-1)^{k-1}E^k F = \frac{(-1)^{k-1}E^{k-1}}{2\sinh z \cosh\frac{z}{2}} \left\{(-1)^k \cosh\left[z\left(H+\frac{1}{2}\right)\right] + \cosh\left[z\left(H-\frac{1}{2}\right)\right]\right\},$$

holding for all $k \in \mathbb{N}$. The above formula is a plain consequence of the supercommutation relations (6.1).

■

Up to a normalization constant, proposition 6.2 determines all coefficients $\alpha_i(z)$ with $i = 1, \ldots, \delta m$.

Proposition 6.3 *The states (6.7) are eigenvectors of the set (6.6), namely*

$$C^{(n)}(z)\, \varphi_z(k, m_l, s_{m_l}, \ldots, 0, 0) = l_n(z)\, \varphi_z(k, m_l, s_{m_l}, \ldots, 0, 0), \tag{6.11}$$

with eigenvalues l_n given by

$$l_n = \frac{\sinh^2\left[z\left(\rho_z - \frac{1}{2}\right)\right]}{\sinh^2 z}, \tag{6.12}$$

where ρ_z is the eigenvalue of $\Delta_q^{(n)}(H)$ on the state $\psi_z(i, s_i, \dots)$, and the value of $i \leq l$ is selected by the condition

$$s_{m_i} \leq n < s_{m_{i+i}}, \qquad s_{m_{l+1}} = N + 1. \tag{6.13}$$

Proof: Notice that

$$C^{(n)}(z)\,\varphi_z(k, m_l, s_{m_l}, \dots, 0, 0) = \left[\Delta_q^{(N)}(E)\right]^{k-m_l} C_h(z)\,\psi(m_l, s_{m_l}, \dots, 0, 0),$$

since $[C^{(h)}(z), \Delta_q^{(N)}(E)] = 0$ for all $n = 1, ..., N$.
If $n \geq s_{m_l}$ we readily get (6.11–6.12) since $\psi_z(m_l, s_{m_l}, \dots, 0, 0)$ is in $\operatorname{Ker} \Delta_q^{(n)}(F)$. Otherwise, if $n < s_{m_l}$ we can note that

$$\left[C^{(n)}(z), \sum_{i=0}^{\delta m} \alpha_i(z) \left[\Delta_q^{(s_{m_l}-1)}(E)\right]^{\delta m - i} (E_{s_{m_l}})^i\right] = 0,$$

so that we can act with $C^{(n)}(z)$ on $\psi_z(m_{l-1}, s_{m_{l-1}}, \dots, 0, 0)$. By iteration we will arrive to a value of i such that condition (6.13) holds and to a function $\psi_z(i, s_i, \dots)$ which fixes the value of ρ_z and so the eigenvalue (6.12). This proves the proposition.

■

Remark 6.4 *We stress the fact that our approach has a simple algebraic interpretation. Indeed, each eigenstate $\varphi_z(k, m_l, s_{m_l}, \dots, 0, 0)$ has to be a basis vector of the tensor product representation*

$$\left[\mathcal{D}^{(j)}\right]^{\otimes N} = \bigoplus_{l=0}^{Nj} c_{j,l}^{(N)}\,\mathcal{D}_l, \tag{6.14}$$

where $\mathcal{D}^{(j)}$ denotes the representation of each site and $\{c_{j,l}^{(N)}\}$ is the set of Clebsch–Gordan coefficients. Our method constructs first the lowest weight vectors $\psi_z(m_l, s_{m_l}, \dots, 0, 0)$ for each $\mathcal{D}_l$, taking account that $l = Nj - m_l$ and then allows us to complete the basis with suitable raising operators.

Thanks to the Schur's Lemma the eigenvalues of the family (6.6) are the values taken by the Casimir (6.2) on each $\mathcal{D}_l$. Furthermore the coefficients $\{c_{j,l}^{(N)}\}$ are related to the spectrum degeneracies; in fact the number of eigenstates of the Hamiltonian (6.5) that belong to the the energy eigenvalue corresponding to the representation $\mathcal{D}_l$ is given by

$$g_{j,l}^{(N)} = c_{j,l}^{(N)}(4l + 1),$$

being the factor $4l + 1$ the degeneracy of each $\mathcal{D}_l$. This latter term could be removed by an external field, while the first one remains.

6.3.1 The $q \to 1$ limit

We now obtain some known results [86] on the Gaudin model with $osp(1|2)$ symmetry considering the limit $q \to 1$.

The family of N commuting observables is $\{C^{(n)}\}_{n=1}^N$:

$$\left[C^{(m)}, C^{(n)}\right] = 0, \quad \forall\, m, n = 1, ..., N,$$

where

$$C^{(m)} = \Delta^{(m)}(C) = \left[\Delta^{(m)}(H)\right]^2 - 2\left[\Delta^{(m)}(E^2), \Delta^{(m)}(F^2)\right] - \left[\Delta^{(m)}(E), \Delta^{(m)}(F)\right].$$

A "physical" non–deformed Gaudin Hamiltonian for the N-bodies system can be choosen as the N-th order coproduct of the Casimir $\Delta^{(N)}(C)$, namely

$$\mathcal{H} = \sum_{i \neq j}^{N} H_i\, H_j - 2\left(E_i^2\, F_j^2 + F_i^2\, E_j^2\right) - \left(E_i\, F_j - F_i\, E_j\right). \tag{6.15}$$

Up to a term proportional to the identity, (6.15) corresponds to the limit $z \to 0$ of the Hamiltonian (6.5), i.e. $\lim_{z\to 0} \mathcal{H}_q = \mathcal{H} + 1/4$. A complete set of independent commuting observables is provided by

$$\left\{\Delta^{(N)}(H),\ C^{(2)}, \dots,\ C^{(N)}\right\}. \tag{6.16}$$

Taking the limit $z \to 0$ in the definition of the states (6.7–6.8–6.9) (i.e. replacing Δ_q with Δ) we obtain the following results:

Proposition 6.5 *The states $\psi(m_l, s_{m_l}, \dots, 0, 0)$ are annihilated by $\Delta^{(s_{m_l})}(F)$ if and only if*

$$\frac{\alpha_{i+1}}{\alpha_i} = \frac{2\,(-1)^{\delta m - i}\left(\tau + \delta m - i - \frac{1}{2}\right) - 1 - 2\tau}{(-1)^{i+1}(1+4j) + 2i + 1 - 4j}, \tag{6.17}$$

where τ is the eigenvalue of $\Delta^{(s_{m_l})}(H)$.

Proposition 6.6 *The states $\varphi(k, m_l, s_{m_l}, \dots, 0, 0)$ are eigenvectors of the set (6.16), namely*

$$C^{(n)}\, \varphi(k, m_l, s_{m_l}, \dots, 0, 0) = l_n\, \varphi(k, m_l, s_{m_l}, \dots, 0, 0), \tag{6.18}$$

with eigenvalues l_n given by

$$l_n = (\rho - i + 1)(\rho - i) + \frac{1}{4}, \tag{6.19}$$

where ρ is the eigenvalue of $\Delta^{(n)}(H)$ on the state $\psi(i, s_i, \dots)$, and the value of $i \leq l$ is selected by the condition

$$s_{m_i} \leq n < s_{m_{i+i}}, \qquad s_{m_{l+1}} = N + 1. \tag{6.20}$$

6.3.2 $\mathcal{U}_q(osp(1|2))$ Gaudin model with $j = 1/2$

Here we consider the particular case of the fundamental representation, namely the spin $j = 1/2$ one ($1 \leq \delta m \leq 2$). This case greatly simplify calculations, allowing a meaningful understanding of the results we have presented in the previous section.

Proposition 6.2 becomes the following one.

Proposition 6.7 *The states (6.8) with $\delta m = 1$ are annihilated by $\Delta_q^{(s_{m_l})}(F)$ if and only if*

$$\alpha_0(z) = 1, \qquad \alpha_1(z) = e^{\frac{z}{2}(\tau - 1)} \frac{\sinh(z\,\tau)}{\sinh z}.$$

The states (6.8) with $\delta m = 2$ are annihilated by $\Delta_q^{(s_{m_l})}(F)$ if and only if

$$\alpha_0(z) = 1, \qquad \alpha_1(z) = -e^{\frac{z\,\tau}{2}} \frac{\cosh\left[z\left(\tau + \frac{1}{2}\right)\right]}{\cosh\left(\frac{z}{2}\right)}, \qquad \alpha_2(z) = e^{z\,\tau} \frac{\sinh(z\,\tau)\cosh\left[z\left(\tau + \frac{1}{2}\right)\right]}{\sinh z \cosh\left(\frac{z}{2}\right)},$$

where $\tau = m_{l-1} - s_{m_l} + 1$.

On the other hand proposition 6.3 reduces to

Proposition 6.8 *The states (6.7) are eigenvectors of the set (6.6), namely*

$$C^{(n)}(z)\,\varphi_z(k, m_l, s_{m_l}, \dots, 0, 0) = l_n(z)\,\varphi_z(k, m_l, s_{m_l}, \dots, 0, 0),$$

with eigenvalues $l_n(z)$ *given by*

$$l_n(z) = \frac{\sinh^2\left[z\left(n - i + \frac{1}{2}\right)\right]}{\sinh^2 z},$$

where the value of $i \le l$ *is selected by the condition*

$$s_{m_i} \le n < s_{m_{i+i}}, \qquad s_{m_{l+1}} = N + 1.$$

In this case it is also possible to determine explicitly the degeneracies of the spectrum. These obviously correspond to those of the spin 1 case of the original $sl(2)$ Gaudin model. Namely, (6.14) now reads:

$$\left[\mathcal{D}^{\left(\frac{1}{2}\right)}\right]^{\otimes N} = \bigoplus_{l=0}^{\frac{N}{2}} c^{(N)}_{\frac{1}{2},l}\,\mathcal{D}_l,$$

and the following result can be proved by means of the character identity.

Proposition 6.9 *The total number of eigenstates* $\varphi(k, m_l, s_{m_l}, \dots, 0, 0)$ *with* $m_l = N/2 - l$ *is given by*

$$c^{(N)}_{\frac{1}{2},l} = \sum_{k=l}^{\left[\frac{N+l}{2}\right]} \binom{N}{2k-l}\binom{2k-l}{k} - \sum_{k=l}^{\left[\frac{N+l-1}{2}\right]} \binom{N}{2k-l+1}\binom{2k-l+1}{k+1}.$$

6.4 The coalgebraic and the Yangian based approaches: a short comparison

We now summarize some features of the long–range systems that can be obtained through the coalgebraic approach (among which the above Gaudin model is only an example), and compare them with the class of models we have obtained in this thesis using the structure of Yangians.

- *Symmetry*
  From the symmetry point of view, complete integrability through the coalgebraic approach is established for rank one superalgebras only. Moreover, complete reducibility of tensor product representations is needed, thus forcing in practice the choice of $osp(1|2)$ as symmetry algebra. Partial results are available for higher rank (super)algebras of the $sl(n)$ and $osp(1|2n)$ series, but the situation is in striking contrast with the full generality of the Yangian based approach, where any $gl(m|n)$ superalgebra can be made the symmetry of the transfer matrix.

- *Boundaries*
  In the coalgebraic approach, each site interacts *in the same way* with all other sites, so that it is not possible to identify the first and last sites of the chain. This rules out the possibility of intriducing non–trivial boundary conditions.

- *Inhomogeneities*
  At the same time, at least in the q–deformed case, it is still unclear whether the introduction of a spectral parameter dependence is possible without destroying integrability.

In particular, and in contrast with the Yangian case, it is still not possible to introduce inhomegeneity parameters in the chain.

This point and the one above can be summarized by saying that, in the class of models built from coalgebras, each site is equivalent to the others. The peculiar mean–field dynamics (and the definition of the related cluster variables, see [93] for a detailed discussion of the classical case) that characterizes these models is, in particular, a straightforward consequence of this fact.

- *Long range vs nearest neighbour interaction*
  The Yangian based models usually display nearest neighbour interactions, at least in the fundamental representation. As a general rule, the range of the interaction changes by changing the representation (usually increasing with its dimension). On the other hand, the coalgebraic apprach always leads to long range interaction for both the hamiltonian and the commuting observables, regardless of the chosen representation.

- *Eigenvectors*
  The algebraic construction of the spectrum and eigenvectors of the coalgebraic systems is reduced to a representation theory problem: no Bethe Ansatz equations are needed. Equations (6.10) and (6.17) can be considered as quantization rules for the quantum numbers labelling the eigenstates of the hamiltonian.

7

Conclusions and open perspectives

We list here the results obtained in the present thesis, together with some related open problems.

- We have presented a general and systematic approach to the construction of integrable transfer matrices with $gl(m|n)$ symmetry, based on the graded Yangian $Y(m|n)$. Integrable spin chains or correlated electrons models can be generated with the usual methods from these transfer matrices, and they share a global $gl(m|n)$ supersymmetry with the transfer matrices they are built from. The key point allowing a unified treatment of all these models – regardless of the choice of m and n, as well as of the representation chosen for the spin variables – is that their integrability only relies on the algebraic structure underlying the construction of the transfer matrix, that closely resemble their non supersymmetric counterparts.

- In order to solve the obtained integrable models, we have developed a graded variant of the analytical Bethe Ansatz approach, in which the spectrum is built by properly "dressing" a reference eigenvalue $\Lambda_0(u)$ (whose eigenvector is the so–called *pseudovacuum*) of the transfer matrix with rational functions depending on a set of generalized quantum numbers, related to the Bethe roots and to the conserved charges of the model. The Bethe equations are then obtained requiring the cancelation of the poles of these "dressed eigenvalues". The results are again independent on the representation, thus allowing the explicit calculation of the spectrum for several kinds of models, ranging from the well–known cases of spin chains where all sites carry the fundamental representation (e.g., the t–J model), to more complicated situations including alternating spin chains, chains with impurities etc. A particularity of superalgebras (that usual algebras do not share) is the existence of different Dynkin diagrams for the same superalgebra. This leads to different presentations of the spectrum of the same transfer matrix, hence to different Bethe equations. Our approach is also universal in the sense that it applies to all Dynkin diagrams of the considered superalgebra.

- There is in principle a huge amount of physical hamiltonians that could fit in our approach: some of them are presented in this work (with a particular focus on the simplest case of $gl(1|2)$), but there is still room for searching interesting cases that can be solved through analytical Bethe Ansatz. The general formulation of a dressing hypothesis for fused models based on typical representations could be an important generalization of our approach, allowing the treatment of integrable spin chains with additional free parameters.

- We generalized our treatment to open supersymmetric spin chains, finding again the spectrum and the Bethe equations. In order to extend the analytical Bethe Ansatz to these integrable systems, we studied the reflection superalgebra and the twisted Yangian as the most natural subalgebras of $Y(m|n)$ related to reflection equations. Thus, after identifying the possible integrable boundary conditions as solutions to the reflection equations, we proved that the same pseudovacuum vector as in the closed case is still an eigenstate of the transfer matrices, both for the reflection algebra and for the twisted super Yangian case. By means of a suitably modified version of the dressing hypothesis, the spectrum of the transfer matrices has been calculated, together with the corresponding Bethe equations. It is worth noticing that, somewhat unexpectedly, the presence of poles in the boundary matrix makes the treatment of the reflection algebra case easier than the closed chain treatment. From the algebraic point of view, some new results about the representations of the reflection superalgebra are presented here for he first time, although a complete classification of the irreducible representations remains to be done.

- In the graded reflection algebra case, we have exploited the algebraic structures to classify the commuting transfer matrices on the basis of their symmetry superalgebras (that, unlike the closed chain case, explicitly depend on the choice of the boundary matrices), in order to shed light on the common physical properties of the hamiltonians that can be generated through them.

- For the twisted Yangian case, the situation is still incomplete: the only boundary matrices we considered in this work are diagonal ones, and a classification of the possible symmetries remains to be done. A promising approach to this problem that could possibly be extended to the superalgebra case is the one presented in the recent work [94]. This generalization is under current investigation.

- Another natural developement of our results is the generalization to the trigonometric case. This has been accomplished for the non–graded case in the recent paper [95], where the analytical Bethe Ansatz approach has been succesfully applied to closed and open spin chains derived from the algebra $U_q(gl(n))$.

- The problem of the construction of the eigenvectors of our transfer matrices is, as a matter of fact, completely left out by our approach. A remarkable amount of information about the excitations of the system in the thermodynamic limit can be extracted from the Bethe equations solely[1], but the calculation of very relevant quantities as the correlation functions needs the knowledge of the eigenvectors as prerequisite. The ultimate goal is thus the formulation of a universal algebraic Bethe Ansatz method for deriving the eigenvectors for any irreducible representation of $Y(n)$ and $Y(m|n)$. Such a process will exclusively rely on the exchange relations emerging from the Yang–Baxter or reflection equation. No results in this direction, of the greatest mathematical and physical relevance, are known at the present moment, and the beforehand knowledge of the Bethe equations that our approach provides could be of great help in such a project.

[1] Moreover, in the thermodynamic limit, it is not necessary to actually solve the Bethe equations, that can be transformed into linear integral equations for the density of the solutions, see e.g. [96].

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