

# Caractérisations de l'aléatoire par les jeux: imprédictibilité et stochasticité.

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# Outline

- 1 Introduction
  - Effective randomness: why?
  - Effective randomness: how?
- 2 The effective randomness zoo
  - Unpredictability notions
  - Typicalness notions
- 3 Randomness and complexity
  - Randomness and Kolmogorov complexity
  - Randomness and compression
- 4 Randomness for computable measures
  - Generalized Bernoulli measures
  - General case

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- **In this thesis: effective randomness for finite and infinite binary sequences**

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These 3 approaches are often referred to as: unpredictability paradigm, typicalness paradigm and incompressibility paradigm, respectively.

Each of these 3 paradigms yields different models/concepts.

- unpredictability paradigm  $\rightarrow$  prediction games
- typicalness paradigm  $\rightarrow$  “statistical” tests
- incompressibility paradigm  $\rightarrow$  Kolmogorov complexity

## Prediction games (intuition)

We consider games where a player tries to guess the bits of a binary sequence. Player wins if his predictions are accurate. The sequence is random if no computer program can make accurate predictions.

## Statistical tests (intuition)

A random sequence should satisfy all the properties of high probabilities, e.g. a random sequence should contain about as many zeros than ones.

We restrict our attention to properties that can be **checked** by computers; for each such properties, we can design a program that tests it (in the above example, a program counting the number of zeros), which we call statistical test. A sequence is random if no test fails on it.

## Kolmogorov complexity (intuition)

A random sequence should contain no pattern whatsoever. Hence (if the sequence is finite) there should be no way to write a short computer program that generates the sequence. We call **Kolmogorov complexity** of a sequence the length of the shortest program that generates it (it is in some sense the ideal compressed form of the sequence) and we say that a finite sequence is random if its Kolmogorov complexity is high.

## In this thesis:

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- stability of randomness notions w.r.t. the probability measure [Chapter 3]

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For **infinite binary sequences**, we will be able to give various definitions of randomness.

Let's play!

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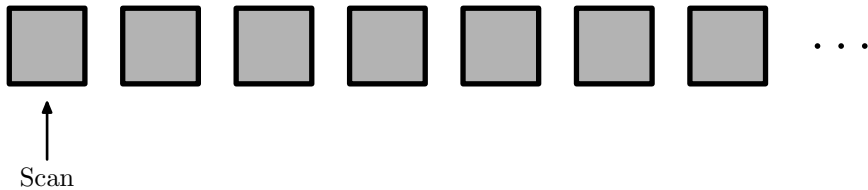
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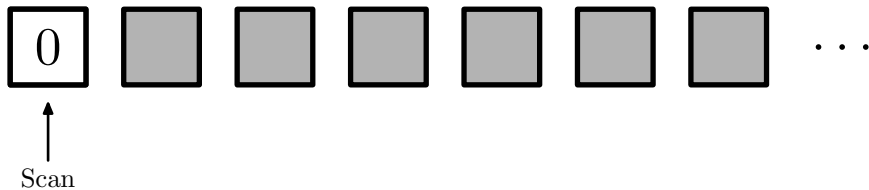
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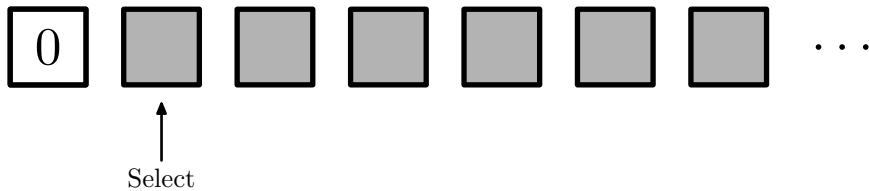
- The bits of the sequence are written on cards, facing down
- The player tries to predict the values of these cards in order. At each move, he can decide to select a bit or simply ask to see the card
- The player wins the infinite game if (1) he selects infinitely many bits during the game (2) the sequence of selected bits is biased i.e. contains more than 50% of zeros or more than 50% of ones

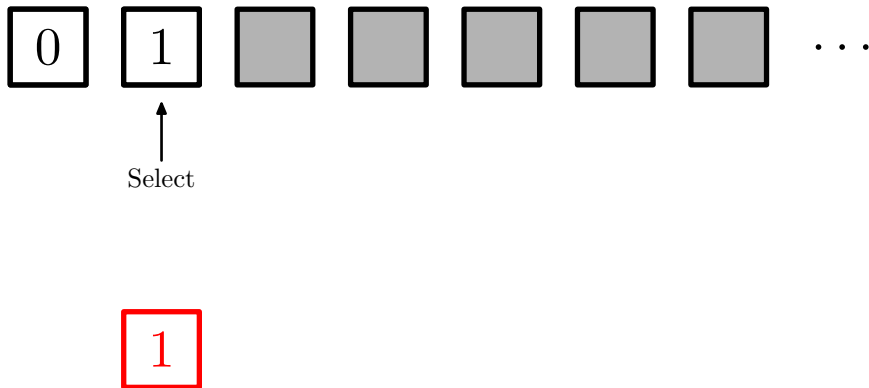


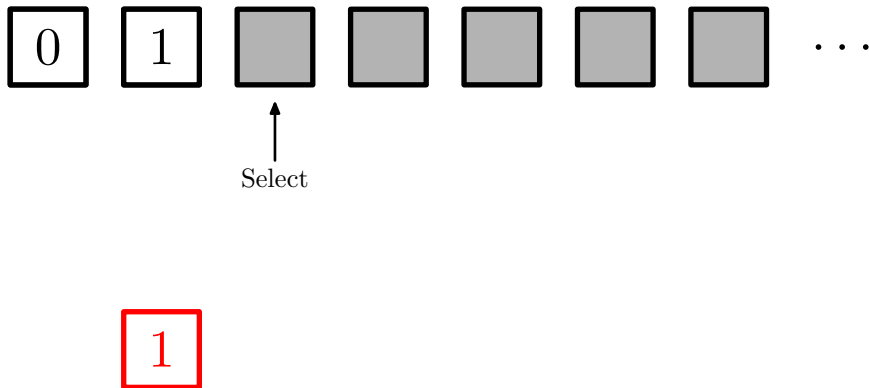


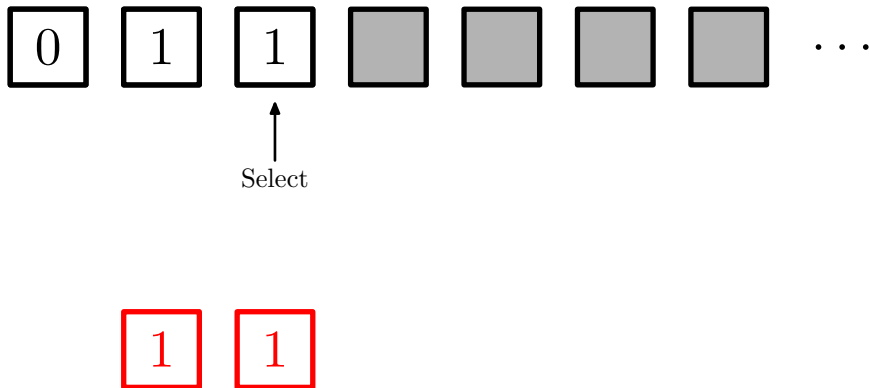


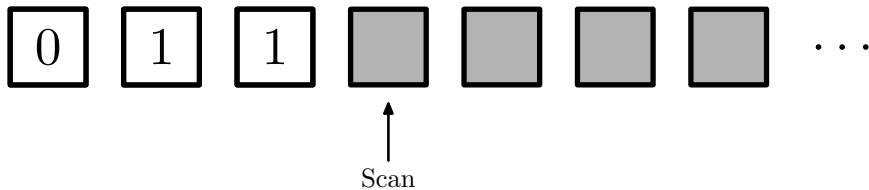




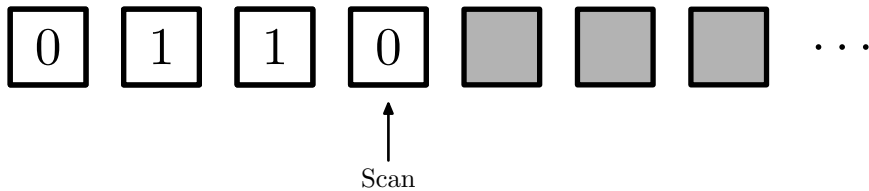


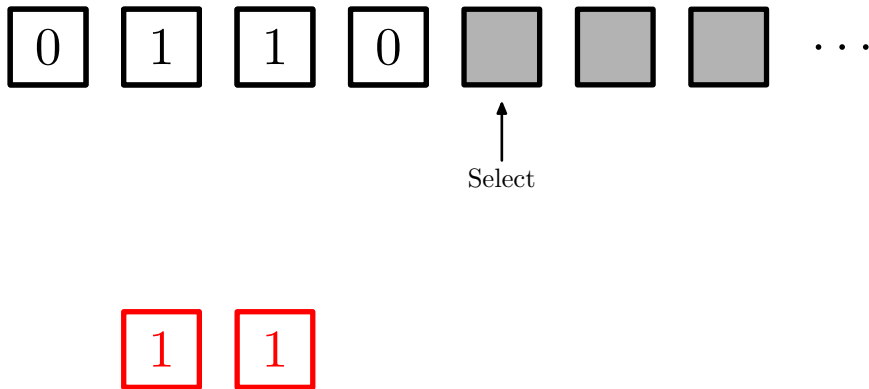


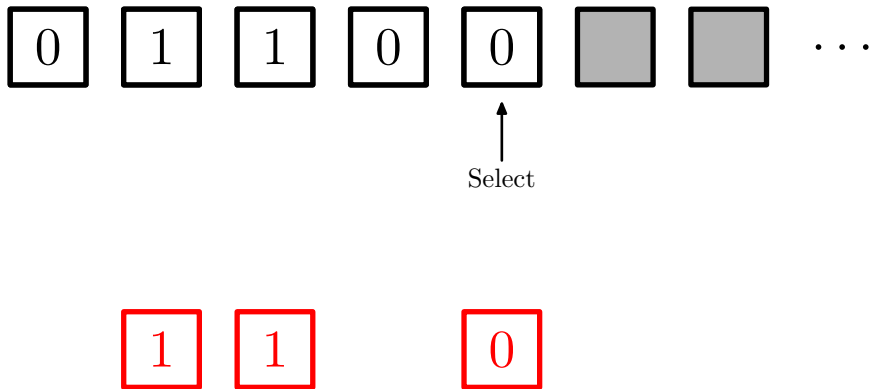


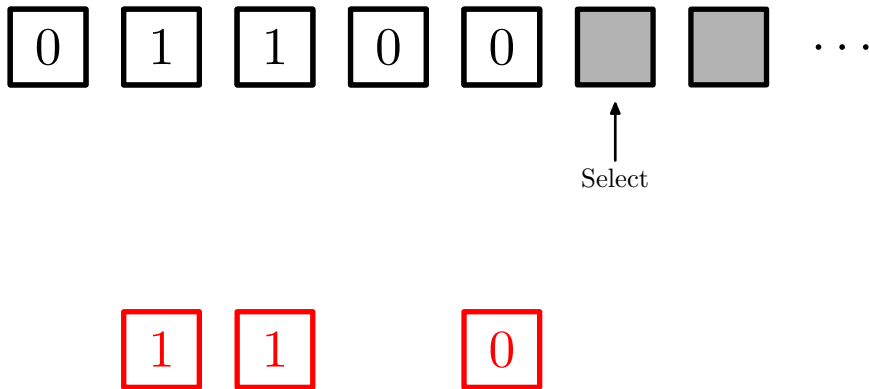


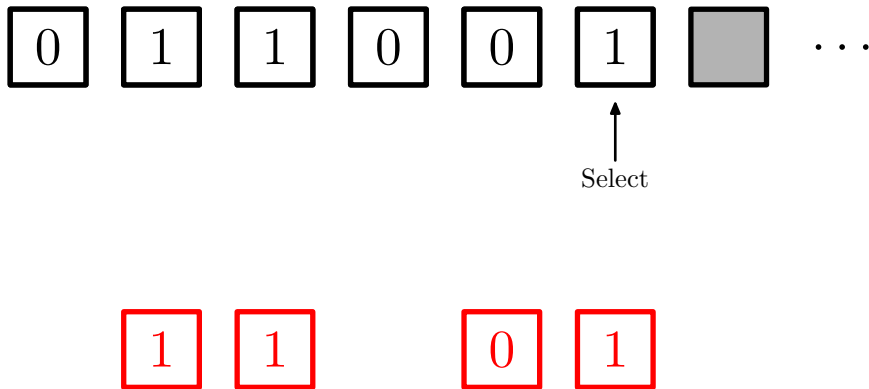


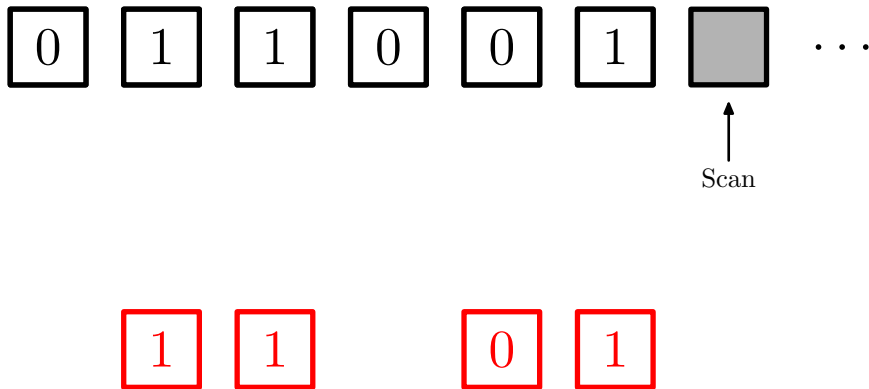


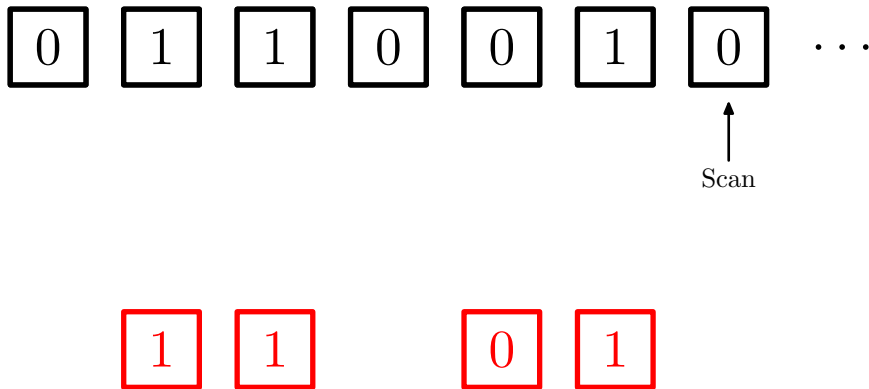












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As argued by Ville, this definition is a bit too weak.

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We now consider a refined version of the previous prediction game. Instead of the binary choice select/read, the player can now bet money on the value of the bits.

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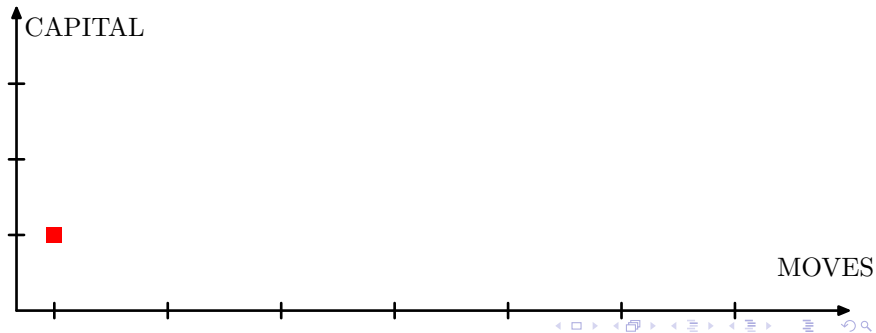
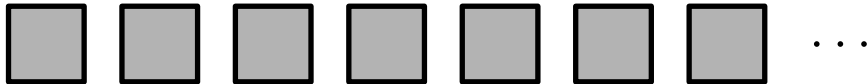
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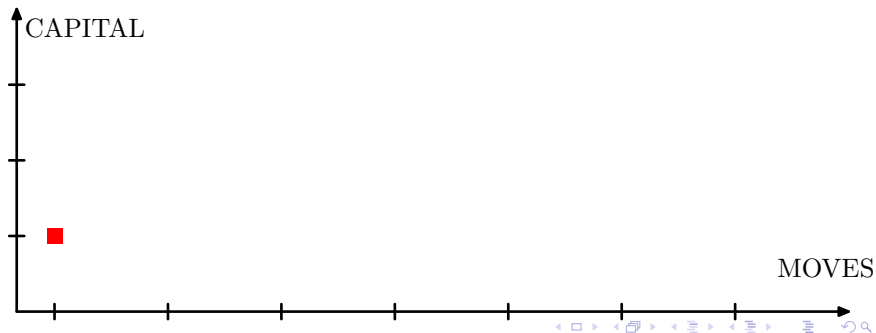
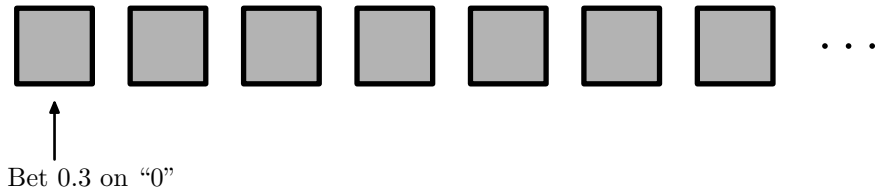
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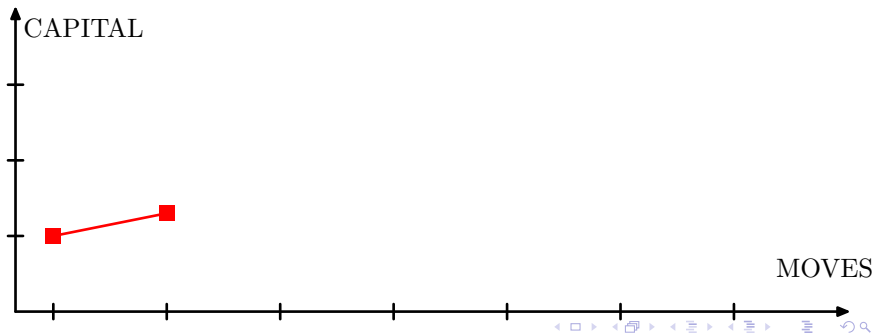
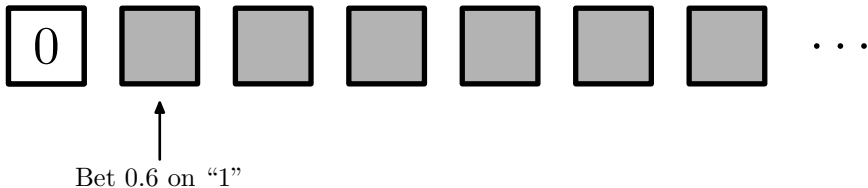
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- The player wins if his capital tends to  $+\infty$  during the game

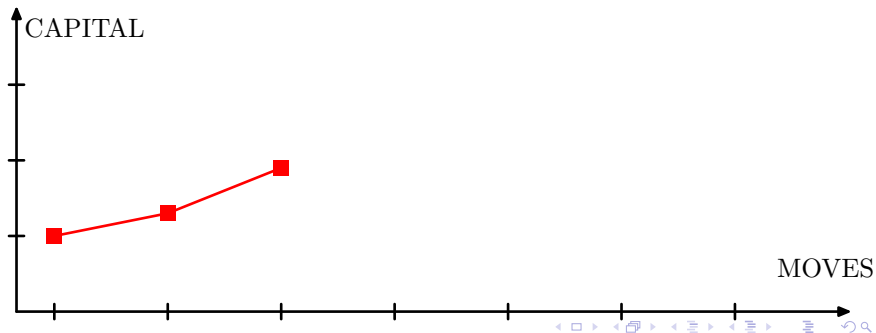
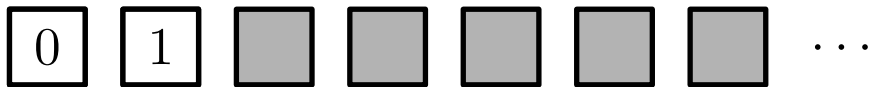


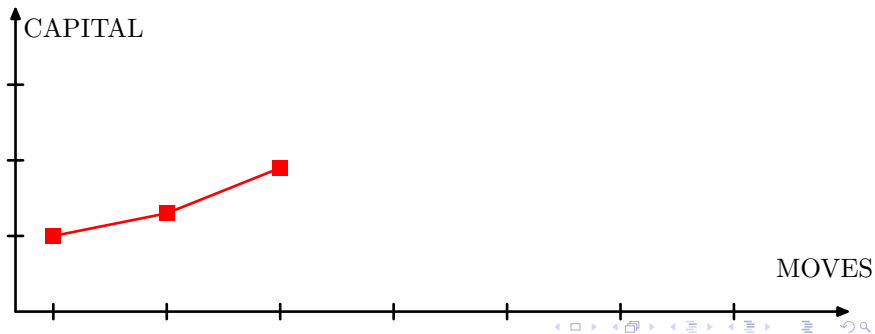
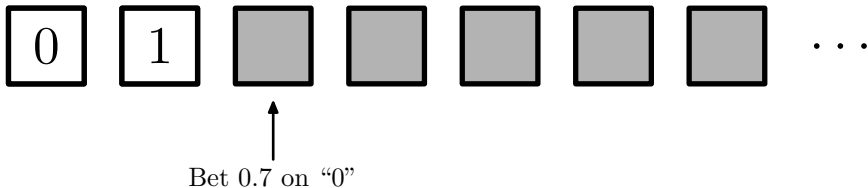


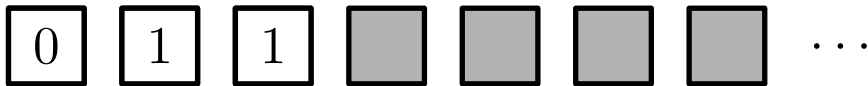




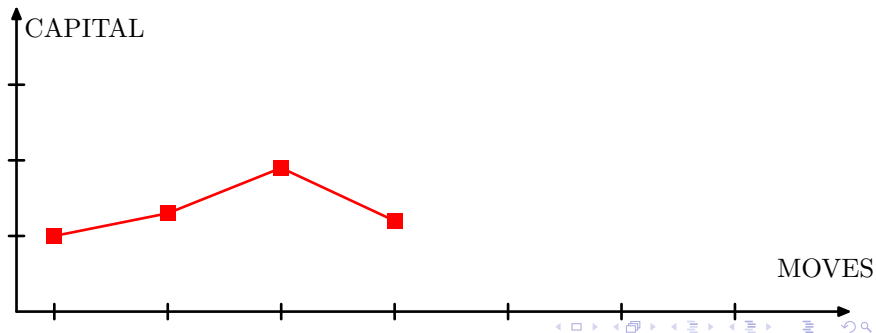


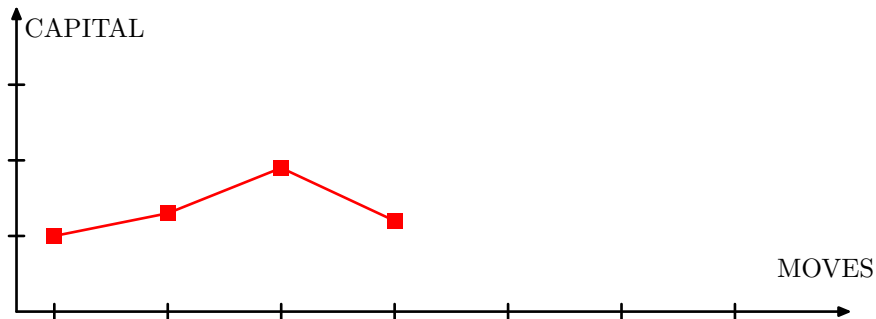
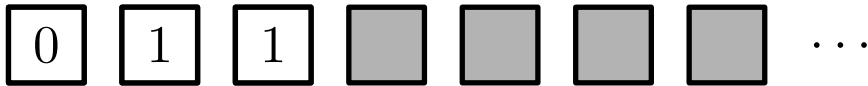


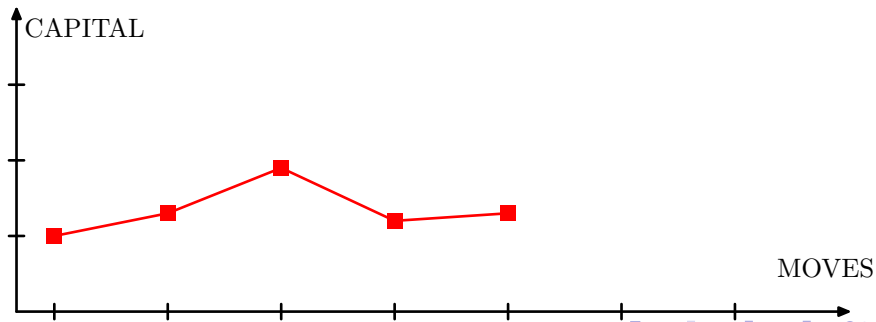
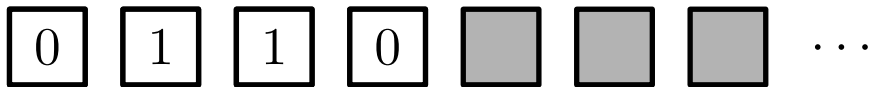


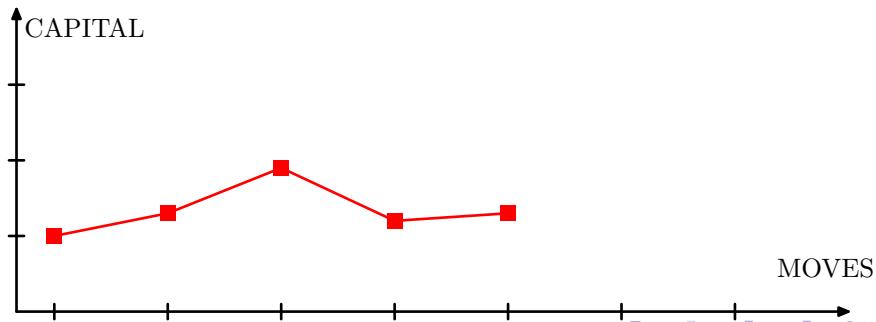
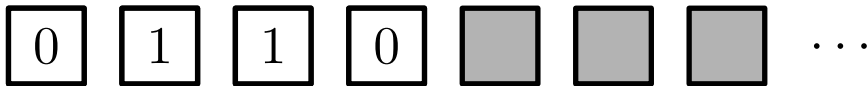


↑  
Bet 0.7 on "0"

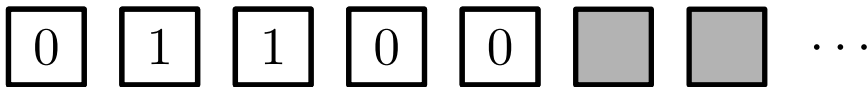




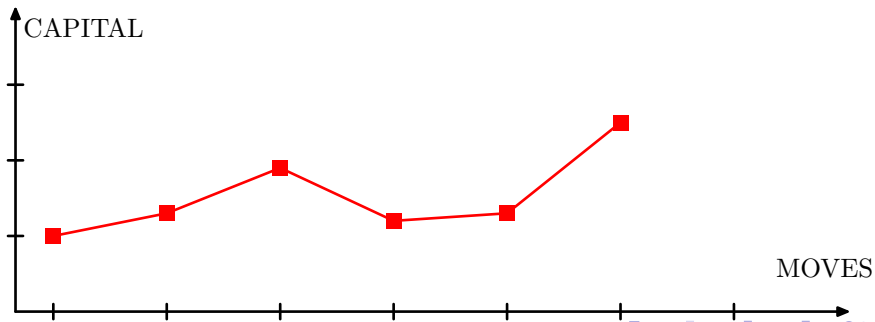


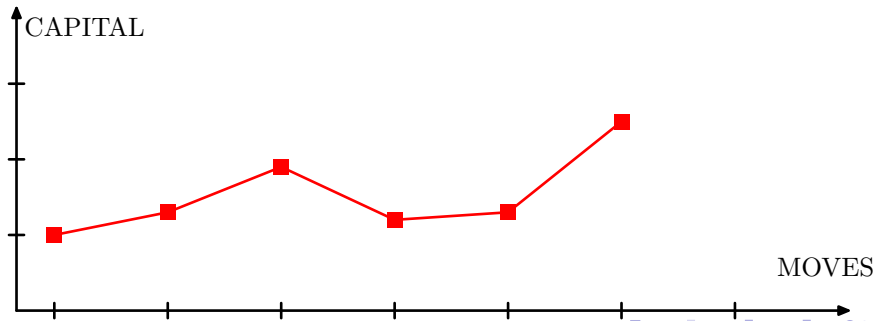
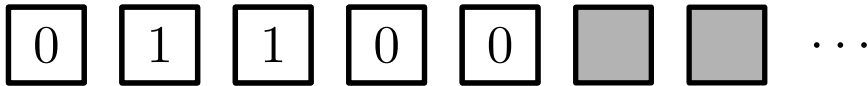


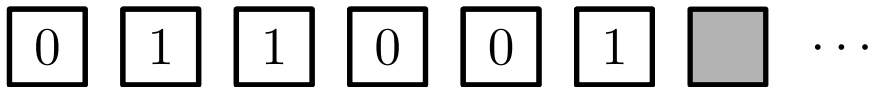




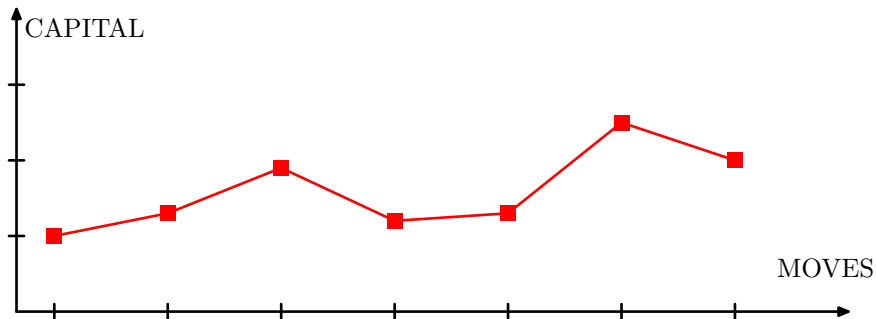
↑  
Bet 1.2 on "0"

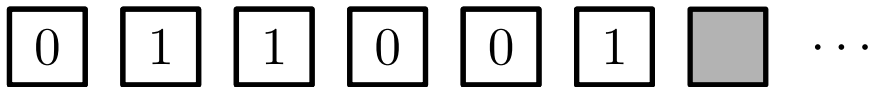




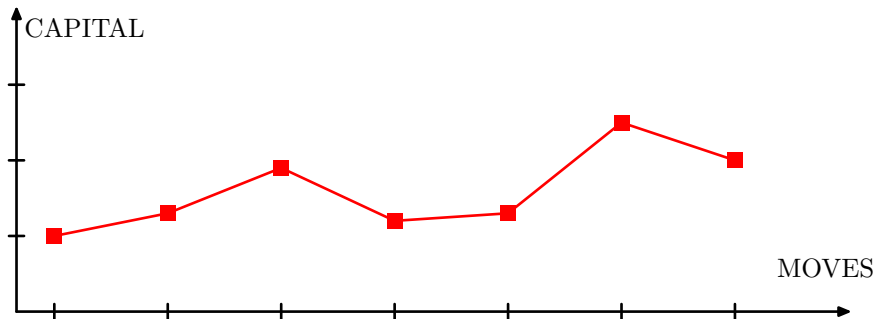


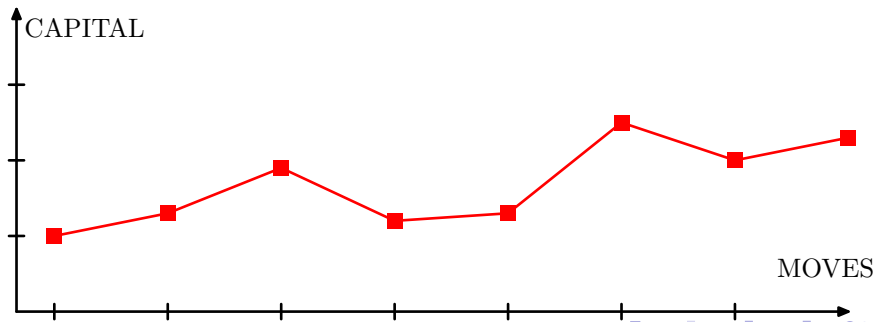
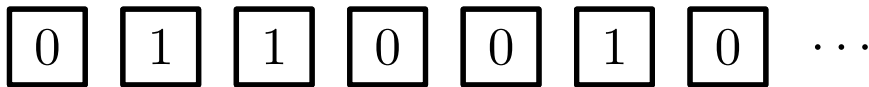
Bet 0.5 on "0"





↑  
Bet 0.3 on "0"





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How does the notion of computable randomness compare to that of Church stochasticity?

## Church stochasticity vs computable randomness

Theorem (Ville 1939)

*Computable randomness is strictly stronger than Church stochasticity*



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### Theorem

Selection of a subsequence with bias  $\delta$

$\Leftrightarrow$  exponentially winning strategy, with exp. factor  $1 - \mathcal{H}(1/2 + \delta)$

## Kolmogorov-Loveland randomness and stochasticity

One can strengthen Church stochasticity and computable randomness by considering games where the Player can guess the bits in any order.

This yields the stronger notions of Kolmogorov-Loveland stochasticity and Kolmogorov-Loveland randomness.

# Statistical tests

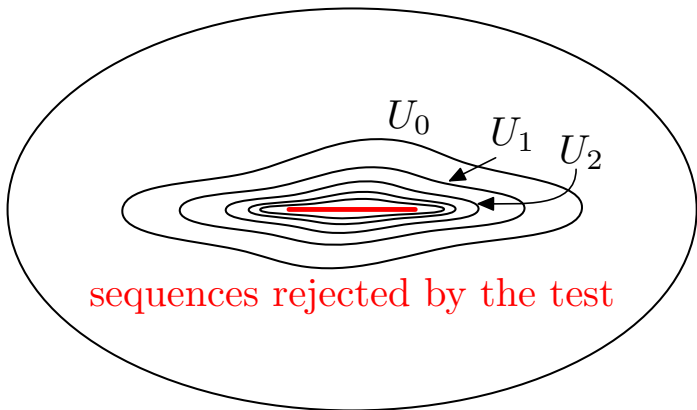
Here we want to formalize the idea that a sequence is non-random if it fails some statistical test.

For us, a statistical test will be a sequence  $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$ , where

- each  $\mathcal{U}_i$  is a set of infinite sequences which can computably generated
- the measure of the  $\mathcal{U}_i$  tends to 0

A sequence  $\alpha$  **fails the test** if it belongs to all  $\mathcal{U}_i$ .

all sequences



## Two types of tests

- Martin-Löf tests: the measure of  $\mathcal{U}_n$  is bounded by a computable function  $\varepsilon(n)$
- Schnorr tests: the measure of  $\mathcal{U}_n$  is computable

### Definition

An infinite sequence is **Martin-Löf random** if it fails no Martin-Löf test.

An infinite sequence is **Schnorr random** if it fails no Schnorr test.

## Theorem

*Martin-Löf randomness implies KL-randomness (hence implies KL-stochasticity, computable randomness, Church stochasticity)*

## Theorem

*Computable randomness implies Schnorr randomness*



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We have discussed unpredictability and typicalness. Let us move on to the last paradigm: incompressibility.

### Incompressibility paradigm

A finite binary sequence is random if it does not have a description shorter than itself.

## Definition

The **Kolmogorov complexity** of a finite binary sequence  $x$  is the length of the shortest program which outputs  $x$ .

Roughly speaking, the complexity of  $x$  lies between 0 and  $size(x)$ .

$$\begin{aligned} complexity(x) \approx 0 &\leftrightarrow x \text{ highly compressible} \\ &\leftrightarrow x \text{ not very random} \end{aligned}$$

$$\begin{aligned} complexity(x) \approx size(x) &\leftrightarrow x \text{ incompressible} \\ &\leftrightarrow x \text{ quite random} \end{aligned}$$

We use two types of Kolmogorov complexity for a string  $x$ :  $C(x)$  (plain complexity) and  $K(x)$  (prefix complexity). They are equal up to a logarithmic term.

# How complex are random sequences?

For Martin-Löf randomness, the situation is well understood:

Theorem (Levin-Schnorr  $\approx$  1970)

*A sequence  $\alpha$  is Martin-Löf random if and only if for all  $n$ :*

$$K(\alpha_0 \dots \alpha_n) \geq n - O(1)$$

For computable and Schnorr randomness, the situation is radically different:

Theorem (Muchnik et al. 1998, Merkle 2003)

*There exists a computably random sequence  $\alpha$  such that for any computable nondecreasing unbounded function (= order function)  $h$ , we have:*

$$C(\alpha_0 \dots \alpha_n) \leq \log n + h(n)$$

Note that this is **very** low: if we remove the term  $h(n)$ , the condition forces  $\alpha$  to be a computable binary sequence!

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- Martin-Löf random sequences are of (almost) maximal complexity.
- Computationally random, Schnorr random, and Church stochastic sequences **can** have very low complexity.
- What about KL-stochastic sequences?

It turns out that KL-stochastic sequences **must** have high complexity.

Theorem (Merkle, Miller, Nies, Reimann, Stephan 2005)

*If a sequence  $\alpha$  is KL-stochastic, then,*

$$\lim_{n \rightarrow +\infty} \frac{K(\alpha_0 \dots \alpha_n)}{n} = 1$$

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Looking at things from another angle: if  $K(\alpha_0 \dots \alpha_n) < sn$  for some  $s < 1$  and infinitely many  $n$ , then there exists a computable non-monotonic selection rule which selects an infinite sequence with bias  $\delta > 0$ .

It turns out that KL-stochastic sequences **must** have high complexity.

Theorem (Merkle, Miller, Nies, Reimann, Stephan 2005)

*If a sequence  $\alpha$  is KL-stochastic, then,*

$$\lim_{n \rightarrow +\infty} \frac{K(\alpha_0 \dots \alpha_n)}{n} = 1$$

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How do  $s$  and  $\delta$  relate? (Asarin, Durand and Vereshchagin for finite sequences).

A precise result for infinite sequences:

### Theorem

If a sequence  $\alpha$  is such that  $K(\alpha_0 \dots \alpha_n) < sn$  for some  $s < 1$  and infinitely many  $n$ , then: there exists a computable non-monotonic selection rule which selects an infinite sequence of bias as close as we want to  $\delta$ , where  $\delta$  is such that  $\mathcal{H}(1/2 + \delta) = s$ .

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The proof involves the game-theoretic argument we saw earlier:

- First, we construct a strategy that succeeds exponentially fast.
- Then, we transform this strategy into a selection rule.

Merkle: there exist computably random, Schnorr random, and Church stochastic sequences of very low complexity. Roughly speaking, this means that there is no necessary condition on the complexity for these notions of randomness.



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Can we find a sufficient one?

Trivially, the Levin-Schnorr condition  $K(\alpha_0 \dots \alpha_n) \geq n - O(1)$  is a sufficient condition. Can we do better than that? That is, some condition of type  $K(\alpha_0 \dots \alpha_n) \geq n - h(n)$  for some unbounded function  $h$ ?

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For Schnorr randomness: **yes**. For Church stochasticity: **no**

## Theorem

There exists an order function  $h$  such that if  $K(\alpha_0 \dots \alpha_n) \geq n - h(n)$  for all  $n$ , then  $\alpha$  is Schnorr random.

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## Theorem

There exists no such function for Church stochasticity.

# Zip!

Like we said, the Kolmogorov complexity  $K(x)$  of a finite sequence  $x$  can be seen as the size of an ideal compression of  $x$ . Unfortunately(?), there is no way to perform effectively this ideal compression.

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One way to overcome this problem is to give up on the hope to find the best compression, and consider instead the compression obtained by a good compressor (Cilibrasi and Vitanyi).

Here we will not study a particular compressor. Rather, we want to give the most general definition of a compressor that captures the idea of compression.

## Definition

A **compressor** is a computable one-to-one function from the set of finite sequences to itself.

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Given a compressor  $\Gamma$ , we get a **computable** upper bound  $C_\Gamma$  of Kolmogorov complexity by setting  $C_\Gamma(x) = |\Gamma(x)|$ .

Similarly we can find compressors  $\Gamma$  that give computable upper bounds of  $K$ , and we then set  $K_\Gamma(x) = |\Gamma(x)|$ .

We now look again at the complexity of random sequences, where this time we use approximations by compression.

For example, what are the sequences  $\alpha$  such that

$$(\forall \Gamma) \quad K_{\Gamma}(\alpha_0 \dots \alpha_n) \geq n - O(1) \quad ?$$

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Martin-Löf sequences satisfy this condition.

**In fact, this condition characterizes Martin-Löf randomness!**

Another very interesting fact is that some notions like Schnorr randomness which seemed rather unrelated to Kolmogorov complexity can be nicely characterized with its approximations by compression:

### Theorem

A sequence  $\alpha$  is Schnorr random if for all  $\Gamma$  and all computable order function  $h$ , one has  $K_{\Gamma}(\alpha_0 \dots \alpha_n) \geq n - h(n) - O(1)$ .



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Some similar characterizations exist for other notions of randomness that are not related to Kolmogorov complexity (weak randomness, computable Hausdorff dimension, etc.)

# Outline

- 1 Introduction
  - Effective randomness: why?
  - Effective randomness: how?
- 2 The effective randomness zoo
  - Unpredictability notions
  - Typicalness notions
- 3 Randomness and complexity
  - Randomness and Kolmogorov complexity
  - Randomness and compression
- 4 Randomness for computable measures
  - Generalized Bernoulli measures
  - General case

So far we have discussed effective randomness w.r.t. the uniform measure, where bits are chosen independently, with a probability distribution  $(1/2, 1/2)$ .

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All randomness notions (Martin-Löf randomness, Schnorr randomness, computable randomness, etc.) can be extended to arbitrary computable probability measures. For stochasticity notions, it is not so simple, as they rely on the Law of Large Numbers, which holds only for very specific measures.

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We are interested in the following problem: how fragile are notions of randomness? Precisely, how much can we modify the measure without changing the randomness notions?

## Generalized Bernoulli measures

Suppose now that each bit  $\alpha_n$  of the sequence is chosen independently from the other, but with a probability distribution  $(1/2 + \delta_n, 1/2 - \delta_n)$ .

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This induces a probability measure, called **generalized Bernoulli measure of parameter  $(1/2 + \delta_n)_{n \in \mathbb{N}}$** .



## Kakutani's theorem

### Theorem (Kakutani 1948)

- 1 A generalized Bernoulli measure of parameter  $(1/2 + \delta_n)_{n \in \mathbb{N}}$  is equivalent (i.e. has the same events of probability 0) to the uniform measure if and only if

$$\sum_{n \in \mathbb{N}} \delta_n^2 < +\infty$$

- 2 When

$$\sum_{n \in \mathbb{N}} \delta_n^2 = +\infty$$

there exists an  $\mathcal{X}$  which has probability 1 for the Bernoulli measure of parameter  $(1/2 + \delta_n)_{n \in \mathbb{N}}$  and probability 0 for the uniform measure.

## Kakutani's theorem: effective versions

### Theorem (Vovk 1987)

- ① *A computable generalized Bernoulli measure of parameter  $(1/2 + \delta_n)_{n \in \mathbb{N}}$  has the same Martin-Löf random sequences as the uniform measure if and only if*

$$\sum_{n \in \mathbb{N}} \delta_n^2 < +\infty$$

- ② *When*

$$\sum_{n \in \mathbb{N}} \delta_n^2 = +\infty$$

*the class of Martin-Löf sequences the Bernoulli measure of parameter  $(1/2 + \delta_n)_{n \in \mathbb{N}}$  and the class of Martin-Löf random sequences for the uniform measure are disjoint.*

## Kakutani's theorem: effective versions

### Theorem

The Kakutani-Vovk criterion holds for computable randomness and Schnorr randomness as well.

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### Theorem

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The proof uses a game theoretic argument: transform a winning strategy for a measure into a winning strategy for the other measure.

Stochasticity on the other hand is robust **beyond** the Kakutani-Vovk criterion.

Theorem (Van Lambalgen, Shen  $\approx$  1988)

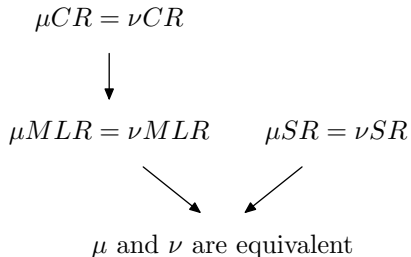
*Let  $\mu$  be a generalized Bernoulli measure of parameter  $(1/2 + \delta_n)_{n \in \mathbb{N}}$  such that  $\lim \delta_n = 0$ . Then, the set of (Church or KL) stochastic sequences has  $\mu$ -probability 1.*

Corollary

*This separates stochasticity notions from randomness notions*

Proof: choose a sequence at random w.r.t. the generalized Bernoulli measure  $\mu$  of parameter  $(1/2 + \frac{1}{\sqrt{n+4}})$ . Then with  $\mu$ -probability 1, we get a stochastic sequence, and with  $\mu$ -probability 1 we get a non-random sequence (by the Kakutani-Vovk criterion).

## General case: a strange hierarchy



## Links with computable analysis

A classical result on the set of infinite binary sequences:

### Theorem

*Let  $\mu$  and  $\nu$  be two measures. The following are equivalent:*

- 1  $\mu$  and  $\nu$  have the same nullsets
- 2  $\mu$  and  $\nu$  have the same  $G_\delta$  nullsets
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Can we effectivize this, replacing “measures” by “**computable measure**”,  $G_\delta$  by “**effective  $G_\delta$** ” and closed by “**effectively closed**”?



For the first part, **yes**

### Theorem

Two computable measures that have the same effective  $G_\delta$  nullsets have the same nullsets.

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### Theorem

Two computable measures that have the same effective  $G_\delta$  nullsets have the same nullsets.

For the second one, **no!**

### Theorem

Two computable measures can have the same effectively closed nullsets without having the same nullsets.

# Conclusion.....

- Very strong links between unpredictability, typicalness, complexity, compressibility, etc.

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- Can we use the results on computable measures to study measure-invariant notions (such as lowness)?
- **KL randomness = Martin-Löf randomness????**

Thank you  
Спасибо  
Merci  
Köszönöm