Robust Control and Observation of LPV Time-Delay Systems

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THESE

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Commande et Observation Robustes des Systèmes LPV Retardés

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Finalement un grand merci à tous les reviewers anonymes qui ont jugé mon travail avec plus ou moins de sévérité qui m'auront permis de progresser dans mes travaux. . . Je me serais bien vengé depuis. . .

Summary

This thesis is concerned with the stability analysis, observation and control of LPV time-delay systems. The main objectives of the thesis are

- the development of adapted and possibly low conservative stability sufficient conditions for LPV time-delay systems.
- the development of new advanced control/observation strategies for such systems using new tools developed in the thesis, such as specific relaxation techniques of Linear and Nonlinear Matrix Inequalities.

For that purpose, this thesis is subdivided is three parts:

- The first part, composed of Chapters 2 and 3, aims at providing a sufficiently detailed state of the art of the representation and stability analysis of both LPV and time-delay systems. In both cases, the importance of LMI in stability analysis is strongly emphasized. Several fundamental results are bridged to each other in order to show the relations between different theories and this constitutes the first part of the contributions of this work.
- The second part, composed of Chapter 4, consists in a presentation of several (new) preliminary results that will be used along the thesis. This part contains most of the contributions of this work.
- Finally, the third part, composed of Chapters 5 and 6, uses results of the second part in order to derive efficient observation, filtering and control strategies for LPV time-delay systems.

Introduction and Structure of the Thesis

Context of the Thesis

This thesis is the fruit of a three years work (2005-2008) spent in the GIPSA-Lab[1] (former LAG[2]) in the SLR[3] Team. The topic of the thesis is on **Robust/LPV Control and Observation of LPV Time-Delay Systems** under supervision of Olivier Sename (Professor at Grenoble-INP[4], France) and Jean-François Lafay (professor at Centrale Nantes, IRCCyN[5] Nantes, France).

This thesis is in the continuity of works of Annas Fattouh [Fattouh, 2000; Fattouh et al., 1998], Olivier Sename [Sename, 1994, 2001; Sename and Fattouh, 2005] and more deeply Jean-François Lafay who was the Olivier Sename's thesis supervisor (the thesis was on the controllability of time-delay systems).

During the thesis the Rhônes-Alpes Region granted me of a scolarship in order to travel and collaborate in a foreign laboratory. I went to School of Electrical and Computer Engineering (ECE) in GeorgiaTech (Georgia Institute of technology) to work with Erik I. Verriest on the topic of time-delay systems with applications in the control of disease epidemics. The collaboration gave rise to a conference paper '**A New Delay-SIR Model for Pulse Vaccination**' [Briat and Verriest, 2008] and potentially to a journal version according to the invitation of the editor of the new Elsevier journal: 'Biomedical Signal Processing and Control'.

Finally, thanks to Emmanuel Witrant (GIPSA Lab), I incorporated the project on the control of unstable modes in plasmas in Tokamaks, cores of the promising future (?) energy production technology exploiting nuclear fusion. The collaboration is done with Erik Olofsson and Per Brunsell (KTH[6]) and S-I. Niculescu (LSS[7]). The work has led to the conference paper [Olofsson et al., 2008].

[1] Grenoble Image Parole Signal Automatique Laboratory - Grenoble Image Speech Signal Control Systems
[2] Laboratoire d'Automatique de Grenoble - Grenoble Control Systems Laboratory
[3] Systèmes Linéaires et Robustesse - Linear Systems and Robustness
[4] Grenoble INstitut Polytechnique - Grenoble Institute of Technology
[5] Institut de Recherche en Communications et Cybernétique de Nantes - Nantes Research Institute in Communications and Cybernetics
[6] Kungliga Tekniska högskolan - Royal Institute of Technology, Sweden
[7] Laboratoire des Signaux et Systèmes - Signals and Systems Laboratory

Introduction and Motivations

At the beginning of the century, Emile Picard (french mathematician) wrote an interesting remark in the proceedings of the 4th International Mathematician Congress in Rome (the complete text will be provided at the end of this section). He noticed that while in classical mechanics equations in which the future is immediately predicted using current information (speed and position), it is not possible to predict the future in the same way when living beings are considered. Indeed, the future evolution would depend on the current information but also on past events. Mathematically speaking, the evolution would consider integral term taken from past to current time and would describe the heredity. In the 1970s such equations began to be studied and these studies give rise to several books in the 1980s. Since then, time-delay systems (which is the modern and usual denomination) have gained more and more interest both in theoretical problems (stability analysis, control and observation...) and applicative problems since they arise in engineering, biology, ecology and economics (examples will be given in Section 3.1). This presence in many different fields has strengthened their importance in modern theories of dynamical systems and control.

Due to the particular structure of these systems, lots of specific approaches have been developed or generalized from more simple cases in order to study their stability, controllability and many other important properties. As a fundamental example, Lyapunov theory has been extended to this type of systems through two celebrated theorems, namely the Lyapunov-Krasovskii and Lyapunov-Razumikhin theorems. From these results lots of advances have been made but many problems remain open.

A great problem in the stability analysis of linear systems is the robustness of the stability. In few words, it consists in determining the stability of a linear system whose constant coefficients belong to a certain interval. Several tools have been developed to study these systems (e.g. μ-analysis) and have led to important results which have been applied successfully to solve challenging engineering problems notably in aerospace. Furthermore, robust stabilization is also an important research framework and is still an open problem.

Another crucial problem is the existence of systems which are not robustly stabilizable. To deal with such a problem it seems necessary to develop a novel approach and here comes LPV control... The idea behind LPV control is to use in the control law the knowledge of the parameters involved in the system. It turns out that, using such a control strategy, the class of systems which are stabilizable is wider than when the values of the parameters are considered as uncertainties.

Moreover, LPV systems can be used to approximate nonlinear systems and hence systematic and generic 'LPV tools' can be applied to derive nonlinear control laws for nonlinear systems. Another interest of LPV control is the design of tunable controllers: external parameters can be added in the design in order to characterize different working modes.

The idea of merging time-delay systems and LPV systems is not new but is rather marginal. Indeed, only few works are based on the stability analysis and control synthesis. No work exists on the observation and few results are provided for the filtering problem. At first sight, it seems straightforward to find solutions to problems involving LPV time-delay systems. Indeed, would it be enough to merge both theories ? Actually it is not so simple, many results in robust stability analysis and robust control developed for finite dimensional systems do not work with time-delay systems. This makes the study of LPV time-delay systems a more complex problem.

Emile Picard's original text [Kolmanovskii and Myshkis, 1999]:

"Les équations différentielles de la mécanique classique sont telles qu'il y en résulte que le mouvement est déterminé par la simple connaissance des positions et des vitesses, c'est à dire par l'état à un instant donné et à l'instant infiniment voisin".

"Les états antérieurs n'y intervenant pas, l'hérédié y est un vain mot. L'application de ces équations où le passé ne se distingue pas de l'avenir, où les mouvements sont de nature réversibles, sont donc inapplicables aux êtres vivants".

"Nous pouvons rêver d'équations fonctionelles plus compliquées que les équations classiques parce qu'elles renfermeront en outre des intégrales prises entre le temps passé très éloigné et le temps actuel, qui apporteront la part de l'hérédité".

Emile Picard, "La mathématique dans ses rapports avec la physique, Actes du IV^{e} congrès international des Mathématiciens, Rome, 1908

English Translation

'Differential equations of classical mechanics are such that the movement is determined by the only knowledge of positions and speeds, that is to say by the state at a given instant and at the instant infinitely nearby.

Since the anterior states are not involved, heredity is a vain word. The application of these equations where the past and future are not distinguishable, where the movements are by nature reversible, are hence unapplicable to living beings.

We may dream about more complex functional equations than classical equations since they shall contain in addition integral terms taken from a distant past time instant and the current time instant, which shall bring the share of heredity.'

Emile Picard, "La mathématique dans ses rapports avec la physique, Proceedings of the IVth Mathematicians' International Symposium, Rome, 1908

Structure of the Thesis

Chapter 1 provides a short summary of the Thesis in French.

Chapter 2 provides an overview of different types of representation for a LPV system. For each model, several adapted stability tests are presented and relative merits are compared.

Chapter 3 gives an insight of different representations of time-delay systems and several physical examples show the interest of such systems. Then a large part is concerned with the stability analysis of these systems in the time domain in which several methods of the literature are presented and compared. A last section addresses the problem of the stability in presence of uncertain delay.

Chapter 4 is devoted to the introduction of preliminary notions and important results used along the thesis. First of all, spaces of delays and parameters are clearly defined.

Second, new methods of relaxation of parameter dependent LMI and matrix inequalities with concave nonlinearity are developed and analyzed. Then a method to compute explicit expression of parameter derivatives in LPV polytopic systems is given using linear algebra. Finally, several Lyapunov-Krasovskii based techniques are given in order to show asymptotic stability of LPV systems.

Chapter 5 presents results in observation and filtering of LPV systems using results provided in Chapter 4. Several types of observers and filters are studied in both certain and uncertain frameworks.

Chapter 6 concludes on the stabilization of LPV time-delay systems. Several structures for the controllers are explored according to the presence of a delayed term in the control law; both state-feedback and dynamic output feedback controllers are synthesized. This chapter also presents a new type of controllers which is called *delay-scheduled controllers* whose gains are smoothly scheduled by the delay value.

Contributions

The contribution of the thesis is plural:

- Methodological contributions
- Theoretical contributions

Methodological contribution

The methodological contribution is based on a common remark by reading journal and conference papers. Why most of the papers concern the stability of time-delay systems only ? Why are there only few papers on the control and observation or filtering ? The main reason comes the fact that, when considering time-delay systems, it is not sufficient to substitute the closed-loop system expression in the stability condition to derive efficient and easy to compute constructive stabilization conditions (taking generally the form of a set of LMIs). This is mainly due to the presence of a high number of decision matrices in the stability conditions.

A global method is to perform a relaxation after substitution of the closed-loop system (which is the direct and efficient method used for finite dimensional systems). We emphasize in this thesis that this may be not the right choice since this alters the efficiency of the initial result. So we preconize to perform a relaxation technique on the initial problem in order to turn the original stability condition into a form which is more suitable for synthesis purposes. Hence a step is added in the design methodology and allow to improve the results. One of the great interests of the relaxation is its adaptability to a wide variety of different LMI stability conditions.

Theoretical contributions

The theoretical contributions are multiple and address several different topics:

- Concave nonlinearities (involving inverse of matrices) in matrix inequality are quite difficult to handle and their simplification (or removal) generally results in conservative conditions. Bounds involving completion by the squares and using the cone complementary algorithm can be used but while the former is too conservative, the latter cannot be used with parameter dependent matrices. To solve this problem we have developed a new exact relaxation which turns the rational dependence into a bilinear one and allows for the application of simple iterative algorithm.

- Several LMI tests have been generalized to the LPV case and the relaxation method have been applied in order to provide new LMI tests better suited for the resolution of design problems.
- A new Lyapunov-Krasovskii functional has been developed in order to consider systems with two delays in which the delays satisfy an algebraic constraint. This functional addresses well the problem of stabilizing a time-delay system with a controller with memory embedding a delay value which is different from the system one.
- A new strategy to control time-delay systems has been introduced and has been called 'delay-scheduled' controllers. This type of controllers are designed using a LPV formulation of time-delay systems. Then using LPV design tools, it is possible to derive controllers whose gain is smoothly scheduled by the delay value, provided that it is measured or estimated. Since the delay is viewed as a parameter, then it is possible to consider uncertainties on the delay and perform robust stabilization in presence of measurement/estimation errors.
- Finally, the last contributions are based on the application of new and adapted stability tests to observation, filtering and control. Such methods will be shown to lead to interesting results.

Contents

List of Figures

List of Tables

List of Publications

Journal Papers

- C. Briat, O. Sename and J.F. Lafay, 'From Stability Analysis to Memory-Resilient Robust Control of Uncertain LTI/LPV Time-Delay Systems - A Discretized Lyapunov-Krasovskii Functional Approach' Submitted to IEEE Transactions on Automatic Control
- C. Briat, O. Sename and J.F. Lafay, 'Memory Resilient Gain-scheduled State-Feedback Control of Uncertain LPV Time-Delay Systems with Time-Varying Delays', Submitted to Systems and Control Letters
- C. Briat and E. Verriest, 'A New Delay-SIR Model for Pulse Vaccination', Accepted with minor changes at Biomedical Signal Processing and Control
- C. Briat, O. Sename and J.F. Lafay, '$\mathcal{H}_\infty$ delay-scheduled control of linear systems with time-varying delays', Accepted at IEEE Transactions on Automatic Control
- C. Briat, O. Sename and J.F. Lafay, 'Delay-Scheduled State-Feedback Design for Time-Delay Systems with Time-Varying Delays - A LPV Approach', Accepted to Systems & Control Letters

International Conference Papers with Proceedings

- C. Briat and J.J. Martinez, "Design of $\mathcal{H}_\infty$ Bounded Non-Fragile Controllers for Discrete-Time Systems", Submitted to 48^{th} *Conference on Decision and Control*, Shanghai, China, 2009.
- C. Briat, O. Sename and J.F. Lafay, "Memory Resilient Gain-scheduled State-Feedback Control of Time-Delay Systems with Time-Varying Delays", Accepted at 6^{th} *IFAC Symposium on Robust Control Design*, Haifa, Israel, 2009.
- C. Briat, O. Sename and J.F. Lafay, "$\mathcal{H}_\infty$ Filtering of Uncertain LPV systems with time-delays", Accepted at 10^{th} *European Control Conference*, Budapest, Hungary, 2009.
- E. Olofsson, E. Witrant, C. Briat, S-I. Niculescu, P. Brunsell, 'Stability Analysis and Model-Based Control in EXTRAP-T2R with Time-Delay Compensation', 47th IEEE Conference on Decision and Control, Cancun, Mexico, 2008
- C. Briat, E. I. Verriest, 'A new delay-SIR Model for Pulse Vaccination', IFAC World Congress, Seoul, South Korea, 2008

- C. Briat, O. Sename and J.F. Lafay, 'Delay-Scheduled State-Feedback Design for Time-Delay Systems with Time-Varying delays', IFAC World Congress, Seoul, South Korea, 2008
- C. Briat, O. Sename and J.F. Lafay, 'Parameter dependent state-feedback control of LPV time-delay systems with time-varying delays using a projection approach', IFAC World Congress, Seoul, South Korea, 2008
- C. Briat, O. Sename and J.F. Lafay, 'A Full-Block S-procedure application to delay-dependent $\mathcal{H}_\infty$ state-feedback control of uncertain time-delay systems', IFAC World Congress, Seoul, South Korea, 2008
- C. Briat, O. Sename and J.F. Lafay, A LFT/$\mathcal{H}_\infty$ state feedback design for linear parameter varying time-delay systems', European Control Conference, Kos, Greece, 2007
- O. Sename, C. Briat, '$\mathcal{H}_\infty$ observer design for uncertain time-delay systems', European Control Conference, Kos, Greece, 2007
- C. Briat, O. Sename and J.F. Lafay, 'Full order LPV/$\mathcal{H}_\infty$ Observers for LPV Time-Delay Systems', IFAC Conference on System, Structure and Control, Foz do Iguacu, Brazil, 2007
- O. Sename, C. Briat, 'Observer-based $\mathcal{H}_\infty$ control for time-delay systems: a new LMI solution', IFAC Conference on Time-Delay Systems, L'Aquila, Italy, 2006

National Conference and Workshop Papers with Proceedings

- C. Briat, O. Sename, J.F. Lafay, 'Filtrage $\mathcal{H}_\infty$/LPV de systèmes LPV incertains à retards' ($\mathcal{H}_\infty$/LPV filtering of uncertain LPV Time-Delay Systems), CIFA Conference, Bucharest, Romania, 2008

National Conference and Workshop Papers without Proceedings

- C. Briat, O. Sename, J.F. Lafay, 'Delay-scheduled controllers for time-delay systems with time-varying delays', GDR MACS MOSAR, Grenoble, France, 2008
- C. Briat, O. Sename, J.F. Lafay, 'Delay-scheduled controllers for time-delay systems with time-varying delays', GDR MACS SAR, Lille, France, 2007
- C. Briat, O. Sename, J.F. Lafay, 'LPV/LFT Control for Time-delay systems with time-varying delays - A delay independent result', GDR MACS SAR, Paris, France, 2006

Notations and Acronyms

$\mathbb{N}$	Set of integers
$\mathbb{Z}$	Set of rational integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$	Set of real numbers, nonnegative real numbers, strictly positive real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{K}^{p\times m}$	Matrix algebra (of dimension $p \times m$) with coefficients in $\mathbb{K}$
$\mathbb{S}^n$	Set of symmetric matrices of dimension n
$\mathbb{S}^n_+$	Cone of positive symmetric semidefinite matrices of dimension n
$\mathbb{S}^n_{++}$	Cone of positive symmetric definite matrices of dimension n
$\mathbb{H}^n$	Set of skew-symmetric matrices of dimension n
j	Imaginary unit (i.e. $j^2 = -1$)
I_n	Identity matrix of dimension n
$\mathbb{1}_{n\times m}$	$n \times m$ matrix with entries equal to 1
$0_{n\times m}$	$n \times m$ matrix with entries equal to 0
A^T	Transpose matrix of A
A^*	Conjugate transpose of A
$A^{\mathcal{S}}$	Hermitian operator such that $A^{\mathcal{S}} := A + A^T$ ($A^{\mathcal{S}} := A + A^*$ for complex matrices)
A^{-1}	Inverse of matrix A
A^{-T}	Transpose of the inverse of A
A^+	Moore-Penrose pseudoinverse of A
$\mathrm{trace}(A)$	Trace of A
$\det(A)$	Determinant of A
$\mathrm{Adj}(A)$	Adjugate matrix of A
$\mathrm{Ker}(A)$	Basis of the null-space of A
$\mathrm{Im}(A)$	Image set of A
$\mathrm{Null}(A)$	Null space of the operator A (i.e. $Im(\mathrm{Ker}[A])$)
$\lambda(A)$	Set of the eigenvalues of A
$\sigma(A)$	Set of singular values of A: $\sqrt{\lambda(A^*A)}$
$\lambda_{max}(A)$, $\lambda_{min}(A)$	Maximal and minimal eigenvalues of A
$\bar{\sigma}(A)$	Maximal singular value of A (i.e. $\sqrt{\lambda_{max}(A^*A)}$)
$\otimes$	Kronecker product

$\|\|w\|\|_q$	q Euclidian norm of vector w, i.e. $\|\|w\|\|_q = (\|w_1\|^q + \ldots + \|w_n\|^q)^{1/q}$
$\|\|w\|\|_{\mathcal{L}_q}$	$\mathcal{L}_q$ norm of signal defined by $\left(\int_0^{+\infty} \|\|w(t)\|\|_q dt\right)^{1/q}$
$\|\|T\|\|_{\mathcal{H}_q}$	$\mathcal{H}_q$ norm of operator/system T
$\|\|T\|\|_{\mathcal{L}_p-\mathcal{L}_q}$	mixed induced-norm of the operator/system T defined by $\frac{\|\|Tw\|\|_{\mathcal{L}_p}}{\|\|w\|\|_{\mathcal{L}_q}}$ with $\|\|w\|\|_{\mathcal{L}_q} \neq 0$
$\|\|M\|\|_q$	Induced q-norm of the matrix M i.e. $\|\|M\|\|_q = \frac{\|\|Mw\|\|_q}{\|\|w\|\|_q}$ for $\|\|w\|\|_q \neq 0$
$\mathrm{col}(\alpha, \beta, \ldots, \delta)$	column vector with component $\alpha, \beta, \ldots, \delta$
$\mathrm{col}_i(x_i)$	column vector with components $x_1, x_2, \ldots$
$\mathcal{C}^1(J, K)$	Set of continuously differentiable functions from set J to K
$\mathcal{F}(J, K)$	Set of functions from J to K
$\mathbb{D}$	Unit open disc
$\bar{\mathbb{D}}$	Closure of $\mathbb{D}$
$\partial\mathbb{D}$	Boundary of $\mathbb{D}$
$h(t)$	Time-Delay
μ^-, μ^+	Bounds on the derivative of the delay such that $\dot{h} \in [\mu^-, \mu^+]$
ρ	vector of parameters
U_ρ	Space of parameter values $U_\rho := \times_{i=1}^p [\rho_i^-, \rho_i^+] \subset \mathbb{R}^p$
U_ν	Set of vertices of the polytope containing $\dot{\rho}$ defined as $U_\nu := \times_{i=1}^p \{\nu_i^-, \nu_i^+\}$ compact of $\mathbb{R}^p$
$\mathrm{hull}[U]$	Convex hull of the set U (smallest convex set containing U)
$\Re(z)$	Real part of z
$\Im(z)$	Imaginary part of z
$\square$	End of proof

LTI	Linear Time-Invariant
LPV	Linear Parameter Varying
LTV	Linear Time-Varying
TS	Takagi-Sugeno
LMI	Linear Matrix Inequality
pLMI	parametrized Linear Matrix Inequality
NMI	Nonlinear Matrix Inequality
BMI	Bilinear Matrix Inequality
LFT	Linear Fractional Transformation
LFR	Linear Fractional Representation
TDS	Time-Delay System
PSD	Positive Symmetric Definite
CCA	Cone Complementary Algorithm
SF	State-Feedback
SOF	Static Output Feedback
DOF	Dynamic Output Feedback
BIBO	Bounded-Input Bounded-Output
SISO	Single Input-Single Output
MIMO	Multi-Input/Multi-Output
SoS	Sum-of-Squares
SDP	Semidefinite Program

Chapter 1

Introduction et Résumé Détaillé

Ce résumé en Français a pour but de donner un bref aperçu du travail effectué pendant ces trois ans de thèse au sein du GIPSA-Lab, anciennement le Laboratoire d'Automatique de Grenoble. Cette thèse a été encadrée par Olivier Sename (GIPSA-Lab) et Jean-François Lafay (IRCCyN, Nantes). Elle porte sur la thématique du contrôle et de l'observation des systèmes à retards dépendant de paramètres.

Lors de ma deuxième année de thèse, j'ai eu l'opportunité de faire un séjour de 6 mois à GeorgiaTech (Atlanta, USA) pour collaborer avec Erik Verriest grâce à une bourse de la région Rhône-Alpes. Cette collaboration a donné lieu à une publication en conférence et une revue internationale.

Les objectifs de la thèse portaient sur l'élaboration de lois de commande et la synthèse d'observateurs pour les systèmes à retard dépendant de paramètres. Cette classe de systèmes mélangeant deux classes de systèmes: les systèmes à retards et les systèmes à paramètres variants.

1.1 Introduction Générale et Motivations

Cette partie est dédiée à la présentation des systèmes abordés pendant de cette thèse.

1.1.1 Systèmes à Retards

Les systèmes à retards font partie de la famille des systèmes de dimension infinie et peuvent être définis par différents formalismes. Le plus utilisé actuellement est celui des équations différentielles fonctionnelles telles que:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x(t-h) \\ x(\theta) & = & \phi(\theta),\ \theta \in [-h_m, 0] \end{array} \tag{1.1}$$

où $x \in \mathbb{R}^n$ est l'état du système et $\phi(\cdot)$ est la condition initiale fonctionnelle. Comme on peut le voir dans l'équation de la dynamique du système, l'évolution de l'état dépend non seulement de l'état courant $x(t)$ mais aussi d'une valeur passée de l'état $x(t-h)$ dont le retard h est (généralement) mal connu et appartient à un intervalle, par exemple $[0, h_m]$ où h_m désigne ici la valeur maximale du retard.

L'analyse de stabilité des équations différentielles retardées est possible en utilisant des généralisations de la théorie de la stabilité de Lyapunov [Lyapunov, 1992]. La stabilité est dans ce cas déterminée par l'existence de fonctionnelles de Lyapunov-Krasovskii ou de fonctions de Lyapunov-Razumikhin. Les fonctionnelles de Lyapunov-Krasovskii conduisent à des résultats moins conservatifs que les fonctions de Lyapunov-Razumikhin [Gu et al., 2003] et sont donc beaucoup plus utilisées dans la litérature. Voici un exemple de fonctionnelle de Lyapunov-Krasovskii qui est utilisée dans cette thèse:

$$\begin{aligned} V(x_t, \dot{x}_t) &= x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta \\ &\quad + \int_{-h_m}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T R \dot{x}(s) ds d\theta \end{aligned} \tag{1.2}$$

où P, Q, R sont des matrices définies positives à déterminer de sorte que la dérivée de V le long des trajectoires du système satisfasse

$$\dot{V} \leq -w(||x(t)||) \tag{1.3}$$

où $w(\cdot)$ est une fonction positive telle que $w(0) = 0$. Cela signifie alors que la fonctionnelle V est décroissante sur les trajectoires du système et V converge vers 0, valeur correspondant à l'équilibre du système. Nous avons donc une condition suffisante pour la stabilité du système à retard. Ainsi toute la difficulté est de trouver une "bonne" fonctionnelle qui soit le moins conservative possible, c'est à dire qui se rapproche le plus d'une condition nécessaire et suffisante.

Il est possible de définir deux types de stabilité pour les systèmes à retards:

La stabilité indépendante du retard qui permet d'analyser le comportement du système pour n'importe quelle valeur du retard de 0 à l'infini.

La stabilité dépendante du retard qui ne considère que des retards bornés.

Par exemple, la fonctionnelle de Lyapunov-Krasovskii (1.2) permet de tester la stabilité du système pour n'importe quel retard compris entre 0 et h_m. On a donc un résultat pour la stabilité dépendante du retard.

Une conséquence importante de l'utilisation de l'approche de Lyapunov-Krasovskii est l'obtention de conditions de stabilité sous la forme d'Inégalités Linéaires Matricielles (Linear Matrix Inequalities - LMI) [Boyd et al., 1994]. Ce type de conditions peut être facilement résolu en utilisant des algorithmes spécialisés comme les algorithmes du point intérieur [Boyd et al., 1994; Nesterov and Nemirovskii, 1994]. Ainsi, étant donné le système, il est possible de tester numériquement si il existe des matrices P, Q, R satisfaisant (1.3).

Par exemple, le test de stabilité du système (1.1) en utilisant (1.2) est énoncé dans le lemme suivant:

Lemme 1.1.1 *Le système (1.1) est asymptotiquement stable pour tout retard constant $h \in [0, h_m]$ si il existe des matrices P, Q, R définies positives telles que la LMI*

$$\begin{bmatrix} A^T P + PA + Q - R & PA_h + R & h_m A^T R \\ \star & -Q - R & h_m A_h^T R \\ \star & \star & -R \end{bmatrix} \prec 0 \tag{1.4}$$

soit satisfaite.

Le symbole $\prec$ signifie que la négativité est considérée en regardant les valeurs propres de la matrice et les symboles $\star$ indiquent les éléments symétriques de la matrice.

D'un autre côté, depuis quelques années, les retards variant dans le temps ont été de plus en plus considérés car ils sont induits, par exemple, dans les réseaux de télécommunications. L'analyse de la stabilité de systèmes avec retards variants dans le temps est beaucoup plus difficile car les systèmes sont désormais variant dans le temps. Ainsi le rôle de la vitesse de variation du retard joue un rôle prépondérant dans la stabilité du système.

L'ensemble des lois de contrôle des systèmes à retards est plus riche que pour les systèmes classiques. En effet, dans ce cas précis, il est possible d'utiliser l'information sur le retard. Si le contrôleur ne comporte pas de partie retardée, on dit qu'il est "sans-mémoire"; à l'inverse s'il en comporte une, on dit qu'il est "avec-mémoire". Ces types de correcteurs seront détaillés un peu plus loin dans ce résumé.

La recherche actuelle sur l'analyse et la commande des systèmes à retards variants dans le temps est active, cela étant dû, entres autres, à l'intérêt pratique et à la difficulté technique de ce contexte. Quelques résultats sont présentés dans les articles [Ariba and Gouaisbaut, 2007; Kharitonov and Niculescu, 2003; Papachristodoulou et al., 2007; Shustin and Fridman, 2007].

1.1.2 Systèmes Linéaires à Paramètres Variants

Les systèmes linéaires à paramètres variants (systèmes LPV) sont une classe étendue des systèmes linéaires classiques. Ce type de systèmes admet une représentation générale de la forme [Packard, 1994; Scherer, 2001; Wu, 2001a]:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho(t))x(t) \\ x(0) & = & x_0 \\ \rho & \in & U_\rho \\ \dot{\rho} & \in & \text{hull}[U_\nu] \end{array} \tag{1.5}$$

où x est l'état du système, x_0 est la condition initiale et $\rho(t)$ est le vecteur de paramètres. Il est généralement admis que les valeurs des paramètres sont bornées. Cependant, les propriétés des trajectoires des paramètres, comme la continuité ou la dérivabilité, ne sont pas imposées. En fonction des cas, les approches sont relativement différentes.

Par la suite, nous allons nous focaliser sur les paramètres dont les trajectoires sont différentiables ainsi l'ensemble hull$[U_\nu]$ est un ensemble compact et connecté.

La stabilité des systèmes LPV peut êre effectuée grâce à une extension des fonctions de Lyapunov utilisées pour les systèmes temps-invariant:

$$V(x,\rho) = x(t)^T P(\rho) x(t) \tag{1.6}$$

qui dépend à la fois de l'état du système et des paramètres. En appliquant le théorème de Lyapunov nous obtenons le résultat suivant [Wu, 2001a]:

Lemme 1.1.2 *Le système (1.5) est asymptotiquement stable si il existe une fonction à valeurs matricielles $P(\rho)$ définie positive pour tout $\rho \in U_\rho$ telle que la LMI dépendant de paramètres*

$$A(\rho)^T P(\rho) + P(\rho)A(\rho) + \sum_i \frac{\partial P(\rho)}{\partial \rho_i}\dot{\rho}_i \prec 0 \tag{1.7}$$

pour tout $(\rho,\dot{\rho}) \in U_\rho \times U_\nu$.

Cette LMI a deux particularités:

1. elle dépend des paramètres ρ et $\dot{\rho}$: on dit qu'elle est semi-infinie;

2. la variable à déterminer est une fonction: on dit que la LMI est de dimension infinie.

Ces deux faits rendent la résolution de cette LMI une tâche plutot difficile qui ne peut pas être directement traitée par des algorithmes.

Comme la LMI est paramétrisée, nous avons affaire en réalité à une infinité de LMIs. De nombreuses méthodes ont été développées afin de résoudre de telles LMIs. Le gridding [Apkarian and Adams, 1998], les méthodes basées sur les "sum-of-squares"[Scherer, 2008], etc...

Afin de simplifier le domaine de recherche de $P(\rho)$ pour le ramener à un problème de dimension finie, l'idée est de projeter la fonction sur une base, par exemple:

$$P(\rho) = P_0 + P_1\rho + \ldots + P_N\rho^N \tag{1.8}$$

Il est assez difficile de choisir une "bonne" base et il n'y a pas de théorie générale pour le faire.

L'aspect intéressant des systèmes LPV n'est pas l'aspect analyse de stabilité des systèmes mais plutot au niveau des possibilités offertes en terme contrôleurs. En effet, pour stabiliser le système il est possible de rechercher un correcteur qui dépend lui aussi des paramètres et permet de stabiliser plus efficacement les systèmes qu'un régulateur classique. On a ainsi une extension de la théorie du "gain-scheduling".

1.1.3 Les systèmes LPV à retards

Les systèmes LPV à retards sont des systèmes qui ont été très peu traités dans la littérature ce qui fait un sujet de recherche très intéressant car ces systèmes se trouvent à l'intersection des domaines des systèmes à retards et LPV. Ils hérient donc des difficultés inhérentes à chaque classe de systèmes mais aussi de nouveaux problèmes émergent. En effet, de nombreux résultats dévelopés dans des thémariques de la commande robuste et LPV ne sont plus applicable pour les systèmes LPV à retards.

La classe générale de systèmes considérée [Wu, 2001b; Zhang and Grigoriadis, 2005] est définie par les relations suivantes:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho,\rho_h)x(t) + A_h(\rho,\rho_h)x(t-h(t)) + E(\rho,\rho_h)w(t)\\ z(t) & = & C(\rho,\rho_h)x(t) + C_h(\rho,\rho_h)x(t-h(t)) + F(\rho,\rho_h)w(t)\\ x(\theta) & = & \phi(\theta), \quad \theta\in[-h_{max},0]\end{array} \tag{1.9}$$

où $x\in\mathbb{R}^n$, $w\in\mathbb{R}^m$, $z\in\mathbb{R}^p$ sont respectivement l'état du système, les entrées exogènes et la sortie contrôlée. Le retard $h(t)$ et les paramètres sont supposés appartenir dans un certain ensemble ayant certaines propriétés intéressantes pour être utilisées afin de réduire le conservatisme des aproches utilisées. Ces ensembles ne sont pas précisés ici pour ne pas complexifier inutilement ce résumé.

Les systèmes LPV retardés ne sont pas uniquement des systèmes abstraits mais peuvent être obtenus après simplification de systèmes non-linéaires. Par exemple, le système de fraisage représenté en Figure 1.1 peut être exprimé de telle sorte que son comportement soit régi par un modèle LPV à retard [Zhang et al., 2002]:

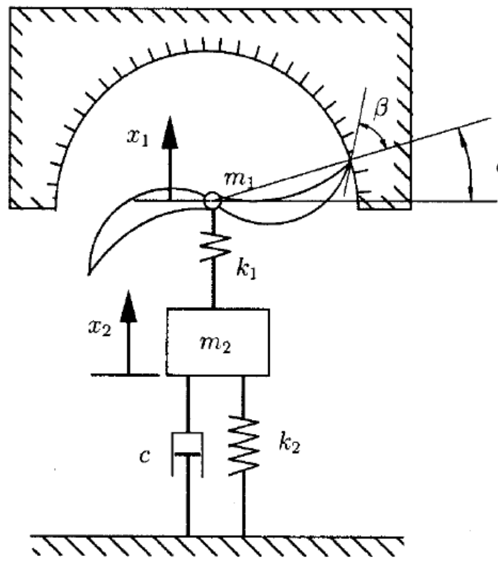

Figure 1.1: Géométrie simplifiée d'un processus de fraisage

$$\dot{x}(t) = (A + A_k k + A_\gamma \gamma + A_{k\gamma} k\gamma)x(t) + (A^h + A_k^h k + A_\gamma^h \gamma + A_{k\gamma}^h k\gamma)x(t-h) \qquad (1.10)$$

où les paramètres sont les coefficients de raideur des ressorts et $\gamma = \cos(2\phi + \beta) \in [-1, 1]$. Une discussion complète intéressante est donnée dans [Zhang et al., 2002].

1.2 Contributions

Les contributions apportées dans cette thèse peuvent être classée en deux parties:

les contributions méthodologiques qui sont relatives à la technique d'approche et la vision des problèmes abordés;

les contributions théoriques représentées par l'amélioration des résultats existants, le développement de nouveaux résultats et idées.

1.2.1 Contributions Méthodologiques

La contribution méthodologique est issue d'une remarque globale faite sur beaucoup de papiers de journaux et de conférences. Pourquoi est-ce que tant de papiers ne traite que le problème de l'analyse de stabilité des systèmes à retards ? Pourquoi y-a-t-il une aussi faible proportion de papiers sur le contrôle et l'observation ? La raison principale vient du fait que, lorsque l'on considère les systèmes à retard, il n'est généralement pas suffisant de remplacer le système en boucle fermée dans la condition de stabilité pour obtenir facilement à la fois une condition suffisante efficace (peu conservative) sur l'existence du contrôleur et donnant lieu à une procédure de construction du contrôleur (solution constructive). Cette difficulté est liée au grand nombre de matrices de décision utilisées pour analyser la stabilité des systèmes à retards.

Une méthode classique est basée sur la relaxation des conditions après la substitution du système en boucle fermée seulement. Cette méthode correspond à celle utilisée pour les systèmes linéaires de dimension finie. Dans cette thèse, nous mettons en avant que cette procédure n'est pas forcément la meilleure puisque qu'elle altère l'efficacité du résultat de stabilité. Nous préconisons donc de faire la relaxation avant d'injecter le système en boucle fermée afin de préparer le résultat de stabilité à être utilisé dans un contexte de stabilisation. Une étape est donc ajoutée dans la méthodologie de synthèse de contrôleurs. Cette procédure peut être appliquée à de nombreux types de résultats sous forme LMIs différents.

1.2.2 Contributions Théoriques

Le contributions théoriques sont réparties dans différents domaines:

- Les non-linéarités concaves (impliquant des inverses de matrices de décisions) dans les inégalités matricielles sont assez difficiles à traiter et leur simplification (ou suppression) résulte généralement en de conservatives conditions. Des bornes basées sur la complétion des carrés ou sur l'algorithme du cône complémentaire peuvent être utilisées mais tandis que la première est conservative, la deuxième ne peut pas être utilisée pour des matrices qui varient en fonction des paramètres (ce qui est assez gênant lorsque l'on travaille avec des systèmes LPV). Pour résoudre ce problème, nous avons introduit une relaxation exacte qui transforme le problème rationnel (qui implique l'inverse d'une matrice) en un problème bilinéaire. Bien que le problème demeure non-linéaire, il est plus facile de résoudre un problème bilinéaire qu'un problème rationnel en utilisant, par exemple, des algorithmes du type "D-K iteration".

- Plusieurs tests LMI ont été généralisés au cas LPV et la méthode de relaxation a été appliquée afin de développer de nouveaux tests LMIs dédiés à la synthèse de lois de commande.

- Un nouveau type de fonctionnelle de Lyapunov-Krasovskii a été développée afin de considérer des systèmes avec deux retards liés par une contrainte d'inégalité. Cette fonctionnelle permet de traiter assez bien le problème de stabilisation d'un système à retard par un contrôleur incorporant un retard qui est différent (mais proche) de celui du système.

- Une nouvelle stratégie de contrôle de systèmes à retards a également été introduite et nommé "controleur séquencé par retard" (delay-scheduled controller). Ces contrôleurs sont calculés à partir d'une reformulation LPV des systèmes linéaires à retards dans laquelle le paramètre est une fonction du retard. Ainsi il est possible de calculer un contrôleur LPV qui sera séquencé par la valeur du retard. Bien entendu, dans ce contexte il est supposé que le retard est connu. Afin de prendre en compte le fait que une erreur sur la connaissance du retard peut persister, la non-fragilité du contrôleur quand il n'utilise qu'une valeur approximative du retard est prise en compte.

- Finalement, les dernières contributions portent sur l'élaboration d'observateurs et filtres pour les systèmes LPV à retards. Ici aussi, différents cas sont traités avec bien sur le cas sans-mémoire, avec mémoire exacte et avec mémoire approximative. Les résultats sont obtenus avec les tests LMIs généraux préliminaires obtenus dans cette thèse.

1.3 Application à l'Observation et au Filtrage

1.3.1 Observation

L'observation des systèmes dynamiques consiste à estimer l'état (ou une partie de l'état) à partir de la connaissance (pas forcément exacte) du modèle du système et d'un ensemble de mesures. Le filtrage est sensiblement identique mais avec les filtres il est théoriquement possible d'estimer d'autres signaux que l'état au sens de la minimisation d'une certaine norme et non au sens de l'estimation asymptotique de l'erreur d'observation.

L'apport de la thèse dans ce domaine a été d'étendre les observateurs présentés dans [Darouach, 2001] au cas des systèmes LPV incertains. L'intérêt de cette méthode est l'approche purement algébrique qui permet de déterminer un ensemble d'observateurs qui satisfont des conditions nécessaires, puis avec un test basé sur des LMIs choisir quel observateur est le meilleur. Dans les cas étudiés, les observateurs sont choisis de telle sorte que l'énergie des perturbations transmises vers l'erreur d'observation soit la plus atténuée possible (minimisation de la norme $\mathcal{L}_2$ induite des perturbations vers l'erreur).

Les observateurs considérés sont des observateurs d'ordre réduit dont l'objectif n'est d'estimer qu'une partie de l'état du système:

$$\begin{array}{rcl}\dot{\xi}(t) &=& M_0(\rho)\xi(t) + M_h(\rho)\xi(t-h(t)) + S(\rho)u(t) + N_0(\rho)y(t) + N_h(\rho)y(t-h(t)) \\ \hat{z} &=& \xi(t) + Hy(t)\end{array} \quad (1.11)$$

Les systèmes observés ont l'allure générale suivante:

$$\begin{array}{rcl}\dot{x}(t) &=& A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ y(t) &=& Cx(t) \\ z(t) &=& Tx(t)\end{array} \quad (1.12)$$

où x est l'état du système, u la commande, w les perturbations, y la sortie mesurée, z la portion de l'état à estimer, ξ l'état de l'observateur et $\hat{z}$ l'estimée de z. Les matrices du systèmes sont supposées incertaines et peuvent être décomposées sous une forme générale

$$Q_0(\rho) + Q_1(\rho)\Delta Q_2(\rho) \quad (1.13)$$

où les matrices $Q_i(\rho)$ sont connues et Δ représente une incertitude structurée ou non, de norme bornée.

L'objectif est donc de choisir les matrices de l'observateur de telle sorte que

1. La dynamique de l'erreur d'observation $e(t) = z(t) - \hat{z}(t)$ est stable

2. $\gamma > 0$ défini tel que

$$\int_0^{+\infty} e(s)^T e(s)ds < \gamma^2 \int_0^{+\infty} w(s)^T w(s)ds \quad (1.14)$$

avec $e(t) = z(t) - \hat{z}(t)$ soit minimal.

Les résultats suivants ont été développés en Section 5.1.1.

Théorème 1.3.1 *Il existe un observateur LPV avec mémoire exacte de la forme (1.11) pour les systèmes de la forme (1.12) si et seulement si les propositions suivantes sont satisfaites:*

1. *La dynamique autonome de l'erreur* $\dot{e}(t) = M_0(\rho)e(t) + M_h(\rho)e(t-h(t))$ *est asymptotiquement stable avec* $e(t) = z(t) - \hat{z}(t)$.
2. $(T - HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T - HC) = 0$
3. $(T - HC)A_h(\rho) - N_h(\rho)C - M_h(\rho)(T - HC) = 0$
4. $(T - HC)B(\rho) - S(\rho) = 0$
5. *L'inégalité* $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ *est satisfaite pour un* $\gamma > 0$ *minimal.*

Lemme 1.3.2 *Il existe une solution* $M_0(\rho), M_h(\rho), N_0(\rho), N_h(\rho), S(\rho), H(\rho)$ *aux équations algébriques données dans le Théorème 1.3.1 si et seulement si la condition de rang ci-dessous est satisfaite:*

$$\text{rank}\begin{bmatrix} T & 0 \\ 0 & T \\ C & 0 \\ 0 & C \\ CA(\rho) & CA_h(\rho) \\ TA(\rho) & TA_h(\rho) \end{bmatrix} = \text{rank}\begin{bmatrix} T & 0 \\ 0 & T \\ C & 0 \\ 0 & C \\ CA(\rho) & CA_h(\rho) \end{bmatrix} \tag{1.15}$$

pour tout $\rho \in U_\rho$.

Le résultat ci-dessus permet de savoir si la structure du système autorise un observateur de la forme (1.11) satisfaisant les égalités algébriques du Théorème 1.3.1. Dans ce cas, il est possible de déduire une paramétrisation des matrices de l'observateur à travers une unique matrice L qui devra être déterminée de telle sorte que l'erreur d'observation soit asymptotiquement stable:

Lemme 1.3.3 *Sous la condition supposée satisfaite du Théorème 1.3.2, les matrices de l'observateur sont données par les expressions* $M_0 = \Theta - L\Xi$, $M_h = \Upsilon - L\Omega$ *et* $H = \Phi - L\Psi$ *où* L *est une matrice libre à déterminer*

$$\begin{aligned}
\Theta &= TAU - \Lambda\Gamma^+\Delta_0 \begin{bmatrix} C \\ CA \end{bmatrix} U & \Phi &= \Lambda\Gamma^+\Delta_H \\
\Xi &= -(I - \Gamma\Gamma^+)\Delta_0 \begin{bmatrix} C \\ CA \end{bmatrix} U & \Psi &= (I - \Gamma\Gamma^+)\Delta_H \\
\Upsilon &= TA_hU - \Lambda\Gamma^+\Delta_h \begin{bmatrix} C \\ CA_h \end{bmatrix} U & S &= FB \\
\Omega &= -(I - \Gamma\Gamma^+)\Delta_h \begin{bmatrix} C \\ CA_h \end{bmatrix} U & F &= T - HC \\
N_0 &= K_0 + M_0H & N_h &= K_h + M_hH
\end{aligned}$$

où la matrice U *est défini telle que*

$$\begin{bmatrix} T \\ \bar{T} \end{bmatrix}^{-1} = \begin{bmatrix} U & V \end{bmatrix}$$

avec la matrice de plein rang colonne $\bar{T}$ *telle que* $\begin{bmatrix} T \\ \bar{T} \end{bmatrix}$. *Les matrices* Δ_i *sont définies de la manière suivante:*

$$\Delta_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \Delta_h = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \Delta_H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$

Le résultat suivant donne une condition suffisante sur l'existence d'un observateur optimal au sens de la minimisation de la norme $\mathcal{L}_2$ induite du transfert des perturbations sur l'erreur d'observation.

Théorème 1.3.4 *Il existe un observateur dépendant des paramètres de la forme (1.11) pour des systèmes de la forme (1.12) tel que le Théorème 1.3.1 soit satisfait s'il existe une fonction matricielle continument différentiable* $P : U_\rho \to \mathbb{S}^r_{++}$, *une fonction matricielle* $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, *des matrices constantes* $Q, R \in \mathbb{S}^r_{++}$, $X \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ *et un scalaire positif* $\gamma > 0$ *tels que la LMI*

$$\begin{bmatrix} -(X+X^T) & \star & \star & \star & \star & \star & \star \\ U_{21}(\rho) & U_{22}(\rho,\nu) & \star & \star & \star & \star & \star \\ U_{31}(\rho) & R & -Q_\mu - R & \star & \star & \star & \star \\ U_{41} & 0 & 0 & -\gamma I_q & \star & \star & \star \\ 0 & I_r & 0 & 0 & -\gamma I_r & \star & \star \\ X & 0 & 0 & 0 & 0 & -P(\rho) & \star \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} \prec 0 \qquad (1.16)$$

soit satisfaite pour tout $(\rho, \nu) \in U\rho \times U_\nu$ *avec*

$$\begin{aligned} U_{21}(\rho) &= \Theta(\rho)^T X - \Xi(\rho)^T \bar{L}(\rho)^T + P(\rho) \\ U_{31}(\rho) &= \Upsilon(\rho)^T X - \Omega(\rho)^T \bar{L}(\rho)^T \\ U_{22}(\rho,\nu) &= \frac{\partial P(\rho)}{\partial \rho} - P(\rho) + Q - R \\ U_{41}(\rho) &= (\rho)E(\rho)^T(T^T X - C^T \bar{H}^T) \end{aligned}$$

et

$$\bar{L}(\rho) = (X^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \qquad (1.17)$$

De plus, le gain L *est donné par* $L(\rho) = X^{-T}\bar{L}(\rho)$ *et l'erreur d'estimation satisfait l'inégalité* $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

La méthode est illustrée avec l'exemple suivant:

Exemple 1.3.5 *Considérons le système proposé dans [Mohammadpour and Grigoriadis, 2007a] avec* $D_{21} = 0$ *qui est la matrice correspondant au transfert de* w *vers* y*:*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1+0.2\rho \\ -2 & -3+0.1\rho \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho & 0.1 \\ -0.2+0.1\rho & -0.3 \end{bmatrix} x_h(t) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x(t) \\ z(t) &= x(t) \end{aligned} \qquad (1.18)$$

Les matrices $Z(\rho)$ and $P(\rho)$ sont choisies pour être de forme polynomiale de degré 2. L'application du Théorème 1.3.4 nous donne $\gamma = 0.01$. L'évolution des erreurs d'observations est donnée sur la Figure 1.3 où il est possible de voir que les erreurs convergent bien vers 0. De plus les matrices de l'observateur sont données ci-dessous:

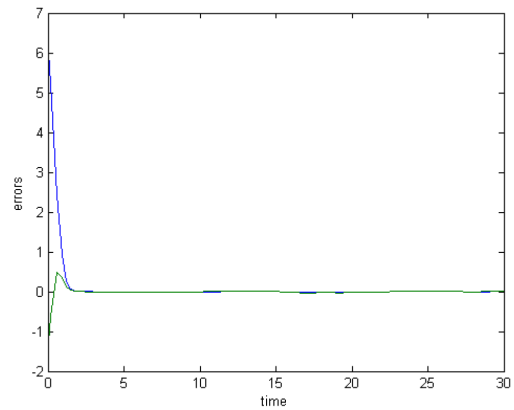

Figure 1.2: Évolution des erreurs d'observation

$$M_0(\rho) = \begin{bmatrix} -0.836\rho^2 - 0.836\rho - 0.667 & -0.078\rho^2 - 0.072\rho + 0.1345 \\ -0.0376\rho^2 - 0.0376\rho - 0.361 & -0.396\rho^2 - 0.406\rho - 0.800 \end{bmatrix}$$

$$M_h(\rho) = \begin{bmatrix} -0.009\rho^2 - 0.0002\rho + 0.00822 & -0.007\rho^2 - 0.0071\rho + 0.014 \\ 0.016\rho^2 - 0.00001\rho - 0.0162 & 0.0134\rho^2 + 0.0134\rho - 0.27 \end{bmatrix}$$

$$N_0(\rho) = \begin{bmatrix} -0.073\rho^2 - 0.063\rho + 0.326 & 0.146\rho^2 + 0.148\rho + 0.620 \\ 0.076\rho^2 + 0.058\rho - 0.684 & -0.152\rho^2 - 0.156\rho - 1.054 \end{bmatrix}$$

$$N_h(\rho) = \begin{bmatrix} 0.001\rho^2 + 0.001\rho + 0.040 & -0.001\rho^2 + 0.019\rho + 0.046 \\ -0.001\rho^2 - 0.27\rho - 0.077 & 0.002\rho^2 - 0.035\rho - 0.088 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.106 & 1.788 \\ 0.798 & 0.404 \end{bmatrix}$$

Pour la simulation, le retard est choisi constant $h = 0.5 < h_{max} = 0.8$. La perturbation en échelon d'amplitude 10 est appliquée à $t = 15s$ et la trajectoire des paramètres est sinusoïdale $\rho(t) = \sin(t)$.

1.3.2 Filtrage

L'objectif du filtrage est proche de celui de l'observation mais demeure un peu différent. L'objectif est de rapprocher la sortie du filtre de celle du système le plus possible, au sens d'une certaine norme.

L'équation générale des filtres est donnée par

$$\begin{array}{rcl} \dot{x}_F(t) & = & A_F(\rho)x_F(t) + A_{Fh}(\rho)x_F(t-h(t)) + B_F(\rho)y(t) \\ z_F(t) & = & C_F(\rho)x_F(t) + C_{Fh}(t-h(t)) + D_F y(t) \end{array} \quad (1.19)$$

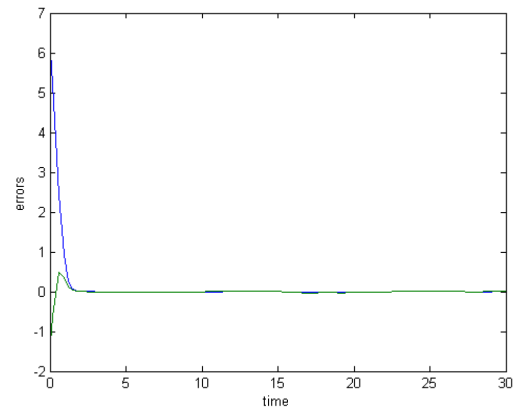

Figure 1.3: Évolution des erreurs d'observation

pour des systèmes de la forme:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t) \\ y(t) & = & C_y(\rho)x(t) + C_{hy}(\rho)x(t-h(t)) + F_y(\rho)w(t) \end{array} \quad (1.20)$$

Les matrices du filtre sont choisies de sorte que:

1. le système étendu formé par le filtre et le système soit stable

2. $\gamma > 0$ défini tel que

$$\int_0^{+\infty} e(s)^T e(s) ds < \gamma^2 \int_0^{+\infty} w(s)^T w(s) ds \quad (1.21)$$

avec $e(t) = z(t) - z_F(t)$ soit minimal.

Voici l'un des théorème donné en Section 5.2.1:

Théorème 1.3.6 *Il existe un filtre de la forme (1.19) s'il existe une fonction matricielle continument différentiable $\tilde{P} : U_\rho \to \mathbb{S}^{2n}_{++}$, des matrices constantes $\tilde{Q}, \tilde{R} \in \mathbb{S}^{2n}_{++}$, $\hat{X} \in \mathbb{R}^{2n \times 2n}$, des fonctions matricielles $\tilde{A}_F, \tilde{A}_{Fh} : U_\rho \to \mathbb{R}^{n \times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n \times m}$, $\tilde{C}_F, \tilde{C}_{Fh} : U_\rho \to \mathbb{R}^{t \times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n \times m}$ et un scalaire $\gamma > 0$ tels que la LMI*

$$\begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho) + \tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}\tilde{R} \\ \star & \tilde{\Psi}_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)\tilde{Q} - \tilde{R} & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} \prec 0 \quad (1.22)$$

soit satisfaite pour tout $\rho \in U_\rho$ *avec* $\tilde{P}(\rho) = \tilde{X}^T P(\rho)\tilde{X}$, $\tilde{Q} = \tilde{X}^T Q\tilde{X}$, $\tilde{R} = \tilde{X}^T R\tilde{X}$ *et*

$$\begin{array}{rclrcl} \Psi_{22}(\rho,\nu) & = & \partial_\rho \tilde{P}(\rho)\nu - \tilde{P}(\rho) + \tilde{Q} - \tilde{R} & \hat{X}_2 & = & X_2 X_4^{-1} X_3 = U^T \Sigma V \quad \text{(SVD)} \\ \hat{X} & = & \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} & \tilde{\mathcal{E}}(\rho) & = & \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho) C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho) C_y(\rho) \\ \end{bmatrix} \\ \tilde{\mathcal{C}}(\rho)^T & = & \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} & \tilde{\mathcal{C}}_h(\rho)^T & = & \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \end{array}$$

$$\begin{array}{rcl} \tilde{\mathcal{A}}(\rho) & = & \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho) C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho) C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\ \tilde{\mathcal{A}}_h(\rho) & = & \begin{bmatrix} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho) C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho) C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix} \end{array}$$

Dans ce cas, les matrices du filtre sont données par les expressions:

$$\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

où $\hat{X}_3 = U\Sigma V$. *De plus l'inégalité* $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ *est satisfaite.*

1.4 Application au Contrôle

La partie "contrôle" est la plus importante dans cette thèse et c'est certainement dans cette partie que les contributions les plus importantes sont présentes. Le système considéré ici est de la forme:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + D(\rho)u(t) + F(\rho)w(t) \\ y(t) & = & C_y(\rho)x(t) + C_{yh}(\rho)x(t-h(t)) + F_y(\rho)w(t) \end{array} \tag{1.23}$$

où z est la sortie à contrôler, u la commande et y la sortie mesurée.

1.4.1 Contrôleurs par retour d'état

Les contrôleurs par retour d'état sont les plus communs dans le contexte des systèmes à retard car ce sont les plus simples à synthétiser. Dans cette thèse trois types de retour d'état ont été considérés pour des systèmes de la forme:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + D(\rho)u(t) + F(\rho)w(t) \end{array} \tag{1.24}$$

1. les contrôleurs par retour d'état sans-mémoire:

$$u(t) = K_0(\rho)x(t) \tag{1.25}$$

2. les contrôleurs par retour d'état avec mémoire-exacte:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t-h(t)) \tag{1.26}$$

3. les contrôleurs par retour d'état avec mémoire-approximative

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \tag{1.27}$$

Les contrôleurs sont donc choisis de telle sorte que:

1. le système en boucle fermée soit asymptotiquement stable;

2. $\gamma > 0$ défini tel que

$$\int_0^{+\infty} z(s)^T z(s)ds < \gamma^2 \int_0^{+\infty} w(s)^T w(s)ds \tag{1.28}$$

   soit minimal.

Il est important de différencier les trois types de contrôleurs. Les contrôleurs sans-mémoire sont les plus simples à utiliser car ils ne nécessitent que l'information "instantanée" de l'état. Cependant, lorsque la valeur du retard peut être connue alors l'utilisation de contrôleurs avec mémoire exacte permet d'obtenir de meilleures performances en boucle fermée. Cependant, comme l'estimation/observation des retards est un problème difficile encore ouvert, les contrôleurs avec une mémoire exacte ne sont pas utilisables en pratique.

L'introduction dans cette thèse de contrôleurs avec une mémoire "approximative" est intéressante car ils permettent d'utiliser seulement une valeur erronée du retard à un instant donné pourvu que l'erreur entre le retard utilisé dans le contrôleur et le retard du système ne soit pas trop grande. De tels contrôleurs ont été considérés dans le passé mais les deux retards étaient considérés comme indépendant alors qu'ils ne le sont pas. En effet, le retard erroné reste dans une boule autour de la trajectoire du retard du système et cette relation doit être prise en compte. Elle est prise en compte grâce à l'introduction d'une nouvelle fonctionelle de Lyapunov-Krasovskii introduite dans la Section 4.7.

Ce théorème démontré dans la Section 6.1.5 donne un résultat sur l'existence d'un contrôleur avec ou sans mémoire:

Théorème 1.4.1 *Il existe un contrôleur de la forme (1.26) qui stabilise le système (1.23) si il existe une fonction matricielle dontinuement différentiable $\tilde{P} : U_\rho \to \mathbb{S}^n_{++}$, des fonctions matricielles $V_0, V_h : U_\rho \to \mathbb{R}^{m\times n}$, des matrices constantes $\tilde{Q}, \tilde{R} \in \mathbb{S}^n_{++}, Y \in \mathbb{R}^{n\times n}$ et un scalaire constant $\gamma > 0$ tels que la LMI*

$$\begin{bmatrix} -(Y+Y^T) & U_{12}(\rho) & U_{13}(\rho) & E(\rho) & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & U_{25}(\rho) & 0 & 0 \\ \star & \star & \tilde{U}_{33} & 0 & U_{26}(\rho) & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \\ & & & & & & \end{bmatrix} \prec 0 \tag{1.29}$$

soit satisfaite pour tout $(\rho,\nu) \in U_\rho \times U_\nu$ *où* $\Psi(\rho,\nu)$ *est défini par*

$$\begin{array}{rcl} U_{12}(\rho) & = & \tilde{P}(\rho) + A(\rho)Y + B(\rho)V_0(\rho) \\ U_{13}(\rho) & = & A_h(\rho)Y + B(\rho)V_h(\rho) \\ U_{25}(\rho) & = & Y^TC(\rho)^T + [D(\rho)V_0(\rho)]^T \\ U_{26}(\rho) & = & Y^TC_h(\rho)^T + [D(\rho)V_h(\rho)]^T \\ \tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho\tilde{P}(\rho)\nu \\ \tilde{U}_{33} & = & -(1-\mu)\tilde{Q} - \tilde{R} \end{array}$$

De plus, le contrôleur correspondant peut être calculé grâce à $K_0(\rho) = V_0(\rho)Y^{-1}$, $K_h(\rho) = V_h(\rho)Y^{-1}$ *et le système en boucle-fermée satisfait* $||z||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

Un contrôleur sans mémoire peut être calculé en fixant $Y_h = 0$ dans la condition LMI.

1.4.2 Contrôleurs séquencés par le retard

Les contrôleurs par retour d'état séquencés par le retard sont une nouveauté introduite dans cette thèse et ont une forme générale

$$u(t) = K(d(t))x(t) \tag{1.30}$$

où l'on voit clairement qu'ils sont structurellement différents des contrôleurs avec mémoire car dans ce cas précis le retard n'est plus vu comme un opérateur mais comme un paramètre. Dans ce cas, on transforme le système à retards en un système LPV incertain où le paramètre n'est rien d'autre qu'une fonction du retard. Ainsi en utilisant des outils développés pour les systèmes LPV, il est possible de développer des contrôleurs séquencés par le retard ou une valeur approchée. Il est important de souligner que, comme le retard est vu comme un paramètre, alors une erreur sur le retard est une variation du paramètre, chose plus facile à traiter que lorsque le retard est vu comme un opérateur. Cette partie est développée dans la Section 6.1.7.

Introduisons d'abord les ensembles:

$$\begin{array}{rcl} H & := & [h_{min}, h_{max}] \\ U & := & [\mu_{min}, \mu_{max}] \\ \hat{H} & := & [h_{min} - \delta, h_{max} + \delta] \\ \hat{U} & := & [\mu_{min} - \nu_{min}, \mu_{max} + \nu_{max}] \end{array}$$

où le premier est l'ensemble des valeurs du retard, le second l'ensemble des valeurs de la dérivée du retard, le troisième est l'ensemble de valeurs du retard utilisé dans le correcteur et le dernier est l'ensemble de valeurs de la dérivée du retard utilisé dans le correcteur.

Le théorème suivant donne le résultat sur la stabilisation par contrôleur séquencé par retard:

Théorème 1.4.2 *Le système (1.23) est stabilisable par un contrôleur séquencé par retard de la forme* $K(\hat{h}) = Y(\hat{h})X^{-1}(\hat{h})$ *si il existe une fonction matricielle* $X : \hat{H} \to \mathbb{S}^n_{++}$, *des fonctions matricielles* $Y : \hat{H} \to \mathbb{R}^{m\times n}$, $\tilde{D} : H \times U \times \hat{H} \times \hat{U} \to \mathbb{S}^n_{++}$, $\tilde{G} : H \times U \times \hat{H} \times \hat{U} \to \mathbb{K}^n$

et une fonction scalaire $\gamma : H \times U \times \hat{H} \times \hat{U} \rightarrow \mathbb{R}_{++}$ *tels que la LMI*

$$\begin{bmatrix} U_{11}(\hat{h},\dot{\hat{h}}) & U_{12}(\hat{h}) & U_{13}(\hat{h},\dot{\hat{h}}) & \alpha A_h \tilde{D}(\xi) & E \\ \star & -\gamma(\xi) I_q & \alpha C_h \tilde{G}^T(\xi) + \bar{C}X(h) & \alpha C_h \tilde{D}(\xi) & F \\ \star & \star & -\dot{\hat{h}}\dfrac{\partial X(\hat{h})}{\partial \hat{h}} - \tilde{D}(\xi) & 0 & 0 \\ \star & \star & \star & -\tilde{D}(\xi) & 0 \\ \star & \star & \star & \star & -\gamma(\xi) I_p \end{bmatrix} \prec 0 \tag{1.31}$$

soit satisfaite $(h, \dot{h}, \delta_h, \dot{\delta}_h) \in H \times U \times [-\delta, \delta] \times [\nu_{min}, \nu_{max}]$, *où* $\xi = col(h, \delta_h, \dot{h}, \dot{\delta}_h)$ *et*

$$\begin{array}{rcl} U_{11}(\hat{h},\dot{\hat{h}}) & = & -\dot{\hat{h}}\dfrac{\partial X(\hat{h})}{\partial \hat{h}} + [X(\hat{h})\bar{A}^T + Y^T(\hat{h})B_u^T]^H \\ U_{12}(\hat{h}) & = & X(\hat{h})\bar{C}^T + Y^T(\hat{h})D_u^T \\ U_{13}(\hat{h},\dot{\hat{h}}) & = & -\dot{\hat{h}}\dfrac{\partial X(\hat{h})}{\partial \hat{h}} + \bar{A}X(\hat{h}) + \alpha A_h \tilde{G}^T(\xi) \\ K(\hat{h}) & = & Y(\hat{h})X(\hat{h})^{-1} \end{array}$$

Afin de trouver une solution à ce problème LMI, il est nécessaire de fixer une structure à $X(\hat{h})$ (par exemple $X(\hat{h}) = X_0 + \hat{h}X_1$) afin de définir la structure de sa dérivée $\dfrac{\partial X(\hat{h})}{\partial \hat{h}}$.

1.4.3 Contrôleurs par retour dynamique de sortie

La synthèse de contrôleurs par retour de sortie dynamique est un problème ouvert dans le cadre des systèmes à retards. En effet, lorsque l'on cherche un retour de sortie dynamique sans-mémoire, il est très difficile d'obtenir des conditions sous forme LMI. Seulement les contrôleurs avec mémoire exacte aboutissent à des conditions LMIs. Dans cette thèse, seulement ces deux types de contrôleurs sont considérés sous deux formes bien distinctes:

- les contrôleurs pleins qui sont non-structurés comme on peut le trouver dans les articles [Apkarian and Adams, 1998; Scherer et al., 1997; Scherer, 2001] dont les conditions peuvent être simplifiées en utilisant un changement de variables spécifique très loin d'être trivial.

- les contrôleurs basés sur observateur qui sont donc structurés [Sename and Briat, 2006] qui sont, un peu paradoxalement, plus difficile à synthétiser de par leur structure imposée.

Comme exemple de résultats, considérons le contrôleur de retour de sortie dynamique suivant:

$$\begin{array}{rcl} \dot{x}_c(t) & = & A_c(\rho)x_c(t) + A_{hc}(\rho)x_c(t-h(t)) + B_c(\rho)y(t) \\ u(t) & = & C_c(\rho)x_c(t) + C_{hc}(\rho)x_c(t-h(t)) + D_c(\rho)y(t) \end{array} \tag{1.32}$$

En utilisant les résultats développés dans cette thèse, nous obtenons le théorème suivant démontré en Section 6.2.2:

Théorème 1.4.3 *Il existe un retour de sortie dynamique de la forme (1.32) pour le système (1.23) s'il existe une fonction matricielle continument différentiable $\tilde{P} : U_\rho \to \mathbb{S}^{2n}_{++}$, des matrices constantes $W_1, X_1 \in \mathbb{S}^n_{++}$, $\tilde{Q}, \tilde{R} \in \mathbb{S}^{2n}_{++}$, une fonction scalaire $\alpha : U_\rho \to \mathbb{R}_{++}$ et un scalaire $\gamma > 0$ tels que la LMI*

$$\begin{bmatrix} -2\tilde{X} & P(\rho)+\mathcal{A}(\rho) & \mathcal{A}_h(\rho) & \mathcal{E}(\rho) & 0 & \tilde{X} & h_{max}\tilde{R} \\ \star & U_{22}(\rho,\nu) & \tilde{R} & 0 & \mathcal{C}(\rho)^T & 0 & 0 \\ \star & \star & U_{33} & 0 & \mathcal{C}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \\ & & & & & & \end{bmatrix} \prec 0 \qquad (1.33)$$

soit satisfaite pour tout $(\rho,\nu) \in U_\rho \times U_\nu$ où $U_{22}(\rho,\nu) = -\tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho \tilde{P}(\rho)\nu$, $U_{33} = -(1-\mu)\tilde{Q} - \tilde{R}$ et

$$\begin{aligned} \tilde{X} &= \begin{bmatrix} W_1 & I \\ I & X_1 \end{bmatrix} \\ \mathcal{A}(\rho) &= \begin{bmatrix} A(\rho)W_1 + B(\rho)\mathcal{C}_c(\rho) & A(\rho) + B(\rho)\mathcal{D}_c(\rho)C_y(\rho) \\ \mathcal{A}_c(\rho) & X_1A(\rho) + \mathcal{B}_c(\rho)C_y(\rho) \end{bmatrix} \\ \mathcal{A}_h(\rho) &= \begin{bmatrix} A_h(\rho)W_1 + B(\rho)\mathcal{C}_c(\rho) & A(\rho) + B(\rho)\mathcal{D}_c(\rho)C_{yh}(\rho) \\ \mathcal{A}_{hc}(\rho) & X_1A_h(\rho) + \mathcal{B}_c(\rho)C_{yh}(\rho) \end{bmatrix} \\ \mathcal{E}(\rho) &= \begin{bmatrix} E(\rho) + B(\rho)\mathcal{D}_c(\rho)F_y(\rho) \\ X_1E(\rho) + \mathcal{B}_c(\rho)F_y(\rho) \end{bmatrix} \\ \mathcal{C}(\rho) &= \begin{bmatrix} C_y(\rho)W_1 + D(\rho)\mathcal{C}_c(\rho) & C_y(\rho) + D(\rho)\mathcal{D}_c(\rho)C_y(\rho) \end{bmatrix} \\ \mathcal{C}_h(\rho) &= \begin{bmatrix} C_h(\rho)W_1 + D(\rho)\mathcal{C}_{yh}(\rho) & C_h(\rho) + D(\rho)\mathcal{D}_c(\rho)C_{yh}(\rho) \end{bmatrix} \\ \mathcal{F}(\rho) &= \begin{bmatrix} F(\rho) + D(\rho)\mathcal{D}_c(\rho)F_y(\rho) \end{bmatrix} \end{aligned}$$

Dans ce cas, le contrôleur correspondant est donné par les expressions:

$$\begin{aligned} \begin{bmatrix} A_c(\rho) & A_{hc}(\rho) & B_c(\rho) \\ C_c(\rho) & C_{hc}(\rho) & D_c(\rho) \end{bmatrix} &= \mathcal{M}_1(\rho)^{-1}\left(\begin{bmatrix} \mathcal{A}_c(\rho) & \mathcal{A}_{hc}(\rho) & \mathcal{B}_c(\rho) \\ \mathcal{C}_c(\rho) & \mathcal{C}_{hc}(\rho) & \mathcal{D}_c(\rho) \end{bmatrix} - \mathcal{M}_2(\rho)\right)\mathcal{M}_3(\rho)^{-1} \\ \mathcal{M}_1(\rho) &= \begin{bmatrix} X_2 & X_1B(\rho) \\ 0 & I \end{bmatrix} \\ \mathcal{M}_2(\rho) &= \begin{bmatrix} X_1A(\rho)W_1 & X_1A_h(\rho)W_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \mathcal{M}_3(\rho) &= \begin{bmatrix} W_2^T & 0 & 0 \\ 0 & W_2^T & 0 \\ C_y(\rho)W_1 & C_{yh}(\rho)W_1 & I \end{bmatrix} \\ X^{-1} &= \begin{bmatrix} X_1 & X_2 \\ \star & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} W_1 & W_2 \\ \star & W_3 \end{bmatrix} \end{aligned}$$

et le système en boucle fermée satisfait $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

1.5 Conclusion

Cette thèse a proposé de nouveaux résultats en rapport au contrôle et à l'observation des systèmes à retards dépendant de paramètres en utilisant des outils de l'automatique moderne. Même si le problème demeure ouvert pour certains cas difficiles, le résultats présentés dans cette thèse ont amémliorés les résultats existants. Le travail a été développé en cinq chapitres.

Les deux premiers chapitres font un état de l'art des systèmes LPV et des systèmes à retards. Le troisième chapitre introduit des résultats préliminaires à partir seront développés les résultats des chapitres suivant portant respectivement sur l'observation/filtrage et le contrôle des systèmes LPV retardés.

Chapter 2

Overview of LPV Systems

Linear Parameter Varying (LPV) systems are a generalization of the general class of Linear Time-Varying (LTV) Systems:

SISO LTV System	SISO LPV System
$\dot{x}(t) = a(t)x(t)$	$\dot{x}(t) = a(\rho(t))(t)$
$x(0) = x_0$	$x(0) = x_0$

The main difference stems from the particularity that for LPV systems the time-dependence is, in some words, 'hidden' into parameters. Generally, when considering LTV systems, two particular cases can occur: either the trajectories of the time-varying coefficients are known (e.g. $a(t) = \sin(t)$) or they are unknown but remain in a known interval of values ($a(t) \in [a_{min}, a_{max}]$ and eventually $\dot{a}(t) \in [d_{min}, d_{max}]$). But it is also possible to consider LPV systems where parameters trajectories are known and exploited to provide matched results. For instance, periodic systems involve parameters with periodic trajectories [Bittanti and Colaneri, 2001; Yakubovich, 1986a,b; Yakubovich et al., 2007]. As we shall see later, the difference between LPV and LTV systems is the possibility of measuring or estimating the time-varying components of the system. As we may see on Figure 2.1, the larger class of finite

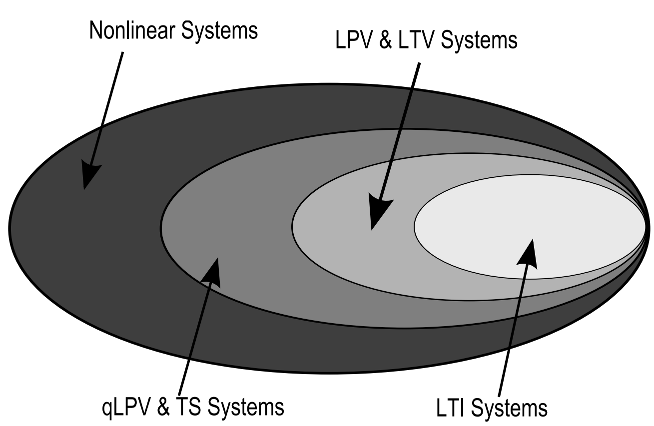

Figure 2.1: Venn diagram of finite dimensional systems

dimensional systems is the class of nonlinear systems. It is possible to approximate nonlinear systems by a specific class of LPV system called *quasi-LPV (qLPV)* systems since the parameters depend on the state. Such systems are obtained via a direct LPV transformation (e.g. $\dot{x}(t) = x(t)^2$ gives $\dot{x}(t) = \rho(t)x(t)$ with $\rho(t) = x(t)$) or using a class of representation referred to as Takagi-Sugeno systems [Castro, 1995; Takagi and Sugeno, 1985]. When the nonlinear system is linearized around a trajectory using Jacobian linearization we obtain a LTV system which can be also classified in the family of LPV systems. Finally, when the nonlinear system is linearized around an operating point, a Linear Time-Invariant (LTI) system is obtained.

The use of the term 'LPV' suggests that the parameters can be known in real time whereas 'LTV' means only that the system is non-stationary but nothing is said about the knowledge of the time-varying components.

Example 2.0.1 *For instance, when piloting an aircraft, the angle of attack is determined by the pilot and therefore can be considered as parameter whose trajectory is unknown in advance. Hence the aircraft can be viewed as a LPV system since the angle of attack can be known. However, if a machine tool is considered where a time-varying parameter is related to the wear of some parts of the machine. In this case, it may be difficult to measure or estimate the time-varying part and hence, a LTV model is better suited to represent such a system than a LPV one.*

A strict analysis does not fall into the context of this introduction and only LPV systems will be considered in the remaining of this chapter. For the interested reader about LTV systems, let us mention for instance the survey on Periodic Systems [Bittanti and Colaneri, 1999] . But, before introducing the interests and motivations for studying LPV systems, let us provide the expression of a generalized LPV system, taking the form of a non-autonomous non-stationary system of linear differential equations with algebraic equalities:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho(t))x(t) + B(\rho(t))u(t) + E(\rho(t))w(t) \\ z(t) & = & C(\rho(t))x(t) + D(\rho(t))u(t) + F(\rho(t))w(t) \\ y(t) & = & C_y(\rho(t))x(t) + F_y(\rho(t))w(t) \end{array} \tag{2.1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n\times n}$, $u \in \mathcal{U} \subset \mathbb{R}^m$, $w \in \mathcal{W} \subset \mathbb{R}^p$, $z \in \mathcal{Z} \subset \mathbb{R}^q$ and $y \in \mathcal{Y} \subset \mathbb{R}^t$ are respectively the state of the system, the control input, the exogenous input, the controlled output and the measured output. For more details on dynamical systems and related fundamental results, the reader should refer to [Khalil, 2002; Scherer and Weiland, 2005; Sontag, 1998]

It is clear from (2.1) that the behavior of output signals depends on input signals and on parameters acting in an internal fashion on the system. It is generally assumed that the parameter dependent matrices have bounded coefficients and this generally requires the boundedness of the parameters $\rho(t)$:

$$\rho(t) \in U_\rho \subset \mathbb{R}^k \text{ for all } t \geq 0 \text{ and } U_\rho \text{ compact}$$

Remark 2.0.2 *If some parameters are unbounded, it is generally possible (except for very special cases) to find a change of variables which defines a new system expression involving new bounded parameters. If a change of variables cannot be found, approaches such as in [Scherer, 2008] can be used.*

From these considerations the questions of stability, controllability and observability are not as 'easy' as in the LTI case and remain important problems beginning to be solved efficiently by recent techniques, mainly using LMIs.

The great interest of LPV systems is their ability to model/aproximate a wide variety of systems, from nonlinear to LTV systems including switched systems; this will be illustrated in Section 2.1. For instance, we may think to an automotive process where the dampers have to be controlled. In this case, possible parameters may be the speed of the car and position/orientation of the car since they are consequences of the driver and road behaviors. It is clear that the behavior of the vehicle varies for different speeds and road configurations. Hence it would be more efficient if the dampers control law would depend on these parameters.

The second interest, illustrated in the latter small scenario, resides in the control of LPV systems: the flexibility and adaptability that LPV control suggests. Indeed, the possibility of using the parameters in the control law gives rise to an interesting opportunity of improving system stability and performances. Coming back to our little scenario, if an engineer wishes to synthesize a control law without any information on the speed and determine a single LTI controller, this falls into the robust control framework and the process may be difficult to stabilize or has poor closed-loop performances. On the other hand, if the speed is measured and 'internally' used in the control law, the stabilization would be an easier task and the closed-loop system would certainly have better performances. This is the advantage of LPV control over robust control, provided that real-time measurement of potential parameters is possible. It is important to note that LPV control techniques can be easily combined with recent results on $\mathcal{H}_\infty$, $\mathcal{H}_2$, μ-norm, to produce enhanced control laws with performances and robustness specifications.

We will conclude this succinct introduction by examples provided in the literature. Since in many cases heavy computations are performed to turn the nonlinear system formulation into a LPV dynamical system, only a simple case is detailed hereunder while others are briefly enumerated with corresponding references.

Inverted Pendulum - robust control and performances This application has been provided in Kajiwara et al. [1999] where a model is given in the LPV form using a change of variable. The inverted pendulum depicted in Figure 2.2 is constituted of two arms moving in the vertical plane. The corresponding LPV model is given by:

$$\frac{d}{dt}\begin{bmatrix} z \\ \dot{z} \\ r_x \\ \dot{\varphi}_1 \end{bmatrix} = A(\rho)\begin{bmatrix} z \\ \dot{z} \\ r_x \\ \dot{\varphi}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{K_a}{T_a} \end{bmatrix} u$$

with

$$A(\rho) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{T_a} \end{bmatrix} + \frac{3}{4\ell_2}g\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} + \rho\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

where φ_1 is the angle of the first arm, $\varphi_2 + \varphi_1$ is the angle of the second arm (with respect to the ground), $r_y = 2\ell_1 \sin(\varphi_1)$, $r_x = 2\ell_1 \cos(\varphi_1)$, ℓ_1 is the half of the length of the arm 1,

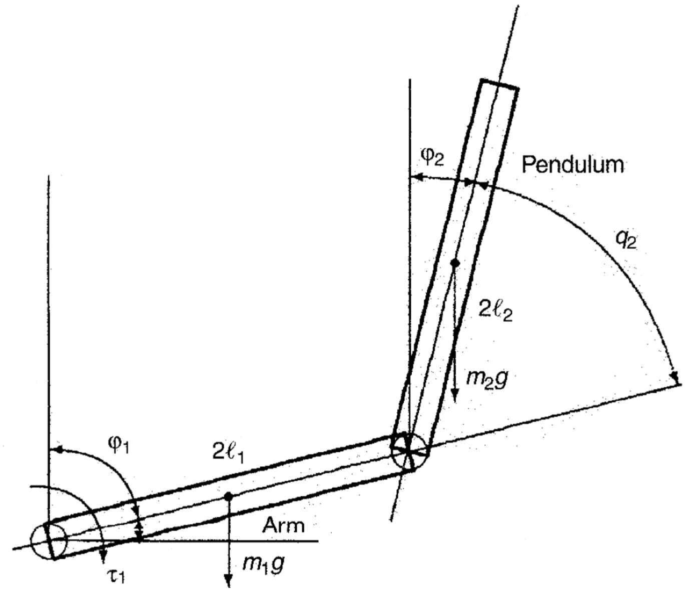

Figure 2.2: Pendulum considered in Kajiwara et al. [1999]

ℓ_2 is the half of the length of the arm 2, g is the gravitational acceleration, the parameter ρ is defined by $\rho = r_y$, K_a, T_a are constant parameters of the actuator (a motor here) and $z := r_x \frac{4}{3}\ell_2\varphi_2$ is the change of variable used to formulate the model as a LPV system.

According to Kajiwara et al. [1999], the obtained control law leads to encouraging results (the paper is from 1999, the beginning of the LPV trend) for the LPV formulation. The LPV approach has led in this application to an enhancement of the stability and performances.

Automotive Suspension System[1] Another application of LPV control is the performance adaptation: indeed, parameters can be introduced in Loop Shaping weighting functions in order to modify in real time the characteristics of the closed-loop systems: the bandwidth, the weight on the control law...

For instance, in [Poussot-Vassal, 2008], control of semi-active suspensions is addressed in view of performing a global chassis control. Since semi-active suspensions, in which the damper coefficient is controlled, can only absorb energy but not supply it, the control input is constrained to belong to a specific set depending on the deflection speed which is the derivative of the difference the sprung mass (z_s) and the unsprung mass z_{us}, i.e. $\dot{z}_s - \dot{z}_{us}$. Figures 2.3 and 2.4 represent different kind of suspension systems with associated characteristics. Ideally, the force produced by the suspension must be positive (negative) if the deflection speed is positive (negative).

Since in the $\mathcal{H}_\infty$ control framework such constraint cannot be explicitly specified, the idea is to use a parameter dependent weighting function on the control input of the form

$$W_u(s, \rho) = \rho(u - v)\frac{1}{s/1000 + 1}$$

[1] Thanks to Charles Poussot-Vassal who provided the material on this topic

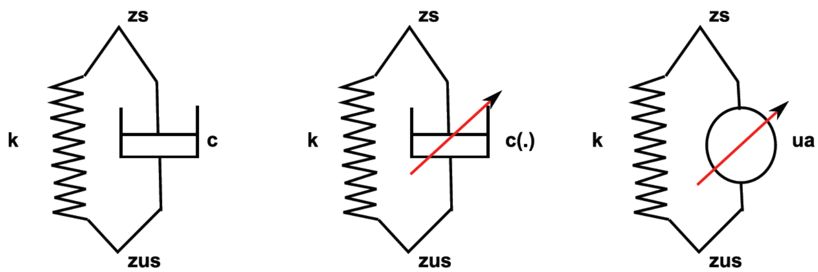

Figure 2.3: Different types of suspensions, from left to right: passive, semi-active and active suspensions

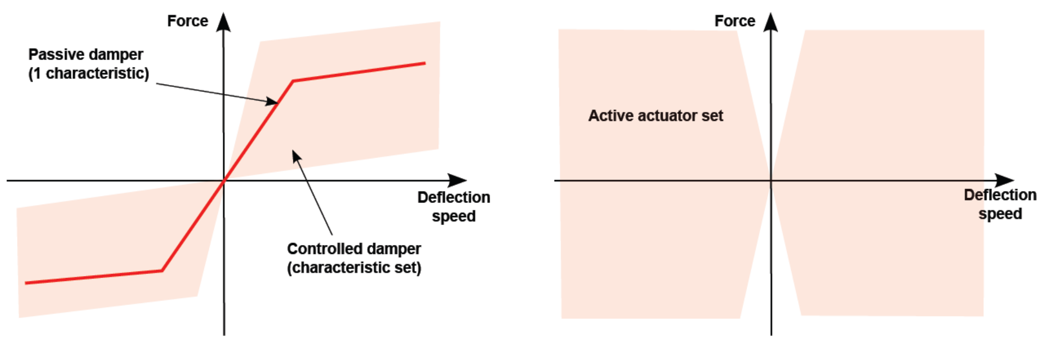

Figure 2.4: Characteristics of passive, semi-active (left) and active (right) suspensions

where u is the computed force and v is the achievable force which satisfies the quadrant constraint. The parameter ρ is chosen to satisfy the following relation

$$\rho(\varepsilon) = 10\frac{\mu\varepsilon^4}{\mu\varepsilon^4 + 1/\mu})$$

for sufficiently large $\mu > 0$, e.g. 10^8. In this case, the parameter belongs to $[0, 10]$ and has the form depicted on Figure 2.5 and the bode diagram of the inverse of the weighting function is plotted on Figure 2.6 where it is shown that if ρ is high (i.e. the computed force is far from the achievable force) the gain applied by the inverse of the filter on the control input is very small. This has the effect of having a control input which is close to 0, value which is always achievable.

This example shows the interest of parameter varying systems and parameter varying control; many other applications of such technique may be developed, for instance let us mention parameter varying bandwidth of the closed-loop system, parameter dependent disturbance rejection where the parameter would correspond to the pulsation of the disturbance, and so on...

A wide range of applications

We give here a non-exhaustive list of application of LPV modeling and control in the literature. In Wei and del Re [2007], the modeling and control of the air path system of

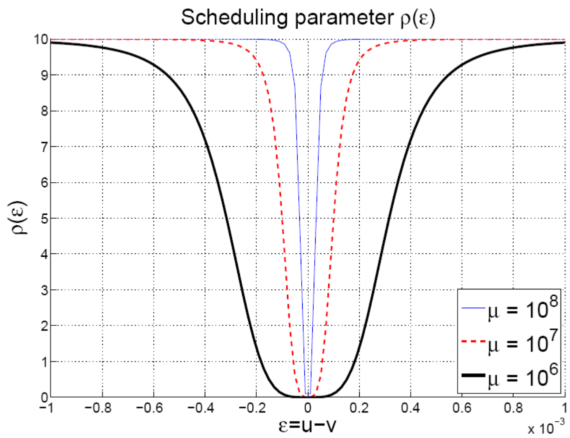

Figure 2.5: Graph of the parameter ρ with respect to $u - v$

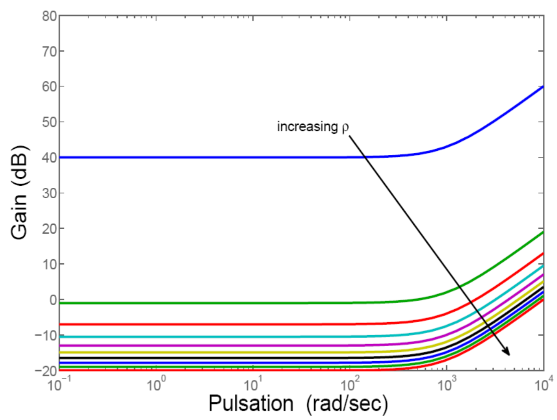

Figure 2.6: Bode diagram of $1/W_u(s, \rho)$ for different values of ρ

diesel engines in view of reducing polluting gas is addressed. This paper shows that a LPV formulation leads to interesting results in terms of simplicity of implementation and system performances. The control of elements in diesel engines is considered in Gauthier et al. [2005, 2007a,b]; He and Yang [2006]; Jung and Glover [2006] where the air flow, the fuel injection and/or the power unit are controlled. In Gilbert et al. [2007]; Reberga et al. [2005], LPV modeling and synthesis are applied to turbofan engines. Electromagnetic actuators are piloted in Forrai et al. [2007] while a robotic application is presented in Kwiatkowski and Werner [2005]. In Liu et al. [2006a,b], LPV controller is applied to power system regulator. In Lim and How [1999]; Tan and Grigoriadis [2000]; White et al. [2007] LPV control is applied in the synthesis of missile autopilot. In Lu et al. [2006], the attitude control of an F-16 Aircraft in response of the pilot orders for different angles of attack is addressed. LPV vehicle suspensions modeling and control is presented in Gaspard et al. [2004]; Poussot-Vassal et al. [2006, 2008a,b] while global chassis control (attitude control) is handled in Gáspár et al. [2007]; Poussot-Vassal et al. [2008c]. Finally, the control of nonuniform sampled-data systems is treated in a LPV fashion in Robert et al. [2006]. This list shows the efficiency and wide applicability of LPV control on theoretical and practical applications and motivates further studies on this topic. It will be shown later in this thesis that LPV methods can be used to control, in a novel fashion, time-delay systems with time-varying delays [Briat et al., 2007a, 2008a].

2.1 Classification of parameters

The behavior of LPV systems highly depends on the behavior of the parameters. Indeed, the global system is defined over a continuum of linear systems induced by a continuum of parameters. If the parameters take discrete values (the set of values is finite) or are piecewise constant continuous, the system would have a specific behavior and, in general, a specific denomination is given for these particular kinds of systems over these peculiar parameter trajectories; this will be deeper detailed further. This motivates the needs for classifying parameters in order to differentiate every behavior and therefore, any system that may arise. Two proper viewpoints can be adopted: either a mathematical one, centered on the analysis on mathematical properties of the parameters trajectories such as continuity and differentiability; or a physical point of view, focusing on the physical meaning of parameters such as measurability and computability. Such a classification aims at discussing on the validity and the meaning of LPV modeling in order to apply control strategies. It is important to note that while the first classification is important for theoretical considerations on the choice of stability results, the second is crucial for a rigorous application of LPV control on physical systems.

2.1.1 Physical Classification

In general, the parameters can be sorted in three types, depending on their meaning and origin.

2.1.1.1 Parameters as functions of states

The parameters may be defined as functions of states, and such cases arise when LPV systems are used to approximate nonlinear ones; for instance

$$\dot{x}(t) = -x(t)^3$$

can be approximated by the LPV system

$$\dot{x}(t) = -\rho(t)^2 x(t)$$

where $\rho(t) := x(t)$.

LPV systems in which states appear in the parameters expressions are called *Quasi-LPV systems*; see [He and Yang, 2006; Jung and Glover, 2006; Liberzon et al., 1999; Shamma and Athans, 1990, 1992; Shin, 2002; Tan and Grigoriadis, 2000; Wei and del Re, 2007; White et al., 2007] for some applications of quasi-LPV systems.

The main difficulty of quasi-LPV comes from the fact that theoretically, the states are unbounded, while by definition, the parameters are. If, by chance, the functions mapping the states to the parameter values are bounded for every state values, the problem would not occur (but this assumption is too strong to be of interest). On the contrary, if the functions are unbounded, then a supplementary condition should be added in order to satisfy the boundedness property of the parameters values. Fortunately, in practice, the states are generally bounded and such problem only occurs in theoretical considerations.

It is worth noting that generally, several LPV systems correspond to a nonlinear system and finding the 'best' LPV model remains a challenging open problem [Bruzelius et al., 2004; Mehendale and Grigoriadis, 2004; Shin, 2002]. Indeed, in the latter example, $\rho(t) = x(t)^2$ would have be chosen. But the latter example is a simple one since the origin (i.e. $x = 0$) is globally asymptotically stable attractive point and hence any parametrization would give an asymptotically stable LPV system. On the contrary, let us consider the Van-der-Pol equation (with reverse vector field) considered in [Bruzelius et al., 2004]:

$$\begin{array}{rcl} \dot{x}_1(t) & = & -x_2(t) \\ \dot{x}_2(t) & = & x_1(t) - a(1 - x_1(t)^2)x_2(t) \end{array}$$

with $a > 0$. It is well-known that this system has an unstable limit cycle: each trajectory starting inside the limit-cycle converges to 0 while each trajectory starting outside diverges. In [Bruzelius et al., 2004], it is shown that a 'good' LPV approximation, giving the exact stability region (i.e. interior of the limit cycle), is difficult to obtain.

2.1.1.2 Internal Parameters

The parameters may be used to represent time-varying parts involved in the system expression (assuming that time-varying terms are bounded), in view of simplifying the stability analysis and/or using them in advanced control laws. For instance, the LTV system:

$$\dot{x}(t) = (a(t) + b\sin(t))x(t) \quad , \; a(t) \text{ bounded over time}$$

can be represented by

$$\dot{x}(t) = (\rho_1(t) + b\rho_2(t))x(t)$$

where $\rho_1(t) := a(t)$ and $\rho_2(t) := \sin(t)$. The term *internal parameters* means that the information used to compute the parameter values is part of the system dynamical model and elapsed time. This is to put in contrast with the last class of parameters exposed in the next section.

2.1.1.3 External parameters

External parameters are involved in control and observation design problems only. Such 'virtual' parameters can be added in the design (for instance in frequency weighting functions in $\mathcal{H}_\infty$ control/observation) in order to modify the behavior of the closed-loop system in real-time. These external signals may stem from a monitoring system and can be used to represent states of emergency, working modes [Lu et al., 2006] or anything else, in view of modifying the behavior of the system, such as the system bandwidth, gains...

Let us consider the SISO LTI system

$$\dot{x}(t) = x(t) + u(t)$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are respectively the system state and the control input. It is proposed to determine a control law such that the closed-loop system has a variable and controlled bandwidth. The following control law is thus suggested:

$$u(t) = -(1 - \rho(t))x(t) + \rho(t)r(t), \ \rho(t) > 0$$

where r is the reference to be tracked. The interconnection yields:

$$\dot{x}(t) = \rho(t)(r(t) - x(t)), \ \rho(t) > 0$$

From the latter expression, the external parameter $\rho(t)$ controls the bandwidth of the closed-loop system and tries to maintain the tracking error to 0. In this scenario, a monitoring system including heuristics would be able to manage the parameter value with respect to high-level data.

2.1.2 Mathematical Classification

On the other hand, the mathematical ordering aims at sorting the parameters behavior by considering mathematical properties of the trajectories. Consequently, these properties will be taken into account in stability tests in order to provide less conservative results than by ignoring these characteristics.

2.1.2.1 Discrete vs. Continuous Valued Parameters

The first idea is to isolate the parameters with respect to the type of values they take (or more precisely the type of the image set of the mapping). Indeed, parameters must be viewed as functions of time $t \in \mathbb{R}_+$:

$$\rho : \mathbb{R}_+ \to \rho(\mathbb{R}_+)$$

where $\rho(\mathbb{R}_+)$ is the image set of $\mathbb{R}_+$ by the vector valued function $\rho(\cdot)$. Recall that the image set of the parameters is always bounded, then one can easily imagine that the image set is continuous or discrete, for instance

$$\rho : t \to \sin(t)$$

maps $t \in \mathbb{T}$ into $[-1, 1]$ continuously while

$$\rho : t \to [\sin(t)]_r$$

where $[\cdot]_r$ is the rounding to the nearest integer operator, maps $\mathbb{T}$ into $\{-1, 0, 1\}$.

The main difference between these image sets is that, while the first one contains an infinite number of values, the second contains only three. Discrete valued image sets are more simple to consider since one has to verify the stability at a finite number of points only. Systems for which parameters take discrete values are called *Switched Systems* (deterministic case) or *Systems with jump parameters* (stochastic case) [Blanchini et al., 2007; Cheng et al., 2006; Colaneri et al., 2008; Daafouz et al., 2002; Ghaoui and Rami, 1997; Hespanha and Morse, 1999; Liberzon et al., 1999; Mariton, 1990; Verriest, 2005; Xie et al., 2002; Xu and Antsaklis, 2002]. It is clear, from the definition of discrete valued image sets, that the parameters trajectories are discontinuous (more precisely they are piecewise constant continuous) while for parameters with continuous image sets, continuity of the trajectories might occur. This brings us to the idea of considering continuity as a second criterium of classification of the parameters.

2.1.2.2 Continuous vs. Discontinuous Parameters

Values of the parameters with continuous image set may evolve within the image set, in two different ways: either in a continuous or a discontinuous fashion.

Definition 2.1.1 *A continuous function f, defined over $\mathbb{R}_+$ such that*

$$f : \mathbb{R}_+ \rightarrow U$$

satisfies the following well-known statement:

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ |t - t_0| \leq \eta \Rightarrow |f(t) - f(t_0)| \leq \varepsilon,\ \forall\ t_0 \in \mathbb{R}_+$$

It is worth noting that there exists a large difference between switched systems (systems with discrete valued parameters) and systems with continuously valued discontinuous parameters. Let us consider for instance the following general parameter discontinuous trajectory

$$\rho(t) = \sum_{i=0}^{+\infty} \rho_i (H(t - t_i) - H(t - t_{i+1}))$$

where $\rho_i \in [\rho^-, \rho^+] \subset \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_i < \ldots < t_{i+1}$ and $H(\cdot)$ is the Heaviside function defined by

$$H(x) := \begin{cases} 0 & \text{if} \quad x \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

Indeed, due to the infinite number of values for ρ_i, systems involving such a parameter trajectory cannot be reduced to a finite number of systems. Hence such systems are of greater complexity than switched systems; these systems are called *LPV systems with arbitrarily fast parameter* and will be detailed further. The advantage of continuous parameters is their potential differentiability and will be the last criterium to classify parameters from a mathematical viewpoint.

2.1.2.3 Differentiable vs. Non-Differentiable Parameters

The final criterium is the first order differentiability of the parameters. By considering bounds on the parameter derivatives, it is then possible to characterize the time-varying nature of the parameters in terms of speed of variation. It is important to note that the speed of variation has a very harmful effect on the stability of LPV systems. This will be detailed in Section 2.3 on the stability analysis of LPV systems.

Definition 2.1.2 *A continuous differentiable function f, defined over $\mathbb{R}_+$ such that*

$$f : \mathbb{R}_+ \rightarrow U$$

satisfies the well-known statement

$$\exists f' : \forall t_0 \in \mathbb{R}_+ : \lim_{\delta t \rightarrow \{0^-, 0^+\}} \frac{f(t_0 + \delta t) - f(t_0)}{t - t_0} = f'(t_0)$$

Note that in the classical definition of the derivative, the limit from each side of 0 must coincide. This is clear that discontinuous functions do not satisfy such a condition and hence have unbounded derivative at discontinuity (with a slight abuse of language). Therefore, no global bounds can be defined for discontinuous parameters. Moreover, from the differentiability property above, the parameter $\rho(t)$ defined by

$$\rho : t \to |\sin(t)|,\ t \geq 0$$

does not admit a derivative at points $t_i = k\pi,\ k \in \mathbb{N}\ \{0\}$. Indeed, the derivative value take the value -1 and 1 respectively by computing the limit from the left and the right: therefore no function f' exists. This is a consequence to the fact that the absolute value function is not differentiable at 0. The non-existence of the function f' is apparently annoying since the global differentiability property is lost due to the presence of a countable infinite number of isolated points. Fortunately, since bounds on the parameters derivatives are necessary only, it is possible to show that this obtrusive troublesome particularity does not introduce any supplementary difficulty.

In these cases (continuous functions with non-smooth points), it is possible to affect two bounded values to the derivative at each point where the function is non-differentiable. For continuous parameters, these two values of the extended derivative are always bounded and it is possible, by extension, to consider that the 'derivative' takes simultaneously all values within a bounded interval (in the preceding example, the interval is $[-1, 1]$). For discontinuous functions, the fact that their derivative values are unbounded is retrieved since the 'derivative' takes all values of $\mathbb{R}$. This 'exotic' version is not without reminding us of the definition of the subgradient in nonsmooth analysis [Clarke, 1983], defined presently in less formal way. Since we are only interested in bounds of the derivative, this definition is sufficient to define them. This gives rise to the following propositions.

Proposition 2.1.3 *For a smooth function $f : \mathbb{R}_+ \to U$, U compact of $\mathbb{R}$, the bounds on the derivative is defined by an interval $[a, b]$ where $a = \min_{t\in\mathbb{R}_+} f'(t)$ and $b = \max_{t\in\mathbb{R}_+} f'(t)$.*

Proposition 2.1.4 *For a continuous nonsmooth (Lipschitz) function we have*

$$a = \min\{a_1, a_2\} \text{ and } b = \max\{b_1, b_2\}$$

where

$$\begin{aligned} a_1 &= \min_{t\in\mathbb{T}-\{t_i\}_i} f'(t) & b_1 &= \max_{t\in\mathbb{T}-\{t_i\}_i} f'(t) \\ a_2 &= \min\{\min\{U_1\}, \ldots, \min\{U_N\}\} & b_2 &= \max\{\max\{U_1\}, \ldots, \max\{U_N\}\} \end{aligned}$$

where $\{t_i\}$ is the set of points where f is nonsmooth and the U_i's are the intervals of values of the derivative at nonsmooth point t_i.

It is important to give an extra discussion on quasi-LPV systems. It is clear that, generally, the functions involving the states of the system are continuously differentiable with respect to them. Then, since the states are also differentiable, it is possible to tackle bounds on the parameter derivatives and these bounds would certainly depend on the bounds on the derivatives of the states. However, bounding derivative of the states is a problematic task. To understand why, let us consider that in the synthesis we fix $\dot{x} \in [a, b]$, where x is the state of a LPV SISO system. Using these values a controller is computed and the closed-loop system exhibits state derivatives going out of the bounds a and b; e.g. $[a-1, b+1]$. This contradictory situation invalidates the synthesis and it cannot be proved exactly that the system is stable for state derivative into $[a-1, b+1]$. Hence the synthesis procedure should be applied again with an enlargement of the bounds of the state derivative bounds, e.g. $[a-2, b+2]$. On the other hand, by expanding to much the derivative bounds (or even considering infinite values), this may result in a too high conservatism in the approach culminating in bad performances of the closed-loop system. This is one of the main difficulty while dealing with quasi-LPV systems which does not occur in any other types of parameters (i.e. internal and external).

2.2 Representation of LPV Systems

The aim of this section is to provide different frameworks used to represent LPV systems with their respective tools for stability analysis.

2.2.1 Several Types of systems...

Amongst the large variety of LPV systems, it is possible to isolate three main types of LPV systems based on the dependence on the parameters:

1. Affine and multi-affine systems
2. Polynomial Systems
3. Rational systems

It is worth noting that every LPV systems can be brought back to one of these latter types by mean of a suitable change of variable (e.g. $\rho_1' \leftarrow e^{\rho_1}$). In the following, we will use the notation ρ instead of $\rho(t)$ to lighten the notation.

2.2.1.1 Affine and Multi-Affine Systems

Affine and multi-affine systems are the most simple LPV systems that can be encountered. Their general expression is given by

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + F(\rho)w(t) \end{array}$$

where

$$\left[\begin{array}{c|c} A(\rho) & E(\rho) \\ \hline C(\rho) & F(\rho) \end{array}\right] = \left[\begin{array}{c|c} A_0 & E_0 \\ \hline C_0 & F_0 \end{array}\right] + \sum_{i=1}^{N} \left[\begin{array}{c|c} A_i & E_i \\ \hline C_i & F_i \end{array}\right] \rho_i$$

Due to the affine dependence, stability of such systems can be determined with a low degree of conservatism (in some cases there is no conservatism). This will be detailed further in Section 2.3.2.

2.2.1.2 Polynomial Systems

Polynomial systems are the immediate generalization of affine systems to a polynomial dependence. Their general expression is given below:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho)x(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + F(\rho)w(t)\end{array}$$

where

$$\left[\begin{array}{c|c} A(\rho) & E(\rho) \\ \hline C(\rho) & F(\rho)\end{array}\right] = \left[\begin{array}{c|c} A_0 & E_0 \\ \hline C_0 & F_0\end{array}\right] + \sum_{i=1}^{N}\left[\begin{array}{c|c} A_i & E_i \\ \hline C_i & F_i\end{array}\right]\rho^{\alpha_i}$$

where $\alpha_i = \left[\begin{array}{ccc}\alpha_i^1 & \dots & \alpha_i^N\end{array}\right]$ and $\rho^{\alpha_i} = \rho_1^{\alpha_i^1}\rho_2^{\alpha_i^2}\dots\rho_N^{\alpha_i^N}$. Such systems are slightly more complicated to analyze, but recently, several approaches brought interesting solutions to stability analysis and control synthesis for this kind of systems. This will be detailed in Section 2.3.3.

2.2.1.3 Rational Systems

The class of rational systems is the last one to be presented:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho)x(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + F(\rho)w(t)\end{array}$$

where $A(\rho),\ E(\rho),\ C(\rho)$ and $F(\rho)$ are matrices with coefficients taking the form of rational functions. Such systems have the advantage to be able to model the largest set of systems and multi-affine/polynomial systems are special case of this kind. Indeed, according to the Padé approximation (see Appendix E.3) in order to approximate any function by rational functions as close as necessary.

2.2.2 But essentially three global frameworks

Even if a LPV system can be classified in several families depending on how the parameters act on the system, only three global techniques are commonly used (at this time) to deal with LPV systems.

2.2.2.1 Polytopic Formulation

Polytopic systems are really spread in robust analysis and robust control. They have been studied in many papers, for instance see [Apkarian and Tuan, 1998; Borges and Peres, 2006; Geromel and Colaneri, 2006; Jungers et al., 2007; Oliveira et al., 2007; Peaucelle et al., 2000].

A time-varying polytopic system is a system governed by the following expressions

$$\begin{array}{rcl}\dot{x}(t) & = & A(\lambda(t))x(t) + E(\lambda(t))w(t) \\ z(t) & = & C(\lambda(t))x(t) + F(\lambda(t))w(t)\end{array} \qquad (2.2)$$

where

$$\left[\begin{array}{cc} A(\lambda(t)) & E(\lambda(t)) \\ C(\lambda(t)) & F(\lambda(t))\end{array}\right] = \sum_{i=1}^{N}\lambda_i(t)\left[\begin{array}{cc} A_i & E_i \\ C_i & F_i\end{array}\right]$$

and $\sum_{i=1}^{N} \lambda_i(t) = 1,\ \lambda_i(t) \geq 0$. The term polytopic comes from the fact that the vector $\lambda(t)$ evolves over the unit simplex (which is a polytope) defined by

$$\Gamma := \left\{ \underset{i}{\mathrm{col}}(\lambda_i(t)) : \sum_{i=1}^{N} \lambda_i(t) = 1,\ \lambda_i(t) \geq 0 \right\} \tag{2.3}$$

This set is depicted on Figure 2.7 for values $N = 2$ and $N = 3$. For $N = 2$, the set takes the form of a segment on a line; for $N = 3$, the set is a triangular closed surface on a plane; and so on...

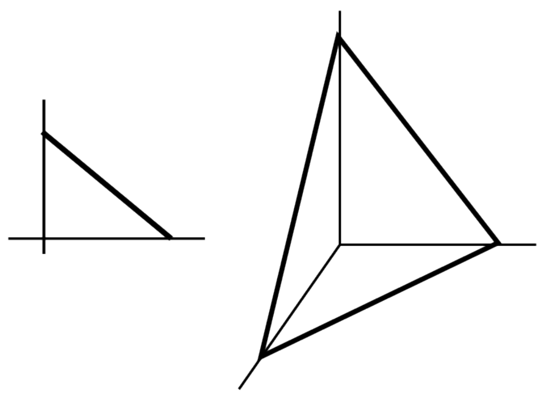

Figure 2.7: Set Γ for $N = 2$ and $N = 3$

The polytope Γ can be explicitly defined from the set of its vertices:

$$\mathcal{V} = \bigcup_{i=1}^{N} \mathcal{V}_i$$

where

$$\mathcal{V}_i = \begin{bmatrix} 0_{(i-1)\times 1} \\ 1 \\ 0_{(N-i)\times 1} \end{bmatrix}$$

Indeed, in this case, the convex hull of $\mathcal{V}$, denoted hull$[\mathcal{V}]$ coincides with Γ. Recall that the convex hull is the convex envelope of $\mathcal{V}$ and is the smallest convex set containing $\mathcal{V}$. The notion of convex hull is illustrated on Figure 2.8.

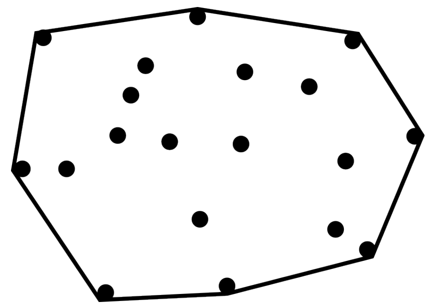

Figure 2.8: Convex hull of a set of points on the plane

Polytopic systems enjoy a nice property exploiting the fact that a polytope is a convex polyhedral and, as we shall see later, the stability of a polytopic system can be characterized by the stability of the 'vertex' systems. It is important to note that any parameter dependent system can be coarsely turned expressed as a polytopic system. Affine and multi-affine systems can be equivalently represented as polytopic systems, this is illustrated in the following example:

Example 2.2.1 *Let us consider the LPV system with two parameters* ρ_1, ρ_2*:*

$$\dot{x}(t) = [A_1\rho_1(t) + A_2\rho_2(t)]\, x(t)$$

with $\rho_i(t) \in [\rho_i^-, \rho_i^+]$ *for* $i = 1, 2$. *The corresponding equivalent polytopic system is then given by*

$$\dot{x} = [A_1[(\lambda_1 + \lambda_3)\rho_1^- + (\lambda_2 + \lambda_4)\rho_1^+] + A_2[(\lambda_1 + \lambda_2)\rho_2^- + (\lambda_3 + \lambda_4)\rho_2^+]]x$$

with $\lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \lambda_4(t) = 1,\ \lambda_i(t) \geq 0.$

It is clear that the polytopic model is not interesting in this case since it involves 4 time-varying parameters instead of 2 for the original system. This is an obvious fact in multi-affine systems. However, the transformation of the above multi-affine system into a polytopic formulation allows to provide somewhat nonconservative stability conditions (depending on the notion of stability which is considered); this will be detailed in Section 2.3.

On the other hand, the transformation of other LPV systems which are not multi-affine may be interesting but remains conservative as demonstrated in the following example.

Example 2.2.2 *Let us consider the polynomially parameter dependent system*

$$\dot{x}(t) = (A_0 + A_1\rho + A_2\rho^2)x(t) \tag{2.4}$$

where $\rho \in [\rho^-, \rho^+]$. *It can be converted into the polytopic system*

$$\dot{x}(t) = [A_0 + A_1 f_1(\lambda(t)) + A_2 f_2(\lambda(t))]x(t) \tag{2.5}$$

with

$$\begin{aligned} f_1(\lambda(t)) &= (\lambda_1(t) + \lambda_3(t))\rho^- + (\lambda_2(t) + \lambda_4(t))\rho^+ \\ f_2(\lambda(t)) &= \lambda_3(t)(\rho^-)^2 + \lambda_4(t)(\rho^+)^2 \end{aligned}$$

Indeed, we have considered

$$\begin{pmatrix} \rho \\ \rho^2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \rho^- \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} \rho^+ \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} \rho^- \\ (\rho^-)^2 \end{pmatrix} + \lambda_4 \begin{pmatrix} \rho^+ \\ (\rho^+)^2 \end{pmatrix} \tag{2.6}$$

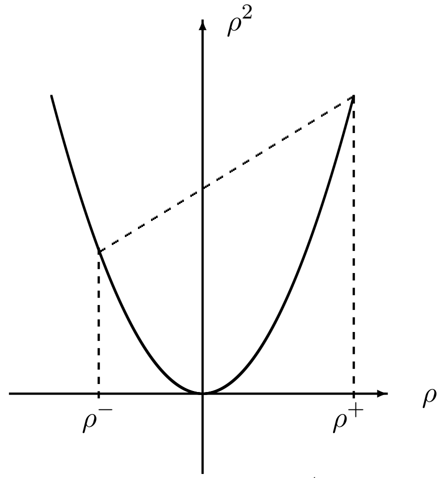

Figure 2.9: Comparison between exact set of values (the parabola) and the approximate set (the interior of the trapezoid)

To see that systems (2.4) and (2.5) are not equivalent it suffices to show that the polytopic parametrization can generate aberrant parameter values. This is easily visualized on Figure 2.9, aberrant values lie inside the trapezoid but not on the parabola. Then dealing with the polytopic systems would results in conservative stability conditions. The drawback of polytopic system as approximants comes from the fact that they decorrelate parameters and functions of them. Indeed, in the previous example, the dependence between ρ and ρ^2 is lost in the parametrization (2.6); only extremal points are correlated.

In order to reduce this conservatism, it is interesting to reduce the size of the polytope. This can be done by adding new vertices in order to shape the non-convex dependence between parameters. For the curve $f(x) = x^2$ it is possible to add new points below the curve to approximate the curve by tangent straight lines as seen on Figure 2.11. Nevertheless, it is not possible to approximate (asymptotically) exactly the parameter set (ρ, ρ^2). Indeed, since the domain has to remain convex, the surface above the curve $f(x) = x^2$ (the epigraph) must be convex too, and thus cannot be reduced more.

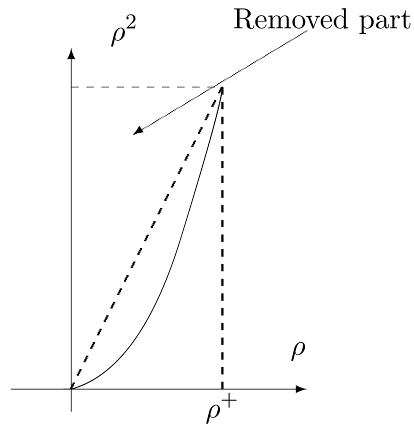

Figure 2.10: Illustration of Polytope Reduction using epigraph reduction

With the assumptions that $\rho^- = 0$ and $\rho^+ > 0$, the polytope can be reduced by removing a part of the epigraph. The surface above the line joining the points $(0, 0)$ and $(\rho^+, (\rho^+)^2)$ can be removed. In this case the new domain is a triangle instead of a rectangle, as depicted in Figure 2.10.

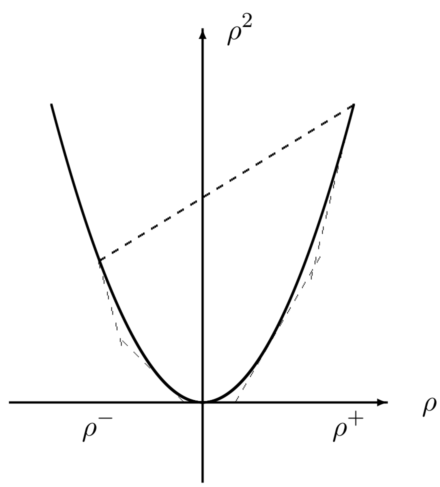

Figure 2.11: Illustration of Polytope Reduction by straight lines

2.2.2.2 Parameter Dependent Formulation

This formulation is the most direct one, the system is considered in his its primal form. The stability analysis or control synthesis are performed directly with specific tools. This formulation is better suited for polynomially parameter dependent systems but can be used with any type of LPV systems:

$$\dot{x}(t) = A(\rho)x(t)$$

where

$$A(\rho) = A_0 + \sum_i A_i \rho^{\alpha_i}$$

with $\alpha_i = \begin{bmatrix} \alpha_i^1 & \dots & \alpha_i^N \end{bmatrix}$ and $\rho^{\alpha_i} = \rho_1^{\alpha_i^1} \rho_2^{\alpha_i^2} \dots \rho_N^{\alpha_i^N}$. It is obvious that the multi-affine case is a special case of this more general formulation. Moreover, even if the definition is given for systems with polynomial dependence on parameters only, it also applies to systems with rational dependence on parameters. However, a more more suitable formulation for such systems is given in the next section.

2.2.2.3 'LFT' Formulation

The last formulation for LPV systems is called, with a slight abuse of language, *LFT systems*. Indeed, the term 'LFT' means 'Linear Fractional Transformation' and is the transformation used to convert a LPV/uncertain system into a Linear Fractional Form (LFR). The interest of this formulation for LPV systems has been emphasized in [Packard, 1994] and this approach has given rise to many papers, let us mention for instance [Apkarian and Adams, 1998; Apkarian and Gahinet, 1995; Scherer, 2001]. The major interest of such a formulation is to embed a large variety of systems in a single class, englobing in a unified way both systems with polynomial and rational dependence on the parameters.

The key idea of this representation is to split the system in two parts: the parameter-varying part and the constant part in view of analyzing them separately. It is worth noting that the idea of separating the system in two connected independent parts is not new. It actually brings us back to the 50's when the nonlinearities on the actuators were dealt with such a representation and lead to Lu're systems [Lur'e, 1951]. In robust stability analysis, such a transformation is extensively used as shown in [Scherer and Weiland, 2005; Zhou et al., 1996] as well as in the well-posedness [Iwasaki and Hara, 1998] or the IQC [Rantzer and Megretski, 1997] frameworks.

As an introductive example, let us consider the LPV system

$$\dot{x}(t) = A(\rho)x(t) \tag{2.7}$$

which is rewritten into an interconnection of two systems

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \\ w(t) & = & \Theta(\rho)z(t) \end{array} \tag{2.8}$$

as depicted in figure 2.12. Note that the matrices of the lower system $(\tilde{A}, B, C, D)$ are constant while the parameter varying part is located in the upper system.

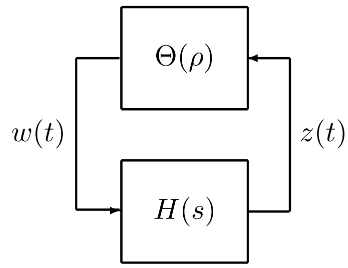

Figure 2.12: System (2.7) written in a 'LFT' form corresponding to description (2.8) where $H(s) = C(sI - \tilde{A})^{-1}B + D$

Example 2.2.3 *Let us consider the LPV system*

$$\dot{x} = \left(\frac{\rho_2}{\rho_1^2 + 1} - 3\right) x \tag{2.9}$$

It is possible to rewrite it in a 'LFT' form as shown below

$$\begin{bmatrix} \dot{x} \\ \hline z_0 \\ z_1 \\ z_2 \end{bmatrix} = \left[\begin{array}{c|c} \tilde{A} & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ \hline w_0 \\ w_1 \\ w_2 \end{bmatrix} \tag{2.10}$$

with

$$\left[\begin{array}{c|c} \tilde{A} & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|ccc} -3 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right] \tag{2.11}$$

and

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \rho_2 & 0 & 0 \\ 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} \tag{2.12}$$

Once the LPV system is split in two parts, the stability of the system or any other property can be determined using theorems applying on system interconnections; this will be detailed in Section 2.3. The matrix $\Theta(\rho)$ is constructed in such a way that it gathers diagonally all the parameters involved in the LPV system (as illustrated in Example 2.2.3):

$$\Theta(\rho) = \text{diag}(I_{n_1} \otimes \rho_1, \ldots, I_{n_p} \otimes \rho_p) \tag{2.13}$$

where n_i is the number of occurrences of parameter ρ_i in $\Theta(\rho)$ and p is the number of distinct parameters. Each parameter is repeated enough times as needed to turn system (2.7) into system (2.8). A complete discussion on the construction of the interconnection is given in [Scherer and Weiland, 2005; Zhou et al., 1996]. It is generally assumed, for simplicity, that $\Theta(\rho)^T\Theta(t) \leq I$ meaning that the parameters ρ belong to the hypercube $[-1, 1]^p$ where p is the number of parameters. It is worth noting that, by a simple change of variable, any bounded real parameter can be modified to belong to the interval $[-1, 1]$. To emphasize the correspondence between both systems, we will turn the LFT formulation into a 'one-block' formulation: from (2.8), we have

$$\begin{aligned} w(t) &= \Theta(\rho)z(t) \\ &= \Theta(\rho)(Cx(t)+Dw(t)) \end{aligned}$$

and then

$$(I-\Theta(\rho)D)w(t)=\Theta(\rho)Cx(t)$$

If the problem is well-posed (i.e. the matrix $I-\Theta(\rho)D$ is nonsingular for all $\rho \in [-1,1]^p$) then we get

$$w(t)=(I-\Theta(\rho)D)^{-1}\Theta(\rho)Cx(t)$$

and finally

$$\dot{x}(t)=(\tilde{A}+B(I-\Theta(\rho)D)^{-1}\Theta(\rho)C)x(t)$$

showing that we have

$$\begin{aligned} A(\rho) &= \tilde{A}+B(I-\Theta(\rho)D)^{-1}\Theta(\rho)C \\ &= \tilde{A}+B\Theta(\rho)(I-D\Theta(\rho))^{-1}C \end{aligned}$$

Example 2.2.4 *We will show here the equivalence between system (2.9) and (2.10)-(2.12). Applying formula $A(\rho)=\tilde{A}+B(I-\Theta(\rho)D)^{-1}\Theta(\rho)C$ yields*

$$\begin{aligned} A(\rho) &= -3+\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\rho_1\\ 0 & \rho_1 & 1\end{bmatrix}^{-1}\begin{bmatrix}\rho_2\\0\\0\end{bmatrix} \\ &= -3+\frac{1}{1+\rho_1^2}\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \rho_1\\ 0 & -\rho_1 & 1\end{bmatrix}\begin{bmatrix}\rho_2\\0\\0\end{bmatrix} \\ &= -3+\frac{\rho_2}{\rho_1^2+1} \end{aligned}$$

2.3 Stability of LPV Systems

In the latter section, three frameworks have been introduced and they cover a wide variety of LPV systems: affine, polynomial and rational systems. These past years, specific tools have been developed to deal with stability analysis of systems belonging to each class and gave rise to interesting results. The aim of the current section is to present these tools and their most important associated results, but first, some preliminary results on stability of LTI systems are necessary.

2.3.1 Notions of stability for LTI and LPV systems

It is convenient, for the reader ease, to introduce several notions of stability of LPV systems. Since LPV systems are defined over a (smooth) continuum of systems, hence the stability may take several forms at the difference of LTI systems. For more details on stability of dynamical systems, the reader should read, for instance, [Khalil, 2002; Scherer and Weiland, 2005; Smale, 2004; Sontag, 1998]. This section is devoted to show the complexity of the stability analysis of LPV systems and introduces ad-hoc notions of stability for this type of systems.

Before giving specific definitions for the stability of LPV systems, it is convenient to introduce two fundamental definitions of stability for uncertain systems. These definitions are also of interest in the framework of LPV systems. In modern systems and control theory, the stability of a dynamical system is determined by mean of a Lyapunov function and Lyapunov's Stability Theory [Lyapunov, 1992]. The key idea behind this theory is that if it is possible to find a nonnegative function, measuring the energy contained into the system, which

- is decreasing over time
- has 0 value at the equilibrium

then the system is said to be stable.

Before defining the different notions of stability for LPV systems, the following fundamental results on stability of dynamical systems are necessary:

Theorem 2.3.1 (Lyapunov Theorem [Lyapunov, 1992]) *Let us consider the general dynamical system*

$$\dot{x}(t) = f(x(t)) \tag{2.14}$$

such that 0 *is an equilibrium point. Then is there exists a function* $V(x(t))$

1. $\eta||x(t)||_2^2 \leq V(x(t)) \leq \varepsilon||x(t)||_2^2$, $\eta, \varepsilon > 0$
2. *the derivative of* V *along the trajectories solution of the system (2.14) satisfies* $\dfrac{\partial V}{\partial x} f(x) \leq -\theta||x(t)||_2^2$, $\theta > 0$ *and* $\dot{V}(0) = 0$

then the system is asymptotically stable and V *is called a Lyapunov function for (2.14).*

Theorem 2.3.2 *Let us consider the following LTI system*

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) \\ x(0) & = & x_0 \end{array} \tag{2.15}$$

The following statements are equivalent:

1. *The system (2.15) is globally uniformly asymptotically stable (i.e. for all* x_0 *we have* $||x(t)|| \to 0$*)*
2. *The system (2.15) is globally uniformly exponentially stable (i.e. for all* x_0 *and for all* $t \geq 0$ *we have* $||x(t)|| \leq \alpha e^{-\beta t}$, $\alpha, \beta > 0$*)*
3. *The matrix* A *is Hurwitz (the eigenvalues of the matrix* A *have negative real part)*
4. *There exist matrices* $P = P^T \succ 0$ *and* $Q = Q^T \succ 0$ *such that the Lyapunov equation*

   $$A^T P + PA + Q = 0$$

   is satisfied.

Proof: *The equivalence between exponential and asymptotic stability is trivial and is then omitted. The properties of global stability and uniform stability hold since the system is linear and time-invariant respectively [Scherer and Weiland, 2005; Sontag, 1998]. It is well known that if the eigenvalues of the matrix A have negative real part then the system is asymptotically stable. Computing the explicit expression of the solution $x(t) = e^{At}x_0$ shows that it converges to 0 if and only if the function is decreasing. This is verified if and only if the eigenvalues lie in the complex open left-half plane. So statements 1 to 3 are equivalent. Now we prove several implications.*

Proof of $4 \Rightarrow 1$

Define the quadratic function $V(x(t)) = x(t)^T P x(t)$ with $P = P^T \succ 0$. Now applying the Lyapunov's theorem 2.3.1 and computing the derivative of V along the solutions of system (2.15) we get the following sufficient condition for stability:

$$A^T P + PA \prec 0$$

which is equivalent to the existence of a matrix $Q = Q^T \succ 0$ such that the Lyapunov equation

$$A^T P + PA + Q = 0 \tag{2.16}$$

is satisfied. Hence the feasibility in $(P, Q) \in \mathbb{S}^n_{++} \times \mathbb{S}^n_{++}$ of the Lyapunov equation implies asymptotic stability.

Proof of $4 \Rightarrow 2$

Suppose (2.16) holds for some $(P, Q) \in \mathbb{S}^n_{++} \times \mathbb{S}^n_{++}$. Hence we have

$$\begin{array}{rcccl} \lambda_{min}(P)||x(t)||_2^2 & \leq & V(x(t)) & \leq & \lambda_{max}(P)||x(t)||_2^2 \\ -\lambda_{max}(Q)||x(t)||_2^2 & \leq & \dfrac{dV(x(t))}{dt} & \leq & -\lambda_{min}(Q)||x(t)||_2^2 \end{array}$$

Therefore we have $||x(t)||_2^2 \leq \lambda_{max}(P)^{-1}V(x(t))$ and

$$\frac{dV(x(t))}{dt} \leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V(x(t))$$

According to the theory of linear differential equations, the solutions are given by

$$V(t) \leq exp\left[\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} t\right] V(0)$$

and hence

$$||x(t)||_2^2 \leq \frac{1}{\lambda_{min}(P)} exp\left[\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} t\right] x(0)^T P x(0)$$

Hence the exponential convergence is proved.

Proof of $3 \Rightarrow 4$

Start with the assumption $\Re[\lambda(A)] < 0$ and consider the Lyapunov equation (2.16). The goal is to show that for any given $Q \in \mathbb{S}^n_{++}$, there exists a solution $P \in \mathbb{S}^n_{++}$ to the Lyapunov's *equation (2.16) provided that* $\Re[\lambda(A)] < 0$. *Let* $Q = Q_0$ *and pre and post multiply (2.16) by* $e^{A^T s}$ *and* e^{As} *respectively:*

$$e^{A^T s} A^T P e^{As} + e^{A^T s} P A e^{As} + e^{A^T s} Q_0 e^{As} = 0$$

This is equivalent to

$$\frac{d}{ds}[e^{A^T s} P e^{As}] + e^{A^T s} Q_0 e^{As} = 0$$

Then summing over $[0, t]$ *we get*

$$\begin{array}{rcl} \int_0^t \frac{d}{ds}[e^{A^T s} P e^{As}] + e^{A^T s} Q_0 e^{As} ds & = & 0 \\ e^{A^T t} P e^{At} - P + \int_0^t e^{A^T s} Q_0 e^{As} ds & = & 0 \end{array}$$

Under the assumption $\Re[\lambda(A)] < 0$, *then we have* $e^{At} \to 0$ *as* t *tends to* $+\infty$, *hence the limit is well defined for the latter equality and we get*

$$P = \int_0^{+\infty} e^{A^T s} Q_0 e^{As} ds$$

Hence, this proves the existence of a solution $P \in \mathbb{S}^n_{++}$ *to (2.16) under the assumption that* A *is Hurwitz.*

Remark 2.3.3 *There exist several techniques to determine conditions under which the Lyapunov equation has solutions, for instance methods based on the expansion approaches such as the Kronecker product approach or the one presented here, methods based on skew-symmetric matrices, and so on. See for instance the book [Gajić and Qureshi, 1995] and references therein.*

Proof of $4 \Rightarrow 3$

The goal is to show that the feasibility of the Lyapunov equation (2.16) implies $\Re[\lambda(A)] < 0$. *First define, the eigenvectors (of* A*)* $e_i \in \mathbb{C}^n - \{0\}$ *associated respectively to the eigenvalue* $\lambda_i \in \mathbb{C}$ *through the relation*

$$Ae_i = \lambda_i e_i$$

Pre and post-multiply (2.16) by e_i^* *and* e_i *we get*

$$\begin{array}{rcl} e_i^* A^T P e_i + e_i^* P A e_i + e_i^* Q e_i & = & 0 \\ (\lambda_i^* + \lambda_i) e_i^* P e_i + e_i^* Q e_i & = & 0 \\ 2\Re[\lambda_i] e_i^* P e_i + e_i^* Q e_i & = & 0 \end{array}$$

Since $e_i^* P e_i > 0$ *and* $e_i^* Q e_i > 0$, *the equality holds if and only if*

$$\Re[\lambda_i] < 0$$

The implication is then proved. □

The notions of stability and Lyapunov function are illustrated in the following

Example 2.3.4 *Let us consider an asymptotically stable LTI system*

$$\dot{x}(t) = Ax(t),\ x(0) = x_0$$

According to Theorem 2.3.2, a necessary and sufficient condition for the stability is the existence of a Lyapunov function of the form:

$$V(x(t)) = x(t)^T P x(t), \quad P = P^T \succ 0 \tag{2.17}$$

It is clear that the function is positive except at $x = 0$ where it is 0. Computing the time derivative of V along the trajectories solution of the system yields

$$\begin{aligned}\dot{V} &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t)\\ &= x(t)^T \left(A^T P + PA\right) x(t)\end{aligned}$$

Since the derivative needs to be negative definite for every $x \neq 0$, then we must have

$$A^T P + PA \prec 0, \quad P = P^T \succ 0$$

Finally, if one can find $P = P^T \succ 0$ such that $A^T P + PA \prec 0$ then the system is asymptotically stable. An explicit solution to such an inequality is provided in the proof of Theorem 2.3.2. For instance, if $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ 0$ then we have

$$A^T P + PA = \begin{bmatrix} 2(p_2 - p_1) & -2p_2 + p_3 \\ \star & -2p_3 \end{bmatrix} \prec 0$$

This is equivalent to satisfying the following nonlinear system of matrix inequalities (a symmetric matrix is positive definite if and only if all its principal minors are positive):

$$\begin{aligned} p_1 &> 0\\ p_3 &> 0\\ p_1 p_3 - p_2^2 &> 0\\ p_1 - p_2 &> 0\\ 4p_3(p_2 - p_1) - (p_3 - 2p_2)^2 &> 0 \end{aligned}$$

A suitable choice is given by

$$\begin{aligned} p_1 &= 3\\ p_2 &= 2\\ p_3 &= 2 \end{aligned}$$

In the framework of uncertain systems, the matrix A depends on (time-invariant/time-varying) uncertain terms δ and is then denoted by $A(\delta)$. Let us focus now, for simplicity, on constant uncertain parameters taking values in a compact set $\mathbf{\Delta} \subset \mathbb{R}^p$ and the uncertain system

$$\begin{aligned} \dot{x}(t) &= A(\delta)x(t)\\ x(0) &= x_0 \end{aligned} \tag{2.18}$$

where x and x_0 are respectively the system state and the initial condition.

Remark 2.3.5 *We also assume that $x = 0$ is an equilibrium point for all $\delta \in \mathbf{\Delta}$. This assumption is fundamental to apply Lyapunov's theory and is responsible of many errors in published papers on stability of nonlinear uncertain systems. When the equilibrium point is nonzero and depends on the value of the uncertain parameters, the following change of variable*

$$\tilde{x}(t) = x(t) - x_e(\delta)$$

transfers the equilibrium point to 0. It is worth noting that this remark does not hold for linear systems which always have an equilibrium point at the origin [Vidyasagar, 1993]. The Lyapunov's theory allows to show the stability of systems without computing the eigenvalue, this makes a powerful tool when dealing with LPV systems which have, by definitions, an infinite number of eigenvalues. Moreover, as we shall see further, the notion of stability of LPV systems is not equivalent to the negativity of the real part of the eigenvalues.

It is possible to define several types of stability for LPV systems, each of them defined by a specific Lyapunov function.

Definition 2.3.6 (Quadratic Stability) *System (2.18) is said to be quadratically stable if there exists a Lyapunov function $V_q(x(t)) = x(t)^T P x(t) > 0$ for every $x \neq 0$ and $V(0) = 0$ such that*

$$\dot{V}_q(t, \delta) = x(t)^T (A(\delta)^T P + P A(\delta)) x(t) < 0$$

for every $x \neq 0$ and $\dot{V}_q(0, \delta) = 0$ for all $\delta \in \mathbf{\Delta}$.

Definition 2.3.7 (Robust Stability) *System (2.18) is said to be robustly stable if there exists a parameter dependent Lyapunov function $V_r(x(t), \delta) = x(t)^T P(\delta) x(t) > 0$ for every $x \neq 0$ and $V(0) = 0$ such that*

$$\dot{V}_r(t, \delta) = x(t)^T (A(\delta)^T P(\delta) + P(\delta) A(\delta)) x(t) < 0$$

for every $x \neq 0$ and $\dot{V}_r(0, \delta) = 0$ for all $\delta \in \mathbf{\Delta}$.

Since the Lyapunov function used to determine robust stability depends on the uncertain constant parameters, it is clear that the robust stability implies quadratic stability. The converse does not hold in general, indeed it may be possible to find uncertain systems which are robustly stable but not stable quadratically. The following example illustrates this fact:

Example 2.3.8 *Let us consider the uncertain system with constant uncertainty $\delta \in [-1, -1/2] \cup [1/2, 1]$:*

$$\dot{x} = A(\delta, \tau) x \tag{2.19}$$

where $\tau > 0$ is a known system parameter and

$$A(\delta, \tau) = \begin{bmatrix} 1 & \delta \\ -(\tau + 2)/\delta & -(\tau - 1) \end{bmatrix}$$

The characteristic polynomial of the system is given by $s^2 + \tau s + 1$ and shows that the eigenvalues of the system do not depend on the uncertain parameter $\delta \in [-1, -1/2] \cup [1/2, 1]$. Moreover, the eigenvalues have strictly negative real part since $\tau > 0$ and this proves that the system is robustly asymptotically stable for any constant uncertainty δ. We aim at showing now that the system is not quadratically stable using reductio ad absurdum.

Let us assume first that the system is quadratically stable, thus there exists a matrix $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ 0$ *such that the LMI*

$$\begin{array}{rcl} L_q(\delta) & := & A(\delta)^T P + PA(\delta) \prec 0 \\ & = & \begin{bmatrix} 2p_1 - 2\dfrac{\tau+2}{\delta}p_2 & p_2 - \dfrac{\tau+2}{\delta}p_3 + p_1\delta - p_2(\tau+1) \\ \star & 2\delta p_2 - 2p_3(\tau+1) \end{bmatrix} \prec 0 \end{array}$$

holds for all $\delta \in [-1, -1/2] \cup [1/2, 1]$. *Choosing* $\delta_0 \in [1/2, 1]$, *we have* $L_q(-\delta_0) \prec 0$ *and* $L_q(\delta_0) \prec 0$ *and hence* $L_q(-\delta_0) + L_q(\delta_0) \prec 0$ *also holds. Computing the sum explicitly yields*

$$\begin{array}{rcl} L_q(-\delta_0) + L_q(\delta_0) & = & [A(-\delta_0) + A(\delta_0)]^T P + P[A(-\delta_0) + A(\delta_0)] \\ & = & \begin{bmatrix} 4p_1 & -\tau p_2 \\ -\tau p_2 & -4(\tau+1)p_3 \end{bmatrix} \end{array}$$

which is not negative definite (since $p_1 > 0$ *by definition) yielding then a contradiction. This shows that the system (2.19) is not quadratically stable.*

Let us consider now a parameter dependent Lyapunov matrix

$$P(\delta) = P_0 + P_1\delta + P_2\delta^2 = \begin{bmatrix} p_1(\delta) & p_2(\delta) \\ p_2(\delta) & p_3(\delta) \end{bmatrix}$$

It is relatively tough to show analytically that such a matrix allows to prove robust stability of system (2.19). However, we will show that the contradiction does not occur with such a Lyapunov matrix $P(\delta)$. *Let* $L_r(\delta) = A(\delta)^T P(\delta) + P(\delta)A(\delta)$ *and compute*

$$\begin{array}{rcl} L_r(\delta_0) + L_r(-\delta_0) & = & \begin{bmatrix} 2(p_1(\delta_0) + p_1(-\delta_0)) & * \\ * & * \end{bmatrix} \\ & = & \begin{bmatrix} 4(p_1^2\delta_0^2 + p_1^0) & * \\ * & * \end{bmatrix} \end{array}$$

with $p_i(\delta) = p_i^2\delta^2 + p_i^1\delta + p_i^0$. *This LMI might be feasible since the only constraint is* $p_1(\delta) > 0$ *for all* $\delta \in [-1, -1/2] \cup [1/2, 1]$ *which allows* $p_1^2\delta_0^2 + p_1^0$ *to take negative values.*

Numerical experiment with $\tau = 2$ *shows that a suitable choice for* $P(\delta)$ *is given by*

$$P(\delta) = \begin{bmatrix} 2.9218 & -0.0017 \\ -0.0017 & 0.0293 \end{bmatrix} + \begin{bmatrix} 0.0157 & 1.1383 \\ 1.1383 & 0.0005 \end{bmatrix}\delta + \begin{bmatrix} 0.0857 & 0.0087 \\ 0.0087 & 0.7601 \end{bmatrix}\delta^2 \quad (2.20)$$

Proposition 2.3.9 *Quadratic stability implies robust stability and quadratic stability is a sufficient condition to stability.*

Proof: *The see that quadratic stability implies robust stability, it suffices to let* $P_i = \bar{P}$, $\bar{P} = \bar{P}^T \succ 0$ *and quadratic stability is then a particular case of robust stability where all the matrices* P_i *are identical. Quadratic stability is a sufficient condition for stability since the Lyapunov function* $V(x) = x^T P x$ *is the most simple one that can be used to determine stability. Thus if stability is ensured for a simple Lyapunov function then it will also ensured using more complex ones.* □

Proposition 2.3.10 *The negativity of the real part of the eigenvalues of the system is a necessary condition for quadratic stability.*

Proof: *The proof is given in Example 2.3.8 where a system having negative real part of eigenvalues but which is not quadratically stable is constructed. On the other hand, it is possible to show that if* $A(\delta)^T P + PA(\delta) \prec 0$ *for some* $P = P^T \succ 0$ *then the eigenvalues of* $A(\delta)$ *have strictly real part for any* $\delta \in \mathbf{\Delta}$*. The procedure is similar as the one used in the proof of Theorem 2.3.2.* $\square$

If the uncertainties were time-varying, the quadratic stability would check the stability for unbounded parameter variation rates while the robust stability would consider bounded parameter variation rates. Indeed, the Lyapunov function derivative becomes in this case

$$\dot{V} = x(t)^T \left(A(\delta)^T P(\delta) + P(\delta)A(\delta) + \sum_{i=1}^{N} \dot{\delta}_i \frac{\partial P(\delta)}{\partial \delta_i} \right) x(t)$$

Hence robust stability of LPV systems is ensured if

$$A(\delta)^T P(\delta) + P(\delta)A(\delta) + \sum_{i=1}^{N} \dot{\delta}_i \frac{\partial P(\delta)}{\partial \delta_i} \prec 0 \tag{2.21}$$

This illustrates the fact that even if an uncertain system with time-varying uncertainties is stable for each frozen uncertainty, the derivative of the Lyapunov function may not be negative definite for some values of δ and $\dot{\delta}$ due to the influence of the term

$$\sum_{i=1}^{N} \dot{\delta}_i \frac{\partial P(\delta)}{\partial \delta_i}$$

This shows the importance of the rate of variation of the uncertainties in the stability of the system.

Proposition 2.3.11 *The negativity of the real parts of the eigenvalues is a necessary and sufficient condition for robust stability provided that the rate of variation of the parameters is sufficiently small.*

Proof: *The proof is based on the Lyapunov inequality (2.21). Indeed, suppose (2.21) holds and let* $\lambda_i(\delta)$ *be the* i^{th} *eigenvalue of the matrix* $A(\delta)$ *associated with the eigenvector* $e_i(\delta) \in \mathbb{C}^n - \{0\}$*. Pre and post-multiply (2.21) by* $e_i(\delta)^*$ *and* $e_i(\delta)$ *respectively we get*

$$(\lambda_i(\delta) + \lambda_i(\delta)^*)e_i(\delta)^* P(\delta)e_i(\delta) + e_i(\delta)^* \left(\sum_{i=1}^{N} \dot{\delta}_i \frac{\partial P(\delta)}{\partial \delta_i} \right) e_i(\delta) < 0$$

Define $z_i(\delta) := e_i(\delta)^* P(\delta) e_i(\delta) > 0$ *and* $y_i(\delta) = e_i(\delta)^* \frac{\partial P(\delta)}{\partial \delta_i} e_i(\delta)$, *the latter inequality rewrites*

$$2\Re\{\lambda_i(\delta)\}z_i + \sum_{i=1}^{N} \dot{\delta}_i y_i(\delta) \prec 0$$

Finally we obtain

$$\Re\{\lambda_i(\delta)\} < -\sum_{i=1}^{N} \dot{\delta}_i \frac{y_i(\delta)}{2z_i(\delta)} \prec 0$$

Since δ_i is bounded, this implies that $\dot{\delta}_i$ has to take both negative and positive values (in general) over time. The minimal value of the right-hand side is then obviously negative and hence we get

$$\Re\{\lambda_i(\delta)\} < -\sum_{i=1}^{N} \kappa_i(\delta)$$

where

$$\kappa_i = \sup_{\dot{\delta}_i} \left[\dot{\delta}_i \frac{y_i(\delta)}{2z_i(\delta)} \right]$$

which is attained at a vertex of the polytope where evolves $\dot{\delta}$. This shows that the larger the bounds on the derivative of δ are, the larger are the real part of the eigenvalues of $A(\delta)$ in absolute value. Moreover, it is important to point out that when the bounds on $\dot{\delta}$ reduce to 0, the LTI case is retrieved with $\Re\{\lambda\} < 0$.

On the other hand, it seems important to illustrate that it is possible to find $P(\delta) = P(\delta)^T \succ 0$ for some $A(\delta)$ such that $\Re\{\lambda_i(A(\delta))\} < \alpha$ and

$$A(\delta)^T P(\delta) + P(\delta)A(\delta) + \sum_{i=1}^{N} \dot{\delta}_i \frac{\partial P(\delta)}{\partial \delta_i} + Q(\delta) = 0$$

is satisfied. However, it is quite tough and will be omitted here. An idea would be to use strong evolution operators of LPV systems which are a generalization of evolution operators for LTV systems [Curtain and Pritchard, 1977; Daleckiĭ and Kreĭn, 1974]. □

The following remark is necessary to clarify the ideas:

Remark 2.3.12 *When considering systems involving uncertainties with infinite variation rates, robust stability in the sense of Definition 2.3.7 cannot be defined. Indeed, suppose that robust stability in the sense of Definition 2.3.7 is sought using a parameter dependent Lyapunov function of the form $V(x,\delta) = x^T P(\delta)x$. Due to the affine dependence of the Lyapunov function derivative $\dot{V}(x,\delta,\dot{\delta})$ on the term $\dot{\delta}$ (with a slight abuse of language since δ is not differentiable at some points), it is necessary and sufficient to consider only the vertices of the polytope in which $\dot{\rho}$ evolves. Since the uncertainties have unbounded parameter variation rate, then the polytope is the whole space $\mathbb{R}^N$ including $\pm\infty$. This implies that the term $\frac{\partial P(\delta)}{\partial \delta}\dot{\delta}$ may reach an infinite value, making the stability condition unfeasible. The only way to make the matrix inequality feasible again is to have $\frac{\partial P(\delta)}{\partial \delta} = 0$ meaning that the Lyapunov function is independent of δ and thus implying $P(\delta) = P_0$. Finally, the Lyapunov function becomes a Lyapunov function for quadratic stability. This shows that it is not possible to define a Lyapunov function depending explicitely/smoothly on the parameters. It seems that only quadratic stability can be defined for such systems but actually it is possible to define piecewise Lyapunov functions. Indeed, in [Xie et al., 1997] is introduced the piecewise Lyapunov function:*

$$V(x) = \max_i \{x(t)^T P_i x(t)\} > 0$$

with $P_i = P_i^T \succ 0$. Such function defines another type of stability improving the quadratic one. However, it leads to quasi-convex problems which are more difficult to solve, especially when a high number of different matrices P_i is considered.

The following example shows the difference between quadratic and robust stability for LPV systems:

Example 2.3.13 *Let us consider the LPV system [Wu et al., 1996]*

$$\dot{x} = A(\rho)x$$

where

$$A(\rho) = \begin{bmatrix} 7 & 12 & \cos(\rho) & \sin(\rho) \\ 6 & 10 & -\sin(\rho) & \cos(\rho) \\ \tau(\gamma+7)\cos(\rho) - 6\tau\sin(\rho) & 12\tau\cos(\rho) - (\gamma+10)\sin(\rho) & -\tau & 0 \\ \tau(\gamma+7)\sin(\rho) + 6\tau\cos(\rho) & 12\tau\sin(\rho) + \tau(10+\gamma)\cos(\rho) & 0 & -\tau \end{bmatrix}$$

For $\tau \geq 17.1169$ and $\gamma > 0$, the matrix $A(\rho)$ has negative eigenvalues for all $\rho \in [-\pi, \pi]$. For similar reasons as for the system in Example 2.3.8, the system is not quadratically stable (i.e. the sum of the right-upper block for $\rho = -\pi$ and $\rho = \pi$ is a zero matrix and the right-upper block matrix is unstable). If ρ is constant then the system is robustly but not quadratically stable. If ρ is time-varying, which effect have the parameters variation rates on the stability of the system.

Let us consider the parameter dependent Lyapunov function

$$V(x, \rho) = x^T P(\rho) x$$

where

$$P(\rho) = P_0 + P_1\cos(\rho) + P_2\sin(\rho) + P_3\cos(\rho)^2 + P_4\sin(\rho)^2$$

A LMI test is performed in order to find the admissible bound on $\dot{\rho}$ (i.e. $|\dot{\rho}| \leq \nu$) with respect to τ such that the system is robustly stable. The results are depicted on Figure 2.13. A best fit approach conjectures that $\nu \sim -0.0198\tau^2 + 1.7402\tau - 24.3626$. In view of interpreting the non-quadratic stability, note that the matrix $A(\rho)$ can be rewritten as:

$$A(\rho) = \begin{bmatrix} 7 & 12 & \cos(\rho) & \sin(\rho) \\ 6 & 10 & -\sin(\rho) & \cos(\rho) \\ 0 & 0 & -\tau & 0 \\ 0 & 0 & 0 & -\tau \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \tau & 0 \\ 0 & \tau \end{bmatrix} K(\rho)$$

with

$$K(\rho) = \begin{bmatrix} \tau(\gamma+7)\cos(\rho) - 6\tau\sin(\rho) & 12\tau\cos(\rho) - (\gamma+10)\sin(\rho) & 0 & 0 \\ \tau(\gamma+7)\sin(\rho) + 6\tau\cos(\rho) & 12\tau\sin(\rho) + \tau(10+\gamma)\cos(\rho) & 0 & 0 \end{bmatrix}$$

The terms $K(\rho)$ and τ play respectively the role of a parameter-dependent state-feedback gain and the bandwidth of the actuators. By transposition of the preceding analysis to stabilization, it is clear that it is not possible to find $K(\rho)$ such that the closed-loop is quadratically stabilizable. On the other hand, it is possible to find $K(\rho)$ such that the system is asymptotically stable provided that ν is sufficiently small (robust stability). From Figure 2.13, we can see

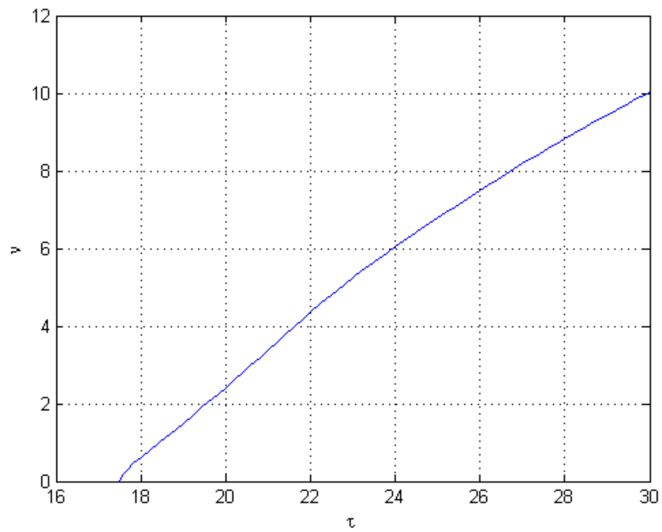

Figure 2.13: Evolution of the maximal parameter derivative value ν with respect to τ that preserves stability

that the larger the bandwidth of the actuator τ is, the larger the allowed bound on parameter derivative ν is.

According to [Wu et al., 1996], the reason for which the system is not quadratically stable is the particular parameter trajectories that allow to the right-upper block to switch arbitrarily fast between values I and $-I$. So regardless of the bandwidth τ of the actuators, the rapidly varying parameter ρ do not allow for parameter-dependent quadratic stabilization. Hence this illustrates why ν is allowed to increase as τ (the bandwidth of the actuator) increases.

The difference between stability for unbounded (quadratic stability) and bounded (robust stability) parameter variation rate has been emphasized in the preceding example. It is important to note that, for the moment, only values of the parameters and bounds on parameters derivative have been considered to study LPV systems stability. The remaining part extends the discussion when the system matrix $A(\rho)$ has nonnegative eigenvalues for some parameter values. We will show that under some (sometimes strong) assumptions, the LPV system may be robustly asymptotically stable even in presence of local parametric instability. This is done by considering additional properties on trajectories of the parameters.

Definition 2.3.14 *Let us consider the LPV system*

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) \\ x(0) & = & x_0 \\ (\rho, \dot{\rho}) & \in & U_\rho \times \mathrm{hull}[U_\nu] \end{array} \tag{2.22}$$

If there exists only a (possibly infinite) countable family of vectors ρ_i for which $A(\rho)$ has at least one eigenvalues with zero real part, then system (2.22) is said to be exponentially stable almost everywhere. This set of values for the parameters is denoted by $\bar{U}_\rho$.

Even if the system has unstable configurations, the system might have a stable behavior. Indeed, this is ensured if the parameter trajectories lie in the set

$$U_\rho^\circ := U_\rho - \bar{U}_\rho$$

It follows that the set U_ρ° is not convex anymore and thus results based on the convexity are not applicable (see for instance Section 2.2.2.1). Moreover, when considering these systems, it is clear that the exponential stability depends on the parameter trajectories (and not only their values). If the trajectories avoid values in $\bar{U}_\rho$ then the system would be quadratically/robustly stable over U_ρ°. On the other hand, if the trajectory stops on an unstable parameter values then the system becomes unstable. This is illustrated on Figure 2.14 and in the following example.

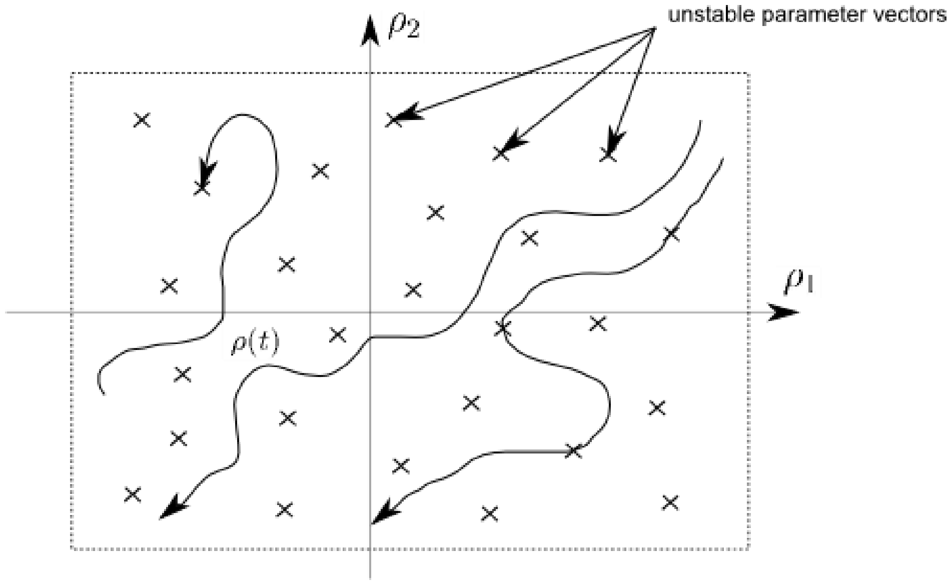

Figure 2.14: Example of trajectories for which the system is unstable (upper trajectory) and exponentially stable (lower trajectories) provided that the trajectories cross singular points sufficiently 'quick'

Example 2.3.15 *Let us consider the scalar system*

$$\begin{array}{rcl} \dot{x}(t) & = & [\cos(\rho(t)) + 1]x(t) \\ x_0 & = & x(0) \neq 0 \end{array}$$

with $\rho(t) \in [0, 2\pi]$. *At singular value 0 for* $\cos(\rho(t))+1$, *that is when the parameter* ρ *has values* $\pi/2$ *and* $3\pi/2$, *the system has a zero eigenvalues. Hence there exists 'unstable' trajectories. For instance the trajectory*

$$\rho(t) = \begin{array}{ll} 2t & \text{if } t \in [0, \pi/4] \\ \pi/2 & \text{if } t > \pi/4 \end{array}$$

makes the system non asymptotically stable since during the period $[0, \pi/4]$ *it does not have enough time to converge to 0.*

On the other hand, if for a particular parameter vector ρ_0, the system matrix $A(\rho_0)$ has at least an eigenvalue with strictly positive real part then the family contains an infinite, but not countable, number of parameter vectors for which the system is unstable (see Figure 2.15). This is a consequence of the continuity of the system eigenvalues with respect to the parameters provided that the evolution of the matrix coefficient is continuous. Finally, we arrive at the conclusion that the stability of the system depends on the values of the parameters and on the behavior of the parameters. If the parameters just cross or avoid the singular (unstable) parameter values, then the system would be quadratically/robustly stable. But if one of the parameter remains at an unstable value permanently, then the system would have an unstable behavior. This brought us to the following idea: if one can characterize the mean duration of the instability then it is possible to characterize asymptotic stability of the system. This is called *average dwell-time* in the switched systems community [Hespanha and Morse, 1999; Lin and Antsaklis, 2009]. This has been generalized to LPV systems in [Hespanha et al., 2001; Mohammadpour and Grigoriadis, 2007b]. However, this only works with discontinuous parameters for technical reasons exposed below.

By average dwell-time argument we mean, if the time spent by the system in an unstable behavior is greater than the time spent in the stable behavior, then the system will be unstable. However, the time spent must be weighted by exponential rates of convergence in order to take into account the speed of the system in each mode. Generally, worst exponential rates are considered, that is we only consider the eigenvalue with maximal real part. Hence, this means that due to the continuity of the eigenvalues with respect to the parameters, the maximal stable exponential rate is close to 0 when the parameters values are close to the boundary between unstable and stable domains. Thus, since the stable exponential rate is close to 0, hence it will be difficult to conclude on the stability with average dwell-time (the stable system part is considered as very slow and hence cannot have any effect on the global stability of the LPV system).

After this brief presentation of different forms of LPV systems stability, some results on their representation and associated tools are provided. If not stated otherwise, in the following, by 'stability' we tacitly means quadratic/robust exponential stability.

2.3.2 Stability of Polytopic Systems

This section is devoted to the stability analysis of LPV polytopic systems. Quadratic and Robust stability are discussed and compared in the polytopic systems framework. In what follows, the following polytopic LPV system is considered

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{N} \lambda_i(t) A_i x(t) \\ x(0) &= x_0 \end{aligned} \tag{2.23}$$

where x is the system state and $\lambda(t) \in \Gamma$ where

$$\Gamma := \left\{ \operatorname*{col}_{i=1}^{N}(\lambda_i(t)) : \sum_{i=1}^{N} \lambda_i(t) = 1,\ \lambda_i(t) \geq 0 \right\} \tag{2.24}$$

A necessary and sufficient condition for robust stability is given below:

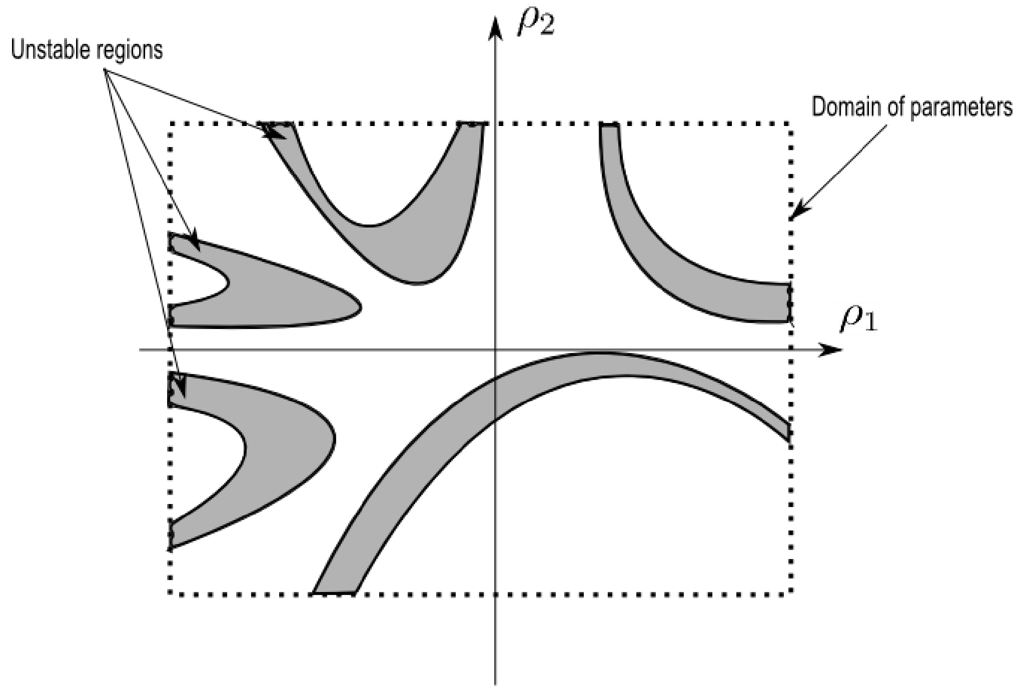

Figure 2.15: Example of stability map of a LPV system with two parameters; the grey regions are unstable regions

Proposition 2.3.16 *The LPV polytopic system (2.23) is quadratically stable if and only if there exists a matrix $P = P^T \succ 0$ such that*

$$A_i^T P + PA_i \prec 0 \tag{2.25}$$

holds for all $i = 1, \ldots, N$.

Proof: *Define the Lyapunov function $V(x(t)) = x(t)^T Px(t)$ with $P = P^T \succ 0$. The time-derivative of the Lyapunov functions computed along trajectories of system (2.2) with $w \equiv 0$ leads to*

$$\dot{V}(x(t)) = x(t)^T (A(\lambda(t))^T P + PA(\lambda(t)))x(t)$$

The quadratic stability of the equilibrium point $x_{eq} = 0$ of system (2.2) is proved if $\dot{V}(x(t)) \prec 0$ for every $x \neq 0$. This yields the following parameter dependent LMI

$$\sum_{i=1}^{N} \lambda_i(t) \left(A_i^T P + PA_i\right) \prec 0 \tag{2.26}$$

for any $\lambda \in \Gamma$.

Sufficiency:

Assume that $A_i^T P + PA_i \prec 0$ for all $i = 1, \ldots, N$. Then it is obvious that (2.26) holds since the sum of negative definite matrices is also a negative definite matrix.

Necessity:

Since (2.26) must be true for every value of $\lambda(t) \in \Gamma$ then it must be true at every vertices $\mathcal{V}_i$ of the polytope and this implies

$$A_i^T P + P A_i \prec 0$$

for all $i = 1, \ldots, N$. $\square$

An interesting fact of the previous result is the transformation of the parameter dependent LMI (2.26) into a set of N LMIs. In other words, a semi-infinite dimensional problem is reduced to a finite dimensional problem (sometimes huge) independent of the parameters vector $\lambda(t)$.

Example 2.3.17 *In Example 2.2.1, the multi-affine system*

$$\dot{x}(t) = (A_1\rho_1(t) + A_2\rho_2(t))x(t)$$

is turned in a polytopic formulation

$$\dot{x} = [A_1[(\lambda_1 + \lambda_3)\rho_1^- + (\lambda_2 + \lambda_4)\rho_1^+] + A_2[(\lambda_1 + \lambda_2)\rho_2^- + (\lambda_3 + \lambda_4)\rho_2^+]]x$$

with $\lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \lambda_4(t) = 1$, $\lambda_i(t) \geq 0$. From this formulation, quadratic stability is ensured if and only if there exists a matrix $P = P^T \succ 0$ such that the set of 4 LMIs is satisfied

$$\begin{array}{rcl}
(A_1\rho_1^- + A_2\rho_2^-)^T P + P(A_1\rho_1^- + A_2\rho_2^-) & \prec & 0 \\
(A_1\rho_1^+ + A_2\rho_2^-)^T P + P(A_1\rho_1^+ + A_2\rho_2^-) & \prec & 0 \\
(A_1\rho_1^- + A_2\rho_2^+)^T P + P(A_1\rho_1^- + A_2\rho_2^+) & \prec & 0 \\
(A_1\rho_1^+ + A_2\rho_2^+)^T P + P(A_1\rho_1^+ + A_2\rho_2^+) & \prec & 0
\end{array}$$

A necessary and sufficient condition to quadratic stability of the multi-affine system is defined by the stability of the system at each vertex of the orthotope $[\rho_1^-, \rho_1^+] \times [\rho_2^-, \rho_2^+]$ using a common Lyapunov function. The exactness of the procedure is a consequence of the fact that an orthotope is also a convex polyhedral and every convex polyhedral can be exactly parametrized over the unit simplex Γ.

Example 2.3.18 *Let us consider again Example 2.2.2 where a LPV system with quadratic dependence on a parameter is turned, in a nonequivalent polytopic description recalled below:*

$$\dot{x}(t) = [A_0 + A_1[(\lambda_1(t) + \lambda_3(t))\rho^- + (\lambda_2(t) + \lambda_4(t))\rho^+] + A_2[\lambda_3(t)(\rho^-)^2 + \lambda_4(t)(\rho^+)^2]]x(t)$$

with $\displaystyle\sum_{i=1}^{4} \lambda_i(t) = 1, \quad \lambda_i(t) \geq 0$. A sufficient condition to stability of such system is hence given by the feasibility of the set of 4 LMIs

$$\begin{array}{rcl}
(A_0 + A_1\rho^-)^T P + P(A_0 + A_1\rho^-) & \prec & 0 \\
(A_0 + A_1\rho^- + A_2(\rho^-)^2)^T P + P(A_0 + A_1\rho^- + A_2(\rho^-)^2) & \prec & 0 \\
(A_0 + A_1\rho^+)^T P + P(A_0 + A_1\rho^+) & \prec & 0 \\
(A_0 + A_1\rho^+ + A_2(\rho)^+)^T P + P(A_0 + A_1\rho^+ + A_2(\rho)^+) & \prec & 0
\end{array}$$

As suggested in the proof and illustrated in the examples above, a necessary and sufficient condition to quadratic stability (or sufficient condition to stability) of (2.26) is the stability of all A_i (A_i have eigenvalues with strictly negative real part for all $i = 1, \ldots, N$). The main difficulty comes from the fact that, even if all the matrices A_i are Hurwitz, a common matrix P satisfying the LMIs may not exist. The robust stability overcomes this problem.

Proposition 2.3.19 *The LPV polytopic system (2.23) is robustly stable if there exists matrices $P_i = P_i^T \succ 0$, a matrix X and a sufficiently large scalar $\sigma > 0$ such that*

$$\begin{bmatrix} -(X+X^T) & P_i + X^T A_i & X^T \\ \star & -\sigma P_i + \mathcal{P}\dot{\lambda}(t) & 0 \\ \star & \star & -P_i/\sigma \end{bmatrix} \prec 0 \tag{2.27}$$

holds for all $i = 1, \ldots, N$ and all $\dot{\lambda} \in \mathcal{S}$ (compact) where $\mathcal{P} := \dfrac{\partial P(\lambda)}{\partial \lambda} = \begin{bmatrix} P_1 & P_2 & \ldots & P_N \end{bmatrix}$.

For simplicity, the compact set $\mathcal{S}$ is not detailed here but is the set where the derivative of $\lambda(t)$ evolves. More details are provided in Section 4.4.

Proof: *The proof is made in three steps, the first step is to provide a relevant parameter dependent Lyapunov function and differentiate it. The second part aims at defining a relaxed LMI in order to linearize the dependence on parameters. Finally, the last step turns a parameter dependent matrix inequality into a set of matrix inequalities that are independent of the parameters.*

Let us consider the parameter dependent Lyapunov function

$$V(x(t), \lambda(t)) = x(t)^T P(\lambda(t)) x(t)$$

where $P(\lambda(t)) = \sum_{i=1}^{N} \lambda_i(t) P_i$, $P_i = P_i^T \succ 0$. The derivative of V along the trajectories solutions of system (2.23) is given by

$$\dot{V}(x(t), \lambda(t), \dot{\lambda}(t)) = x(t)^T \left(A(\lambda(t))^T P(\lambda(t)) + P(\lambda(t)) A(\lambda(t)) + \mathcal{P}\dot{\lambda}(t) \right) x(t)$$

where $A(\lambda(t)) = \sum_{i=1}^{N} \lambda_i(t) A_i$. Since $\dot{V}(\cdot,\cdot,\cdot)$ must be negative (except for $x = 0$) for all $x \in \mathbb{R}^n$, $\lambda \in \Gamma$ and $\dot{\lambda} \in \mathcal{S}$ we must have

$$A(\lambda(t))^T P(\lambda(t)) + P(\lambda(t)) A(\lambda(t)) + \mathcal{P}\dot{\lambda}(t) \prec 0 \tag{2.28}$$

The idea would be to use the same proof as for quadratic stability to provide a sufficient condition for robust stability. However, the equivalence holds if only if the dependence on the parameters is affine. Due to the product $A(\lambda(t))^T P(\lambda(t))$ in LMI (2.28) the dependence is not affine anymore but quadratic. The idea now is to turn LMI (2.28) into an equivalent formulation where these quadratic terms are removed.

Let us consider the following LMI where X is a constant full matrix of appropriate dimensions

$$\begin{bmatrix} -(X+X^T) & P(\lambda) + X^T A(\lambda) & X^T \\ \star & -\sigma P(\lambda) + \mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix} \prec 0 \tag{2.29}$$

We aim now at showing that LMI (2.29) implies (2.28). Note that (2.29) can be rewritten in the expanded form

$$\underbrace{\begin{bmatrix} 0 & P(\lambda) & 0 \\ \star & -\sigma P(\lambda)+\mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix}}_{\Psi} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X^T \begin{bmatrix} -I & A(\lambda) & I \end{bmatrix} + \begin{bmatrix} -I \\ A(\lambda)^T \\ I \end{bmatrix} X \begin{bmatrix} I & 0 & 0 \end{bmatrix} \prec 0$$

Since the matrix X is unconstrained (free) then the projection lemma applies (see Appendix D.18). A basis of the null-space of $U_1 := \begin{bmatrix} -I & A(\lambda) & I \end{bmatrix}$ and $U_2 := \begin{bmatrix} I & 0 & 0 \end{bmatrix}$ are given respectively by

$$\mathrm{Ker}[U_1] = \begin{bmatrix} A(\lambda) & I \\ I & 0 \\ I & I \end{bmatrix} \qquad \mathrm{Ker}[U_2] = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$$

Finally the projection lemma yields the following two underlying LMIs

$$\begin{aligned} \mathrm{Ker}[U_1]^T \Psi \mathrm{Ker}[U_1] &= \mathrm{Ker}[U_1]^T \begin{bmatrix} 0 & P(\lambda) & 0 \\ \star & -\sigma P(\lambda)+\mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix} \mathrm{Ker}[U_1] \prec 0 \\ &= \begin{bmatrix} A(\lambda)^T P(\lambda) + P(\lambda)A(\lambda) + \sigma P(\lambda) + \mathcal{P}\dot{\lambda} & P(\lambda) \\ \star & -P(\lambda)/\sigma \end{bmatrix} \prec 0 \\ \mathrm{Ker}[U_2]^T \Psi \mathrm{Ker}[U_2] &= \mathrm{Ker}[U_2]^T \begin{bmatrix} 0 & P(\lambda) & 0 \\ \star & -\sigma P(\lambda)+\mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix} \mathrm{Ker}[U_2] \prec 0 \\ &= \begin{bmatrix} -\sigma P(\lambda)+\mathcal{P}\dot{\lambda} & 0 \\ \star & -P(\lambda)/\sigma \end{bmatrix} \prec 0 \end{aligned}$$

A Schur complement on the first LMI yields

$$A(\lambda)^T P(\lambda) + P(\lambda)A(\lambda) + \mathcal{P}\dot{\lambda} \prec 0$$

which is identical to (2.28). The second LMI is satisfied if and only if $-\sigma P(\lambda) + \mathcal{P}\dot{\lambda} \prec 0$ and this inequality is verified if σ is sufficiently large. This means that (2.29) implies (2.28). The final part of the proof is the transformation of the parameter dependent matrix inequality (2.29) into a set of N matrix inequalities (2.27). This is done in the same way as for quadratic stability. □

It is worth noting that condition (2.27) is not a LMI condition due to the unknown scalar term $\sigma > 0$. Nevertheless, if σ is fixed, the condition becomes a LMI. This is called a quasi-convex program. Moreover, σ is not completely unknown since it must be sufficiently large. In this case, it is possible to determine its value using an increasing linear search. Note also that in the case of constant λ, the term $\mathcal{P}\dot{\lambda}$ is 0 and hence a suitable and simple choice for σ is 1.

Remark 2.3.20 *The principle of the polytopic formulation is based on the fact that the system and stability conditions (here in a LMI form) have affine dependence on the parameters. If, for some reason, the affine dependence is lost the stability of the system is not equivalent to (or implied by) the feasibility of the LMI at each vertex. In [Apkarian and Tuan, 1998; Jungers et al., 2007; Oliveira et al., 2007] some results are provided for which the dependence is lost and some sufficient conditions are expressed to relax the parameter dependent LMI conditions. This is also introduced in Section 4.2.*

In terms of computational complexity, let us consider that a system has N parameters, hence the number of LMIs to be solved simultaneously is then given by $\#(LMIs) = 2^N$. This can be very time and memory consuming for some applications.

2.3.3 Stability of Polynomially Parameter Dependent Systems

The most simple and intuitive description of polynomially parameter dependent systems or systems with polynomial of functions of parameters (e.g. $\cos(\rho), e^{\rho}\dots$) is to deal directly with a primal formulation:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho(t))x(t) \\ x(0) & = & x_0 \\ \rho & \in & U_\rho \subsetneqq \mathbb{R}^N \end{array} \tag{2.30}$$

as done in Section 2.3.1. In order to avoid repetition on stability of such systems, we will focus on how to express stability conditions and how solving them. The reader should refer to Section 2.3.1 to get preliminary results. We only recall here LMIs used to define quadratic (rate-independent) and robust (rate dependent) stability and then a discussion is provided on relaxation techniques.

Lemma 2.3.21 *System (2.30) is quadratically stable if and only if there exists a matrix $P = P^T \succ 0$ such that*

$$A(\rho)^T P + PA(\rho) \prec 0 \tag{2.31}$$

holds for all $\rho \in U_\rho$.

Proof: *The proof is an application of the Lyapunov stability theory with $V(x) = x^T Px$ as a Lyapunov function.* □

Lemma 2.3.22 *System (2.30) is robustly stable if and only if there exists a continuously differentiable matrix function $P(\rho) = P(\rho)^T \succ 0$ such that*

$$A(\rho)^T P(\rho) + P(\rho)A(\rho) + \sum_{i=1}^{N} \nu_i \frac{\partial P(\rho)}{\partial \rho_i} \prec 0$$

holds for all $\rho \in U_\rho$ and all $\nu = \operatorname*{col}_{i=1}^{N}(\nu_i) \in U_\nu$ where U_ν is the set of vertices of the polytope in which the derivative of the parameters $\dot{\rho}$ evolves.

Proof: *The proof is an application of the Lyapunov stability theory with $V(x, \rho) = x^T P(\rho)x$ as a Lyapunov function. After differentiation, the term $\dot{\rho}$ enters affinely in the LMI and hence a polytopic formulation is equivalent for the description of the dependence on $\dot{\rho}$. Hence it suffices to consider the vertices of the polytope to consider any value for the derivative of the parameters within the polytope.* □

The LMI for quadratic stability is technically called a *semi-infinite dimensional LMI* due to the dependence on parameters. Indeed, a continuum of LMIs is parametrized by ρ. This means that it must be satisfied for all $\rho \in U_\rho$ and the verification of such a LMI constraint is a challenging problem due to the infinite number of values of ρ.

The LMIs for robust stability is technically called an *infinite dimensional semi-infinite LMI*. The term 'infinite dimensional' comes from the fact that the unknown variable $P(\rho)$ to be determined is a function (and thus belong to an infinite dimensional space) and the term 'semi-infinite' comes from the fact that the LMI must be satisfied for all $(\rho, \dot{\rho}) \in U_\rho \times U_\nu$. Solving this LMI is also challenging due to the difficulty of finding a matrix function $P(\rho)$.

The remaining of this section aims at showing different relaxations schemes allowing to turn these difficult LMI problems into more tractable LMI conditions. Roughly speaking, primal LMIs are relaxed into a set of finite number of finite dimensional LMIs which is easier to solve with convex optimization tools. First of all, a method to relax the infinite-dimensional part into a finite dimensional problem is provided. It is based on projections of functions on a particular basis of functions. Second, methods to relax the semi-infinite part of the LMIs are introduced. Some of these methods work for every parameter dependent LMIs independently of the type of LPV system (affine, polynomial or rational). However, these (more or less) recent results are rather complicated and remain technically difficult due to large theoretical background. Nevertheless, they will be explained in broad strokes with a sufficient number of references if precisions are sought. Three methods will be introduced: the relaxation by discretization (or commonly called 'gridding'), the 'Sum-of-Squares' approach and the global polynomial optimization. They will be illustrated through examples and a discussion on advantages and drawbacks will be provided. It is important to mention that lots of works have been provided to relax semi-infinite dimensional part especially for LMIs with quadratic parameter dependence but not only [Apkarian and Tuan, 1998; Féron et al., 1996; Lim and How, 2002; Oliveira and Peres, 2002; Oliveira et al., 2007; Sato and Peaucelle, 2007; Scherer, 2005; Scherer and Hol, 2006; Trofino and De Souza, 1999; Tuan and Apkarian, 1998; Tuan et al., 2001a].

2.3.3.1 Relaxation of matrix functions

The relaxation of the infinite dimensional part can be reduced to a finite dimensional problem by projecting the function on a finite basis of function; for instance let us consider a polynomial basis

$$f_{\alpha_i}(\rho) = \rho^{\alpha_i}, \; i = 1, \ldots, N_b$$

Therefore a choice for the matrix $P(\rho)$ can be

$$P(\rho) = \sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho)$$

where the matrices $P_i = P_i^T$ have to be determined. Therefore the robust stability conditions becomes

Corollary 2.3.23 *System (2.30) is robustly stable if and only if there exist matrices $P_i = P_i^T$ such that such that LMIs*

$$\begin{aligned} A(\rho)^T \left(\sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho) \right) + \left(\sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho) \right) A(\rho) + \sum_{i=1}^{N} \nu_i \left(\sum_{i=1}^{N_b} P_i \frac{\partial f_{\alpha_i}(\rho)}{\partial \rho_i} \right) &\prec 0 \\ \sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho) &\succ 0 \end{aligned}$$

holds for all $\rho \in U_\rho$ and all $\nu = \underset{i=1}{\overset{N}{\text{col}}}(\nu_i) \in U_\nu$ where U_ν is the set of vertices of the polytope in which the derivative of the parameters $\dot{\rho}$ evolves.

We have explicitly turned an infinite dimensional problem into a finite dimensional problem where only N_b matrices are sought. The main difficulty of this relaxation stems from the difficulty of finding both 'good' types and number of basis functions. The central idea, generally admitted, is to mimic the behavior of the system and reproduce the same parameter dependence for $P(\rho)$. An iterative procedure to find the number of basis function might be the best technique but remains time consuming.

2.3.3.2 Relaxation of parametrized LMIs by discretization (gridding)

This LMI relaxation is applicable for any type of parametrized LMIs provided that the coefficients of the LMIs are finite for any values in the parameter set. The discretization is the most intuitive and simple way to make the problem finite dimensional. It proposes to replace the initial semi-infinite problem by a discretized version involving a finite number of finite dimensional LMIs. This is illustrated in the following example.

Example 2.3.24 *The following generic problem is considered. Let $L(x, \rho)$ be a real symmetric matrix in the unknown variable $x \in \mathcal{X} \subset \mathbb{R}^n$ where the parameter vector ρ belongs to some compact subset U_ρ of $\mathbb{R}^N$. The problem aimed to be solved is:*

$$\begin{array}{ll} \text{Solve} & L(x,\rho) \prec 0 \text{ for all } \rho \in U_\rho \\ s.t. & x \in \mathcal{X} \end{array}$$

The gridding approach proposes to simplify the latter problem into a discretized version. Let $\bar{U}_\rho := \{\rho^1, \ldots, \rho^k\}$ be a set of distinct points belonging to U_ρ (i.e. $\rho^j \in U_\rho$ for all $j = 1, \ldots, k$. Hence the problem reduces to

$$\begin{array}{ll} \text{Solve} & L(x,\rho) \prec 0 \text{ for all } \rho \in \bar{U}_\rho \\ s.t. & x \in \mathcal{X} \end{array}$$

This approach is based on the claim that, by discretizing the parameter space, there exists a density of the grid for which most of critical points are considered. By critical points, we mean, the points in U_ρ for which the LMI is unfeasible in $\mathcal{X}$. However, the density which has to be considered is unknown and its determination remains a difficult problem. Indeed, if one wants to find a 'good' density, the location of unfeasible regions in the parameter domain is a crucial information. Unfortunately, this information is not accessible since the knowledge of unfeasible regions is equivalent to the knowledge of the (un)feasibility of the problem which is actually sought. This paradox shows that probably no method to find a 'perfect' gridding would be developed someday. However, probabilistic approaches have been developed to provide probabilistic certificate of feasibility, see for instance [Calafiore et al., 2000; Tempo et al., 1997].

Remark 2.3.25 *It is important to point out that this relaxation is an inner approximation of the original problem. Indeed, the finite number of values for which the feasibility is tested is strictly lower than the real set (which contains an infinite number of values). Hence the feasibility of the relaxed problem is a necessary condition only for the feasibility of the initial problem. This fact makes this relaxation to be a possible interesting certificate for unfeasibility.*

Example 2.3.26 *Let us consider the trivial LPV system*

$$\dot{x}(t) = (\rho - 1)^2 x(t)$$

with $\rho \in [0, 2]$. *It is clear that for* $\rho = 1$, *the eigenvalue of the system is 0. Hence if the discretization do not consider explicitly these two values, the system would be considered as asymptotically stable. It seems very difficult in this case to prove exactly (i.e. find a 'good' grid) that the system is asymptotically stable and moreover, this cannot be viewed in simulations since the parameters have to stay, for a long time, at critical values to observe instability.*

Example 2.3.27 *Let us consider now the system*

$$\dot{x}(t) = (\rho^2 - 1 + \varepsilon)x(t) \qquad 0 \leq \varepsilon \leq 1$$

with $\rho \in [-1, 1]$, *there exist an infinite number of parameters for which the system is unstable:* $\rho \in [-1, -\sqrt{1-\varepsilon}] \cup [\sqrt{1-\varepsilon}, 1]$. *The Lebesgue measure of the interval of values of* ρ, *for which the system is unstable, is* $2(1-\sqrt{1-\varepsilon})$ *and taking a gridding of 5 equally spaced points suffices to prove instability of the system. The largest* ε *is, the easiest is the proof of instability (the measure of the interval grows up). On the contrary, the smallest* ε *is, the hardest is the proof of instability.*

A second drawback of the approach is the uncertain location of the eigenvalues of the LMI between gridding points. It is worth noting that the discretization grid can be chosen to be nonuniform over the whole parameter space. Indeed, in theory of interpolation, it has been shown, in many works, that an uniform discretization may be far from the best choice. For instance, in Lagrange polynomial interpolation, if the points are equally spaced, the interpolated function oscillates above and below the real curve: this is called the Runge's phenomenon [Runge, 1901]. This can be a problem since the eigenvalues may change sign between gridding points. It has been shown that if the gridding points coincide with zeros the Chebyshev polynomials, the resulting interpolation polynomial minimizes the Runge's problem [Burden and Faires, 2004]. For interpolation with function and their derivatives, Hermite interpolation polynomials should be considered instead [Burden and Faires, 2004]. Although these methods give ideas on the discretization scheme, they lead to complicated expression for unknown functions since the order of polynomials approximately equals to the number of discretization points. For more details about these topics, see for instance [Abramowitz and Stegun, 1972; Bartels et al., 1998].

In terms of computational complexity, let us consider by simplicity that the system has p parameters whose parameter space are discretized in $N+1$ points. This means that the total number of points is $(N+1)^{N_p}$. Hence, the number of LMI to be solved simultaneously is equal to the number of points, and thus we have $\#(LMIs) = (N+1)^{N_p}$. Generally, this number is quite large since the number of gridding points must be sufficiently large to be 'sure' to capture the behavior of the system.

2.3.3.3 Relaxation of Parametrized LMIs using methods based on Sum-of-Squares (SoS)

We show here, in a very simple way, what is the sum-of-squares relaxation; where does it come from and how to use it in the framework of parameter dependent LMIs. The interested reader should refer to [Gatermann and Parrilo, 2004; Helton, 2002; Parrilo, 2000; Prajna et al., 2004; Scherer and Hol, 2006] and references therein to get more details. This method applies only for polynomially parameter dependent LMIs (or possibly to some vary special cases of rationally parameter dependent LMIs).

The idea is to describe the set of parameter values by a set of polynomial inequalities. Then using an interesting variation of the $\mathcal{S}$-procedure (see Appendix D.10) constraints are injected in the LMIs. In such a method, the scalar variables introduced by the $\mathcal{S}$-procedure are not constant anymore but vary with respect to parameters, allowing a more tight relaxation. Finally, it is aimed to show that the obtained LMI is Sum-of-Squares (SoS). Indeed, showing the LMI to be SoS shows its positive definiteness. Moreover, testing if a polynomial is SoS can be cast as a semidefinite programming problem (SDP problem), this is an important fact demonstrating the interest of such an approach.

Theorem 2.3.28 *Let $p(x)$ be a univariate polynomial of order N. $p(x)$ is nonnegative if and only if it is sum-of-squares, i.e. there exists N polynomials $h_i(x)$ such that $p(x) = \sum_{i=1}^{N} h_i(x)^2$. Moreover, the degree of $p(x)$ is even and the coefficient of the higher power is positive.*

Proof ***Necessity:***

The necessity is obvious. Suppose that $p(x)$ is SOS thus it writes $p(x) = \sum_i q_i(x)^2$ which is obviously nonnegative.

Sufficiency:

Since $p(x) = p_n x^n + \ldots + p_1 x + p_0 \geq 0$ is univariate then it can be factorized as

$$p(x) = p_n \prod_i (x - r_i)^{n_i} \prod_k (x - \alpha_k + j\beta_k)^{m_j} (x - \alpha_k - j\beta_k)^{m_j}$$

where r_i and $\alpha_k \pm j\beta_k$ are respectively all real and complex roots of $p(x)$ with respective order of multiplicity n_i and m_j. It is clear that a univariate polynomial is nonnegative if and only if $p_n > 0$ and the orders of multiplicity of real roots are even. Let $n_i = 2n'_j$ and noting that

$$(x - \alpha_k + j\beta_k)(x - \alpha_k + j\beta_k) = (x - \alpha_k)^2 + \beta_k^2$$

then we have

$$p(x) = p_n \prod_i (x - r_i)^{2n'_i} \prod_k \left[(x - \alpha_k)^2 + \beta_k^2 \right]^{m_j}$$

In virtue of the property that products of sums of squares are sums of squares (the set of SoS is closed under multiplication), and that all the expression above are SoS, it follows that $p(x)$ is SoS. □

To illustrate the fact that the nonnegativity of a polynomial can be expressed as a SDP, let us consider a SoS nonnegative multivariate (n variables) polynomial $p(x)$ of degree $2d$.

Then we have

$$\begin{aligned} p(x) &= \sum_i q_i(x)^2 \geq 0 \\ &= \sum_i z(x)^T L_i^T L_i z(x) \geq 0 \qquad \text{with } q_i(x) = L_i z(x) \\ &= \sum_i z(x)^T Q_i z(x) \geq 0 \qquad \text{with } Q_i = L_i^T L_i \\ &= z(x)^T Q z(x) \geq 0 \qquad \text{with } Q = \sum_i Q_i \end{aligned}$$

where $z(x)$ is a vector containing monomial of degree up to d whose number of components equals at most $\begin{pmatrix} n+d \\ d \end{pmatrix}$ and the number of squares equals rank$[Q]$. Hence the positive definiteness of Q implies the polynomial $p(x)$ to be Sum-of-Squares.
This can be easily transposed to the matrix case:

Theorem 2.3.29 *Let $P(x)$ be a matrix univariate polynomial of order N. $P(x)$ is non-negative if and only if it is SoS, i.e. there exists N matrix polynomials $H_i(x)$ such that $P(x) = \sum_{i=1}^{N} H_i(x)^T H_i(x)$.*

In the univariate case, the positivity of the polynomial is equivalent to the existence of a SoS decomposition. This is also true for quadratic polynomials and quartic polynomials in two variables. On the other hand, in the general multivariate case, a positive definite polynomial is not necessarily SoS in general. Fortunately, the set of SoS multivariate polynomials is dense in the set of nonnegative polynomials and allows SoS approach to be applied successfully in many problems. The following example describes a nonnegative polynomial which is not SoS but whose nonnegativity can be expressed as a SoS decomposition problem through an equivalent test.

Example 2.3.30 *The Motzkin's polynomial*

$$m(x) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3)$$

is globally nonnegative but cannot be written as a SoS. It is depicted on Figure 2.16 showing that it vanishes at $|x_1| = |x_2| = 1$ To see that it is globally nonnegative, let us consider the triplet $(1, x_1^2 x_2^4, x_1^4 x_2^2)$ and in virtue of the arithmetic-geometric mean inequality (i.e. the arithmetic mean is greater or equal to the geometric mean [Cauchy, 1821; Jensen, 1806]) then we have

$$\begin{aligned} & \frac{1 + x_1^2 x_2^4 + x_1^4 x_2^2}{3} && \geq \sqrt[3]{x_1^6 x_2^6} \\ \Rightarrow \quad & 1 + x_1^2 x_2^4 + x_1^4 x_2^2 - 3x_1^2 x_2^2 && \geq 0 \\ \Rightarrow \quad & 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3) && \geq 0 \end{aligned}$$

It is relatively tough to show that the Motzkin's polynomial is not SoS. On the other hand, we will show that its nonnegativity can be cast as a SDP problem anyway by turning the nonnegativity analysis of $m(x)$ into an equivalent problem involving another polynomial which

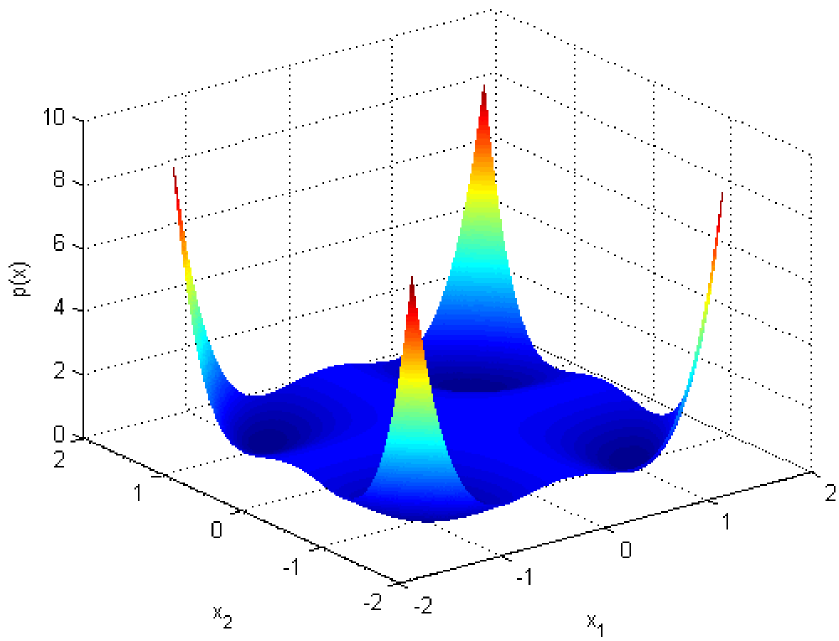

Figure 2.16: Motzkin's polynomial

is SoS. First multiply the Motzkin's polynomial $m(x)$ by the positive polynomial $x_1^2 + x_2^2 + 1$ and we get

$$m'(x) = (x_1^2 + x_2^2 + 1)m(x)$$

It is clear that nonnegativity of $m'(x)$ and $m(x)$ are equivalent. Hence by solving a SDP the following SOS decomposition of $m'(x)$ is obtained:

$$\begin{aligned} m'(x) &= (x_1^2 + x_2^2 + 1)(1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3)) \\ &= (x_1^2 x_2 - x_2)^2 + (x_1 x_2^2 - x)^2 + (x_1^2 x_2^2 - 1)^2 + \frac{1}{4}(x_1 x_2^3 - x_1^3 x_2)^2 \\ &\quad + \frac{3}{4}(x_1 x_2^3 + x_1^3 x_2 - 2x_1 x_2)^2 \end{aligned}$$

The SoS approach is explained in the remaining of the section. Let $\mathcal{M}(\rho) \succ 0$ be a parameter dependent LMI to be satisfied for every value of ρ in an orthotope $\mathcal{I}$ explicitly defined by

$$\mathcal{I} := [\rho_1^-, \rho_1^+] \times \ldots \times [\rho_p^-, \rho_p^+]$$

This orthotope can defined through a set of polynomial inequalities (a semi-algebraic set):

$$\mathcal{I} = \{\rho : \; g_i(\rho) \geq 0, \; i = 1, \ldots, p\}$$

The following example describes the construction of such polynomials.

Example 2.3.31 *For example, let $(\rho_1, \rho_2) \in \mathcal{I}_2 := [-1, 1] \times [2, 3]$ hence we have*

$$\begin{aligned} g_1(\rho_1) &= -\rho_1^2 + 1 \\ g_2(\rho_2) &= -\rho_2^2 + 5\rho_2 - 6 \end{aligned}$$

The expression of $\mathcal{I}$ through polynomial inequalities is not unique. In the example, above we have chosen to define one polynomial of degree 2 for each parameter. It would also be possible to define 4 polynomials of degree 1 or also 2 polynomials of degree 4, and so on. Supplementary constraints can be added in order to specify other relations between parameters. All these constraints can be combined into a more general semi-algebraic set, say $\mathcal{I}'$. Hence, by invoking the full version of the $\mathcal{S}$-procedure we claim that

$$\mathcal{M}'(\rho) = \mathcal{M}(\rho) - \sum_{i=1}^{N} g_i(\rho) Z_i - \varepsilon I \succ 0 \tag{2.32}$$

where matrices $Z_i = Z_i^T \succ 0$ and a scalar $\varepsilon > 0$ are sought. The idea is to show that if $\mathcal{M}'(\rho)$ is SoS (i.e. $\mathcal{M}'(\rho) \succeq 0$) for all $\rho \in \mathcal{I}$ (or $\mathcal{I}'$) and in this case we should have

$$\begin{aligned} \mathcal{M}'(\rho) \succeq 0 \quad &\Leftrightarrow \quad \mathcal{M}(\rho) - \sum_{i=1}^{N} g_i(\rho) Z_i - \varepsilon I \succeq 0 \\ &\Rightarrow \quad \mathcal{M}(\rho) \succeq \sum_{i=1}^{N} g_i(\rho) Z_i + \varepsilon I \succeq \varepsilon I \succ 0 \end{aligned}$$

The second step of the method is based on the expression of the parameter dependent LMI in a quadratic form. Let $\mathcal{B}(\rho)$ be a basis of the multivariate matrix valued polynomial $\mathcal{M}'(\rho)$ such that its *spectral factorization* is given by

$$\mathcal{M}'(\rho) = \mathcal{B}(\rho)^T \mathcal{Q} \mathcal{B}(\rho)$$

where $\mathcal{Q}$ is a constant symmetric matrix. Now by stating that $\mathcal{Q} \succ 0$ then this implies that $\mathcal{M}'(\rho)$ is sum-of-squares. Therefore, the goal is to find matrices $Z_i = Z_i^T \succ 0$ such that $\mathcal{Q} \succ 0$.

Using this formulation, it may happen that no solution is found even though $\mathcal{M}(\rho) \succ 0$ for all $\rho \in \mathcal{I}$. So, the next idea is to replace the constant positive definite matrices Z_i by matrices $Z_i(\rho)$ depending on the parameters which are SoS. This adds extra degrees of freedom and allows to reduce the conservatism of the approach.

Finally let

$$\mathcal{B}_2(\rho)^T \mathcal{Q}' \mathcal{B}_2(\rho)$$

be the spectral factorization of

$$\mathcal{M}(\rho) - \sum_{i=1}^{N} g_i(\rho) Z_i(\rho) \tag{2.33}$$

where $\mathcal{B}_2(\rho)$ is a quadratic basis for (2.33) and $\mathcal{Q}'$ is constant symmetric matrix.

It is also possible to add other degrees of freedom based on the kernel of quadratic forms, indeed there exist matrices $\mathcal{K}$ such that

$$\mathcal{B}_2(\rho)^T \mathcal{K} \mathcal{B}_2(\rho) = 0$$

where $\mathcal{K}$ is constant symmetric matrix. This constraint allows to take into account relations between monomials in the basis $\mathcal{B}_2(\rho)$. Thus determining that

$$\mathcal{Q}' + \mathcal{K} - \varepsilon I \succeq 0$$

for some $\varepsilon > 0$ we have

$$\begin{aligned}
\mathcal{Q}' + \mathcal{K} - \varepsilon I \succeq 0 \quad &\Rightarrow \quad \mathcal{B}_2(\rho)^T(\mathcal{Q}' + \mathcal{K})\mathcal{B}_2(\rho) \succeq \varepsilon I, \quad \text{for all } \rho \in \mathcal{I} \\
&\Leftrightarrow \quad \mathcal{B}_2(\rho)^T \mathcal{Q}' \mathcal{B}_2(\rho) \succeq \varepsilon I, \quad \text{for all } \rho \in \mathcal{I} \\
&\Leftrightarrow \quad \mathcal{M} - \sum_{i=1}^{N} g_i(\rho) Z_i(\rho) \succeq \varepsilon I, \quad \text{for all } \rho \in \mathcal{I} \\
&\Leftrightarrow \quad \mathcal{M} \succeq \sum_{i=1}^{N} g_i(\rho) Z_i(\rho) \succeq \varepsilon I \succ 0, \quad \text{for all } \rho \in \mathcal{I}
\end{aligned}$$

This method leads to more and more precise results by growing up the degree of the matrix valued polynomials $Z_i(\rho)$. Moreover it has been shown that it asymptotically converges to a necessary and sufficient condition when the degree of $Z_i(\rho)$ grows (non conservative condition). Fortunately, the nonconservative condition is generally attained for reasonable degree values.

This approach suffers from a high computational complexity since

1. the number of variables grows up very quickly while raising the degree of SoS polynomials

2. the size of the LMIs grows up quickly with respect to the order of polynomials involved in the problem formulation

See for instance [Dietz et al., 2006] for a brief analysis of the increase of the number of decision variables on a particular case. This is a common fact that good relaxations for parameter dependent LMIs lead to expensive test from a computational point of view.
The following example ends the part of SOS relaxation.

Example 2.3.32 *Let us consider the matrix*

$$\mathcal{M}(\rho) := \begin{bmatrix} -(\rho^2 - 4) & 1 \\ 1 & -(\rho^2 - 4) \end{bmatrix}$$

and $\rho \in [-1, 1]$. The goal is to prove, using SoS, that $\mathcal{M}(\rho) \succ 0$ for all $\rho \in [-1, 1]$. It is clear that $\mathcal{M}(\rho)$ is not globally positive definite (i.e. for all $\rho \in \mathbb{R}$). To see this, remember that for a univariate polynomial positive definiteness is equivalent to the existence of SoS decomposition. Hence, if we show that $\mathcal{M}(\rho)$ is not SoS then it is not positive definite on $\mathbb{R}$. A spectral decomposition on $\mathcal{M}(\rho)$ yields

$$\mathcal{M}(\rho) = \mathcal{B}(\rho)^T \left[\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \mathcal{B}(\rho)$$

where $\mathcal{B}(\rho) = \left[\begin{array}{c} I_2 \\ \hline \rho I_2 \end{array}\right]$. In the multivariate case we

$$\mathcal{M}(\rho) \succ 0 \text{ for all } \rho \in \mathbb{R} \Leftrightarrow \mathcal{Q} \succ 0$$

The latter matrix is not globally positive definite since the $(2,2)$ *right-lower block is negative definite. Now define the set* $\mathcal{I} := [-1, 1]$. $\mathcal{I}$ *can be defined as a semi-algebraic set:*

$$\mathcal{I} = \left\{x \in \mathbb{R} : \; g(x) := -x^2 + 1 \geq 0\right\}$$

Introduce

$$\mathcal{M}(\rho) - g(\rho)Z = \mathcal{B}(\rho)^T \mathcal{Q} \mathcal{B}(\rho)$$

where $\mathcal{Q} = \left[\begin{array}{cc|cc} 4-z_1 & 1-z_2 & 0 & 0 \\ 1-z_2 & 4-z_3 & 0 & 0 \\ \hline 0 & 0 & -1+z_1 & z_2 \\ 0 & 0 & z_2 & -1+z_3 \end{array}\right]$ *and* $Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \succ 0$. *We can see that the positive definite matrix* Z *appears positively in the right-lower block and could make it positive definite. Now we seek* $Z = Z^T \succ 0$ *such that* $\mathcal{Q} \succ 0$. *Or equivalently if and only if*

$$\begin{array}{rcl} \begin{bmatrix} 4-z_1 & 1-z_2 \\ 1-z_2 & 4-z_3 \end{bmatrix} & \succ & 0 \\ \begin{bmatrix} -1+z_1 & z_2 \\ z_2 & -1+z_3 \end{bmatrix} & \succ & 0 \end{array}$$

Note that the problem is affine in the variable Z *and hence can be solved using SDP. From these inequalities we get*

$$\begin{array}{rcl} Z & \succ & I_2 \\ \begin{bmatrix} -1+z_1 & z_2 \\ z_2 & -1+z_3 \end{bmatrix} & \succ & 0 \end{array}$$

Choosing $Z = 2I_2$ *we obtain*

$$\begin{bmatrix} 4-z_1 & 1-z_2 \\ 1-z_2 & 4-z_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The eigenvalues of the latter matrix are respectively $\{1, 3\}$ *showing that* $\mathcal{Q} \succ 0$. *Hence we have* $\mathcal{M}(\rho) - g(\rho)Z \succ 0$ *and finally*

$$\mathcal{M}(\rho) \succ g(\rho)Z \succeq 0$$

If $\mathcal{Q}$ *was not found positive definite, then* Z *would have been chosen as a function of* ρ*, and the procedure applied again. This shows that the SoS approach can be used to elaborate finite dimensional LMI conditions for the positive (negative) definiteness of parameter dependent matrices in which parameters evolve in a bounded compact set.*

2.3.3.4 Global Polynomial Optimization and the Problem of Moments

This approach is dual to the sum-of-squares relaxation. Since the matrix case can be straightforwardly turned into the scalar case, we will focus here on the scalar case only for illustration purpose. The reader should refer to [Henrion and Lasserre, 2004, 2006; Lasserre, 2001, 2007] and references therein to get more details. This method is based on measure theory and aims at turning the initial optimization problem over $\mathbb{R}^n$ into another optimization problem over a measure space. Although, the optimization over measure spaces is a rather complicated

problem, such a reformulation is very general and allows to solve a wide type of optimization problems, including polynomial optimization problems, using SDP.
Consider the optimization problem

$$\begin{array}{l} \inf c(x) \text{ s.t.} \\ x \in \mathbb{R}^n \\ g_i(x) \geq 0 \end{array} \tag{2.34}$$

where $c(x) = \sum_{i=1}^{N} \beta_i x^{\alpha_i}$ and $g_i(x)$ are scalar multivariate polynomials with $\alpha_i = \begin{bmatrix} \alpha_i^1 & \dots & \alpha_i^n \end{bmatrix}$ and $x^{\alpha_i} = x_1^{\alpha_i^1} x_2^{\alpha_i^2} \dots x_n^{\alpha_i^n}$.

Proposition 2.3.33 *Assuming that the set*

$$\mathcal{X} := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \text{ for all } i = 1, \dots, N\}$$

is non empty, then the optimization problem (2.34) is equivalent to the following optimization problem

$$\begin{array}{l} \inf\limits_{\mu} \displaystyle\int_{\mathcal{X}} c(x) d\mu(x) \text{ s.t.} \\ \displaystyle\int_{\mathcal{X}} d\mu(x) = 1 \end{array}$$

where μ is a probability measure over $\mathcal{X}$.

Proof: To see the equivalence, note that

$$\begin{array}{rcl} \displaystyle\int_{\mathcal{X}} c(x) d\mu(x) & \geq & \inf\limits_{x \in \mathcal{X}} c(x) \displaystyle\int_{\mathcal{X}} d\mu(x) \\ & \geq & \inf\limits_{x \in \mathcal{X}} c(x) \end{array}$$

Hence the infimum of (2.34) is lower than the infimum of (2.3.33). Then suppose that $x^* \in \mathcal{X}$ is a global minimizer of $c(x)$ then the corresponding measure is

$$\mu^*(x) = \delta(x - x^*)$$

where δ is the Dirac measure. With this appropriate choice, the infimums of both problems coincide. $\square$

This shows that the global minimum of problem (2.34) coincides with the global minimum of problem (2.3.33). Now, the aim is to explain how the measure μ is found since an optimization problem over a measure space is a nontrivial problem. First note, that a measure is uniquely characterized by its moments defined by:

$$m_{\alpha_i}(\mu) = \int_{\mathcal{X}} x^{\alpha_i} d\mu(x)$$

where $\alpha_i = \begin{bmatrix} \alpha_i^1 & \dots & \alpha_i^n \end{bmatrix}$ and $x^{\alpha_i} = x_1^{\alpha_i^1} x_2^{\alpha_i^2} \dots x_n^{\alpha_i^n}$.
Using the moment formulation, the modified cost writes:

$$\int_{\mathcal{X}} c(x) d\mu(x) = \sum_{i=1}^{N} \beta_i m_{\alpha_i}(\mu)$$

and then the optimization problem becomes

$$\min \sum_{i=1}^{N} \beta_i m_{\alpha_i}$$

such that

$$\begin{array}{lcl} m_{[0\ 0\ \dots\ 0]} & = & 1 \\ M_k(m) & \succeq & 0 \\ M_{k-d_i}(g_i m) & \succeq & 0 \end{array}$$

where $2d_i$ or $2d_i - 1$ is the degree of polynomial $g_i(x)$. $M_k(m) \succ 0$ and $M_{k-d_i}(g_i m) \prec 0$ are LMIs constraints in m (the moments) corresponding to respective truncations of moment and localizing matrices (matrices defining the constraints corresponding to the $g_i(x)$ in terms of moments). The following example should clarify the idea behind the above reformulation.

Example 2.3.34 *Let us consider the following polynomial optimization problem*

$$\begin{array}{ll} \inf_{x \in \mathbb{R}^2} & 2x_1 + 2x_1^2 - x_1 x_2 \quad \text{s.t.} \\ & g_1(x) := 2x_1^2 - x_2 \geq 0 \\ & g_2(x) := -x_1^2 - x_2^2 + 4 \geq 0 \end{array} \tag{2.35}$$

It is clear that the semi-algebraic set

$$\mathcal{X} := \left\{ x \in \mathbb{R}^2 : \ g_1(x) \geq 0, \ g_2(x) \geq 0 \right\}$$

is non convex since it consists in the closed-interior of a ball minus the epigraph of a parabola. It is illustrated in Figure 2.17. Turning the optimization into the measure formulation, we

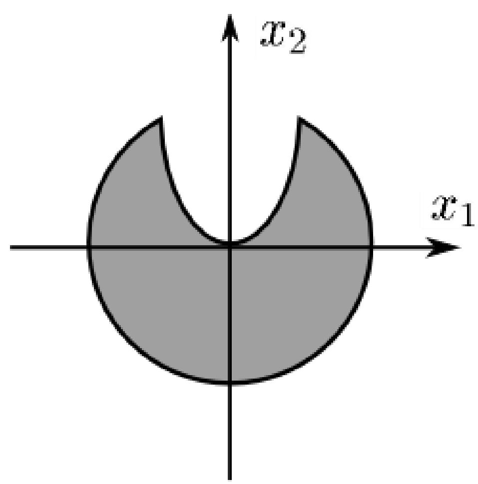

Figure 2.17: Representation of the nonconvex set $\mathcal{X} := \left\{ x \in \mathbb{R}^2 : \ g_1(x) \geq 0, \ g_2(x) \geq 0 \right\}$ considered in the polynomial optimization problem (2.35)

get

$$\begin{array}{l} \inf_m 2m_{10} + 2m_{20} - m_{12} \ \text{s.t.} \\ 2m_{20} - m_{01} \geq 0 \\ -m_{20} - m_{02} + 4 \geq 0 \\ m_{00} = 1 \end{array}$$

where $m_{ij} = \int_{\mathcal{X}} x_1^i x_2^j d\mu$. *Moreover, let us define the following rank-one matrix:*

$$N_1(x) := \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{bmatrix} \succeq 0$$

Computing the integral of $N_1(x)$ *over* $\mathcal{X}$ *with measure* $d\mu(x)$ *we get*

$$\int_{\mathcal{X}} N_1(x) d\mu(x) = M_1(m) \succeq 0$$

where $M_1(m) = \left[\begin{array}{c|cc} 1 & m_{10} & m_{01} \\ \hline m_{10} & m_{20} & m_{11} \\ m_{01} & m_{11} & m_{02} \end{array}\right] \succeq 0$ *This leads to the first approximation of the polynomial optimization problem*

$$\begin{array}{l} \inf 2m_{10} + 2m_{20} - m_{12} \text{ s.t.} \\ 2m_{20} - m_{01} \geq 0 \\ -m_{20} - m_{02} + 4 \geq 0 \\ m_{00} = 1 \\ M_1(m) \succeq 0 \end{array}$$

In order to derive tighter relaxations, note that the matrices $g_1(x)N_1(x)$ *and* $g_2(x)N_1(x)$ *are positive semidefinite since* $N_1(x) \succeq 0$ *and* $g_1(x), g_2(x) \geq 0$. *Hence we obtain,*

$$\begin{aligned} g_1(x)N_1(x) &= (2x_1^2 - x_2) \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1^2 - x_2 & 2x_1^3 - x_2x_1 & 2x_1^2x_2 - x_2^2 \\ 2x_1^3 - x_2x_1 & 2x_1^4 - x_2x_1^2 & 2x_1^3x_2 - x_2^2x_1 \\ 2x_1^4 - x_2x_1^2 & 2x_1^3x_2 - x_2^2x_1 & 2x_1^2x_2^2 - x_2^3 \end{bmatrix} \succeq 0 \\ g_2(x)N_1(x) &= (-x_1^2 - x_2^2 + 4) \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{bmatrix} \\ &= \begin{bmatrix} -x_1^2 - x_2^2 + 4 & -x_1^3 - x_2^2x_1 + 4x_1 & -x_1^4 - x_2^2x_1^2 + 4x_1^2 \\ -x_1^3 - x_2^2x_1 + 4x_1 & -x_1^4 - x_2^2x_1^2 + 4x_1^2 & -x_1^3x_2 - x_2^3x_1 + 4x_1x_2 \\ -x_1^4 - x_2^2x_1^2 + 4x_1^2 & -x_1^3x_2 - x_2^3x_1 + 4x_1x_2 & -x_1^2x_2^2 - x_2^4 + 4x_2^2 \end{bmatrix} \succeq 0 \end{aligned}$$

Computing the integral of $g_1(x)N_1(x)$ *and* $g_2(x)N_1(x)$ *over* $\mathcal{X}$ *with measure* $d\mu(x)$ *leads respectively to matrices* $M_1(g_1m)$ *and* $M_1(g_2m)$ *writing*

$$\begin{aligned} M_1(g_1m) &= \left[\begin{array}{c|cc} 2m_{20} - m_{01} & 2m_{30} - m_{11} & 2m_{21} - m_{02} \\ \hline 2m_{30} - m_{11} & 2m_{40} - m_{21} & 2m_{31} - m_{12} \\ 2m_{21} - m_{02} & 2m_{31} - m_{12} & 2m_{22} - m_{03} \end{array}\right] \succeq 0 \\ M_1(g_2m) &= \begin{bmatrix} -m_{20} - m_{02} + 4 & -m_{30} - m_{12} + 4m_{10} & -m_{40} - m_{22} + 4m_{20} \\ -m_{30} - m_{12} + 4m_{10} & -m_{40} - m_{22} + m_{20} & -m_{31} - m_{13} + 4m_{11} \\ -m_{40} - m_{22} + 4m_{20} & -m_{31} - m_{13} + 4m_{11} & -m_{22} - m_{04} + 4m_{02} \end{bmatrix} \succeq 0 \end{aligned}$$

Since higher order moments are present (up to order 4), we construct the higher order relaxation matrix $M_2(m)$ obtained from the matrix $N_2(x) = L_2(x)L_2(x)^T$:

$$L_2(x) = \left[\begin{array}{c} 1 \\ \hline x_1 \\ x_2 \\ \hline x_1^2 \\ x_1x_2 \\ x_2^2 \end{array}\right] \qquad M_2(m) = \left[\begin{array}{c|cc|ccc} 1 & m_{10} & m_{01} & m_{20} & m_{11} & m_{02} \\ \hline m_{10} & m_{20} & m_{11} & m_{30} & m_{21} & m_{12} \\ m_{01} & m_{11} & m_{02} & m_{21} & m_{12} & m_{03} \\ \hline m_{20} & m_{30} & m_{21} & m_{40} & m_{31} & m_{22} \\ m_{11} & m_{21} & m_{12} & m_{31} & m_{22} & m_{13} \\ m_{02} & m_{12} & m_{03} & m_{22} & m_{13} & m_{33} \end{array}\right] \succeq 0$$

Finally, the optimization problem becomes

$$\begin{array}{l} \inf_m 2m_{10} + 2m_{20} - m_{12} \text{ s.t.} \\ 2m_{20} - m_{01} \geq 0 \\ -m_{20} - m_{02} + 4 \geq 0 \\ m_{00} = 1 \\ M_1(g_1 m) \succeq 0 \\ M_1(g_2 m) \succeq 0 \\ M_2(m) \succeq 0 \end{array}$$

With a similar procedure, it is possible to construct higher order relaxations until obtain satisfying results.

It has been shown that the global minimum found using the moment-based relaxation asymptotically converges to the actual global minimum when the order of relaxation k tends to $+\infty$. Fortunately, as in the sum-of-squares approach, the global minimizer is found for small values of k. In order to point out the duality between these two methods, just memorize that raising k corresponds to raise the degree of SoS polynomials.

The generalization to parameter dependent LMIs is obtained remarking that a parameter dependent symmetric matrix $\mathcal{M}(\rho)$ is negative definite if and only if all its principal minors are strictly negative. This brings back the matrix problem to a multiple polynomial scalar problem. Indeed, for a polynomially parameter dependent symmetric matrix of dimension k, there are k principal minors taking the form of polynomials, which is exactly the form presented in this section.

However, the formulation of LMI problem is not trivial and this is illustrated in the following example.

Example 2.3.35 *Let $L(\rho, M) \prec 0$ be a parameter dependent LMI aimed to be satisfied where $M \in \mathcal{M}$ represents decision matrices and the parameter vector ρ belongs to a compact set U_ρ. We define the following optimization problem:*

$$\begin{array}{ll} \inf & -t \\ & f_i(\rho, M, t) > 0 \\ & M \in \mathcal{M} \\ & \rho \in U_\rho \end{array}$$

where $f_i(\rho, M, t))$ are all minors of $L(\rho, M) - tI \succ 0$. The scalar t allows to determine whether the LMI is satisfied or not. If $t < 0$ then the problem is feasible and there exists $M \in \mathcal{M}$ such that $L(\rho, M) \prec 0$ for all $\rho \in U_\rho$. Moreover, the parameter vector for which the minimum is *attained is also returned by the optimization procedure. On the other hand, if $t > 0$ then this means that there exists a parameter vector for which $L(\rho, M) \nprec 0$ and the parameter vector for which maximal eigenvalue of $L(\rho, M)$ is attained is returned.*

Consider the scalar inequality $f(\rho) = \rho^2 - 4$ where $\rho \in [-1, 1]$. It is clear that $f(\rho) < 0$ over that domain. Now consider the optimization problem:

$$\begin{array}{ll} \inf & -t \\ & \rho^2 - 4 - t > 0 \\ & \rho \in [-1, 1] \end{array}$$

It is simple to show that $t_{opt} = -3$ for $\rho = \pm 1$. Therefore inequality $\rho^2 - 4 < 0$ is satisfied for all $\rho \in [-1, 1]$. Now consider the second optimization problem:

$$\begin{array}{ll} \inf & -t \\ & \rho^2 - 4 - t > 0 \\ & \rho \in [0, 3] \end{array}$$

In this case, $t_{opt} = 5$ for $\rho = 3$. Finally consider the problem of finding k such that $\rho^2 - 4 + k < 0$. In this case, the constraint $t < 0$ must be added in order to obtain coherent results. The optimization problem is thus defined by

$$\begin{array}{ll} \inf & -t \\ & t < 0 \\ & \rho^2 - 4 + k - t > 0 \\ & \rho \in [0, 3] \end{array}$$

We obtain $k = -5 - \varepsilon$ and $t = -\varepsilon$ for sufficiently small $\varepsilon > 0$. This example illustrates that the moment approach can be used in order to prove stability of LPV systems and find suboptimal stabilizing controllers.

This approach is well-dedicated for small to medium size problems. Indeed, the dimension of LMIs grows quickly, slowing dramatically the resolution using classical SDP solvers. Hence the computational complexity is globally the same as of sum-of-squares relaxations.

It is worth noting that, as a by-product, such method can be used to find solutions to BMIs using either scalarization or directly by considering Polynomial Matrix Inequalities (PMI) [Henrion and Lasserre, 2006]. Nevertheless, although the theory for matrix valued optimization problems is ready, it is still experimentally at a very preliminary level [Henrion, 2008].

2.3.4 Stability of 'LFT' systems

The stability of 'LFT' systems is still an active research topic. Indeed, 'LFT' systems provide an unified way to model LPV systems with every type of parameter dependence: affine, polynomial and rational. By rewriting LPV systems in 'LFT' form, the initial system is split in two interconnected subsystems: one constant and one time-varying. The stability of the LPV systems is then determined using results on the stability interconnected systems. Most of these results are summarized in this section.

Let us recall first the LPV system is a 'LFT' formulation:

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \\ w(t) & = & \Theta(\rho)z(t) \end{array} \tag{2.36}$$

The parameter matrix $\Theta(\rho)$ is not detailed here since its structure is not fixed a priori and depend on stability analysis methods. It is important to note that all the tools provided in that section have been initially developed for robust stability analysis of linear systems. Due to the genericity of the 'LFT' procedure, these tools apply naturally to the LPV case.

2.3.4.1 Passivity

The passivity is a very strong result for LPV systems. It can only be used with positive $\Theta(\rho)$ and the LTI system must satisfy a very constraining inequality. Let $H(s) = C(sI - \tilde{A})^{-1}B + D$ be the transfer function from w to z corresponding to system (2.36) and assume that $\Theta(\rho)$ is diagonal with bounded nonnegative components. We have the following definition (see for instance [Khalil, 2002; Scherer and Weiland, 2005; Wyatt et al., 1981])

Definition 2.3.36 *System $H(s)$ is (strictly) passive if and only if*

$$H(j\omega) + H(j\omega)^*(\succ 0) \succeq 0, \qquad \text{for all } \omega \in \mathbb{R} \tag{2.37}$$

This means that, in the SISO case, that the Nyquist plot of $H(j\omega)$ must lie within the complex open right half-plane (which is very constraining for system of order greater than 1). We need the following result:

Proposition 2.3.37 *If a strictly passive system is interconnected with a passive system, then the resulting system is passive.*

As any passive system is asymptotically stable, then the stability of the interconnection is shown. Hence System (2.36) is asymptotically stable if $\Theta(\rho)$ is a passive operator and $H(s)$ a strictly passive one. $\Theta(\rho)$ is passive since it is diagonal and has nonnegative elements and $H(s)$ is strictly passive if strict inequality (2.37) holds. The following examples illustrate the approach.

Example 2.3.38 *Let us consider the SISO LPV system*

$$\dot{x} = -(2 - \rho)x \tag{2.38}$$

where $\rho \in [0, 1]$. It is clear that the system is quadratically stable since there exists $p > 0$ such that $-(2 + \rho)p < 0$ for all $\rho \in [0, 1]$. The LPV system is then rewritten into the 'LFT' form

$$\begin{array}{rcl} \dot{x} & = & -2x + w \\ z & = & x \\ w & = & \rho x \end{array}$$

The transfer function $H(s)$ corresponding to the LTI system is then given by $H(s) = \dfrac{1}{s+2}$ and is strictly passive since

$$\begin{array}{rcl} H(j\omega) + H(j\omega)^* & = & \dfrac{1}{j\omega + 2} + \dfrac{1}{-j\omega + 2} \\ & = & \dfrac{4}{\omega^2 + 4} > 0, \qquad \text{for all } \omega \in \mathbb{R} \end{array}$$

Hence the LPV system (2.38) is asymptotically stable.

Example 2.3.39 *Let us consider the SISO LPV system*

$$\dot{x} = -(2+\rho)x \tag{2.39}$$

where $\rho \in [0,1]$. This system is also quadratically stable and the 'LFT' formulation is then given by

$$\begin{aligned} \dot{x} &= -2x - w \\ z &= x \\ w &= \rho x \end{aligned}$$

The transfer function $H(s)$ corresponding to the LTI system is then given by $H(s) = \dfrac{-1}{s+2}$ and is strictly passive if and only if $H(j\omega) + H(j\omega)^* \succ 0, \quad \text{for all } \omega \in \mathbb{R}$. However

$$\begin{aligned} H(j\omega) + H(j\omega)^* &= \frac{-1}{j\omega+2} + \frac{-1}{-j\omega+2}, \qquad \text{for all } \omega \in \mathbb{R} \\ &= \frac{-4}{\omega^2+4} \not\succ, \qquad \text{for all } \omega \in \mathbb{R} \end{aligned}$$

Since $H(s)$ is not strictly passive then asymptotic stability of system (2.39) cannot be proved by passivity.

Example 2.3.39 shows that if a system has non minimum phase, the passivity may fail even in the more simple case of a 1^{st} order system. The fact that a very few systems are (strictly) passive implies that the stability analysis of LPV systems in 'LFT' form is very restrictive and is not considered in the literature. However, passivity is of interest in many applicative problems, e.g. teleoperation [Anderson and Spong, 1989; Hokayem and Spong, 2006; Niemeyer, 1996], control of Port Hamiltonian Systems [Ortega et al., 2008] and so on [Khalil, 2002; van der Schaft, 1996].

In the above examples, the sign analysis of the sum $H(j\omega) + H(j\omega)^*$ is performed analytically due to its simple expression. However, if the transfer function is more complex (i.e. higher order systems and/or MIMO systems), an analytical analysis is far tougher. Fortunately, a LMI test has been provided, for instance, in [Scherer and Weiland, 2005] allowing to an easy test for MIMO systems.

Theorem 2.3.40 *A system (A_s, B_s, C_s, D_s) is strictly passive if and only if there exists a matrix $P = P^T \succ 0$ such the LMI*

$$\begin{bmatrix} A_s^T P + P A_s & P B_s - C_s^T \\ \star & -(D_s + D_s^T) \end{bmatrix} \prec 0$$

is feasible.

The origin of this LMI is detailed in Appendix D.5.

2.3.4.2 Small-Gain Theorem

The small-gain theorem is an enhancement of the passivity based stability analysis of interconnections since it takes into account variations of energy between inputs and outputs of dynamical systems involved in the interconnection. A simple energy analysis of loop-signals suggests that asymptotic stability of the interconnection is equivalent to finiteness of the energy of the loop-signals involved in the interconnection. Hence the problem remains to determine whether the energy of these signals is finite or not.

Definition 2.3.41 *The energy gain (or $\mathcal{L}_2$-gain or $\mathcal{L}_2$-induced norm) of a time-invariant operator T is defined by the relation*

$$\begin{aligned}\gamma_{\mathcal{L}_2} &:= ||T||_{\mathcal{L}_2-\mathcal{L}_2}\\ &:= \sup_{w\in\mathcal{L}_2,\ w\neq 0} \frac{||Tw||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}}\end{aligned}$$

where $\mathcal{L}_2$ is set of bounded energy signal (see appendix B for more details). For instance, unitary energy inputs give at most $\gamma_{\mathcal{L}_2}$ energy outputs. Moreover, the time-invariant operator is asymptotically stable if and only if $\gamma_{\mathcal{L}_2} < +\infty$.

Definition 2.3.42 *The $\mathcal{H}_\infty$-norm of a linear time-invariant operator T is given by*

$$\begin{aligned}\gamma_{\mathcal{H}_\infty} &:= ||T||_{\mathcal{H}_\infty}\\ &:= \sup_{w\in\mathbb{R}} \bar{\sigma}(T(j\omega))\end{aligned}$$

where $\bar{\sigma}(T)$ is the maximal singular value of the transfer matrix $T(s)$ (see appendix A.6 for more details on singular values and singular values decomposition).

In the LTI case, the $\mathcal{H}_\infty$-norm of a time-invariant operator coincides with the $\mathcal{L}_2$-induced norm (see for instance [Doyle et al., 1990]). As suggested by the definitions, if a LTI system is asymptotically stable then it has finite $\mathcal{H}_\infty$-norm.
As an illustration of the approach, let us consider the interconnection of two SISO transfer functions $H_1(s)$ and $H_2(s)$ as shown in Figure 2.18.

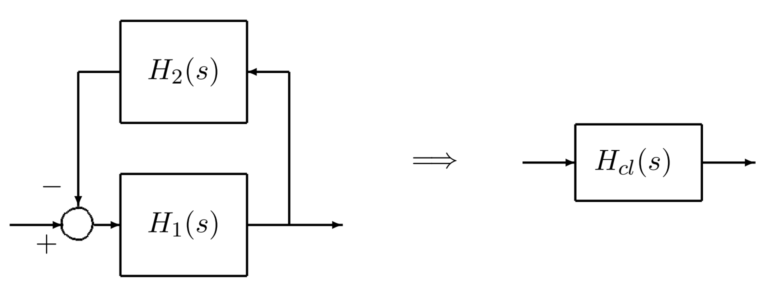

Figure 2.18: Interconnection of two SISO transfer functions

The closed-loop transfer function is then given by the expression

$$H_{cl}(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

It is clear that the closed-loop system is asymptotically stable if and only if $H_1(s)H_2(s) \neq -1$ for all $s \in \mathbb{C}^+$. From this consideration, by imposing the condition

$$\sup_{s\in\mathbb{C}^+} |H_1(s)H_2(s)| < 1$$

it is ensured that $H_1(s)H_2(s) \neq -1$ for all $s \in \mathbb{C}^+$. Finally, noting that

$$\sup_{s\in\mathbb{C}^+} |H_1(s)H_2(s)| < 1$$

is equivalent to

$$||H_1H_2||_{\mathcal{H}_\infty} < 1$$

we get a sufficient condition for stability in term of $\mathcal{H}_\infty$-norm analysis.

Theorem 2.3.43 (Small-Gain Theorem [Zhou et al., 1996]) *The LPV system (2.36) is asymptotically stable if the inequality*

$$\frac{||\Theta(\rho)Hw||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} < 1$$

holds where $\Theta(\rho)$ is a full-matrix depending on the parameters such that $\Theta(\rho)^T\Theta(\rho) \preceq I$ and H is the LTI operator mapping w to z

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \end{array}$$

from system (2.36).

It is clear that the sufficient condition $||\Theta(\rho)Hw||_{\mathcal{L}_2} < ||w||_{\mathcal{L}_2}$ may be tough to verify due to the time-varying nature of the matrix $\Theta(\rho)$. Hence, in virtue of the submultiplicative property of the $\mathcal{H}_\infty$ norm (or the $\mathcal{L}_2$ induced norm), i.e.

$$||H_1H_2||_{\mathcal{H}_\infty} < ||H_1||_{\mathcal{H}_\infty} \cdot ||H_2||_{\mathcal{H}_\infty} \tag{2.40}$$

then a more conservative sufficient condition is given by

$$||\Theta(\rho)||_{\mathcal{L}_2-\mathcal{L}_2} \cdot ||H||_{\mathcal{H}_\infty} < 1$$

This condition is sufficient only since, by considering the norm, we loose all information on the phase of $H_1(s)H_2(s)$. Indeed, the sup constraint restricts the bode magnitude plot of $H_1(j\omega)H_2(j\omega)$ to evolve inside the unit disk ignoring the value of the phase. This results evidently in a conservative (hence sufficient) stability condition. The following examples illustrate non-equivalence between these results on asymptotic stability.

Example 2.3.44 *Let us consider two asymptotically stable LTI SISO system $H_1(s)$ and $H_2(s)$ interconnected as depicted on Figure 2.18 and defined by*

$$\begin{array}{rcl} H_1(s) & = & \dfrac{10}{(s+1)(s+2)} \\ H_2(s) & = & \dfrac{10}{(s+3)(s+4)} \end{array}$$

Since both $H_1(s)$ and $H_2(s)$ are asymptotically stable then $H_1(s)H_2(s)$ is asymptotically stable too. Then we have

$$\begin{aligned} ||H_1H_2||_{\mathcal{H}_\infty} &= \sup_{s\in\mathbb{C}^+} |H_1(s)H_2(s)| \\ &= \sup_{\omega\in\mathbb{R}} |H_1(j\omega)H_2(j\omega)| \qquad \text{(by the maximum modulus principle (see Appendix E.4))} \\ &= H_1(0)H_2(0) \\ &= 100/24 > 1 \end{aligned}$$

Hence according to the small-gain theorem the interconnection is not asymptotically stable even though we have

$$H_{bf}(s) = \frac{1}{1+H_1(s)H_2(s)} = \frac{s^4+10s^3+35s^2+50s+24}{s^4+10s^3+35s^2+50s+124}$$

which has poles $\{-4.8747+2.0950i, -4.8747-2.0950i, -0.1253+2.0950i, -0.1253-2.0950i\}$ *with negative real part, showing that the interconnection is asymptotically stable.*

In the latter example, the equality $||H_1H_2||_{\mathcal{H}_\infty} = ||H_1||_{\mathcal{H}_\infty} \cdot ||H_2||_{\mathcal{H}_\infty}$ holds since the transfer functions $H_1(s)$ and $H_2(s)$ reach their maximum modulus value at the same argument $s = 0$. The following example presents a case for which this equality does not hold:

Example 2.3.45 *Let us consider two asymptotically stable LTI SISO system $H_1(s)$ and $H_2(s)$ interconnected as depicted on Figure 2.18 and defined by*

$$\begin{aligned} H_1(s) &= \frac{1}{s^2+0.1s+10} \\ H_2(s) &= \frac{10}{(s+3)(s+4)} \end{aligned}$$

In this case we have

$$\begin{aligned} ||H_1||_{\mathcal{H}_\infty} &= \frac{10^3}{\sqrt{99975}} \text{ at } \omega = \frac{\sqrt{39.98}}{2} \\ ||H_2||_{\mathcal{H}_\infty} &= \frac{10}{12} \text{ at } \omega = 0 \\ ||H_1H_2||_{\mathcal{H}_\infty} &= 0.7084 \text{ at } \omega = 3.1608 \end{aligned}$$

This shows that while the Nyquist plot of $H_1(j\omega)H_2(j\omega)$ remains within the unit disk (asymptotic stability of the interconnection). On the other hand, the inequality based on the sub-multiplicative property of the $\mathcal{H}_\infty$-norm gives $\dfrac{10^4}{12\sqrt{99975}}$, which is approximately 2.6356, and does not allow to conclude on the stability of the interconnection.

Let us now come back to LPV system (2.36). Since, by definition $\Theta(\rho)^T\Theta(\rho) \preceq I$, we have $||\Theta(\rho)||_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$. To see this, let $\tilde{z}(t) = \Theta(\rho)\bar{z}(t)$ and then the energy of $\tilde{z}(t)$ writes

$$\begin{aligned} \int_0^{+\infty} \tilde{z}(s)^T\tilde{z}(s)ds &= \int_0^{+\infty} \bar{z}(s)^T\Theta(\rho(s))^T\Theta(\rho(s))\bar{z}(s)ds \\ &\leq \int_0^{+\infty} \bar{z}(s)^T\bar{z}(s)ds \end{aligned}$$

Finally the stability condition reduces to

$$||H||_{\mathcal{H}_\infty} < 1$$

and can be verified using semidefinite programming through a LMI feasibility test. Indeed, instead of the initial $\mathcal{H}_\infty$-norm computation using bisection algorithm [Zhou et al., 1996] or Hamiltonian matrix [Doyle et al., 1990], the bounded real lemma (see Appendix D.8 and [Scherer and Weiland, 2005; Scherer et al., 1997; Skelton et al., 1997]) allows to compute the $\mathcal{L}_2$-induced norm of linear (possibly time/parameter varying) systems through an optimization problem involving a LMI:

Lemma 2.3.46 (Small gain Theorem - Bounded Real Lemma) *The interconnected system (2.36) is asymptotically stable if there exist a matrix $P = P^T \succ 0$ and a scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & PB & C^T \\ \star & -(1-\varepsilon)I & D^T \\ \star & \star & -(1-\varepsilon)I \end{bmatrix} \prec 0$$

Moreover, we have $||H||_{\mathcal{H}_\infty} < 1$.

The small-gain condition is a simple stability test but is however rather conservative. First of all, it does not consider the phase and secondly no information is looked out on how the elements are interconnected, the shape of the intersecting elements are not considered but only their maximal modulus value. Indeed, as illustrated in Example 2.3.44, if the maximum values do not occur at the same frequency, the submultiplicative inequality is conservative. This is far more complicated when dealing with nonlinear or non-stationary elements. The last example illustrates this.

Example 2.3.47 *As an example note that*

$$\begin{array}{rcl} ||2\sin(t)||_{\mathcal{L}_\infty} & < & 2 \\ \left|\left|\dfrac{1}{2+\cos(t)}\right|\right|_{\mathcal{L}_\infty} & < & 1 \end{array}$$

Hence we have

$$||2\sin(t)||_{\mathcal{L}_\infty} \cdot \left|\left|\frac{1}{2+\cos(t)}\right|\right|_{\mathcal{L}_\infty} < 2$$

but actually we have $\left|\left|\dfrac{2\sin(t)}{2+\cos(t)}\right|\right|_{\mathcal{L}_\infty} < 2/3$ which shows that the application of the submultiplicative property may result in very conservative bounds (conditions).

Figure 2.19 provides a geometric representation of the conservatism of the small-gain theorem.

Figure 2.19: Illustration of the conservatism induced by the use of the $\mathcal{H}_\infty$-norm. Although, the pieces of puzzle fit together, the consideration of the $\mathcal{H}_\infty$ norm says the contrary.

In order to explain Figure 2.19 assume that the free piece is an operator P and let O be the center of the piece. Since the piece is two-dimensional we assume that it belongs to a two-dimensional normed vector space. In what follows we will consider that the free piece is an operator and the remaining of the puzzle constitutes the other operator. The interconnection of systems is substituted to an interconnection of pieces of puzzle. Moreover, the asymptotic stability of the interconnection is replaced by the possibility of placing the piece at this place. The image shows that the piece interlocks perfectly. We show hereafter that by ignoring the shape of the piece and reducing it to a single value (a norm), the piece and the remaining cannot be shown to fit together.
The operator P maps any vector $v(\theta)$ such that

$$v(\theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

to a new vector $v'(\theta)$ whose 2-norm equals the length between the center O and the boundary of the piece in the orientation $\theta \in [0, 2\pi]$ (orientation 0 points to the right). Therefore for every $v(\theta)$ we have

$$Pv(\theta) = v'(\theta) = \lambda(\theta)v(\theta)$$

since only the norm of the vector is changed. As a comparison the $\mathcal{H}_\infty$ of an operator is the largest energy gain that is applied to an input signal entering this operator. It is the modification of the norm of the input signal where the norm is the energy. It is clear that by considering the norm

$$||P|| = \sup_{\theta \in [0,2\pi]} \frac{||Pv(\theta)||_2}{||v(\theta)||_2} \tag{2.41}$$

the farthest point on the boundary from 0 is taken into account and the piece is considered as circle shaped with radius $||P||$. In this condition no puzzle can be completed and hence more sophisticated techniques should be employed to determine if some pieces correspond to each other.

The puzzle analogy shows that the shape (the structure) of interconnected elements should play a crucial role in the stability analysis. It is clear that if the matrix $\Theta(\rho)$ is full, then a priori only norm-information can be extracted. On the other hand, if this matrix has a specific and known form it is possible to refine the small-gain theorem in a new version.

2.3.4.3 Scaled-Small Gain Theorem

The aim of the scaled-small gain lemma is to reduce the conservatism of the small-gain theorem by considering the structure of the parameter-varying matrix gain $\Theta(\rho)$. It is generally assumed that it has a diagonal structure

$$\Theta(\rho) = \mathrm{diag}(I_{n_1} \otimes \rho_1, \ldots, I_{n_p} \otimes \rho_p) \tag{2.42}$$

Let us introduce the set of D-scalings defined by

$$\mathcal{S}_D(\Theta) := \left\{ L \in \mathbb{S}^{\bar{n}}_{++} : \ \Theta(\rho)L^{1/2} = L^{1/2}\Theta(\rho) \text{ for all } \rho \in [-1,1]^p \right\} \tag{2.43}$$

where $\bar{n} = \sum_i n_i$ and $L^{1/2}$ denotes the positive square-root of L. For more details on the scalings the reader should refer to [Apkarian and Gahinet, 1995] or Appendix D.12.

The key idea is to define a matrix $L \in \mathcal{S}(\Theta)$ to embed information on the structure of the parameter matrix, through a commutation property. This additional matrix will then be introduced in the bounded-real lemma and provides an extra degree of freedom leading to a reduction of the conservatism [Packard and Doyle, 1993].

Example 2.3.48 *Consider the following parameter matrix* $\Theta(\rho) = \mathrm{diag}(\rho_1 I_5, \rho_2 I_2)$ *then a suitable matrix* $L \in \mathcal{S}(\Theta)$ *is given by*

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$$

where $L_1 \in \mathbb{S}^5_{++}$ *and* $L_2 \in \mathbb{S}^2_{++}$.

Since $L \in \mathcal{S}_D(\Theta)$ is positive definite, let us define a dual parameter matrix

$$\tilde{\Theta}(\rho) = L^{1/2}\Theta(\rho)L^{-1/2}$$

It is clear that, in virtue of the definition of set $\mathcal{S}_D(\Theta)$, the following identity holds

$$\tilde{\Theta}(\rho) = \Theta(\rho)$$

In what follows, we aim at showing that the feasibility of the scaled-bounded real lemma implies asymptotic stability of the interconnection. To this aim, let w_2 and z_2 be $\mathcal{L}_2$ signals satisfying $w_2(t) = \tilde{\Theta}(\rho)z_2(t)$. First, let us show that operator $\tilde{\Theta}(\rho)$ has unitary energy gain. Suppose that it is has energy gain of $\gamma_\theta > 0$, then the following integral quadratic function must be nonnegative.

$$\int_0^{+\infty} \begin{bmatrix} w_2(s) \\ z_2(s) \end{bmatrix}^T \begin{bmatrix} \gamma_\theta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w_2(s) \\ z_2(s) \end{bmatrix} ds = \int_0^{+\infty} z(s)^T \Xi(\rho, L) z(s) ds$$

with

$$\Xi(\rho, L) = \gamma_\theta^2 I - L^{-1/2}\Theta(\rho)^T L\Theta(\rho)L^{-1/2}$$

and $z_2(t)/w_2(t)$ are respectively the input and output of operator $\tilde{\Theta}(\rho)$. The latter integral quadratic form is nonnegative for all z if and only if

$$\gamma_\theta^2 I - L^{-1/2}\Theta(\rho)^T L\Theta(\rho)L^{-1/2} \succeq 0$$

and, according to the definition of the set $\mathcal{S}_D(\Theta)$ by (2.43), if and only if

$$\gamma_\theta^2 I - \Theta(\rho)^T\Theta(\rho) \succeq 0 \tag{2.44}$$

Since $\Theta(\rho)^T\Theta(\rho) \preceq I$ then $\gamma_\theta = 1$ is the minimal value such that (2.44) holds. This shows that it is not aberrant to consider $\tilde{\Theta}(\rho)$ instead of $\Theta(\rho)$.

Finally, in virtue of these considerations, if the interconnection

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Ew(t) \\ z(t) & = & Cx(t) + Dw(t) \\ w(t) & = & \tilde{\Theta}(\rho)z(t) \end{array} \tag{2.45}$$

is asymptotically stable, then (2.36) is asymptotically stable. It is worth noting that, by introducing notations $z_2(t) = L^{-1/2}z(t)$ and $w(t) = L^{1/2}w_2(t)$ we get the following system:

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + BL^{1/2}w_2(t) \\ z_2(t) & = & L^{-1/2}(Cx(t) + DL^{1/2}w(t)) \\ w_2(t) & = & \Theta(\rho)z_2(t) \end{array} \tag{2.46}$$

Finally, applying the bounded-real lemma on scaled system (2.46) we get the matrix inequality in P and $L^{1/2}$:

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & PBL^{1/2} & C^T L^{-1/2} \\ \star & -I & L^{1/2}D^T L^{-1/2} \\ \star & \star & -I \end{bmatrix} \prec 0$$

Performing a congruence transformation with respect to matrix $\operatorname{diag}(I, L^{1/2}, L^{1/2})$ yields the following result:

Lemma 2.3.49 (Scaled-Bounded Real Lemma) *System (2.36) is asymptotically stable if there exist $P = P^T \succ 0$ and $L \in \mathcal{S}_D(\Theta)$ such that*

$$\begin{bmatrix} A^T P + PA & PBL & C^T \\ \star & -L & LD^T \\ \star & \star & -L \end{bmatrix} \prec 0$$

Note that, if $L = I$, the condition of the small-gain theorem is retrieved.

Another vision of the scaled-small gain lemma, is the problem of finding a linear bijective change of variable for signals involved in the interconnection, such that the system behavior is unchanged (role of the commutation). A suitable change of variable is then given by matrices $L^{1/2}$ and $L^{-1/2}$. Several different approaches can be used to establish such a result, for instance using the $\mathcal{S}$-procedure (see the next section and Appendix D.13), or the bounding lemma (see appendix D.15).

The scaled-small gain theorem leads to less conservative result than the small-gain but only considers the structure of the parameter varying matrix $\Theta(\rho)$. This is the reason why, for a small number of uncertainties, the result is necessary and sufficient as proved in [Packard and Doyle, 1993]. In this paper, it is shown that D-scaling provides a necessary and sufficient condition if and only if the sum of the number repeated scalar blocks and the unrepeated full-blocks is lower than 3. For larger uncertainties, not enough information is taken into account on how the two subsystems are interconnected. Indeed, this information is destroyed again by the use of norms which gathers multiple data (each entry of matrices) into one unique positive scalar value (see the Examples 2.3.44, 2.3.45 and 2.3.47).

The next idea would be to find a better framework in which the shape of the operators can be better characterized and considered, avoiding then the use of coarse norms (e.g. the $\mathcal{H}_\infty$). The next section on the full-block $\mathcal{S}$-procedure and the notion of well-posedness of feedback systems, partially solves this problem.

2.3.4.4 Full-Block $\mathcal{S}$-procedure and Well-Posedness of Feedback Systems

Both recent results have brought many improvements in the field of LPV system analysis and LPV control. We have chosen to present them simultaneously since they are two facets of the same theory but are proved using different fundamental theories.

Full-Block $\mathcal{S}$-procedure

The full-block $\mathcal{S}$-procedure has been developed in several research papers [Scherer, 1996, 1997, 1999, 2001; Scherer and Hol, 2006] and has been applied to several topics [Münz et al., 2008; Wu, 2003]. In [Briat et al., 2008b], we have developed a delay-dependent stabilization test using the full-block $\mathcal{S}$-procedure and is an extension of [Wu, 2003] where delay-independent stability is considered only.

This approach is based on the theory of dissipativity of dynamical systems (see appendix D.1 and [Scherer and Weiland, 2005] for more details on dissipativity) but to avoid too much (and sometimes tough) explanations, the fundamental result of the full-block $\mathcal{S}$-procedure will be retrieved here through a simple application of the $\mathcal{S}$-procedure (see appendix D.10 and [Boyd et al., 1994]).

Let us consider system (2.36) where $\Theta(\rho)$ has possibly a full structure. We also relax the image set of the parameters to a more general compact set, we hence assume that

$$\rho \in U_\rho := \times_{i=1}^{p} [\rho_i^-,\ \rho_i^+]$$

The key idea of the full-block $\mathcal{S}$-procedure is to characterize the parameter matrix $\Theta(\rho)$ in a more complex way allowing for a tighter approximation of the set in which the parameter matrix $\Theta(\rho)$ evolves. Namely, the set is characterized by an ellipsoid (instead of a simple ball) possibly not including zero. This is performed using an integral quadratic constraint (IQC). Indeed assume that there exists a bounded matrix function of time $M_\Theta(t) = M_\Theta(t)^T$ such that

$$\int_0^t \begin{bmatrix} z(s) \\ w(s) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} z(s) \\ w(s) \end{bmatrix} ds \succ 0 \tag{2.47}$$

for all $t > 0$ and $w(t) = \Theta(\rho)z(t)$. The latter IQC is equivalent to

$$\int_0^t z(s)^T \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix} z(s) ds \succ 0 \tag{2.48}$$

It is clear that the matrix M_Θ has, a priori, no imposed inertia. It will be shown at the end of this section that it is possible to define specific matrices M_Θ for which passivity, small-gain and scaled small-gain results are retrieved.
Hence, the following Lyapunov function (viewed here as a storage function) is considered

$$V(x(t)) = x(t)^T P x(t) > 0$$

with constraint on input and output signals w and z taking the form of the IQC (2.47). Thus, by invoking the $\mathcal{S}$-procedure (see Appendix D.10) or the theory of dissipative systems (see [Scherer and Weiland, 2005; Willems, 1972] or Appendix D.1), the following function is constructed

$$\begin{aligned} H(x,w) &= x(t)^T P x(t) - \int_0^t \begin{bmatrix} z(s) \\ w(s) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} z(s) \\ w(s) \end{bmatrix} ds \\ &= x(t)^T P x(t) - \int_0^t z(s)^T \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix} z(s) ds \end{aligned}$$

Since, the integrand of (2.48) is a quadratic form and the parameters ρ are allowed to have any trajectory in the set U_ρ, then inequality (2.48) holds if and only if

$$\begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix}^T M_\Theta(t) \begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix} \succ 0$$

for any trajectories of $\rho(t) \in U_\rho$ and all $t > 0$. Finally, computing the time-derivative of H leads to the result:

Theorem 2.3.50 *System (2.36) is asymptotically stable if and only if there exist a matrix $P = P^T \succ 0$ and a bounded matrix function $M_\Theta : \mathbb{R}^+ \to \mathbb{S}^{n_w+n_z}$ such that the LMIs*

$$\begin{aligned} \begin{bmatrix} A^T P + PA & PE \\ \star & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T M_\Theta(t) \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} &\prec 0 \\ \begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix}^T M_\Theta(t) \begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix} &\succ 0 \end{aligned}$$

hold for all $t > 0$ and for all $\rho \in U_\rho$.

Proof: *A complete proof with meaningful discussions can be found in [Scherer, 1997, 1999, 2001].* □

The main difficulty of such a result resides in the computation of LMI (2.49). Even if M_Θ is chosen constant, we are faced to a problem involving infinitely many inequalities since the inequality should be satisfied for any parameter trajectories. Methods for dealing with such parameter dependent LMIs have been introduced in Sections 2.3.3.2, 2.3.3.3 and 2.3.3.4 where gridding, SOS and global polynomial optimization approaches are introduced.

It is important to point out that, due to the losslessness of the $\mathcal{S}$-procedure (see Appendix D.10 and [Boyd et al., 1994]), the conservativeness of the approach stems from the choice of $M_\Theta(s)$ satisfying LMI (2.49).

Well-Posedness Approach

The well-posedness approach is now compared to the full-block $\mathcal{S}$-procedure. This approach has been initially introduced in [Iwasaki and Hara, 1998] and deployed in many frameworks [Gouaisbaut and Peaucelle, 2006a, 2007; Iwasaki, 1998, 2000; Langbort et al., 2004; Peaucelle and Arzelier, 2005; Peaucelle et al., 2007]. The key idea behind well-posedness is the notion of topological separation [Safonov, 2000; Teel, 1996; Zames, 1966] and is explained in what follows.

Consider two interconnected linear operators $H_1 \in \mathbb{R}^{n_z \times n_w}$ and $H_2 \in \mathbb{R}^{n_w \times n_z}$ such that

$$\begin{array}{rcl} z & = & H_1(w+u_1) \\ w & = & H_2(z+u_2) \end{array} \tag{2.49}$$

where u_1, u_2 are exogenous signals as described in Figure 2.20.

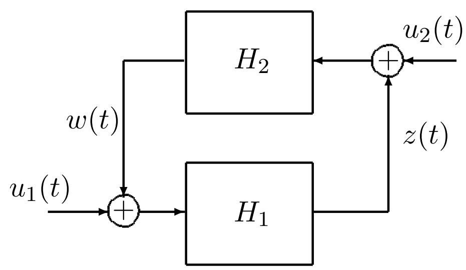

Figure 2.20: Setup of the well-posedness framework

It is convenient to introduce here the definition of well-posedness:

Definition 2.3.51 *The interconnection depicted in Figure 2.20 is said to be well-posed if and only if the loop-signals z, w are uniquely defined by the input signals u_1, u_2 and initial values of the loop-signals $(z(0, w(0))$. In other terms, it is equivalent to the existence of positive scalars $\gamma, \eta > 0$ such that the energy of loop signals is bounded by a function of the energy of input signals and initial values of the loop signals, i.e.*

$$\left|\left|\begin{pmatrix} z \\ w \end{pmatrix}\right|\right|_{\mathcal{L}_2} \leq \gamma \left|\left|\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right|\right|_{\mathcal{L}_2} + \eta \left|\left|\begin{pmatrix} z(0) \\ w(0) \end{pmatrix}\right|\right|_2$$

The latter definition says that if for any finite energy input signals, we get finite energy loop-signals then the system is well-posed. Recall that the notion of stability is not defined yet since operators H_1 and H_2 are linear operators.

The idea behind well-posedness is to prove well-posedness of the interconnection when the interconnection describes a dynamical system. In this case, well-posedness is equivalent to asymptotic stability or equivalently $\mathcal{L}_2 - \mathcal{L}_2$ stability. The following example shows how a dynamical system can be represented in an interconnection as of Figure 2.20.

Example 2.3.52 *Let us consider the trivial linear dynamical system described by $\dot{x} = A(x(t) + v(t))$ where $v(t)$ is an external input.. First, note that by imposing $H_1 = A$ and $H_2 = s^{-1}$ where s is the Laplace variable, then the interconnection depicted in Figure 2.21 is equivalent to system $\dot{x} = A(x(t) + v(t))$.*

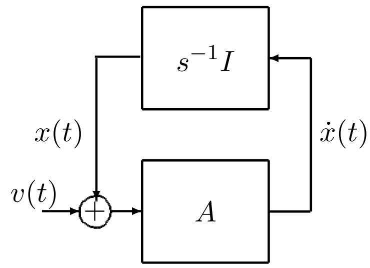

Figure 2.21: Representation of a linear time invariant dynamical system in the well-posedness framework

Now suppose that the interconnection is well-posed: the future evolution of $x(t)$ for $t > t_0$ is uniquely defined by $x(t_0)$ and signal $v(t)$ for all $s \in \mathbb{C}^+$. We aim now to illustrate that well-posedness in this case is equivalent to asymptotic stability. From Figure 2.21 we have

$$\dot{x} = A(x + v)$$

which is equivalent to

$$sx = A(x + v) + x(t_0)$$

and thus

$$(sI - A)x = Av + x(t_0)$$

Therefore if $x(t)$ is uniquely defined by $v(t)$ and $x(t_0)$ for $t > t_0$, this means that $(sI - A)$ is nonsingular for all $s \in \mathbb{C}^+$. This condition is equivalent to saying that A has no eigenvalues in the complex right-half plane and that A is a Hurwitz matrix.

On the other hand, suppose that A is Hurwitz then this means that $sI - A$ is non singular for all $s \in \mathbb{C}^+$ and hence x is uniquely defined by $v(t)$ and $x(t_0)$. Thus the interconnection is well-posed.

We aim now at introducing how well-posedness can be proved efficiently (using numerical tools), at least for linear dynamical systems. This is performed through nice geometrical arguments. Coming back to the setup depicted in Figure 2.20, let $\mathcal{G}_1$ and $\mathcal{G}_2^-$ be respectively the graph of H_1 and the inverse graph of H_2 defined as:

$$\begin{aligned}
\mathcal{G}_1 &:= \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{R}^{n_w + n_z} : \ z = H_1 w \right\} = \operatorname{Im} \begin{pmatrix} H_1 \\ I \end{pmatrix} \\
\mathcal{G}_2^- &:= \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{R}^{n_w + n_z} : \ w = H_2 z \right\} = \operatorname{Im} \begin{pmatrix} I \\ H_2 \end{pmatrix}
\end{aligned}$$

where n_w and n_z denote respectively the dimension of w and z. We have the following important result:

Proposition 2.3.53 *Interconnection (2.49) is well-posed if and only if the following relation holds:*

$$\mathcal{G}_1 \cap \mathcal{G}_2^- = \{0\}$$

In order to visualize this crucial result, let us consider the case where $z = H_1(w+u_1)$ and $w = H_2(z+u_2)$, $H_1 \in \mathbb{R}^{n_z \times n_w}$ and $H_2 \in \mathbb{R}^{n_w \times n_z}$. The graphs are then given by

$$\begin{array}{rcl} \mathcal{G}_1 & = & \mathrm{Im}\begin{pmatrix} H_1 \\ 1 \end{pmatrix} \\ \mathcal{G}_2^- & = & \mathrm{Im}\begin{pmatrix} 1 \\ H_2 \end{pmatrix} \end{array}$$

We aim now at finding the intersection of these sets and we get the following system of linear equations

$$\begin{array}{rcl} H_1 w - z & = & 0 \\ w - H_2 z & = & 0 \end{array} \tag{2.50}$$

which, compacted to a matrix form, becomes

$$H \begin{bmatrix} z \\ w \end{bmatrix} = 0 \tag{2.51}$$

with $H = \begin{bmatrix} -I_{n_z} & H_1 \\ -H_2 & I_{n_w} \end{bmatrix}$. If $\det(H) = 0$ then there exists a infinite number of vectors $\begin{bmatrix} z \\ w \end{bmatrix}$ such that (2.51) is satisfied and thus the interconnection is not well-posed. Moreover, in this case we have $\det(I - H_2 H_1) = 0$ and the null-space is spanned by $\begin{pmatrix} H_1 \\ I \end{pmatrix}$. If the matrix H is non singular, then the null-space reduces to the singleton $\{0\}$ and the system is well-posed since the intersection of the graphs is $\{0\}$. It is important to point out that the following relation holds in any case

$$\mathcal{G}_1 \cap \mathcal{G}_2^- = \mathrm{Null}\begin{pmatrix} -I_{n_z} & H_1 \\ -H_2 & I_{n_w} \end{pmatrix}$$

Therefore, the problem of determining if an interconnection of systems is well-posed is crucial in the framework of interconnected dynamical systems and reduces to the analysis of the intersection of graphs (or equivalently to matrix algebra on linear mappings H_1 and H_2). The idea now is to find a simple way to prove that the graphs do not intersect except at 0. In what follows, the framework is restricted to linear mappings and in this case, the graphs become convex sets, which is an interesting property. First, recall a fundamental result on convex analysis called the *Separating Hyperplane Theorem*; [Boyd and Vandenberghe, 2004, p. 46].

Theorem 2.3.54 (Separating Hyperplane Theorem) *Suppose C_1 and C_2 are two convex sets that do not intersect (i.e. $C_1 \cap C_2 = \emptyset$). Then there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C_1$ and $a^T x \geq b$ for all $x \in C_2$. In other words, the affine function $a^T x - b$ is nonpositive on C_1 and nonnegative on C_2. The hyperplane $\{x : \ a^T x = b\}$ is called a separating hyperplane for the sets C_1 and C_2, or is said to separate the sets C_1 and C_2.*

This results says that two convex sets are disjoints if and only if one can find a linear function which is positive on one set and negative on the other one. The latter result applied to the separation of graphs $\mathcal{G}_1$ and $\mathcal{G}_2^-$ leads to the following theorem proved in [Iwasaki, 2000; Iwasaki and Hara, 1998]:

Theorem 2.3.55 (Quadratic Separation Theorem) *The following statements are equivalent:*

1. *The interconnection is well-posed*

2. $\det(I - H_2 H_1) \neq 0$

3. *There exist* $M = M^T$ *such that*

   (a) $\begin{bmatrix} I & H_1 \end{bmatrix} M \begin{bmatrix} I \\ H_1^* \end{bmatrix} \prec 0$

   (b) $\begin{bmatrix} H_2 & I \end{bmatrix} M \begin{bmatrix} H_2^* \\ I \end{bmatrix} \succeq 0$

The following example illustrates the method.

Example 2.3.56 *Let us consider again the LTI system of Example 2.3.52 and define* $H_1 := A$ *and* $H_2 := s^{-1}I$. *From Theorem 2.3.55, the system is asymptotically (exponentially) stable if and only if the LMIs hold*

$$\begin{bmatrix} I & A \end{bmatrix} M \begin{bmatrix} I \\ A^T \end{bmatrix} \prec 0 \qquad \text{and} \qquad \begin{bmatrix} s^{-1}I & I \end{bmatrix} M \begin{bmatrix} s^{-*}I \\ I \end{bmatrix} \succeq 0$$

As in Example 2.3.52, the well-posedness of the interconnection is sought for all $s \in \mathbb{C}^+$. *Indeed, this would mean that* A *has no eigenvalues in* $\mathbb{C}^+$ *implying that the system is asymptotically stable. It is clear that if* M *is chosen to be*

$$M := \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$$

with $X = X^T \succ 0$, *then*

$$\begin{bmatrix} s^{-1}I & I \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} s^{-*}I \\ I \end{bmatrix} = (s^{-1} + s^{-*})X = 2\Re[s^{-1}]X \succeq 0 \qquad \text{since } s \in \mathbb{C}^+$$

Note that the matrix M *is the only one which can be define implicitly the set of complex number with nonnegative real part. No quadratic function can be positive on the closed complex right-half plane and negative of the open complex right-half plane.*

Therefore, the stability of system $\dot{x} = Ax$ *is ensured if and only if*

$$\begin{bmatrix} I & A \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} I \\ A^T \end{bmatrix} \prec 0$$

which is equivalent to

$$AX + XA^T \prec 0 \Leftrightarrow PA + A^T P \prec 0 \tag{2.52}$$

where $P = X^{-1}$. *The well-known LMI condition obtained from Lyapunov theorem is retrieved. Moreover the equivalence with Lyapunov inequality shows that the choice of the matrix* M *provides a necessary and sufficient condition in the well-posedness framework. This can be also shown using the results on the losslessness of D-scalings [Packard and Doyle, 1993] extended to a more general framework. This is explained in the following.*

List of scalings

We aim now at defining several scalings/separators that may be used in both full-block $\mathcal{S}$-procedure and well-posedness approaches; [Iwasaki and Hara, 1998]. First of all, let us introduce the P-separator. Suppose that H_2 is block diagonal and satisfies

$$\begin{bmatrix} H_2 & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ \star & M_{22} \end{bmatrix} \begin{bmatrix} H_2^* \\ I \end{bmatrix} \succeq 0$$

with fixed matrices M_{ij}. Then under the assumption that $M_{22} \succ 0$ and $M_{11} - M^{12} M_{22}^{-1} M_{12}^T \prec 0$, the P-separator defined as ($\otimes$ denotes the Kronecker product):

$$P \otimes M = \begin{bmatrix} P \otimes M_{11} & P \otimes M_{12} \\ \star & P \otimes M_{22} \end{bmatrix}$$

provides a nonconservative condition if $2c + f \leq 3$ where c is the number of repeated scalar blocks in H_2 and f is the number of unrepeated full-blocks in H_2 [Iwasaki and Hara, 1998; Packard and Doyle, 1993]. For instance, in example 2.3.56, $c = 1$ (only s^{-1} is repeated) and $f = 0$ (no full-blocks). Hence a necessary and sufficient condition to stability of system $\dot{x} = Ax$ is obtained considering

$$M = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$$

Example 2.3.57 *Let us consider again Example 2.3.52. The set of values of $s \in \mathbb{C}^+$ can be defined in an implicit way*

$$\mathbb{C}^+ := \left\{ s \in \mathbb{C} : \begin{bmatrix} s^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s^{-*} \\ 1 \end{bmatrix} \geq 0 \right\}$$

Using the P-separator we get the matrix $M = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$, $X = X^T \succ 0$ and since we have one repeated scalar block then the P-separator provides a nonconservative stability condition.

The following (non-exhaustive) list enumerates specific scalings/separators for different types of operators H_2:

1. The constant scaling $M = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ for positive operators results in a passivity based test.

2. The constant scaling $M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ for unitary norm bounded operators results in a test based on the bounded real lemma.

3. The constant D-scalings $M = \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}$ for $D = D^T > 0$ (for unitary norm bounded operators) are the most simple ones and the result of the scaled-bounded real lemma are retrieved.

4. The constant D-G scalings $M = \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix}$ for $D = D^T > 0$ and $G + G^* = 0$ (for unitary norm bounded operators) are a generalization of the constant D-scalings to a more general case.

5. LFT scalings: $M = \begin{bmatrix} R & S \\ S^* & Q \end{bmatrix}$ with $R \prec 0$ and $\begin{bmatrix} \Theta_k & I \end{bmatrix} M \begin{bmatrix} \Theta_k^T \\ I \end{bmatrix} \succeq 0$ for $k = 1, \dots, 2^\alpha$ (time-invariant and time-varying parameters evolving on a set with vertices Θ_k).

6. Vertex separators $M = \begin{bmatrix} R & S \\ S^* & Q \end{bmatrix}$ with $R_{ii} \preceq 0$ and $\begin{bmatrix} \Theta_k & I \end{bmatrix} M \begin{bmatrix} \Theta_k^T \\ I \end{bmatrix} \succeq 0$ for $k = 1, \dots, 2^\alpha$ (time-invariant and time-varying parameters).

These separators lead to less and less conservative results despite of increasing the computational complexity. Moreover, the LFT scalings and vertex separators work for parameters only and not operators.

After this brief description of well-posedness, we wish now to supply a LPV system description in the well-posedness framework. Let us rewrite the LPV system (2.36) into the form:

$$\begin{bmatrix} z(t) \\ \dot{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} D & C \\ B & A \end{bmatrix}}_{H_1} \begin{bmatrix} w(t) \\ x(t) \end{bmatrix}$$

with $\begin{bmatrix} w(t) \\ x(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \Theta(\rho) & 0 \\ 0 & s^{-1}I \end{bmatrix}}_{H_2} \begin{bmatrix} z(t) \\ \dot{x}(t) \end{bmatrix}$ for all $s \in \mathbb{C}^+$.

Following the previous results, well-posedness of the latter system is equivalent to the asymptotic (exponential) stability of system (2.36). In this case, M can be chosen as

$$M := \left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0 \\ 0 & 0 & 0 & P \\ \hline M_{12}^T & 0 & M_{22} & 0 \\ 0 & P & 0 & 0 \end{array}\right]$$

where $P = P^T \succ 0$, $M_{11} = M_{11}^T$, M_{12} and $M_{22} = M_{22}^T$ are free matrices to be determined. Note that the matrix M contains both a P-separator involving the matrix $P = P^T \succ 0$ (for the stability condition) and a full-separator $M = [M_{ij}]_{i,j}$ (for the parameter consideration). Applying Theorem 2.3.55, we get

$$\left[\begin{array}{cc|cc} I & 0 & D & C \\ 0 & I & B & A \end{array}\right] \left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0 \\ 0 & 0 & 0 & P \\ \hline M_{12}^T & 0 & M_{22} & 0 \\ 0 & P & 0 & 0 \end{array}\right] \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hline D^T & B^T \\ C^T & A^T \end{array}\right] \prec 0$$
$$\left[\begin{array}{cc|cc} \Theta(\rho) & 0 & I & 0 \\ 0 & s^{-1}I & 0 & I \end{array}\right] \left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0 \\ 0 & 0 & 0 & P \\ \hline M_{12}^T & 0 & M_{22} & 0 \\ 0 & P & 0 & 0 \end{array}\right] \left[\begin{array}{cc} \Theta(\rho) & 0 \\ 0 & s^{-1}I \\ \hline I & 0 \\ 0 & I \end{array}\right] \succeq 0$$

Expanding the relations, we get

$$\begin{array}{rcl}\Re[s^{-1}]P & \succeq & 0\\ \begin{bmatrix}\Theta(\rho) & I\end{bmatrix}\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{bmatrix}\Theta(\rho)^T\\ I\end{bmatrix} & \succeq & 0\\ \begin{bmatrix}0 & CP\\ \star & AP+PA^T\end{bmatrix}+\begin{bmatrix}I & D\\ 0 & B\end{bmatrix}\begin{bmatrix}M_{11} & M_{12}\\ \star & M_{22}\end{bmatrix}\begin{bmatrix}I & 0\\ D^T & B^T\end{bmatrix} & \prec & 0\end{array}$$

Inequality (2.53) is satisfied by assumption therefore only inequalities (2.53) and (2.53) have to be (numerically) checked.

Equivalence of well-posedness and full-block $\mathcal{S}$-procedure approaches

In order to bridge results from full-block $\mathcal{S}$-procedure and well-posedness, we will show that inequalities (2.53) and (2.53) are equivalent to (2.49) and (2.49) which are obtained from the full-block $\mathcal{S}$-procedure . Note that (2.53) can be rewritten into the form

$$\begin{bmatrix}0 & C\\ I & A\\ \hline I & D\\ 0 & B\end{bmatrix}^T\left[\begin{array}{cc|cc}0 & P & 0 & 0\\ P & 0 & 0 & 0\\ \hline 0 & 0 & M_{11} & M_{12}\\ 0 & 0 & M_{12}^T & M_{22}\end{array}\right]\begin{bmatrix}0 & C\\ I & A\\ \hline I & D\\ 0 & B\end{bmatrix}\prec 0 \tag{2.53}$$

It is possible to show that the dualization lemma applies (see Appendix D.14 or [Iwasaki and Hara, 1998; Scherer and Weiland, 2005; Wu, 2003]) and then LMI (2.53) is equivalent to

$$\begin{bmatrix}-A & -B\\ I & 0\\ \hline -C & -D\\ 0 & I\end{bmatrix}^T\left[\begin{array}{cc|cc}0 & \tilde{P} & 0 & 0\\ \tilde{P} & 0 & 0 & 0\\ \hline 0 & 0 & \tilde{M}_{11} & \tilde{M}_{12}\\ 0 & 0 & \tilde{M}_{12}^T & \tilde{M}_{22}\end{array}\right]\begin{bmatrix}-A & -B\\ I & 0\\ \hline -C & -D\\ 0 & I\end{bmatrix}\succ 0$$

where $\tilde{P}=P^{-1}$ and $\begin{bmatrix}\tilde{M}_{11} & \tilde{M}_{12}\\ \tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix}=\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}^{-1}$. By expanding the latter inequality we get

$$\begin{bmatrix}-A^T\tilde{P}-\tilde{P}A & -\tilde{P}B\\ \star & 0\end{bmatrix}+\begin{bmatrix}-C & -D\\ 0 & I\end{bmatrix}^T\begin{bmatrix}\tilde{M}_{11} & \tilde{M}_{12}\\ \tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix}\begin{bmatrix}-C & -D\\ 0 & I\end{bmatrix}\succ 0$$

or equivalently

$$\begin{bmatrix}-A^T\tilde{P}-\tilde{P}A & -\tilde{P}B\\ \star & 0\end{bmatrix}+\begin{bmatrix}C & D\\ 0 & I\end{bmatrix}^T\begin{bmatrix}\tilde{M}_{11} & -\tilde{M}_{12}\\ -\tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix}\begin{bmatrix}C & D\\ 0 & I\end{bmatrix}\succ 0 \tag{2.54}$$

Moreover, in virtue of the dualization lemma again, we have

$$\begin{bmatrix}\Theta(\rho) & I\end{bmatrix}\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{bmatrix}\Theta(\rho)^T\\ I\end{bmatrix}\succeq 0\Longleftrightarrow\begin{bmatrix}-I & \Theta(\rho)^T\end{bmatrix}\begin{bmatrix}\tilde{M}_{11} & \tilde{M}_{12}\\ \tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix}\begin{bmatrix}-I\\ \Theta(\rho)\end{bmatrix}\prec 0$$

and equivalently

$$\begin{bmatrix}I & \Theta(\rho)^T\end{bmatrix}\begin{bmatrix}\tilde{M}_{11} & -\tilde{M}_{12}\\ -\tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix}\begin{bmatrix}I\\ \Theta(\rho)\end{bmatrix}\prec 0 \tag{2.55}$$

Finally, by multiplying inequalities (2.54) and (2.55) by -1, we get

$$\begin{array}{rl} \begin{bmatrix} A^T\tilde{P}+\tilde{P}A & \tilde{P}B \\ \star & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} -\tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & -\tilde{M}_{22} \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} & \prec 0 \\ \begin{bmatrix} I & \Theta(\rho)^T \end{bmatrix} \begin{bmatrix} -\tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & -\tilde{M}_{22} \end{bmatrix} \begin{bmatrix} I \\ \Theta(\rho) \end{bmatrix} & \succ 0 \end{array}$$

By identification these latter relations are identical to (2.49) and (2.49) obtained by application of the full-block $\mathcal{S}$-procedure where $M_\Theta = \begin{bmatrix} -\tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & -\tilde{M}_{22} \end{bmatrix}$ and $\tilde{P}$ plays the role of the Lyapunov matrix used to define the quadratic Lyapunov function.

This emphasizes the similarities between the results obtained from the full-block $\mathcal{S}$-procedure and the well-posedness approach. It is important to point out that in every methods presented up to here, only quadratic stability was considered and may result in conservative stability conditions. Robust stability is addressed here in the framework of well-posedness of feedback systems. The procedure used here can be applied to any approach presented in preceding sections. The main reasons for presenting robust stability for LFT systems at this stage only is due to the simplicity of the well-posedness approach in this context. Moreover, as we shall see later, it is possible to connect these results to parameter dependent Lyapunov functions results introduced in Section 2.3.3.

LPV systems as implicit systems

The following method has been developed in [Iwasaki, 1998] but some other methods have been developed in order to define a LPV system as an implicit system [Masubuchi and Suzuki, 2008; Scherer, 2001]. It is convenient to introduce the following result on well-posedness of implicit systems. This is the generalization of well-posedness theory for dynamical system governed by expressions of the form

$$\begin{bmatrix} \mathcal{A}-sI & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \zeta = 0 \tag{2.56}$$

where s is the Laplace variable and ζ are signals involved in the system. Such an expression describes a linear-time invariant dynamical system coupled with a static equality between signals. This type of systems is not without recalling singular systems in which static relations are captured in a matrix E factoring the time-derivative of x (e.g. $E\dot{x} = Ax$). As an illustration of the formalism, system (2.56) represents the wide class of systems governed by equations:

$$\begin{array}{rcl} \dot{x} & = & \mathcal{A}x + \mathcal{B}w \\ 0 & = & \mathcal{C}x + \mathcal{D}w \end{array}$$

when $\zeta = \text{col}(x, w)$ which in turn can be rewritten in the singular system form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

Some fundamental definitions on implicit systems are recalled here for informational purpose.

Definition 2.3.58 *The implicit system (2.56) is said to be regular if the following conditions hold:*

1. *There is no impulsive solution, i.e. the system is impulse free [Verghese et al., 1981];*
2. *for each $x(0^-)$, the solution, if any, is unique.*

One of the particularities of such systems is that under certain circumstances (i.e. according to system matrices), the state trajectories may contain Dirac pulses of theoretically infinite amplitude. Moreover, it is also possible that no solutions exist or may be non unique. It is then important to characterize the regularity of implicit systems. The following lemma, proved in [Iwasaki, 1998], gives a necessary and sufficient condition for the system to be regular.

Lemma 2.3.59 *Implicit system (2.56) is regular if and only if $\mathcal{D}$ has full column rank.*

Definition 2.3.60 *System (2.56) is said to be stable if it is regular and for each $x(0^-)$ the solution, if any, converges to zero as t tends to $+\infty$.*

The latter definition generalizes the notion of stability of linear differential systems to linear differential systems with static equalities constraints. The following lemmas [Iwasaki, 1998] provide necessary and sufficient conditions for stability and robust stability of (uncertain) implicit systems.

Lemma 2.3.61 *Consider implicit system (2.56). The following statements are equivalent:*

1. *The system is stable;*
2. *The matrix $\begin{bmatrix} \mathcal{A} - sI & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ has full-column rank for all $s \in \mathbb{C}^+ \cup \{\infty\}$;*
3. *$\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$ has full-column rank for all $s \in \mathbb{C}^+ \cup \{\infty\}$.*

Let us consider now the uncertain implicit system governed by

$$\begin{bmatrix} \mathcal{A} - sI & \mathcal{B} & 0 \\ \mathcal{C} & \mathcal{D} & \Delta \end{bmatrix} \zeta = 0 \tag{2.57}$$

where $\Delta \in \mathbf{\Delta}$ is an unknown but constant matrix and ζ contains all signals involved the uncertain system. Under some technical assumptions and results which are not detailed here [Iwasaki, 1998], we have the following result on robust stability.

Lemma 2.3.62 *Consider the implicit system (2.57) where $\Delta \in \mathbf{\Delta}$. The following statements are equivalent:*

1. *Implicit system (2.57) is stable for all $\Delta \in \mathbf{\Delta}$.*
2. *for each $\omega \in \mathbb{R} \cup \{\infty\}$, there exists an Hermitian matrix $\Pi(j\omega)$ such that*

$$\begin{array}{rcl} [\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}]^T \Pi(j\omega) [\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}] & \prec & 0 \\ \Delta^T \Pi(j\omega) \Delta & \succeq & 0, \qquad \text{for all } \Delta \in \mathbf{\Delta} \end{array}$$

These inequalities have to be satisfied for all $\omega \in \mathbb{R} \cup \{\infty\}$ and may be difficult to solve. The reader should refer to Sections 2.3.3.1, 2.3.3 and 2.3.3.3 for more details on such parameter dependent LMIs. On the other hand if a constant matrix $\Pi(j\omega) := \Pi_0$ is sought then the variable ω can be eliminated using an extension of the Kalman-Yakubovich-Popov lemma [Iwasaki, 1998]. In [Scherer, 2008], another interesting approach (leading to the same result) to remove the variable ω is also developed.

Finally, the following sufficient condition, given in terms of quadratic separation, is obtained for stability analysis of uncertain implicit systems.

Lemma 2.3.63 *Consider the uncertain implicit system (2.57). If there exist $P = P^T$ and $\Pi \in \mathbf{\Pi}$ such that*

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & P \\ 0 & \Pi & 0 \\ P & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix} \prec 0$$

where $\mathbf{\Pi} := \left\{\Pi : \ \Delta^T \Pi \Delta \succeq 0, \ \forall \ \Delta \in \mathbf{\Delta}\right\}$ then the system is stable for all constant $\Delta \in \mathbf{\Delta}$ provided that there exists at least one Δ_0 such that (2.57) is stable. Moreover, if $P \succ 0$ then the system is stable for all time-varying $\Delta(t) \in \mathbf{\Delta}$ even if there is no Δ_0 such that (2.57) is stable.

From these results we are able to provide a robust stability test for LPV systems. First of all, let us show how parameter variations can be taken into account. Differentiating z and w channels in (2.36) yields

$$\begin{array}{rcl} \dot{z} &=& C\dot{x} + D\dot{w} \\ &=& C\tilde{A}x + CBw + D\dot{w} \\ \dot{w} &=& \dot{\Theta}z + \Theta\dot{z} \end{array}$$

Finally defining $\phi = \dot{\Theta}z$ we have

$$\left[\begin{array}{c} \dot{x} \\ \dot{w} \\ \hline z \\ \dot{z} \\ z \\ \Theta z \\ \Theta\dot{z} \\ \dot{\Theta}z \end{array}\right] = \left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} x \\ w \\ \hline \dot{w} \\ \phi \end{array}\right]$$

Hence letting $\bar{\Theta} = \text{diag}(\Theta, \Theta, \dot{\Theta}) \in \bar{\mathbf{\Theta}}$, $\Delta = \begin{bmatrix} I \\ \bar{\Theta} \end{bmatrix}$ and $\zeta = \text{col}(x, w, \dot{w}, \phi, -z, -\dot{z}, -z)$ then form (2.57) is retrieved; i.e.

$$\left[\begin{array}{cc|cc|ccc} \tilde{A} - sI & B & 0 & 0 & 0 & 0 & 0 \\ 0 & -sI & I & 0 & 0 & 0 & 0 \\ \hline C & D & 0 & 0 & I & 0 & 0 \\ C\tilde{A} & CB & D & 0 & 0 & I & 0 \\ C & D & 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & \Theta & 0 & 0 \\ 0 & 0 & I & -I & 0 & \Theta & 0 \\ 0 & 0 & 0 & I & 0 & 0 & \dot{\Theta} \end{array}\right] \left[\begin{array}{c} x \\ w \\ \hline \dot{w} \\ \phi \\ \hline -z \\ -\dot{z} \\ -z \end{array}\right] = 0$$

This leads to the following theorem [Iwasaki, 1998]:

Theorem 2.3.64 *LPV system (2.36) is robustly stable for all* $\bar{\Theta} = \mathrm{diag}(\Theta, \Theta, \dot{\Theta}) \in \bar{\boldsymbol{\Theta}}$ *if there exist real symmetric matrices* P *and* $\bar{\Pi}$ *that*

$$\left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]^T \left[\begin{array}{c|c|c} 0 & 0 & P \\ 0 & \bar{\Pi} & 0 \\ P & 0 & 0 \end{array}\right] \left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right] \prec 0$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta(\rho) & 0 & 0 \\ 0 & \Theta(\rho) & 0 \\ 0 & 0 & \Theta(\dot{\rho}) \end{bmatrix}^T \bar{\Pi} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta(\rho) & 0 & 0 \\ 0 & \Theta(\rho) & 0 \\ 0 & 0 & \Theta(\dot{\rho}) \end{bmatrix} \succeq 0$$

provided that $\tilde{A}$ *is Hurwitz.*

The interest of the following theorem is that the symmetric matrix P is not necessarily positive definite and it provides a less conservative conditions than by considering usual positivity requirement. We aim at showing now that this stability condition can be interpreted in terms of a parameter dependent Lyapunov function depending on $\Theta(\rho)$. First of all, the LPV system (2.36) is rewritten in the compact form

$$\dot{x} = A_\Theta x := [\tilde{A} + B(I - \Theta(\rho)D)^{-1}\Theta(\rho)C]x$$

It is well-known that this system is stable if $(I - \Theta(\rho)D)$ is invertible for all $\Theta(\rho) \in \boldsymbol{\Theta}$ and if there exists a parameter dependent Lyapunov function $V(x, \Theta) = x^T P_\Theta x$ such that $P_\Theta = P_\Theta^T \succ 0$ and

$$\dot{P}_\Theta + P_\Theta A_\Theta + A_\Theta^T P_\Theta \prec 0$$

for all $\Theta \in \boldsymbol{\Theta}$. Let N_Θ be the matrix $N_\Theta := (I - \Theta D)^{-1}\Theta C$ such that we have $w = N_\Theta x$. Differentiating w yields

$$\dot{w} = \dot{N}_\Theta x + N_\Theta \dot{x}$$

where $\dot{N}_\Theta = (I - \Theta D)^{-1}\dot{\Theta}(I - \Theta D)^{-1}C$. Now construct a parameter dependent Lyapunov function $V(x) = x^T P_\Theta x$ with

$$P_\Theta = \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} \tag{2.58}$$

Then we have

$$\dot{P}_\Theta = \begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} + \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix}$$

and hence the Lyapunov inequality is given by

$$\dot{V}(x) = x^T(\dot{P}_\Theta + A_\Theta^T P_\Theta + P_\Theta A_\Theta)x < 0$$

for all $x \neq 0$. Moreover note that

$$\begin{aligned}
\dot{x} &= A_\Theta x \\
&= \tilde{A}x + Bw \\
&= (\tilde{A} + BN_\Theta)x \\
\dot{w} &= \dot{N}_\Theta x + N_\Theta A_\Theta x \\
&= \dot{N}_\Theta x + N_\Theta(\tilde{A}x + Bw) \\
&= \left(\dot{N}_\Theta + N_\Theta(\tilde{A} + BN_\Theta)\right) x
\end{aligned}$$

Hence $\dot{V}$ becomes

$$\begin{aligned}
\dot{V}(t) &= x^T \left(\begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} + \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix} + \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} (\tilde{A} + BN_\Theta) \right. \\
&\quad \left. + (\tilde{A} + BN_\Theta)^T \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} \right) x < 0 \\
&= x^T \left(\begin{bmatrix} \tilde{A} + BN_\Theta \\ \dot{N}_\Theta + N_\Theta(\tilde{A} + BN_\Theta) \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} \right)^{\mathcal{S}} x < 0
\end{aligned}$$

The following equalities hold

$$\begin{aligned}
\begin{bmatrix} \tilde{A} + BN_\Theta \\ \dot{N}_\Theta + N_\Theta(\tilde{A} + BN_\Theta) \end{bmatrix} x &= \begin{bmatrix} \tilde{A} & B & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix} \\
\begin{bmatrix} I \\ N_\Theta \end{bmatrix} x &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix}
\end{aligned}$$

And finally we get

$$\dot{V}(t) = \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{A}^T & 0 \\ B^T & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} \tilde{A} & B & 0 \\ 0 & 0 & I \end{bmatrix} \right) \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix} < 0$$

It is worth noting that in the latter condition, no information is taken into account about the parameters and their derivative. This is captured by the following static relations:

$$\begin{aligned}
w &= N_\Theta x \\
&= (I - \Theta D)^{-1}\Theta C x \\
\Rightarrow \quad 0 &= \Theta C x + (\Theta D - I)w \\
w &= \Theta z = \Theta(Cx + Dw) \\
\Rightarrow \quad \dot{w} &= \dot{\Theta} z + \Theta C \dot{x} + \Theta D \dot{w} \\
&= \eta + \Theta C \tilde{A} x + \Theta C B w + \Theta D \dot{w} \\
\Rightarrow \quad 0 &= \Theta C \tilde{A} x + \Theta C B w + (\Theta D - I)\dot{w} \\
\eta &= \dot{\Theta}(Cx + Dw) \\
\Rightarrow \quad 0 &= \dot{\Theta} C x + \dot{\Theta} D w - \eta
\end{aligned}$$

where $\eta = \dot{\Theta} z$. Hence gathering the

$$\begin{array}{lcl} \Theta C x + (\Theta D - I) w & = & 0 \\ \Theta C \tilde{A} x + \Theta C B w + (\Theta D - I) \dot{w} & = & 0 \\ \dot{\Theta} C x + \dot{\Theta} D w - \eta & & \end{array}$$

into a compact matrix form yields

$$\underbrace{\begin{bmatrix} \Theta C & \Theta D - I & 0 & 0 \\ \Theta C \tilde{A} & \Theta C B & \Theta D - I & I \\ \dot{\Theta} C & \dot{\Theta} D & 0 & -I \end{bmatrix}}_{\Psi(\Theta)} \begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix} = 0 \tag{2.59}$$

Then it follows that the Lyapunov inequality becomes

$$\begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix}^T \left(\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \right)^{\mathcal{S}} \begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix} < 0 \tag{2.60}$$

for all signals $\mathrm{col}(x, w, \dot{w}, \eta) \neq 0$ such that (2.59) holds. Now rewrite (2.59) as a matrix product

$$\Psi(\Theta) = \begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix}$$

It follows that the Lyapunov inequality is equivalent to (2.60) for all non zero vector $\mathrm{col}(x, w, \dot{w}, \eta)$ such that

$$\begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix} = 0$$

holds for some $\bar{\Theta} \in \bar{\boldsymbol{\Theta}}$ with $\bar{\Theta} = \mathrm{diag}(\Theta, \Theta, \dot{\Theta})$. This problem falls into the framework of the generalized Finsler's lemma (see Appendix D.17). It follows that the Lyapunov inequality feasibility is equivalent to the existence of symmetric matrices P and $\bar{\Pi}$ such that

$$\left(\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \right)^{\mathcal{S}} + \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix}^T \bar{\Pi} \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix} \prec 0$$

$$\mathrm{Ker}\left(\begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix}\right)^T \bar{\Pi}\mathrm{Ker}\left(\begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix}\right) \succeq 0$$

hold.

The first LMI is identical to

$$\left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]^T \left[\begin{array}{c|c|c} 0 & 0 & P \\ 0 & \bar{\Pi} & 0 \\ P & 0 & 0 \end{array}\right] \left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right] \prec 0 \tag{2.61}$$

while the second one is identical to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \dot{\Theta} \end{bmatrix}^T \cdot \bar{\Pi} \cdot \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \dot{\Theta} \end{bmatrix} \succeq 0 \tag{2.62}$$

These LMIs (2.61) and (2.62) are equivalent to LMIs provided in Theorem 2.3.64. This shows that the feasibility of the Lyapunov inequality implies the feasibility of LMIs of Theorem 2.3.64. By following the development backaward, this shows that feasibility of LMIs of Theorem 2.3.64 implies the existence of a parameter Lyapunov function $V(x) = x^T P_\Theta x$ where P_Θ is defined in (2.58).

We have shown in this section that full-block $\mathcal{S}$-procedure and well-posedness approach are equivalent. Moreover, they embed preceding methods such as passivity and small-gain results. Using a variation of the well-posedness results extended to implicit systems a robust stability test has been developed. Moreover, the well-posedness allows for an explicit construction of a (parameter dependent) Lyapunov function proving the stability of the LPV system. It is worth noting that this Lyapunov function has the same dependence on parameters than the system.

In Lemma 2.3.62, the separator $\Pi(j\omega)$ depends on the frequency variable ω and is relaxed to a constant matrix in order to provide tractable conditions. However, this simplification introduces some conservatism in the approach and it would be interesting to keep this dependence on ω in order to characterize, in the frequency domain, additional information on the parameters. Next sections are devoted to the introduction of methods in which constraints in the frequency domain are allowed: the first one to be presented is the extension of the full-block $\mathcal{S}$-procedure to allow for frequency dependent scaling while the second one uses Integral Quadratic Constraints (IQC) which try to confine the stability conditions into a least conservatism domain.

2.3.4.5 Frequency-Dependent D-Scalings

The use of frequency-dependent scalings with full-block $\mathcal{S}$-procedure is very recent and has been proposed in [Scherer and Köse, 2007a,b]. The idea is to replace the constant D-scalings by frequency-dependent scalings playing the role of dynamic filters, which will characterize the uncertainties/parameters in the frequency domain. Indeed, constant D-scalings allow to characterize the $\mathcal{H}_\infty$ (or induced $\mathcal{L}_2$ norm) over the whole frequency domain and results in conservative conditions if the parameters belong to a specific frequency domain (note that in certain cases, D-scalings are lossless as emphasized in [Iwasaki and Hara, 1998; Packard and Doyle, 1993] and Section 2.3.4.4 in the list of scalings).

Let us consider system (2.36) and suppose that

$$\Theta = \text{diag}(\Theta_1(\rho), \ldots, \Theta_q(\rho))$$

and $||\Theta(\rho)||_{\mathcal{H}_\infty} \leq 1$. Frequency dependent D-scalings will consider the set $\mathcal{Q}$ of matrices structured as

$$Q(s) = \text{diag}(q_1(s)I, \ldots, q_m(s)I) \tag{2.63}$$

in correspondence with the structure of $\Theta(\rho)$ where the components q_i are SISO transfer functions, real valued and bounded on the imaginary axis $\mathbb{C}_0$. The stability of the LPV system is then guaranteed if there exists some multiplier $Q \in \mathcal{Q}$ for which

$$\begin{bmatrix} H(s) \\ I \end{bmatrix}^* \begin{bmatrix} Q(s) & 0 \\ 0 & -Q(s) \end{bmatrix} \begin{bmatrix} H(s) \\ I \end{bmatrix} \prec 0, \qquad Q(s) \succ 0 \text{ on } \mathbb{C}_0 \tag{2.64}$$

The key idea is to approximate any filter by a finite basis of elementary filters of the form

$$\begin{array}{rcl} f_{1,\kappa}(s) & = & \begin{bmatrix} 1 & f_1(s) & f_2(s) & \ldots & f_\kappa(s) \end{bmatrix} \\ f_{2,\kappa}(s) & = & \begin{bmatrix} 1 & f_1(s)^* & f_2(s)^* & \ldots & f_\kappa(s)^* \end{bmatrix} \end{array}$$

where $f_{1,\kappa}(s)$ and $f_{2,\kappa}(s)$ are respectively stable and anti-stable rows with $f(s)^* = f(-s)^T$. Let us recall that an anti-stable transfer function has all its poles in $\mathbb{C}^+$. Hence for sufficiently large κ any filter stable (anti-stable) can be uniformly approximated on $\mathbb{C}_0$ by $f_{1,\kappa}(s)l_1$ ($f_{2,\kappa}(s)l_2$) for suitable real-valued columns vectors l_1 (l_2) (see [Boyd and Barrat, 1991; Pinkus, 1985; Scherer, 1995]). This implies that $Q(s)$ can be approximated by

$$\Psi_1(s)^* M \Psi_1(s) = \Psi_2(s)^* M \Psi_2(s) \tag{2.65}$$

where $\Psi_j := \text{diag}(I \otimes f_{j,\kappa}^T, \ldots, I \otimes f_{j,\kappa}^T)$ and M is a symmetric matrix such that $M := \text{diag}(I \otimes M^1, \ldots, I \otimes M_m)$ in which the M^i's have to be determined. We give here the main stability result which has been initially introduced in [Scherer and Köse, 2007a,b]:

Theorem 2.3.65 *A is stable and (2.64) holds for Q represented as (2.65) if and only if the following LMIs are feasible:*

$$\begin{bmatrix} I & 0 \\ A_p & B_p \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} 0 & X & 0 \\ \star & 0 & 0 \\ \star & \star & \text{diag}(M, -M) \end{bmatrix} \begin{bmatrix} I & 0 \\ A_p & B_p \\ C_p & D_p \end{bmatrix} \prec 0$$

$$\begin{bmatrix} I & 0 \\ A_{\Psi_1} & B_{\Psi_1} \\ C_{\Psi_1} & D_{\Psi_1} \end{bmatrix}^T \begin{bmatrix} 0 & Y & 0 \\ \star & 0 & 0 \\ \star & \star & M \end{bmatrix} \begin{bmatrix} I & 0 \\ A_p & B_p \\ C_p & D_p \end{bmatrix} \succ 0$$

$$\begin{bmatrix} X_{11} - Y & X_{13} \\ \star & X_{33} \end{bmatrix} \succ 0$$

where $\left[\begin{array}{c|c} A_{\Psi_1} & B_{\Psi_1} \\ \hline C_{\Psi_1} & D_{\Psi_1} \end{array}\right]$ *is a minimal realization of* Ψ_1 *and*

$$\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] := \left[\begin{array}{ccc|c} A_{\Psi_1} & 0 & B_{\Psi_1}C & D \\ 0 & A_{\Psi_2} & 0 & B_{\Psi_2} \\ 0 & 0 & A & B \\ \hline C_{\Psi_1} & 0 & D_{\Psi_1} & D_{\Psi_1}D \\ 0 & C_{\Psi_2} & 0 & D_{\Psi_2} \end{array}\right]$$

is a minimal realization of $\begin{bmatrix} \Psi_1 G \\ \Psi_2 \end{bmatrix}$.

The idea in this approach is to choose a basis of transfer functions from which the matrix $Q(s)$ will be approximated. The conservatism of the approach thus depends on the complexity (the completeness) of the basis and on the type of scalings used (here D scalings). This result is very close to results based on integral quadratic constraints which are presented in the next section.

2.3.4.6 Analysis via Integral Quadratic Constraints (IQC)

This section is devoted to IQC analysis and is provided for informative purposes only [Rantzer and Megretski, 1997]. The key ideas, which are very similar to the well-posedness and full-block $\mathcal{S}$-procedure, are briefly explained hereafter.

The central idea of the IQC framework is identical to the well-posedness: the loop signals must be uniquely defined by the inputs; for bounded energy inputs, we get bounded energy loop signals ($\mathcal{L}_2$ internal stability). The first step is to define any blocks and signals involved in the interconnection by means of integral quadratic constraints of the form

$$\int_{-\infty}^{+\infty} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}^T M_q \left(\begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \right) dt \geq 0$$

where M_q is a bounded self-adjoint operator on the $\mathcal{L}_2$ space. With such an IQC, it is possible to capture and characterize many behaviors of operators and signals (see [Rantzer and Megretski, 1997] for a nonexhaustive list of such IQCs). Using Parseval equality (see Appendix D.21), the latter IQC has a frequency dependent counterpart

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* M_q(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

where $\hat{z}$ and $\hat{w}$ denote respectively the Fourier transform of z and w.

The aim of the IQC is to study stability of interconnected systems by defining all signals and operators involved in the interconnection using IQCs, expressed as well in the frequency domain as in the time-domain. These IQCs include extra degree of freedom and this is the reason why the larger the number of IQCs is, the smaller is the conservatism. Moreover, it is worth noting that a wide class of operators and signals can be characterized using IQC: periodic signals, constant signals, norm-bounded operators, constant and time-varying uncertainties, static nonlinearities or even operators with memory (such as delay operators)...

Example 2.3.66 *Let us consider the LPV system under 'LFT' form:*

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \\ w(t) & = & \Theta(t)z(t) \end{array}$$

where $\Theta(t)$ is a diagonal matrix gathering the parameters involved in the system. According to the type of set where the parameters evolve, it is possible to define an IQC to define such sets. For instance, if the parameters evolve within the internal $[-\alpha, \alpha]$, then the signals w and z satisfy the following IQC

$$\int_{-\infty}^{+\infty} \left[\begin{array}{c} z(\theta) \\ w(\theta) \end{array} \right]^T \left[\begin{array}{cc} \alpha^2 Q & 0 \\ 0 & -Q \end{array} \right] \left[\begin{array}{c} z(\theta) \\ w(\theta) \end{array} \right] d\theta$$

for some $Q = Q^T \prec 0$. More generally, we can retrieve the results of the full-block $\mathcal{S}$-procedure by considering that the values of parameter matrix $\Theta(\rho)$ evolve within an ellipsoid, i.e. if we have

$$\left[\begin{array}{c} I \\ \Theta(\rho) \end{array} \right]^T \left[\begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[\begin{array}{c} I \\ \Theta(\rho) \end{array} \right] \prec 0$$

By pre and post multiplying the latter inequality by $z(t)^T$ and $z(t)$ and noting that $w(t) = \Theta(\rho)z(t)$ we have

$$\left[\begin{array}{c} z(t) \\ w(t) \end{array} \right]^T \left[\begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[\begin{array}{c} z(t) \\ w(t) \end{array} \right] \prec 0$$

Taking the integral from $-\infty$ to $+\infty$ we get

$$\int_{-\infty}^{+\infty} \left[\begin{array}{c} z(t) \\ w(t) \end{array} \right]^T \left[\begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[\begin{array}{c} z(t) \\ w(t) \end{array} \right] dt \prec 0$$

which is an IQC corresponding to the supply-rate of full-block $\mathcal{S}$-procedure approach or the multiplier used in the well-posedness approach.

Once all signals and operators have been defined through IQCs, then by invoking the Kalmna-Yakubovicth-Popov lemma (see Appendix D.3), it is possible to obtain a LMI where the sum of all IQC's are involved in. The methodology is illustrated hereafter by considering the stability analysis of a LPV system.

Let us consider system (2.36) with transfer function H mapping w to z. We assume that signals z and w satisfy all the following IQCs:

$$\int_{-\infty}^{+\infty} \left[\begin{array}{c} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{array} \right]^* \Pi_q(j\omega) \left[\begin{array}{c} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{array} \right] d\omega \geq 0$$

for all $q = 1, \ldots, N$ when $\Pi_q(j\omega)$ are Hermitian frequency dependent matrices defining the IQC's. In this case, there exist matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and a set of symmetric real matrices $M_1, \ldots, M_N$ to be determined such that

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \Pi_q(j\omega) \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} = \begin{bmatrix} \tilde{C}(j\omega I - \tilde{A})^{-1}\tilde{B} + \tilde{D} \\ I \end{bmatrix}^* M_q \begin{bmatrix} \tilde{C}(j\omega I - \tilde{A})^{-1}\tilde{B} + \tilde{D} \\ I \end{bmatrix}$$

for all $q = 1, \ldots, N$. By application of the Kalman-Yakubovitch-Popov Lemma (see appendix D.3 and references [Rantzer, 1996; Scherer and Weiland, 2005; Willems, 1971; Yakubovitch, 1974]), it follows that there exists a matrix $P = P^T \succ 0$ such that

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \star & 0 \end{bmatrix} + \sum_{i=1}^{N} \begin{bmatrix} \tilde{C}^T & 0 \\ \tilde{D}^T & I \end{bmatrix} M_q \begin{bmatrix} \tilde{C} & \tilde{D} \\ 0 & I \end{bmatrix} \prec 0$$

For instance, let $N = 1$ and define $\Pi_1 = \begin{bmatrix} -M & 0 \\ 0 & M \end{bmatrix}$ where $M = M^T \succ 0$ is a matrix to be determined. If all the parameters ρ take values in the interval $[-1, 1]$ and signals w and z are defined such that $w = \Theta(\rho)z$. Then it is clear that

$$\int_{-\infty}^{+\infty} \begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix}^* \Pi_1 \begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix} d\omega \geq 0 \tag{2.66}$$

Thus Π_1 defines an IQC for the loop signals z and w. Since Π_1 does not depend on the frequency then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (A, B, C, D)$ and hence we get the LMI

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0 \tag{2.67}$$

which is equivalent to the existence of $P = P^T \succ 0$ and $\tilde{M} = \tilde{M} \succ 0$ such that

$$\begin{bmatrix} A^T P + PA & PB & C^T M \\ B^T P & -M & D^T M \\ MC & MD & -M \end{bmatrix} \prec 0 \tag{2.68}$$

Above, we may recognize the scaled-bounded real lemma and this points out that results obtained from full-block multipliers and well-posedness can be retrieved with an appropriate choice of the multipliers $\Pi_q(j\omega)$. The main difference between the full-block $\mathcal{S}$-procedure extended to frequency dependent D-scalings and the IQC approach resides in the choice of the filters: $Q(s)$ and $\Pi(j\omega)$ for the full-block $\mathcal{S}$-procedure and the IQC analysis respectively. In the full-block $\mathcal{S}$-procedure a basis is chosen and a suitable filter is computed by SDP while in the IQC framework the filter is computed by hand and then degrees of freedom are inserted in the IQCs which are solved numerically. It is worth mentioning that, at this stage, only D-scalings have been extended to depend on the frequency but one can easily imagine to extend a more general case of scalings leading then to a framework, closer to IQC analysis. In such a case, we can strongly think that filters initially computed for IQC can be used in the full-block $\mathcal{S}$-procedure .

This concludes the part on stability analysis of LPV systems in 'LFT' formulation. Several methods have been presented and relations between results emphasized. The methods provide less and less conservative results. It is important to note that while IQC analysis is currently one of the most powerful technique for stability analysis, it is generally difficult to derive stabilization conditions in terms of LMIs without restricting too much the type of IQCs. On the other hand, the full-block $\mathcal{S}$-procedure is well-dedicated to LPV control of LPV systems as emphasized in [Scherer, 2001; Scherer and Köse, 2007a] and (almost) always results in LMI conditions.

2.4 Chapter Conclusion

This chapter has provided an overview of LPV systems. First, a precise definition of parameters is given and several classes of parameters have been isolated: discontinuous, continuous and differentiable continuous parameters. It is highlighted that some classes enjoy nice properties which can be exploited to provide more precise stability and synthesis tools, leading, for instance, to different notions of stability.

Second, three types of LPV systems are presented: polytopic LPV systems, polynomial LPV systems and 'LFT' systems. While the first one is particularly adapted for systems with an affine dependence on the parameters, it leads generally to a conservative representation of systems with non-affine parameter dependence. Examples are given to show the interest of such a representation. On the second hand, polynomial systems are better suited to deal with more general representation excluding rational dependence. Finally, 'LFT' systems are the most powerful representation since they allow to consider any type of parameter dependence, including rational relations.

Third, stability analysis techniques for each type of LPV systems are presented. It is shown that LMIs have a crucial role in stability analysis of LPV systems. Indeed, they provide an efficient and simple way to deal with the stability of LPV systems as well as for LTI systems. However, due to the time-varying nature of LPV systems the LMIs are also parameter-varying and hence more difficult to verify.

It has been shown that in the polytopic framework this infinite set of LMIs can be equivalently characterized by a finite set by considering the LMIs at the vertices of the polytope only. This is a powerful property that makes the polytopic approach widely used in the literature.

On the other hand, when considering polynomial systems, LMIs are far more complicated: they include infinite-dimensional decision variables (decision variables which are functions) and we are confronted to parameter dependent LMIs. In this case, relaxations play a central role in order to reduce this computationally untractable problem into a tractable one. The infinite-dimensional variables are projected over a chosen basis (generally polynomial) in order to bring back the problem to a finite-dimensional one. Since the parameter dependence is nonlinear, it is not sufficient (in general) in this case to consider the LMI at the vertices of the set of the parameters, except for very special cases. This is why several relaxation approaches have been developed. Among others, the gridding, Sum-of-Squares and polynomial optimization approaches. The first one proposes to grid the space of parameters and to consider the LMI at these points only. Although simple, this method is shown to be computationally very expensive and is only a necessary condition for stability. The second one, is based on recent results on Sum-of-Squares polynomials and is very efficient but may be very expensive from a computational point of view. The third one is based on the application of a recent result on polynomial optimization problems solved by a sequence of LMI relaxations. The two latter methods are in fact equivalent but are based on different frameworks.

Finally, stability analysis of LPV systems under 'LFT' form is developed. Several approaches are presented and the different results are linked to each others using underlying theories and by emphasizing common results. Passivity, small-gain and scaled-small gain results are described and generalized through the full-block $\mathcal{S}$-procedure and the dissipativity framework. The well-posedness approach based on topological separation is then shown to be equivalent to the full-block $\mathcal{S}$-procedure . A generalization of the full-block $\mathcal{S}$-procedure involving frequency-dependent scalings is then provided and is put in contrast with the IQC approach which consider Integral Quadratic Constraints in order to specify the types of signals involved in the interconnection.

Chapter 3

Overview of Time-Delay Systems

Time-delay systems (also called *Hereditary Systems*) are a particular case of infinite dimensional systems in which the current evolution of the state is affected not only by current signal values but also by past values. Such systems have suggested more and more interest these past years due to their applicability to communication networks and many other systems. The other interest of time-delay resides in their ability to model transport, diffusion, propagation phenomena [Niculescu, 2001]. They can be viewed as an approximation of distributed systems governed by partial differential equations. For instance, it has been shown that the Dirichlet's control problem (boundary control of systems governed by partial differential equations) can be approximated by delay differential systems; see for instance Hayami [1951]; Moussa [1996]; Niculescu [2001].

This chapter provides some background on time-delay systems, mainly on stability analysis. The chapter will focus especially on the stability analysis of delay differential equations using several modern techniques such as Lyapunov-Krasovskii functionals, the interconnection of systems and Integral Quadratic Constraints (IQC).

Section 3.1 will provide different formalisms to represent time-delay systems where especially functional differential equations are detailed. Several types of time-delay systems are isolated, depending on the type of delays (constant or time-varying) and how they act on system signals.

Section 3.2 is devoted to stability analysis of time-delay systems. Indeed, this large amount of works has led to a wide arsenal of techniques for modeling, stability analysis and control design that need to be introduced to give an insight on the whole field. Only key and original results will be explained due to space limitations and redundant approaches will be avoided. Indeed, many results, although formulated differently, are completely equivalent and in this case a single version will be provided with references to equivalent approaches.

The readers discovering the field of time-delay systems are heavily encouraged to read this chapter carefully to get the necessary background to read this thesis. The interested readers will find several references in each section to deepen their understanding of the domain.

Most of this chapter is based on the books [Dugard and Verriest, 1998; Gu et al., 2003; Kolmanovskii and Myshkis, 1999; Niculescu, 2001] and several published papers which will be cited when needed.

3.1 Representation of Time-Delay Systems

Three different representations are commonly used for modeling time-delay systems:

1. Differential equation with coefficients in a ring of operators:

   This framework has been developed early to study time-delay systems in [Conte and Perdon, 1995, 1996; Conte et al., 1997; Kamen, 1978; Morse, 1976; Perdon and Conte, 1999; Picard and Lafay, 1996; Sename et al., 1995].

   A linear time-delay system is governed by a following linear differential equation with coefficient in a module, e.g.

   $$\dot{x}(t) = A(\nabla)x(t)$$

   where in the general case $\nabla = \underset{i}{\text{col}}(\nabla_i)$ is the vector of delay operators such that $x(t - h_i) = \nabla_i x(t)$. In this case, the coefficient of the A matrix is a multivariate polynomial in the variable ∇. Since the inverse of ∇ (the predictive operator $x(t + h_i) = \nabla_i^{-1} x(t)$) is undefined from a causality point of view, the operators ∇_i of the matrix A belong thus to a ring.

2. Differential equation on an infinite dimensional abstract linear space:

   This type of representation stems from the application of infinite dimensional systems theory to the case of time-delay system. This type of system is completely characterized by the state

   $$\tilde{x} = \begin{bmatrix} x(t) \\ x_t(s) \end{bmatrix}$$

   for all $s \in [-h, 0]$ and $x_t(s) = x(t + s)$. The state-space is then the Hilbert space

   $$\mathbb{R}^n \times \mathcal{L}_2([-h, 0], \mathbb{R}^n)$$

   One can easily see that the state of the system contains a point in an Euclidian space $x(t)$ and a function of bounded energy , $x_t(s)$, the latter belonging to an infinite dimensional linear space. This motivates the denomination of 'Infinite Dimensional Abstract Linear Space' [Bensoussan et al., 2006; Curtain et al., 1994; Iftime et al., 2005; Meinsma and Zwart, 2000]. In that state space, the system rewrites

   $$\frac{d}{dt}\begin{bmatrix} y(t) \\ x_t(\cdot) \end{bmatrix} = \mathcal{A}\begin{bmatrix} y(t) \\ x_t(\cdot) \end{bmatrix}$$

   where the operator $\mathcal{A}$ is given by

   $$\mathcal{A}\begin{bmatrix} y(t) \\ x_t(\cdot) \end{bmatrix} = \begin{bmatrix} Ay(t) + A_h x_t(-h) \\ \dfrac{dx_t(\theta)}{d\theta} \end{bmatrix}$$

   The operator $\mathcal{A}$ is the infinite dimensional counterpart of the finite dimensional operator A in linear systems described by $\dot{x} = Ax$, and many tools involved in the theory of finite dimensional systems have been extended to infinite dimensional systems (e.g. the exponential of matrix, eigenvalues and eigenfunctions, the fundamental matrix or also the explicit solution). The readers should refer to [Bensoussan et al., 2006] to get more details on infinite dimensional systems and a complete characterization of time-delay systems as systems in a Banach functional space.

3. Functional Differential equation: evolution in a finite Euclidian space or in a functional space.

Since only functional differential equations will be used throughout this thesis, only this one will be deeper explained.

3.1.1 Functional Differential Equations

The most spread representation is by the mean of functional differential equations [Bellman and Cooke, 1963; Gu et al., 2003; K.Hale and Lunel, 1991; Kolmanovskii and Myshkis, 1962; Niculescu, 2001]: several types of time-delay systems can be considered according to the worldly accepted denomination introduced by Kamenskii [Kolmanovskii and Myshkis, 1999]:

1. System with discrete delay acting on the state x, inputs u or/and outputs y, e.g.

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x(t-h_x) + Bu(t) + B_h u(t-h_u) \\ y(t) & = & C_h x(t-h_y) \end{array}$$

   where h_x, h_u and h_y are respectively the delay state, the input delay and the measurement delay.

2. Distributed delay systems where the delay acts on state x or inputs u in a distributed fashion, e.g.

$$\dot{x}(t) = Ax(t) + \int_{-h_x}^{0} A_h(\theta)x(t+\theta)d\theta + Bu(t) + \int_{-h_u}^{0} B_h(\theta)xu(t+\theta)d\theta$$

3. Neutral delay systems where the delay acts on the higher-order state-derivative, e.g.

$$\dot{x}(t) - F\dot{x}(t-h) = Ax(t)$$

The following paragraphs are devoted to a brief emphasis of the difference between these classes of systems through illustrative application examples. These examples are borrowed from [Briat and Verriest, 2008; Kolmanovskii and Myshkis, 1999; Niculescu, 2001; Verriest and Pepe, 2007].

Systems with discrete delays

Systems with discrete delays are systems which remember locally past signals values, at some specific past time instants. An interesting example presented in [Niculescu, 2001] considers an irreversible chemical reaction producing a material B from a material A. Such a reaction is neither instantaneous nor complete and in order to resolve enhance the quantity of reacted products, a classical technique is to use a recycle stream. The streaming process does not take place instantaneously and the whole process (i.e. reaction + streaming) can be modeled by a system of nonlinear delay differential equations with discrete delay:

$$\begin{aligned}\dot{A}(t) &= \frac{q}{V}[\lambda A_0 + (1-\lambda)A(t-\tau) - A(t)] - K_0 e^{-\frac{Q}{T}} A(t) \\ \dot{T}(t) &= \frac{1}{V}[\lambda T_0 + (1-\lambda)T(t-\tau) - T(t)]\frac{\Delta H}{C\rho} - K_0 e^{-\frac{Q}{T}} A(t) - \frac{1}{VC\rho}U(T(t) - T_w)\end{aligned}$$

where $A(t)$ is the concentration of the component A, $T(t)$ is the temperature (A_0, T_0 correspond to these values at initial time $t = 0$) and $\lambda \in [0,1]$ is the recycle coefficient, $(1-\lambda)q$ is the recycle flow rate of the unreacted A and τ is the transport delay. The others terms are constants of the system.

The description of economic behaviors is another application of functional differential equations [Belair and Mackey, 1989; Kolmanovskii and Myshkis, 1999, 1962; Niculescu, 2001]. For instance, the following discrete delay model has been used for describing interactions between consumer memory and price fluctuations on commodity market:

$$\ddot{x}(t) + \frac{1}{R}\dot{x}(t) + \dot{x}(t-\tau) + \frac{Q}{R}x(t) + \frac{1}{R}x(t-\tau) = 0$$

where x denotes the relative variation of the market price of the commodity and Q, R, τ are parameters of the model. In particular, τ is the time that must elapse before a decision to alter production is translated into an actual change in supply. Actually, this model is obtained by differentiating the following dynamical model involving a distributed delay:

$$\dot{x}(t) + \frac{Q}{R}\int_{-\infty}^{0} e^{-\theta/R}x(t+\theta)d\theta + x(t-\tau) = 0$$

Note that this operation cannot be always performed, more details on this procedure can be found for instance in [Verriest, 1999].

Other applications of time-delay systems with discrete delays arise in heat exchanger dynamics, traffic modeling, teleoperation systems, networks such as internet, modeling of rivers, population dynamics... Delays also appear in neural networks, any systems with delayed measurement, system controlled by delayed feedback and in this case, delays are a consequence of technological constraints.

The reader should refer to the following papers/books and references therein to get more details on pointwise delay systems:

Stability analysis: [Bliman, 2001; Chiasson and Loiseau, 2007; Dugard and Verriest, 1998; Fridman and Shaked, 2001; Gouaisbaut and Peaucelle, 2006b; Goubet-Batholoméus et al., 1997; Gu et al., 2003; Han, 2005a, 2008; Han and Gu, 2001; He et al., 2004; Jun and Safonov, 2001; Kao and Rantzer, 2007; K.Hale and Lunel, 1991; Kharitonov and Melchor-Aguila, 2003; Kharitonov and Niculescu, 2003; Kolmanovskii and Myshkis, 1962; Michiels and Niculescu, 2007; Moon et al., 2001; Niculescu, 2001; Park et al., 1998; Richard, 2000; Sipahi and Olgac, 2006; Verriest and Ivanov, 1991, 1994a,b; Xu and Lam, 2005; Zhang et al., 1999, 2001]

Control Design: [Dugard and Verriest, 1998; Ivanescu et al., 2000; Meinsma and Mirkin, 2005; Michiels and Niculescu, 2007; Michiels et al., 2005; Mirkhin, 2003; Mondié and

Michiels, 2003; Niculescu, 2001; Seuret et al., 2009a; Suplin et al., 2006; Verriest, 2000; Verriest et al., 2002; Verriest and Ivanov, 1991, 1994a,b; Verriest and Pepe, 2007; Verriest et al., 2004; Witrant et al., 2005; Wu, 2003; Xie et al., 1992; Xu et al., 2006]

Observers: [Darouach, 2001; Fattouh, 2000; Fattouh and Sename, 2004; Fattouh et al., 1999, 2000a,c; Germani et al., 2001; Picard et al., 1996; Sename, 1997, 2001; Sename et al., 2001]

Distributed delay Systems

Distributed delay systems are systems where the delay does not have a local effect as in pointwise delay systems but acts in a distributed fashion over a whole interval. For instance, consider the following SIR-model [Anderson and May, 1982, 2002; Hethcote, 2002; van den Driessche, 1999; Wickwire, 1977] used in epidemiology [Briat and Verriest, 2008]

$$\begin{aligned} \dot{S}(t) &= -\beta S(t)I(t) \\ \dot{I}(t) &= \beta S(t)I(t) - \beta \int_h^{\infty} \gamma(\tau)S(t-\tau)I(t-\tau)d\tau \\ \dot{R}(t) &= \beta \int_h^{\infty} \gamma(\tau)S(t-\tau)I(t-\tau)d\tau \end{aligned}$$

where S is the number of susceptible people, I the number of infectious people and R the number of recovered people. The distributed delay here taking value over $[h, +\infty]$ is the time spent by infectious people before recovering from the disease. This delay me be different from a person to another but obeys a probability density represented by $\gamma(\tau)$ which tends to 0 at infinity and whose integral over $[h, +\infty]$ equals 1. It is assumed here that once recovered from the disease, people become resistant and therefore remain within the set of recovered people. It can be easily shown that $\dot{S} + \dot{R} + \dot{I} = 0$, showing that the system is Hamiltonian (energy preserving) and hence stable.

Another example of systems governed by distributed delay differential equations are combustion models [Crocco, 1951; Fiagbedzi and Pearson, 1987; Fleifil et al., 1974, 2000; Niculescu, 2001; Zheng and Frank, 2002] involved in propulsion and power-generation. Delay in such models can have destabilizing effects but it has been shown these recent years that this delay can be used in advantageous manner. The following example is taken from [Niculescu, 2001; Niculescu et al., 2000].

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega_i^2 x = c_1 x(t-\tau_c) - c_2 \int_0^{t-\tau_c} x(\xi)d\xi$$

For more details and some results on systems with distributed delays, the readers should refer to [Chiasson and Loiseau, 2007; Fattouh et al., 2000b; Fiagbedzi and Pearson, 1987; Fridman and Shaked, 2001; Gu et al., 1999, 2003; Ivanescu et al., 1999; K.Hale and Lunel, 1991; Kolmanovskii and Richard, 1997; Münz and Allgöwer, 2007; Münz et al., 2008; Niculescu, 2001; Richard, 2000; Tchangani et al., 1997; Verriest, 1995, 1999; Zheng and Frank, 2002] and references therein.

Neutral Delay Systems

Finally, neutral delay systems, arising for instance in the analysis of the coupling between transmission lines and population dynamics, are systems where discrete delays act on the higher order derivative of the dynamical system. (See [Brayton, 1966; K.Hale and Lunel, 1991; Kuang, 1993]. The origin of the term 'neutral' is unclear while the other terms are easy to understand.

An example of dynamical system governed by neutral delay equation is the evolution of forests. The model is based on a refinement of the delay-free logistic (or Pearl-Verhulst equation [Murray, 2002; Pearl, 1930; Verhulst, 1938]) where effects as soil depletion and erosion have been introduced

$$\dot{x}(t) = rx(t)\left[1 - \frac{x(t-\tau) + c\dot{x}(t-\tau)}{K}\right] \tag{3.1}$$

where x is the population, r is the intrinsic growth rate and K the environmental carrying capacity. See Gopalsamy and Zhang [1988]; Pielou [1977]; Verriest and Pepe [2007] for more details.

More information on neutral delay systems can be found in [Bliman, 2002; Brayton, 1966; Fridman, 2001; Gopalsamy and Zhang, 1988; Han, 2002, 2005b; K.Hale and Lunel, 1991; Kolmanovskii and Myshkis, 1962; Niculescu, 2001; Picard et al., 1998; Verriest and Pepe, 2007] and references therein.

3.1.2 Constant Delays vs. Time-Varying Delays and Quenching Phenomenon

In the latter examples of time-delay systems represented in term of a functional differential equations, the delay is assumed to be constant. In some applications (networks, sampled-data control...) the delay is time-varying, making the system non-stationary. At first sight, it may appear as a technical detail but, actually, it leads to a phenomena called *Quenching* (see [Louisell, 1999; Papachristodoulou et al., 2007]). Indeed, as well as for uncertain LTI and LTV/LPV systems for which there is a gap of stability when where the rate of variation of the parameters play a central role (see Section 2.3.1), there is also a stability gap between systems with constant and time-varying delays. Indeed, it is possible to find systems which are stable for constant delay $h \in [h_1, h_2]$ but unstable for time-varying delay belonging to the same interval. In such a phenomenon, the bound on the delay derivative plays an important role [Kharitonov and Niculescu, 2003; Papachristodoulou et al., 2007] similarly to as the rate of variation of the parameters in LTV/LPV systems.

In some systems, the delay may be a known function of time or depend on some parameters. Moreover, methods to estimate the delay in real time are currently developed [Belkoura et al., 2007, 2008; Drakunov et al., 2006; Veysset et al., 2006]. In these cases, one can imagine to use this information to study stability and design specific control laws.

It is also possible to define systems in which the delay is a function of the state. This makes the stability analysis of the system extremely harder and only very few (uncomplete) results have been provided on that topic. See for instance [Bartha, 2001; Feldstein et al., 2005; Louihi and Hbid, 2007; Luzianina et al., 2000, 2001; Verriest, 2002; Walther, 2003] and references therein.

3.2 Stability Analysis of Time-Delay Systems

The stability analysis of time-delay systems is a very studied problem and has led to lots of approaches which can be classified in two main framework: the frequency-domain and time-domain analysis [Gu et al., 2003; Niculescu, 2001]. While the first one deals with characteristic quasipolynomial (the generalization of characteristic polynomial to TDS [Gu et al., 2003]) of the system, the second one considers directly the state-space domain and matrices. Before entering in more details, some preliminary definitions are necessary.

Definition 3.2.1 (Delay-Independent Stability) *If a time-delay system is stable for any delay values belonging to $\mathbb{R}_+$, the system is said to be delay-independent stable.*

The term *delay-independent stable* has been introduced for the first time in [Kamen et al., 1985] and became commonly used in the time-delay community.

Example 3.2.2 *A delay-independent stable time-delay system with constant time-delay is given by*

$$\dot{x}(t) = \begin{bmatrix} -5 & 1 \\ 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x(t-h) \tag{3.2}$$

It seems obvious that if a system is delay-independent stable, then it is necessary stable for $h = 0$ and $h \to +\infty$ which means that A and $A+A_h$ are necessary Hurwitz (all the eigenvalues lie in the open left-half plane). On the second hand, for any value of h from 0 to $+\infty$), the system must be stable too. It is shown in [Gu et al., 2003] that a supplementary sufficient condition is given by

$$\bar{\rho}[(j\omega - A)^{-1}A_h] < 1, \qquad \forall \omega \in \mathbb{R}$$

where $\bar{\rho}(\cdot)$ denotes the spectral radius (i.e. $\max_i |\lambda_i(\cdot)|$). By verifying these conditions we find

$$\begin{aligned} \lambda(A) &= \{-5, -5\} \\ \lambda(A + A_h) &= \left\{ \frac{-13 \pm \sqrt{3}}{2} \right\} \\ \bar{\rho}[(j\omega - A)^{-1}A_h] &\sim 0.4739 < 1 \end{aligned}$$

The system is confirmed to be delay-independent stable.

Definition 3.2.3 (Delay-Dependent Stability) *If a time-delay system is stable for all delay values belonging to a compact subset D of $\mathbb{R}_+$ then the system is said to be delay-dependent stable.*

Example 3.2.4 *A well-known system being delay-dependent stable [Gouaisbaut and Peaucelle, 2006b] is given by*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h) \tag{3.3}$$

and is stable for any constant delay belonging to $[0, 6.17]$. To see this note that $A + A_h$ is Hurwitz and hence the system is stable for zero delay. On the other hand, $A - A_h$ is not Hurwitz (has eigenvalues $\{-1, 0.1\}$) and shows that for some values of the delay the system has positive eigenvalues. This is explained further in [Gu et al., 2003] where quasipolynomials-based methods are introduced.

When the lower bound of the interval of delay is 0, the term 'delay-margin' is often referred to the upper bound of the interval. It is possible to find systems for which the lower bound of the interval is non zero and in this case these systems are referred to as *systems with non-small delay* or *interval-delay systems*.

Example 3.2.5

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-h)$$

This system is not stable for $h = 0$ since the matrix $A + A_h$ is not Hurwitz. Indeed, in [Gouaisbaut and Peaucelle, 2006a; Gu et al., 2003], it is shown that the system is stable for all constant delay in the interval $[0.10016826, 1.7178]$.

Other systems may exhibit (almost) periodicity in the intervals of stability: there exists a (finite or infinite) countable sequence of disjoint intervals for which the system is stable. Such a behavior most often occurs in systems with several delays. Moreover, in the case of multiple delay systems the stability map (the set of delays for which the system is stable) can be very complicated as presented for instance in [Knospe and Roozbehani, 2006; Sipahi and Olgac, 2005, 2006]. The following example is borrowed from [Knospe and Roozbehani, 2006].

Example 3.2.6 *Let us consider the system with 2 delays*

$$\dot{x}(t) = \begin{bmatrix} -3.0881 & 2.6698 \\ -9.7383 & 2.8318 \end{bmatrix} x(t) + \begin{bmatrix} 0.5645 & 0.0178 \\ 1.2597 & 0.8020 \end{bmatrix} x(t-h_1) + \begin{bmatrix} 0.4176 & 0.0144 \\ 0.9432 & 0.5976 \end{bmatrix} x(t-h_2) \tag{3.4}$$

The stability map for this system is depicted on Figure 3.1. On this figure, it is possible to see that the there are notches which show that the stability set is not as regular as for system with single delay. The boxes are approximations of the stability set obtained using method of [Knospe and Roozbehani, 2006].

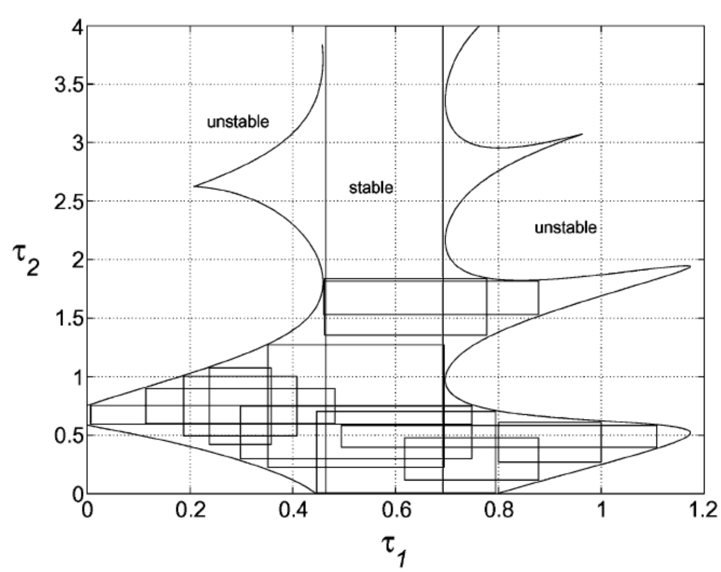

Figure 3.1: Stability regions of system (3.4) w.r.t. to delay values (source: [Knospe and Roozbehani, 2006])

In the case of time-varying delays, the stability may depend or not on the rate of variation of the delay (the derivative of the delay) and in these cases, a similar vocabulary has been introduced.

Definition 3.2.7 (Rate-Dependent Stability) *For a stable time-delay system with time-varying delays, if the stability depends on the rate of variation, then the system is said to be rate-dependent stable.*

Definition 3.2.8 (Rate-Independent Stability) *For a stable time-delay system with time-varying delays, if the stability does not depend on the rate of variation, then the system is said to be rate-independent stable.*

In most of the cases the bound on the rate of variation of the delay is closely related to the delay-margin, the greater the absolute value of the rate is, the lower is the delay-margin. Papachristodoulou et al. [2007] have shown that system

$$\dot{x}(t) = -x(t - h(t)) \tag{3.5}$$

is unstable for a delay-rate bound greater than approximately 0.86 even though for a constant delay, the system is stable for $h < \pi/2$.

In [Kharitonov and Niculescu, 2003], analytical methods are provided to deal with uncertain delays around a known constant value. With such an approach it is possible to quantify and give bounds on the variation of the delay. For instance, the relevant example considered in Kharitonov and Niculescu [2003]

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t))$$

is stable for a delay equal to 1. Using the method of [Kharitonov and Niculescu, 2003] where the time-varying delay is written as

$$h(t) = h_0 + \eta(t) \qquad \dot{h}(t) = \dot{\eta}(t)$$

it is shown that the stability is preserved for every $|\eta(t)| \leq \eta_0$ and $|\dot{\eta}(t)| \leq \dot{\eta}_0$ such that

$$\eta_0 < \frac{1}{640}\mu_0 \qquad \dot{\eta}_0 < 1 - 8\mu_0$$

with $\mu_0 \in (0, 1/40)$. From these inequalities we can see that the larger $\dot{\eta}_0$ is, the smaller η_0 must be to preserve stability. This illustrates the effect of a time-varying delay on the stability of time-delay systems.

3.2.1 Time-Domain Stability Analysis

Several frequency domain approaches have been provided in literature and lead for more or less difficult stability analysis techniques for time-delay systems with constant delay. These methods cannot be applied (except for very special cases) to systems with time-varying delays or even to time-varying systems, uncertain systems with time-varying uncertainties and nonlinear systems (except locally). The advantage of time-domain approaches compared to frequency-domain techniques is their wide applicability to any type of systems. Many approaches have been developed these past years and amongst them, the extension of Lyapunov theory and Lyapunov functions play a central role.

This section is devoted to a presentation of many time-domain approaches. On the first hand, the extensions of Lyapunov theory through the celebrated Lyapunov-Krasovskii and Lyapunov-Razumikhin theorems are introduced. On the second hand, an historical review is developed in which the use of model transformations is introduced and justified. The concept of additional dynamics is then shown as a consequence of model transformations and as a limitation of some approaches. Still in the context of model transformations, the problem of the bounding of cross-terms is explained and solved in different manners exposed chronologically. To conclude on the part on extensions of Lyapunov's theory, recent results does not involving model transformations are provided.

Finally more 'exotic' stability tests not directly based on extension of Lyapunov's theory but relying on well-posedness theory (Section 2.3.4.4), integral quadratic constraint theory (IQC) (Section 2.3.4.6) or even small-gain theorems (Sections 2.3.4.2 and 2.3.4.3) are introduced as an opening to new promising methods.

Remark 3.2.9 *All the definitions of stability of finite-dimensional systems can be generalized to time-delay systems by introducing the continuous norm $||\cdot||_c$ defined by*

$$||\phi||_c := \max_{a\leq\theta\leq b} ||\phi(\theta)||_2$$

where $\phi \in \mathcal{C}_a([a,b],\mathbb{R}^n)$ which is the set of absolutely continuous functions mapping $[a,b]$ to $\mathbb{R}^n$.

3.2.1.1 On the extension of Lyapunov Theory

Throughout this part, we will focus on the stability analysis of the general single delayed system

$$\begin{array}{rcl} \dot{x}(t) & = & f(x_t,t) \\ x_{t_0} & = & \phi \end{array} \tag{3.6}$$

where $x_t(\theta) = x(t+\theta)$ and $\phi \in \mathcal{C}([-h,\ 0],\mathbb{R})$ is the functional initial condition. We also assume that $x(t) = 0$ identically is a solution to (3.6), that will be referred to as the *trivial solution*.

As in the study of systems without delay, the Lyapunov method is an effective approach. For a system without delay, it consists in the construction of a Lyapunov function $V(t,x(t))$, which is some sense is a potential measure quantifying the deviation of the state $x(t)$ from the trivial solution 0. Since, for a delay-free system, $x(t)$ is needed to specify the system future evolution beyond t, and since in a time-delay system the 'state' at time required for the same purpose is the value of $x(\theta)$ in the interval $\theta \in [t-h,t]$ (i.e. x_t), it is natural to expect that for a time-delay system, the corresponding Lyapunov function be a functional $V(t,x_t)$ depending on x_t, which also should measure the deviation of x_t from the trivial solution 0. Such a functional is known as the Lyapunov-Krasovskii functional.

More specifically, let $V(t,\phi)$ be differentiable, and let $x_t(\tau,\phi)$ be a solution of (3.6) at time t with the initial condition $x_\tau = \phi$. We may calculate the derivative of $V(t,x_t)$ with respect to

t and evaluate it at $t = \tau$. This gives rise to

$$\begin{aligned}\dot{V}(\tau,\phi) &= \left.\frac{d}{dt}V(t,x_t)\right|_{t=\tau,\ x_t=\phi} \\ &= \limsup_{\Delta t\to 0}\frac{1}{\Delta t}[V(\tau+\Delta t, x_{t+\Delta t}(\tau,\phi)) - V(\tau,\phi)]\end{aligned}$$

Intuitively, a nonpositive $\dot{V}$ indicates that x_t does not grow with t, which in turn means that the system under consideration is stable in light of remark 3.2.9. The more precise statement of this observation is the following theorem.

Theorem 3.2.10 (Lyapunov-Krasovskii Stability Theorem)
Suppose $f : \mathbb{R}\times\mathcal{C}([-h,\ 0],\mathbb{R}^n) \to \mathbb{R}^n$ in (3.6) maps $\mathbb{R}\times$(bounded sets of $\mathcal{C}([-h,\ 0],\mathbb{R}^n)$) into bounded sets of $\mathbb{R}^n$, and u, v, $w : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exists a continuous differentiable functional $V : \mathbb{R}\times\mathcal{C} \to \mathbb{R}$ such that

$$u(||\phi(0)|| \leq V(t,\phi) \leq v(||\phi||_c)$$

and

$$\dot{V}(t,\phi) \leq -w(||\phi(0)||)$$

then the trivial solution of (3.6) is uniformly stable. If $w(s) > 0$ for $s > 0$, then it is uniformly asymptotically stable. If, in addition, $\lim_{s\to+\infty} u(s) = +\infty$, then it is globally uniformly asymptotically stable.

Complete Lyapunov-Krasovskii Functional

In the special case of linear time-delay systems, it is possible to give a generic 'complete' Lyapunov-Krasovskii functional [Fridman, 2006a; Gu et al., 2003; Papachristodoulou et al., 2007]. The term *complete* means that, if computed exactly, it provides necessary and sufficient conditions to the delay-dependent stability for such systems. Let us consider the following LTI time-delay system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_h x(t-h) \\ x(\theta) &= \phi(\theta), \quad \theta\in[-h,0]\end{aligned} \tag{3.7}$$

where $x \in \S \subset \mathbb{R}^n$, $\phi \in \mathcal{C}_{[-h,\ 0]}$ and $h \in \mathbb{R}_+$ are respectively the system state, the functional initial condition and the constant time-delay.

Theorem 3.2.11 *The system (3.7) is delay-dependent asymptotically stable for a constant time-delay h if and only if there exist a constant matrix $P = P^T \in \mathbb{R}^{n\times n}$, a scalar $\varepsilon > 0$ and continuously differentiable matrix functions*

$$\begin{aligned}Q(\xi) &: [-h,0] \to \mathbb{R}^{n\times n} \\ R(\xi,\eta) &= R(\eta,\xi)^T, \quad \text{with } R(\xi,\eta) : [-h,0]^2 \to \mathbb{R}^{n\times n} \\ S(\xi) &= S(\xi)^T : [-h,0] \to \mathbb{R}^{n\times n}\end{aligned}$$

such that

$$\begin{aligned}V(x_t) &= x(t)^T P x(t) + 2x(t)^T\int_{-r}^{0} Q(\xi)x(t+\xi)d\xi + \int_{-r}^{0} x(t+\xi)^T S(\xi)x(t+\xi)d\xi \\ &+ \int_{-r}^{0}\left[\int_{-r}^{0} x(t+\xi)^T R(\xi,\eta)x(t+\eta)d\eta\right]d\xi \geq \varepsilon||x(t)||^2\end{aligned}$$

is a Lyapunov-Krasovskii functional. Moreover its time derivative satisfies

$$\begin{aligned}
\dot{V}(x_t) &= x(t)^T[PA + A^T P + Q(0) + Q^T(0) + S(0)]x(t) - x(t-h)^T S(-h)x(t-h) \\
&- \int_{-h}^{0} x(t+\xi)^T \dot{S}(\xi)x(t+\xi)d\xi + 2x(t)^T[PA_h - Q(-h)]x(t-h) \\
&- \int_{-h}^{0} d\xi \int_{-h}^{0} x(t+\xi)^T \left[\frac{\partial}{\partial \xi}R(\xi,\eta) + \frac{\partial}{\partial \eta}R(\xi,\eta)\right] x(t+\eta)d\eta \\
&+2x(t)^T \int_{-h}^{0} [A^T Q(\xi) - \dot{Q}(\xi) + R(0,\xi)]x(t+\xi)d\xi \\
&+2x(t)^T \int_{-h}^{0} [A_h^T Q(\xi) - R(-h,\xi)]x(t+\xi)d\xi \leq -\varepsilon||x(t)||^2
\end{aligned}$$

In practice, it is numerically difficult to check the existence of such a quadratic functional. Indeed, it describes an infinite dimensional problem since decision variables are functions (i.e. Q, R, S). To overcome this problem a discretization scheme may be adopted [Fridman, 2006b; Gu et al., 2003; Han and Gu, 2001] or a Sum-of-Squares based relaxation [Papachristodoulou and Prajna, 2002; Papachristodoulou et al., 2005, 2007; Prajna et al., 2004]. Section 4.6.1 will be devoted to a particular discretized Lyapunov-Krasovskii functional.

Note that the Lyapunov-Krasovskii functional requires the state variable $x(t)$ in the interval $[-h, 0]$ and involves the manipulation of functionals, this consequently makes the application of the Lyapunov-Krasovskii theorem rather difficult. This difficulty may sometimes be circumvented using the Razumikhin theorem, an alternative result invoking only functions rather than functionals. The key idea behind the Razumikhin theorem also focuses on a function $V(x)$ representative of the size of $x(t)$. For such a function,

$$\bar{V}(x_t) = \max_{\theta\in[-h,0]} V(x(t+\theta))$$

serves to measure the size of x_t. If $V(x(t)) < \bar{V}(x_t)$, then $\dot{V}(x) > 0$ does not make $\bar{V}(x_t)$ grow. Indeed, for $\bar{V}(x_t)$ to not grow, it is only necessary that $\dot{V}(x(t))$ is not positive whenever $V(x(t)) = \bar{V}(x_t)$. The precise statement is as follows.

Theorem 3.2.12 (Lyapunov-Razumikhin Stability Theorem)
Suppose $f : \mathbb{R} \times \mathcal{C}([-h,\ 0], \mathbb{R}^n) \to \mathbb{R}^n$ *in (3.6) takes* $\mathbb{R} \times$ *(bounded sets of* $\mathcal{C}([-h,\ 0], \mathbb{R}^n)$*) into bounded sets of* $\mathbb{R}^n$*, and* $u,\ v,\ w : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ *are continuous nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*, and* $u(0) = v(0) = 0$*,* v *strictly increasing.*

If there exists a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$u(||x||) \leq V(t,x) \leq v(||x||), \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n$$

and the derivative of V *along the solution* $x(t)$ *of (3.6) satisfies*

$$\dot{V}(t, x(t)) \leq -w(||x(t)||) \text{ whenever } V(t+\theta, x(t+\theta)) \leq V(t, x(t)) \tag{3.8}$$

for $\theta \in [-h, 0]$*, then the system (3.6) is uniformly stable.*

Moreover, if $w(s) > 0$ for $s > 0$, and there exists a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that condition (3.8) is strengthened to

$$\dot{V}(t, x(t)) \leq -w(||x(t)||) \quad \text{if} \quad V(t+\theta, x(t+\theta)) \leq p(V(t, x(t)))$$

for $\theta \in [-h, 0]$, then the system (3.6) is uniformly asymptotically stable. If in addition $\lim_{s\to+\infty} u(s) = +\infty$, then the system (3.6) is globally uniformly asymptotically stable.

Lyapunov-Krasovskii and Lyapunov-Razumikhin are the most famous results concerning stability of time-delay systems in the time-domain. However, there exists several others results, see for instance [Barnea, 1969]. In Sections 3.2.1.4 and 3.2.1.5 different stability tests will be derived using both theorems.

3.2.1.2 About model transformations

Model-transformations have been introduced early in the stability analysis of time-delay systems. They allow to turn a time-delay system into a new system, which is referred to as a *comparison system*. Finally, the stability of the original system is determined through the stability analysis of the comparison model. They are generally used to remove annoying terms in the equations or to turn the expression of the system in a more convenient form. Comparison systems may be of different types, (uncertain) finite dimensional linear systems (see [Gu et al., 2003; Knospe and Roozbehani, 2006, 2003; Roozbehani and Knospe, 2005; Zhang et al., 1999, 2001]), time-delay systems (see [Fridman and Shaked, 2001; Gu et al., 2003]). In our papers [Briat et al., 2007a, 2008a], a time-delay system is turned into an uncertain finite dimensional LPV systems from which a new control strategy is developed; this will developed in Section 6.1.7.

Some model transformations are introduced here, although the list is non exhaustive due to the important work that has been done in that field, it will be focused on two initial first-order model transformations [Goubet-Batholoméus et al., 1997; Kolmanovskii and Richard, 1997, 1999; Kolmanovskii et al., 1998; Li and de Souza, 1996; Niculescu, 1999; Niculescu and Chen, 1999; Su, 1994; Su and Huang, 1992] and a recent one [Fridman, 2001; Fridman and Shaked, 2001] which will be detailed in the following. The motivation for which only three model transformations have been chosen to be presented, comes from the fact that both first ones are simple but may induce some conservatism. It will be shown in Section 3.2.1.3 than the third one is less conservative than the others despite of its apparent complexity.

First of all, let us consider the linear time-delay system

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + A_h x(t-h) \\ x_0 &=& \phi \end{array} \tag{3.9}$$

where A, A_h are given $n \times n$ real matrices and ϕ is the functional initial condition.

The three model transformations to be analyzed are given below:

Newton-Leibniz formula : The Newton-Leibniz formula is the oldest model transformation which has been introduced [Goubet-Batholoméus et al., 1997; Kolmanovskii and

Richard, 1997, 1999; Li and de Souza, 1996; Su, 1994; Su and Huang, 1992] and is still in use for different purposes [Gu et al., 2003; He et al., 2004; Niculescu, 2001]:

$$x(t-h) = x(t) - \int_{t-h}^{t} \dot{x}(\theta)d\theta$$

It allows to turn the time-delay system with discrete delay (3.9) into the following system with distributed delay:

$$\dot{x}(t) = (A + A_h)x(t) - A_h \int_{t-h}^{t} [Ax(s) + A_h x(s-h)]ds$$

Parametrized Leibniz-Newton formula : This model transformation [Kolmanovskii et al., 1998; Niculescu, 1999; Niculescu and Chen, 1999] improves the result obtained from the Leibniz-Newton formula by introducing a free parameter C to be chosen adequately:

$$Cx(t-h) = Cx(t) - C\int_{t-h}^{t} \dot{x}(\theta)d\theta$$

where $C \in \mathbb{R}^{n\times n}$ is a free matrix. It allows to turn the time-delay system with discrete delay into a system with distributed delay:

$$\dot{x}(t) = (A + C)x(t) + (A_h - C)x(t-h) - C\int_{t-h}^{t} [Ax(s) + A_h x(s-h)]ds$$

Note for particular values for C, previous system expressions are recovered:

- $C = 0$: the original system is recovered
- $C = A_h$: the system obtained from Leibniz-Newton formula is recovered.

This model transformation [Fridman, 2001; Fridman and Shaked, 2001] allows to turn a time-delay system into a singular system with distributed delay

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{t-h}^{t} \begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} ds$$

where $y(t) = \dot{x}(t)$. By substituting the expression of $y(t)$ defined by the second row in the first row, the original system (3.9) is retrieved.

Many other model transformations have been provided in the literature and the readers should refer, for instance, to [Gouaisbaut and Peaucelle, 2006a,b, 2007; Goubet-Batholoméus et al., 1997; Gu et al., 2003; Kao and Rantzer, 2007; Knospe and Roozbehani, 2006, 2003; Kolmanovskii and Myshkis, 1962; Kolmanovskii and Richard, 1997, 1998; Kolmanovskii et al., 1998; Li and de Souza, 1996; Niculescu, 2001, 1997, 1999; Niculescu and Chen, 1999; Roozbehani and Knospe, 2005; Su, 1994; Su and Huang, 1992; Zhang et al., 1999]. Many of model transformations introduced in the latter references have not been introduced in same spirit as the model-transformations detailed above, but in view of turning the system into another form in order to analyze it using different tools. This will be detailed in Sections 3.2.1.4 to 3.2.1.8.

3.2.1.3 Additional Dynamics

Stability tests obtained from comparison systems are, in most of the cases, outer approximations of the original system only. This means that if the comparison model is stable then the original system is stable too but the converse does not necessary hold. The following development is borrowed from [Gu and Niculescu, 1999, 2000; Gu et al., 2003; Verriest, 1999] and some precisions on additional dynamics can also be found in [Kharitonov and Melchor-Aguila, 2003] and references therein.

For instance the simpler model transformation (i.e. the Leibniz-Newton formula) leads to the comparison system

$$\dot{z}(t) = (A + A_h)z(t) - A_h \int_{t-h}^{t} [Az(t) + A_h z(s-h)]ds$$

where the instantaneous state is set to z to emphasize the difference between the original and comparison model. The characteristic quasipolynomial of the latter comparison system is then given by

$$\Delta_c(s) := \det\left[s^2 I - (A + A_h)s + A_h A(1 - e^{-sh}) + A_h^2 e^{-sh}(1 - e^{-sh})\right]$$

while the quasipolynomial of the original system is

$$\Delta_o := \det(sI - A - A_h e^{-sh})$$

Therefore, as the quasipolynomial of both systems are different, it seems evident that the qualitative behavior of both systems might be different. To see this, note that the quasipolynomial of the transformed system can be factorized as

$$\Delta_c(s) = \Delta_o(s)\Delta_a(s)$$

where

$$\Delta_a(s) := \det\left(I - \frac{1 - e^{-sh}}{s} A_h\right)$$

Hence the quasipolynomial of the transformed system exhibits supplementary zeros which are responsible of *additional dynamics*. It is clear that if the real part of these zeros are nonnegative, the comparison system is unstable, even if the original system is stable (zeros of $\Delta_o(s)$ have negative real part). Some results on this stability analysis are presented below. Note that

$$\Delta_a(s) = \prod_{i=1}^{n} \left(1 - \lambda_i \frac{1 - e^{-sh}}{s}\right)$$

where λ_i is the i^{th} eigenvalue of matrix A_h and let $s = s_{ik}$, $k = 1, 2, 3, \ldots$ be all the solutions of the equation

$$1 - \lambda_i \frac{1 - e^{-sh}}{s} = 0$$

Then s_{ik}, $i = 1, \ldots, n$, $k = 1, 2, 3, \ldots$ are all the additional poles of the comparison system.

Descriptor Model Transformation : Proposition 3.2.13 *For any given A_h, all the additional poles satisfy*

$$\lim_{h\to 0^+} \Re(s_{ik}) = -\infty$$

As a result, all the additional poles have negative real part for sufficiently small h. As h increases, some of the additional poles may cross the imaginary axis. It turns out that the exact crossing value can be analytically calculated.

Theorem 3.2.14 *Corresponding to an eigenvalue λ_i of A_h, $\Im(\lambda_i) \neq 0$, there is an additional pole $s_{i,k}$ on the imaginary axis if and only if the time delay satisfies*

$$h = h_{i,k} = \frac{k\pi + \arg(\lambda_i)}{\Im(\lambda_i)} > 0, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Corresponding to a positive real eigenvalue λ_i of A_h, there is an additional pole on the imaginary axis if and only if

$$h = \frac{1}{\lambda_i}$$

No additional poles corresponding to a negative real eigenvalue λ_i of A_h will reach the imaginary axis for any finite delay.

Therefore, if all the eigenvalues of the matrix A_h are real and negative, then the original and comparison system are equivalent for this particular model transformation. On the other hand, if the matrix A_h does not satisfy this strong assumption, it seems necessary to use another model transformation. Here comes the parametrized model transformation for which it can be shown that the additional poles are solutions of the equation

$$\det\left(I - C\frac{1 - e^{-sh}}{s}\right) = 0$$

This shows that for a judicious choice of the free matrix C, no unstable dynamics are generated which emphasizes the interest of the parametrized model transformation.

Finally, let us consider now the descriptor model transformation, the comparison model is governed by

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{t-h}^{t} \begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix}\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} ds$$

The corresponding characteristic polynomial is

$$\begin{aligned} \Delta_{cd}(s) &:= \det\left(\begin{bmatrix} sI & -I \\ -(A + A_h) & I + A_h\dfrac{1 - e^{-sh}}{s} \end{bmatrix}\right) \\ &:= \det\left(sI - A - A_h e^{-sh}\right) \end{aligned}$$

and is identical to the quasipolynomial of the original system (using the determinant formula (see Appendix A.1)). This is the great advantage of this model transformation. However the system is changed in a singular system with distributed delay which may introduce some difficulties in the stability analysis.

3.2.1.4 Stability Analysis: Lyapunov-Razumikhin Functions

This section is devoted to simple stability tests using Lyapunov-Razumikhin theorem and both delay-dependent and delay-independent tests are provided. Let us consider here a general linear time-delay system of the form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h(t-h) \\ x(t+\theta) & = & \phi(\theta), \quad \theta \in [-\bar{h}, 0] \\ h & \in & [0, \bar{h}] \end{array} \tag{3.10}$$

Delay-independent stability test via Lyapunov-Razumikhin theorem

A simple test on delay-independent stability using the quadratic Lyapunov-Razumikhin function

$$V(x(t)) = x(t)^T P x(t)$$

is provided here. The time-derivative of V along the trajectories solutions of system (3.10) is given by

$$\dot{V}(x(t)) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PA_h \\ A_h^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

Applying the Lyapunov-Razumikhin Theorem 3.2.12, $\dot{V}(x(t))$ must be negative whenever $V(x(t+\theta)) < pV(x(t))$ for some $p > 1$ and for all $\theta \in [-h, 0]$. Since the latter inequality holds for all $\theta \in [-h, 0]$ then we have $V(x(t-h)) < pV(x(t))$ and by application of the $\mathcal{S}$-procedure (see [Boyd et al., 1994] or appendix D.10), we get

$$\begin{bmatrix} A^T P + PA + \tau p P & PA_h \\ A_h^T P & -\tau P \end{bmatrix} \prec 0 \tag{3.11}$$

with $\tau > 0$. Finally let $p = 1 + \delta$, for a small $\delta > 0$, we get the following result

Theorem 3.2.15 *System (3.10) is asymptotically stable independent of delay if there exists $P = P^T \succ 0$ and a scalar $\tau > 0$ such that*

$$\begin{bmatrix} A^T P + PA + \tau P & PA_h \\ A_h^T P & -\tau P \end{bmatrix} \prec 0 \tag{3.12}$$

Note that the feasibility of matrix inequality (3.12) implies the feasibility of matrix inequality (3.11).

It is clear that the latter inequality provides a delay-independent stability test since the matrix inequality does not depend on the delay. Moreover, it is worth noting, that (3.12) is not a LMI due to bilinear term τP but fall into the framework of generalized eigenvalues problem [Boyd et al., 1994; Gu et al., 2003; Nesterov and Nemirovskii, 1994]. Nevertheless, the problem is quasi-convex since if τ is fixed, then (3.12) becomes a LMI. This means that a suitable value for τ can be found using an iterative line search.

Delay-dependent stability test via Lyapunov-Razumikhin theorem

We give an example of delay-dependent result obtained from the application of the Lyapunov-Razumikhin theorem 3.2.12. The proof can be found in [Gu et al., 2003] and is omitted since it requires preliminary results on stability of distributed delay which are not of interest. However, it is important to say that it is based on the Leibniz-Newton model transformation.

Theorem 3.2.16 *System (3.10) is delay-dependent asymptotically stable is there exists $P = P^T \succ 0$ and scalars $\alpha, \alpha_0, \alpha_1 > 0$ such that*

$$\begin{bmatrix} M & P(\alpha I - A_h)A & P(\alpha I - A_h)A_h \\ \star & -\alpha_0 P - \alpha\bar{h}A_0^T P A_0 & -\alpha\bar{h}A^T P A_h \\ \star & \star & -\alpha_1 P - \alpha\bar{h}A_h^T P A_h \end{bmatrix} \prec 0$$

holds with $M = \frac{1}{\bar{h}}\left[P(A + A_h) + (A + A_h)^T P\right] + (\alpha_0 + \alpha_1)P$

A discussion on the choice of scalars $\alpha, \alpha_i,\ i = 0, 1$ is provided in [Gu et al., 2003]. As previously, the computation of $P, \alpha, \alpha_i, i = 0, 1$ is not an easy task since the resulting condition is not a LMI. The problem is quasi-convex and an iterative procedure should be performed in order to find suitable values for $\alpha, \alpha_i,\ i = 0, 1$. However, this iterative procedure is more difficult than in the delay-independent case since the search has to be performed over a three-dimensional space (instead of a one-dimensional), which is more involved from a algorithmic and computational point of view.

3.2.1.5 Stability Analysis: Lyapunov-Krasovskii Functionals

Despite of the simplicity of Lyapunov-Razumikhin functions, they generally lead to nonlinear matrix inequalities and to conservative results due to the use of non-equivalent model transformations. The use of Lyapunov-Krasovskii functionals, even if historically were used with identical model-transformations, have led to more and more accurate LMI results by applying either more precise bounding techniques of cross-terms [Park, 1999; Park et al., 1998], more exact model transformations [Fridman, 2001; Fridman and Shaked, 2001] or also other methods without any model transformations [Briat et al., 2009; Gouaisbaut and Peaucelle, 2006b; Han, 2005a; Xu and Lam, 2007; Xu et al., 2006].

This section is devoted to the introduction (in a chronological order) of different results on delay-independent and delay-dependent stability based on Lyapunov-Krasovskii Theorem 3.2.10. First, a simple delay-dependent stability test will be provided and then a delay-dependent stability test will be developed. The delay-dependent stability test is based on the Leibniz-Newton model transformation and induces cross terms in the equations. These terms involving products of signals at time t and the integral of same signals over the $[t - h, t]$ are of great difficulty. Different bounds have been provided in the literature to avoid/overcome these difficulties and are of interest since they had led to more and more accurate results. Finally, several other stability tests not based on model transformation and avoiding them are introduced.

Delay-Independent stability test via Lyapunov-Krasovskii theorem

Consider the Lyapunov-Krasovskii functional given by

$$V(x_t) = x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta \tag{3.13}$$

where $P, Q \in \mathbb{S}_{++}^n$ are constant decision matrices. Computing the derivative of the Lyapunov-

Krasovskii functional $V(x_t)$ along the trajectories solutions of system (3.9) yields

$$\begin{aligned}\dot{V}(x_t) &= \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) + x(t)^T Qx(t) - x(t-h)^T Qx(t-h) \\ &= [Ax(t) + A_h x(t-h)]^T Px(t) + x(t)^T P[Ax(t) + A_h x(t-h)] \\ &\quad + x(t)^T Qx(t) - x(t-h)^T Qx(t-h) \\ &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^T P + PA + Q & PA_h \\ A_h^T P & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}\end{aligned}$$

By enforcing the latter quadratic form to be negative definite we get

$$\begin{bmatrix} A^T P + PA + Q & PA_h \\ A_h^T P & -Q \end{bmatrix} \prec 0$$

Moreover by continuity of the eigenvalues this is equivalent to say that

$$\begin{bmatrix} A^T P + PA + Q + \varepsilon I & PA_h \\ A_h^T P & -Q \end{bmatrix} \prec 0$$

for some $\varepsilon > 0$. This implies that $\dot{V}(t) \leq -\varepsilon ||x(t)||^2$ and the Lyapunov-Krasovskii Theorem is satisfied. We then obtain the following result which was first proven in [Verriest and Ivanov, 1991, 1994a]:

Theorem 3.2.17 *System (3.10) is asymptotically stable for any delay if there exist matrices $P = P^T \succ 0$ and $Q = Q^T \succ 0$ such that*

$$\begin{bmatrix} A^T P + PA + Q & PA_h \\ \star & -Q \end{bmatrix} \prec 0$$

holds.

It is worth noting that the structure reminds the matrix inequality obtained from the application of the Lyapunov-Razumikhin Theorem, but is LMI in the current case. Moreover, this test is less conservative than the delay-independent Lyapunov-Razumikhin test since matrix Q is free and independent of P in the Lyapunov-Krasovskii test while the matrix is τP is the Lyapunov-Razumikhin test and strongly correlated to P. As a conclusion the Lyapunov-Krasovskii based test includes the Lyapunov-Razumikhin test as a particular case $Q = \tau P$.

Delay-Dependent stability test via Lyapunov-Krasovskii theorem

Many studies have dealt with the problem of determination of the delay-margin for time-delay systems. The aim of this paragraph is to provided an evolutive point of view of methods used to determine the delay-margin of a time-delay through the use of the Lyapunov-Krasovskii theorem 3.2.10. In this objective, model transformations have played a central role (and sometimes still play an important role in certain approaches). Despite of their conservatism they have facilitated the derivation of delay-dependent stability conditions. However, additional dynamics are not the only difficulties they induce, they also generate cross-terms in the the mathematical proofs of the stability conditions. While additional dynamics are a 'hidden problem' which is not viewed directly in the mathematical proof of stability tests, cross-terms are mathematical difficulties that have needed to be overcome or avoided.

Let us consider the following Lyapunov-Krasovskii functional

$$\begin{array}{rcl} V(x_t,\dot{x}_t) & = & V_1(x_t)+V_2(x_t)+V_3(x_t,\dot{x}_t) \\ V_1(x_t) & = & x(t)^TPx(t) \\ V_2(x_t) & = & \int_{t-h}^{t} x(\theta)^TQx(\theta)d\theta \\ V_3(x_t,\dot{x}_t) & = & \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^TZ\dot{x}(\eta)d\eta d\theta \end{array}$$

with $P=P^T, Q=Q^T, Z=Z^T\succ 0$. According to Leibniz-Newton transformation, system (3.9) is turned into

$$\dot{x}(t)=(A+A_h)x(t)-\int_{t-h}^{t} x(\theta)d\theta$$

Computing the derivative of V along the trajectories solutions of the latter system yields

$$\begin{array}{rcl} \dot{V}_1(x_t) & = & \dot{x}(t)^TPx(t)+x(t)^TP\dot{x}(t) \\ & = & x(t)^T[(A+Ah)^TP+P(A+A_h)]x(t)\underbrace{-2x(t)^TPA_h\int_{t-h}^{t}\dot{x}(\theta)d\theta}_{\text{cross term}} \\ & = & x(t)^T[(A+Ah)^TP+P(A+A_h)]x(t) \\ & & \underbrace{-2x(t)^TPA_h\int_{t-h}^{t}[Ax(\theta)+A_hx(\theta-h)]d\theta}_{\text{cross term}} \\ \dot{V}_2(x_t) & = & x(t)^TQx(t)-x(t-h)^TQx(t-h) \\ \dot{V}_3(x_t,\dot{x}_t) & = & h\dot{x}(t)^TZ\dot{x}(t)-\int_{t-h}^{t}\dot{x}(\theta)^TZ\dot{x}(\theta)d\theta \end{array}$$

It is possible to see that a cross-term appears in $\dot{V}_1$ and is a coupling between the state at time t and an integral of $Ax(\theta)+A_hx(\theta-h)$ over $[t-h,t]$. This annoying term must be bounded in order to decouple the integral term from $x(t)$. A simple bound can be provided by noting that

$$\int_{t-h}^{t}\begin{bmatrix} x(t) \\ x(\theta) \\ x(\theta-h)\end{bmatrix}^T\begin{bmatrix} PA_h \\ A^T \\ A_h^T\end{bmatrix}Z\begin{bmatrix} PA_h \\ A^T \\ A_h^T\end{bmatrix}^T\begin{bmatrix} x(t) \\ x(\theta) \\ x(\theta-h)\end{bmatrix}d\theta\geq 0$$

for some $Z=Z^T\succ 0$ and hence

$$\begin{array}{rcl} -2x(t)^TPA_h\int_{t-h}^{t}\dot{x}(\theta)d\theta & = & -2x(t)^TPA_h\int_{t-h}^{t}Ax(\theta)+A_hx(\theta-h)d\theta \\ & \leq & \int_{t-h}^{t}x(t)^TPA_hZ^{-1}A_h^TPx(t)d\theta+\int_{t-h}^{t}\dot{x}(\theta)^TZ\dot{x}(\theta)d\theta \\ & \leq & hx(t)^TPA_hZ^{-1}A_h^TPx(t)+\int_{t-h}^{t}\dot{x}(\theta)^TZ\dot{x}(\theta)d\theta \end{array}$$

And thus we have

$$\begin{aligned}\dot{V} &\leq x(t)^T[(A+Ah)^T P + P(A+A_h) + Q]x(t) + hx(t)^T PA_h Z^{-1} A_h^T Px(t) \\ &\quad -x(t-h)^T Qx(t-h) + h\dot{x}(t)^T Z\dot{x}(t)\end{aligned}$$

Finally since

$$h\dot{x}(t)^T Z\dot{x}(t) = h \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^T ZA & A^T ZA_h \\ \star & A_h^T ZA_h \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

we get

$$\dot{V} \leq \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} \Psi & hA^T ZA_h \\ \star & -Q + hA_h^T ZA_h \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

where $\Psi = (A+Ah)^T P + P(A+A_h) + Q + hA^T ZA + hPA_h Z^{-1} A_h^T P$ and therefore system (3.9) is delay-dependent stable with delay margin h if there exists symmetric positive definite matrices P, Q, Z such that the LMI

$$\begin{bmatrix} (A+Ah)^T P + P(A+A_h) + Q + hA^T ZA & hA^T ZA_h & +hPA_h \\ \star & -Q + hA_h^T ZA_h & 0 \\ \star & \star & -hZ \end{bmatrix} \prec 0$$

holds.

Through the use of the Lyapunov-Krasovskii theorem 3.2.10 and the Leibniz-Newton model transformation we have developed a delay-dependent stability test. This model transformation has introduced cross terms which have been bounded by a technique based on a completion of the squares. This bound allowed to compensate the integral term coming the differentiation of V_3 and then remove the annoying integral term

$$\int_{t-h}^{t} \dot{x}(\theta)^T Z\dot{x}(\theta) d\theta$$

from the expression of $\dot{V}$.

Obviously, the bound on cross-terms is very conservative since, while the left-hand side of the inequality may be negative, the right-hand side is always nonnegative. One of the great improvement of the Lyapunov-Krasovskii based methods was the introduction of better bounds on cross-terms. Some additional material is detailed in Appendix E.2 on bounding cross-terms.

Park's Bounding Method

A seminal result on time-delay system (from my point of view) is provided here and has been introduced in [Park, 1999; Park et al., 1998]. The idea was based on a more accurate bounding of cross terms in the derivative of the Lyapunov-Krasovskii functional (see appendix E.2 or [Park, 1999; Park et al., 1998]).

In [Park, 1999], the authors introduced the following lemma:

Lemma 3.2.18 *Assume that $a(\alpha) \in \mathbb{R}^{n_x}$ and $b(\alpha) \in \mathbb{R}^{n_y}$ are given for $\alpha \in \Omega$. Then, for any positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$ and any matrix $M \in \mathbb{R}^{n_y \times n_y}$, the following holds*

$$-2\int_{\Omega} b(\alpha)^T a(\alpha) d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \Psi \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha$$

with

$$\Psi = \begin{bmatrix} X & XM \\ M^T X & (M^T X + I)X^{-1}(XM + I) \end{bmatrix}$$

The bound provided in the latter lemma is able to provide a tighter bound on the cross term radically improving the contemporary results. The following Lyapunov-Krasovskii functional

$$V(x_t, \dot{x}_t) = x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T A_h^T R A_h \dot{x}(\eta) d\eta d\theta \qquad (3.14)$$

used along with the Park's bounding theorem leads to the theorem

Theorem 3.2.19 *System (3.10) is asymptotically delay-dependent stable for all $h \in [0, \bar{h}]$ if there exist $P = P^T \succ 0$, $Q = Q^T \succ 0$, $R = R^T \succ 0$, $V = V^T \succ 0$ and W such that*

$$\begin{bmatrix} M_{11} & -W^T A_h & A^T A_h^T V & \bar{h}(W^T + P) \\ \star & -Q & A_h^T A_h^T V & 0 \\ \star & \star & -V & 0 \\ \star & \star & \star & -V \end{bmatrix} \prec 0$$

holds with $M_{11} = (A + A_h)^T P + P(A + A_h) + W^T A_h + A_h^T W + V$.

Although this technique allows to consequently reduce the conservatism of the method by finding a more accurate bound on cross-terms, it is still limited by the use of the Leibniz-Newton model-transformation (which may introduce additional dynamics) and hence it would be more convenient to use Park's bounding method with a model transformation which does not generate additional dynamics.

Descriptor Model Transformation This model transformation has been introduced in [Fridman, 2001; Fridman and Shaked, 2001] and as shown in Section 3.2.1.3, it does not introduce any additional dynamics. It is briefly recalled here for system (3.10):

$$\mathcal{E}\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \mathcal{A}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \mathcal{A}_h \int_{t-h}^{t} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} ds$$

where $\mathcal{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix}$ and $\mathcal{A}_h = \begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix}$. One of the earliest results in this framework considers the Lyapunov-Krasovskii functional

$$V(x_t, y_t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T E^T P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{-h}^{0}\int_{t+\theta} y(s)^T R y(s) ds d\theta$$

where $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$, $E^T P = P^T E$, $P_1 = P_1^T \succ 0$ and $R = R^T \succ 0$. It is proved in [Fridman and Shaked, 2001] that such a Lyapunov-Krasovskii functional leads to the following theorem

Theorem 3.2.20 *System (3.10) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exists matrices $P_1 = P_1^T \succ 0$, $R = R^T \succ 0$, P_2, P_3 such that the LMI*

$$\begin{bmatrix} (A+A_h)^T P_2 + P_2^T(A+A_h) & P_1 - P_2^T + (A+A_h)^T P_3 & \bar{h}P_2^T A_h \\ \star & -P_3 - P_3^T + hR & \bar{h}P_3^T A_h \\ \star & \star & -\bar{h}R \end{bmatrix} \prec 0$$

holds.

This results is based on a bounding technique of cross terms involving a positive matrix as on page 119. However, results of [Fridman and Shaked, 2002b] involves Park's bounding technique and leads to less conservative stability conditions coupled with complete Lyapunov-Krasovskii functional [Fridman, 2006a]. Although this method is interesting and leads to results of quality, it still leads to cross terms which are difficult to bound and result in conservative conditions from an absolute point of view.

Method of Free Weighting Matrices

The following approach has been introduced in [He et al., 2004] and consists in injecting additional constraints into the LMI in order to tackle relations between signals involved in the system. These constraints involve additional free variables adding extra-degree of freedom into the LMI and this motivates the denomination of *free weighting matrices approach*. The Lyapunov-Krasovskii functional used in [He et al., 2004] is

$$V(x_t, \dot{x}_t) = x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T R \dot{x}(\eta) d\eta d\theta \tag{3.15}$$

and is very similar to (3.14).

It is important to note that the following equality holds for all signals $\dot{x}, x, x_h$ governed by the expression of system (3.9).

$$\begin{array}{lcl} 2\left[x(t)^T N_1 + x(t-h)^T N_2 + \dot{x}(t) N_3\right] \cdot \left[x(t) - x(t-h) - \displaystyle\int_{t-h}^{t} \dot{x}(s) ds\right] & = & 0 \\ 2\left[x(t)^T T_1 + x(t-h)^T T_2 + \dot{x}(t) T_3\right] \cdot \left[\dot{x}(t) - Ax(t) - A_h x(t-h)\right] & = & 0 \\ \bar{h}\begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix}^T X \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix} - \displaystyle\int_{t-h}^{t} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix}^T X \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix} d\theta & \geq & 0 \\ \text{for any matrices } N_i,\ T_i \text{ and } X = X^T \succeq 0 & & \end{array}$$

Indeed, the first constraint defines the Leibniz-Newton integral formula, the second constraint defines the model of the system and the last one defines $\bar{h}$ as the maximal value of the time-delay h. The key idea in this method is to differentiate the Lyapunov-Krasovskii functional without substituting $\dot{x}$ by its explicit value. The constraints are then added and this leads to a quadratic form in $\mathrm{col}(\dot{x}(t), x(t), x(t-h))$ involving an integral quadratic term with vector $\mathrm{col}(\dot{x}(t), x(t), x(t-h), \dot{x}(s))$. By an appropriate choice of the matrix X the integral term can be neglected and finally by a Schur complement the following result is obtained:

Theorem 3.2.21 *System (3.10) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exists matrices $P = P^T \succ 0$, $Q = Q^T \succ 0$, $R = R^T \succ 0$, $X = X^T \succeq 0$*

$N_1, N_2, N_3, T_1, T_2, T_3$ *such that the LMI*

$$\begin{bmatrix} Q + N_1^{\mathcal{S}} - (T_1A)^{\mathcal{S}} & N_2^T - N_1 - A^T T_2^T - T_1 A_h & P + N_3^T + T_1 - A^T T_3^T & \bar{h}N_1 \\ \star & -Q - N_2^{\mathcal{S}} - (T_2A_h)^{\mathcal{S}} & -N_3^T + T_2 - A_h^T T_3^T & \bar{h}N_2 \\ \star & \star & \bar{h}R + T_3^{\mathcal{S}} & \bar{h}N_3 \\ \star & \star & \star & -\bar{h}R \end{bmatrix} \prec 0 \tag{3.16}$$

holds.

While the addition of free variables is an advantage in the stability analysis (especially for robust stability analysis [He et al., 2004]), it becomes a drawback in synthesis problems since these decision variables are coupled to the system matrices (hence to the controller of observers gain) preventing to find a linearizing change of variable. A usual method consists in assuming a common simplification $T_i = \varepsilon_i K$ where ε_i are chosen fixed scalars and K is a decision matrix which considerably reduces the efficiency of the approach.

Approach using Jensen's inequality

We describe here a result which is not based on any model transformation but relies on the use the Jensen's inequality (see Appendix E.1) avoiding the cross-terms. It has been provided in different papers for instance in [Gouaisbaut and Peaucelle, 2006b; Han, 2005a].

Let us consider the Lyapunov-Krasovskii functional (3.15) and computing its time-derivative along the trajectories solutions of system (3.9) gives

$$\begin{aligned} \dot{V} &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^TP + PA + Q & PA_h \\ A_h^TP & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} + h\dot{x}(t)^T R\dot{x}(t) \\ &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^TP + PA + Q + hA^TRA & PA_h + hA^TRA_h \\ A_h^TP + hA_h^TRA & -Q + hA_h^TRA_h \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ &\quad - \int_{t-h}^{t} \dot{x}(\theta)^T R\dot{x}(\theta)d\theta \end{aligned}$$

At first sight the integral term could be neglected (bounded above by 0) but this will result in a too conservative condition. A more tight solution is the use of the Jensen's inequality on this integral term. The Jensen's inequality allows to establish the following bound on the integral term

$$-\int_{t-h}^{t} \dot{x}(s)^T R\dot{x}(s)ds \le -\bar{h}^{-1}\left(\int_{t-h}^{t} \dot{x}(s)ds\right)^T R\left(\int_{t-h}^{t} \dot{x}(s)ds\right)$$

leading then to the following result:

Theorem 3.2.22 *System (3.10) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exists matrices $P = P^T \succ 0$, $Q = Q^T \succ 0$, $R = R^T \succ 0$ such that the LMI*

$$\begin{bmatrix} A^TP + PA + Q - \bar{h}^{-1}R + \bar{h}A^TRA & PA_h + \bar{h}^{-1}R + \bar{h}A^TRA_h \\ \star & -Q - \bar{h}^{-1}R + \bar{h}A_h^TRA_h \end{bmatrix} \prec 0 \tag{3.17}$$

holds.

As a remark, it is important to note that this result is identical to the method of free weighting matrices presented in the previous paragraph on page on page 123. Indeed, LMI (3.16) can be written as

$$\Psi + U^T Z V + V^T Z^T U \prec 0 \tag{3.18}$$

where

$$\Psi = \begin{bmatrix} Q & 0 & P & 0 \\ \star & -Q & 0 & 0 \\ \star & \star & \bar{h}R & 0 \\ \star & \star & \star & -\bar{h}R \end{bmatrix} \qquad Z = \begin{bmatrix} T_1 & N_1 \\ T_2 & N_2 \\ T_3 & N_3 \end{bmatrix} \qquad U = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} -A & -A_h & I & 0 \\ I & -I & 0 & \bar{h}I \end{bmatrix}$$

Here the matrix Z is an unconstrained matrix and hence the projection lemma applies (see appendix D.18). It states that there exist at least one solution Z to (3.18) if and only if the two following underlying LMIs hold

$$\begin{array}{rcl} \mathrm{Ker}[U]^T \Psi \mathrm{Ker}[U] & \prec & 0 \\ \mathrm{Ker}[V]^T \Psi \mathrm{Ker}[V] & \prec & 0 \end{array}$$

These basis of null-spaces can be expressed as

$$\mathrm{Ker}[U] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \qquad \mathrm{Ker}[V] = \begin{bmatrix} I & 0 \\ 0 & I \\ A & A_h \\ -\bar{h}^{-1} & \bar{h}^{-1}I \end{bmatrix}$$

The leads to

$$\mathrm{Ker}[U]^T \Psi \mathrm{Ker}[U] = -\bar{h}R$$

which is negative definite by definition of $R \succ 0$. This means that feasibility of (3.18) is equivalent to the feasibility of the second underlying LMI:

$$\mathrm{Ker}[V]^T \Psi \mathrm{Ker}[V] = \begin{bmatrix} A^T P + PA + Q - \bar{h}^{-1}R + \bar{h}A^T RA & PA_h + \bar{h}^{-1}R + \bar{h}A^T RA_h \\ \star & -Q - \bar{h}^{-1}R + \bar{h}A_h^T RA_h \end{bmatrix}$$

The latter LMI is identical to (3.17) showing that the approaches are equivalent. The advantage of formulation (3.16) is the decoupling between Lyapunov matrices P, Q, R and data matrices A, A_h which allows to provide interesting robust stability result for polytopic type uncertainties [Gouaisbaut and Peaucelle, 2006b; He et al., 2004]. For stability analysis, criterion (3.17) is more interesting since it has low computational complexity (due to the absence of 'slack' variables and therefore a lower number of decision matrices).

Actually, many results in time-delay systems are related to each others modulo congruence transformations, Schur complements or through the use of other theorems. This is emphasized in [Xu and Lam, 2007]. Other Lyapunov-Krasovskii based approaches avoiding model transformation have been provided in many research papers, see for instance [Xu et al., 2006].

3.2.1.6 Stability Analysis: (Scaled) Small-Gain Theorem

Different results based on Lyapunov-Krasovskii functionals have been introduced. It is aimed here at showing that similar results can be retrieved through the use of (scaled) small-gain theorem. Indeed, it is possible to provide delay-independent and delay-dependent stability tests based on the use of the small-gain theorem as emphasized for instance in [Zhang et al., 2001].

We will consider in the following the operators:

$$\begin{array}{rcl} \mathscr{D}_h & : & x(t) \rightarrow x(t-h) \\ \mathscr{S}_h & : & x(t) \rightarrow \displaystyle\int_{t-h}^{t} x(s)ds \end{array}$$

Delay-Independent Stability Test using Scaled Small-Gain Theorem

This paragraph is devoted to delay-independent stability test using scaled-small gain theorem. First, the system (3.9) must be rewritten as an interconnection of two subsystems (i.e. a linear finite dimensional systems and the delay operator $\mathscr{D}_h$) according to the framework of small-gain theorem. Hence (3.9) is rewritten in an 'LFT' form:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h w(t) \\ z(t) & = & x(t) \\ w(t) & = & \mathscr{D}_h(z(t)) \end{array} \tag{3.19}$$

It has been shown that the operator $\mathscr{D}_h(\cdot)$ is asymptotically stable and therefore has finite $\mathcal{H}_\infty$-norm. Indeed, if the input of the operator has finite $\mathcal{L}_2$-norm then the output, which is the delayed input with a constant delay, will have finite energy too; this shows stability. In order to determine the value of the $\mathcal{H}_\infty$-norm of $\mathscr{D}_h(\cdot)$ it suffices to compute the ratio of the output energy over the input energy:

$$\begin{array}{rcl} \displaystyle\int_0^{+\infty} w(\theta)^T w(\theta) d\theta & = & \displaystyle\int_0^{+\infty} z(\theta-h)^T z(\theta-h) d\theta \\ & = & \displaystyle\int_{-h}^{+\infty} z(\theta')^T z(\theta') d\theta' \end{array}$$

with the change of variable $\theta' = \theta - h$. Hence assuming zero initial conditions (i.e. $z(t) = 0$ for all $t < 0$) we get

$$\int_0^{+\infty} w(\theta)^T w(\theta) d\theta \quad = \quad \int_0^{+\infty} z(\theta')^T z(\theta') d\theta'$$

showing that the operator $\mathscr{D}_h(\cdot)$ has unitary $\mathcal{H}_\infty$-norm. Using this result it is possible to apply the small-gain theorem in the dissipativity framework and to this aim we define the Hamiltonian function:

$$H(x_t) = S(x) - \int_0^t s(x(\tau), x(\tau-h)) d\tau$$

where $S(x) = x(t)^T P x(t)$ is the storage function with supply-rate:

$$s(x(t), x(t-h)) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

In the dissipativity framework, if the derivative $\dot{H}$ of the Hamiltonian function H is negative definite then this means that the interconnected system (3.19) is asymptotically stable and hence (3.9) is delay-independent stable. Differentiating H along the trajectories solution of system (3.19) gives

$$\dot{H} := \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \begin{bmatrix} A^T P + PA + L & PA_h \\ \star & -L \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \prec 0$$

This leads to the following theorem:

Theorem 3.2.23 *System (3.10) is delay-independent asymptotically stable if there exist matrices $P = P^T \succ 0$ and $L = L^T \succ 0$ such that the LMI*

$$\begin{bmatrix} A^T P + PA + L & PA_h \\ \star & -L \end{bmatrix} \prec 0$$

holds.

It is easy to recognize the LMI obtained by application of the Lyapunov-Krasovskii theorem with Lyapunov-Krasovskii functional

$$V(x_t) = x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T L x(\theta) d\theta \tag{3.20}$$

as detailed in paragraph on page 118. This suggests that the Hamiltonian function H coincides with the above Lyapunov-Krasovskii functional. This is proved in what follows.

Comparison with the Delay-Independent Lyapunov-Krasovskii Functional

First of all rewrite H as

$$\begin{aligned} H(x_t) &= x(t)^T P x(t) + \int_0^t (x(s)^T L x(s) - x(s-h)^T L x(s-h)) ds \\ &= x(t)^T P x(t) + \int_0^t \int_{s-h}^{s} Y(\tau) d\tau ds \end{aligned}$$

where $Y(t) = \frac{d}{dt}(x(t)^T L x(t))$. Defining $\tau' = \tau - s + h$ we get

$$H(x_t) = x(t)^T P x(t) + \int_0^t \int_0^h Y(\tau' + s - h) d\tau' ds$$

Now exchanging the order of integration yields

$$\begin{aligned} H(x_t) &= x(t)^T P x(t) + \int_0^h \int_0^t Y(\tau' + s - h) ds d\tau' \\ &= x(t)^T P x(t) + \int_0^h \left(x(\tau' + t - h)^T L x(\tau' + t - h) - x(\tau' - h)^T L x(\tau' - h) \right) d\tau' \end{aligned}$$

Assuming zero initial conditions (i.e. $x(s) = 0$ for all $s \leq 0$ thus $Y(s) = 0$ for all $s \leq 0$) then we have

$$\int_0^h x(\tau' - h)^T L x(\tau' - h) d\tau' = 0$$

and hence $H(x_t)$ reduces to

$$H(x_t) = x(t)^T P x(t) + \int_0^h x(\tau' + t - h)^T L x(\tau' + t - h) d\tau'$$

Finally let $\theta = \tau' + t - h$, we obtain

$$H(x_t) = x(t)^T P x(t) + \int_{t-h}^t x(\theta)^T L x(\theta) d\theta$$

and Lyapunov-Krasovskii functional (3.20) is retrieved.

In [Zhang et al., 2001] the relation between Lyapunov-Krasovskii and small-gain results for time-delay, in general, is also emphasized. In [Bliman, 2001], less delay-independent stability tests are provided, based on extension of Lyapunov-Krasovskii functions which can also be viewed as an extension of small-gain based results introduced in this paragraph.

Delay-Dependent Stability Test using Scaled Small-Gain Theorem

While a delay-independent test can be obtained using $\mathscr{D}_h$, $\mathscr{S}_h$ can be used to derive a delay-dependent test. According to operator $\mathscr{S}_h(\cdot)$, system (3.9) is rewritten as

$$\begin{array}{rcl} \dot{x}(t) & = & (A + A_h)x(t) - A_h w(t) \\ z(t) & = & (A + A_h)x(t) - A_h w(t) \\ w(t) & = & \mathscr{S}_h(z(t)) \end{array} \tag{3.21}$$

This reformulation is identical to the Leibniz-Newton model transformation (see Section 3.2.1.2) and then adds additional dynamics (see Section 3.2.1.3). Hence systems (3.9) and (3.21) are not equivalent. The operator $\mathscr{S}_h$ is a stable LTI system; therefore it has finite $\mathcal{H}_\infty$ norm. First, note that the corresponding transfer function is given by

$$\hat{\mathscr{S}_h}(s) = \frac{1 - e^{-sh}}{s}$$

The $\mathcal{H}_\infty$ norm γ_∞ is defined as

$$\begin{array}{rcl} \gamma_\infty & := & \sup\limits_{s \in \mathbb{C}^+} \left| \dfrac{1 - e^{-sh}}{s} \right| = \sup\limits_{\omega \in \mathbb{R}} \left| \dfrac{1 - e^{-j\omega h}}{j\omega} \right| \\ & = & \sup\limits_{\omega \in \mathbb{R}} \dfrac{\left|1 - e^{-j\omega h}\right|}{\omega} = \lim\limits_{\omega \to 0^+} \dfrac{\left|1 - e^{-j\omega h}\right|}{\omega} \\ & = & h \le \bar{h} \end{array}$$

For any $h \in [0, \bar{h}]$, the worst-case $\mathcal{H}_\infty$ norm of the operator $\mathscr{S}_h$ is $\bar{h}$. This interesting fact allows to express delay-dependent result from scaled small-gain theorems. Define the storage function $S(x) = x^T P x$ and the supply-rate

$$s(\dot{x}(t), x(t), x(t-h)) = \begin{bmatrix} \dot{x}(s) \\ x(s) - x(s-h) \end{bmatrix}^T \begin{bmatrix} -\bar{h}^2 L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ x(s) - x(s-h) \end{bmatrix} \tag{3.22}$$

to construct the Hamiltonian function

$$H(\dot{x}, x_t) = S(x) - \int_0^t s(\dot{x}(\tau), x(\tau), x(\tau - h)) d\tau \tag{3.23}$$

Finally differentiating H along the trajectories solution of system (3.21) leads to the following theorem.

Theorem 3.2.24 *System (3.10) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exist matrices $P = P^T \succ 0$ and $L = L^T \succ 0$ such that the LMI*

$$\begin{bmatrix} (A+A_h)^T P + P(A+A_h) & -PA_h & \bar{h}(A+A_h)^T L \\ \star & -L & -\bar{h}A_h^T L \\ \star & \star & -L \end{bmatrix} \prec 0 \tag{3.24}$$

holds.

Proof: *The proof is only sketched. Differentiating the Hamiltonian function and using the Leibniz-Newton formula we obtain*

$$\begin{aligned} \dot{H} \quad \leq \quad & \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} (A+A_h)^T P + P(A+A_h) & -PA_h \\ \star & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ & + \begin{bmatrix} x(t) \\ x(t) - x(t-h) \end{bmatrix}^T \begin{bmatrix} \bar{h}^2 L & 0 \\ \star & -L \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-h) \end{bmatrix} \end{aligned} \tag{3.25}$$

with $y(t) = \displaystyle\int_{t-h}^{t} \dot{x}(s)ds$. In virtue of the Leibniz-Newton formula we have

$$\begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-h) \end{bmatrix} = \begin{bmatrix} A + A_h & -A_h \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{3.26}$$

Hence, substituting (3.26) into (3.25) and using the Schur complement we get the LMI of Theorem 3.2.24. $\square$

Similarly as for the delay-independent test, it is possible to show the existence of a relationship between the latter result and a Lyapunov-Krasovskii functional.

Connection with a Lyapunov-Krasovskii functional

To connects these results, we will show that the supply-rate is equal to a functional that can be added to the function $S(x) = x^T Px$. First let us consider the opposite of the supply rate $s(\cdot)$ (3.22) can be rewritten as

$$\begin{aligned} -s(\cdot) \quad &= \quad \int_0^t \begin{bmatrix} \dot{x}(s) \\ \int_{s-h}^{s} \dot{x}(\theta)d\theta \end{bmatrix}^T \begin{bmatrix} \bar{h}^2 L & 0 \\ 0 & -L \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ \int_{s-h}^{s} \dot{x}(\theta)d\theta \end{bmatrix} ds \\ &= \quad \bar{h}^2 \int_0^t \dot{x}(s)^T L \dot{x}(s)ds - \int_0^t \left(\int_{s-h}^{s} \dot{x}(\theta)d\theta \right)^T L \left(\int_{s-h}^{s} \dot{x}(\theta)d\theta \right) \end{aligned}$$

Moreover, let

$$\begin{aligned} V_i \quad &= \quad \bar{h} \int_{-\bar{h}}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T L \dot{x}(s)dsd\theta \\ &= \quad \bar{h}^2 \int_0^t \dot{x}(s)^T L \dot{x}(s)ds - \bar{h} \int_{t-\bar{h}}^{t} \int_0^{\theta} \dot{x}(s)^T L \dot{x}(s)dsd\theta \end{aligned}$$

We can see that the first term in the right-hand side are identical, hence it remains to show the relation between the terms

$$\mathcal{I}_1 := \int_0^t \left(\int_{s-h}^s \dot{x}(\theta)d\theta\right)^T L \left(\int_{s-h}^s \dot{x}(\theta)d\theta\right)$$

and

$$\mathcal{I}_2 := \bar{h}\int_{t-\bar{h}}^t \int_0^\theta \dot{x}(s)^T L\dot{x}(s)dsd\theta$$

Invoking the Jensen's inequality (Appendix E.1) we get the relation

$$\mathcal{I}_2 \leq \mathcal{I}_1$$

This shows that the same result can be obtained using the scaled-bounded real and a particular Lyapunov-Krasovskii functional provided that the Jensen's Inequality is used. Some other connections between Lyapunov-Krasovskii functionals and small-gain results are also provided, for instance, in [Bliman, 2000; Zhang et al., 2001].

3.2.1.7 Stability Analysis: Padé Approximants

Still in the family of approaches considering a time-delay system into an interconnection of two subsystems, namely a finite dimensional system and a delay operator, the method provided in [Zhang et al., 1999] is of great interest. This method actually holds only for constant delay but leads to very interesting delay-dependent stability results that deserve to be presented. The system that will be considered is given below

$$\begin{array}{rcl}\dot{x}(t) &=& Ax(t) + A_h x(t-h)\\ x(\theta) &=& \phi(\theta),\ \theta\in[-h,0]\end{array} \tag{3.27}$$

It is rewritten as in an 'LFT' form:

$$\begin{array}{rcl}\dot{x}(t) &=& Ax(t) + A_h w(t)\\ z(t) &=& x(t)\\ w(t) &=& \mathscr{D}_h(z(t)) - z(t)\end{array} \tag{3.28}$$

Since $\mathcal{D}_h$ is a time-invariant linear operator, the corresponding transfer function is given by

$$H_d(s) := \frac{W(s)}{Z(s)} = e^{-sh} - 1$$

Due to the complexity of exponential term, it is proposed in Zhang et al. [1999] to approximate the delay operator by a parameter dependent filter coinciding with the Padé approximant of e^{-sh} (see Appendix E.3). The idea of using Padé approximants to deal with time-delay is not new and the reader should refer for instance to [Lam, 1990; Saff and Varga, 1975]) but the current solution is interesting since it involves LMIs.

Formally, Padé approximants aim at approximating continuous functions, over a certain domain, by a rational function and this is the reason why they are interesting tools in systems and control theory. Indeed, as a transfer function should be (strictly) proper, power series for instance cannot be applied as approximants but Padé approximants can.

From this approximation, the system can be rewritten into an interconnection of a finite dimensional LTI system and a parameter dependent filter (i.e. the Padé approximation).

Let us consider system (3.27) and define the matrices $\bar{A} = A + A_h$ and $A_h = HF$ where H, F are full-rank factors of A_h. Let $\Psi(s,h) = \det(sI - A - A_h e^{-sh})$ be the characteristic quasipolynomial of (3.27). It is well-known that system (3.27) is asymptotically stable for all $h \in [0, \bar{h}]$ if and only if

$$\Psi(j\omega, h) \neq 0, \ \forall \omega \geq 0, \ h \in [0, \bar{h}]$$

Assuming that $G = F(sI - \bar{A})^{-1}H$ and $\Phi(hs) = (e^{-hs} - 1)I$, it is possible to rewrite the system as an interconnection of these subsystems and the delay-dependent stability condition becomes equivalent to the following statement:

$$\det[I - G(j\omega)\Phi(j\omega h)] \neq 0, \ \forall \omega \geq 0, \ h \in [0, \bar{h}]$$

Since this statement is very difficult to be checked exactly, the idea is then to provide an inner and outer approximation of the set defining the set of delay-operators for each delay from 0 to $\bar{h}$

$$\Omega_A(\omega, \bar{h}) := \left\{ e^{-j\omega h} : \ h \in [0, \ \bar{h}] \right\}$$

Using the Padé approximation, the inner and outer sets are given by

$$\begin{array}{rcl} \Omega_B(\omega, \bar{h}) & := & \left\{ R_m(j\theta\alpha_m\omega) : \ \theta \in [0, \bar{h}] \right\} \\ \Omega_C(\omega, \bar{h}) & := & \left\{ R_m(j\theta\omega) : \ \theta \in [0, \bar{h}] \right\} \end{array}$$

where $R_m(s) = \dfrac{N_m(s)}{N_m(-s)}$ is the m^{th} order ($m \geq 3$) Padé approximation of e^s and

$$\alpha_m := \frac{1}{2\pi} \min\{\omega > 0 : \ R_m(j\omega) = 1\}$$

The following lemma, proved in [Zhang et al., 1999], is useful for comprehensive purpose

Lemma 3.2.25 *For every integer $m \geq 3$, the following statements hold:*

1. *All poles of $R_m(s)$ are in the open left half complex plane.*
2. $\Omega_C(\omega, \bar{h}) \subseteq \Omega_A(\omega, \bar{h}) \subseteq \Omega_B(\omega, \bar{h}), \ \forall \omega \geq 0.$
3. $\lim_{m \to +\infty} \alpha_m = 1$

This result says that the Padé approximmation $R_m(s)$ is a stable operator but, overall that the greater the order is, the better the approximation of the set is. Indeed, the condition

$$\det[I - G(j\omega)R_m(j\theta\omega)] \neq 0, \ \forall \omega \geq 0, \ \theta \in [0, \bar{h}]$$

is a necessary condition for stability since $\Omega_C(\omega, \bar{h})$ is included in $\Omega_A(\omega, \bar{h})$. On the other hand, since $\Omega_B(\omega, \bar{h})$ contains $\Omega_A(\omega, \bar{h})$, therefore

$$\det[I - G(j\omega)R_m(j\theta\alpha_m\omega)] \neq 0, \ \forall \omega \geq 0, \ \theta \in [0, \bar{h}]$$

is a sufficient condition only. But when $m \to +\infty$ then $\alpha_m \to 1$ and hence the sets $\Omega_B(\omega,\bar{h})$ and $\Omega_C(\omega,\bar{h})$ converge to each other, to finally coincide with $\Omega_A(\omega,\bar{h})$ showing that, at infinity, the stability of the interconnected system over $\Omega_A(\omega,\bar{h})$, $\Omega_B(\omega,\bar{h})$ and $\Omega_C(\omega,\bar{h})$ are equivalent.

Since we are interested in a delay-dependent stability sufficient condition, the set $\Omega_B(\omega,\bar{h})$ is considered. Let (A_P, B_P, C_P, D_P) be the minimal realization of $P(s) := (R_m(\alpha_m s) - 1)I$ and denote n_P be the order of A_P. Note that in $P(s)$ the set $\Omega_B(\omega,\bar{h})$ is considered due to the presence of α_m. Also introduce $A_s := \bar{A} + HD_PD$, $B_s := B_PF$ and $C_s := HC_P$. Using this formulation, Zhang et al. [1999] provide this very interesting result:

Theorem 3.2.26 *System (3.27) is delay-dependent asymptotically stable for all $h \in [0,\bar{h}]$ if there exist matrices $X_0 \in \mathbb{S}^n_{++}$, $X_{22} \in \mathbb{S}^{n_P}_{++}$ and $X_1 \in \mathbb{R}^{n\times n}$, $X_{12} \in \mathbb{R}^{n_P\times n_P}$ such that*

$$\Pi(0) \prec 0, \qquad \Pi(\bar{h}) \prec 0$$

and

$$\begin{bmatrix} X_0 + \bar{h}X_1 & \bar{h}X_{12} \\ \star & \bar{h}X_{22} \end{bmatrix} \succ 0$$

where

$$\Pi(\theta) := \begin{bmatrix} \Pi_{11}(\theta) & \Pi_{12}(\theta) \\ \star & \Pi_{22}(\theta) \end{bmatrix}$$

with

$$\begin{array}{rcl} \Pi_{11}(\theta) & := & (X_0 + \theta X_1)A_s + X_{12}B_s + A_s^T(X_0 + \theta X_1)^T + B_s^T X_{12}^T \\ \Pi_{12}(\theta) & := & (X_0 + \theta X_1)C_s + X_{12}A_P + \theta A_s^T X_{12} + B_s^T X_{22} \\ \Pi_{22}(\theta) & := & \theta X_{12}^T C_s + \theta C_s^T X_{12} + X_{22}A_P + A_P^T X_{22} \end{array}$$

Considering the system (3.3) and using the latter theorem with $m = 5$ the delay margin is estimated as $\bar{h} = 6.150$ while the actual delay margin is 6.172. The computed delay-margin is very close to the theoretical one. This result drastically improved contemporary ones and is still competitive with recent works. Many results based on 'complete' discretized Lyapunov-Krasovskii functionals lead to similar result but with a larger computational complexity.

It is worth noting that in this approach a model transformation is used (expressed through the operator $e^{-sh} - 1$) but does not introduce any conservatism (i.e. additional dynamics). The only constraint imposed by the method is the asymptotic stability of the system for zero delay (since the matrix $\bar{A}$ needs to be Hurwitz). This is not a problem since stability over an interval including 0 is sought. For a more general approach using similar results, the reader should refer to [Knospe and Roozbehani, 2006, 2003; Roozbehani and Knospe, 2005].

3.2.1.8 Stability Analysis: Integral Quadratic Constraints

The approach based on Integral Quadratic Constraints (IQC) [Rantzer and Megretski, 1997] has led to more and more works since they provide an efficient way to study stability of a wide variety of systems, including time-delay systems [Fu et al., 1998; Jun and Safonov, 2001, 2002; Kao and Rantzer, 2007]. The key idea behind IQC analysis is the $\mathcal{L}_2$ stability of an interconnected system. Indeed, if for any exogenous $\mathcal{L}_2$ inputs, the loop-signals have bounded energy this means that the interconnection of systems is stable. The reader should refer to Section 2.3.4.6 for some brief explanations on IQC method.

Part of the results of [Kao and Rantzer, 2007] (in the constant-delay case) is presented here. Indeed, Kao and Rantzer [2007] has provided very efficient criteria for stability analysis of time-delay systems which lead efficient results, sometimes very close to the theoretical ones (at least for constant time-delays). Let us consider the delay-operators

$$\begin{array}{lcl} x(t)-x(t-h) & := & \mathscr{S}_h(x(t)) \\ x(t-h) & := & \mathscr{D}_h(x(t)) \end{array}$$

Note that these delay operators are equivalent to those proposed, for instance, in [Zhang et al., 1999] where Padé approximants are used. However the operators above can be extended to the time-varying delay case while the use of Padé approximation restricts the approach to constant delay case. This suggests that the IQC approach provided by Kao and Rantzer [2007] can be viewed as a generalization of the approach of [Zhang et al., 1999] to the time-varying delay case, although different techniques are used to study stability. Another comparison can be made between results that can be obtained with scaled-small gain, IQC techniques [Jun and Safonov, 2001, 2002] and Lyapunov-Krasovskii functionals which lead to similar (sometimes identical) stability tests.

Using these operators, time-delay system (3.27) can be rewritten as an interconnection of two subsystems:

$$\begin{array}{rcl} \dot{x}(t) & = & (A+A_h)x(t)-A_hw(t) \\ z(t) & = & x(t) \\ w(t) & = & \mathscr{S}_h(z(t)) \end{array}$$

In the IQC analysis, the operators involved in the interconnections are defined by their input/output behavior through IQC. The following propositions introduce one IQC for each operator:

Proposition 3.2.27 *The operator $\mathscr{D}_h$ satisfies the IQC defined by*

$$\int_{-\infty}^{+\infty}\begin{bmatrix} v(t) \\ \mathscr{D}_h(v(t)) \end{bmatrix}^T\begin{bmatrix} X_1 & 0 \\ 0 & -X_1 \end{bmatrix}\begin{bmatrix} v(t) \\ \mathscr{D}_h(v(t)) \end{bmatrix}dt \geq 0$$

for any $X_1 = X_1^T \succeq 0$.

Proposition 3.2.28 *Suppose $h \in [0, \bar{h}]$, then the operator $\mathscr{S}_h$ satisfies any IQC defined by*

$$\int_{-\infty}^{+\infty}\begin{bmatrix} v(t) \\ \mathscr{S}_h(v(t)) \end{bmatrix}^T\begin{bmatrix} |\psi(j\omega)|^2X_2 & 0 \\ 0 & -X_2 \end{bmatrix}\begin{bmatrix} v(t) \\ \mathscr{S}_h(v(t)) \end{bmatrix}dt \geq 0$$

for any $X_2 = X_2^T \succeq 0$ and where $|\psi(j\omega)| \geq g(\omega)+\delta$, for all $\omega \in \mathbb{R}$. The function $g(\omega)$ is defined below

$$g(\omega) := \begin{cases} 2 & \text{if } |\omega| > \dfrac{\pi}{\bar{h}} \\ 2\left|\sin\left(\dfrac{\omega\bar{h}}{2}\right)\right| & \text{if } |\omega| \leq \dfrac{\pi}{\bar{h}} \end{cases}$$

μ	0	0.1	0.2	0.5	0.8	0.999
[Kim, 2001]	1	0.974	0.883	0.655	0.322	0.001
[Wu et al., 2004]	4.4772	3.604	3.033	2.008	1.364	1.001
[Fridman and Shaked, 2002a]	4.4772	3.604	3.033	2.008	1.364	1.001
[Kao and Rantzer, 2007]	6.117	4.4714	3.807	2.280	1.608	1.360

Table 3.1: Comparison of different stability margins of system (3.29) with respect to the upper bound μ on the derivative of the delay $h(t)$

A good example of $\psi(s)$ satisfying the above conditions is

$$\psi(s) = 2\frac{\bar{h}^2 s^2 + c\bar{h}s}{\bar{h}^2 s^2 + a\bar{h} + b} + \delta$$

where $a = \sqrt{6.5 + 2b}$, $b = \sqrt{50}$, $c = \sqrt{12.5}$ and δ is an arbitrary small positive number.

Using these IQCs, the criterium obtained from the KYP Lemma (see Appendix D.3 and Section 2.3.4.6) leads to a very accurate computation of the theoretical the delay margin for system (3.3). This result is very effective since the model transformation used to rewrite the time-delay system as an interconnection of a linear system and the delay operator $\mathscr{S}_h(x(t))$ does not introduce any additional dynamics. Hence the interconnected system is completely equivalent to the original system. Moreover, the characterization of the operator $\mathscr{S}_h$ in terms of IQCs is sufficiently tight to remove most of the conservatism for system (3.3).

This makes, at this time and from my point of view, the best numerical tool to analyze stability of a time-delay system since, compared to approaches such as discretized functionals (see [Gu et al., 2003]) or Padé approximation (see [Zhang et al., 1999]), the computational complexity is very low and the method allows for an easy extension to time-varying delays.

Example 3.2.29 *As an example let us consider the system*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)) \tag{3.29}$$

where the delay satisfies $\dot{h} \leq \mu$. It is shown in [Kao and Rantzer, 2007] that the IQC approach presented above leads to the results of Table 3.2.29 using the IQCβ toolbox [Jonsson et al., 2004]. Clearly, the result obtained for $\mu = 0$ is very close to the theoretical one and is computed with only two decision variable introduced by the use of two IQCs. This demonstrates the possibilities of the approach both in terms of computational complexity and efficiency.

3.2.1.9 Stability Analysis: Well-Posedness Approach

The section on stability analysis of time-delay systems concludes with the stability analysis through well-posedness analysis of interconnections; see Section 2.3.4.4, on page 80 or [Iwasaki and Hara, 1998] for more details on well-posedness of feedback systems.

The result provided here is borrowed from [Gouaisbaut and Peaucelle, 2006a] and is an application of results on well-posedness to the interconnection of an uncertain matrix and an implicit linear transformation [Peaucelle et al., 2007]. Let us consider the interconnected system:

$$\begin{array}{rcl} w & = & \Delta(z+v) \\ Ez & = & H(w+u) \end{array} \tag{3.30}$$

where w, z are loop signals, u, v exogenous input signals and Δ an uncertain matrix. The corresponding set-up is depicted in Figure 3.2.

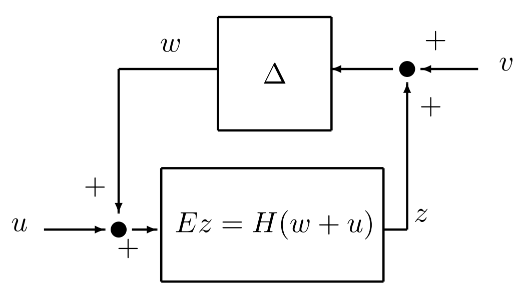

Figure 3.2: Interconnection of an uncertain matrix Δ and the implicit linear transformation $Ez = H(w+u)$

The following result on stability of (3.30) has been proved in [Peaucelle et al., 2007].

Theorem 3.2.30 *The closed-loop system (3.30) is well-posed if and only if there exists an Hermitian matrix $X = X^*$ such that*

$$\begin{bmatrix} EE^{\circledast} & -H \end{bmatrix}_{\perp}^{*} X \begin{bmatrix} EE^{\circledast} & -H \end{bmatrix}_{\perp} \succ 0 \tag{3.31}$$

$$\begin{bmatrix} 0 & I \\ \Delta E_{\perp} & \Delta E^{\circledast} \end{bmatrix}^{*} X \begin{bmatrix} 0 & I \\ \Delta E_{\perp} & \Delta E^{\circledast} \end{bmatrix} \preceq 0 \quad \text{for all } \Delta \in \boldsymbol{\Delta} \tag{3.32}$$

where $\boldsymbol{\Delta}$ is the set of uncertainties, E° denotes a full-rank matrix whose columns span the same space as the columns of E and $E^{\circledast} = E^{\circ *}$. Moreover, if E and H are real, the equivalence still holds for X restricted to be real.

We aim here at developing a simple delay-dependent stability result from the latter theorem. Define

$$E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ I_n & 0 & -I_n \end{bmatrix} \qquad H = \begin{bmatrix} A & A_h & 0 \\ I_n & 0 & 0 \\ -I_n & I_n & hI_n \end{bmatrix}$$

$$\Delta(s) = \begin{bmatrix} s^{-1}I_n & 0 & 0 \\ 0 & e^{-sh}I_n & 0 \\ 0 & 0 & \dfrac{1-e^{-sh}}{sh}I_n \end{bmatrix}$$

Substituting these matrices into (3.30) it can be shown that system (3.27) is retrieved. It is sought to find sufficient condition for the stability of system (3.27) using Theorem 3.2.30.

Note that inequality (3.32) is always verified if X is chosen as

$$X = \begin{bmatrix} 0 & 0 & 0 & -P & 0 & 0 \\ \star & -Q & 0 & 0 & 0 & 0 \\ \star & \star & -R & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & Q & 0 \\ \star & \star & \star & \star & \star & R \end{bmatrix}$$

where $P = P^T \succ 0$, $Q = Q^T \succ 0$ and $R = R^T \succ 0$. In this case, (3.31) is equivalent to LMI (3.17) obtained with Lyapunov-Krasovskii functional (3.15). This result has been extended, in a similar fashion as of [Bliman, 2002], to obtain a more accurate delay margin in [Gouaisbaut and Peaucelle, 2006a,b] by considering higher order derivatives. However this has been provided for constant delays only and Ariba and Gouaisbaut [2007] have extended the results to the time-varying delay case.

The key idea in well-posedness based results is the use of Taylor expansions to approximate the time-varying delay operator and the greater the order of the Taylor expansion is, the smaller the conservatism is. While the approach of Zhang et al. [1999] considers the frequency domain, the approach of Ariba and Gouaisbaut [2007]; Gouaisbaut and Peaucelle [2006b] lies in the time-domain and hence allows for the study of systems with time-varying delays.

3.2.2 Robustness with respect to delay uncertainty

Stability with respect to delay uncertainty is still unsolved completely. Some papers are devoted to or use results on robust stability analysis with respect to delay uncertainty [Fridman, 2006a; Kharitonov and Niculescu, 2003; Michiels et al., 2005; Sename and Briat, 2006; Seuret et al., 2009b; Verriest et al., 2002]. The idea (interest) behind of robust stability of systems with uncertain delay is multiple:

- Assuming that the stability of the system (3.33) is known for a nominal delay value h_0 the maximal deviation δ from this nominal value for which the system remains is stable is sought. Therefore the system will be shown to be stable for any delay belonging to $[h_0 - \delta^+, h_0 - \delta^-]$. In the case of a time-varying delay, the bound on the derivative of the variation η can also be considered.

$$\dot{x}(t) = Ax(t) + A_h(x - h_0 + \theta(t)), \quad \theta(t) \in [\delta^-, \delta^+],\ |\dot{\theta}| < \eta \tag{3.33}$$

- Assuming that a controlled time-delay system (with delay h) by a controller with memory but involving a different time-delay value, say h_c, takes the form (3.34). The implemented delay h_c can be decomposed into a sum of the real delay h and an uncertain value θ, representing the knowledge error on the delay value. In this case, the closed-loop system (3.34) involves two-delays which are interrelated by an inequality. Here also, the delays can be chosen time-varying and bounds on the derivatives η, ν can be taken into account:

$$\begin{array}{rcl} \dot{x} & = & Ax(t) + A_h^1 x(t - h(t)) + A_h^2 x(t - h(t) - \theta(t)) \\ & & h(t) \in [0, h_{max}],\ |\dot{h}| < \eta,\ \theta(t) \in [-\delta, \delta],\ |\dot{\theta}| < \upsilon \end{array} \tag{3.34}$$

In both case, some solutions exist and can be expressed in both frequency and time domains: the frequency domain approaches are restricted to deal with constant time-delay (except for special cases) while time-domain are not. In the following, we aim at providing several methods covering the latter scenarii.

3.2.2.1 Frequency domain: Matrix Pencil approach

The approach provided here has been introduced in the nice paper proposed by [Kharitonov and Niculescu, 2003]. The idea is to analyze the stability of perturbed delay system, assuming the stability of the nominal one. The interest of this approach is to provide *necessary and sufficient* conditions in terms of generalized eigenvalue distribution of some (finite dimensional) constant matrix pencil.

Let us consider system (3.33) with constant delays $h-\theta$ which is assumed to be stable for $\theta = 0$. Hence this means that the characteristic quasipolynomial

$$\det(sI_n - A - A_h e^{-sh}) = 0$$

has no solutions with $\Re(s) \geq 0$. Consider now

$$\det(sI_n - A - A_h e^{-s(h-\theta)}) = 0$$

and in this case we are interested to find all terms $\zeta := h - \theta$ such that

$$\det\left(j\omega I - A - A_h e^{-j\omega\zeta}\right) \neq 0, \qquad \forall \omega \in \mathbb{R} \tag{3.35}$$

Note that if (3.35) is guaranteed for all $\zeta \geq 0$ then the system is delay independent stable and else we have a delay-dependent stability result. The following theorem proved in [Kharitonov and Niculescu, 2003] is based on matrix pencils [Chen et al., 1995; Niculescu, 2001] and provides a necessary and sufficient condition to stability of uncertain system (3.35).

Theorem 3.2.31 *The linear time-delay system (3.33) with constant delay perturbation θ is robustly stable if and only if the nominal system (3.33) is stable (i.e. for $\theta = 0$) and the following inequality hold*

$$h - \inf\{\beta :\ (\beta,\alpha) \in \Pi_{h,+}\} < \theta < h - \sup\{\beta :\ (\beta,\alpha) \in \Pi_{h,-}\}$$

where

$$\Pi(z) = z\begin{bmatrix} I_p & 0 \\ 0 & \phi_\otimes(A_h, I_n)\end{bmatrix} + \begin{bmatrix} 0 & -I_p \\ \phi_\otimes(I_n, A_h^T) & \phi_\oplus(A, A^T)\end{bmatrix}$$

$$\begin{aligned}
\Pi_{h,+} &= \Big\{(h_{k_i}, \alpha_k) :\ h_{k_i} = \frac{\alpha_k}{\omega_{k_i}} > h :\ e^{-j\alpha_k} \in \tilde{\sigma}(\Pi),\ j\omega_{k_i} \in \tilde{\sigma}\left(A + e^{-j\alpha_k}A_h\right) - \{0\}, \\
&\qquad\qquad 1 \leq k \leq 2p,\ 1 \leq i \leq n\Big\} \\
\Pi_{h,-} &= \Big\{(h_{k_i}, \alpha_k) :\ h_{k_i} = \frac{\alpha_k}{\omega_{k_i}} < h :\ e^{-j\alpha_k} \in \tilde{\sigma}(\Pi),\ j\omega_{k_i} \in \tilde{\sigma}\left(A + e^{-j\alpha_k}A_h\right) - \{0\}, \\
&\qquad\qquad 1 \leq k \leq 2p,\ 1 \leq i \leq n\Big\}
\end{aligned}$$

where $\tilde{\sigma}(\cdot)$ denotes the set of (generalized) eigenvalues of corresponding matrix (pencil) and $\phi_\otimes, \phi_\oplus$ correspond to the following special matrix tensor product and sum (see Appendix A.5 or [Niculescu, 2001]).

This result might be used to analyze stability for systems of the form where two delays are interrelated by an equality/inequality and this deserves future attention. . .

3.2.2.2 Frequency domain: Rouché's Theorem

The Rouché's Theorem, a celebrated result of complex analysis [Levinson and Redheffer, 1970] allows to compute a bound on the maximal deviation from a delay nominal value for systems of the form (3.34). It provides a sufficient condition only and a bound can be easily deduced from the computation of norms of some multivariable transfer functions. It has been used successfully in [Dugard and Verriest, 1998; Sename and Briat, 2006; Verriest et al., 2002].

The Rouché's Theorem [Levinson and Redheffer, 1970] is recalled for reader ease and the proof is provided in Appendix E.6:

Theorem 3.2.32 *Given two functions f and g analytic (holomorphic) inside and on a contour γ. If $|g(z)| < |f(z)|$ for all z on γ, then f and $f+g$ have the same number of roots inside γ.*

Let us consider system (3.34) with constant delay. We tacitly assume that it is asymptotically stable system for $h = h_c$, i.e.

$$\dot{x}(t) = Ax(t) + (A_h^1 + A_h^2)x(t-h)$$

is asymptotically stable. Since we have $h_c = h + \theta$ hence we can write

$$e^{-sh_c} = e^{-sh} + (e^{-s(h+\theta)} - e^{-sh}) = e^{-sh}(1 - \Delta(s))$$

where $\Delta(s) = 1 - e^{-s\theta}$. The characteristic quasipolynomial of the closed-loop system is given by

$$\begin{array}{rcl}\chi(s) & = & \det(sI - A - A_h^1 e^{-sh} - A_h^2 e^{-sh_c}) \\ & = & \det(sI - A - A_h^1 e^{-sh} - A_h^2 e^{-sh}(1-\Delta(s))) \\ & = & \det((sI - A - A_h^1 e^{-sh} - A_h^2 e^{-sh}) + A_h^{(2)} e^{-sh}\Delta(s)) \\ & = & \det(\Psi(s))\det(I + \Psi(s)^{-1}A_h^2 e^{-sh}\Delta(s))\end{array}$$

where $\Psi(s) = sI - A - \left(A_h^1 + A_h^2\right)e^{-sh}$. Since the 'exact' design gives a stable system then $\det(\Psi(s))$ does not change sign when s sweeps the imaginary axis. Then the perturbed closed-loop remains stable if $\det(1 + \Psi(s)^{-1}A_h^2\Delta(s))$ does not change sign for all $s = j\omega$, $\omega \in \mathbb{R}$.

Invoking Rouché's theorem (see appendix E.6) it follows that a stability condition is

$$\left|\left|\Psi(s)^{-1}A_h^2 e^{-sh}\Delta(s)\right|\right|_\infty < 1$$

First recall that $|\Delta(s)| \leq |\delta_h s|$ for all $s = j\omega$, $\omega \in \mathbb{R}$ and where δ_h is an upper bound on the absolute value of delay uncertainty. Finally we have

$$\left|\left|\Psi(s)^{-1}A_h^2 e^{-sh}\Delta(s)\right|\right|_\infty \leq \delta_h \left|\left|\Psi(s)^{-1}A_h^2 e^{-sh}s\right|\right|_\infty$$

and gives the following bound preserving stability

$$\delta_h < 1/||\Psi(s)^{-1}A_h^{(2)}e^{-sh}s||_\infty$$

Hence, for any $\theta \in [-\delta_h, \delta_h]$, the determinant has fixed sign, implying the absence of zero crossings, and henceforth the stability of the perturbed system (provided the nominal one is stable). This approach allows to give an analytic bound on the delay error value but, in the stabilization framework, it is difficult to address a robust stabilization problem directly since the analysis has to be done a posteriori (on the closed-loop system). For this reason, the development of an iterative algorithm which optimizes δ_h seems to be a difficult task.

3.2.2.3 Time-Domain: Small Gain Theorem

Time-domain methods have interesting properties, first they allow for time-varying delays and second it is possible to consider the uncertainty on the delay in the synthesis framework, guaranteeing a prescribed bound on the delay uncertainty. The first method to be investigated is an application of the small-gain theorem. Note that it is possible to rewrite system (3.34) using Leibniz-Newton transformation as

$$\begin{array}{rcl}\dot{x}(t) & = & Ax(t) + (A_h^1 + A_h^2)x(t-h(t)) - \delta_h A_h^2 w(t) \\ z(t) & = & \dot{x}(t) \\ w(t) & = & \dfrac{1}{\delta_h}\displaystyle\int_{t-h_c(t)}^{t-h(t)} z(s)ds\end{array}$$

With a similar argument as the one in [Gu et al., 2003], the $\mathcal{H}_\infty$ norm of the integral operator can be bounded by δ_h and hence a simple application of the scaled small gain theorem allows to provide a robustness analysis by considering the Hamiltonian function

$$H(x_t) = S(x_t) - \int_0^t \begin{bmatrix} x(s-h(s)) - x(s-h_x(s)) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} L & 0 \\ 0 & -L \end{bmatrix} (\star)^T$$

where $S(x)$ is the storage function and the integral is the supply-rate condition related to the scaled-small gain. Note that any Lyapunov-Krasovskii functional may play the role of the storage function $S(x)$. Similar results have been provided in [Gu et al., 2003; Jiang and Han, 2006, 2005] where a time-varying delay is approximated by a constant one and where the uncertainty represents the time-varying part.

3.2.2.4 Time-Domain: Lyapunov-Krasovskii functionals

Since the scaled-small gain theorem may lead to conservative results it would be more convenient to use a Lyapunov-Krasovskii approach to deal with such a problem of stabilization with incorrect delay value. Considering again system (3.34), the Lyapunov-Krasovskii functional

$$\begin{array}{rcl}V(x_t) & = & V_n(x_t) + V_u(x_t) \\ V_n(x_t) & = & x(t)^T P x(t) + \displaystyle\int_{t-h}^{t} x(s)^T Q x(s) ds + \int_{-h}^{0}\int_{t+\beta}^{t} \dot{x}(s)^T R x(s) ds d\beta \\ V_u(x_t) & = & \displaystyle\int_{t-h_c}^{t} x(s)^T S x(s) ds + \int_{\delta^-}^{\delta^+}\int_{t+\beta-h}^{t} \dot{x}(s) T \dot{x}(s) ds d\beta\end{array}$$

can lead to a robust stability analysis criterium for system (3.34). This will be detailed in Section 4.7. Note that Kharitonov and Niculescu [2003] have also provided a solution in terms of a complete Lyapunov-Krasovskii functional for such a goal in the single delay framework.

3.3 Chapter Conclusion

A brief first section has been devoted to different types of time-delay systems representations: systems over a ring, infinite-dimensional systems over a abstract space and functional differential equations. The latter representation has been chosen to be considered since many tools are available to study them (e.g. Lyapunov-Krasovskii theorem) and can be easily extended to LPV case.

There exist several frequency domain methods that work for system with constant time-delays and some of these methods might provide necessary and sufficient conditions for stability. Since these methods cannot be extended to time-varying systems and systems with time-varying delays, time-domain approaches have been privileged in this chapter since they extend to LPV time-delay systems. Even if all examples of criteria have been developed for systems with constant delays, most of them can be, more or less easily, extended to time-varying delays (except Padé approximation which is actually extended, in a somewhat certain sense, either in [Gouaisbaut and Peaucelle, 2006a] or [Kao and Rantzer, 2007] as explained in the above section).

Among time-domain techniques, fundamental theorems extending Lyapunov's theory have been provided and illustrated through examples of stability tests. While Lyapunov-Razumikhin Theorem is a simple test involving to the use of function, the Lyapunov-Krasovskii employs functionals. However, while Lyapunov-Razumikhin tests are not LMIs, Lyapunov-Krasovskii tests are shown to be LMIs and thus provide more general results. As an example, the Lyapunov-Razumikhin delay-independent stability test is a particular case of the Lyapunov-Krasovskii one. The evolution of Lyapunov-Krasovskii criteria has been discussed by a successive introduction of model transformations, additional dynamics and the problem of cross-terms.

Scaled-small gain theorem can be used to develop stability criteria for time-delay systems and a connection between Lyapunov-Krasovskii results has been emphasized. Moreover, these results have also been derived in the IQC framework in [Jun and Safonov, 2001].

A technique based on a an approximation of the delay element by Padé approximants has been presented and shown as an interesting and efficient technique but unfortunately limited to constant delay-case.

In order to relieve this lack, IQC techniques using the efficient input/output behavior point of view provide a very tight solution to the stability analysis of time-delay systems using same operators as in [Zhang et al., 1999] but extended in the time-varying case. A recent result based on well-posedness has been introduced and is related to recent results based on Lyapunov-Krasovskii functionals.

Finally, results on robust stability of systems with respect to delay uncertainty have been provided as an anticipation of future use in this thesis. Both frequency and time domain techniques have been provided as point of comparison.

All the time-domain techniques have not been introduced in this section and, as an insight, the reader should refer to [Briat et al., 2007a, 2008a; Gouaisbaut and Peaucelle, 2007; Han and Gu, 2001; He et al., 2007; Jiang and Han, 2006, 2005; Kharitonov and Niculescu, 2003; Knospe and Roozbehani, 2006, 2003; Michiels et al., 2005; Roozbehani and Knospe, 2005] and references therein for other techniques. Among them it is important to distinguish range-stability analysis which addresses the problem of finding a compact set of delay value (possibly excluding 0) for which the system is stable (similarly as for robustness analysis with delay uncertainty). Most of these results are based on Lyapunov-Krasovskii functionals or approximation of delay elements.

Chapter 4

Definitions and Preliminary Results

THIS CHAPTER aims at introducing some basic concepts and fundamental results used along the thesis. Section 4.1 provides redundant notions such as delay and parameter spaces and the class of LPV time-delay systems under consideration throughout this thesis. These definitions are quite standard and their relevance will be briefly emphasized as a justification. Finally an example of LPV time-delay systems is given in order to motivate the interest of our work on this kind of systems.

Section 4.2 will provide a relaxation method for polynomially parameter dependent Linear Matrix Inequalities. Indeed, it is well known that parametrized LMIs consist in an infinite (uncountable) number of LMIs that have to be satisfied. When the dependence is affine, a convexity argument, as in the polytopic approach (see Section 2.3.2), allows to conclude on the feasibility of the whole set of LMIs only by considering a particular finite set of LMIs (more precisely the LMIs evaluated at the vertices of the convex polyhedral set containing parameters values). On the other hand, when the dependence is polynomial, it is not necessary and sufficient to consider only the vertices of the set of values of the parameters [Apkarian and Tuan, 1998; Oliveira and Peres, 2006; Trofino and De Souza, 1999; Tuan and Apkarian, 1998; Tuan et al., 2001a] except for very special simple or conservative cases. Indeed, such a relaxation can only be considered under certain strong assumptions on the degree of polynomials and involved matrices. There exists different approaches to solve very efficiently and accurately this type of problems (see Sections 2.3.3.2, 2.3.3.3 and 2.3.3.4). We will provide here an approach based on spectral factorization of parameter dependent matrices and the Finsler's lemma (Appendix D.16). This approach will turn the polynomially parameter dependent LMI into a slightly more conservative LMI involving 'slack' variables. Such an LMI will have the interesting feature of having an affine parameter dependence on which convex relaxations can be applied without any conservatism. Such an approach has been introduced in [Briat et al., 2008a; Sato, 2006; Sato and Peaucelle, 2007].

Section 4.3 is devoted to the development of a new relaxation for concave nonlinearity of the form $-\alpha^T\beta^{-1}\alpha$ with $\beta = \beta^T \succ 0$ in negative definite LMIs. Several approach to deal with such non linearities have been provided in the literature: the hyperplane bound and an application of the cone complementary algorithm. While the first one is too conservative since it corresponds to the linearization of the nonlinearity around some fixed point, the second one cannot be applied on parameter-varying matrices. These two limitations motivated us to introduce a new method based on the introduction of a 'slack' variable with the drawback of keeping a nonlinear structure of the problem (the problem becomes BMI). However, even if the structure remains complex and cannot be efficiently solved by interior point algorithms as LMIs, it has a nicer form that the initial problem and can be efficiently solved with iterative LMIs procedures. A discussion is then provided in order to explain the algorithm, its initialization step and optimality gap compared to the initial problem.

Section 4.4 aims at providing a simple algebraic approach in order to compute bounds on the rate of variation of parameters in the polytopic framework. Indeed, in the literature, most of the approaches consider LPV polytopic systems with unbounded parameter variation rates which is rather conservative since it considers constant Lyapunov functions concluding then on quadratic stability. When a general parameter dependent system is turned into a polytopic formulation, the values and the dependence is 'hidden' into the new parameters since a 'mixing' of all parameters is performed. From this consideration it is difficult to make a correspondence between the derivative of the initial parameters and the derivative of the polytopic parameters. This section provides then a simple methodology to compute these bounds.

Section 4.5 aims at providing a simple stability/performances test expressed through parameter dependent LMIs for LPV time-delay systems. This approach is based on the generalization of a simple Lyapunov-Krasovskii functional introduced for instance in [Gouaisbaut and Peaucelle, 2006b; Han, 2005a]. This result has the benefit of being interesting from a computational point of view since it involves a few matrix variables and no model transformation is employed. However, it is difficult to use it for synthesis purposes and this motivates the development of an associated relaxation leading to another LMI which can be efficiently used for design objectives.

Section 4.6 extends the 'simple' approach to a discretized version of a more complex Lyapunov-Krasovskii functional. The same relaxation scheme is then applied in order to get an LMI adapted to design objectives.

Finally, Section 4.7 develops a new Lyapunov-Krasovskii functional for systems with two delays where the delays satisfy an algebraic constraint. Such a configuration occurs whenever a time-delay system is controlled/observed by a controller/observer implementing a delay which is different from the system one. In this case it is important to take into account this specific problem in order to ensure robustness of the closed-loop stability/performances.

4.1 Definitions

This section is devoted to the introduction of the definitions which will be used along the thesis. First of all, delay spaces under consideration will be defined. Restrictions on these sets will be introduced and justified through simple examples. Second, parameters sets will be introduced and a particular class, the delayed parameters, will be introduced and their properties analyzed (continuity, differentiability, set of values...). Finally, the class of LPV systems which will be analyzed in the thesis will be introduced with an example of a milling process borrowed from [Zhang et al., 2002].

4.1.1 Delay Spaces

Even if only one type of delays will be considered in this thesis, it seems important to define common sets that can be encountered in the literature. Each delay-space considers a particular stability result: delay-dependent/independent and rate dependent/independent. Due to the large diversity of these spaces, only some of them are described below:

$$\mathscr{H}_1 := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+, [h_{min}, h_{max}]) : \; |\dot{h}| < \mu \right\}$$

which defines bounded delay with bounded derivative. It is assumed that when $\mu = 0$ then the delay is constant. We will denote further the set $\mathscr{H}_1^\circ$ the particular case when $h_{min} = 0$:

$$\mathscr{H}_1^\circ := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+, [0, h_{max}]) : \; |\dot{h}| < \mu \right\} \tag{4.1}$$

The set

$$\mathscr{H}_2 := \{ h : \mathbb{R}_+ \to [h_{min}, h_{max}] \}$$

defines the set of bounded delays with unbounded derivatives. However, unbounded derivatives may lead to causality problems as commented in [Ivanescu et al., 2003]. We will denote further the set $\mathscr{H}_2^\circ$ the particular case when $h_{min} = 0$. Then,

$$\mathscr{H}_3 := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+) : \; |\dot{h}| < \mu \right\}$$

defines the set of unbounded delays with bounded derivatives. Finally,

$$\mathscr{H}_4 := \{ h : \mathbb{R}_+ \to \mathbb{R}_+ \}$$

corresponds to the set of unbounded delays with unbounded derivatives.

Among them, the most relevant and useful sets are $\mathscr{H}_1$ and $\mathscr{H}_2$. In many cases, $\mathscr{H}_2$ is useful when no information is available on the rate of variation of the delay. On the second hand, when dealing with delays with bounded derivatives the Lyapunov-Krasovskii functional approach can only be used whenever the delay derivative in less than 1 (or in some cases between -1 and 1), which is very constraining since it appears to be difficult to deal with between delay derivatives between 1 and $+\infty$. Model transformations can be used in order to deal with such cases; see for instance [Gu et al., 2003; Jiang and Han, 2005; Shustin and Fridman, 2007].

The argument that the delay derivative must be greater than -1 can be justified by considering input delay systems and is not of interest in the case of state-delayed system. However this will be explained for completeness. To see this, consider the problem of Figure 4.1 where an transmitter sends data to a receiver continuously (the data is a continuous flow). The data are driven through a medium of length ℓ with with a finite variable speed $v(t)$ depending on the time instant of emission (as in a network where the speed of propagation depends on the occupation of the servers). Hence, the time of transmission is given by $h(t) = \ell v(t)$. When a data is transmitted at times t and $t + \delta t$, they will be received at times $t + h(t)$ and $t + \delta t + h(t + \delta t)$ respectively. The causality principle claims that if a data value is emitted at time t, it will reach the receiver before the data emitted at time $t + \delta t$ for every $\delta t > 0$. This is translated in the formal expression

$$t + h(t) < t + \delta t + h(t + \delta t)$$

then we have

$$-\delta t < h(t + \delta t) - h(t)$$

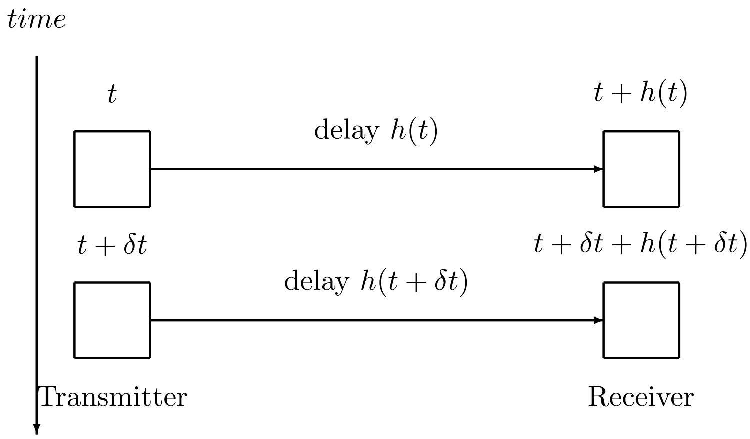

Figure 4.1: Illustration of continuous data transmission between two entities

and thus

$$-1 < \frac{h(t+\delta t) - h(t)}{\delta t}$$

Since the inequality is true for every $\delta t > 0$ then we get

$$-1 < \dot{h}(t)$$

This condition ensures that once emitted the data will be received in a correct order. Note that it might not be the case when considering the control of system over a network using packet switching. In such a case, since the data might not follow the same path, it is not guaranteed that the data will be received in a correct order (this is the reason why the TCP protocol implements a packet counter allowing to reorganize the packets once received).

The second idea, which is important for state-delayed systems, is to look at the evolution of the function $f(t) = t - h(t)$ compared to t. It is clear that $f(t) \leq t$ which means that $h(t) \geq 0$ but it is also interesting to have $f(t)$ increasing. Indeed, having $f(t)$ increasing means that there exists an inverse function $f^{-1}(\cdot)$ and in some applications and computations this property is important. If for some time values t, $f(t)$ is locally decreasing, then this means that there exist $t_1 < t_2$ such that $t_2 - h(t_2) = t_1 - h(t_1)$. This would mean that the same data is considered at different times which may be incorrect (depending on the context).

Let $t_2 = t_1 + \delta t$ with $\delta t > 0$ and thus we have $t_1 + \delta t - h(t_1 + \delta t) = t_1 - h(t_1)$ which is equivalent to

$$\delta t - h(t_1 + \delta t) = -h(t_1)$$

and finally

$$1 = \frac{h(t_1 + \delta t) - h(t_1)}{\delta t}$$

If δt tends to 0, we get

$$1 = \dot{h}(t)$$

This shows that if the delay derivative reaches 1 at some time-instants, then the same data will be used at these different times. If this has to be avoided, by continuity, it suffices to restrict $\dot{h}$ to satisfy the inequality

$$\dot{h}(t) < 1 \tag{4.2}$$

Such a function $t - h(t)$ satisfying this property is depicted on Figure 4.2.

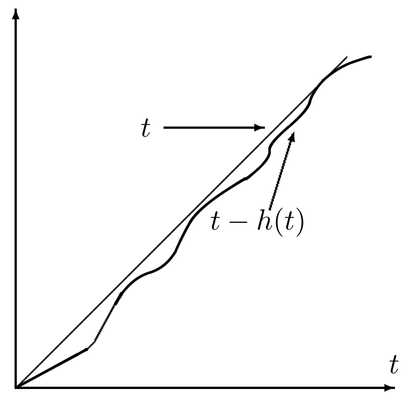

Figure 4.2: Illustration of the nondecreasingness of the function $t - h(t)$

In some specific applications, for instance control of systems with varying sampling-rate (sampled-data systems) [Fridman et al., 2004; Suplin et al., 2007] time delay systems having a derivative equals to 1 almost everywhere are used. Indeed, systems with zero-order hold on the input are turned into input time-varying delay systems where the time-varying delay describes the zero-order hold with variable sampling period. Since a zero-order hold maintains a specific value during a certain amount of time (the period) it seems obvious that $\dot{h}(t) = 1$ all over the period (in this case all works as if the time was frozen over the period).

4.1.2 Parameter Spaces

This section is devoted to the description of the considered parameter sets. Only common sets will be briefly introduced since more details have been given in Section 2.1. Along this thesis we will mainly focus on continuous parameters (smooth and nonsmooth). In some applications, delayed parameters are encountered and basic properties (continuity and differentiability) of such parameters will be discussed hereafter. First of all, let us introduce the following sets:

$$U_\rho := \times_{i=1}^{N_p}[\rho_i^-, \rho_i^+] \text{ compact of } \mathbb{R}^{N_p}$$

where $N_p > 0$ is the number of parameters and

$$U_\nu := \times_{i=1}^{N_p}\{\nu_i^-, \nu_i^+\}$$

The set U_ρ is the set of values taken by the parameters and is supposed to be a bounded orthotope of $\mathbb{R}^{N_p}$. On the second hand, the set U_ν is a discrete set of $\mathbb{R}^{N_p}$ containing 2^{N_p} values. It contains the set of vertices of the orthotope where the parameter derivative values evolve. Hence this orthotope is defined as the convex hull of the points contained in U_ν and is denoted hull$[U_\nu]$.

In some applications, the delay might act on some parameters or the use of particular Lyapunov-Krasovskii functionals introduces delayed parameters [Zhang et al., 2002]. Thus it seems important to introduce this important case. Obviously, the set of values of the delayed parameters must be included into the set of non-delayed parameters. In an absolute point of view they coincide but it will be shown in the following that this set of delayed parameters values can be smaller than the set of initial parameters.

Let us consider the case $N_p = 1$ (for simplicity) and define the delayed parameter as $\rho(t-h(t))$. Moreover, without loss of generality let $\nu := \nu^+ = -\nu^-$ and $\bar{\rho} = \rho^+ = -\rho^-$. Since the parameter is supposed to be continuous and differentiable almost everywhere then it satisfies the so-called Lipschitz condition

$$|\rho(t_2) - \rho(t_1)| \leq \nu|t_2 - t_1|$$

for any $t_1 \neq t_2$, $t_1, t_2 \in \mathbb{R}_+$. Hence assuming that $t_2 > t_1$ then we have

$$-\nu(t_2 - t_1) \leq \rho(t_2) - \rho(t_1) \leq \nu(t_2 - t_1)$$

Let $t_2 = t$ and $t_1 = t - h(t)$ then we obtain

$$-\nu h(t) \leq \rho(t) - \rho(t - h(t)) \leq \nu h(t)$$

and hence we obtain

$$\rho(t) - \nu h(t) \leq \rho(t - h) \leq \rho(t) + \nu h(t)$$

Since in most of the cases the current value of the delay $h(t) \in [h_{min}, h_{max}]$ is generally unknown then it is more convenient to consider

$$\rho(t) - \nu h_{max} \leq \rho(t - h) \leq \rho(t) + \nu h_{max}$$

This shows that the set of values of the delayed parameters depends on the rate of variation of the parameters ν and the maximal delay value h_{max}. Hence for sufficiently small ν and h_{max} the set of values of the delayed parameters does not coincide with U_ν and is reduced to a neighborhood of the value of $\rho(t)$ at time t. This neighborhood, in the one dimensional case, is an interval centered around $\rho(t)$ with radius νh.

Proposition 4.1.1 *If $\nu h_{max} \geq 2\bar{\rho}$ then the set of value of $\rho(t - h)$ coincides with U_ρ for every $t \geq 0$.*

A direct analysis shows that if the parameters are discontinuous (i.e. unbounded derivatives) and/or the delay is unbounded (i.e. $h_{max} = +\infty$), then the set of delayed-parameters coincide with the set of non-delayed parameters.

Proposition 4.1.2 *If $\nu h_{max} < 2\bar{\rho}$ then the set of value of $\rho(t-h)$ is included in U_ρ for every $t \geq 0$ and is depicted in Figure 4.3.*

The set of generalized parameters (ρ, ρ_h) where ρ_h is the delayed parameter is a polyhedral with 6 vertices and 6 edges. Moreover the set of values of the delayed parameter ρ_h can be parametrized by ρ:

$$U_{\rho_h}(\rho) := \{y \in \mathbb{R} : \ |y - \rho| \leq \nu h\} \cap U_\rho$$

Hence the set of all values for ρ_h is given by

$$\bar{U}_{\rho_h} := \{y \in \mathbb{R} : \ |y - x| \leq \nu h, \ x \in U_\rho\} \cap U_\rho$$

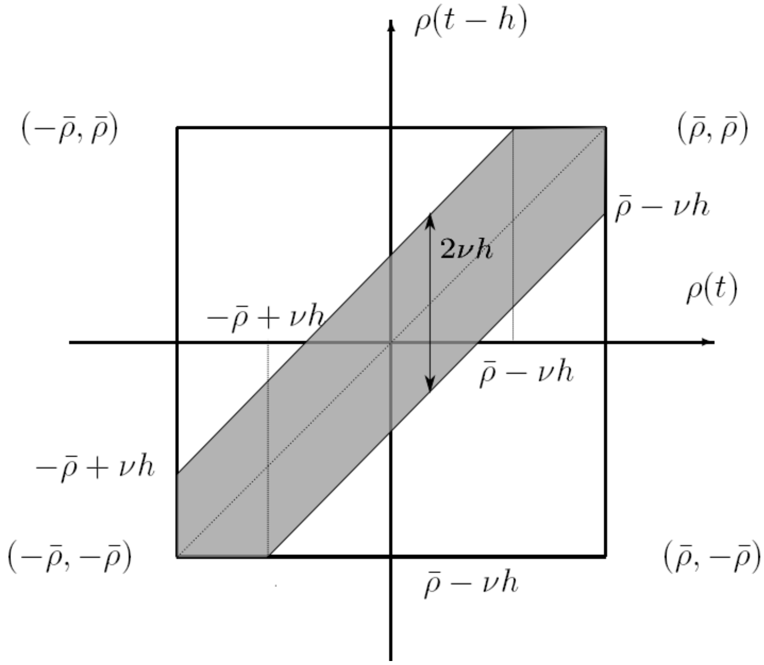

Figure 4.3: Set of the values of $\rho(t-h)$ (in grey) with respect to the set of value of $\rho(t)$ (the horizontal interval $[-\bar\rho,\bar\rho]$

and the whole set $\bar U_\rho$ of values of (ρ,ρ_h) is defined by

$$\bar U_\rho := \{(\rho_1,\rho_2):\ \rho_1\in U_\rho,\ \rho_2\in U_{\rho_h}(\rho_1)\}$$

Let us consider now the derivative of the delayed parameters for the particular case of continuous parameters. In the case of constant delay, the set of delayed parameter derivative values coincides with the set $[-\nu,\nu]=\text{hull}[\{-\nu,\nu\}]$ since the delay is constant, i.e.

$$\frac{d}{dt}\rho(t-h)=\rho'(t-h)\in[-\nu,\nu]$$

However, in the case of a time-varying delay two cases may occur according to the type of the rate of variation (bounded or unbounded) of the delay. Assume first that the rate is bounded and then we have

$$\frac{d}{dt}\rho(t-h(t))=(1-\dot h(t))\rho'(t-h(t))$$

and hence we have

$$-(1+\mu)\nu\le\frac{d}{dt}\rho(t-h(t))\le(1+\mu)\nu$$

This shows that the set of values of the rate of variation of delayed parameters is larger than for the nondelayed ones. Finally, if the delay derivative is unbounded then the rate of variation of delayed parameters is unbounded too.

4.1.3 Class of LPV Time-Delay Systems

Throughout this thesis, the following class of LPV time-delay systems [Wu, 2001b; Zhang and Grigoriadis, 2005] will be considered if not stated otherwise:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t) \\ x(\theta) & = & \phi(\theta), \quad \theta \in [-h_{max}, 0] \end{array} \tag{4.3}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $z \in \mathbb{R}^p$ are respectively the system state, the exogenous input and the controlled output. Such a class captures a wide class of LPV time-delay systems. Moreover, the delay is assumed to belong to $\mathscr{H}_1^\circ$ with $h_{min} = 0$ and the N_p parameters $\rho \in U_\rho$, $\dot{\rho} \in U_\nu$.

Such systems arise in many nonlinear physical systems with delay approximated using LPV systems. For instance, in [Zhang et al., 2002] a milling process is modeled as a LPV time-delay systems as shown below: The corresponding model is given by the expressions

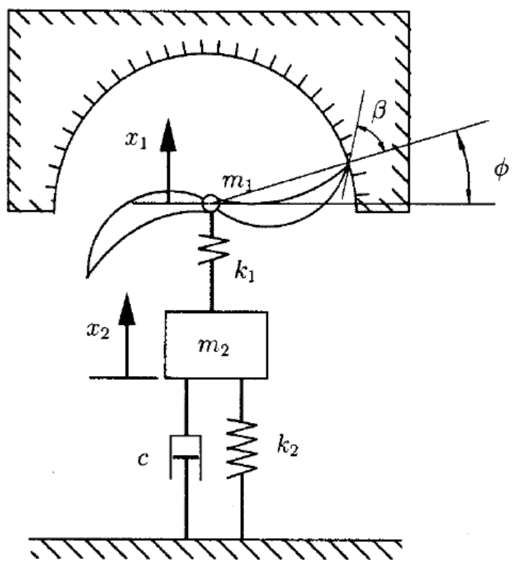

Figure 4.4: Simplified geometry of a milling process

$$\begin{array}{l} m_1\ddot{x}_1 + k_1(x_1 - x_2) = f \\ m_2\ddot{x}_2 + c\dot{x}_2 + k_2(x_2 - x_1) + k_2 x_2 = 0 \\ f = k\sin(\phi+\beta)z(t) \\ z(t) = z_a + \sin(\phi)[x_1(t-h) - x_1(t)] \end{array}$$

where k_1 and k_2 are the stiffness coefficients of the two springs, c is the damping coefficient, m_1 is the mass of the cutter, m_2 is the mass of the 'spindle'. The displacements of the blade and tool are x_1 and x_2 respectively. The angle β depends on the particular material and tool used, and is constant. The angle ϕ denotes the angular position of the blade and k is the cutting stiffness. z_a is the average chip thickness (here assumed, without loss of generality, $z_a = 0$) and $h = \pi/\omega$ is the delay between successive passes of the blades. This system can be modeled as a LPV system with delay of the form

$$\dot{x}(t) = (A + A_k k + A_\gamma \gamma + A_{k\gamma} k\gamma)x(t) + (A^h + A^h_k k + A^h_\gamma \gamma + A^h_{k\gamma} k\gamma)x(t-h) \tag{4.4}$$

where the parameters are the stiffness $k = k_1 = k_2$ and $\gamma = \cos(2\phi + \beta) \in [-1, 1]$. An interesting discussion about this process is provided in [Zhang et al., 2002].

4.2 Relaxation of Polynomially Parameter Dependent LMIs

In this section, we present a method of relaxation of polynomially parameter dependent LMIs which has been also provided in [Sato, 2006; Sato and Peaucelle, 2007]. It will be used to deal with parameter dependent LMIs (also called 'robust LMIs') that arise, for instance, in the (robust) stability analysis of uncertain and LPV systems.

Since several years, many results on relaxation of polynomially parameter dependent LMIs have been provided in the literature. Even if many were applied to polynomial of degree 2, arising for instance in gain-scheduled state-feedback controller for polytopic systems, most of them can be applied to polynomial of higher degree. For instance, let us mention the following works on this topic [Apkarian and Tuan, 1998; Geromel and Colaneri, 2006; Oliveira and Peres, 2002, 2006; Oliveira et al., 2007; Scherer, 2008; Tuan and Apkarian, 1998, 2002].

The approach provided in this section is close to the Sum-of-Squares relaxation in the sense that the matrix of polynomials is represented in a spectral form (See Section 2.3.3.3). But at the difference of the classical SOS approach, this method does not involve any choice or decision of the designer (such as the degree of polynomials) except the choice of the basis in which the polynomial is expressed (by basis we mean the outer factor of the spectral form). We will also see that this method linearizes the dependence on the parameters, and thus turns a polynomially parameter dependent LMI into a conservative affine version involving a slack variable.
Let us consider the parameter dependent LMI $\mathcal{M}(x, \rho)$ defined by

$$\mathcal{M}(x, \rho) := \mathcal{M}_0(x) + \sum_{i=1}^{N} \mathcal{M}_i(x) u_i(\rho) \tag{4.5}$$

where $\mathcal{M}_i(x) \in \mathbb{S}^n$, $x \in \mathbb{R}^d$ denotes the vector of decision variables and $u_i(\rho)$ are monomials in $\rho = \underset{i}{\mathrm{col}}(\rho_i) \in U_\rho$.

The following result details the transformation of the polynomially parameter dependent LMI into an affine form:

Theorem 4.2.1 *Let us consider a polynomially parameter dependent matrix inequality of the form (4.5). It can be written into a spectral form*

$$\Theta_\perp(\rho)^T \mathcal{M}(x) \Theta_\perp(\rho) \prec 0 \tag{4.6}$$

where $\mathcal{M}$ is a parameter independent symmetric matrix constructed from $\mathcal{M}_0(x)$, $\mathcal{M}_i(x)$ and $\Theta_\perp(\theta)$ a rectangular matrix gathering monomials occurring in the parameter dependent LMIs (e.g. $\Theta_\perp(\rho) = \mathrm{col}(1, u_1(\rho), \ldots, u_n(\rho))$). Then (4.6) is feasible in $x \in \mathbb{R}^d$ for all $\rho \in U_\rho$ if there exists $x \in \mathbb{R}^d$ and a matrix $\mathcal{P}$ of appropriate dimensions such that

$$\mathcal{M}(x) + \mathcal{P}^T \Theta(\rho) + \Theta(\rho)^T \mathcal{P} \prec 0$$

holds for all $\rho \in U_\rho$ and $\Theta(\rho)\Theta_\perp(\rho) = 0$. Moreover, with an appropriate choice of $\Theta_\perp(\rho)$ then $\Theta(\rho) = \sum_{i=1}^{N} \Theta_i \rho_i$ is affine in ρ_i.

Proof: *The proof is a simple application of the Finsler's lemma to the parameter dependent LMIs. Consider first the parametrized LMI in its spectral form (4.6) and then invoking the Finsler's lemma (see Appendix D.16), we can claim that this is equivalent to the existence of* $\mathcal{P}(\rho)$ *such that*

$$\mathcal{M}(x) + \mathcal{P}(\rho)^T\Theta(\rho) + \Theta(\rho)^T\mathcal{P}(\rho) \prec 0$$

holds. However, since the aim of the procedure is the linearization of the parameter dependence then by restricting $\mathcal{P}$ *to be parameter independent we get*

$$\mathcal{M}(x) + \mathcal{P}(\rho)^T\Theta(\rho) + \Theta(\rho)^T\mathcal{P}(\rho) \prec 0 \Rightarrow \Theta_\perp(\rho)^T\mathcal{M}(x)\Theta_\perp(\rho) \prec 0$$

We aim to show now that, for any polynomial, it is possible to construct a corresponding $\Theta(\rho)$ *to* $\Theta_\perp(\rho)$ *such that* $\Theta(\rho)$ *is affine in* ρ. *To show this, note that the trivial basis for univariate polynomials* $\Theta_\perp(\rho) = \mathrm{col}(1, \rho, \rho^2, \ldots, \rho^n)$ *admits*

$$\Theta(\rho) = \begin{bmatrix} -\rho & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & -\rho & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -\rho & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -\rho & 1 & \ldots & 0 \\ \vdots & & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix}$$

Hence since the trivial basis is the kernel of an affine parameter dependent matrix then it is possible to find an affine $\Theta(\rho)$ *for every univariate polynomial. This generalizes directly to the multivariate case.* $\square$

It is worth noting that using the trivial functions $u_i(\rho) = \rho^i$ in $\Theta_\perp(\rho)$ (for the univariate case) is not the best choice in general. Indeed, the most intuitive choice is to choose $\Theta_\perp(\rho) = col(1, \rho_1, \ldots, \rho_N, \rho_1^2, \ldots)$ which will give an affine $\Theta(\rho)$ but with increased complexity since the dimension of $\mathcal{M}$ is larger than in the case of taking the $u_i(\rho)$'s. Hence, it is important to point out the properties of a nontrivial basis $\Theta_\perp(\rho)$ (reduced dimension) for which $\Theta(\rho)$ is affine. Actually, if the polynomials (may not be exclusively monomials) $v_i(\rho)$ components of $\Theta_\perp(\rho)$ are chosen to satisfy

$$v_i(\rho) = \sum_j^n p_{ij}(\rho)v_j(\rho)$$

where the $p_{ij}(\rho)$'s are affine polynomials in ρ and N is the size of the basis, then there exists an affine $\Theta(\rho)$. The latter equality can be rewritten into the compact form

$$v(\rho) = P(\rho)v(\rho)$$

where $v(\rho) = \underset{i}{\mathrm{col}}(v_i(\rho))$ and $P(\rho) = \begin{bmatrix} p_{11}(\rho) & \ldots & p_{1N}(\rho) \\ \vdots & \ddots & \vdots \\ p_{N1}(\rho) & \ldots & p_{NN}(\rho) \end{bmatrix}$ or equivalently

$$(I - P(\rho))v(\rho) = 0$$

which is kind of a generalized eigenvalue problem. It is worth mentioning that the computational complexity of the procedure depends on the number N of functions $u_i(\rho)$. Hence the problem results, for a given $\mathcal{M}(X,\rho)$, in finding the minimal N such that

$$\begin{aligned}
&\det(I - P(\rho)) = 0\\
&v(\rho) \in \mathrm{Ker}[I - P(\rho)]\\
&\mathcal{M}(x,\rho) := \sum_{i=1}^{N} \mathcal{M}_i(x) v_i(\rho)
\end{aligned}$$

for some $\mathcal{M}_i(x)$ and for all $\rho \in U_\rho$ with $P(\rho)$ affine in ρ. Indeed, if this condition is satisfied this means that there exists a $\Theta_\perp(\rho)$ which is a basis of the null space of an affine matrix $\Theta(\rho)$. This optimization problem is non trivial since it is a semi-infinite dimensional problem where the cost is the dimension of a basis. This gives rise to interesting optimization problem that will not be treated here but belongs to further works and investigations.

Coming back to theorem (4.2.1), it is possible to derive an important result for LMI involving quadratic polynomial dependence, useful in polytopic systems.

Corollary 4.2.2 *The following parameter dependent matrix inequality is feasible*

$$\mathcal{M}(\lambda) = \mathcal{M}_0 + \sum_{i=1}^{N} \lambda_i \mathcal{M}_i + \sum_{i,j=1}^{N} \lambda_i \lambda_j \mathcal{M}_{ij} \prec 0 \tag{4.7}$$

provided that $\sum_{i=1}^{N} \lambda_i = 1,\ \lambda_i \geq 0$ *if there exists* $\mathcal{Z}$ *such that*

$$\tilde{\mathcal{M}} + \mathcal{Z}^T \Pi(\lambda) + \Pi(\lambda)^T \mathcal{Z} < 0$$

is feasible for all $\lambda \in \Lambda$ *where*

$$\Pi(\lambda) = \begin{bmatrix} -\lambda_1 I & I & 0 & \dots & 0\\ -\lambda_2 I & 0 & I & \dots & 0\\ \vdots & \vdots & & \ddots & 0\\ -\lambda_N I & 0 & 0 & \dots & I \end{bmatrix}$$

$$\tilde{\mathcal{M}} = \begin{bmatrix} \mathcal{M}_0 & \mathcal{M}_1/2 & \dots & \mathcal{M}_N/2\\ \star & \mathcal{M}_{11} & \dots & (\mathcal{M}_{1N} + \mathcal{M}_{N1})/2\\ \vdots & \vdots & \ddots & \vdots\\ \star & \star & \dots & \mathcal{M}_{NN} \end{bmatrix}$$

The latter LMI is feasible if and only if

$$\tilde{\mathcal{M}} + \mathcal{Z}^T \Pi_i + \Pi_i^T \mathcal{Z} < 0$$

where $\Pi(\lambda) = \sum_{i=1}^{N} \lambda_i \Pi_i$.

The following example shows the interest of the approach:

Example 4.2.3 *Let us consider the univariate polynomial*

$$p(x) = -x^4 + 4x^3 + 43x^2 - 58x - 240$$

whose graph is depicted on Figure 4.5

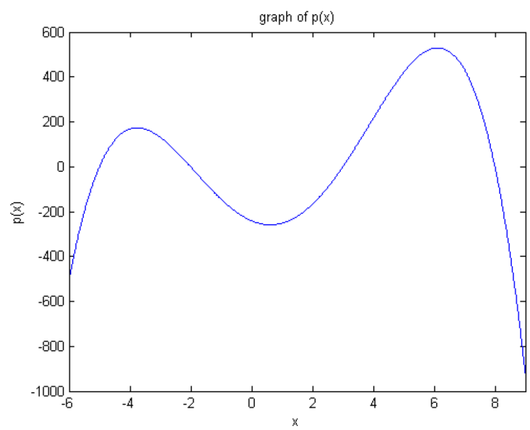

Figure 4.5: Graph of the polynomial $p(x)$ over $x \in [-6, 9]$

The goal is to find the supremum of $p(x)$ over the interval $[-6, 9]$, hence we are looking for the minimal value of γ such that

$$p(x) \leq \gamma \ \forall x \in [-6, 9]$$

which is equivalent to the following optimization problem

$$\begin{array}{l} \min \gamma \ s.t. \\ p(x) - \gamma \leq 0 \\ x \in [-6, 9] \end{array}$$

First of all, $p(x) - \gamma$ is rewritten in the spectral form (the repartition of the terms along anti-diagonals is arbitrary):

$$p(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} -\gamma - 240 & -29 & 20 \\ \star & 3 & 2 \\ \star & \star & -1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \leq 0$$

Applying Theorem 4.2.1, we get the following LMI

$$\begin{bmatrix} -\gamma - 240 & -29 & 20 \\ \star & 3 & 2 \\ \star & \star & -1 \end{bmatrix} + N^T \Theta(x) + \Theta(x)^T N \preceq 0$$

where N is a free matrix variable belonging to $\mathbb{R}^{2\times 3}$ and $\Theta(x)$ is defined such that

$$\Theta(x) \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = 0$$

A suitable choice is given by

$$\Theta(x) = \begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \end{bmatrix}$$

Finally we get the parameter dependent LMI with linear dependence in x:

$$\mathcal{P}(\gamma, x) = \begin{bmatrix} -\gamma - 240 & -29 & 20 \\ \star & 3 & 2 \\ \star & \star & -1 \end{bmatrix} + N^T \begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \end{bmatrix} + \begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \end{bmatrix}^T N \preceq 0$$

Hence with a polytopic argument, the optimization problem becomes

$$\begin{array}{l} \min \gamma \ s.t. \\ \mathcal{P}(\gamma, -6) \preceq 0 \\ \mathcal{P}(\gamma, 9) \preceq 0 \end{array}$$

Solving this SDP we get $\gamma_{opt} = 529.6340928975$ *found with*

$$N = \begin{bmatrix} -10.0347 & 5.4752 & 0.2318 \\ -3.6163 & -0.0356 & -0.03792 \end{bmatrix}$$

The theoretical result is given by $s := \sup_{x \in [-6,9]} p(x) = 529.63265619463$ *and the computation error is*

$$\varepsilon := \gamma - s = 0.001436702914$$

We can see that the computed maximum is very close to the theoretical one. This shows that this relaxation may lead to interesting results.

4.3 Relaxation of Concave Nonlinearity

Concave nonlinearities are the most difficult nonlinearities to handle in the LMI framework. They may appear in many problems especially when congruence transformations are performed and occur for instance in the problems studied in [Briat et al., 2008c; Chen and Zheng, 2006; Daafouz et al., 2002; Gao and Wang, 2003; Geromel et al., 2009] and certainly in many others. First of all, known solutions will be presented and explained and finally the new 'exact' relaxation will be provided. Indeed, it is well known that, even if the following problem in ε, α and β is nonlinear

$$\varepsilon + \alpha^T \beta^{-1} \alpha \prec 0, \quad \varepsilon = \varepsilon^T \prec 0, \quad \beta = \beta^T \succ 0$$

the problem is convex since the nonlinearity $\alpha^T \beta^{-1} \alpha$ is convex. A Schur complement (Appendix D.4) on this matrix inequality yields the matrix

$$\begin{bmatrix} \varepsilon & \alpha^T \\ \alpha & -\beta \end{bmatrix} \prec 0$$

which is affine (and then convex) in the decision variable. But the question is what happens when the sign '+' is turned into a sign '-' ? In such as a case, the convex nonlinearity becomes concave and the Schur complement does not apply anymore. The following section aims at providing solutions on the relaxation of such nonlinearity.

Let us consider now the following nonlinear matrix inequality

$$\varepsilon - \alpha^T \beta^{-1} \alpha \prec 0, \quad \varepsilon = \varepsilon^T, \quad \beta = \beta^T \succ 0 \tag{4.8}$$

Note that the negative definiteness of ε is not assumed anymore. This suggests that the nonlinear term is necessary for the negative definiteness of the sum. Indeed, if $\varepsilon \prec 0$ there exists a trivial (conservative) bound on the nonlinear term which is 0 (since the nonlinear term is positive semidefinite). Moreover, the matrix α is not necessarily square and these facts show the wide adaptability of the proposed approach.

The following result has been often used in the literature to bound the nonlinear term, e.g. in [Daafouz et al., 2002].

Lemma 4.3.1 *The following inequality*

$$-\alpha^T \beta^{-1} \alpha \preceq -\alpha - \alpha^T + \beta$$

holds.

Proof: *Since $\beta \succ 0$, then define the inequality*

$$(I - \beta^{-1}\alpha)^T \beta (I - \beta^{-1}\alpha) \succeq 0$$

and thus we have

$$\begin{array}{rcl} \beta - \alpha^T - \alpha + \alpha^T \beta^{-1} \alpha & \succeq & 0 \\ & \text{which implies} & \\ -\alpha^T \beta^{-1} \alpha & \preceq & \beta - \alpha^T - \alpha \end{array}$$

This concludes the proof. □

A direct extension of the latter results yields

Lemma 4.3.2 *The following relation*

$$-\alpha^T \beta^{-1} \alpha \preceq -\omega(\alpha + \alpha^T) + \omega^2 \beta$$

holds for any $\omega > 0$.

Using these lemmas, the nonlinearity is bounded by an hyperplane as seen on figure 4.6 where the scalar case is considered with $\omega = 1$. Actually these results are a simple linearization of the nonlinearity around some particular point. Hence, such a bound will be conservative when the computed matrices are far from the linearization point. In the second result, ω has the role of a tuning parameter which 'moves' the linearization point in order to decrease the conservatism of the bound.

In [Chen and Zheng, 2006; Gao and Wang, 2003] it has been proposed to use the Cone Complementary Algorithm [Ghaoui et al., 1997] as a relaxation result. Initially, this algorithm was developed to deal with static output feedback design or more generally to matrix inequality based problems involving both matrices and their inverse. At the light of [Chen and Zheng, 2006; Gao and Wang, 2003], it turns out that it can also be applied efficiently

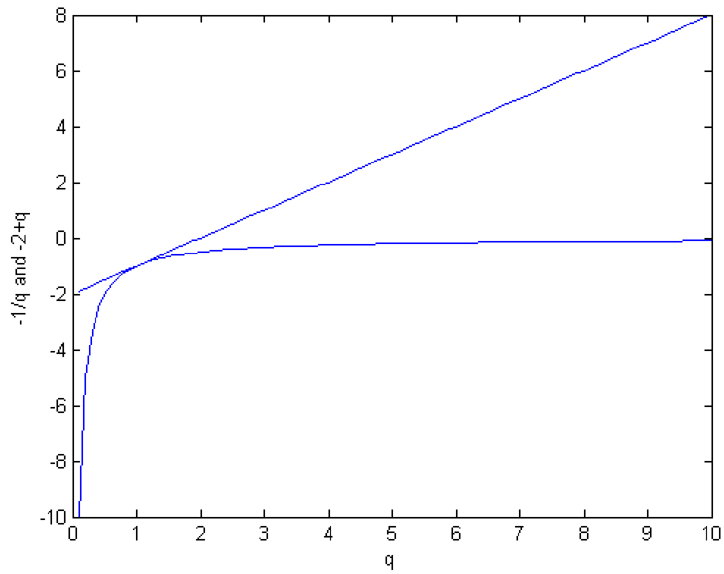

Figure 4.6: Evolution of the concave nonlinearity and the linear bound in the scalar case with fixed $\alpha = p = 1$, $\beta = q$ and $\omega = 1$

to relax concave nonlinearity of the form $-\alpha^T \beta^{-1} \alpha$. To adapt this algorithm to a relaxation scheme, let $\upsilon \leq \alpha^T \bar{\beta} \alpha$ and an inversion produces $\bar{\upsilon} \geq \bar{\alpha} \beta \bar{\alpha}^T$ where $\bar{\alpha}\alpha = I$, $\bar{\beta}\beta = I$ and $\bar{\upsilon}\upsilon = I$. Finally, we get the problem of finding $\mathcal{X} = (\varepsilon, \alpha, \beta, \upsilon, \bar{\alpha}, \bar{\beta}, \bar{\upsilon})$ such that

$$\begin{array}{lcl} \varepsilon - \upsilon & \prec & 0 \\ \begin{bmatrix} \bar{\upsilon} & \bar{\alpha}^T \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} & \succeq & 0 \\ \bar{\alpha}\alpha & = & I \\ \bar{\beta}\beta & = & I \\ \bar{\upsilon}\upsilon & = & I \end{array} \tag{4.9}$$

which is a nonconvex problem due to nonlinear equalities. It is clear that the latter problem is identical to the initial one. The Cone Complementary Algorithm allows to solve problem (4.8) by the following iterative procedure based on a relaxed version of the optimization problem:

Algorithm 4.3.3 ***Adapted Cone Complementary Algorithm:***

1. *Initialize* $i = 0$, *and* $\mathcal{X}_0 := (\varepsilon_0, \alpha_0, \beta_0, \upsilon_0, \bar{\alpha}_0, \bar{\beta}_0, \bar{\upsilon}_0)$ *solution of*

$$\varepsilon - \upsilon \prec 0 \qquad \begin{bmatrix} \bar{\upsilon} & \bar{\alpha}^T \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} \succeq 0$$

2. *Find* $\mathcal{X}_{i+1}$ *solution of*

$$\gamma_{i+1} := min_{\mathcal{X}} \operatorname{trace}(\upsilon_i \bar{\upsilon} + \bar{\upsilon}_i \upsilon + \alpha_i \bar{\alpha} + \bar{\alpha}_i \alpha + \beta_i \bar{\beta} + \bar{\beta}_i \beta)$$

such that

$$\varepsilon - \upsilon \prec 0 \qquad \begin{bmatrix} \bar{\upsilon} & \bar{\alpha}^T \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} \succeq 0$$

3. *if $\gamma_{i+1} = 6n$ then STOP: Solution Found*
   else if $i > i_{max}$ then STOP: Infeasible problem (too many iterations)
   else $i = i + 1$, goto Step 2.

Although this algorithm does not converge systematically to a global optimum of the optimization problem, it gives quite good results in practice. However, this efficient approach suffers from two drawbacks:

1. It can be applied with square matrices only since the procedure needs the inversion of the matrix α.

2. It can only deal with constant matrices since they are needed to be inverted and the inverse of parameter dependent matrices cannot be expressed in a linear fashion with respect to the unknown matrices. As an examples, the inverse of the matrix $P(\rho) = P_0 + \rho P_1$ is defined by

$$P(\rho)^{-1} = P_0^{-1} - P_0^{-1}(P_0^{-1} - P_1^{-1}\rho^{-1})^{-1}P_0^{-1}$$

   and cannot be expressed linearly for instance $P(\rho)^{-1} = S_0 + S_1 u(\rho)$ where $u(\rho)$ is a particular function.

The parameter dependent matrix case should be treated with the lemma 4.3.1 and 4.3.2 with a possibly parameter varying $\omega(\rho)$. However, due to the high conservatism of this bound, we have been brought to develop the following result to overcome these problems. Such a result has been published in [Briat et al., 2008c] and allows for a 'good' relaxation of the nonlinearity by bilinearities.

Theorem 4.3.4 *Consider a symmetric positive definite matrix function $\beta(\cdot)$, a matrix (non necessarily square) function $\alpha(\cdot)$ and a symmetric matrix function $\varepsilon(\cdot)$ then the following propositions are equivalent:*

a) $\varepsilon(\cdot) - \alpha^T(\cdot)\beta^{-1}(\cdot)\alpha(\cdot) \prec 0$

b) There exists a matrix function of appropriate dimensions $\eta(\cdot)$ such that

$$\begin{bmatrix} \varepsilon(\cdot) + \alpha(\cdot)^T\eta(\cdot) + \eta(\cdot)^T\alpha(\cdot) & \star \\ \beta(\cdot)\eta(\cdot) & -\beta(\cdot) \end{bmatrix} \prec 0$$

Proof: *b)⇒ a)*

First we suppose there exists $\eta(\cdot)$ such that (4.3.4) holds. Hence using Schur complement there exists $\eta(\cdot)$ such that

$$\varepsilon(\cdot) + [\eta^T(\cdot)\alpha(\cdot)]^H + \eta^T(\cdot)\beta(\cdot)\eta(\cdot) \prec 0$$

Completing the squares, this is equivalent to

$$\varepsilon(\cdot) + \zeta^T(\cdot)\beta^{-1}(\cdot)\zeta(\cdot) - \alpha^T(\cdot)\beta^{-1}(\cdot)\alpha(\cdot) \prec 0$$

with $\zeta(\cdot) = \alpha(\cdot) + \beta(\cdot)\eta(\cdot)$. Finally we obtain

$$\varepsilon(\cdot) - \alpha^T(\cdot)\beta^{-1}(\cdot)\alpha(\cdot) \prec -\zeta^T(\cdot)\beta^{-1}(\cdot)\zeta(\cdot) \tag{4.10}$$

Since $\beta(\cdot) > 0$ then the right-hand side of equation (4.10) is negative semidefinite for all $\eta(\cdot)$. Then we can conclude that if there exist a $\eta(\cdot)$ such that (4.10) is satisfied then a) is true. Moreover when $\zeta(\cdot)$ vanishes identically then no conservatism is induced and the bound equals the nonlinear term. That means that when $\eta(\cdot) = -\beta^{-1}(\cdot)\alpha(\cdot)$ the relaxation is exact.

a)⇒b)

First consider the matrix

$$\Theta(\cdot) = \begin{bmatrix} \varepsilon(\cdot) & \alpha(\cdot)^T \\ \alpha(\cdot) & \beta(\cdot) \end{bmatrix}$$

with $\beta(\cdot) \succ 0$ and $\varepsilon(\cdot) - \alpha(\cdot)^T\beta(\cdot)^{-1}\alpha(\cdot) \prec 0$. Let $\dim(\varepsilon(\cdot)) = n$ and $\dim(\beta(\cdot)) = l$ and note that $\Theta(\cdot)$ may be rewritten as

$$\Theta(\cdot) = \begin{bmatrix} \delta(\cdot)^{1/2} & \alpha(\cdot)^T\beta(\cdot)^{-1/2} \\ 0 & \beta(\cdot)^{1/2} \end{bmatrix} \begin{bmatrix} -I_n & 0 \\ 0 & I_l \end{bmatrix} \begin{bmatrix} \delta(\cdot)^{1/2} & 0 \\ \beta(\cdot)^{-1/2}\alpha(\cdot) & \beta(\cdot)^{1/2} \end{bmatrix}$$

where $\delta^{1/2}(\cdot)$ and $\beta^{1/2}(\cdot)$ define the symmetric positive definite square root of matrices $\delta(\cdot)$ and $\beta(\cdot)$ with $\delta(\cdot) = -\varepsilon(\cdot) + \alpha(\cdot)^T\beta^{-1}(\cdot)\alpha(\cdot)$. From this equality it is clear the matrix Θ has n negative eigenvalues and l positive eigenvalues since $\Theta(\cdot)$ is congruent to $\mathrm{diag}(-I_n, I_l)$. Then there exists a subspace with maximal rank of the form

$$\Lambda(\cdot) = \begin{bmatrix} \theta(\cdot) \\ \eta(\cdot) \end{bmatrix} \qquad \textit{with } \operatorname{rank} \Lambda(\cdot) = n \tag{4.11}$$

with $\theta(\cdot) \in \mathbb{R}^{n\times n}$ and $\eta(\cdot) \in \mathbb{R}^{l\times n}$ such that $\Lambda(\cdot)^T\Theta(\cdot)\Lambda(\cdot) < 0$. Expand the latter inequality leads to (dropping the dependence $(\cdot)$):

$$\theta^T\varepsilon\theta + \theta^T\alpha^T\eta + \eta^T\alpha\theta + \eta^T\beta\eta \prec 0 \tag{4.12}$$

Rearranging the terms using the fact that $\beta(\cdot) \succ 0$ is symmetric leads to

$$\theta^T(\varepsilon - \alpha^T\beta^{-1}\alpha)\theta + (\alpha\theta + \beta\eta)^T\beta^{-1}(\alpha\theta + \beta\eta) \prec 0 \tag{4.13}$$

Since $\varepsilon - \alpha^T\beta^{-1}\alpha \prec 0$ and $\beta(\cdot) \succ 0$ then it implies that $\theta^T(\varepsilon - \alpha^T\beta^{-1}\alpha)\theta \prec 0$. Hence θ is of full rank. Now let $\mathcal{K}$ be the set such that

$$\mathcal{K} := \{\kappa : \ \theta^T(\varepsilon - \alpha^T\beta^{-1}\alpha)\theta + \kappa^T\beta^{-1}\kappa \prec 0\} \tag{4.14}$$

It is clear that the set $\mathcal{K}$ is nonempty since it includes $\kappa = 0$. It is not reduced to a singleton since it exists a neighborhood $\mathcal{N}$ centered around $\kappa = 0$ for which (4.14) is satisfied for all $\kappa \in \mathcal{N}$. Now we will show that for all nonsingular θ there exist values for κ (and hence values for η) for which (4.14) holds. First note that $\beta \succ 0$ is nonsingular, then the equation

$$\alpha\theta + \beta\eta = \kappa \tag{4.15}$$

for given θ and κ has the solution $\eta = \beta^{-1}(\kappa - \alpha\theta)$. Hence this means that for given $\varepsilon, \beta, \alpha, \theta$ such that $\varepsilon - \alpha^T\beta^{-1}\alpha \prec 0$, $\beta \succ 0$, $rank(\theta) = n$, there exist η such that (4.13) is satisfied. The existence of such a η is thus shown.

Now fix $\theta = I$ for simplicity and consider (4.13) we obtain

$$\varepsilon + \alpha^T\eta + \eta^T\alpha + \eta^T\beta\eta \prec 0 \tag{4.16}$$

Apply the Schur's Lemma to obtain

$$\begin{bmatrix} \varepsilon + \alpha^T\eta + \eta^T\alpha & \eta^T\beta \\ \beta\eta & -\beta \end{bmatrix} \prec 0 \tag{4.17}$$

This concludes the proof. □

This theorem has the benefit to allow for the consideration of parameter varying matrices and nonsquare α, this is a great improvement compared to previous methods. Moreover, it involves only feasibility problems and this can be directly extended to optimization problems. This is not the case for the cone complementary algorithm which already involves an optimization problem (i.e. the trace cost is aimed to be minimized). Hence, if minimal $\mathcal{L}_2$-performances are sought then we would be in presence of a multi-objective optimization problem (the costs would be a weighted sum of the trace and the norm) which is not a trivial problem. The tradeoff between the costs shall be done with care, in order to not too penalize the trace cost which is the most important one.

Since (4.3.4) is bilinear (BMI) then no efficient algorithm is available to solve it in reasonable time. Indeed, noting that by fixing the value of η the problem is convex in ε, α and β and vice-versa (this is a quasiconvex problem), it seems interesting to develop such an algorithm matching this particular form of BMI. Due to this property an algorithm in two steps can be used to find a solution iteratively such as the D-K iteration algorithm used in μ-synthesis [Apkarian et al., 1993; Balas et al., 1998]. It is important to precise that the D-K algorithm is a general algorithm but there exist lots of algorithms to solve BMIs, see for instance Henrion and Lasserre [2006]; Ibaraki and Tomizuka [2001]; Tuan et al. [1999].

Since any iterative optimization procedure needs to find an initial feasible point in order to converge to a local/global minimum, the remaining problem is to find this initial feasible point. In the proof of theorem 4.3.4, it is shown that the relaxation is exact if and only if $\eta = -\beta^{-1}\alpha$ and hence finding an initial η_0 is equivalent to finding an initial α_0 and β_0. If all the matrices are square then lemmas 4.3.1 and 4.3.2 can be used to find an initial feasible point. If α is rectangular, then a nondeterministic approach can be used to find a 'good' (random) value for η_0.

Finally, if parameter dependent matrices $\varepsilon(\rho)$ and $\alpha(\rho)$ are considered, then according to the exact relation $\eta(\rho) = -\beta^{-1}\alpha(\rho)$, the matrix $\eta(\rho)$ has the same parameter dependence as $\alpha(\rho)$. For simplicity of initialization, it is possible to define a constant η_0 which does not depend on the parameters but when the optimization procedure in η is launched a second time then η shall be defined as parameter dependent.

Algorithm 4.3.5

1. *Let $i = 0$, fix $\eta_i(\rho)$*
2. *Solve for $(\varepsilon_i(\rho), \alpha_i(\rho), \beta_i(\rho))$ solutions of*

$$\begin{bmatrix} \varepsilon(\rho) + \eta_i(\rho)\alpha(\rho) + \alpha(\rho)^T\eta_i(\rho) & \eta_i(\rho)^T\beta(\rho) \\ \star & -\beta(\rho) \end{bmatrix} \prec 0 \tag{4.18}$$

3. *Let $i = i + 1$, solve for η_i solution of*

$$\begin{bmatrix} \varepsilon_{i-1}(\rho) + \eta_i(\rho)\alpha_{i-1}(\rho) + \alpha_{i-1}(\rho)^T\eta_i(\rho) & \eta_i(\rho)^T\beta_{i-1}(\rho) \\ \star & -\beta_{i-1}(\rho) \end{bmatrix} \prec 0 \tag{4.19}$$

4. If stopping criterion is satisfied then STOP else go to Step 2.

It will be shown in Section 6.1.3 that such an algorithm might lead to good results in a small number of iterations (between 1 and 4).

4.4 Polytopic Systems and Bounded-Parameter Variation Rates

In many papers, only parameter dependent polytopic system with arbitrary fast-varying parameter variation rate (unbounded rate of variation) are considered (see for instance [Oliveira et al., 2007]). However, some of them consider robust stability instead of quadratic stability [de Souza and Trofino, 2005]. In this section, an easy way to consider bounded parameter variation rate in the polytopic domain is introduced. The main difference between stability conditions expressed for arbitrary fast varying system and bounded rate parameters, is the presence, or not, of parameter derivatives into these conditions. The main difficulty is that derivatives of the polytopic variables have a non-straightforward relation with parameter derivatives. As an example, let us consider the following polytopic LPV system with $N = 2^s$ polytopic variables where s is a positive integer:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\lambda)x(t) + E(\lambda)w(t) \\ z & = & C(\lambda)x(t) + F(\lambda)w(t)\end{array}$$

The robust bounded-real lemma (see Section 2.3.2) is then given by the LMI condition

$$\begin{bmatrix} P(\lambda)A(\lambda) + A(\lambda)^T P(\lambda) + \mathcal{P}[\dot{\lambda}(t) \otimes I] & P(\lambda)E(\lambda) & C(\lambda)^T \\ \star & -\gamma I & F(\lambda)^T \\ \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{4.20}$$

where $\mathcal{P} = \begin{bmatrix} P_1 & P_2 & \dots & P_N \end{bmatrix}$. Now rewrite the matrix $A(\lambda)$ as the following:

$$A(\lambda) = \sum_{i=1}^{N} \lambda_i(t) V_i A_i$$

where the time-varying parameters are given by $\rho(t) = \sum_{i=1}^{N} \lambda_i(t) V_i$ where the V_i are the vertices of the polytope in which $\rho(t)$ evolve and $\lambda_i(t)$ the time-varying polytopic coordinates evolving over the unit simplex Γ:

$$\Gamma := \left\{ \lambda(t) \in [0,\ 1]^N : \ \sum_{i=1}^{N} \lambda_i(t) = 1,\ t \geq 0 \right\}$$

The extremal values of $\rho(t)$ are the V_i, $i = 1, \dots, N$ but, on the other hand, provided that bounds on the rate of variation are known, then it is possible to define a polytope containing the parameter derivatives, i.e. $\dot{\rho}(t) \in \text{hull}[D]$. Indeed, differentiating the parameters $\rho(t)$ we get

$$\dot{\rho}(t) = \sum_{i=1}^{N} \dot{\lambda}_i(t) V_i$$

and from this expression, the relation between the values $\dot{\lambda}_i(t)$ and D is unclear. What are the extremal values for $\dot{\lambda}_i(t)$? A way to find them is to define

$$\dot{\rho}(t) := \sum_{i=1}^{N} \lambda'_i(t) D_i$$

where D_i, $i = 1, \ldots, N$ are the vertices of the polytope containing all possible values of the parameter derivatives ($D = \text{hull}[D_i]$) and $\lambda'_i(t)$ the time-varying polytopic coordinates evolving over the unit simplex Γ. In this case, we have the following equality

$$\sum_{i=1}^{N} \dot{\lambda}_i(t) V_i = \sum_{i=1}^{N} \lambda'_i(t) D_i$$

which is equivalently written in a compact matrix form

$$V\dot{\lambda}(t) = D\lambda'(t) \tag{4.21}$$

with $V = \begin{bmatrix} V_1 & V_2 & \ldots & V_N \end{bmatrix}$, $D = \begin{bmatrix} D_1 & D_2 & \ldots & D_N \end{bmatrix}$, $\dot{\lambda}(t) = \text{col}(\dot{\lambda}_i(t))$ and $\lambda' = \text{col}(\lambda'_i(t))$. Note that we have the following equality constraints

$$\begin{array}{rcl} \displaystyle\sum_{i=1}^{N} \lambda'_i(t) & = & 1 \\ \displaystyle\sum_{i=1}^{N} \dot{\lambda}_i(t) & = & 0 \end{array}$$

Combined to (4.21), we get

$$\left[\begin{array}{c} V \\ \hline 1 \quad 1 \quad \ldots \quad 1 \end{array}\right] \dot{\lambda}(t) = \left[\begin{array}{c} D \\ \hline 1 \quad 1 \quad \ldots \quad 1 \end{array}\right] \lambda'(t) - \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]$$

which is rewritten compactly as

$$\bar{V}\dot{\lambda}(t) = \bar{D}\lambda'(t) - C$$

with $\bar{V} = \left[\begin{array}{c} V \\ \hline 1 \quad 1 \quad \ldots \quad 1 \end{array}\right]$, $\bar{D} = \left[\begin{array}{c} D \\ \hline 1 \quad 1 \quad \ldots \quad 1 \end{array}\right]$ and $C = \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]$. Such an equation has solutions in $\dot{\lambda}(t)$ if and only if one of the following statements holds (see Appendix A.8 or Skelton et al. [1997]):

1. $(I - \bar{V}\bar{V}^+)(\bar{D}\lambda'(t) - C) = 0$
2. $\bar{D}\lambda'(t) - C = \bar{V}\bar{V}^+(\bar{D}\lambda'(t) - C)$

In this case the set of solutions is given by

$$\dot{\lambda}(t) = \bar{V}^+(\bar{D}\lambda'(t) - C) + (I - \bar{V}^+\bar{V})Z \tag{4.22}$$

where Z is an arbitrary matrix with appropriate dimensions. It is clear that $V \in \mathbb{R}^{\log_2(N) \times N}$ and then $\text{rank}[V] = \dim(\rho) = \log_2(N)$. Finally, due to the structure of V, we have $\text{rank}[\bar{V}] =$

$\log_2(N)+1$, hence $\bar{V}$ is a full row rank matrix and admits a right pseudoinverse $\bar{V}^+$ such that $\bar{V}\bar{V}^+ = I$. This shows that the first statement above holds for every $\bar{D}\Lambda'(t) - C$ and hence all the solutions of the problem write:

$$\dot{\lambda}(t) = \bar{V}^+(\bar{D}\lambda'(t) - C) + (I - \bar{V}^+\bar{V})Z$$

for a free matrix Z of appropriate dimensions. It is worth noting that the solution is affine in $\lambda'(t)$ (which seems logical since the equation is linear) and that Z can be removed from the solution since only a solution is needed. Z can be tuned in order to modulate the values of the vector $-\bar{V}^+C$ but it is not of great interest and then the term Z can be set to 0.

Finally, substituting $\dot{\lambda}(t) = M\lambda'(t) + N$ with $M = \bar{V}^+(\bar{D}$ and $N = -\bar{V}^+\bar{D}C$ into the LMI (4.20) we get a new condition in terms of the λ_i and λ_i':

$$\begin{bmatrix} P(\lambda)A(\lambda) + A(\lambda)^T P(\lambda) + \mathcal{P}[(M\lambda'(t) + N) \otimes I] & P(\lambda)E(\lambda) & C(\lambda)^T \\ \star & -\gamma I & F(\lambda)^T \\ \star & \star & -\gamma I \end{bmatrix} \prec 0$$

The following example illustrates the approach.

Example 4.4.1 *Consider a two parameter problem with* $(\rho_1, \rho_2) \in [-1, 1] \times [-2, 3]$ *and* $(\dot{\rho}_1, \dot{\rho}_2) \in [-2, 3] \times [-5, 6]$. *We have the following matrices*

$$V = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & -2 & 3 & 3 \\ -5 & 6 & -5 & 6 \end{bmatrix}$$

Thus we can choose

$$\bar{V}^+ = \frac{1}{10} \begin{bmatrix} -2.5 & -1 & 3 \\ -2.5 & 1 & 2 \\ 2.5 & -1 & 3 \\ 2.5 & 1 & 2 \end{bmatrix}$$

then

$$\bar{V}^+\bar{D} = \begin{bmatrix} 1.3 & 0.2 & 0.05 & -1.05 \\ 0.2 & 1.3 & -1.05 & 0.05 \\ 0.3 & -0.8 & 1.55 & 0.45 \\ -0.8 & 0.3 & 0.45 & 1.55 \end{bmatrix} \quad \text{and} \quad \bar{V}^+C = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.2 \end{bmatrix}$$

Finally using (4.22) we get

$$\dot{\lambda}(t) = \begin{bmatrix} 1.3 & 0.2 & 0.05 & -1.05 \\ 0.2 & 1.3 & -1.05 & 0.05 \\ 0.3 & -0.8 & 1.55 & 0.45 \\ -0.8 & 0.3 & 0.45 & 1.55 \end{bmatrix} \lambda'(t) - \begin{bmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.2 \end{bmatrix}$$

This ends the section on computing the bounds on polytopic parameter derivatives in terms of another polytopic variables.

4.5 $\mathcal{H}_\infty$ Performances Test via Simple Lyapunov-Krasovskii functional and Related Relaxations

In this section, simple Lyapunov-Krasovskii functionals are considered as in [Gouaisbaut and Peaucelle, 2006b; Han, 2005a]. Fundamental results are recalled and generalized for LPV systems with time-varying delays. The type of Lyapunov-Krasovskii functionals proposed in these papers allows to avoid any model transformations or any bounding of cross terms. The only conservatism of the method comes from the initial choice of the Lyapunov-Krasovskii functionals (which is not complete) and the use of the Jensen's inequality (see [Gu et al., 2003] or Appendix E.1) used to bound an integral term in the Lyapunov-Krasovskii functional derivative. The main advantage of these functionals is based on their simplicity and the small number of involved Lyapunov-Krasovskii variables, thus minimizing products between data matrices and decision variables, making them potentially interesting criteria for stabilization problem.

As we shall see later, in the case of a simple Lyapunov-Krasovskii functional, two matrix products occur and thus a relaxation scheme must be performed in order to get tractable LMI condition for the stabilization problem. In the framework of a discretized Lyapunov-Krasovskii functional, many couplings would appear corresponding to the order of discretization that has been considered.

We will consider in this section the following LPV time-delay system:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + E(\rho)w(t)\\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t)\end{array}\tag{4.23}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ are respectively the system state, the exogenous inputs and the controlled outputs. The delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$, the parameters ρ satisfy $\rho \in U_\rho$ and $\dot{\rho} \in \text{hull}[U_\nu]$.

4.5.1 Simple Lyapunov-Krasovskii functional

The main result of this subsection is based on the use of the following parameter dependent Lyapunov-Krasovskii functional [Gouaisbaut and Peaucelle, 2006b; Han, 2005a]:

$$\begin{array}{rcl}V(t) & = & V_1(t) + V_2(t) + V_3(t)\\ V_1(t) & = & x(t)^T P(\rho) x(t)^T\\ V_2(t) & = & \displaystyle\int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta\\ V_3(t) & = & \displaystyle\int_{-h_{max}}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T h_{max} R \dot{x}(\eta) d\eta d\theta\end{array}\tag{4.24}$$

from which the following results is derived:

Lemma 4.5.1 *System (4.23) is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^n$, $Q, R \in \mathbb{S}_{++}^n$*

and $\gamma > 0$ *such that the LMI*

$$\begin{bmatrix} \Psi_{11}(\rho,\nu) & P(\rho)A_h(\rho)+R & P(\rho)E(\rho) & C(\rho)^T & h_{max}A(\rho)^TR \\ \star & -(1-\mu)Q-R & 0 & C_h(\rho)^T & h_{max}A_h(\rho)^TR \\ \star & \star & -\gamma I_m & F(\rho)^T & h_{max}E(\rho)^TR \\ \star & \star & \star & -\gamma I_p & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.25}$$

with

$$\Psi_{11}(\rho,\nu = A(\rho)^TP(\rho)+PA(\rho)+\partial_\rho P(\rho)\nu + Q - R \tag{4.26}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$.

Proof: *Computing the time-derivative of (4.24) along the trajectories solutions of system (4.3) leads to*

$$\begin{aligned} \dot V_1(t) &= \dot x(t)^TP(\rho)x(t) + x(t)P(\rho)\dot x(t) + x(t)^T\partial_\rho P(\rho)\dot\rho x(t) \\ \dot V_2(t) &= x(t)^TQx(t) - (1-\dot h)x(t-h(t))^TQx(t-h(t)) \\ \dot V_3(t) &= h_{max}^2\dot x(t)^TR\dot x(t) - \int_{t-h_{max}}^t \dot x(\theta)^Th_{max}R\dot x(\theta)d\theta \end{aligned}$$

Since $|\dot h| < 1$, *we have* $-(1-\dot h) \leq -(1-\mu)$ *and since* $h(t) \leq h_{max}$ *then*

$$-\int_{t-h_{max}}^T \dot x(\theta)^Th_{max}R\dot x(\theta)d\theta \leq -\int_{t-h(t)}^t \dot x(\theta)^Th_{max}R\dot x(\theta)d\theta$$

Finally using the Jensen's inequality (see Appendix E.1) it is possible to bound the integral term in $\dot V_3(t)$ *as follows:*

$$\begin{aligned} \dot V_3(t) &\leq h_{max}^2\dot x(t)^TR\dot x(t) - \int_{t-h(t)}^t \dot x(\theta)^th_{max}R\dot x(\theta)d\theta \\ &\leq h_{max}^2\dot x(t)^TR\dot x(t) - \frac{h_{max}}{h(t)}\left(\int_{t-h(t)}^t \dot x(\theta)d\theta\right)^T R\left(\int_{t-h(t)}^t \dot x(\theta)d\theta\right) \\ &= h_{max}^2\dot x(t)^TR\dot x(t) - \frac{h_{max}}{h(t)}\left[x(t)-x(t-h(t))\right]^T R\left[x(t)-x(t-h(t))\right] \end{aligned}$$

It is proved now that the previous expression is well-posed when $h(t)$ *tends to zero. First let* t_i *be the time-instants for which* $h(t_i) = 0$ *and we aim at proving that when* $t \to t_i$ *then the quantity*

$$\frac{1}{h(t)}\left(x(t)-x(t-h(t))\right)^T R\left(x(t)-x(t-h(t))\right) \tag{4.27}$$

remains bounded. Rewrite it in the form

$$h(t)\left(\frac{x(t)-x(t-h(t))}{h(t)}\right)^T R\left(\frac{x(t)-x(t-h(t))}{h(t)}\right)$$

When $t \to t_i$ *we have* $\dfrac{x(t)-x(t-h(t))}{h(t)} \to \dot x(t_i)$ *since* $x(t)$ *is differentiable. Moreover, as* $\dot x(t)$ *is finite for all* $t \in \mathbb{R}^+$ *this proves that (4.27) remains bounded when* $t \to t_i$. *Finally bounding* $-\dfrac{h_{max}}{h(t)}$ *by* -1 *we get*

$$\dot V_3(t) \leq h_{max}^2\dot x(t)^TR\dot x(t) - \left[x(t)-x(t-h(t))\right]^T R\left[x(t)-x(t-h(t))\right]$$

Gathering all the derivative terms $\dot{V}_i$ we get the following quadratic inequality:

$$\dot{V}(t) \leq X(t)^T \Psi(\rho,\dot{\rho}) X(t) < 0$$

with

$$\Psi(\rho,\dot{\rho}) = \begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h(\rho)+R & P(\rho)E(\rho) \\ \star & -(1-\mu)Q-R & 0 \\ \star & \star & 0 \end{bmatrix} + h_{max}^2 \mathcal{T}(\rho)^T R \mathcal{T}(\rho)$$

$$\begin{array}{rcl} X(t) & = & \mathrm{col}(x(t), x(t-h(t)), w(t)) \\ \mathcal{T} & = & \begin{bmatrix} A(\rho) & A_h(\rho) & E(\rho) \end{bmatrix} \\ \Psi_{11}(\rho,\dot{\rho}) & = & A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R \end{array}$$

$\mathcal{L}_2$ performances are introduced in the criterium through the Hamiltonian function H defined by

$$H(t) = V(t) - \int_0^t \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta$$

If the hamiltonian function satisfies $\dot{H} < 0$ for all nonzero $X(t)$ then have

$$\lim_{t\to+\infty} H(t) = \lim_{t\to+\infty} V(t) - \int_0^t \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta < 0$$

Assuming that the system is asymptotically stable ($\lim_{t\to+\infty} V(t) = 0$) then we get

$$\lim_{t\to+\infty} H(t) = -\int_0^{+\infty} \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta < 0$$

Finally we have

$$\int_0^{+\infty} \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta > 0$$

which is equivalent to

$$\gamma ||w||_{\mathcal{L}_2}^2 - \gamma^{-1} ||z||_{\mathcal{L}_2}^2 > 0$$

and thus

$$\frac{||z||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} < \gamma^2$$

Expanding $z(t)$ into the expression of $\dot{H}$ leads to

$$\dot{H} \leq \dot{V} - \gamma w(t)^T w(t) + \gamma^{-1} X(t)^T \begin{bmatrix} C(\rho)^T \\ C_h(\rho)^T \\ F(\rho)^T \end{bmatrix} \begin{bmatrix} C(\rho) & C_h(\rho) & F(\rho) \end{bmatrix} X(t)$$

Finally performing a Schur complement onto term

$$-\begin{bmatrix} C(\rho)^T & h_{max} A(\rho)^T R \\ C_h(\rho)^T & h_{max} A_h(\rho)^T R \\ F(\rho)^T & h_{max} E(\rho)^T R \end{bmatrix} \begin{bmatrix} -\gamma^{-1} I & 0 \\ 0 & -R^{-1} \end{bmatrix} \begin{bmatrix} C(\rho) & C_h(\rho) & F(\rho) \\ h_{max} R A(\rho) & h_{max} R A_h(\rho) & h_{max} R E(\rho) \end{bmatrix}$$

leads to LMI (4.25). Finally, noting that $\dot{\rho} \in \mathrm{hull}[U\nu]$ enters affinely in the LMI, it suffices to check the LMI only at the vertices which are the elements of U_ν. This concludes the proof.
□

4.5.2 Associated Relaxation

It is clear from the expression of LMI (4.25) that this criterium is not suited for stabilization purposes due to the multiple product terms PA and RA. Indeed, by introducing the closed-loop system state-space into LMI conditions, due to coupling terms, the linearization is an impossible task without considering (strong) assumptions. In many problems, the common simplification would be to consider 'proportional' matrices in the sense that

$$R = \varepsilon P \qquad \text{or} \qquad P(\rho) = \varepsilon(\rho)R$$

where $\varepsilon > 0$ is a chosen fixed scalar. It is clear that such a simplification is very conservative since the initial space of decision

$$(P(\rho), Q, R) \in \mathbb{S}^n_{++} \times \mathbb{S}^n_{++} \times \mathbb{S}^n_{++}$$

is reduced to

$$(P, Q, \varepsilon) \in \mathbb{S}^n_{++} \times \mathbb{S}^n_{++} \times \mathbb{R}_{++} \qquad \text{or} \qquad (\varepsilon(\rho), Q, R) \in \mathbb{R}_{++} \times \mathbb{S}^n_{++} \times \mathbb{S}^n_{++}$$

The idea we propose here is, rather simplifying the stabilization conditions after introducing the closed-loop system expression, we turn the initial LMI condition into a form which better fits the stabilization problem [Tuan et al., 2001b]. Roughly speaking, a LMI is efficient for a stabilization problem if there is only one coupling between a decision matrix and system variables. This decision matrix is not a Lyapunov variable but is a 'slack' variable introduced by applying the Finsler's Lemma (see Appendix D.16). Using this lemma we obtain the following relaxation to LMI (4.25):

Lemma 4.5.2 *System (4.23) is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^n_{++}$, constant matrices $Q, R \in \mathbb{S}^n_{++}$, a matrix function $X : U_\rho \to \mathbb{R}^{n\times n}$ and $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} -X(\rho)^H & \Xi_{12} & X(\rho)^T A_h(\rho) & X(\rho)^T E(\rho) & 0 & X(\rho)^T & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -Q_\mu - R & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_m & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.28}$$

with $Q_\mu = (1-\mu)Q$, $\Xi_{12} = P(\rho) + X(\rho)^T A(\rho)$ and

$$\Psi_{22}(\rho,\nu) = \partial_\rho P(\rho)\nu - P(\rho) + Q - R \tag{4.29}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$.

Proof: *The proof is inspired from Tuan et al. [2001b]. Rewrite (4.28) as*

$$\mathcal{M}(\rho,\nu) + \left[\mathcal{P}(\rho)^T X(\rho,\rho_h)\mathcal{Q}\right]^H \prec 0$$

with

$$\begin{array}{rcl}\mathcal{M}(\rho,\nu) & = & \begin{bmatrix} 0 & P(\rho) & 0 & 0 & 0 & 0 & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q(\rho_h)-R & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \\ \mathcal{P}(\rho) & = & \begin{bmatrix} -I & A(\rho) & A_h(\rho) & E(\rho) & 0 & I & 0 \end{bmatrix} \\ \mathcal{Q} & = & \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

Noting that explicit basis of the null-space of $\mathcal{P}$ *and* $\mathcal{Q}$ *are given by*

$$\mathrm{Ker}(\mathcal{P}(\rho)) = \begin{bmatrix} A(\rho) & A_h(\rho) & E(\rho) & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \qquad \mathrm{Ker}(\mathcal{Q}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and applying the projection lemma (see Appendix D.18) we get the two underlying LMIs

$$\begin{bmatrix} \Psi_{11}(\rho) & P(\rho)A_h(\rho)+R & P(\rho)E(\rho) & C(\rho)^T & P(\rho) & h_{max}A(\rho)^TR \\ \star & -(1-\mu)Q-R & 0 & C_h(\rho)^T & 0 & h_{max}A_h(\rho)^TR \\ \star & \star & -\gamma I & F(\rho)^T & 0 & h_{max}E(\rho)^TR \\ \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & -P(\rho) & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.30}$$

$$\begin{bmatrix} \Psi_{22}(\rho,\nu) & R & C(\rho)^T & 0 & 0 & 0 \\ \star & -(1-\mu)Q-R & C_h(\rho)^T & 0 & 0 & 0 \\ \star & \star & -\gamma I & F(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.31}$$

LMI (4.30) is equivalent to (4.28) modulo a Schur complement (see Appendix D.4). Hence this shows that feasibility of (4.28) implies feasibility of (4.30) and (4.31). This concludes the proof. □

Although (4.28) implies (4.25), it also implies LMI (4.31) which is not always satisfied. Thus conservatism is induced while imposing supplementary constraints: among others the left-upper block gives $-P(\rho)+Q(\rho)-R+\partial_\rho P(\rho)\nu<0$ and the 2×2 right-bottom block gives $-P(\rho)+h_{max}^2R<0$ (invoking the Schur's complement) which restrict the initial set for decision variables.

4.5.3 Reduced Simple Lyapunov-Krasovskii functional

Another result based on a simple Lyapunov-Krasovskii functional is provided. This results aims at reducing the computational complexity of the stability test obtained from Lyapunov-Krasovskii functional (4.24) when the matrices A_h and C_h take the following form:

Assumption 4.5.3 *The matrices A_h and C_h are assumed to have the following form:*

$$A_h(\rho) = \begin{bmatrix} A_h^{11}(\rho) & 0 \\ A_h^{21}(\rho) & 0 \end{bmatrix} \qquad C_h(\rho) = \begin{bmatrix} C_h'(\rho) & 0 \end{bmatrix}$$

Indeed, as illustrated above, the second part of the state is not affected by the delay and thus this state information can be removed from the part of the Lyapunov-Krasovskii functional dealing with the stability analysis of the delayed part. It is interesting to note that such a representation occurs in the filtering problem of time-delay systems using a memoryless filter [Zhang and Han, 2008] or by controlling a time-delay system using a memoryless dynamic controller. Indeed, in each of this case, the second part of the state is either the filter or controller state, which are not affected by the delay (provided that no delay acts on the control input of the system).

It is possible to write $A_h(\rho) = A_h'(\rho)Z$ and $C_h(\rho) = C_h'(\rho)Z$ where

$$Z = \begin{bmatrix} I & 0 \end{bmatrix} \qquad A_h'(\rho) = \begin{bmatrix} A_h^{11} \\ A_h^{21} \end{bmatrix}$$

Hence there is no increase of conservatism by considering the Lyapunov-Krasovskii functional

$$\begin{array}{rcl} V(t) & = & V_1(t) + V_2(t) + V_3(t) \\ V_1(t) & = & x(t)^T P(\rho)x(t) \\ V_2(t) & = & \displaystyle\int_{t-h(t)}^{t} x(\theta)^T Z^T Q(\rho(\theta))Zx(t\theta)d\theta \\ V_3(t) & = & \displaystyle\int_{-h_{max}}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T Z^T RZ\dot{x}(\eta)d\eta d\theta \end{array} \tag{4.32}$$

which gives rise to the following result:

Lemma 4.5.4 *System (4.23) with assumption 4.5.3 is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \le \gamma||w||_{\mathcal{L}_2}$ if there exist matrix a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^n$, constant matrices $Q, R \in \mathbb{S}_{++}^n$ and a scalar $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} \Psi_{11}'(\rho,\nu) & P(\rho)A_h'(\rho) + R & P(\rho)E(\rho) & C(\rho)^T & h_{max}A(\rho)^T Z^T R \\ \star & -(1-\mu)Q(\rho_h) - R & 0 & C_h'(\rho)^T & h_{max}A_h'(\rho)^T Z^T R \\ \star & \star & -\gamma I_m & F(\rho)^T & h_{max}E(\rho)^T Z^T R \\ \star & \star & \star & -\gamma I_p & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.33}$$

with

$$\Psi_{11}'(\rho,\nu = A(\rho)^T P(\rho) + PA(\rho) + \partial_\rho P(\rho)\nu + Z^T(Q(\rho) - R)Z \tag{4.34}$$

holds for all $(\rho,\rho_h,\nu) \in U_\rho \times U_{\rho_h} \times U_\nu$.

Proof: *Similarly as in the proof of Lemma 4.5.1, the time-derivative of the Lyapunov-Krasovskii functional (4.32) can be expressed and bounded as follows*

$$\begin{array}{rcl}\dot{V}_1(t) & = & \dot{x}(t)^T P(\rho)x(t) + \dot{x}(t)P(\rho)x(t)^T + x(t)^T \partial_\rho P(\rho)\dot{\rho}x(t) \\ \dot{V}_2(t) & \leq & x(t)^T Z^T Q(\rho)Zx(t) - (1-\mu)x(t-h(t))^T Z^T Q(\rho_h)Zx(t-h(t)) \\ \dot{V}_3(t) & \leq & h_{max}^2 \dot{x}(t)^T Z^T RZ\dot{x}(t) - (x(t)-x(t-h(t)))^T Z^T RZ(x(t)-x(t-h(t)))\end{array}$$

Gathering all the derivative terms $\dot{V}_i$ we get the following quadratic inequality:

$$\dot{V}(t) \leq X(t)^T \Psi'(\rho,\nu)X(t) < 0$$

$$\begin{array}{rcl}\Psi'(\rho,\nu) & = & \begin{bmatrix} \Psi'_{11}(\rho,\dot{\rho}) & P(\rho)A'_h(\rho) + Z^T R & P(\rho)E(\rho) \\ \star & -(1-\mu)Q - R & 0 \\ \star & \star & 0 \end{bmatrix} + h_{max}^2 \mathcal{T}(\rho)^T R \mathcal{T}(\rho) \\ X(t) & = & \mathrm{col}(x(t), Zx(t-h(t)), w(t)) \\ \mathcal{T}(\rho) & = & \begin{bmatrix} A(\rho) & A'_h(\rho) & E(\rho) \end{bmatrix} \\ \Psi_{11}(\rho,\dot{\rho}) & = & A(\rho)^T P(\rho) + P(\rho)A(\rho) + Z^T QZ - Z^T RZ\end{array}$$

Adding the constraint

$$\int_0^t \gamma w(\eta)^T w(\eta) - \gamma^{-1} z(\eta)^T z(\eta) d\eta > 0$$

to the Lyapunov function in view of constructing the Hamiltonian function we get

$$\dot{H} \leq \dot{V} - \gamma w(t)^T w(t) + \gamma^{-1} X(t)^T \begin{bmatrix} C(\rho)^T \\ C'_h(\rho)^T \\ F(\rho)^T \end{bmatrix} \begin{bmatrix} C(\rho) & C'_h(\rho) & F(\rho) \end{bmatrix} X(t)$$

Finally performing a Schur complement onto term

$$-\begin{bmatrix} C(\rho)^T & h_{max}A(\rho)^T Z^T R \\ C'_h(\rho)^T & h_{max}A'_h(\rho)^T Z^T R \\ F(\rho)^T & h_{max}E(\rho)^T Z^T R \end{bmatrix} \begin{bmatrix} -\gamma^{-1}I & 0 \\ 0 & -R^{-1} \end{bmatrix} \begin{bmatrix} C(\rho)^T & h_{max}A(\rho)^T Z^T R \\ C'_h(\rho)^T & h_{max}A'_h(\rho)^T Z^T R \\ F(\rho)^T & h_{max}E(\rho)^T Z^T R \end{bmatrix}^T$$

leads to LMI (4.33). Finally, noting that $\dot{\rho} \in \mathrm{hull}[U\nu]$ enters affinely in the LMI, it suffices to check the LMI only at the vertices which are the elements of $U\nu$. This concludes the proof. □

4.5.4 Associated Relaxation

Similarly as for lemma 4.5.1, it is convenient to construct a relaxed result which will be of interest further in the thesis.

Lemma 4.5.5 *System(4.23) with assumption 4.5.3 is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function*

$P : U\rho \to \mathbb{S}^n_{++}$, a matrix function $X : U_\rho \to \mathbb{R}^{n\times n}$, constant matrices $Q, R \in \mathbb{S}^n_{++}$ and $\gamma > 0$ such that the LMI

$$\begin{bmatrix} -X(\rho)^H & \Xi_{12} & X(\rho)^T A_h(\rho) & X(\rho)^T E(\rho) & 0 & X(\rho)^T & h_{max} Z^T R \\ \star & \Psi'_{22}(\rho,\nu) & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -Q_\mu - R & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_m & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max} Z^T R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.35}$$

with $\Xi_{12} = P(\rho) + X(\rho)^T A(\rho)$, $Q_\mu = (1-\mu)Q$ and

$$\Psi'_{22}(\rho,\nu) = \partial_\rho P(\rho)\nu - P(\rho) + Z^T(Q - R)Z \tag{4.36}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$.

Proof: *The proof is similar to the proof of lemma 4.5.2.* □

This concludes the section on results based on simple Lyapunov-Krasovskii functionals. The interest of such functionals, despite of their conservatism, is the avoidance of any model transformation and any bounding of cross-terms. The next section generalizes these functionals to a more general form in order to obtain less conservative results.

4.6 Discretized Lyapunov-Krasovskii Functional for systems with time varying delay and Associated Relaxation

The current section aims at improving previous results based on simple Lyapunov-Krasovskii functionals of the form (4.24) and (4.32). It is clear that, compared to complete Lyapunov-Krasovskii functionals defined in [Fridman, 2006a; Gu et al., 2003; Han, 2005b], the conservatism comes from the fact that the matrices Q and R are constant with respect to the integration parameter. Moreover, another advantage of the discretization approach is to divide the delay into smaller fragments in order to reduce the conservatism induced by the Jensen's inequality. To see this, let us consider the following example:

Example 4.6.1 *In this example we will consider the function $\dot{x}(\theta) = \theta$ and we will analyze the gap between the following integral*

$$\begin{aligned} \mathcal{I}_1 &:= -\int_{t-h}^{t} \dot{x}(\theta)^2 d\theta \\ \mathcal{I}_2 &:= -\frac{1}{h}\left(\int_{t-h}^{t} \dot{x}(\theta) d\theta\right)^2 \end{aligned}$$

were $h > 0$ and $t \in \mathbb{R}_+$. Then we have

$$\begin{aligned}
\mathcal{I}_1 &= -\int_{t-h}^{t} \theta^2 d\theta \\
&= \frac{1}{3}((t-h)^3 - t^3) \\
&= \frac{1}{3}(3th^2 - 3t^2h - h^3) \\
\mathcal{I}_2 &= -\frac{1}{h}\left(\int_{t-h}^{t} \dot{x}(\theta)d\theta\right)^2 \\
&= -\frac{1}{4h}(t^2 - (t-h)^2)^2 \\
&= \frac{1}{4}(4th^2 - 4t^2h - h^3)
\end{aligned}$$

The Jensen's inequality claims that $\mathcal{I}_2 \geq \mathcal{I}_1$ and hence the conservatism gap is given by the positive difference between $\mathcal{I}_2$ and $\mathcal{I}_1$, namely $\delta\mathcal{I}_{21}$:

$$\begin{aligned}
\delta\mathcal{I}_{21} &:= \mathcal{I}_2 - \mathcal{I}_1 \\
&= \frac{1}{4}(4th^2 - 4t^2h - h^3) - \frac{1}{3}(3th^2 - 3t^2h - h^3) \\
&= \frac{1}{12}h^3
\end{aligned}$$

This shows that the gap between the initial integral term and its corresponding bounds varies proportionally to the cube of the delay value. Hence, this suggests that by considering smaller delay values it might be possible to reduce the conservatism of the approach. First of all, decompose $\mathcal{I}_1$ into

$$\mathcal{I}_1 = \int_{t-h}^{t-h/2} \dot{x}(\theta)^2 d\theta + \int_{t-h/2}^{t} \dot{x}(\theta)^2 d\theta$$

Let us consider the sum of the Jensen's bound of each integral term

$$\mathcal{I}_3 := -\frac{2}{h}\left[\left(\int_{t-h}^{t-h/2} \dot{x}(\theta)d\theta\right)^2 + \left(\int_{t-h/2}^{t} \dot{x}(\theta)d\theta\right)^2\right]$$

Using the explicit expression of $\dot{x}(\theta)$ we get

$$\begin{aligned}
\mathcal{I}_3 &= -\frac{2}{4h}\left[\left((t-h/2)^2 - (t-h)^2\right)^2 + \left(t^2 - (t-h/2)^2\right)^2\right] \\
&= -t^2h + th^2 - \frac{5}{16}h^3
\end{aligned}$$

The corresponding gap $\delta\mathcal{I}_{31} := \mathcal{I}_3 - \mathcal{I}_1$ is then given by

$$\delta\mathcal{I}_{31} = \frac{1}{48}h^3$$

By fragmenting the delay up to order 3 we get

$$\begin{aligned}
\mathcal{I}_4 &:= -\frac{3}{4h}\left[\left((t-2h/3)^2 - (t-h)^2\right)^2 + \left((t-h/3)^2 - (t-2h/3)^2\right) + \left(t^2 - (t-h/3)^2\right)^2\right] \\
&= -t^2h + th^2 - \frac{35}{108}h^3
\end{aligned}$$

and the resulting gap $\delta\mathcal{I}_{41} := \mathcal{I}_4 - \mathcal{I}_1$ is given by

$$\delta\mathcal{I}_{41} = \frac{1}{108}h^3$$

This example shows that by increasing the order of the fragmentation it should possible to reduce the conservatism brought by the use of the Jensen's inequality. It is interesting to note that since the gap evolves as a polynomial of degree 3 and for each fragmentation the degree will remain to 3 (this is an intrinsic property related to the fact that $\dot{x}(\theta)$ is of degree 1). Fragmenting the delay will decrease the coefficient, only meaning that for a greater order of fragmentation the conservatism will be reduced. This has been also noticed in [Gouaisbaut and Peaucelle, 2006a,b; Han, 2008]. As a conjectural result, it can be shown that

$$\begin{array}{rcl} \delta\mathcal{I}_{N1} & := & \mathcal{I}_N - \mathcal{I}_1 \\ & = & \dfrac{1}{12N^2}h^3 \end{array}$$

where N is the fragmentation order and $\mathcal{I}_N$ is given by the expression

$$\mathcal{I}_N := -\frac{N}{4h}\sum_{i=0}^{N-1}\left[\left(t - \frac{N-i-1}{N}h\right)^2 - \left(t - \frac{N-i}{N}h\right)^2\right]^2$$

Although this reduction of conservatism is shown here in a special case, this is a general fact.

4.6.1 Discretized Lyapunov-Krasovskii functional

According to latter remarks, we introduce the following Lyapunov-Krasovskii functional which is a generalization of [Han, 2008]:

$$\begin{array}{rcl} V(x_t,\dot{x}_t) & = & V_1(x(t)) + V_2(x_t) + V_3(\dot{x}_t) \\ V_1(x(t)) & = & x(t)^T P(\rho) x(t) \\ V_2(x_t) & = & \displaystyle\sum_{i=0}^{N-1}\int_{t-(i+1)h_N(t)}^{t-ih_N(t)} x(\theta)^T Q_i x(\theta) d\theta \\ V_3(\dot{x}_t) & = & \displaystyle\sum_{i=0}^{N-1}\int_{-(i+1)\bar{h}}^{-i\bar{h}}\int_{t+\theta}^{t} \dot{x}(\eta)^T \bar{h} R_i \dot{x}(\eta) d\eta d\theta \end{array} \quad (4.37)$$

with $h_N(t) \triangleq \dfrac{h(t)}{N}$ and $\bar{h} \triangleq \dfrac{h_{max}}{N}$. This Lyapunov-Krasovskii functional gives the following result:

Lemma 4.6.2 *System (4.23) is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix $P : U_\rho \to \mathbb{S}_{++}^n$, constant matrices $Q_i, R_i \in \mathbb{S}_{++}^n$, $i \in \{0,\ldots,N-1\}$ and a scalar $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} \mathcal{M}_{11} & \Gamma_2(\rho)^T & \bar{h}\Gamma_1(\rho)^T R_0 & \ldots & \bar{h}\Gamma_1(\rho)^T R_{N-1} \\ \star & -\gamma I & 0 & \ldots & 0 \\ \star & \star & -\bar{h}R_0 & & \\ \star & \star & & \ddots & \\ \star & \star & & & -\bar{h}R_{N-1} \end{bmatrix} \prec 0 \quad (4.38)$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ *where* $\mu_N = \mu/N$

$$\mathcal{M}_{11} = \left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \ldots & P(\rho)A_h(\rho) & P(\rho)E(\rho) \\ \star & N_1^{(1)} & R_1 & 0 & \ldots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \ldots & 0 & 0 & -\gamma I \end{array}\right] \tag{4.39}$$

$$\begin{array}{rcl} M_{11} & = & A(\rho)^T P(\rho) + P(\rho)A(\rho) + \partial_\rho P(\rho)\nu + Q_0 - R_0 \\ N_i^{(1)} & = & -(1-i\mu_N)Q_{i-1} + (1+i\mu_N)Q_i - R_{i-1} - R_i \\ N^{(2)} & = & -(1-\mu)Q_{N-1} - R_{N-1} \\ \Gamma_1(\rho) & = & \left[\begin{array}{cccccc|c} A(\rho) & 0 & 0 & 0 & \ldots & A_h(\rho) & E(\rho) \end{array}\right] \\ \Gamma_2(\rho) & = & \left[\begin{array}{ccccc|c} C(\rho) & 0 & 0 & \ldots & C_h(\rho) & F(\rho) \end{array}\right] \end{array}$$

Proof: *Computing the derivative of (4.37) along the trajectories solutions of system (4.3) and with similar arguments as for the proof of lemma 4.5.1 we get:*

$$\begin{array}{rcl} \dot{V}(t) & \leq & Y(t)^T \left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \ldots & P(\rho)A_h(\rho) & P(\rho)E(\rho) \\ \star & N_1^{(1)} & R_1 & 0 & \ldots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \ldots & 0 & 0 & 0 \end{array}\right] Y(t) \\ & & +\bar{h}\sum_{i=0}^{N-1} Y(t)^T \Gamma_1^T(\rho)\Gamma_1(\rho)Y(t) \end{array}$$

with

$$\begin{array}{rcl} M_{11} & = & A(\rho)^T P(\rho) + P(\rho)A(\rho) + \partial_\rho P(\rho) + Q_0(\rho_0) - R_0 \\ N_i^{(1)} & = & -(1-i\mu_N)Q_{i-1} + (1+i\mu_N)Q_i - R_{i-1} - R_i \\ N^{(2)} & = & -(1-\mu)Q_{N-1} - R_{N-1} \\ \Gamma_1(\rho) & = & \left[\begin{array}{cccccc|c} A(\rho) & 0 & 0 & 0 & \ldots & A_h(\rho) & E(\rho) \end{array}\right] \\ Y(t) & = & \mathrm{col}(x(t), x_1(t), x_2(t), \ldots, x_N(t), w(t)) \\ x_i(t) & = & x(t - ih_n(t)) \end{array}$$

The time-derivative of the Hamiltonian function is negative definite if and only if

$$\dot{H}(t) \leq Y(t)^T \left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \dots & P(\rho)A_h(\rho) & P(\rho)E(\rho) \\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right] Y(t)$$
$$+\bar{h}\sum_{i=0}^{N-1} Y(t)^T\Gamma_1^T(\rho)\Gamma_1(\rho)Y(t) + \gamma^{-1}Y(t)^T\Gamma_2^T(\rho)\Gamma_2(\rho)Y(t)$$

with

$$\Gamma_2(\rho) = \left[\begin{array}{ccccc|c} C(\rho) & 0 & 0 & \dots & C_h(\rho) & F(\rho) \end{array}\right]$$

Then in virtue of the Schur complement with respect to terms

$$+\bar{h}\sum_{i=0}^{N-1} Y(t)^T\Gamma_1^T(\rho)\Gamma_1(\rho)Y(t) + \gamma^{-1}Y(t)^T\Gamma_2^T(\rho)\Gamma_2(\rho)Y(t)$$

we get

$$\begin{bmatrix} \mathcal{M}_{11} & \Gamma_2(\rho)^T & \bar{h}\Gamma_1(\rho)^T R_0 & \dots & \bar{h}\Gamma_1(\rho)^T R_{N-1} \\ \star & -\gamma I & 0 & \dots & 0 \\ \star & \star & -\bar{h}R_0 & & \\ \star & \star & & \ddots & \\ \star & \star & & & -\bar{h}R_{N-1} \end{bmatrix} \prec 0$$

with

$$\mathcal{M}_{11} = \left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \dots & P(\rho)A_h(\rho) & P(\rho)E(\rho) \\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right]$$

This concludes the proof. □

This result allows to obtain less conservative results than by using lemma 4.5.1 since:

1. extra degrees of freedom are added by fragmenting the delay which is equivalent to choose piecewise constant continuous functions $Q(\theta)$ and $R(\theta)$
2. the fragmentation of the delay reduces the conservatism of the Jensen's inequality

It is also important to notice that similar results are obtained in [Gouaisbaut and Peaucelle, 2006b; Peaucelle et al., 2007]. However, these results are based on translation of the state by

N	1	2	3	4
h_{max} Lemma 4.6.2	4.4721	5.7175	5.9678	6.0569
nb vars Lemma 4.6.2	9	15	21	27
h_{max} [Gouaisbaut and Peaucelle, 2006b]	4.4721	5.71	5.91	6.03
nb vars [Gouaisbaut and Peaucelle, 2006b]	9	50	147	324
h_{max} [Peaucelle et al., 2007]	4.4721	5.71	5.96	6.05
nb vars [Peaucelle et al., 2007]	9	16	27	42

Table 4.1: Comparison of the results obtained using different methods based on delay fragmentation

fragmented time-invariant delays which makes the problems more difficult when time-varying delays are considered but not unsolvable [Ariba et al., 2008]. The approach provided here is not based on any translation of the state and hence the problem of time-varying delays does not hold. The derived results are based on the application of the Lyapunov-Krasovskii's theorem using the functional (4.37), as done in [Han, 2008].

Example 4.6.3 *Let us consider the time-delay system [Gouaisbaut and Peaucelle, 2006b]*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h) \tag{4.40}$$

where the delay h is constant. The analytical maximal delay value for which the system is asymptotically stable is $h_{analytical} = 6.17$. Table 4.1 provides results using lemma 4.6.2. For $N = 1$, lemma 4.6.2 coincides with lemma 4.5.1.

On the other hand, by increasing N, the bound on the delay margin is less and less conservative which illustrates the effect of the delay fragmentation. Compared to results of [Gouaisbaut and Peaucelle, 2006b], the results are roughly identical for each fragmentation order. However, the number of decision variables is larger using the methods provided in [Gouaisbaut and Peaucelle, 2006b; Peaucelle et al., 2007] for each fragmentation number. On the other hand, since in [Gouaisbaut and Peaucelle, 2006b] the state of the system is augmented in order to gather every fragmented delayed state, then the number of decision matrices grows very quickly. Besides, this is an underlying problem of the Lyapunov approach where the Lyapunov matrix grows in $\dfrac{n(n+1)}{2}$ which is numerically expensive for large values of n. A way to avoid this problem is to consider an alternative approach based on nonsmooth analysis and optimization as described for instance in [Apkarian and Noll, 2006, 2007; Apkarian et al., 2007; Burke et al., 2006; Clarke, 1983; Clarke et al., 1998; Lewis, 2005, 2007]. Indeed, the number of decision variables with lemma 4.6.2 is given by

$$\frac{1}{2}(2N+1)n(n+1) \tag{4.41}$$

and a size of LMI constraint (4.38)

$$n(2N+1) \times n(2N+1) \tag{4.42}$$

where n is the dimension of the system and N the order of fragmentation. For instance, the number of variables is 27 for a system dimension $n = 2$ and a discretization order of $N = 4$;

as shown in the previous example. On the other hand, the approach in [Gouaisbaut and Peaucelle, 2006b] results in a number of decision variables

$$\frac{1}{2}Nn(1+2N)(Nn+1)$$

and a size of the principal LMI of

$$2Nn \times 2Nn$$

For instance, the number of variables is 27 for a system dimension $n = 2$ and a discretization order of $N = 4$; as shown in the previous example. The method of [Peaucelle et al., 2007] also results in a larger number of decision matrices.

When solving LMI problems with interior point algorithms [Henrion, 2008; Nesterov and Nemirovskii, 1994], the complexity (and thus the time of computation) of the algorithm highly depend on the size of LMIs. Hence, the size of LMIs is an important criterium to compare different methods. Actually, the LMI (4.38) can be reduced to a lower size by a Schur complement (see Appendix D.4) which results in a LMI of size $n(N+1) \times n(N+1)$ which is smaller than the LMI obtained in [Gouaisbaut and Peaucelle, 2006a].

4.6.2 Associated Relaxation

As for lemma 4.5.1, due to the multiple products between Lyapunov matrices $P(\rho)$ and R_i for $i = 0, \ldots, N-1$ and data matrices A, A_h and E, the linearization procedure is a difficult task in the control synthesis problems. A relaxed version is provided in order to decouple these terms by introducing a slack variable.

Lemma 4.6.4 *System (4.23) is asymptotically stable for all $h \in \mathcal{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function $P : U\rho \to \mathbb{S}_{++}^n$, a matrix function $X : U_\rho \to \mathbb{R}^{n\times n}$, constant matrices $Q_i, R_i \in \mathbb{S}_{++}^n$, $i \in \{0, \ldots, N-1\}$ and a scalar $\gamma > 0$ such that the LMI*

$$\left[\begin{array}{cccc|ccc} -X(\rho)^H & U_{12}(\rho) & 0 & X(\rho)^T & \bar{h}R_0 & \ldots & \bar{h}R_{N-1} \\ \star & U_{22}(\rho,\nu) & U_{23}(\rho) & 0 & 0 & \ldots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \ldots & 0 \\ \star & \star & \star & -P(\rho) & -\bar{h}R_0 & \ldots & -\bar{h}R_{N-1} \\ \hline \star & \star & \star & \star & \multicolumn{3}{c}{-\operatorname{diag}_i R_i} \end{array}\right] \prec 0 \tag{4.43}$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ and where

$$U_{22} = \left[\begin{array}{cccccc|c} U'_{11} & R_0 & 0 & 0 & \ldots & 0 & 0 \\ \star & N_1^{(1)} & R_1 & 0 & \ldots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \ldots & 0 & 0 & -\gamma I \end{array}\right] \tag{4.44}$$

$$
\begin{array}{rcl}
U'_{11} & = & \partial_\rho P(\rho)\dot{\rho} - P(\rho) + Q_0 - R_0 \\
N_i^{(1)} & = & -(1-i\mu_N)Q_{i-1} + (1+i\mu_N)Q_i - R_{i-1} - R_i \\
N^{(2)} & = & -(1-\mu)Q_{N-1} - R_{N-1} \\
U_{12}(\rho) & = & \left[\begin{array}{cccccccc} P(\rho)+X(\rho)^T A(\rho) & 0 & 0 & X(\rho)^T A_h(\rho) & \ldots & 0 & X(\rho)^T E(\rho) \end{array}\right] \\
U_{23}(\rho) & = & \left[\begin{array}{ccccc|c} C(\rho) & 0 & \ldots & 0 & C_h(\rho) & F(\rho) \end{array}\right]^T
\end{array}
$$

Proof: *The proof is similar to the proof of Lemma 4.5.2.* □

4.7 Simple Lyapunov-Krasovskii functional for systems with delay uncertainty

We consider here LPV time-delay systems of the form

$$
\begin{array}{rcl}
\dot{x}(t) & = & A(\rho)x(t) + A_h^1(\rho)x(t-h(t)) + A_h^2(\rho)x(t-h_c(t)) + E(\rho)w(t) \\
z(t) & = & C(\rho)x(t) + C_h^1(\rho)x(t-h(t)) + C_h^2(\rho)x(t-h_c(t)) + F(\rho)w(t)
\end{array} \tag{4.45}
$$

where the delays $h(t)$ and $h_c(t)$ are assumed to satisfy the relation $h_c(t) = h(t) + \theta(t)$ where $\theta(t) \in [-\delta, \delta]$, $\delta > 0$. The problem addressed in this section is the development of a delay-dependent test for a time-delay system involving two-delays which are related by an algebraic equation. This problem arises when stabilization, observation or filtering of a time-delay systems by a process (controller, observer or filter) involving a delay which is different from the system is sought. In this problem the objectives can be:

1. Given h_{max}, find the maximal uncertainty bound δ for which the system remains stable
2. Given δ find the delay value h_{max} for which the system remains stable

When dealing with performances criterium such as $\mathcal{H}_\infty$ level γ. Other combinations can be considered:

1. Given h_{max} and γ, find the maximal uncertainty bound δ such that the LMI conditions remain feasible
2. Given δ and γ, find the delay value h_{max} such that the LMI conditions remain feasible
3. Given h_{max} and δ, find the minimal $\mathcal{L}_2$ performances index γ for which the LMI conditions remain feasible.

The following sections address the problem of finding a Lyapunov-Krasovskii functional capturing both the stability/performances of system (4.45) and the algebraic equality $h_c(t) = h(t) + \theta(t)$. The last equality makes of this problem a new open problem which is has not been addressed in the literature (to our best knowledge).

4.7.1 Lyapunov-Krasovskii functional

The main idea in this problem is to capture both the maximal delay value for h but also capture the fact that the relation $h_c(t) = h(t) + \theta(t)$ exists with $\theta(t) \in [-\delta, \delta]$.

If a Lyapunov-Krasovskii functional of the form

$$\begin{array}{rcl} V(x_t, \dot{x}_t) & = & V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(\dot{x}_t) + V_5(\dot{x}_t) \\ V_1(x_t) & = & x(t)^T P x(t) \\ V_2(x_t) & = & \int_{t-h(t)}^{t} x(\theta)^T Q_1 x(\theta) d\theta \\ V_3(x_t) & = & \int_{t-h_c(t)}^{t} x(\theta)^T Q_2 x(\theta) d\theta \\ V_4(\dot{x}_t) & = & \int_{h_{max}}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T R_1 \dot{x}(\eta) d\eta d\theta \\ V_5(\dot{x}_t) & = & \int_{h_{max}+\delta}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T R_2 \dot{x}(\eta) d\eta d\theta \end{array} \tag{4.46}$$

were considered, it is clear that only the condition $h_{c_{max}} = h_{max} + \delta$ would be taken into account, but the 'global' constraint $h(t) = h_c(t) + \theta(t)$ would not. In such a case, the delays would be considered as independent and only their maximal amplitude (i.e. h_{max} and $h_{max} + \delta$) would be mutually dependent. This shows that a new specific Lyapunov-Krasovskii functional should be considered instead:

$$\begin{array}{rcl} V(x_t) & = & V_n(x_t) + V_u(x_t) \\ \text{where} & & \\ V_n(x_t) & = & x(t)^T P(\rho) x(t) + \int_{t-h(t)}^{t} x(s)^T Q_1 x(s) ds + \int_{-h_{max}}^{0} \int_{t+\beta}^{t} \dot{x}(s)^T h_{max} R_1 x(s) ds d\beta \\ V_u(x_t) & = & \int_{t-h_c(t)}^{t} x(s)^T Q_2 x(s) ds + \int_{-\delta}^{\delta} \int_{t+\beta-h(t)}^{t} \dot{x}(s) R_2 \dot{x}(s) ds d\beta \end{array} \tag{4.47}$$

The main difference compared to previous functionals is presence of the term last term of $V_u(x_t)$ which introduces terms in $t - h(t) + \delta$ and $t - h(t) - \delta$ which can be bounded by terms involving $h_c(t)$. This will be better explained in the proof of the following theorem:

Theorem 4.7.1 *System (4.45) is delay-dependent stable with $h(t) \in [0, h_{max}]$, $h_c(t) = h_c(t) + \theta(t)$, $\theta(t) \in [-\delta, \delta]$, $|\dot{h}(t)| < \mu$ and $|\dot{h}_c(t)| < \mu_c$ such that $||z||_{\mathcal{L}_2} \le \gamma ||w||_{\mathcal{L}_2}$ if there exists a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^n_{++}$ and symmetric matrices $Q_1, Q_2, R_1, R_2 \succ 0$ and a scalar $\gamma > 0$ such that*

$$\begin{bmatrix} \Psi_{11}(\rho, \dot{\rho}) & P(\rho)A_h(\rho) + R_1 & P(\rho)E(\rho) & h_{max}A(\rho)^T R_1 & A(\rho)^T R_2 & C(\rho)^T \\ \star & -(1-\mu)(Q_1 + Q_2) - R_1 & 0 & h_{max}A_h(\rho)^T R_1 & A_h(\rho)^T R_2 & C_h(\rho)^T \\ \star & \star & -\gamma I & h_{max}E(\rho)^T R_1 & E(\rho)^T R_2 & F(\rho)^T \\ \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & -(2\delta)^{-1} R_2 & 0 \\ \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{4.48}$$

$$\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) & h_{max}A(\rho)^TR_1 & A(\rho)^TR_2 & C(\rho)^T \\ \star & \Psi_{22} & (1-\mu)R_2/\delta & 0 & h_{max}A_h^1(\rho)^TR_1 & A_h^1(\rho)^TR_2 & C_h^1(\rho)^T \\ \star & \star & \Psi_{33} & 0 & h_{max}A_h^2(\rho)^TR_1 & A_h^2(\rho)^TR_2 & C_h^2(\rho)^T \\ \star & \star & \star & -\gamma I & h_{max}E(\rho)^TR_1 & E(\rho)^TR_2 & F(\rho)^T \\ \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & \star & -\dfrac{R_2}{2\delta} & 0 \\ \star & \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{4.49}$$

hold for all $\rho \in U_\rho$ *and* $\nu = \mathrm{col}(\nu_i) \in U_\nu$ *where*

$$\begin{aligned} \Psi_{11}(\rho,\nu) &= A(\rho)^TP(\rho)+P(\rho)A(\rho)+Q_1+Q_2+\sum_{i=1}^{N}\frac{\partial P}{\partial \rho_i}\nu_i - R_1 \\ \Psi_{22} &= -(1-\mu)(Q_1+R_2/\delta)-R_1 \\ \Psi_{33} &= -(1-\mu_c)Q_2-(1-\mu)R_2/\delta \\ A_h &= A_h^1+A_h^2 \\ C_h &= C_h^1+C_h^2 \end{aligned}$$

Proof: *Differentiating (4.47) along the trajectories solutions of the system (4.45) yields:*

$$\begin{aligned} \dot{V}_n &\le Y(t)^T\left(\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho})-Q_2 & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) \\ \star & -(1-\dot{h})Q_1-R_1 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix}\right. \\ &\quad \left. +h_{max}^2\begin{bmatrix} A(\rho)^T \\ A_h^1(\rho)^T \\ A_h^2(\rho)^T \\ E(\rho)^T \end{bmatrix} R_1 \begin{bmatrix} A(\rho) & A_h^1(\rho) & A_h^2(\rho) & E(\rho) \end{bmatrix}\right) Y(t) \qquad (4.50) \\ \dot{V}_u &= x(t)^TQ_2x(t)-(1-\dot{h}_c)x(t-h_c(t))^TQ_2x(t-h_c(t))+2\delta\dot{x}(t)^TR_2\dot{x}(t) \\ &\quad -(1-\dot{h}(t))\int_{t-\delta-h(t)}^{t+\delta-h(t)}\dot{x}(s)^TR_2\dot{x}(s)ds \end{aligned}$$

where $\Psi_{11}(\rho,\dot{\rho}) = A(\rho)^TP(\rho)+P(\rho)A(\rho)+Q_1+Q_2+\dfrac{\partial P}{\partial \rho}\dot{\rho}-R_1$ *and* $Y(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_c(t)) \\ w(t) \end{bmatrix}$.

Moreover, note that we have the inequality

$$\begin{aligned} -\int_{t-\delta-h(t)}^{t+\delta-h(t)}\dot{x}(s)^TR_2\dot{x}(s)ds &\le -\,\mathrm{sgn}(h(t)-h_c(t))\int_{t-h(t)}^{t-h_c(t)}\dot{x}(s)^TR_2\dot{x}(s)ds \\ &\le -\frac{1}{|h(t)-h_c(t)|}\left(\int_{t-h(t)}^{t-h_c(t)}\dot{x}(s)ds\right)^T R_2\left(\int_{t-h(t)}^{t-h_c(t)}\dot{x}(s)ds\right) \\ &\le -\frac{1}{\delta}\left(\int_{t-h(t)}^{t-h_c(t)}\dot{x}(s)ds\right)^T R_2\left(\int_{t-h(t)}^{t-h_c(t)}\dot{x}(s)ds\right) \end{aligned} \tag{4.51}$$

This shows that two cases must be treated separately:

1. *either when $h_c(t_i) = h(t_i)$ for some $t_i \geq 0$ and in this case $x(t_i - h(t_i)) = x(t_i - h_c(t_i))$, or*

2. *when $h_c(t) \neq h(t)$ for all $t \geq 0$ and $t \neq t_i$.*

Case. 1: *When $h_c(t_i) = h(t_i)$ the derivative of the Lyapunov-Krasovskii functional reduces to*

$$\begin{aligned}\dot{V} \leq\ & X(t_i)^T \left(\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h(\rho) + R_1 & P(\rho)E(\rho) \\ \star & -(1-\dot{h}(t_i))(Q_1+Q_2) & 0 \\ \star & \star & 0 \end{bmatrix} \right. \\ & + \left. \begin{bmatrix} h_{max}A(\rho)^T & h_{max}A(\rho)^T \\ h_{max}A_h(\rho)^T & h_{max}A_h(\rho)^T \\ h_{max}E(\rho)^T & h_{max}E(\rho)^T \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & 2\delta R_2 \end{bmatrix} \begin{bmatrix} h_{max}A(\rho)^T & h_{max}A(\rho)^T \\ h_{max}A_h(\rho)^T & h_{max}A_h(\rho)^T \\ h_{max}E(\rho)^T & h_{max}E(\rho)^T \end{bmatrix}^T \right) X(t_i)\end{aligned}$$

where $X(t) = \mathrm{col}(x(t), x(t-h(t)), w(t))$ and $A_h(\rho) = A_h^1(\rho) + A_h^2(\rho)$. And finally, a Schur complement yields LMI

$$X(t_i)^T \begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h(\rho) + R_1 & P(\rho)E(\rho) & h_{max}A(\rho)^T R_1 & A(\rho)^T R_2 \\ \star & -(1-\dot{h}(t_i))(Q_1+Q_2) - R_1 & 0 & h_{max}A_h(\rho)^T R_1 & A_h(\rho)^T R_2 \\ \star & \star & 0 & h_{max}E(\rho)^T R_1 & E(\rho)^T R_2 \\ \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & -(2\delta)^{-1}R_2 \end{bmatrix} X(t_i) \prec 0$$

Adding the input/output constraint

$$-\gamma w(t_i)^T w(t_i) + \gamma^{-1} z(t_i)^T z(t_i) = -\gamma w(t_i)^T w(t_i) + \gamma^{-1} X(t_i)^T \begin{bmatrix} C(\rho)^T \\ C_h(\rho)^T \\ F(\rho)^T \end{bmatrix} \begin{bmatrix} C(\rho)^T \\ C_h(\rho)^T \\ F(\rho)^T \end{bmatrix}^T X(t_i)$$

with $C_h(\rho) = C_h^1(\rho) + C_h^2(\rho)$. A Schur complement leads to the final LMI for the case 1:

$$\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h(\rho) + R_1 & P(\rho)E(\rho) & h_{max}A(\rho)^T R_1 & A(\rho)^T R_2 & C(\rho)^T \\ \star & -(1-\dot{h}(t_i))(Q_1+Q_2) - R_1 & 0 & h_{max}A_h(\rho)^T R_1 & A_h(\rho)^T R_2 & C_h(\rho)^T \\ \star & \star & -\gamma I & h_{max}E(\rho)^T R_1 & E(\rho)^T R_2 & F(\rho)^T \\ \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & -(2\delta)^{-1}R_2 & 0 \\ \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} \prec 0$$

Case.* *2: *When $t \geq 0$ and $t \neq t_i$ then we have*

$$\begin{array}{rcl}\dot{V} & \leq & Y(t)^T \left(\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) \\ \star & -(1-\dot{h}(t))Q_1 - R_1 & 0 & 0 \\ \star & \star & -(1-\dot{h}_c(t))Q_2 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \right. \\ & & + \left. \begin{bmatrix} A(\rho)^T & A(\rho)^T \\ A_h^1(\rho)^T & A_h^1(\rho)^T \\ A_h^2(\rho)^T & A_h^2(\rho)^T \\ E(\rho)^T & E(\rho)^T \end{bmatrix} \begin{bmatrix} h_{max}R_1 & 0 \\ 0 & 2\delta R_2 \end{bmatrix} \begin{bmatrix} A(\rho) & A_h^1(\rho) & A_h^2(\rho) & E(\rho) \\ A(\rho) & A_h(\rho) & A_h^2(\rho) & E(\rho) \end{bmatrix} \right) Y(t) \\ & & -\dfrac{(1-\dot{h}(t))}{\delta} Y(t)^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ \star & R_2 & -R_2 & 0 \\ \star & \star & R_2 & 0 \\ \star & \star & \star & 0 \end{bmatrix} Y(t)\end{array}$$

and this leads to

$$\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) & h_{max}A(\rho)^TR_1 & A(\rho)^TR_2 \\ \star & \Psi_{22} & (1-\dot{h}(t))R_2/\delta & 0 & h_{max}A_h^1(\rho)^TR_1 & A_h^1(\rho)^TR_2 \\ \star & \star & \Psi_{33} & 0 & h_{max}A_h^2(\rho)^TR_1 & A_h^2(\rho)^TR_2 \\ \star & \star & \star & 0 & h_{max}E(\rho)^TR_1 & E(\rho)^TR_2 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & \star & -(2\delta)^{-1}R_2 \end{bmatrix} \prec 0$$

where $\Psi_{22} = -(1-\dot{h}(t))Q_1 - (1-\dot{h})R_2/\delta - R_1$ and $\Psi_{33} = -(1-\dot{h}_c(t))Q_2 - (1-\dot{h}(t))R_2/\delta$. Finally, adding the same input/ouput constraint as for the case 1, yields

$$\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) & h_{max}A(\rho)^TR_1 & A(\rho)^TR_2 & C(\rho)^T \\ \star & \Psi_{22} & (1-\dot{h}(t))R_2/\delta & 0 & h_{max}A_h^1(\rho)^TR_1 & A_h^1(\rho)^TR_2 & C_h^1(\rho)^T \\ \star & \star & \Psi_{33} & 0 & h_{max}A_h^2(\rho)^TR_1 & A_h^2(\rho)^TR_2 & C_h^2(\rho)^T \\ \star & \star & \star & -\gamma I & h_{max}E(\rho)^TR_1 & E(\rho)^TR_2 & F(\rho)^T \\ \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & \star & -(2\delta)^{-1}R_2 & 0 \\ \star & \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} \prec 0$$

Bounding $-(1-\dot{h}(t)) \leq -(1-\mu)$ and $-(1-\dot{h}(t)) \leq -(1-\mu_c)$ leads to the proposed result. Finally considering that $\dot{\rho}$ belongs to the polytope $\mathrm{hull}(U_\nu)$, the dependence on $\dot{\rho}$ is relaxed. $\square$

4.7.2 Associated Relaxation

Similarly as previously, we provide a relaxed version of the results.

Theorem 4.7.2 *System (4.45) is delay-dependent stable with $h(t) \in [0, h_{max}]$, $h_c(t) = h_c(t) + \theta(t)$, $\theta(t) \in [-\delta, \delta]$, $|\dot{h}(t)| < \mu$ and $|\dot{h}_c(t)| < \mu_c$ such that $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ if there exists a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^n_{++}$ and symmetric matrices $Q_1, Q_2, R_1, R_2 \succ 0$, a matrix $X : U_\rho \to \mathbb{R}^{n\times n}$ and a scalar $\gamma > 0$ such that*

$$\begin{bmatrix} -X(\rho)^H & P(\rho)+X(\rho)^T A(\rho) & X(\rho)^T A_h(\rho) & X(\rho)^T E(\rho) & 0 & X(\rho)^T & h_{max}R_1 & R_2 \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & C(\rho)^T & 0 & 0 & 0 \\ \star & \star & \Theta_{22} & 0 & C_h(\rho)^T & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I & F(\rho)^T & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & \star & \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \prec 0$$

and

$$\begin{bmatrix} \Pi_{11}(\rho,\nu) & \Pi_{12}(\rho) \\ \star & \Pi_{22}(\rho) \end{bmatrix} \prec 0$$

hold for all $\rho \in U_\rho$ and where

$$\Pi_{11}(\rho,\nu) = \begin{bmatrix} -X(\rho)^H & P(\rho)+X(\rho)^T A(\rho) & X(\rho)^T A_h^1(\rho) & X(\rho)^T A_h^2(\rho) & X(\rho)^T E(\rho) \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & 0 \\ \star & \star & \Psi_{22} & (1-\mu)R_2/\delta & 0 \\ \star & \star & \star & \Psi_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix}$$

$$\Pi_{12}(\rho) = \begin{bmatrix} 0 & X(\rho)^T & h_{max}R_1 & R_2 \\ C(\rho)^T & 0 & 0 & 0 \\ C_h^1(\rho)^T & 0 & 0 & 0 \\ C_h^2(\rho)^T & 0 & 0 & 0 \\ F(\rho)^T & 0 & 0 & 0 \end{bmatrix} \qquad \Pi_{22}(\rho) = \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & -R_1 & 0 \\ \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix}$$

$$\begin{aligned} \Theta_{11}(\rho,\nu) &= -P(\rho) + Q_1 + Q_2 + \sum_{i=1}^{N} \frac{\partial P}{\partial \rho_i}\nu_i - R_1 & \Theta_{22} &= -(1-\mu)(Q_1+Q_2) - R_1 \\ \Psi_{22} &= -(1-\mu)(Q_1 + R_2/\delta) - R_1 & \Psi_{33} &= -(1-\mu_c)Q_2 - (1-\mu)R_2/\delta \\ A_h &= A_h^1 + A_h^2 & C_h &= C_h^1 + C_h^2 \end{aligned}$$

Proof: *The proof is similar as for other relaxations.* □

4.8 Chapter Conclusion

In this chapter, preliminary results which will be used in the remaining of the thesis have been presented. First of all, fundamental definitions for the set of the delay and the parameters have been detailed and explained.

Second, a method to relax polynomially parameter dependent LMIs have been provided. It has the benefit of turning the initial LMI condition into a new LMI condition whose dependence is linear only with respect to the parameters, at the expense of an increase of the computational complexity through the addition of a 'slack' variable.

Then a novel relaxation for concave nonlinearity has been presented which finds its interest where the cone complementary algorithm cannot apply, i.e. when the involved matrices are not constant. This method will be applied successfully in Section 6.1.3.

The following section has been devoted to the computation of the bounds on parameter derivatives in the polytopic case and allows to deal easily with robust stability and synthesis in the LPV polytopic approach.

A simple Lyapunov-Krasovskii has been introduced with its associated relaxation. This functional has proven its efficiency despite of its simplicity and this has motivated its use in this thesis. The associated relaxation finds its interest in the design problems which greatly simplifies the problem.

In order to improve latter results based on a simple functional, the following section has been devoted to a discretized version of this functional where the decision matrices are functions. Using this 'complete' version it is possible to refine the results until reach theoretical delay margin. Its associated relaxation allows to transfer the quality of the results from the stability analysis to design purposes

Finally, a new Lyapunov-Krasovskii functional has been provided in order to analyze the stability of a system with two delays which are coupled through an algebraic inequality. Such case occurs when a time-delay systems is observed or controlled by an observer or a controller with memory but implementing a delay different from the system one. This will be used in Sections and 5.1.2 and 6.1.6.

Chapter 5

Observation and Filtering of LPV time-delay systems

One of the objectives of systems theory is to provide tools on observation and filtering of systems. The objectives of observation and filtering is to estimate unmeasured signals or clean signals from eventual noises and/or disturbances provided that a model of the system is available. However, conceptual differences remain between the notion of filters and observers and will be emphasized in the introduction of this chapter.

An observer aims at estimating signals of a system by finding observer matrices such that the state estimation error is asymptotically stable. This means that, for any initial conditions of the observer, the autonomous observation error will tend to zero, in other words, the autonomous linear differential equation governing the observation error is asymptotically (exponentially) stable. Moreover, it is important to note that a good observer should be able to observe whatever the value of the state of the system is and hence the observation error should be independent of the system state. However, this can be handled in the certain case only since in the uncertain case the observation error depends on the current state. However, it is possible to construct nonlinear observers which makes the error converge to zero even if the state is nonzero [Boutayeb and Darouach, 2003; Gu and Poon, 2001]. As a final remark, the use of observers is better suited for control purposes since the observer estimates the system state, allowing the use of a state-feedback.

On the other hand, the filtering of system does not require any stability of a 'filtering error' but aims at guaranteeing a minimal attenuation, in some norm sense, of a residual computed from the difference of a desired estimated signal and the estimate under action of disturbances. In this case, the quality of the estimation would depend on the current state of the system.

At first sight, it seems that filtering is less relevant than observation but actually each way has its own benefits and drawbacks. While many observation approaches work well for LTI and certain systems, when dealing with LPV systems, the problem is far more difficult. Moreover, the class of systems that can be treated by observation theories is not as wide as for filtering ones. Filtering approaches can address a large variety of systems and the resulting problem is generally more simple to handle and for this reason, only filtering of LPV systems is generally provided in the literature [Mohammadpour and Grigoriadis, 2006a,b, 2007a,b, 2008].

When dealing with time-delay systems, the diversity of observers and filters is slightly larger than for finite-dimensional linear dynamical systems. Indeed, it is possible to consider the additional information on the delay when it is available. This gives rise to the notion of filters/observers with memory and memoryless observers/filters. It may seems uninteresting to design memoryless observers and filters but actually, for two reasons, it is important to consider them. First of all, implementing a delay in the observer/filter needs memory space which can be incompatible with embedded applications; secondly, the real-time estimation of the value of the current delay of a physical system remains a challenging open problem [Belkoura et al., 2007, 2008; Drakunov et al., 2006].

In this chapter, we will be interested in both observation and filtering of LPV time-delay systems. Observers that will be designed for the LPV case are based on an algebraic approach, initially developed for LTI time-delay systems [Darouach, 2001, 2005]. Reduced-order as well as full-order observers will be designed for LPV time-delay systems. Necessary and sufficient conditions for their existence will be provided in terms of algebraic matrix equalities and the stability of a functional differential equation. The computation of observer matrices will be performed through the computation of solutions of LMIs. An example of filter design for uncertain LPV systems will also be introduced and is a generalization of the method presented in [Tuan et al., 2001b, 2003] to time-delay systems and it will be shown that interesting performances are achieved.

It is important to point out that, in both filtering and observation case, only memory processes with exact delay value is addressed and generally no robustness analysis is given in presence of uncertainty on implemented delay. In [Sename and Briat, 2006; Verriest et al., 2002], a robustness analysis is performed a posteriori in the case of constant delay according to an application of the Rouché's theorem (see Section 3.2.2 and Appendix E.6). In this section, robust filtering/observation with respect to delay uncertainty and parametric uncertainties will be addressed and therefore the designed processes will remain stable even in presence of (time-varying) delay-uncertainty provided that the delay implementation error remains in a ball whose radius is a priori fixed or maximized by an optimization algorithm based on LMIs. This problem has never been addressed in the literature and is one of the main points on this section. In [Briat et al., 2007c], an Luenberger observer has been developed for LPV time-delay systems using a free weighting approach [He et al., 2004]; the results are not presented here since we will focus on more interesting observer synthesis techniques.

Hereunder, a non exhaustive bibliography on observation and filtering of time-delay systems and LPV systems is given for informational purpose:

- For pioneering works on observation of delay systems see [Bhat and Koivo, 1976; Fattouh, 2000; Fattouh et al., 1998; Gressang and Lamont, 1975; H.-Hashemi and Leondes, 1979; Lee and Olbrot, 1981; Ogunnaike, 1981; Pearson and Fiagbedzi, 1989; Sename, 2001]

- Concerning observers for nonlinear delay systems see [Germani et al., 1998, 1999, 2001, 2002; Pepe, 2001]

- Recent works on observation of linear time-delay systems [Chen, 2007; Koenig and Marx, 2004; Koenig et al., 2004, 2006; Picard and Lafay, 1996; Picard et al., 1996; Sename, 1997; Sename and Briat, 2006; Sename et al., 2001; Verriest et al., 2002]

- Recent works on the filtering of time-delay systems [DeSouza et al., 1999; Fridman et al., 2003a,b; Zhang and Han, 2008]
- Filtering of LPV systems [Borges and Peres, 2006; Tuan et al., 2001b, 2003]
- Filters for LPV time-delay systems [Mohammadpour and Grigoriadis, 2006a,b, 2007a, 2008; Wu et al., 2006]

5.1 Observation of Unperturbed LPV Time-Delay Systems

This section is devoted to the design of observers and filters for LPV time-delay systems without uncertainties. Several approaches will be provided depending on the type of filter/observer (with or without memory) and the knowledge of the delay (exactly known, approximately known and unknown).

The observers designed in this section are based on the extension to the LPV case of the method of Darouach [2001, 2005]. Throughout this section on observers the following class of LPV time-delay system will be considered:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) & = & Tx(t) \\ y(t) & = & Cx(t) \end{array} \tag{5.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, $w \in \mathbb{R}^q$ and $z \in \mathbb{R}^r$ are respectively the system state, the system control input, the system measurements, the system exogenous inputs and the signal to be estimated. In this framework, the observer aims at estimating as close as possible the signal $z(t)$ which is a linear combination of the state variables of the system. The matrix T is assumed to have full row-rank and whenever $\text{rank}(T) = n$ then $T = I$. The delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$ recalled below:

$$\mathscr{H}_1^\circ := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+, [0, h_{max}]) : \ |\dot{h}| < \mu \right\}$$

The corresponding general observer is governed by the following expressions:

$$\begin{array}{rcl} \dot{\xi}(t) & = & M_0(\rho)\xi(t) + M_h(\rho)\xi(t-d(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-d(t)) + S(\rho)u(t) \\ \hat{z}(t) & = & \xi(t) + Hy(t) \end{array} \tag{5.2}$$

where $\xi \in \mathbb{R}^r$, $\hat{z} \in \mathbb{R}^r$ are respectively the observer state and the estimated output. The delay $d(t)$ is unconstrained at this stage and precisions will be given in each forthcoming sections depending on the context and on the type of observer considered. The matrices $M_0(\rho)$, $M_h(\rho)$, $N_0(\rho)$, $M_H(\rho)$ and H are matrices of appropriate dimensions which define the observer. Note that H is a constant matrix and $\hat{z}$ is a linear combination of the observer state and the measurement vector y.

It is worth mentioning that when dealing with such an observer, it is difficult to consider a disturbance term on the measured output since during the design procedure, the measured output needs to be differentiated. If it would depend on the disturbance w, then a term $\dot{w}$ would appear in the equations and then the disturbance vector should be augmented in order to contain both w and $\dot{w}$ (e.g. $\tilde{w} = \text{col}(w, \dot{w})$). This is a straightforward generalization of the current method and is then not explored in the thesis.

Definition 5.1.1 *If $r = n$ then the observer is called full-order observer while if $T = T_r \in \mathbb{R}^{r\times n}$ such as* $\text{rank}(T_r) = r < n$ *then the observer is called reduced-order observer.*

The aim of the observer is to decouple the system state from the error $e(t) = z(t) - \hat{z}(t)$ as in [Darouach, 2001], that is we should have an equation of the form

$$\dot{e}(t) = f(e_t) + g(\eta(t))$$

where $f(\cdot)$ is a functional and $g(\cdot)$ is a function gathering other signals (such as disturbances) excluding the state of the system. In this case, it is clear that if $f(\cdot)$ describes a stable vector field then the observation error has stable dynamics. Moreover, for every trajectories of the system, the error will have the same behavior in terms of rate of convergence, response time... We will see that this ideal behavior can be only be achieved when the delay implemented in the observer is identical to the delay involved in the system and when the system is perfectly known (no uncertainties). Therefore such a behavior cannot be obtained from a practical point of view.

It is important to note that when the observation error cannot be isolated from the state of the system, only stable LPV time-delay system can be observed. Indeed, suppose that the error obeys the following dynamical model

$$\dot{e}(t) = f(e_t) + g(x_t) + h(\eta(t))$$

where $f(\cdot)$, $g(\cdot)$ are functionals and $h(\cdot)$ is a function gathering other signals. From this expression even if $f(\cdot)$ is a stable vector field, then the error will remains bounded around 0 if and only if the other terms are bounded too (BIBO stability). However, if the system is unstable then we might have $g(x_t) \to +\infty$ as $t \to +\infty$ and hence $e(t) \to +\infty$ as $t \to +\infty$. Such a behavior arises when dealing with observer involving a delay which is different from the system one or using memoryless observers. An immediate choice would be to consider the term $g(x_t)$ as a disturbance term and in many frameworks it would be correct, for instance in a pure stabilization or α-stabilization problems where it is assumed that the system is stable or controlled (i.e. $x(t)$ does not tends to $+\infty$ as t goes to ∞).

In the $\mathcal{H}_\infty$ problem where an observer minimizing the influence of the disturbances onto the observation error (in the $\mathcal{L}_2$ sense) is sought, we cope with two possibilities:

1. either the vector of disturbances is augmented to contain both the initial disturbances vector $\eta(t)$ and the term $g(x_t)$ but in this case a loss of information occurs since the relation between the disturbances $\eta(t)$ and $x(t)$ is not taken into account; or

2. the system is augmented in order to contain both the state of the system and the observation error and in this case, the $\mathcal{H}_\infty$ analysis/synthesis is more tight.

This is the latter approach which will be considered throughout this section on observers.

5.1.1 Observer with exact delay value - simple Lyapunov-Krasovskii functional case

In this section, the problem of observation of a LPV time-delay system with an observer involving a delay identical to the system one is solved; see [Darouach, 2001] for the LTI case.

Even if this observer may be not implementable, the design approach is interesting and can be extended to implementable cases. The observer to be designed is then given by the equations:

$$\begin{array}{rcl}\dot{\xi}(t) & = & M_0(\rho)\xi(t) + M_h(\rho)\xi(t-h(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-h(t)) + S(\rho)u(t) \\ \hat{z}(t) & = & \xi(t) + Hy(t)\end{array} \tag{5.3}$$

where $\xi \in \mathbb{R}^r$, $\hat{z} \in \mathbb{R}^r$ are respectively the observer state and the estimated output. The following theorem provides a necessary and sufficient condition to the existence of such an observer.

Theorem 5.1.2 *There exists an LPV/$\mathcal{H}_\infty$ observer with memory (5.3) for system of the form (5.1) if and only if the following statements hold:*

1. *The autonomous error dynamical expression $\dot{e}(t) = M_0(\rho)e(t) + M_h(\rho)e(t-h(t))$ is asymptotically stable where $e(t) = z(t) - \hat{z}(t)$ and $h \in \mathscr{H}_1^\circ$.*
2. $(T - HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T - HC) = 0$
3. $(T - HC)A_h(\rho) - N_h(\rho)C - M_h(\rho)(T - HC) = 0$
4. $(T - HC)B(\rho) - S(\rho) = 0$
5. *The inequality $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ holds for some $\gamma > 0$*

Proof: *First let $e(t) = z(t) - \hat{z}(t)$ be the estimation error. The latter equality reduces to*

$$e(t) = (T - HC)x(t) - \xi(t) \tag{5.4}$$

according to the definition of $\hat{z}(t)$ in (5.3). Computing the time derivative of $e(t)$ we get

$$\begin{array}{rcl}\dot{e}(t) & = & (T-HC)\dot{x}(t) - \dot{\xi}(t) \\ & = & (T-HC)\left[A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t)\right] \\ & & -\left[M_0(\rho)\xi(t) + M_h(\rho)\xi(t-h(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-h(t))\right. \\ & & \left.+S(\rho)u(t)\right] \\ & = & [(T-HC)A(\rho) - N_0(\rho)C]x(t) + [(T-HC)A_h(\rho) - N_h(\rho)C]x(t-h(t)) \\ & & +[(T-HC)B(\rho) - S(\rho)]u(t) - M_0(\rho)\xi(t) - M_h(\rho)\xi(t-h(t)) \\ & & +(T-HC)E(\rho)w(t)\end{array}$$

Using $\xi(t) = (T - HC)x(t) - e(t)$ obtained from (5.4) we get

$$\begin{array}{rcl}\dot{e}(t) & = & [(T-HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T-HC)]x(t) \\ & & +[(T-HC)A_h(\rho) - N_h(\rho)C - M_h(\rho)(T-HC)]x(t-h(t)) \\ & & +[(T-HC)B(\rho) - S(\rho)]u(t) + M_0(\rho)e(t) + M_h(\rho)e(t-h(t)) \\ & & +(T-HC)E(\rho)w(t)\end{array}$$

According to the discussion at the beginning of Section 5.1, we wish to obtain an error e whose dynamical model is independent of the the control input, the current and delayed state of the system. Hence by imposing

$$(T-HC)A(\rho)x(t) - N_0(\rho)C - M_0(\rho)(T-HC) = 0 \tag{5.5}$$

$$(T-HC)A_h(\rho)x(t-h(t)) - N_h(\rho)C - M_h(\rho)(T-HC) = 0 \tag{5.6}$$

$$(T-HC)B(\rho) - S(\rho) = 0 \tag{5.7}$$

the error dynamical model reduces to

$$\dot{e}(t) = M_0(\rho)e(t) + M_h(\rho)e(t-h(t)) + (T-HC)E(\rho)w(t) \tag{5.8}$$

and is actually independent of the system state and control input. Finally, if the latter dynamical model defines stable dynamics then it is possible to find a $\gamma > 0$ such that $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$. This concludes the proof. $\square$

A theorem providing necessary and sufficient conditions for the existence of an observer of the form (5.4) for systems (5.1) has been developed. It is worth noting that such a design can be extended to $\mathcal{H}_2$, $\mathcal{L}_\infty$-$\mathcal{L}_\infty$ problems and so on. However, such a result is not constructive and then Theorem 5.1.2 cannot be directly used for synthesis purposes. It can be divided in two parts:

1. the first one involves nonlinear algebraic equations (statements 2 to 4) which are 'static'
2. the second part involves dynamic related conditions related to the stability of a system and its worst-case energy gain

The first step of the solution is to explicitly define the set of all matrices satisfying statements 2 to 5. This is performed in the following lemma where it considered that the matrix $H(\rho)$ depends on the parameters while it should be constant. This condition will be relaxed when the LMI conditions for gain computation will be provided.

Lemma 5.1.3 *There exists a solution $M_0(\rho), M_h(\rho), N_0(\rho), N_h(\rho), S(\rho), H(\rho)$ to equations (5.5), (5.6) and (5.7) if and only if the following rank equality holds*

$$\text{rank}\begin{bmatrix} T & 0 \\ 0 & T \\ C & 0 \\ 0 & C \\ CA(\rho) & CA_h(\rho) \\ TA(\rho) & TA_h(\rho) \end{bmatrix} = \text{rank}\begin{bmatrix} T & 0 \\ 0 & T \\ C & 0 \\ 0 & C \\ CA(\rho) & CA_h(\rho) \end{bmatrix} \tag{5.9}$$

for all $\rho \in U_\rho$.

Proof: *Equation (5.7) is explicit since it suffices to find H then S is obtained by the explicit expression*

$$S(\rho) = (T - HC)B(\rho)$$

On the other hand, the two equalities (5.5) and (5.6) are nonlinear due to terms $M_0(\rho)(T - HC)$ and $M_h(\rho)(T-HC)$. However rewriting them into the form

$$\begin{array}{rcl} (T - H(\rho)C)A(\rho)x(t) + (M_0(\rho)H(\rho) - N_0(\rho))C - M_0(\rho)T & = & 0 \\ (T - H(\rho)C)A_h(\rho)x(t) + (M_h(\rho)H(\rho) - N_h(\rho))C - M_h(\rho)T & = & 0 \end{array}$$

shows that the change of variable

$$\begin{array}{rcl} K_0(\rho) & = & N_0(\rho) - M_0(\rho)H(\rho) \\ K_h(\rho) & = & N_h(\rho) - M_h(\rho)H(\rho) \end{array} \tag{5.10}$$

linearizes the expressions into

$$\begin{array}{rcl}(T-H(\rho)C)A(\rho)x(t)-K_0(\rho)C-M_0(\rho)T & = & 0\\ (T-H(\rho)C)A_h(\rho)x(t)-K_h(\rho)C-M_h(\rho)T & = & 0\end{array} \tag{5.11}$$

It is important to note that the change of variable is bijective and hence no conservatism is introduced. Indeed, the set of matrices $(M_0(\rho), M_h(\rho), K_0(\rho), K_h(\rho), H(\rho))$ *defines in a unique way the set* $(M_0(\rho), M_h(\rho), N_0(\rho), N_h(\rho), H(\rho))$ *due to the change of variable (5.10). Rewriting equalities (5.11) in a more compact matrix expression leads to*

$$\nabla(\rho)\Gamma(\rho) = \Lambda(\rho)$$

where

$$\begin{array}{rcl}\nabla(\rho) & = & \left[\begin{array}{ccccc} M_0(\rho) & M_h(\rho) & K_0(\rho) & K_h(\rho) & H(\rho)\end{array}\right] : U_\rho \to \mathbb{R}^{r\times 2r+3m}\\ \Gamma(\rho) & = & \left[\begin{array}{cc} T & 0\\ 0 & T\\ C & 0\\ 0 & C\\ CA(\rho) & CA_h(\rho)\\ \end{array}\right] \qquad \Lambda(\rho) = \left[\begin{array}{cc} TA(\rho) & TA_h(\rho)\end{array}\right]\end{array}$$

According to [Darouach, 2001; Koenig et al., 2006; Lancaster and Tismenetsky, 1985; Mitra and Mitra, 1971], there exist solutions with $H(\rho)$ *to this expression if and only if*

$$\text{rank}\left[\begin{array}{c}\Gamma(\rho)\\ \Lambda(\rho)\end{array}\right] = \text{rank}\,\Gamma(\rho)$$

which is exactly (5.9). This concludes the proof. □

Whenever Lemma 5.1.3 is satisfied then it is confirmed that there exists at least one solution to equations (5.5), (5.6) and (5.7). The number of solution is either 1 or is infinite. We are interested in the case of an infinite number of solutions since it is not guaranteed that the unique solution gives a stable error dynamical model.

Remark 5.1.4 *A sufficient condition for an infinite number of solutions is that the number of unknown variables (the number of coefficients in the unknown matrices) exceeds the number of equations (the number of coefficients in matrices of dimension equals to the order of the observer). Hence, it suffices that the following condition*

$$2\dim(z)\dim(x) \leq \dim(z)^2 + 3\dim(z)\dim(y) \tag{5.12}$$

holds. From this inequality it is possible to give more relevant conditions for the existence of an infinite number of solutions, indeed we must have

$$\begin{array}{rcl}\dim(y) & \geq & \frac{2}{3}(\dim(x)-\dim(z))\\ \dim(z) & \geq & \dim(x) - \frac{3}{2}\dim(y)\end{array} \tag{5.13}$$

The first inequality provides a lower bound on the number of sensors that must be used for a given system dimension and observer order. The second inequality provides the minimal observer order that can be used for some given system dimension and output dimension. It is worth noting that the problem may be unsolvable since no consideration on the stability of the error is taken into account in these conditions.

When the number of solution is infinite, the objective (and interest) is to parametrize the set of solution. The following lemma provides such a parametrization provided that lemma 5.1.3 is satisfied.

Lemma 5.1.5 *Under conditions of Theorem 5.1.3, the observer matrices are given by the expressions $M_0 = \Theta - L\Xi$, $M_h = \Upsilon - L\Omega$ and $H = \Phi - L\Psi$ where L is a free matrix of appropriate dimensions and*

$$\begin{array}{rclcrcl}
\Theta & = & TAU - \Lambda\Gamma^+\Delta_0\begin{bmatrix} C \\ CA \end{bmatrix}U & & \Xi & = & -(I-\Gamma\Gamma^+)\Delta_0\begin{bmatrix} C \\ CA \end{bmatrix}U \\
\Upsilon & = & TA_hU - \Lambda\Gamma^+\Delta_h\begin{bmatrix} C \\ CA_h \end{bmatrix}U & & \Omega & = & -(I-\Gamma\Gamma^+)\Delta_h\begin{bmatrix} C \\ CA_h \end{bmatrix}U \\
\Phi & = & \Lambda\Gamma^+\Delta_H & & \Psi & = & (I-\Gamma\Gamma^+)\Delta_H \\
N_0 & = & K_0 + M_0H & & N_h & = & K_h + M_hH \\
F & = & T - HC & & S & = & FB
\end{array}$$

where U is defined such that

$$\begin{bmatrix} T \\ \bar{T} \end{bmatrix}^{-1} = \begin{bmatrix} U & V \end{bmatrix}$$

, $\bar{T}$ is a full column rank matrix such that $\begin{bmatrix} T \\ \bar{T} \end{bmatrix}$ is nonsingular and

$$\Delta_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_m & 0 \\ 0 & 0 \\ 0 & I_m \end{bmatrix} \Delta_h = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_m & 0 \\ 0 & I_m \end{bmatrix} \Delta_H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_m \end{bmatrix}$$

Proof: *Provided that lemma 5.1.3 is verified, all the solutions of equation $\nabla(\rho)\Gamma(\rho) = \Lambda(\rho)$ are given by the expression (see Appendix A.8 or [Darouach, 2001; Skelton et al., 1997]):*

$$\nabla_s(\rho) = \Lambda(\rho)\Gamma^+(\rho) - L(\rho)(I - \Gamma(\rho)\Gamma^+(\rho))$$

where Γ^+ is the Moore-Penrose pseudoinverse of Γ and $L(\rho)$ is a free variable giving the parametrization of the set of solutions which will be referred to as the generalized observer gain. It is of interest to express these relations as functions of the generalized gain $L(\rho)$ which leads to

$$\begin{array}{rcl}
\begin{bmatrix} K_0(\rho) & H(\rho) \end{bmatrix} & = & \nabla_s(\rho)\Delta_0 \\
\begin{bmatrix} K_h(\rho) & H(\rho) \end{bmatrix} & = & \nabla_s(\rho)\Delta_h \\
H(\rho) & = & \nabla_s(\rho)\Delta_H
\end{array}$$

where Δ_0, Δ_h and Δ_H are defined above. Hence (5.5) and (5.6) can be rewritten into the form:

$$\begin{array}{rcl} M_0(\rho)T & = & TA(\rho) - \begin{bmatrix} K_0(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} \\ M_h(\rho)T & = & TA_h(\rho) - \begin{bmatrix} K_h(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA_h(\rho) \end{bmatrix} \end{array}$$

Since T is a full row rank matrix then there exists a full row rank matrix $\bar{T}$ such that

$$\det \begin{bmatrix} T \\ \bar{T} \end{bmatrix} \neq 0$$

Hence the latter matrix is invertible and its inverse is denoted by $\begin{bmatrix} U & V \end{bmatrix}$. Then by right multiplying expressions (5.14) and (5.14) by $\begin{bmatrix} U & V \end{bmatrix}$ we get

$$\begin{array}{rcl} M_0(\rho) & = & TA(\rho)U - \begin{bmatrix} K_0(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} U \\ M_h(\rho) & = & TA_h(\rho)U - \begin{bmatrix} K_h(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA_h(\rho) \end{bmatrix} U \end{array}$$

which are explicit formulae for observer matrices $M_0(\rho)$ and $M_h(\rho)$. Hence $M_0(\rho) = \Theta(\rho) - L(\rho)\Xi(\rho)$, $M_h(\rho) = \Upsilon(\rho) - L(\rho)\Omega(\rho)$ and $H(\rho) = \Phi(\rho) - L(\rho)\Psi(\rho)$ with matrices defined in theorem 5.1.5. This concludes the proof. $\square$

The problem of finding five distinct matrices ($M_0(\cdot)$, $M_h(\cdot)$, $N_0(\cdot)$, $N_h(\cdot)$, $H(\rho)$) under the algebraic equality constraints (5.5), (5.6) and (5.7) has been turned into a problem of finding a free 'generalized' gain $L(\rho)$ which parametrizes the set of all solutions to equations (5.5), (5.6) and (5.7).

This transformation is the keypoint of this algebraic approach and makes the final problem to be the determination of a 'good' generalized gain. It is clear that some elements in the set of all observer matrices would give unstable error dynamics. Hence a 'good' choice is synonym to a choice giving good convergence properties, good disturbances rejection. We have chosen in this thesis to consider the $\mathcal{L}_2$-induced norm of the transfer from the disturbances $w(t)$ to the observation error $e(t)$ as a criterium to minimize for the choice of $L(\rho)$ (i.e. we aim at finding $L(\rho)$ such that $||e||_{\mathcal{L}_2} \leq \gamma ||e||_{\mathcal{L}_2}$ where $\gamma > 0$ is as small as possible). It is clear that other performances criteria could be used such as $\mathcal{H}_2$ or $\mathcal{L}_\infty$ induced norm. Such a search is difficult to perform analytically and it will be shown a bit later that such an optimization problem can be cast as an SDP.

Now noting that by considering lemma 5.1.5 the state estimation error dynamics are governed by the expression

$$\dot{e}(t) = (\Theta(\rho) - L(\rho)\Xi(\rho))e(t) + (\Upsilon(\rho) - L(\rho)\Omega(\rho))e_h(t) + FE(\rho)w(t) \tag{5.14}$$

with $F = T - HC = T - (\Phi - L\Psi)C$.

It is important to point out that if $FE = 0$ then the observer totally decouples the state estimation error e from the exogenous inputs w and thus the state estimation error is autonomous. Observers having this property are called *unknown input observers* and some additional material can be found in [Koenig and Marx, 2004; Koenig et al., 2004; Sename, 1997; Sename et al., 2001] and references therein.

In the following we will consider that $FE \neq 0$ and the objective is to minimize the impact of the disturbances $w(t)$ onto the error $e(t)$ (in the $\mathcal{L}_2$ sense) by an appropriate choice of the matrix $L(\rho)$. Note that if there exists $L(\rho)$ such that $||FE|| = 0$ or is close to 0, the algorithms would find it out.

Finally, according to the latter results on the family of observers with infinite cardinal, the following theorem provides a constructive sufficient condition on the existence of an optimal observer minimizing the $\mathcal{L}_2$-induced norm of the transfer from $w(t)$ to $e(t)$:

Theorem 5.1.6 *There exists a parameter dependent observer (5.3) for LPV time-delay system (5.1) such that Theorem 5.1.2 is satisfied for all $h \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^r$, a matrix function $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, constant matrices $Q, R \in \mathbb{S}_{++}^r$, $X \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ and a positive scalar $\gamma > 0$ such that the following matrix inequality*

$$\begin{bmatrix} -(X+X^T) & \star & \star & \star & \star & \star & \star \\ U_{21}(\rho) & U_{22}(\rho,\nu) & \star & \star & \star & \star & \star \\ U_{31}(\rho) & R & -Q_\mu - R & \star & \star & \star & \star \\ U_{41} & 0 & 0 & -\gamma I_q & \star & \star & \star \\ 0 & I_r & 0 & 0 & -\gamma I_r & \star & \star \\ X & 0 & 0 & 0 & 0 & -P(\rho) & \star \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} \prec 0 \tag{5.15}$$

holds for all $(\rho,\nu) \in U\rho \times U_\nu$ with

$$\begin{array}{rcl} U_{21}(\rho) & = & \Theta(\rho)^T X - \Xi(\rho)^T \bar{L}(\rho)^T + P(\rho) \\ U_{31}(\rho) & = & \Upsilon(\rho)^T X - \Omega(\rho)^T \bar{L}(\rho)^T \\ U_{22}(\rho,\nu) & = & \dfrac{\partial P(\rho)}{\partial \rho} - P(\rho) + Q - R \\ U_{41}(\rho) & = & (\rho)E(\rho)^T(T^T X - C^T \bar{H}^T) \end{array}$$

and

$$\bar{L}(\rho) = (X^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \tag{5.16}$$

Moreover, the gain is given by $L(\rho) = X^{-T}\bar{L}(\rho)$ and we have $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

Proof: *Since the dynamical model of the observation error is a LPV time-delay system, in order to derive constructive sufficient conditions for its stability, it is possible to consider Lemma 4.5.2 which consider such systems by providing a relaxation to the simple Lyapunov-Krasovskii functional. Substituting the model of the estimation error (5.14) into LMI of Lemma 4.5.2 where the matrices C, C_h and F are respectively set to I, 0 and 0 (in order to minimize the impact of the disturbances w onto the observation error e only) leads to*

$$\begin{bmatrix} -(X+X^T) & \star & \star & \star & \star & \star & \star \\ V_{21}(\rho) & V_{22}(\rho,\nu) & \star & \star & \star & \star & \star \\ V_{31} & R & -Q_\mu - R & \star & \star & \star & \star \\ V_{41} & 0 & 0 & -\gamma I_q & \star & \star & \star \\ 0 & I_r & 0 & 0 & -\gamma I_r & \star & \star \\ X & 0 & 0 & 0 & 0 & -P(\rho) & \star \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} \prec 0$$

with

$$\begin{array}{rcl} V_{21}(\rho) & = & (\Theta(\rho) - L(\rho)\Xi(\rho))^T X + P(\rho) \\ V_{31}(\rho) & = & V_{31}(\rho) = (\Upsilon(\rho) - L(\rho)\Omega(\rho))^T X \\ V_{41} & = & E(\rho)^T [T - (\Phi(\rho) - L(\rho)\Psi(\rho))C]^T X \\ V_{22}(\rho,\nu) & = & \dfrac{\partial P}{\partial \rho} - P(\rho) + Q - R \end{array}$$

By considering the change of variable $\bar{L}(\rho) = X^T L(\rho)$, the problem is linearized and results in the following LMI:

$$\begin{bmatrix} -(X+X^T) & \star & \star & \star & \star & \star & \star \\ W_{21}(\rho) & V_{22}(\rho,\nu) & \star & \star & \star & \star & \star \\ W_{31}(\rho) & R & -Q_\mu - R & \star & \star & \star & \star \\ W_{41}(\rho) & 0 & 0 & -\gamma I_q & \star & \star & \star \\ 0 & I_r & 0 & 0 & -\gamma I_r & \star & \star \\ X & 0 & 0 & 0 & 0 & -P(\rho) & \star \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} \prec 0 \tag{5.17}$$

with

$$\begin{array}{rcl} W_{21}(\rho) & = & \Theta(\rho)^T X - \Xi(\rho)^T \bar{L}(\rho)^T + P(\rho) \\ W_{31}(\rho) & = & V_{31}(\rho) = \Upsilon(\rho)^T X - \Omega(\rho)^T \bar{L}(\rho)^T \\ W_{41} & = & E(\rho)^T [X^T(T - \Phi(\rho)C) + \bar{L}(\rho)\Psi(\rho)C]^T \end{array}$$

Actually the problem is still unsolved since in the reconstruction of the observer, the matrix H may depend on ρ. Indeed, in the definition of the observer (5.2), H is a constant matrix but in the construction procedure provided in Lemma 5.1.5, H is allowed to be parameter varying which is an aberrant result. Hence, an extra constraint is needed in order to enforce H as a constant matrix while using Lemma 5.1.5. This is developed in the following. Since from lemma 5.1.5, H satisfies the relation

$$H = \Phi(\rho) - L(\rho)\Psi(\rho) \tag{5.18}$$

which implies

$$\begin{array}{rcl} L(\rho)\Psi(\rho) & = & \Phi(\rho) - H \\ \bar{L}(\rho)\Psi(\rho) & = & X^T\Phi(\rho) - \bar{H} \end{array}$$

with $\bar{H} = X^T H$, $\bar{L}(\rho) = X^T L(\rho)$ and since $\Psi(\rho)$ is a full column rank matrix then the solution of the equality is given by

$$\bar{L}(\rho) = (X^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \tag{5.19}$$

for any $Z(\rho)$ of appropriate dimensions (see Appendix A.8). This expression will guarantee that for any matrix $Z(\rho)$, the resulting H will be parameter independent. Moreover by replacing the new expression of $\bar{L}$ into the expressions

$$\begin{array}{c} X^T\Theta(\rho) - \bar{L}(\rho)\Xi(\rho) \\ X^T\Upsilon(\rho) - \bar{L}(\rho)\Omega(\rho) \end{array}$$

of LMI (5.17) then we ensure that H is a constant in the expression of matrices $M_0(\rho)$ and $M_h(\rho)$. Finally, the problem reformulated in a LMI optimization problem (5.15) where the matrix $Z(\rho)$ is the new parametrizing (decision) matrix. □

Remark 5.1.7 *It is important to note that the choice of the structure of the matrices $P(\rho)$ and $Z(\rho)$ is crucial in such a problem. Actually, according to [Apkarian and Adams, 1998], the idea is to 'mimic' the dependence of the system on the parameters but no complete theory is available to choose the structure of parameter dependent matrices.*

On the other hand, it is possible to derive a detectability test by eliminating the matrix $Z(\rho)$ from (5.15) using the projection lemma (see Appendix D.18). However, this test will only provide a sufficient optimal condition which is independent of the controller. Following Appendix A.9, it is possible to construct an optimal gain $Z(\rho)$ from the existence condition obtained using the projection lemma. It is important to notice that the the optimal gain is non-unique according to Appendix A.9. The resulting controller may depend on the derivative of the parameters making the observer non-implementable in practice.

Finally, the analysis of the detectability of the system for given structure of $P(\rho)$ and $Z(\rho)$ is a difficult problem due to the time-varying nature of the parameters and the delay. Hence, no criteria as rank conditions are allowed in this case.

To conclude on this remark, a rigorous analysis of the choice of the structure of $Z(\rho)$ or the impact of the choice of $Z(\rho)$ on the existence of the controller is a very difficult problem in the case of LPV time-delay systems with time-varying delays. The development of the solution of such a problem needs new technical tools which are, to my best knowledge, unavailable at this time.

This section concludes on the following example:

Example 5.1.8 *Let us consider the system proposed in [Mohammadpour and Grigoriadis, 2007a] with $D_{21} = 0$ which is the transfer from w to y:*

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1+0.2\rho \\ -2 & -3+0.1\rho \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho & 0.1 \\ -0.2+0.1\rho & -0.3 \end{bmatrix} x_h(t) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\
y(t) &= \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x(t) \\
z(t) &= x(t)
\end{aligned}
$$

The matrices $Z(\rho)$ and $P(\rho)$ are chosen to be polynomial of degree 2. For simulation purpose, the delay is assumed to be constant and set to $h = 0.5 < h_{max} = 0.8$. A step disturbance $w(t)$ of magnitude 10 is applied on the system at time $t = 15s$ and the parameter trajectory is given by $\rho(t) = \sin(t)$. Applying Theorem 5.1.6, we compute an observer for which we have $||e||_{\mathcal{L}_2} \leq 0.01||w||_{\mathcal{L}_2}$. Figure 5.1.8 shows the evolution of the observation errors where we can see that the errors converge to 0 and remain close to even in presence of disturbances. Since heavy symbolic computation are necessary to compute such an observer (e.g. pseudo-inverse of parameter dependent matrices...) the solutions for matrices are rational functions with high degrees but by analyzing the zeroes and the poles of each coefficient, it appears that several pairs of zeroes/poles are very near. Hence using a least mean square approximation of these

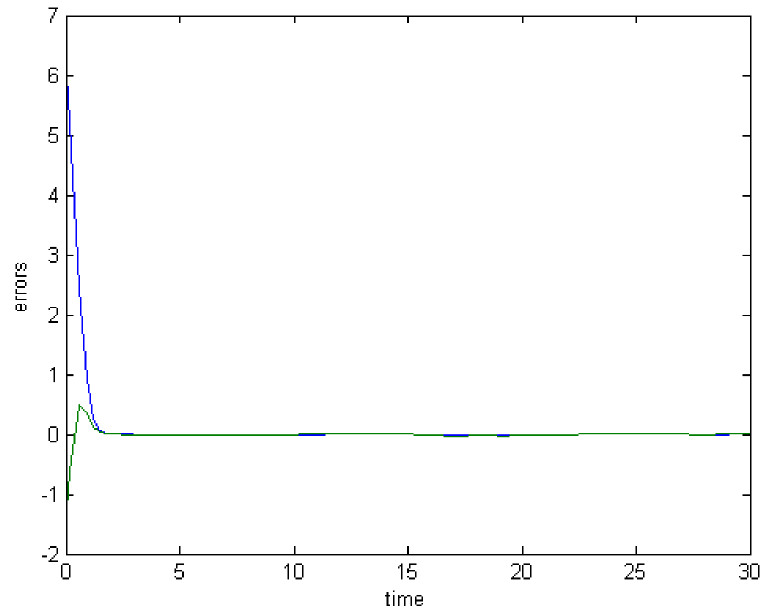

Figure 5.1: Evolution of the observation errors

polynomial coefficients we get the following observer matrices (energy error between initial and approximants) less than 10^{-6}*):*

$$M_0(\rho) = \begin{bmatrix} -0.836\rho^2 - 0.836\rho - 0.667 & -0.078\rho^2 - 0.072\rho + 0.1345 \\ -0.0376\rho^2 - 0.0376\rho - 0.361 & -0.396\rho^2 - 0.406\rho - 0.800 \end{bmatrix}$$

$$M_h(\rho) = \begin{bmatrix} -0.009\rho^2 - 0.0002\rho + 0.00822 & -0.007\rho^2 - 0.0071\rho + 0.014 \\ 0.016\rho^2 - 0.00001\rho - 0.0162 & 0.0134\rho^2 + 0.0134\rho - 0.27 \end{bmatrix}$$

$$N_0(\rho) = \begin{bmatrix} -0.073\rho^2 - 0.063\rho + 0.326 & 0.146\rho^2 + 0.148\rho + 0.620 \\ 0.076\rho^2 + 0.058\rho - 0.684 & -0.152\rho^2 - 0.156\rho - 1.054 \end{bmatrix}$$

$$N_h(\rho) = \begin{bmatrix} 0.001\rho^2 + 0.001\rho + 0.040 & -0.001\rho^2 + 0.019\rho + 0.046 \\ -0.001\rho^2 - 0.27\rho - 0.077 & 0.002\rho^2 - 0.035\rho - 0.088 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.106 & 1.788 \\ 0.798 & 0.404 \end{bmatrix}$$

As a conclusion of the approach, observers designed with this approach lead to interesting results due to their good performances. As the model of the system is exact such observers can be designed on unstable systems and the delay-margin of the observation error can be larger than the delay-margin of the system. However, such properties are not of interest since in practice the system is not known exactly: uncertainties on the delay and on the coefficient of the system are generally encountered. These problems are (partially) answered in the following sections.

5.1.2 Observer with approximate delay value

From the dynamical equations of the observer, it is clear that the exact knowledge of the delay is a crucial condition to the design of the observer of the latter sections. However, estimating or measuring the delay in real time is a challenging open problem [Belkoura et al., 2007, 2008; Drakunov et al., 2006]. Therefore, it seems convenient to consider the case when the delay is not exactly known and thus the design of an observer with approximate delay value.

The considered observer is given by:

$$\begin{array}{rcl}\dot{\xi}(t) & = & M_0(\rho)\xi(t) + M_h(\rho)\xi(t-d(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-d(t)) \\ \hat{z}(t) & = & \xi(t) + Hy(t)\end{array} \tag{5.20}$$

where $d(t)$ is the delay implemented in the observer. The idea is to impose a relationship between the real and implemented delays:

$$d(t) = h(t) + \varepsilon(t)$$

where $\varepsilon(t) \in [-\delta, \delta]$, $\delta > 0$ denotes a bounded error. Whenever the delays are locally equal (in time), then the error dynamical model is identical to (5.8). On the other hand, if the delays are different then we have the following extended model:

$$\begin{array}{rcl}\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix} & = & \mathcal{A}(\rho)\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} + \mathcal{A}_h(\rho)\begin{bmatrix}x(t-h(t))\\ e(t-h(t))\end{bmatrix} + \mathcal{A}_d(\rho)\begin{bmatrix}x(t-d(t))\\ e(t-d(t))\end{bmatrix} + \mathcal{E}(\rho)w(t) \\ \mathcal{A}(\rho) & = & \begin{bmatrix}A(\rho) & 0\\ (T-HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T-HC) & M_0(\rho)\end{bmatrix} \\ \mathcal{A}_h(\rho) & = & \begin{bmatrix}A_h(\rho) & 0\\ (T-HC)A_h(\rho) & 0\end{bmatrix} \\ \mathcal{A}_d(\rho) & = & \begin{bmatrix}0 & 0\\ -M_h(\rho)(T-HC) - N_h(\rho)C & M_h(\rho)\end{bmatrix} \\ \mathcal{E}(\rho) & = & \begin{bmatrix}E(\rho)\\ (T-HC)E(\rho)\end{bmatrix}\end{array} \tag{5.21}$$

where we have assumed without loss of generality that the control input is 0 (i.e. $u(t) \equiv 0$) since the solution $S(\rho)$ of the observer gain is trivial.

Conditions of Lemmas 5.1.3 and 5.1.5 are supposed to be fulfilled and thus we have

$$\begin{array}{rcl}(T-HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T-HC) & = & 0\\ (T-HC)A_h(\rho) - M_h(\rho)(T-HC) - N_h(\rho)C & = & 0\end{array}$$

Then matrices $\mathcal{A}(\rho)$ and $\mathcal{A}_d(\rho)$ in model (5.21) can be rewritten as

$$\begin{array}{rcl}\mathcal{A}(\rho) & = & \begin{bmatrix}A(\rho) & 0\\ 0 & M_0(\rho)\end{bmatrix}\\ \mathcal{A}_d(\rho) & = & \begin{bmatrix}0 & 0\\ -(T-HC)A_h(\rho) & M_h(\rho)\end{bmatrix}\end{array}$$

Similarly as in the latter section, it is possible to provide nonconstructive necessary and sufficient conditions taking the form of the following theorem:

Theorem 5.1.9 *There exists an LPV/$\mathcal{H}_\infty$ observer with memory of the form (5.20) for system of the form (5.2) if and only if the following statements hold:*

1. *The autonomous error dynamical expression* $\dot{\eta}(t) = \mathcal{A}(\rho)\eta(t) + \mathcal{A}_h(\rho)e(t-h(t)) + \mathcal{A}_d(\rho)\eta(t-d(t))$ *is asymptotically stable where* $\eta(t) = \mathrm{col}(x(t), e(t))$ *with* $e(t) = z(t) - \hat{z}(t)$, $d(t) = h(t) + \varepsilon(t)$, $|\varepsilon(t)| \leq \delta$ *and* $h \in \mathscr{H}_1^\circ$.

2. $(T - HC)A(\rho)x(t) - N_0(\rho)C - M_0(\rho)(T - HC) = 0$

3. $(T - HC)A_h(\rho)x(t - h(t)) - N_hC - M_h(\rho)(T - HC) = 0$

4. *The inequality* $||e||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ *holds for some* $\gamma > 0$

Due to the form of the dynamical model of the observation error, we can easily recognize the general structure considered in Section 4.7. From the results of this section, the following Theorem is derived:

Theorem 5.1.10 *There exists a parameter dependent observer of the form (5.20) such that Theorem 5.1.9 holds for all* $h \in \mathscr{H}_1^\circ$, $d(t) = h(t) + \varepsilon(t)$ *with* $\varepsilon(t) \in [-\delta, \delta]$ *is satisfied if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}_{++}^r$, *a matrix function* $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, *constant matrices* $Q_i, R_i \in \mathbb{S}_{++}^{r+n}$, $i = 1, 2$, $X_1 \in \mathbb{R}^{n\times n}$, $X_2 \in \mathbb{R}^{n\times r}$, $X_3 \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ *and a positive scalar* $\gamma > 0$ *such that the following LMIs*

$$\begin{bmatrix} -X^H & P(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_d(\rho)+\tilde{\mathcal{A}}_h(\rho) & \bar{\mathcal{E}}(\rho) & 0 & X^T & h_{max}R_1 & R_2 \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & \mathcal{I}^T & 0 & 0 & 0 \\ \star & \star & \Theta_{22} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & \star & \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \prec 0 \tag{5.22}$$

and

$$\begin{bmatrix} \Pi_{11}(\rho,\nu) & \Pi_{12}(\rho) \\ \star & \Pi_{22}(\rho) \end{bmatrix} \prec 0 \tag{5.23}$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ *and where*

$$\Pi_{11}(\rho,\nu) = \begin{bmatrix} -X^H & P(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{A}}_d(\rho) & \bar{\mathcal{E}}(\rho) \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & 0 \\ \star & \star & \Psi_{22} & (1-\mu)R_2/\delta & 0 \\ \star & \star & \star & \Psi_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix}$$

$$\Pi_{12}(\rho) = \begin{bmatrix} 0 & X^T & h_{max}R_1 & R_2 \\ \mathcal{I}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Pi_{22}(\rho) = \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & -R_1 & 0 \\ \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix}$$

$$X = \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix}$$

$$\Theta_{11}(\rho,\nu) = -P(\rho)+Q_1+Q_2+\sum_{i=1}^{N}\frac{\partial P}{\partial \rho_i}(\rho)\nu_i - R_1$$

$$\Theta_{22} = -(1-\mu)(Q_1+Q_2)-R_1$$

$$\Psi_{22} = -(1-\mu)(Q_1+R_2/\delta)-R_1$$

$$\Psi_{33} = -(1-\mu_c)Q_2-(1-\mu)R_2/\delta$$

$$\mathcal{I} = \begin{bmatrix} 0 & I_r \end{bmatrix}$$

$$\bar{L}(\rho) = (X_3^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(\rho)(I-\Psi(\rho)\Psi(\rho)^+)$$

$$\tilde{\mathcal{A}}(\rho) = \begin{bmatrix} X_1^TA(\rho) & 0 \\ X_2^TA(\rho) & X_3^T\Theta(\rho)-\bar{L}(\rho)\Xi(\rho) \end{bmatrix}$$

$$\tilde{\mathcal{A}}_h(\rho) = \begin{bmatrix} X_1^TA_h(\rho) & 0 \\ X_2^TA_h(\rho)+X_3^T(T-\Phi(\rho)C)A_h(\rho)+\bar{L}(\rho)\Psi(\rho)CAh(\rho) & 0 \end{bmatrix}$$

$$\tilde{\mathcal{A}}_d(\rho) = \begin{bmatrix} 0 & 0 \\ -X_3^T(T-\Phi(\rho)C)-\bar{L}(\rho)\Psi(\rho)C & X_3^T\Upsilon(\rho)-\bar{L}(\rho)\Omega(\rho) \end{bmatrix}$$

$$\bar{\mathcal{E}}(\rho) = \begin{bmatrix} X_1^TE(\rho)^T \\ (X_2^TT-\bar{H}C)E(\rho)^T \end{bmatrix}$$

Moreover, the gain is given by $L(\rho) = X_3^{-T}\bar{L}(\rho)$ *and we have* $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

Proof: *Let us consider LMIs (4.48) and (4.49) of lemma 4.7.1. Let us define the matrices*

$$\tilde{\mathcal{A}}(\rho) = X^T\mathcal{A}(\rho) = \begin{bmatrix} X_1^TA(\rho) & 0 \\ X_2^TA(\rho) & X_3^T\Theta(\rho)-\bar{L}(\rho)\Xi(\rho) \end{bmatrix}$$

$$\tilde{\mathcal{A}}_h(\rho) = X^T\mathcal{A}_h(\rho) = \begin{bmatrix} X_1^TA_h(\rho) & 0 \\ X_2^TA_h(\rho)+X_3^T(T-\Phi(\rho)C)A_h(\rho)+\bar{L}(\rho)\Psi(\rho)CAh(\rho) & 0 \end{bmatrix}$$

$$\tilde{\mathcal{A}}_d(\rho) = X^T\mathcal{A}_d(\rho) = \begin{bmatrix} 0 & 0 \\ -X_3^T(T-\Phi(\rho)C)-\bar{L}(\rho)\Psi(\rho)C & X_3^T\Upsilon(\rho)-\bar{L}(\rho)\Omega(\rho) \end{bmatrix}$$

$$\tilde{\mathcal{E}}(\rho) = X^T\mathcal{E}(\rho) = \begin{bmatrix} X_1^TE(\rho) \\ X_2^TE(\rho)+X_3^T(T-\Phi(\rho)C)+\bar{L}(\rho)\Psi(\rho)C \end{bmatrix}$$

Now substituting these expressions in the LMIs (4.48) and (4.49) of Lemma 4.7.1 we get LMIs (5.22) and (5.23). Since the matrix H has to be chosen independent of the parameter

ρ it suffices to parametrize $\bar{L}$ by $Z(\rho)$ as

$$\bar{L}(\rho) = (X_3^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+)$$

for some $Z(\rho)$ of appropriate dimensions. Now denoting $\bar{\mathcal{E}}(\rho) = \begin{bmatrix} X_1^T E(\rho)^T \\ (X_2^T T - \bar{H}C)E(\rho)^T \end{bmatrix}$ concludes the proof. □

5.1.3 Memoryless Observer

As a final design technique, we consider here the case where the delay is not known and therefore no information on the delay can be used in the observer. This motivates the choice of the following observer:

$$\begin{array}{rcl} \dot{\xi}(t) & = & M(\rho)\xi(t) + N(\rho)y(t) \\ \hat{z}(t) & = & \xi(t) + Hy(t) \end{array} \tag{5.24}$$

In this case the extended system containing both the dynamical model of the system and the observer is then given by

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} & = & \mathcal{A}(\rho)\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \mathcal{A}_h(\rho)\begin{bmatrix} x(t-h(t)) \\ e(t-h(t)) \end{bmatrix} + \mathcal{E}(\rho)w(t) \\ \mathcal{A}(\rho) & = & \begin{bmatrix} A(\rho) & 0 \\ (T-HC)A(\rho) - M(\rho)(T-HC) - N(\rho)C & M(\rho) \end{bmatrix} \\ \mathcal{A}_h(\rho) & = & \begin{bmatrix} A_h(\rho) & 0 \\ (T(HC)A_h(\rho) & 0 \end{bmatrix} \\ \mathcal{E}(\rho) & = & \begin{bmatrix} E(\rho) \\ (T-HC)E(\rho) \end{bmatrix} \end{array} \tag{5.25}$$

From this expression, it is possible to provide the following theorem:

Theorem 5.1.11 *There exists an LPV/$\mathcal{H}_\infty$ observer with memory of the form (5.24) for system of the form (5.1) if and only if the following statements hold:*

1. *The unforced extended dynamical system $\dot{\zeta}(t) = \mathcal{A}(\rho)(\rho)\zeta(t) + \mathcal{A}(\rho)_h(\rho)\zeta(t-h(t))$ is asymptotically stable where $\zeta(t) = \mathrm{col}(x(t), e(t))$ and $e(t) = z(t) - \hat{z}(t)$*
2. *$(T-HC)A(\rho) - M(\rho)(T-HC) - N(\rho)C = 0$*
3. *The inequality $||e||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ holds for some $\gamma > 0$*

The next results are the memoryless counterparts of Lemmas 5.1.3 and 5.1.5 dealing with observers with memory.

Lemma 5.1.12 *There exists a solution $M(\rho), N(\rho), H(\rho)$ to the equation of statement 2 if and only if the following rank equality holds*

$$\mathrm{rank}\begin{bmatrix} T \\ C \\ CA(\rho) \\ TA(\rho) \end{bmatrix} = \mathrm{rank}\begin{bmatrix} T \\ C \\ CA(\rho) \end{bmatrix} \tag{5.26}$$

Proof: *The proof is similar as for lemma 5.1.3.* □

In the case when lemma 5.1.12 is verified then it is possible to find matrices $M(\rho)$ and $N(\rho)$ such that equation of theorem 5.1.11, statement 2 is verified.

Lemma 5.1.13 *Under condition of Lemma 5.1.12, the observer matrices are parametrized with respect to a free matrix $L(\rho)$ according to the following expressions*

$$\begin{array}{rcl}
M(\rho) & = & \Theta(\rho) - L(\rho)\Xi(\rho) \\
H & = & \Phi(\rho) - L(\rho)\Omega(\rho) \\
\Theta(\rho) & = & TA(\rho)U - \Lambda(\rho)\Gamma(\rho)^{+}\Delta_M \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} U \\
\Xi(\rho) & = & -(I - \Gamma(\rho)\Gamma(\rho)^{+})\Delta_M \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} U \\
\Phi(\rho) & = & \Lambda(\rho)\Gamma(\rho)^{+}\Delta_H \\
\Psi(\rho) & = & -(I - \Gamma(\rho)\Gamma(\rho)^{+})\Delta_H \\
S(\rho) & = & (T - HC)B(\rho) \\
N(\rho) & = & K(\rho) + M(\rho)H \\
K(\rho) & = & \left[\Lambda(\rho)\Gamma(\rho)^{+} + L(\rho)s(I - \Gamma(\rho)\Gamma(\rho)^{+})\right]\Delta_K
\end{array}$$

and

$$\Delta_M = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \qquad \Delta_K = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \qquad \Delta_H = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

Proof: *The proof is similar as for lemma 5.1.5* □

Whenever Lemma 5.1.12 is satisfied and according to matrix definitions of lemma 5.1.13, system (5.25) rewrites

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \mathcal{A}(\rho) \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \mathcal{A}_h(\rho) Z \begin{bmatrix} x(t-h(t)) \\ e(t-h(t)) \end{bmatrix} + \mathcal{B}(\rho)u(t) + \mathcal{E}(\rho)w(t) \qquad (5.27)$$

$$\begin{array}{rclcrcl}
\mathcal{A}(\rho) & = & \begin{bmatrix} A(\rho) & 0 \\ 0 & \Theta(\rho) - L(\rho)\Xi(\rho) \end{bmatrix} & & \mathcal{A}_h(\rho) & = & \begin{bmatrix} A_h(\rho) \\ \left[T - \Phi(\rho)C + L(\rho)\Omega(\rho)C\right] A_h(\rho) \end{bmatrix} \\
\mathcal{B}(\rho) & = & \begin{bmatrix} B(\rho) \\ 0 \end{bmatrix} & & \mathcal{E}(\rho) & = & \begin{bmatrix} E(\rho) \\ \left[T - \Phi(\rho)C + L(\rho)\Omega(\rho)\right] E(\rho) \end{bmatrix} \\
Y & = & \begin{bmatrix} I_n & 0 \end{bmatrix} & & & &
\end{array}$$

Finally we have the following theorem:

Theorem 5.1.14 *There exists a parameter dependent observer of the form (5.24) such that theorem 5.1.11 for all $h \in \mathscr{H}_1^{\circ}$ is satisfied if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^r$, a matrix function $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, constant matrices $Q, R \in \mathbb{S}_{++}^{r+n}$, $X_1 \in \mathbb{R}^{n\times n}$, $X_2 \in \mathbb{R}^{n\times r}$, $X_3 \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ and a positive scalar $\gamma > 0$ such that* *the following LMI*

$$\begin{bmatrix} -X^H & P(\rho)+X^T\tilde{\mathcal{A}}(\rho) & X^T\tilde{\mathcal{A}}_h(\rho) & X^T\tilde{\mathcal{E}}(\rho) & 0 & X^T & h_{max}Y^TR \\ \star & \Psi'_{22}(\rho,\nu) & R & 0 & \mathcal{I}^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_m & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}Y^TR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

holds for all $(\rho,\nu)\in U_\rho\times U_\nu$ *where*

$$\begin{aligned} \Psi'_{22}(\rho,\nu) &= \frac{\partial}{\partial\rho}P(\rho)\nu - P(\rho) + Y^T(Q-R)Y \qquad Y = \begin{bmatrix} I_n & 0 \end{bmatrix} \\ X &= \begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix} \\ \mathcal{I} &= \begin{bmatrix} 0 & I_r \end{bmatrix} \qquad \bar{L}(\rho) = (X_3^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(I-\Psi(\rho)\Psi(\rho)^+) \end{aligned}$$

Moreover the generalized observer gain $L(\rho)$ *is given by the relation* $L(\rho) = X_3^{-T}\bar{L}(\rho)$ *and we have* $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$.

Proof: *Due to the structure of* $\mathcal{A}_h(\rho)$ *it is clear that such a problem falls into the framework of Section 4.5.3 which considers the stability of time-delay systems where the delay acts on only a specific subpart of the system state (i.e. on the state of the system only). Hence injecting the extended system into LMI (4.35) we get with*

$$\begin{bmatrix} -X^H & P(\rho)+X^T\mathcal{A}(\rho) & X^T\mathcal{A}_h(\rho) & X^T\mathcal{E}(\rho) & 0 & X^T & h_{max}Y^TR \\ \star & \Psi'_{22}(\rho,\nu) & R & 0 & \mathcal{I}^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_m & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}Y^TR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

with $R\in\mathbb{S}^n_{++}$, $Z = \begin{bmatrix} I_n & 0 \end{bmatrix}$, $\mathcal{I} = \begin{bmatrix} 0 & I_r \end{bmatrix}$ *and*

$$\Psi'_{22}(\rho,\nu) = \partial_\rho P(\rho)\nu - P(\rho) + Y^T(Q-R)Y$$

Choosing $X = \begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix}$ *then we have the following relations:*

$$\begin{aligned} \tilde{\mathcal{A}}(\rho) &= X^T\mathcal{A}(\rho) = \begin{bmatrix} X_1^TA(\rho) & 0 \\ X_2^TA(\rho) & X_3^T\Theta(\rho)-\bar{L}(\rho)\Xi(\rho) \end{bmatrix} \\ \tilde{\mathcal{A}}_h(\rho) &= X^T\mathcal{A}_h(\rho) = \begin{bmatrix} X_1^TA_h(\rho) \\ X_2^TA_h(\rho)+X_3^T(T-\Phi(\rho)C)+\bar{L}(\rho)\Omega(\rho)CA_h(\rho) \end{bmatrix} \\ \tilde{\mathcal{E}}(\rho) &= X^T\mathcal{E}(\rho) = \begin{bmatrix} X_1^TE(\rho) \\ X_2^TE(\rho)+X_3^T(T-\Phi(\rho)C)+\bar{L}(\rho)\Omega(\rho)E(\rho) \end{bmatrix} \end{aligned}$$

where $\bar{L}(\rho) = X_3^TL(\rho)$.

An interesting fact of such a Lyapunov-Krasovskii functional of the form (4.24) is the embedding of an information on the structure of the system (the delay does not act on some part of the state) and allows to reduce the number of decision variables. Finally, since a constant H matrix is sought (as in proof of Theorem 5.1.6), then by choosing $\bar{L}$ such that

$$\bar{L}(\rho) = (X_3^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+)$$

where $Z(\rho)$ is a free matrix with appropriate dimension and $\bar{H} = X_3^T H$. This completes the proof. □

5.2 Filtering of uncertain LPV Time-Delay Systems

This section is devoted to the filtering of LPV time-delay systems and we are interested in finding a LPV filter of the form

$$\begin{array}{rcl} \dot{x}_F(t) & = & A_F(\rho)x(t) + A_{Fh}(\rho)x(t-d(t)) + B_F(\rho)y(t) \\ z_F(t) & = & C_F(\rho)x(t) + C_{Fh}(\rho)x(t-d(t)) + D_F(\rho)y(t) \end{array} \tag{5.28}$$

for systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & (A(\rho) + \Delta A(\rho,t))x(t) + (A_h(\rho) + \Delta A_h(\rho,t))x(t) + (E(\rho) + \Delta E(\rho,t))w(t) \\ z(t) & = & (C(\rho)x(t) + \Delta C(\rho,t))x(t) + (C_h(\rho)x(t) + \Delta C_h(\rho,t))x(t-h(t)) \\ & & +(F(\rho)x(t) + \Delta F(\rho,t))w(t) \\ y(t) & = & (C_y(\rho)x(t) + \Delta C_y(\rho,t))x(t) + (C_{yh}(\rho)x(t) + \Delta C_{yh}(\rho,t))x(t-h(t)) \\ & & +(F_y(\rho)x(t) + \Delta F_y(\rho,t))w(t) \end{array} \tag{5.29}$$

where $x \in \mathbb{R}^n,\ x_F \in \mathbb{R}^r,\ w \in \mathbb{R}^m,\ y \in \mathbb{R}^p,\ z, z_F \in \mathbb{R}^t$ are respectively the system state, the filter state, the system measurements, the system exogenous inputs, the signal to be estimated and its estimate. The time-varying delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$ and the filter delay $d(t)$ is unconstrained at this time.

Definition 5.2.1 *When $r < n$ the filter is said to be a reduced-order filter while if $r = n$ it is a full-order filter.*

We will consider in the following only full-order filters (i.e. $r = n$). It is possible to generalize the results to the case of reduced-order filters by considering, for instance, the approach of [Tuan et al., 2001b].

The uncertain terms are assumed to obey

$$\begin{bmatrix} \Delta A(\rho,t)) & \Delta A_h(\rho,t)) & \Delta E(\rho,t)) \\ \Delta C(\rho,t)) & \Delta C_h(\rho,t)) & \Delta F(\rho,t)) \\ \Delta C_y(\rho,t)) & \Delta C_{yh}(\rho,t)) & \Delta F_y(\rho,t)) \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ G_z(\rho) \\ G_y(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_x(\rho) & H_{xh}(\rho) & H_w(\rho) \end{bmatrix} \tag{5.30}$$

with $\Delta^T\Delta \preceq I$ where all matrices are of appropriate dimensions provided that the uncertain terms are all defined.

5.2.1 Design of robust filters with exact delay-value - simple Lyapunov-Krasovskii functional

This section is devoted to the design of filter with exact delay-value (i.e. $d(t) = h(t)$ for all $t \geq 0$). Even if such a filter are difficult to realize they allow to give a lower bound on the best achievable $\mathcal{H}_\infty$ norm. Using such a filter, the extended system describing the evolution of both the system and filter states is given by

$$\begin{array}{rcl}\begin{bmatrix} \dot{x}(t) \\ \dot{x}_F(t) \end{bmatrix} & = & (\mathcal{A}(\rho) + \Delta\mathcal{A}(\rho,t))\begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + (\mathcal{A}_h(\rho) + \Delta\mathcal{A}_h(\rho,t))\begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix} \\ & & +(\mathcal{E}(\rho) + \Delta\mathcal{E}(\rho,t))w(t) \\ e(t) & = & z(t) - z_F(t) \\ & = & (\mathcal{C}(\rho) + \Delta\mathcal{C}(\rho,t))\begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + (\mathcal{C}_h(\rho) + \Delta\mathcal{C}_h(\rho,t))\begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix} \\ & & +(\mathcal{F}(\rho) + \Delta\mathcal{F}(\rho,t))w(t) \end{array} \tag{5.31}$$

where

$$\begin{array}{rcl}\mathcal{A}(\rho) & = & \begin{bmatrix} A(\rho) & 0 \\ B_F(\rho)C_y(\rho) & A_F(\rho) \end{bmatrix} \\ \Delta\mathcal{A}(\rho,t) & = & \begin{bmatrix} \Delta A(\rho,t) & 0 \\ B_F(\rho)\Delta C_y(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix}\Delta(t)\begin{bmatrix} H_x(\rho) & 0 \end{bmatrix} \\ \mathcal{A}_h(\rho) & = & \begin{bmatrix} A_h(\rho) & 0 \\ B_F(\rho)C_{yh}(\rho) & A_{Fh}(\rho) \end{bmatrix} \\ \Delta\mathcal{A}_h(\rho,t) & = & \begin{bmatrix} \Delta A_h(\rho,t) & 0 \\ B_F(\rho)\Delta C_{yh}(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix}\Delta(t)\begin{bmatrix} H_{xh}(\rho) & 0 \end{bmatrix} \\ \mathcal{E}(\rho) & = & \begin{bmatrix} E(\rho) \\ B_F(\rho)F_y(\rho) \end{bmatrix} \\ \Delta\mathcal{E}(\rho,t) & = & \begin{bmatrix} \Delta E(\rho,t) & 0 \\ B_F(\rho)\Delta E(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix}\Delta(t)H_w(\rho) \\ \mathcal{C}(\rho) & = & \begin{bmatrix} C(\rho) - D_F(\rho)C_y(\rho) & -C_F(\rho) \end{bmatrix} \\ \Delta\mathcal{C}(\rho,t) & = & \begin{bmatrix} \Delta C(\rho,t) - D_F(\rho)\Delta C_y(\rho,t) & 0 \end{bmatrix} = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t)\begin{bmatrix} H_x(\rho) & 0 \end{bmatrix} \\ \mathcal{C}_h(\rho) & = & \begin{bmatrix} C_h(\rho) - D_F(\rho)C_{yh}(\rho) & -C_{Fh}(\rho) \end{bmatrix} \\ \Delta\mathcal{C}_h(\rho,t) & = & \begin{bmatrix} \Delta C_h(\rho,t) - D_F(\rho)\Delta C_{yh}(\rho,t) & 0 \end{bmatrix} = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t)\begin{bmatrix} H_{xh}(\rho) & 0 \end{bmatrix} \\ \mathcal{F}(\rho) & = & F(\rho) - D_F(\rho)F_y(\rho) \\ \Delta\mathcal{F}(\rho,t) & = & \Delta F(\rho) - D_F(\rho)\Delta F_y(\rho) = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t)H_w(\rho) \end{array}$$

This leads to the following theorem which is a consequence of the relaxation theorem developed in Section 4.5.

Theorem 5.2.2 *There exists a full-order filter of the form (5.28) with $d(t) = h(t)$, $h(t) \in \mathscr{H}_1^\circ$ if there exists a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}^{2n}_{++}$, symmetric matrices $\tilde{Q}, \tilde{R} \in \mathbb{S}^{2n}_{++}$, $\hat{X} \in \mathbb{R}^{2n\times 2n}$, matrix functions $\tilde{A}_F, \tilde{A}_{Fh} : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$,*

$\tilde{C}_F, \tilde{C}_{Fh} : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ *and scalars* $\gamma, \varepsilon > 0$ *such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \tilde{\mathcal{G}}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{5.32}$$

holds for all $\rho \in U_\rho$ *with* $\Psi_{22}(\rho,\nu) = \partial_\rho\tilde{P}(\rho)\nu - \tilde{P}(\rho) + \tilde{Q} - \tilde{R}$, $\tilde{P}(\rho) = \tilde{X}^T P(\rho)\tilde{X}$, $\tilde{Q} = \tilde{X}^T Q\tilde{X}$, $\tilde{R} = \tilde{X}^T R\tilde{X}$,

$$\Psi(\rho,\nu) = \begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho) + \tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}\tilde{R} \\ \star & \tilde{\Psi}_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)\tilde{Q} - \tilde{R} & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

$$\tilde{\mathcal{A}}(\rho) = \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \qquad \hat{X} = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix}$$

$$\tilde{\mathcal{A}}_h(\rho) = \begin{bmatrix} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix}$$

$$\tilde{\mathcal{E}}(\rho) = \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \end{bmatrix} \qquad \tilde{\mathcal{C}}(\rho)^T = \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix}$$

$$\tilde{\mathcal{C}}_h(\rho)^T = \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \qquad \hat{X}_2 = X_2X_4^{-1}X_3 = U^T\Sigma V \qquad \text{(SVD)}$$

$$\mathcal{H}(\rho)^T = \begin{bmatrix} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho)^T \\ 0 \\ \hline H_w(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix} \qquad \tilde{\mathcal{G}}(\rho)^T = \begin{bmatrix} \hat{X}_1^T G_x(\rho) + \tilde{B}_F(\rho)G_y(\rho) \\ \hat{X}^T G_x + \tilde{B}_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline G_z(\rho) - D_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix}$$

Moreover the filter matrices are computed using

$$\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

where $\hat{X}_3 = U\Sigma V$ *and we have* $||e||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$.

Proof: *Substitute the model (5.31) into LMI (4.28) we get*

$$\begin{bmatrix} -X^H & P(\rho)+X^T\bar{A}(\rho) & X^T\bar{A}_h(\rho) & X^T\bar{E}(\rho) & 0 & X^T & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & \bar{C}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & \bar{C}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \bar{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

with

$$\begin{array}{rcl} \Psi_{22}(\rho,\nu) & = & \partial_\rho P(\rho)\nu - P(\rho) + Q - R \\ \bar{A}(\rho) & = & \mathcal{A}(\rho) + \Delta\mathcal{A}(\rho) \\ \bar{A}_h(\rho) & = & \mathcal{A}_h(\rho) + \Delta\mathcal{A}_h(\rho) \\ \bar{E}(\rho) & = & \mathcal{E}(\rho) + \Delta\mathcal{E}(\rho) \\ \bar{C}(\rho) & = & \mathcal{C}(\rho) + \Delta\mathcal{C}(\rho) \\ \bar{C}_h(\rho) & = & \mathcal{C}_h(\rho) + \Delta\mathcal{C}_h(\rho) \\ \bar{F}(\rho) & = & \mathcal{F}(\rho) + \Delta\mathcal{F}(\rho) \end{array}$$

The latter inequality can be rewritten in the following form

$$\mathcal{M}(\rho,\nu) + \mathcal{D}^T\mathcal{G}(\rho)^T\Delta(t)\mathcal{H}(\rho) + \mathcal{H}(\rho)^T\Delta(t)^T\mathcal{G}(\rho)\mathcal{D} \prec 0$$

where $\mathcal{D} = \mathrm{diag}(X, I, \ldots, I)$,

$$\mathcal{M}(\rho,\nu) = \begin{bmatrix} -X^H & P(\rho)+X^T\mathcal{A}(\rho) & X^T\mathcal{A}_h(\rho) & X^T\mathcal{E}(\rho) & 0 & X^T & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & \mathcal{C}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & \mathcal{C}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix}$$

$$\mathcal{H}(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho)^T \\ 0 \\ \hline H_w(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad \mathcal{G}(\rho)^T = \left[\begin{array}{c} G_x(\rho) \\ B_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline G_z(\rho) - D_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$$

Then invoking the bounding lemma (see Appendix D.15), we get the following equivalent matrix inequality

$$\begin{bmatrix} \mathcal{M}(\rho,\nu)+\varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \mathcal{G}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{5.33}$$

where $\varepsilon > 0$ is unknown variable to be designed. Since the latter matrix inequality is nonlinear, it cannot be solved efficiently in a reasonable time. The remaining of the proof is devoted to the linearization of such an inequality. To this aim, let us define the matrix $\tilde{X} = \begin{bmatrix} I_n & 0 \\ 0 & X_4^{-1}X_3 \end{bmatrix}$ then we get

$$\begin{array}{lclcl}
\tilde{\mathcal{A}}(\rho) & = & \tilde{X}^T X^T \mathcal{A}(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1^T A(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) & = & \tilde{X}^T X^T \mathcal{A}_h(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1^T A_h(\rho)+\tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho)+\tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) & = & \tilde{X}^T X^T \mathcal{E}(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1 E(\rho)+\tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho)+\tilde{B}_F(\rho)C_y(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T & = & \tilde{X}^T \mathcal{C}(\rho) & = & \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T & = & \tilde{X}^T \mathcal{C}_h(\rho) & = & \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
\tilde{G}_1(\rho) & = & \tilde{X}^T X^T \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} & = & \begin{bmatrix} \hat{X}_1^T G_x(\rho)+\tilde{B}_F(\rho)G_y(\rho) \\ \hat{X}^T G_x+\tilde{B}_F(\rho)G_y(\rho) \end{bmatrix} \\
\hat{X} & = & \tilde{X}^T X \tilde{X} & = & \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} = \begin{bmatrix} X_1 & X_2X_4^{-1}X_3 \\ X_3^T X_4^{-1}X_3 & X_3^T X_4^{-1}X_3 \end{bmatrix} \\
\tilde{A}_F(\rho) & = & X_3^T A_F(\rho)X_4^{-1}X_3 & & \\
\tilde{A}_{Fh}(\rho) & = & X_3^T A_{Fh}(\rho)X_4^{-1}X_3 & & \\
\tilde{B}_F(\rho) & = & X_3^T B_F(\rho) & & \\
\tilde{C}_F(\rho) & = & C_F(\rho)X_4^{-1}X_3 & & \\
\tilde{C}_{Fh}(\rho) & = & C_{Fh}(\rho)X_4^{-1}X_3 & &
\end{array}$$

Then perform a congruence transformation on (5.33) with respect to $\text{diag}(\tilde{X},\tilde{X},\tilde{X},I_q,I_r,\tilde{X},\tilde{X},I)$ *we get LMI (5.32). Now let us focus on the computation of the filter matrices. Note that*

$$\begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \\ \tilde{C}_F(\rho) & \tilde{C}_{Fh}(\rho) & \tilde{D}_F(\rho) \end{bmatrix} = \begin{bmatrix} X_3^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} \begin{bmatrix} X_4^{-1}X_3 & 0 & 0 \\ 0 & X_4^{-1}X_3 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Thus it suffices to construct back the matrix X in order to compute the observer gain. A singular values decomposition (SVD, see Appendix A.6) on $\hat{X}_3$ allows to compute the matrices X_3 and X_4 which are necessary to construct the filter matrices. Indeed, we have $\hat{X}_2 = U^T\Sigma V$ and hence

$$\begin{array}{rcl} X_2 & = & U^T \\ X_4 & = & \Sigma^{-1} \\ X_3 & = & V \end{array}$$

and finally we have

$$\begin{aligned}\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} &= \begin{bmatrix} V^T & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \\ \tilde{C}_F(\rho) & \tilde{C}_{Fh}(\rho) & \tilde{D}_F(\rho) \end{bmatrix} \begin{bmatrix} \Sigma V & 0 & 0 \\ 0 & \Sigma V & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)V^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)V^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)V^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)V^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}\end{aligned}$$

□

5.2.2 Design of robust memoryless filters

This last section is devoted to the synthesis of robust memoryless filters. The resulting synthesis condition are based on the application of the reduced Lyapunov-Krasovskii functional introduced in Section 4.5.3 which applies on systems where the delay acts only on a subpart of the state. In this case, only the state of the system is affected by the delay.

Theorem 5.2.3 *There exists a full-order filter of the form (5.28) with $A_{Fh} = 0$ and $C_{Fh} = 0$ with $h(t) \in \mathscr{H}_1^\circ$ if there exists a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}_{++}^{2n}$, symmetric matrices $\tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^{2n}$, $\hat{X} \in \mathbb{R}^{2n\times 2n}$, matrix functions $\tilde{A}_F : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$, $\tilde{C}_F : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ and scalars $\gamma, \varepsilon > 0$ such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \mathcal{G}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{5.34}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\tilde{\Psi}'_{22}(\rho,\nu) = \partial_\rho\tilde{P}(\rho)\nu - \tilde{P}(\rho) + Z^T(Q(\rho) - R)Z$, $\tilde{\mathcal{C}}_h(\rho)^T = C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T$, $\hat{X}_3 = X_2X_4^{-1}X_3 = U^T\Sigma V$ (SVD),

$$\Psi(\rho,\nu) = \begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho) + \tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}Z^TR \\ \star & \tilde{\Psi}'_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q - R & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_m & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}Z^TR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix}$$

$$\hat{X} = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \qquad \tilde{\mathcal{A}}(\rho) = \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix}$$

$$\tilde{\mathcal{A}}_h(\rho) = \begin{bmatrix} \hat{X}_1^T A_h + \tilde{B}_F(\rho)C_{yh}(\rho) \\ \hat{X}_2^T A_h + \tilde{B}_F(\rho)C_{yh}(\rho) \end{bmatrix} \qquad \tilde{\mathcal{E}}(\rho) = \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \end{bmatrix}$$

$$\tilde{\mathcal{C}}(\rho)^T = \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix}$$

$$\mathcal{H}(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho) \\ \hline H_w(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right] \qquad \mathcal{G}(\rho)^T = \left[\begin{array}{c} \hat{X}_1^T G_x(\rho) + \tilde{B}_F(\rho)G_y(\rho) \\ \hat{X}^T G_x + \tilde{B}_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ \hline G_z(\rho) - \tilde{D}_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right]$$

Moreover the filter matrices are computed using

$$\begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)V^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)V^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

5.2.3 Example

We will show the effectiveness of the approach compared to existing ones through the following example. Let us consider the following system Mohammadpour and Grigoriadis [2007a]:

$$\begin{array}{rcl} \dot{x} & = & \begin{bmatrix} 0 & 1+0.2\rho \\ -2 & -3+0.1\rho \end{bmatrix} x + \begin{bmatrix} 0.2\rho & 0.1 \\ -0.2+0.1\rho & -0.3 \end{bmatrix} x_h \\ & & + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w \\ z & = & \begin{bmatrix} 0.3 & 1.5 \\ -0.45 & 0.75 \end{bmatrix} x + \begin{bmatrix} 0.5\rho & -0.5 \end{bmatrix} w \\ y & = & \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1+0.1\rho \end{bmatrix} w \end{array} \tag{5.35}$$

where $\rho(t) = sin(t) \in [-1, 1]$ and $\dot{\rho}(t) \in [-1, 1]$. Parameter dependent matrices are expressed onto a basis defined by the functions

$$f_0(\rho) = 1 \qquad f_1(\rho) = \rho \tag{5.36}$$

We use Theorems 5.2.2 and 5.2.3 with an uniform gridding of 11 points over the whole parameter space and the results are verified on a denser grid (around 100 points).

Results of Mohammadpour and Grigoriadis [2007a] are depicted in Figure 5.2. In Figure 5.3, the evolution of the worst case performance for the delayed filter computed with Theorem 5.2.3 and the memoryless one computed with Theorem 5.2.2. As a first analysis, the delayed filter gives better performance than the memoryless one which seems obvious since the information on the delay is used in the delayed filter. As a comparison with the results in Mohammadpour and Grigoriadis [2007a], our results are less conservative and then improves the existing ones (see result of Mohammadpour and Grigoriadis [2007a] in Figure 5.2 and proposed results in Figure 5.3). It is possible to see that for small delay values both solutions leads to very similar results. The main difference appears for larger delay values for which the worst case disturbance gain is drastically different. Figure 5.4 shows the evolution of the

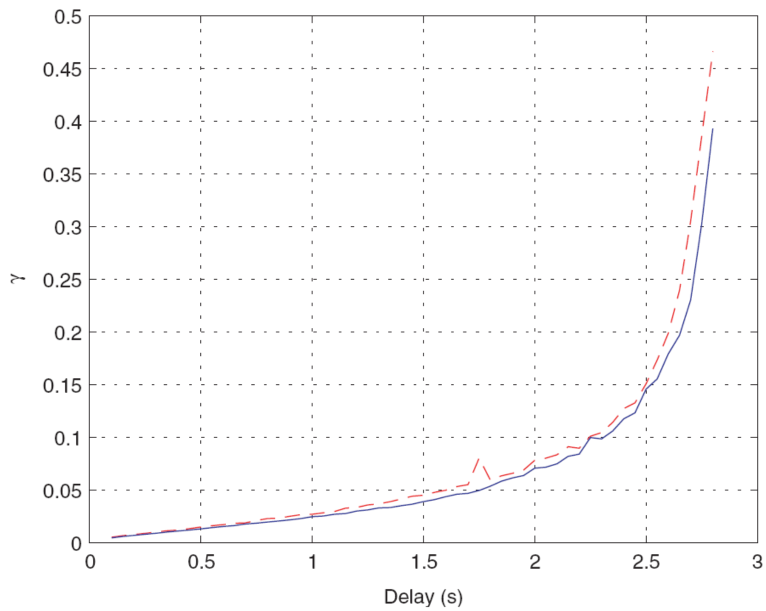

Figure 5.2: Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain) in Mohammadpour and Grigoriadis [2007a]

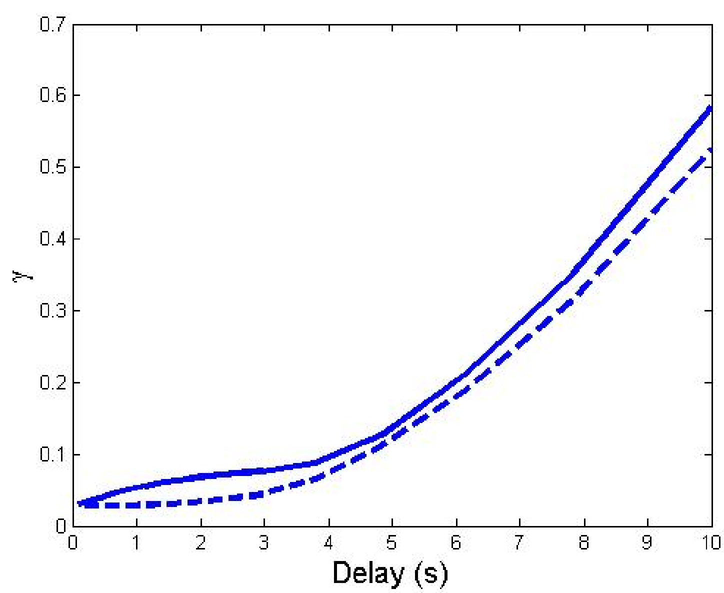

Figure 5.3: Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain)

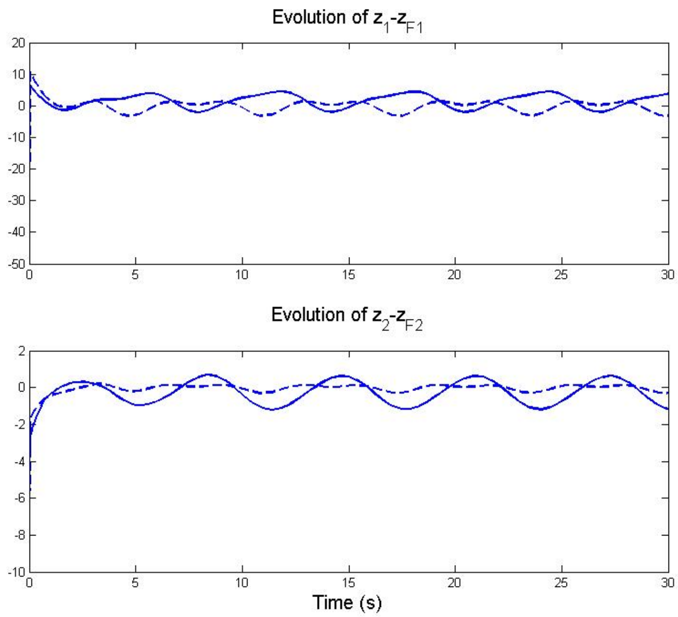

Figure 5.4: Evolution of $z(t) - z_F(t)$ for the delayed filter (dashed) and the memoryless filter (plain)

error $z(t) - z_F(t)$ for a delay $h = 3$ and a step disturbance of amplitude 20. We can easily see that the delayed filter gives better results than the memoryless one. Consider now system (5.35) with uncertainties defined by: according to the notation (5.30). The evolution of the worst-case performance $\mathcal{L}_2$-gain is depicted in Figure 5.5.

Figure 5.6 shows the evolution of the error $z(t) - z_F(t)$ for a delay $h = 4.5$, $\Delta(t) = sin(10t)I_2$ and a step disturbance of amplitude 20. The delayed filter achieves a $\mathcal{L}_2$ performance gain of $\gamma_{del} = 0.59$ and the memoryless of $\gamma_{ml} = 0.78$.

5.3 Chapter Conclusion

This chapter has been devoted to the design of observer and filters for both unperturbed and uncertain LPV time-delay systems.

Three types of observers have been synthesized: observers with memory, with both exact and approximate delay value knowledge, and memoryless observers. They have been developed using an algebraic approach generalized from [Darouach, 2001] to the LPV framework. The matrices of the observers are chosen such that the system state and the control input do not affect the evolution of the observation error and that the disturbances are attenuated in the $\mathcal{L}_2$ sense. The set of observers decoupling the error from the system state and the control input is parametrized through an algebraic equation involving a free parameter to be chosen. This parameter is chosen as a solution of an LMI optimization problem where the $\mathcal{L}_2$ gain from the disturbances to the observation error is minimized. This approach is better suited for certain systems since the matrices acting on the system state in the observation error dynamical model can be set to zero by an appropriate choice of the observer matrices. How-

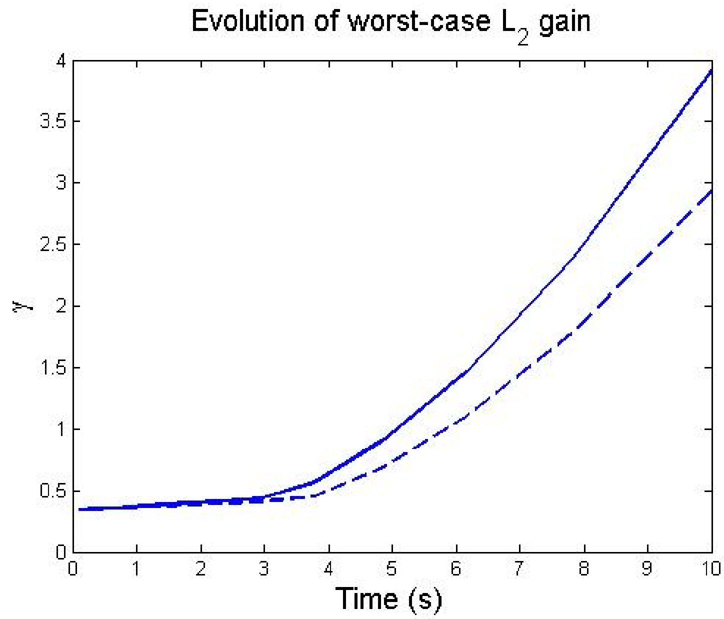

Figure 5.5: Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain)

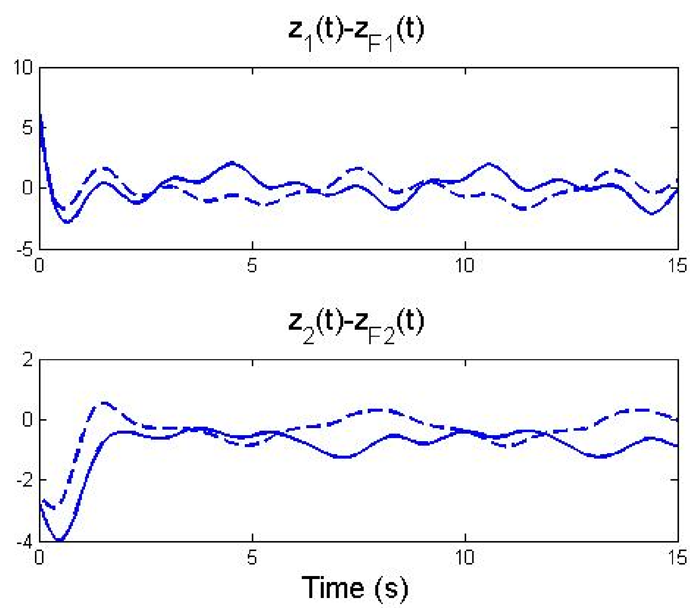

Figure 5.6: Evolution of $z(t) - z_F(t)$ for the delayed filter (dashed) and the memoryless filter (plain)

ever, in the uncertain case, these matrices cannot be set to zero due to uncertain terms and, inevitably, the observer has much poorer performances. Such observers have been developed but are not exposed in this thesis for brevity.

On the second hand, the problem of designing filters for uncertain systems has been addressed in the second section of the chapter. Two types of filters have been considered: memoryless filters and filters with memory (exact delay value knowledge). The design of filters is more simple and more direct than observer design, since it is not sought to obtain an estimation error which is independent of the state variables of the system. The constructive existence conditions are given in terms of a convex optimization problem involving LMIs where the $\mathcal{L}_2$ attenuation gain from the disturbances to the filtering error is minimized. Other filters can also be derived using this very general approach.

Chapter 6

Control of LPV Time-Delay Systems

THIS CHAPTER is devoted to the control of (uncertain) LPV time-delay systems. Despite of its apparent simplicity the control of LPV time-delay systems is still an open problem. Indeed, in the LMI based approaches, conservatism induced by relaxations (as bounding techniques, model transformations, relaxations of nonlinear terms. . .) is responsible of bad results. A major problem is the presence of multiple products (e.g. KR, KP where P, K, R are decision variables) occurring very often in many design concerning time-delay systems and preventing the linearization of BMIs into LMIs. For instance, in the descriptor approach [Fridman, 2001], the coupled terms KP_1 and KP_2 (when considering a state-feedback control law) must be relaxed and then the relaxation $P_2 = \varepsilon P_1$ is usually performed where ε is a fixed chosen scalar. This type of relaxation is also needed when the design is done using the free-weighting approach [He et al., 2004]. Most of the approaches are done using the same procedure as follows:

1. Elaborate a stability/performance test based on some method for the open-loop system
2. Substitute the closed-loop system into the LMI conditions
3. Simplify the obtained BMIs
4. Linearize by congruence and change of variable to obtain LMIs.

In this chapter we will propose another strategy by adding a step into this methodology:

1. Develop a stability/performances test for an open loop system
2. If the obtained conditions involve potential coupled terms, they are relaxed using for instance the Finsler's lemma (see Appendix D.16) in order to remove these coupled terms.
3. Substitute the closed-loop system expression in the relaxed version of the stability/performances LMI test.
4. Linearize immediately/use of congruence transformations.

It will be shown in this chapter that this methodology gives rise to good results, not only for LTI systems but also for LPV system. Since the number of results for uncertain LTI systems is larger than those for LPV systems, the methods will be compared with both LTI and LPV methods.

It is worth mentioning that even if the relaxed version of the test without coupled terms is not equivalent to the original test, the conservatism is generally not worse than using classical relaxations and is a good point of the provided methods. Moreover, the relaxation is use allows for generalizations to discretized versions of Lyapunov-Krasovskii functionals.

A new approach for the control of time-delay systems with time-varying delays is developed in this chapter. This method allows to find a memoryless controller where the gains of the controller are smoothly scheduled by the delay value or an approximate one. Due to the similarity with gain-scheduled controller synthesis in the LPV framework, this type of controllers is referred to as *delay-scheduled controllers*. Such a controller is hence midway between memoryless and with memory since it embeds an information on the delay value without any delayed terms in the control law expression.

Several of our results have been detailed in the following papers:

- [Briat et al., 2007a] a delay-scheduled state-feedback strategy is designed based on a specific model transformation. In this paper, the computed controller is LPV depends on the value of the delay in a LFT fashion. A paper version [Briat et al., 2007b] is still under review at IEEE Transactions on Automatic Control (2nd round).

- [Briat et al., 2008a] an enhanced delay-scheduled controller approach is developed where the model transformation has been improved and the controller is not in LFT form. The results are then less conservative. A journal version is in revision at Systems and Control Letters.

- A full-block $\mathcal{S}$-procedure approach is provided in [Briat et al., 2008b] where the control of uncertain time-delay systems is solved.

- LPV control for time-delay systems is detailed in [Briat et al., 2008c] where a projection approach is used to provide constructive sufficient conditions for a stabilizing controller.

Some key references to modern control techniques for time-delay systems are recalled below (see also Chapter 3):

Robust control of LTI time-delay systems: [Fu et al., 1998], [Souza and Li, 1999], [Ivanescu et al., 2000], [Moon et al., 2001], [Fridman and Shaked, 2002a], [Wu, 2003], [Suplin et al., 2004], [Jiang and Han, 2005], [Fridman, 2006a], [Suplin et al., 2006], [Fridman, 2006b], [Fridman and Shaked, 2006], [Jiang and Han, 2006], [Xu et al., 2006], [Chen, 2007].

LPV control of LPV/LTI time-delay systems: [Wu and Grigoriadis, 2001], [Wu, 2001b], [Zhang and Grigoriadis, 2005]

The first section will be concerned to the synthesis of state-feedback control laws, both memoryless and with memory state-feedback controllers synthesis will be explored. Moreover, the uncertainty on the delay-knowledge using state-feedback with memory will be taken into account through a specific Lyapunov-Krasovskii functional. Finally the synthesis of delayed-scheduled state-feedback will be solved. The last section will be devoted to the synthesis of dynamic output feedback controllers. Both observer-based and full-block controllers will be synthesized.

6.1 State-Feedback Control laws

In this section the stabilization of general uncertain LPV time-delay systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & (A(\rho)+\Delta A(\rho,t))x(t)+(A_h(\rho)+\Delta A_h(\rho,t))x(t-h(t)) \\ & & +(B(\rho)+\Delta B(\rho,t))u(t)+(E(\rho)+\Delta E(\rho,t))w(t) \\ z(t) & = & C(\rho)x(t)+C_h(\rho)x(t-h(t))+D(\rho)u(t)+F(\rho)w(t) \end{array} \tag{6.1}$$

using a general control law of the form

$$u(t)=K(\rho)x(t)+K_h(\rho)x(t-d(t)) \tag{6.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ and $h(t) \in \mathscr{H}_1^\circ$ are respectively the state of the system, the control input, the disturbances, the controlled outputs and the system delay. The set $\mathscr{H}_1^\circ$ is given by

$$\mathscr{H}_1^\circ := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+,[0,h_{max}]) : \ |\dot{h}| < \mu \right\}$$

The controller delay $d(t)=h(t)+\varepsilon(t)$, $\varepsilon(t) \in [-\delta,\delta]$ is not defined a priori and may admit fast variations. The uncertain terms are given as:

$$\left[\begin{array}{cccc} \Delta A(\rho) & \Delta A_h(\rho) & \Delta B(\rho) & \Delta E(\rho) \end{array} \right] = G(\rho)\Delta(t) \left[\begin{array}{cccc} H_A(\rho) & H_{A_h}(\rho) & H_B(\rho) & H_E(\rho) \end{array} \right]$$

where matrices $G(\rho), H_A(\rho), H_{A_h}(\rho), H_B(\rho), H_E(\rho)$ are full rank matrices and $\Delta(t)^T\Delta(t) \preceq I$.

Definition 6.1.1 *Whenever $K_h(\cdot)=0$, the controller is said to be memoryless while when $K_h(\cdot) \neq 0$, it is said to be with memory.*

Problem 6.1.2 *The problem is to find a control law of the form (6.2) which asymptotically stabilizes system (6.1) for all $h \in \mathscr{H}_1^\circ$ and all $\Delta(t)$ such that $\Delta(t)^T\Delta(t) \preceq I$ and minimizes $\gamma > 0$ such that*

$$||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

In the following the following main problems will be addressed:

1. The stabilization of the system by a *memoryless state-feedback* control law of the form $u(t)=K(\rho)x(t)$.

2. The stabilization of the system by a state-feedback control law with memory of the form $u(t)=K(\rho)x(t)+K_h(\rho)x(t-h(t))$ with *exact delay value.*

3. The stabilization of the system by a state-feedback control law with memory of the form $u(t)=K(\rho)x(t)+K_h(\rho)x(t-h(t))$ with *approximate delay value.*

4. The stabilization of the system using *delay-scheduled controllers.*

In this chapter, the following terminology will be used: by *relaxed* we mean that the initial stability and performances test has been modified in order to remove all potential coupled terms. The term *simple* means that the Lyapunov-Krasovskii functional is 'simple' in the sense that the structure of the functional involves constant matrices only. This term has to be put in parallel with the term *complete*. Finally, the term *discretized* means that a (complete or not) Lyapunov-Krasovskii functional involving function as decision variables has been discretized in order to get tractable stability conditions [Fridman, 2006b; Gu et al., 2003; Seuret et al., 2009a].

6.1.1 Memoryless State-Feedback Design - Relaxed Simple Lyapunov-Krasovskii functional

The most simple state-feedback controller that can be designed is the memoryless one (i.e. when $K_h(\cdot) = 0$), where only the instantaneous state $x(t)$ is used to compute the control input $u(t)$. In such a case, the closed-loop system is governed by

$$\begin{array}{rcl} \dot{x}(t) & = & (A(\rho) + \Delta A(\rho,t) + [B(\rho) + \Delta B(\rho,t)]K(\rho))x(t) \\ & & +(A_h(\rho) + \Delta A_h(\rho,t))x(t-h(t)) + (E(\rho) + \Delta E(\rho,t))w(t) \\ z(t) & = & (C(\rho) + D(\rho)K(\rho))x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t) \end{array} \tag{6.3}$$

The solution is given using the relaxed version of the LMI condition obtained from the simple Lyapunov-Krasovskii functional as described in Section 4.5.2. Hence after a substitution of the closed-loop into the relaxed LMI, a simple existence test of the controller can be provided in terms of parameter dependent LMIs.

Theorem 6.1.3 *There exists a state-feedback control law of the form $u(t) = K(\rho)x(t)$ which asymptotically stabilizes system (6.1) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}_{++}^n$, constant matrices $Y \in \mathbb{R}^{n\times n}, \tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^n$, a scalar $\gamma > 0$ and scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the LMIs*

$$\begin{bmatrix} \tilde{U}_{11}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} & 0 \\ \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 & Y^TH_{A_h}(\rho)^T \\ \star & \star & -\gamma I_pF(\rho)^T & 0 & 0 & H_E(\rho)^T & \\ \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\ \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} & 0 \\ \star & \star & \star & \star & \star & \tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0$$

and

$$\mathrm{Ker}[\mathcal{U}_1(\rho)]^T\Xi(\rho,\nu)\mathrm{Ker}[\mathcal{U}_1(\rho)] \prec 0$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\Xi(\rho,\nu)$ is defined by

$$\begin{bmatrix} \tilde{U}_{11}(\rho) & \tilde{P}(\rho)+A(\rho)Y & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} & 0 \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & Y^TC(\rho)^T & 0 & 0 & Y^TH_A(\rho)^T \\ \star & \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 & Y^TH_{A_h}(\rho)^T \\ \star & \star & \star & -\gamma I_pF(\rho)^T & 0 & 0 & H_E(\rho)^T & \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & \tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\varepsilon(\rho)I \end{bmatrix}$$

$$\begin{array}{rcl}\tilde{U}_{11}(\rho) & = & -(Y+Y^T)+\varepsilon(\rho)G(\rho)G(\rho)^T \\ \tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho)+\tilde{Q}-\tilde{R}+\partial_\rho\tilde{P}(\rho)\nu \\ \tilde{U}_{33} & = & -(1-\mu)\tilde{Q}-\tilde{R} \\ \mathcal{U}_1(\rho) & = & \begin{bmatrix} B(\rho)^T & 0 & 0 & 0 & D(\rho)^T & 0 & 0 & H_B(\rho)^T \end{bmatrix} \\ \mathcal{U}_2 & = & \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

In this case a suitable controller is given by the expression:

$$\begin{array}{rcl}K(\rho,\dot{\rho}) & = & -\tau\mathcal{U}_1(\rho)\Psi(\rho,\dot{\rho})\mathcal{U}_2^T(\mathcal{U}_2\Psi(\rho,\dot{\rho})\mathcal{U}_2^T)^{-1}Y^{-1} \\ \tau & > & 0\end{array} \tag{6.4}$$

such that $\Psi(\rho,\dot{\rho})=(\tau\mathcal{U}_1(\rho)^T\mathcal{U}_1(\rho)-\Xi(\rho,\dot{\rho}))^{-1}\succ 0$ *or by solving the LMI*

$$\tilde{\Xi}(\rho,\nu)+\bar{B}(\rho)K(\rho)Y\bar{C}+[\bar{B}(\rho)K(\rho)Y\bar{C}]^T\prec 0 \tag{6.5}$$

in $K(\rho)$. *Moreover with such a control law, the closed-loop system satisfies* $||z||_{\mathcal{L}_2}\le\gamma||w||_{\mathcal{L}_2}$.

Proof: *The proof is based on an application of lemma 4.5.2. Substituting matrices of the closed-loop system (6.3) into LMI (4.28), we get*

$$\bar{\Psi}(\rho,\nu)+\bar{\mathcal{G}}(\rho)^T\Delta(t)\bar{\mathcal{H}}(\rho)+(\star)^T\prec 0 \tag{6.6}$$

where $\bar{\Psi}(\rho,\nu)$ *is defined by*

$$\begin{bmatrix} -X^H & U_{12}(\rho) & X^TA_h(\rho) & X^TE(\rho) & 0 & X^T & h_{max}R \\ \star & U_{22}(\rho,\nu) & R & 0 & U_{25}(\rho)^T & 0 & 0 \\ \star & \star & U_{33}(\rho) & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix}$$

with

$$\begin{array}{rcl}U_{12}(\rho) & = & P(\rho)+X^T(A(\rho)+B(\rho)K(\rho)) \\ U_{25}(\rho) & = & (C(\rho)+D(\rho)K(\rho)) \\ U_{22}(\rho,\nu) & = & -P(\rho)+Q-R+\partial_\rho P(\rho)\nu \\ U_{33} & = & -(1-\mu)Q-R \\ \bar{\mathcal{G}}(\rho) & = & \begin{bmatrix} G(\rho)^TX & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \bar{\mathcal{H}}(\rho) & = & \begin{bmatrix} 0 & H_A(\rho)+H_B(\rho)K(\rho) & H_{A_h}(\rho) & H_E(\rho) & 0 & 0 & 0 \end{bmatrix}\end{array}$$

Due to the structure of LMI (6.6) it is possible to apply the bounding lemma (see Appendix D.15) and hence we obtain the following LMI

$$\bar{\Psi}(\rho,\nu)+\varepsilon(\rho)\bar{\mathcal{G}}(\rho)^T\bar{\mathcal{G}}(\rho)+\varepsilon(\rho)^{-1}\bar{\mathcal{H}}(\rho)^T\bar{\mathcal{H}}(\rho)^T\prec 0$$

which involves an additional scalar function $\varepsilon(\rho)$. *Then performing a congruence transformation with respect to matrix* $\mathrm{diag}(Y,Y,Y,I,I,Y,Y)$ *where* $Y=X^{-1}$ *and using the change of variable* $V(\rho)=K(\rho)Y$ *we get the inequality:*

$$\Psi(\rho,\nu)+\varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho)+\varepsilon(\rho)^{-1}\mathcal{H}(\rho)^T\mathcal{H}(\rho)\prec 0 \tag{6.7}$$

where $\Psi(\rho,\nu)$ is defined by

$$\begin{bmatrix} -(Y+Y^T) & \tilde{U}_{12}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & U_{25}(\rho) & 0 & 0 \\ \star & \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

in which $\tilde{P}(\rho) = Y^TP(\rho)Y$, $\tilde{Q} = Y^TQY$, $\tilde{R} = Y^TRY$

$$\begin{array}{rclrcl} \tilde{U}_{12}(\rho) & = & \tilde{P}(\rho) + A(\rho)Y + B(\rho)V(\rho) & \tilde{U}_{25}(\rho) & = & [C(\rho)Y + D(\rho)V(\rho)]^T \\ \tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho\tilde{P}(\rho)\nu & \tilde{U}_{33} & = & -(1-\mu)\tilde{Q} - \tilde{R} \end{array}$$

$$\begin{array}{rcl} \mathcal{G}(\rho) & = & \begin{bmatrix} G(\rho)^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathcal{H}(\rho) & = & \begin{bmatrix} 0 & H_A(\rho)Y + H_B(\rho)V(\rho) & H_{A_h}(\rho)Y & H_E(\rho) & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The latter inequality is not a LMI due to the term $\varepsilon(\rho)^{-1}\mathcal{H}(\rho)^T\mathcal{H}(\rho)$ but using the Schur complement(see Appendix D.4) we get the following equivalent LMI formulation:

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho) & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho) \end{bmatrix} \prec 0$$

Now rewrite the latter LMI as

$$\Xi(\rho,\nu) + \mathcal{U}_1(\rho)^TV(\rho)\mathcal{U}_2 + (\star)^T \prec 0 \tag{6.8}$$

where $\Xi(\rho,\nu)$ is defined by

$$\begin{bmatrix} \tilde{U}_{11}(\rho) & \tilde{U}_{12}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} & 0 \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & Y^TC(\rho)^T & 0 & 0 & Y^TH_A(\rho)^T \\ \star & \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 & Y^TH_{A_h}(\rho)^T \\ \star & \star & \star & -\gamma I_pF(\rho)^T & 0 & 0 & H_E(\rho)^T & \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & \tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\varepsilon(\rho)I \end{bmatrix}$$

with

$$\begin{array}{rcl} \tilde{U}_{11}(\rho) & = & -(Y+Y^T) + \varepsilon(\rho)G(\rho)G(\rho)^T \\ \tilde{U}_{12}(\rho) & = & \tilde{P}(\rho) + A(\rho)Y \\ \mathcal{U}_1(\rho) & = & \begin{bmatrix} B(\rho)^T & 0 & 0 & 0 & D(\rho)^T & 0 & 0 & H_B(\rho)^T \end{bmatrix} \\ \mathcal{U}_2 & = & \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since $V(\rho)$ is a free variable then the projection lemma applies (see appendix D.18) and we get conditions of Theorem 6.1.3. The controller can be constructed using either

$$\Xi(\rho,\nu) + \mathcal{U}_1(\rho)^TV(\rho)\mathcal{U}_2 + (\star)^T \prec 0$$

or by applying the algebraic relations given in Appendix A.9. □

The latter theorem is a theorem stating the existence of a parameter dependent matrix gain $K(\rho)$ such that system (6.3) is asymptotically stable and $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$. The advantage of such a result is the possibility of constructing the controller from algebraic equations involving only known matrices computed from the solution of the LMI problem. The explicit formula leads to a controller which ensures exactly the predicted performances. On the other hand, such a controller may depend on the parameter derivative (as emphasized by the relaxation (6.4)) making the controller (in most of the cases) unimplementable in practice. Three solutions are offered to overcome this difficulty:

1. Choose a constant matrix P: this removes the parameter derivative term but increasing the conservatism of the approach by tolerating arbitrarily fast varying parameters (quadratic stability).

2. Construct the controller using SDP (6.5): in this case a specific structure must be affected to the controller (for instance polynomial in ρ) which may result in a deterioration of performances. Moreover, since the structure of the controller is chosen by the designer after solving for the other matrices (i.e. $\tilde{P}, \tilde{Q}, \tilde{R}, Y$), then the SDP may have no solution if the controller is not sufficiently complex.

This nonequivalence is a consequence of the parameter varying nature of the matrices involved in the LMIs and the will of considering robust stability. The following result solves this problem of non-equivalence between the set of LMIs of Theorem 6.1.3 and the SDP (6.5) by providing a global approach where only one LMI has to be solved.

Theorem 6.1.4 *There exists a state-feedback control law of the form $u(t) = K(\rho)x(t)$ which asymptotically stabilizes system (6.1) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}_{++}^n$, a matrix function $V(\rho) : U_\rho \to \mathbb{R}^{m\times n}$, constant matrices $Y \in \mathbb{R}^{n\times n}, \tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^n$, a scalar $\gamma > 0$ and a scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho) & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\Psi(\rho,\nu)$ is defined by

$$\begin{bmatrix} -Y^H & \tilde{U}_{12}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & U_{25}(\rho) & 0 & 0 \\ \star & \star & \tilde{U}_{33} & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

$$\begin{array}{rcl}\tilde{U}_{12}(\rho) & = & \tilde{P}(\rho)+A(\rho)Y+B(\rho)V(\rho)\\ \tilde{U}_{25}(\rho) & = & [C(\rho)Y+D(\rho)V(\rho)]^T\\ \mathcal{G}(\rho) & = & \begin{bmatrix} G(\rho)^T & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\\ \mathcal{H}(\rho) & = & \begin{bmatrix} 0 & H_A(\rho)Y+H_B(\rho)V(\rho) & H_{A_h}(\rho) & H_E(\rho) & 0 & 0 & 0\end{bmatrix}\\ \tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho)+\tilde{Q}-\tilde{R}+\partial_\rho P(\rho)\nu\\ \tilde{U}_{33} & = & -(1-\mu)\tilde{Q}-\tilde{R}\end{array}$$

In this case, a suitable control law is given by $u(t)=K(\rho)x(t)$ *where* $K(\rho)=V(\rho)Y^{-1}$ *and the closed-loop system (6.3) satisfies*

$$||z||_{\mathcal{L}_2}\leq\gamma||w||_{\mathcal{L}_2}$$

Proof: The proof follows the same lines as for the proof of Theorem 6.1.3 but stops just before the application of the projection lemma. □

As for Theorem 6.1.3, the structure of the controller is fixed by the designer through the choice of the structure of $V(\rho)$ and may result in conservative results if the structure is sufficiently complex or not. On the other hand, Theorem 6.1.4 is easier to use since the controller synthesis is made in one step only while the number of steps for controller computation using Theorem 6.1.3 is two. The interest of Theorem 6.1.3 is to provide in the first step (the solution of the projected inequalities) the minimal γ that can be expected using this Lyapunov-Krasovskii functional (modulo the conservatism induced by the relaxation) whatever the structure of the controller is. Hence, this result may be used to tune the complexity of the controller using Theorem 6.1.4.

Remark 6.1.5 *Another result has been developed in [Briat et al., 2008b] for uncertain LTI time-delay systems using the full-block* $\mathcal{S}$*-procedure approach [Scherer, 2001; Wu, 2003]. The results of [Briat et al., 2008b] can be extended to the LPV framework by authorizing a parameter dependent Lyapunov function and parameter dependent scalings.*

6.1.2 Memoryless State-Feedback Design - Relaxed Discretized Lyapunov-Krasovskii functional

The Lyapunov-Krasovskii functional used to derive conditions of Theorems 6.1.3 and 6.1.4 is simple (in the sense that the decision matrices are in small finite number). Thus latter results can be enhanced by considering more complex functionals. Due to the difficulty to find numerically such functions, the matrix functions are then approximated and the obtained functional is called 'the discretized version' of such a functional.

The following result is obtained by the use of the relaxation of the discretized Lyapunov-Krasovskii functional described in Theorem 4.6.4 of Section 4.6.2. The applied methodology is as usual: substitute the closed-loop system in the LMI and then turn the BMI problem into a LMI one through the use of congruence transformation.

Theorem 6.1.6 *There exists a state-feedback control law of the form* $u(t)=K(\rho)x(t)$ *which asymptotically stabilizes system (6.1) with* $h(t)\in\mathscr{H}_1^\circ$ *if there exist a continuously differentiable matrix function* $\tilde{P}:U_\rho\to\mathbb{S}^n_{++}$*, constant matrices* $Y\in\mathbb{R}^{n\times n},\tilde{Q}_i,\tilde{R}_i\in\mathbb{S}^n_{++}$*,*

$i = 0, \ldots, N-1$, a matrix function $V : U_\rho \to \mathbb{R}^{m\times n}$, a scalar $\gamma > 0$ and a scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the LMIs

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho) & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{6.9}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ and where

$$\Psi(\rho,\nu) = \left[\begin{array}{cccc|ccc} -Y^H & \tilde{U}_{12}(\rho) & 0 & Y & \bar{h}\tilde{R}_0 & \ldots & \bar{h}\tilde{R}_{N-1} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{U}_{23}(\rho) & 0 & 0 & \ldots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \ldots & 0 \\ \star & \star & \star & -\tilde{P}(\rho) & -\bar{h}\tilde{R}_0 & \ldots & -\bar{h}\tilde{R}_{N-1} \\ \hline \star & \star & \star & \star & & -\operatorname{diag}_i \tilde{R}_i & \end{array}\right]$$

$$\tilde{U}_{22} = \begin{bmatrix} \tilde{U}'_{11} & \tilde{R}_0 & 0 & 0 & \ldots & 0 & 0 \\ \star & \tilde{N}_1^{(1)} & \tilde{R}_1 & 0 & \ldots & 0 & 0 \\ \star & \star & \tilde{N}_2^{(1)} & \tilde{R}_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \tilde{R}_{N-1} & 0 \\ & & & & & \tilde{N}^{(2)} & 0 \\ \star & \star & \star & \ldots & 0 & 0 & -\gamma I \end{bmatrix}$$

and $\bar{h} = h_{max}/N$, $\mu_N = \mu/N$,

$$\begin{aligned} \tilde{U}'_{11} &= \partial_\rho \tilde{P}(\rho)\nu - \tilde{P}(\rho) + \tilde{Q}_0 - \tilde{R}_0 \\ \tilde{N}_i^{(1)} &= -(1 - i\mu_N)\tilde{Q}_{i-1} + (1 + i\mu_N)\tilde{Q}_i - \tilde{R}_{i-1} - \tilde{R}_i \\ \tilde{N}^{(2)} &= -(1-\mu)\tilde{Q}_{N-1} - \tilde{R}_{N-1} \\ \tilde{U}_{12}(\rho) &= \begin{bmatrix} \tilde{P}(\rho) + A(\rho)Y + B(\rho)V(\rho) & 0 & \ldots & 0 & A_h(\rho)Y & \ldots & 0 & E(\rho) \end{bmatrix} \\ \tilde{U}_{23}(\rho) &= \begin{bmatrix} C(\rho)Y + D(\rho)V(\rho) & 0 & \ldots & 0 & C_h(\rho) & F(\rho) \end{bmatrix}^T \\ \mathcal{G}(\rho) &= \left[\begin{array}{c|cccccc|c|ccc} 0 & G(\rho)^T & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{array}\right] \\ \mathcal{H}(\rho) &= \left[\begin{array}{c|cccccc|c|ccc} 0 & H_A(\rho)Y + H_B(\rho)V(\rho) & 0 & \ldots & 0 & H_{A_h}(\rho)Y & H_E(\rho) & 0 & 0 & \ldots & 0 \end{array}\right] \end{aligned}$$

Proof: *The proof is similar as for the proof of theorem 6.1.4 but using lemma 4.6.4.* □

6.1.3 Memoryless State-Feedback Design - Simple Lyapunov-Krasovskii functional

The approach developed in this section is based on the properties of the adjoint system of a time-delay system [Bensoussan et al., 2006; Suplin et al., 2006]. The interest of adjoint systems is to allow for the computation of controllers without any congruence transformation on the matrix inequalities. While this is not always interesting for finite dimensional linear systems, it is of great importance for time-delay systems for which a large number of decision matrices are involved in the stability conditions. Indeed, linearizing congruence transformations on matrix inequalities might not exist in time-delay system framework (for instance there exists no linearizing congruence transformation for state-feedback design using LMI (4.25) of Lemma 4.5.1). The use of adjoint systems partially overcomes this problem.

6.1.3.1 About adjoint systems of LPV systems

The first property of adjoint system of a LTI system is that the stability of the adjoint is equivalent to the stability of the original system. Moreover, the $\mathcal{H}_\infty$-norm is also preserved by considering the adjoint. However, does that statement hold when the system is time-varying (LTV or LPV) ? Actually, this is not a trivial equation since the outputs are computed by integrating time-varying matrices and then for a given input, the outputs of the original and the adjoint systems are different. Thus they have different $\mathcal{L}_2$ norms.

However, in the light of the use of the dualization lemma (see Appendix D.14) for LTV/LPV systems expressed under LFT forms, it turns out that the $\mathcal{L}_2$-induced norm is preserved by considering the adjoint. However, the worst-case input signal (the signal for which the $\mathcal{L}_2$-induced norm is effectively attained) will be different for the original and the adjoint system.

Let us consider the system

$$\begin{array}{rcl} \dot{x}(t) & = & \mathcal{A}(\rho)x(t) + \mathcal{B}(\rho)w_1(t) \\ z_1(t) & = & \mathcal{C}(\rho)x(t) + \mathcal{D}(\rho)w_1(t) \end{array} \tag{6.10}$$

which is rewritten in an 'LFT' form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B_0w_0(t) + B_1w_1(t) \\ z_0(t) & = & C_0x(t) + D_{00}w_0(t) + D_{01}w_1(t) \\ z_1(t) & = & C_1x(t) + D_{10}w_0(t) + D_{11}w_1(t) \\ w_0(t) & = & \Theta(\rho)z_0(t) \end{array} \tag{6.11}$$

For such a system, the adjoint is given by the expression:

$$\begin{array}{rcl} \dot{\tilde{x}}(t) & = & A^T\tilde{x}(t) + C_0^T\tilde{w}_0(t) + C_1^T\tilde{w}_1(t) \\ \tilde{z}_0(t) & = & B_0^T\tilde{x}(t) + D_{00}^T\tilde{w}_0(t) + D_{10}^T\tilde{w}_1(t) \\ \tilde{z}_1(t) & = & B_1^T\tilde{x}(t) + D_{01}^T\tilde{w}_0(t) + D_{11}^T\tilde{w}_1(t) \\ \tilde{w}_0(t) & = & \Theta(\rho)^T\tilde{z}_0(t) \end{array} \tag{6.12}$$

Since any LPV system can be turned into an equivalent 'LFT' system, this approach is very general to demonstrate that the $\mathcal{L}_2$-norm of (6.11) and (6.12) coincides. The following results shows the identity:

Theorem 6.1.7 *Let us consider system (6.10) and (6.11), then the following statements are equivalent:*

1. *The LPV system is quadratically stable if and only if there exist $P \in \mathbb{S}^n_{++}$, $F \in \mathbb{S}^{2n_0}$ and a scalar $\gamma > 0$ such that following LMIs*

$$\left[\begin{array}{ccc} I & 0 & 0 \\ A & B_0 & B_1 \\ \hline 0 & I & 0 \\ C_0 & D_{00} & D_{01} \\ \hline 0 & 0 & I \\ C_1 & D_{10} & D_{11} \end{array}\right]^T \left[\begin{array}{cc|c|cc} 0 & P & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & F & 0 & 0 \\ \hline 0 & 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & 0 & \gamma^{-1}I \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ A & B_0 & B_1 \\ \hline 0 & I & 0 \\ C_0 & D_{00} & D_{01} \\ \hline 0 & 0 & I \\ C_1 & D_{10} & D_{11} \end{array}\right] \prec 0 \tag{6.13}$$

$$\begin{bmatrix} \Theta(\rho) \\ I \end{bmatrix}^T F \begin{bmatrix} \Theta(\rho) \\ I \end{bmatrix} \succ 0 \tag{6.14}$$

holds for all $\rho \in U_\rho$. *In this case, the system satisfies* $||z||_{\mathcal{L}_2} < \gamma ||w||_{\mathcal{L}_2}$.

2. *LPV system is quadratically asymptotically stable if and only if there exist* $\tilde{P} \in \mathbb{S}^n_{++}$, $\tilde{F} \in \mathbb{S}^{2n_0}$ *and a scalar* $\gamma > 0$ *such that following LMIs*

$$\left[\begin{array}{ccc} A^T & C_0^T & C_1^T \\ I & 0 & 0 \\ \hline B_0^T & D_{00}^T & D_{10}^T \\ 0 & I & 0 \\ \hline B_1^T & D_{01}^T & D_{11}^T \\ 0 & 0 & I \end{array}\right]^T \left[\begin{array}{cc|c|cc} 0 & \tilde{P} & 0 & 0 & 0 \\ \tilde{P} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{F} & 0 & 0 \\ \hline 0 & 0 & 0 & -\gamma^{-1}I & 0 \\ 0 & 0 & 0 & 0 & \gamma I \end{array}\right] \left[\begin{array}{ccc} A^T & C_0^T & C_1^T \\ I & 0 & 0 \\ \hline B_0^T & D_{00}^T & D_{10}^T \\ 0 & I & 0 \\ \hline B_1^T & D_{01}^T & D_{11}^T \\ 0 & 0 & I \end{array}\right] \succ 0$$

$$\begin{bmatrix} -I \\ \Theta(\rho)^T \end{bmatrix}^T F \begin{bmatrix} -I \\ \Theta(\rho)^T \end{bmatrix} \succ 0 \tag{6.15}$$

holds for all $\rho \in U_\rho$. *In this case, the system satisfies* $||z||_{\mathcal{L}_2} < \gamma ||w||_{\mathcal{L}_2}$.

Moreover, we have the following relations between the matrices:

$$\begin{aligned} \tilde{P} &= P^{-1} \\ \tilde{F} &= F^{-1} \end{aligned}$$

Proof: *Statement 1 can be obtained by applying the full-block* $\mathcal{S}$*-procedure on LFT system (6.11) (Appendix D.13, Section 2.3.4.4 or [Scherer, 2001]). Statement 2. can be proved applying the dualization lemma (Appendix D.14 or [Scherer, 2001]) on LMIs (6.13) and (6.14).* □

Actually, it is difficult to see that it suffices to replace the original system matrices by adjoint matrices into the matrix inequality (6.13) to obtain (6.15). This motivates the introduction of the following corollary where we have assumed that we have $\Theta(\rho)^T\Theta(\rho) \leq I$ and $F = \text{diag}(-I_{n_0}, I_{n_0})$:

Corollary 6.1.8 *Let us consider system (6.10) and (6.11), then the following statements are equivalent:*

1. *The LPV system is quadratically stable if there exist* $P \in \mathbb{S}^n_{++}$ *and a scalar* $\gamma > 0$ *such that following LMIs*

$$\begin{bmatrix} PA + A^TP & PB_0 & PB_1 & C_1^T & C_0^T \\ \star & -I & 0 & D_{10}^T & D_{00}^T \\ \star & \star & -\gamma I & D_{11}^T & D_{01}^T \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} \prec 0$$

2. *The LPV system is quadratically stable if there exist* $\tilde{P} \in \mathbb{S}^n_{++}$ *and a scalar* $\gamma > 0$ *such that following LMIs*

$$\begin{bmatrix} PA^T + AP & PC_0^T & PC_1^T & B_1 & B_0 \\ \star & I & 0 & D_{01} & D_{00} \\ \star & \star & -\gamma I & D_{11}^T & D_{10} \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -I \\ \end{bmatrix} \prec 0$$

Proof: *The proof is done by expanding the inequalities and the above matrix inequalities are obtained modulo Schur complement arguments (see Appendix D.4).* □

From this result, it is possible to conclude that the $\mathcal{L}_2$-induced norm is identical for a time-varying system and its adjoint since the same LMI structure is feasible for both system. Roughly speaking, it suffices to substitute the adjoint system in the original stability condition. This suggests it might also be the case for time-delay systems. This has been done in the case of LTI system in [Suplin et al., 2006] and implies that it also holds for uncertain LTI time-delay systems with constant uncertainties. In the case of time-varying uncertainties it has been shown in [Wu, 2003] using the dualization lemma (see Appendix D.14) that the delay-independent stability with $\mathcal{L}_2$ performances is preserved by considering the adjoint system. Although the dualization lemma provides an efficient and strong theoretical way to deal correctly with adjoint systems, the rank condition (see Appendix D.14) is unfortunately rarely satisfied when considering time-delay systems and this makes the use of the adjoint a difficult problem in the context of LPV time-delay systems.

6.1.3.2 LPV Control of LPV time-delay systems using adjoint

One of our papers [Briat et al., 2008b], provides a solution to the state-feedback stabilization problem of uncertain time-delay systems. It is shown that adjoint of delay systems may involve delayed uncertainties and delayed loop inputs creating then difficulties and leading to some conservatism when the delayed state is affected by uncertainties. A solution is provided using the projection lemma (see Appendix D.18) and the cone-complementary algorithm [Ghaoui et al., 1997] used here to relax a non-convex (even concave) term in a matrix inequality similarly as in [Chen and Zheng, 2006]. Since this paper only deals with constant uncertain systems but not LPV, this will not be explained here but such an approach can be generalized to the LPV framework by introducing parameter dependent matrices in the Lyapunov-Krasovskii functionals and authorizing the scalings (separators) to be parameter dependent. On the other hand, this makes the cone complementary algorithm unapplicable since this algorithm can only be applied on constant matrices while we are in presence of parameter dependent matrices. In such a case, the algorithm provided in Section 4.3 shall be used.

In what follows, we propose a method to solve this problem which has been proposed in [Briat et al., 2008c]. The idea of the method is the following: first of all the LMI (4.25) of Lemma 4.5.1 (Section 4.5.1), obtained from a simple parameter dependent Lyapunov-Krasovskii functional, is considered. This LMI has two coupled terms $P(\rho)A(\rho)$ and $RA(\rho)$ which means that if the closed-loop is substituted into, then exact linearization by congruence transformations is not possible (i.e. $P(\rho)(A(\rho)+B(\rho)K(\rho))$ and $R(A(\rho)+B(\rho)K(\rho))$). Since we wish to avoid the too conservative simplification $P(\rho)=\alpha(\rho)R$ for instance, another way is considered. This way is the use of the projection lemma whose action is to remove the controller matrix from the inequalities. If the projection lemma were applied directly on the original system then it would lead to a projection with respect to a basis of the kernel of matrices

$$\begin{array}{rcl} M_1 & = & \left[\begin{array}{ccccc} I & 0 & 0 & 0 & 0 \end{array} \right] \\ M_2(\rho) & = & \left[\begin{array}{ccccc} B(\rho)^T P(\rho) & 0 & 0 & D(\rho)^T & h_{max} B(\rho)^T R \end{array} \right] \end{array}$$

since we have an inequality of the form

$$\Psi_o(\rho,\dot{\rho}) + M_2(\rho)^T K(\rho) M_1 + (\star)^T \prec 0.$$

From the expression of $M_2(\rho)$ we can see that

$$\mathrm{Ker}[M_2(\rho)] = J(\rho)Z(\rho)$$

with

$$\begin{array}{rcl} J(\rho) & = & \mathrm{diag}(P(\rho)^{-1}, I, I, I, h_{max}^{-1} R^{-1}) \\ Z(\rho) & = & \mathrm{Ker} \left[\begin{array}{ccccc} B(\rho)^T & 0 & 0 & D(\rho)^T & B(\rho)^T \end{array} \right] \end{array}$$

and hence a congruence transformation with respect to $\mathrm{diag}(P(\rho)^{-1}, I, I, I, h_{max}^{-1} R^{-1})$ has to be performed and leads to nonlinear terms in the resulting conditions. Moreover, these nonlinear terms cannot be relaxed since the kernel $Z(\rho)$ surrounds the matrix $\mathcal{NL}(\cdot) := J(\rho)^T \Psi_0(\rho,\dot{\rho}) J(\rho)$ containing the (non)linear terms $X(\rho)$, Q, R, $X(\rho)QX(\rho)$, $X(\rho)RX(\rho)$, $R^{-1}, X(\rho)R$, ρ, $\dot{\rho}$:

$$Z(\rho)^T \mathcal{NL}(\dot{)} Z(\rho) \prec 0$$

where $X(\rho) = P(\rho)^{-1}$ Such a configuration prevents any congruence transformations which have been used to linearize the inequality and the high number of nonlinear terms indicates that considering the original system with this stability/performance test is not a good idea.

Let us consider now the adjoint system instead: in this case, the projection must be done with respect to a basis of the kernel of matrices

$$\begin{array}{rcl} M_1(\rho) & = & \left[\begin{array}{ccccc} P(\rho) & 0 & 0 & 0 & h_{max} R \end{array} \right] \\ M_2(\rho) & = & \left[\begin{array}{ccccc} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{array} \right] \end{array}$$

since we have inequality

$$\Psi_a(\rho,\dot{\rho}) + M_2(\rho)^T K(\rho) M_1 + (\star)^T \prec 0$$

We can see that no congruence transformation is needed and it is possible to project immediately: this is the interest of the use of the adjoint. After that, since the matrix $P(\rho)$ and R are nonsingular, there exist an infinite number of values for $\mathrm{Ker}[M_1(\rho)]$ belonging to a set which can be defined implicitly. The next step of the approach resides in the choice of a 'good' kernel basis for $M_1(\rho)$. It is shown that a good basis is given by

$$\left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -h_{max}^{-1} R^{-1} P(\rho) & 0 & 0 & 0 \end{array} \right]$$

and such a choice limits the number of nonlinearities: there is only one concave nonlinearity of the form $-h_{max}^2 P(\rho)R^{-1}P(\rho)$. Concave nonlinearities are one of the most difficult nonlinearities that can be encountered in convex programming. The remaining of the approach consists in relaxing exactly this concave nonlinearity by a BMI involving a 'slack' variable (see Section 4.3) which is more simple to solve than the 'rational' matrix inequality involving the matrix R and its inverse. Using this, we get the following result:

Theorem 6.1.9 *There exists a state-feedback control law of the form $u(t) = K(\rho)x(t)$ which asymptotically stabilizes nominal system (6.1) (with $C_h(\cdot) = 0$ and $\Delta = 0$) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^n$, constant matrices $Q, R \in \mathbb{S}_{++}^n$, a scalar $\gamma > 0$ and scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the matrix inequalities*

$$\begin{bmatrix} Q - R + \partial_\rho P(\rho)\nu - h_{max}^{-2} P(\rho)R^{-1}P(\rho) & R & 0 & E(\rho) \\ \star & -(1-\mu)Q - R & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \\ \end{bmatrix} \prec 0 \tag{6.16}$$

$$\mathrm{Ker}[U(\rho)]^T \Psi(\rho,\nu) \mathrm{Ker}[U(\rho)] \prec 0$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ with

$$\begin{aligned} \Psi(\rho,\nu) &= \begin{bmatrix} \Psi_{11}(\rho,\nu) & P(\rho)A_h(\rho)^T + R & P(\rho)C(\rho)^T & E & h_{max}A(\rho)R \\ \star & -(1-\mu)Q - R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \\ U(\rho) &= \begin{bmatrix} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{bmatrix} \end{aligned}$$

where $\Psi_{11}(\rho,\nu) = A(\rho)P(\rho) + P(\rho)A(\rho)P^T + Q - R + \partial_\rho P(\rho)\nu$ Moreover, in this case a suitable control law can be computed by solving the following SDP in $K(\rho)$

$$\Psi(\rho,\nu) + U(\rho)^T K(\rho)V(\rho) + (\star)^T \prec 0$$

with $V(\rho) = \begin{bmatrix} P(\rho) & 0 & 0 & 0 & h_{max}R \end{bmatrix}$ and the closed-loop system satisfies $||z||_{\mathcal{L}_2} \le \gamma ||w||_{\mathcal{L}_2}$.

Proof: *The proof is based on an application of Lemma 4.5.2 which considers the stability and $\mathcal{L}_2$ performance for general time-delay systems using a simple Lyapunov-Krasovskii functional. Substituting matrices of the closed-loop system (6.3) into LMI (4.28) with $C_h(\cdot) = 0$ and $\Delta = 0$ we get:*

$$\begin{bmatrix} \tilde{\Phi}_{11}(\rho,\nu) & P(\rho)A_h(\rho)^T + R & P(\rho)C_{cl}(\rho)^T & E(\rho) & h_{max}A_{cl}(\rho)R \\ \star & -(1-\mu)Q - R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C_{cl}(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

with $\tilde{\Phi}_{11}(\rho,\nu) = A_{cl}(\rho)P(\rho) + P(\rho)A_{cl}(\rho)^T + Q - R + \partial_\rho P(\rho)\nu$, $A_{cl}(\rho) = A(\rho) + B(\rho)K(\rho)$ and $C_{cl}(\rho) = C(\rho) + D(\rho)K(\rho)$. The latter inequality can be rewritten as

$$\Psi(\rho,\nu) + U(\rho)^T K(\rho)V(\rho) + V(\rho)^T K(\rho)^T U(\rho) \prec 0 \tag{6.17}$$

where $\Psi(\rho,\nu)$ is defined by

$$\begin{bmatrix} \Psi_{11}(\rho,\nu) & P(\rho)A_h(\rho)^T+R & P(\rho)C(\rho)^T & E & h_{max}A(\rho)R \\ \star & -(1-\mu)Q-R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix}$$

$$U(\rho) = \begin{bmatrix} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{bmatrix} \qquad V(\rho) = \begin{bmatrix} P(\rho) & 0 & 0 & 0 & h_{max}R \end{bmatrix}$$

The projection lemma applies and we get the following underlying matrix inequalities:

$$\begin{array}{rcl} \mathrm{Ker}[U(\rho)]^T\Psi(\rho,\nu)\mathrm{Ker}[U(\rho)] & \prec & 0 \\ \mathrm{Ker}[V(\rho)]^T\Psi(\rho,\nu)\mathrm{Ker}[V(\rho)] & \prec & 0 \end{array}$$

While $\mathrm{Ker}[U(\rho)]$ *cannot be computed exactly in the general case,* $\mathrm{Ker}[V(\rho)]$ *can since it involves unknown decision matrices $P(\rho)$ and R whose properties are known. The whole null-space of $V(\rho)$ is spanned by*

$$\mathrm{Ker}[V(\rho)] = \begin{bmatrix} P_1(\rho) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ P_2(\rho) & 0 & 0 & 0 \end{bmatrix}$$

where $P_1(\rho)$ and $P_2(\rho)$ are such that $P(\rho)P_1(\rho)+h_{max}P_2(\rho)R=0$. Since the matrices $P(\rho)$ and R are positive definite (nonsingular) then there exists an infinite number of solutions $(P_1(\rho),P_2(\rho))$. Choosing $P_1(\rho)=I$ and $P_2(\rho)=-h_{max}^{-1}R^{-1}P(\rho)$ we get the following basis for the nullspace of $V(\rho)$:

$$\mathrm{Ker}[V(\rho)] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -h_{max}^{-1}R^{-1}P(\rho) & 0 & 0 & 0 \end{bmatrix}$$

Finally applying the projection lemma we get inequality $\mathrm{Ker}[V(\rho)]^T\Psi(\rho,\nu)\mathrm{Ker}[V(\rho)] \prec 0$ *which is equivalent to*

$$\begin{bmatrix} Q-R+\partial_\rho P(\rho)\nu-h_{max}^{-2}P(\rho)R^{-1}P(\rho) & R & 0 & E(\rho) \\ \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} \prec 0$$

We thus obtain a sufficient condition for the existence of a stabilizing controller. The computation of the controller can be done by SDP. Indeed, after solving the existence conditions, the variables $P(\rho),Q,R,\gamma$ are known and hence the matrix inequality (6.17) is linear in $K(\rho)$ and is a LMI problem. □

It is worth mentioning that matrix inequality (6.16) is strongly nonconvex due to the term $-h_{max}^{-2}P(\rho)R^{-1}P(\rho)$ which is a concave nonlinearity. In [Briat et al., 2008b; Chen and

Zheng, 2006; Gao and Wang, 2003] such a nonlinearity is relaxed by considering the inverse of matrix P (which is parameter independent in their case) and hence such a problem can be solved using the cone complementary algorithm [El-Ghaoui and Gahinet, 1993]. However, in the present case, such a relaxation scheme cannot be considered due to the parameter dependence of $P(\rho)$ which is a matrix function. Hence, new relaxation schemes should be developed.

The first one, just mentioned for completeness (see also Section 4.3), proposes to bound the concave function by an hyperplane (which is an affine function). This is done using a completion by the squares (see Section 4.3) and we get

$$-h_{max}^{-2}P(\rho)R^{-1}P(\rho) \preceq -2P(\rho) + h_{max}^2 R$$

Actually this method is very conservative since it corresponds to the linearization of the nonlinearity around a certain point and hence the approximation is correct in a neighborhood of the linearization point only. This motivates the development of the more general relaxation described in Section 4.3. Such a relaxation turns the rational nonlinearity into a bilinear nonlinearity in which the products involve a 'slack' variable.

Theorem 6.1.10 *There exists a state-feedback control law of the form $u(t) = K(\rho)x(t)$ which asymptotically stabilizes nominal system (6.1) (with $C_h(\cdot) = 0$ and $\Delta = 0$) for all $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^n$, a matrix function $\Lambda : U_\rho \to \mathbb{R}^{n\times n}$, constant matrices $Q, R \in \mathbb{S}_{++}^n$, a scalar $\gamma > 0$ and scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the matrix inequalities*

$$\begin{bmatrix} \Upsilon_{11}(\rho,\nu) & R & 0 & E(\rho) & \Lambda(\rho)^T R \\ \star & -(1-\mu)Q - R & 0 & 0 & 0 \\ \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\star & -h_{max}^2 R \\ \end{bmatrix} \prec 0 \tag{6.18}$$

$$\mathrm{Ker}[U(\rho)]^T \Psi(\rho,\nu)\mathrm{Ker}[U(\rho)] \prec 0 \tag{6.19}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\Upsilon_{11}(\rho,\nu) = Q - R + \partial_\rho P(\rho)\nu + \Lambda(\rho)^T P(\rho) + P(\rho)\Lambda(\rho)^T$, $\Psi_{11}(\rho,\nu) = A(\rho)P(\rho) + (\star)^T + Q - R + \partial_\rho P(\rho)\nu$,

$$\Psi(\rho,\nu) = \begin{bmatrix} \Psi_{11}(\rho,\nu) & P(\rho)A_h(\rho)^T + R & P(\rho)C(\rho)^T & E & h_{max}A(\rho)R \\ \star & -(1-\mu)Q - R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix}$$

Moreover, in this case a suitable control law can be computed by solving the following SDP in $K(\rho)$

$$\Psi(\rho,\nu) + U(\rho)^T K(\rho)V(\rho) + (\star)^T \prec 0$$

with $V(\rho) = \begin{bmatrix} P(\rho) & 0 & 0 & 0 & h_{max}R \end{bmatrix}$, $U(\rho) = \begin{bmatrix} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{bmatrix}$ and the closed-loop system satisfies $||z||_{\mathcal{L}_2} \le \gamma||w||_{\mathcal{L}_2}$.

Proof: *The relaxation is done using Theorem 4.3.4 on the matrix inequality*

$$\begin{bmatrix} \Phi_{11} - h_{max}^2 P(\rho) R^{-1} P(\rho) & \Phi_{12}(\rho) \\ \star & \Phi_{22} \end{bmatrix} \prec 0$$

where

$$\begin{bmatrix} \Phi_{11}(\rho) & \Phi_{12}(\rho) \\ \star & \Phi_{22} \end{bmatrix} = \left[\begin{array}{c|ccc} Q - R + \partial_\rho P(\rho)\nu & R & 0 & E(\rho) \\ \hline \star & -(1-\mu)Q - R & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{array}\right]$$

and $\eta(\rho) = \Lambda(\rho)$, $\alpha(\rho) = P(\rho)$ *and* $\beta = h_{max}^{-2} R$. $\square$

Although this approach preserves the nonlinearity of the problem, the numerical difficulty is reduced due to the fact that the problem is bilinear only (while before it was rational). Hence more simple algorithmic tools can be used to obtain local optimal solutions. One of the interest of this 'slack' variable is to decouple Lyapunov matrices products ($PR^{-1}P$) which allows then to solve for them simultaneously. Indeed in the first nonlinear problem $P(\rho)$ and R needed to be solved separately. Moreover, matrices R and R^{-1} appear in the same inequality which complicates the resolution. Note that several algorithms have been provided in the literature to solve for matrix inequalities where matrices and their inverse are involved [Ghaoui et al., 1997; Iwasaki and Skelton, 1995b; Skelton et al., 1997]. So, even if the problem is still nonlinear, the nonlinearities are much 'nicer'.

The following algorithm describes how to solve this nonlinear optimization problem:

Algorithm 6.1.11

1. *Generate an initial symmetric constant matrix* Λ_0 *such that* $\Lambda_0^T P + P\Lambda_0 \prec 0$, *choose a common structure for* $P(\rho)$ *and* $\Lambda(\rho)$ *e.g.* $Z(\rho) = Z_0 + Z_1\rho + Z\rho^2$ *with* $Z(\rho) = \{P(\rho), \Lambda(\rho)\}$.

2. *Solve the optimization problem*

$$\begin{array}{l} \min\ \gamma \\ \text{such that } P(\rho), Q, R \succ 0,\ \gamma > 0 \\ (6.18) \text{ and } (6.19) \end{array}$$

   If the problem is unfeasible then go to step 1.

3. *Solve the optimization problem*

$$\begin{array}{l} \min\limits_{\gamma, \Lambda(\rho), Q}\ \gamma \\ \text{such that } Q \succ 0,\ \gamma > 0 \\ (6.18) \text{ and } (6.19) \end{array}$$

4. *If stopping criterion is satisfied then STOP else go to step 2.*

Although this algorithm does not guarantee any global convergence, if the stabilization problem is feasible it turns out that it is easy to find an initial feasible point Λ_0 which can be defined here by $\Lambda_0 = -\varepsilon I$ with $\varepsilon > 0$. Moreover, several experiments seem to emphasize that a small number of iterations are sufficient to converge to a local optimum. Advantages of such an approach is to deal directly with initial bounded real lemma without any relaxation at the expense of a larger computational complexity. For more details on this relaxation, the readers should refer to Section 4.3.

Example 6.1.12 *In this example we will compare the proposed method expressed through Theorem 6.1.10 with an existent one proposed in [Zhang and Grigoriadis, 2005]. Let us consider the system*

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1+\phi sin(t) \\ -2 & -3+\delta sin(t) \end{bmatrix} x(t) + \begin{bmatrix} \phi sin(t) & 0.1 \\ -0.2+\delta sin(t) & -0.3 \end{bmatrix} x_h(t) \\
&\quad + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} w(t) + \begin{bmatrix} \phi sin(t) \\ 0.1+\delta sin(t) \end{bmatrix} u(t) \\
z(t) &= \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t)
\end{aligned} \tag{6.20}$$

which is borrowed from [Wu and Grigoriadis, 2001] and has been modified by [Zhang and Grigoriadis, 2005].

Case $\phi = 0.2$ ***and*** $\delta = 0.1$***:***

Choosing $\rho(t) = sin(t)$ *as parameter, it can be easily deduced that* $\rho, \dot{\rho} \in [-1, 1]$. *The parameter space is gridded over* $N_p = 40$ *points uniformly spaced.*

Choosing, as in [Zhang and Grigoriadis, 2005], $h_M = 0.5$, $\mu = 0.5$, $P(\rho) = P_c$ *and* $\Lambda(\rho) = \Lambda_c$ *(quadratic stability), we find* $\gamma^* = 1.8492$ *in 4 iterations of the algorithm for which the initial point has been randomly chosen. It is important to note that the first iteration gives a maximal bound on* γ *of 1.89 which is also a better result than those obtained before (See [Wu and Grigoriadis, 2001; Zhang and Grigoriadis, 2005]), for instance in Zhang and Grigoriadis [2005], an optimal value* $\gamma = 3.09$ *is found. In our case, the resulting a controller is given by* $K(\rho) = K_0 + K_1\rho + K_2\rho^2$ *where* $K_0 = \begin{bmatrix} -5.9172 & -16.3288 \end{bmatrix}$, $K_1 = \begin{bmatrix} -53.1109 - 32.4388 \end{bmatrix}$ *and* $K_2 = \begin{bmatrix} -8.4071 & 3.0878 \end{bmatrix}$.

It is worth noting that after computing the controller, the $\mathcal{L}_2$*-induced norm achieved is now* $\gamma_r = 2.2777$ *corresponding to an increase of* 23.17%. *Better performances should be obtained while considering a more complex controller form but we are limited by the fact that we do not consider rational controllers.*

The values of each coefficient of the gain $K(\rho)$ *w.r.t. parameter values are represented at the top of figure 6.1. The bottom of figure 6.1 describes the gain computed by the method of [Zhang and Grigoriadis, 2005].*

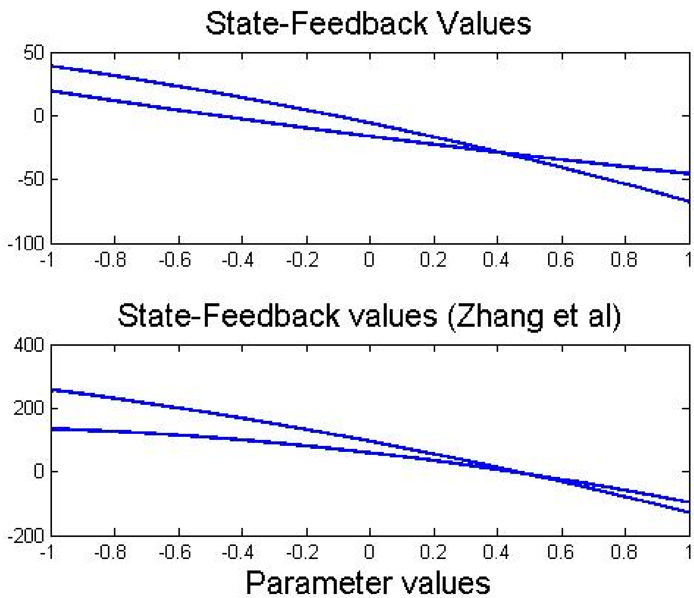

Figure 6.1: Simulation 1 - Gains controller evolution with respect to the parameter value - theorem 6.1.10 (top) and method of [Zhang and Grigoriadis, 2005]

Note that despite of lower controller gain values, we obtain better results than in the previous approaches, this is a great advantage of the proposed method.

For simulation purposes let $h(t) = 0.5|\sin(t)|$ and $\rho(t) = sin(t)$ and we will differentiate two cases: the stabilization with non-zero initial conditions and zero inputs and the stabilization with zero initial conditions and non-zero inputs.

Simulation 1: Stabilization ($x(0) \neq 0$ and $w(t) = 0$):

We obtain results depicted in Figures 6.2-6.4. We can see that the rate of convergence is very near but using our method the necessary input energy to make the system converge to 0 is less than in the case of [Zhang and Grigoriadis, 2005].

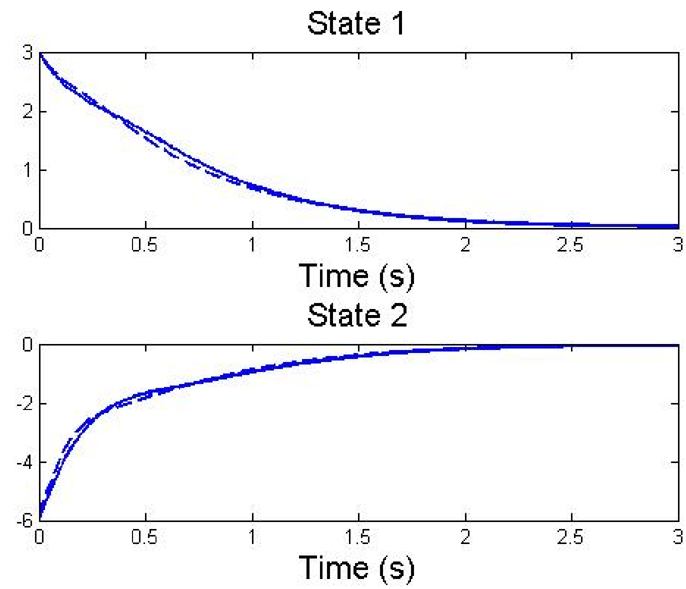

Figure 6.2: Simulation 1 - State evolution - theorem 6.1.10 in full and [Zhang and Grigoriadis, 2005] in dashed

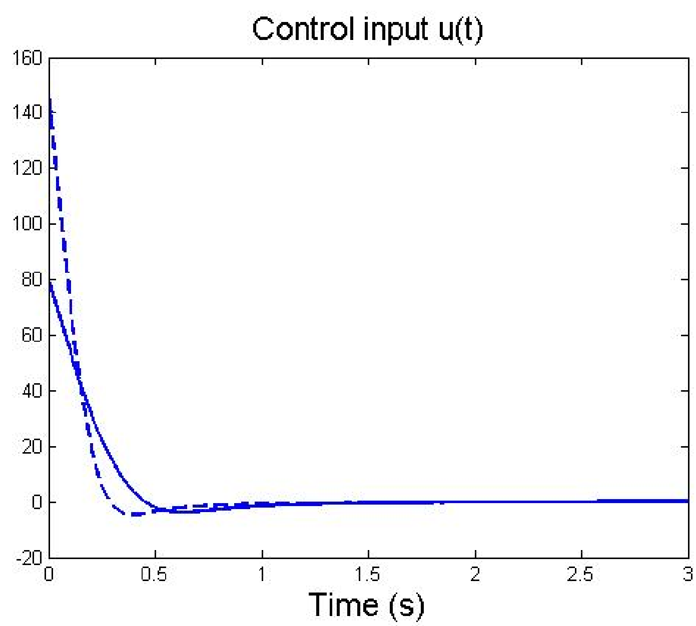

Figure 6.3: Simulation 1 - Control input evolution - theorem 6.1.10 in full and [Zhang and Grigoriadis, 2005] in dashed

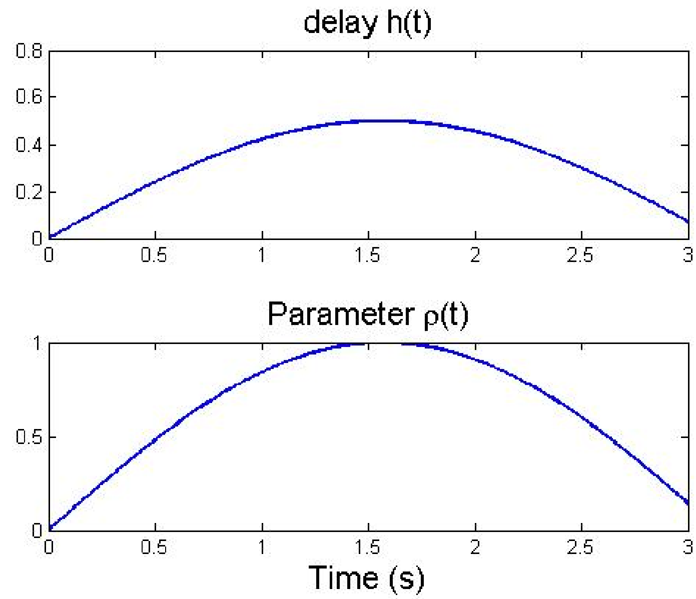

Figure 6.4: Simulation 1 - Delay and parameter evolution

Simulation 2: Disturbance attenuation ($x(0) = 0$ ***and*** $w(t) \neq 0$***)***

We consider here a unit step disturbance and we obtain the following results depicted in Figures 6.5-6.7. We can see that our control input has smaller bounds and that the second state is less affected by the disturbance than by using method of [Zhang and Grigoriadis, 2005]. Remember than the control output z contains the control input and the second state only, this is the reason why the first state is more sensitive to the disturbance than in [Zhang and Grigoriadis, 2005].

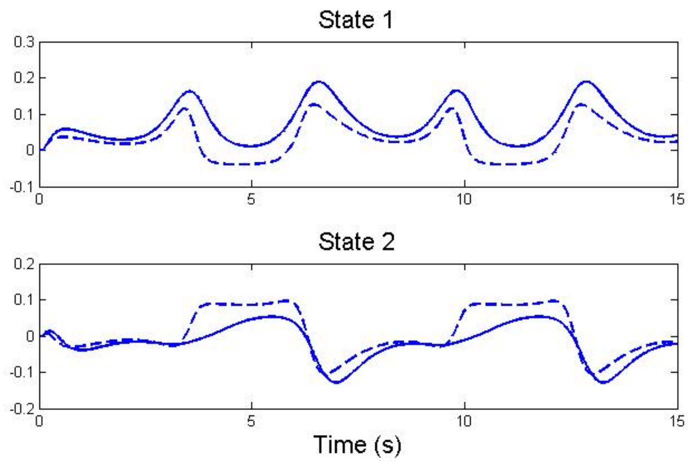

Figure 6.5: Simulation 2 - State evolution - theorem 6.1.10 in full and [Zhang and Grigoriadis, 2005] in dashed

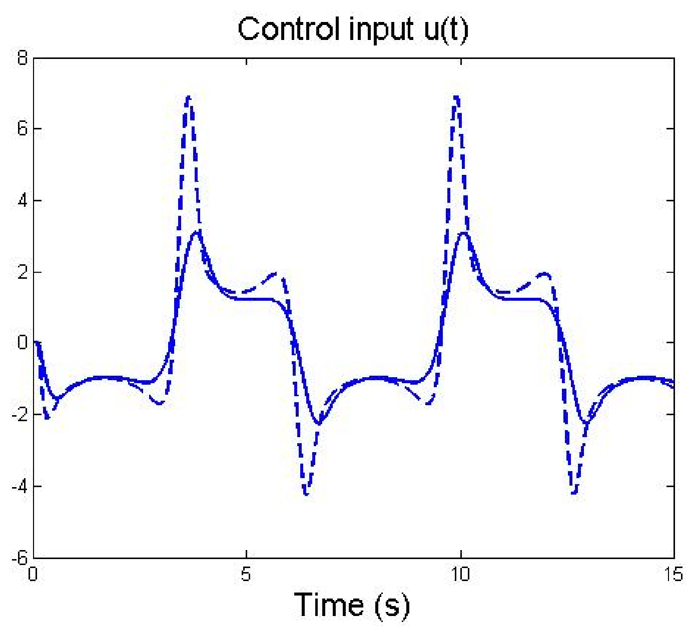

Figure 6.6: Simulation 2 - Control input evolution - theorem 6.1.10 in full and [Zhang and Grigoriadis, 2005] in dashed

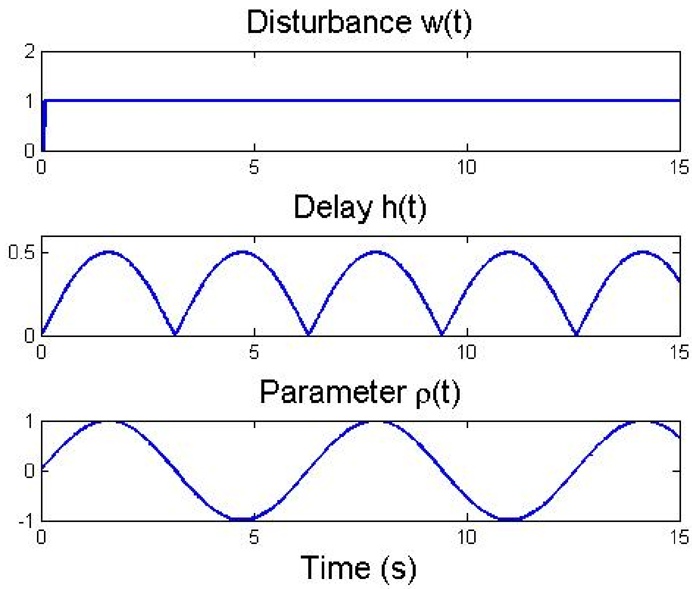

Figure 6.7: Simulation 2 - Delay and parameter evolution

Then we check, the delay upper bound for which a parameter dependent stabilizing controller exists and guarantees $\gamma^* < 10$ with $\mu = 0.5$ and we find $h_M = 79.1511$, for $\gamma^* < 2$ we find $h_M = 1.750$. In [Zhang and Grigoriadis, 2005], the delay upper bound for which a stabilizing controller exist is $h_M = 1.65$. This shows that our result is less conservative.
Case $\phi = 2$ and $\delta = 1$:

Using the results of [Zhang and Grigoriadis, 2005] no solution is found. With lemma 6.1.10, we find that there exists a controller such that the closed-loop system has a $\mathcal{L}_2$-induced norm lower than $\gamma = 6.4498$.

6.1.4 Memoryless state-feedback - Polytopic approach

Let us consider the polytopic LPV time-delay system:

$$\begin{array}{rcl} \dot{x}(t) & = & \sum_{i=1}^{N} (A_i x(t) + A_{hi} x(t - h_i(t)) + B_i u(t) + E_i w(t)) \\ z(t) & = & \sum_{i=1}^{N} (C_i x(t) + C_{hi} x(t - h_i(t)) + D_i u(t) + F_i w(t)) \end{array} \tag{6.21}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ and $h(t) \in \mathscr{H}_1^\circ$ are respectively the state of the system, the control input, the disturbances, the controlled outputs and the delay of the system. The goal is to stabilize the system with a LPV polytopic state-feedback control law of the form:

$$u(t) = \sum_{i=1}^{N} K_i x(t)$$

where the K_i are the gains to be designed. The parameters λ_i are assumed to evolve within a unitary polytope (unit simplex) characterized by

$$\Lambda := \left\{ \lambda_i(t) \in [0,1],\ \lambda_i(t) \geq 0,\ \sum_i \lambda_i(t) = 1 \right\}$$

When robust stability is addressed it is convenient to define the set in which the parameters derivative evolve

$$U_s := \left\{ \dot{\lambda}_i(t),\ \sum_{i=1} \dot{\lambda}_i(t) = 0 \right\} \subset \mathbb{R}^N$$

The idea of the approach is to define a parameter dependent Lyapunov-Krasovskii functional similar to those used before. Then we use a relaxation in order to remove multiple products and we substitute the closed-loop system into the relaxed stability/performances conditions. Since the whole polytopic approach is based on the linear dependence on the parameters, it is not possible here to consider only the vertices since there are quadratic terms in $\lambda(t)$ in the LMIs due to the terms $B(\lambda)K(\lambda)$ and $D(\lambda)K(\lambda)$. We provide here a solution based on the linearizing result introduced in Section 4.2 and more precisely given in Corollary 4.2.2.

Theorem 6.1.13 *There exists a state-feedback control law of the form* $u(t) = \sum_{i=1}^{N} K_i x(t)$ *which asymptotically stabilizes the system (6.21) for all* $h \in \mathscr{H}_1^\circ$ *if there exist matrices* $\tilde{P}_i, \tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^n, Y \in \mathbb{R}^{n\times n}, V_i \in \mathbb{R}^{m\times n}$ *and a scalar* $\gamma > 0$ *such that the parameter dependent LMI:*

$$\Omega_0 + \sum_{i=1}^{N} \lambda_i \Omega_i + \sum_{i,j=1}^{N,N} \lambda_i \lambda_j \Omega_{ij} \prec 0 \tag{6.22}$$

holds for all λ_i *such that* $\sum_{i=1}^{N}\lambda_i = 1$, $\lambda_i(t) \geq 0$, $\dot{\lambda} \in U_s$ *and where*

$$\Omega_0 = \begin{bmatrix} -Y^H & 0 & 0 & 0 & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}^0(\dot{\lambda}) & \tilde{R} & 0 & 0 & 0 & 0 \\ \star & \star & \tilde{U}_{33}^0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_p & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

$$\Omega_i = \begin{bmatrix} 0 & \tilde{P}_i + A_iY & A_{hi}Y & E_i & 0 & 0 & 0 \\ \star & \tilde{U}_{22}^i & 0 & 0 & [C_iY]^T & 0 & 0 \\ \star & \star & 0 & 0 & C_{hi}^T & 0 & 0 \\ \star & \star & \star & 0 & F_i^T & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}_i & 0 \\ \star & \star & \star & \star & \star & \star & 0 \end{bmatrix}$$

$$\Omega_{ij} = \begin{bmatrix} 0 & B_iV_j & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & [D_iV_j]^T & 0 & 0 \\ \star & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 \end{bmatrix}$$

with $\tilde{U}_{22}(\dot{\lambda})^0 = \sum_{i=1}^{N}\dot{\lambda}_i(t)P_i + \tilde{Q} - \tilde{R}$, $\tilde{U}_{22}^i = -P_i + \tilde{Q} - \tilde{R}$ *and* $\tilde{U}_{33} = -(1-\mu)\tilde{Q} - \tilde{R}$. *In this case, the controller matrices are given by* $K_i = V_iY^{-1}$ *and the closed-loop system satisfies* $||z||_{\mathcal{L}_2} \geq \gamma||w||_{\mathcal{L}_2}$.

Proof: *Consider the following Lyapunov-Krasovskii functional*

$$\begin{aligned} V(x_t,\dot{x}_t) &= V^1(x_t) + V^2(x_t) + V^3(\dot{x}_t) \\ V^1(x_t) &= \sum_{i=1}^{N} x(t)^T P_i x(t) \\ V^2(x_t) &= \int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta \\ V^3(x_t) &= \int_{-h_{max}}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T (h_{max}R) x(\eta) d\eta d\theta \end{aligned}$$

Since the form is very similar to the Lyapunov-Krasovskii functionals developed in Section 4.5, get the following LMI

$$\Omega_0 + \sum_{i=1}^{N}\lambda_i\Omega_i + \sum_{i,j=1}^{N,N}\lambda_i\lambda_j\Omega_{ij} \prec 0 \tag{6.23}$$

which is the polytopic LPV counterpart of LMI (4.35) on which a congruence transformation with respect to the matrix $\text{diag}(Y, Y, Y, I, I, Y, Y)$ *with* $Y = X^{-1}$ *and the change of variable* $V_i = K_iY$ *have been performed.* □

A direct way to solve LMI would be to impose

$$\begin{array}{rcl} \Omega_0 & \prec & 0 \\ \Omega_i & \prec & 0 \\ \Omega_{ij} & \prec & 0 \end{array}$$

for $i, j = 1, \ldots, N$. Despite of its simplicity this method is very conservative. This is the reason why relaxations like SOS-relaxation, polynomial optimization and linearization approaches should be employed instead. The reader should refer to sections 4.2 and 2.3.3.1 to get more explanations about these approaches. We have chosen to employ here the linearization approach detailed in Section 4.2.

The principle is to turn the polynomial parameter dependence into a new affine parameter dependent LMI involving 'slack' variables. This new LMI is not equivalent to the first one but still leads to interesting results:

Theorem 6.1.14 *There exists a state-feedback control law of the form* $u(t) = \sum_{i=1}^{N} K_i x(t)$ *which asymptotically stabilizes the system (6.21) for all* $h \in \mathscr{H}_1^{\circ}$ *if there exist matrices* $\tilde{P}_i, \tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^n, Y \in \mathbb{R}^{n\times n}, V_i \in \mathbb{R}^{m\times n}$*, a scalar* $\gamma > 0$ *and a matrix* $\mathcal{Z}$ *such that the LMIs*

$$\tilde{\mathcal{K}} + \mathcal{Z}^T \Pi_i + \Pi_i^T \mathcal{Z} < 0$$

hold for all $(\lambda, \dot{\lambda}) \in \Lambda \times U_s$*,* $i = 1, \ldots, N$*,* $\Pi(\lambda) = \sum_{i=1}^{N} \Pi_i \lambda_i$ *and where*

$$\Pi(\lambda) = \begin{bmatrix} -\lambda_1 I & I & 0 & \ldots & 0 \\ -\lambda_2 I & 0 & I & \ldots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -\lambda_N I & 0 & 0 & \ldots & I \end{bmatrix} \qquad \tilde{\mathcal{K}} = \begin{bmatrix} \Omega_0 & \Omega_1/2 & \ldots & \Omega_{N-1}/2 \\ \star & \Omega_{11} & \ldots & (\Omega_{1N} + \Omega_{N1})/2 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \ldots & \Omega_{NN} \end{bmatrix}$$

In this case suitable controller matrices are given by $K_i = V_i Y^{-1}$ *and the closed-loop system satisfies* $||z||_{\mathcal{L}_2} \geq \gamma ||w||_{\mathcal{L}_2}$

Proof: This is a straightforward application of Corollary 4.2.2 to LMI (6.22). □

6.1.5 Hereditary State-Feedback Controller Design - exact delay value case

We consider in this section the design of state-feedback control laws embedding a delayed information:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \tag{6.24}$$

It is clear that such a control law will lead to better results than memoryless control laws. We will consider first that the delay used in the controller is identical to the delay involved in the system dynamical model. The next section will be devoted to the case when the delay of the controller and the system are different.

The approach of this section is similar to the one proposed for the design of memoryless control laws (see Theorem 6.1.4 for a similar proof) and leads to the following theorem:

Theorem 6.1.15 *There exists a stabilizing control law of the form (6.24) for system (6.1) with $h \in \mathscr{H}_1^\circ$ if there exists a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}_{++}^n$, matrix functions $V_0, V_h : U_\rho \to \mathbb{R}^{m\times n}$, constant matrices $\tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^n, Y \in \mathbb{R}^{n\times n}$, a constant scalar $\gamma > 0$ and a scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G} & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{6.25}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\Psi(\rho,\nu)$ is defined by

$$\begin{bmatrix} -(Y+Y^T) & U_{12}(\rho) & U_{13}(\rho) & E(\rho) & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & U_{25}(\rho) & 0 & 0 \\ \star & \star & \tilde{U}_{33} & 0 & U_{26}(\rho) & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

and

$$\begin{array}{lll} U_{12}(\rho) & = & \tilde{P}(\rho) + A(\rho)Y + B(\rho)V_0(\rho) \\ U_{25}(\rho) & = & Y^TC(\rho)^T + [D(\rho)V_0(\rho)]^T \\ \mathcal{G}(\rho) & = & \begin{bmatrix} G(\rho)^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \tilde{U}_{33} & = & -(1-\mu)\tilde{Q} - \tilde{R} \end{array} \qquad \begin{array}{lll} U_{13}(\rho) & = & A_h(\rho)Y + B(\rho)V_h(\rho) \\ U_{26}(\rho) & = & Y^TC_h(\rho)^T + [D(\rho)V_h(\rho)]^T \\ \tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho\tilde{P}(\rho)\nu \end{array}$$

$$\mathcal{H} = \begin{bmatrix} 0 & H_A(\rho)Y + H_B(\rho)V_0(\rho) & H_{A_h}(\rho)Y + B(\rho)V_h(\rho) & H_E(\rho) & 0 & 0 & 0 \end{bmatrix}$$

Moreover a suitable control gains are given by $K_0(\rho) = V_0(\rho)Y^{-1}$ and $K_h(\rho) = V_h(\rho)Y^{-1}$ and the closed-loop satisfies $||z||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

6.1.6 Hereditary State-Feedback Controller Design - approximate delay value case

This section is devoted to the design of control of the form (6.2) in which the delay $d(t)$ is different from the delay $h(t)$ of the system. In this case, we have the following control law:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t-d(t)) \tag{6.26}$$

The approach is again similar to the others, the main difference lies in the choice of the Lyapunov-Krasovskii functional. Since the closed-loop will involve two delayed terms ($x(t-h(t))$ and $x(t-d(t))$) which are coupled together by the algebraic equality $d(t) = h(t) + \varepsilon(t)$ with $|\varepsilon(t)| \leq \delta$. This equality constitutes a difficulty in the design since the Lyapunov-Krasovskii functional must embed this information in order to characterize correctly the system stability. This is done using the Lyapunov-Krasovskii defined in Section 4.7 and the algorithm to obtain relaxed LMIs as design solutions. Using this we get the Theorem:

Theorem 6.1.16 *There exists a state-feedback control law of the form (6.26) if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^n_{++}$, matrix functions $V_0, V_h : U_\rho \to \mathbb{R}^{m\times n}$, $\tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2 \in \mathbb{S}^n_{++}$, a scalar $\gamma > 0$ and a scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ if the following LMIs*

$$\begin{bmatrix} U_{11}(\rho) & U_{12}(\rho) & U_{13}(\rho) & E(\rho) & 0 & \tilde{X} & h_{max}\tilde{R}_1 & \tilde{R}_2 & 0 \\ \star & U_{22}(\rho,\nu) & \tilde{R}_1 & 0 & U_{25}(\rho) & 0 & 0 & 0 & U_{29}(\rho) \\ \star & \star & U_{33} & 0 & U_{35}(\rho) & 0 & 0 & 0 & U_{39}(\rho) \\ \star & \star & \star & -\gamma I & F(\rho)^T & 0 & 0 & 0 & H_E(\rho)^T \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 & 0 \\ \star & \star & \star & \star & \star & \star & -\tilde{R}_1 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta} & 0 \\ \star & \star & \star & \star & \star & \star & \star & 0 & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{6.27}$$

$$\begin{bmatrix} \Pi_{11}(\rho,\nu) & \Pi_{12}(\rho) & \Pi_{13}(\rho) \\ \star & \Pi_{22}(\rho) & 0 \\ \star & \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $U_{22}(\rho,\nu) = -\tilde{P}(\rho) + \tilde{Q}_1 + \tilde{Q}_2 + \sum_{i=1}^{N} \dfrac{\partial \tilde{P}}{\partial \rho_i}\nu_i - \tilde{R}_1$ and

$$\begin{array}{rclcrcl} U_{11}(\rho) & = & -\tilde{X}(\rho)^H + \varepsilon(\rho)G(\rho)G(\rho)^T & & U_{12}(\rho) & = & \tilde{P}(\rho) + A(\rho)\tilde{X} + B(\rho)V_0(\rho) \\ U_{13}(\rho) & = & A_h(\rho)\tilde{X} + B(\rho)V_h(\rho) & & U_{33} & = & -(1-\mu)(\tilde{Q}_1 + \tilde{Q}_2) - \tilde{R}_1 \\ U_{25}(\rho) & = & U_{25}(\rho)[C(\rho)\tilde{X} + D(\rho)V_0(\rho)]^T & & U_{29}(\rho) & = & [H_A(\rho)\tilde{X} + H_B(\rho)V_0(\rho)]^T \\ U_{35}(\rho) & = & [C_h(\rho)X + D(\rho)V_h(\rho)]^T & & U_{39}(\rho) & = & [H_{A_h}(\rho)\tilde{X} + H_B(\rho)V_h(\rho)]^T \end{array}$$

$\Pi_{11}(\rho,\nu)$ *is defined by*

$$\begin{bmatrix} -\tilde{X}(\rho)^H + \varepsilon(\rho)G(\rho)G(\rho)^T & \tilde{P}(\rho) + A(\rho)\tilde{X} + B(\rho)V_0(\rho) & A_h(\rho)\tilde{X} & B(\rho)V_h(\rho) & E(\rho) \\ \star & \tilde{\Theta}_{11}(\rho,\nu) & \tilde{R}_1 & 0 & 0 \\ \star & \star & \tilde{\Psi}_{22} & (1-\mu)\tilde{R}_2/\delta & 0 \\ \star & \star & \star & \tilde{\Psi}_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix}$$

and $\tilde{\Psi}_{22} = -(1-\mu_h)(\tilde{Q}_1 + \tilde{R}_2/\delta) - \tilde{R}_1$, $\tilde{\Psi}_{33} = -(1-\mu_d)\tilde{Q}_2 - (1-\mu)\tilde{R}_2/\delta$

$$\Pi_{12}(\rho) = \begin{bmatrix} 0 & \tilde{X}(\rho) & h_{max}\tilde{R}_1 & \tilde{R}_2 \\ \left[C(\rho)\tilde{X} + D(\rho)V_0(\rho)\right]^T & 0 & 0 & 0 \\ \left[C_h(\rho)\tilde{X}\right]^T & 0 & 0 & 0 \\ [D(\rho)V_h(\rho)]^T & 0 & 0 & 0 \\ F(\rho)^T & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi_{13}(\rho) = \begin{bmatrix} 0 \\ \left[H_A(\rho)\tilde{X} + H_B(\rho)V_0(\rho)\right]^T \\ 0 \\ [H_B(\rho)V_h(\rho)]^T \\ 0 \end{bmatrix} \quad \Pi_{22}(\rho) = \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 \\ \star & \star & -\tilde{R}_1 & 0 \\ \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta} \end{bmatrix}$$

6.1.7 Delay-Scheduled State-Feedback Controllers

This section is devoted to the development of a new technique to control time-delay systems with time-varying delays provided that the delay can be measured or estimated in real-time. The difference between state-feedback with memory and delay-scheduled state-feedback controllers comes from the fact that the former uses the delayed state into the control law expression while the latter the instantaneous state only. On the other hand, while the former uses constant gains (in the LTI case), the latter involves a matrix gain which depends on the delay value, as in the LPV framework. Hence, a delay-scheduled state-feedback control law is defined by

$$u(t) = K(\hat{h})x(t) \tag{6.28}$$

Since the gain scheduling technique is a well-established method in the LPV framework through different approaches such as LPV polytopic systems, polynomial systems and 'LFT' systems, it seems important to develop an application of LPV theory to time-delay systems. This section will consider the following time-delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h(t-h(t)) + Bu(t) + Ew(t) \\ z(t) & = & Cx(t) + C_hx(t-h(t)) + Du(t) + Fw(t) \end{array} \tag{6.29}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ are respectively the system state, the control input, the exogenous inputs and the controlled outputs.

In [Briat et al., 2007a], a model transformation has been introduced in order to turn a time-delay system into an uncertain LPV system. However, this model transformations suffer from two main problems: the first one is the singularity of the $\mathcal{L}_2$ norm of the operator for 0 delay values. The second one is the conservatism induced by the computation of the $\mathcal{L}_2$-induced norm of that operator. The model transformation presented below authorizes zero delay values and the $\mathcal{L}_2$-induced norm computation is tighter.

Let the operator

$$\begin{array}{rcl} \mathcal{D}_h : \mathcal{L}_2 & \rightarrow & \mathcal{L}_2 \\ \eta(t) & \rightarrow & \dfrac{1}{\sqrt{h(t)h_{max}}}\displaystyle\int_{t-h(t)}^{t}\eta(s)ds \end{array}$$

This operator enjoys the following properties:

1. $\mathcal{D}_h$ is $\mathcal{L}_2 - \mathcal{L}_2$ stable.
2. $\mathcal{D}_h$ has an induced $\mathcal{L}_2 - \mathcal{L}_2$ norm lower than 1.

Proof: *Let us prove first that for a $\mathcal{L}_2$ input signal we get a $\mathcal{L}_2$ output signal. Assume that $\eta(t)$ is continuous and denote by $\eta_p(t)$ the signal satisfying $d\eta_p(t)/dt = \eta(t)$ then we have*

$$\mathcal{D}_h(\eta(t)) = \frac{\eta_p(t) - \eta_p(t-h(t))}{\sqrt{h(t)h_{max}}} \tag{6.30}$$

Note that as $h(t)$ is always positive then (6.30) is bounded since $\eta(t)$ is continuous and belongs to $\mathcal{L}_2$ (and hence to $\mathcal{L}_\infty$). The main problem is when the delay reaches 0. Suppose now that there exist a (possibly infinite) family of time instants $t_{i+1} > t_i \geq 0$ such that $h(t_i) = 0$. Since $\eta_p(t)$ is continuously differentiable and hence we have

$$lim_{t\rightarrow t_i}\frac{\eta_p(t) - \eta_p(t-h(t))}{\sqrt{h(t)h_{max}}} = \sqrt{\frac{h(t_i)}{h_{max}}}\eta(t_i)$$

As $\eta(t)$ is continuous and belongs to $\mathcal{L}_2$, we can state that $\eta(t_i)$ is always finite and then the output signal remains bounded even if the delay reaches zero. This proves that $\mathcal{D}_h$has a finite $\mathcal{L}_\infty$-induced norm (no singularities). Let us prove now that it has a finite induced $\mathcal{L}_2$-norm using a similar method as in Gu et al. [2003]:

$$||\mathcal{D}_h(\eta)||_{\mathcal{L}_2}^2 := \int_0^{+\infty}\frac{dt}{h(t)h_{max}}\int_{t-h(t)}^{t}\eta^T(\theta)d\theta\cdot\int_{t-h(t)}^{t}\eta(\theta)d\theta$$

Then using the Jensen's inequality (see [Gu et al., 2003]) we obtain

$$||\mathcal{D}_h(\eta)||_{\mathcal{L}_2}^2 \leq \int_0^{+\infty}\frac{dt}{h_{max}}\int_{t-h(t)}^{t}\eta^T(\theta)\eta(\theta)d\theta \tag{6.31}$$

To solve the problem we will exchange the order of integration under the assumption $\eta(\theta) = 0$ when $\theta \leq 0$. First note that the domain is contained in $t - h_M \leq \theta \leq t$, $\theta \geq 0$ and is bounded by lines $\theta = t$ and $\theta = p(t) := t - h(t)$. Since $p(\theta)$ is a non-decreasing function then the set of segments where $\theta = p(t)$ is constant is countable. Hence for almost all θ the function $p(t)$ is increasing and the inverse $t = q(\theta) := p^{-1}(\theta) :=$ is well-defined and then we have

$$\begin{aligned} ||\mathcal{D}_h(\eta)||_{\mathcal{L}_2}^2 &\leq \frac{1}{h_{max}}\int_0^{+\infty}\eta^T(\theta)\eta(\theta)d\theta\int_\theta^{q(\theta)}dt \\ &= \frac{1}{h_{max}}\int_0^{+\infty}\eta^T(\theta)\eta(\theta)(q(\theta)-\theta)d\theta \end{aligned}$$

Hence using the fact that $\theta = t - h(t)$ and that $t = q(\theta)$ then we have the equality $\theta = q(\theta) - h(q(\theta))$ and hence we have $q(\theta) - \theta = h(q(\theta))$. This leads to

$$\begin{aligned} ||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} &\leq \frac{1}{h_{max}} \int_0^{+\infty} \eta^T(\theta)\eta(\theta)h(q(\theta))d\theta \\ &\leq ||\eta||^2_{\mathcal{L}_2} \end{aligned}$$

We have then proved that $\mathcal{D}_h$ defines a $\mathcal{L}_2 - \mathcal{L}_2$ stable operator with an $\mathcal{L}_2$-induced norm lower than 1. □

We show now how to use this operator to transform a time-delay system into an uncertain LPV system. Consider system (6.29) and note that $x_h(t) = x(t - h(t)) = \mathcal{D}_h(\dot{x}(t))$ then substituting this expression into system (6.29) we get the following 'LFT' system:

$$\begin{array}{rcl} \dot{y}(t) &=& \bar{A}y(t) - \alpha(t)A_h w_0(t) + B_u u(t) + Ew(t) \\ z_0(t) &=& \dot{y}(t) \\ z(t) &=& \bar{C}y(t) - \alpha(t)C_h w_0(t) + D_u u(t) + Fw(t) \\ w_0(t) &=& \mathcal{D}_h(z_0(t)) \\ \bar{A} &=& A + A_h \\ \bar{C} &=& C + C_h \end{array} \tag{6.32}$$

where $\alpha(t) = \sqrt{h(t)h_M}$ and $y(t)$ is the new state of the system emphasizing that the transformed model is not always equivalent to the original one.

This system is then obviously:

- uncertain due to the presence of the "unknown" structured norm bounded LTV dynamic operator $\mathcal{D}_h$. For this part we will use results of robust stability analysis and robust synthesis.

- parameter varying (even affine in $\alpha(t)$). We will use parameter dependent Lyapunov functions to tackle this time-varying part.

It is clear that this system is not equivalent to (6.29) due to the model transformation introducing additional dynamics (see Section 3.2.1.3 and [Gu and Niculescu, 1999, 2000; Gu et al., 2003]). Just note that additional dynamics may be a source of conservatism in stability analysis. Nevertheless, in the stabilization problem this is less problematic since we aim to stabilize the system and hence we stabilize these additional dynamics (assuming they are stabilizable).

Before introducing the main results of this section based on this model transformation it is necessary to introduce the following sets

$$\begin{array}{rcl} H &:=& [h_{min}, h_{max}] \\ U &:=& [\mu_{min}, \mu_{max}] \\ \hat{H} &:=& [h_{min} - \delta, h_{max} + \delta] \\ \hat{U} &:=& [\mu_{min} - \nu_{min}, \mu_{max} + \nu_{max}] \end{array}$$

The set H corresponds to the set of values of the delay, the set U defines the set of values of the delay derivative. The sets $\hat{H}$ and $\hat{U}$ represent respectively the set of values of the measured delay and its derivative. It is worth mentioning that the measurement error belongs to $[-\delta, \delta]$ while its derivative remains within $[\nu_{min}, \nu_{max}]$.

6.1.7.1 Stability and $\mathcal{L}_2$ performances analysis

This section is devoted to the stability analysis of the transformed system using robust and LPV stability analysis tools. The robustness with respect to the operator $\mathcal{D}_h$ will be ensured using the full-block $\mathcal{S}$-procedure [Scherer, 1996, 1999, 2001] while the stability with respect to the parameter varying part will be tackled using a parameter dependent Lyapunov function. The full-block $\mathcal{S}$-procedure is used with parameter dependent D scalings, $D(\cdot)$ being the decision variables, as shown below:

Lemma 6.1.17 *System (6.32) without control input (i.e. $u(t) = 0$) is asymptotically stable for $h \in \mathscr{H}$ and satisfies the $\mathcal{H}_\infty$-norm property $||z||_2/||w||_2 < \gamma(h, \dot{h})$ if there exist a smooth matrix function $P : H \to \mathbb{S}^n_{++}$, matrix functions $D : H \times U \to \mathbb{S}^n_{++}$ and a function $\gamma : H \times U \to \mathbb{R}_{++}$ such that the LMI*

$$\begin{bmatrix} [\bar{A}^T P(h)]^H + \dfrac{dP}{dh}\dot{h} & -\alpha P(h) A_h & P(h)E & \bar{C}^T & \bar{A}^T D(h,\dot{h}) \\ \star & -D(h) & 0 & -\alpha C_h^T & -\alpha A_h^T D(h,\dot{h}) \\ \star & \star & -\gamma(h,\dot{h}) I_p & F^T & E^T D(h,\dot{h}) \\ \star & \star & \star & -\gamma(h,\dot{h}) I_q & 0 \\ \star & \star & \star & \star & -D(h,\dot{h}) \end{bmatrix} \prec 0 \quad (6.33)$$

holds for all $h \in H$ and $\dot{h} \in U$ with $\alpha = \sqrt{h_{max}h}$.

Proof: *Let us consider system (6.32), it is possible to apply the full-block $\mathcal{S}$-procedure in order to develop an efficient stability test. Combining with $\mathcal{L}_2$ performances we obtain the following LMI*

$$\begin{aligned} &\begin{bmatrix} \dfrac{\partial P}{\partial h}\dot{h} + \bar{A}^T P(h) + P(h)\bar{A} & -\alpha P A_h & PE \\ \star & 0 & 0 \\ \star & \star & -\gamma(h,\dot{h}) I \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \bar{A}^T \\ I & -\alpha A_h^T \\ 0 & E^T \end{bmatrix} \mho(h,\dot{h}) \begin{bmatrix} 0 & I & 0 \\ \bar{A} & -\alpha A_h & E \end{bmatrix} \\ &+\gamma^{-1}(h,\dot{h}) \begin{bmatrix} \bar{C}^T \\ -\alpha A_h^T \\ F^T \end{bmatrix} \begin{bmatrix} \bar{C}^T \\ -\alpha A_h^T \\ F^T \end{bmatrix}^T \prec 0 \end{aligned} \quad (6.34)$$

where $\mho(h, \dot{h})$ satisfies

$$\int_0^t \begin{bmatrix} \mathcal{D}_h(\eta) \\ \eta \end{bmatrix}^T \mho(h,\dot{h}) \begin{bmatrix} \mathcal{D}_h(\eta) \\ \eta \end{bmatrix} ds > 0 \quad \text{for all } \eta \in \mathcal{L}_2$$

The separator $\mho(h,\dot{h}) = \mho^*(h,\dot{h})$ is chosen noting $||\mathcal{D}_h||_\infty < 1$ then $\mathcal{D}_h$ may satisfy

$$\int_0^t \begin{bmatrix} \mathcal{D}_h(\eta) \\ \eta \end{bmatrix}^T \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mho_1} \begin{bmatrix} \mathcal{D}_h(\eta) \\ \eta \end{bmatrix} ds > 0 \quad \text{for all } \eta \in \mathcal{L}_2$$

Hence a set of separators can be parametrized as $\mho = \mho_1 \otimes D$ where $D = D^* > 0$. But the set of separators is limited to be Hermitian and the signal are real valued then the separator becomes

$$\mho(h,\dot{h}) := \begin{bmatrix} -D(h,\dot{h}) & 0 \\ \star & D(h,\dot{h}) \end{bmatrix} \tag{6.35}$$

where $D : H \times U \to \mathbb{S}^n_{++}$. Then expand (6.34) and perform a Schur complement on quadratic term

$$-\begin{bmatrix} \bar{C}^T & \bar{A}^T D(h,\dot{h}) \\ -\alpha C_h & -\alpha A_h D(h,\dot{h}) \\ F & E^T D(h,\dot{h}) \end{bmatrix} \begin{bmatrix} -\gamma^{-1}(h,\dot{h}) I_q & 0 \\ 0 & -D^{-1}(h,\dot{h}) \end{bmatrix} \begin{bmatrix} \bar{C}^T & \bar{A}^T D(h,\dot{h}) \\ -\alpha C_h & -\alpha A_h D(h,\dot{h}) \\ F & E^T D(h,\dot{h}) \end{bmatrix}^T$$

leads to inequality (6.33). □

The LMI provided in the latter theorem can be easily solved using classical LMI solvers. Moreover, if the parameter dependence is linear then a polytopic relaxation will be exact. However, if the dependence is polynomial then a more complex relaxation scheme should be adopted. For more details about these relaxations, the readers should refer to Sections 2.3.3.2, 2.3.3.3, 2.3.3.4 and 4.2.

6.1.7.2 Delay-Scheduled state-feedback design

We provide in that section the computation of a delay-scheduled state-feedback of the form (6.28) for system (6.29). In this case, the closed-loop system is then given by

$$\begin{array}{rcl} \dot{y}(t) & = & \bar{A}_{cl}(h,\delta_h)y(t) - A_h\alpha(t)w_0(t) + Ew(t) \\ z(t) & = & \bar{C}_{cl}(h,\delta_h)y(t) - C_h\alpha(t)w_0(t) + Fw(t) \\ z_0(t) & = & \dot{y}(t) \\ w_0(t) & = & \mathcal{D}_h(z_0(t)) \end{array} \tag{6.36}$$

with $\hat{h} = h + \delta_h$, a state feedback of the form $K(h + \delta_h)$ and closed-loop system matrices $\bar{A}_{cl}(h,\delta_h) = \bar{A} + B_u K(\hat{h})$, $\bar{C}_{cl}(h,\delta_h) = \bar{C} + D_u K(\hat{h})$. As shown in previous sections, there exist several ways to compute this controller:

1. Use an approach involving congruence transformations and change of variable. Using this approach, it is possible to fix a desired form to the controller.

2. Elaborate a stabilizability test (independent of the controller) based on the projection lemma (see Appendix D.18). A suitable controller is then deduced either through a LMI problem or an explicit algebraic equality.

We will only provide here a solution based on a change of variable but a solution based on the projection lemma can also be employed (see Section 6.1.1 for details, differences and interests of these approaches). This approach allows to fix the controller structure which can be independent of the delay derivative. However, the result may be conservative since it is difficult to choose adequately the controller structure. The approach using the projection lemma is interesting since it allows to compute the minimal $\mathcal{L}_2$ performances gain that can be reached using this approach but the controller which is computed from algebraic equations might depend on the delay-derivative.

Theorem 6.1.18 *The system (6.32) is stabilizable with a delay-scheduled state feedback $K(\hat{h}) = Y(\hat{h})X^{-1}(\hat{h})$ if there exists a smooth matrix function $X : \hat{H} \to \mathbb{S}^n_{++}$, matrix functions $Y : \hat{H} \to \mathbb{R}^{m\times n}$, $\tilde{D} : H \times U \times \hat{H} \times \hat{U} \to \mathbb{S}^n_{++}$ and a scalar function $\gamma : H \times U \times \hat{H} \times \hat{U} \to \mathbb{R}_{++}$ such that the LMI*

$$\begin{bmatrix} U_{11}(\hat{h},\dot{\hat{h}}) & U_{12}(\hat{h}) & U_{13}(\hat{h},\dot{\hat{h}}) & \alpha A_h\tilde{D}(\xi) & E \\ \star & -\gamma(\xi)I_q & \alpha\bar{C}X(h) & \alpha C_h\tilde{D}(\xi) & F \\ \star & \star & -\dot{\hat{h}}\dfrac{\partial X(\hat{h})}{\partial\hat{h}} - \tilde{D}(\xi) & 0 & 0 \\ \star & \star & \star & -\tilde{D}(\xi) & 0 \\ \star & \star & \star & \star & -\gamma(\xi)I_p \end{bmatrix} \prec 0 \tag{6.37}$$

holds for all $h \in H$, $\dot{h} \in U$, $\delta_h \in \Delta$ and $\dot{\delta}_h \in \Delta_\nu$, where $\xi = col(h,\delta_h,\dot{h},\dot{\delta}_h)$ and

$$\begin{aligned} U_{11}(\hat{h},\dot{\hat{h}}) &= -\dot{\hat{h}}\frac{\partial X(\hat{h})}{\partial\hat{h}} + [X(\hat{h})\bar{A}^T + Y^T(\hat{h})B_u^T]^H \\ U_{12}(\hat{h}) &= X(\hat{h})\bar{C}^T + Y^T(\hat{h})D_u^T \\ U_{13}(\hat{h},\dot{\hat{h}}) &= -\dot{\hat{h}}\frac{\partial X(\hat{h})}{\partial\hat{h}} + \bar{A}X(\hat{h}) \\ K(\hat{h}) &= Y(\hat{h})X(\hat{h})^{-1} \end{aligned}$$

Proof: *First note that the real unknown delay is $h(t)$ and the estimated one is $\hat{h}(t) = h(t)+\delta_h(t)$. X must depend on $\hat{h}(t)$ only since the controller gain is a function of X. Indeed, if X depends on $h(t)$, then the controller would also depend on $h(t)$ which is not possible since $h(t)$ is unknown. Nevertheless, other variables may depend on all the parameters (i.e. $h(t),\delta_h(t),\dot{h}(t),\dot{\delta}_h(t)$). From here let $\xi = col(h,\delta_h,\dot{h},\dot{\delta}_h)$ for simplicity. First note that LMI (6.34) can be rewritten in the following form*

$$\begin{bmatrix} I & 0 & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & I & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & 0 & I \\ \bar{C} & -\alpha C_h & F \end{bmatrix}^T M(h,\dot{h}) \underbrace{\begin{bmatrix} I & 0 & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & I & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & 0 & I \\ \bar{C} & -\alpha C_h & F \end{bmatrix}}_{\mathscr{S}} \prec 0 \tag{6.38}$$

where $M(h,\dot{h}) = \begin{bmatrix} \dot{h}\dfrac{dP(h)}{dh} & P(h) \\ P(h) & 0 \end{bmatrix} \oplus \mho(h,\dot{h}) \oplus [-\gamma(h,\dot{h})I_p] \oplus [\gamma^{-1}(h,\dot{h})I_q]$. First inject the closed-loop system into (6.38). Note that $\dim(M) = 4n+p+q$ and $n^-(M) = 2n+p$

(where $n^-(M)$ is the number of negative eigenvalues of the symmetric matrix M, $n = \dim(x)$, $p = \dim(w)$ and $q = \dim(z)$) and the latter equals the rank of the subspace $\mathscr{S}$ (defined in (6.38)). Then it is possible to apply the dualization lemma and we obtain

$$\begin{bmatrix} -\bar{A}_{cl}^T(\hat{h}) & -\bar{C}^T(\hat{h}) & 0 \\ I_n & 0 & I_n \\ \hline \alpha A_h^T & \alpha C_h^T & 0 \\ 0 & 0 & -I_n \\ \hline -E^T & -F^T & 0 \\ 0 & I_q & 0 \end{bmatrix}^T M^{-1}(\xi) \underbrace{\begin{bmatrix} -\bar{A}_{cl}^T(\hat{h}) & -\bar{C}^T(\hat{h}) & 0 \\ I_n & 0 & I_n \\ \hline \alpha A_h^T & \alpha C_h^T & 0 \\ 0 & 0 & -I_n \\ \hline -E^T & -F^T & 0 \\ 0 & I_q & 0 \end{bmatrix}}_{S^+} \succ 0 \tag{6.39}$$

where $M^{-1}(\xi) = \begin{bmatrix} \frac{dP(\hat{h})}{dt} & P(\hat{h}) \\ \star & 0 \end{bmatrix}^{-1} \oplus \mho^{-1}(\xi) \oplus [-\gamma^{-1}(\xi)] \oplus [\gamma(\xi)]$.

Let $X = P^{-1}$ and then

$$\frac{dX(\hat{h})}{dt} = -X\frac{dP(\hat{h})}{dt}X$$

and thus

$$\begin{bmatrix} \dfrac{dP(\hat{h})}{dt} & P(\hat{h}) \\ \star & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & X(\hat{h}) \\ \star & \dfrac{dX(\hat{h})}{dt} \end{bmatrix}$$

Denote also $\mho^{-1}(\xi) = \begin{bmatrix} -\tilde{D}(\xi) & \tilde{G}^T(\xi) \\ \star & \tilde{D}(\xi) \end{bmatrix}$ *with $\tilde{D} \in \mathbb{S}^n_{++}$. Moreover $\mho^{-1}(\cdot)$ satisfies the inequality*

$$\begin{bmatrix} -I_n \\ \mathcal{D}_h^T(\cdot) \end{bmatrix}^T \begin{bmatrix} -\tilde{D}(\xi) & \tilde{G}^T(\xi) \\ \star & \tilde{D}(\xi) \end{bmatrix} \begin{bmatrix} -I_n \\ \mathcal{D}_h^T(\cdot) \end{bmatrix} \prec 0 \tag{6.40}$$

Then expand (6.39) and notice that $\tilde{R}(h, \dot{h}) < 0$, the Schur complement can be used on the quadratic term:

$$-\begin{bmatrix} \alpha A_h\tilde{D} & E \\ \alpha C_h\tilde{D} & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{D}^{-1} & 0 \\ 0 & \gamma^{-1}(\xi)I_p \end{bmatrix} \begin{bmatrix} \alpha A_h\tilde{D} & E \\ \alpha C_h\tilde{D} & F \\ 0 & 0 \end{bmatrix}^T \tag{6.41}$$

Finally multiplying the LMI by -1 (to get a negative definite inequality) we obtain inequality (6.37) in which $Y(\hat{h}) = K(\hat{h})X(\hat{h})$ is a linearizing change of variable. □

We have expressed the stability and stabilizability problems as polynomially parametrized LMIs (6.33) and (6.37). Moreover the $\mathcal{L}_2$-induced norm is expressed as a positive function of the parameters and its minimization is not a well-defined problem since we cannot minimize a function. We detail below how to turn this problem into a tractable one.

Since the cost to be minimized needs to be unique for every parameters, the idea is here, to provide an idea on how to turn the semi-infinite number of cost (defined for each value of the parameters) into a single one. This step is performed by an integration procedure with respect to some specific measures.

Let us illustrate this on the elementary cost $\gamma(\xi)$, $\xi = \mathrm{col}(h, \delta_h, \dot{h}, \dot{\delta}_h)$. It is possible to define several 'general' costs $\mathcal{J}(\cdot)$:

$$\mathcal{J}_\theta(\gamma) \quad := \quad \int_{H\times U\times \bar{H}\times \hat{U}} \gamma(\xi) d\theta(\xi) \tag{6.42}$$

where $d\theta(\xi)$ is a probability measure over $H \times U \times \bar{H} \times \hat{U}$ $\left(\text{i.e. } \int_{H\times U\times \bar{H}\times \hat{U}} d\theta\xi = 1\right)$.

We propose here some interesting values of the measure $d\theta(\cdot,\cdot)$:

- $d\theta_1(\xi) = \mu(H \times U \times \bar{H} \times \hat{U})^{-1}$ where $\mu(\cdot)$ is the Lebesgue measure.
- $d\theta_2(\xi) = \delta(\prod_{i=1}^{f}(\xi - \xi_i))$ with $\delta(t)$ is the Dirac distribution.
- $d\theta_3(\xi) = p(\xi)$ where $p(\cdot)$ denotes for instance a probability density function.

The first one minimizes the volume below the hypersurface defined by the application $\gamma : H \times U \times \bar{H} \times \hat{U} \to \mathbb{R}_+$ with equal preference for any parameter values. The second one aims to minimize the $\mathcal{H}_\infty$-norm, specifically for certain delay, errors and their derivative values. This may be interesting for systems with discrete valued delays. The third one is dedicated when we have a stochastic model of the delay (and eventually a model for its derivative) attempts for instance to minimize in priority the $\mathcal{H}_\infty$-norm for high probable delay values.

6.2 Dynamic Output Feedback Control laws

This section is devoted to the stabilization of time-delay systems by dynamic output-feedback. Two different laws control will be developed:

1. 'observer based control laws' which means that the controller is composed by an observer which estimate the system state and a state-feedback control law which generate the control input from the estimated state;
2. full-block controllers where all the matrices are sought such that the closed-loop system is asymptotically stable.

Each approach has its own benefits and drawbacks and rather than enumerating them just retain that:

1. The main difficulty in the synthesis of observer-based control laws is the fact that, first of all, it is not possible to exactly linearize the conditions by congruence transformations and change of variables due to a low number of degrees of freedom (two for observer based-control laws). However, the obtained controller is rather simple and then easy to implement.
2. In the full-block output-feedback control law framework, congruence transformations and change of variable are possible. Moreover, this approach leads to exact LMI conditions when dealing with output-feedback with memory (when the delay is exactly

known). Nevertheless, such a case almost never occurs since the delay is generally not exactly known except in some vary special cases (for instance when the delay represents a variable sampling period [Fridman et al., 2004; Suplin et al., 2007]). This is the reason why memoryless controller are often preferred but are more difficult to design due to the presence of bilinear terms (non-linearizable) in the resulting conditions.

In [Sename and Briat, 2006], the problem of finding a observer-based control law for LTI time-delay systems is derived through iterative LMI conditions. The result is provided in the delay-independent framework only. This section will consider delay-dependent results with both memoryless and with memory controllers. It is also possible to elaborate delay-scheduled dynamic output feedback control laws based on the approach detailed in Section 6.1.7 but tractable conditions can only be obtained using more simple scalings than D scalings. Otherwise, iterative LMI procedures would deal with such problems.

6.2.1 Memoryless observer based control laws

This section aims at developing sufficient conditions to the existence of a memoryless observer-based control law of the form

$$\begin{array}{rcl}\dot{\xi}(t) & = & A(\rho)\xi(t) + B(\rho)u(t) + L(\rho)(y(t) - C_y(\rho)\xi(t)) \\ u(t) & = & -K(\rho)\xi(t)\end{array} \tag{6.43}$$

for LPV time-delay systems

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + D(\rho)u(t) + F(\rho)w(t) \\ y(t) & = & C_y(\rho)x(t) + F_y(\rho)w(t)\end{array} \tag{6.44}$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $w \in \mathbb{R}^q$ and $z \in \mathbb{R}^r$ are respectively the system state, the controller state, the control input, the measured output, the exogenous inputs and the controlled outputs. The delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$ and the parameters $\rho \in U_\nu$ with $\dot{\rho} \in \text{hull}[U_\nu]$. In such a case, the closed-loop system can be expressed by the equations

$$\begin{array}{rcl}\begin{bmatrix}\dot{e}(t) \\ \dot{x}(t)\end{bmatrix} & = & \begin{bmatrix} A(\rho) - L(\rho)C_y(\rho) & 0 \\ B(\rho)K(\rho) & A(\rho) - B(\rho)K(\rho)\end{bmatrix}\begin{bmatrix} e(t) \\ x(t)\end{bmatrix} \\ & & + \begin{bmatrix} 0 & A_h(\rho) \\ 0 & A_h(\rho)\end{bmatrix}\begin{bmatrix} e(t-h(t)) \\ x(t-h(t))\end{bmatrix} + \begin{bmatrix} E(\rho) - L(\rho)F_y(\rho) \\ E(\rho)\end{bmatrix} w(t)\end{array}$$

and we define an extended output vector $\tilde{z}(t) = \begin{bmatrix} Te(t) \\ z(t)\end{bmatrix}$ with T full row rank. The role of the matrix T is to weight the observation error in order to reduce the impact of the disturbances on the observation error.

Before introducing the main result of the section, known methodologies will be briefly introduced here. A common methodology is to assume that the Lyapunov matrix multiplied with system matrices are block-diagonal and each block corresponds to a specific part of the augmented system (i.e. the observation error and the system-state).

Remark 6.2.1 *It is worth mentioning that since the design matrices $K(\rho)$ and $L(\rho)$ are not multiplied in the same fashion with system matrices ($L(\rho)$ is free from the left while $K(\rho)$ is free from the right) then this suggests that congruence transformations would lead to nonlinear terms without possibility of linearization. Hence, in order to overcome this problem, a commutation approach has been introduced [Chen, 2007; Daafouz et al., 2002] where a block of Lyapunov matrix is constrained such that it commutes with a system matrix. For instance, the Lyapunov matrix X is constrained such that it commutes with the system measurement matrix C_y, i.e. $C_yX = \hat{X}C_y$ where $\text{rank}[C_y] = p$. In this case, the change of variable $\hat{L} = L\hat{X}$ is allowed but this considerably increases the conservatism of the approach. However, in the $\mathcal{H}_\infty$ observer-based control problem, this is more difficult since the observer gain appears in different places (e.g. $A - LC_y$ and $E - LF_y$) and hence this approach fails in the studied problem.*

In the presented method no congruence transformations are applied but we use a simple approach based on linear bounds on nonlinear terms. Using this approach we get the following result:

Theorem 6.2.2 *There exists an observer-based control law of the form (6.43) which asymptotically stabilizes system (6.44) for all $h \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix $P : U_\rho \to \mathbb{S}_{++}^n$, matrix functions $X_0, X_c : U_\rho \to \mathbb{R}^{n\times n}$, $K : U_\rho \to \mathbb{R}^{m\times n}$, $L_o : U_\rho \to \mathbb{R}^{n\times p}$, constant matrices $Q, R \in \mathbb{S}_{++}^n$, scalar functions $\alpha_1, \alpha_2 : U_\rho \to \mathbb{R}_{++}$ and a constant scalar $\gamma > 0$ such that the following LMI*

$$\left[\begin{array}{cccc|c} -(X+X^T) & \Omega_2(\rho)^T & \Omega_3(\rho)^T & \Omega_5(\rho)^T & \\ \star & \Omega_4(\rho,\dot{\rho}) & \Omega_6(\rho)^T & 0 & \Omega_c^T \\ \star & \star & \Omega_8(\rho) & 0 & \\ \star & \star & \star & \Omega_{10}(\rho) & \\ \hline \star & \star & \star & \star & -\Omega_d \end{array}\right] \prec 0$$

holds for all $(\rho,\nu)\in U_\rho\times U_\nu$ *where*

$$\begin{aligned}
\Omega_2(\rho) &= \begin{bmatrix} P(\rho)+\begin{bmatrix} A(\rho)^TX_o(\rho)-C_y(\rho)^TL_o(\rho)^T & 0\\ 0 & A(\rho)^TX_c\end{bmatrix}\\ \begin{bmatrix} 0 & 0\\ A_h(\rho)^TX_o(\rho) & A_h(\rho)^TX_c(\rho)\end{bmatrix}\end{bmatrix}\\
\Omega_3(\rho) &= \begin{bmatrix} E(\rho)^TX_o(\rho)-F_y(\rho)^TL_o(\rho)^T\\ E(\rho)^TX_c(\rho)\\ 0\\ 0\end{bmatrix}\\
\Omega_5(\rho) &= \begin{bmatrix} X\\ h_{max}R\end{bmatrix}\\
\Omega_6(\rho) &= \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ T & 0 & 0 & 0\\ D(\rho)K(\rho) & C(\rho)-D(\rho)K(\rho) & 0 & C_h(\rho)\end{bmatrix}\\
\Omega_8(\rho) &= \begin{bmatrix} -\gamma(\rho)I_w & F(\rho)^T\\ \star & -\gamma(\rho)I_z\end{bmatrix}\\
\Omega_{10}(\rho) &= \begin{bmatrix} -P(\rho) & -h_{max}R\\ \star & -R\end{bmatrix}\\
\Omega_c(\rho) &= \left[\begin{array}{cccc} 0 & 0 & 0 & 0\\ \alpha_1(\rho)X_c(\rho)^TB(\rho) & \alpha_1(\rho)X_c(\rho)^TB(\rho) & 0 & 0\\ \hline 0 & 0 & K(\rho)^T & 0\\ 0 & 0 & 0 & K(\rho)^T\\ \hline 0 & 0 & & \\ 0 & 0 & & \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array}\right]\\
\Omega_d(\rho) &= \begin{bmatrix} \alpha_1(\rho)I & 0 & 0 & 0\\ \star & \alpha_2(\rho)I & 0 & 0\\ \star & \star & \alpha_1(\rho)I & 0\\ \star & \star & \star & \alpha_2(\rho)I\end{bmatrix}
\end{aligned}$$

Moreover the observer gain $L(\rho)=X_o(\rho)^{-T}L_o(\rho)$ *and the closed-loop satisfies* $||\tilde{z}||_{\mathcal{L}_2}\leq\gamma||w||_{\mathcal{L}_2}$.

Proof: *First of all we assume that the matrix* X *is structured as follows:*

$$X=diag(X_o,X_c)$$

Since we are interested in a simple stabilization test, we will consider the Lyapunov-Krasovskii functional of Section 4.5.1 whose relaxation is provided in Section 4.5.2. After substitution of the extended system in the LMI of Lemma 4.5.2 we get

$$\Xi=\begin{bmatrix} -(X(\rho)+X(\rho)^T) & \Xi_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T\\ \Xi_2(\rho) & \Xi_4(\rho,\dot{\rho}) & \Xi_6(\rho)^T & 0\\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0\\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho)\end{bmatrix}$$

$$\Xi_2(\rho) = \left[\begin{array}{c} P(\rho) + \left[\begin{array}{cc} A(\rho)^T X_o(\rho) - C_y(\rho)^T L_o(\rho)^T & -K(\rho)^T B(\rho)^T X_c(\rho) \\ 0 & A(\rho)^T X_c - K(\rho)^T B(\rho)^T X_c(\rho) \end{array} \right] \\ \left[\begin{array}{cc} 0 & 0 \\ A_h(\rho)^T X_o(\rho) & A_h(\rho)^T X_c(\rho) \end{array} \right] \end{array} \right]$$

$$\Xi_3(\rho) = \left[\begin{array}{c} E(\rho)^T X_o(\rho) - F_y(\rho)^T L_o(\rho)^T \\ E(\rho)^T X_c(\rho) \\ 0 \\ 0 \end{array} \right]$$

$$\Xi_5(\rho) = \left[\begin{array}{c} X \\ h_{max} R \end{array} \right]$$

$$\Xi_6(\rho) = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ T & 0 & 0 & 0 \\ D(\rho)K(\rho) & C(\rho) - D(\rho)K(\rho) & 0 & C_h(\rho) \end{array} \right]$$

$$\Xi_8(\rho) = \left[\begin{array}{cc} -\gamma(\rho) I_w & F(\rho)^T \\ \star & -\gamma(\rho) I_z \end{array} \right]$$

$$\Xi_{10}(\rho) = \left[\begin{array}{cc} -P(\rho) & -h_{max} R \\ \star & -R \end{array} \right]$$

$$\Xi_4(\rho, \dot\rho) = \left[\begin{array}{cc} \dfrac{\partial P(\rho)}{\partial \rho}\dot\rho - P(\rho) + Q - R & R \\ \star & -(1-\mu)Q - R \end{array} \right]$$

The main difficulty comes from the bilinear term $X_c(\rho)^T B(\rho) K(\rho)$. It is worth mentioning that in this case it is not possible to find a linearizing congruence transformation. However, it is possible to use the well-known bound on cross-terms heavily used in time-delay systems (see Appendix E.2):

$$\begin{array}{rcl} -2x_3^T X_c(\rho)^T B(\rho) K(\rho) x_2 & \leq & \alpha_1(\rho) x_3^T X_c(\rho)^T B(\rho) B(\rho)^T X_c(\rho) x_3 + \alpha_1(\rho)^{-1} x_2^T K(\rho)^T K(\rho) x_2 \\ -2x_4^T X_c(\rho)^T B(\rho) K(\rho) x_2 & \leq & \alpha_2(\rho) x_4^T X_c(\rho)^T B(\rho) B(\rho)^T X_c(\rho) x_4 + \alpha_2(\rho)^{-1} x_2^T K(\rho)^T K(\rho) x_2 \end{array}$$

for any real valued vectors x_2, x_3, x_4 of appropriate dimensions and real valued positive scalar functions $\alpha_1(\cdot), \alpha_2(\cdot)$. Using these inequalities it is possible to show that the following inequality implies $\Xi \prec 0$:

$$\Upsilon = \left[\begin{array}{cccc} -(X + X^T) + Y_1 & \Upsilon_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T \\ \Upsilon_2(\rho) & \Upsilon_4(\rho, \dot\rho) & \Xi_6(\rho)^T & 0 \\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0 \\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho) \end{array} \right] \prec 0$$

with

$$Y_1 = \left[\begin{array}{cc} 0 & 0 \\ 0 & (\alpha_2(\rho) + \alpha_1(\rho))[X_c(\rho)^T B(\rho) B(\rho)^T X_c(\rho)] \end{array} \right]$$

$$\Upsilon_2(\rho) = \begin{bmatrix} P(\rho) + \begin{bmatrix} A(\rho)^T X_o(\rho) - C_y(\rho)^T L_o(\rho)^T & 0 \\ 0 & A(\rho)^T X_c \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ A_h(\rho)^T X_o(\rho) & A_h(\rho)^T X_c(\rho) \end{bmatrix} \end{bmatrix}$$

$$\begin{aligned} \Upsilon_4(\rho, \dot{\rho}) &= \Xi_4(\rho, \dot{\rho}) + Y_2 \\ Y_2 &= \begin{bmatrix} \alpha_1(\rho)^{-1} K(\rho)^T K(\rho) & 0 \\ 0 & \alpha_2(\rho)^{-1} K(\rho)^T K(\rho) \end{bmatrix} \end{aligned}$$

Finally since

$$\begin{aligned} Y_1 &= \begin{bmatrix} 0 & 0 \\ \alpha_1(\rho) X_c(\rho)^T B(\rho) & \alpha_2(\rho) X_c(\rho)^T B(\rho) \end{bmatrix} \begin{bmatrix} \alpha_1(\rho)^{-1} I & 0 \\ 0 & \alpha_2(\rho)^{-1} I \end{bmatrix} (\star)^T \\ Y_2 &= \begin{bmatrix} K(\rho)^T & 0 \\ 0 & K(\rho)^T \end{bmatrix} \begin{bmatrix} \alpha_1(\rho)^{-1} I & 0 \\ 0 & \alpha_2(\rho)^{-1} I \end{bmatrix} (\star)^T \end{aligned}$$

where $(\star)^T$ *stands for the symmetric part of the quadratic term, then* Υ *may be rewritten into the form*

$$\begin{aligned} \Upsilon &= \begin{bmatrix} -(X + X^T) & \Upsilon_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T \\ \Upsilon_2(\rho) & \Xi_4(\rho, \dot{\rho}) & \Xi_6(\rho)^T & 0 \\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0 \\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho) \end{bmatrix} \\ &+ \underbrace{\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \alpha_1(\rho) X_c(\rho)^T B(\rho) & \alpha_1(\rho) X_c(\rho)^T B(\rho) & 0 & 0 \\ \hline 0 & 0 & K(\rho)^T & 0 \\ 0 & 0 & 0 & K(\rho)^T \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\Upsilon_c(\rho)} \Upsilon_d(\rho)^{-1} (\star)^T \prec 0 \end{aligned}$$

with $\Upsilon_d(\rho)^{-1} = \begin{bmatrix} \alpha_1(\rho)^{-1} I & 0 & 0 & 0 \\ \star & \alpha_2(\rho)^{-1} I & 0 & 0 \\ \star & \star & \alpha_1(\rho)^{-1} I & 0 \\ \star & \star & \star & \alpha_2(\rho)^{-1} I \end{bmatrix}$. *And finally applying Schur complement we get*

$$\left[\begin{array}{cccc|c} -(X + X^T) & \Upsilon_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T & \\ \Upsilon_2(\rho) & \Xi_4(\rho, \dot{\rho}) & \Xi_6(\rho)^T & 0 & \Upsilon_c^T \\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0 & \\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho) & \\ \hline & & \Upsilon_c & & \Upsilon_d \end{array}\right] \prec 0$$

which is linear in $X_o, X_c, L_o, K, P, Q, R$ $\square$

Remark 6.2.3 *The procedure is similar as for the design of observer-based control law with memory:*

$$\begin{array}{rcl}\dot{\xi}(t) & = & A(\rho)\xi(t) + A_h(\rho)\xi(t-h(t)) + B(\rho)u(t) \\ & & s + L(\rho)(y(t) - C_y(\rho)\xi(t) - C_{yh}(\rho)\xi(t-h(t))) \\ u(t) & = & -K(\rho)\xi(t) - K_h(\rho)\xi(t-h(t))\end{array} \tag{6.45}$$

Indeed, in this case, the extended system would be

$$\begin{array}{rcl}\begin{bmatrix} \dot{e}(t) \\ \dot{x}(t) \end{bmatrix} & = & \begin{bmatrix} A(\rho) - L(\rho)C_y(\rho) & 0 \\ B(\rho)K(\rho) & A(\rho) - B(\rho)K(\rho) \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \\ & & + \begin{bmatrix} A_h(\rho) - L(\rho)C_{yh}(\rho) & 0 \\ B(\rho)K_h(\rho) & A_h(\rho) - B(\rho)K_h(\rho) \end{bmatrix} \begin{bmatrix} e(t-h(t)) \\ x(t-h(t)) \end{bmatrix} + \begin{bmatrix} E - LF_y \\ E \end{bmatrix} w(t)\end{array}$$

6.2.2 Dynamic Output Feedback with memory design - exact delay case

This section is devoted to the design of a dynamic output feedback controller with memory. The delay is assumed to be exactly known. The advantage of such controllers resides in the existence of congruence transformations and linearizing change of variables. However, they are difficult to implement in practice due to the imprecision on the delay value knowledge. Section 3.2.2 presents methods allowing to deal a posteriori on delay uncertainty that can be used in order to give a bound on the maximal error on the delay value knowledge that can be tolerated.

The class of systems under consideration is given by:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + D(\rho)u(t) + F(\rho)w(t) \\ y(t) & = & C_y(\rho)x(t) + C_{yh}(\rho)x(t-h(t)) + F_y(\rho)w(t)\end{array} \tag{6.46}$$

for which the following stabilizing controllers have to be designed

$$\begin{array}{rcl}\dot{x}_c(t) & = & A_c(\rho)x_c(t) + A_{hc}(\rho)x_c(t-h(t)) + B_c(\rho)y(t) \\ u(t) & = & C_c(\rho)x_c(t) + C_{hc}(\rho)x_c(t-h(t)) + D_c(\rho)y(t)\end{array} \tag{6.47}$$

where $x \in \mathbb{R}^n$, $x_c \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $w \in \mathbb{R}^q$ and $z \in \mathbb{R}^r$ are respectively the system state, the controller state, the control input, the measured output, the exogenous inputs and the controlled outputs. The delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$ and the parameters $\rho \in U_\nu$ with $\dot{\rho} \in \text{hull}[U_\nu]$. The closed-loop system is given by

$$\begin{array}{rcl}\dot{\bar{x}}(t) & = & \underbrace{\begin{bmatrix} A + BD_cC_y & BC_c \\ B_cC_y & A_c \end{bmatrix}}_{A_{cl}} \bar{x}(t) + \underbrace{\begin{bmatrix} A_h + BD_cC_{yh} & BC_{hc} \\ B_cC_{yh} & A_{hc} \end{bmatrix}}_{A_{hcl}} \bar{x}(t-h(t)) \\ & & + \underbrace{\begin{bmatrix} E + BD_cF_y \\ B_cF_y \end{bmatrix}}_{E_{cl}} w(t) \\ z(t) & = & \underbrace{\begin{bmatrix} C + DD_cC_y & DC_c \end{bmatrix}}_{C_{cl}} \bar{x}(t) + \underbrace{\begin{bmatrix} C_h + DD_cC_{yh} & DC_{hc} \end{bmatrix}}_{C_{hcl}} \bar{x}(t-h(t)) \\ & & + \underbrace{(F + DD_cF_y)}_{F_{cl}} w(t)\end{array}$$

with $\bar{x}(t) = col(x(t), x_c(t))$ and where the dependence on the parameters has been dropped in order to improve the clarity.

The methodology to develop the main theorem is a bit different than for the other methods and is inspired from [Scherer and Weiland, 2005; Scherer et al., 1997]. The method is based on a LMI relaxation of a Lyapunov-Krasovskii based approach. After substitution of the closed-loop system, a congruence transformation and a linearization change of variable are performed.

Theorem 6.2.4 *There exists a dynamic output feedback of the form (6.47) for system (6.46) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}^{2n}_{++}$, constant matrices $W_1, X_1 \in \mathbb{S}^n_{++}$, $\tilde{Q}, \tilde{R} \in \mathbb{S}^{2n}_{++}$, a scalar function $\alpha : U_\rho \to \mathbb{R}_{++}$ and a scalar $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} -2\tilde{X} & P(\rho)+\mathcal{A}(\rho) & \mathcal{A}_h(\rho) & \mathcal{E}(\rho) & 0 & \tilde{X} & h_{max}\tilde{R} \\ \star & U_{22}(\rho,\nu) & \tilde{R} & 0 & \mathcal{C}(\rho)^T & 0 & 0 \\ \star & \star & U_{33} & 0 & \mathcal{C}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \\ & & & & & & \end{bmatrix} \prec 0$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ with $U_{22}(\rho,\dot{\rho}) = U_{22}(\rho,\nu) - \tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho \tilde{P}(\rho)\nu$, $U_{33} = -(1-\mu)\tilde{Q} - \tilde{R}$ and

$$\begin{aligned}
\tilde{X} &= \begin{bmatrix} W_1 & I \\ I & X_1 \end{bmatrix} \\
\mathcal{A}(\rho) &= \begin{bmatrix} A(\rho)W_1 + B(\rho)\mathcal{C}_c(\rho) & A(\rho) + B(\rho)\mathcal{D}_c(\rho)C_y(\rho) \\ \mathcal{A}_c(\rho) & X_1A(\rho) + \mathcal{B}_c(\rho)C_y(\rho) \end{bmatrix} \\
\mathcal{A}_h(\rho) &= \begin{bmatrix} A_h(\rho)W_1 + B(\rho)\mathcal{C}_c(\rho) & A(\rho) + B(\rho)\mathcal{D}_c(\rho)C_{yh}(\rho) \\ \mathcal{A}_{hc}(\rho) & X_1A_h(\rho) + \mathcal{B}_c(\rho)C_{yh}(\rho) \end{bmatrix} \\
\mathcal{E}(\rho) &= \begin{bmatrix} E(\rho) + B(\rho)\mathcal{D}_c(\rho)F_y(\rho) \\ X_1E(\rho) + \mathcal{B}_c(\rho)F_y(\rho) \end{bmatrix} \\
\mathcal{C}(\rho) &= \begin{bmatrix} C_y(\rho)W_1 + D(\rho)\mathcal{C}_c(\rho) & C_y(\rho) + D(\rho)\mathcal{D}_c(\rho)C_y(\rho) \end{bmatrix} \\
\mathcal{C}_h(\rho) &= \begin{bmatrix} C_h(\rho)W_1 + D(\rho)\mathcal{C}_{yh}(\rho) & C_h(\rho) + D(\rho)\mathcal{D}_c(\rho)C_{yh}(\rho) \end{bmatrix} \\
\mathcal{F}(\rho) &= \begin{bmatrix} F(\rho) + D(\rho)\mathcal{D}_c(\rho)F_y(\rho) \end{bmatrix}
\end{aligned}$$

In this case the corresponding controller is given by

$$\begin{aligned}
\begin{bmatrix} A_c(\rho) & A_{hc}(\rho) & B_c(\rho) \\ C_c(\rho) & C_{hc}(\rho) & D_c(\rho) \end{bmatrix} &= \mathcal{M}_1(\rho)^{-1}\left(\begin{bmatrix} \mathcal{A}_c(\rho) & \mathcal{A}_{hc}(\rho) & \mathcal{B}_c(\rho) \\ \mathcal{C}_c(\rho) & \mathcal{C}_{hc}(\rho) & \mathcal{D}_c(\rho) \end{bmatrix} - \mathcal{M}_2(\rho)\right)\mathcal{M}_3(\rho)^{-1} \\
\mathcal{M}_1(\rho) &= \begin{bmatrix} X_2 & X_1B(\rho) \\ 0 & I \end{bmatrix} \\
\mathcal{M}_2(\rho) &= \begin{bmatrix} X_1A(\rho)W_1 & X_1A_h(\rho)W_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathcal{M}_3(\rho) &= \begin{bmatrix} W_2^T & 0 & 0 \\ 0 & W_2^T & 0 \\ C_y(\rho)W_1 & C_{yh}(\rho)W_1 & I \end{bmatrix} \\
X^{-1} &= \begin{bmatrix} X_1 & X_2 \\ \star & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} W_1 & W_2 \\ \star & W_3 \end{bmatrix}
\end{aligned}$$

and the closed-loop system satisfies $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Proof: *First of all we rewrite the closed-loop system under the form*

$$\begin{aligned}
\left[\begin{array}{c|c|c} A_{cl} & A_{hcl} & E_{cl} \\ \hline C_{cl} & C_{hcl} & F_{cl} \end{array}\right] &= \Theta + \left[\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & D \end{array}\right]\Omega\left[\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ C_y & 0 & C_{yh} & 0 & F_y \end{array}\right] \\
\Theta &= \left[\begin{array}{cc|cc|c} A & 0 & A_h & 0 & E \\ 0 & 0 & 0 & 0 & 0 \\ \hline C & 0 & C_h & 0 & F \end{array}\right] \\
\Omega &= \begin{bmatrix} A_c & A_{hc} & B_c \\ C_c & C_{hc} & D_c \end{bmatrix}
\end{aligned}$$

For simplicity we restrict X to be a symmetric positive definite matrix such that

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \qquad W := X^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix}$$

By injecting the closed-loop system in LMI (4.35) of Theorem 4.5.5 we get

$$\begin{bmatrix}
-2X & P(\rho)+X^TA_{cl}(\rho) & X^TA_{hcl}(\rho) & X^TE_{cl}(\rho) & 0 & X & h_{max}R \\
\star & U_{22}(\rho,\dot{\rho}) & R & 0 & C_{cl}(\rho)^T & 0 & 0 \\
\star & \star & U_{33} & 0 & C_{hcl}(\rho)^T & 0 & 0 \\
\star & \star & \star & -\gamma I & F_cl(\rho)^T & 0 & 0 \\
\star & \star & \star & \star & -\gamma I & 0 & 0 \\
\star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\
\star & \star & \star & \star & \star & \star & -R
\end{bmatrix} \prec 0$$

with $U_{22}(\rho,\dot{\rho}) = -P(\rho)+Q-R+\partial_\rho P(\rho)\nu$, $U_{33} = -(1-\mu)Q-R$. *To linearize this inequality, a congruence transformation is performed with respect to the matrix* $\text{diag}(Z^T, Z^T, Z^T, I, I, Z^T, Z^T)$ *where*

$$Z := \begin{bmatrix} W_1 & I \\ W_2^T & 0 \end{bmatrix}$$

Then we get

$$\begin{bmatrix} -2Z^TXZ & V_{12}(\rho) & V_{13}(\rho) & Z^TX^TE_{cl}(\rho) & 0 & Z^TXZ & h_{max}\tilde{R} \\ \star & V_{22}(\rho,\nu) & \tilde{R} & 0 & Z^TC_{cl}(\rho)^T & 0 & 0 \\ \star & \star & V_{33} & 0 & Z^TC_{hcl}(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F_cl(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} \prec 0$$

with $V_{33} = -[(1-\mu)\tilde{Q} + \tilde{R}]$, $\tilde{P}(\rho) = Z^TP(\rho)Z$, $\tilde{Q} = Z^TQZ$, $\tilde{R} = Z^TRZ$ *and*

$$\begin{aligned} V_{12}(\rho) &= \tilde{P}(\rho) + Z^TX^TA_{cl}(\rho)Z \\ V_{13}(\rho) &= V_{13}(\rho)Z^TX^TA_{hcl}(\rho)Z \\ V_{22}(\rho,\nu) &= V_{22}(\rho,\nu) - \tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho\tilde{P}(\rho)\nu)Z \end{aligned}$$

Note that

$$Z^TX = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} \qquad Z^TXZ = \begin{bmatrix} W_1 & I \\ I & X_1 \end{bmatrix}$$

and then defining

$$\mathcal{Z} = \left[\begin{array}{c|c|c} Z^TXA_{cl}Z & Z^TXA_{hcl}Z & Z^TXE_{cl} \\ \hline C_{cl}Z & C_{hcl}Z & F_{cl} \end{array}\right]$$

we get

$$\begin{aligned} \mathcal{Z} &= \left[\begin{array}{cc|cc|c} AW_1 & A & A_hW_1 & A & E \\ 0 & X_1A & 0 & X_1A_h & X_1E \\ \hline C_yW_1 & C_y & C_hW_1 & C_h & F \end{array}\right] + \Theta_1 \begin{bmatrix} \mathcal{A}_c & \mathcal{A}_{hc} & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{C}_{hc} & \mathcal{D}_c \end{bmatrix} \\ \Theta_1 &= \left[\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & D \end{array}\right] \qquad \Theta_2 = \left[\begin{array}{cc|cc|c} I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & C_y & 0 & C_{yh} & F_y \end{array}\right] \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \mathcal{A}_c & \mathcal{A}_{hc} & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{C}_{hc} & \mathcal{D}_c \end{bmatrix} &= \begin{bmatrix} X_1AW_1 & X_1AhW_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Omega_1 \begin{bmatrix} A_c & A_{hc} & B_c \\ C_c & C_{hc} & D_c \end{bmatrix} \\ \Omega_1 &= \begin{bmatrix} X_2 & X_1B \\ 0 & I \end{bmatrix} \\ \Omega_2 &= \begin{bmatrix} W_2^T & 0 & 0 \\ 0 & W_2^T & 0 \\ C_yW_1 & C_{yh}W_1 & I \end{bmatrix} \end{aligned}$$

Finally we get

$$\mathcal{Z} = \left[\begin{array}{cc|cc|c} AW_1 + B\mathcal{C}_c & A + B\mathcal{D}_cC_y & A_hW_1 + B\mathcal{C}_c & A + B\mathcal{D}_cC_{yh} & E + B\mathcal{D}_cF_y \\ \mathcal{A}_c & X_1A + \mathcal{B}_cC_y & \mathcal{A}_{hc} & X_1A_h + \mathcal{B}_cC_{yh} & X_1E + \mathcal{B}_cF_y \\ \hline C_yW_1 + D\mathcal{C}_c & C_y + D\mathcal{D}_cC_y & C_hW_1 + D\mathcal{C}_{yh} & C_h + D\mathcal{D}_cC_{yh} & F + D\mathcal{D}_cF_y \end{array}\right]$$

which shows that the equations are linearized with respect to the new variables ($\mathcal{A}_c$, $\mathcal{A}_{hc}$, $\mathcal{B}_c$, $\mathcal{C}_c$, $\mathcal{C}_{ch}$, $\mathcal{D}_c$). *Finally replacing the linearized values into the inequality leads to the result. The construction of the controller is performed by the inversion of the change of variable.* □

Remark 6.2.5 *The design of a memoryless controller of the form*

$$\begin{array}{rcl} x_c(t) & = & A_c x_c(t) + B_c y(t) \\ u(t) & = & C_c x_c(t) + D_c y(t) \end{array} \tag{6.48}$$

is more involved since the matrix $\mathcal{Z}$ is in this case defined by

$$\begin{array}{rcl} \mathcal{Z} & = & \left[\begin{array}{cc|cc|c} AW_1 & A & A_hW_1 & A_h & E \\ 0 & X_1A & X_1A_hW_1 & X_1A_h & X_1E \\ \hline CW_1 & C & C_hW_1 & C_h & F \end{array}\right] \\ & & + \left[\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & D \end{array}\right] \left[\begin{array}{cc} \mathcal{A}_c & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{D}_c \end{array}\right] \left[\begin{array}{cc|cc|c} I & 0 & 0 & 0 & 0 \\ 0 & C_y & 0 & 0 & D_y \end{array}\right] \end{array} \tag{6.49}$$

is nonlinear due to the term $X_1A_hW_1$. The change of variable is given by

$$\left[\begin{array}{cc} \mathcal{A}_c & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{D}_c \end{array}\right] = \left[\begin{array}{cc} X_1AW_1 & 0 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} X_2 & X_1B \\ 0 & I \end{array}\right] \left[\begin{array}{cc} A_c & B_c \\ C_c & D_c \end{array}\right] \left[\begin{array}{cc} W_2^T & 0 \\ C_yW_1 & I \end{array}\right] \tag{6.50}$$

Finally we would have

$$\mathcal{Z} = \left[\begin{array}{cc|cc|c} AW_1 + B\mathcal{C}_c & A + B\mathcal{D}_cC_y & A_hW_1 & A & E + B\mathcal{D}_cF_y \\ \mathcal{A}_c & X_1A + \mathcal{B}_cC_y & X_1A_hW_1 & X_1A_h & X_1E + \mathcal{B}_cF_y \\ \hline C_yW_1 + D\mathcal{C}_c & C_y + D\mathcal{D}_cC_y & C_hW_1 & C_h & F + D\mathcal{D}_cF_y \end{array}\right] \tag{6.51}$$

and the problem would be nonconvex. However it can be relaxed using the same bounding technique as for the observer based control law:

$$2x^TX_1A_hW_1y \leq x^TXXx + y^TW_1A_h^TA_hW_1y \tag{6.52}$$

6.3 Chapter Conclusion

We have developed in this chapter several control laws to stabilize LPV time-delay systems with $\mathcal{L}_2$ performances optimization. Both state-feedback and dynamic output feedback control laws have been developed in both memoryless and with-memory structures. We have emphasized the interest of the relaxations of LMI with multiple coupling in the synthesis problem in terms of computational complexity and conservativeness. Although the bilinear approach gives better results it is difficult to extend it to the case of discretized Lyapunov-Krasovskii functional due to the high number of products between system data matrices and decision variables: for a discretization of order N it would result in the introduction of N 'slack' variables and hence $2N$ bilinearities which complexifies the initialization of the iterative LMI algorithm. A new type of controllers has been introduced, the 'delay-scheduled' state-feedback controllers whose gain is smoothly scheduled by the delay value, in a similar way as for gain-scheduling strategies used in the LPV control framework.

It is important to note that the design of such control laws is still an open problem in the framework of time-delay systems.

Conclusion and Future Works

Summary and Main Contributions

This thesis has considered the control and observation of LPV time-delay systems using a part of the arsenal of modern control tools. Even if the problem remains open for several complex cases, the results presented in this thesis has brought several results in this domain. The work has been presented in five chapters.

- In the first chapter, a state of the art on LPV systems is presented in which different types of representation coupled with their specific stability tests have been introduced.
- The second chapter, a (non-exhaustive) state of the art of time-delay systems is addressed with a particular focusing on time-domain methods, especially Lyapunov-Krasovskii functionals, small-gain, well-posedness and IQC based methods.
- The third chapter gathers parts of the theoretical contributions of this work. Two methods of relaxations for parameter dependent matrix inequalities and for matrix inequalities with particular concave nonlinearities are presented. Known Lyapunov-Krasovskii functionals are generalized to the LPV case and relaxed using a specific approach in order to get 'easy-to-use' condition in the synthesis framework. Finally, a new Lyapunov-Krasovskii functional has been expressed in order to consider the special case of systems with two delays, in which the delays satisfy an equality, arising in the problem of stabilization of a time-delay system with a controller implementing a different delay.
- The fourth chapter uses results of chapter three in order to construct observers and filters which have been shown to lead with interesting results.
- The fifth and last chapter used results of chapter 3 in order to derive different control laws: memoryless /with memory state-feedback/dynamic output feedback controllers. Moreover, a new design technique based on a LPV representation of time-delay systems has been applied to construct a new type of controller called 'delay-scheduled' controller where the controller gain depend on the delay. Using this technique, the robustness analysis with respect to delay knowledge uncertainty can be performed easily since the delay is not viewed anymore as an operator but as a scheduling parameter.

Future Works

As a perspective of the results developed in this thesis we can mention:

- The provided results only considers systems with are stable/stabilizable/detectable for zero delay (i.e. $A + A_h$ Hurwitz) and hence they may be conservative while considering systems which are not stable/stabilizable/detectable for zero delay but only from $h_{min} \neq 0$. Hence, it seems interesting and important to consider delay-range stability [He et al., 2007; Jiang and Han, 2005; Knospe and Roozbehani, 2006, 2003; Roozbehani and Knospe, 2005]. Note also that only few results exists on discretized Lyapunov-Krasovskii functionals for such systems.

- Two types of controllers have been developed in this thesis: state-feedback and dynamic output feedback control laws. It seems important to extend these results to the static-output feedback case [Li et al., 1998; Michiels et al., 2004; Peaucelle and Arzelier, 2005; Sename and Lafay, 1993; Seuret et al., 2009a; Syrmos et al., 1995]. It is worth mentioning that despite of its simplicity, the static output feedback case is difficult to develop to the NP-hardness of its necessary and sufficient existence condition [Fu, 2004]. The method proposed in [Prempain and Postlethwaite, 2005] deserves attention and shall be generalized to time-delay systems and LPV systems. Moreover, delayed static-output feedback control is able to stabilize systems which are not stabilizable by instantaneous static-output feedback as noticed in [Niculescu and Abdallah, 2000]. In such a control law, the delay is an extra degree of freedom.

- Since many control systems have bounded inputs, it may be interesting to develop control laws in presence of saturations on the inputs [da Silva and Tarbouriech, 2005; Ferreres and Biannic, 2007; Henrion and Tarbouriech, 1999; Henrion et al., 2005; Wu and Lu, 2004; Wu and Soto, 2004].

- The generalization of such approach to :
  - input/output delayed systems, distributed and neutral delay systems
  - systems with delayed parameters
  - systems with parameter dependent delays

- The extension of the current work to more complex parameter-dependent Lyapunov-Krasovskii functional, e.g.

$$\begin{array}{rcl} V(x_t, \dot{x}_t) & = & V_1(x,\rho) + V_2(x_t,\rho_t) \\ V_1(x,\rho) & = & x(t)^T P(\rho) x(t) \\ V_2(x_t,\rho_t) & = & \displaystyle\int_{t-h(t)}^{t} x(\theta)^T Q(\rho(\theta)) x(\theta) d\theta \end{array}$$

  in order to reduce the conservatism of the approach.

- The application of such control strategies on physical systems, currently, the stabilization of unstable modes in fusion plasmas [Olofsson et al., 2008].

Chapter 7

Appendix

A Technical Results in Linear Algebra

This appendix is devoted to the introduction of some fundamentals on matrix algebra. It is supposed that matrix multiplication and inversion are known. Determinants of block matrices, notion of eigenvalues and eigenvectors, inverse of block matrices, notion of order in the set of symmetric matrices, singular value decomposition, Moore-Penrose pseudo-inverse and the resolution of specific matrix equalities and inequalities will be considered.

A.1 Determinant Formulae

We give here several important relations concerning the determinant. For a square matrix $A \in \mathbb{C}^{n\times n}$, its determinant is denoted $\det(A)$. If A and B are both square matrices of same dimensions, then it can be shown that

$$\det(AB) = \det(A)\det(B) = \det(BA)$$

Another well-known fact is

$$\det\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right) = \det(A)\det(D) \tag{A.1}$$

where both A and D are square. If A is square and nonsingular, then we can use the latter relations and the equality:

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

to get the equality

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(D - CA^{-1}B)$$

which is known as the *Schur (determinant) complement* or the *Schur formula*. This formula has been introduced in the papers [Schur, 1917a,b] which have been translated to English in [Gohberg, 1986]). For more details see [Zhang, 2005].

Similarly, if D is nonsingular, we can show

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(D)\det(A - BD^{-1}C)$$

If $A = I$ and $D = I$ and BC is a square matrix, we arrive at the following very useful identity

$$\det(I - BC) = \det(I - CB)$$

A.2 Eigenvalues of Matrices

Definition A.1 *For a square matrix $M \in \mathbb{R}^{n\times n}$, the spectrum of M (the set of eigenvalues of M) is denoted $\lambda(A) = \underset{i}{\mathrm{col}}(\lambda_i)$ each one of these zeroes the characteristic polynomial defined as*

$$\chi_M(\lambda) = \det(\lambda I - M)$$

where $\det(M)$ *is the determinant of M.*

We have the following relations:

$$\sum_{i=1}^{n} \lambda_i = \mathrm{trace}(A)$$
$$\prod_{i=1}^{n} \lambda_i = \det(A)$$

For the special cases $n = 1, 2$ and 3 the characteristic polynomial is given by the expressions

$n = 1$**:** $\quad \chi_M(\lambda) = \lambda - M$

$n = 2$**:** $\quad \chi_M(\lambda) = \lambda^2 - \mathrm{trace}(M)\lambda + \det(M)$

$n = 3$**:** $\quad \chi_M(\lambda) = \lambda^3 - \mathrm{trace}(M)\lambda^2 + \mathrm{trace}[\mathrm{Adj}(M)]\lambda + \det(M)$

where $\mathrm{trace}(M)$ and $\mathrm{Adj}(M)$ are respectively the trace and the adjugate matrix of M. Let us consider now symmetric matrices, i.e. matrices such that $M = M^*$ (or $M = M^T$ if M is a real matrix). It can be shown that in this case all the eigenvalues of M are real [Bhatia, 1997]. Moreover, we have the following definition:

Definition A.2 *The eigenvectors of a symmetric square matrix M are defined to be the nonzero full column rank matrices v_i such that*

$$(A - \lambda_i)v_i = 0$$

In this case, the matrix $M' = PMP^{-1}$ exhibits all the eigenvalues of M on the diagonal:

$$M' = \underset{i}{\mathrm{diag}}(\lambda_i I_{m_i})$$

where the eigenvalues are repeated as many times as their order of multiplicity m_i. The matrix P is defined as $P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$. This decomposition is called **spectral decomposition**.

Remark A.3 *This suggests that for a symmetric matrix M with eigenvalues λ_j with order of multiplicity m_j, there exists m_j eigenvectors v_k such that $(A - \lambda_j)v = 0$. It is important to emphasize that it is not the always the case for general matrices. In such a case, the matrix may be non-diagonalizable but can be reduced to a Jordan matrix. Any algebra book or course should detail this correctly.*

The fact that every symmetric matrix can be diagonalized in an orthonormal basis is an interesting fact and makes symmetric matrices a useful tools in many fields. The interest of symmetric matrices is the ability to generalize the notion of positive and negative number to the matrix case. Indeed, since the eigenvalues of symmetric matrices are all real then it is possible to define positive and negative matrices, hence a relation of order in the cone of symmetric matrices.

Definition A.4 *A symmetric matrix M is said to be positive (semi)definite if all its eigenvalues are positive (nonnegative). This is denoted by $M \succ 0 (\succeq 0)$.*

Definition A.5 *A symmetric matrix M is said to be negative (semi)definite if all its eigenvalues are negative (nonpositive). This is denoted by $M \prec 0 (\preceq 0)$.*

The notion of positivity and negativity of a symmetric matrix M is related to its associated quadratic form $x^T M x$ where x is a real vector.

Proposition A.6 *A $n \times n$ symmetric matrix M is positive (semi)definite if and only if the quadratic form $x^T M x > 0 (\geq 0)$ for all $x \in \mathbb{R}^n - \{0\}$.*

Proof:
Sufficiency:
Suppose all the eigenvalues of M are nonnegative. Define now the quadratic form $Q(x) = x^T M x$ and since M is nonnegative, then in virtue of the Cholesky decomposition of symmetric nonnegative matrices we have $Q(x) = x^T L^T L x$ which is equal to $||Lx||_2$ and is obviously nonnegative.

Necessity:
Suppose now that $Q(x) = x^T M x \geq 0$ for all $x \in \mathbb{R}^n$. A well-known result says that if a quadratic form is positive semidefinite then it is sum-of-squares (see Section 2.3.3.3) and writes as $Q(x) = \sum_i q_i(x)^2$. Now introduce line vectors L_i such that $q_i(x) = L_i x$ and therefore we have $q_i(x)^2 = x^T L_i^T L_i x$. Finally denoting

$$L := \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$

then we have $Q(x) = x^T L^T L x$ where $L^T L \succeq 0$. This concludes the proof. □

Proposition A.7 *A $n \times n$ symmetric matrix M is negative (semi)definite if and only if the quadratic form $x^T M x < 0 (\leq 0)$ for all $x \in \mathbb{R}^n - \{0\}$.*

A.3 Exponential of Matrices

Definition A.8 *The exponential of a square matrix A is given by the expression*

$$e^M = \exp(M) := \sum_{i=1}^{+\infty} \frac{M^i}{i!}$$

Theorem A.9 (Cayley-Hamilton Theorem) *Any square matrix $M \in \mathbb{R}^{n\times n}$ satisfies the equality*

$$\chi_M(M) = 0$$

where $\chi_M(\lambda)$ is the characteristic polynomial of M.

This theorem shows that for a matrix M of dimension n, M^n can be computed as a linear combination of all other lower powers M^k, $0 \leq k < n$. For instance for $n = 2$ we have

$$M^2 = \text{trace}(M)M - \det(M)I$$

It allows to compute any powers of M using a linear combination of all powers of M from 0 to $n-1$. For instance,

$$\begin{array}{rcl} M^3 & = & \text{trace}(M)M^2 - \det(M)M \\ & = & \text{trace}(M)(\text{trace}(M)M - \det(M)) - \det(M)M \\ & = & [\text{trace}(M)^2 - \det(M)]M - \det(M) \end{array}$$

One of the most important applications of this theorem is the obtention of the rank condition for controllability and observability of linear systems. Indeed, since any power of M can be expressed in through a linear combination of powers from 0 to $n-1$ therefore any sum of power of matrices can be written is such a manner.

Proposition A.10 *The exponential of a matrix M, by virtue of the Cayley-Hamilton theorem, can be expressed as*

$$\exp(M) = \sum_{i=0}^{n-1} \alpha_i M^i$$

where the α_i satisfies the linear system

$$\sum_{i=0}^{n-1} \alpha_i \lambda_j^i = e^{\lambda_j} \qquad \text{for all } j = 1, \ldots, n$$

The infinite sum has been amazingly converted into a finite sum where the coefficients are determined by solving a system of linear equations.

A.4 Generalities on Block-Matrices

Let us consider the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Definition A.11 *Assuming that M is square and invertible then the inverse of M is given by*

$$\begin{aligned} M^{-1} &= \begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \end{aligned}$$

These formulae are called Banachiewicz inversion formulae *[Banachiewicz, 1937]. For more details see [Zhang, 2005].*

The first formula is well-defined if A is invertible while the second when D is invertible. By identification of the blocks we get the well-known matrix inversion lemma which has been first introduced in [Duncan, 1917]:

Lemma A.12 (Duncan inversion formulae)

$$(A-BD^{-1}C)^{-1} = A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1}$$

or also

$$(A-BDC)^{-1} = A^{-1}+A^{-1}B(D^{-1}-CA^{-1}B)^{-1}CA^{-1}$$

We also have the identity:

$$A^{-1}B(D-CA^{-1}B)^{-1} = (A-BD^{-1}C)^{-1}BD^{-1} \tag{A.2}$$

For more details about these formulae and refer to [Zhang, 2005].

A.5 Kronecker operators and Matrix Tensor Sum and Product

This sections aims at providing some elementary definitions about Kronecker product and sum. The Kronecker product is defined by

$$A\otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B & \\ \vdots & & \ddots & \vdots \\ a_{p1}B & \dots & a_{pn}B & \end{bmatrix}$$

We have the following relations where α is a scalar:

$$\begin{aligned} 1\otimes A = A\otimes 1 &= A \\ A\otimes(B+\alpha C) &= A\otimes B+\alpha A\otimes C \\ A\otimes(B\otimes C) &= (A\otimes B)\otimes C \\ (A\otimes B)(C\otimes C) &= (AC)\otimes(BD) \\ (A\otimes B)^{-1} &= A^{-1}\otimes B^{-1} \\ (A\otimes B)^T &= A^T\otimes B^T \\ \lambda(A\otimes B) &= \{\nu_i\mu_j\ \forall(i,j)\} \text{ where } \lambda(A)=\nu,\ \lambda(B)=\mu \\ \mathrm{trace}(A\otimes B) &= \mathrm{trace}(A)\,\mathrm{trace}(B) \\ \det(A\otimes B) &= \det(A)^n\det(B)^n,\ n=\dim(A) \\ \mathrm{rank}(A\otimes B) &= \mathrm{rank}(A)\,\mathrm{rank}(B) \end{aligned}$$

The Kronecker sum of two matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ is defined by

$$A \oplus B = A \otimes I_m + I_n \otimes B$$

Moreover we have the following properties

$$\begin{array}{rcl} e^{A\oplus B} & = & e^A \otimes e^B \\ \lambda(A\oplus B) & = & \lambda(A) \cup \lambda(B) \end{array}$$

It is convenient to introduce the tensor product and sum $\phi_\otimes, \phi_\oplus : \mathbb{C}^{m\times m} \times \mathbb{C}^{m\times m} \to \mathbb{C}^{p\times p}$. Let us consider $P, Q \in \mathbb{C}^{m\times m}$ with $m \geq 2$ and

$$\begin{array}{lcl} p = m^2 & : & \left\{ \begin{array}{rcl} \phi_\otimes(P,Q) & = & P\otimes Q \\ \phi_\oplus(P,Q) & = & P\oplus Q \end{array}\right. \\ p = \dfrac{m(m-1)}{2} & : & \left\{ \begin{array}{rcl} \phi_\otimes(P,Q) & = & P\tilde{\otimes} Q \\ \phi_\oplus(P,Q) & = & P\tilde{\oplus} Q \end{array}\right. \end{array}$$

where $\oplus$ and $\otimes$ are Kronecker sum and product define above. On the other hand, the operators $\tilde{\oplus}$ and $\tilde{\otimes}$ are defined as follows [Qiu and Davidson, 1991]:

$$P\tilde{\otimes}Q = [c_{i,j}] \in \mathbb{C}^{p\times p}$$

where $c_{i,j} = (p_{i_1,j_1}q_{i_2,j_2} + p_{i_2,j_2}q_{i_1,j_1} - p_{i_2,j_1}q_{i_1,j_2} - p_{i_1,j_2}q_{i_2,j_1}$ where (i_1,i_2) is the i^{th} pair of sequence $(1,2),(1,3),\ldots,(1,m),(2,3),\ldots,(2,m),\ldots,(m,m)$ and (j_1,j_2) is generated by duality. For $P\tilde{\oplus}Q$ the classical definition is extended in

$$P\tilde{\oplus}Q = P\tilde{\oplus}I_m + I_m\tilde{\oplus}Q$$

Algebraic properties of these tensor product and sum can be found in [Marcus, 1973; Qiu and Davidson, 1991].

A.6 Singular-Values Decomposition

The eigenvalue decomposition of a square matrix is the problem in finding a basis in which the matrix has an expression where the eigenvalues are located on the diagonal: this is the spectral decomposition. We provide here a kind of generalization of such a procedure when the matrix M is not necessarily square: this is called the singular-value decomposition. A unitary matrix U is defined as $U^*U = I = UU^*$ where the superscript * denotes the complex conjugate transpose.

Theorem A.13 *Let $M \in \mathbb{C}^{k\times n}$ be a matrix of rank r. Then there exist unitary matrices U and V such that*

$$M = U\Sigma V^*$$

where U and V satisfy

$$MM^*U = U\Sigma\Sigma^* \qquad M^*MV = V\Sigma^*\Sigma$$

and Σ has the canonical structure

$$\Sigma = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Sigma_0 = \mathrm{diag}(\sigma_1,\ldots,\sigma_r) \prec 0$$

The numbers $\sigma_i > 0$, $i = 1,\ldots,r$ are called the nonzero singular values of M.

Proof: *The proof is given in Skelton et al. [1997] and for more on singular value decomposition see Horn and Johnson [1990] or many other books on linear algebra.* □

A.7 Moore-Penrose Pseudoinverse

Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular matrix (i.e. $\det(M) \neq 0$), then there exists a matrix inverse denoted M^{-1} such that $MM^{-1} = M^{-1}M = I$. We provide here the generalization of this procedure to rectangular matrices. It has been shown that any $n \times m$ matrix M can be expressed as a singular value decomposition $M = U\Sigma V^*$.

Theorem A.14 *For every matrix $M \in \mathbb{R}^{n\times m}$, there exist a unique matrix $M^+ \in \mathbb{R}^{m\times n}$, the Moore-Penrose pseudoinverse of M, which satisfies the relation below:*

$$\begin{array}{ll} MM^+M = M & M^+MM^+ = M^+ \\ (MM^+)^* = MM^+ & (M^+M)^* = M^+M \end{array}$$

Moreover, M^+ is given by

$$M^+ := V \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$$

Moreover consider the matrix $M \in \mathbb{R}^{n\times m}$ then

- if M has full row rank n then $M^+ = M^*(MM^*)^{-1}$
- if M has full column rank m then $M^+ = (M^*M)^{-1}M^*$

A.8 Solving $AX = B$

The solution X of equation $AX = B$ is trivial when A is a nonsingular matrix. We aim here at showing that there exists an explicit expression to X when A is a rectangular matrix sharing specific assumptions with B.

Theorem A.15 *Let $A \in \mathbb{R}^{n_1\times n_2}$, $X \in \mathbb{R}^{n_2\times n_3}$ and $B \in \mathbb{R}^{n_1\times n_3}$. Then the following statements are equivalent:*

1. *The equation $AX = B$ has a solution X.*
2. *A and B satisfy $AA^+B = B$.*
3. *A and B satisfy $(I - AA^+)B = 0$.*

In this case all solutions are

$$X = A^+B + (I - A^+A)Z$$

where $Z \in \mathbb{R}^{n_1\times n_3}$ is arbitrary and A^+ is the Moore-Penrose pseudoinverse of A.

A.9 Solving $BXC + (BXC)^* + Q \prec 0$

Such equation arises in the many design problems and it is important to provide material about it.

Theorem A.16 *Let matrices $B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{k\times n}$ and $Q = Q^* \in \mathbb{H}^n$ be given. Then the following statements are equivalent:*

1. *There exists a X satisfying*

$$BXC + (BXC)^* + Q < 0$$

2. *The following two conditions hold*

$$\begin{array}{rcl} \mathrm{Ker}[B]Q\mathrm{Ker}[B]^* & \prec & 0 \quad \text{or} \quad BB^* \succ 0 \\ \mathrm{Ker}[C]^*Q\mathrm{Ker}[C] & \prec & 0 \quad \text{or} \quad C^*C \succ 0 \end{array}$$

Suppose the above statements hold. Let r_b and r_c be the ranks of B and C, respectively, and (B_ℓ, B_r) and (C_ℓ, C_r) be any full rank factors of B and C (i.e. $B = B_\ell B_r$ and $C = C_\ell C_r$). Then all matrices X in statement 1. are given by

$$X = B_r^+ K C_\ell^+ Z - B_r^+ B_r Z C_\ell C_\ell^+$$

where Z is an arbitrary matrix and

$$\begin{array}{rcl} K & := & -R^{-1}B_\ell^*\Phi C_r^*(C_r\Phi C_r^*)^{-1} + S^{1/2}L(C_r\Phi C_r^*)^{-1/2} \\ S & := & R^{-1} - R^{-1}B_\ell^* - R^{-1}B_\ell^*[\Phi - \Phi C_r^*(C_r^\Phi C_r^*)^{-1}C_r\Phi]B_\ell R^{-1} \end{array}$$

where L is an arbitrary matrix such that $||L|| < 1$ (i.e. $\bar{\sigma}(L) < 1$) and R is an arbitrary positive definite matrix such that

$$\Phi := (B_\ell R^{-1} B_\ell^* - Q)^{-1} \succ 0$$

The solution X is quite complicated and can be approximated by a more simple expression. If one of statements above holds, then more simple solutions are given by each of the expressions [Iwasaki and Skelton, 1995a]:

$$\begin{array}{rcl} X_B & := & -\tau_B B^*\Psi_B C^T(C\Psi_B C^*)^{-1} \\ X_C & := & -\tau_C(B^*\Psi_C B)^{-1}B^*\Psi_C C^* \end{array}$$

where $\tau_B, \tau_C > 0$ are sufficiently large scalars such that

$$\begin{array}{rcl} \Psi_B & := & (\tau_B BB^* - Q)^{-1} \succ 0 \\ \Psi_C & := & (\tau_C C^*C - Q)^{-1} \succ 0 \end{array}$$

B $\mathcal{L}_q$ and $\mathcal{H}_q$ Spaces

This appendix is devoted to the introduction of very important signals and systems spaces.

Let us consider here linear systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Ew(t) \\ z(t) & = & Cx(t) + Fw(t) \end{array} \tag{B.3}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^p$ and $z \in \mathcal{Z} \subset \mathbb{R}^q$ are respectively the state, the inputs and the outputs of the system.

B.1 Norms for Signals

The functions of time $x(t), w(t)$ and $z(t)$ are generally referred as *signals* since these functions, whatever they represent (temperature, speed, position...), are considered in an abstract space where the physical signification is not useful anymore. This is the reason why spaces of signals must be considered and these spaces are called $\mathcal{L}_q^n$ defined hereunder:

$$\mathcal{L}_q := \left\{ u \in \mathcal{F}([0,+\infty), \mathbb{R}^n) : \left(\int_0^{+\infty} ||u(t)||_q^q dt \right)^{1/q} < \infty \right\}$$

Only signals with support $[0, +\infty]$ are considered here by simplicity but is possible to define such sets for signals evolving on the more general support $[t_0, t_1]$.

It is possible to associate a norm to each one of this signals set and is denoted by $||\cdot||_{\mathcal{L}_q}$ and called $\mathcal{L}_q$ norm. Recall that a norm satisfies all the following properties:

1. $||u||_{\mathcal{L}_q} \geq 0$
2. $||u||_{\mathcal{L}_q} = 0 \Leftrightarrow u(t) = 0$ for all $t \geq 0$
3. $||\alpha u||_{\mathcal{L}_q} = |\alpha| \cdot ||u||_{\mathcal{L}_q}$ where α is a constant
4. $||u + v||_{\mathcal{L}_q} \leq ||u||_{\mathcal{L}_q} + ||v||_{\mathcal{L}_q}$

The $\mathcal{L}_q$ norm is then defined as

$$||u||_{\mathcal{L}_q} := \left(\int_0^{+\infty} ||u(t)||_q^q dt \right)^{1/q} \tag{B.4}$$

and hence a signal $u(t)$ belongs to the space $\mathcal{L}_q^n$ if and only if its $\mathcal{L}_q$-norm is bounded. There are three main norms for signals

$\mathcal{L}_1$-norm The 1-norm of a signal $u(t)$ is the integral of its absolute value

$$||u||_{\mathcal{L}_1} := \int_0^{+\infty} ||u(t)||_1 dt \tag{B.5}$$

In some papers and books, the $\mathcal{L}_1$ norm is treated as an electrical consumption.

$\mathcal{L}_2$-norm The 2-norm of a signal $u(t)$ is

$$||u||_{\mathcal{L}_2} := \left(\int_0^{+\infty} ||u(t)||_2^2 dt \right)^{1/2} \tag{B.6}$$

From a physical point of view, the $\mathcal{L}_2$ norm represents the energy of a signal.

$\mathcal{L}_\infty$-norm The $\mathcal{L}_\infty$ norm of a signal is the larger upper bound of the absolute value

$$||u||_{\mathcal{L}_\infty} := \max_i \sup_{t \in [0,+\infty)} |u_i(t)| \tag{B.7}$$

The $\mathcal{L}_\infty$ norm of a signal is the maximum value that the signals under some input values. This norm is useful when the amplitude of signals needs to be constrained. Sometimes the following alternative definition for the $\mathcal{L}_\infty$-norm is utilized:

$$||u||_{\mathcal{L}_\infty} := \sup_{t \in [0,+\infty)} \sqrt{u(t)^T u(t)} \tag{B.8}$$

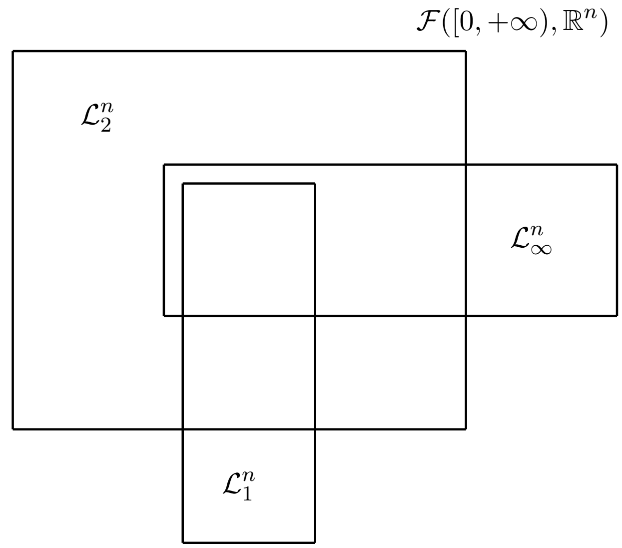

Figure 7.1: Inclusion of Signal Sets

Note that in the scalar case, the two definitions coincide. However, the first version denotes the maximal amplitude of a component of a signal while the second one coincides with the maximal power of a signal.

Remark B.1 *If the second property of norms (i.e. there exists $u(t) \neq 0$ such that $||u||_{\mathcal{L}_q} = 0$) is not satisfied, the term semi-norm is used instead of norm. For instance the power of a signal is a semi-norm and is referred in the literature as the power semi-norm:*

$$||y||_P := \lim_{T \to +\infty} \frac{1}{T} \int_0^T y(t)^* y(t) dt \tag{B.9}$$

These spaces enjoy a non trivial inclusion relationship described by the Venn diagram depicted in Figure 7.1 inspired from Doyle et al. [1990].

B.2 Norms for Systems

While physical magnitudes can be viewed as signals only, the relation between these signals and how they evolve in time (dynamical behavior) is called *system*. A system may be viewed as a physical process but also as a (linear) operator mapping a function space to another. For instance, system (B.3) maps the Euclidian space $\mathcal{W}$ to $\mathcal{Z}$. Note that these spaces are not function spaces but Euclidian space containing values taken by input and output signals. However, by considering these Euclidian spaces, only few information is considered and it is possible (and more interesting) to capture greater information on the operator by considering function spaces instead. This is the role of $\mathcal{L}_q$ spaces: rather than considering function spaces where no constraints apply on elements, $\mathcal{L}_q$ spaces consider elements with a specific (desired) behavior, allowing to tackle more information on the system and its related signals.

By considering these norms, it seems interesting to develop a similar framework for systems and this brings us to the notion of norms for systems denoted $\mathcal{H}_q$. The letter $\mathcal{H}$ stands for

Hardy space and is defined for functions holomorphic over $\mathbb{D}$ as

$$\mathcal{H}_q := \left\{ f \in \mathcal{F}(\mathbb{D}, \mathbb{C}) : f \text{ holomorphic over } \mathbb{D} \text{ and } \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} [f(re^{\imath\theta})]^p d\theta \right)^{1/p} < \infty \right\} \tag{B.10}$$

For $0 < p < q < \infty$, it can be shown that $\mathcal{H}_q$ is a subset of $\mathcal{H}_p$. Variations of the latter definition exists for other domains than the unit open disc $\mathbb{D}$, in our case the domain which has to be considered in the right-half plane $\mathbb{C}^+$.

Definition B.2

$\mathcal{H}_2$-norm *The $\mathcal{H}_2$-norm of a signal $u(t)$ is*

$$||H||_{\mathcal{H}_2} := \left(\text{trace} \frac{1}{2\pi} \int_0^{+\infty} H(j\omega)H(j\omega)^* d\omega \right)^{1/2} \tag{B.11}$$

The $\mathcal{H}_2$ norm can be viewed as the energy of the impulse response of the system.

$\mathcal{H}_\infty$-norm *The $\mathcal{H}_\infty$-norm of a signal is the least upper bound of its absolute value*

$$||H||_{\mathcal{H}_\infty} := \max \bar{\sigma}(H(j\omega)) \tag{B.12}$$

The $\mathcal{H}_\infty$ norm can be viewed as the maximal energy gain from the inputs to the outputs.

An important property of the $\mathcal{H}_\infty$-norm is the submultiplicative property:

$$||M_1 M_2||_{\mathcal{H}_\infty} \leq ||M_1||_{\mathcal{H}_\infty} ||M_2||_{\mathcal{H}_\infty} \tag{B.13}$$

which has important consequences in robustness analysis and robust control synthesis. Note that some system norms do not satisfy such a property, for instance the $\mathcal{H}_2$ does not.

It is worth noting that the $\mathcal{H}_2$ norm is the same as in Definition B.2. By the way, both definitions coincide in the SISO case but the *induced-norm* version (norm of a system induced by the norm of the input and output spaces) holds in the MIMO case and therefore defines a generalization of Definition B.2. This is the reason why this extended version is called *Generalized $\mathcal{H}_2$ norm* or $\mathcal{L}_2 - \mathcal{L}_\infty$ *induced norm*.

To conclude this section, the great interest of induced norms are their domain of validity. Indeed, the system norms are generally expressed in terms of functions over the frequency domain (restricting the validity of the definitions over LTI systems only) and the signal norms over the time domain (quite general framework). Hence, this duality allows for the computation of norms of systems which are neither time-invariant nor linear by considering the quotient of norms of the output signals space over norms of the input signal spaces. This opens the doors to gain analysis of time-varying, parameter varying, nonlinear and distributed systems. Therefore, the energy gain of time-varying system will then be referred to as its $\mathcal{L}_2$-gain. The correspondence between signal and system norms is summarized in Table 7.1.

	$\|\|w\|\|_{\mathcal{L}_2}$	$\|\|w\|\|_{\mathcal{L}_\infty}$
$\|\|z\|\|_{\mathcal{L}_2}$	$\|\|H\|\|_{\mathcal{H}_\infty}$	∞
$\|\|z\|\|_{\mathcal{L}_\infty}$	$\|\|H\|\|_{\mathcal{H}_2}$	$\|\|H\|\|_{\mathcal{H}_1}$

Table 7.1: Correspondence between norms of signals and systems

C Linear Matrix Inequalities

This appendix aims at providing a brief overview of Linear Matrix Inequalities (LMIs). A brief history is given, then some preliminary definitions and methods to solve them are introduced. For a short story of automatic control see [Åström, 1999].

C.1 Story

Historically, the first LMI appeared in the pioneering work of Lyapunov (actually its Ph.D thesis in 1890) which was on the 'General Stability of Motion' where is exposed what is called 'the Lyapunov's theory'. His thesis has been translated from Russian to French at the impulse of Henry Poincaré and has been finally translated from French to English [Lyapunov, 1992]. In this work, the stability of a linear time-invariant dynamical systems $\dot{x} = Ax$ is equivalent to the feasibility of the Linear Matrix Inequality:

$$A^T P + PA \prec 0 \qquad P = P^T \succ 0$$

In his work Lyapunov introduced notions that are still in use in modern control theory and since then, many results have been grafted over it. Indeed, in 1940, Lur'e, Postnikov et al. applied Lyapunov' s theory to control problems involving nonlinearity in the actuator. This has lead to Lur'e systems which are defined as

$$\dot{x} = Ax + B\phi(x) \tag{C.14}$$

where $\phi(\cdot)$ is a nonlinear function of x. Although the stability criteria were not in an LMI form (in reality they were polynomially frequency dependent inequalities), they actually were equivalent to an LMI formulation. The bridge, which was unknown at this time, between frequency dependent inequalities and LMI has been emphasized in an important result derived by Yakubovich, Popov, Kalman, Anderson... and is called the Positive Real Lemma (some precision on it and its link to passivity are introduced in Appendix D.5). This positive real lemma, reduces the solution of an LMI into simple graphical criterion in the complex plane (which is linked to Popov, circle and Tsypkin criteria). In 1962, Kalman derived one of the most important work of this century: the 'Kalman-Yakubovitch-Popov' Lemma which bridges completely graphical tests in the complex plane and a family of LMIs (see Appendix D.3) and allows by now to switch easily from frequency domain to time-domain criteria.

In 1970, Willems focused on solving algebraic equations such as Lyapunov's or Ricatti's equations (ARE), rather than LMIs. Indeed, the solvability was not well established at this time and the numerical algebra was developed to solve algebraic equations rather than LMIs. To understand the power of LMIs, it has been necessary to develop complex mathematical tools and algorithms to solve them.

In 1919, the 'Ellipsoid Algorithm' of Khachiyan was the first algorithm to exhibit a polynomial complexity (polynomial bound on worst-case iteration count) for Linear Programming. Linear Programming problems are optimization problems where the optimization cost and constraints are all affine in the unknown variables. In 1984, Karmarkar introduced 'Interior Point' methods for LP which has led to lower complexity and better efficiency than ellipsoidal methods.

The particularity of LMIs is that, although the cost and the constraints are affine on the unknown variables, the inequalities are not componentwise but represent the location of the eigenvalues of the matrix inequality. Therefore, the problem is obviously non linear since the location of the eigenvalues of a symmetric matrix depend on the sign of its principal minors. By computing the principal minors of a LMI, it appears that we obtain a set of polynomial scalars inequality and therefore is nonlinear. However, although the optimization problem is non linear, it can be shown that the optimization problem is a convex problem, one of the most studied field in optimization [Boyd and Vandenberghe, 2004] and hence now LMI benefits of a huge arsenal of solid tools.

In 1988, Nesterov, Nemirovskii and Alizadeh [Nesterov and Nemirovskii, 1994] extend IP methods for Semidefinite Programming (SDP) which is the class of problems where LMIs belong. Since then, IP methods have been heavily developed and is now the most powerful tools to solve numerically LMIs.

Finally, in 1994, the research effort on application of LMI to control culminated in [Boyd et al., 1994] where many other authors brought important contributions, for instance Apkarian, Bernussou, Gahinet, Geromel, Peres...

Since then many solvers for SDP have been developed for instance SeDuMi [Sturm, 1999, 2001], DSDP, SDPT3... Since all this solvers have been developed for the mathematical framework of SDP and since the representation of LMI in the field of automatic control is based on a matrix representation, softwares called 'parsers' have been developed as interface between these notations, for instance SeDuMi Interface and the best one: Yalmip [Löfberg, 2004].

C.2 Definitions

A Linear Matrix Inequality (LMI) is an inequality of the form

$$\mathcal{L}(x) := \mathcal{L}_0 + \sum_{i=1}^{m} \mathcal{L}_i x_i \succ 0 \tag{C.15}$$

where $x \in \mathbb{R}^m$ is the variable and the symmetric matrix $\mathcal{L}_i \in \mathbb{S}^n$, $i = 1, \ldots, n$ are given data. The inequality symbol $\succ$ means that $\mathcal{L}(x)$ is positive definite (i.e. $y^T \mathcal{L}(x) y > 0$ for all $y \in \mathbb{R}^n - \{0\}$. This inequality is equivalent to m polynomial inequalities corresponding to the leading minor of $\mathcal{L}(x)$.

The LMI (C.15) is a convex constraint on x: the subset $\{x \in \mathbb{R}^m : \mathcal{L}(x) \succ 0\}$ is convex.

Multiple LMI $\mathcal{L}^{(1)}(x) \succ 0, \ldots, \mathcal{L}^{(q)}(x) \succ$ can be expressed as a single LMI $\text{diag}(\mathcal{L}^{(1)}(x) \succ 0, \ldots, \mathcal{L}^{(q)}(x)) \succ 0$. This shows that the intersection of LMI constraints is also a LMI constraint. This can be connected with the property that the intersection of convex sets is also a convex set.

Notation (C.15) is the 'mathematical' notation while the following is the notation used in the field of automatic control and system theory

$$A^T P + PA \prec 0 \qquad P = P^T \succ 0 \tag{C.16}$$

where the matrix $P = P^T \succ 0$ is the variable and $A \in \mathbb{R}^{n \times n}$ a given data. It is not possible to give a general formulation of LMIs where matrices are variable since there is a large variety of different forms. Nevertheless, any LMI in 'matrix variable' form can be written into the mathematical form (but the converse is not necessarily true). To write this, just decompose $P = P^T \succ 0$ over a basis of symmetric matrices of dimension n denoted by P_i. Hence $P := P(x) = \sum_{i=1}^{m} P_i x_i$ with $m = \dfrac{n(n+1)}{2}$. Finally by identification we get $\mathcal{L}_0$ and $\mathcal{L}_i = -A^T P_i - P_i A$.

Definition C.1 *An LMI $\mathcal{M}(x) \preceq 0$ is feasible if and only if there exists x such that $\mathcal{M}(x) \preceq 0$. It is said to be strictly feasible if and only if there exists x such that $\mathcal{M}(x) \prec 0$.*

C.3 How to solve them ?

Several approaches allowing the determination of the solution of LMIs are presented here.

Algebraic Methods

In order to solve simple LMI, algebraic methods can be used using linear algebra. This is possible when dealing with only few decision matrices. For instance, let us consider the well-known Lyapunov stability LMI condition for linear time-invariant systems:

$$A^T P + PA \prec 0 \tag{C.17}$$

Assume there exists $P = P^T \succ 0$ such that the LMI is satisfied, i.e. the system $\dot{x}(t) = Ax(t)$ is asymptotically stable (all the eigenvalues of A lie in the left half complex plane). Let $P_0 = \int_0^{+\infty} e^{A^T t} Q e^{At} dt$ with some matrix $Q = Q^T \succ 0$ and inject the expression of P_0 into the latter LMI, we get

$$\begin{aligned} \int_0^{+\infty} A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A dt &= \int_0^{+\infty} \frac{d}{dt}\left[e^{A^T t} Q e^{At}\right] dt \\ &= \lim_{t\to+\infty} e^{A^T t} Q e^{At} - Q \end{aligned}$$

Since the system is asymptotically stable then $\lim_{t\to+\infty} e^{A^T t} Q e^{At} = 0$ an then a parametrization of the solutions of the LMI (C.17) is given by P_0. It is clear that this method may become very complicated while dealing with LMIs of high dimensions and with a large number of decision matrices. There exist many other methods as detailed for instance in [Gajić and Qureshi, 1995].

Algorithms

At this time, Interior Points algorithms are mainly used. Simple algorithms are presented in [Boyd et al., 1994] while the complete theory of IP algorithms with barrier function, in an unified framework, is detailed in the (very difficult to understand in detail for nonspecialists) book [Nesterov and Nemirovskii, 1994]. The idea of barrier function is briefly explained here:

Consider the optimization problem:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^T x \\ \text{s.t.} & F_i(x) \succeq 0, \; i = 1, \ldots, p \end{array} \tag{C.18}$$

where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $F_i(x)$ are respectively the cost vector, the decision variable and the LMIs constraints. It is important to note that any LMI optimization problem can be rewritten in the latter form. The idea of interior point algorithm with barrier function, is to turn a constrained optimization problem into an unconstrained one. By introducing the set

$$\mathcal{X}_f := \{x \in \mathbb{R}^n : \; F_i(x) \succeq 0, \; i = 1, \ldots, p\} \tag{C.19}$$

the optimization problem (C.18) is equivalent to

$$\min_{x\in\mathcal{X}_f} \qquad c^T x \tag{C.20}$$

The key idea to define implicitly the set $\mathcal{X}_f$ (since it is difficult and time consuming to define it explicitly) is to define a function which is small in the interior of $\mathcal{X}_f$ and tends to infinity for each sequence of points converging to the boundary of $\mathcal{X}_f$. This function is called a *barrier function*. It also important, for mathematical purpose, that this barrier function be analytic (differentiable), convex and self-concordant. Indeed, if the barrier function is convex then the optimization problem will be convex and hence the theory of convex optimization applies. The differentiability of the barrier function (actually it must be C^3) allows for the computation of gradient and hessian in the iterative optimization procedure. Finally, the self-concordance of a barrier function is a property, which has been introduced specifically in the framework of SDP optimization, which guarantees nice convergence properties of the Newton algorithm used to solve these unconstrained optimization problems. This notion has been introduced in the book [Nesterov and Nemirovskii, 1994] and the definition is given below:

Let $F(x)$ by function which is convex and analytic. It is said to be self-concordant with parameter a if

$$|D^3F(x)[h,h,h]| \leq 2a^{-1/2}(D^2F(x)[h,h])^{3/2}$$

in a metric defined by the hessian itself and

$$|DF(x)[h]| \leq b(D^2F(x)[h,h])^{3/2}$$

where $D^kF(x)[h_1,\ldots,h_k]$ is the k^{th} differential of F taken at x along the collection of direction $[h_1,\ldots,h_k]$. The first inequality defines the Lipschitz continuity of the Hessian of the barrier with respect to the local Euclidian metric defined by the Hessian itself. The second inequality defines the Lipschitz continuity of the barrier itself with respect to the same local Euclidian structure. The signification of term self-concordant is not easy to understand. The first idea could be that the absolute value of the third derivative is bounded by a function of the second one. This establishes a link between them and shows that the third order term in the Taylor expansion can always be bounded by the second order term. Another idea is that the third order derivative can be approximated by an expression involving the the Hessian.

A good barrier function for SDP is the logarithmic barrier

$$f(x) = -\log\det F(x) = \log\det F(x)^{-1}$$

This function is analytic, convex and self-concordant on $\{x : \ F(x) \succ 0\}$.

Finally the constrained optimization problem (C.18) (and equivalently (C.20)) is converted into the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} \qquad c^T x + \log\det F^{-1}(x) \tag{C.21}$$

The Newton algorithm is then used to find the solution of the optimization problem (C.21). It can be shown that the optimum of (C.21) coincides with the optimum of (C.20) and therefore no modification of the problem is done when adding the self-concordant barrier function to the cost.

The Newton algorithm aims to find zeros of functions, say $f(x)$ and the iteration procedure is

$$x_{k+1} = x_k - [\nabla^2 f(x)]^{-1} \nabla f(x) \tag{C.22}$$

where $\nabla^2 f(x)$ and $\nabla f(x)$ are respectively the Hessian and the gradient of f evaluated at x. Despite of its apparent simplicity, this iteration procedure converges quadratically provided that the initial condition x_0 belongs satisfies

$$\frac{L}{2m^2}||f'(x_0)||_2 < 1 \tag{C.23}$$

where L is the Lipschitz continuity constant of the Hessian and m is defined as $h^T f''(x)h \geq m||h||_2^2$. It can be shown that in the case of unconstrained optimization with self-concordant barrier functions, the Newton procedure can compute very efficiently the global optimum of optimization problems (C.18)-(C.21).

In [Nesterov and Nemirovskii, 1994], it is shown that for every allowable x_i (i.e. $x_i \in \mathcal{X}_f$) the next value x_{i+1} remains in $\mathcal{X}_f$ (is allowable too) and $f(x_{i+1}) \leq f(x_i)$. Then for a good initialization of the iterative procedure, it suffices to find a point in $\mathcal{X}_f$. For this purpose, most solvers implement an initialization procedure resulting in the determination of an initial feasible point from which the optimum of the optimization problem can be easily computed.

D Technical Results in Robust Analysis, Control and LMIs

This appendix aims at providing a catalog of important definitions and theorems extensively used in the literature. Let us consider a multivariable finite dimensional linear time-invariant systems of the form:

$$Z(s) = H(s)W(s) \tag{D.24}$$

where s stands for the Laplace variable, $H(s)$ the transfer function of the system and $W(s)$, $Z(s)$ are respectively the input and the output. Assume that (D.24) admits the following minimal realization Σ_l:

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bw(t) \\ z(t) &=& Cx(t) + Dw(t) \end{array} \tag{D.25}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^p$ and $z \in \mathcal{Z} \subset \mathbb{R}^q$ are respectively the state, the inputs and the outputs.

D.1 Dissipative Systems and Supply Rates

The dissipativity is a theory devoted to study the stability of non-autonomous dynamical systems and has been introduced by Willems [Willems, 1972]. The main principle of the theory is really simple: if the system stores less energy than we supply to it, then this means that the difference of energy is dissipated by the system.

Let us consider the general system Σ governed by the equations

$$\begin{array}{rcl} \dot{x}(t) &=& f(x,w) \\ z(t) &=& h(x,w) \end{array} \tag{D.26}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^p$ and $z \in \mathcal{Z} \subset \mathbb{R}^q$ are respectively the state, the inputs and the outputs.

Let $s(w,z)$ be a mapping from $\mathcal{W} \times \mathcal{Z} \to \mathbb{R}$. It is assumed that for any $t_0, t_1 \in \mathbb{R}$ and for all input-output pairs (w,z) satisfying (D.26), the function $s(w,z)$ is absolutely integrable (i.e. $\int_{t_0}^{t_1} |s(w(t),z(t))|dt < \infty$). This mapping is referred to as the *supply function* and its meaning will be detailed just after the following definition:

Definition D.1 *The system (D.26) with supply function s is said to be dissipative if there exists a function $V : \mathcal{X} \to \mathbb{R}$ such that*

$$V(x(t_0)) + \int_{t_0}^{t_1} s(w(t),z(t))dt \geq V(x(t_1)) \tag{D.27}$$

for all $t_0 \leq t_1$ and all signals (w,x,z) which satisfies (D.26). The pair (Σ, s) is said to be conservative if the equality holds for all $t_0 \leq t_1$ and all signals (w,x,z) which satisfies (D.26). In such as case, V is called a Storage Function. Storage Function

The supply-rate s should be interpreted as the supply delivered to the system. This means that $s(w,z)$ represents the rate at which supply circulates into the system if the pair (w,z) is generated. Hence, when the integral $\int_0^T s(w(t),z(t))dt$ is positive then the work is done on the system while the work is done by the system when the integral is negative. The function V is called the storage function and generalizes the notion of an energy for a dissipative system.

Thanks to this interpretation, inequality (D.27) says that for any interval $[t_0, t_1]$, the change of internal storage $V(x(t_1)) - V(x(t_0))$ will never exceed the amount of supply that flows into the system. This means that part of what is supplied is stored while the remaining part is dissipated.

For more details on dissipativity and dissipative systems, please refer to [Scherer and Weiland, 2004; Willems, 1972].

D.2 Linear Dissipative Systems and Quadratic Supply Rates

We detail here the special case of linear system governed by expressions (D.25). Suppose that $x^* = 0$ is the point of neutral storage and consider quadratic supply functions $s : \mathcal{W} \times \mathcal{Z} \to \mathcal{R}$ defined by

$$s(w(t),z(t)) = \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \tag{D.28}$$

We provide here the essential result about dissipativity of linear systems with quadratic supply rate.

Theorem D.2 *Suppose that system Σ_l defined by (D.25) is controllable and let the supply function be defined by (D.28). Then the following statements are equivalent:*

1. *(Σ_l, s) is dissipative*
2. *(Σ_l, s) admits a quadratic storage function $V(x) = x^T P x$ with $P = P^T$*
3. *There exists $P = P^T$ such that*

$$F(P) := \begin{bmatrix} A^T P + PA & PB \\ \star & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \preceq 0$$

4. *For all $\omega \in \mathbb{R}$ with $\det(j\omega I - A) \neq 0$, the tranfer function $H(s) = C(sI - A)^{-1}E + F$ satisfies*
$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} \succeq 0$$

The proof can be found in [Scherer and Weiland, 2005].

D.3 Kalman-Yakubovich-Popov Lemma

The Kalman-Yakubovich-Popov lemma shows that, amongst others, the frequency condition given by Popov in [Popov, 1961] is equivalent to the existence of a Lyapunov function [Kalman, 1963]. Initially, this result has been developed to solve the problems of stability of nonlinear systems initiated by Lur'e in [Lur'e, 1951]. The Kalman's result strengthens and generalizes the Yakubovich's result proved in [Yakubovitch, 1962].

The Lur'e problem was to analyze the stability of the (closed-loop) system

$$\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + b\varphi(y(t)) \\ \dot{x}_2(t) &= -\varphi(y(t)) \\ y(t) &= c^T x_1(t) + \rho x_2(t) \end{aligned} \tag{D.29}$$

where $\rho > 0$ and $\varphi(\cdot)$ is a nonlinear continuous function which satisfies the sector condition $\psi \in \mathcal{A}_\kappa$ where
$$\mathcal{A}_\kappa := \left\{\varphi : \ 0 < y\varphi(y) < \kappa y^2, \ \varphi(0) = 0\right\}$$
In [Popov, 1961], the following result is proved:

Theorem D.3 (Theorem of Popov) *Assume that A is stable and that $\rho > 0$ then (D.29) is globally asymptotically stable if the condition*

$$\mathrm{Re}(2\alpha\rho + j\omega\beta)[c^T(j\omega I - A)^{-1}b + \rho/j\omega] \geq 0 \tag{D.30}$$

holds for all real ω with $2\alpha\rho = 1$ and some $\beta \geq 0$.

Popov also studied, but did not resolve, the question of existence of a Lyapunov function which assures that (D.29) is globally asymptotically stable when (D.30) holds. To this aim, he also introduced the following candidate Lyapunov function for (D.29):

$$V(x_1, y) = x_1^T P x_1 + \alpha(y - h^T x_1)^2 + \beta \int_0^y \varphi(s)ds \tag{D.31}$$

with α, β real.

The result introduced by Kalman in [Kalman, 1963] allows to solve the problem of Lur'e by showing that the condition (D.30) is equivalent to the existence of a Lyapunov function of the form (D.31):

Lemma D.4 (Kalman-Yakubovich-Popov Lemma) *Let us consider the real constant $\gamma \geq 0$ and the SISO transfer function $H(j\omega) = c^T(j\omega I - A)^{-1}b$ such that the pair (A, b) is completely controllable then a symmetric matrix P and a vector q satisfying*

1. $A^T P + PA = -qq^T$

2. $Pb - c = \sqrt{\gamma}q$

exist if and only if

$$\gamma + 2\mathrm{Re}[c^T(j\omega I - A)^{-1}b] \geq 0$$

Moreover the set $\{x : \ x^T P x = 0\}$ *is the unobservable subspace for the pair* (A, b)*.*

The latter lemma allows to state the result:

Theorem D.5 (Main Theorem of [Kalman, 1963]) *Consider system (D.29) where* A *is stable,* (A, b) *is completely controllable and* (A, c^T) *is completely observable. We seek a suitable Lyapunov function* V *of the form (D.31).*

1. $V > 0$ *and* $\dot{V} \leq 0$ *for any* $\varphi(\cdot) \in \mathcal{A}_\infty$ *if and only if there exist real constant* $\alpha \geq 0$*,* $\beta \geq 0$*,* $\alpha + \beta \geq 0$ *and (D.30) holds.*

2. *Suppose* V *satisfies the preceding conditions. Then* V *is a Lyapunov function which assures global asymptotic stability of (D.29) if and only if either (a)* $\alpha \neq 0$ *or (b)* $\alpha = 0$ *and the equality sign in (D.30) occurs at those values of* ω *where* $\mathrm{Re}[c^T(j\omega I - A)^{-1}b] \geq 0$

3. *There is an constructive procedure for computing* V*.*

Finally, the constants α, β can be computed using:

$$\begin{array}{lcl} \sqrt{\gamma}q & = & Pb - \alpha\rho c - \beta A^T c/2 \\ \gamma & = & \beta(\rho + c^T b) \end{array}$$

The KYP lemma used to prove the latter theorem has become more famous that the theorem itself. Since then, it has been generalized to a more general framework using the notion of dissipativity (among others). However, despite of its different form, it still establishes a relation between a frequency domain condition and the existence of a Lyapunov function. Some important considerations are provided in [Rantzer, 1996; Scherer and Weiland, 2005; Willems, 1971; Yakubovitch, 1974] and references therein.

Lemma D.6 (Modern Kalman-Yakubovich-Popov Lemma) *For any triple of matrices* $A \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^{n\times m}$ *and* $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \in \mathbb{S}^{n+m}$*, the following statements are equivalent:*

1. *There exists a symmetric matrix* $P = P^T$ *such that*

$$M + \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \prec 0$$

2. $M_{22} \prec 0$ *and for all* $\omega \in \mathbb{R}$ *and complex vectors* $\mathrm{col}(x, w) \neq 0$

$$\begin{bmatrix} A - j\omega I & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \qquad \text{implies} \qquad \begin{bmatrix} x \\ w \end{bmatrix}^* M \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

If (A, E) is controllable, the corresponding equivalence also holds for non-strict inequalities.

Finally, if

$$M = -\begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}$$

then statement 2 is equivalent to the condition that for all $\omega \in \mathbb{R}$, with $\det(j\omega I - A) \neq 0$ we have

$$\begin{bmatrix} I \\ C(j\omega I - A)^{-1}B + D \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ C(j\omega I - A)^{-1}B + D \end{bmatrix} \succ 0$$

This lemma proves that geometric considerations is the complex plane (such as Popov' criterion, circle criterion…) have time-domain counterparts in terms of Lyapunov functions. These conditions may in turn be expressed through linear matrix inequalities, or equivalently, by algebraic Ricatti inequalities. Another great interest of this result is to allow for the generalization of some definitions from the time-invariant to the time-varying case. The final important fact is that, it turns a semi-infinite matrix inequality (due to the frequency variable $\omega \in [0, +\infty)$) into a finite dimensional matrix inequality involving a finite dimensional variable $P = P^T \succ 0$.

Several other versions of the KYP-lemma has been also developed in different frameworks, see for instance [Hencey and Alleyne, 2007; Iwasaki et al., 1998; Paré et al., 1999].

D.4 Schur complement

The term *Schur complement* has been introduced by Emilie Virginia Haynsworth in [Haynsworth, 1968] and in the same article she proved the inertia addivity formula which is called now *Haynsworth inertia additivity formula*. In some words, she proved that the inertia is additive on the Schur complement and is a direct consequence of the Guttman rank additivity formula [Guttman, 1946]. For more details, please refer to [Zhang, 2005].

In the context of LMIs the inertia additivity formula can be written into the form [Boyd et al., 1994]

Lemma D.7 *The following statements are equivalent:*

1. $\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \prec 0$
2. $M_{11} \prec 0$ *and* $M_{22} - M_{12}^T M_{11}^{-1} M_{12} \prec 0$
3. $M_{22} \prec 0$ *and* $M_{11} - M_{12} M_{22}^{-1} M_{12}^T \prec 0$

This lemma allows to exhibit convex linear matrix inequalities from nonlinear matrix inequality. Indeed, it is difficult to see that statements 2 and 3 provide convex inequalities. But according to this lemma, they can be turned into an affine LMI.

It is important to say that when a matrix is positive definite then all its Schur complement must be positive definite. The following example illustrates a trap of the Schur complement.

Example D.8 *Let us for instance consider the following LMI:*

$$\begin{bmatrix} -E^T P E - Q & A^T P & Q \\ \star & -P & 0 \\ \star & \star & -Q \end{bmatrix} \prec 0$$

where $P = P^T \succ 0$, $Q = Q^T \succ 0$ and E, A are square. Is this LMI can be satisfied ? First of all, the diagonal terms must be negative definite: this is the case, due to the assumptions on the matrices P and Q. However, by performing the Schur complement with respect to the right-lower block we obtain the two underlying inequalities:

$$\begin{array}{rl} -Q & \prec 0 \\ \begin{bmatrix} -E^TPE & A^TP \\ \star & -P \end{bmatrix} & \prec 0 \end{array}$$

While the first equality is satisfied, the second may be not satisfied if E is not of full rank since in this case the term E^TPE would have zero eigenvalues. This is the case when considering discrete-time singular systems of the form $Ex(k+1) = Ax(k)$.

This example shows that Schur complements should be used with care.

There also exists a non-strict version of the Schur complement [Boyd et al., 1994].

Lemma D.9 *The following statements are equivalent:*

1. $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \succeq 0$

2. *The following relations hold*

$$R \succeq 0, \qquad M_{11} - M_{12}M_{22}^{+}M_{12}^T \succeq 0, \qquad S(I - M_{22}M_{22}^{+}) = 0$$

where M_{22}^{+} is the Moore-Penrose pseudoinverse of M_{22}.

D.5 Positive real lemma

The positive real lemma is highly related to the passivity of a system and has played a crucial role in questions related to the stability of interconnected systems and synthesis of passive electrical networks.

An LMI formulation to passivity can be derived using the dissipativity framework by considering the supply function $s(w, z) = z^Tw + w^Tz$. This leads to:

Lemma D.10 *System (D.25) is passive (or positive real) if and only if there exists a matrix $P \in \mathbb{S}_{++}^n$ such that*

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ \star & -(D + D^T) \end{bmatrix} \prec 0 \tag{D.32}$$

Then for all $\omega \in \mathbb{R}$ with $\det(j\omega I - A) \neq 0$ one has $H(j\omega)^* + H(j\omega) \succeq 0$.

Moreover, $V(x) = x^TPx$ defines a quadratic storage function.

Proof: *The proof is an application of the Kalman-Yakubovich-Popov lemma with quadratic supply function $s(w, z) = w^Tz + z^Tw$.* □

D.6 $\mathcal{H}_2$ Performances

The $\mathcal{H}_2$ norm of a system measures the output energy in the impulse responses of the system. The $\mathcal{H}_2$ norm of a system described by $H(s)$ is given (under certain assumptions) by

$$||H||_{\mathcal{H}_2}^2 = \frac{1}{2\pi} \operatorname{trace} \int_{-\infty}^{+\infty} H(j\omega)H(j\omega)^* d\omega$$

Lemma D.11 *Suppose system (D.25) with $D = 0$ is asymptotically stable. Then $||H||_{\mathcal{H}_2} < \nu$ if and only if there exists $P \in \mathbb{S}_{++}^n$, $Z \in \mathbb{S}_{++}^q$ and $\nu > 0$ such that*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -I_p \end{bmatrix} \prec 0 \qquad \begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} \succ 0 \qquad \operatorname{trace}(Z) < \nu^2$$

Proof: *Let e_i be the i^{th} vector of the standard basis of the input space $\mathbb{R}^p$, we define the input $w(t)$ as*

$$w(t) = \sum_{i=1}^{p} w_i(t)$$

where $w_i(t) = \delta(t)e_i$ and $\delta(t)$ is the Dirac distribution. Define the impulse responses with zero initial condition

$$\begin{aligned} z_i(t) &= C\int_0^t e^{A(t-s)}Bw_i(s)ds \\ &= Ce^{At}Be_i \end{aligned}$$

The total energy (the sum of the energy of each z_i) equals the $\mathcal{H}_2$-norm. Hence we have

$$\begin{aligned} \sum_{i=1}^{p} ||z_i||_{\mathcal{L}_2}^2 &= \sum_{i=1}^{p} \int_0^{+\infty} e_i^T B^T e^{A^T t} C^T C e^{At} B e_i \\ &= \operatorname{trace}\left(\int_0^{+\infty} B^T e^{A^T t} C^T C e^{At} B dt\right) \\ &= \operatorname{trace}\left(\int_0^{+\infty} Ce^{At} BB^T e^{A^T t} C^T dt\right) \end{aligned}$$

since $\operatorname{trace}(AB) = \operatorname{trace}(BA)$. *Using Parseval equality (Appendix D.21) it is possible to show that*

$$\begin{aligned} \sum_{i=1}^{p} ||z_i||_{\mathcal{L}_2}^2 &= \frac{1}{2\pi} \operatorname{trace}\left(\int_{-\infty}^{+\infty} H(j\omega)H(j\omega)^* d\omega\right) \\ &= ||H||_{\mathcal{H}_2}^2 \end{aligned}$$

Now letting $W = \int_0^{+\infty} e^{At}BB^T e^{A^T t} dt$ be the controllability grammian which satisfies

$$AW + WA^T + BB^T = 0$$

This can be retrieved using results of Appendix C.3. Note also that

$$||H||_{\mathcal{H}_2}^2 = \operatorname{trace}[CWC^T] = \nu^2$$

Since $\text{rank}[B] = p \leq n$*, hence* $BB^T \succeq 0$ *and* $W \succeq 0$*. Since* A *is Hurwitz, this means that there exists* $X \succeq W$ *such that*

$$AX + XA^T + BB^T \prec 0 \qquad CXC^T \prec Z \qquad \text{trace}\, Z < \nu^2$$

Pre and post multiplying by $P := X^{-1}$ *we get*

$$PA + A^T P + PBB^T P \prec 0 \qquad CP^{-1}C^T \prec Z \qquad \text{trace}\, Z < \nu^2$$

A Schur complement yields LMIs

$$\begin{bmatrix} PA + A^T P & PB \\ \star & -I_p \end{bmatrix} \prec 0 \qquad \begin{bmatrix} Z & C \\ \star & P \end{bmatrix} \succ 0 \qquad \text{trace}[Z] < \nu^2$$

This concludes the proof. $\square$

D.7 Generalized $\mathcal{H}_2$ performances

The generalized $\mathcal{H}_2$ performance is defined as the $\mathcal{L}_2 - \mathcal{L}_\infty$ induced norm of a system. The system is then defined as an operator from the set of signals of bounded energy to set of signals with finite amplitude (energy to peak norm). In the scalar case, $\mathcal{L}_2 - \mathcal{L}_\infty$ induced norm coincides with the $\mathcal{H}_2$ norm which is the reason for calling it the generalized $\mathcal{H}_2$ norm.

Lemma D.12 *Suppose system (D.25) with* $F = 0$ *is asymptotically stable. Then* $||H||_{\mathcal{L}_2,\mathcal{L}_\infty} < \nu$ *if and only if there exists* $P \in \mathbb{S}^n_{++}$ *and* $\nu > 0$ *such that*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -I_p \end{bmatrix} \prec 0 \qquad \begin{bmatrix} P & C^T \\ C & \nu^2 I_q \end{bmatrix} \succ 0 \tag{D.33}$$

Proof: *First, let us consider the input behavior of the system in the* $\mathcal{L}_2$*-norm sense. To this aim define the storage function* $V(x) = x^T Px$ *with the supply-rate* $s(w, z) = w^T w$*. The dissipativity constraint yields*

$$\dot{V} - s(w, z) < 0$$

or equivalently

$$\begin{bmatrix} A^T P + PA & PB \\ \star & -I_p \end{bmatrix} \prec 0$$

Moreover we have

$$\begin{array}{rcl} x(t)^T Px(t) & \leq & \int_0^t w(s)^T w(s) ds \\ & = & ||w||^2_{\mathcal{L}_2} \end{array}$$

The output z *satisfies*

$$\begin{array}{rcl} z(t)^T z(t) & = & x(t)^T C^T Cx(t) \\ & \leq & \nu^2 x(t)^T Px(t) \\ & \leq & \nu^2 ||w||^2_{\mathcal{L}_2} \end{array} \tag{D.34}$$

for some $\nu > 0$ *satisfying* $P - \nu^{-2} C^T C \succ 0$ *or equivalently*

$$\begin{bmatrix} P & C^T \\ \star & \nu^2 I_q \end{bmatrix} \succ 0$$

Finally taking the supremum on the left hand side of (D.34) we get

$$\sup_{t\geq 0} z(t)^T z(t) \leq \nu^2 ||w||^2_{\mathcal{L}_2}$$

and thus

$$||z||_{\mathcal{L}_\infty} \leq \nu ||w||_{\mathcal{L}_2}$$

This concludes the proof. □

D.8 Bounded-Real Lemma - $\mathcal{H}_\infty$ Performances

The bounded real lemma is a well known lemma allowing for the computation of the $\mathcal{H}_\infty$ norm of a linear system. It can be obtained in the dissipativity framework while considering the supply function $s(w,z) = \gamma w^T w - \gamma^{-1} z^T z$.

Lemma D.13 *System (D.25) is asymptotically stable if and only if there exists $P \in \mathbb{S}^n_{++}$ and $\gamma > 0$ such that*

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ \star & -\gamma I & D^T \\ \star & \star & -\gamma I \end{bmatrix} \prec 0 \qquad \text{(D.35)}$$

Then for all $\omega \in \mathbb{R}$ with $\det(j\omega I - A) \neq 0$ one has $H(j\omega)^* H(j\omega) \preceq \gamma^2 I$. Moreover, $V(x) = x^T P x$ defines a quadratic storage function.

Proof: *The proof is a trivial application of the Kalman-Yakubovich-Popov lemma with quadratic supply function $s(w,z) = \gamma w^T w - \gamma^{-1} z^T z$.* □

This result is extremely important and has led to important improvements in systems and control theory. As a first interpretation, it is equivalent to the following input/output signals inequality:

$$||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

meaning that γ is the energy gain ($\mathcal{L}_2$-gain) of the system. This means that for an input of unit energy, the energy of the output is less than γ. Moreover, in the time-invariant case, it is possible to show that the $\mathcal{L}_2$ induced norm coincides with the $\mathcal{H}_\infty$ norm of the system.

Property D.14 (Submultiplicativity) *The bounded real lemma is a useful tool in robust control since the $\mathcal{H}_\infty$ norm is sub-multiplicative which means that for two asymptotically stable transfer functions $M_1(s)$ and $M_2(s)$, the following relation holds:*

$$||M_1(s)M_2(s)||_\infty \leq ||M_1(s)||_\infty \cdot ||M_2(s)||_\infty$$

This can be modified to have the useful implication for $\alpha, \beta > 0$:

$$||M_1(s)||_\infty < \beta/\alpha \text{ and } ||M_2(s)||_\infty < \alpha \Rightarrow ||M_1(s)M_2(s)||_\infty < \beta$$

which is the basis of small-gain theorem (see Appendix D.11).

D.9 $\mathcal{L}_\infty - \mathcal{L}_\infty$ Performances

The $\mathcal{L}_\infty$ induced norm is also called *peak-to-peak norm* since it considers the system as an operator mapping the space of signals with finite amplitude (power) to itself (modulo the dimension of the space). This norm is also called $\mathcal{L}_1$-norm and is defined using the second version of the $\mathcal{L}_\infty$-norm (B.8) corresponding to the maximal power.

Lemma D.15 *Consider system (D.25) then if there exists $P = P^T > 0$ and scalars $\beta, \beta, \delta > 0$ such that*

$$\begin{bmatrix} A^TP + PA + \alpha P & PB \\ B^TP & -\beta I \end{bmatrix} \preceq 0 \qquad \begin{bmatrix} \alpha P & 0 & C^T \\ 0 & (\delta-\beta)I & D^T \\ C & D & \delta I \end{bmatrix} \succ 0$$

then the peak-to-peak norm of the system is lower than δ, that is $||H||_{\mathcal{L}_\infty - \mathcal{L}_\infty} < \delta$.

Proof: *The first inequality implies*

$$x(t)^T[A^TP + PA + \alpha P]x(t) + x(t)^TPBw(t) + w(t)^TB^TPw(t) - \beta w(t)^Tw(t) < 0$$

or equivalently

$$\frac{d}{dt}x(t)^TPx(t) + \alpha x(t)^TPx(t) - \beta w(t)^Tw(t) < 0$$

Denoting $V = x(t)^TPx(t)$ hence we have the linear differential inequality

$$\dot{V} + \alpha V - \beta w^Tw < 0$$

and hence we have

$$V < e^{-\alpha t}V(0) + \beta\int_0^t e^{-\alpha(t-s)}w(s)^Tw(s)ds$$

Assuming $x(0) = 0$ and $w(t)^Tw(t) \leq 1$ yields

$$\begin{array}{rcl} V & \leq & \beta\displaystyle\int_0^t e^{-\alpha(t-s)}ds \\ & = & \dfrac{\beta}{\alpha}\left(1 - e^{-\lambda t}\right) \leq \dfrac{\beta}{\alpha} \end{array}$$

Taking a Schur complement of the second inequality yields that

$$\begin{bmatrix} \alpha P & 0 \\ \star & (\delta-\beta)I_p \end{bmatrix} - \frac{1}{\delta}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} \succ 0$$

and hence

$$\alpha x(t)^TPx(t) + (\delta-\beta)w(t)^Tw(t) - \frac{1}{\delta}z(t)^Tz(t) < 0$$

or equivalently

$$z(t)^Tz(t) < \delta\left(x(t)^TPx(t) + (\delta-\beta)w(t)^Tw(t)\right)$$

With the assumption $w(t)^Tw(t) \leq 1$ we thus have

$$\begin{array}{rcl} z(t)^Tz(t) & < & \delta(\beta + \delta - \beta) \\ & = & \delta^2 \end{array}$$

Consequently, the peak-to-peak gain is smaller than δ □

Remark D.16 *It is important to note that this result is a sufficient condition only, that is, the minimal δ is only an upper bound on the real peak-to-peak gain. Moreover, it is also important to note that the conditions are not convex (actually quasiconvex) due to the term αP.*

D.10 $\mathcal{S}$-procedure

The $\mathcal{S}$-procedure allows to deal easily with implications in the LMI framework (but not only). Indeed, we aim to express the following problem

$$\text{for all } \xi \in \mathbb{R}^n \text{ such that } \xi^T M_i \xi \leq 0,\ i = 1, \ldots, N \Rightarrow \xi^T M_0 \xi < 0 \tag{D.36}$$

as an LMI problem.

Lemma D.17 ($\mathcal{S}$-procedure) *If there exist scalars $\tau_1, \ldots, \tau_N \geq 0$ such that*

$$M_0 - \sum_{i=1}^{N} \tau_i M_i \prec 0 \tag{D.37}$$

then (D.36) holds. The converse is not true in general unless $N = 1$ for real valued problems or $N = 2$ for complex valued problems.

Despite of its conservatism, it is a very useful tool in robust analysis and control theory and plays a crucial role in the full-block $\mathcal{S}$-procedure (in some sense) [Scherer, 2001], IQC techniques [Rantzer and Megretski, 1997], Lur'e systems [Lur'e and Postnikov, 1944]...

Historically, the first result of this kind was obtained by Finsler in [Finsler, 1937] (see Appendix D.16) and was later generalized by Hestenes and McShane in [Hestenes and MacShane, 1940]. In the field of automatic control, the idea was certainly first used by Lur'e and Postnikov in [Lur'e and Postnikov, 1944].

The theoretical background was developed some 30 years later by Yakubovich: in the early 70's he proved a theorem known as the $\mathcal{S}$-lemma [Yakubovich, 1971, 1977] using an old theoretical result of Dines [Dines, 1941] on the convexity of homogeneous quadratic mappings. The simplicity of the method allowed rapid advances in control theory. Later, Megretski and Treil extended the results to infinite dimensional spaces giving rise to more general applications [Megretski and Treil, 1993]. Articles written since then mainly discuss some new applications, not new extensions to the theory. Yakubovich himself presented some applications [Yakubovich, 1979], which were followed by many others [Boyd et al., 1994], including contemporary ones [Goldfarb and Iyengar, 2003; Luo, 2003; Rantzer and Megretski, 1997; Scherer, 1997], spanning over a broad range of engineering, financial mathematics and abstract dynamical systems.

Although the result emerged mainly from practice, Yakubovich himself was aware of the theoretical implications of the $\mathcal{S}$-lemma [Fradkov and Yakubovich, 1979]. The theoretical line was then continued by others (see e.g., [Boyd et al., 1994], or recently, [Derinkuyu and Pinar, 2005; Luo et al., 2004; Sturm and Zhang, 2003] but apart from a few exceptions such as [Ben-Tal and Nemirovskii, 2001; Boyd et al., 1994; Iwasaki et al., 1998; Luo, 2003] or [Jönsson, 2001] the results did not reach the control community. The term $\mathcal{S}$-method was coined by Aizerman and Gantmacher in their book [Aizerman and Gantmacher, 1963], but later it changed to $\mathcal{S}$-procedure.

The $\mathcal{S}$-method tries to decide the stability of a system of linear differential equations by constructing a Lyapunov matrix. During the process an auxiliary matrix $\mathcal{S}$ (for stability) is introduced. This construction leads to a system of quadratic equations (the Lure'e resolving equations, [Lur'e and Postnikov, 1944]). If that quadratic system can be solved then a suitable Lyapunov function can be constructed.

The term $\mathcal{S}$-lemma refers to results stating that such a system can be solved under some conditions; the first such result is due to Yakubovich [Yakubovich, 1971]. For a complete survey of the $\mathcal{S}$-lemma, see [Pólik and Terlaky, 2007].

D.11 Small Gain Theorem

Let us consider the interconnection depicted on figure 7.2

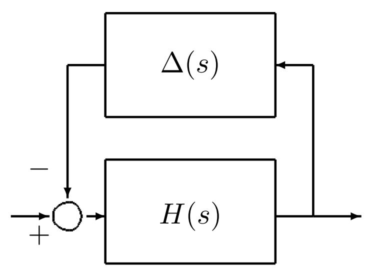

Figure 7.2: Interconnection of systems

It is clear that the closed-loop system $H_{cl}(s) = \dfrac{H(s)}{1+\Delta(s)H(s)}$ and hence the $\mathcal{H}_\infty$ norm of the closed-loop system is bounded if and only if

$$\Delta(j\omega)H(j\omega) \neq 1$$

for all $\omega \in \mathbb{R}$. A sufficient condition can be given in terms of the $\mathcal{H}_\infty$-norm of the product $\Delta(s)H(s)$:

$$||\Delta(s)H(s)||_{\mathcal{H}_\infty} < 1$$

Indeed, the norm of the whole transfer is bounded by

$$||H_{cl}||_{\mathcal{H}_\infty} \leq \frac{||H||_{\mathcal{H}_\infty}}{1-||\Delta H||_{\mathcal{H}_\infty}} \tag{D.38}$$

The latter implies that $||H_{cl}||_{\mathcal{H}_\infty}$ is bounded provided that $||\Delta(s)H(s)||_{\mathcal{H}_\infty} < 1$.

It is well-known that the closed-loop system is stable if and only if $||\Delta H||_{\mathcal{H}_\infty} < 1$. It is not difficult to verify that ΔH satisfies this property by using standard LMI arguments (see the computation of $\mathcal{H}_\infty$-norm in Appendix D.8). Let us assume, for some reasons, that the computation of the $\mathcal{H}_\infty$-norm of the product ΔH cannot be performed. Since the $\mathcal{H}_\infty$-norm is submultiplicative then the inequality

$$||H_{cl}||_{\mathcal{H}_\infty} \leq \frac{||H||_{\mathcal{H}_\infty}}{1-||H||_{\mathcal{H}_\infty}\cdot||\Delta||_{\mathcal{H}_\infty}}$$

implies (D.38) and hence the closed-loop is asymptotically stable if $||H||_{\mathcal{H}_\infty} \cdot ||\Delta||_{\mathcal{H}_\infty} < 1$.

We are able to state the small-gain theorem in a very general fashion:

Theorem D.18 (Small-Gain Theorem) *Let us consider a general interconnection of two blocks, say a transfer function $H(s)$ and a general block $\Delta \in \mathbb{R}^{n_\delta \times n\delta}$ such that $||\Delta||_{\mathcal{H}_\infty} < 1$. The interconnected system is stable if the $\mathcal{H}_\infty$-norm of $H(s)$ satisfies*

$$||H(s)||_{\mathcal{H}_\infty} < 1$$

Moreover, if Δ is unstructured with elements in $\mathbb{C}$, the latter condition is necessary and sufficient.

Proof: *A complete and rigorous proof is given in [Zhou et al., 1996].* □

Assuming that $H(s)$ admits realization (A, B, C, D), it is easy to determine the stability sufficient condition is given by the *Small-Gain Thorem*:

Theorem D.19 (LMI expression of the Small-Gain Theorem) *The closed-loop system is stable if there exist $P = P^T \succ 0$ and $\alpha \in [0, 1)$ such that the LMI holds*

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ \star & -\alpha I & D^T \\ \star & \star & -\alpha I \end{bmatrix} \prec 0$$

D.12 Scalings and Scaled-Small Gain theorem

In order to reduce the conservatism of the small-gain theorem which takes into account norms only, some *scalings* may be introduced in the loop. These scalings do not modify the interconnection but allow for a reduction of conservatism. Let us consider an uncertain square matrix Δ containing, for simplicity, unknown real valued parameters ρ_i and full-blocks gathered on the diagonal:

$$\begin{aligned} \Delta &= \operatorname{diag}(\Delta_s, \Delta_f) \\ \Delta_s &:= \operatorname{diag}_i(\rho_i I_{s_i}) \\ \Delta_f &:= \operatorname{diag}_i(F_i) \end{aligned}$$

where s_i is the number of occurrence of scalar parameter ρ_i and F_i are full-blocks. The idea is to capture the structure of the uncertain matrix Δ by a matrix commutation property

$$L\Delta = \Delta L$$

which can also be defined by an identity relation

$$\Delta = L^{-1} \Delta L$$

The set of scalings corresponding to the uncertain structure Δ is defined by

$$\mathcal{S}(\Delta) := \{L \in \mathbb{S}_{++} : \ L\Delta = \Delta L\}$$

This set enjoys the following properties:

1. $I \in \mathcal{S}(\Delta)$ and therefore the small-gain is a particular case (more conservative) of this approach.
2. $L \in \mathcal{S}(\Delta) \Longrightarrow L^T \in \mathcal{S}(\Delta)$
3. $L \in \mathcal{S}(\Delta) \Longrightarrow L^{-1} \in \mathcal{S}(\Delta)$
4. $L_1 \in \mathcal{S}(\Delta), L_2 \in \mathcal{S}(\Delta) \Longrightarrow L_1L_2\Delta = \Delta L_1L_2$ note that the matrix L_1L_2 is not necessarily symmetric.
5. $\mathcal{S}(\Delta)$ is a convex subset of $\mathbb{R}^k$ where k is the dimension of Δ.

The structure of $L \in S(\Delta)$ can be expressed easily by

$$\begin{array}{rcl} L & = & \text{diag}(L_s, L_f) \\ L_s & = & \underset{i}{\text{diag}}(L_i^s),\ L_i^s \in \mathbb{S}_{++}^{s_i} \\ L_f & = & \underset{i}{\text{diag}}(l_i I_{n_i}) \end{array}$$

where n_i is the size of square full-block F_i. Using this scaling it is possible modify the small-gain theorem into another refined version called *Scaled-Small Gain Theorem*

Theorem D.20 *The closed-loop system is stable if there exist $P = P^T \succ 0$ and $L \in \mathcal{S}(D)$ such that the following LMI holds*

$$\begin{bmatrix} A^TP + PA & PB & C^TL \\ \star & -L & D^TL \\ \star & \star & -L \end{bmatrix} \prec 0 \tag{D.39}$$

Despite of the conservatism reduction, this result is still conservative since it stills considers the norm of the operator and it would be more interesting to capture a more complex (complete) set of uncertainty. Actually the scaled-small gain can be obtained in the dissipativity framework by considering a supply-function:

$$s(w(t), z(t)) = w(t)^T Lw(t) - z(t)^T Lz(t)$$

for some L which satisfies

$$L - \Delta^T L\Delta \succeq 0 \tag{D.40}$$

Thus this means that Δ is contained into a ball centered about 0 but using the scaling L, a degree of freedom is added and is expected to reduce the conservatism.

Remark D.21 *It is important to note that D-scalings may provide a nonconservative stability condition under the assumption the the number of full-blocks n_f and the number of repeated scalar blocks n_s in the uncertainty Δ satisfy the inequality*

$$n_s + nf \leq 3$$

See for instance the very complete paper [Packard and Doyle, 1993] on the complex structured singular value.

However, two main difficulties remains:

1. 0 is always included in the set of values for Δ
2. Δ is again restricted to lie within a ball

The full-block $\mathcal{S}$-procedure solves this problems by translating the uncertainty and considering ellipsoids instead of balls.

D.13 Full-Block $\mathcal{S}$-procedure

The Full-Block $\mathcal{S}$-procedure unifies all the frameworks of scalings into a single one, where small-gain and scaled-small gain results are particular cases only. It allows to consider a large class of uncertainties possibly not including 0 and provides bounds taking the form of ellipsoids instead of balls.

The full-block $\mathcal{S}$-procedure considers a full-block supply-function of the form

$$s(w(t), z(t)) = \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}$$

such that

$$\int_0^{+\infty} s(w(t), z(t))dt \geq 0$$

We have the following theorem:

Theorem D.22 *The closed-loop system is stable if there exist $P = P^T \succ 0$, $Q = Q^T \prec 0$ and $R = R^T \succeq 0$ and S such that the LMIs*

$$\begin{bmatrix} A^TP + PA & PB \\ \star & 0 \end{bmatrix} + \begin{bmatrix} 0 & C^T \\ I & D^T \end{bmatrix} \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \prec 0$$

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \succeq 0 \tag{D.41}$$

hold.

Proof: *A proof is given in Section 2.3.4.4 or in [Scherer, 1999].* □

To see that the full-block $\mathcal{S}$-procedure allows to consider more general uncertainties, let us consider the LMI (D.41) and try to express in a form similar to (D.40).

(D.41) is equivalent to

$$\Delta^T Q\Delta + \Delta^T S + S^T \Delta + R \succeq 0$$

Completing the squares we get

$$(\Delta^T + S^T Q^{-1})Q(\Delta + Q^{-1}S) + R - S^T Q^{-1}S \succeq 0$$

where it is possible to see that

1. the uncertainty is translated by $Q^{-1}S$
2. the latter expression is clearly the expression of a general ellipsoid.

Moreover, for a given ellipsoid (Δ_0, Z, W):

$$(\Delta - \Delta_0)^T(-Z)(\Delta - \Delta_0) \preceq W$$

it is possible to determine the values Q, S, R using the relations:

$$\begin{aligned} Q &= Z \\ S &= Q\Delta_0 \\ R &= W - \Delta_0^T Z\Delta_0 \end{aligned}$$

D.14 Dualization Lemma

The dualization lemma (which has been introduced simultaneously but separately in [Scherer, 1999] and [Iwasaki and Hara, 1998] allows to turn an LMI into another equivalent one provided that some strong assumptions are satisfied. First we need the following result:

Proposition D.23 *Let $P \in \mathbb{R}^{n\times n}$ a nonsingular matrix and define two matrices $S_k \in \mathbb{R}^{n\times k}$ and $S_\ell \in \mathbb{R}^{n\times \ell}$ satisfying* $\det\left(\begin{bmatrix} S_k & S_\ell \end{bmatrix}\right)$ *(i.e. forms a basis of $\mathbb{R}^n$) with $n = k + \ell$. We have the following fact:*

If $S_k^T P S_k \prec 0$ and $S_\ell^T P S_\ell \succ 0$ then P has exactly k negative and ℓ positive eigenvalues.

Conversely, if P has exactly k negative and ℓ positive eigenvalues, then there exist $S_k \in \mathbb{R}^{n\times k}$ and $S_\ell \in \mathbb{R}^{n\times \ell}$ satisfying $\det\left(\begin{bmatrix} S_k & S_\ell \end{bmatrix}\right)$.

Proof: *The proof is provided in [Scherer and Weiland, 2005].* □

Lemma D.24 (Dualization Lemma) *Let $M \in \mathbb{S}^n$ nonsingular and $S \in \mathbb{R}^{n\times p}$ with* $\operatorname{rank}(S) = p < n$ *such that the number $n^-(M)$ of negative eigenvalues of M satisfies $n^-(M) = rank(S) = p$. In this case, the following statements are equivalent:*

1. *The LMI $S^T M S \prec 0$ holds*
2. *The LMI $S^{\perp T} M^{-1} S^{\perp} \prec 0$ holds where $S^\perp$ is a basis of the orthogonal complement of $Im(S)$ (i.e. $S^T S^\perp = 0$).*

Proof: *The proof is provided in [Scherer and Weiland, 2005].* □

At first sight, this result may seem superfluous, but actually it is very useful in the robust/LPV control context. Indeed, when using multipliers to study systems expressed through LFR, it has the property of decoupling data matrices from multipliers and Lyapunov matrix, making the problem convex [Scherer, 1999; Wu, 2003].

However, the rank constraint is a very strong condition and such lemma is difficult to apply in general. For instance, by considering time-delay systems and the Lyapunov-Krasovskii theorems, the rank condition is generally not satisfied due to the presence of a high number of Lyapunov matrices.

D.15 Bounding Lemma

The bounding lemma [de Souza and Li, 1999; Khargonekar et al., 2001; Petersen, 1987; Xie et al., 1992] is used to remove uncertainties from matrix inequalities in the robust analysis/control framework. It deals with both real and complex parameter uncertainties. We provide here the real version of the result:

Lemma D.25 *Let $\Psi \in \mathbb{S}^n$ a symmetric matrix and $P \in \mathbb{R}^{m\times n}, Q \in \mathbb{R}^{p\times n}$ and $\Delta(t) \in \mathbf{\Delta}$ be an uncertain matrix (possibly time-varying) satisfying*

$$\mathbf{\Delta}(R) := \{\Delta(t) \in \mathbb{R}^{m\times p} : p \leq m,\ \Delta^T\Delta \leq R,\ R > 0\}$$

then the following statements are equivalent:

1. *The LMI*
$$\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^T P \prec 0 \tag{D.42}$$
holds for all $\Delta(t) \in \mathbf{\Delta}(R)$

2. *There exists a scalar* $\varepsilon > 0$ *such that the LMI*
$$\Psi + \varepsilon P^T P + \varepsilon^{-1} Q^T R Q \prec 0$$
holds.

Proof: It seems interesting to provide the proof of this result. It is actually an old (and interesting result) and then the proof is not easy to find since, generally, provided references are not the original one. The original paper where this result has been provided for the first time is [good question, I am still looking for it]. Moreover, most of the technical results involved in the proof can be found in [Khargonekar et al., 2001; Petersen, 1987]. Without loss of generality, let us consider for simplicity here that $R = I$.

Sufficiency:

Assume that matrices Ψ, P *and* Q *contain decision matrices such that (D.42) is an LMI. Assume there exist* Ψ_0, P_0, Q_0 *and* $\varepsilon_0 > 0$ *such that* $\Psi_0 + \varepsilon_0 P_0^T P_0 + \varepsilon^{-1} Q_0^T Q_0 \prec 0$ *holds.*

We immediately need the following well-known fact:

Proposition D.26 *For any matrices* X *and* Y *with appropriate dimensions, we have* $X^TY + Y^TX \preceq \beta X^TX + \beta^{-1}Y^TY$, for any $\beta > 0$. *The latter inequality is a consequence of the inequality* $(\beta^{-1/2}X - \beta Y)^T(\beta^{-1/2}X - \beta^{1/2}Y) \succeq 0$.

Whatever the inertia of the matrix inequality $\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^TP$, *there always a scalar* $\varepsilon > 0$ *such that*

$$\begin{array}{rcl}\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^TP & \preceq & \Psi + \varepsilon P^TP + \varepsilon^{-1}Q^T\Delta^T\Delta Q \\ & preceq & \Psi + \varepsilon P^TP + \varepsilon^{-1}Q^TQ \qquad \text{for some } \varepsilon > 0\end{array}$$

Hence by assumption, the left-hand side is negative definite if we choose $\varepsilon = \varepsilon_0$, $P = P_0$, $Q = Q_0$ *and* $\Psi = \Psi_0$. *The sufficiency is shown.*

Necessity:

Before showing the necessity we need the following results proved in [Petersen, 1987].

Lemma D.27 *Given any* $x \in \mathbb{R}^n$ *we have*

$$\max_{\Delta(t)\in\mathbf{\Delta}(\mathbf{I})} \left\{(x^TM_1M_2\Delta(t)M_3x)^2\right\} = x^TM_1M_2M_2^TM_1xx^TM_3^TM_3x$$

where $M_1 = M_1^T$.

Lemma D.28 *Let* X, Y *and* Z *be given* $r \times r$ *matrices such that* $X \succeq 0$, $Y \prec 0$ *and* $Z \succeq 0$. *Furthermore, assume that*
$$(\xi^TY\xi)^2 - 4(\xi^TX\xi\xi^TZ\xi) > 0$$
for all $\xi \in \mathbb{R}^r$ *with* $\xi \neq 0$. *Then there exists a constant* $\lambda > 0$ *such that*

$$\lambda^2X + \lambda Y + Z \prec 0$$

The proof of necessity follows the same lines as the proof of Theorem 2.3 of [Petersen, 1987] and is recalled here. Assume that there exists Ψ_0, Q_0 and P_0 such that

$$\Psi_0 + P_0^T \Delta(t) Q_0 + Q_0^T \Delta(t)^T P_0 \prec 0$$

holds. Assume also the LMI is satisfied for the nominal system (i.e. $\Delta(t) = 0$), therefore $\Psi_0 \in \mathbb{S}^n_{--}$ and we have

$$\begin{array}{lcl} \Psi_0 & \prec & -P_0^T \Delta(t) Q_0 - Q_0^T \Delta(t)^T P_0 \\ x^T \Psi_0 x & < & -2x^T P_0^T \Delta(t) Q_0 x, \text{ for all} x \in \mathbb{R}^n \\ x^T \Psi_0 x & < & -2 \max\limits_{\Delta(t) \in \boldsymbol{\Delta}(I)} \{x^T P_0^T \Delta(t) Q_0 x\} \\ (x^T \Psi_0 x)^2 & > & 4 \max\limits_{\Delta(t) \in \boldsymbol{\Delta}(\mathbf{I})} \{(x^T P_0^T \Delta(t) Q_0 x)^2\} \end{array}$$

By application of Lemma D.27 with $M_1 = I$, $M_2 = P^T$ and $M_3 = Q$, we get

$$\begin{array}{r} (x^T \Psi x)^2 \quad > 4x^T P^T P x x^T Q^T Q x \\ (x^T \Psi x)^2 - 4x^T P^T P x x^T Q^T Q x > 0 \end{array}$$

Note that $P_0^T P_0 \succeq 0$, $Q_0^T Q_0 \succeq$ and $\Psi_0 \prec 0$ hence Lemma D.28 applies with $Y = \Psi_0$, $X = P_0^T P_0$ and $Z = Q_0^T Q_0$. Therefore there exists $\lambda > 0$ such that

$$\lambda^2 P_0^T P_0 + \lambda \Psi_0 + Q_0^T Q_0 \prec 0$$

Finally, multiplying the latter inequality by λ^{-1} and letting $\varepsilon = \lambda^{-1}$ we get inequality

$$\Psi_0 + \varepsilon P_0^T P_0 + \varepsilon^{-1} Q_0^T Q_0 \prec 0$$

This concludes the proof of sufficiency. □

There also exist a 'dual' version of the previous lemma where the uncertainty satisfies $\Delta(t)\Delta(t)^T < R$ in the case $m \leq p$. In this case we obtain

Lemma D.29 *Let $\Psi \in \mathbb{S}^n$ a symmetric matrix and $P \in \mathbb{R}^{m \times n}, Q \in \mathbb{R}^{p \times n}$ and $\Delta(t) \in \boldsymbol{\Delta}'$ be an uncertain matrix (possibly time-varying) with*

$$\boldsymbol{\Delta}' := \{\Delta(t) \in \mathbb{R}^{m \times p} : m \leq p, \ \Delta\Delta^T \leq R, \ R > 0\}$$

then the following statements are equivalent:

1. *The LMI*
$$\Psi + P^T \Delta(t) Q + Q^T \Delta(t)^T P \prec 0$$
holds for all $\Delta(t) \in \boldsymbol{\Delta}'$

2. *There exists a scalar $\varepsilon > 0$ such that the LMI*
$$\Psi + \varepsilon P^T R P + \varepsilon^{-1} Q^T Q \prec 0$$
holds.

The bounding lemma can neither be used to deal with rational uncertainties nor dynamical operators (such as dynamical systems or infinite dimensional operators...). This is the main drawback of the bounding lemma but, on the other hand, it provides simple and easy to use results in many cases and this motivates its utilization in many works. The bounding-lemma provides the same result as the scaled-small gain for one single full uncertainty block. We aim to show now that with this framework it is possible to retrieve small-gain and the full-block multiplier results. Extensions have also been provided in [Shcherbakov and Topunov, 2008].

Equivalence with scaled-small gain

In the scaled-small gain result, the uncertainty are assumed to satisfy the commutative relation

$$L\Delta = L\Delta,\ L = L^T \succ 0 \tag{D.43}$$

and therefore we have $\Delta = L^{-1}\Delta L$. Finally, we get the following result:

Lemma D.30 *Let $\Psi \in \mathbb{S}^n$ a symmetric matrix and $P \in \mathbb{R}^{m\times n}, Q \in \mathbb{R}^{m\times n}$ and $\Delta(t) \in \boldsymbol{\Delta}$ be an uncertain matrix (possibly time-varying) with*

$$\boldsymbol{\Delta}_1 := \{\Delta(t) \in \mathbb{R}^{m\times m} :\ \Delta^T\Delta \leq I\}$$

then the following statements are equivalent:

1. *The LMI*
$$\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^TP \prec 0$$
holds for all $\Delta(t) \in \boldsymbol{\Delta}$

2. *The LMI*
$$\Psi + P^TL^{-1}\Delta(t)LQ + Q^TL^T\Delta(t)^TL^{-T}P \prec 0$$
holds for all $\Delta(t) \in \boldsymbol{\Delta}_1$ and some $L \in \mathcal{S}(\Delta)$.

3. *There exists a scalar $\tilde{L} \in \mathcal{S}(\Delta)$ such that the LMI*
$$\begin{bmatrix} \Psi + P^T\tilde{L}P & Q^T\tilde{L} \\ \star & -\tilde{L} \end{bmatrix} \prec 0 \tag{D.44}$$
holds.

Proof: *The equivalence between the first and second statement is done by replacing Δ by $L^{-1}\Delta L$. The third statement is obtained by similar argument than for obtaining statement two of lemma D.29. Then a change of variable $\tilde{L} \leftarrow \varepsilon L$ and a Schur's complement leads to LMI (D.44).* □

To see clearly the equivalence with the scaled-small gain, let us consider system

$$\dot{x} = (A + B\Delta C)x$$

which can be rewritten as an interconnection depicted in Figure 7.2 where $H(s) = C(sI - A)^{-1}B$. The robust stability of the system is ensured if there exists $Z = Z^T \succ 0$ such that the LMI

$$(A + B\Delta C)^TZ + Z(A + B\Delta C) \prec 0$$

holds. This LMI can be rewritten in the form

$$\Psi + \mathcal{P}^T\Delta(t)\mathcal{Q} + \mathcal{Q}^T\Delta(t)^T\mathcal{P} \prec 0$$

where $\Psi = A^TZ + ZA$, $\mathcal{P}^T = ZB$ and $\mathcal{Q} = C$. Apply lemma D.30, we obtain

$$\begin{bmatrix} \Psi + Z^T\tilde{L}Z & Q^T\tilde{L} \\ \star & -\tilde{L} \end{bmatrix} \prec 0$$

which is identical to

$$\begin{bmatrix} A^TP + PA + PB\tilde{L}B^TP & C^T \\ \star & -\tilde{L} \end{bmatrix} \prec 0$$

A Schur complement on the latter inequality and letting $\tilde{L}' = \tilde{L}$ (see properties of the set $\mathcal{S}(\Delta)$) yields

$$\begin{bmatrix} A^TP + PA & PB & C^T\tilde{L}' \\ \star & -\tilde{L}' & 0 \\ \star & \star & -\tilde{L}' \end{bmatrix} \prec 0$$

which is exactly the scaled bounded real lemma.

Equivalence with full-block $\mathcal{S}$-procedure

Let us consider the set of uncertainties $\boldsymbol{\Delta}_q$ defined by

$$\boldsymbol{\Delta}_q := \left\{ \Delta \in \mathbb{R}^{m\times p} : p \le m, \begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} U & V \\ \star & W \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \preceq 0 \right\}$$

Now consider equation (D.42) and rewrite it into

$$\Psi + \begin{bmatrix} P^T & Q^T \end{bmatrix} \begin{bmatrix} 0 & \Delta(t) \\ \Delta(t)^T & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \prec 0 \tag{D.45}$$

We need to transform the quadratic inequality defining the set $\boldsymbol{\Delta}_q$. Note that in virtue of the dualization lemma ([Scherer, 1999] or Appendix D.14) we have

$$\begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} xU & V \\ \star & W \end{bmatrix}^{-1} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} \prec 0$$

Let $\begin{bmatrix} U & V \\ \star & W \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{U} & \tilde{V} \\ \star & \tilde{W} \end{bmatrix}$ and expand the latter inequality

$$\begin{aligned} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} \tilde{U} & \tilde{V} \\ \star & \tilde{W} \end{bmatrix} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} &= \tilde{U} - \tilde{V}\Delta^T - \Delta\tilde{V}^T + \Delta\tilde{W}\Delta^T \prec 0 \\ &= (\Delta - \tilde{V}\tilde{W}^{-1})\tilde{W}(\Delta^T - \tilde{W}^{-1}\tilde{V}^T) + \tilde{U} - \tilde{V}\tilde{W}\tilde{V}^T \prec 0 \end{aligned}$$

Since $\tilde{W} \succ 0$ and $\tilde{U} \prec 0$ then $\tilde{U} - \tilde{V}\tilde{W}\tilde{V}^T \prec 0$. Let $U' = \tilde{U} - \tilde{V}\tilde{W}\tilde{V}^T$, $V' = \tilde{V}\tilde{W}^{-1}$ and $W' = \tilde{W}^{-1}$ hence the latter inequality is equivalent to

$$(\Delta - V')(W')^{-1}(\Delta^T - V'^T) + U' \prec 0$$

A Schur complement yields

$$\begin{bmatrix} U' & \Delta - V' \\ \star & -W' \end{bmatrix} \prec 0$$

and finally we have

$$\begin{bmatrix} 0 & \Delta \\ \star & 0 \end{bmatrix} \prec \begin{bmatrix} -U' & V' \\ \star & W' \end{bmatrix}$$

Now substitute the bound on matrix $\begin{bmatrix} 0 & \Delta \\ \star & 0 \end{bmatrix}$ into inequality (D.45) leads to

$$\Psi + \begin{bmatrix} P^T & Q^T \end{bmatrix} \begin{bmatrix} -U' & V' \\ \star & W' \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \prec 0$$

Despite of the apparent difference with results obtained from the full-block $\mathcal{S}$-procedure, they are actually identical. We have used a linearization procedure which has turned the quadratic definition of the uncertainty set into a linear definition. This linear definition has been used to bound the uncertainty into the LMI. A similar result has been provided in Scherer [1996].

D.16 Finsler's Lemma

The Finsler's lemma [Finsler, 1937; Jacobson, 1977; Pólik and Terlaky, 2007; Skelton et al., 1997] is a very useful tool in robust control to deal with LMI-defined constraints. Initially provided in [Finsler, 1937], the lemma was stated as follows

Lemma D.31 *Let S_1 and S_2 by symmetric matrices of the same dimension such that $x^T S_2 x = 0$ with $x \neq 0$ implies $x^T S_1 x > 0$ then there exists $y \in \mathbb{R}$ such that $S_1 + y S_2$ is positive definite.*

In control theory, it is defined in a very general manner which consists in an assembly of related results:

Lemma D.32 *The following statements are equivalent:*

1. *$x^T M x < 0$ for all $x \in \mathcal{X} := \{x \in \mathbb{R}^n - \{0\} : Bx = 0\}$*

2. *There exists a scalar $\tau \in \mathbb{R}$ such that $M - \tau B^T B \prec 0$ and if such τ exists, it must satisfy*
$$\tau > \tau_{min} := \lambda_{max}([D^T(M - MB_\perp(B_\perp^T M B_\perp)^{-1} B_\perp M)D]$$
where $D := (B_r B_l^T)^{-1/2} B_l^+$ with (B_r, B_l) is any full rank factor of B (i.e. $B = B_l B_r$) and $B_\perp$ is any basis of the nullspace of B.

3. *There exists a symmetric matrix X such that $M - B^T X B \prec 0$ holds.*

4. *There exists an **unconstrained** matrix N such that*
$$M + N^T B + B^T N \prec 0$$

5. *The LMI $B_\perp^T M B_\perp \prec 0$ holds where $B_\perp$ is any basis of the nullspace of B.*

6. *There exists a matrix $W \in \mathbb{S}_+^{n+m}$ and a scalar $\tau > 0$ such that*
$$\begin{bmatrix} M & B^T \\ B & -\tau I_m \end{bmatrix} \prec W \qquad \text{rank}(W) = m$$

In statements 1) and 2), we can recognize the original Finsler's lemma where $S_1 = M$ and $S_2 = B^T B$. Statement 3) is the 'matrix version' of the Finsler's lemma which is defined in [Skelton et al., 1997]. Statements 3) and 4) can be shown equivalent using elementary algebra. Statement 5) can be retrieved from 4) using the projection lemma (Appendix D.18) or conversely, 4) can be obtained from 5) through the creation lemma (inverse procedure of the elimination/projection lemma). Finally, statement 6) has been recently added in Kim and Moon [2006] to deal with reduced-order output feedback and constrained controllers [Kim and Moon, 2006; Kim et al., 2007] (e.g. decentralized controllers).

Remark D.33 *Let us point out that if the matrix N is constrained (has a specific structure) then the equivalence is lost and statement 4 implies the others only.*

D.17 Generalization of Finsler's lemma

A generalization of the Finsler's lemma has been provided in [Iwasaki, 1998; Scherer, 1997] and is recalled here. Indeed, the Finsler's lemma is generally applicable when the matrix B is known and hence the basis of the null-space can be easily computed. This generalization allows for the use of unknown matrices.

Lemma D.34 *Let matrices $M = M^T$, B and a compact subset of real matrices $\mathcal{K}$ be given. The following statements are equivalent:*

1. *for each $K \in \mathcal{K}$*
$$x^T M x < 0, \qquad \forall\ x \neq 0 \text{ s.t. } KFx = 0$$
2. *there exists $Z = Z^T$ such that*
$$\begin{array}{lcl} M + F^T Z F & \prec & 0 \\ \mathrm{Ker}[K]^T Z \mathrm{Ker}[K] & \succeq & 0 \qquad \forall\ K \in \mathcal{K} \end{array} \tag{D.46}$$

Proof: *Suppose 1) holds. Choose $K \in \mathcal{K}$ arbitrarily then in virtue of the Finsler's lemma (Appendix D.16) there exists a real scalar τ such that*

$$M + \tau F^T K^T K F \prec 0$$

Since $\mathcal{K}$ is compact then τ can be chosen independently of K. Hence we have

$$M \prec -F^T S F \qquad \forall\ S \in \{\tau K^T K :\ K \in \mathcal{K}\}$$

It has been shown in [Iwasaki, 1998] that the latter inequality is equivalent to the existence of a symmetric matrix Z such that

$$M + F^T Z F \prec 0 \qquad \text{and } -Z \preceq \tau K^T K, \quad \forall\ K \in \mathcal{K}$$

Then performing a congruence transformation on the second inequality with respect to Ker$[K]$ *yields*

$$\mathrm{Ker}[K]^T Z \mathrm{Ker}[K] \succeq 0 \qquad \forall\ K \in \mathcal{K}$$

Suppose now 2) holds. Set $x \neq 0$ and $K \in \mathcal{K}$ such that $KFx = 0$. Then it is possible to find η such that $Fx = \mathrm{Ker}[K]\eta$ and hence we have

$$\begin{array}{rcl} x^T(M + F^T Z F)x & < & 0 \\ x^T M x & < & -x^T F^T Z F x < -\eta^T \mathrm{Ker}[K]^T Z \mathrm{Ker}[K]\eta \leq 0 \end{array}$$

and we get 1). □

D.18 Projection Lemma

The projection lemma is used to remove a decision matrix and gives a necessary and sufficient condition to the existence of such a matrix. Generally, the controller matrix is removed to obtain LMIs instead of a BMI [Apkarian and Gahinet, 1995; Scherer, 1999]. It is also called the *elimination lemma* since it is used to eliminate decision matrices. The reverse operation is generally referred to as the *creation lemma* and is useful in robust analysis [Gouaisbaut and Peaucelle, 2006b].

Lemma D.35 *Let $\Psi \in \mathbb{S}^n$ and P, Q matrices of appropriate dimensions, then the following statements are equivalent:*

1. *There exists an* ***unconstrained*** *matrix Ω such that*

$$\Psi + P^T\Omega Q + Q^T\Omega^T P \prec 0$$

2. *The two following underlying LMIs hold*

$$\begin{array}{rcl} \mathrm{Ker}[P]^T\Psi\mathrm{Ker}[P] & \prec & 0 \\ \mathrm{Ker}[Q]^T\Psi\mathrm{Ker}[Q] & \prec & 0 \end{array}$$

3. *There exists two scalars $\tau_1, \tau_2 \in \mathbb{R}$ such that*

$$\begin{array}{rcl} \Psi - \tau_1 P^T P & \prec & 0 \\ \Psi - \tau_2 Q^T Q & \prec & 0 \end{array}$$

Proof: The proof is based on the proof of Gahinet and Apkarian. First it is aimed to show that 1) is equivalent to 2). The equivalence between 2) and 3) is a consequence of the Finsler Lemma (Appendix D.16).

1) ⇒ 2) *This implication is trivial. It suffices to pre and post-multiply by both* Ker$[P]$ *and* Ker$[Q]$*. This yields statement 2).*

1) ⇒ 2)

This part of the proof is more involved but is based on elementary linear algebra and LMIs. Let us consider the following matrix

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{PQ} & \mathcal{B}_P & \mathcal{B}_Q \end{bmatrix}$$

in which $\mathcal{K}_{PQ}$ is a basis of the kernel of $[P] \cap [Q]$. The matrices $\mathcal{K}_P$ and $\mathcal{K}_Q$ are defined such that $\mathcal{N}_P = \begin{bmatrix} \mathcal{K}_{PQ} & \mathcal{K}_P \end{bmatrix}$ and $\mathcal{N}_Q = \begin{bmatrix} \mathcal{K}_{PQ} & \mathcal{K}_Q \end{bmatrix}$ are respectively the basis of the null-spaces of P and Q respectively. Finally, define $\mathcal{K}_r$ such that

$$\mathcal{T} := \begin{bmatrix} \mathcal{K}_{PQ} & \mathcal{K}_P & \mathcal{K}_Q & \mathcal{K}_r \end{bmatrix}$$

is invertible. In this case,

$$\Psi + P^T \Omega Q + Q^T \Omega^T P \prec 0$$

is equivalent to

$$\mathcal{T}^T \left[\Psi + P^T \Omega Q + Q^T \Omega^T P\right] \mathcal{T} \prec 0$$

Define $P\mathcal{T} = \begin{bmatrix} 0 & 0 & P_1 & P_2 \end{bmatrix}$ *and* $Q\mathcal{T} = \begin{bmatrix} 0 & Q_1 & 0 & Q_2 \end{bmatrix}$ *then we have*

$$\Psi + P^T \Omega Q + Q^T \Omega^T P = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \star & \Psi_{22} & \Psi_{23} + \Upsilon_{11}^T & \Psi_{24} + \Upsilon_{21}^T \\ \star & \star & \Psi_{33} & \Psi_{34} + \Upsilon_{12} \\ \star & \star & \star & \Psi_{44} + \Upsilon_{22} + \Upsilon_{22}^T \end{bmatrix}$$

where

$$\mathcal{T} \Omega \mathcal{T} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \star & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \star & \star & \Psi_{33} & \Psi_{34} \\ \star & \star & \star & \Psi_{44} \end{bmatrix}$$

and

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{bmatrix} := \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} \Omega \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$$

Using the Schur complement, we get the two underlying LMIs

$$\Lambda := \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \star & \Psi_{22} & \Psi_{23} + \Upsilon_{11}^T \\ \star & \star & \Psi_{33} \end{bmatrix} \succ 0$$

$$\Psi_{44} + \Upsilon_{22} + \Upsilon_{22}^T - \begin{bmatrix} \Psi_{14} \\ \Psi_{24} + \Upsilon_{21}^T \\ \Psi_{34} + \Upsilon_{12} \end{bmatrix}^T \Lambda^{-1} \begin{bmatrix} \Psi_{14} \\ \Psi_{24} + \Upsilon_{21}^T \\ \Psi_{34} + \Upsilon_{12} \end{bmatrix} \prec 0$$

Note that given Υ_{11}*,* Υ_{12} *and* Υ_{21} *such that* $\Lambda \prec 0$*, it is always possible to find* Υ_{22} *such that the second LMI is satisfied.* □

The assumption that Ω is unconstrained plays a central role in the proof and in the equivalence between the two statements. This means that when dealing with constrained controllers having a given structure, equivalence is lost and statement 2 may admit a solution while statement 1 does not (but this is not always the case). For instance, in some papers, the authors remove uncertain or symmetric terms invoking the projection lemma. This is uncorrect since, for the first case, the projection lemma provides an existence condition of the removed matrix and it is not sought to find an uncertainty for which the condition is satisfied…the feasibility of the LMI must be satisfied for all uncertain terms belonging in a known defined set; in the second case, the matrix is symmetric and hence constrained which does not fall into the projection lemma conditions of application.

D.19 Completion Lemma

This theorem shows that it is possible to construct a matrix and its inverse from only block of each only. It has consequences in the construction of Lyapunov matrices in the dynamic output feedback synthesis problem [Packard et al., 1991].

Theorem D.36 *Let $X \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{S}^n_{++}$. There exist $X_2 \in \mathbb{R}^{n\times r}$, $X_3 \in \mathbb{R}^{r\times r}$, $Y_2 \in \mathbb{R}^{n\times r}$ and $Y_3 \in \mathbb{R}^{r\times r}$ such that*

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} \succ 0 \qquad \text{and} \qquad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \succeq 0 \qquad \text{and} \qquad \text{rank}\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r$$

Proof: *The proof is based on simple linear algebra.*

Sufficiency:

From LMI

$$\begin{bmatrix} X & I_n \\ \star & Y \end{bmatrix}$$

we can state that $X - Y^{-1} \succeq 0$. Then it is possible to set a matrix $X_2 \in \mathbb{R}^{n\times r}$ such that $X_2 X_2^T = X - Y^{-1}$ and hence we have $X - X_2 X_2^T \succ 0$ which is equivalent to the LMI

$$\begin{bmatrix} X & \tilde{X}_2 \\ \star & I_r \end{bmatrix} \succ 0$$

Performing a congruence with respect to $\text{diag}(I_n, \tilde{X}_3^T)$ *with $\tilde{X}_3$ nonsingular yields*

$$\begin{bmatrix} X & X_2 \\ \star & X_3 \end{bmatrix} \succ 0$$

with $X_2 = \tilde{X}_2 \tilde{X}_3$ and $X_3 = \tilde{X}_3^T \tilde{X}_3$. This shows that it is possible to complete the matrix with X_2 and X_3 such that the completed matrix is positive definite. According to the Banachiewicz inversion formula (see [Banachiewicz, 1937] or Appendix A) it is then possible to define the inverse of this matrix. This proves sufficiency.

Necessity:

Using the Banachiewicz inversion formula we can state that

$$Y = X^{-1} + X^{-1} X_2 (X_3 - X_2^T X^{-1} X_2)^{-1} X_2^T X^{-1}$$

. Moreover, since $\text{rank}[X_3 - X_2^T X^{-1} X_2] = r$ *then we have*

$$X^{-1} X_2 (X_3 - X_2^T X^{-1} X_2)^{-1} X_2^T X^{-1} \in \mathbb{S}^n_+$$

and this implies $Y \succeq X^{-1}$ and $\text{rank}[Y - X^{-1}] \leq r$. *Finally, we can conclude that*

$$\begin{bmatrix} X & I_n \\ \star & Y \end{bmatrix} \succeq 0$$

and $\text{rank}[X] + \text{rank}[Y - X^{-1}] \leq n + r$. *According to the Guttman rank additivity formula [Guttman, 1946] the rank condition is equivalent to*

$$\text{rank}\begin{bmatrix} X & I_n \\ \star & Y \end{bmatrix} \leq n + r$$

Necessity is proved. □

Due to the rank condition, the problem is not convex. This problem occurs when fixed-order controllers of order $r < n$ are sought and is responsible of the difficulty of synthesizing such controllers. However, it turns out that when full-order controllers are considered, the problem becomes convex. Indeed, when $r = n$, we have the following corollary which does not contain any rank condition anymore:

Corollary D.37 *Let $X \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{S}^n_{++}$. There exist $X_2, X_3, Y_2, Y_3 \in \mathbb{R}^{n\times r}$ such that*

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} \succ 0 \qquad \text{and} \qquad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \succ 0$$

As a final remark on the completion lemma for $r = n$, it is important to say that for any given $X, Y \succ 0$, the set of admissible values for X_2, X_3, Y_2 and Y_3 is not reduced to a singleton but contains an infinite number of solution. To see this, let us consider the matrix equation

$$\begin{bmatrix} X & X_2 \\ \star & X_3 \end{bmatrix} \begin{bmatrix} Y & Y_2 \\ \star & Y_3 \end{bmatrix} = I_{2n}$$

or equivalently

$$\begin{array}{llll} \text{a) } XY + X_2Y_2 & = I_n & \text{b) } XY_2 + X_2Y_3 & = 0 \\ \text{c) } X_2^TY + X_3Y_2^T & = 0 & \text{d) } X_2^TY_2 + X_3Y_3 & = I_n \end{array}$$

Assume that the matrices X, Y are known and that X_2 is invertible, then

$$\begin{array}{rrcl} \text{a)} \;\Rightarrow & Y_2^T & = & X_2^{-1}(I_n - XY) \\ & Y_2 & = & (I_n - YX)X_2^{-T} \end{array}$$

Since $\text{rank}[Y - X^{-1}] = n$ then $I_n - XY$ is nonsingular implying that Y_2 is invertible.

$$\begin{array}{rrcl} \text{b)} \;\Rightarrow & Y_3 & = & -X_2^{-1}XY_2 \\ & & = & -X_2^{-1}X(I_n - YX)X_2^{-T} \\ \text{c)} \;\Rightarrow & X_3 & = & -X_2^TYY_2^{-T} \\ & & = & -X_2^TY(I_n - YX)^{-1}X_2 \end{array}$$

Then substituting the explicit values of Y_2, Y_3 and X_3 into $X_2^TY_2 + X_3Y_3$ should be equal to I_n. If it does not, this means that X_2 should be chosen as nonsingular. If it does, this would mean that any nonsingular X_2 can be chosen to complete the matrices.

$$\begin{array}{rcl} X_2^TY_2 + X_3Y_3 & = & X_2^T(I_n - YX)X_2^{-T} + X_2^TY(I_n - YX)^{-1}X_2X_2^{-1}X(I_n - YX)X_2^{-T} \\ & = & X_2^T\left[I_n + Y(I_n - YX)^{-1}X\right][I_n - YX]\,X_2^{-T} \end{array}$$

Using the Duncan inversion formulae (Appendix A or [Duncan, 1917]), it is easy to show that

$$(I_n - XY)^{-1} = I_n + Y(I_n - YX)^{-1}X$$

and hence

$$X_2^T Y_2 + X_3 Y_3 = I_n$$

This shows that by choosing any nonsingular matrix X_2, it is possible to complete the matrices adequately. The same reasoning holds when Y_2 is chosen first instead of X_2. Hence, the completed matrices are given by

$$\begin{array}{rcl}\begin{bmatrix} X & X_2 \\ \star & X_3 \end{bmatrix} & = & \begin{bmatrix} X & X_2 \\ \star & -X_2^T Y(I_n - YX)^{-1} X_2 \end{bmatrix} \\ \begin{bmatrix} Y & Y_2 \\ \star & Y_3 \end{bmatrix} & = & \begin{bmatrix} Y & (I_n - YX)X_2^{-T} \\ \star & -X_2^{-1} X(I_n - YX)X_2^{-T} \end{bmatrix}\end{array}$$

From a computational point of view, it is important to compute matrices X and Y whose product is far from identity in order to have the eigenvalues of the matrix $I_n - XY$ far from 0. Hence, it is interesting to replace the LMI

$$\begin{bmatrix} X & I_n \\ \star & Y \end{bmatrix} \succ 0 \qquad \text{by} \qquad \begin{bmatrix} X & \alpha I_n \\ \star & Y \end{bmatrix} \succ 0$$

for some positive scalar α. By maximizing α, the minimal eigenvalue of XY is maximized and hence pushes it away from 1. This allows to avoid bad conditioning when inverting $I - YX$.

D.20 Application of the Projection Lemma

This appendix shows an application of the Projection Lemma in the context of the synthesis of a parameter dependent dynamic output feedback. The synthesis is performed using the scaled-small gain theorem.

Let us consider the following LPV system in 'LFT' form

$$\begin{array}{rcl} \dot{x} & = & Ax(t) + B_0 w(t) + B_1 u(t) \\ z(t) & = & C_0 x(t) + D_{00} w(t) + D_{01} u(t) \\ y(t) & = & C_1 x(t) + D_{10} w(t) \end{array}$$

where x, u, w, z and y are respectively the system state, the control input, the parameters input, the parameters output and the measured output. We seek a controller of the form:

$$\begin{bmatrix} \dot{x}_c(t) \\ z_c(t) \\ u(t) \end{bmatrix} = \Omega \begin{bmatrix} x_c(t) \\ w_c(t) \\ y(t) \end{bmatrix}$$

where x_c, w_c and z_c are respectively the controller state, the parameter input and the parameter output. The parameters input and output are defined by

$$\begin{bmatrix} w(t) \\ w_c(t) \end{bmatrix} = \mathrm{diag}(\Theta(\rho), \Theta(\rho)) \begin{bmatrix} z(t) \\ z_c(t) \end{bmatrix}$$

From this description, the system is scheduled by the parameters through the signals w and z while the controller is scheduled through the signals w_c and z_c. We introduce the scaling L is defined such that

$$L\,\mathrm{diag}(\Theta(\rho), \Theta(\rho)) = \mathrm{diag}(\Theta(\rho), \Theta(\rho))L$$

It is possible to rewrite the system as

$$\left[\begin{array}{c} \dot{x} \\ \dot{x}_c \\ \hline z \\ z_c \\ \hline x_c \\ w_c \\ y \end{array}\right] = \left[\begin{array}{cc|cc|ccc} A & 0 & B_0 & 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hline C_0 & 0 & D_{00} & 0 & 0 & 0 & D_{01} \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ C_1 & 0 & D_{10} & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ x_c \\ \hline w \\ w_c \\ \hline \dot{x}_c \\ z_c \\ u \end{array}\right]$$

The closed-loop system is given by

$$\begin{bmatrix} \dot{\bar{x}}(t) \\ \bar{z}(t) \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{B}_1\Omega\bar{C}_1 & \bar{B}_0 + \bar{B}_1\Omega\bar{D}_{10} \\ \bar{C}_0 + \bar{D}_{01}\Omega\bar{C}_1 & \bar{D}_{00} + \bar{D}_{01}\Omega\bar{D}_{10} \end{bmatrix}$$

where

$$\begin{array}{lllllllll} \bar{A} & = & \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} & \bar{B}_0 & = & \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} & \bar{B}_1 & = & \begin{bmatrix} 0 & 0 & B_1 \\ I & 0 & 0 \end{bmatrix} \\ \bar{C}_0 & = & \begin{bmatrix} C_0 & 0 \\ 0 & 0 \end{bmatrix} & \bar{D}_{00} & = & \begin{bmatrix} D_{00} & 0 \\ 0 & 0 \end{bmatrix} & \bar{D}_{01} & = & \begin{bmatrix} 0 & 0 & D_{01} \\ 0 & I & 0 \end{bmatrix} \\ \bar{C}_1 & = & \begin{bmatrix} 0 & I \\ 0 & 0 \\ C_1 & 0 \end{bmatrix} & \bar{D}_{10} & = & \begin{bmatrix} 0 & 0 \\ 0 & I \\ D_{10} & 0 \end{bmatrix} & & & \end{array}$$

The stability of the closed-loop system is ensured, in virtue of the scaled-small gain theorem if the following nonlinear matrix inequality is satisfied

$$\begin{bmatrix} (\bar{A} + \bar{B}_1\Omega\bar{C}_1)^T P + P(\bar{A} + \bar{B}_1\Omega\bar{C}_1) & P(\bar{B}_0 + \bar{B}_1\Omega\bar{D}_{10}) & (\bar{C}_0 + \bar{D}_{01}\Omega\bar{C}_1)^T \\ \star & -L & (\bar{D}_{00} + \bar{D}_{01}\Omega\bar{D}_{10})^T \\ \star & \star & -L^{-1} \end{bmatrix} \prec 0$$

which can be rewritten into

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} & P\bar{B}_0 & \bar{C}_0^T \\ \star & -L & \bar{D}_{00}^T \\ \star & \star & -L^{-1} \end{bmatrix} + \begin{bmatrix} P\bar{B}_1 \\ 0 \\ \bar{D}_{01} \end{bmatrix} \Omega \begin{bmatrix} \bar{C}_1 & \bar{D}_{00} & 0 \end{bmatrix} + (\star)^T \prec 0$$

Let

$$\begin{array}{lllll} P & = & \begin{bmatrix} P_{11} & P_{12} \\ \star & P_{22} \end{bmatrix} & X = P^{-1} & = \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \\ L & = & \begin{bmatrix} L_{11} & L_{12} \\ \star & L_{22} \end{bmatrix} & J = L^{-1} & = \begin{bmatrix} J_{11} & J_{12} \\ \star & J_{22} \end{bmatrix} \end{array}$$

A basis of the null space of $\begin{bmatrix} \bar{C}_1 & \bar{D}_{10} & 0 \end{bmatrix}$ is given by

$$\operatorname{Ker} \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ C_1 & 0 & D_{10} & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} N_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]$$

with $C_1N_1 + D_{10}N_2 = 0$ and a basis of the null space of $\begin{bmatrix} P\bar{B}_1 \\ 0 \\ \bar{D}_{01} \end{bmatrix}^T$ is given by

$$\mathrm{Ker}\left[\begin{array}{ccc} P\begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & P\begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & D_{01} \\ 0 & I & 0 \end{array}\right]^T = \mathrm{diag}(X, I, I)\left[\begin{array}{ccc} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

where $B_1^T M_1 + D_{01}^T M_2 = 0$. Hence, in virtue of the projection lemma we get the two underlying matrix inequalities:

$$\left[\begin{array}{ccc} N_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T \begin{bmatrix} \bar{A}^T P + P\bar{A} & P\bar{B}_0 & \bar{C}_0^T \\ \star & -L & \bar{D}_{00}^T \\ \star & \star & -L^{-1} \end{bmatrix} \left[\begin{array}{ccc} N_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0$$

$$\left[\begin{array}{ccc} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]^T \begin{bmatrix} X\bar{A}^T + \bar{A}X & \bar{B}_0 & X\bar{C}_0^T \\ \star & -L & \bar{D}_{00}^T \\ \star & \star & -L^{-1} \end{bmatrix} \left[\begin{array}{ccc} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \prec 0$$

Removing lines and columns corresponding to zero lines and columns to null-spaces leads to

$$\left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T \left[\begin{array}{c|c|cc} A^T P_{11} + P_{11}A & P\bar{B}_0 & \bar{C}_0^T & 0 \\ \hline \star & -L_{11} & D_{00}^T & 0 \\ \hline \star & \star & -J_{11} & -J_{12} \\ \star & \star & \star & -J_{22} \end{array}\right] \left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0$$

$$\left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \end{array}\right]^T \left[\begin{array}{c|cc|c} X_{11}A^T + X_{11}A & \bar{B}_0 & 0 & X_{11}\bar{C}_0^T \\ \hline \star & -L_{11} & -L_{12} & \bar{D}_{00}^T \\ \star & \star & -L_{22} & 0 \\ \hline \star & \star & \star & -J_{11} \end{array}\right] \left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \end{array}\right] \prec 0$$

Reorganize columns and rows yields

$$\left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T \left[\begin{array}{c|c|cc} A^T P_{11} + P_{11}A & P\bar{B}_0 & \bar{C}_0^T & 0 \\ \hline \star & -L_{11} & D_{00}^T & 0 \\ \hline \star & \star & -J_{11} & -J_{12} \\ \star & \star & \star & -J_{22} \end{array}\right] \left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0$$

$$\left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline M_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T \left[\begin{array}{c|c|cc} X_{11}A^T + X_{11}A & X_{11}\bar{C}_0^T & \bar{B}_0 & 0 \\ \hline \star & -J_{11} & D_{00} & 0 \\ \hline \star & \star & -L_{11} & -L_{12} \\ \star & \star & \star & -L_{22} \end{array}\right] \left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline M_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0$$

Finally applying Schur's complement (see Appendix D.4), we get

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^T \left(\begin{bmatrix} A^T P_{11} + P_{11} A & P\bar{B}_0 \\ \star & -L_{11} \end{bmatrix} + \begin{bmatrix} \bar{C}_0^T & 0 \\ D_{00}^T & 0 \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ \star & L_{22} \end{bmatrix} (\star)^T \right) \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \prec 0$$
$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^T \left(\begin{bmatrix} X_{11} A^T + X_{11} A & X_{11}\bar{C}_0^T \\ \star & -J_{11} \end{bmatrix} + \begin{bmatrix} \bar{B}_0 & 0 \\ D_{00} & 0 \end{bmatrix} \begin{bmatrix} J_{11} & J_{12} \\ \star & J_{22} \end{bmatrix} (\star)^T \right) \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \prec 0$$

and equivalently

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^T \left(\begin{bmatrix} A^T P_{11} + P_{11} A & P\bar{B}_0 \\ \star & -L_{11} \end{bmatrix} + \begin{bmatrix} \bar{C}_0^T \\ D_{00}^T \end{bmatrix} L_{11} \begin{bmatrix} \bar{C}_0^T \\ D_{00}^T \end{bmatrix}^T \right) \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \prec 0$$
$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^T \left(\begin{bmatrix} X_{11} A^T + X_{11} A & X_{11}\bar{C}_0^T \\ \star & -J_{11} \end{bmatrix} + \begin{bmatrix} \bar{B}_0 \\ D_{00} \end{bmatrix} J_{11} \begin{bmatrix} \bar{B}_0 \\ D_{00} \end{bmatrix}^T \right) \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \prec 0$$

The above matrix inequalities are LMIs. Indeed, by considering only one block of each matrix and their inverse, the condition is LMI. Moreover, the whole matrices P, X, L, J can be constructed from these blocks using singular value decomposition (Appendix A.6) and completion lemma (Appendix D.19).

To illustrate this, we will construct P. Note first that it is possible to construct P_{12} and X_{12} from P_{11} and X_{11} using the singular value decomposition. Indeed, we have

$$P_{11}X_{11} + P_{12}X_{12}^T = I$$

and perform a singular value decomposition on $I - P_{11}X_{11} = U^T \Sigma V$, by identification we can choose

$$P_{12} = U^T\Sigma^{1/2} \text{ and } X_{12} = V^T\Sigma^{1/2}$$

Finally, P is the solution of the algebraic equation

$$P \begin{bmatrix} X_{11} & I \\ X_{12}^T & 0 \end{bmatrix} = \begin{bmatrix} I & P_{11} \\ 0 & P_{12}^T \end{bmatrix}$$

In an identical way, the other matrices can be computed.

D.21 Parseval's Theorem

The Parseval's theorem allows to bridge the energy of a signal in the time-domain to an expression into the frequency domain. It has been first proven by Marc-Antoine Parseval des Chênes in his thesis [Parseval des Chênes, 1806] pertaining on the resolution of linear second order partial differential equations with constant coefficients. This equality is heavily used in IQC analysis [Rantzer and Megretski, 1997] where it is used to connect time-domain properties and frequency domain properties of signals.

Theorem D.38 *Let $x(t)$ be a $\mathcal{L}_2$ signal and define its Fourier transform as $X(\omega)$ where $\omega = 2\pi f$ and f is the frequency then the following equality holds*

$$\int_{-\infty}^{+\infty} ||x(t)||_2^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} ||X(\omega)||_2^2 d\omega$$

Proof:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} ||x(t)||_2^2 dt &= \int_{-\infty}^{+\infty} x(t)^* x(t) dt \\
&= \int_{-\infty}^{+\infty} \left[\left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega')^* e^{-j\omega' t} d\omega' \right) \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega \right) \right] dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[X(\omega') \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(X(\omega) \left(\int_{-\infty}^{+\infty} e^{j(\omega-\omega')t} dt \right) \right) d\omega \right] d\omega'
\end{aligned}
$$

Note that $\int_{-\infty}^{+\infty} e^{j(\omega-\omega')t} dt = 2\pi\delta(\omega-\omega')$ *by the definition of the Dirac pulse* δ *and the Fourier transform. This leads to*

$$
\begin{aligned}
\int_{-\infty}^{+\infty} ||x(t)||_2^2 dt &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[X(\omega') \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(X(\omega) \cdot 2\pi\delta(\omega-\omega') \right) d\omega \right] d\omega' \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[X(\omega') \int_{-\infty}^{+\infty} \left(X(\omega)\delta(\omega-\omega') \right) d\omega \right] d\omega' \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)^* X(\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} ||X(\omega)||_2^2 d\omega
\end{aligned}
$$

□

We have the following corollary where a symmetric matrix M is inserted in the energy expression:

Theorem D.39 *Let* $x(t)$ *be a* $\mathcal{L}_2$ *signal and define its Fourier transform as* $X(\omega)$ *where* $\omega = 2\pi f$ *and* f *is the frequency then the following equality holds*

$$
\int_{-\infty}^{+\infty} x(t)^* M x(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)^* M X(\omega) d\omega
$$

Proof: *The proof follows the same lines as for the standard version of the Parseval's theorem.*
□

It is possible to consider a more complete form for the Parseval's theorem which consider a frequency weighting through the use of the frequency dependent weighting matrix $\hat{M}(j\omega)$:

Theorem D.40 *Let* $x(t)$ *be a* $\mathcal{L}_2$ *signal and define its Fourier transform as* $X(\omega)$ *where* $\omega = 2\pi f$ *and* f *is the frequency then the following equality holds*

$$
\int_{-\infty}^{+\infty} \sigma(x(t), x_f(t))^* dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)^* \hat{M}(j\omega) X(\omega) d\omega
$$

where $\sigma(x(t), x_f(t))$ *is a quadratic form and* $\dot{x}_f(t) = A_f x_f(t) + B_f x(t)$.

For example, if $\sigma(x(t), x_f(t)) = \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix}^T M \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix}$ then $\hat{M}(j\omega)$ is given by

$$
\hat{M}(j\omega) = \begin{bmatrix} I & B_f^*(j\omega I - A_f)^{-*} \end{bmatrix} M \begin{bmatrix} I \\ (j\omega I - A_f)^{-1} B_f \end{bmatrix}
$$

E Technical Results in Time-Delay Systems

In this Appendix, we will give the reader further results used in the time-delay stability analysis framework. We will consider, in the following, the time-delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h(x(t-h(t))) + Ew(t) \\ z(t) & = & Cx(t) + C_h x(t-h(t)) + Fw(t) \end{array} \tag{E.47}$$

E.1 Jensen's Inequality

This inequality has been first proven in [Jensen, 1806] and has been inspired from the inequality of arithmetic and geometric means introduced by Cauchy in [Cauchy, 1821]. The Jensen's result has found many applications in statistical physics, information theory through the *Gibbs' inequality*, in statistics through the well-known *Rao-Blackwell Theorem* [Blackwell, 1947; Durrett, 2005; Øksendal, 2003; Rao, 1945, 1973], in automatic control and stability analysis of dynamical systems [Gouaisbaut and Peaucelle, 2006b; Gu et al., 2003] and almost surely in many other fields.

Definition E.1 *Let ϕ be a convex function and $f(x)$ is integrable over $[a,b]$, $a<b$. Then the following inequality holds*

$$\phi\left(\int_a^b f(x)dx\right) \le (b-a)\int_a^b \phi(f(x))dx$$

The Jensen's inequality is often used in the $\mathcal{H}_\infty$ norm analytical computation of integral operators in time-delay systems framework. It is also used in approaches based on Lyapunov-Krasovskii functionals as an efficient bounding technique. An example of application is given below:

$$\left(\int_{t-h}^t \dot{x}(\theta)d\theta\right)^T P \left(\int_{t-h}^t \dot{x}(\theta)d\theta\right) \le h\int_{t-h}^t \dot{x}(\theta)^T P\dot{x}(\theta)d\theta$$

with $P = P^T \succ 0$. The convex function is $\phi(z) = z^T Pz$ and $f(t) = \dot{x}(t)$.

E.2 Bounding of cross-terms

The use of model-transformations for stability analysis and control synthesis of time-delay systems may lead to annoying terms referred to as *cross-terms*, generally involving products between signals and integrals. A common cross-term is

$$-2x(t)^T A^T PA_h \int_{t-h}^t \dot{x}(s)ds$$

and appears, for instance, when the Newton-Leibniz model-transformation is used with a quadratic Lyapunov-Razumikhin function of the form $V(x(t)) = x(t)^T Px(t)$.

Proposition E.2 *For any $Z = Z^T \succ 0$ we have*

$$\begin{array}{rcl} \pm 2x(t)^T A^T PA_h \displaystyle\int_{t-h}^t \dot{x}(s)ds & \le & x(t)^T A^T ZAx(t) + \left(\displaystyle\int_{t-h}^t \dot{x}(s)ds\right)^T A_h^T PZ^{-1}PA_h \left(\displaystyle\int_{t-h}^t \dot{x}(s)ds\right) \\ & \le & x(t)^T A^T ZAx(t) + h\displaystyle\int_{t-h}^t \dot{x}(s)^T A_h^T PZ^{-1}PA_h\dot{x}(s)ds \\ & \le & hx(t)^T A^T ZAx(t) + \displaystyle\int_{t-h}^t \dot{x}(s)^T A_h^T PZ^{-1}PA_h\dot{x}(s)ds \end{array}$$

Proof: *The idea is to use completion by the squares, the first line is obtained by writing*

$$\begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h \int_{t-h}^{t} \dot{x}(s)ds \end{bmatrix}^T \begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h \int_{t-h}^{t} \dot{x}(s)ds \end{bmatrix} \geq 0$$

for some $Z = Z^T \succ 0$. *Expand the latter expression leads to first inequality. Then apply Jensen's inequality onto the quadratic integral term leads to second inequality. Finally, the last inequality is obtained by completion of the squares too but in another fashion:*

$$\int_{t-h}^{t} \begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h\dot{x}(s)ds \end{bmatrix}^T \begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h\dot{x}(s) \end{bmatrix} \geq 0$$

Expanding the latter quadratic form leads to the last inequality. □

The latter bounding technique is relatively inaccurate since the cross terms may admits negative values while the right-hand side term is always positive. This drove Park to introduce a new bound [Park, 1999; Park et al., 1998] and is generally referred to as *Park's bound*. The idea is to use a more complete completion by the squares and is given below in the original Park's terminology:

Lemma E.3 *Assume that* $a(\alpha) \in \mathbb{R}^{n_x}$ *and* $b(\alpha) \in \mathbb{R}^{n_y}$ *are given for* $\alpha \in \Omega$. *Then, for any positive definite matrix* $X \in \mathbb{R}^{n_x \times n_x}$ *and any matrix* $M \in \mathbb{R}^{n_y \times n_y}$, *the following holds*

$$-2\int_{\Omega} b(\alpha)^T a(\alpha)d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & XM \\ M^TX & (M^TX+I)X^{-1}(XM+I) \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}$$

This model transformation has led to a great improvement of results at this time (see comparison with contemporary results in Park [1999]; Park et al. [1998]). The obtained result is presented in Section 3.2.1.5. Inspired from the latter bound, another one has been employed in Moon et al. [2001] and is sometimes referred as Moon's inequality.

Lemma E.4 *Assume that* $a(\cdot) \in \mathbb{R}^{n_a}$, $b(\cdot) \in \mathbb{R}^{n_b}$ *and* $\mathcal{N}(\cdot) \in \mathbb{R}^{n_a \times n_a}$ *are defined on the interval* Ω. *Then, for any matrices* $X \in \mathbb{R}^{n_a \times n_a}$, $Y \in \mathbb{R}^{n_a \times n_b}$ *and* $Z \in \mathbb{R}^{n_b \times n_b}$, *the following holds*

$$-2\int_{\Omega} a(\alpha)^T \mathcal{N} b(\alpha)d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & Y-\mathcal{N} \\ \star & Z \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}$$

where

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0$$

Proof: *See Moon et al. [2001].* □

Although this result is less accurate than the Park's bound, its more simple form allows for easy design techniques than using Park's inequality.

E.3 Padé Approximants

This appendix introduces the Padé approximation of a continuous function. This approximation is of great interest in the framework of time-delay systems [Zhang et al., 1999]. System (E.47) with constant time-delay h can be rewritten as an interconnection of two subsystems:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h w_0(t) \\ z_0(t) & = & x(t) \\ z(t) & = & Cx(t) + C_h w_0(t) \\ w_0(t) & = & z_0(t-h) \end{array}$$

which can be written in the frequency domain as

$$\begin{array}{rcl} H_1(s) & = & C(sI-A)^{-1}B \\ H_2(s) & = & e^{-sh} \end{array}$$

using the interconnection of Figure 7.2.

In order to analyze stability of the interconnection it may be interesting to approximate the operator e^{-sh} by a proper (stable) transfer function. Power series cannot be used since the transfer function would be not proper. The Padé approximants play here an important role by approximating a function by a rational function with arbitrary degree for the denominator and numerator.

Let us consider a function $f(x)$ which is sought to be approximated by a rational function $R_{m,n}(x)$ defined as

$$R_{m,n}(x) := \frac{P_m(x)}{Q_n(x)} = \frac{\sum_{i=0}^m a_i x^i}{\sum_{i=0}^n b_i x^i}$$

where polynomials $P_m(x)$ and $Q_n(x)$ are of degree m and n respectively. These polynomials can be found using a relation linking the truncated power series of $f(x)$ and polynomials $P_m(x)$ and $Q_n(x)$. The truncated power series $Z_m(x)$ of $f(x)$ of degree m is given by

$$Z_m(x) := \sum_{i=0}^m c_i x^i$$

In this case we look for a_i and b_i such that

$$\sum_{i=0}^m c_i x^i = \frac{P_m(x)}{Q_n(x)}$$

or equivalently

$$Q_n(x) \sum_{i=0}^m c_i x^i = P_m(x)$$

This results into an homogenous system of $n+m+1$ equations with $n+m+2$ unknowns and so admits infinitely many solutions. However, it can be shown that the generated rational functions $R_{m,n}(x)$ are all the same (the obtained polynomials are not prime at a constant factor). Table 7.2 summarizes few of Padé approximants for the exponential function e^z with complex argument z:

$m \backslash n$	1	2	3
0	$\dfrac{1}{1-z}$	$\dfrac{1}{1-z+\frac{1}{2}z^2}$	$\dfrac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3}$
1	$\dfrac{1+\frac{1}{2}z}{1-\frac{1}{2}z^2}$	$\dfrac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$	$\dfrac{1+\frac{1}{4}z}{1-\frac{3}{4}z+\frac{1}{4}z^2-\frac{1}{24}z^3}$
2	$\dfrac{1+\frac{2}{3}z+\frac{1}{6}z^3}{1-\frac{1}{3}z}$	$\dfrac{1+\frac{1}{2}z+\frac{1}{12}z^2}{1-\frac{1}{2}z+\frac{1}{12}z^2}$	$\dfrac{1+\frac{2}{5}z+\frac{1}{20}z^2}{1-\frac{3}{5}z+\frac{3}{20}z^2-\frac{1}{60}z^3}$
3	$\dfrac{1+\frac{3}{4}z+\frac{1}{4}z^2+\frac{1}{24}z^3}{1-\frac{1}{4}z}$	$\dfrac{1+\frac{3}{5}z+\frac{3}{20}z^2+\frac{1}{60}z^3}{1-\frac{2}{5}z+\frac{1}{20}z^2}$	$\dfrac{1+\frac{1}{2}z+\frac{1}{10}z^2+\frac{1}{120}z^3}{1-\frac{1}{2}z+\frac{1}{10}z^2-\frac{1}{120}z^3}$

Table 7.2: First Padé's approximants of the function e^s

The column $n = 0$ has been omitted since it coincides with the truncated power series.

A particularity of Padé approximants of the exponential is the regularity of the numerator and the denominator when $m = n$. Indeed, denote $N_m(z)$ the numerator of $R_{m,m}(z)$ and then we have

$$R_m(z) := R_{m,m}(z) = \frac{N_m(z)}{N_m(-z)}$$

It is proved in [Zhang et al., 1999] that the proper transfer function is asymptotically stable for all $m \geq 0$, that is the polynomial $N_m(-z)$ has all its roots in the complex left-half plane.

E.4 Maximum Modulus Principle

The maximum modulus principle is an interesting result very useful in complex analysis which is necessary to study the bounded of some function norms. See for instance [Levinson and Redheffer, 1970].

Definition E.5 *A complex function is said to be* holomorphic *at $z = z_0$ if it is complex differentiable at $z = z_0$, i.e. if the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

A widely used synonym for 'holomorphic' is *analytical* which is very used by physicists, engineers... Mathematicians prefer the term 'holomorphic function' or 'holomorphic map'.

Definition E.6 *A complex function is holomorphic on some set U if it is holomorphic at every point $z_0 \in U$.*

Theorem E.7 *Let f be a holomorphic function on come connected open subset $D \subset \mathbb{C}$ and taking complex values. If z_0 is a point such that*

$$|f(z_0)| \geq |f(z)| \tag{E.48}$$

for all z in any neighborhood of z_0, then the function f is constant on D.

This can viewed otherwise, if f is an holomorphic function f over a connected open subset D, then its modulus cannot $|f|$ exhibit a true local maximum on D. Hence the maximum modulus is attained on the boundary of ∂D. This has strong consequences in system theory, as illustrated in the following example:

Example E.8 *This example shows how the maximum modulus principle can be used in order to prove the stability of a system. Let us consider for simplicity a SISO system $H(s) = N(s)/D(s)$ where $N(s)$ and $D(s)$ are arbitrary. The system is proper if the degree of $N(s)$ is lower than the degree of $D(s)$ and it is asymptotically stable if all the zeros of $D(s)$ have negative real part ($H(s)$ has all its poles located in $\mathbb{C}^-$). Hence this means that the modulus of $H(s)$ denoted by $|H(s)|$ is bounded for all $\mathbb{C}^+ \cup \mathbb{C}^0$. By the maximum modulus principle the maximum cannot be reached in the interior of $\mathbb{C}^+$ hence it suffices to consider the boundary $\partial\mathbb{C}^+$ only to check the boundedness of $|H(s)|$ over $\mathbb{C}^+$ only. Noting that $\partial\mathbb{C}^+ = \mathbb{C}^0 \cup \{+\infty\}$ (the boundary of $\mathbb{C}^+$ is constituted of the imaginary axis $\mathbb{C}^0$ and a point at infinity) then this means that if $|H(s)|$ is bounded over $\partial\mathbb{C}^+$ we have*

- *$|H(j\omega)| < +\infty$ for all $\omega \in \mathbb{R}$ and hence $H(s)$ has no poles on the imaginary axis.*
- *$|H(+\infty)| < +\infty$ then the transfer function $H(s)$ is proper.*

This implies that the stability of a system can be checked only by verifying the boundedness of the transfer function over the boundary of $\mathbb{C}^+$. This can be easily generalized to MIMO systems by considering the modulus of singular values. This is the definition of the $\mathcal{H}_\infty$-norm and this justifies that the following equality for a strictly proper MIMO transfer function:

$$\sup_{s \in \mathbb{C}^+} \bar{\sigma}(H(s)) = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(H(j\omega))$$

For more information about the maximum modulus principle please refer for instance to [Krantz, 2001; Levinson and Redheffer, 1970; Solomentsev, 2001; Titchmarsh, 1939].

E.5 Argument principle

The winding number of a closed-curve C in the plane around a given point z_0 is number representing the total number of times that curve travels counterclockwise around the point z_0. The winding number depends on the orientation of the curve, and is negative if the curve travels around the point clockwise. This notion has given rise to the celebrated Nyquist criterion and is also a first step towards the elaboration of the Rouché's theorem presented in Appendix E.6.

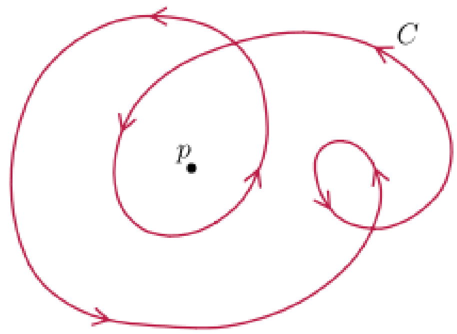

Figure 7.3: This curve has winding number two around the point p

Theorem E.9 (Nyquist Criterion) *Let us consider an open-loop transfer function $H_{ol}(s)$ with N unstable poles. The corresponding closed-loop system is asymptotically stable if and only if the open-loop transfer function $H_{ol}(s)$ travels N times around the critical point -1 counterclockwise when s sweeps the imaginary axis.*

More generally we have the following theorem which is also called the argument principle:

Theorem E.10 *Let $f(z)$ be a function and C be a closed contour on $\mathbb{C}$ such that no poles and zeros are on C but C may contain any poles and zeros ($f(z)$ is meromorphic inside C), then the following formula holds:*

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi j(N - P) \tag{E.49}$$

denote respectively the number of zeros and poles of $f(z)$ inside the contour C, with each zero and pole counted as many times as its multiplicity and order respectively.

More generally, suppose that C is a curve, oriented counter-clockwise, which is contractible to a point inside an open set D in the complex plane. For each point $z \in D$, let $n(C, z)$ be the winding number of C around the point z. Then

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi j \left(\sum_a n(C, a) - \sum_b c(C, b) \right) \tag{E.50}$$

where the first summation is over all zeros a of f counted with their multiplicities, and the second summation is over the poles b of f. This makes a connection between the maximal principle and the winding number of a function of a complex variable. For more information about this please refer to [Levinson and Redheffer, 1970].

E.6 Proof of Rouché's theorem

This theorem is important in complex analysis and has important consequence in the stability analysis of time-delay systems. It can be used in order to get some information on the number of zeros of a function on a compact set without computing them.

The theorem is recalled below for readability:

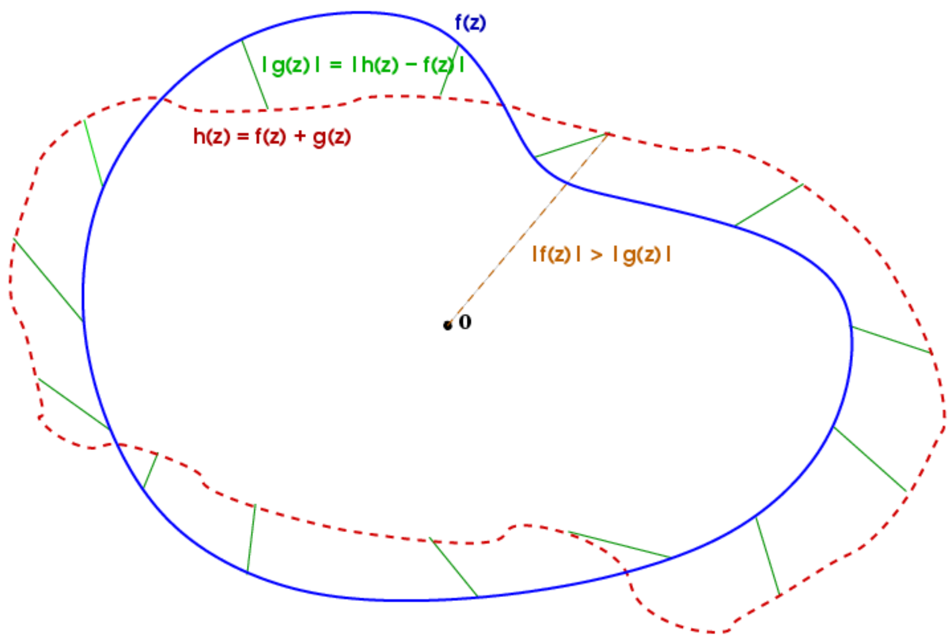

Figure 7.4: Illustration of the meaning of the Rouché's theorem

Theorem E.11 *Given two functions f and g analytic (holomorphic) inside and on a contour C. If $|g(z)| < |f(z)|$ for all z on C, then f and $f+g$ have the same number of roots inside C.*

Let us define the function h such that $h = f + g$. It is holomorphic since it is the sum of two holomorphic functions. From the argument principle (see appendix E.5), we have

$$N_h - P_h = I_h(C, 0) = \frac{1}{2\pi j} \oint_C \frac{h'(z)}{h(z)} dz$$

where N_h is the number of zeroes of h inside C, P_h is the number of poles, and $I_h(C, 0)$ is the winding number of $h(C)$ about 0. Since h is analytic inside and on C, it follows that $P_h = 0$ and then

$$N_h = I_h(C, 0) = \frac{1}{2\pi j} \oint_C \frac{h'(z)}{h(z)} dz$$

One has that $\dfrac{h'(z)}{h(z)} = \mathcal{D}[\log(h(z))]$, where $\mathcal{D}$ denotes the complex derivative. Keeping in mind that $h = f + g$, we find

$$\begin{aligned}
N_h &= \frac{1}{2\pi j}\oint_C \frac{h'(z)}{h(z)} \\
&= \frac{1}{2\pi j}\oint_C \mathcal{D}[\log(h(z))]dz \\
&= \frac{1}{2\pi j}\oint_C \mathcal{D}[\log(f(z)+g(z))] \\
&= \frac{1}{2\pi j}\oint_C \mathcal{D}\left[\log\left(f(z)\left(1+\frac{g(z)}{f(z)}\right)\right)\right]dz \\
&= \frac{1}{2\pi j}\oint_C \frac{f'(z)}{f(z)} + \frac{1}{2\pi j}\oint_C \frac{\mathcal{D}(1+g(z)/f(z))}{1+g(z)/f(z)}dz \\
&= I_f(C,0) + I_{1+g(z)/f(z)}(C,0)
\end{aligned}$$

The winding number of $1+g/f$ over C is zero. This is because we supposed that $|g(z)| < |f(z)|$, so g/f is constrained to a circle of radius 1, and adding 1 to g/f shifts it away from zero, and thus $1+g/f$ is constrained to a circle of radius 1 about 1, and C under $1+g/f$ cannot wind around 0. Finally we get

$$N_h = I_f(C,0)$$

which equals to N_f or the number of zeros of f. This concludes the proof.□

Example E.12 *An example of application is the determination of the number of roots of a* 3^{th} *order polynomial, say* z^3+z^2-1*, contained in the disk* $|z|<2$*. The idea is to remove the higher order term to use it as a bound on the rest of the polynomial. Indeed, define* $f(z)=z^3$ *and* $g(z)=z^2-1$*, the contour is defined by* $|z|=2$*. Hence for all* z *on this contour we have* $|g(z)|\le 5$ *and* $|f(z)|=8$ *showing that we have* $|g(z)|<|f(z)|$ *for any* z *such that* $|z|=2$*. This shows that* z^3 *and* z^3+z^2-1 *have the same number of zeros in the disc* $|z|<2$*, which is 3.*

For more information about the Rouché's theorem, please refer to [Levinson and Redheffer, 1970].

Bibliography

M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, New York, USA, 1972.

M.A. Aizerman and F.R. Gantmacher. *Absolute Stability of Regulator Systems.* Holden-Day Series in Information Systems, Holden-Day, Inc, San Francisco 1964. Originally published as 'Absolutnaya Ustoichivost' Reguliruyemykh Sistem' by the Academy of Sciences of the USSR, Moscow, 1963.

R.J. Anderson and M.W. Spong. Bilateral control of teleoperators with time-delay. *IEEE Transations on Automatic Control*, 34(5):494–501, 1989.

R.M. Anderson and R.M. May. Directly transmitted infectious diseases: control by vaccination. *Science*, 215:1053–1060, 1982.

R.M. Anderson and R.M. May. *Infectious Diseases of Humans: Dynamics and Control.* Oxford University Press, 2002.

P. Apkarian and R.J. Adams. Advanced gain-scheduling techniques for uncertain systems. *IEEE Transactions on Automatic Control*, 6:21–32, 1998.

P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers. *IEEE Transactions on Automatic Control*, 5:853–864, 1995.

P. Apkarian and D. Noll. IQC analysis and synthesis via nonsmooth optimization. *Systems & Control Letters*, 55:971–981, 2006.

P. Apkarian and D. Noll. Nonsmooth optimization for multiband frequency domain control design. *Automatica*, 43:724–731, 2007.

P. Apkarian and H.D. Tuan. Parametrized LMIs in control theory. In *Conference on Decision and Control, Tampa, Florida*, 1998.

P. Apkarian, J. Chretien, P. Gahinet, and J.M. Biannic. μ-synthesis by D-K iterations with constant scalings. In *European Control Conference*, 1993.

P. Apkarian, D. Noll, and A. Rondepierre. Nonsmooth optimization algorithm for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis. In *46th IEEE Conference on Decision and Control, New Orleans, Louisiana, 2007*, 2007.

Y. Ariba and F. Gouaisbaut. Delay-dependent stability analysis of linear systems with time-varying delay. In *46th IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007.

Y. Ariba, F. Gouaisbaut, and D. Peaucelle. Stability analysis of time-varying delay systems in quadratic separation framework. In *the International conference on mathematical problems in engineering , aerospace and sciences (ICNPAA'08), June 25-27 2008, Genoa, Italy*, 2008.

K.J. Åström. Automatic control - the hidden technology. In *Proceedings of 5th European Control Conference, Karlsruhe, Germany*, 1999.

G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith. *μ Analysis and Sysnthesis Toolbox.* The MathWorks, Natick, USA, 1998.

T. Banachiewicz. Zur berechnung der determinanten, wie auch der inversen, und zur darauf basierten auflosung der systeme linearer gleichungen. *Acta Astronomica, Serie C*, 3:41–67, 1937.

D.I. Barnea. A method and new results for stability and instability of autonomous functional differential equations. *SIAM Journal on Applied Mathematics*, 17(4):681–697, 1969.

R. H. Bartels, J. C. Beatty, and B. A. Barsky. *An Introduction to Splines for Use in Computer Graphics and Geometric Modelling.* Morgan Kaufmann, San Francisco, USA, 1998.

M. Bartha. Convergence of solutions for an equation with state-dependent delay. *Journal of Mathematical Analysis and Applications*, 254:410–432, 2001.

J. Belair and M. C. Mackey. Consumer memory an dprice fluctuations in commodity markets: An integrodifferential model. *Journal of Dynamics and Differential Equations*, 1(3):299–325, 1989.

L. Belkoura, J.P. Richard, and M. Fliess. Real time identification of time-delay systems. In *IFAC Workshop on Time-Delay Systems*, Nantes, France, 2007.

L. Belkoura, J.P. Richard, and M. Fliess. A convolution approach for delay systems identification. In *IFAC World Congress*, Seoul, Korea, 2008.

R.E. Bellman and K.L. Cooke. *Differential Difference Equations.* Academic press, New York, USA, 1963.

A. Ben-Tal and A. Nemirovskii. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications.* SIAM Series on Optimization, Philadelphia, 2001.

A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems - 2^{nd} Edition.* Springer, 2006.

K.P.M. Bhat and H.N. Koivo. An observer theory for time-delay systems. *IEEE Transactions on Automatic Control*, 216(2):166–169, 1976.

R. Bhatia. *Matrix Analysis.* Springer, 1997.

S. Bittanti and P. Colaneri. *Periodic control.* in Encyclopedia of Electrical and Electronics Engineering, J. Webster, Ed. New York: Wiley, vol. 16, pp. 59-73., 1999.

S. Bittanti and P. Colaneri. *Periodic Control Systems 2001.* Proc. IFAC Workshop, S. Bittanti and P. Colaneri (Eds.), Cernobbio-Como, Italy, Aug. 27-28, 2001.

D. Blackwell. Conditional expectation and unbiased sequential estimation. *Ann. Math. Statist.*, 18:105–110, 1947.

F. Blanchini, S. Miani, and C. Savorgnan. Stability results for linear parameter varying and switching systems. *Automatica*, 43:1817–1823, 2007.

P-A Bliman. LMI characterization of the strong delay-independent stability of linear delay systems via quadratic Lyapunov-Krasovskii functionals. *Systems & Control Letters*, 43: 263–274, 2001.

P-A. Bliman. Lyapunov equation for the stability of linear delay systems of retarded and neutral type. *IEEE Transactions on Automatic Control*, 17:327–335, 2002.

P.A. Bliman. Stability of nonlinear delay systems: delay-independent small gain theorem and frequency domain interpretation of the Lyapunov-krasovskii method. Research report, INRIA Rocquencourt, 2000.

R. A. Borges and P. L. D. Peres. $\mathcal{H}_\infty$ LPV filtering for linear systems with arbitrarily time-varying parameters in polytopic domains. In *Conference on Decision and Control, San Diego, USA*, 2006.

M. Boutayeb and M. Darouach. Comments on 'a robust state observer scheme'. *IEEE Transactions on Automatic Control*, 48(7):1292–1293, 2003.

S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, Cambridge, MA, USA, 2004.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory.* PA, SIAM, Philadelphia, 1994.

S.P. Boyd and G.H. Barrat. *Linear controller design - Limits of Performances.* Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1991.

R.K. Brayton. Bifurcation of periodic solutions in a nonlinear difference-differential equation of neutral type. *Quart. Appl. Math*, 24:215–224, 1966.

C. Briat and E. I. Verriest. A new delay-SIR model for pulse vaccination. In *IFAC World Congress, Korea, Seoul*, 2008.

C. Briat, O. Sename, and J-F. Lafay. A LFT/$\mathcal{H}_\infty$ state-feedback design for linear parameter varying time delay systems. In *European Control Conference 2007, Kos, Greece*, 2007a.

C. Briat, O. Sename, and J-F. Lafay. Some $\mathcal{H}_\infty$ LFT control approaches for LPV and LTI systems with interval time-varying delays. Internal report, GIPSA-Lab, Grenoble, France, 2007b.

C. Briat, O. Sename, and J-F. Lafay. Full order LPV/$\mathcal{H}_\infty$ observers for LPV time-delay systems. In *3rd IFAC Symposium on System Structure and Control 2007, Foz do Iguacu, Brazil*, 2007c.

C. Briat, O. Sename, and J.F. Lafay. Delay-scheduled state-feedback design for time-delay systems with time-varying delays. In *IFAC World Congress, Korea, Seoul*, 2008a.

C. Briat, O. Sename, and J.F. Lafay. A full-block $\mathcal{S}$-procedure application to delay-dependent $\mathcal{H}_infty$ state-feedback control of uncertain time-delay systems. In *IFAC World Congress, Korea, Seoul*, 2008b.

C. Briat, O. Sename, and J.F. Lafay. Parameter dependent state-feedback control of LPV time delay systems with time varying delays using a projection approach. In *IFAC World Congress, Korea, Seoul*, 2008c.

C. Briat, O. Sename, and J. F. Lafay. $\mathcal{H}_\infty$ filtering of uncertain LPV systems with time-delays (accepted). In *10th European Control Conference, 2009, Budapest, Hungary*, 2009.

F. Bruzelius, S. Petterssona, and C. Breitholz. Linear parameter varying descriptions of nonlinear systems. In *American Control Conference, Boston, Massachussetts*, 2004.

R.L. Burden and J. Douglas Faires. *Numerical Analysis 8th Edition.* Thomson Brooks/Cole, 2004.

J.V. Burke, D. Henrion, A.S. Lewis, and M.L. Overton. Stabilization via nonsmooth, nonconvex optimization. *IEEE Transactions on Automatic Control*, 51(11):1760–1769, 2006.

G.C. Calafiore, F. Dabbene, and R. Tempo. Randomized algorithms for probabilistic robustness with real and complex structured uncertainty. *Systems & Control Letters*, 45(12): 2218–2235, 2000.

J. Castro. Fuzzy logic controllers are universal approximator. *IEEE Transactions on Systems, Man, Cybernetics*, 25:629–635, 1995.

A-L. Cauchy. *Analyse Algébrique (French) Algebraic Analysis.* Edition Jacques Gabay, France, 1821.

J. Chen, C. Gu, and C.N. Nett. A new method for computing delay margins for stability of linear systems. *Systems and Control Letters*, 26:107–117, 1995.

J.D. Chen. Robust $\mathcal{H}_\infty$ output dynamic observer-based control of uncertain time-delay systems. *Chaos, Solitons and Fractals*, 31:391–403, 2007.

W. H. Chen and W. X. Zheng. On improved robust stabilization of uncertain systems with unknown input delays. *Automatica*, 42:1067–1072, 2006.

D. Cheng, Y. Lin, and Y. Wang. Accessibility of switched linear systems. *IEEE Trans. on Automatic Control*, 51(9):1486–1491, 2006.

J. Chiasson and J.J. Loiseau. *Applications of Time-Delay Systems.* Springer-Verlag, Berlin, Germany, 2007.

F.H. Clarke. *Optimization and Nonsmooth Analysis.* Wiley, New York, USA, 1983.

F.H. Clarke, Y.S. Ledyaev, R.J. Stern, and P.R. Wolenski. *Nonsmooth Analysis and Control Theory.* Springer, 1998.

P. Colaneri, J.C. Geromel, and A. Astolfi. Stabilization of continuous-time switched nonlinear systems. *System and Control Letters*, 57:95–103, 2008.

G. Conte and A.M. Perdon. The decoupling problem for systems over a ring. In *34th IEEE Conference on Decision and Control*, pages 2041–2045, New Orleans, USA, 1995.

G. Conte and A.M. Perdon. Noninteracting control problems for delay-diffrerential systems via systems over rings. In *Colloque Analyse et Commande des systèmes avec retards*, pages 101–114, Nantes, France, 1996.

G. Conte, A.M. Perdon, and A. Lombardo. The decoupling problem with weak ouptput controllability for systems over a ring. In *36th IEEE Conference on Decision and Control*, pages 313–317, San Diego, USA, 1997.

L. Crocco. Aspects of combustion stability in liquid propellant rocket motors, part I. fundamentals - low frequency instability with monopropellants. *Journal of American Rocket Society*, 21:163–178, 1951.

R. . Curtain, H. Logemann, S. Townley, and H. Zwart. Well-posedness, stabilizability and admissibility for pritchard-slamon systems. *Journal of Methematical Systems, Estimation and Control*, 4(4):1–38, 1994.

R.F. Curtain and A.J. Pritchard. *Functional Analysis in Modern Applied Mathematics*. Academic Press, New York, 1977.

J. M. G. da Silva and S. Tarbouriech. Antiwindup design with guaranteed regions of stability - an LMI" based approach. *IEEE Transactions on Automatic Control*, 50(1):106–111, 2005.

J. Daafouz, P. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, 47(11):1883–1887, 2002.

J.L. Daleckiĭ and M.G. Kreĭn. *Stability of Solutions of Differential Equations in Banach Space*. AMS Publisher, 1974.

M. Darouach. Linear functional observers for systems with delays in state variables. *IEEE Transactions on Automatic Control*, 46(3):491–496, 2001.

M. Darouach. Reduced-order observers for linear neutral delay systems. *IEEE Transactions on Automatic Control*, 50(9):1407–1413, 2005.

C.E. de Souza and X. Li. Delay-dependent robust $\mathcal{H}_\infty$ control of uncertain linear state-delayed systems. *Automatica*, 35:1313–1321, 1999.

C.E. de Souza and A. Trofino. Gain-scheduled $\mathcal{H}_2$ controller synthesis for linear paremeter varying systems via parameter-dependent Lyapunov functions. *International Journal of Robust and Nonlinear Control*, 16:243–257, 2005.

K. Derinkuyu and M.C. Pinar. On the $\mathcal{S}$-procedure and some variants. *Mathematical Methods of OR*, –:–, 2005.

C. E. DeSouza, R. M. Palhares, and P. L. D. Peres. Robust $\mathcal{H}_\infty$ filtering for uncertain linear systems with multiple time-varying state delays: An LMI approach. In *Proc. 38th IEEE Confer. on Decision & Control*, pages 2023–2028, Phoenix, Arizona, USA, 1999.

S. G. Dietz, C. W. Scherer, and W. Huygen. Linear parameter-varying controller synthesis using matrix sum-of squares. In *Proceedings of the XII Latin-American congress on automatic control*, 2006.

L.L. Dines. On the mapping of quadratic forms. *Bulletin of the American Mathematical Society*, 47:494–498, 1941.

J. Doyle, B. Francis, and A. Tanenbaum. *Feedback Control Theory*. Macmillan Publishing Co., 1990.

S. Drakunov, S. Perruquetti, J.P. Richard, and L. Belkoura. Delay identification in time-delay systems using variable structure control. *Annual Reviews in Control*, 30(2):143–158, 2006.

L. Dugard and E. I. Verriest. *(Eds) Stability and control of time-delay systems*, volume 228 of *LNCIS*. Springer Verlag, 1998.

W.J. Duncan. Some devices for the solution of large sets of simultaneous linear equations. (with an appendix on the reciprocation of partitioned matrices.). *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Seventh Series*, 35:660–670, 1917.

R. Durrett. *Probability - Theory and Examples (3rd Edition)*. Brooks/Cole, 2005.

L. El-Ghaoui and P. Gahinet. Rank minimization under LMI constraints - a framework for output feedback problems. In *European Control Conference*, 1993.

A. Fattouh. *Robust Observation and Digital Control for systems with time-delays*. PhD thesis, I.N.P.G - Laboratoire d'Automatique de Grenoble, Grenoble, France, 2000. in french.

A. Fattouh and O. Sename. A model matching solution of robust observer design for time-delay systems. In S. Niculescu and K. Gu, editors, *Advances in Time-Delay Systems*, volume 38 of *LNCSE*, pages 137–154. Springer-Verlag, 2004. presented at the 14th Int. Symp. on Mathematical Theory of Networks and System, Perpignan, France, 2000.

A. Fattouh, O. Sename, and J.-M. Dion. $\mathcal{H}_\infty$ observer design for time-delay systems. In *Proc. 37th IEEE Confer. on Decision & Control*, pages 4545–4546, Tampa, Florida, USA, 1998.

A. Fattouh, O. Sename, and J.-M. Dion. Robust observer design for time-delay systems: A riccati equation approach. *Kybernetika*, 35(6):753–764, 1999.

A. Fattouh, O. Sename, and J.-M. Dion. An LMI approach to robust observer design for linear time-delay systems. In *Proc. 39th IEEE Confer. on Decision & Control*, Sydney, Australia, December, 12-15, 2000a.

A. Fattouh, O. Sename, and J.-M. Dion. $\mathcal{H}_\infty$ controller and observer design for linear systems with point and distributed time-delays: An LMI approach. In *2nd IFAC Workshop on Linear Time Delay Systems*, Ancône, Italy, 2000b.

A. Fattouh, O. Sename, and J.-M. Dion. Robust observer design for linear uncertain time-delay systems: A factorization approach. In *14th Int. Symp. on Mathematical Theory of Networks and Systems*, Perpignan, France, June, 19-23, 2000c.

A. Feldstein, K.W. Neves, and S. Thompson. Sharpness results for state dependent delay differential equations: An overview. *Applied Numerical Analysis*, 56:472–487, 2005.

G. Ferreres and J. M. Biannic. Convex design of a robust antiwindup controller for an LFT model. *IEEE Transactions on Automatic Control*, 52(11):2173–2177, 2007.

Y.A. Fiagbedzi and A.E. Pearson. A multistage reduction technique for feedback stabilizing distributed time-lag systems. *Automatica*, 23:311–326, 1987.

P. Finsler. Über das vorkommen definiter und semi-definiter formen in scharen quadratischer formen. *Commentarii Mathematici Helvetici*, 9:188–192, 1937.

M. Fleifil, A.M. Annaswamy, Z. Ghoniem, and A.F. Ghoniem. Response of a laminar premixed flame to flow oscillations: A kinematic model and thermoacosutic instability result. *Combustion and Flame*, 106:487–510, 1974.

M. Fleifil, J.P. Hathout, A.M. Annaswamy, and A.F. Ghoniem. Reduced order modeling of heat disease dynamics and active control of time-delay instability. In *38th AIAA Aerospace Sciences Meeting, Reno, Nevada, USA*, 2000.

A. Forrai, T. Ueda, and T. Yumura. Electromagnetic actuator control: A linear parameter-varying (LPV) approach. *IEEE Trans. on Industrial Electronics*, 54(3):1430–1441, 2007.

A.L. Fradkov and V.A. Yakubovich. The $\mathcal{S}$-procedure and a duality relation in nonconvex problems of quadratic programming. *Vestnik Leningrad University*, 5:101–109, 1979.

E. Fridman. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, 43(4):309–319, 2001.

E. Fridman. Stability of systems with uncertain delays: a new 'complete' Lyapunov-Krasovskii Functional. *IEEE Transactions on Automatic Control*, 51:885–890, 2006a.

E. Fridman. Descriptor discretized Lyapunov functional method: Analysis and design. *IEEE Transactions on Automatic Control*, 51:890–897, 2006b.

E. Fridman and U. Shaked. New bounded real lemma representations for time-delay systems and their applications. *IEEE Transactions on Automatic Control*, 46(12):1913–1979, 2001.

E. Fridman and U. Shaked. An improved stabilization method for linear time-delay systems. *IEEE Transactions on Automatic Control*, 47(11):1931–1937, 2002a.

E. Fridman and U. Shaked. A descriptor approach to $\mathcal{H}_\infty$ control of linear time-delay systems. *IEEE Transactions on Automatic Control*, 47(2):253–270, 2002b.

E. Fridman and U. Shaked. Input-output approach to stability and $\mathcal{L}_2$-gain analysis of systems with time-varying delays. *System & Control Letters*, 55:1041–1053, 2006.

E. Fridman, U. Shaked, and L. Xie. Robust $\mathcal{H}_\infty$ filtering of linear systems with time varying delay. *IEEE Transactions on Automatic Control*, 48(1):159–165, 2003a.

E. Fridman, U. Shaked, and L. Xie. Robust $\mathcal{H}_2$ filtering of linear systems with time delays. *International Journal of Robust and Nonlinear Control*, 13(10):983–1010, 2003b.

E. Fridman, A. Seuret, and J. P. Richard. Robust sampled-data stabilization of linear systems: An input delay approach. *Automatica*, 40:1441–1446, 2004.

E. Féron, P. Apkarian, and P. Gahinet. Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions. *IEEE Transactions on Automatic Control*, 41: 1041–1046, 1996.

M. Fu. Pole placement via static output feedback is NP-hard. *IEEE Transactions on Automatic Control*, 49(5):855–857, 2004.

M. Fu, H. Li, and S. I. Niculescu. *Robust Stability and stabilization of time-delay systems via intergal quadractic constraint approach*, chapter 4, pages 101–116. *In Dugard and Verriest(Eds)*, 1998.

P. Gahinet and P. Apkarian. A linear matrix inequality approach to $\mathcal{H}_\infty$ control. URL `citeseer.ist.psu.edu/124688.html`.

Z. Gajić and M.T.J. Qureshi. *Lyapunov Matrix Equation in System Stability and Control*. Academic Press, 1995.

H. Gao and C. Wang. Comments and further results on 'a descriptor system approach to $\mathcal{H}_\infty$ control of linear time-delay systems'. *IEEE Transactions on Automatic Control*, 48(3): 520–525, 2003.

P. Gáspár, Z. Szabó, J. Bokor, C. Poussot-Vassal, O. Sename, and L. Dugard. Toward global chassis control by integrating the brake and suspension systems. In *Proceedings of the 5th IFAC Symposium on Advances in Automotive Control (AAC)*, Aptos, California, USA, august 2007.

P. Gaspard, I. Szaszi, and J. Bokor. Active suspension design using LPV control. In *IFAC Symposium on Advances in Automotive Control*, University of Salerno, Italy, April 2004.

K. Gatermann and P.A. Parrilo. Symmetry groups, semidefinite programs and sum of squares. *Journal of Pure and Applied Algebra*, 192:95–128, 2004.

C. Gauthier, O. Sename, L. Dugard, and G. Meissonier. Modeling of a diesel engine common rail injection system. In *16th IFAC World Congress*, Prague, Czech Republic, 2005.

C. Gauthier, O. Sename, L. Dugard, and G. Meissonier. A $\mathcal{H}_\infty$ linear parameter varying (LPV) controller for a diesel engine common rail injection system. In *European Control Conference*, Kos, Greece, 2007a.

C. Gauthier, O. Sename, L. Dugard, and G. Meissonnier. An LFT approach to $\mathcal{H}_\infty$ control design for diesel engine common rail injection system. *Oil & Gas Science and Technology - Rev. IFP*, 62:513–522, 2007b.

A. Germani, C. Manes, and P. Pepe. A state observer for nonlinear delay systems. In *Proc. 37th IEEE Confer. on Decision & Control*, pages 355–360, Tampa, Florida, USA, 1998.

A. Germani, C. Manes, and P. Pepe. An observer for M.I.M.O. nonlinear delay systems. In *IFAC 14th World Congress*, pages 243–248, Beijing, China, 1999.

A. Germani, C. Manes, and P. Pepe. An asymptotic state observer for a class of nonlinear delay systems. *Kybernetika*, 37(4):459–478, 2001.

A. Germani, C. Manes, and P. Pepe. A new approach to state observation of nonlinear systems with delayed-ouputs. *IEEE Transactions on Automatic Control*, 47(1):96–101, 2002.

J. C. Geromel and P. Colaneri. Robust stability of time-varying polytopic systems. *Systems and Control Letters*, 55:81–85, 2006.

J.C. Geromel, A.P.C. Gonçalves, and A.R. Fioravanti. Dynamic output feedback control of discrete-time markov jump linear systems through linear matrix inequalities. *SIAM Journal on Control and Optimization*, 48(2):573–593, 2009.

L. El Ghaoui and M. Ait Rami. Robust state-feedback stabilization of jump linear systems via LMIs. *International Journal of Robust and Nonlinear Control*, 6(9-10):1015–1022, 1997.

L. El Ghaoui, F. Oustry, and M. Ait Rami. A Cone Complementary linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, 42:1171–1176, 1997.

W. Gilbert, D. Henrion, and J. Bernussou. Polynomial LPV synthesis applied to turbofan engines. In *IFAC symposium on Automatic Control in Aerospace*, Toulouse, France, 2007.

I. Gohberg. *Schur Methods in Operator Theory and Signal Processing. Operator Theory: Advances and Applications*. Birkhauser Verlag, Basel, 1986.

D. Goldfarb and G. Iyengar. Robust portfolio selection problems. *Mathematics of OR*, 28: 1–38, 2003.

K. Gopalsamy and B.G. Zhang. On a neutral delay-logistic equation. *Dynamics and stability of systems*, 2:183–195, 1988.

F. Gouaisbaut and D. Peaucelle. Stability of time-delay systems with non-small delay. In *Conference on Decision and Control, San Diego, California*, 2006a.

F. Gouaisbaut and D. Peaucelle. Delay dependent robust stability of time delay-systems. In 5^{th} *IFAC Symposium on Robust Control Design*, Toulouse, France, 2006b.

F. Gouaisbaut and D. Peaucelle. Robust stability of time-delay systems with interval delays. In 46^{th} *IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007, 2007.

A. Goubet-Batholoméus, M. Dambrine, and J.P. Richard. Stability of perturbed systems with time-varying delays. *Systems & Control Letters*, 31:155–163, 1997.

R. V. Gressang and G. B. Lamont. Observers for systems characterized by semigroups. *IEEE Transactions on Automatic Control*, 20(6):523–528, 1975.

D. W. Gu and F. W. Poon. A robust state observer scheme. *IEEE Transactions on Automatic Control*, 46:1958–1963, 2001.

K. Gu and S-I. Niculescu. Additional dynamics in transformed time-delay systems. In *Conference on Decision and Control, Phoenix, Arizona*, 1999.

K. Gu and S-I. Niculescu. Further remarks on additional dynamics in various model transformation of linear delay systems. In *Conference on Decision and Control, Chicago, Illinois*, 2000.

K. Gu, A.C.J. Luo, and S-I. Niculescu. Discretized Lyapunov functional for systems with distributed delay. In *European Control Conference*, Karlsruhe, Germany, 1999.

K. Gu, V.L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhäuser, 2003.

L. Guttman. Enlargement methods for computing the inverse matrix. *The Annals of Mathematical Statistics*, 17:336–343, 1946.

H. H.-Hashemi and C. T. Leondes. Observer theory for systems with time delay. *Int. Journal Systems Sci.*, 10(7):797–806, 1979.

Q.L. Han. Robust stability of uncertain delay differential systems of neutral type. *Automatica*, 38:719–723, 2002.

Q.L. Han. Absolute stability of time-delay systems with sector-bounded nonlinearity. *Automatica*, 41:2171–2176, 2005a.

Q.L. Han. On stability of linear neutral systems with mixed time delays: a discretized Lyapunov functional approach. *Automatica*, 41:1209–1218, 2005b.

Q.L. Han. A delay decomposition approach to stability of linear neutral systems. In *17th IFAC World Congress, Seoul, South Korea*, 2008.

Q.L. Han and K. Gu. Stability of linear systems with time-varying delay: A generalized discretized Lyapunov functional approach. *Asian Journal of Control*, 3:170–180, 2001.

S. Hayami. On the propagation of flood waves. *Disaster Prevention Institute - Bulletin no. 1 - Kyoto University, Kyoto, Japan*, 1, 1951.

E.V. Haynworth. Some devices for the solution of large sets of simultaneous linear equations. (with an appendix on the reciprocation of partitioned matrices.). *Determination of the inertia of a partitioned Hermitian matrix*, 1:73–81, 1968.

B. He and M . Yang. Robust LPV control of diesel auxiliary power unit for series hybrid electric vehicles. *IEEE Trans. on Power Electronics*, 21(3):791–798, 2006.

Y. He, M. Wu, and J-H. She adn G-P. Liu. Parameter-dependent Lyapunov functional for stability of time-delay systems with polytopic type uncertainties. *IEEE Transactions on Automatic Control*, 49:828–832, 2004.

Y. He, Q.-G. Wang, C. Lin, and M. Wu. Delay-range-dependent stability for systems with time-varying delays. *Automatica*, 43:371–376, 2007.

J.W. Helton. 'Positive' noncommutative polynomials are sum of squares. *Annals of Mathematics*, 156:675–694, 2002.

B. Hencey and A.G. Alleyne. A KYP lemma for LMI regions. *IEEE Transactions on Automatic Control*, 52(10):1926–1930, 2007.

D. Henrion. LMI, optimization and polynomial methods. Course, Supelec, Paris, 2008.

D. Henrion and J.B. Lasserre. Solving nonconvex optimization problems. *IEEE Control Systems Magazine*, 2004.

D. Henrion and J.B. Lasserre. Convergent relaxations of polynomial matrix inequalities and static output feedback. *IEEE Transactions on Automatic Control*, 51(2):192–202, 2006.

D. Henrion and S. Tarbouriech. LMI relaxations for robust stability of linear systems with staurating controls. *Automatica*, 35:1599–1604, 1999.

D. Henrion, S. Tarbouriech, and C. Kučera. Control of linear systems subject to time-domain constraints with polynomial pole placement and LMIs. *IEEE Transactions on Automatic Control*, 50(9):1360–1364, 2005.

J. Hespanha, O. Yakimenko, I. Kaminer, and A. Pascoal. Linear parametrically varying systems with brief instabilities: an application to integrated vision/imu navigation. In *40th Conf. on Decision and Control. Vol. 3, pp. 2361-2371*, Orlando, FL, USA, 2001.

J.P. Hespanha and A.S. Morse. Stability of switched systems with average dwell-time. In *38th Conference on Decision and Control*, Phoenix, Arizona, USA, 1999.

M.R. Hestenes and E.J. MacShane. A theorem on quadratic forms and its applications in the calculus of variations. *Transactions on the American Mathematical Society*, 47:501–512, 1940.

H. W. Hethcote. The mathematics of infectious diseases. *SIAM Review*, 42 (4):599–653, 2002.

P.F. Hokayem and M.W. Spong. Bilateral teleoperation: An historical survey. *Automatica*, 42:2035–2057, 2006.

R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambrdige University Press, Cambridge, 1990.

S. Ibaraki and M. Tomizuka. Rank minimization aproach for solving BMI problems with random search. In *American Control Conference, Arlington, Virginia, USA, 2001*, 2001.

O. V. Iftime, H. J. Zwart, and R. F. Curtain. Representation of all solutions of the control algebraic Ricatti equations for infinite-dimensional systems. *International Journal of Control*, 78(7):505–520, 2005.

D. Ivanescu, S.-I. Niculescu, J.-M. Dion, and L. Dugard. Control of some distributed delay systems using generalized Popov theory. In *Proc. 14th IFAC World Congress*, pages 265–270, Beijing, China, 1999.

D. Ivanescu, J.-M. Dion, L. Dugard, and S.-I. Niculescu. Dynamical compensation for time-delay systems: an LMI approach. *Int. J. of Robust and Nonlinear control*, 10:611–628, 2000.

D. Ivanescu, S.-I. Niculescu, L. Dugard, J.-M. Dion, and E.I. Verriest. On delay dependent stability for linear neutral systems. *Automatica*, 39(2):255–261, 2003.

T. Iwasaki. LPV system analysis with quadratic separator. In *Conference on on Decision and Control*, 1998.

T. Iwasaki. *Control Synthesis for well-posedness of feedback systems, Chapter 14 of Advances on LMI Methods in Control, Editors: L. El Ghaoui and S.-I. Niculescu.* SIAM, Natick, USA, 2000.

T. Iwasaki and S. Hara. Well-posedness of feedback systems: insight into exact robustness analysis and approximate computations. *IEEE Transactions on Automatic Control*, 43: 619–630, 1998.

T. Iwasaki and R.E. Skelton. A unified approach to fixed order controller design via linear matrix inequalities. *Mathematical Problems in Engineering*, 1:59–75, 1995a.

T. Iwasaki and R.E. Skelton. The x-y centering algorithm for the dual-LMI problem. *International Journal of Control*, 62:1257–1252, 1995b.

T. Iwasaki, G. Meinsma, and M. Fu. Generalized $\mathcal{S}$-procedure and finite frequency KYP lemma. *Mathematical Problems in Engineering*, 6:305–320, 1998.

D.H. Jacobson. *Extensions of Linear-Quadratic Control, Optimization and Matrix Theory.* Academic Press, New York, 1977.

J.L.W.V. Jensen. Sur les fonctions convexes et les inégalites entre les valeurs moyennes. *Acta Mathematica*, 30:175–193, 1806.

X. Jiang and Q.-L. Han. Delay-dependent robust stability for uncertain linear systems with interval time-varying delay. *Automatica*, 42:1059–1065, 2006.

X. Jiang and Q.L. Han. On $\mathcal{H}_\infty$ control for linear systems with interval time-varying delay. *Automatica*, 41:2099–2106, 2005.

U. Jonsson, C. Y. Kao, A. Megretski, and A. Rantzer. A guide to iqcβ: A MATLAB toolbox for robust stability and performance analysis. Technical report, 2004. URL `http://www.ee.unimelb.edu.au/staff/cykao/IQCM_v08_2004.pdf`.

U. T. Jönsson. A lecture on the $\mathcal{S}$-procedure. Technical report, Division of Optimization and Systems Theory, Royal Institute of Technology, Stockholm, Sweden, 2001.

M. Jun and M.G. Safonov. IQC robustness analysis for time-delay systems. *International Journal of Robust and Nonlinear Control*, 11:1455–1468, 2001.

M. Jun and M.G. Safonov. Rational multiplier IQCs for uncertain time-delays and LMI stability conditions. *IEEE Transactions on Automatic Control*, 47(11):1871–1875, 2002.

M. Jung and K. Glover. Calibrate linear parameter-varying control of a turbocharged diesel engine. *IEEE Trans. on Control Systems Technology*, 14(1):45–62, 2006.

M. Jungers, P.L.D. Peres, E.B. Castelan, E.R. De Pieri, and H. Abou-Kandil. Nash strategy parameter dependent control for polytopic systems. In 3^{rd} *IFAC Symposium on Systems, Strcuture and Control, 2007, Brazil*, 2007.

H. Kajiwara, P. Apkarian, and P. Gahinet. LPV techniques for control of an inverted pendulum. *IEEE Control System Magazine*, 19:44–54, 1999.

R.E. Kalman. Lyapunov functions for the problem of Lur'e in automatic control. *Proceedings of the National Academy of Sciences of the United States of America*, 49(2):201–205, 1963.

E.W. Kamen. Lectures on algebraic system theory: Linear systems over rings. Contractor Report 3016, NASA, 1978.

E.W. Kamen, P.P. Khargonekar, and A. Tannenbaum. Stabilization of time-delay systems using finite dimensional compensators. *IEEE Transactions on Automatic Control*, 30(1): 75–78, 1985.

C. Y. Kao and A. Rantzer. Stability analysis of systems with uncertain time-varying delays. *Automatica*, 43:959–970, 2007.

J. K.Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, USA, 1991.

H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, Upper Saddle River, New Jersey, USA, 2002.

P.P. Khargonekar, I.A. Petersen, and K. Zhou. Robust stabilization of uncertain linear systems: Quadratic satbilizability and $\mathcal{H}_\infty$ control theory. *Journal of Mathematical Analysis and Applications*, 254:410–432, 2001.

V.L. Kharitonov and D. Melchor-Aguila. Lyapunov-krasovskii functionals for additional dynamics. *International Journal of Robust and Nonlinear Control*, 13:793–804, 2003.

V.L. Kharitonov and S.I. Niculescu. On the stability of systems with uncertain delay. *IEEE Transactions on Automatic Control*, 48:127–132, 2003.

J. H. Kim. Delay and its time-derivative dependent robust stability of time-delayed linear systems with uncertainty. *IEEE Transactions on Automatic Control*, 46(5):789–792, 2001.

S.J. Kim and Y.H. Moon. Structurally constrained $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control: A rank-constrained LMI approach. *Automatica*, 42:1583–1588, 2006.

S.J. Kim, Y.H. Moon, and S. Kwon. Solving rank constrained LMI problems with application to reduced-order output feedback stabilization. *IEEE Transactions on Automatic Control*, 52(9):1737–1741, 2007.

C.R. Knospe and M. Roozbehani. Stability of linear systems with interval time delays excluding zero. *IEEE Transactions on Automatic Control*, 51:1271–1288, 2006.

C.R. Knospe and Mardavij Roozbehani. Stability of linear systems with interval time-delay. In *IEEE American Control Conference*, Denver, Colorado, 2003.

D. Koenig and B. Marx. Design of observers for descriptor systems with delayed state and unknwon inputs. In *American Control Conference, Boston, Massachusetts, USA*, 2004.

D. Koenig, B. Marx, and O. Sename. Unknown inputs proportional integral observers for descriptors systems with multiple delays and unknown inputs. In *American Control Conference, Boston, Massachusetts, USA*, 2004.

D. Koenig, D. Jacquet, and S. Mammar. Delay dependent $\mathcal{H}_\infty$ observer for linear delay descriptor systems. In *American Control Conference 2006, Minneapolis, USA*, 2006.

V. Kolmanovskii and A. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

V.B. Kolmanovskii and A.D. Myshkis. *Applied Theory of functional differential equations*. Kluwer, 1962.

V.B. Kolmanovskii and J.-P. Richard. Stability of some systems with distributed delays. *JESA special issue on 'Analysis and control of time-delay systems with delays*, 31:971–982, 1997.

V.B. Kolmanovskii and J.-P. Richard. Stability of some systems with delays. *IEEE Transactions on Automatic Control*, 44:984–989, 1998.

V.B. Kolmanovskii and J.P. Richard. Stability of some linear systems with delays. *IEEE Transactions on Automatic Control*, 44:985–989, 1999.

V.B. Kolmanovskii, J.-P. Richard, and A. Ph. Tchangani. Some model transformation for the stability study of linear systems with delays. In *Proc. IFAC Workshop on Linear Time-Delay Systems, pp. 75-80*, Grenoble, France, 1998.

S.G. Krantz. The maximum-modulus principle - boundary maximum modulus principle theorem. *Handbook of Complex Variables*, pages 76–77, 2001.

Y. Kuang. *Delay Differential Equations with Applications in Population Dynamics*. Academic, San Diego, CA, USA, 1993.

A. Kwiatkowski and H. Werner. LPV control of a 2-DOF robot using parameter reduction. In *European Control Conference*, Seville, Spain, 2005.

J. Lam. Convergence of a class of padé approximations for time-delay systems. *International Journal of Control*, 52(4):989–1008, 1990.

P. Lancaster and M. Tismenetsky. *The Theory of Matrices*. Academic, Orlanda, Flordia, USA, 1985.

C. Langbort, R.S. Chandra, and R. D'Andrea. Distributed control design for systems interconnected over arbitrary graph. *IEEE Transactions on Automatic Control*, 49(9):1502–1519, 2004.

J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.*, 11(3):796–817, 2001.

J.B. Lasserre. Semidefinite programming for gradient and hessian computation in maximum entripy estimation. In 46^{th} *IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007, 2007.

E.B. Lee and A.W. Olbrot. Observability and related structural results for linear hereditary systems. *Int. Journal of Control*, 34:1061–1078, 1981.

N. Levinson and R.M. Redheffer. *Complex Variables.* Holden-Day, Baltimore, USA, 1970.

A.S. Lewis. Eigenvalues and nonsmooth optimization. *in Foundations of Computational Mathematics, Santander 2005, L.M. Pardo, A. Pinkus, E. Suli and M.J. Todd (eds)*, 2005.

A.S. Lewis. Nonsmooth optimization and robust control. *Annual Reviews in Control*, 31: 167–177, 2007.

J. Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. URL `http://control.ee.ethz.ch/~joloef/yalmip.php`.

X. Li and C.E. de Souza. Criteria for robust stability of uncertain linear systems with time-varying state delays. In *13th IFAC World Congress. Vol. H pp.137-142*, San Francisco, CA, USA, 1996.

X. Li, C. E. de Souza, and A. Trofino. Delay dependent robust stabilization of uncertain linear state-delayed systems via static output feedback. In *IFAC Workshop on Linear Time-Delay Systems*, pages 1–6, Grenoble, France, 1998.

D. Liberzon, J .P. Hespanha, and A. S. Morse. Stability of switched systems: a lie-algebraic condition. *System and Control Letters*, 37:117–122, 1999.

S. Lim and J. How. Application of improved $\mathcal{L}_2$ gain synthesis on LPV missile autopilot design. In *American Control Conference*, San Diego, CA, USA, 1999.

S. Lim and J.P. How. Analysis of linear parameter-varying systems using a nonsmooth dissipative framework. *International Journal of Robust and Nonlinear Control*, 2002.

H. Lin and P.J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2):308–322, 2009.

Q. Liu, V. Vittal, and N Elia. LPV supplementary damping controller design thyristor controlled series capacacitor (TCSC) device. *IEEE Trans. on Power Systems*, 21(3):1242–1249, 2006a.

Q. Liu, V. Vittal, and N Elia. Expansion of system operating range by an interpolated LPV FACTS controller using multiple Lyapunov functions. *IEEE Trans. on Power Systems*, 21(3):1311–1320, 2006b.

M. Louihi and M.L. Hbid. Exponential stability for a class of state-dependent delay equations via the crandall-liggett approach. *J. Math. Anal. Appl.*, 329:1045–1063, 2007.

J. Louisell. New examples of quenching in delay differential equations having time-varying delay. In *4th European Control Conference*, 1999.

B. Lu, F. Wu, and S. Kim. Switching LPV control of an F-16 aircraft via controller state reset. *IEEE Trans. on Control Systems Technology*, 14(2):267–277, 2006.

Z-Q. Luo. Applications of convex optimization in signal processing and digital communications. *Math. Programming (Series B)*, 97:177–207, 2003.

Z-Q. Luo, J.F. Sturm, and S. Zhang. Multivariate nonnegative quadratic mappings. *SIAM Journal on Optimization*, 14:1140–1162, 2004.

A.I. Lur'e. *Some nonlinear problems in the Theory of Automatic Control (Nekotorye Nelineinye Zadachi Teorii Automaticheskogo Regulirovaniya)*. Gostekhizdat, Moscow, 1951.

A.I. Lur'e and V.N. Postnikov. On the theory of stability of control systems. *Prikl. Mat. i Mekh (Applied mathematics and mechanics)*, 8(3):3–13, 1944.

T. Luzianina, K. Engelborghs, and D. Roose. Diiferential equations with state-dependent delay - a numerical study. In *16th IMACS World Congress*, 2000.

T. Luzianina, K. Engelborghs, and D. Roose. Numerical bifurcation analysis of differential equations with state-dependent delay. *International Journal of Bifurcation and Chaos*, 11(3):737–753, 2001.

A.M. Lyapunov. *General Problem of the Stability of Motion (Translated from Russian to French by E. Davaux and from French to English by A.T. Fuller)*. Taylor & Francis, London, UK, 1992.

M. Marcus. *Finite dimensional multilinear algebra. Part I.* Marcel Dekker, New York, USA, 1973.

M. Mariton. *Jump Linear Systems in Automatic Control*. Marcel Dekker, Inc, 1990.

I. Masubuchi and A. Suzuki. Gain-scheduled controller synthesis based on new LMIs for dissipativity of descriptor LPV systems. In *17th IFAC World Congress, Seoul, South Korea*, 2008.

A. Megretski and S. Treil. Power distribution in optimization and robustness of uncertain systems. *Journal of Mathematical Systems, Estimation and Control*, 3:301–319, 1993.

C. S. Mehendale and K. M. Grigoriadis. A new approach to LPV gain-scheduling design and implementation. In *43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas*, 2004.

G. Meinsma and L. Mirkin. $\mathcal{H}_\infty$ control of systems with multiple i/o delays via decomposition to adobe problems. *IEEE Transactions on Automatic Control*, 50:199–211, 2005.

G. Meinsma and H. Zwart. On $\mathcal{H}_\infty$ control for dead-time systems. *IEEE Transactions on Automatic Control*, 45(2):272–285, Feb. 2000.

W. Michiels and S.I. Niculescu. *Stability and stabilization of time-delay systems. An eigenvalue based approach.* SIAM Publication, Philadelphia, USA, 2007.

W. Michiels, S.-I. Niculescu, and L. Moreau. Using delays and time-varying gains to improve the static output feedback of linear systems: a comparison. *IMA Journal of Mathematical Control and Information*, 21(4):393–418, 2004.

W. Michiels, V. Van Assche, and S. I. Niculescu. Stabilization of time-delay systems with a controlled time-varying delay and applications. *IEEE Transactions on Automatic Control*, 50(4):493–504, 2005.

L. Mirkhin. On the extraction of dead-time controllers and estimators from delay-free parametrizations. *IEEE Transactions on Automatic Control*, 48(4):543–553, 2003.

C.R. Mitra and S.K. Mitra. *Generalized Inverse of Matrices and its applications.* Wiley, New York, USA, 1971.

J. Mohammadpour and K. M. Grigoriadis. Rate-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filetring for time varying state delayed LPV systems. In *Conference on Decision and Control, San Diego, USA*, 2006a.

J. Mohammadpour and K. M. Grigoriadis. Delay-dependent $\mathcal{H}_\infty$ filtering for a class of time-delayed LPV systems. In *American Control Conference, Minneapolis, USA*, 2006b.

J. Mohammadpour and K. M. Grigoriadis. Less conservative results of delay-depdendent $\mathcal{H}_\infty$ filtering for a class of time-delayed LPV systems. *International Journal of Control*, 80(2): 281–291, 2007a.

J. Mohammadpour and K. M. Grigoriadis. Stability and performances analysis of time-delayed linear parameter varying systems with brief instability. In *46th Conf. on Decision and Control. Vol. 3, pp. 2779-2784*, New Orleans, LA, USA, 2007b.

J. Mohammadpour and K. M. Grigoriadis. Delay-dependent $\mathcal{H}_\infty$ filtering for time-delayed lpv systems. *Systems and Control Letters*, 57:290–299, 2008.

S. Mondié and W. Michiels. Finite spectrum assignement of unstable time-delay systems with a safe implementation. *IEEE Transactions on Automatic Control*, 48:2207–2212, 2003.

Y.S. Moon, P. Park, W.H. Kwon, and Y.S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, 74:1447–1455, 2001.

A.S. Morse. Ring models for delay differential systems. *Automatica*, 12:529–531, 1976.

R. Moussa. Analytical Hayami solution for the diffusive wave flood routing problem with lateral inflow. *Hydrological Processes*, 10(9):1209–1227, 1996.

U. Münz and F. Allgöwer. $\mathcal{L}_2$-gain based controller design for linear systems with distributed delays and rational delay kernels. In *7th IFAC Workshop on Time-Delay Systems, Nantes, France*, 2007.

U. Münz, J. M. Rieber, and F. Allgöwer. Robust stability of distributed delay systems. In *17th IFAC World Congress, Seoul, South Korea*, 2008.

J.D. Murray. *Mathematical Biology Part I. An Introduction. 3rd Edition.* Springer, 2002.

Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithm in Convex Programming.* PA, SIAM, Philadelphia, 1994.

S.-I. Niculescu. *Delay effects on stability. A robust control approach*, volume 269. Springer-Verlag: Heidelbeg, 2001.

S.-I. Niculescu and C.T. Abdallah. Delay effects on static output feedback stabilization. In *39th IEEE Conference on Decision and Control*, Sydney, Australia, 2000.

S.I. Niculescu. *Time-delay systems. Qualitative aspects on the stability and stabilization (in French).* Diderot, Nouveaux Essais, Paris, France, 1997.

S.I. Niculescu. On some frequency-sweeping tests for delay-dependent stability: A model transformation case study. In *13th IFAC World Congress. Vol. H pp.137-142*, San Francisco, CA, USA, 1999.

S.I. Niculescu and J. Chen. Frequency sweeping tests for asymptotic stability: A model transformation for multiple delays. In *38th IEEE Conf. on Decision and Control pp.4678-4683*, Phoenix, AZ, USA, 1999.

S.I. Niculescu, A.M. Annaswamy, J.P. Hathout, and A.F. Ghoniem. Control of time-delay induced instabilities in combustion systems. Technical report, Internal report HeuDiaDyc'00, 2000.

G. Niemeyer. *Using wave variables in time delayed force reflecting teleoperation.* PhD thesis, Dept. Aeronautics Astronautics, MIT., Cambridge, 1996.

B.A. Ogunnaike. A new approach to observer design for time-delay systems. *Int. Journal of Control*, 33(3):519–542, 1981.

B. Øksendal. *Stochastic Differential Equations - An Introduction with Applications (6th Edition).* Springer, 2003.

R.C.L.F. Oliveira and P.L.D. Peres. Parameter-dependent LMIs in robust analysis: Characterization of homogenus polynomially parameter-dependent solutions via LMI relaxations. *IEEE Transactions on Automatic Control*, 57(7):1334–1340, 2002.

R.C.L.F. Oliveira and P.L.D. Peres. LMI conditions for robust stability analysis based on polynomially parameter-dependent Lyapunov functions. *Systems & Control Letters*, 55: 52–61, 2006.

R.C.L.F. Oliveira, V.F. Montagner, P.L.D. Peres, and P.-A. Bliman. LMI relaxations for $\mathcal{H}_\infty$ control of time-varying polytopic systems by means of parameter dependent quadratically stabilizing gains. In 3^{rd} *IFAC Symposium on System, Structure and Control, Foz do Iguassu, Brasil*, 2007.

E. Olofsson, E. Witrant, C; Briat, and S.I. Niculescu. Stability analysis and model-based control in EXTRAP-T2R with time-delay compensation. In *47th IEEE Conference on Decision and Control (Submitted)*, Cancun, Mexico, 2008.

R. Ortega, A.J. van der Schaft, F. Castaños, and A. Astolfi. Control by interconnection and standard passivity-based control of port-hamiltonian systems. *IEEE Transations on Automatic Control*, 53(11):2527–2541, 2008.

A. Packard. Gain scheduling via Linear Fractional Transformations. *Systems and Control Letters*, 22:79–92, 1994.

A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29:71–109, 1993.

A. Packard, K. Zhou, P. Pandey, and G. Becker. A collection of robust control problems leading to LMI's. In *30th conference on Decision and Control*, pages 1245–1250, Brighton, England, December 1991.

A. Papachristodoulou and S. Prajna. On the construction of Lyapunov functions using the sum-of-square decomposition. In *41th IEEE Conference on Decision and Control*, Las Vegas, NV, USA, 2002.

A. Papachristodoulou, M. Peet, and S. Lall. Constructing Lyapunov-Kraasovskii functionals for linear time delay systems. In *IEEE American Control Conference*, 2005.

A. Papachristodoulou, M. M. Peet, and S.-I.Niculescu. Stability analysis of linear systems with time-varying delays: Delay uncertainty and quenching. In *46th Conference on Decision and Control*, New Orleans, LA, USA, 2007, 2007.

T. Paré, A. Hassibi, and J. How. A KYP lemma and invariance principle for systems with multiple hysteresis nonlinearities. Technical report, Standford University, California, USA, 1999. URL `http://sun-valley.stanford.edu/~tpare/hyskyp_ijc.pdf`.

P. Park. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Transactions on Automatic Control*, 44(4):876–877, 1999.

P. Park, Y.M.Moon, and W.H. Kwon. A delay-dependent robust stability criterion for uncertain time-delay systems. In *43th IEEE American Control Conference*, Philadelphia, PA, USA, 1998.

P. Parrilo. *Structured Semidefinite Programs and Semilagebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, Pasadena, California, 2000.

M.A. Parseval des Chênes. Mémoire sur les séries et sur l'intégration complète d'une équation aux differences partielle linéaire du second ordre, à coefficiens constans (french). *Sciences, mathématiques et Physiques (Savans étrangers)*, 1:638–648, 1806.

R. Pearl. *The biology of Population Growth*. Knopf, New York, 1930.

A. E. Pearson and Y. A. Fiagbedzi. An observer for time lag systems. *IEEE Transactions on Automatic Control*, 34(7):775–777, 1989.

D. Peaucelle and D. Arzelier. Ellipsoidal sets for resilient and robust static output feedback. *IEEE Transactions on Automatic Control*, 50:899–904, 2005.

D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou. Robust stability of time-varying polytopic systems. *Systems and Control Letters*, 40:21–30, 2000.

D. Peaucelle, D. Arzelier, D. Henrion, and F. Gouaisbaut. Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation. *Automatica*, 43: 795–804, 2007.

P. Pepe. Approximated delayless observers for a class of nonlinear time delay systems. In *Proc. 40th IEEE Confer. on Decision & Control*, Orlando, Florida,USA, 2001.

A.M. Perdon and G. Conte. Invited session : Algebraic and ring-theoretic methods in the study of time-delay systems. In *Proc. 38th IEEE Confer. on Decision & Control*, Phoenix, Arizona, USA, 1999.

I.A. Petersen. A stabilization algorithm for a class of uncertain linear systems. *Systems & Control Letters*, 8:351–357, 1987.

P. Picard and J.F. Lafay. Weak observability and observers for linear systems with delays. In *MTNS 96*, Saint Louis, USA, 1996.

P. Picard, O. Sename, and J.F. Lafay. Observers and observability indices for linear systems with delays. In *CESA 96, IEEE Conference on Computational Engineering in Systems Applications*, volume 1, pages 81–86, Lille, France, 1996.

P. Picard, O. Sename, and J.F. Lafay. Weak controllability and controllability indices for linear neutral systems. *Mathematical Computers and Simulation,*, 45:223–233, 1998.

E.C. Pielou. *Mathematical Ecology.* Wiley Interscience, New York, USA, 1977.

A. Pinkus. n-widths in apprximmation theory. In *Volume 7 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer*, Berlin, Germany, 1985.

I. Pólik and T. Terlaky. A survey of the s-lemma. *SIAM Review*, 49(3):371–418, 2007.

V.M. Popov. Absolute stability of nonlinear systems of automatic control. *Automation and Remote Control (Translated from Automatica i Telemekhanka)*, 22(8):857–875, 1961.

C. Poussot-Vassal. *Global Chassis Control using a Multivariable Robust LPV control approach.* PhD thesis, Grenoble-INP, 2008. URL `ElectronicversionatOneraDCDD`.

C. Poussot-Vassal, O. Sename, L. Dugard, P. Gáspár, Z. Szabó, and J. Bokor. Multi-objective qLPV $\mathcal{H}_\infty/\mathcal{H}_2$ control of a half vehicle. In *Proceedings of the 10th Mini-conference on Vehicle System Dynamics, Identification and Anomalies (VSDIA)*, Budapest, Hungary, november 2006.

C. Poussot-Vassal, A. Drivet O. Sename, L. Dugard, and R. Ramirez-Mendoza. A self tuning suspension controller for multi-body quarter vehicle model. In *Proceedings of the 17th IFAC World Congress (WC)*, Seoul, South Korea, july 2008a.

C. Poussot-Vassal, O. Sename, L. Dugard, P. Gáspár, Z. Szabó, and J. Bokor. New semi-active suspension control strategy through LPV technique. *to appear in Control Engineering Practice (Accepted)*, 2008b.

C. Poussot-Vassal, O. Sename, L. Dugard, P. Gáspár, Z. Szabó, and J. Bokor. Attitude and handling improvements through gain-scheduled suspensions and brakes control. In *Proceedings of the 17th IFAC World Congress (WC)*, Seoul, South Korea, july 2008c.

S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo. *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, 2004.

E. Prempain and I. Postlethwaite. Static $\mathcal{H}_\infty$ loop shaping control of a fly-by-wire helicopter. *Automatica*, 41:1517–1528, 2005.

L. Qiu and E.J. Davidson. The stability robustness determination of state space models with real unstructured perturbations. *Math. Control Signals Syst.*, 4:247–267, 1991.

A. Rantzer. On the Kalman-Yakubovicth-Popov lemma. *System & Control Letters*, 28(1), 1996.

A. Rantzer and A. Megretski. System analysis via Integral Quadratic Constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, 1997.

C.R. Rao. Information and accuracy attainable in the estimation of statistical parameters. *Bull. Cal. Math. Soc.*, 37:81–91, 1945.

C.R. Rao. *Linear Statistical Inference and its Applications.* Wiley, 1973.

L. Reberga, D. Henrion, and F. Vary. LPV modeling of a turbofan engine. In *IFAC World Congress on Automatic Control*, Prague, Szech Republic, 2005.

J-P. Richard. Linear time delay systems: some recent advances and open problems. In *2nd IFAC Workshop on Linear Time Delay Systems*, Ancona, Italy, 2000.

D. Robert, O. Sename, and D. Simon. Synthesis of a sampling period dependent controller using LPV approach. In *5th IFAC Symposium on Robust Control Design ROCOND'06*, Toulouse, France, 2006.

M. Roozbehani and C. R. Knospe. Robust stability and $\mathcal{H}_\infty$ performance analysis of interval-dependent time-delay system. In *American Control Conference, Portland, USA*, 2005.

C. Runge. Über empirische funktionen und die interpolation zwischen äquidistanten ordinaten. *Zeitschrift für Mathematik und Physik*, 46:224–243, 1901.

E.B. Saff and R.S. Varga. On the zeros and poles of Padé approximmants to e^z. *Numeric. Methods*, 25:1–14, 1975.

M. G. Safonov. *Stability, Robustness of Multivariable Feedback Systems.* MIT Press, Cambridge, MA, USA, 2000.

M. Sato. Filter design for LPV systems using quadratically parameter-dependent Lyapunov functions. *Automatica*, 42:2017–2023, 2006.

M. Sato and D. Peaucelle. Robust stability/peformance analaysis for uncertain linear systems via multiple slack variable approach: Polynomial ltipd systems. In *46rd IEEE Conference on Decision and Control, New Orleans, LA, USA*, 2007.

C. Scherer. Robust generalized $\mathcal{H}_2$ control for uncertain and LPV systems with general scalings. In *Conference on Decision and Control*, 1996.

C. Scherer. A full-block $\mathcal{S}$-procedure with applications. In *Conference on Decision and Control*, 1997.

C. Scherer. Robust mixed control and LPV control with full-block scalings. *Advances in LMI Methods in Control, SIAM*, 1999.

C. Scherer and S. Weiland. Linear matrix inequalities in control. Technical report, Delft Center for Systems and Control (Delft University of Technology) and Department of Electrical Engineering (Eindhoven University of Technology), 2004.

C. Scherer and S. Weiland. *Linear Matrix Inequalities in Control*. 2005.

C. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Transaction on Automatic Control*, 42(7):896–911, 1997.

C. W. Scherer. LPV control and full-block multipliers. *Automatica*, 37:361–375, 2001.

C. W. Scherer. LMI relaxations in robust control (to appear). *European Journal of Control (preprint available at http://www.dcsc.tudelft.nl/ cscherer/pub.html)*, 2008.

C. W. Scherer and I. Emre Köse. Gain-scheduling with dynamic D-scalings. In *46th Conference on Decision and Control*, New Orleans, LA, USA, 2007, 2007a.

C. W. Scherer and I. Emre Köse. On robust controller synthesis with dynamic D-scalings. In *46th Conference on Decision and Control*, New Orleans, LA, USA, 2007, 2007b.

C.W. Scherer. Multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ control. *IEEE Transactions on Automatic Control*, 40: 1054–1962, 1995.

C.W. Scherer. Relaxations for robust linear matrix inequality problems with verifications for exactness. *SIAM Journal on Matrix Analysis and Applications*, 27(2):365–395, 2005.

C.W. Scherer and C.W.J. Hol. Matrix sum-of-squares relaxations for robust semi-definite programs. *Mathematical Programming Series B (available at `http://www.dcsc.tudelft.nl/ cscherer/pub.html`)*, 107:189–211, 2006.

J. Schur. Uber potenzreihen, die im innern des einheitskreises beschrankt sind [i]. *Journal fur die reine und angewandte Mathematik*, 147:205–232, 1917a.

J. Schur. Uber potenzreihen, die im innern des einheitskreises beschrankt sind [ii]. *Journal fur die reine und angewandte Mathematik*, 148:122–145, 1917b.

O. Sename. *Sur la commandabilité et le découplage des systèmes linéaires à retards*. PhD thesis, Ecole Centrale Nantes, France, 1994.

O. Sename. Unknown input robust observers for time-delay systems. In *36th IEEE Conference on Decision and Control*, pages 1629–1630, San Diego, California, USA, 1997.

O. Sename. New trends in design of observers for time-delay systems. *Kybernetika*, 37(4): 427–458, 2001.

O. Sename and C. Briat. Observer-based $\mathcal{H}_\infty$ control for time-delay systems: a new LMI solution. In *IFAC TDS Conference, L'Aquila, Italy*, 2006.

O. Sename and A. Fattouh. Robust $\mathcal{H}_\infty$ control of a bilateral teleoperation system under communication time-delay. In *IFAC World Congress*, Prague, Czech Republic, 2005.

O. Sename and J.F. Lafay. A sufficient condition for static decoupling without prediction of linear time-invariant systems with delays. In *ECC 93, European Control Conference*, pages 673–678, Groningen, The Netherlands, 1993.

O. Sename, J.F. Lafay, and R. Rabah. Controllability indices of linear systems with delays. *Kybernetika*, 6:559–580, 1995.

O. Sename, A. Fattouh, and J-M. Dion. Further results on unknown input observers design for time-delay systems. In *40th IEEE Conference on Decision and Control*, Orlando, Florida, USA, 2001.

A. Seuret, C. Edwards, S.K. Spurgeon, and E. Fridman. Static output feedback sliding mode control design via an artificial stabilizing delay. *IEEE Transactions on Automatic Control*, 54(2):256–265, 2009a.

A. Seuret, C. Edwards, S.K. Spurgeon, and E. Fridman. Static output feedback sliding mode control design via an artificial stabilizing delay. *IEEE Transactions on Automatic Control*, 54(2):256–265, 2009b.

J. Shamma and M. Athans. Analysis of nonlinear gain scheduled control systems. *IEEE Transactions on Automatic Control*, 35:898–907, 1990.

J. S. Shamma and M. Athans. Gain scheduling: potential hazards and possible remedies. *IEEE Contr. Syst. Magazine*, 12(3):101–107, 1992.

P.S. Shcherbakov and M.V. Topunov. Extensions of petersen's lemma on matrix uncertainty. In *IFAC World Congress, Seoul, Korea*, 2008.

J. Y. Shin. Analysis of linear parameter varying system models based on reachable set. In *American Control Conference*, Anchorage, AK, USA, 2002, 2002.

E. Shustin and E. Fridman. On delay-derivative-dependent stability of systems with fast-varying delays. *Automatica*, 43:1649–1655, 2007.

R. Sipahi and N. Olgac. Complete stability robustness of third-order LTI multiple time-delay systems. *Automatica*, 41:1413–1422, 2005.

R. Sipahi and N. Olgac. A unique methodology for the syability robustness analysis of multiple time delay systems. *Systems & Control Letters*, 55:819–825, 2006.

R.E. Skelton, T. Iwasaki, and K.M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, 1997.

S. Smale. *Differential Equations, Dynamical Systems & an introduction to Chaos*. Academic Press, 2004.

E.D. Solomentsev. Maximum-modulus principle. *Hazewinkel, Michiel, Encyclopaedia of Mathematics*, 2001.

E. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, New York, USA, 1998.

C. E. De Souza and X. Li. Delay-dependent $\mathcal{H}_\infty$ control of uncertain linear state-delayed systems. *Automatica*, 35:1313–1321, 1999.

F-F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Mathematics of Operation Research*, 28:246–267, 2003.

J. F. Sturm. Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software - Special issue on Interior Point Methods*, 11-12:625–653, 1999.

J. F. Sturm. Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(12):625–653, 2001.

J.-H. Su. Further results on the robust stability of linear systems with a single time delay. *Systems & Control Letters*, 23:375–379, 1994.

T.J. Su and C.G. Huang. Robust stability of delay dependence for linear unceratin systems. *IEEE Transactions on Automatic Control*, 37:1656–1659, 1992.

V. Suplin, E. Fridman, and U. Shaked. A projection approach to $\mathcal{H}_\infty$ control of time delay systems. In *43th IEEE Conference on Decision and Control*, Atlantis, Bahamas, 2004.

V. Suplin, E. Fridman, and U. Shaked. $\mathcal{H}_\infty$ control of linear uncertain time-delay systems - a projection approach. *IEEE Transactions on Automatic Control*, 51:680–685, 2006.

V. Suplin, E. Fridman, and U. Shaked. Sampled-data $\mathcal{H}_\infty$ control and filtering: Non uniform uncertain sampling. *Automatica*, 43:1072–1083, 2007.

V. L. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis. Static output feedback: A survey. Survey, School of Engineering - Department of Electrical and Computer Engineering - University of New Mexico, 1995.

T. Takagi and M. Sugeno. Fuzzy identification of systems and its application to modelling and control. *IEEE Transactions on Systems, Man, Cybernetics*, 15(1):116–132, 1985.

K. Tan and K.M. Grigoriadis. $\mathcal{L}_2$-$\mathcal{L}_2$ and $\mathcal{L}_2$-$\mathcal{L}_\infty$ output feedback control of time-delayed LPV systems. In *Conference on on Decision and Control*, 2000.

A. P. Tchangani, M. Dambrine, and J.-P. Richard. Stability of linear differential equations with distributed delay. In *Proc. 36th Confer. on Decision & Control*, pages 3779–3784, San Diego, California, USA, 1997.

A. R. Teel. On graphs, conic relations, and input-output stability of nonlinear feedback systems. *IEEE Transactions on Automatic Control*, 41:702–709, 1996.

R. Tempo, E.W. Bai, and D. Dabbene. Probabilistic robustness analysis: Explicit bounds for the minimum number of samples. *Systems & Control Letters*, 30:237–242, 1997.

E.C. Titchmarsh. *The Theory of Functions (2nd Ed)*. Oxford University Press, 1939.

A. Trofino and C.E. De Souza. Bi-quadratic stability of uncertain linear systems. In *38th Conference on Decision and Control, Phoenix, Arizona, USA*, 1999.

H.D. Tuan and P. Apkarian. Relaxations of parametrized LMIs with control applications. In *Conference on Decision and Control, Tampa, Florida*, 1998.

H.D. Tuan and P Apkarian. Monotonic relaxations for robust control: New characterizations. *IEEE Transactions on Automatic Control*, 47(2):378–384, 2002.

H.D. Tuan, P. Apkarian, and Y. Nakashima. A new lagrangian dual global optimization algorithm for solving bilinear matrix inequalities. In *American Control Conference, San Diego, California, USA, 1999*, 1999.

H.D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto. Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on Fuzzy Systems*, 9(2):324–332, 2001a.

H.D Tuan, P. Apkarian, and T.Q Nguyen. Robust and reduced order filtering: new LMI-based characterizations and methods. *IEEE Transactions on Signal Processing*, 49:2875–2984, 2001b.

H.D Tuan, P. Apkarian, and T.Q Nguyen. Robust filtering for uncertain nonlinearly parametrized plants. *IEEE Transactions on Signal Processing*, 51:1806–1815, 2003.

P. van den Driessche. Time delay in epidemic models. *Mathematical Approaches for Emerging and Reemerging Infectious Diseases: An Introduction*, 1999.

A.J. van der Schaft. *$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinear Control, Lect. Notes in Control and Information Sciences, Vol. 218,p. 168 (2nd revised and enlarged edition, Springer-Verlag, London, 2000 (Springer Communications and Control Engineering series))*. Springer-Verlag, Berlin, 1996.

G. C. Verghese, B. C. Lévy, and T. Kailath. A generalized state-space for singular systems. *IEEE Transactions on Automatic Control*, 26:811–831, 1981.

P.F. Verhulst. Notice sur la loi que la population suit dans son accroissement (french). *Correspondance mathématique et physique*, 10:113–121, 1938.

E. I. Verriest. Stability of systems with distributed delays. In *IFAC Conf. System Structure and Control*, pages 294–299, Nantes, France, 1995.

E. I. Verriest. Linear systems with rational distributed delay: reduction and stability. In *European Control Conference, ECC'99*, Karlsruhe, Germany, 1999.

E. I. Verriest. Robust stability and stabilization: from linear to nonlinear. In *2nd IFAC Workshop on Linear Time Delay Systems*, Ancône, Italy, 2000.

E. I Verriest, O. Sename, and P. Pepe. Robust observer-controller for delay-differential systems. In *IEEE Conference on Decision and Control*, Las Vegas, USA, 2002.

E.I. Verriest. Stability of systems with state-dependent and random delays. *IMA Journal of Mathematical Control and Information*, 19(1-2):103–114, 2002.

E.I. Verriest. Optimal control for switched point delay systems with refractory period. In *16-th IFAC World Congress, Prague, Czeck Republic*, 2005.

E.I. Verriest and A.F. Ivanov. Robust stabilization of systems with delayed feedback. In *Proceedings of the second International Symposium on Implicit and Robust Systems, Warszawa, Poland*, 1991.

E.I. Verriest and A.F. Ivanov. Robust stability of systems with delayed feedback. *Circuits, Systems and Signal Processing*, 13(2-3):213–222, 1994a.

E.I. Verriest and A.F. Ivanov. Robust stability of delay-difference equations. *Systems and Networks: Mathematical Theory and Applications, Ed. U. Helmke, R. Mennicken and J. Saurer, University of Regensburg*, pages 725–726, 1994b.

E.I. Verriest and P. Pepe. Time optimal and optimal impulsive control for coupled differential difference point delay systems with an application in forestry. In *IFAC Workshop on time-delay systems*, Nantes, France, 2007.

E.I. Verriest, F. Delmotte, and M. Egerstedt. Optimal impulsive control for point delay systems with refractory period. In *Proceedings of the 5-th IFAC Workshop on Time Delay Systems, Leuven, Belgium, September 2004*, 2004.

F. Veysset, L. Belkoura P. Coton, and J.P. Richard. Delay system identification applied to the longitudinal flight of an aircraft through a vertical gust. In *MED'06, 14th IEEE Mediterranean Conference on Control and Automatique*, Ancona, Italy, 2006.

M. Vidyasagar. *Nonlinear systems analysis, 2nd ed.* Prentice-Hall, Upper Saddle River, NJ, USA, 1993.

H.O. Walther. Differentiable semiflows for differential equations with state-dependent delays. Technical report, Universitatis Lagellonicae Acta Mathematica, 2003.

X. Wei and L. del Re. Gain scheduled $\mathcal{H}_\infty$ control for air path systems of diesel engines using LPV techniques. *IEEE Trans. on Control Systems Technology*, 15(3):406–415, 2007.

B. A. White, L. Bruyere, and A. Tsourdos. Missile autopilot design using quasi-LPV polynomial eigenstructure assignment. *IEEE Trans. on Aerospace and Electronics Systems*, 43(4): 1470–1483, 2007.

K. Wickwire. Mathematical models for the control of pests and infectious diseases: A survey. *Theoretical Population Biology*, pages 182–238, 1977.

J.C. Willems. *The analysis of feedback systems.* MIT Press, Cambridge, MA, USA, 1971.

J.C. Willems. Dissipative dynamical systems i & ii. *Rational Mechanics and Analysis*, 45(5): 321–393, 1972.

E. Witrant, D. Georges, C. Canudas De Wit, and O. Sename. Stabilization of network controlled systems with a predictive approach. In 1^{st} *Workshop on Networked Control System and Fault Tolerant Control, Ajaccio, France*, 2005.

F. Wu. A generalized LPV system analysis and control synthesis framework. *International Journal of Control*, 74:745–759, 2001a.

F. Wu. Delay dependent induced $\mathcal{L}_2$-norm analysis and control for LPV systems with state delays. In *ASME International Mechanical Engineering Congress and Exposition*, 2001b.

F. Wu. Robust quadratic performance for time-delayed uncertain linear systems. *International Journal of Robust and Nonlinear Systems*, 13:153–172, 2003.

F. Wu and K.M. Grigoriadis. LPV systems with parameter-varying time delays: analysis and control. *Automatica*, 37:221–229, 2001.

F. Wu and B. Lu. Anti-windup control design for exponentially unstable LTI systems with actuator saturation. *Systems & Control Letters*, 52:305–322, 2004.

F. Wu and M. Soto. Extended anti-windup control schemes for LTI and LFT systems with actuator saturations. *International Journal of Robust and Nonlinear Control*, 14:1255–1281, 2004.

F. Wu, X.H. Yang, A. Packard, and G. Becker. Induced $\mathcal{L}_2$-norm control for LPV systems with bounded parameter variation rates. *International Journal of Robust and Nonlinear Control*, 6(9-10):983–998, 1996.

L. Wu, P. Shi, C. Wang, and H. Gao. Delay-dependent robust $\mathcal{H}_\infty$ and $\mathcal{L}_2$-$\mathcal{L}_\infty$ filtering for LPV systems with both discrete and distributed delays. In *IEE proc. Control Theory and Applications. Vol. 153, p.483-492*, 2006.

M. Wu, Y. He, J. H. She, and G. P. Liu. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, 40:1435–1439, 2004.

J. L. Wyatt, L.O Chua, J.W. Gannett, I.C. Gokar, and D.N. Green. Energy concepts in the state-space theory of nonlinear n-ports, i. passivity. *IEEE Transactions on Circuits and Systems*, 28(1):48–61, 1981.

G. Xie, D. Zheng, and L. Wang. Controllability of switched linear systems. *IEEE Trans. on Automatic Control*, 47(8):1401–1405, 2002.

L. Xie, M. Fu, and C.E. de Souza. $\mathcal{H}_\infty$ control and quadratic stabilization of systems with parameter uncertainty via output feedback. *IEEE Transactions on Automatic Control*, 37(8):1253–1256, 1992.

L. Xie, S. Shishkin, and M. Fu. Piecewise Lyapunov functions for robust stability of linear time-varying systems. *Systems & Control Letters*, 31(3):165–171, 1997.

S. Xu and J. Lam. Improved delay dependent stability criteria for time-delay systems. *IEEE Transactions on Automatic Control*, 50:384–387, 2005.

S. Xu and J. Lam. On equivalence and efficiency of certain stability criteria for time-delay systems. *IEEE Transactions on Automatic Control*, 52(1):95–101, 2007.

S. Xu, J. Lam, and Y. Zhou. New results on delay-dependent robust $\mathcal{H}_\infty$ control for systems with time-varying delays. *Automatica*, 42(2):343–348, 2006.

X. Xu and P. J. Antsaklis. Optimal control of switched autonomous systems. In *Conference on Decision and Control*, Las Vegas, NV, USA, 2002.

V.A. Yakubovich. $\mathcal{S}$-procedure in nonlinear control theory (in russian). *Vestnik Leningrad University*, 1:62–77, 1971.

V.A. Yakubovich. $\mathcal{S}$-procedure in nonlinear control theory (english translation). *Vestnik Leningrad University*, 4:73–93, 1977.

V.A. Yakubovich. Minimization of quadratic functionals under quadratic constraints and the necessity of a frequency condition in the quadratic criterion for absolute stability of nonlinear control systems. *Soviet. Math. Doklady*, 14:593–597, 1979.

V.A. Yakubovich. A frequency theorem for periodic systems. *Soviet Math. Dokl.*, 33(3): 360–363, 1986a.

V.A. Yakubovich. Linear quadratic optimization problem and the frequency theorem for periodic systems. *Siberian Math. Journ.*, 21(4):614–630, 1986b.

V.A. Yakubovich, A.L. Fradkov, D.J. Hill, and A.V. Proskurnikov. Dissipativity of T-periodic linear systems. *IEEE Transactions on Automatic Control*, 52(6):1039–1047, 2007.

V.A. Yakubovitch. The solution to certain matrix inequalities in automatic control. *Soviet Math Dokl.*, 3:620–623, 1962.

V.A. Yakubovitch. A frequency theorem for the case in which the state and control spaces are Hilbert spaces with an application to some problems of synthesis of optimal controls - Part I-II. *Sibirskii Mat. Zh. vol. 15, no 3, pp. 639-668, 1974; English translation in Siberian Math. J*, 1974.

G. Zames. On the input output stability of time-varying nonlinear feecback systems, part i : Conditions using concepts of loop gain, conicity and positivity. *IEEE Trans. on Aut. Control*, 11(2):228–238, 1966.

F. Zhang. *The Schur Complement and Its Applications.* Springer, 2005.

F. Zhang and K.M. Grigoriadis. Delay-dependent stability analysis and $\mathcal{H}_\infty$ control for state-delayed LPV system. In *Conference on Control and Automation*, 2005.

J. Zhang, C.R. Knospe, and P. Tsiotras. Stability of linear time-delay systems: A delay-dependent criterion with a tight conservatism bound. In *38th IEEE Conf. on Decision and Control pp.4678-4683*, Phoenix, AZ, USA, 1999.

J. Zhang, C.R. Knospe, and P. Tsiotras. Stability of time-delay systems: Equivalence between Lyapunov and scaled small-gain conditions. *IEEE Transactions on Automatic Control*, 46: 482–486, 2001.

X. Zhang, P. Tsiotras, and C. Knospe. Stability analysis of LPV time-delayed systems. *Int. Journal of Control*, 75:538–558, 2002.

X-M. Zhang and Q-L. Han. Robust $\mathcal{H}_\infty$ filtering for a class of uncertain linear systems with time-varying delay. *Automatica*, 44:157–166, 2008.

F. Zheng and P.M. Frank. Robust control of uncertain distributed delay systems with application to the stabilization of combustion in rocket motor chambers. *Automatica*, 38: 487–497, 2002.

K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, Upper Saddle River, New Jersey, USA, 1996.