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Thèse

FLUIDES COMPLEXES EN FILMS MINCES

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Résumé

Cette thèse est consacrée à la modélisation, à l'analyse mathématique et à la simulation numérique d'écoulements de divers fluides complexes dans des domaines de faible épaisseur. En effet, les modèles de fluides newtoniens ne sont pas toujours suffisants pour décrire de manière réaliste les écoulements considérés. Plusieurs phénomènes peuvent être pris en compte :

1. le caractère complexe des fluides eux-mêmes, comme pour des fluides non-newtoniens ;
2. l'hétérogénéité de l'écoulement, dans le cas de mélanges de fluides par exemple.

Il est important d'analyser comment ces modèles peuvent être simplifiés dans le cas de domaines minces, et d'étudier rigoureusement les modèles approchés.

Dans une première partie, des écoulements de fluides non newtoniens visco-élastiques représentés par une loi de comportement de type Oldroyd-B couplée aux équations de Navier-Stokes sont étudiés. Dans le cas de géométries minces, un modèle approché a été proposé. On justifie la validité de cette approximation ; la démonstration repose sur des estimations et des résultats de régularité fins.

Dans une deuxième partie, on considère un modèle d'écoulement piezovisqueux utilisé en lubrification hydrodynamique. Ce modèle fait aussi intervenir la déformation élastohydrodynamique du domaine (déformation du type Hertz), et l'aspect diphasique de la cavitation, qui est décrit par le modèle d'Elrod-Adams (en pression-saturation). On montre l'existence d'une solution à ce problème pour des lois pression-viscosité réalistes.

Dans une troisième partie, on introduit un modèle diphasique à interface diffuse, permettant de rendre compte de phénomènes plus fins tels que les gouttes. Pour cela, un paramètre d'ordre est introduit (fraction volumique d'une phase dans le mélange), gouverné par le modèle de Cahn-Hilliard. Un système approché est obtenu de manière heuristique pour un domaine de faible épaisseur. On étudie les propriétés mathématiques de ce système, et on montre un résultat d'existence, avec prise en compte ou non de la tension de surface.

Dans la dernière partie, un schéma numérique est mis en place pour simuler le modèle décrit précédemment d'écoulements diphasiques en domaines minces. Il permet de prendre en compte différents phénomènes physiques, comme de grandes variations de la viscosité ou la présence de recirculations à l'intérieur d'une goutte, ainsi que de simuler des mélanges dans le cadre d'écoulements lubrifiés.

Abstract

This thesis is devoted to the modelling, the mathematical analysis and the numerical simulation of various complex flows in thin film situations. In fact, Newtonian models are not always sufficient to describe the physical flows in a realistic way. Many phenomena may be taken into account, such as:

1. the complexity of the fluids themselves, as for non Newtonian fluids;
2. the heterogeneity of the flow, for example in the case of mixtures.

It is of importance to analyse how these models can be simplified in thin domains, and to study rigorously the approximate models.

In the first part, we study non Newtonian viscoelastic flows represented by a behavior law of Oldroyd type coupled with the Navier-Stokes equations. We justify the validity of this approximation; the proof is based on elaborate estimates and regularity results.

In the second part, we consider a piezoviscous model used in hydrodynamical lubrication. This model also involves the elastohydrodynamical deformation of the domain (Hertz-type deformation) and the diphasic aspect of cavitation through the Elrod-Adams model (pressure-saturation model). We prove the existence of a solution to this problem for realistic pressure-viscosity laws.

In the third part, we introduce a diffuse-interface diphasic model which allows to model complex phenomena such as drops. To this end, we use an order parameter (volumic fraction of one phase in the mixture) governed by the Cahn-Hilliard model. An approximate system is obtained in a heuristical way for a thin domain. We study the mathematical properties of this system, and we prove an existence result with or without surface tension.

In the last part, a numerical scheme is introduced in order to simulate the previous model for diphasic flows in thin domains. It allows to take into account several physical features, such as great variations of the viscosity or recirculations in drops, as well as to simulate mixtures in the lubricated flows setting.

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Introduction

1.1 Sur les écoulements complexes en film mince : Problématique

1.1.1 Introduction

Dans de nombreux domaines, les modèles de fluides newtoniens ne sont pas suffisants pour rendre compte des aspects complexes des écoulements considérés. Les physiciens et les mathématiciens ont donc été amenés à développer des modèles plus sophistiqués, permettant de prendre en compte diverses propriétés des fluides. Dans cette optique, plusieurs approches sont possibles.

- D'une part, le caractère complexe des fluides étudiés peut être pris en compte globalement, en utilisant des modèles de *fluides non-newtoniens* (par exemple visco-élastiques), ou en tenant compte de la *compressibilité des fluides.*

- D'autre part, l'hétérogénéité de l'écoulement lui-même peut être introduite dans la modélisation, dans le cas de *mélanges de fluides* par exemple (présence de plusieurs phases dans l'écoulement), ou lors d'*écoulements turbulents.*

Par ailleurs, la complexité des équations de Navier-Stokes complètes (en trois dimensions) a poussé les physiciens et les mathématiciens à fabriquer de nouveaux modèles plus simples, qui sont des approximations satisfaisantes des équations de Navier-Stokes dans certains cas, en particulier pour certaines géométries. C'est le cas pour des domaines anisotropes, où une des dimensions du domaine est très inférieure aux autres. Différents modèles plus simples sont alors obtenus, en fonction des ordres de grandeur des paramètres caractéristiques du problème et des paramètres choisis comme négligeables ou prépondérants. Lorsque les équations de Navier-Stokes dans un domaine anisotropique (ou domaine mince) sont couplées à un modèle prenant en compte le caractère complexe du fluide considéré ou de l'écoulement, la même approche peut être utilisée.

Du point de vue des applications en ingénierie, cette simplification des équations est cruciale, car elle diminue le coût des calculs et permet de réaliser des simulations numériques des phénomènes physiques. Dans le cas de domaines minces, cette simplification peut se faire par différents aspects :

- la réduction à un nombre de dimensions inférieur, c'est-à-dire l'obtention d'un modèle bidimensionnel pour un phénomène physique tridimensionnel,
- la simplification des équations elles-mêmes, en prenant en compte le fait que certains termes sont négligeables,
- la possibilité de découpler les équations afin de résoudre des équations plus simples, par exemple le découplage du calcul de deux grandeurs physiques telles que la pression et le champ de vitesse.

Les problèmes mathématiques survenant en mécanique des films minces apparaissent à deux niveaux. D'une part, il s'agit de justifier les équations obtenues souvent heuristiquement à partir d'un modèle tridimensionnel. D'autre part, l'étude de ces équations, dont la structure est différente de celles tridimensionnelles, est elle-même l'objet de travaux mathématiques.

1.1.2 Quelques modèles classiques en film mince

Les différents modèles usuels en film mince sont issus d'approximations des équations de Navier-Stokes. Enonçons de manière générale ces équations sur le champ de vitesses $\boldsymbol{u}$, la pression p, en prenant en compte divers termes sources :

$$\begin{cases} \rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) - \eta\Delta\boldsymbol{u} + \nabla p = \mathcal{F}_\sigma + \mathcal{F}_g + \mathcal{F}_c + \mathcal{F}_\kappa, \\ + \text{ conditions aux limites sur } \boldsymbol{u}. \end{cases}$$

où η est la viscosité, ρ la densité. Le terme $\mathcal{F}_\sigma$ correspond au terme d'extra-contrainte, dans le cas non-newtonien, $\mathcal{F}_g$ est le terme de gravité, $\mathcal{F}_c$ correspond aux forces de Coriolis, et $\mathcal{F}_\kappa$ désigne la prise en compte de la tension de surface. Donnons quelques exemples de modélisations *dans le cas de domaines minces*, en précisant quels termes sont prépondérants selon les approches :

✗ Pour les applications usuelles en lubrification, le phénomène le plus important est la prise en compte du cisaillement (imposé par le mouvement relatif de deux surfaces entre lesquelles le fluide s'écoule, voir Figure 1.1), qui apparaît à travers les conditions aux limites sur $\boldsymbol{u}$ (conditions d'adhérence du fluide sur les parois).

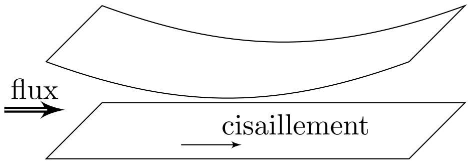

Figure 1.1: Allure du domaine pour des mécanismes lubrifiés

Dans ce cas, il est usuel de négliger les termes sources (gravité, force de Coriolis), ainsi que les effets inertiels et transitoires. L'anisotropie du domaine permet de négliger les variations verticales de la pression. Enfin, si le fluide est newtonien et homogène, la tension de surface est faible. Le modèle obtenu est l'*équation de Reynolds*, qui peut s'écrire comme une équation sur la pression uniquement. Nous donnons plus de détails sur l'équation de Reynolds dans la Section 1.2.1.

✗ Un second exemple, utilisé en météorologie pour la simulation des mouvements de l'atmosphère et ceux de l'océan, consiste à prendre en compte les effets de la gravité, et éventuellement ceux des forces de Coriolis. Ce modèle est connu sous le nom d'équations de Saint Venant, ou encore *équations en eau peu profonde* (shallow water) [SV71], [GP01]. De manière plus précise, celles-ci ne décrivent pas la vitesse du fluide en tout point, mais concernent une vitesse moyennée sur l'épaisseur du domaine. Il s'agit d'un modèle à frontière libre, où la hauteur de la surface supérieure est une inconnue du problème (voir figure 1.2).

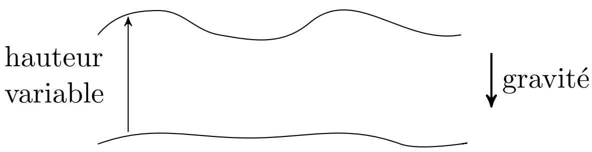

Figure 1.2: Allure du domaine pour les équations en eau peu profonde

Dans ce cas, des conditions aux limites de Bernoulli sont imposées sur la surface libre (indiquant que la vitesse normale à la surface est nulle), prenant éventuellement en compte la tension de surface (condition limite sur $\boldsymbol{\sigma}$). Sur la surface inférieure, des conditions de type Navier, prenant en compte le frottement, sont parfois considérées. C'est un problème essentiellement hyperbolique, transitoire, dont la justification mathématique a été effectuée récemment [BN07].

✗ Un troisième exemple d'application existe dans le cadre de la *microfluidique*, c'est-à-dire à des échelles micrométriques. De telles échelles sont pertinentes pour la modélisation de nombreux phénomènes, dans des domaines d'application aussi divers que la biologie, la chimie ou la science des matériaux, et permettent en particulier d'étudier le transport de gouttes dans un microcanal (par exemple dans une géométrie telle que celle de la Figure 1.3).

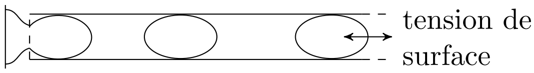

Figure 1.3: Allure du domaine pour le transport de gouttes en microcanaux

Dans de tels problèmes, la tension de surface à l'interface entre les deux fluides est à prendre en compte. Celle-ci est modélisée par un terme lié à la courbure de l'interface, et fait intervenir la fonction caractéristique de l'interface ϕ, appelée fonction "level-set", qui vérifie une équation de transport. Dans ce cas, la densité et la viscosité du bifluide sont considérées comme variables en fonction de ϕ. Puisque la tension de surface et les effets visqueux sont prépondérants, l'équation de Stokes stationnaire avec tension de surface modélise de façon satisfaisante le comportement des fluides dans des microcanaux [Poi40], [KBA05] (les termes de convection, les termes instationnaires et les termes de gravité et Coriolis sont négligés).

✗ Pour des applications industrielles de type *moulage de pièces* ou injection de matières plastiques entre deux plaques proches, le modèle de Hele-Shaw est approprié. Dans le modèle initial, les deux plaques supérieure et inférieure sont fixes, séparées par une distance h, et un fluide est injecté par un orifice situé sur une de ces plaques (voir Figure 1.4). Le flux d'entrée du fluide est imposé.

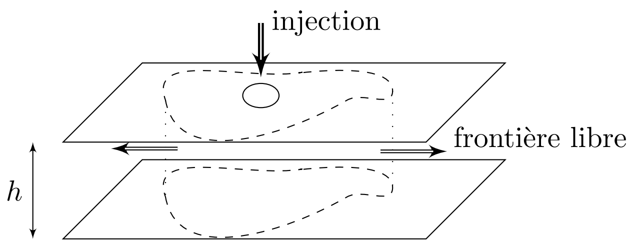

Figure 1.4: Allure du domaine pour le moulage de pièces

La position du domaine occupé par le fluide n'est pas connue, il s'agit d'un problème à frontière libre. Les ordres de grandeur en jeu pour ces applications sont tels que les termes de convection, la tension de surface et le terme instationnaire dans l'équation de Navier-Stokes sont négligés. L'aspect instationnaire vient du fait que le domaine occupé par le fluide dépend du temps. De plus, la frontière libre peut être supposée verticale. En ce qui concerne les conditions aux limites sur la vitesse sur les surfaces fixes hors de la zone d'injection, des conditions de non-glissement sont imposées.

Dans [BBT95], une généralisation du modèle de Hele-Shaw est étudiée, pour des conditions d'injection latérale entre des plaques non parallèles dont la position respective peut dépendre du temps, et est définie par une fonction connue variable $h(x,t)$. La

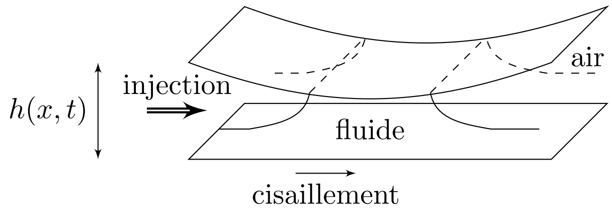

Figure 1.5: Allure du domaine pour la cavitation en lubrification

vitesse relative provoque un effet de cisaillement (Figure 1.5), et l'hypothèse d'une frontière libre "verticale" ne peut être conservée. Celle-ci peut être décrite par une fonction $\theta(x,t)$ telle que l'épaisseur du fluide soir $h(x,t)\theta(x,t)$. On obtient alors des équations proches de celles utilisées en lubrification pour décrire le phénomène de cavitation (*modèle d'Elrod-Adams*). Ce modèle est présenté plus en détail en section 1.2.2, dans le paragraphe sur la cavitation.

1.1.3 Quelques modèles classiques pour les écoulements complexes

Les fluides newtoniens sont des fluides pour lesquels le tenseur des contraintes visqueuses est une fonction linéaire connue du tenseur de déformation. Cependant, de nombreux fluides présentent un comportement plus complexe, qui peut être décrit par des lois de comportement sur la viscosité, ou par des lois plus complexes sur la contrainte. D'autre part, la présence de deux fluides ou de deux phases confère également à l'écoulement un caractère non-newtonien. Nous donnons quelques exemples des différents modèles pouvant être considérés :

✗ Les fluides quasi-newtoniens sont des fluides dont la viscosité η s'exprime comme une fonction non-linéaire du tenseur de déformations $D(\boldsymbol{u})$. Parmi ces fluides, on distingue différentes lois, comme les lois de Carreau, lois de puissance... De telles lois permettent par exemple de modéliser le comportement pseudoplastique d'un fluide. Ces modèles ont été étudiés d'un point de vue numérique dans [BN90], [San93], travaux dans lesquels des estimations d'erreur et des taux de convergences sont obtenus. Des résultats d'existence et de régularité d'une solution sont dus à Blavier et Mikelić [BM95], dans le cas stationnaire, puis à Marušić-Paloka [MP02] dans le cas instationnaire.

✗ Les fluides à seuil, comme ceux de Bingham, ont des lois de comportement qualitativement différentes selon la contrainte. Il s'agit de fluides ayant le comportement d'un corps rigide pour des contraintes faibles, et ayant un comportement visqueux au-delà d'un certain seuil pour le tenseur des contraintes. Ce type de modèle permet par exemple de simuler les écoulements de boues, de laves... Existence et unicité d'une solution sont montrées dans le cas stationnaire dans [Kim87], puis dans le cas instationnaire dans [Com92].

✗ Des modèles plus complexes peuvent être considérés, pour lesquels la contrainte $\boldsymbol{\sigma}$ est solution d'une équation aux dérivées partielles (ce sont les lois dites différentielles), ou bien d'une équation intégrale (lois intégrales). De tels modèles permettent de représenter le comportement des fluides visco-élastiques. Les lois intégrales sont obtenues par des considérations moléculaires ou de mécanique des milieux continus. Celles-ci représentent bien le fait que les fluides visco-élastiques sont des fluides à mémoire, mais leur étude mathématique présente de très grandes difficultés et elles sont très coûteuses en temps de calcul pour la simulation numérique. Certaines de ces lois peuvent être écrites sous forme de lois différentielles, pour lesquels un grand nombre d'outils mathématiques sont disponibles. Nous nous intéressons ici à une de ces lois différentielles, utilisée dans le modèle d'Oldroyd-B (voir plus de détails en Section 1.2.3) :

  - Du point de vue mathématique, le système des équations de Navier-Stokes couplées à de telles lois a été étudié par Renardy [Ren85], qui a montré un théorème d'existence pour le cas stationnaire. Ces résultats ont été généralisés au cas instationnaire par [GS90], [FCGO98] [MT04], où les auteurs montrent des résultats d'existence locale en temps, ou à données petites. Dans le cas d'un fluide diphasique, un résultat d'existence locale en temps est montré pour ce modèle dans [Chu04].
  - Le premier résultat global en temps est dû à Lions et Masmoudi [LM00].

✗ Dans le cas de mélanges de deux fluides ou de deux phases différentes, les paramètres physiques du bifluide tels que la viscosité, ou éventuellement la densité, ne sont plus constants, et dépendent de la composition locale de celui-ci. Il existe divers modèles permettant de prendre en compte cet aspect, selon le point de vue adopté : les modèles dits "à interface ponctuelle", qui supposent que l'épaisseur de l'interface entre les deux fluides est nulle ([OF03]), et les modèles dits "à interface diffuse", plus récents, prenant en compte des interactions chimiques à l'interface, et dont l'étude mathématique est due à Boyer ([Boy99], [Boy02], [Boy01]).

1.1.4 Position du problème

Cette thèse se situe dans le cadre de l'étude de différents modèles de fluides complexes pour des écoulements de faible épaisseur :

- Lors de l'injection de polymères entre deux plaques proches, le caractère visco-élastique des polymères est pris en compte par une loi de comportement appropriée (loi d'Oldroyd-B). L'étude mathématique de cette loi en film mince est effectuée dans le chapitre 2, où nous nous attachons à justifier rigoureusement le passage en film mince pour le système Navier-Stokes/Oldroyd.

- De nombreux problèmes complexes sont à prendre en compte en lubrification hydrodynamique, en particulier le phénomène de cavitation, qui correspond à l'apparition de bulles de gaz dans un lubrifiant liquide. D'un point de vue mathématique, il s'agit de traiter un problème diphasique. Ce phénomène est traité dans le chapitre 3, dans lequel nous présentons l'analyse mathématique d'un modèle dérivé de l'équation de Reynolds, qui décrit le comportement d'un fluide piezovisqueux en élastohydrodynamique, et la possible apparition de cavitation.

- Enfin, dans les chapitres 4 et 5, nous présentons les choix d'adimensionnement permettant l'obtention formelle du modèle limite étudié (système Reynolds/Cahn-Hilliard) et nous effectuons l'analyse mathématique des équations limites obtenues, ainsi que des simulations numériques sur ce système.

1.2 Etat de l'art

1.2.1 L'équation de Reynolds

L'équation de Reynolds a été introduite dans [Rey86], étude dans laquelle Reynolds obtient par un raisonnement heuristique l'équation qui porte son nom. L'idée de ce travail est d'utiliser l'hypothèse de film mince (c'est-à-dire que la distance séparant les deux surfaces entre lesquelles circule le fluide est très petite devant les dimensions des surfaces) pour simplifier les équations de Navier-Stokes en négligeant les variations de la pression dans la direction transverse aux deux surfaces, ainsi que certains termes dans l'équation de l'hydrodynamique. Dans le cas de l'équation de Reynolds unidimensionnelle, Sommerfeld a obtenu la première solution analytique de l'équation de Reynolds en 1904 [Som04]. D'un point de vue mathématique, des justifications partielles de cette approximation (analogue à la théorie des plaques en élasticité) ont été données en 1950 par Wannier [Wan50], puis en 1959 par Elrod [Elr60] et par Cimatti en 1983 [Cim83], basées sur des méthodes de développements asymptotiques formels. La démonstration rigoureuse de la convergence

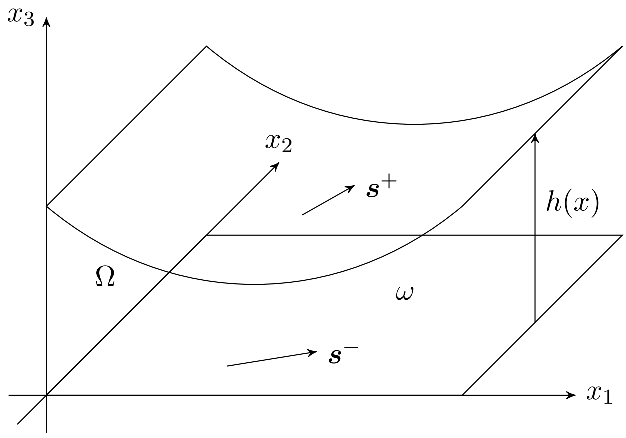

Figure 1.6: Géométrie usuelle dans laquelle l'équation de Reynolds est pertinente

des équations de Stokes vers l'équation de Reynolds a été établie par Bayada et Chambat en 1986 [BC86b], et pour les équations de Navier-Stokes par Assemien, Bayada, Chambat dans [ABC94], et Nazarov et Videman dans [NV04].

L'équation de Reynolds permet de déterminer la distribution de pression p dans un espace mince rempli de fluide entre deux surfaces. Elle s'écrit de manière générale, en trois dimensions, pour un domaine $\Omega = \{(x,z), x \in \omega, 0 < z < h(x)\}$ de la forme indiquée sur la Figure 1.6 :

$$\operatorname{div}_x\left(\frac{\rho h^3}{6\eta}\nabla_x p\right) = \operatorname{div}_x\left(\rho h(\boldsymbol{s}^+ + \boldsymbol{s}^-)\right) + \frac{\partial}{\partial t}\left(\frac{\rho h}{2}\right),$$

où $x = (x_1, x_2)$, ρ est la masse volumique, η la viscosité, $h(x)$ la hauteur adimensionnée du contact, supposée strictement positive (pas de contact). $\boldsymbol{s}^\pm$ désignent les composantes horizontales des vitesses de chacune des surfaces dans les directions (x_1, x_2). Souvent, cette équation peut être simplifiée. Dans le cas d'un régime établi, stationnaire, d'un fluide de densité constante, et en supposant que la vitesse de cisaillement est dirigée dans le sens des x_1 (i.e. $\boldsymbol{s}^+ + \boldsymbol{s}^- = (s,0) = \boldsymbol{s}$), l'équation de Reynolds se réduit à :

$$\operatorname{div}_x\left(\frac{h^3}{6\eta}\nabla_x p\right) = \operatorname{div}_x(h\boldsymbol{s}).$$

La transition Navier-Stokes – Reynolds

Indiquons en quelques mots comment l'équation de Reynolds est obtenue à partir de l'équation de Navier-Stokes [BC86b]. Nous considérons un domaine $\Omega_\varepsilon = \{(x, x_3) \in \mathbb{R}^{d+1}, x \in \omega, 0 < x_3 < \varepsilon h(x)\}$, où $\omega \subset \mathbb{R}^d$, $d = 1$ ou 2. Ici ε est un petit paramètre permettant de modéliser l'hypothèse d'écoulement de faible épaisseur (film mince). Les conditions aux limites choisies sur le champ de vitesses $\boldsymbol{u} = (u_1, u_2, u_3)$ permettent de modéliser à la fois des phénomènes de cisaillement et des phénomènes d'injection :

$$\boldsymbol{u}|_{x_3=0} = \boldsymbol{s} = (s, 0, 0), \qquad \boldsymbol{u}|_{x_3=\varepsilon h(x)} = (0,0,0), \qquad \int_{\partial\omega\times(0,\varepsilon h(x))} \boldsymbol{u}\cdot\boldsymbol{n} = Q_\varepsilon(x), \tag{1.1}$$

où Q_ε est un flux donné.

En introduisant le changement de variable

$$x = x, \qquad z = \frac{x_3}{\varepsilon},$$

les équations de Navier-Stokes sont réécrites avec des coefficients dépendant de ε, mais dans un domaine fixe renormalisé

$$\Omega = \{(x,z) \in \mathbb{R}^{d+1}, x \in \omega, 0 < z < h(x)\}.$$

De manière formelle, nous écrivons le premier terme des développements asymptotiques pour le champ de vitesses $\boldsymbol{u}$ et pour la pression p :

$$u_1 = u_1^*, \qquad u_2 = u_2^*, \qquad u_3 = \varepsilon u_3^*, \qquad p = \frac{1}{\varepsilon^2}p^*.$$

En injectant ces relations dans les équations de Navier-Stokes, il vient (en notant $\mathcal{F}$ les termes sources), pour $i \in \{1,2\}$:

$$\begin{aligned}
&\rho\Big(\partial_t\, u_i^* + \sum_{j=1}^{2} u_j^*\partial_{x_j}u_i^* + \varepsilon u_3^*\frac{1}{\varepsilon}\partial_z u_i^*\Big) - \eta\Big(\sum_{j=1}^{2}\partial_{x_j}^2 u_i^* + \frac{1}{\varepsilon^2}\partial_z^2 u_i^*\Big) + \partial_{x_i}\frac{p^*}{\varepsilon^2} = \mathcal{F}_i,\\
&\rho\Big(\partial_t\,\varepsilon u_3^* + \sum_{j=1}^{2} u_j^*\partial_{x_j}(\varepsilon u_3^*) + \varepsilon u_3^*\frac{1}{\varepsilon}\partial_z(\varepsilon u_3^*)\Big) - \eta\Big(\sum_{j=1}^{2}\partial_{x_j}^2 u_3^* + \frac{1}{\varepsilon^2}\partial_z^2(\varepsilon u_3^*)\Big) + \frac{1}{\varepsilon}\partial_z\frac{p^*}{\varepsilon^2} = \mathcal{F}_3,\\
&\sum_{j=1}^{2}\partial_{x_j}u_j^* + \frac{1}{\varepsilon}\partial_z(\varepsilon u_3^*) = 0,
\end{aligned}$$

ce système se simplifie, en supposant que η et ρ ne dépendent pas de ε et en ne retenant que les termes prépondérants (d'ordre le plus élevé par rapport à ε), en :

$$\begin{aligned}
&-\eta\partial_z^2 u_1^* + \partial_{x_1}p^* = 0,\\
&-\eta\partial_z^2 u_2^* + \partial_{x_2}p^* = 0,\\
&\partial_z p^* = 0,
\end{aligned} \tag{1.2}$$

$$\partial_{x_1} u_1^* + \partial_{x_2} u_2^* + \partial_z u_3^* = 0.$$

En intégrant deux fois les deux premières équations de (1.2) par rapport à z, et en utilisant les conditions aux limites (1.1), il vient

$$u_i^* = \frac{1}{2\eta} \partial_{x_i} p^* z(z-h) + s_i \left(1 - \frac{z}{h}\right), \qquad \text{pour } i \in \{1,2\}. \tag{1.3}$$

La vitesse verticale u_3^* est déduite de l'équation de la divergence en fonction de u_1^*, u_2^*. La condition d'incompressibilité et les conditions aux limites (1.1) donnent $u_3^*(x, h(x)) = 0 = -\int_0^{h(x)} \mathrm{div}_x\, \boldsymbol{u}^*\, dz$, où $\mathrm{div}_x\, \boldsymbol{u}^* = \partial_{x_1} u_1^*(x,z) + \partial_{x_2} u_2^*(x,z)$. De cette relation, on déduit l'équation de Reynolds, qui s'écrit sur la pression uniquement :

$$\mathrm{div}_x \left(\frac{h^3}{6\eta} \nabla_x p^* \right) = \mathrm{div}_x (h\boldsymbol{s}).$$

Les conditions aux limites

Il est important de noter que sur les bords latéraux, la valeur de la vitesse n'est pas imposée, mais seulement le flux. En effet, toutes données égales par ailleurs, les situations correspondant à des données de vitesses latérales différentes mais de même flux induisent la même équation de Reynolds (voir [BC86b]). Cette remarque est naturelle au vu de la structure de l'équation de Reynolds, qui est une équation de conservation du flux.

D'autre part, les équations de Navier-Stokes sont usuellement munies de conditions aux limites sur la vitesse. Nous expliquons en quelques mots ce que cela implique pour les conditions en pression.

- *Conditions de Neumann sur la pression :* les conditions de Dirichlet imposées sur la vitesse correspondent à des conditions de Neumann sur la pression. En effet, (1.3) implique que pour $x \in \partial\omega$:

  $$Q(x) = \int_0^{h(x)} u_1^*|_{x\in\partial\omega} = \frac{1}{2\eta} \partial_x p^*(x)|_{x\in\partial\omega} \int_0^{h(x)} z(z-h(x))dz + \int_0^{h(x)} s \left(1 - \frac{z}{h(x)}\right) dz,$$

  ce qui donne une relation entre $\partial_x p^*|_{x\in\partial\omega}$ et Q. Ces conditions permettent de modéliser des phénomènes d'injection, que nous allons considérer par la suite.

- *Conditions de Dirichlet sur la pression :* ce sont les conditions utilisées le plus souvent par les mécaniciens, mais elles ne correspondent pas au passage à la limite entre Navier-Stokes et Reynolds en partant de conditions de Dirichlet sur la vitesse. Il a été montré dans [CMP94] que les équations de Navier-Stokes peuvent être munies de conditions aux limites en pression. A partir de ces travaux, Bayada et

Chambat [BC89] ont généralisé la procédure asymptotique menant à l'équation de Reynolds pour des conditions de Dirichlet en pression, et une condition aux limites de Dirichlet est obtenue pour la pression dans le modèle asymptotique, ce qui permet de justifier le modèle utilisé par les ingénieurs.

1.2.2 Sur les fluides multiphasiques et leur comportement en film mince

Les modèles à interface ponctuelle

Une première classe de modèles concerne les modèles appelés à interface ponctuelle, car l'interface est supposée d'épaisseur nulle, et il n'y a pas de zone de mélange. La position de l'interface étant inconnue, il s'agit d'un problème à frontière libre. En général, cela correspond à des modèles où seuls les effets hydrodynamiques de l'écoulement sont pris en compte, sans tenir compte des effets chimiques à l'interface.

En film mince, la transition des équations de Navier-Stokes à l'équation de Reynolds a été étudiée dans le cas de fluides multicouches, afin de prendre en compte la différence de comportement d'un lubrifiant loin et près des surfaces. Les premiers travaux portent sur le cas où la position de l'interface entre les deux fluides de viscosité différente est supposée connue. Sous cette hypothèse, Tichy [Tic95] a obtenu une équation de Reynolds modifiée, dépendant des viscosités des deux fluides. Saint Jean Paulin et Taous [SJPT90] ont établi rigoureusement une équation de Reynolds globale pour deux fluides newtoniens non miscibles de viscosités différentes. Dans ces deux cas, la continuité de la vitesse et de la contrainte à l'interface est imposée, mais le fait que la vitesse doit être tangente à l'interface n'est pas respecté.

Si l'on veut effectivement prendre en compte la condition de non pénétration à l'interface, la frontière entre les deux fluides de viscosité différentes n'est pas connue. Il s'agit alors d'un problème à frontière libre, dans lequel la viscosité η vérifie une équation de transport. Pour une géométrie analogue à celle décrite Figure 1.6, et en supposant qu'il n'y a pas de cisaillement, Mikelić et Paoli [MP97][1] ont repris une approche asymptotique (quand l'épaisseur ε du domaine entre les deux fluides tend vers zéro) telle que celle décrite en Section 1.2.1, en supposant que la frontière libre est le graphe d'une fonction.L'évolution de la saturation $S(x,t)$ d'un des deux fluides dans le mélange permet de représenter celle de l'interface, grâce à l'hypothèse de graphe. Ils ont montré que si le problème initial est de dimension 2, l'interface limite (quand ε tend vers zéro) décrite par S vérifie une équation de Buckley-Leverett généralisée :

$$h\frac{\partial S}{\partial t} + Q\frac{\partial}{\partial x} f(S) = 0,$$

[1] en prenant comme point de départ des travaux de Nouri, Poupaud et Demay [NPD97]

où Q correspond au flux entrant, et f est une fonction connue. Cette équation est couplée à l'équation de Reynolds qui s'écrit :

$$\frac{\partial}{\partial x}\left(\frac{h^3}{6\eta}G(S)\frac{\partial p}{\partial x}\right) = s\frac{\partial}{\partial x}\left(h\,H(S)\right),$$

où G et H sont des fonctions connues.

Dans [Pao03], Paoli a généralisé les résultats précédents au cas avec cisaillement, puis Bayada, Martin et Vázquez [BMV06] ont étudié le système limite et montré l'existence et l'unicité d'une solution à ce système. Ils ont de plus comparé numériquement les résultats obtenus par le couplage Reynolds et Buckley-Leverett avec ceux obtenus avec le modèle heuristique de cavitation d'Elrod-Adams.

Les modèles à interface diffuse

Les modèles de type Cahn-Hilliard, à l'inverse des précédents, modélisent les phénomènes physico-chimiques à travers l'interface entre deux phases. Ils supposent en particulier que l'épaisseur de l'interface n'est pas nulle, et qu'il existe une zone, de petite épaisseur, dans laquelle les grandeurs physiques (par exemple les concentrations de chaque phase) évoluent continûment. Cette approche s'explique simplement : à l'échelle moléculaire, les deux phases ne peuvent être rigoureusement séparées, et il y a donc au moins une zone d'épaisseur de quelques tailles moléculaires dans laquelle les deux phases sont mélangées. Ces modèles sont donc pertinents, et permettent en particulier de simuler le comportement de phénomènes de l'ordre de grandeur de l'épaisseur de l'interface. De nombreuses études physiques leur sont consacrées [AMW98]. Ces modèles sont également intéressants par leur mise en oeuvre numérique assez naturelle, et permettent de simuler des phénomènes impliquant de larges déformations des interfaces et des changements topologiques.

Les premiers modèles ont été proposés par Cahn et Hilliard [CH58]. Ils sont basés sur l'introduction d'une énergie libre, dite de Cahn-Hilliard, qui rend compte de l'excès d'énergie à l'interface dû aux fortes variations de la composition du mélange dans cette zone. Ainsi, l'épaisseur de l'interface est prise en compte *via* cette énergie. Les premiers travaux mathématiques sur ces modèles sont dus à Elliott et Garcke [EG96], qui ont montré un résultat d'existence.

Ces modèles peuvent être enrichis en rajoutant les effets hydrodynamiques de l'écoulement, tout en gardant la modélisation à interface diffuse par l'énergie libre de Cahn-Hilliard. L'aspect hydrodynamique est pris en compte par l'intermédiaire d'un champ de vitesse moyen, sur lequel on écrit les équations de Navier-Stokes, avec un terme de force prenant en compte la tension de surface. Ces modèles ont été proposés par Chella et Viñals [CV96], et par Anderson, McFadden et Wheeler [AMW98]. Les premiers travaux mathématiques sur ces modèles sont dus à Boyer [Boy99], qui a montré l'existence, l'unicité et la régularité d'une solution au système couplé entre les équations de Navier-Stokes et l'équation de Cahn-Hilliard hydrodynamique.

Décrivons en quelques mots les termes significatifs intervenant dans ce genre de modèles. Pour décrire le mélange, on introduit les champs de vitesses de chacun des fluides $\boldsymbol{u}_1$ et $\boldsymbol{u}_2$, ainsi qu'un paramètre d'ordre $\varphi(t,x)$, qui est une renormalisation (entre -1 et 1) de la fraction volumique d'une des phases dans le mélange, et qui quantifie la proportion de l'un des deux fluides au point (t,x). A partir du paramètre d'ordre, on définit par moyenne pondérée la densité totale du mélange en fonction des densités ρ_1 et ρ_2 des deux fluides par la relation suivante :

$$\rho = \frac{1+\varphi}{2}\rho_1 + \frac{1-\varphi}{2}\rho_2. \tag{1.4}$$

Notons que, si chaque phase est incompressible, le mélange, lui, ne l'est pas. Cependant, on peut définir une vitesse moyenne $\boldsymbol{u}$, telle que la condition

$$\operatorname{div} \boldsymbol{u} = 0$$

soit conservée[2]. Il suffit de choisir une moyenne volumique définie par :

$$\boldsymbol{u} = \frac{1+\varphi}{2}\boldsymbol{u}_1 + \frac{1-\varphi}{2}\boldsymbol{u}_2.$$

En écrivant l'équation de conservation de la masse pour chacune des phases, on obtient une équation sur le paramètre d'ordre :

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi + \operatorname{div}\left(\frac{1-\varphi^2}{2}\boldsymbol{w}\right) = 0,$$

où $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ est la vitesse relative. La deuxième étape consiste à écrire les équations de Navier-Stokes pour chacune des phases, en tenant compte de la dépendance de la viscosité par rapport à φ, par une relation du même type que (1.4), et des forces extérieures. Dans les termes prenant en compte les potentiels chimiques de chaque phase, le potentiel F de Cahn-Hilliard intervient :

$$\mu = -\alpha^2\Delta\varphi + F'(\varphi),$$

où α est un paramètre sans dimension relié à l'épaisseur de l'interface, et F est un potentiel en double puits, dont les deux minima correspondent aux deux phases pures (voir Figure 1.7). Ce terme tend à réduire la taille des zones interfaciales, ce qui correspond au cas où les deux phases sont séparées.

[2] pour plus de détails, voir [LT98]

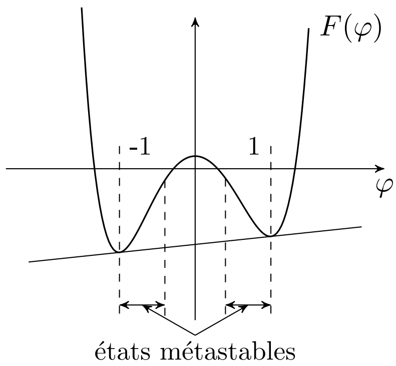

Figure 1.7: Allure du potentiel en double puits $F(\varphi)$

Le couplage avec les équations de Navier-Stokes est ensuite introduit, et $\boldsymbol{w}$ est supposée négligeable devant $\boldsymbol{u}$. En considérant de surcroît que les deux fluides sont homogènes, c'est-à-dire que $\rho_1 = \rho_2$, les équations se simplifient, et le système devient (sous forme adimensionnée) :

$$\begin{aligned} &\partial_t \varphi + \boldsymbol{u} \cdot \nabla\varphi + \frac{1}{\mathcal{P}e} \operatorname{div}\left(B(\varphi)\nabla\mu\right) = 0, \quad \text{avec } \mu = -\alpha^2\Delta\varphi + F'(\varphi), \\ &\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla\boldsymbol{u} - \frac{2}{\mathcal{R}e} \operatorname{div}\left(\eta(\varphi)D(\boldsymbol{u})\right) + \nabla p = \mathcal{F} + \kappa\mu \cdot \nabla\varphi, \end{aligned} \tag{1.5}$$

où $\mathcal{F}$ désigne les forces extérieures, κ le nombre capillaire, $\mathcal{P}e$ et $\mathcal{R}e$ les nombres de Péclet et Reynolds. Dans le cas où $\rho_1 \neq \rho_2$, le modèle est plus complexe, et des difficultés mathématiques supplémentaires apparaissent. Cette situation a été traitée en partie par Boyer dans [Boy01] (dans le cas où ρ_1 n'est pas trop loin de ρ_2), mais il n'y a pas de résultat d'existence dans le cas général. Par ailleurs, le passage d'un modèle à interface diffuse à un modèle à interface ponctuelle est un problème ouvert en général.

Dans le contexte des films minces, les modèles à interface diffuse n'ont pas encore été étudiés. C'est le sujet traité dans le chapitre 4 de ce travail.Un modèle asymptotique est obtenu heuristiquement, pour lequel nous montrons l'existence et la régularité d'une solution.

Un cas particulier d'écoulement multiphasique en film mince : la cavitation

On constate expérimentalement dans de nombreux mécanismes lubrifiés que l'écoulement n'est pas homogène et qu'il se divise en deux zones :

- Une zone non cavitée, ou saturée, notée Ω_+, dans laquelle la pression reste strictement supérieure à la pression de vapeur saturante. Dans cette zone, le film est complet, l'interstice entre les deux surfaces est rempli de lubrifiant.

- Une zone cavitée, notée $\Omega_0 = \Omega \backslash \Omega_+$, dans laquelle la pression est égale à la pression de saturation. Dans cette zone, l'interstice entre les deux surfaces est rempli d'un mélange de liquide et de gaz.

Ce phénomène dit de cavitation a été caractérisé par Dowson et Taylor [DT79] comme la rupture d'un film fluide continu par la présence d'un gaz ou de vapeur. L'équation de Reynolds n'étant plus valable dans la zone cavitée Ω_0, il est nécessaire de proposer un modèle permettant une description complète de l'écoulement.

Observons que la pression étant indépendante de la direction normale à l'écoulement, il est équivalent en ce qui la concerne de considérer une partition (Ω_+, Ω_0) de l'écoulement complet ou une partition (ω_+, ω_0) sur une section de l'écoulement. Cette partition (ω_+, ω_0) correspond à une interface non diffuse par rapport à la pression. Cependant, la caractérisation de l'interface entre Ω_+ et ω_0 du point de vue physique n'est pas claire. D'une certaine façon, Ω_0 peut être considéré comme analogue à une "mushy region"[3] comme celle apparaissant dans le problème de Stefan correspondant à un mélange de glace et d'eau à température constante.

Chacun des modèles de cavitation est alors caractérisé par une condition supplémentaire à imposer sur la pression dans la zone Ω_0, ou bien une technique numérique adaptée afin de déterminer la position de la frontière Σ entre les zones cavitées et non cavitées.

Un des modèles les plus étudiés est le modèle de Reynolds (ou de Swift-Stieber), qui s'obtient en écrivant que le gradient de la pression s'annule à l'interface et qui peut être modélisé par une inéquation variationnelle. L'existence et l'unicité d'une solution sont étudiées en particulier dans [Sta72], [Cim77]. Cependant, ce modèle n'est pas conservatif, ce qui du point de vue physique n'est pas satisfaisant [BC86a], et a motivé l'étude de nouveaux modèles, plus "riches" d'un point de vue physique et mathématique.

Le modèle d'Elrod-Adams [EA75] a été proposé par Floberg, Jakobsson et Olsson. En plus de la pression, il introduit une inconnue supplémentaire θ, comprise entre 0 et 1, définie comme la proportion locale de liquide en chaque point du domaine. Les différentes zones du domaine (voir Figure 1.8) sont alors caractérisées par :

- dans les zones saturées Ω_+ (ou ω_+), $p > p_s$ et $\theta = 1$;
- dans les zones non saturées Ω_0 (ou ω_0), $p = p_s$ et $0 \leq \theta \leq 1$.

Du point de vue mathématique, on obtient un problème hyperbolique-parabolique, dont l'existence d'une solution a été montrée pour la première fois dans [BC83] pour un palier infiniment long, puis par Vázquez dans [VC94] dans le cas bidimensionnel. Pour des conditions limites périodiques, ce problème a également été étudié dans [AO03]. Ce

[3] voir par exemple [Gup03], [Vis98], [BK94]

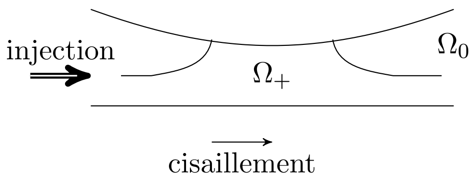

Figure 1.8: Zones saturées et non saturées en cavitation

modèle est introduit dans le chapitre 3 afin de modéliser un phénomène de cavitation pour un fluide piezovisqueux en élastohydrodynamique.

1.2.3 Sur les fluides non newtoniens et leur comportement en film mince

Les fluides quasi-newtoniens et les fluides visco-élastiques

Les travaux mentionnés précédemment dans les sections 1.2.1 et 1.2.2 concernent des fluides newtoniens. Cependant, de nombreux fluides intervenant dans des applications industrielles ou biologiques ne sont pas modélisés de façon satisfaisante par des modèles newtoniens, car leur comportement n'est pas purement visqueux ou linéaire. Il est donc nécessaire de prendre en compte l'aspect non newtonien de ces fluides, c'est-à-dire, en termes mathématiques, la variation non-linéaire du tenseur des contraintes visqueuses en fonction du tenseur de déformations. Signalons par exemple que de nombreux fluides biologiques, et en particulier le sang, ont un tel comportement. Dans le domaine de la lubrification, il est d'un grand intérêt de pouvoir contrôler les caractéristiques d'un fluide pour garantir certaines propriétés dans de larges plages de fonctionnement (température, contraintes...), ce qui se fait le plus souvent par l'ajout d'additifs. Le comportement des fluides devient alors non newtonien. Un autre grand domaine d'application concerne les polymères. On cherche alors à simuler ce qu'il se passe au moment de l'injection d'un matériau dans un moule, et en particulier à caractériser les conditions de remplissage.

Dès lors que la géométrie de l'écoulement est anisotrope, on cherche à obtenir comme dans le cas newtonien des modèles simplifiés. Il existe pour les modèles cités dans la section 1.1.3 des résultats relatifs à leur comportement en film mince.

✗ En ce qui concerne les fluides quasi-newtoniens, la justification du passage en film mince a été effectuée par Bourgeat, Mikelić et Tapiero [BMT93] pour une viscosité η vérifiant une loi de puissance en fonction du tenseur de déformation $D(\boldsymbol{u})$. Ces résultats ont été généralisés par exemple au cas instationnaire dans [CM98], ou à différentes conditions aux limites [BEM04]. Dans le cas où la viscosité est une fonction non-linéaire plus générale du tenseur de déformations, Sac-Epée et Taous [SET05] ont montré la convergence du modèle vers une équation limite. Soulignons que pour ces modèles, la modification de la viscosité ne change pas la structure de l'équation limite, qui est une équation de Reynolds généralisée.

✗ Les fluides de Bingham, ou fluides à seuil, sont des fluides dont le tenseur de contraintes $\boldsymbol{\sigma}$ suit une relation en fonction du tenseur de déformation $D(\boldsymbol{u})$ de la forme :

$$\begin{cases} \boldsymbol{\sigma} \leq g & \Leftrightarrow D(\boldsymbol{u}) = 0, \\ \boldsymbol{\sigma} > g & \Leftrightarrow \mu D(\boldsymbol{u}) = (1 - \frac{g}{f(\|\boldsymbol{\sigma}\|)})\boldsymbol{\sigma}, \end{cases}$$

où μ et g sont des constantes dépendant du fluide, et f est une fonction non linéaire donnée. Bunoiu et Kesavan ont effectué dans [BK04] le passage rigoureux en film mince, et ils ont montré que la loi limite obtenue était également une loi de type Bingham, couplée avec l'équation de Reynolds.

✗ Afin de prendre en compte non seulement les aspects visqueux de l'écoulement, mais également les effets élastiques, nous nous intéressons aux fluides visco-élastiques, pour lesquels la contrainte $\boldsymbol{\sigma}$ vérifie des lois plus complexes, qui ne permettent pas d'obtenir à la limite en film mince une équation de Reynolds généralisée "simple".

Il existe de nombreux modèles différentiels, parfois très sophistiqués, permettant de modéliser le comportement de fluides visco-élastiques. Un des modèles les plus simples d'un point de vue physique, mais déjà complexe du point de vue mathématique, est le modèle d'Oldroyd-B, qui a été introduit par Oldroyd [Old50]. L'idée est de considérer que chaque "cellule" de fluide est une combinaison de masses et de ressorts. En réalité, ce modèle peut être obtenu à partir d'un modèle bien plus complexe, le modèle FENE (Finite Extensible Non-linear Elastic), qui est un modèle moléculaire considérant un ensemble d'haltères hookéennes en suspension dans un solvant newtonien.

Selon les modèles de "cellules" choisis, différents modèles différentiels sont obtenus. Des considérations physiques permettent d'obtenir une équation sur la contrainte $\boldsymbol{\sigma}$ de la forme :

$$\lambda \frac{D\boldsymbol{\sigma}}{Dt} + f(\boldsymbol{\sigma})\boldsymbol{\sigma} = 2\eta r D(\boldsymbol{u}),$$

où $D(\boldsymbol{u})$ est le tenseur de déformation, λ un temps de relaxation relatif aux propriétés rhéologiques du fluide (lié au nombre de Déborah $\mathcal{D}e$), et r un paramètre décrivant la proportion relative des comportements purement visqueux et purement élastiques. La notation $\dfrac{D\cdot}{Dt}$ désigne une dérivée invariante par toute transformation euclidienne, de sorte à ne pas être liée au repère. Selon les dérivées invariantes choisies, on obtient divers modèles. La formulation la plus générale est la suivante, pour un tenseur quelconque $\boldsymbol{M}$:

$$\frac{D\boldsymbol{M}}{Dt} = \frac{d\boldsymbol{M}}{dt} - W(\boldsymbol{u}) \cdot \boldsymbol{M} + \boldsymbol{M} \cdot W(\boldsymbol{u}) - a(D(\boldsymbol{u}) \cdot \boldsymbol{M} + \boldsymbol{M} \cdot D(\boldsymbol{u})), \tag{1.6}$$

où $W(\boldsymbol{u})$ est le tenseur de vorticité, et a peut prendre différentes valeurs dans $[-1, 1]$.

Lorsque le paramètre r est pris égal à 1, diverses formes du modèle de Maxwell généralisé sont retrouvées. Dans le cas où f est l'identité, il s'agit du modèle de Maxwell classique. Si l'on choisit en revanche une forme linéarisée de f, des lois de Phan-Thien-Tanner sont obtenues [TW98]. Au contraire, le choix de $r = 0$ permet de retrouver un modèle newtonien. Le modèle d'Oldroyd-B correspond au cas où f est l'identité et où $0 < r < 1$.

Les travaux pour obtenir des modèles en film mince pour les fluides visco-élastiques sont essentiellement heuristiques. Une première approche, souvent utilisée dans la littérature physique, consiste à prendre le paramètre définissant l'épaisseur de l'écoulement comme le petit paramètre principal, et à considérer le nombre de Déborah comme un paramètre de perturbation. C'est l'approche utilisée par Tichy dans [Tic96] pour le cas du modèle de Maxwell avec $a = 1$. Par la suite, dans le cas où le nombre de Déborah est du même ordre de grandeur que l'épaisseur du domaine, le modèle de Maxwell a été considéré par Sawyer et Tichy [ST98] et Huang, Li, Meng et Wen [HLMW02] tandis que le modèle de Phan-Thien-Tanner a été traité par Akyildiz et Bellout [TAB04]. Dans tous ces cas, une équation de Reynolds non-linéaire est obtenue, qui permet de calculer la pression dans le film mince.

Dans le cas du modèle d'Oldroyd-B, en supposant le nombre de Déborah du même ordre de grandeur que l'épaisseur du domaine et en imposant aux contraintes un comportement permettant d'équilibrer les parties visqueuses et élastiques dans les équations, Bayada, Chupin et Martin [BCM07] ont proposé un modèle limite obtenu heuristiquement, dont ils ont montré l'existence et l'unicité. Le but du chapitre 2 de cette thèse est de compléter cette étude, en justifiant rigoureusement du point de vue de l'analyse sa validité mathématique.

Les fluides piezovisqueux en élastohydrodynamique

La *piezoviscosité* est un autre aspect du caractère non-newtonien d'un écoulement et apparaît dans la plupart des mécanismes lubrifiés soumis à des contraintes sévères. Dans ce cas, la viscosité du lubrifiant n'est pas constante, mais varie de manière exponentielle avec la pression [Bla01]. L'introduction de cette caractéristique dans les équations tridimensionnelles de l'écoulement n'a pratiquement jamais été étudiée du point de vue mathématique [HMNR03]. Bien que parfois contestée [Sze98], l'équation des films minces associée est très couramment utilisée, en particulier avec une loi pression-viscosité de type loi de Barus :

$$\eta(p) = \eta_0 e^{\alpha p},$$

où η_0 est la viscosité du fluide dans des régimes de fonctionnement standard et α est un coefficient piezovisqueux donné.

Pour ce type de relation, les pressions sont très importantes, et induisent une déformation élastique des surfaces en contact dont il faut tenir compte. En pratique, le modèle utilisé est un modèle élastohydrodynamique, qui couple la piezoviscosité et une expression de la déformation (modèle de Hertz pour les contacts ponctuels):

$$h(p)(x_1, x_2) = h_0(x_1, x_2) + (k \star p)(x_1, x_2),$$

où h est la hauteur du domaine, h_0 la contribution rigide de la hauteur, k est un noyau intégral correspondant à la contribution élastique non locale due aux grandes pressions, et $f \star g$ désigne la convolution des deux fonctions f et g. Dans les conditions physiques décrites ci-dessus, le phénomène de cavitation ne peut être négligé, et ce modèle doit être couplé à un modèle de cavitation. Des résultats d'existence, de régularité et d'unicité ont été montrés sous certaines hypothèses pour un premier modèle de cavitation, par Hu [Hu90], Rodrigues [Rod93], puis pour un modèle plus réaliste par Durany, Vázquez [DV94].

1.3 Description des résultats obtenus

1.3.1 Résumé du Chapitre 2

Fluides visco-élastiques en domaines minces

Dans ce chapitre, nous nous intéressons à des écoulements de fluides visco-élastiques décrits par la loi d'Oldroyd-B, dans des domaines minces. Plus précisément nous justifions la pertinence mathématique du modèle limite obtenu de manière heuristique dans [BCM07] lorsque l'épaisseur du domaine tend vers zéro.

Ce travail s'inscrit donc à la suite des travaux portant sur la justification des modèles limites en films minces décrits dans la section 1.2.1, pour des modèles dont l'intérêt a été évoqué dans la section 1.2.3.

Par rapport aux travaux précédents mentionnés dans la section 1.2.3, soulignons les difficultés rencontrées pour ce modèle :

- L'analyse mathématique des équations en domaine non mince est ardue, et peu de travaux y sont consacrés. Il est donc difficile d'avoir un "point de départ" pour l'analyse de la convergence en film mince.
- Le modèle obtenu à la limite est plus complexe que ceux obtenus auparavant, dans la mesure où il ne porte pas seulement sur une équation en pression, mais sur un système couplé vitesse/pression. Ce modèle limite présente donc également des difficultés d'analyse mathématique.

▶ **Section** 2.2

La loi d'Oldroyd a été introduite dans la Section 1.2.3. C'est une loi de comportement sur le tenseur des contraintes $\boldsymbol{\sigma}$, comprenant un terme de transport à la vitesse $\boldsymbol{u}$ et des termes non-linéaires regroupés dans g :

$$\lambda(\partial_t \boldsymbol{\sigma} + \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} + g(\boldsymbol{\sigma}, \nabla \boldsymbol{u})) + \boldsymbol{\sigma} = 2r\eta D(\boldsymbol{u}), \tag{1.7}$$

où η est la viscosité du fluide, λ le temps de relaxation et r un paramètre décrivant la proportion relative des comportements visqueux et élastiques. $D(\boldsymbol{u})$ est la partie symétrique de $\nabla \boldsymbol{u}$ et $W(\boldsymbol{u})$ sa partie antisymétrique. En toute généralité, les termes non-linéaires s'écrivent :

$$g(\boldsymbol{\sigma}, \nabla \boldsymbol{u}) = -W(\boldsymbol{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot W(\boldsymbol{u}) + a(\boldsymbol{\sigma} \cdot D(\boldsymbol{u}) + D(\boldsymbol{u}) \cdot \boldsymbol{\sigma}). \tag{1.8}$$

Dans cette relation, les différentes valeurs de a ($a = 1, -1, 0$) correspondent à différentes "dérivées objectives"[4].

L'équation de comportement précédente peut être couplée aux équations de Navier-Stokes pour modéliser l'écoulement d'un fluide visco-élastique de la manière suivante :

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \cdot \boldsymbol{u}) - (1-r)\eta\Delta\boldsymbol{u} + \nabla p = \nabla \cdot \boldsymbol{\sigma}, \tag{1.9}$$

où ρ désigne la densité du fluide. De plus, la condition d'incompressibilité s'écrit de manière usuelle :

$$\operatorname{div} \boldsymbol{u} = 0. \tag{1.10}$$

Dans un domaine $\Omega^\varepsilon = \{(x,y) \in \mathbb{R}^{d+1}, x \in \omega, 0 < y < \varepsilon h(x)\}$, où $\omega \subset \mathbb{R}^d$, $d = 1$ ou 2, l'existence d'une solution globale en temps au système couplé (1.7)-(1.9)-(1.10) est montrée par Lions et Masmoudi dans [LM00], uniquement pour le cas où le paramètre a introduit dans (1.8) est égal à zéro. C'est à cause de cette limitation sur le théorème d'existence dans le domaine non mince que nous nous restreignons aussi au cas $a = 0$ dans ce travail. D'autres résultats d'existence ont été montrés sur ce système (citons en particulier les travaux de Guillopé et Saut [GS90], et de Fernández-Cara, Guillén et Ortega [FCGO98]), mais il s'agit dans tous les cas de résultats locaux en temps (c'est-à-dire que la solution existe sur un intervalle $[0, T_{\Omega^\varepsilon}]$), ou de manière équivalente de résultats à données petites (c'est-à-dire que les données du problèmes sont supposées être majorées par une constante C_{Ω^ε} dépendant de Ω^ε). Ces théorèmes d'existence ne peuvent pas être utilisés comme point de départ de notre étude. En effet, dans la mesure où nous allons nous intéresser à la limite $\varepsilon \to 0$ du système de départ, il est important de contrôler la dépendance en ε du temps T_{Ω^ε}, afin de vérifier que celui-ci ne tend pas vers zéro lorsque ε tend vers zéro. Compte tenu des techniques de construction de T_{Ω^ε} et des estimations que nous avons obtenu, il ne nous a pas été possible de prouver que T_{Ω^ε} ne tend pas vers zéro.

Il est à noter également que les théorèmes d'existence de [LM00] sont obtenus par régularisation. Ce travail concerne donc uniquement les solutions du système (1.7)-(1.9)-(1.10) définies comme limites de solutions de problèmes régularisés. Nous choisissons ici une régularisation qui consiste à ajouter un terme $\delta\Delta\boldsymbol{\sigma}$ à l'équation (1.7), où δ est un petit paramètre que nous ferons tendre vers zéro.

Le système (1.7)-(1.9)-(1.10) est associé à des conditions aux limites de deux types différents. On impose des conditions d'adhérence sur les bords inférieur et supérieur du domaine, en prenant en compte le cisaillement par le mouvement relatif des deux surfaces. En ce qui concerne les conditions aux limites latérales, nous avons signalé

[4] respectivement dérivée convectée supérieure ou inférieure, dérivée de Jaumann

dans la section 1.2.1 que la seule donnée pertinente est le flux. Celui-ci est donc imposé, et la valeur de la vitesse sur les bords latéraux est laissée libre, et sera choisie ultérieurement afin d'éviter l'existence de couches limites. Remarquons que ces conditions aux limites permettent en particulier de simuler des phénomènes de lubrification (cisaillement prépondérant), ou d'injection de fluide dans un domaine. Enfin, pour la contrainte $\boldsymbol{\sigma}$, comme celle-ci vérifie (sans régularisation) une équation de transport, il faut imposer une condition de Dirichlet sur les bords correspondant aux caractéristiques entrantes. A nouveau, le choix de cette valeur sera effectué ultérieurement (conditions aux limites “bien préparées”).

▶ **Section** 2.3

Afin de travailler dans un domaine $\Omega = \{(x,z) \in \mathbb{R}^{d+1}, x \in \omega, 0 < z < h(x)\}$ indépendant de ε, nous effectuons le changement de variables $z = \dfrac{y}{\varepsilon}$, et nous réécrivons le système, avec des coefficients dépendant à présent de ε. Dans les applications que nous envisageons (lubrification, injection de fluides à vitesse modérée), il est usuel de considérer que la vitesse verticale est d'ordre ε, et que la pression est d'ordre $1/\varepsilon^2$ (ces considérations sont développées dans le cas newtonien dans [BC86b]). D'autre part, le tenseur des contraintes est supposé de l'ordre de $1/\varepsilon$ et le temps de relaxation de l'ordre de ε ($\lambda = \varepsilon\lambda^*$) afin d'équilibrer les effets visqueux et élastiques dans la loi d'Oldroyd. Ces remarques conduisent à introduire les développements asymptotiques suivants, où $\boldsymbol{u}^*$, p^* et $\boldsymbol{\sigma}^*$ sont solutions du système limite obtenu dans [BCM07] :

$$u_1 = u_1^* + v_1 \text{ et } u_2 = \varepsilon u_2^* + \varepsilon v_2,$$
$$p = \frac{1}{\varepsilon^2} p^* + \frac{1}{\varepsilon^2} q,$$
$$\boldsymbol{\sigma} = \frac{1}{\varepsilon}\boldsymbol{\sigma}^* + \frac{1}{\varepsilon}\boldsymbol{\tau}.$$

L'objet de ce travail est de montrer que les restes du développement $\boldsymbol{v} = (v_1, v_2)$, q et $\boldsymbol{\tau}$ tendent vers zéro lorsque ε et le paramètre de régularisation δ tendent vers zéro.

En injectant ces développements dans le système, nous obtenons un système non linéaire en $\boldsymbol{v}$, q et $\boldsymbol{\tau}$ comportant également des termes linéaires et des termes constants. Plus précisément, il s'agit d'un système sur les restes similaire à (1.7)-(1.9)-(1.10), mais dont le second membre contient des termes linéaires en $\boldsymbol{v}$, q, $\boldsymbol{\tau}$ et des termes constants dépendants de $\boldsymbol{u}^*$, p^* et $\boldsymbol{\sigma}^*$.

▶ **Section** 2.4

Dans cette sous-section, nous précisons le système vérifié par les quantités limites

$\boldsymbol{u}^*$, p^* et $\boldsymbol{\sigma}^*$. Celles-ci sont choisies comme les solutions du système qui annule les contributions aux termes constants du second membre du système précédent d'ordre le plus élevé (c'est-à-dire les termes d'ordre ε^s avec s minimal).

De manière usuelle dans le cas newtonien, le système peut être réécrit à cette étape de sorte à découpler vitesse et pression, et obtenir l'équation de Reynolds. C'est aussi le cas de certains fluides non-newtoniens, pour lesquels une équation de Reynolds modifiée est obtenue (voir par exemple [SET05]). Cependant, dans le cas visco-élastique, l'équation ne se met pas sous forme découplée de manière simple. Nous obtenons un système couplant la composante horizontale u_1^* du champ de vitesses et la pression p^*:

$$\begin{aligned} -\eta(1-r)\partial_z^2 u_1^* - \partial_z \left(\frac{\eta r \partial_z u_1^*}{1+{\lambda^*}^2 |\partial_z u_1^*|^2} \right) + \partial_x p^* = 0, \\ \text{avec } \partial_z p^* = 0 \quad \text{ et } \partial_x \left(\int_0^{h(x)} u_1^*(x,z) dz \right) = 0. \end{aligned} \tag{1.11}$$

Dans [BCM07], un résultat d'existence et d'unicité a été montré pour ce système. Cependant, afin de pouvoir donner un sens par la suite aux termes intervenant dans les estimations d'énergie, nous montrons un résultat de régularité plus fort :

Théorème 1.1. *Soit $r < 2/9$, et supposons que $h \in H^k(\omega)$ pour $k \geq 1$. Alors la solution $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$ du système limite vérifie :*

$$\begin{gathered} p^* \in C^{k+1}(\bar{\omega}), \quad u_1^*, \partial_z u_1^*, \partial_z^2 u_1^* \in C^{k+1}(\bar{\Omega}), \quad \boldsymbol{\sigma}^*, \partial_x \boldsymbol{\sigma}^* \in C^{k+1}(\bar{\Omega}), \\ \partial_x u_1^* \in C^k(\bar{\Omega}), \quad u_2^*, \partial_z u_2^*, \partial_z^2 u_2^* \in C^k(\bar{\Omega}), \quad \partial_x \boldsymbol{\sigma}^* \in C^k(\bar{\Omega}), \\ \partial_x u_2^* \in C^{k-1}(\bar{\Omega}). \end{gathered}$$

Idée de la preuve :

En introduisant la fonction $\phi(t) = \eta(1-r)t + \partial_z \left(\frac{\eta r t}{1+{\lambda^*}^2 t^2} \right)$, la première équation de (1.11) devient $\partial_z (\phi(\partial_z u_1^*)) = \partial_x p^*$. Nous procédons alors comme pour obtenir l'équation de Reynolds, en utilisant le fait que p^* est indépendant de z et en intégrant l'équation par rapport à z. Cependant, la constante d'intégration ne peut pas être donnée de manière explicite, mais elle est déterminée par les conditions aux limites sur u_1^*. Après une seconde intégration par rapport à z, et en utilisant la condition d'incompressibilité, une équation différentielle ordinaire du premier degré sur $\partial_x p^*$ est obtenue, qui peut être interprétée comme une équation de Reynolds implicite généralisée. L'application du théorème de Cauchy-Lipschitz permet de déduire l'existence de $\partial_x p^*$ et d'obtenir la régularité de $\partial_x p^*$ en fonction de h. $\square$

▶ **Section** 2.5
Nous montrons la convergence des restes $\boldsymbol{v}$, q et $\boldsymbol{\tau}$ vers zéro. Nous énonçons le théorème principal de ce chapitre :

Théorème 1.2. *Si* $|\partial_z u_1^*|_\infty$ *et* $|\boldsymbol{\sigma}^*|_\infty, |\partial_z \boldsymbol{\sigma}^*|_\infty$ *sont assez petits, alors les convergences suivantes sont vérifiées à sous-suite près lorsque* ε *et le paramètre de régularisation* δ *tendent vers zéro :*

$$\begin{gathered}
(u_1, \partial_z u_1) \to (u_1^*, \partial_z u_1^*), \quad \partial_x u_1 \rightharpoonup \partial_x u_1^* \quad \textit{dans } L^2(0,T;L^2(\Omega)), \\
(u_2, \partial_z u_2) \to (0,0), \quad \partial_x u_2 \rightharpoonup 0 \quad \textit{dans } L^2(0,T;L^2(\Omega)), \\
\varepsilon \boldsymbol{\sigma} \to \boldsymbol{\sigma}^* \quad \textit{dans } L^2(0,T;L^2(\Omega)), \\
(u_1, u_2, \varepsilon\boldsymbol{\sigma}) \rightharpoonup^\star (u_1^*, 0, \boldsymbol{\sigma}^*) \quad \textit{dans } L^\infty(0,T;L^2(\Omega)), \\
\varepsilon^2 p \to p^* \quad \textit{dans } \mathcal{D}'(0,T;L^2(\Omega)).
\end{gathered}$$

Précisons que la constante de petitesse pour $|\partial_z u_1^*|_\infty$ et $|\boldsymbol{\sigma}^*|_\infty, |\partial_z \boldsymbol{\sigma}^*|_\infty$ est explicite en fonction des paramètres λ^*, η, et r. De plus, dans le cas plus simple où h est constant, nous montrons que cette condition est vérifiée pour $s\lambda^* < \dfrac{h}{12}$.

Idée de la preuve :

En utilisant $\boldsymbol{v}$ comme fonction test dans l'équation de Navier-Stokes, et $\boldsymbol{\tau}$ dans l'équation d'Oldroyd, il vient des estimations de la forme :

$$\left| \begin{pmatrix} \partial_x \\ \frac{1}{\varepsilon}\partial_z \end{pmatrix} \begin{pmatrix} v_1 \\ \varepsilon v_2 \end{pmatrix} \right|_2^2 + \frac{1}{\varepsilon^2} |\boldsymbol{\tau}|_2^2 \le \int_\Omega (Q_1 + L_1 + C_1) \cdot \boldsymbol{v} + (Q_2 + L_2 + C_2) \cdot \boldsymbol{\tau} \tag{1.12}$$

où Q_i, L_i et C_i sont respectivement des termes quadratiques, linéaires et constants en $\boldsymbol{v}$ et $\boldsymbol{\tau}$. Nous détaillons sur trois exemples "types" de termes pourquoi nous imposons des conditions limites bien préparées à la fois sur $\boldsymbol{u}$ et sur $\boldsymbol{\sigma}$, et où les hypothèses annoncées dans le théorème 1.2 sont nécessaires.

▷ Pour les termes quadratiques de convection en $\boldsymbol{v}$, une intégration par parties et la condition d'incompressibilité donnent :

$$\int_\Omega \boldsymbol{v} \cdot \nabla v_1 \, v_1 = -\int_\Omega \boldsymbol{v} \cdot \nabla v_1 \, v_1 + \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} v_1^2.$$

Les conditions aux limites bien préparées assurent que $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}^*|_{\partial\Omega}$, et il en découle que $\boldsymbol{v}|_{\partial\Omega} = 0$, ce qui permet de prouver que le terme de bord est égal à zéro, et que $\displaystyle\int_\Omega \boldsymbol{v} \cdot \nabla v_1 \, v_1 = 0$.

- ▷ Les termes quadratiques en $\boldsymbol{\tau}$ sont traités en utilisant que sur les bords où la vitesse est entrante, $\boldsymbol{\sigma}|_{U_{\text{in}}} = \boldsymbol{\sigma}^*|_{U_{\text{in}}}$, et donc que $\boldsymbol{\tau}|_{U_{\text{in}}} = 0$. D'où

$$-\int_\Omega \boldsymbol{u}^* \cdot \nabla\tau_{11}\,\tau_{11} = -\frac{1}{2}\int_{\partial\Omega} \boldsymbol{u}^* \cdot n\,\tau_{11}^2 = \underbrace{-\frac{1}{2}\int_{U_{\text{out}}} \boldsymbol{u}^* \cdot n\,\tau_{11}^2}_{\leq 0} - \frac{1}{2}\int_{U_{\text{in}}} \boldsymbol{u}^* \cdot n\,\underbrace{\tau_{11}^2}_{=0}.$$

- ▷ Pour les termes linéaires d'ordre 1, des conditions de petitesse sont imposées, afin de pouvoir "absorber les termes de droite par les termes de gauche" dans (1.12) :

$$-\lambda^*\int_\Omega \partial_x v_2\,\sigma_{12}^*\,\tau_{11} \leq \lambda^*|\sigma_{12}^*|_\infty\,|\varepsilon\partial_x v_2|_2\,\left|\frac{1}{\varepsilon}\tau_{11}\right|_2 \leq \frac{1}{2\varepsilon^2}|\tau_{11}|_2^2 + \underbrace{\frac{\lambda^{*2}|\sigma_{12}^*|_\infty^2}{2}}_{<1}|\varepsilon\partial_x v_2|_2^2$$

  Dans le cas des termes linéaires d'ordre ε, la petitesse de ε suffit.

- ▷ Les termes constants sont traités de manière simple avec les inégalités de Poincaré et de Young.

Les estimations *a priori* ainsi obtenues sont uniformes en δ, il est donc possible de passer à la limite $\delta \to 0$ dans celles-ci. De plus, le passage à la limite $\varepsilon \to 0$ donne les convergences vers zéro des restes $\boldsymbol{v}$, q et $\boldsymbol{\tau}$, et donc les convergences annoncées de $(\boldsymbol{u}, p, \boldsymbol{\sigma})$ vers $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$ dans les espaces fonctionnels adéquats.

En ce qui concerne la pression, une estimation *a priori* est également obtenue sur le gradient de la pression, en utilisant des résultats classiques d'interpolation pour les espaces de Sobolev et des injections de Sobolev anisotropiques. On conclut grâce à un lemme usuel, qui permet de contrôler la norme L^2 à partir de la norme H^{-1} du gradient d'une fonction. □

1.3.2 Résumé du Chapitre 3

Problèmes piezovisqueux élastohydrodynamiques en lubrification avec le modèle de cavitation d'Elrod-Adams

Dans ce chapitre, nous abordons un autre aspect de la complexité des écoulements en film mince, qui prend à la fois en compte le caractère non-newtonien de certains fluides, induit par la piezoviscosité, et le comportement diphasique de l'écoulement, dans la mesure où la cavitation est introduite. De plus, dans le cadre des applications visées (lubrification), l'interaction fluide-structure est également considérée, et la fonction décrivant l'épaisseur du domaine est une fonction non locale de la pression.

Les problèmes piezovisqueux (c'est-à-dire dans lesquels la viscosité dépend de la pression) et élastohydrodynamiques (c'est-à-dire prenant en compte l'interaction fluide-structure) en lubrification ont déjà été étudiés pour différents modèles de cavitation. Cette étude s'inscrit donc dans la lignée de nombreux travaux, qu'elle complète ou généralise :

(1) Un premier modèle de cavitation, basé sur une inégalité variationnelle, a été étudié par Hu [Hu90] et Rodrigues [Rod93]. Les auteurs obtiennent une inéquation variationnelle, pour laquelle un résultat d'existence et d'unicité est montré. Dans ces travaux, une approche de type point fixe est utilisée et, pour vérifier la compacité, une hypothèse de petitesse sur les données est imposée.

(2) Lorsque l'on considère le modèle d'Elrod-Adams pour modéliser la cavitation, qui est un modèle en pression-saturation, des résultats d'existence ont été obtenus par Bayada, El Alaoui et Vázquez dans [BTV96]. A nouveau, la compacité est conséquence d'une hypothèse de petitesse sur les données.

(3) Par ailleurs, Bayada et Bellout ([Bel03] dans le cas uni-dimensionnel, [BB05] pour le cas multi-dimensionnel) ont observé qu'une telle hypothèse de petitesse n'était pas très réaliste. De plus, des résultats numériques satisfaisants sont obtenus pour de larges gammes de paramètres. Ils proposent une hypothèse moins restrictive sur le comportement de la viscosité en fonction de la pression à l'infini en remplacement de l'hypothèse de petitesse sur les données ; cette nouvelle hypothèse sur la viscosité est compatible avec les cas physiques pertinents. Les auteurs ont montré sans autre hypothèse restrictive un résultat d'existence pour des problèmes piezovisqueux élastohydrodynamiques avec le modèle de cavitation basé sur une inégalité variationnelle.

Ce travail se propose d'appliquer la même approche avec le modèle d'Elrod-Adams, c'est-à-dire d'obtenir un résultat d'existence en supposant un comportement asymptotique spécifique pour la viscosité, mais sans condition de données petites. Nous nous plaçons dans le cas multi-dimensionnel, et les techniques développées dans [Bel03] spécifiques au cas mono-dimensionnel ne peuvent plus être appliquées. Par rapport à [BB05], nous considérons un système à deux inconnues (pression-saturation), et les conditions aux limites sont différentes : la modélisation d'une injection de fluide sur une partie de la frontière est prise en compte, ce qui correspond dans le cas de la lubrification à l'existence d'une rainure d'alimentation.

► **Section** 3.2

Le problème "de base" de la lubrification avec des conditions d'injection, sur une partie de la frontière s'écrit dans le domaine $\Omega = \{(x,z) \in \mathbb{R}^{d+1}, x \in \omega \subset \mathbb{R}^d, 0 \leq z \leq h(x),$

pour $d = 1$ ou 2 :

$$\mathrm{div}_x \left(\frac{h^3}{6\eta} \nabla_x p \right) = \mathrm{div}_x(h\boldsymbol{s}), \qquad \text{avec } \partial_z p = 0,$$

où $\boldsymbol{s} = (s, 0)$ est une vitesse de cisaillement donnée et η désigne la viscosité. En ce qui concerne les conditions aux limites, le flux entrant est imposé sur un bord du domaine, ce qui correspond à une condition de Neumann sur la pression, et des conditions de Dirichlet sont imposées sur le reste du bord du domaine.

Dans le cas de grandes valeurs de la pression (voir section 1.2.3), l'interaction du fluide avec la structure environnante est prise en compte en introduisant une dépendance de h en fonction de la pression :

$$h(p) = h_0 + k \star p,$$

où k est un noyau correspondant au contact sphère-plan, qui s'écrit

$$k(x, y) = \frac{k_0}{\sqrt{x^2 + y^2}},$$

et h_0 et k_0 sont des constantes données.

De plus, la viscosité varie également en fonction de la pression. Nous avons choisi la loi de Barus:

$$\eta(p) = \eta_0 e^{\alpha p},$$

où η_0 et α sont des constantes positives. En prenant en compte tous ces aspects, l'équation de Reynolds sans cavitation se réécrit dans Ω :

$$\mathrm{div}_x \left(\frac{1}{6\eta_0} e^{-\alpha p} h^3(p) \nabla_x p \right) = \mathrm{div}_x(h(p)\boldsymbol{s}), \qquad \text{et } h(p) = h_0 + k \star p.$$

La prise en compte de la cavitation se traduit par l'introduction de la quantité $\theta \in \mathcal{H}(p)$, où $\mathcal{H}$ dénote le graphe de Heaviside, correspondant à la proportion locale du fluide dans l'air. Le problème devient :

Trouver (p, θ) tel que :

$$\begin{gathered} \mathrm{div}_x \left(\frac{1}{6\eta_0} e^{-\alpha p} h^3(p) \nabla_x p \right) = \mathrm{div}_x(\theta\, h(p)\boldsymbol{s}), \\ p \geq 0, \qquad 0 \leq \theta \leq 1. \end{gathered} \tag{1.13}$$

En définissant un espace fonctionnel V adapté aux conditions aux limites, nous écrivons une formulation faible de ce problème. Notons G_0 le flux entrant sur la partie du

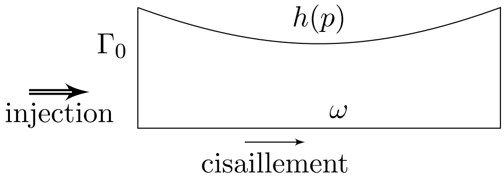

Figure 1.9: Allure du domaine en lubrification avec conditions d'injection

bord Γ_0 (voir Figure 1.9), et l'équation devient :

$$\int_\Omega \frac{h^3(p)}{6\eta(p)} \nabla_x p \cdot \nabla_x \varphi = s \int_\Omega h(p)\theta \, \nabla_x \varphi + \int_{\Gamma_0} G_0 \varphi, \quad \forall \varphi \in V, \qquad \text{avec } \theta \in \mathcal{H}(p).$$

En outre, afin de nous ramener au cas isovisqueux, nous effectuons dans l'équation le changement de variable suivant (transformation de Grübin), en introduisant la "pression réduite" P telle que :

$$P = a(p) = \int_0^p \frac{ds}{\eta(s)}, \qquad \text{en tout point } (x, y) \in \Omega.$$

En accord avec les cas physiques observés, et en particulier avec le cas des viscosités vérifiant la loi de Barus, nous supposons que

$$A = \lim_{p \to +\infty} a(p) = \int_0^{+\infty} \frac{ds}{\eta(s)} < +\infty.$$

Il est donc important de noter que la fonction γ inverse de a admet une asymptote verticale en A. La nouvelle hauteur $H(P)$ s'écrit en fonction de γ

$$H(P) = h_0 + k \star \gamma(P).$$

Pour s'affranchir du problème de la divergence de γ en A, la fonction γ est régularisée par troncature en $A - \varepsilon$, où $\varepsilon > 0$ est un petit paramètre. De plus, la fonction de saturation θ (fonction de Heaviside) est également régularisée, par une fonction θ_δ, où $\delta > 0$ est un petit paramètre.

▶ **Section** 3.3

Dans cette section, nous présentons un résultat d'existence pour le problème régularisé suivant :

$$\int_\Omega H^3(P_{\delta\varepsilon})\nabla P_{\delta\varepsilon}\cdot\nabla\varphi = 6s\int_\Omega H(P_{\delta\varepsilon})\theta_\delta\partial_x\varphi + \int_{\Gamma_0} G_0\varphi, \quad \forall\varphi\in V, \qquad (1.14)$$
$$\text{avec } H(P_{\delta\varepsilon}) = h_0 + k\star\gamma_\varepsilon(P_{\delta\varepsilon}).$$

Théorème 1.3. *Pour δ et ε fixés, il existe une solution $P_{\delta\varepsilon}$ du problème régularisé, vérifiant $P_{\delta\varepsilon}\in V$ et $P_{\delta\varepsilon}\geq 0$ avec $\|P_{\delta\varepsilon}\|_{H^1(\Omega)}\leq R$, où R est une constante indépendante de δ et ε.*

De plus, $P_{\delta\varepsilon}$ vérifie $P_{\delta\varepsilon}\in L^\infty(\Omega)$ et $\gamma_\varepsilon(P_{\delta\varepsilon})\in L^1(\Omega)$.

Idée de la preuve :

La démonstration est inspirée des travaux de Bayada, El Alaoui Talibi et Vázquez [BTV96], mais est adaptée aux conditions aux limites nouvelles imposées sur p (il s'agit du terme de flux imposé G_0). Le résultat d'existence dans $H^1(\Omega)$ est montré par une méthode de point fixe. Soulignons que ce théorème n'impose pas que $R < A$, et donc s'affranchit de conditions de petitesse sur les données utilisée dans [BTV96].

En ce qui concerne l'estimation L^∞, l'idée consiste à définir A_k l'ensemble des points de Ω où $P_{\delta\varepsilon}\geq k$, et à montrer qu'il existe k^* pour lequel $A_{k^*}=\emptyset$.

La régularité de $\gamma_\varepsilon(P_{\delta\varepsilon})$ est montrée en utilisant un raisonnement par l'absurde et en utilisant la monotonie de γ. □

▶ **Section** 3.4

Dans cette partie, nous définissons une limite P de $P_{\delta\varepsilon}$ lorsque ε puis δ tendent vers zéro, et nous montrons que celle-ci vérifie des inégalités adéquates, et tout d'abord que $\|P\|_{L^\infty}$ reste borné par A (pour que $\gamma(P)$ soit bien défini). Nous en déduisons que p est borné dans L^1 indépendamment de ε et δ. Nous définissons également θ une limite de θ_δ lorsque δ tend vers zéro.

Nous énonçons le théorème principal de ce chapitre :

Théorème 1.4. *Sous l'hypothèse réaliste*

$$\eta(p)\underset{+\infty}{\sim}(p+Q)^\beta, \qquad \textit{pour } 1<\beta<\frac{3}{2}, \qquad (1.15)$$

le couple (P,θ) est solution du problème suivant :

$$\int_\Omega H^3(P)\nabla P\cdot\nabla\varphi = 6s\int_\Omega H(P)\theta\,\partial_x\varphi + \int_{\Gamma_0} G_0\varphi, \quad \forall\varphi\in V,$$
$$\textit{avec } \theta\in\mathcal{H}(P) \textit{ et } H(P) = h_0 + k\star\gamma(P).$$

Idée de la preuve :

Les résultats de la Section 2 ne permettent pas de passer à la limite lorsque ε puis δ tendent vers zéro dans les termes non linéaires $H^3(P_{\delta\varepsilon})\nabla P_{\delta\varepsilon}$. Des estimations supplémentaires sont nécessaires.

Comme $P_{\delta\varepsilon}$ est borné dans H^1, $\nabla P_{\delta\varepsilon}$ converge faiblement dans L^2, et donc il suffit de montrer que $H^3(P_{\delta\varepsilon})$ converge fortement dans L^2, et donc que $H(P_{\delta\varepsilon})$ converge fortement dans L^6. Par ailleurs, $H(P_{\delta\varepsilon})$ est donné par $H(P_{\delta\varepsilon}) = h_0 + k \star \gamma_\varepsilon(P_{\delta\varepsilon})$, et k est un noyau régularisant (qui s'écrit $\dfrac{1}{\sqrt{|\xi|^2}}$ en variables de Fourier). Il suffit donc finalement de montrer que $k \star \gamma_\varepsilon(P_{\delta\varepsilon})$ est borné dans $W^{1,6}$ (pour avoir la convergence forte dans L^6), et pour cela, il est suffisant de prouver que $\gamma_\varepsilon(P_{\delta\varepsilon})$ est borné dans L^6, grâce aux propriétés régularisantes de k. Pour cela, nous montrons qu'il existe σ tel que $\gamma_\varepsilon(P_{\delta\varepsilon})^\sigma$ est borné dans H^1.

Cette propriété s'obtient en utilisant une fonction test ψ judicieusement choisie dans la formulation faible, telle que $\psi'(P_{\delta\varepsilon}) = \gamma'_\varepsilon(P_{\delta\varepsilon})\gamma_\varepsilon^{2(\sigma-1)}(P_{\delta\varepsilon})$. Par ailleurs, nous montrons que

$$\int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon}) \leq C.$$

En choisissant γ_ε telle que $\gamma'_\varepsilon(P_{\delta\varepsilon}) = \gamma_\varepsilon^\beta(P_{\delta\varepsilon})$, et en ajustant σ en fonction de β, on montre l'estimation voulue. Il découle de la définition de γ_ε que la condition $\gamma'_\varepsilon(P_{\delta\varepsilon}) = \gamma_\varepsilon^\beta(P_{\delta\varepsilon})$ est vérifiée lorsque η est telle que $\eta(p) = p^\beta$ pour p suffisamment grand. La condition à l'infini (1.15) énoncée dans la proposition correspond donc à cette restriction.

1.3.3 Résumé du Chapitre 4

Ecoulements diphasiques en film mince

Dans ce chapitre, nous nous intéressons à une autre modélisation des écoulements diphasiques, qui prend en compte non seulement les effets hydrodynamiques à l'interface entre les deux fluides, mais qui permet également de modéliser les effet chimiques à l'interface. A partir des modélisations proposées par Chella et Viñals [CV96], nous effectuons un passage à la limite heuristique et obtenons un système "film mince" couplé entre l'équation de Reynolds et l'équation de Cahn-Hilliard hydrodynamique. Nous étudions ce modèle, et montrons l'existence d'une solution sous certaines conditions de petitesse sur les données.

Par rapport aux travaux mentionnés dans la section 1.2.2, le système étudié dans ce chapitre présente des difficultés nouvelles :

- D'une part, même si l'équation de Reynolds seule est plus simple d'un point de vue mathématique que les équations de Navier-Stokes, le couplage dû à l'équation de Cahn-Hilliard induit une équation de Reynolds modifiée, dont la régularité n'est pas immédiate.
- D'autre part, le choix des conditions limites de ce travail est nouveau : le phénomène d'injection est pris en compte. La perte de la conservation de la quantité de chaque fluide dans le domaine implique des difficultés d'analyse mathématique.

► **Section** 4.2

Pour modéliser l'aspect diphasique, nous introduisons un paramètre d'ordre φ, qui correspond à la proportion d'un des fluides dans le mélange. L'évolution de ce paramètre est décrite par l'équation de Cahn-Hilliard hydrodynamique, qui comprend à la fois un terme de transport à la vitesse $\boldsymbol{u}$ et un terme de diffusion. En ce qui concerne la diffusion, celle-ci s'exprime par l'intermédiaire d'un potentiel F appelé potentiel de Cahn-Hilliard. La loi s'écrit donc :

$$\partial_t \varphi + \boldsymbol{u} \cdot \nabla\varphi + \mathrm{div}(B(\varphi)\nabla\mu) = 0, \qquad \text{où } \mu = -\alpha^2\Delta\varphi + F'(\varphi). \tag{1.16}$$

Notre approche consiste à réitérer la démarche effectuée pour obtenir l'équation de Reynolds dans le cas d'une viscosité non constante dépendant de φ, et à obtenir une équation de Reynolds généralisée sur la pression p et le champ de vitesse $\boldsymbol{u} = (u, v)$, dont les coefficients d, e, f, g sont variables et dépendent de φ.

$$\partial_x(d(\varphi)\partial_x p(x)) = \partial_x(e(\varphi)), \quad u(x,y) = f(\varphi)\partial_x p + g(\varphi), \quad v(x,y) = -\int_0^y \partial_x u(x,z)dz. \tag{1.17}$$

Nous nous intéressons également au cas où la tension de surface est prise en compte dans les équations de Navier-Stokes, et où l'ordre de grandeur du coefficient de capillarité est tel que ce terme persiste dans l'équation de Reynolds. Nous montrons que moyennant la définition d'une pression modifiée, celle-ci vérifie également une équation de type 1.17.

Nous précisons les conditions aux limites retenues pour ce système. Nous imposons des conditions limites non homogènes de non-glissement sur les parois supérieures et inférieures du domaine. En ce qui concerne les conditions sur les parois latérales, un flux entrant est imposé. Pour ce qui est du paramètre d'ordre, les travaux précédents portant sur l'équation de Cahn-Hilliard [Boy99], [Chu03] imposaient une conservation au cours du temps de la quantité de chaque fluide à l'intérieur du domaine, ce qui empêchait par exemple la modélisation de phénomènes tels que le remplissage par un fluide d'une cuve initialement remplie d'air. Dans ce travail, nous choisissons des conditions aux limites compatibles avec de tels phénomènes d'injection, en imposant une condition de Dirichlet sur le paramètre d'ordre (c'est-à-dire que la composition du mélange injecté est supposée connue) sur un bord latéral du domaine. Les conditions classiques d'absence de diffusion sur les autres bords sont conservées.

▶ **Section** 4.3
Dans cette partie, nous définissons les espaces fonctionnels adaptés aux conditions aux limites que nous avons imposées. De plus, nous explicitons les hypothèses mathématiques requises sur les fonctions B et F intervenant dans l'équation de Cahn-Hilliard[5] ; ces conditions sont satisfaites dans les cas physiques réalistes.

Par ailleurs, nous définissons des relèvements des conditions aux limites à la fois pour la vitesse et pour le paramètre d'ordre, qui vérifient certaines conditions (de régularité, de petitesse) qui seront requises dans la suite du travail. L'existence de tels relèvements est prouvée.

Enfin, nous énonçons des résultats classiques d'injection de Sobolev, pour lesquels la dépendance des constantes d'injection en fonction de la taille du domaine est explicitée.

▶ **Section** 4.4
L'équation (1.17) montre que vitesse et pression peuvent être exprimées explicitement en fonction du paramètre d'ordre. C'est pourquoi nous étudions dans cette partie la régularité de p et $\boldsymbol{u}$ pour $\varphi \in H^1(\Omega)$. Plus précisément, nous montrons le résultat suivant :

Théorème 1.5. *Pour $\varphi \in H^1(\Omega)$, la solution $(p, \boldsymbol{u})$ de l'équation (1.17) satisfait*

$$\partial_x p \in H^1(0,L) \cap L^\infty(0,L), \quad u \in H^1(\Omega) \cap L^\infty(\Omega), \quad v \in L^2(\Omega),\ \partial_y v \in L^2(\Omega). \tag{1.18}$$

Idée de la preuve :

Comme
$$\partial_x(d(\varphi)\partial_x p) = \partial_x(e(\varphi)),$$
il suffit d'établir la régularité des coefficients d et e en fonction de φ, et de vérifier que l'opérateur $\partial_x(d(\varphi)\partial_x \cdot)$ est coercif.

Pour la régularité des coefficients, ceux-ci dépendent de φ par l'intermédiaire de la viscosité $\eta(\varphi)$. En utilisant la régularité de $\eta(\varphi)$, l'inégalité de Cauchy-Schwarz, et des

[5]Ces conditions sont les mêmes que celles imposées dans les travaux de Boyer [Boy99] sur le couplage Navier-Stokes/Cahn-Hilliard.

estimations de traces dans des espaces de Sobolev, nous montrons que pour $\varphi \in H^1(\Omega)$ les coefficients $d(\varphi)$ et $e(\varphi)$ sont dans $H^1(0,L) \cap L^\infty(0,L)$.

D'autre part, en ce qui concerne la coercivité, il suffit de montrer que le coefficient $d(\varphi)$ est minoré par une constante strictement positive. Pour cela, nous utilisons une démarche similaire à la preuve de l'inégalité de Cauchy-Schwarz, en introduisant un polynôme dont les coefficients dépendent de $\eta(\varphi)$, et dont nous prouvons qu'il est toujours strictement positif. Le fait que le discriminant de ce polynôme est strictement négatif montre directement la minoration de $d(\varphi)$ voulue.

De plus, nous montrons des estimations de p, $\boldsymbol{u}$ en norme adéquate en fonction de la norme H^1 de φ, pour lesquelles nous prenons le soin de préciser la dépendance des constantes intervenant en fonction de la taille du domaine. □

▶ **Section** 4.5

Obtention des estimations *a priori* : Idées principales

▷ Nous utilisons le procédé de Galerkin. La première étape consiste à définir une base de fonctions ψ_i vérifiant des conditions aux limites adéquates. La fonction φ est approchée par φ_n, combinaison linéaire des ψ_i pour $1 \leq i \leq n$. Par ailleurs, afin de pouvoir choisir μ comme fonction test dans l'équation en φ par la suite, μ est définie par un projecteur sur l'espace de fonctions adéquat de dimension finie.

▷ L'obtention d'estimations *a priori* sur φ_n entraîne l'apparition de nouvelles difficultés par rapport aux travaux de Boyer [Boy99] et Chupin [Chu03] :

- La première différence vient du fait que, en raison de l'injection, la quantité de chaque fluide n'est pas supposée constante, c'est-à-dire que la moyenne $m(\varphi)$ dans le domaine n'est plus conservée au cours du temps. Cela pose des difficultés mathématiques supplémentaires, et il n'est pas possible ici d'utiliser des inégalités de type Poincaré sur $\varphi - m(\varphi)$. Nous travaillons sur des termes de la forme $\varphi - \varphi_l$, où φ_l est la valeur de φ sur le bord où il y a injection.
- D'autre part, les conditions aux limites sur la vitesse sont également différentes, dans la mesure où un flux d'entrée est imposé sur les bords latéraux (la vitesse n'est pas supposée être tangente aux bords sur toute la frontière du domaine). De nouveaux termes de bord apparaissent dans les intégrations par parties.
- Les termes de bord supplémentaires sont délicats à traiter dans la mesure où les théorèmes de traces font "perdre" de la régularité (dans le sens où la norme L^2 d'une fonction sur le bord n'est majorée que par la norme $H^{1/2}$ sur tout le domaine).

- Dans les travaux de Boyer, le terme de convection de l'équation de Cahn-Hilliard $\boldsymbol{u}\cdot\nabla\varphi$ s'annule avec le terme de tension de surface $\mu\cdot\nabla\varphi$, ce qui n'est plus le cas dans notre travail, lorsque le coefficient de capillarité est supposé négligeable. Il faut donc traiter le terme de convection. Par ailleurs, l'équation de Reynolds étant anisotropique, la régularité obtenue sur la composante verticale de la vitesse est plus faible que celle obtenue sur la composante horizontale (et donc que celle obtenue dans le cas où l'on étudie les équations de Navier-Stokes complètes). Cette régularité plus faible sur v induit des difficultés supplémentaires.

▶ **Section** 4.6
Dans cette section, nous énonçons et montrons le théorème principal de ce chapitre.

Théorème 1.6. *Pour une condition initiale sur φ suffisamment régulière, sous une hypothèse de petitesse sur L, il existe une solution $(p,\boldsymbol{u},\varphi,\mu)$ au système* (1.17)-(1.16) *vérifiant* (1.18) *et :*

$$\varphi\in L^\infty(0,\infty;H^1(\Omega))\cap L^2_{loc}(0,\infty;H^3(\Omega)),\qquad \mu\in L^2_{loc}(0,\infty;H^1(\Omega)). \tag{1.19}$$

Idée de la preuve :

▷ Nous utilisons les estimations *a priori* montrées dans la section précédente et obtenons des estimations de la forme :

$$\frac{d}{dt}\left(|\nabla\varphi|_2^2\right)+|\nabla\varphi|_2^2+|\Delta\varphi|_2^2+|\nabla\mu|_2^2\le C_1f(|\nabla\varphi|_2^2)+C_2|\Delta\varphi|_2^2+C_3|\nabla\mu|_2^2+C_4,$$

où f contient des termes linéaires et non linéaires. Pour conclure, nous sommes amenés à supposer que les constantes C_i ne sont pas trop grandes, ce qui entraîne la condition sur L énoncée.

▷ Ces convergences ne sont pas suffisantes pour passer à la limite sur n et montrer la convergence de la condition initiale. Il faut également estimer la dérivée temporelle de φ afin d'obtenir une convergence dans un espace $\mathcal{C}(0,T;X)$. A ce stade, nous concluons sur les résultats de convergence des approximations de Galerkin.

□

▶ **Section** 4.7

Dans cette section, le cas où le terme de tension de surface est conservé est abordé. Une approche similaire au cas précédent est utilisée ; cependant, nous procédons comme dans [Boy02], et utilisons le fait que le terme de tension de surface dans l'équation de Navier-Stokes s'annule avec le terme de transport de l'équation de Cahn-Hilliard. Des estimations *a priori* similaires au cas sans tension de surface donnent de la régularité sur φ, μ. La régularité de $\boldsymbol{u}$, p est tirée de l'équation de Reynolds, dans laquelle les termes supplémentaires dus à la tension de surface sont estimés.

1.3.4 Résumé du Chapitre 5

Etude numérique d'écoulements diphasiques en film mince

Dans ce chapitre, nous cherchons à simuler des écoulements diphasiques dans des domaines minces, en prenant en compte la diffusion à l'interface entre les deux phases et les effets de capillarité. Nous adaptons un schéma numérique développé par Boyer [Boy02] pour les équations de Navier-Stokes couplées à l'équation de Cahn-Hilliard hydrodynamique au cas du couplage équation de Reynolds/équation de Cahn-Hilliard hydrodynamique introduit dans le chapitre précédent. Nous présentons ensuite des simulations numériques, pour différentes applications. En particulier, nous montrons que le modèle choisi permet de simuler de nouveaux aspects du phénomène de cavitation, dans la mesure où le modèle autorise la présence de plusieurs couches de chaque fluide.

▶ **Section** 5.2

Dans cette section, nous présentons succinctement les équations permettant de modéliser un mélange de deux fluides dans un domaine mince, telles qu'elles ont été introduites dans le Chapitre 4. Lors de la procédure d'adimensionnement permettant de prendre en compte l'anisotropie du domaine, on a montré que le système limite obtenu est composé de l'équation de Reynolds à viscosité variable couplée à l'équation de Cahn-Hilliard hydrodynamique, données par (1.16) et (1.17).

▶ **Section** 5.3

Cette partie est consacrée à la présentation du schéma numérique.

▷ Comme ce travail prend en compte des domaines non rectangulaires, et que la discrétisation se fait par différences finies, la première étape consiste à effectuer un changement de variables afin de se ramener à un domaine rectangulaire, pour lequel l'utilisation de différences finies est adaptée. En définissant le domaine non rescalé $\Omega = \{(x, y) \in \mathbb{R}^2, x \in (0, L), y \in (0, h(x))$ et le domaine rescalé $\Omega = \{(x, z) \in$

$\mathbb{R}^2, x \in (0, L), z \in (0, 1)$, on obtient les correspondances suivantes :

$$\partial_x \cdot \longleftrightarrow \partial_x \cdot -z\frac{h'(x)}{h(x)}\partial_z \cdot, \qquad \partial_y \cdot \longleftrightarrow \frac{1}{h(x)}\partial_z \cdot .$$

Ce changement de variables induit des termes supplémentaires dans les équations, et celles-ci sont réécrites.

- ▷ Les deux équations de Reynolds et de Cahn-Hilliard sont traitées en deux temps : pour un paramètre d'ordre φ donné, la pression p puis le champ de vitesses $\boldsymbol{u}$ sont calculés par l'équation de Reynolds, ensuite φ et le potentiel chimique μ sont recalculés pour le champ de vitesse obtenu précédemment.
- ▷ L'équation de Reynolds fait intervenir des termes intégraux en fonction de la viscosité (elle-même fonction de φ), qui doivent être calculés.La prise en compte des termes de tension de surface est également détaillée.
- ▷ Pour l'équation de Cahn-Hilliard, la discrétisation temporelle se fait avec un pas de temps variable δt.
  - ⋆ Dans un premier temps, connaissant les valeurs φ^n, μ^n au temps t^*, $\varphi^{n+1/2}$ et $\mu^{n+1/2}$ sont calculées comme solutions de l'équation de Cahn-Hilliard sans terme de convection, en utilisant un θ-schéma. De manière plus précise, $\varphi^{n+1/2}$ et $\mu^{n+1/2}$ sont solutions de

$$\begin{cases} \dfrac{\varphi^{n+1/2} - \varphi^n}{\delta t} - \dfrac{1}{\mathcal{P}e} \operatorname{div}\Big(B(\varphi^n)\nabla(\theta\mu^{n+1/2} + (1-\theta)\mu^n)\Big) = 0, \\ \theta\mu^{n+1/2} + (1-\theta)\mu^n = -\alpha^2\Delta(\theta\varphi^{n+1/2} + (1-\theta)\varphi^n) + F'(\theta\varphi^{n+1/2} + (1-\theta)\varphi^n). \end{cases}$$

    Le paramètre θ est choisi supérieur à 0.5 pour assurer la stabilité du schéma, mais suffisamment proche de 0.5 pour garder une bonne précision. La valeur choisie est $\theta = 0.6$. Ce système non-linéaire est résolu par une méthode de point fixe, et, en pratique, peu d'itérations sont nécessaires pour que la méthode converge.
  - ⋆ En ce qui concerne la partie convective, φ^{n+1} est calculé en fonction de $\varphi^{n+1/2}$ (et par la suite, μ^{n+1} est déduit de φ^{n+1} par la relation $\mu^{n+1} = -\alpha^2\Delta\varphi^{n+1} + F'(\varphi^{n+1})$. En introduisant l'opérateur de convection $K\cdot = \boldsymbol{u}\cdot\nabla\cdot$, le schéma de Runge-Kutta d'ordre trois utilisé s'écrit:

$$\varphi^{n+1} - \varphi^{n+1/2} = -\delta t K(\varphi^{n+1/2}) + \frac{1}{2}\delta t^2 K^2(\varphi^{n+1/2}) - \frac{1}{6}\delta t^3 K^3(\varphi^{n+1/2}).$$

- ▷ En ce qui concerne la discrétisation spatiale, nous décrivons en quel point de chaque cellule sont choisies les différentes variables. Les conditions limites sont traitées par l'introduction d'inconnues artificielles autour du domaine. L'opérateur de convection doit être discrétisé par un schéma L^∞-stable, dans la mesure où les valeurs physiques du paramètre d'ordre φ sont comprises entre -1 et 1. De ce fait, nous utilisons des limiteurs de flux, comme ceux proposés dans [GR91], et utilisés pour l'équation de Cahn-Hilliard dans [BCF04]. Ce schéma induit une condition de stabilité de type C.F.L. (Courant-Friedrich-Levy) classique.

► **Section** 5.4

Nous présentons dans cette partie différents cas tests permettant de valider le programme développé.

- ▷ Pour ce qui est de l'équation de Reynolds, nous présentons quelques cas tests permettant de vérifier l'allure des courbes de pression en fonction de la forme du domaine (en général convergent-divergent, plus ou moins "raide"). De plus, nous étudions la dépendance de ces courbes en fonction du flux d'entrée imposé.
- ▷ Dans la mesure où le code développé prend en compte les différences de viscosité entre les deux fluides, nous présentons deux tests permettant de mettre en évidence l'influence qualitative des rapports de viscosité utilisés sur l'allure de l'interface entre les deux fluides.
- ▷ Afin de tester la partie diffusive de l'équation de Cahn-Hilliard, nous introduisons des coefficients de diffusion anisotropes dans les directions x et z, et nous montrons les résultats obtenus lorsque l'une des directions de diffusion est négligée. De plus, cette partie permet de simuler numériquement différents choix d'adimensionnement pour les coefficients liés à l'épaisseur de l'interface ou au frottement.
- ▷ Un dernier cas test consiste à s'intéresser au transport d'une goutte dans un canal. En traçant le champ de vitesse relatif, nous mettons en évidence l'existence de recirculations à l'intérieur de la goutte.

► **Section** 5.5

Les équations choisies permettent de simuler des phénomènes apparaissant en lubrification, dans la mesure où nous pouvons travailler dans des domaines de type convergent-divergent, et nous prenons en compte le cisaillement. Ainsi, nous nous intéressons à la modélisation de la cavitation par le modèle de Cahn-Hilliard (à interface diffuse). Nous présentons des premiers résultats, permettant de mettre en évidence l'existence de zones saturées ainsi que l'apparition de zones à trois couches, où une bulle d'air est capturée entre deux couches de lubrifiant.

Bibliographie

[ABC94] A. Assemien, G. Bayada, et M. Chambat. Inertial effects in the asymptotic behavior of a thin film flow. *Asymptotic Anal.*, 9(3):177–208, 1994.

[AMW98] D. M. Anderson, G. B. McFadden, et A.A. Wheeler. Diffuse-interface methods in fluid mechanics. *Ann. Rev. Fluid Mech.*, 30:139–165, 1998.

[AO03] S. J. Alvarez et R. Oujja. On the uniqueness of the solution of an evolution free boundary problem in theory of lubrication. *Nonlinear Anal.*, 54(5):845–872, 2003.

[BB05] G. Bayada et H. Bellout. An unconditional existence result for the quasi-variational elastohydrodynamic free boundary value problem. *J. Differential Equations*, 216(1):134–152, 2005.

[BBT95] G. Bayada, M. Boukrouche, et M. El-A. Talibi. The transient lubrication problem as a generalized Hele-Shaw type problem. *Z. Anal. Anwendungen*, 14(1):59–87, 1995.

[BC86a] G. Bayada et M. Chambat. Sur quelques modélisations de la zone de cavitation en lubrification hydrodynamique. *J. Méc. Théor. Appl.*, 5(5):703–729, 1986.

[BC86b] G. Bayada et M. Chambat. The transition between the Stokes equations and the Reynolds equation: a mathematical proof. *Appl. Math. Optim.*, 14(1):73–93, 1986.

[BC89] G. Bayada et M. Chambat. Modélisation de la jonction d'un écoulement tridimensionnel et d'un film mince bidimensionnel. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(1):81–84, 1989.

[BC83] G. Bayada et M. Chambat. Analysis of a free boundary problem in partial lubrication. *Quart. Appl. Math.*, 40(4):369–375, 1982/83.

[BCF04] F. Boyer, L. Chupin, et P. Fabrie. Numerical study of viscoelastic mixtures through a Cahn-Hilliard flow model. *Eur. J. Mech. B Fluids*, 23(5):759–780, 2004.

[BCM07] G. Bayada, L. Chupin, et S. Martin. Viscoelastic fluids in a thin domain. *Quart. Appl. Math.*, 65(4):625–651, 2007.

[Bel03] H. Bellout. Existence of a solution to the line contact problem of elasto-hydrodynamic lubrication. *European J. Appl. Math.*, 14(3):279–290, 2003.

[BEM04] M. Boukrouche et R. El Mir. Asymptotic analysis of a non-Newtonian fluid in a thin domain with Tresca law. *Nonlinear Anal.*, 59(1-2):85–105, 2004.

[BK94] M. Bertsch et M. H. A. Klaver. The Stefan problem with mushy regions: differentiability of the interfaces. *Ann. Mat. Pura Appl. (4)*, 166:27–61, 1994.

[BK04] R. Bunoiu et S. Kesavan. Asymptotic behaviour of a Bingham fluid in thin layers. *J. Math. Anal. Appl.*, 293(2):405–418, 2004.

[Bla01] S. Blair. The pressure-viscosity coefficient of a perfluorinated polyether over a wide temperature range. *J. of Tribology*, 123(1):50–53, 2001.

[BM95] E. Blavier et A. Mikelić. On the stationary quasi-Newtonian flow obeying a power-law. *Math. Methods Appl. Sci.*, 18(12):927–948, 1995.

[BMT93] A. Bourgeat, A. Mikelić, et R. Tapiéro. Dérivation des équations moyennées décrivant un écoulement non newtonien dans un domaine de faible épaisseur. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(9):965–970, 1993.

[BMV06] G. Bayada, S. Martin, et C. Vázquez. About a generalized Buckley-Leverett equation and lubrication multifluid flow. *European J. Appl. Math.*, 17(5):491–524, 2006.

[BN90] J. Baranger et K. Najib. Analyse numérique des écoulements quasi-newtoniens dont la viscosité obéit à la loi puissance ou la loi de Carreau. *Numer. Math.*, 58(1):35–49, 1990.

[BN07] D. Bresch et P. Noble. Mathematical justification of a shallow water model. *Methods Appl. Anal.*, 14(2):87–118, 2007.

[Boy99] F. Boyer. Mathematical study of multi-phase flow under shear through order parameter formulation. *Asymptot. Anal.*, 20(2):175–212, 1999.

[Boy01] F. Boyer. Nonhomogeneous Cahn-Hilliard fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(2):225–259, 2001.

[Boy02] F. Boyer. A theoretical and numerical model for the study of incompressible mixture flows. *Computers and Fluids*, 31(1):41–68, 2002.

[BTV96] G. Bayada, M. El-A. Talibi, et C. Vázquez. Existence of solutions for elastohydrodynamic piezoviscous lubrication problems with a new model of cavitation. *European J. Appl. Math.*, 7(1):63–73, 1996.

[CH58] J. W. Cahn et J. E. Hilliard. Free energy of nonuniform system. I. Interfacial free energy. *J. Chem. Phys.*, 28(2):258–267, 1958.

[Chu03] L. Chupin. Existence result for a mixture of non Newtonian flows with stress diffusion using the Cahn-Hilliard formulation. *Discrete Contin. Dyn. Syst. Ser. B*, 3(1):45–68, 2003.

[Chu04] L. Chupin. Some theoretical results concerning diphasic viscoelastic flows of the Oldroyd kind. *Adv. Differential Equations*, 9(9-10):1039–1078, 2004.

[Cim77] G. Cimatti. On a problem of the theory of lubrication governed by a variational inequality. *Appl. Math. Optim.*, 3(2/3):227–242, 1977.

[Cim83] G. Cimatti. How the Reynolds equation is related to the Stokes equations. *Appl. Math. Optim.*, 10(3):267–274, 1983.

[CM98] T. Clopeau et A. Mikelić. On the non-stationary quasi-Newtonian flow through a thin slab. Dans *Navier-Stokes equations: theory and numerical methods (Varenna, 1997)*, volume 388 de *Pitman Res. Notes Math. Ser.*, pages 1–15. Longman, 1998.

[CMP94] C. Conca, F. Murat, et O. Pironneau. The Stokes and Navier-Stokes equations with boundary conditions involving the pressure. *Japan. J. Math. (N.S.)*, 20(2):279–318, 1994.

[Com92] E. Comparini. A one-dimensional Bingham flow. *J. Math. Anal. Appl.*, 169(1):127–139, 1992.

[CV96] R. Chella et J. Viñals. Mixing of a two-phase fluid by cavity flow. *Phys. Rev. E*, 53 (B)(4):3832–3840, 1996.

[DT79] D. Dowson et C. M. Taylor. Cavitation in bearings. *Ann. Rev. Fluid Mech*, 11:35–66, 1979.

[DV94] J. Durany et C. Vazquez. Mathematical analysis of an elastohydrodynamic lubrication problem with cavitation. *Appl. Anal.*, 53(1-2):135–142, 1994.

[EA75] H. G. Elrod et M. L. Adams. A computer program for cavitation. *Cavitation and related phenomena in lubrication - Proceedings - Mech. Eng. Publ. ltd*, pages 37–42, 1975.

[EG96] C. M. Elliott et H. Garcke. On the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.*, 27(2):404–423, 1996.

[Elr60] H. G. Elrod. A derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties. *Quart. Appl. Math.*, 17:349–359, 1960.

[FCGO98] E. Fernández-Cara, F. Guillén, et R. R. Ortega. Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):1–29, 1998.

[GP01] J.-F. Gerbeau et B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1):89–102, 2001.

[GR91] E. Godlewski et P.-A. Raviart. *Hyperbolic systems of conservation laws*, volume 3/4 de *Mathématiques & Applications (Paris) [Mathematics and Applications]*. Ellipses, Paris, 1991.

[GS90] C. Guillopé et J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849–869, 1990.

[Gup03] S. C. Gupta. *The classical Stefan problem*, volume 45 de *North-Holland Series in Applied Mathematics and Mechanics*. Elsevier Science B.V., Amsterdam, 2003. Basic concepts, modelling and analysis.

[HLMW02] P. Huang, Z.-H. Li, Y.-G. Meng, et S.-Z. Wen. Study on thin film lubrication with second-order fluid. *J. of Tribology*, 124(3):547–552, 2002.

[HMNR03] J. Hron, J. Málek, J. Nečas, et K. R. Rajagopal. Numerical simulations and global existence of solutions of two-dimensional flows of fluids with pressure- and shear-dependent viscosities. *Math. Comput. Simulation*, 61(3-6):297–315, 2003.

[Hu90] B. Hu. A quasi-variational inequality arising in elastohydrodynamics. *SIAM J. Math. Anal.*, 21(1):18–36, 1990.

[KBA05] G. Karniadakis, A. Beskok, et N. Aluru. *Microflows and nanoflows*, volume 29 de *Interdisciplinary Applied Mathematics*. Springer, New York, 2005. Fundamentals and simulation, With a foreword by Chih-Ming Ho.

[Kim87] J. U. Kim. On the initial-boundary value problem for a Bingham fluid in a three-dimensional domain. *Trans. Amer. Math. Soc.*, 304(2):751–770, 1987.

[LM00] P. L. Lions et N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[LT98] J. Lowengrub et L. Truskinovsky. Quasi-incompressible Cahn-Hilliard fluids and topological transitions. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 454(1978):2617–2654, 1998.

[MP97] A. Mikelić et L. Paoli. On the derivation of the Buckley-Leverett model from the two fluid Navier-Stokes equations in a thin domain. *Comput. Geosci.*, 1(1):59–83, 1997.

[MP02] E. Marušić-Paloka. Existence and regularity of purely viscous polymer flow. *Appl. Anal.*, 81(4):939–950, 2002.

[MT04] L. Molinet et R. Talhouk. On the global and periodic regular flows of viscoelastic fluids with a differential constitutive law. *Nonlinear Differential Equations Appl.*, 11(3):349–359, 2004.

[NPD97] A. Nouri, F. Poupaud, et Y. Demay. An existence theorem for the multifluid Stokes problem. *Quart. Appl. Math.*, 55(3):421–435, 1997.

[NV04] S. Nazarov et J. H. Videman. Reynolds type equation for a thin flow under intensive transverse percolation. *Math. Nachr.*, 269/270:189–209, 2004.

[OF03] S. Osher et R. Fedkiw. *Level set methods and dynamic implicit surfaces*, volume 153 de *Applied Mathematical Sciences*. Springer-Verlag, New York, 2003.

[Old50] J. G. Oldroyd. On the formulation of rheological equations of state. *Proc. Roy. Soc. London. Ser. A.*, 200:523–541, 1950.

[Pao03] L. Paoli. Asymptotic behavior of a two fluid flow in a thin domain: from Stokes equations to Buckley-Leverett equation and Reynolds law. *Asymptot. Anal.*, 34(2):93–120, 2003.

[Poi40] J.-M. Poiseuille. Recherches expérimentales sur le mouvement des liquides dans les tubes de très petits diamètres. *C. R. Hebdo. Acad. Sci.*, 11:961–967 & 1041–1048, 1840.

[Ren85] M. Renardy. Existence of slow steady flows of viscoelastic fluids with differential constitutive equations. *Z. Angew. Math. Mech.*, 65(9):449–451, 1985.

[Rey86] O. Reynolds. On the theory of lubrication and its application to Mr Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil. *Proc. Roy. Soc. London*, 40:191–203, 1886.

[Rod93] J.-F. Rodrigues. Remarks on the Reynolds problem of elastohydrodynamic lubrication. *European J. Appl. Math.*, 4(1):83–96, 1993.

[San93] D. Sandri. Sur l'approximation numérique des écoulements quasi-newtoniens dont la viscosité suit la loi puissance ou la loi de Carreau. *RAIRO Modél. Math. Anal. Numér.*, 27(2):131–155, 1993.

[SET05] J.-M. Sac-Epée et K. Taous. On a wide class of nonlinear models for non-Newtonian fluids with mixed boundary conditions in thin domains. *Asymptot. Anal.*, 44(1-2):151–171, 2005.

[SJPT90] J. Saint Jean Paulin et K. Taous. About derivation of Reynolds law from Navier-Stokes equation for two non-miscible fluids. Dans *Mathematical Modelling in Lubrication*, pages 99–104. Universidade de Vigo, 1990.

[Som04] A. Sommerfeld. Zur hydrodynamischen Theorie der Schmiermittelreibung. *Z. Math. Phys.*, 40:97–155, 1904.

[ST98] W. G. Sawyer et J. A. Tichy. Non-Newtonian lubrication with the second-order fluid. *J. of Tribology*, 120(3):622–628, 1998.

[Sta72] G. Stampacchia. On a problem of numerical analysis connected with the theory of variational inequalities. Dans *Symposia Mathematica, Vol. X (Convegno di Analisi Numerica, INDAM, Rome, 1972)*, pages 281–293. Academic Press, London, 1972.

[SV71] A. J.-C. de Saint Venant. Théorie du mouvement non-permanent des eaux, avec applications aux crues des rivières et à l'introduction des marées dans leur lit. *C. R. Acad. Sci.*, 73:147–154, 1871.

[Sze98] A. Z. Szeri. *Fluid Film Lubrication: Theory and Design.* Cambridge University Press, 1998.

[TAB04] F. Talay Akyildiz et H. Bellout. Viscoelastic lubrication with Phan-Thein-Tanner fluid (PTT). *ASME J. Tribol.*, 126:288–291, 2004.

[Tic95] J. A. Tichy. A surface layer model for thin film lubrication. *Tribology Transactions*, 38(3):577–582, 1995.

[Tic96] J. A. Tichy. Non-Newtonian lubrication with the convected Maxwell model. *ASME J. Tribol.*, 118:344–348, 1996.

[TW98] R. I. Tanner et K. Walters. *Rheology: an historical perspective*, volume 7 de *Rheology series.* Elsevier, 1998.

[VC94] C. Vázquez Cendón. Existence and uniqueness of solution for a lubrication problem with cavitation in a journal bearing with axial supply. *Adv. Math. Sci. Appl.*, 4(2):313–331, 1994.

[Vis98] A. Visintin. Introduction to the models of phase transitions. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 1(1):1–47, 1998.

[Wan50] G. H. Wannier. A contribution to the hydrodynamics of lubrication. *Quart. Appl. Math.*, 8:1–32, 1950.

2

Viscoelastic fluids in thin domains

ABSTRACT *The present paper deals with the rigorous computation of the constitutive law of a non-Newtonian viscoelastic Oldroyd-B fluid in a thin gap in the context of the lubrication.*

2.1 Introduction

This paper concerns the study of a viscoelastic fluid flow in a thin gap, the motion of which is imposed due to non homogeneous boundary conditions.

When a Newtonian flow is contained between two close given surfaces in relative motion, it is well known that it is possible to replace the Stokes or Navier-Stokes equations governing the fluid's motion by a simpler asymptotic model. The asymptotic pressure is proved to be independent of the normal direction to the close surfaces and obeys the Reynolds thin film equation whose coefficients include the velocities, the geometrical description of the surrounding surfaces and some rheological characteristics of the fluid. As a following step, the computation of this pressure allows an asymptotic velocity of the fluid to be easily computed. Such asymptotic procedure first proposed in a formal way by Reynolds [BC86] has been rigorously confirmed for Newtonian stationary flow [ABC94], and then generalized in a lot of situations covering numerous applications for both compressible fluid [MPS05], unsteady cases [BCC99], multifluid flows [Pao03].

It is well known however that in numerous applications, the fluid to be considered is a non-Newtonian one. This is the case for numerous biological fluids, modern lubricants in engineering applications due to the additives they contain, polymers in injection or molding process. In all of these applications, there are situations in which the flow is anisotropic. It is usual to take account of this geometrical effect in order to simplify the three-dimensional equations of motion, trying to recover a two dimensional Reynolds-like equation with respect to the pressure only. Such procedures are mostly heuristic ones. Nevertheless, some mathematical works appeared in the literature to justify them. They include thin film asymptotic studies of Bingham flow [BK04], quasi Newtonian flow (Carreau's law, power law or Williamson's law, in which various stress-velocity relations are chosen: [BMT93], [BT95], [SET05]) and also micro polar ones [BL96]. It has been possible to obtain rigorously some thin film approximation for such fluids using a so called generalized Reynolds equation for the pressure.

However, in the previous examples, elasticity effects are neglected. Introduction of such viscoelastic behavior is characterized by the Deborah number which is related to the relaxation time. One of the most popular laws is the Oldroyd-B model whose constitutive equation is an interpolation between purely viscous and purely elastic models, thus introducing an additional parameter which describes the relative proportion of both behaviors. A formal procedure has been proposed in [BCM07]. However, the asymptotic system so obtained lacks the usual characteristic of classical generalized Reynolds equation as it has not been possible to gain an equation in the asymptotic pressure only. Both velocity u^* and pressure p^* are coupled by a non linear system.

It is the goal of this work to justify rigorously this asymptotic system. Section 2.2 is devoted to the precise statement of the 3-D problem. One difficulty has been to find an existence theorem for the general Oldroyd-B model, acting as a starting point for the mathematical procedure. Most of the existence theorems, however, deal with small data or small time assumptions. To control this kind of property with respect to the smallness of the gap appears somewhat difficult. So we are led to consider a more particular Oldroyd-B model, for which unconditional existence theorem has been proved [LM00]. Moreover, a specific attention is devoted to the boundary conditions to be introduced both on the velocity and on the stress. The goal is to use "well prepared" boundary conditions so as to prevent boundary layer on the lateral side of the domain.

In Section 2.3, after suitable scaling procedure, asymptotic expansions of both pressure, viscosity and stress are introduced, taking into account the previous formal results from [BCM07]. Section 2.4 is mainly concerned with the proof of some additional regularity properties for the formal asymptotic solution. Assuming some restrictions on the rheological parameters, it will be proved that it is possible to gain a C^k regularity for p^*, $k > 1$, which in turn improves the regularity of u^* and the stress tensor σ^*. This result is obtained by introducing a differential Cauchy system satisfied by the derivative of p^*. Finally, section 2.5, is devoted to the convergence to zero of the second term of the asymptotic expansions, which in turn proves the convergence of the solution of the real 3-D problem towards u^*, p^*, σ^* (Theorems 2.15 and 2.17).

2.2 Introduction of the problem and known results

2.2.1 Formulation of the problem

We consider unsteady incompressible flows of viscoelastic fluids, which are ruled by Oldroyd's law, in a thin domain $\hat{\Omega}^\varepsilon = \{(x,y) \in \mathbb{R}^n, x \in \omega \text{ and } 0 < y < \varepsilon h(x)\}$, where ω is an $(n-1)$-dimensional domain, with $n = 2$ or $n = 3$ ($x = x_1$ or $x = (x_1, x_2)$), as in Figure 2.1.

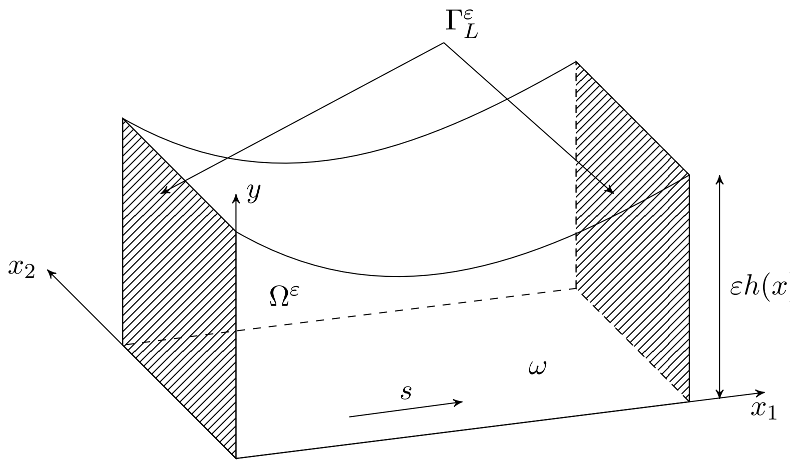

Figure 2.1: Domain $\hat{\Omega}_\varepsilon$

The following hypotheses on h are required:

$$\forall x \in \omega,\ 0 < h_0 \leq h(x) \leq h_M, \quad \text{and} \quad h \in \mathcal{C}^1(\bar{\omega}).$$

Let $\hat{\boldsymbol{u}}^\varepsilon = (\hat{u}_1^\varepsilon, \hat{u}_2^\varepsilon, \hat{u}_3^\varepsilon)$ be the velocity field in the three-dimensional case, or $\hat{\boldsymbol{u}}^\varepsilon = (\hat{u}_1^\varepsilon, \hat{u}_2^\varepsilon)$ in the two-dimensional case, $\hat{p}^\varepsilon$ the pressure, and $\hat{\boldsymbol{\sigma}}^\varepsilon$ the stress symmetric tensor in the domain $\hat{\Omega}^\varepsilon$. Bold letters stand for vectorial or tensorial functions, the notation $\hat{f}$ corresponds to a function f defined in the domain $\hat{\Omega}^\varepsilon$, and the superscript $^\varepsilon$ denotes the dependence on ε.

Formulation of the problem The formulation of the problem reads as follows, in $(0, \infty) \times \hat{\Omega}^\varepsilon$:

$$\left\{ \begin{array}{rcl} \rho\, \partial_t \hat{\boldsymbol{u}}^\varepsilon + \rho\, \hat{\boldsymbol{u}}^\varepsilon \cdot \nabla \hat{\boldsymbol{u}}^\varepsilon - (1-r)\nu\, \Delta \hat{\boldsymbol{u}}^\varepsilon + \nabla \hat{p}^\varepsilon & = & \nabla \cdot \hat{\boldsymbol{\sigma}}^\varepsilon\,, \\ \nabla \cdot \hat{\boldsymbol{u}}^\varepsilon & = & 0\,, \\ \lambda\ (\partial_t \hat{\boldsymbol{\sigma}}^\varepsilon + \hat{\boldsymbol{u}}^\varepsilon \cdot \nabla \hat{\boldsymbol{\sigma}}^\varepsilon + g(\hat{\boldsymbol{\sigma}}^\varepsilon, \nabla \hat{\boldsymbol{u}}^\varepsilon)) + \hat{\boldsymbol{\sigma}}^\varepsilon & = & 2r\nu D(\hat{\boldsymbol{u}}^\varepsilon)\,, \end{array} \right. \tag{2.1}$$

where the nonlinear terms $g(\hat{\boldsymbol{\sigma}}^\varepsilon, \nabla\hat{\boldsymbol{u}}^\varepsilon)$, the vorticity tensor $W(\hat{\boldsymbol{u}}^\varepsilon)$ and the deformation tensor $D(\hat{\boldsymbol{u}}^\varepsilon)$ are given by:

$$g(\hat{\boldsymbol{\sigma}}^\varepsilon, \nabla\hat{\boldsymbol{u}}^\varepsilon) = -W(\hat{\boldsymbol{u}}^\varepsilon)\cdot\hat{\boldsymbol{\sigma}}^\varepsilon + \hat{\boldsymbol{\sigma}}^\varepsilon\cdot W(\hat{\boldsymbol{u}}^\varepsilon),$$
$$W(\hat{\boldsymbol{u}}^\varepsilon) = \frac{\nabla\hat{\boldsymbol{u}}^\varepsilon - {}^t\nabla\hat{\boldsymbol{u}}^\varepsilon}{2} \quad \text{and} \quad D(\hat{\boldsymbol{u}}^\varepsilon) = \frac{\nabla\hat{\boldsymbol{u}}^\varepsilon + {}^t\nabla\hat{\boldsymbol{u}}^\varepsilon}{2}.$$

In this formulation, the physical parameters are the viscosity ν, the density ρ, and the relaxation time λ. The parameter λ is related to the viscoelastic behavior and the Deborah number. The parameter $r \in [0, 1)$ describes the relative proportion of the viscous and elastic behavior.

Initial conditions This problem is considered with the following initial conditions:

$$\hat{\boldsymbol{u}}^\varepsilon|_{t=0} = \hat{\boldsymbol{u}}_0^\varepsilon, \quad \hat{\boldsymbol{\sigma}}^\varepsilon|_{t=0} = \hat{\boldsymbol{\sigma}}_0^\varepsilon, \tag{2.2}$$

for $\hat{\boldsymbol{u}}_0^\varepsilon \in \boldsymbol{L}^2(\hat{\Omega}^\varepsilon)$, $\hat{\boldsymbol{\sigma}}_0^\varepsilon \in \boldsymbol{L}^2(\hat{\Omega}^\varepsilon)$. The bold notation $\boldsymbol{L}^2(\hat{\Omega}^\varepsilon)$ denotes the set of vectorial or tensorial functions whose all components belong to $L^2(\hat{\Omega}^\varepsilon)$.

Boundary conditions Dirichlet boundary conditions are set on top and bottom of the domain, and the conditions on the lateral part of the boundary $\hat{\Gamma}_L^\varepsilon$, defined by

$$\hat{\Gamma}_L^\varepsilon = \{(x, y) \in \mathbb{R}^n, x \in \partial\omega \text{ and } 0 < y < \varepsilon h(x)\},$$

will be specified later (in section 2.4.2). Therefore, it is possible to write the boundary conditions in a shortened way:

$$\hat{\boldsymbol{u}}^\varepsilon|_{\partial\hat{\Omega}^\varepsilon} = \hat{\boldsymbol{J}}^\varepsilon, \tag{2.3}$$

where $\hat{\boldsymbol{J}}^\varepsilon$ is a given function such that $\hat{\boldsymbol{J}}^\varepsilon \in \boldsymbol{H}^{1/2}(\partial\hat{\Omega}^\varepsilon)$ and satisfying $\hat{\boldsymbol{J}}^\varepsilon|_{y=h^\varepsilon} = 0$, $\hat{\boldsymbol{J}}^\varepsilon|_{y=0} = (s, 0)$. This function will be fully determined in Subsection 2.4.2.

Since $\hat{\boldsymbol{\sigma}}^\varepsilon$ satisfies a transport equation in the domain $\hat{\Omega}^\varepsilon$, it remains to impose boundary conditions on $\hat{\boldsymbol{\sigma}}^\varepsilon$ on the part of the boundary where $\hat{\boldsymbol{u}}^\varepsilon$ is an incoming velocity. Let us define $\hat{\Gamma}_+^\varepsilon$ the part of $\hat{\Gamma}_L^\varepsilon$ such that $\hat{\boldsymbol{J}}^\varepsilon|_{\hat{\Gamma}_+^\varepsilon} \cdot n < 0$, and $\hat{\Gamma}_-^\varepsilon = \hat{\Gamma}_L^\varepsilon \setminus \hat{\Gamma}_+^\varepsilon$. We set

$$\hat{\boldsymbol{\sigma}}^\varepsilon|_{\hat{\Gamma}_+^\varepsilon} = \hat{\boldsymbol{\theta}}^\varepsilon, \tag{2.4}$$

where $\hat{\boldsymbol{\theta}}^\varepsilon$ is a given function in $\boldsymbol{H}^{1/2}(\hat{\Gamma}_+^\varepsilon)$ which will also be determined in Subsection 2.4.2.

Moreover, since the pressure is defined up to a constant, the mean pressure is chosen to be zero: $\int\limits_{\hat\Omega^\varepsilon} \hat p^\varepsilon = 0$.

Notations Let us introduce the following function space:

$$V = \{\hat{\boldsymbol\varphi} \in \boldsymbol{H}^1_0(\hat\Omega^\varepsilon),\ \nabla\cdot\hat{\boldsymbol\varphi} = 0\},$$

and the following notations, that will be used in the later. For $\hat f$ defined in $\hat\Omega^\varepsilon$:

- $|\hat f|$ denotes the L^2-norm in $\hat\Omega^\varepsilon$,
- $|\hat f|_p$ denotes the L^p-norm in $\hat\Omega^\varepsilon$, for $2 < p \le +\infty$,
- the spaces $\mathcal{C}^m(\overline{\hat\Omega^\varepsilon})$ for $m \ge 1$ are equipped with the norms $\|\hat f\|_{\mathcal{C}^m} = |\hat f|_\infty + \sum\limits_{i=1}^{m} |\hat f^{(i)}|_\infty$.

For $\hat f$ defined in $\mathbb{R}^+ \times \hat\Omega^\varepsilon$, $\|\hat f\|_{L^\alpha(L^\beta)}$ denotes the norm of the space $L^\alpha(0,\infty, L^\beta(\hat\Omega^\varepsilon))$, with $1 \le \alpha,\ \beta \le \infty$.

2.2.2 Existence theorem in the thin domain

Theorem 2.1. *For $\varepsilon > 0$ fixed, problem (2.1)-(2.3) admits a weak solution*

$$\hat{\boldsymbol u}^\varepsilon \in L^2_{loc}(0,\infty,\boldsymbol{H}^1(\hat\Omega^\varepsilon))\,, \quad \hat p^\varepsilon \in L^2_{loc}(0,\infty,L^2(\hat\Omega^\varepsilon))\,, \quad \hat{\boldsymbol\sigma}^\varepsilon \in \mathcal{C}(0,\infty,\boldsymbol{L}^2(\hat\Omega^\varepsilon))\,.$$

Proof. This result is proved in [LM00]. □

Remark 2.2. *Let us emphasize that for the following, it is essential to know the global (in time) existence of a solution for problem (2.1)-(2.3). Other existence theorems have been proved for this problem, for example in [GS90], [FCGO98], [Chu04], but these theorems are either local in time (on a time interval $[0, T^\varepsilon]$), or a small data assumption is needed. In this work, these theorems cannot be used, since there is no control on the behavior of T^ε (or equivalently of the data) when ε tends to zero, in particular T^ε may tend to zero. Consequently, this work is restricted to the specific case treated in [LM00], taking one parameter of the Oldroyd model to be zero. In all generality, the non-linear term reads $g(\boldsymbol\sigma, \nabla\boldsymbol u) = -W(\boldsymbol u)\cdot\boldsymbol\sigma + \boldsymbol\sigma\cdot W(\boldsymbol u) - a\,(\boldsymbol\sigma\cdot D(\boldsymbol u) + D(\boldsymbol u)\cdot\boldsymbol\sigma)$, which is called objective derivative. Here the parameter a is taken to be zero. This case corresponds to the so-called Jaumann derivative.*

Remark 2.3. *The following computations are made in the two-dimensional case (i.e. $\omega = (0, L)$ is a one-dimensional domain) for the sake of simplicity. However, note that except for the regularity results on the limit problem in Section 2.4.3, all estimates are independent of the dimension, thus the following computations should apply to the three-dimensional case (provided that the regularity of the limit problem is obtained otherwise).*

Regularizing the system In the proof of the previous theorem, the existence of a solution is achieved by regularization. Therefore, this study only concerns solutions obtained as the limit of a regularized problem approximating (2.1), in which an additional term $-\eta\Delta\hat{\boldsymbol{\sigma}}^{\varepsilon\eta}$ is added to the Oldroyd equation, with $\eta > 0$ a small parameter. Here a regularization of the form $-\eta\Delta(\hat{\boldsymbol{\sigma}}^{\varepsilon\eta} - \hat{\boldsymbol{G}})$ is chosen, with $\hat{\boldsymbol{G}}$ a symmetric tensor in $\boldsymbol{H}^2(\hat{\Omega}^\varepsilon)$ independent of η and ε which will be specified later. After obtaining the needed energy estimates uniformly in η, we will let η tend to zero. This approach allows to multiply the Oldroyd equation by $\hat{\boldsymbol{\sigma}}^{\varepsilon\eta}$, since $\hat{\boldsymbol{\sigma}}^{\varepsilon\eta}$ is regular enough. Of course, one can choose another regularization which leads to energy estimates which are uniform in the regularization parameter.

Furthermore, because of the regularizing term, boundary conditions on the whole boundary are needed. Let us write $\hat{\boldsymbol{\sigma}}^{\varepsilon\eta}|_{\partial\hat{\Omega}^\varepsilon} = \hat{\boldsymbol{\theta}}^{\varepsilon\eta}$, where $\hat{\boldsymbol{\theta}}^{\varepsilon\eta}$ is now a function of $\boldsymbol{H}^{1/2}(\partial\hat{\Omega}^\varepsilon)$, which will be determined later by equation (2.12).

2.3 Asymptotic expansions

2.3.1 Renormalization of the domain

After introducing a new variable $z = \dfrac{y}{\varepsilon}$, the system (2.1) can be rewritten in a fixed re-scaled domain:

$$\Omega = \{(x,z) \in \mathbb{R}^n, x \in \omega \text{ and } 0 < z < h(x)\}.$$

For a function $\hat{f}$ defined in Ω^ε, f is defined in Ω by $f(x,z) = \hat{f}(x,\varepsilon z)$. For a function $f \in L^p(\Omega)$, $|f|_p$ still denotes the L^p-norm in Ω, with similar notations for the other norms. Moreover, the regularizing term $\eta\Delta\boldsymbol{\sigma}^{\varepsilon\eta}$ is introduced. Denoting $\boldsymbol{\sigma}^{\varepsilon\eta} = \begin{pmatrix} \sigma_{11}^{\varepsilon\eta} & \sigma_{12}^{\varepsilon\eta} \\ \sigma_{12}^{\varepsilon\eta} & \sigma_{22}^{\varepsilon\eta} \end{pmatrix}$, and similar notations for the components of $\boldsymbol{G}$, the problem reads, in $(0,\infty) \times \Omega$:

$$\begin{cases} \rho\ \delta_t u_1^{\varepsilon\eta} - (1-r)\nu\,\Delta_\varepsilon u_1^{\varepsilon\eta} + \partial_x p^{\varepsilon\eta} - \partial_x \sigma_{11}^{\varepsilon\eta} - \dfrac{1}{\varepsilon}\partial_z \sigma_{12}^{\varepsilon\eta} = 0\,, \\ \rho\ \delta_t u_2^{\varepsilon\eta} - (1-r)\nu\,\Delta_\varepsilon u_2^{\varepsilon\eta} + \dfrac{1}{\varepsilon}\partial_z p^{\varepsilon\eta} - \partial_x \sigma_{12}^{\varepsilon\eta} - \dfrac{1}{\varepsilon}\partial_z \sigma_{22}^{\varepsilon\eta} = 0\,, \\ \nabla_\varepsilon \cdot \boldsymbol{u}^{\varepsilon\eta} = 0\,, \\ \lambda\ \left(\delta_t \sigma_{11}^{\varepsilon\eta} - \tilde{N}(\boldsymbol{u}^{\varepsilon\eta}, \sigma_{12}^{\varepsilon\eta})\right) + \sigma_{11}^{\varepsilon\eta} - \eta\Delta_\varepsilon(\sigma_{11}^{\varepsilon\eta} - G_{11}) - 2r\nu\partial_x u_1^{\varepsilon\eta} = 0\,, \\ \lambda\ \left(\delta_t \sigma_{12}^{\varepsilon\eta} + \dfrac{1}{2}\tilde{N}(\boldsymbol{u}^{\varepsilon\eta}, \sigma_{11}^{\varepsilon\eta} - \sigma_{22}^{\varepsilon\eta})\right) + \sigma_{12}^{\varepsilon\eta} - \eta\Delta_\varepsilon(\sigma_{12}^{\varepsilon\eta} - G_{12}) - r\nu\left(\partial_x u_2^{\varepsilon\eta} + \dfrac{1}{\varepsilon}\partial_z u_1^{\varepsilon\eta}\right) = 0\,, \\ \lambda\ \left(\delta_t \sigma_{22}^{\varepsilon\eta} + \tilde{N}(\boldsymbol{u}^{\varepsilon\eta}, \sigma_{12}^{\varepsilon\eta})\right) + \sigma_{22}^{\varepsilon\eta} - \eta\Delta_\varepsilon(\sigma_{22}^{\varepsilon\eta} - G_{22}) - 2r\nu\dfrac{1}{\varepsilon}\partial_z u_2^{\varepsilon\eta} = 0\,, \end{cases} \tag{2.5}$$

where the convective derivative δ_t is given by $\delta_t = \partial_t + \boldsymbol{u}^{\varepsilon\eta} \cdot \nabla_\varepsilon$. The derivation operators are defined as follows: $\nabla_\varepsilon = \left(\partial_x, \frac{1}{\varepsilon}\partial_z\right)$ and $\Delta_\varepsilon = \partial_x^2 + \frac{1}{\varepsilon^2}\partial_z^2$. The non-linear terms $\tilde{N}$ are given by $\tilde{N}(\boldsymbol{u}, f) = \left(\partial_x u_2 - \frac{1}{\varepsilon}\partial_z u_1\right) f$.

2.3.2 Introduction of the asymptotic expansions

It has been proposed in [BCM07] that when η, ε tend to zero, $(\boldsymbol{u}^{\varepsilon\eta}, p^{\varepsilon\eta}, \boldsymbol{\sigma}^{\varepsilon\eta})$ tends formally to a triplet $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$ satisfying a system that will be given later in (2.10). This analysis leads to the introduction of the following asymptotic expansions:

$$u_1^{\varepsilon\eta} = u_1^* + v_1^{\varepsilon\eta} \text{ and } u_2^{\varepsilon\eta} = \varepsilon u_2^* + \varepsilon v_2^{\varepsilon\eta}, \tag{2.6}$$

$$p^{\varepsilon\eta} = \frac{1}{\varepsilon^2}p^* + \frac{1}{\varepsilon^2}q^{\varepsilon\eta}, \tag{2.7}$$

$$\boldsymbol{\sigma}^{\varepsilon\eta} = \frac{1}{\varepsilon}\boldsymbol{\sigma}^* + \frac{1}{\varepsilon}\boldsymbol{\tau}^{\varepsilon\eta}, \tag{2.8}$$

with $\boldsymbol{\sigma}^* = \begin{pmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{pmatrix}$, and $\boldsymbol{\tau}^{\varepsilon\eta} = \begin{pmatrix} \tau_{11}^{\varepsilon\eta} & \tau_{12}^{\varepsilon\eta} \\ \tau_{12}^{\varepsilon\eta} & \tau_{22}^{\varepsilon\eta} \end{pmatrix}$. If denoting $\boldsymbol{u}^* = (u_1^*, u_2^*)$, and $\boldsymbol{v}^{\varepsilon\eta} = (v_1^{\varepsilon\eta}, v_2^{\varepsilon\eta})$, (2.6) becomes $\boldsymbol{u}^{\varepsilon\eta} = \boldsymbol{u}^* + \boldsymbol{v}^{\varepsilon\eta}$.

The scaling orders chosen for the pressure and the different components of the velocity field and of the stress tensor are motivated by some mathematical and physical remarks. Classically, the pressure has to be of order $1/\varepsilon^2$ if the horizontal velocity is of order 1 (see [BC86] for the rigorous explanation). On the other hand, the stress tensor has to be of order $1/\varepsilon$ and the Deborah number λ of order ε in order to balance the Newtonian and non-Newtonian contribution in Oldroyd equation (see [BCM07]). Hence; let $\lambda = \varepsilon\lambda^*$.

A wise choice of the function $\boldsymbol{G}$ in the regularizing term is $\boldsymbol{G} = \boldsymbol{\sigma}^*$. The regularity of $\boldsymbol{G}$ in $\boldsymbol{H}^2(\Omega)$ is proved by Theorem 2.7 (where it is proved that $\partial_x^2\boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{C}}^0(\bar{\Omega})$, $\partial_x\partial_z\boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{C}}^0(\bar{\Omega})$ and $\partial_z^2\boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{C}}^1(\bar{\Omega})$, thus $\Delta\boldsymbol{\sigma}^* \in \boldsymbol{L}^2(\Omega)$). A formal substitution of (2.6), (2.7), (2.8) into (2.5) leads to the following system:

$$\begin{cases} \rho\ \mathrm{d}_t v_1^{\varepsilon\eta} - (1-r)\nu\,\Delta_\varepsilon v_1^{\varepsilon\eta} + \dfrac{1}{\varepsilon^2}\partial_x q^{\varepsilon\eta} - \dfrac{1}{\varepsilon}\partial_x \tau_{11}^{\varepsilon\eta} - \dfrac{1}{\varepsilon^2}\partial_z \tau_{12}^{\varepsilon\eta} = \tilde{L}_1^{\varepsilon\eta} + \dfrac{1}{\varepsilon}C_1 + \dfrac{1}{\varepsilon^2}C_1', \\ \rho\ \mathrm{d}_t v_2^{\varepsilon\eta} - (1-r)\nu\,\Delta_\varepsilon v_2^{\varepsilon\eta} + \dfrac{1}{\varepsilon^4}\partial_z q^{\varepsilon\eta} - \dfrac{1}{\varepsilon^2}\partial_x \tau_{12}^{\varepsilon\eta} - \dfrac{1}{\varepsilon^3}\partial_z \tau_{22}^{\varepsilon\eta} = \dfrac{1}{\varepsilon^2}\tilde{L}_2^{\varepsilon\eta} + \dfrac{1}{\varepsilon^3}C_2 + \dfrac{1}{\varepsilon^4}C_2', \\ \nabla\cdot \boldsymbol{v}^{\varepsilon\eta} = \nabla\cdot\boldsymbol{u}^*, \\ \lambda^*\ \left(\mathrm{d}_t\tau_{11}^{\varepsilon\eta} - N(\boldsymbol{v}^{\varepsilon\eta},\tau_{12}^{\varepsilon\eta})\right) + \dfrac{1}{\varepsilon}\tau_{11}^{\varepsilon\eta} - \eta\Delta_\varepsilon\tau_{11}^{\varepsilon\eta} - 2r\nu\partial_x v_1^{\varepsilon\eta} = \tilde{L}_{11}^{\varepsilon\eta} + \dfrac{1}{\varepsilon}\tilde{L}_{11}'^{\varepsilon\eta}, \\ \lambda^*\ \left(\mathrm{d}_t\tau_{12}^{\varepsilon\eta} + \dfrac{1}{2}N(\boldsymbol{v}^{\varepsilon\eta},\tau_{11}^{\varepsilon\eta}-\tau_{22}^{\varepsilon\eta})\right) + \dfrac{1}{\varepsilon}\tau_{12}^{\varepsilon\eta} - \eta\Delta_\varepsilon\tau_{12}^{\varepsilon\eta} - r\nu\left(\partial_x v_2^{\varepsilon\eta} + \dfrac{1}{\varepsilon}\partial_z v_1^{\varepsilon\eta}\right) = \tilde{L}_{12}^{\varepsilon\eta} + \dfrac{1}{\varepsilon}\tilde{L}_{12}'^{\varepsilon\eta}, \\ \lambda^*\ \left(\mathrm{d}_t\tau_{22}^{\varepsilon\eta} + N(\boldsymbol{v}^{\varepsilon\eta},\tau_{12}^{\varepsilon\eta})\right) + \dfrac{1}{\varepsilon}\tau_{22}^{\varepsilon\eta} - \eta\Delta_\varepsilon\tau_{22}^{\varepsilon\eta} - \dfrac{2r\nu}{\varepsilon}\partial_z v_2^{\varepsilon\eta} = \tilde{L}_{22}^{\varepsilon\eta} + \dfrac{1}{\varepsilon}\tilde{L}_{22}'^{\varepsilon\eta}, \end{cases} \tag{2.9}$$

with the following notations: $\mathrm{d}_t = \partial_t + \boldsymbol{v}^{\varepsilon\eta}\cdot\nabla$ is the so-called convective derivative, the non-linear terms $N(\boldsymbol{v}^{\varepsilon\eta}, f) = \left(\varepsilon\partial_x v_2^{\varepsilon\eta} - \dfrac{1}{\varepsilon}\partial_z v_1^{\varepsilon\eta}\right) f$ for $f\in L^2(\Omega)$ and the following linear (with respect to $\boldsymbol{v}^{\varepsilon\eta}$) and constant terms

$$\begin{aligned} \tilde{L}_1^{\varepsilon\eta} &= \underbrace{-\rho\ \boldsymbol{v}^{\varepsilon\eta}\cdot\nabla u_1^* - \rho\ \boldsymbol{u}^*\cdot\nabla v_1^{\varepsilon\eta}}_{\mathcal{L}_1^{\varepsilon\eta}} - \rho\ \partial_t u_1^* - \rho\ \boldsymbol{u}^*\cdot\nabla u_1^* + (1-r)\nu\partial_x^2 u_1^*, \\ C_1 &= \partial_x\sigma_{11}^*, \\ C_1' &= (1-r)\nu\partial_z^2 u_1^* - \partial_x p^* + \partial_z\sigma_{12}^*; \\ \tilde{L}_2^{\varepsilon\eta} &= \underbrace{-\rho\ \varepsilon^2\boldsymbol{v}^{\varepsilon\eta}\cdot\nabla u_2^* - \rho\ \varepsilon^2\boldsymbol{u}^*\cdot\nabla v_2^{\varepsilon\eta}}_{\mathcal{L}_2^{\varepsilon\eta}} \\ &\qquad - \rho\ \varepsilon^2\partial_t u_2^* - \rho\ \varepsilon^2\boldsymbol{u}^*\cdot\nabla u_2^* + \varepsilon^2(1-r)\nu\partial_x^2 u_2^* + (1-r)\nu\partial_z^2 u_2^* + \partial_x\sigma_{12}^*, \\ C_2 &= \partial_z\sigma_{22}^*, \\ C_2' &= \partial_z p^*. \end{aligned}$$

For the Oldroyd equation, the following linear (with respect to $\boldsymbol{v}$ and $\boldsymbol{\tau}$) and constant terms appear:

$$\begin{aligned} \tilde{L}_{11}^{\varepsilon\eta} =& \mathcal{L}_{11}^{\varepsilon\eta} + \lambda^*\left(-\partial_t\sigma_{11}^* - \boldsymbol{u}^*\cdot\nabla\sigma_{11}^* + \varepsilon\partial_x u_2^*\sigma_{12}^*\right) + 2r\nu\partial_x u_1^*, \\ \text{with}\quad & \mathcal{L}_{11}^{\varepsilon\eta} = \lambda^*\left(\varepsilon\partial_x u_2^*\tau_{12}^{\varepsilon\eta} + \varepsilon\partial_x v_2^{\varepsilon\eta}\sigma_{12}^* - \boldsymbol{v}^{\varepsilon\eta}\cdot\nabla\sigma_{11}^* - \boldsymbol{u}^*\cdot\nabla\tau_{11}^{\varepsilon\eta}\right), \\ \tilde{L}_{11}'^{\varepsilon\eta} =& \underbrace{-\lambda^*\left(\partial_z u_1^*\tau_{12}^{\varepsilon\eta} + \partial_z v_1^{\varepsilon\eta}\sigma_{12}^*\right)}_{\mathcal{L}_{11}'^{\varepsilon\eta}} - \lambda^*\partial_z u_1^*\sigma_{12}^* - \sigma_{11}^*; \\ \tilde{L}_{22}^{\varepsilon\eta} =& \mathcal{L}_{22}^{\varepsilon\eta} - \lambda^*\left(\partial_t\sigma_{22}^* + \boldsymbol{u}^*\cdot\nabla\sigma_{22}^* + \varepsilon\partial_x u_2^*\sigma_{12}^*\right) + 2r\nu\partial_z u_2^*, \\ \text{with}\quad & \mathcal{L}_{22}^{\varepsilon\eta} = -\lambda^*\left(\varepsilon\partial_x u_2^*\tau_{12}^{\varepsilon\eta} + \varepsilon\partial_x v_2\sigma_{12}^* + \boldsymbol{v}^{\varepsilon\eta}\cdot\nabla\sigma_{22}^* + \boldsymbol{u}^*\cdot\nabla\tau_{22}^{\varepsilon\eta}\right), \end{aligned}$$

$$\tilde{L}'^{\varepsilon\eta}_{22} = \underbrace{\lambda^* \left(\partial_z u_1^* \tau_{12}^{\varepsilon\eta} + \partial_z v_1^{\varepsilon\eta} \sigma_{12}^*\right)}_{\mathcal{L}'^{\varepsilon\eta}_{22}} + \lambda^* \partial_z u_1^* \sigma_{12}^* - \sigma_{22}^*$$

$$\tilde{L}^{\varepsilon\eta}_{12} = \underbrace{-\frac{\lambda^*}{2} \left(\varepsilon \partial_x u_2^* (\tau_{11}^{\varepsilon\eta} - \tau_{22}^{\varepsilon\eta}) + \varepsilon \partial_x v_2^{\varepsilon\eta} (\sigma_{11}^* - \sigma_{22}^*) + 2\boldsymbol{v}^{\varepsilon\eta} \cdot \nabla \sigma_{12}^* + 2\boldsymbol{u}^* \cdot \nabla \tau_{12}^{\varepsilon\eta}\right)}_{\mathcal{L}^{\varepsilon\eta}_{12}}$$
$$- \frac{\lambda^*}{2} \left(2\partial_t \sigma_{12}^* + 2\boldsymbol{u}^* \cdot \nabla \sigma_{12}^* + \partial_x u_2^* (\sigma_{11}^* - \sigma_{22}^*)\right) + r\nu\varepsilon \partial_x u_2^*,$$

$$\tilde{L}'^{\varepsilon\eta}_{12} = \underbrace{-\frac{\lambda^*}{2} \left(\partial_z u_1^* (\tau_{11}^{\varepsilon\eta} - \tau_{22}^{\varepsilon\eta}) + \partial_z v_1^{\varepsilon\eta} (\sigma_{11}^* - \sigma_{22}^*)\right)}_{\mathcal{L}'^{\varepsilon\eta}_{12}} + \frac{\lambda^*}{2} \partial_z u_1^* (\sigma_{11}^* - \sigma_{22}^*) - \sigma_{12}^* + r\nu \partial_z u_1^*;$$

Note that the first order derivatives of $\boldsymbol{\sigma}^*$ occur in the terms $\tilde{L}^{\varepsilon\eta}$ and $C^{\varepsilon\eta}$. It will be shown in Theorem 2.7 that $\boldsymbol{\sigma}^*$ has sufficient regularity.

Let us observe also that equations (2.9) are similar to (2.5), except for the linear terms on the right. Thus the energy estimates will be obtained similarly for both systems, multiplying Navier-Stokes equation by the velocity and Oldroyd equation by the stress tensor, and integrating over Ω.

2.4 Limit equations

2.4.1 Limit system

In an heuristic way, the following system of equations satisfied by $\boldsymbol{u}^*$, p^*, $\boldsymbol{\sigma}^*$ is inferred from (2.9): $\boldsymbol{u}^*$, p^*, $\boldsymbol{\sigma}^*$ are steady-state functions solutions of:

$$\begin{cases} (1-r)\nu \partial_z^2 u_1^* - \partial_x p^* + \partial_z \sigma_{12}^* = 0, \\ \partial_z p^* = 0, \\ \nabla \cdot \boldsymbol{u}^* = 0, \\ \lambda^* \partial_z u_1^* \sigma_{12}^* + \sigma_{11}^* = 0, \\ -\dfrac{\lambda^*}{2} \partial_z u_1^* (\sigma_{11}^* - \sigma_{22}^*) + \sigma_{12}^* = r\nu \partial_z u_1^*, \\ -\lambda^* \partial_z u_1^* \sigma_{12}^* + \sigma_{22}^* = 0. \end{cases} \tag{2.10}$$

This system is equipped with the following boundary conditions (Dirichlet condition on the upper and lower part of the boundary, flux imposed on the lateral part of the boundary):

$$\begin{cases} \boldsymbol{u}^* = 0\,, & \text{for } z = h(x), \\ \boldsymbol{u}^* = (s,0)\,, & \text{for } z = 0, \\ \int\limits_0^{h(x)} \boldsymbol{u}^* \,\mathrm{d}z \cdot n = \Phi_0 & \text{on } \Gamma_L. \end{cases} \tag{2.11}$$

The compatibility condition reads $\int\limits_{\partial\omega} \Phi_0 = 0$. Moreover, since p^* is defined up to a constant, the mean pressure is taken to be zero: $\int\limits_{\Omega} p^* = 0$.

Remark 2.4. *Each equation of the previous system (2.10) is obtained by canceling the constant part (i.e. the part independent of $\boldsymbol{v}^{\varepsilon\eta}$, $q^{\varepsilon\eta}$, $\tau^{\varepsilon\eta}$) of respectively C_1', C_2', $\nabla\cdot\boldsymbol{u}^*$, $\tilde{L}_{11}'^{\varepsilon\eta}$, $\tilde{L}_{12}'^{\varepsilon\eta}$, $\tilde{L}_{22}'^{\varepsilon\eta}$.*

2.4.2 Determination of the boundary conditions

Remark 2.5. *The lateral boundary conditions on $\boldsymbol{u}^*$ do not depend on the ones on $\boldsymbol{u}^{\varepsilon\eta}$, but only on the flux. Therefore, different boundary conditions on $\boldsymbol{u}^{\varepsilon\eta}$ corresponding to the same flux lead to the same limit problem. This is a classical fact when passing from a two-dimensional problem to a one-dimensional problem (or similarly from a three-dimensional problem to a two-dimensional one), and has already been observed in [BC86] for example. Here, in order to avoid boundary layers, $\boldsymbol{u}^{\varepsilon\eta} = \boldsymbol{u}^*$ is imposed on the lateral part of the boundary.*
Similarly, any value of $\boldsymbol{\sigma}^{\varepsilon\eta}$ on the boundary leads to the same limit problem. Again, in order to avoid boundary layers, well-prepared boundary conditions are also chosen for $\boldsymbol{\sigma}^{\varepsilon\eta}$.

The previous remark allows us to define precisely the function $\boldsymbol{J}^\varepsilon$ introduced in (2.3). Since $\boldsymbol{u}^*|_{\Gamma_L} \in \boldsymbol{H}^{1/2}(\Gamma_L)$, it is possible to construct $\boldsymbol{J}^\varepsilon \in \boldsymbol{H}^{1/2}(\partial\Omega)$ satisfying $\boldsymbol{J}^\varepsilon|_{z=h} = 0$, $\boldsymbol{J}^\varepsilon|_{z=0} = (s,0)$ and $\boldsymbol{J}^\varepsilon|_{\Gamma_L} = \boldsymbol{u}^*|_{\Gamma_L}$. Therefore, the boundary conditions on $\boldsymbol{u}^{\varepsilon\eta}$ become

$$\begin{cases} \boldsymbol{u}^{\varepsilon\eta} = 0\,, & \text{for } z = h(x), \\ \boldsymbol{u}^{\varepsilon\eta} = (s,0)\,, & \text{for } z = 0, \\ \boldsymbol{u}^{\varepsilon\eta} = \boldsymbol{u}^* & \text{on } \Gamma_L. \end{cases}$$

Thus $\boldsymbol{u}^{\varepsilon\eta}|_{\partial\Omega} = \boldsymbol{u}^*|_{\partial\Omega}$, and $\boldsymbol{v}^{\varepsilon\eta}$ will satisfy zero boundary conditions: $\boldsymbol{v}^{\varepsilon\eta}|_{\partial\Omega} = 0$.
Moreover, since $\boldsymbol{\sigma}^* \in \boldsymbol{H}^1(\Omega)$ (see Theorem 2.7 for this regularity result), $\boldsymbol{\theta}^\varepsilon$ can be defined as follows:

$$\boldsymbol{\theta}^\varepsilon = \boldsymbol{\sigma}^*|_{\Gamma_+} \in \boldsymbol{H}^{1/2}(\Gamma_+). \tag{2.12}$$

Therefore

$$\boldsymbol{\sigma}^{\varepsilon\eta}|_{\Gamma_+} = \boldsymbol{\sigma}^*|_{\Gamma_+},$$

and this implies that $\boldsymbol{\tau}^{\varepsilon\eta}|_{\Gamma_+} = 0$.

On the other part Γ_- of the boundary, $\boldsymbol{\sigma}^{\varepsilon\eta}$ is chosen such that $\boldsymbol{\sigma}^{\varepsilon\eta} \cdot n|_{\Gamma_-} = \boldsymbol{\sigma}^* \cdot n|_{\Gamma_-}$, for example $\boldsymbol{\sigma}^{\varepsilon\eta}|_{\Gamma_-} = \boldsymbol{\sigma}^*|_{\Gamma_-}$.

2.4.3 Existence of a solution to the limit problem

System (2.10)-(2.11) has already been studied in [BCM07].

Theorem 2.6. *Assume that $r < 8/9$. Then system (2.10)-(2.11) has a unique solution satisfying*

$$\boldsymbol{u}^* \in \boldsymbol{L}^2(\Omega), \quad \partial_z \boldsymbol{u}^* \in \boldsymbol{L}^2(\Omega), \quad p^* \in H^1(\omega), \quad \boldsymbol{\sigma}^* \in \boldsymbol{L}^2(\Omega). \tag{2.13}$$

Proof. This result has been proved in [BCM07]. □

This existence result is not sufficient for this study. Therefore, the following stronger regularity result is proved on the limit problem (2.10)-(2.11).

Theorem 2.7. *Assume $r < 2/9$. If $h \in H^k(\omega)$, for $k \in \mathbb{N}^*$, then the unique solution $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$ of the system (2.10)-(2.11) satisfies*

$$\begin{gathered}
p^* \in \mathcal{C}^{k+1}(\bar{\omega}), \quad u_1^*,\, \partial_z u_1^*,\, \partial_z^2 u_1^* \in \mathcal{C}^{k+1}(\bar{\Omega}), \quad \boldsymbol{\sigma}^*,\, \partial_z \boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{C}}^{k+1}(\bar{\Omega}), \\
\partial_x u_1^* \in \mathcal{C}^{k}(\bar{\Omega}), \quad u_2^*,\, \partial_z u_2^*, \partial_z^2 u_2^* \in \mathcal{C}^{k}(\bar{\Omega}), \quad \partial_x \boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{C}}^{k}(\bar{\Omega}), \\
\partial_x u_2^* \in \mathcal{C}^{k-1}(\bar{\Omega}).
\end{gathered} \tag{2.14}$$

Proof. Let us observe that system (2.10) can be expressed as a system on u_1^*, p^* only. Using (2.10), σ_{11}^*, σ_{22}^* can be expressed as functions of σ_{12}^* and $\partial_z u_1^*$. Indeed, from the fourth and the last equations of (2.10), it follows that

$$\sigma_{22}^* = -\sigma_{11}^* = \lambda^* \partial_z u_1^* \sigma_{12}^*. \tag{2.15}$$

Moreover, the divergence equation can be rewritten in order to eliminate u_2^*. Integrating this equation between $z = 0$ and $z = h$, and using the fact that $u_2^*|_{z=0} = u_2^*|_{z=h} = u_1^*|_{z=h} = 0$, it follows:

$$\partial_x \left(\int_0^h u_1^* \,\mathrm{d}z \right) = 0. \tag{2.16}$$

Thus, the system in u_1^*, p^* can be written in the following form:

$$\begin{cases} -\nu(1-r)\partial_z^2 u_1^* - \partial_z \sigma_{12}^* + \partial_x p^* = 0, \qquad \text{with} \quad \sigma_{12}^* = \dfrac{\nu r \partial_z u_1^*}{1+\lambda^{*2}|\partial_z u_1^*|^2}, \\ \partial_z p^* = 0, \\ \partial_x \left(\displaystyle\int_0^h u_1^* \, \mathrm{d}z \right) = 0, \end{cases} \tag{2.17}$$

equipped with the boundary conditions stated in (2.11) and the condition $\int\limits_\Omega p^* = 0$.

For the sake of readability, the superscripts * are omitted in the rest of this section.

Denote $q = \partial_x p$. Let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ defined by $\phi(t) = \nu(1-r)t + \dfrac{\nu r t}{1+\lambda^2 t^2}$. The first equation of (2.17) becomes $q = \partial_z(\phi(\partial_z u_1))$.

A simple study of function ϕ allows us to show the following properties:

$$0 < \nu \left(1 - \frac{9r}{8}\right) < |\phi'|_\infty < \nu, \qquad \text{and} \qquad \phi(t) \xrightarrow[t \to \pm\infty]{} \pm\infty. \tag{2.18}$$

Therefore the function ϕ is invertible, and $\psi = \phi^{-1}$ belongs to $\mathcal{C}^\infty(\mathbb{R})$. Moreover, ψ is an increasing function as ϕ. After integrating $q = \partial_z(\phi(\partial_z u_1))$ with respect to z between 0 and z, the first equation of (2.17) becomes:

$$\phi(\partial_z u_1(x,z)) = q(x)\, z + \kappa(x),$$

where $\kappa(x)$ is an integration constant. Therefore, it follows that

$$\partial_z u_1(x,z) = \psi(q(x)\, z + \kappa(x)).$$

Since $u_1|_{z=0} = s$, the integration between 0 and z of the previous equation yields:

$$u_1(x,z) = s + \int_0^z \psi(q(x)t + \kappa(x))\mathrm{d}t. \tag{2.19}$$

The boundary condition $u_1|_{z=h(x)} = 0$ implies also:

$$\int_0^{h(x)} \psi(q(x)t + \kappa(x)) + s = 0. \tag{2.20}$$

For $(h,q,s,\kappa) \in \mathbb{R}^4$, let us introduce $F(h,q,s,\kappa) = \displaystyle\int_0^h \psi(qt+\kappa) + s$.

Lemma 2.8. *For any* $(h, q, s) \in \mathbb{R}^3$ *there exists a unique* $\kappa \in \mathbb{R}$ *such that*

$$F(h, q, s, \kappa) = 0.$$

Proof.
- If such a κ exists, it is unique from the implicit function theorem, since for all $(h, q, s, \kappa) \in \mathbb{R}^4$

$$\frac{\partial F}{\partial \kappa}(h, q, s, \kappa) = \int_0^h \psi'(qt + \kappa)\mathrm{d}t > 0.$$

- The following limits are computed, using the fact that $\lim_{t \to \pm\infty} \psi(t) = \pm\infty$:

$$\lim_{\kappa \to +\infty} F(h, q, s, \kappa) = +\infty \quad \text{and} \quad \lim_{\kappa \to -\infty} F(h, q, s, \kappa) = -\infty.$$

  Therefore, there exists $\kappa \in \mathbb{R}$ such that $F(h, q, s, \kappa) = 0$. Let us denote $K(h, q, s) = \kappa$. By the implicit function theorem, $K \in \mathcal{C}^\infty(\mathbb{R}^3)$.

□

Therefore, the following expression holds true for $(h, q, s) \in \mathbb{R}^3$:

$$F(h, q, s, K(h, q, s)) = 0. \tag{2.21}$$

It is now possible to obtain an information on the sign of $\partial_q K$. Indeed, differentiating the expression (2.21) with respect to q, it follows

$$\partial_q F + \partial_\kappa F \, \partial_q K = 0.$$

For $h > 0$, since $\partial_q F = \displaystyle\int_0^h t\psi'(qt + \kappa)\mathrm{d}t > 0$ and $\partial_a F = \displaystyle\int_0^h \psi'(qt + \kappa)\mathrm{d}t > 0$, we deduce that $\partial_q K$ is strictly negative.

Now, by using equation (2.16) and the expression (2.19) for u, it follows:

$$\int_0^{h(x)} \int_0^z \partial_x \Big(\psi(q(x)t + K(h(x), q(x), s)) \Big) \mathrm{d}t \, \mathrm{d}z = 0.$$

or if changing the direction of integration

$$\int_0^{h(x)} (h(x) - t)\partial_x \Big(\psi(q(x)t + K(h(x), q(x), s)) \Big) \mathrm{d}t = 0.$$

This can be rewritten as

$$\begin{aligned} q'(x) \int_0^{h(x)} (h(x)-t) \Big((t+\partial_q K(h(x),q(x),s)\Big) \psi'\big(q(x)t+K(h(x),q(x),s)\big)\mathrm{d}t \\ = -\int_0^{h(x)} (h(x)-t) \Big(h'(x)\partial_h K(h(x),q(x),s)\Big) \psi'\big(q(x)t+K(h(x),q(x),s)\big)\mathrm{d}t, \end{aligned}$$

which can be seen as an ordinary differential equation in q. Let

$$\begin{aligned} U(x,q) &= \int_0^{h(x)} \Big(h(x)-t\Big) \Big(t+\partial_q K(h(x),q,s)\Big) \psi'\big(qt+K(h(x),q,s)\big)\mathrm{d}t, \\ V(x,q) &= \int_0^{h(x)} \Big(h(x)-t\Big) \Big(h'(x)\partial_h K(h(x),q,s)\Big) \psi'\big(qt+K(h(x),q,s)\big)\mathrm{d}t. \end{aligned}$$

The differential equation becomes $U(x,q(x))\,q'(x) = -V(x,q(x))$ for $x \in \omega$. Note that this equation is in some sense a generalized Reynolds equation for the pressure.

Lemma 2.9. *Let $r < 2/9$. Then $U(x,q) < 0$ for any $(x,q) \in \omega \times \mathbb{R}$.*

Proof. Let $(x,q) \in \omega \times \mathbb{R}$. Equation (2.20) and definition (2.21) of K imply:

$$\int_0^{h(x)} \psi(qt+K(h(x),q,s))\mathrm{d}t = -s,$$

which becomes, after derivation with respect to q,

$$\int_0^{h(x)} \Big(t+\partial_q K(h(x),q,s)\Big)\psi'\big(qt+K(h(x),q,s)\big)\mathrm{d}t = 0. \tag{2.22}$$

With the notation $K'(x,q) = \partial_q K(h(x),q,s)$, (2.22) implies

$$K'(x,q) = -\frac{\displaystyle\int_0^{h(x)} t\,\psi'\big(qt+K(h(x),q,s)\big)\mathrm{d}t}{\displaystyle\int_0^{h(x)} \psi'\big(qt+K(h(x),q,s)\big)\mathrm{d}t}.$$

Now, using this expression, $U(x,q)$ can be simplified:

$$U(x,q) = \int_0^{h(x)} -t\Big(t+\partial_q K(h(x),q,s)\Big) \psi'\big(qt+K(h(x),q,s)\big)\mathrm{d}t. \tag{2.23}$$

Recalling the estimate of $|\phi|_\infty$ in (2.18), it follows that for any $t \in \mathbb{R}$:

$$\frac{1}{\nu} < \psi'(t) = \frac{1}{\phi'(\psi(t))} < \frac{1}{\nu(1-9r/8)}$$

Let $m = \dfrac{1}{\nu}$, $M = \dfrac{1}{\nu(1-9r/8)}$. Then

$$-\frac{bh(x)}{2m} \leq K'(x,q) \leq -\frac{ah(x)}{2M}.$$

Now, (2.23) implies that:

$$h(x)^3\left(\frac{m}{3}-\frac{M}{4}\right) = \int_0^{h(x)} tm\left(t - \frac{Mh(x)}{2m}\right)$$
$$\leq -U(x,q) \leq \int_0^{h(x)} tM\left(t-\frac{mh(x)}{2M}\right) = h(x)^3\left(\frac{M}{3}-\frac{m}{4}\right).$$

In order to prove that U remains strictly negative, it suffices to prove that $0 < \dfrac{m}{3} - \dfrac{M}{4}$, i.e. that $\dfrac{m}{M} > \dfrac{3}{4}$, which is satisfied under the condition $r < \dfrac{2}{9}$. □

It is possible to apply Picard-Lindelöf theorem (or Cauchy-Lipschitz theorem) to the ordinary differential equation $-U(x,q(x))\,q'(x) = V(x,q(x))$, as U remains strictly negative by Lemma 2.9. Since ψ and K are $\mathcal{C}^\infty$-functions, the regularity of q' is determined by the regularity of q and h. By hypothesis, h belongs to $H^k(\omega)$, with $k \in \mathbb{N}$, hence $h \in L^2(\omega)$. Moreover, Theorem 2.6 implies that $q \in L^2(\omega)$. Thus $q' \in L^2(\omega)$, which means $q \in H^1(\omega)$. Iterating this process as long as h is regular, $h \in H^k(\omega)$ and $q \in H^k(\omega)$ implies that $q' \in H^k(\omega)$, thus $\partial_x p = q \in H^{k+1}(\omega)$, and $p \in H^{k+2}(\omega)$. By the classical Sobolev embedding, p belongs to $\mathcal{C}^{k+1}(\bar{\omega})$. Note that this embedding is true when $n = 2$, since ω is a $(n-1)$-dimensional domain.

At last, recalling the expression (2.19), it follows that $u_1 \in \mathcal{C}^{k+1}(\bar{\omega})$, and, taking the first and second derivatives of (2.19) with respect to z, that $\partial_z u_1$ and $\partial_z^2 u_1$ also belong to $\mathcal{C}^{k+1}(\bar{\omega})$.

As observed in the introduction of the proof, $\boldsymbol{\sigma}$ and u_2 are given as functions of p, u_1, and the needed regularity follows. □

Remark 2.10. *Since in practical applications, h is very regular ($h \in C^\infty(\bar{\omega})$), the previous theorem gives as much regularity as wanted. In particular, the following result will be useful subsequently.*

Corollary 2.11. *Assume $r < 2/9$. If $h \in H^1(\omega)$, then the unique solution $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$ of* *the system* (2.10)-(2.11) *satisfies*

$$
\begin{gathered}
p^* \in \mathcal{C}^2(\bar{\omega}), \quad u_1^*,\, \partial_z u_1^*,\, \partial_z^2 u_1^* \in \mathcal{C}^2(\bar{\Omega}), \quad \boldsymbol{\sigma}^*,\, \partial_z \boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{C}}^2(\bar{\Omega}), \\
\partial_x u_1^* \in \mathcal{C}^1(\bar{\Omega}), \quad u_2^*,\, \partial_z u_2^*, \partial_z^2 u_2^* \in \mathcal{C}^1(\bar{\Omega}), \quad \partial_x \boldsymbol{\sigma}^* \in \boldsymbol{\mathcal{C}}^1(\bar{\Omega}), \\
\partial_x u_2^* \in \mathcal{C}^0(\bar{\Omega}).
\end{gathered} \tag{2.24}
$$

Proof. It suffices to take $k = 1$ in theorem 2.7. □

2.5 Convergence of the remainders

2.5.1 Equations on the remainders

From now on, the superscripts ${}^{\varepsilon\eta}$ are dropped although the functions still depend on ε and η. Using the equations (2.10), system (2.9) becomes

$$\rho\, \mathrm{d}_t v_1 - (1-r)\nu\, \Delta_\varepsilon v_1 + \frac{1}{\varepsilon^2}\partial_x q - \frac{1}{\varepsilon}\partial_x \tau_{11} - \frac{1}{\varepsilon^2}\partial_z \tau_{12} = L_1 + \frac{1}{\varepsilon} C_1, \tag{2.25a}$$

$$\rho\, \mathrm{d}_t v_2 - (1-r)\nu\, \Delta_\varepsilon v_2 + \frac{1}{\varepsilon^4}\partial_x q - \frac{1}{\varepsilon^2}\partial_x \tau_{12} - \frac{1}{\varepsilon^3}\partial_z \tau_{22} = \frac{1}{\varepsilon^2} L_2 + \frac{1}{\varepsilon^3} C_2, \tag{2.25b}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2.25c}$$

$$\lambda^* \mathrm{d}_t \tau_{11} - \lambda^* N(\boldsymbol{v}, \tau_{12}) + \frac{1}{\varepsilon}\tau_{11} - \eta \Delta_\varepsilon \tau_{11} - 2r\nu \partial_x v_1 = L_{11} + \frac{1}{\varepsilon} L'_{11} + \eta \Delta_\varepsilon \sigma_{11}^*, \tag{2.25d}$$

$$\lambda^* \mathrm{d}_t \tau_{12} + \frac{\lambda^*}{2} N(\boldsymbol{v}, \tau_{11} - \tau_{22}) + \frac{1}{\varepsilon}\tau_{12} - \eta \Delta_\varepsilon \tau_{12} - r\nu \left(\partial_x v_2 + \frac{1}{\varepsilon}\partial_z v_1 \right) = L_{12} + \frac{1}{\varepsilon} L'_{12} + \eta \Delta_\varepsilon \sigma_{12}^*, \tag{2.25e}$$

$$\lambda^* \mathrm{d}_t \tau_{22} + \lambda^* N(\boldsymbol{v}, \tau_{12}) + \frac{1}{\varepsilon}\tau_{22} - \eta \Delta_\varepsilon \tau_{22} - \frac{2r\nu}{\varepsilon}\partial_z v_2 = L_{22} + \frac{1}{\varepsilon} L'_{22} + \eta \Delta_\varepsilon \sigma_{22}^*, \tag{2.25f}$$

with the new quantities

$$
\begin{aligned}
L_1 &= \mathcal{L}_1 - \rho\, \boldsymbol{u}^* \cdot \nabla u_1^* + (1-r)\nu \partial_x^2 u_1^*, \\
L_2 &= \mathcal{L}_2 - \rho\, \varepsilon^2 \boldsymbol{u}^* \cdot \nabla u_2^* + (1-r)\nu \partial_x^2 u_2^* + (1-r)\nu \partial_z u_2^* + \partial_x \sigma_{12}^*, \\
L_{11} &= \mathcal{L}_{11} + \lambda^* \left(-\boldsymbol{u}^* \cdot \nabla \sigma_{11}^* + \varepsilon \partial_x u_2^* \sigma_{12}^* \right) + 2r\nu \partial_x u_1^*, \\
L'_{11} &= \mathcal{L}'_{11}, \\
L_{12} &= \mathcal{L}_{12} - \frac{\lambda^*}{2} \left(2\boldsymbol{u}^* \cdot \nabla \sigma_{12}^* + \partial_x u_2^* (\sigma_{11}^* - \sigma_{22}^*) \right) + r\nu\varepsilon \partial_x u_2^*, \\
L'_{12} &= \mathcal{L}'_{12}, \\
L_{22} &= \mathcal{L}_{22} - \lambda^* \left(\boldsymbol{u}^* \cdot \nabla \sigma_{22}^* + \varepsilon \partial_x u_2^* \sigma_{12}^* \right) + 2r\nu \partial_z u_2^*, \\
L'_{22} &= \mathcal{L}'_{22}.
\end{aligned}
$$

and with the initial and boundary conditions

$$\boldsymbol{v}|_{t=0} = \boldsymbol{u}_0 - \boldsymbol{u}^*, \qquad \boldsymbol{\tau}|_{t=0} = \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}^*, \qquad \boldsymbol{v}|_{\partial\Omega} = 0, \qquad \boldsymbol{\tau}|_{\Gamma_+} = 0. \tag{2.26}$$

Let us observe that both initial conditions $\boldsymbol{v}|_{t=0}$ and $\boldsymbol{\tau}|_{t=0}$ belong to $\boldsymbol{L}^2(\Omega)$. $\boldsymbol{v}$, q and $\boldsymbol{\tau}$ are defined by (2.6), (2.7), (2.8). From the existence theorem 2.1 for $(\boldsymbol{u}, p, \boldsymbol{\sigma})$ and theorem 2.6 for $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$, it follows that system (2.25) admits a solution $(\boldsymbol{v}, q, \boldsymbol{\tau}) \in L^2_{\text{loc}}(0, \infty, \boldsymbol{H}^1(\Omega)) \times L^2_{\text{loc}}(0, \infty, L^2(\Omega)) \times \mathcal{C}(0, \infty, \boldsymbol{L}^2(\Omega))$ for $r < 8/9$.

2.5.2 Convergence of the velocity and the stress tensor

Before starting the *a priori* estimates, let us explain how the non-linear terms in (2.25) are handled. The non-linear terms $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ of Navier-Stokes equation and $\boldsymbol{v} \cdot \nabla \boldsymbol{\tau}$ of Oldroyd equation are treated with the following Lemma 2.12. On the other hand, the non-linear terms $N(\boldsymbol{v}, \boldsymbol{\tau}) = \left(\varepsilon \partial_x v_2 - \frac{1}{\varepsilon} \partial_z v_1\right) \boldsymbol{\tau}$ in (2.25d)-(2.25f) are zero when multiplied by $\boldsymbol{\tau}$ and integrated over Ω.

Lemma 2.12. *Let $\boldsymbol{n}$ be the exterior normal of the domain Ω. Let $\boldsymbol{\phi} \in \boldsymbol{H}^1(\Omega)$ be a vector field satisfying $\nabla \cdot \boldsymbol{\phi} = 0$ and $\boldsymbol{\phi} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$. Let $w \in H^1(\Omega)$. Then*

$$\int_\Omega \boldsymbol{\phi} \cdot \nabla w \, w = 0.$$

Proof. By integration by parts:

$$\int_\Omega \boldsymbol{\phi} \cdot \nabla w \, w = -\int_\Omega \underbrace{\nabla \cdot \boldsymbol{\phi}}_{=0} \cdot w^2 - \int_\Omega \boldsymbol{\phi} \cdot \nabla w \, w + \int_{\partial\Omega} \underbrace{\boldsymbol{\phi} \cdot \boldsymbol{n}}_{=0} \, w^2 = 0.$$

□

The classical approach consists in obtaining *a priori* estimates for $\boldsymbol{v}$.

Proposition 2.13. *Let $(\boldsymbol{v}, q, \boldsymbol{\tau})$ be a solution of (2.25). Then $\boldsymbol{v} = (v_1,\, v_2)$ satisfy the following inequality for ε small enough:*

$$r\nu\rho \frac{d}{dt}\left(|v_1|^2 + |\varepsilon v_2|^2\right) + \frac{3}{2} r(1-r)\nu^2 \left(|\nabla_\varepsilon v_1|^2 + |\varepsilon \nabla_\varepsilon v_2|^2\right) \leq -\mathcal{D}_1 - \mathcal{D}_2 + C, \tag{2.27}$$

where $\mathcal{D}_1 = \dfrac{2r\nu}{\varepsilon} \displaystyle\int_\Omega \tau_{11}\, \partial_x v_1 + \dfrac{2r\nu}{\varepsilon^2} \displaystyle\int_\Omega \tau_{12}\, \partial_z v_1$, $\mathcal{D}_2 = 2r\nu \displaystyle\int_\Omega \tau_{12}\, \partial_x v_2 + \dfrac{2r\nu}{\varepsilon} \displaystyle\int_\Omega \tau_{22}\, \partial_z v_2$ *and C is a constant independent of ε.*

Proof. The proof consists in obtaining classical *a priori* estimates on both v_1 and v_2.

Step 1. Let us multiply (2.25a) by v_1 and integrate over Ω. Observe that v_1 is regular enough to do so. Since $\boldsymbol{v}|_{\partial\Omega} = 0$, the boundary terms in the integration by parts are all zero. For example $-\int\limits_\Omega \Delta_\varepsilon v_1\, v_1 = \int\limits_\Omega |\nabla_\varepsilon v_1|^2$. Moreover, the convection terms $\int\limits_\Omega \boldsymbol{v}\cdot\nabla v_1\, v_1$ contained in $\int\limits_\Omega \mathrm{d}_t v_1\, v_1$ are equal to zero by Lemma 2.12, since $\nabla\cdot\boldsymbol{v} = 0$ and $\boldsymbol{v}|_{\partial\Omega} = 0$. It follows:

$$\begin{aligned}\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_1|^2 + (1-r)\nu|\nabla_\varepsilon v_1|^2 - \frac{1}{\varepsilon^2}\int\limits_\Omega q\,\partial_x v_1 \\ = \underbrace{-\frac{1}{\varepsilon}\int\limits_\Omega \tau_{11}\,\partial_x v_1 - \frac{1}{\varepsilon^2}\int\limits_\Omega \tau_{12}\,\partial_z v_1}_{-\mathcal{D}_1/2r\nu} + \int\limits_\Omega L_1\, v_1 + \frac{1}{\varepsilon}\int\limits_\Omega C_1\, v_1.\end{aligned} \tag{2.28}$$

It remains to estimate the terms $\int\limits_\Omega L_1\, v_1$ and $\int\limits_\Omega C_1\, v_1$.

Main idea Estimates of the form: $\int\limits_\Omega L_1\, v_1 + \frac{1}{\varepsilon}\int\limits_\Omega C_1\, v_1 \le C + \kappa_1|\nabla_\varepsilon v_1|^2 + \kappa_2|\partial_z v_2|^2$ will be proved, where C is a constant independent of ε and where the constants κ_1, κ_2 satisfy $\kappa_1, \kappa_2 < (1-r)\nu/4$. These constants will be specified later in the proof.

In the following, C, c_i and M_i will denote some constants independent of ε and η, which might depend on $|\Omega|$, on the physical parameters of the problem and on $\boldsymbol{u}^*$, $\boldsymbol{\sigma}^*$ in sufficiently regular norms.

- Let us estimate first the linear (with respect to $\boldsymbol{v}$) term $\mathcal{L}_1$ of L_1. To this end, the Poincaré inequality is useful: for $f \in L^2(\Omega)$, with $f|_{z=h} = 0$, $|f| \le C_P|\partial_z f|$. The constant C_P only depends on Ω.

  $\star$ $\rho\int\limits_\Omega v_1\,\partial_x u_1^*\, v_1 \le \rho|\partial_x u_1^*|_\infty\,|v_1|^2 \le \rho\,\varepsilon^2 C_P^2|\partial_x u_1^*|_\infty\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2 =: M_1\varepsilon^2\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2.$

  Note that by Theorem 2.7, $\partial_x u_1^* \in L^\infty(\Omega)$. In the following, all the regularity results used in the estimates also follow from Theorem 2.7.

  $\star$ For the next term, the Poincaré inequality is combined with Young's inequality:

$$\begin{aligned}\rho\int\limits_\Omega v_2\,\partial_z u_1^*\, v_1 &\le \rho|\partial_z u_1^*|_\infty\,|v_2|\,|v_1| \le \rho\, C_P^2|\partial_z u_1^*|_\infty\,|\partial_z v_2|\,|\partial_z v_1| \\ &\le \underbrace{\rho\, C_P^2|\partial_z u_1^*|_\infty}_{=:M_2}\left(\frac{\varepsilon}{2}|\partial_z v_2|^2 + \frac{\varepsilon}{2}\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2\right).\end{aligned}$$

⋆ In a similar way:

$$\rho \int_\Omega \boldsymbol{u}^* \cdot \nabla v_1 \, v_1 \leq \underbrace{\rho \, C_P |u_1^*|_\infty}_{=:M_3} \left(\frac{\varepsilon}{2} |\partial_x v_1|^2 + \frac{\varepsilon}{2} \left| \frac{1}{\varepsilon} \partial_z v_1 \right|^2 \right) + \varepsilon^2 \underbrace{\rho \, C_P |u_2^*|_\infty}_{=:M_4} \left| \frac{1}{\varepsilon} \partial_z v_1 \right|^2 .$$

Observe that it was not possible here to apply Lemma 2.12, since $\boldsymbol{u}^* \cdot \boldsymbol{n}|_{\partial\Omega} \neq 0$.

- It remains the easier terms of L_1 and C_1 (the ones which do not depend on $\boldsymbol{v}$).

⋆ The first term is treated using again Poincaré and Young's inequalities:

$$\begin{aligned} \rho \int_\Omega \boldsymbol{u}^* \cdot \nabla u_1^* \, v_1 &\leq \rho \, C_P |\boldsymbol{u}^*|_\infty \, |\nabla u_1^*| \, |\partial_z v_1| \\ &\leq \frac{1}{2} (\rho \, C_P |\boldsymbol{u}^*|_\infty \, |\nabla u_1^*|)^2 + \frac{\varepsilon^2}{2} \left| \frac{1}{\varepsilon} \partial_z v_1 \right|^2 \leq C + \frac{\varepsilon^2}{2} \left| \frac{1}{\varepsilon} \partial_z v_1 \right|^2 . \end{aligned}$$

⋆ Similarly, $(1-r)\nu \displaystyle\int_\Omega \partial_x^2 u_1^* \, v_1 \leq C + \frac{\varepsilon^2}{2} \left| \frac{1}{\varepsilon} \partial_z v_1 \right|^2 .$

⋆ The last term is estimated as follows, using Young's inequality:

$$\frac{1}{\varepsilon} \int_\Omega \partial_x \sigma_{11}^* \, v_1 \leq \frac{1}{4c} |\partial_x \sigma_{11}^*|^2 + c \left| \frac{1}{\varepsilon} \partial_z v_1 \right|^2 \leq C + c \left| \frac{1}{\varepsilon} \partial_z v_1 \right|^2 ,$$

where c is a positive constant independent of ε that can be chosen arbitrarily.

Now, let us choose ε and c small enough such that all constants satisfy:

$$M_1 \varepsilon^2, \frac{M_2 \varepsilon}{2}, \frac{M_3 \varepsilon}{2}, M_4 \varepsilon^2, \frac{\varepsilon^2}{2}, c \leq \frac{(1-r)\nu}{36}. \tag{2.29}$$

Step 2. Let us multiply (2.25b) by $\varepsilon^2 v_2$ and integrate over Ω. Again, the boundary terms in the integrations by parts vanish, since $v_2|_{\partial\Omega} = 0$, and the convection terms are equal to zero since $\nabla \cdot \boldsymbol{v} = 0$ and $\boldsymbol{v}|_{\partial\Omega} = 0$ (by Lemma 2.12). It follows:

$$\begin{aligned} \frac{\rho \, \varepsilon^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} |v_2|^2 + (1-r)\nu |\varepsilon \nabla_\varepsilon v_2|^2 - \frac{1}{\varepsilon^2} \int_\Omega q \, \partial_z v_2 & \\ = \underbrace{- \int_\Omega \tau_{12} \, \partial_x v_2 - \frac{1}{\varepsilon} \int_\Omega \tau_{22} \, \partial_z v_2}_{-\mathcal{D}_2 / 2r\nu} + \int_\Omega L_2 \, v_2 + \frac{1}{\varepsilon} \int_\Omega C_2 \, v_2 . & \end{aligned} \tag{2.30}$$

Each term of $\int_\Omega L_2 \, v_2$ and $\int_\Omega C_2 \, v_2$ is estimated with the help of Poincaré and Young's inequalities as in the previous step.

$\star$ $\displaystyle \varepsilon^2\rho\int_\Omega \boldsymbol{v}\cdot\nabla u_2^*\, v_2 \leq \varepsilon^2 \underbrace{\rho C_P^2|\partial_x u_2^*|_\infty}_{=:M_5}\left(\frac{\varepsilon}{2}\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2+\frac{\varepsilon}{2}|\partial_z v_2|^2\right)+\varepsilon^2\underbrace{\rho C_P^2|\partial_z u_2^*|_\infty}_{=:M_6}|\partial_z v_2|^2.$

$\star$ $\displaystyle \varepsilon^2\rho\int_\Omega \boldsymbol{u}^*\cdot\nabla v_2\, v_2 \leq \varepsilon \underbrace{\rho C_P|u_1^*|_\infty}_{=:M_7}\left(|\varepsilon\partial_x v_2|^2+|\partial_z v_2|^2\right)+\varepsilon^2\underbrace{\rho C_P|u_2^*|_\infty}_{=:M_8}|\partial_z v_2|^2.$

$\star$ $\displaystyle \varepsilon^2\rho\int_\Omega \boldsymbol{u}^*\cdot\nabla u_2^*\, v_2 \leq \frac{1}{2}\varepsilon^2\rho|\boldsymbol{u}^*|_\infty^2|\nabla u_2^*|^2+\varepsilon^2\underbrace{\frac{1}{2}C_P^2}_{=:M_9}|\partial_z v_2|^2 \leq C+\varepsilon^2 M_9|\partial_z v_2|^2.$

$\star$ By integration by parts (all boundary terms are equal to zero since $v_2|_{\partial\Omega}=0$) and Young inequality as before:

$$(1-r)\nu\varepsilon^2\int_\Omega \partial_x^2 u_2^*\, v_2 = -(1-r)\nu\varepsilon^2\int_\Omega \partial_x u_2^*\,\partial_x v_2 \leq \varepsilon\underbrace{(1-r)\nu}_{=:M_{10}}\left(\frac{1}{2}|\partial_x u_2^*|^2+\frac{1}{2}|\varepsilon\partial_x v_2|^2\right).$$

$\star$ $\displaystyle (1-r)\nu\int_\Omega \partial_z^2 u_2^*\, r_2 \leq \frac{1}{4c_1}(1-r)^2\nu^2C_P^2|\partial_z^2 u_2^*|^2+c_1|\partial_z v_2|^2 \leq C+c_1|\partial_z v_2|^2$, where c_1 is a arbitrary positive constant.

$\star$ $\displaystyle \int_\Omega \partial_x\sigma_{12}^*\, v_2 \leq \frac{C_P^2}{4c_1}|\partial_x\sigma_{12}^*|^2+c_1|\partial_z v_2|^2 \leq C+c_1|\partial_z v_2|^2.$

$\star$ The C_2 term is treated with integration by parts (again, no boundary terms since $v_2|_{\partial\Omega}=v_1|_{\partial\Omega}=0$) and the divergence equation. The term is then treated as the previous one:

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega \partial_z\sigma_{22}^*\, v_2 &= -\frac{1}{\varepsilon}\int_\Omega \sigma_{22}^*\,\partial_z v_2 = \frac{1}{\varepsilon}\int_\Omega \sigma_{22}^*\,\partial_x v_1 = -\frac{1}{\varepsilon}\int_\Omega \partial_x\sigma_{22}^* v_1\\ &\leq C_P|\partial_x\sigma_{22}^*|\left|\frac{1}{\varepsilon}\partial_z v_1\right| \leq \frac{C_P^2}{4c_2}|\partial_x\sigma_{22}^*|^2+c_2\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2 \leq C+c_2\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2.\end{aligned}$$

Now, let us choose ε, c_1 and c_2 small enough such that

$$\frac{M_5\varepsilon^3}{2}, M_6\varepsilon^2, M_7\varepsilon, M_8\varepsilon, M_9\varepsilon, \frac{M_{10}\varepsilon}{2}, c_1, c_2 \leq \frac{(1-r)\nu}{36}. \tag{2.31}$$

Step 3. After summing (2.28) and (2.30), and multiplying by $2r\nu$, we obtain for ε small enough (satisfying (2.29) and (2.31)):

$$\begin{aligned} r\nu\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(|v_1|^2+|\varepsilon v_2|^2\right)+\frac{3}{2}r(1-r)\nu^2\left(|\nabla_\varepsilon v_1|^2+|\varepsilon\nabla_\varepsilon v_2|^2\right)&\\ -\frac{2r\nu}{\varepsilon^2}\int_\Omega q\left(\partial_x v_1+\partial_z v_2\right) \leq -\mathcal{D}_1-\mathcal{D}_2+C,&\end{aligned}$$

where C is a constant independent of ε. From the divergence equation $\nabla\cdot\boldsymbol{v} = \partial_x v_1 + \partial_z v_2 = 0$ it follows that the pressure term $\int\limits_\Omega q\ (\partial_x v_1 + \partial_z v_2) = 0$, and equation (2.27) is obtained.

□

Proposition 2.14. *Let us suppose that*

$$\lambda^*|\partial_z u_1^*|_\infty \le 1/12, \qquad \lambda^*|\sigma_{12}^*|_\infty \le \chi, \qquad \lambda^*(|\sigma_{11}^*|_\infty + |\sigma_{22}^*|_\infty) \le \chi,$$
$$2\lambda^*|\partial_z \sigma_{12}^*|_\infty \le \chi, \qquad \lambda^*|\partial_z \sigma_{11}^*|_\infty \le \chi,$$

where $\chi = \dfrac{\nu}{6}\sqrt{r(1-r)}$*. Then for* ε *small enough,* τ_{11}*,* τ_{12}*,* τ_{22} *solution of* (2.25) *satisfy the following inequality:*

$$\begin{aligned}\frac{\lambda^*}{2\varepsilon}\frac{d}{dt}\left(|\tau_{11}|^2 + 2|\tau_{12}|^2 + |\tau_{22}|^2\right) + \frac{1}{2}\left(\left|\frac{1}{\varepsilon}\tau_{11}\right|^2 + 2\left|\frac{1}{\varepsilon}\tau_{12}\right|^2 + \left|\frac{1}{\varepsilon}\tau_{22}\right|^2\right)\\ + \frac{\eta}{\varepsilon}\left(|\nabla_\varepsilon\tau_{11}|^2 + 2|\nabla_\varepsilon\tau_{12}|^2 + |\nabla_\varepsilon\tau_{22}|^2\right) \le \mathcal{D}_1 + \mathcal{D}_2 + r(1-r)\nu^2\left(|\nabla_\varepsilon v_1|^2 + |\varepsilon\nabla_\varepsilon v_2|^2\right) + C,\end{aligned} \tag{2.32}$$

where C *is a constant independent of* ε*.*

Proof. As in the previous proposition, classical *a priori* estimates on τ_{11}, τ_{12} and τ_{22} are obtained, and the remaining terms are estimated accurately.

Step 1. Let us multiply (2.25d) by $\dfrac{\tau_{11}}{\varepsilon}$ and integrate over Ω. Again, the convection terms $\int\limits_\Omega \boldsymbol{v}\cdot\nabla\tau_{11}\,\tau_{11}$ contained in $\int\limits_\Omega \mathrm{d}_t\tau_{11}\,\tau_{11}$ are equal to zero by Lemma 2.12, since $\nabla\cdot\boldsymbol{v} = 0$ and $\boldsymbol{v}|_{\partial\Omega} = 0$ (see (2.26)). Moreover, there is no boundary term in the integration by parts since the boundary conditions on $\boldsymbol{\sigma}$ have been chosen such that $\boldsymbol{\tau}\cdot n|_{\partial\Omega} = 0$ (see also (2.26)). It follows:

$$\begin{aligned}\frac{\lambda^*}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}|\tau_{11}|^2 - \frac{\lambda^*}{\varepsilon}\int\limits_\Omega N(\boldsymbol{v},\tau_{12})\,\tau_{11} + \left|\frac{1}{\varepsilon}\tau_{11}\right|^2 + \frac{\eta}{\varepsilon}|\nabla_\varepsilon\tau_{11}|^2\\ = \frac{2r\nu}{\varepsilon}\int\limits_\Omega \partial_x v_1\,\tau_{11} + \frac{1}{\varepsilon}\int\limits_\Omega L_{11}\,\tau_{11} + \frac{1}{\varepsilon^2}\int\limits_\Omega L_{11}'\,\tau_{11}.\end{aligned} \tag{2.33}$$

- The terms of $\displaystyle\int\limits_\Omega \mathcal{L}_{11}\,\tau_{11}$ are estimated as follows:

$$\star\ \lambda^*\int\limits_\Omega \partial_x u_2^*\,\tau_{12}\,\tau_{11} \le \underbrace{\lambda^*|\partial_x u_2^*|_\infty}_{=:M_{11}}\left(\frac{\varepsilon^2}{2}\left|\frac{1}{\varepsilon}\tau_{11}\right|^2 + \frac{\varepsilon^2}{2}\left|\frac{1}{\varepsilon}\tau_{12}\right|^2\right).$$

⋆ In a same way: $\displaystyle \frac{\lambda^*}{\varepsilon}\int_\Omega v_1\,\partial_x\sigma_{11}^*\,\tau_{11} \leq \underbrace{\lambda^*|\partial_x\sigma_{11}^*|_\infty C_P}_{=:M_{12}}\left(\frac{\varepsilon}{2}\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2+\frac{\varepsilon}{2}\left|\frac{1}{\varepsilon}\tau_{11}\right|^2\right).$

Let us choose ε small enough such that:

$$\frac{M_{11}\varepsilon^2}{2} \leq \frac{1}{24} \qquad \text{and} \qquad \frac{M_{12}\varepsilon}{2} \leq \min\left\{\frac{r(1-r)\nu}{6},\frac{1}{24}\right\}.$$

⋆ $\displaystyle \lambda^*\int_\Omega \partial_x v_2\,\sigma_{12}^*\,\tau_{11} \leq \lambda^*|\sigma_{12}^*|_\infty\,|\varepsilon\partial_x v_2|\left|\frac{1}{\varepsilon}\tau_{11}\right| \leq \lambda^*|\sigma_{12}^*|_\infty\left(\frac{1}{4c_3}|\varepsilon\partial_x v_2|^2+c_3\left|\frac{1}{\varepsilon}\tau_{11}\right|^2\right).$

Here, it is not possible to choose c_3 such that both coefficients are less than $r(1-r)\nu/6$ and $1/24$. Therefore, a condition on $\lambda^*|\sigma_{12}^*|_\infty$ is imposed such that:

$$\frac{\lambda^*|\sigma_{12}^*|_\infty}{4c_3} \leq \frac{r(1-r)\nu}{6} \qquad \text{and} \qquad \lambda^*|\sigma_{12}^*|_\infty c_3 \leq \frac{1}{24}.$$

Choosing c_3 satisfying $\lambda^*|\sigma_{12}^*|_\infty c_3 = 1/24$, the condition on $\lambda^*|\sigma_{12}^*|_\infty$ becomes:

$$\lambda^*|\sigma_{12}^*|_\infty \leq \frac{\nu}{6}\sqrt{r(1-r)} =: \chi.$$

⋆ Similarly the following term can be estimated:

$$\frac{\lambda^*}{\varepsilon}\int_\Omega v_2\,\partial_z\sigma_{11}^*\,\tau_{11} \leq \lambda^*|\partial_z\sigma_{11}^*|_\infty\,|\partial_z v_2|\left|\frac{1}{\varepsilon}\tau_{11}\right| \leq \lambda^*|\partial_z\sigma_{11}^*|_\infty\left(\frac{1}{4c_3}|\partial_z v_2|^2+c_3\left|\frac{1}{\varepsilon}\tau_{11}\right|^2\right).$$

The same reasoning as before allows us to control both terms providing that $\lambda^*|\partial_z\sigma_{11}^*|_\infty \leq \chi$.

⋆ In order to treat the term $-\lambda^*\int_\Omega \boldsymbol{u}^*\cdot\nabla\tau_{11}\,\tau_{11}$, it is not possible to apply Lemma 2.12, since $\boldsymbol{u}^*\cdot n|_{\partial\Omega} \neq 0$. However, integration by parts implies that

$$-\lambda^*\int_\Omega \boldsymbol{u}^*\cdot\nabla\tau_{11}\,\tau_{11} = -\frac{\lambda^*}{2}\int_{\partial\Omega}\boldsymbol{u}^*\cdot n\,\tau_{11}^2.$$

On ω, since $\boldsymbol{u}^* = (s,0)$ (see (2.11)), it follows $\boldsymbol{u}^*\cdot n = 0$. Thus it remains to consider the boundary integral on Γ_L. This boundary integral is split into two integrals on Γ_+ and Γ_-. On Γ_-, we have $\boldsymbol{u}^*\cdot n > 0$, thus $-\frac{\lambda^*}{2}\int_{\Gamma_-}\boldsymbol{u}^*\cdot n\,\tau_{11}^2 \leq 0$, and this term is trivially bounded by zero. On Γ_+, the boundary conditions are chosen in subsection 2.4.2 such that $\boldsymbol{\tau}|_{\Gamma_+} = 0$, therefore $-\frac{\lambda^*}{2}\int_{\Gamma_+}\boldsymbol{u}^*\cdot n\,\tau_{11}^2 = 0$.

- All other terms of $\int_\Omega L_{11}\,\tau_{11}$ are easier to manage, since they are linear in τ_{11}, and they are treated with Young and Poincaré inequalities in the same way as the ones in v_1, v_2.
- For the terms of $\int_\Omega L'_{11}\,\tau_{11}$, we proceed as before:

$$\frac{\lambda^*}{\varepsilon^2}\int_\Omega \partial_z u_1^*\,\tau_{12}\,\tau_{11} \le \lambda^*|\partial_z u_1^*|_\infty \left|\frac{1}{\varepsilon}\tau_{12}\right| \left|\frac{1}{\varepsilon}\tau_{11}\right| \le \lambda^*|\partial_z u_1^*|_\infty \left(\frac{1}{2}\left|\frac{1}{\varepsilon}\tau_{12}\right|^2 + \frac{1}{2}\left|\frac{1}{\varepsilon}\tau_{11}\right|^2\right).$$

Choosing $\lambda^*|\partial_z u_1^*|_\infty \le 1/12$, both terms are bounded by $1/24$.

$$\frac{\lambda^*}{\varepsilon^2}\int_\Omega \partial_z v_1\,\sigma_{12}^*\,\tau_{11} \le \lambda^*|\sigma_{12}^*|_\infty \left|\frac{1}{\varepsilon}\partial_z v_1\right| \left|\frac{1}{\varepsilon}\tau_{11}\right| \le \lambda^*|\sigma_{12}^*|_\infty \left(\frac{1}{4c_3}\left|\frac{1}{\varepsilon}\partial_z v_1\right|^2 + c_3\left|\frac{1}{\varepsilon}\tau_{11}\right|^2\right).$$

Imposing $\lambda^*|\sigma_{12}^*|_\infty \le \chi$ is enough to ensure that the coefficients are less than $r(1-r)\nu/6$ and $1/24$.

Step 2. Now, multiplying equation (2.25e) by $\frac{2\tau_{12}}{\varepsilon}$ and integrating over Ω, with the same reasoning as in the previous step it follows:

$$\begin{aligned}\frac{\lambda^*}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}|\tau_{12}|^2 + \frac{\lambda^*}{\varepsilon}\int_\Omega N(\boldsymbol{v},\tau_{11}-\tau_{22})\,\tau_{12} + 2\left|\frac{1}{\varepsilon}\tau_{12}\right|^2 + \frac{2\eta}{\varepsilon}|\nabla_\varepsilon\tau_{12}|^2 \\ = \frac{2r\nu}{\varepsilon}\int_\Omega \left(\partial_x v_2 + \frac{1}{\varepsilon}\partial_z v_1\right)\tau_{12} + \frac{2}{\varepsilon}\int_\Omega L_{12}\,\tau_{12} + \frac{2}{\varepsilon^2}\int_\Omega L'_{12}\,\tau_{12}\end{aligned} \tag{2.34}$$

The terms in L_{12} and L'_{12} are of the same type as the ones in L_{11} and L'_{11}, and are treated very similarly to them, applying Young's inequality, and assuming smallness conditions on ε. Thus, let us only write the terms needing additional assumptions.

⋆ $\lambda^*\int_\Omega \partial_x v_2\,(\sigma_{11}^*-\sigma_{22}^*)\,\tau_{12} \le \lambda^*(|\sigma_{11}^*|_\infty+|\sigma_{22}^*|_\infty)\,|\varepsilon\partial_x v_2|\,\left|\frac{1}{\varepsilon}\tau_{12}\right|$, and it is enough to assume that $\lambda^*(|\sigma_{11}^*|_\infty + |\sigma_{22}^*|_\infty) \le \chi$.

⋆ $\frac{2\lambda^*}{\varepsilon}\int_\Omega v_2\,\partial_z\sigma_{12}^*\,\tau_{12} \le 2\lambda^*|\partial_z\sigma_{12}^*|_\infty\,|\partial_z v_2|\,\left|\frac{1}{\varepsilon}\tau_{12}\right|$, and we assume that $2\lambda^*|\partial_z\sigma_{12}^*|_\infty \le \chi$.

⋆ $\frac{\lambda^*}{\varepsilon^2}\int_\Omega \partial_z u^*\,(\tau_{11}-\tau_{22})\,\tau_{12} \le \lambda^*|\partial_z u_1^*|_\infty \left(\left|\frac{1}{\varepsilon}\tau_{11}\right| + \left|\frac{1}{\varepsilon}\tau_{22}\right|\right)\left|\frac{1}{\varepsilon}\tau_{12}\right|$, it has already been assumed that $\lambda^*|\partial_z u_1^*|_\infty \le 1/12$.

⋆ $\frac{\lambda^*}{\varepsilon^2}\int_\Omega \partial_z v_1\,(\sigma_{11}^*-\sigma_{22}^*)\,\tau_{12} \le \lambda^*(|\sigma_{11}^*|_\infty + |\sigma_{22}^*|_\infty)\left|\frac{1}{\varepsilon}\partial_z v_1\right|\left|\frac{1}{\varepsilon}\tau_{12}\right|$, it has already been assumed that $\lambda^*(|\sigma^*_{11}|_\infty + |\sigma^*_{22}|_\infty) \le \chi$.

Step 3. Multiplying (2.25f) by $\frac{\tau_{22}}{\varepsilon}$, and estimating the terms just as the ones in τ_{11}, it follows

$$\frac{\lambda^*}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}|\tau_{22}|^2 + \frac{\lambda^*}{\varepsilon}\int_\Omega N(\boldsymbol{v},\tau_{12})\,\tau_{22} + \left|\frac{1}{\varepsilon}\tau_{22}\right|^2 + \frac{\eta}{\varepsilon}|\nabla_\varepsilon\tau_{22}|^2 \\ = \frac{2r\nu}{\varepsilon^2}\int_\Omega \partial_z v_2\, c + \frac{1}{\varepsilon}\int_\Omega L_{22}\,\tau_{22} + \frac{1}{\varepsilon^2}\int_\Omega L'_{22}\,\tau_{22}. \tag{2.35}$$

Assuming that $\lambda|\sigma^*_{12}|_\infty \le \chi$, $\lambda^*|\partial_z\sigma^*_{11}|_\infty \le \chi$ and $\lambda^*|\partial_z u^*_1|_\infty \le 1/12$, all the terms $\frac{1}{\varepsilon}\int_\Omega L_{22}\,\tau_{22}$ and $\frac{1}{\varepsilon^2}\int_\Omega L'_{22}\,\tau_{22}$ are bounded and estimated as in Step 1.

Step 4. Summing (2.33), (2.34) and (2.35), and noticing that

$$-\int_\Omega N(\boldsymbol{v},\tau_{12})\,\tau_{11} + \int_\Omega N(\boldsymbol{v},\tau_{11}-\tau_{22})\,\tau_{12} + \int_\Omega N(\boldsymbol{v},\tau_{12})\,\tau_{22} \\ = \int_\Omega \left(\varepsilon\partial_x v_2 - \frac{1}{\varepsilon}\partial_z v_1\right)(-\tau_{12}\,\tau_{11} + (\tau_{11}-\tau_{22})\,\tau_{12} + \tau_{12}\,\tau_{22}) = 0,$$

it follows that for ε small enough

$$\frac{\lambda^*}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tau_{11}|^2 + 2|\tau_{12}|^2 + |\tau_{22}|^2\right) + \frac{1}{2}\left(\left|\frac{1}{\varepsilon}\tau_{11}\right|^2 + 2\left|\frac{1}{\varepsilon}\tau_{12}\right|^2 + \left|\frac{1}{\varepsilon}\tau_{22}\right|^2\right) \\ + \frac{\eta}{\varepsilon}\left(|\nabla_\varepsilon\tau_{11}|^2 + 2|\nabla_\varepsilon\tau_{12}|^2 + |\nabla_\varepsilon\tau_{22}|^2\right) \le \mathcal{D}_1 + \mathcal{D}_2 + r(1-r)\nu^2\left(|\nabla_\varepsilon v_1|^2 + |\varepsilon\nabla_\varepsilon v_2|^2\right) + C,$$

where we recognized the terms $\mathcal{D}_1 + \mathcal{D}_2$, and where C is a constant independent of ε.

□

From now on, let us come back to the notation with the superscripts $^{\varepsilon\eta}$, denoting the dependence on ε and η.

Theorem 2.15. *Suppose that the solution $\boldsymbol{u}^*, \boldsymbol{\sigma}^*$ of system (2.10)-(2.11) satisfies the following smallness assumptions*

$$\lambda^*|\partial_z u^*_1|_\infty \le 1/12, \qquad \lambda^*|\sigma^*_{12}|_\infty \le \chi, \qquad \lambda^*(|\sigma^*_{11}|_\infty + |\sigma^*_{22}|_\infty) \le \chi, \\ 2\lambda^*|\partial_z\sigma^*_{12}|_\infty \le \chi, \qquad \lambda^*|\partial_z\sigma^*_{11}|_\infty \le \chi, \tag{2.36}$$

where $\chi = \frac{\nu}{6}\sqrt{r(1-r)}$. Then the following convergences hold true up to subsequences *when η and then ε tend to zero:*

$$u_1^{\varepsilon\eta} \to u_1^*, \quad \partial_z u_1^{\varepsilon\eta} \to \partial_z u_1^*, \quad \partial_x u_1^{\varepsilon\eta} \rightharpoonup \partial_x u_1^* \qquad \textit{in } L^2(0,T,L^2(\Omega)), \tag{2.37}$$

$$u_2^{\varepsilon\eta} \to 0, \quad \partial_z u_2^{\varepsilon\eta} \to 0, \quad \partial_x u_2^{\varepsilon\eta} \rightharpoonup 0 \qquad \textit{in } L^2(0,T,L^2(\Omega)), \tag{2.38}$$

$$\varepsilon\boldsymbol{\sigma}^{\varepsilon\eta} \to \boldsymbol{\sigma}^* \qquad \textit{in } L^2(0,T,L^2(\Omega)), \tag{2.39}$$

$$u_1^{\varepsilon\eta} \rightharpoonup^* u_1^*, \quad u_2^{\varepsilon\eta} \rightharpoonup^* 0, \quad \varepsilon\boldsymbol{\sigma}^{\varepsilon\eta} \to \boldsymbol{\sigma}^* \qquad \textit{in } L^\infty(0,T,L^2(\Omega)). \tag{2.40}$$

Proof. Summing (2.27), (2.32) implies that for ε small enough (i.e. if assumption (2.36) is satisfied):

$$\begin{aligned} r\nu\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(|v_1^{\varepsilon\eta}|^2 + |\varepsilon v_2^{\varepsilon\eta}|^2\right) &+ \frac{\lambda^*}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tau_{11}^{\varepsilon\eta}|^2 + 2|\tau_{12}^{\varepsilon\eta}|^2 + |\tau_{22}^{\varepsilon\eta}|^2\right) \\ &+ \frac{\eta}{\varepsilon}\left(|\nabla_\varepsilon\tau_{11}^{\varepsilon\eta}|^2 + 2|\nabla_\varepsilon\tau_{12}^{\varepsilon\eta}|^2 + |\nabla_\varepsilon\tau_{22}^{\varepsilon\eta}|^2\right) + \frac{1}{2}\left|\frac{1}{\varepsilon}\tau_{11}^{\varepsilon\eta}\right|^2 + \left|\frac{1}{\varepsilon}\tau_{12}^{\varepsilon\eta}\right|^2 + \frac{1}{2}\left|\frac{1}{\varepsilon}\tau_{22}^{\varepsilon\eta}\right|^2 \\ &+ \frac{r(1-r)\nu^2}{2}\left(|\partial_x v_1^{\varepsilon\eta}|^2 + \left|\frac{1}{\varepsilon}\partial_z v_1^{\varepsilon\eta}\right|^2 + |\varepsilon\partial_x v_2^{\varepsilon\eta}|^2 + |\partial_z v_2^{\varepsilon\eta}|^2\right) \le C. \end{aligned} \tag{2.41}$$

From this inequality, it follows that $\boldsymbol{v}^{\varepsilon\eta}$ converges to $\boldsymbol{v}^\varepsilon$ in $L^2(0,T;\boldsymbol{H}^1(\Omega))$ and $\boldsymbol{\tau}^{\varepsilon\eta}$ converges $\boldsymbol{\tau}^\varepsilon$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$, as η tends to zero. $\boldsymbol{v}^\varepsilon$ and $\boldsymbol{\tau}^\varepsilon$ are the solutions solutions of (2.25) without the terms $\eta\Delta\boldsymbol{\tau}^{\varepsilon\eta}$. Indeed, recalling the weak formulation of the system (2.25), it suffices to notice that Hölder's inequality allows us to treat the term $\eta\Delta\boldsymbol{\tau}^{\varepsilon\eta}$:

$$\eta\int_\Omega \nabla_\varepsilon\boldsymbol{\tau}^{\varepsilon\eta}\cdot\nabla_\varepsilon\boldsymbol{\phi} \le \eta^{1/2}\Big(\underbrace{\eta|\nabla_\varepsilon\boldsymbol{\tau}^{\varepsilon\eta}|^2}_{\le C} + |\nabla_\varepsilon\boldsymbol{\phi}|^2\Big) \xrightarrow[\eta\to 0]{} 0, \quad \forall\boldsymbol{\phi}\in\boldsymbol{H}_0^1(\Omega).$$

Moreover, $\boldsymbol{v}^\varepsilon$ and $\boldsymbol{\tau}^\varepsilon$ satisfy the following estimate:

$$\begin{aligned} r\nu\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(|v_1^\varepsilon|^2 + |\varepsilon v_2^\varepsilon|^2\right) + \frac{\lambda^*}{2\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tau_{11}^\varepsilon|^2 + 2|\tau_{12}^\varepsilon|^2 + |\tau_{22}^\varepsilon|^2\right) + \frac{1}{2}\left|\frac{1}{\varepsilon}\tau_{11}^\varepsilon\right|^2 + \left|\frac{1}{\varepsilon}\tau_{12}^\varepsilon\right|^2 + \frac{1}{2}\left|\frac{1}{\varepsilon}\tau_{22}^\varepsilon\right|^2 \\ + \frac{1}{2}r(1-r)\nu^2\left(|\partial_x v_1^\varepsilon|^2 + \left|\frac{1}{\varepsilon}\partial_z v_1^\varepsilon\right|^2 + |\varepsilon\partial_x v_2^\varepsilon|^2 + |\partial_z v_2^\varepsilon|^2\right) \le C. \end{aligned} \tag{2.42}$$

It remains to pass to the limit as ε tends to zero. After integrating (2.42) between 0 and T, it yields that

▷ $\|v_1^\varepsilon\|_{L^2(L^2)} \le \|\partial_z v_1^\varepsilon\|_{L^2(L^2)} \le C\varepsilon$, thus the following convergence results hold true in $L^2(0,T,L^2(\Omega))$ as ε tends to zero:

$$v_1^\varepsilon \to 0 \quad \text{and} \quad \partial_z v_1^\varepsilon \to 0. \tag{2.43}$$

From these convergence results, it follows that $u_1^\varepsilon = u_1^* + v_1^\varepsilon \to u_1^*$ in $L^2(0,T,L^2(\Omega))$ and $\partial_z u_1^\varepsilon \to \partial_z u_1^*$ in $L^2(0,T,L^2(\Omega))$.

- ▷ $\|\partial_x v_1^\varepsilon\|_{L^2(L^2)} \leq C$, thus $\partial_x v_1^\varepsilon$ converges weakly in $L^2(0,T,L^2(\Omega))$. Now, since it is already known that $u_1^\varepsilon \to u_1^*$, it follows that $\partial_x u_1^\varepsilon \rightharpoonup \partial_x u_1^*$ in $L^2(0,T,L^2(\Omega))$.

- ▷ Similarly $\|v_2^\varepsilon\|_{L^2(L^2)} \leq \|\partial_z v_2^\varepsilon\|_{L^2(L^2)} \leq C$, thus $\varepsilon v_2^\varepsilon$ and $\varepsilon \partial_z v_2^\varepsilon$ converge strongly to zero in $L^2(0,T,L^2(\Omega))$, and thus $u_2^\varepsilon = \varepsilon u_2^* + \varepsilon v_2^\varepsilon \to 0$ in $L^2(0,T,L^2(\Omega))$, and $\partial_z u_2^\varepsilon \to 0$ in $L^2(0,T,L^2(\Omega))$.

- ▷ $\|\partial_x v_2^\varepsilon\|_{L^2(L^2)} \leq \dfrac{C}{\varepsilon}$, thus $\partial_x u_2^\varepsilon$ converges weakly in $L^2(0,T,L^2(\Omega))$. Since $u_2^\varepsilon \to 0$, it implies that $\partial_x u_2^\varepsilon \rightharpoonup 0$ in $L^2(0,T,L^2(\Omega))$.

- ▷ $\|\tau_{11}^\varepsilon\|_{L^2(L^2)}, \|\tau_{12}^\varepsilon\|_{L^2(L^2)}, \|\tau_{22}^\varepsilon\|_{L^2(L^2)} \leq C\varepsilon$, therefore $\tau_{11}^\varepsilon, \tau_{12}^\varepsilon, \tau_{22}^\varepsilon \to 0$ in $L^2(0,T,L^2(\Omega))$. Thus $\varepsilon\sigma_{11}^\varepsilon = \sigma_{11}^* + \tau_{11}^\varepsilon \to \sigma_{11}^*$ in $L^2(0,T,L^2(\Omega))$, and in the same way $\varepsilon\sigma_{12}^\varepsilon \to \sigma_{12}^*$ in $L^2(0,T,L^2(\Omega))$, $\varepsilon\sigma_{22}^\varepsilon \to \sigma_{22}^*$ in $L^2(0,T,L^2(\Omega))$.

- ▷ From the terms with the derivatives in time, using the fact that $\boldsymbol{v}^\varepsilon|_{t=0} = \boldsymbol{u}_0^\varepsilon - \boldsymbol{u}^* \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{\tau}^\varepsilon|_{t=0} = \boldsymbol{\sigma}_0^\varepsilon - \boldsymbol{\sigma}^* \in \boldsymbol{L}^2(\Omega)$ are bounded independently of ε, we can conclude that
$$\|\boldsymbol{v}^\varepsilon\|_{L^\infty(L^2)} \leq C \qquad \text{and} \qquad \|\boldsymbol{\tau}^\varepsilon\|_{L^\infty(L^2)} \leq C\sqrt{\varepsilon}.$$
These estimates and the uniqueness of the limit imply that v_1^ε and $\varepsilon v_2^\varepsilon$ converge weakly-* in $L^\infty(0,T,L^2(\Omega))$ toward zero, and that $\boldsymbol{\tau}^\varepsilon$ converges strongly in $L^\infty(0,T,\boldsymbol{L}^2(\Omega))$ toward zero, which proves the last estimate (2.40).

□

Note that in a simplified case (with a simpler geometry), hypothesis (2.36) is satisfied under a small data assumption on the physical parameters.

Remark 2.16. *When h is constant with respect to x, p^* is also independent of x, so that equation (2.10) reduces to*
$$-(1-r)\partial_z^2 u_1^* - r\frac{\partial}{\partial z}\left(\frac{\partial_z u_1^*}{1+\lambda^{*2}|\partial_z u_1^*|^2}\right) = 0.$$
It has been shown in [BCF04] for example that for $r < 8/9$ this equation admits a unique solution $u_1^* = s(1-\frac{z}{h})$.
Now, it follows that $\sigma_{12}^* = \dfrac{r\nu\partial_z u_1^*}{1+\lambda^{*2}|\partial_z u_1^*|^2} = \dfrac{-r\nu s}{h+\lambda^{*2}s^2/h}$, *and* $\sigma_{11}^* = -\sigma_{22}^* = -\lambda^*\partial_z u_1^*\sigma_{12}^* = \dfrac{-r\nu s^2\lambda^*}{h^2+\lambda^{*2}s^2}$.

In this case, hypothesis (2.36) becomes more simple. Since $\partial_z u_1^* = -s/h$, σ_{11}^* and σ_{12}^* are constant with respect to z, so that the last two conditions are trivially verified. Using the fact that $r < 8/9$, it leads to a smallness condition on $s\lambda^*$ with respect to h ($s\lambda^* \leq h/12$ is enough in order to satisfy all conditions).
Observe that this condition is not optimal, but it shows that in the simplified case when $h(x)$ is constant, a simple choice of the parameters s, λ^* and h satisfies hypothesis (2.36).

2.5.3 Convergence results for the pressure

It remains to prove the convergence of the pressure.

Theorem 2.17. *Under the same smallness assumption (2.36), the following convergence result holds true for p:*

$$\varepsilon^2 p \underset{\varepsilon \to 0}{\rightarrow} p^* \quad \text{in } \mathcal{D}'(0,T,L^2(\Omega)). \tag{2.44}$$

Proof. Throughout the proof, C will denote some generic constants independent of ε. Let $\varepsilon \leq 1$. Let us integrate over $\Omega_T = \Omega \times (0,T)$ equation (2.25a) multiplied by $\varepsilon^2 \phi_1$, for any function $\phi_1 \in H_0^1(\Omega)$. It follows:

$$\begin{aligned}
&\rho\varepsilon^2 \iint\limits_{(0,T)\times\Omega} \partial_t v_1 \phi_1 + \rho\varepsilon^2 \iint\limits_{(0,T)\times\Omega} v_1 \partial_x v_1 \phi_1 + \rho\varepsilon \iint\limits_{(0,T)\times\Omega} v_2 \partial_z v_1 \phi_1 + (1-r)\nu\varepsilon^2 \iint\limits_{(0,T)\times\Omega} \partial_x v_1 \, \partial_x \phi_1 \\
&\quad + (1-r)\nu \iint\limits_{(0,T)\times\Omega} \partial_z v_1 \, \partial_z \phi_1 + \iint\limits_{(0,T)\times\Omega} \partial_x q \, \phi_1 = -\varepsilon \iint\limits_{(0,T)\times\Omega} \tau_{11} \partial_x \phi_1 - \iint\limits_{(0,T)\times\Omega} \tau_{12} \partial_z \phi_1 \\
&\quad + \varepsilon^2 \iint\limits_{(0,T)\times\Omega} L_1 \phi_1 + \varepsilon \iint\limits_{(0,T)\times\Omega} C_1 \phi_1, \quad \forall \phi_1 \in H_0^1(\Omega).
\end{aligned} \tag{2.45}$$

Using the fact that ϕ_1 is independent of t, the first term becomes

$$\rho\varepsilon^2 \iint\limits_{(0,T)\times\Omega} \partial_t v_1 \phi_1 = \rho\varepsilon^2 \int\limits_{\Omega} \phi_1 \int\limits_0^T \partial_t v_1 = \rho\varepsilon^2 \int\limits_{\Omega} \phi_1 (v_1(T) - v_1(0)),$$

where $v_1(0) = u_{1_0} - u_1^*$ denotes the value of v_1 at time $t = 0$. Now, introducing

$$\pi = \int\limits_0^T q \, \mathrm{d}t,$$

and using integration by parts for the pressure term (the boundary term is zero since

$\phi_1 \in H_0^1(\Omega)$), (2.45) becomes: $\forall \phi_1 \in H_0^1(\Omega)$,

$$\rho\varepsilon^2 \int_\Omega \phi_1(v_1(T)-v_1(0)) + \rho\varepsilon^2 \iint_{(0,T)\times\Omega} v_1\partial_x v_1\phi_1 + \rho\varepsilon \iint_{(0,T)\times\Omega} r_2\partial_z v_1\phi_1$$
$$+ (1-r)\nu\varepsilon^2 \iint_{(0,T)\times\Omega} \partial_x v_1\,\partial_x\phi_1 + (1-r)\nu \iint_{(0,T)\times\Omega} \partial_z v_1\,\partial_z\phi_1 - \int_\Omega \pi\,\partial_x\phi_1$$
$$= -\varepsilon \iint_{(0,T)\times\Omega} \tau_{11}\partial_x\phi_1 - \iint_{(0,T)\times\Omega} \tau_{12}\partial_z\phi_1 + \varepsilon^2 \iint_{(0,T)\times\Omega} L_1\phi_1 + \varepsilon \iint_{(0,T)\times\Omega} C_1\phi_1.$$

It remains to estimate all terms independent of π. The non-linear terms are to be handled with care, since $\phi_1 \notin L^\infty(\Omega)$. Proceeding as in [BCC99], Hölder's inequality with exponents $2+\delta$, δ' and 2 leads:

$$\left| \iint_{(0,T)\times\Omega} v_1\partial_x v_1\phi_1 \right| \le |\phi_1|_{\delta'} \int_0^T |v_1|_{2+\delta}\,|\partial_x v_1|, \tag{2.46}$$

where $\dfrac{1}{2+\delta} + \dfrac{1}{2} + \dfrac{1}{\delta'} = 1$ (which implies that $\delta' = \dfrac{2(2+\delta)}{\delta}$). According to interpolation theory, $\left[L^2, L^4\right]_\theta = L^{2+\delta}$ for $\theta = \dfrac{\delta}{2+\delta}$, which yields the following estimate:

$$|v_1|_{2+\delta} \le C|v_1|_4^\theta\,|v_1|^{1-\theta}.$$

Moreover Lemma 3.2 of [ABC94] states that for $v_1 \in H_0^1(\Omega)$, we have:

$$|v_1|_4 \le \sqrt{2}|\partial_x v_1|^{1/4}\,|\partial_z v_1|^{3/4}.$$

Using the two last inequalities and the Poincaré inequality, (2.46) becomes

$$\rho\varepsilon^2 \left| \iint_{(0,T)\times\Omega} v_1\partial_x v_1\phi_1 \right| \le \rho\varepsilon^2|\phi_1|_{\delta'} C \int_0^T |\partial_x v_1|^{\theta/4}\,|\partial_z v_1|^{3\theta/4}|\partial_z v_1|^{1-\theta}|\partial_x v_1|,$$

and Hölder's inequality implies that

$$\rho\varepsilon^2 \iint_{(0,T)\times\Omega} v_1\partial_x v_1\phi_1 \le \rho\varepsilon^2|\phi_1|_{\delta'} C\|\partial_x v_1\|_{L^2(\Omega_T)}^{1+\theta/4}\,\|\partial_z v_1\|_{L^2(\Omega_T)}^{1-\theta/4}.$$

Now, choose θ (and thus δ) such that $\delta' \ge 6$. It suffices to take $\theta \le \frac{1}{3}$, for example take

$\theta = \dfrac{1}{3}$. Then $\delta' = 6$, and the usual Sobolev embeddings read $H^1(\Omega) \hookrightarrow L^6(\Omega)$ (which is true in dimension 2 or 3). Therefore, the last estimate becomes

$$\rho\varepsilon^2 \iint\limits_{(0,T)\times\Omega} v_1 \partial_x v_1 \phi_1 \le \rho\varepsilon^2 C \|\phi_1\|_{H^1} \|\partial_x v_1\|^{13/12}_{L^2(\Omega_T)} \|\partial_z v_1\|^{11/12}_{L^2(\Omega_T)}.$$

Recalling that $\|\partial_z v_1\|_{L^2(L^2)} \le C\varepsilon$ and $\|\partial_x v_1\|_{L^2(L^2)} \le C$, we conclude

$$\rho\varepsilon^2 \iint\limits_{(0,T)\times\Omega} v_1 \partial_x v_1 \phi_1 \le \rho\varepsilon^2 C \|\phi_1\|_{H^1} \varepsilon^{11/12} = \rho\varepsilon^{2+11/12} C \|\phi_1\|_{H^1} \le \ C\varepsilon \|\phi_1\|_{H^1}.$$

In a similar way, we obtain

$$\rho\varepsilon \iint\limits_{(0,T)\times\Omega} r_2 \partial_z v_1 \phi_1 \le \rho\varepsilon^{2-1/12} C \|\phi_1\|_{H^1} \le \widetilde{C}\varepsilon \|\phi_1\|_{H^1}.$$

For the term $\rho\varepsilon^2 \int\limits_\Omega \phi_1(v_1(T) - v_1(0))$, we apply Cauchy-Schwarz inequality. $v_1(0)$ is bounded, and for $v_1(T)$, we use the Poincaré inequality. It follows, using the fact that $|\partial_z v_1| \le C\varepsilon$:

$$\begin{aligned}\rho\varepsilon^2 \int\limits_\Omega \phi_1(v_1(T) - v_1(0)) &\le (C|v_1| + C)\varepsilon^2 \|\phi_1\|_{H^1} \le (C|\partial_z v_1| + C)\varepsilon^2 \|\phi_1\|_{H^1} \\ &\le C\varepsilon^2 \|\phi_1\|_{H^1} \le C\varepsilon \|\phi_1\|_{H^1}.\end{aligned}$$

For the other linear terms, a simple application of Cauchy-Schwarz inequality allows to obtain similar estimates. Indeed, it suffices to use the estimate (2.42) in order to estimate the L^2-norm of $\partial_x v_1$, $\partial_z v_1$, τ_{11}, τ_{12}, L_1, C_1. For example, since $|\partial_x v_1| \le C$, we have the following estimate:

$$\rho\varepsilon^2 \int\limits_\Omega \partial_x v_1 \, \partial_x \phi_1 \le \rho\varepsilon^2 |\partial_x v_1| \, |\partial_x \phi_1| \le C\varepsilon^2 \|\phi_1\|_{H^1}.$$

For the terms L_1 and C_1, C_1 and the constant part of L_1 are obviously bounded uniformly in ε. It remains to estimate the linear term $\mathcal{L}_1$ of L_1. Recalling its definition and using the Poincaré inequality in the second estimate:

$$|\mathcal{L}_1| \le C\left(|v_1| + |v_2| + |\partial_x v_1| + |\partial_z v_1|\right) \le C\left(|\partial_z v_1| + |\partial_x v_1| + |\partial_z v_2|\right).$$

Using again (2.42), the boundedness of $\mathcal{L}_1$ follows:

$$|\mathcal{L}_1| \leq C.$$

Hence $\forall \phi_1 \in H_0^1(\Omega)$:

$$\int_\Omega \partial_x \pi \, \phi_1 \leq C \left(\varepsilon + \varepsilon^2 |\partial_x v_1| + |\partial_z v_1| + \varepsilon |\tau_{11}| + |\tau_{12}| + \varepsilon^2 |L_1| + \varepsilon |C_1|\right) \|\phi_1\|_{H^1} \leq C\varepsilon \|\phi_1\|_{H^1}.$$

The same approach with (2.25b) gives a similar estimate, for all $\phi_2 \in H_0^1(\Omega)$:

$$\int_\Omega \partial_z \pi \, \varphi_2 \leq C \left(\varepsilon + \varepsilon^4 |\partial_x v_2| + \varepsilon^2 |\partial_z v_2| + \varepsilon^2 |\tau_{12}| + \varepsilon |\tau_{22}| + \varepsilon^2 |L_2| + \varepsilon |C_2|\right) \|\phi_2\|_{H^1} \leq C\varepsilon \|\phi_2\|_{H^1}.$$

Thus we can conclude that $\|\nabla \pi\|_{L^\infty(H^{-1})} \leq C\varepsilon$.
Now recall that for $f \in L_0^2(\Omega)$, the following applies: $|f| \leq \|\nabla q\|_{H^{-1}}$ (see for example [Tem79]). Since $p \in L_0^2(\Omega)$ and $p^* \in L_0^2(\Omega)$, q lies in $L_0^2(\Omega)$. From the definition of π as function of q, it is clear that $\pi \in L_0^2(\Omega)$.
This allows to deduce

$$|\pi|_{L^\infty(L^2)} \leq \|\nabla \pi\|_{L^\infty(H^{-1})} \leq C\varepsilon \to 0,$$

thus π tends to zero in $L^\infty(0, T, L_0^2(\Omega))$ when $\varepsilon \to 0$. Now, since $q = \dfrac{\partial \pi}{\partial t}$, it follows that q tends to zero in $\mathcal{D}'(0, T, L_0^2(\Omega))$, and therefore:

$$\varepsilon^2 p \underset{\varepsilon \to 0}{\to} p^* \quad \text{in } \mathcal{D}'(0, T, L^2(\Omega)).$$

This concludes the proof. □

2.5.4 Open problems

This work concerns only the solutions of the problem (2.5) that are obtained as the limit of the regularized problem we chose (with an additional term $-\eta \Delta \boldsymbol{\sigma}$). Since there is no uniqueness result for problem (2.5), it is not known how other solutions behave.

Formally, the passing to the limit can be done for $a \neq 0$ (see [BCM07]), and a similar limit problem (involving the parameter a, but of the same structure). However, the proof of the existence theorem in $\hat{\Omega}^\varepsilon$ strongly relies on the fact that $a = 0$. No global results are proved in the case $a \neq 0$.

At last, since the computations are independent of the dimension of the domain Ω, the result should be true in the three-dimensional case. The limit problem on $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$

reads:

$$\begin{cases} (1-r)\nu\partial_z^2 u_1^* - \partial_x p^* + \partial_z \sigma_{13}^* = 0, \\ (1-r)\nu\partial_z^2 u_2^* - \partial_x p^* + \partial_z \sigma_{23}^* = 0, \\ \partial_z p^* = 0, \\ \nabla \cdot \boldsymbol{u}^* = 0, \\ -\lambda^* \partial_z u_1^* \sigma_{13}^* + \sigma_{11}^* = 0, \\ -\dfrac{\lambda^*}{2}\partial_z u_1^* \sigma_{13}^* - \partial_z u_2^* \sigma_{23}^* + \sigma_{12}^* = 0, \\ -\lambda^* \partial_z u_2^* \sigma_{23}^* + \sigma_{22}^* = 0, \\ \dfrac{\lambda^*}{2}\partial_z u_2^*(\sigma_{33}^* - \sigma_{22}^*) - \dfrac{\lambda^*}{2}\partial_z u_1^* \sigma_{12}^* + \sigma_{23}^* = r\nu\partial_z u_2^*, \\ \lambda^* \left(\partial_z u_1^* \sigma_{13}^* + \partial_z u_2^* \sigma_{23}^*\right) + \sigma_{33}^* = 0, \\ \dfrac{\lambda^*}{2}\partial_z u_1^*(\sigma_{33}^* - \sigma_{11}^*) - \dfrac{\lambda^*}{2}\partial_z u_2^* \sigma_{12}^* + \sigma_{12}^* = r\nu\partial_z u_1^*. \end{cases} \tag{2.47}$$

Bibliography

[ABC94] A. Assemien, G. Bayada, and M. Chambat. Inertial effects in the asymptotic behavior of a thin film flow. *Asymptotic Anal.*, 9(3):177–208, 1994.

[BC86] G. Bayada and M. Chambat. The transition between the Stokes equations and the Reynolds equation: a mathematical proof. *Appl. Math. Optim.*, 14(1):73–93, 1986.

[BCC99] G. Bayada, M. Chambat, and I. Ciuperca. Asymptotic Navier-Stokes equations in a thin moving boundary domain. *Asymptot. Anal.*, 21(2):117–132, 1999.

[BCF04] F. Boyer, L. Chupin, and P. Fabrie. Numerical study of viscoelastic mixtures through a Cahn-Hilliard flow model. *Eur. J. Mech. B Fluids*, 23(5):759–780, 2004.

[BCM07] G. Bayada, L. Chupin, and S. Martin. Viscoelastic fluids in a thin domain. *Quart. Appl. Math.*, 65(4):625–651, 2007.

[BK04] R. Bunoiu and S. Kesavan. Asymptotic behaviour of a Bingham fluid in thin layers. *J. Math. Anal. Appl.*, 293(2):405–418, 2004.

[BL96] G. Bayada and G. Lukaszewicz. On micropolar fluids in the theory of lubrication. Rigorous derivation of an analogue of the Reynolds equation. *Internat. J. Engrg. Sci.*, 34(13):1477–1490, 1996.

[BMT93] A. Bourgeat, A. Mikelić, and R. Tapiéro. Dérivation des équations moyennées décrivant un écoulement non newtonien dans un domaine de faible épaisseur. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(9):965–970, 1993.

[BT95] F. Boughanim and R. Tapiéro. Derivation of the two-dimensional Carreau law for a quasi-Newtonian fluid flow through a thin slab. *Appl. Anal.*, 57(3-4):243–269, 1995.

[Chu04] L. Chupin. Some theoretical results concerning diphasic viscoelastic flows of the Oldroyd kind. *Adv. Differential Equations*, 9(9-10):1039–1078, 2004.

[FCGO98] E. Fernández-Cara, F. Guillén, and R. R. Ortega. Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):1–29, 1998.

[GS90] C. Guillopé and J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849–869, 1990.

[LM00] P. L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[MPS05] E. Marušić-Paloka and M. Starčević. Rigorous justification of the Reynolds equations for gas lubrication. *C. R. Mécanique*, 33(7):534–541, 2005.

[Pao03] L. Paoli. Asymptotic behavior of a two fluid flow in a thin domain: from Stokes equations to Buckley-Leverett equation and Reynolds law. *Asymptot. Anal.*, 34(2):93–120, 2003.

[SET05] J.-M. Sac-Epée and K. Taous. On a wide class of nonlinear models for non-Newtonian fluids with mixed boundary conditions in thin domains. *Asymptot. Anal.*, 44(1-2):151–171, 2005.

[Tem79] R. Temam. *Navier-Stokes equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, revised edition, 1979.

3

Elastohydrodynamic piezoviscous lubrication problems with Elrod-Adams model of cavitation

Article paru dans
Differential and Integral Equations

Abstract *An unconditional existence result of a solution for a steady fluid-structure problem is stated. More precisely, we consider an incompressible fluid in a thin film, ruled by the Reynolds equation coupled with a surface deformation modelled by a non-linear non local Hertz law. The viscosity is supposed to depend non-linearly on the fluid pressure. Due to the apparition of a mushy region, the two-phase flow satisfies a free boundary problem defined by a pressure-saturation model.*

Such a problem has been studied with simpler free boundary models (variational inequality), or with boundary conditions imposing small data assumptions. We show that up to a realistic hypothesis on the asymptotic pressure-viscosity behaviour it is possible to obtain an unconditional solution of the problem.

3.1 Introduction

The knowledge of the pressure in a lubricated device is a key problem to compute operational characteristic of devices such as bearings, seals, magnetic recorder heads... Mathematically speaking, it means to solve the Reynolds equation ([FND$^+$97]). At first glance, it is a classical elliptic equation in which coefficients are related to the viscosity μ of the fluid, the gap h between the surrounding surfaces and some velocities data. However it is well known that in real operational conditions, the pressure inside the fluid is so high that the viscosity is no longer constant while the surrounding elastic surfaces are deformed. This fluid-structure interaction is often described by the Hertz integral model ([Sze98]).

Moreover, the fluid cannot be considered as an homogeneous one. Thus a free boundary between a full film area and a mushy region made by a mixture of oil and air (the cavitation region) must be included in the model. The most usual one in the mathematical literature is based upon a first kind of variational inequality ([GLT76], [Rod87]). Considering all these aspects leads to a much more complicated Reynolds EHD (elastohydrodynamic) “equation” which is a quasi-variational non local non linear inequality. Existence theorem and uniqueness results have been obtained by Oden an Wu [OW85], Rodriguez [Rod93], Hu [Hu90]. Most often, the proof of the existence is obtained by a fixed point approach using both L^∞ and H^1 estimates, as well as a small data assumption to obtain compactness results.

More recently, it has been observed (Bellout [Bel03], Bayada and Bellout [BB05]) that such small data assumption can be avoided if a specific viscosity-pressure behaviour is assumed. From a practical point of view, this behaviour is much more reasonable than the small data assumption: satisfactory numerical computation results are obtained for a very large range of data while the specific viscosity-pressure behaviour retained does not contradict any experiments ([Ver02]).

Another step in the complexity of the model was introduced as it was observed ([EA75], [Flo73]) that the previously used variational inequality model describing the cavitation does not fulfill a mass flow conservation property. Moreover, it cannot be used to describe some phenomena like starvation since only data on the pressure can be used in the variational inequality model in a satisfactory way. Based upon a generalization of the free boundary in the dam problem ([Chi84], [BKS78]), the new mathematical model addresses a two-unknown system (pressure and saturation) and a hyperbolic-elliptic Reynolds equation. This model is a full conservative one and allows both data on the pressure and input flow to be dealt with. An existence theorem and uniqueness properties have been obtained in [BC86], [AC94] for basic isoviscous fluid and rigid surfaces. Generalization to the full piezoviscous EHD problem appears in [BTV96] in which an existence theorem using a small data assumption has been obtained considering only data for the pressure.

The purpose of the present paper is to prove that for this new cavitation model, the small data assumption can be avoided while boundary data both on input flow and pressure can be introduced. To be observed also is the fact that while a small data assumption allows various approaches to be used (see [BTV96]), the present work relies strongly on the Grübin transform (see section 3.2) and does not seem to be generalizable to other approaches.

In section 3.2 a precise statement of the problem is given and some related regularized systems are introduced. Section 3.3 is devoted to obtaining some estimates. Some of them are very close, although different, to the one used for the small data case. At last in section 3.4 new estimates and the introduction of a specific viscosity-pressure relation allow to prove the existence of a solution to the problem (Theorem 3.14).

3.2 Formulation and regularization of the problem

3.2.1 Statement of the problem

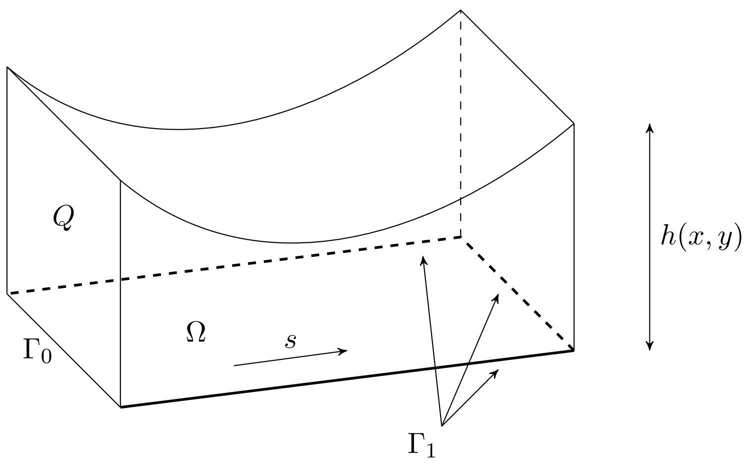

Figure 3.1: Domain Q

Let $\Omega = (0, L) \times (0, 1)$ a rectangular two-dimensional domain, with its boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\Gamma_0 = \{(0, y), y \in (0, 1)\}$. Let Q be the three-dimensional domain given by $Q = \{(x, y, z), (x, y) \in \Omega, z \in (0, h(x, y))\}$ (see Figure 3.1).
We consider a Newtonian fluid in the domain Q, with a given input parameter $\mathcal{G}_0$ on $\Gamma_0 \times (0, h(x, y))$, and a given velocity $\boldsymbol{s} = (s, 0)$ on Ω with $s \geq 0$. Moreover, let us introduce $G_0(y) = \displaystyle\int_0^{h(0,y)} \mathcal{G}_0(0, y, z)\mathrm{d}z$.
In a thin domain (i.e. h small with respect to the other dimensions), it is possible to reduce the Navier-Stokes equations to the Reynolds equation, which is an equation in Ω on the pressure only. In order to take into account the phenomenon of cavitation, we introduce the Elrod-Adams model.
This model considers that the cavitation zone is characterized by :
- a constant pressure, supposed to be equal to zero,
- an homogeneous blend of air and fluid, for example oil.

It introduces a function θ, defined in Ω, corresponding to the local proportion of the fluid in an elementary domain around the point $M(x, y)$, for $(x, y) \in \Omega$ (see Figure 3.2).

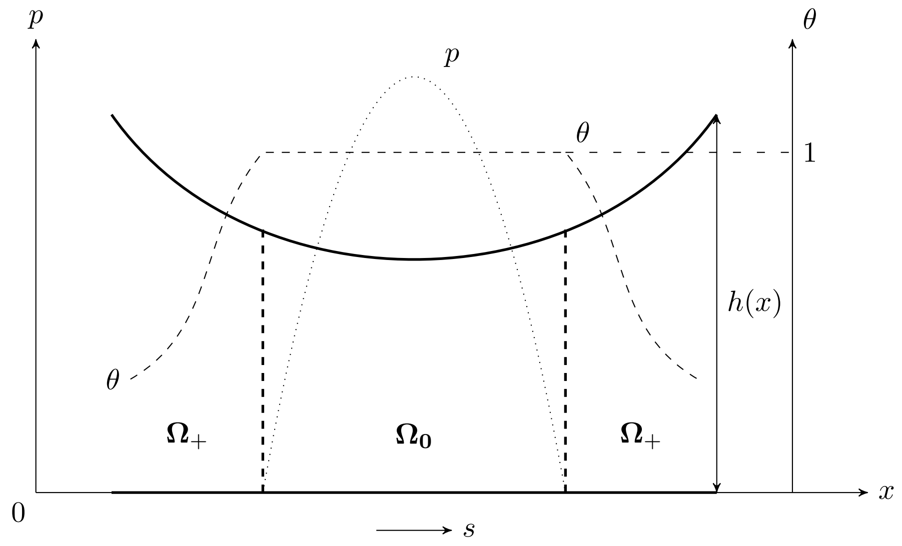

Figure 3.2: Partition of Ω and profiles of p and θ in the one-dimensional case

If the pressure p is equal to the saturation pressure, p must be positive, so that it is possible to define an unknown partition of Ω into a part Ω_+ where $p > 0$, and a part Ω_0 where $p = 0$ (cavitation zone, with a blend of air and oil). Therefore, the function $\theta \in L^\infty(\Omega)$ satisfies natural conditions :

$$\begin{cases} \theta = 1 & \text{in } \Omega_+ \\ 0 \leq \theta \leq 1 & \text{in } \Omega_0 \end{cases}$$

Physically, for high pressures, the viscosity of the fluid depends on the pressure p. Let us denote it by $\eta(p)$. In all generality, we suppose η to be a positive continuous function of p.

Moreover, we consider the height of the fluid $h(p, x, y)$ to be given by:

$$h(p,x,y) = h_0(x,y) + \int_\Omega k(x-s, y-t)\, p(s,t)\, \mathrm{d}s\, \mathrm{d}t, \quad (x,y) \in \Omega,\ p \in L^2(\Omega),$$

where the kernel k is defined by $k(x,y) = \dfrac{k_0}{\sqrt{x^2+y^2}}$, with k_0 a non-negative constant. This kernel corresponds physically to a point contact. The function h_0 is supposed to be regular and positive, such that $h_0 \geq m > 0$, where m is a constant. An important regularity property of this kernel is used in Section 3.4 and stated in Lemma 3.13.

Let us impose the following boundary condition: $p|_{\Gamma_1} = 0$. On Γ_0, the flow G_0 is supposed to be given as a positive function, with $G_0 \in L^\infty(\Gamma_0)$. It is now natural to define the following functional spaces:

$$V = \left\{ \phi \in H^1(\Omega),\, \phi|_{\Gamma_1} = 0 \right\},$$
$$V^+ = \{ \phi \in V,\, \phi \geq 0 \}.$$

When h tends to zero, it has been proved that the three-dimensional equations reduce to an equation on p in Ω. The strong formulation of the problem can be written as follows (see [BC86] for more details): *Find p and θ such that:*

$$\begin{cases} \operatorname{div}\left(\dfrac{h^3(p)}{\eta(p)} \nabla p \right) = 6s \dfrac{\partial(\theta\, h(p))}{\partial x}, & \text{in } \mathcal{D}'(\Omega). \\ \theta \in \mathcal{H}(p), & \end{cases}$$

where $\mathcal{H}(p)$ is the Heaviside graph, which means that

$$0 \leq \theta \leq 1 \qquad \text{and} \qquad p(1-\theta) = 0 \;\; \text{almost everywhere.}$$

Thus the weak formulation of this problem is: *Find $p \in V^+$ and $\theta \in L^\infty(\Omega)$ such that:*

$$(\mathcal{P}) \begin{cases} \displaystyle\int_\Omega \frac{h^3(p)}{\eta(p)} \nabla p \cdot \nabla \varphi = 6s \int_\Omega h(p)\, \theta\, \partial_x \varphi + \int_{\Gamma_0} G_0\, \varphi\,, \quad \forall \varphi \in V, \\ \theta \in \mathcal{H}(p). \end{cases}$$

Remark 3.1. *It is possible to interpret physically the local input flow G_0 as follows. The weak formulation $(\mathcal{P})$ implicitly contains the following relation between the input flow G_0 and the pressure:*

$$-G_0(y) = \begin{cases} 6s\, \theta_0\, h(p,0,y) & \text{if } p(0,y) = 0, \\ 6s\, h(p,0,y) - \dfrac{h^3(p,0,y)}{\eta(p)} \dfrac{\partial p}{\partial n} & \text{if } p(0,y) \neq 0. \end{cases}$$

It is to be noticed that if $s < 0$, since θ_0, h and G_0 are positive, only the second case can occur, and thus $G_0(y) = 6s\, h(p,0,y) - \dfrac{h^3(p,0,y)}{\eta(p)} \dfrac{\partial p}{\partial n}$.

3.2.2 The problem for the reduced pressure

A classical approach consists in introducing a change of functions that reduces the problem to one close to an isoviscous case (see [BTV96], in which this approach and the direct one without such change of functions are compared. It is shown that similar results are obtained in both cases).

Thus let us use the following change of functions (*Grübin transform*):

$$P(x,y) = a(p(x,y)) = \int_0^{p(x,y)} \frac{\mathrm{d}s}{\eta(s)}, \ (x,y) \in \Omega.$$

P is called *reduced pressure* (see Figure 3.3).

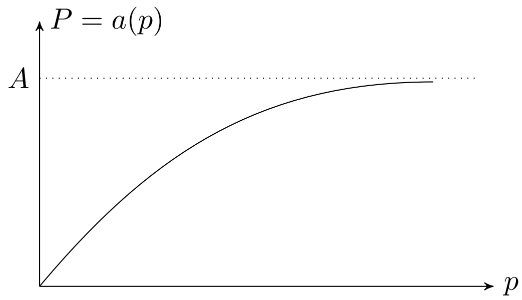

Figure 3.3: Profile of the reduced pressure P

Let A be defined by:

$$A = \int_0^{+\infty} \frac{\mathrm{d}s}{\eta(s)}.$$

The case $A = +\infty$ has already been treated in [Rod93]. However, it has been proved experimentally that A has a finite value. In particular, for a viscosity given by Barus law:

$$\eta(p) = \eta_0 e^{\alpha p}, \quad \text{with } \eta_0 > 0, \ \alpha > 0,$$

the quantity A is finite ($A = \dfrac{\eta_0}{\alpha}$). Therefore, we are concerned in this paper with fluids with a viscosity satisfying $A < +\infty$.

Remark 3.2. *Let us emphasize that the Barus law is a first approximation used by the physicists, and that its validity for high values of the pressure is questionable. In particular, it is more realistic from a physical point of view (see [Bla01] for example) to suppose a polynomial growth at infinity, which corresponds to the hypothesis* (3.5) *made in this work. This observation does not change the fact that A is finite.*

Furthermore, let the function γ be the inverse of the function a (as shown in Figure 3.4). Thus

$$p(x,y) = \gamma(P(x,y))$$

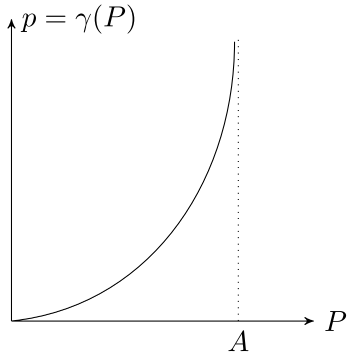

Figure 3.4: Profile of $\gamma(P)$

The weak formulation becomes: *Find* $P \in V^+$ *and* $\theta \in L^\infty(\Omega)$ *such that:*

$$(\mathcal{P}') \begin{cases} \int\limits_\Omega H^3(P)\nabla P \cdot \nabla\varphi = 6s \int\limits_\Omega H(P)\,\theta\,\partial_x\varphi + \int\limits_{\Gamma_0} G_0\,\varphi\,, \\ \forall\varphi \in V, \\ \theta \in \mathcal{H}(P), \end{cases}$$

with

$$H(P,x,y) = h_0(x,y) + \int\limits_\Omega k(x-s,y-t)\,\gamma(P(s,t))\,\mathrm{d}s\,\mathrm{d}t.$$

The purpose of this paper will be to prove an existence theorem (Theorem 3.14) for the weak formulation $(\mathcal{P}')$.

3.2.3 Introduction of a regularized problem

First, in order to regularize the Heaviside function, let us introduce Z_δ a continuous approximation of θ (Figure 3.5) such that, for $\delta > 0$:

$$Z_\delta(t) = \begin{cases} 1 & \text{if } t > \delta, \\ 0 & \text{if } t < 0, \\ \dfrac{t}{\delta} & \text{if } 0 \le t \le \delta. \end{cases}$$

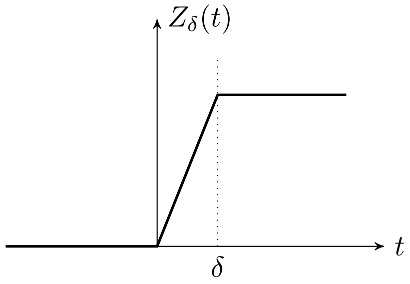

Figure 3.5: Regularization of θ

Now it remains to regularize the function γ, which is done by truncation (Figure 3.6). For $\varepsilon > 0$:

$$\gamma_\varepsilon(s) = \begin{cases} \gamma(s) & \text{if } 0 \leq s \leq A - \varepsilon, \\ \gamma(A-\varepsilon) & \text{if } s \geq A - \varepsilon. \end{cases}$$

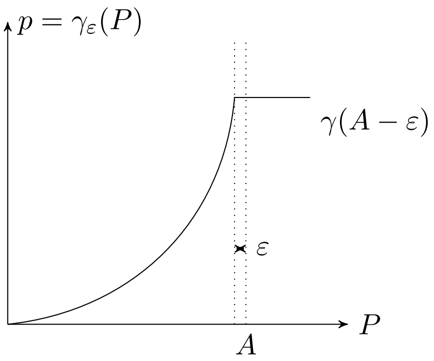

Figure 3.6: Regularization of γ

The regularized problem is: *Find $P_{\delta\varepsilon} \in V^+$ such that:*

$$(\mathcal{P}_{\delta\varepsilon}) \quad \int_\Omega H_\varepsilon^3(P_{\delta\varepsilon}) \nabla P_{\delta\varepsilon} \cdot \nabla\varphi = 6s \int_\Omega H_\varepsilon(P_{\delta\varepsilon})\, Z_\delta(P_{\delta\varepsilon})\, \partial_x \varphi + \int_{\Gamma_0} G_0\, \varphi\,, \quad \forall \varphi \in V,$$

with

$$H_\varepsilon(q,x,y) = h_0(x,y) + \int_\Omega k(x-s, y-t)\, \gamma_\varepsilon(q(s,t))\, \mathrm{d}s\, \mathrm{d}t. \tag{3.1}$$

3.3 Existence result and first estimates for the regularized problem

3.3.1 Existence of a solution

In this section, the existence of a solution for the problem $(\mathcal{P}_{\delta\varepsilon})$ is established, for fixed δ and ε.

Theorem 3.3. *For fixed $\delta > 0$ and $\varepsilon > 0$, there exists $P_{\delta\varepsilon} \in V^+$ solution of $(\mathcal{P}_{\delta\varepsilon})$. Moreover, $P_{\delta\varepsilon}$ satisfies:*

$$\|P_{\delta\varepsilon}\|_{H^1(\Omega)} \leq R,$$

where R is a constant independent of δ and ε.

Proof. Let us emphasize that this result will be shown without any condition on the data, by means of a fixed point method.
For fixed $P_{\delta\varepsilon} \in L^2(\Omega)$, let us introduce the following problem: *Find $q \in V^+$ such that:*

$$(\mathcal{Q}) \quad \int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})\nabla q\, \nabla\varphi = 6s\int_\Omega H_\varepsilon(P_{\delta\varepsilon})\, Z_\delta(q)\, \partial_x\varphi + \int_{\Gamma_0} G_0\, \varphi\,, \quad \forall\varphi \in V.$$

Step 1: Since $H_\varepsilon \geq h_0(x,y) > 0$, it is a classical mixed Dirichlet-Neumann problem, for which the existence and uniqueness of a solution are well known.
Step 2: Let us now derive estimates for the solution q of $(\mathcal{Q})$. Choosing $\varphi = q \in V$ in the weak formulation, it follows:

$$\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla q|^2 = 6s\int_\Omega H_\varepsilon(P_{\delta\varepsilon})\, Z_\delta(q)\, \partial_x q + \int_{\Gamma_0} G_0\, q\,. \tag{3.2}$$

Let $h_{0m} = \min\limits_{(x,y)\in\Omega} h_0(x,y) > 0$. Therefore, (3.1) and the positivity of $P_{\delta\varepsilon}$ (thus of $\gamma_\varepsilon(P_{\delta\varepsilon})$) and of k yield $H_\varepsilon(P_{\delta\varepsilon}) \geq h_{0m}$. The left-hand side term can be estimated in the following way:

$$h_{0m}\, \|H_\varepsilon(P_{\delta\varepsilon})\nabla q\|^2_{L^2(\Omega)} \leq \int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla q|^2.$$

Moreover, using that $\|Z_\delta\|_{L^\infty} \leq 1$, and applying Cauchy-Schwarz inequality to the first right-hand side term in (3.2), it follows:

$$6s\int_\Omega H_\varepsilon(P_{\delta\varepsilon})\, Z_\delta(q)\, \partial_x q \leq 6|s|\int_\Omega |H_\varepsilon(P_{\delta\varepsilon})|\, |\partial_x q| \leq 6|s|\, |\Omega|^{1/2}\, \|H_\varepsilon(P_{\delta\varepsilon})\nabla q\|_{L^2(\Omega)}.$$

It remains the second right-hand side term. Let $G = \|G_0(y)\|_{L^2(\Gamma_0)}$, hence the following applies:

$$\int_{\Gamma_0} G_0\, q \leq G\, \|q\|_{L^2(\Gamma_0)} \leq G\, \|q\|_{L^2(\Gamma)} \leq G\, \|q\|_{H^{1/2}(\Gamma)} \leq C\, G\, \|q\|_{H^1(\Omega)}$$

$$\leq C\, G\, \|\nabla q\|_{L^2(\Omega)} \leq \frac{C}{h_{0m}}\, G\, \|H_\varepsilon(P_{\delta\varepsilon}) \nabla q\|_{L^2(\Omega)},$$

where C denotes several constants obtained from trace theorems in Sobolev spaces and from Poincaré inequality. These constants are independent of both η and ε.

At last, equation (3.2) becomes:

$$h_{0m}\, \|H_\varepsilon(P_{\delta\varepsilon}) \nabla q\|^2_{L^2(\Omega)} \leq 6|s|\, |\Omega|^{1/2}\, \|H_\varepsilon(P_{\delta\varepsilon}) \nabla q\|_{L^2(\Omega)} + \frac{C\, G}{h_{0m}} \|H_\varepsilon(P_{\delta\varepsilon}) \nabla q\|_{L^2(\Omega)}.$$

This implies that:

$$h_{0m}\, \|H_\varepsilon(P_{\delta\varepsilon}) \nabla q\|_{L^2(\Omega)} \leq 6|s|\, |\Omega|^{1/2} + \frac{C\, G}{h_{0m}},$$

where $|\Omega|$ denotes the measure of Ω. The last estimate means that

$$\|\nabla q\|_{L^2(\Omega)} \leq \frac{6|s|\, |\Omega|^{1/2}\, h_{0m} + C\, G}{h_{0m}^3}.$$

Let us emphasize that this estimate is independent of η and ε.

Step 3: It remains to check out the positivity of q.

Let us choose $\varphi = q^- \in V$ (since $q \in H^1(\Omega)$, the negative part $q^- \in H^1(\Omega)$).We obtain

$$\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon}) \nabla(q^+ - q^-) \cdot \nabla q^- = 6s \int_\Omega H_\varepsilon(P_{\delta\varepsilon})\, Z_\delta(q)\, \partial_x q^- + \int_{\Gamma_0} G_0\, q^-.$$

The term $\nabla q^+ \nabla q^-$ is zero almost everywhere, and so is the term $Z_\delta(q)\, \partial_x q^-$. Indeed, if $q \geq 0$, $q^- = 0$, and if $q < 0$, $Z_\delta(q) = 0$. It remains:

$$- \int_\Omega H_\varepsilon^3(P_{\delta\varepsilon}) |\nabla q^-|^2 = \int_{\Gamma_0} G_0\, q^-.$$

Since $G_0(y) \geq 0$, we have $\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon}) |\nabla q^-|^2 \leq 0$, and thus $\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon}) |\nabla q^-|^2 = 0$. Hence q^- is constant almost everywhere. Furthermore $q|_{\Gamma_1} = 0$, therefore $q^- = 0$, which proves that $q \geq 0$.

Step 4: In order to conclude the proof of Theorem 3.3, it remains to apply Schauder fixed point theorem. Now, let

$$R = \frac{6|s|\,|\Omega|^{1/2}\,h_{0m} + C\,G}{h_{0m}^3} \quad \text{and} \quad B_R = \left\{ f \in L^2(\Omega),\, 0 \le \|f\|_{L^2} \le R \right\},$$

and let us define $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ by $T(P_{\delta\varepsilon}) = q$, where q is solution of $(\mathcal{Q})$. T is well defined, and we proved that $T(B_R) \subset B_R$. Moreover, T is continuous, since the function $P_{\delta\varepsilon} \mapsto H_\varepsilon(P_{\delta\varepsilon})$ is continuous. Let $(q^n)_{n\in\mathbb{N}}$ be a sequence of solutions of $(\mathcal{Q})$. We have:

$$\|\nabla q^n\|_{L^2} \le R.$$

Since $q^n|_{\Gamma_1} = 0$, the Poincaré inequality yields

$$\|q^n\|_{L^2} \le C\|\nabla q^n\|_{L^2} \le CR.$$

Using the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, it follows that q^n converges strongly in $L^2(\Omega)$. Finally, applying Schauder fixed point theorem, it follows that the problem $(\mathcal{P}_{\delta\varepsilon})$ admits a solution $P_{\delta\varepsilon} \in B_R$ satisfying additionally $P_{\delta\varepsilon} \in V^+$ and

$$\|P_{\delta\varepsilon}\|_{H^1(\Omega)} \le R.$$

This concludes the proof. □

Remark 3.4. *Let us emphasize that in previous works (in particular [BTV96]), similar H^1-bounds have been obtained provided some smallness assumption on the data. In the present paper, the constant R is not supposed to satisfy any smallness condition, and in particular is not supposed to be less than A. Therefore it will be shown separately that $P_{\delta\varepsilon}$ remains bounded by A.*

3.3.2 Classical estimates

In this section, we will obtain first estimates on $P_{\delta\varepsilon}$ and $\gamma_\varepsilon(P_{\delta\varepsilon})$. These estimates will be useful in order to prove the convergence of $\gamma_\varepsilon(P_{\delta\varepsilon})$ toward the expected function. However, it will not be enough to pass to the limit, and better estimates will be obtained in the next section.
Let us start with an L^∞ bound for $P_{\delta\varepsilon}$.

Proposition 3.5. *The solution $P_{\delta\varepsilon}$ of the problem $(\mathcal{P}_{\delta\varepsilon})$ satisfies the following inequality:*

$$\|P_{\delta\varepsilon}\|_{L^\infty(\Omega)} \le \frac{8|\Omega|^{1/6}C}{H_{m\delta\varepsilon}^2}\left(6|s| + \frac{G'\,C}{H_{m\delta\varepsilon}}\right) \le \frac{8|\Omega|^{1/6}C}{h_{0m}^2}\left(6|s| + \frac{G'\,C}{h_{0m}}\right)$$

where $H_{m\delta\varepsilon} = \min\limits_{(x,y)\in\Omega} H_\varepsilon(P_{\delta\varepsilon}(x,y)) \geq h_{0m}$ *and* $G' = \|G_0\|_{L^\infty(\Gamma_0)}$.

Proof. The key point in the proof is to use a lemma by Kinderlehrer-Stampacchia [KS80]. However, due to the boundary term on Γ_0, a specific treatment is to be used.

Let $k > 0$. Let $P_{\delta\varepsilon}^{(k)}$ be the function defined by

$$P_{\delta\varepsilon}^{(k)} = \begin{cases} P_{\delta\varepsilon} - k & \text{if } P_{\delta\varepsilon} \geq k \\ 0 & \text{if } P_{\delta\varepsilon} \leq k \end{cases}$$

and A_k the set $A_k = \{(x,y) \in \bar{\Omega}, P_{\delta\varepsilon}(x,y) \geq k\}$. It is easy to check that $P_{\delta\varepsilon}^{(k)}$ lies in V^+ and that

$$\nabla P_{\delta\varepsilon}^{(k)} = \begin{cases} \nabla P_{\delta\varepsilon} & \text{in } \mathring{A}_k, \\ 0 & \text{in } \bar{\Omega} \setminus A_k. \end{cases} \tag{3.3}$$

Obviously, we have a similar relation for $\partial_x P_{\delta\varepsilon}^{(k)}$.

Choosing $\varphi = P_{\delta\varepsilon}^{(k)}$ as a test function in $(\mathcal{P}_{\delta\varepsilon})$, we have:

$$\int_{A_k} H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}^{(k)}|^2 = 6s \int_{A_k} H_\varepsilon(P_{\delta\varepsilon})\, Z_\delta(P_{\delta\varepsilon})\, \partial_x P_{\delta\varepsilon}^{(k)} + \int_{\Gamma_0} G_0\, P_{\delta\varepsilon}^{(k)}.$$

Now, the last term can be estimated in the following way:

$$\begin{aligned} \int_{\Gamma_0} G_0\, P_{\delta\varepsilon}^{(k)} &\leq \|G_0\|_{L^\infty(\Gamma_0)} \int_{\Gamma_0 \cap A_k} P_{\delta\varepsilon}^{(k)} \leq \|G_0\|_{L^\infty(\Gamma_0)} \int_{A_k} P_{\delta\varepsilon}^{(k)} \\ &\leq \|G_0\|_{L^\infty(\Gamma_0)} \int_{A_k} \nabla P_{\delta\varepsilon}^{(k)}, \end{aligned}$$

using Poincaré inequality in $L^1(A_k)$. Thus, since $\|G_0\|_{L^\infty(\Gamma_0)}$ is a constant independent of ε and δ we can conclude that

$$\begin{aligned} \int_{A_k} H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}^{(k)}|^2 \leq 6|s| &\int_{A_k} \frac{1}{H_{m\delta\varepsilon}^{1/2}} \left(H_\varepsilon^3(P_{\delta\varepsilon})\,|\nabla P_{\delta\varepsilon}^{(k)}|^2\right)^{1/2} \\ &+ C \int_{A_k} \frac{1}{H_{m\delta\varepsilon}^{3/2}} \left(H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}^{(k)}|^2\right)^{1/2}. \end{aligned}$$

Hence, using Cauchy-Schwarz inequality, and since $H_\varepsilon(P_{\delta\varepsilon}) \geq H_{m\delta\varepsilon}$:

$$\int_{A_k} H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}^{(k)}|^2 \leq \left(\frac{6|s|}{H_{m\delta\varepsilon}^{1/2}} + \frac{C}{H_{m\delta\varepsilon}^{3/2}}\right)|A_k|^{1/2}\left(\int_{A_k} H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}^{(k)}|^2\right)^{1/2}.$$

It follows that:

$$\left(\int_{A_k} H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}^{(k)}|^2\right)^{1/2} \leq \left(\frac{6|s|}{H_{m\delta\varepsilon}^{1/2}} + \frac{C\,G'}{H_{m\delta\varepsilon}^{3/2}}\right)|A_k|^{1/2}.$$

Finally we get:

$$\int_{A_k} |\nabla P_{\delta\varepsilon}^{(k)}|^2 \leq \frac{1}{H_{m\delta\varepsilon}^3}\left(\frac{6|s|}{H_{m\delta\varepsilon}^{1/2}} + \frac{C\,G'}{H_{m\delta\varepsilon}^{3/2}}\right)^2 |A_k| \leq \frac{1}{H_{m\delta\varepsilon}^4}\left(6|s| + \frac{C\,G'}{H_{m\delta\varepsilon}}\right)^2 |A_k|.$$

Moreover, classical Sobolev embeddings ($H^1(\Omega) \subset L^3(\Omega)$) imply that:

$$\int_{A_k} |\nabla P_{\delta\varepsilon}^{(k)}|^2 = \int_\Omega |\nabla P_{\delta\varepsilon}^{(k)}|^2 \geq C\left(\int_\Omega |P_{\delta\varepsilon}^{(k)}|^3\right)^{2/3} = C\left(\int_{A_k} |P_{\delta\varepsilon}^{(k)}|^3\right)^{2/3},$$

where C depends only on Ω, and thus does not depend on k. Now, let $\ell > k$. Clearly $A_\ell \subset A_k$, therefore

$$\left(\int_{A_k} |P_{\delta\varepsilon}^{(k)}|^3\right)^{2/3} \geq \left(\int_{A_\ell} |P_{\delta\varepsilon}^{(k)}|^3\right)^{2/3} \geq \left(\int_{A_\ell} |\ell - k|^3\right)^{2/3} \geq (\ell - k)^2 |A_\ell|^{2/3},$$

since in A_ℓ, $P \geq \ell$, thus $P_{\delta\varepsilon}^{(k)} = P - k \geq \ell - k$.

Let us denote $\phi(\ell) = |A_\ell|$. Previous computations imply that:

$$\phi(\ell)^{2/3} \leq \frac{1}{(\ell - k)^2}\frac{C}{H_{m\delta\varepsilon}^4}\left(6|s| + \frac{C\,G'}{H_{m\delta\varepsilon}}\right)^2 \phi(k),$$

hence

$$\phi(\ell) \leq \frac{1}{(\ell - k)^3}\frac{C}{H_{m\delta\varepsilon}^6}\left(6|s| + \frac{C\,G'}{H_{m\delta\varepsilon}}\right)^3 \phi(k)^{3/2},$$

where C denotes different constants independent of ε and δ. Applying a lemma by Kinderlehrer and Stampacchia, given for example in [KS80, Lemma B.1], we conclude that:

$$\phi(d) = 0 \quad \text{for} \quad d^3 = \frac{2^9|\Omega|^{1/2}C}{H_{m\delta\varepsilon}^6}\left(6|s| + \frac{C\,G'}{H_{m\delta\varepsilon}}\right)^3.$$

Now, since $\phi(d) = 0 \longleftrightarrow |A_d| = 0 \longleftrightarrow P_{\delta\varepsilon} < d$ in Ω, the previous relation implies that:

$$\|P_{\delta\varepsilon}\|_{L^\infty(\Omega)} \leq \frac{8|\Omega|^{1/6}C}{H_{m\delta\varepsilon}^2}\left(6|s| + \frac{C\,G'}{H_{m\delta\varepsilon}}\right). \tag{3.4}$$

Moreover, since $H_{m\delta\varepsilon}$ is bounded from below by h_{0m}, the second part of the desired inequality follows immediately from the first one. □

To end this section, let us prove an L^1 estimate uniformly with respect to both ε and δ for $p_{\delta\varepsilon}$. This estimate will be used in order to show an L^1 bound on p.

Theorem 3.6. *There exists a constant C independent of ε and δ such that*

$$\int_\Omega p_{\delta\varepsilon} = \int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon}) \leq C.$$

Proof. Because of the definition of the kernel k, we know that $k(x-s, y-t) \geq \dfrac{1}{2\sqrt{2}|\Omega|}$.
Thus $H_{m\delta\varepsilon}$ satisfies

$$H_{m\delta\varepsilon} \geq (2\sqrt{2}|\Omega|)^{-1}\int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon}).$$

From (3.4), and using the fact that $H_{m\delta\varepsilon} \geq h_{0m}$, it follows that

$$\|P_{\delta\varepsilon}\|_{L^\infty(\Omega)} \leq \frac{C}{H_{m\delta\varepsilon}^2},$$

thus

$$\int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon}) \leq C\|P_{\delta\varepsilon}\|_{L^\infty(\Omega)}^{-2},$$

where C denotes some constants independent of ε and η.
Now, if we suppose that $\int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon})$ tends to infinity when ε tends to zero, $\|P_{\delta\varepsilon}\|_{L^\infty(\Omega)}$ would tend to zero. Thus $\|P_{\delta\varepsilon}\|_{L^\infty(\Omega)} \leq \dfrac{A}{2}$ when ε tends to zero. But, from the definition of γ_ε and the monotonicity of γ we have, for ε small enough

$$\int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon}) \leq \int_\Omega \gamma(P_{\delta\varepsilon}) \leq \int_\Omega \gamma\left(\frac{A}{2}\right).$$

This leads to a contradiction and concludes the proof. □

3.4 Passing to the limit - Additional estimates

3.4.1 First estimates for the reduced problem

Since $(P_{\delta\varepsilon})_{\delta,\varepsilon}$ is bounded in $H^1(\Omega)$, there exists P_δ such that it converges strongly to P_δ up to a subsequence for $\varepsilon \to 0$. In a similar way, there exists P such that $(P_\delta)_\delta$ converges strongly to P up to a subsequence for $\delta \to 0$. The following theorem states some immediate estimates on a limit P of $(P_{\delta\varepsilon})_{\delta,\varepsilon}$.

Theorem 3.7. *Let P constructed as above. P satisfies:*

$$\|P\|_{L^\infty(\Omega)} \leq A.$$

Moreover there exists a constant C such that

$$\|p\|_{L^1(\Omega)} = \int_\Omega \gamma(P) \leq C.$$

Before proving this theorem, let us state the following lemma, whose proof can be find in Bayada and Bellout [BB05, Lemma 6].

Lemma 3.8. *Let $E =]-M, M[\times]-M, M[\subset \mathbb{R}^2$ and let v_n be a sequence of functions $L^2(E)$ which converges almost everywhere to v. If*

$$\Omega_\tau = \{(x,y) \in E,\, v(x,y) \geq A + \tau\},$$
$$\Omega_\tau^n = \{(x,y) \in E,\, v_n(x,y) \geq A\}$$

and $|\Omega_\tau| \neq 0$ then there exists $n_0 > 0$ such that $\forall n \geq n_0$, $|\Omega_\tau^n| \geq \frac{1}{2}|\Omega_\tau|$.

Proof. (of Theorem 3.7) The first estimate is obtained by contradiction. Let us assume that there exists $\tau > 0$ such that $\Omega_\tau = \{(x,y) \in \Omega,\, P(x,y) \geq A + \tau\}$ has a non-zero measure. Then, applying Lemma 3.8 to $P_{\delta\varepsilon}$, it follows that

$$\int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon}) \geq \frac{1}{2}|\Omega_\tau|\gamma(A-\varepsilon) \xrightarrow[\varepsilon,\delta\to 0]{} +\infty,$$

which is in contradiction with Theorem 3.6.
For the second estimate, let $\tau > 0$, and

$$P_{\delta\varepsilon}^\tau(x,y) = \inf(P_{\delta\varepsilon}(x,y), A-\tau).$$

Since $P_{\delta\varepsilon}$ converges strongly to P in $L^2(\Omega)$, $P_{\delta\varepsilon}^{\tau}$ converges strongly to

$$P^{\tau}(x,y) = \inf(P(x,y), A-\tau)$$

in $L^2(\Omega)$. Now, it is clear that $\gamma_\varepsilon(P_{\delta\varepsilon}^{\tau}) \leq \gamma_\varepsilon(P_{\delta\varepsilon})$, since γ_ε is increasing, thus

$$\int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon}^{\tau}) \leq C$$

by Theorem 3.6. Moreover, for ε small enough, $\gamma_\varepsilon(P_{\delta\varepsilon}^{\tau}) = \gamma(P_{\delta\varepsilon}^{\tau}) \leq \gamma(A-\tau) \leq C$, with C a constant independent of ε and η, but which depends on τ. Thus for fixed τ, $\gamma_\varepsilon(P_{\delta\varepsilon}^{\tau})$ converges to $\gamma(P^{\tau})$ in $L^1(\Omega)$, and

$$\int_\Omega \gamma(P^{\tau}) \leq C.$$

Now, letting τ go to zero, we obtain from the monotone convergence theorem that

$$\int_\Omega \gamma(P) \leq C,$$

since $\gamma(P^{\tau}) \xrightarrow[\tau\to 0]{} \gamma(P)$. □

3.4.2 Additional estimates

It remains to pass to the limit in the non-linear terms of $(\mathcal{P}_{\delta\varepsilon})$. Let us explain in the following the main steps of the proof.

Main idea for passing to the limit

It is well-known ([BTV96]) that the estimates obtained in the previous section are not enough to prove an unconditional existence result for the problem $(\mathcal{P})$.

In order to treat the non-linear term when passing to the limit in the equation when $\varepsilon \to 0$ and $\delta \to 0$, stronger estimates on H_ε have to be proved. For the term $H_\varepsilon^3 \nabla P_{\delta\varepsilon}$, since $\nabla P_{\delta\varepsilon}$ converges weakly in $L^2(\Omega)$, it suffices that H_ε converges strongly in $L^6(\Omega)$. To this purpose, we will show that H_ε is bounded in $W^{1,6}(\Omega)$.

To this end, we will see that it is enough to show that $\gamma_\varepsilon(P_{\delta\varepsilon})$ is bounded in $L^6(\Omega)$, since the convolution kernel k in H_ε has a regularizing effect. To prove this, we will introduce the function $\gamma_\varepsilon(P_{\delta\varepsilon})^{\sigma}$, for some $\sigma > 0$, and prove that this function is bounded in $H^1(\Omega)$. Thus, we will be able to conclude that $\gamma_\varepsilon(P_{\delta\varepsilon})$ is bounded in $L^{\sigma r}(\Omega)$, for any $r \geq 2$, and therefore at least in $L^6(\Omega)$ (see Proposition 3.9).

However, since $\gamma_\varepsilon(P_{\delta\varepsilon})^\sigma$ will be used as a test function in the weak formulation, it will be necessary to introduce a cut-off function ψ and consider $\gamma_\varepsilon(P_{\delta\varepsilon})^\sigma\psi(P_{\delta\varepsilon})$.

Detailed estimates

In order to obtain the needed estimates, let us introduce an additional hypothesis on the asymptotic behaviour of the piezoviscosity law. More precisely, we suppose that:

$$\exists p^* > 0, \quad \eta(p) = (p+p_0)^\beta \quad \text{for } p \geq p^*, \quad \text{with} \quad \beta > 1, \quad p_0 \geq 0, \tag{3.5}$$

where p_0 and β are constants. Actually we could suppose only that $\eta(p) \underset{+\infty}{\sim} (p+p_0)^\beta$, which in particular allows to consider Barus law for finite values of p and an asymptotic behaviour of this sort (see Introduction for physical explanation).
The following proposition is the key of the needed estimate on H_ε.

Proposition 3.9. *Let $\eta(p)$ satisfy the condition (3.5). For $1 < \beta < \dfrac{3}{2}$, $\gamma_\varepsilon(P_{\delta\varepsilon})$ satisfies*

$$\|\gamma_\varepsilon(P_{\delta\varepsilon})\|_{L^6(\Omega)} \leq C,$$

where C is independent of ε and δ.

Before starting the proof, let us introduce the following functions and notations. Defining $a_1 = \displaystyle\int_0^{p^*} \frac{\mathrm{d}s}{\eta(s)}$, hypothesis (3.5) implies that $A = a_1 + \dfrac{(p^*+p_0)^{1-\beta}}{\beta-1}$. Let us denote $A = a_1 + a_2$. Moreover, let ε be small enough, so that $\varepsilon < \dfrac{a_2}{3}$.
Then we introduce the function

$$f_\varepsilon(P_{\delta\varepsilon}) = (p_0 + \gamma_\varepsilon(P_{\delta\varepsilon}))^\alpha\, \psi(P_{\delta\varepsilon}), \tag{3.6}$$

where α will be chosen below and where $\psi(t) \in C^2(\mathbb{R})$ is a cut-off function defined by $\psi'(t) \geq 0$ and

$$\psi(t) = \begin{cases} 0 & \text{for } t < a_1 + \dfrac{a_2}{3}, \\ 1 & \text{for } a_1 + \dfrac{2a_2}{3} < t. \end{cases} \tag{3.7}$$

Let us observe that the function $f_\varepsilon \in V$ defined in this way is an admissible test function for the problem $(\mathcal{P}_{\delta\varepsilon})$, since $P_{\delta\varepsilon}|_{\Gamma_1} = 0$, and thus on Γ_1 we have $\psi(P_{\delta\varepsilon}) = 0$.

Proof. The result of Proposition 3.9 will be proved under the following condition on the parameters :

$$2 - \alpha - \beta \geq 0, \quad 1 < \beta < \frac{3}{2}, \quad 1 + \alpha - \beta > 0. \tag{3.8}$$

Let us observe that the set of all α and β satisfying the condition (3.8) is non-empty. In particular for any $\beta \in \left]1, \frac{3}{2}\right[$, there exists an α such that (α, β) satisfies the condition (3.8).

Introducing the function $g_{\delta\varepsilon} = \gamma_\varepsilon(P_{\delta\varepsilon})^\sigma\, \psi(P_{\delta\varepsilon})$ for $\sigma > 0$, we will show that $\|g_{\delta\varepsilon}\|_{H^1(\Omega)}$ is bounded. Moreover, let us denote (see Figure 3.7)

$$\Omega_1 = \{(x,y) \in \Omega,\ P_{\delta\varepsilon}(x,y) \leq a_1 + \frac{2a_2}{3}\}$$
$$\Omega_2 = \{(x,y) \in \Omega,\ a_1 + \frac{2a_2}{3} < P_{\delta\varepsilon}(x,y) < A - \varepsilon\}$$
$$\Omega_3 = \{(x,y) \in \Omega,\ A - \varepsilon \leq P_{\delta\varepsilon}(x,y)\}$$

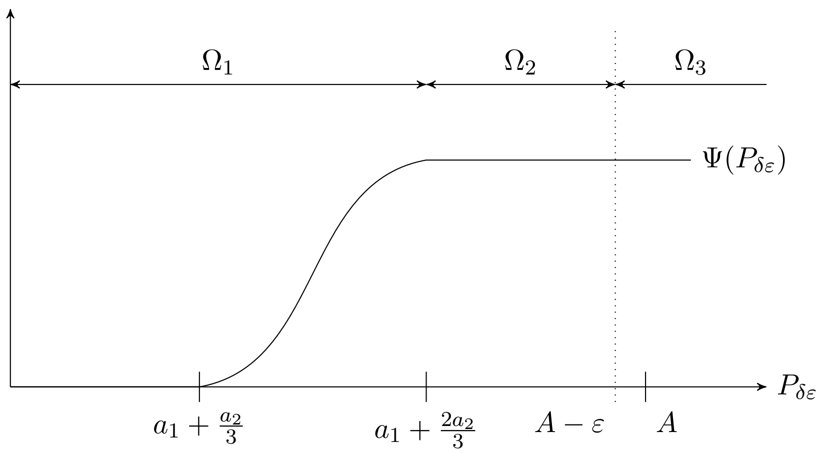

Figure 3.7: Partition of Ω and profile of $\psi(P_{\delta\varepsilon})$

Then $\bar{\Omega} = \bar{\Omega}_1 \amalg \bar{\Omega}_2 \amalg \bar{\Omega}_3$, since these three sets are pairwise disjoints.
Now, expanding $|\nabla g_{\delta\varepsilon}(P_{\delta\varepsilon})|^2$, it follows that

$$\int_\Omega |\nabla g_{\delta\varepsilon}(P_{\delta\varepsilon})|^2 = \sigma^2 \int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon})^{2(\sigma-1)}(\gamma_\varepsilon')^2 |\nabla P_{\delta\varepsilon}|^2 \psi(P_{\delta\varepsilon})^2$$
$$+ 2\sigma \int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon})^{2\sigma-1}(\gamma_\varepsilon')^2 |\nabla P_{\delta\varepsilon}|^2 \psi'(P_{\delta\varepsilon})\, \psi(P_{\delta\varepsilon}) + \int_\Omega \gamma_\varepsilon(P_{\delta\varepsilon})^{2\sigma} |\nabla P_{\delta\varepsilon}|^2 \psi'(P_{\delta\varepsilon})^2.$$

- In Ω_1, each of these three terms are bounded independently of ε and δ, since far from $A - \varepsilon$, $P_{\delta\varepsilon}$ is bounded, and so is $\gamma_\varepsilon(P_{\delta\varepsilon})$.

- In Ω_3, $\gamma_\varepsilon(P_{\delta\varepsilon}) = \gamma(A - \varepsilon)$ is constant, hence $\gamma_\varepsilon' = 0$. Moreover $\psi \equiv 1$, thus $\psi' \equiv 0$.

Therefore $\int_{\Omega_3} |\nabla g_{\delta\varepsilon}(P_{\delta\varepsilon})|^2 = 0.$

- In Ω_2, we have again $\psi \equiv 1$. It remains

$$\int_{\Omega} |\nabla g_{\delta\varepsilon}(P_{\delta\varepsilon})|^2 \leq C + \sigma^2 \int_{\Omega_2} \gamma_\varepsilon(P_{\delta\varepsilon})^{2(\sigma-1)}(\gamma'_\varepsilon(P_{\delta\varepsilon}))^2|\nabla P_{\delta\varepsilon}|^2.$$

Now, since $f_\varepsilon = (p_0 + \gamma_\varepsilon(P_{\delta\varepsilon}))^\alpha \; \psi(P_{\delta\varepsilon}) = (p_0 + \gamma_\varepsilon(P_{\delta\varepsilon}))^\alpha$ in Ω_2, we have $\gamma_\varepsilon(P_{\delta\varepsilon}) = f_\varepsilon^{1/\alpha} - p_0$, and thus

$$\gamma'_\varepsilon(P_{\delta\varepsilon}) = \frac{1}{\alpha} f_\varepsilon^{\frac{1-\alpha}{\alpha}} \; f'_\varepsilon = \frac{1}{\alpha} \left(p_0 + \gamma_\varepsilon(P_{\delta\varepsilon})\right)^{1-\alpha} \; f'_\varepsilon \tag{3.9}$$

On the other hand, hypothesis (3.5) implies that

$$a(p) = a_1 + \int_{p^*}^{p} (s+p_0)^{-\beta} \mathrm{d}s = A + \frac{(p+p_0)^{1-\beta}}{1-\beta}.$$

Therefore

$$\gamma_\varepsilon(P_{\delta\varepsilon}) = ((1-\beta)(P_{\delta\varepsilon} - A))^{\frac{1}{1-\beta}} - p_0,$$

hence

$$\gamma'_\varepsilon(P_{\delta\varepsilon}) = ((1-\beta)(P_{\delta\varepsilon} - A))^{\frac{\beta}{1-\beta}} = (\gamma_\varepsilon(P_{\delta\varepsilon}) + p_0)^\beta \tag{3.10}$$

Using the two expressions of $\gamma'_\varepsilon(P_{\delta\varepsilon})$ obtained in (3.9) and (3.10), we get that

$$(\gamma'_\varepsilon(P_{\delta\varepsilon}))^2 \leq \frac{1}{\alpha} \left(p_0 + \gamma_\varepsilon(P_{\delta\varepsilon})\right)^{1-\alpha+\beta} \; f'_\varepsilon(P_{\delta\varepsilon}),$$

and conclude that

$$\int_{\Omega} |\nabla g_{\delta\varepsilon}(P_{\delta\varepsilon})|^2 \leq C + \sigma^2 \int_{\Omega_2} \frac{1}{\alpha} \left(p_0 + \gamma_\varepsilon(P_{\delta\varepsilon})\right)^{1-\alpha+\beta+2(\sigma-1)} \; f'_\varepsilon(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2.$$

Choosing $\sigma = \dfrac{1+\alpha-\beta}{2} > 0$, it follows that

$$\int_{\Omega} |\nabla g_{\delta\varepsilon}(P_{\delta\varepsilon})|^2 \leq C + \sigma^2 \int_{\Omega_2} \frac{1}{\alpha} f'_\varepsilon |\nabla P_{\delta\varepsilon}|^2.$$

Now, using Proposition 3.10 below, we can conclude that

$$\int_\Omega |\nabla g_{\delta\varepsilon}(P_{\delta\varepsilon})|^2 \leq C, \tag{3.11}$$

thus $\|g_{\delta\varepsilon}\|_{H^1(\Omega)}$ is bounded, and this implies that

$$\|\gamma_\varepsilon(P_{\delta\varepsilon})^\sigma\|_{H^1(\Omega)} \leq C.$$

Finally, using Sobolev embeddings, it follows that $\gamma_\varepsilon(P_{\delta\varepsilon})$ is bounded in $L^{\sigma r}(\Omega)$ for any $r \geq 2$, thus in $L^6(\Omega)$ for r big enough ($r = 6/\sigma$). Therefore, we proved that

$$\|\gamma_\varepsilon(P_{\delta\varepsilon})\|_{L^6(\Omega)} \leq C.$$

□

Now, let us present the proof of the following result, which has been used in the previous proof in order to establish (3.11).

Proposition 3.10. *Suppose that there exists a constant $c^* > 0$ such that ψ satisfies*

$$\psi'(t) \leq c^*\psi(t), \quad \forall t > a_1 + \frac{a_2}{2}, \tag{3.12}$$

and suppose that $0 < \alpha < 1$ and $2 - \alpha - \beta \geq 0$. Then the following inequality holds true:

$$\int_\Omega |\nabla P_{\delta\varepsilon}|^2 f'_\varepsilon(P_{\delta\varepsilon}) \leq C,$$

where C is a constant independent of ε and δ.

Remark 3.11. *Let us observe that any C^2-function ψ satisfying the condition (3.7) satisfies also the condition (3.12). Indeed, ψ' is a C^1-function on $\left[a_1 + \frac{a_2}{2}, A\right]$, thus bounded. Moreover, for $t \in \left[a_1 + \frac{a_2}{2}, A\right]$, $\psi(t) \geq \psi(a_1 + \frac{a_2}{2})$.*

Proof. *Step 1:* Let us obtain a bound for the term $\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2 f'_\varepsilon(P_{\delta\varepsilon})$ independent of η and ε. To this end, choosing $\varphi = f_\varepsilon(P_{\delta\varepsilon}) \in V$ as a test function in $(\mathcal{P}_{\delta\varepsilon})$, we obtain

$$\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2 f'_\varepsilon(P_{\delta\varepsilon}) = 6s\int_\Omega H_\varepsilon(P_{\delta\varepsilon})Z_\delta(P_{\delta\varepsilon})f'_\varepsilon(P_{\delta\varepsilon})\partial_x P_{\delta\varepsilon} + \int_{\Gamma_0} G_0 f_\varepsilon(P_{\delta\varepsilon})$$

and, using Young inequality,

$$\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2\, f_\varepsilon'(P_{\delta\varepsilon}) \le \frac{1}{2}\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2\, f_\varepsilon'(P_{\delta\varepsilon}) + 18s^2\int_\Omega \frac{1}{H_\varepsilon(P_{\delta\varepsilon})} f_\varepsilon'(P_{\delta\varepsilon}) + \int_{\Gamma_0} G_0 f_\varepsilon(P_{\delta\varepsilon}).$$

Thus the following estimate holds true:

$$\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2\, f_\varepsilon'(P_{\delta\varepsilon}) \le 36s^2\int_\Omega \frac{1}{H_\varepsilon(P_{\delta\varepsilon})} f_\varepsilon'(P_{\delta\varepsilon}) + 2\int_{\Gamma_0} G_0 f_\varepsilon(P_{\delta\varepsilon}). \tag{3.13}$$

On the other hand, the trace operator is continuous from $W^{1,1}(\Omega)$ to $L^1(\Gamma)$ (see for example [Neč67]). Let us denote $\bar{G}_0$ the extension of G_0 to Ω such that $\bar{G}_0(x,y) = G_0(y)$. Thus, using Poincaré inequality in $L^1(\Omega)$ for $f_\varepsilon(P_{\delta\varepsilon})|_{\Gamma_1} = 0$, it follows

$$\begin{aligned}\|G_0 f_\varepsilon(P_{\delta\varepsilon})\|_{L^1(\Gamma_0)} &\le \|G_0 f_\varepsilon(P_{\delta\varepsilon})\|_{L^1(\Gamma)} \le C\|\bar{G}_0 f_\varepsilon(P_{\delta\varepsilon})\|_{W^{1,1}(\Omega)}\\ &\le C\|\bar{G}_0\|_{L^\infty(\Omega)}\|f_\varepsilon(P_{\delta\varepsilon})\|_{W^{1,1}(\Omega)} \le C\|\bar{G}_0\|_{L^\infty(\Omega)}\|\nabla f_\varepsilon(P_{\delta\varepsilon})\|_{L^1(\Omega)}\\ &\le C\|\bar{G}_0\|_{L^\infty(\Omega)}\int_\Omega |f_\varepsilon'(P_{\delta\varepsilon})||\nabla P_{\delta\varepsilon}|.\end{aligned}$$

Now, using Cauchy-Schwarz and Young inequalities, we have:

$$2\int_{\Gamma_0} G_0 f_\varepsilon(P_{\delta\varepsilon}) \le C\left(\frac{1}{2}\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2\, f_\varepsilon'(P_{\delta\varepsilon}) + \frac{1}{2}\int_\Omega \frac{1}{H_\varepsilon^3(P_{\delta\varepsilon})} f_\varepsilon'(P_{\delta\varepsilon})\right),$$

where C denotes a constant independent of ε and δ. Using this relation in (3.13), together with the definition of h_{0m}, it follows that

$$\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2\, f_\varepsilon'(P_{\delta\varepsilon}) \le \frac{C}{h_{0m}}\int_\Omega f_\varepsilon'(P_{\delta\varepsilon}) + \frac{C\|\bar{G}_0\|_{L^\infty}^2}{h_{0m}^3}\int_\Omega \bar{f_\varepsilon}'(P_{\delta\varepsilon})$$

and thus

$$\int_\Omega H_\varepsilon^3(P_{\delta\varepsilon})|\nabla P_{\delta\varepsilon}|^2\, f_\varepsilon'(P_{\delta\varepsilon}) \le C\int_\Omega f_\varepsilon'(P_{\delta\varepsilon}). \tag{3.14}$$

Step 2: Let us recall the following lemma (see [BB05, p. 147]).

Lemma 3.12. *Suppose that $\eta(p)$ satisfies (3.5). Let f_ε be defined as in (3.6) and ψ as* *in (3.7). Then there exist some constants C and M independent of ε and δ such that*

$$f'_\varepsilon(t) \leq M + C(Q + \gamma_\varepsilon(t))\psi(t) \quad \forall t > 0. \tag{3.15}$$

Now, using (3.14) and (3.15):

$$\int_\Omega |\nabla P_{\delta\varepsilon}|^2 f'_\varepsilon(P_{\delta\varepsilon}) \leq C \int_\Omega M + C(Q + \gamma_\varepsilon(P_{\delta\varepsilon}))\psi(P_{\delta\varepsilon})$$

where we used again that $H_\varepsilon \geq h_{0m}$ in order to get rid of the term $H^3_\varepsilon(P_{\delta\varepsilon})$. Therefore, using the fact that $\gamma_\varepsilon(P_{\delta\varepsilon})$ is bounded in $L^1(\Omega)$ (Theorem 3.6), and that $\psi(t)$ is a function in $C^2(\mathbb{R})$, we obtain

$$\int_\Omega |\nabla P_{\delta\varepsilon}|^2 f'_\varepsilon(P_{\delta\varepsilon}) \leq C, \tag{3.16}$$

which concludes the proof. □

3.4.3 Passing to the limit

In this section, we state the existence theorem. In the proof, it is shown that Proposition 3.9 provides the key estimate in order to pass to the limit. The following property of the kernel k will be used.

Lemma 3.13. *Let $k_0 \geq 0$, and let k be defined on $\mathbb{R}^2 \setminus (0,0)$ by*

$$k(x,y) = \frac{k_0}{\sqrt{x^2+y^2}}, \qquad \forall (x,y) \in \mathbb{R}^2 \setminus (0,0).$$

Let $f \in L^p(\Omega)$, for $1 < p < \infty$. Then the convolution $k \star f$ belongs to $W^{1,p}(\Omega)$. Moreover, there exists a constant C such that for any $f \in L^p(\Omega)$:

$$\|k \star f\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Proof. This result is a direct application of [Ste70, II,4.2, Th. 3]. Since $k \in L^1_{\text{loc}}(\mathbb{R})$, it is well known that $k \star f \in L^6(\Omega)$. Now let us compute

$$\nabla(k \star f) = \nabla k \star f = \frac{x}{|x|^3} \star f.$$

Let $\omega(x) = \frac{x}{|x|}$. The cancellation property (24) of [Ste70, p. 39] and the smoothness property (25) are satisfied by this function. We can write

$$\nabla(k \star f) = \frac{\omega(x)}{|x|^2} \star f.$$

Thus [Ste70, II,4.2, Th. 3] can be applied with $T(f) = \lim_{\varepsilon \to 0} T_\varepsilon(f)$, for

$$T_\varepsilon(f)(x) = \int\limits_{|y| \geq \varepsilon} \frac{\omega(y)}{|y|^2} f(x-y)dy.$$

It follows that $T(f) \in L^p(\Omega)$, which means that $\nabla(k \star f) \in L^p(\Omega)$, and thus $k \star f \in W^{1,p}(\Omega)$. □

Now let us state the main theorem.

Theorem 3.14. *Let P be defined as a limit of $P_{\delta\varepsilon}$ as in Theorem 3.7 and θ be the limit of $Z_\delta(P_\delta)$ for $\delta \to 0$. Under hypothesis (3.5) and (3.8), (P, θ) solves the following problem:*

$$(\mathcal{P}') \begin{cases} \int\limits_\Omega H^3(P)\nabla P \cdot \nabla\varphi = 6s \int\limits_\Omega H(P)\,\theta\,\partial_x\varphi + \int\limits_{\Gamma_0} G_0\,\varphi\,, \quad \forall\varphi \in V, \\ \theta \in \mathcal{H}(P), \end{cases}$$

with

$$H(P,x,y) = h_0(x) + \int\limits_\Omega k(x-s, y-t)\,\gamma(P(s,t))\,ds\,dt.$$

Proof. It follows from Lemma 3.13 that if $\gamma_\varepsilon(P_{\delta\varepsilon})$ is bounded in $L^6(\Omega)$, then $H_\varepsilon(P_{\delta\varepsilon})$ given by (3.1) is bounded in $W^{1,6}(\Omega)$, and thus $H^3_\varepsilon(P_{\delta\varepsilon})$ is bounded in $H^1(\Omega)$.
Hence we have the following convergences, for $\varepsilon \to 0$ and then $\eta \to 0$:

$$\begin{aligned} P_{\delta\varepsilon} \to P \text{ in } L^2(\Omega), &\qquad \text{and} \qquad Z_\delta(P_{\delta\varepsilon}) \rightharpoonup \theta \text{ in } L^\infty(\Omega), \\ H^3_\varepsilon(P_{\delta\varepsilon}) \rightharpoonup J \text{ in } H^1(\Omega), &\qquad \text{and thus} \quad H^3_\varepsilon(P_{\delta\varepsilon}) \to J \text{ in } L^2(\Omega). \end{aligned}$$

Now, we showed in the proof of Theorem 3.7 that $\gamma_\varepsilon(P_{\delta\varepsilon})$ converges to $\gamma(P)$ in $L^1(\Omega)$. Thus, by the uniqueness of the limit, it follows that $J = H^3(P)$.
Therefore, it is possible to pass to the limit in every term of problem $(\mathcal{P}_{\delta\varepsilon})$. It remains to prove that $\theta \in \mathcal{H}(P)$. We have

$$0 \leq \int\limits_\Omega P_{\delta\varepsilon}(1 - Z_\delta(P_{\delta\varepsilon})) \leq \int\limits_{\{P_{\delta\varepsilon} \leq \delta\}} P_{\delta\varepsilon}(1 - Z_\delta(P_{\delta\varepsilon})) \leq \int\limits_{\{P_{\delta\varepsilon} \leq \delta\}} \delta \leq \delta|\Omega|.$$

Letting ε and δ tend to zero, it follows that

$$\int\limits_\Omega P(1-\theta) = 0,$$

and since $P \geq 0$, $1-\theta \geq 0$, we conclude that $P(1-\theta) = 0$ almost everywhere. Thus, (P, θ) is a solution of the problem $(\mathcal{P}')$. This concludes the proof. □

Remark 3.15. *Now, if we consider the problem in one dimension, it is possible to prove that problems* $(\mathcal{P})$ *and* $(\mathcal{P}')$ *are equivalent. Indeed, since the embedding* $H^1 \hookrightarrow C^0$ *is compact in one-dimensional space, the weak convergence of* $P_{\delta\varepsilon}$ *implies actually that* $P_{\delta\varepsilon}$ *converges uniformly to* P*. Thus,* $\gamma_\varepsilon(P_{\delta\varepsilon})$ *also converges uniformly to* $\gamma(P)$*, thus* $P < A$ *and problems* $(\mathcal{P})$ *and* $(\mathcal{P}')$ *are equivalent.*
However, in the two-dimensional case, we are not able to prove that problems $(\mathcal{P})$ *and* $(\mathcal{P}')$ *are equivalent. The estimate* $\|P\|_{L^\infty(\Omega)} \leq A$ *we obtained previously is not enough to prove the existence of a solution of* $(\mathcal{P})$*, since* p *can be infinite and thus does not lie in* $H^1(\Omega)$*. In fact, since physically the pressure* p *cannot be infinite, it is relevant to have studied problem* $(\mathcal{P}')$*.*

Bibliography

[AC94] S. J. Alvarez and J. Carrillo. A free boundary problem in theory of lubrication. *Comm. Partial Differential Equations*, 19(11-12):1743–1761, 1994.

[BB05] G. Bayada and H. Bellout. An unconditional existence result for the quasi-variational elastohydrodynamic free boundary value problem. *J. Differential Equations*, 216(1):134–152, 2005.

[BC86] G. Bayada and M. Chambat. Sur quelques modélisations de la zone de cavitation en lubrification hydrodynamique. *J. Méc. Théor. Appl.*, 5(5):703–729, 1986.

[Bel03] H. Bellout. Existence of a solution to the line contact problem of elastohydrodynamic lubrication. *European J. Appl. Math.*, 14(3):279–290, 2003.

[BKS78] H. Brezis, D. Kinderlehrer, and G. Stampacchia. Sur une nouvelle formulation du probléme de l'écoulement à travers une digue. *C. R. Acad. Sci. Paris Sér. A-B*, 287(9):A711–A714, 1978.

[Bla01] S. Blair. The pressure-viscosity coefficient of a perfluorinated polyether over a wide temperature range. *J. of Tribology*, 123(1):50–53, 2001.

[BTV96] G. Bayada, M. El-A. Talibi, and C. Vázquez. Existence of solutions for elastohydrodynamic piezoviscous lubrication problems with a new model of cavitation. *European J. Appl. Math.*, 7(1):63–73, 1996.

[Chi84] M. Chipot. *Variational inequalities and flow in porous media*, volume 52 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[EA75] H. G. Elrod and M. L. Adams. A computer program for cavitation. *Cavitation and related phenomena in lubrication - Proceedings - Mech. Eng. Publ. ltd*, pages 37–42, 1975.

[Flo73] L. Floberg. Lubrication of two rotating cylinders at variable lubricant supply with reference to the tensile strength of the liquid lubricant. *ASME J. Lub. Technol.*, 95:155–165, 1973.

[FND$^+$97] J. Frene, D. Nicolas, B. Deugueurce, D. Berthe, and M. Godet. *Hydordynamic lubrication: bearings and thrust bearings*. Elsevier Science, 1997.

[GLT76] R. Glowinski, J.-L. Lions, and R. Trémolières. *Analyse numérique des inéquations variationnelles. Tome 1.* Dunod, Paris, 1976. Théorie générale premiéres applications, Méthodes Mathématiques de l'Informatique, 5.

[Hu90] B. Hu. A quasi-variational inequality arising in elastohydrodynamics. *SIAM J. Math. Anal.*, 21(1):18–36, 1990.

[KS80] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.

[Neč67] J. Nečas. *Les méthodes directes en théorie des équations elliptiques*. Masson et Cie, Éditeurs, Paris, 1967.

[OW85] J. T. Oden and S. R. Wu. Existence of solutions to the Reynolds equation of eslastohydrodynamic lubrication. *Int. Jour. Eng. Sci.*, 31:207–215, 1985.

[Rod87] J.-F. Rodrigues. *Obstacle problems in mathematical physics*, volume 134 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. , Notas de Matemática [Mathematical Notes], 114.

[Rod93] J.-F. Rodrigues. Remarks on the Reynolds problem of elastohydrodynamic lubrication. *European J. Appl. Math.*, 4(1):83–96, 1993.

[Ste70] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[Sze98] A. Z. Szeri. *Fluid Film Lubrication: Theory and Design*. Cambridge University Press, 1998.

[Ver02] P. Vergne. *Comportement rhéologique des lubrifiants et lubrification, Approches expérimentales.* PhD Thesis, INSA de Lyon, 2002.

4

Diphasic flows in thin films

Abstract *In this chapter, we are interested in a model for diphasic fluids which takes into account not only the hydrodynamical effects at the interface between the two fluids, but also the chemical effects. We introduce an heuristic limit problem in "thin films", which is a system coupling the Reynolds equation and the hydrodynamical Cahn-Hilliard equation. We study this model, and prove an existence result under some smallness condition on the data.*

4.1 Introduction

In many applications, the geometry of the flow is anisotropic (i.e. one dimension is small with respect to the others), e.g. in lubrication problems. In this case, the flow of a fluid between two close surfaces in relative motion is described by an asymptotic approximation of the Navier-Stokes equations, the Reynolds equation. This equation makes it possible to uncouple the pressure and the velocity in the Newtonian setting. Indeed, the pressure in thin domains is considered to be independent of the direction in which the domain is thin. Thus an equation on the pressure only is obtained, and the velocity can be deduced from the pressure. This approach was introduced by Reynolds, and has been rigorously justified in [BC86] for the Stokes equation, and generalized afterwards in many works: for the steady-case Navier-Stokes equations [ABC94], for the unsteady case [BCC99], for compressible fluids with the perfect gases law [MPS05]... It is of interest to investigate how this approach can be used for the case of a two fluid flow. A partial answer to this question has been given in [Pao03], for a different model of the diphasic aspect than the one considered in this work.

In the case of two-phase fluids (or of a two fluid flow), several models are known. The most recent one in lubrication applications is to introduce a variable viscosity η, which is either equal to the viscosity η_1 of one fluid or the viscosity η_2 of the other fluid (that is to say that the fluids are considered to be non-miscible). The behavior of η is described by a transport equation. In that case, when supposing the interface between the two fluids is the graph of a function, the asymptotic equations corresponding to the thin film approximation can be interpreted as a generalized Buckley-Leverett equation, which governs the behavior of the saturation (i.e. the proportion of one fluid in the mixture) inside the gap, coupled with a generalized Reynolds equation, which governs the behavior of the pressure. These equations are investigated in [Pao03] without shear effects, and in [BMV06] with the shear effects. One of the main disadvantages of the method is that the fluid interface is supposed to be the graph of a function, which hinders for example the formation of bubbles. In addition, this kind of models only takes into account hydrodynamical effects between the two phases, and the surface tension is neglected.

The second class of models to describe diphasic flows, which has been used up to now only on the whole Navier-Stokes equations, is the class of the so-called diffuse interface models. They are based on chemical properties at the interface between the two fluids, which enable an exchange between the two phases. In this paper, a Cahn-Hilliard equation enhanced with a transport term is used, which involves an interaction potential. Thus this model describes both the chemical and the hydrodynamical properties of the flow. To this end, an order parameter φ is introduced, for example the volumic fraction of one phase in the mixture. The surface tension can be taken into account *via* an additional term depending on φ in the Navier-Stokes equations. This kind of model has been studied for the complete Navier-Stokes equations in [Boy99], and for viscoelastic fluids in [Chu03].

In this chapter, we introduce in a heuristic way an asymptotic system (i.e. a thin film approximation) for a diphasic fluid in a thin film modelled by the Cahn-Hilliard equation. Proceeding in a similar way as for the Newtonian case, the Navier-Stokes equations is approximated by a modified Reynolds equation, in which the viscosity is not constant anymore. Depending on the scaling order of the capillarity term, the additional surface tension term in the Navier-Stokes equations remains or not in the Reynolds equation. For the Cahn-Hilliard equation, we choose the scaling order of the physical parameters such that the structure of the equation is preserved. Therefore, we study from a theoretical point of view the Reynolds (with or without the surface tension term)/Cahn-Hilliard system, and prove the existence and the regularity of a solution under a smallness assumption on the initial data and the geometry.

Let us describe briefly the main steps of the mathematical analysis. First, we study the Reynolds equation without the surface tension term and investigate the regularity of the pressure and the velocity as functions of the order parameter. Next, we prove the existence of a solution to the system Reynolds/Cahn-Hilliard, by using a Galerkin process, which consists in introducing finite dimension approximations of φ. After obtaining *a priori* estimates for these approximations, we conclude that they converge to a solution of the system Reynolds/Cahn-Hilliard. The last part consists in adapting this proof to the case with surface tension.

This chapter is organized as follows. In Section 4.2, we describe the diffuse-interface model and explain how it is obtained heuristically as an asymptotic limit from the Navier-Stokes and Cahn-Hilliard equation. Moreover, we describe two models depending on the order of magnitude of the capillarity coefficient. In Section 4.3, notations are introduced and lift operators for the boundary values are built. Moreover, some specific results on trace estimates and Poincaré inequalities are presented. They are used in the following sections for obtaining the *a priori* estimates. Section 4.4 is dedicated to the study of the Reynolds equation, by proving the regularity of the pressure and the velocity as functions of φ. The Galerkin process and the *a priori* estimates on the order parameter are presented in Section 4.5. Further, convergence results of the Galerkin approximations are obtained in Section 4.6. At last, in Section 4.7, the process is adapted to the case when the surface tension is taken into account.

4.2 The model and the governing equations

In this section, we will first present how a fluid is described in a thin domain by the Reynolds equation. Next, we introduce the hydrodynamical Cahn-Hilliard model for any fluid. Lastly, we combine both aspects for modelling a diphasic fluid in a thin domain.

4.2.1 Modeling of one fluid in a thin domain

For $\varepsilon > 0$, consider a thin domain $\hat{\Omega}^\varepsilon = \big\{(x, y) \in \mathbb{R}^2,\ 0 < x < L,\ 0 < y < \varepsilon h(x)\big\}$, with h a regular mapping from $[0, L]$ to $\mathbb{R}_+^*$. An incompressible fluid flow is described by the Navier-Stokes equations on the velocity $\hat{\boldsymbol{u}}^\varepsilon = (\hat{u}^\varepsilon, \hat{v}^\varepsilon)$ and the pressure $\hat{p}^\varepsilon$, depending on the physical parameters of the fluid (the density ρ, the viscosity η), and the external forces $\mathcal{F}$ (for example the gravity $\rho\,\boldsymbol{g}$):

$$\rho\left(\partial_t \hat{\boldsymbol{u}}^\varepsilon + \hat{\boldsymbol{u}}^\varepsilon \cdot \nabla \hat{\boldsymbol{u}}^\varepsilon\right) - \operatorname{div}\left(\eta D(\hat{\boldsymbol{u}}^\varepsilon)\right) + \nabla \hat{p}^\varepsilon = \mathcal{F}, \qquad \operatorname{div} \hat{\boldsymbol{u}}^\varepsilon = 0. \tag{4.1}$$

We use boundary conditions on $\hat{\boldsymbol{u}}^\varepsilon$ suitable for lubrication applications: in order to take the shear effects into account, Dirichlet boundary conditions are imposed on the velocity on $\{y = 0\}$ and $\{y = \varepsilon h(x)\}$. Without loss of generality, the shear velocity $s \geq 0$ is supposed to be positive, and the boundary conditions read:

$$\forall x \in [0, L] \quad \hat{u}^\varepsilon(x, 0) = s \quad \text{and} \quad \hat{u}^\varepsilon(x, \varepsilon h(x)) = \hat{v}^\varepsilon(x, 0) = \hat{v}^\varepsilon(x, \varepsilon h(x)) = 0. \tag{4.2}$$

It has been showed in [BC86] that the boundary conditions on $\hat{\boldsymbol{u}}^\varepsilon$ for the Navier-Stokes equations on the lateral part of the boundary only occur in the limit problem (i.e. as the thickness of the domain ε tends to 0) by means of the input flow: indeed, any boundary condition corresponding to a same input flow will lead to the same limit problem (as $\varepsilon \to 0$). Therefore the lateral boundary conditions on $\hat{\boldsymbol{u}}^\varepsilon$ are not given explicitly, only the input flow $q = \displaystyle\int_0^{h(0)} \hat{\boldsymbol{u}}^\varepsilon|_{x=0} \cdot \boldsymbol{n}$ needs to be prescribed. Observe that according to the divergence-free condition and the boundary conditions on $\hat{\boldsymbol{u}}^\varepsilon$, this flow is constant on any "vertical" section of the domain:

$$\begin{aligned}\partial_x \left(\int_0^{h(x)} \hat{u}^\varepsilon(x, z) dz \right) &= \underbrace{h'(x)\hat{u}^\varepsilon(x, h(x))}_{=0} + \int_0^{h(x)} \partial_x \hat{u}^\varepsilon(x, z) dz = - \int_0^{h(x)} \partial_z \hat{v}^\varepsilon(x, z) dz \\ &= -\hat{v}^\varepsilon(x, h(x)) + \hat{v}^\varepsilon(x, 0) = 0,\end{aligned}$$

thus

$$q = \int_0^{h(x)} \hat{u}^\varepsilon(x, z) dz, \qquad \forall x \in (0, L).$$

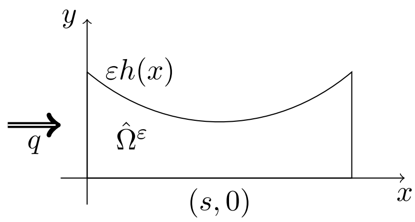

Figure 4.1: Domain $\hat{\Omega}^\varepsilon$ and boundary conditions on the velocity

We introduce the following change of variables

$$x = x, \qquad z = \frac{y}{\varepsilon},$$

and the rescaled domain

$$\Omega = \left\{(x,z) \in \mathbb{R}^2,\ 0 < x < L,\ 0 < z < h(x)\right\}$$

For a function $\hat{f}$ defined in $\hat{\Omega}^\varepsilon$, f is defined by $f(x,z) = \hat{f}(x,\varepsilon z)$. The equations (4.1) on $\boldsymbol{u}^\varepsilon$ and p^ε are rewritten in Ω. Moreover, we choose the following scaling orders

$$u^\varepsilon = u^*, \qquad v^\varepsilon = \varepsilon v^*, \qquad p^\varepsilon = \frac{1}{\varepsilon^2} p^*,$$

and the following equation on $\boldsymbol{u}$, p is obtained as the limit of (4.1) as ε tends to zero:

$$\partial_z \left(\eta\, \partial_z u^*\right) = \partial_x p^*, \qquad \partial_z p^* = 0, \qquad \partial_x u^* + \partial_z v^* = 0. \tag{4.3}$$

Observe that due to the orders of magnitude, the unsteady term and the convection term disappear.

It is well-known that after integrating twice the first equation of (4.3) with respect to z, and making use of the boundary conditions (4.2), and of the fact that $\partial_z p^* = 0$, u^* can be expressed as a function of p^*. The incompressibility condition enables to obtain an equation on the pressure only, the Reynolds equation:

$$\partial_x \left(\frac{h^3}{12\eta} \partial_x p^*\right) = s\partial_x \left(\frac{h}{2}\right). \tag{4.4}$$

The first boundary condition on p^* is deduced from the ones on $\boldsymbol{u}^*$. Indeed, the choice of the input flow q corresponds to a Neumann condition on p^* at $x = 0$: denoting $\partial_x p^*(0) = w_{\text{in}}$, it follows from (4.5) that

$$q = \int_0^{h(0)} u^*(0,Z)dZ = -w_{\text{in}} \frac{h(0)^3}{12\eta} + \frac{sh(0)}{2},$$

which determines w_{in} as a function of q. Moreover, the solution of (4.4) with the Neumann boundary condition $\partial_x p^*(0) = w_{\text{in}}$ is defined up to a constant. We can thus choose $p^*(L) = 0$ to gain a well-defined pressure p^*. At last, u^* and v^* are given by:

$$u^*(x,z) = \frac{z(z-h)}{2\eta} \partial_x p^* + s\left(1 - \frac{z}{h}\right) \qquad \text{and} \qquad v^*(x,z) = -\int_0^z \partial_x u^*(x,Z)\, dZ. \tag{4.5}$$

4.2.2 Modeling of a mixture in a thin domain

Modeling of a mixture in any domain

Since we want to study the mixture of two fluids, we introduce an order parameter φ[1] (describing the volumic fraction of one fluid in the flow). All physical parameters can be written as functions of φ. FRom a physical point of view, the viscosity $\eta(\varphi)$ of the mixture is given as function of the viscosities of the two fluids η_1 and η_2 by:

$$\frac{1}{\eta(\varphi)} = \frac{1+\varphi}{2\eta_1} + \frac{1-\varphi}{2\eta_2} \qquad \text{for } \varphi \in [-1,1], \tag{4.6}$$

so that $\varphi = 1$ and $\varphi = -1$ correspond respectively to the fluids of viscosity η_1 and η_2 only. However, since we do not prove mathematically that φ remains in the interval $[-1,1]$, we will only impose the following condition on η:

$$\eta \in \mathcal{C}^1(\mathbb{R}), \qquad \text{and} \qquad 0 < \eta_m \leq \eta(\varphi) \leq \eta_M, \forall \varphi \in \mathbb{R}.$$

In particular, $\eta(\varphi)$ can be defined as in (4.6) for $\varphi \in [-1,1]$, and extended to a regular bounded strictly positive function to $\mathbb{R}$.

In a similar way, the density ρ of the mixture can be defined as a function of φ. However, the non homogeneous case $\rho_1 \neq \rho_2$ induces further difficulties (see [Boy01]) due to the loss of the local conservation equation for the density. We do not wish to take these effects into account in this paper. Therefore, we restrict ourselves to the case $\rho_1 = \rho_2$ (as in [Boy99] for example).

In order to describe the evolution of φ, we introduce the Cahn-Hilliard equation, which is composed of both a transport term, taking the mechanical effects into account, and a diffusive term modelling the chemical effects. The Cahn-Hilliard equation reads:

$$\partial_t \varphi + \boldsymbol{u} \cdot \nabla\varphi - \frac{1}{\mathcal{P}e} \operatorname{div}\left(\mathcal{B}(\varphi)\nabla\mu\right) = 0, \tag{4.7}$$

$$\mu = -\alpha^2 \Delta\varphi + F'(\varphi). \tag{4.8}$$

The variable μ is the chemical potential, $\mathcal{B}(\varphi)$ is called mobility, $\mathcal{P}e$ is the Péclet number, α is an non-dimensional parameter measuring the thickness of the diffuse interface, and the function F is called Cahn-Hilliard potential. From a physical point of view, F must have a double-well structure, each of the wells representing one of the two fluids. A

[1] From a physical point of view, the relevant values of φ are $[-1,1]$. However, previous works (e.g. [Boy99]) showed that it is not easy to prove mathematically that φ remains in this interval.

rational choice for F is given by a logarithmic form[2]

$$F(\xi) = 1 - \xi^2 + c\left((1+\xi)\log(1+\xi) + (1-\xi)\log(1-\xi)\right),$$

or its polynomial approximation

$$F(\xi) = (1-\xi^2)^2.$$

General assumptions on the Cahn-Hilliard potential

These physically realistic potentials share several mathematical properties. In the following, we prove mathematical results for potentials F having these properties. More precisely, the function F is supposed to be regular (e.g. of class $\mathcal{C}^2$). Since F is a physical potential, it is bounded from below. Moreover, only the derivative of F occurs in the equations, therefore the addition of a constant does not change the equations. It is thus realistic to suppose there exists a constant $F_0 > 0$ such that for all $\xi \in \mathbb{R}$

$$F(\xi) \geq F_0.$$

As for the convexity of the potential, it corresponds to the stability of the mixture. Usual potentials contain some stable and unstable regions (see for example Figure 4.2). In order to include such cases, we do not impose a condition as strong as the convexity but only:

$$\exists F_5 \geq 0, \quad F''(\xi) \geq -F_5, \quad \forall \xi \in \mathbb{R}. \tag{4.9}$$

Moreover, in a two-dimensional domain, the following hypothesis on the growth of the potential near the points $\varphi = 1$ and $\varphi = -1$ is imposed:

$$\begin{gathered} \exists F_1,\, F_2 > 0,\ \exists p \in [1,+\infty),\ \forall \xi \in \mathbb{R}, \\ |F'(\xi)| \leq F_1|\xi|^p + F_2 \text{ and } |F''(\xi)| \leq F_1|\xi|^{p-1} + F_2. \end{gathered} \tag{4.10}$$

This hypothesis is satisfied for any polynomial function. At last, we state a generalization of the convexity:

$$\forall \gamma \in \mathbb{R},\ \exists F_3(\gamma) > 0,\ F_4(\gamma) \geq 0,\ \forall \xi \in \mathbb{R},\ (\xi - \gamma)F'(\xi) \geq F_3(\gamma)F(\xi) - F_4(\gamma). \tag{4.11}$$

This hypothesis is also satisfied by a function of the form $F(\varphi) = \dfrac{\varphi^4}{4} - \dfrac{\varphi^2}{2}$ (as in Figure 4.2) and such a function can be used as a model case.

As far as the mobility $\mathcal{B}$ is concerned, it is supposed to be regular, positive, and

[2] For more details, we refer to [Doi97] or [GSMS83].

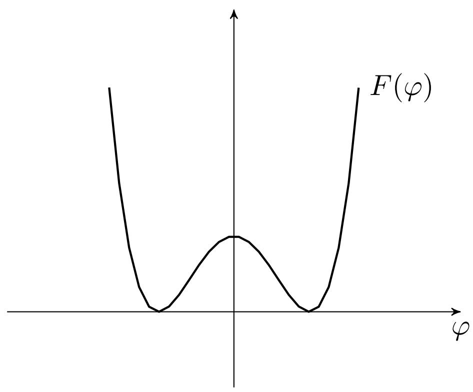

Figure 4.2: Possible appearance of the potential $F(\varphi)$

bounded from above and from below:

$$\mathcal{B} \in \mathcal{C}^2(\mathbb{R}), \quad \forall \xi \in \mathbb{R}, \quad 0 < \mathcal{B}_m \leq \mathcal{B}(\xi) \leq \mathcal{B}_M. \tag{4.12}$$

Let us mention that other types of functions $\mathcal{B}$ can be considered, in particular the degenerate case $\mathcal{B}(\xi) = (1 - \xi^2)^\sigma$, with $\sigma \geq 0$, which is the situation arising in physical applications. This case has been mostly studied in [Boy99], but introduces further mathematical difficulties.

This equation is equipped with boundary conditions on φ and μ. Here we are interested in modelling injection phenomena, therefore we consider a Dirichlet condition on φ on some part of the boundary. Let us emphasize that these boundary conditions are different from those considered in [Boy99], [Chu03]. Let us define two parts Γ_l (where the injection takes place) and Γ_0 of the boundary $\Gamma = \partial\Omega = \Gamma_l \cup \Gamma_0$ (see Figure 4.3).

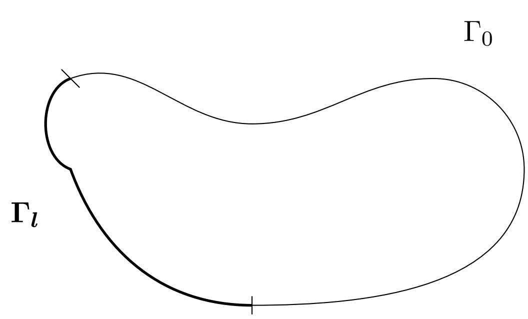

Figure 4.3: Notations for the boundary $\Gamma = \Gamma_l \cup \Gamma_0$

Let $\boldsymbol{n}$ denote the exterior normal to the domain. The boundary conditions read

$$\left.\frac{\partial \mu}{\partial \boldsymbol{n}}\right|_{\Gamma_0} = 0, \qquad \left.\frac{\partial \varphi}{\partial \boldsymbol{n}}\right|_{\Gamma_0} = 0, \qquad \text{and} \qquad \varphi|_{\Gamma_l} = \varphi_l, \qquad \mu|_{\Gamma_l} = 0, \tag{4.13}$$

for a given $\varphi_l \in H^{5/2}(\Gamma_l)$.
At last, let us define the initial condition: $\varphi(t=0) = \varphi_0 \in H^3(\Omega)$, where φ_0 is supposed to be satisfying the same boundary conditions as φ.

It is possible to take surface tension effects into account by adding an additional term $\kappa\, \mu \nabla \varphi$, where κ is the capillarity coefficient, to the external forces $\mathcal{F}$. From a mathematical point of view, the Navier-Stokes (with surface tension) / Cahn-Hilliard system has been studied in [Boy99], with homogeneous Neumann boundary conditions.

Modeling of diphasic flows in a thin domain

In order to couple the Cahn-Hilliard equation with the thin film aspect presented at the beginning of the section, let us come back to the previous form of the domain, and determine the partition $\Gamma = \Gamma_l \cup \Gamma_0$ of the boundary (see Figure 4.4):

$$\Gamma_l = \{(x,y) \in \Gamma, x = 0\} \quad \text{and} \quad \Gamma_0 = \Gamma \setminus \Gamma_l.$$

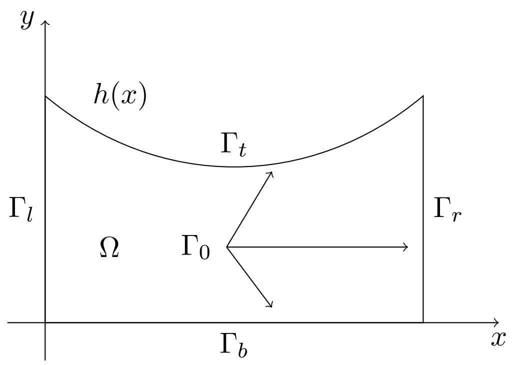

Figure 4.4: Domain Ω and notations for the boundary

As before, the flow in the domain $\hat{\Omega}^\varepsilon$ is described by the Navier-Stokes equations (4.1), where the viscosity η is not constant anymore but depends on the order parameter φ. The boundary conditions (4.2) are unchanged. Because of the non-constant viscosity, the coefficients in the Reynolds equation depend on η. Let us introduce the following coefficients, that will be useful in the following:

$$a(x,z) = \int_0^z \frac{dZ}{\eta(\varphi(x,Z))}, \quad b(x,z) = \int_0^z \frac{Z dZ}{\eta(\varphi(x,Z))}, \quad c(x,z) = \int_0^z \frac{Z^2 dZ}{\eta(\varphi(x,Z))}, \tag{4.14}$$

and

$$\widetilde{a}(x) = a(x, h(x)), \qquad \widetilde{b}(x) = b(x, h(x)), \qquad \widetilde{c}(x) = c(x, h(x)),$$

for all $(x, z) \in \Omega$. We define also:

$$\widetilde{d}(x) = \left(\widetilde{c}(x) - \frac{\widetilde{b}(x)^2}{\widetilde{a}(x)}\right) \quad \text{and} \quad \widetilde{e}(x) = \frac{\widetilde{b}(x)}{\widetilde{a}(x)}. \tag{4.15}$$

Formally, we can pass to the limit (as ε tends to zero) similarly to the previous section (but this limit has not been justified rigorously yet). We have to choose the order of scaling of the additional parameters (for example κ). To this purpose, we consider two different cases.

Case 1 : $\boldsymbol{\kappa = \varepsilon\kappa^*}$.

▷ *Asymptotic limit of the Navier-Stokes equations*

Since the capillarity coefficient vanishes when passing the to the limit $\varepsilon \to 0$, we obtain formally the same system (4.3), where η is not constant anymore. After integrating twice the first equation of (4.3) and using the boundary conditions, we find, $\forall (x, z) \in \Omega$,

$$u^*(x, z) = \left(b(x, z) - \frac{\widetilde{b}(x)}{\widetilde{a}(x)} a(x, z)\right) \partial_x p^*(x) + \left(1 - \frac{a(x, z)}{\widetilde{a}(x)}\right) s \tag{4.16}$$

where the coefficients are given by (4.14).

As before, we use the fact that $\boldsymbol{u}^*$ is divergence-free and the boundary conditions to obtain

$$\int_0^{h(x)} \partial_x u^*(x, z)\, dz = \partial_x \int_0^{h(x)} u^*(x, z)\, dz = 0. \tag{4.17}$$

After integrating (4.16), we have

$$\partial_x \left(\widetilde{d}(x) \partial_x p^*(x)\right) = s \partial_x \left(\widetilde{e}(x)\right) \tag{4.18}$$

where the coefficients are given by (4.15).

The velocity $\boldsymbol{u}^* = (u^*, v^*)$ is determined from the pressure by:

$$\begin{aligned} u^*(x, z) &= \left(b(x, z) - \frac{a(x, z)\widetilde{b}(x)}{\widetilde{a}(x)}\right) \partial_x p^*(x) + \left(1 - \frac{a(x, z)}{\widetilde{a}(x)}\right) s, \\ v^*(x, z) &= -\int_0^z \partial_x u^*(x, Z)\, dZ. \end{aligned} \tag{4.19}$$

▷ *Asymptotic limit of the Cahn-Hilliard equation*

In order to choose the scaling order of the parameters in the Cahn-Hilliard equation, let us distinguish the anisotropic coefficients $\mathcal{B}_x$, $\mathcal{B}_z$, α_x and α_z by writing the Cahn-Hilliard equation (4.7)-(4.8) in the domain $\hat{\Omega}^\varepsilon$ in the following form:

$$\partial_t \hat{\varphi}^\varepsilon + \hat{u}^\varepsilon \partial_x \hat{\varphi}^\varepsilon + \hat{v}^\varepsilon \partial_z \hat{\varphi}^\varepsilon + \frac{1}{\mathcal{P}e} \operatorname{div} \left(\begin{pmatrix} \mathcal{B}_x(\varphi) & 0 \\ 0 & \mathcal{B}_z(\varphi) \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_z \end{pmatrix} \hat{\mu}^\varepsilon \right) = 0,$$
$$\hat{\mu}^\varepsilon = -\operatorname{div} \left(\begin{pmatrix} \alpha_x^2 & 0 \\ 0 & \alpha_z^2 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_z \end{pmatrix} \hat{\varphi}^\varepsilon \right) + F'(\hat{\varphi}^\varepsilon).$$

The system on φ^ε, μ^ε, u^ε, v^ε is rewritten in Ω, using the same change of variables as before. Moreover, let us choose the following scaling orders for the unknowns:

$$u^\varepsilon = u^*, \qquad v^\varepsilon = \varepsilon v^*, \qquad \varphi^\varepsilon = \varphi^*, \qquad \mu^\varepsilon = \mu^*,$$

and for the coefficients $\mathcal{B}_x$, $\mathcal{B}_z$, α_x and α_z, let us assume that they write

$$\mathcal{B}_x = \mathcal{B}_x^*, \quad \mathcal{B}_z = \varepsilon^2 \mathcal{B}_z^*, \quad \alpha_x = \alpha_x^*, \quad \alpha_z = \varepsilon^2 \alpha_z^*.$$

The system reads:

$$\partial_t \varphi^* + u^* \partial_x \varphi^* + v^* \partial_z \varphi^* + \frac{1}{\mathcal{P}e} \operatorname{div} \left(\begin{pmatrix} \mathcal{B}_x^*(\varphi^*) & 0 \\ 0 & \mathcal{B}_z^*(\varphi^*) \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_z \end{pmatrix} \mu^* \right) = 0, \tag{4.20}$$
$$\mu^* = -\operatorname{div}_\varepsilon \left(\begin{pmatrix} \alpha_x^{*2} & 0 \\ 0 & \alpha_z^{*2} \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_z \end{pmatrix} \varphi^* \right) + F'(\varphi^*). \tag{4.21}$$

Observe that with these scaling orders, the Cahn-Hilliard equation remains unchanged, when defining $\mathcal{B}^*(\varphi^*) = \begin{pmatrix} \mathcal{B}_x^*(\varphi^*) & 0 \\ 0 & \mathcal{B}_z^*(\varphi^*) \end{pmatrix}$, $\alpha^{*2} = \begin{pmatrix} \alpha_x^{*2} & 0 \\ 0 & \alpha_z^{*2} \end{pmatrix}$.

Remark 4.1. *The choices of scaling orders for α can be justified from a physical point of view . Indeed, since α is related to the thickness of the interface, the anisotropy of the domain can lead to an anisotropy of the interface. As far as the mobility is concerned, it is related to the friction coefficient, and the scaling orders of $\mathcal{B}_x$ and $\mathcal{B}_z$ can also be different. However, these choices are made mostly from a mathematical point of view. Indeed, when neglecting some terms in the Cahn-Hilliard equation, further difficulties arise in the mathematical analysis.*

For the sake of simplicity, let us choose $\mathcal{B}_x^*(\varphi) = \mathcal{B}_z^*(\varphi)$ and $\alpha_x^* = \alpha_z^*$. Therefore, the matrices $\mathcal{B}^*(\varphi)$ and α^{*2} are of the form $\mathcal{B}^*(\varphi^*) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha^{*2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This assumption makes the notations simpler, but all the work in the following could be done with matrices.

▷ *Coupling the two equations*

Therefore the whole system (Reynolds and Cahn-Hilliard equations) reads, in the case where the capillarity coefficient κ is of order ε:

$$\boxed{\begin{cases} \partial_x(\widetilde{d}\,\partial_x p^*) = s\,\partial_x \widetilde{e} \\ u^* = \left(b - \dfrac{a\,\widetilde{b}}{\widetilde{a}}\right)\partial_x p^* + s\left(1 - \dfrac{a}{\widetilde{a}}\right) \\ v^*(\cdot, z) = -\displaystyle\int_0^z \partial_x u^*(\cdot, Z)dZ \\ \partial_t\,\varphi^* + u^*\,\partial_x\varphi^* + v^*\,\partial_z\varphi^* - \dfrac{1}{\mathcal{P}e}\operatorname{div}(\mathcal{B}^*(\varphi^*)\nabla\mu^*) = 0 \\ \mu^* = -\alpha^{*2}\Delta\varphi^* + F'(\varphi^*). \end{cases}} \tag{4.22}$$

The coefficients a, b, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{d}$, $\widetilde{e}$ are explicit functions of φ (given by (4.14), (4.15)), as well as the functions $\mathcal{B}^*$, F. The quantities s, q, $\mathcal{P}e$, α^* are physical constants. The boundary conditions read

$$u^*(x,0) = s, \qquad u^*(x,h(x)) = v^*(x,0) = v^*(x,h(x)) = 0, \tag{4.23}$$

$$\int_0^{h(0)} \boldsymbol{u}^*|_{x=0}\cdot\boldsymbol{n} = q, \tag{4.24}$$

$$\partial_x p^*(0) = w_{\text{in}}, \qquad p^*(L) = 0, \tag{4.25}$$

$$\frac{\partial\varphi^*}{\partial\boldsymbol{n}}|_{\Gamma_0} = \frac{\partial\mu^*}{\partial\boldsymbol{n}}|_{\Gamma_0} = 0, \qquad \varphi^*|_{\Gamma_l} = \varphi_l,\ \mu^*|_{\Gamma_l} = 0, \tag{4.26}$$

and w_{in} is given as a function of the input flow q, the shear velocity s and $\widetilde{a}_0 = \widetilde{a}(0)$, $\widetilde{b}_0 = \widetilde{b}(0)$:

$$w_{\text{in}} = \frac{q - s\left(h(0) - 1/\widetilde{a}_0\int_0^{h(0)} a(0,Z)\,dZ\right)}{\int_0^{h(0)} b(0,Z)\,dZ - \widetilde{b}_0/\widetilde{a}_0\int_0^{h(0)} a(0,Z)\,dZ}.$$

Case 2 : $\boldsymbol{\kappa = \kappa^*}$.

▷ *Asymptotic limit of the Navier-Stokes equations*

In this case, the two first equations of (4.3) are modified, since some terms due to the surface tension remain when passing to the limit $\varepsilon \to 0$:

$$\begin{cases} -\partial_z(\eta(\varphi^*)\,\partial_z u^*) + \partial_x p^* = \kappa^*\,\mu^*\,\partial_x\varphi^* \\ \partial_z p^* = \kappa^*\,\mu^*\,\partial_z\varphi^*. \end{cases}$$

The boundary conditions are unchanged. In order to write a modified Reynolds equation similar to (4.18), we introduce

$$p^*_\kappa = p^* - \kappa^* \int_0^z \mu^*\,\partial_Z\varphi^*\ dZ,$$

so that the second equation reads $\partial_z p^*_\kappa = 0$. With this new pressure p^*_κ, the first equation becomes:

$$-\partial_z(\eta(\varphi^*)\,\partial_z u^*) + \partial_x p^*_\kappa = \kappa^* \left(\mu^*\,\partial_x\,\varphi^* - \partial_x\left(\int_0^z \mu^*\,\partial_Z\varphi^*\ dZ\right)\right) =: \kappa^*\,k[\varphi^*]. \quad (4.27)$$

Again, after integrating twice with respect to z, we obtain, using the boundary conditions (4.2):

$$u^* = \left(b - \frac{a\widetilde{b}}{\widetilde{a}}\right)\partial_x p^*_\kappa + s\left(1 - \frac{a}{\widetilde{a}}\right) \quad (4.28)$$
$$+\,\kappa\left(\frac{a}{\widetilde{a}}\int_0^h \frac{1}{\eta(\varphi^*)}\int_0^Z k[\varphi^*(\cdot,\xi)]\,d\xi\,dZ - \int_0^z \frac{1}{\eta(\varphi^*)}\int_0^Z k[\varphi^*(\cdot,\xi)]\,d\xi\,dZ\right).$$

Let

$$\mathcal{K}^*(x,z) = \int_0^z \frac{1}{\eta(\varphi^*(x,Z))}\int_0^Z k[\varphi^*(x,\xi)]\,d\xi\,dZ, \qquad \text{and} \qquad \widetilde{\mathcal{K}}^*(x) = \mathcal{K}^*(x,h(x)). \quad (4.29)$$

It is possible to compute $\int_0^h u^*(\cdot,Z)\ dZ$ as a function of $\mathcal{K}^*$:

$$\int_0^h u^*(\cdot,Z)\ dZ = -\widetilde{d}\partial_x p^*_\kappa + s\widetilde{e} + \kappa\left(\left(h - \frac{\widetilde{b}}{\widetilde{a}}\right)\widetilde{\mathcal{K}}^* - \int_0^h \mathcal{K}^*(\cdot,Z)\,dZ\right).$$

At last, using the fact that $\partial_x\left(\int_0^h u^*(\cdot,Z)\ dZ\right) = 0$, we deduce a modified Reynolds equation on p^*_κ:

$$\partial_x(\widetilde{d}\,\partial_x p^*_\kappa) = s\,\partial_x\widetilde{e} + \kappa^*\,\partial_x\left(\left(h - \frac{\widetilde{b}}{\widetilde{a}}\right)\widetilde{\mathcal{K}}^* - \int_0^h \mathcal{K}^*(\cdot,Z)\,dZ\right). \quad (4.30)$$

Moreover, the velocity is given by:

$$\begin{cases} u^* = \left(b - \dfrac{a\widetilde{b}}{\widetilde{a}}\right) \partial_x p^*_\kappa + s\left(1 - \dfrac{a}{\widetilde{a}}\right) + \kappa^* \left(\dfrac{a}{\widetilde{a}}\widetilde{\mathcal{K}}^* - \mathcal{K}^*\right) \\ v^*(\cdot, z) = -\displaystyle\int_0^z \partial_x u^*(\cdot, Z)\, dZ. \end{cases} \tag{4.31}$$

▷ *Coupling with the the Cahn-Hilliard equation*

The scaling for the Cahn-Hilliard equation is the same as for the previous case.

Therefore the whole system (Reynolds and Cahn-Hilliard equations) reads, in the case where the capillarity coefficient κ is of order 1:

$$\boxed{\begin{cases} \partial_x(\widetilde{d}\,\partial_x p^*_\kappa) = s\,\partial_x \widetilde{e} + \kappa^*\,\partial_x \left(\left(h - \dfrac{\widetilde{b}}{\widetilde{a}}\right)\widetilde{\mathcal{K}}^* - \displaystyle\int_0^h \mathcal{K}^*(\cdot, Z)\, dZ\right) \\ u^* = \left(b - \dfrac{a\widetilde{b}}{\widetilde{a}}\right) \partial_x p^*_\kappa + s\left(1 - \dfrac{a}{\widetilde{a}}\right) + \kappa^* \left(\dfrac{a}{\widetilde{a}}\widetilde{\mathcal{K}}^* - \mathcal{K}^*\right) \\ v^*(\cdot, z) = -\displaystyle\int_0^z \partial_x u^*(\cdot, Z) dZ \\ \partial_t \varphi^* + u^*\,\partial_x \varphi^* + v^*\,\partial_z \varphi^* - \dfrac{1}{\mathcal{P}e}\,\mathrm{div}\,(\mathcal{B}^* \nabla \mu^*) = 0 \\ \mu^* = -\,\mathrm{div}\left(\alpha^{*2} \nabla \varphi^*\right) + F'(\varphi^*). \end{cases}} \tag{4.32}$$

The coefficients a, b, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{d}$, $\widetilde{e}$, $\mathcal{K}^*$, $\widetilde{\mathcal{K}}^*$ are explicit functions of φ (given by (4.14), (4.15), (4.29)), as well as the functions $\mathcal{B}^*$, F. The quantities s, κ, $\mathcal{P}e$, α^* are physical constants. The boundary conditions (4.23), (4.24), (4.25), (4.26) still hold.

4.3 Preliminary results

4.3.1 Notations and classical results

- In the sequel, we will work with the two systems (4.22)and (4.32). For the sake of readability, we will drop the subscripts *.

- The function h is supposed to belong to $\mathcal{C}^2(\mathbb{R})$, with

$$\begin{aligned} &\forall x \in [0, L], \qquad 0 < h_m \le h(x) \le h_M, \\ &\forall x \in [0, L], \qquad 0 < h'_m \le h'(x) \le h'_M. \end{aligned}$$

Observe that the regularity of h ensures that the domain Ω defined by:

$$\Omega = \{(x,z) \in \mathbb{R}^2,\, x \in (0,L),\, z \in (0,h(x))\},$$

satisfies the segment property and cone property (see [Ada75, § 4.2 and 4.3]).

- C denotes any constant depending only on the physical parameters of the problem and on Ω. Moreover, let us define the quantity

$$\sigma := \frac{h_M}{h_m}.$$

In order to control the dependence of the constants when the size of the domain becomes small, we introduce the following notation: constants independent of the domain are denoted by $\bar{C}$, as well as the constants depending on Ω only through σ (i.e. at fixed σ, the constants $\bar{C}$ remain fixed, even if the size of the domain is changed).

- For $f \in L^1(\Omega)$, we define the mean value of f, denoted by $m(f) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f$.

- For the usual Sobolev spaces, we denote $|\cdot|_p$ the L^p-norm in Ω, and by $\|\cdot\|_s$ the H^s-norm in Ω.

- Let us define the following function spaces:

$$\Phi = \{\phi \in \mathcal{D}(\bar{\Omega}),\ \frac{\partial \phi}{\partial \boldsymbol{n}}|_{\Gamma_0} = 0,\ \phi|_{\Gamma_l} = 0\}, \quad \Phi^s = \overline{\Phi}^{H^s(\Omega)},$$
$$\Phi_l = \{\phi \in \mathcal{D}(\bar{\Omega}),\ \frac{\partial \phi}{\partial \boldsymbol{n}}|_{\Gamma_0} = 0\},$$
$$\Phi_l^s = \overline{\{\phi \in \Phi_l,\, \phi|_{\Gamma_l} = \varphi_l\}}^{H^s(\Omega)} \quad \text{for } s \leq 3.$$

Embedding results in $\mathcal{C}(0,T;X)$

Let us state a classical proposition proved in [Sim87] which allows to obtain a strong convergence result from a weak convergence of a function and of its time derivative:

Proposition 4.2. *Let $X \subset Y \subset Z$ three Hilbert spaces, and suppose that the embedding $X \hookrightarrow Y$ is compact.*

i) For any $T > 0$, $p_1, p_2 \in]1,+\infty[$, the embedding

$$\left\{ f \in L^{p_1}(0,T;X),\ \frac{df}{dt} \in L^{p_2}(0,T;Z) \right\} \hookrightarrow L^{p_1}(0,T;Y)$$

is compact.

ii) For any $T > 0$ and any $p > 1$, the embedding

$$\left\{ f \in L^\infty(0,T;X),\ \frac{df}{dt} \in L^p(0,T;Z) \right\} \hookrightarrow \mathcal{C}(0,T;Y)$$

is compact.

iii) For any $T > 0$, we have the following continuous embedding

$$\left\{ f \in L^2(0,T;X),\ \frac{df}{dt} \in L^2(0,T;Y) \right\} \hookrightarrow \mathcal{C}(0,T;[X,Y]_{1/2}),$$

where $[X,Y]_{1/2}$ denotes the interpolation as defined e.g. in [Tem79, II.2.1].

Multiplicative algebra

In order to deal with the nonlinear terms, the following proposition (see [Hör97]) will also be useful:

Proposition 4.3. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$. Then the mapping $(f,g) \to fg$ is continuous from $H^{s_1}(\Omega) \times H^{s_2}(\Omega)$ into $H^s(\Omega)$ if*

$$s_1 + s_2 \geq 0, \quad s = \min\left\{ s_1, s_2, s_1 + s_2 - \frac{d}{2} - \varepsilon \right\},\ \varepsilon > 0.$$

The number ε can be chosen zero if $s_1 \neq \frac{d}{2}$, $s_2 \neq \frac{d}{2}$ and $\min\left\{s_1, s_2, s_1 + s_2 - \frac{d}{2}\right\} \neq \frac{d}{2}$.

Sobolev embeddings

Let us specify the constants in the usual Sobolev embeddings. These results are proved in [Ada75].

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^2$ satisfying the segment property. Then $H^1(\Omega) \hookrightarrow L^q(\Omega)$, for any $2 \leq q < +\infty$. Moreover, the embedding constant can be specified: if $f \in H^1(\Omega)$, then*

$$|f|_q \leq \bar{C}|\Omega|^{1/q}\|f\|_1, \tag{4.33}$$

where $\bar{C}$ only depends on q.

Proof. We proceed as in [Ada75, Cor. 5.13]. Let $s = \dfrac{2q}{2+q} < 2$. It is showed that $H^1(\Omega) \hookrightarrow H^s(\Omega)$ with

$$\|f\|_s \leq |\Omega|^{1/s-1/2}\|f\|_1 = |\Omega|^{1/q}\|f\|_1.$$

Moreover, $H^s(\Omega) \hookrightarrow L^q(\Omega)$. This result has been stated in [Ada75, Lemma 5.10] for a domain satisfying the cone property, in which case the embedding constant may depend on the cone of the cone property. However, in the proof of this result, it is first showed that for domains satisfying the segment property, the embedding constant only depends on q. □

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^2$ satisfying the cone property. Then $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Moreover let $R = \min(h_m, L)$. If $f \in H^2(\Omega)$, then*

$$|f|_\infty \leq \bar{C}(R^{-2/3}|\Omega|^{5/6} + R^{1/3}|\Omega|^{1/3})\|f\|_2. \tag{4.34}$$

Let us denote $C_\infty := \bar{C}(R^{-2/3}|\Omega|^{5/6} + R^{1/3}|\Omega|^{1/3})$. In particular, the embedding constant C_∞ remains bounded as $|\Omega| \to 0$.

Proof. We proceed as in [Ada75, Lemma 5.15]. Proposition 4.4 yields that $H^2(\Omega) \hookrightarrow W^{1,3}(\Omega)$. Moreover, $W^{1,3}(\Omega) \hookrightarrow L^\infty(\Omega)$. Let us state the embedding constants explicitly. Let $f \in H^2(\Omega)$. It is proved in [Ada75, Lemma 5.15] that for all $x \in \Omega$,

$$\bar{C}R^2|f(x)| \leq \bar{C}R^{4/3}|f|_3 + \bar{C}R^{7/3}|\nabla f|_3,$$

i.e.

$$|f|_\infty \leq \bar{C}(R^{-2/3}|f|_3 + R^{1/3}|\nabla f|_3).$$

Now, using (4.33) with $q = 3$, it follows:

$$|f|_\infty \leq \bar{C}(R^{-2/3}|\Omega|^{1/3}\|f\|_1 + R^{1/3}|\Omega|^{1/3}\|\nabla f\|_1).$$

Applying again (4.33) with $q = 2$ to $\|f\|_1 = |f|_2 + |\nabla f|_2$, we have:

$$|f|_\infty \leq \bar{C}(R^{-2/3}|\Omega|^{1/3+1/2}\|f\|_2 + R^{1/3}|\Omega|^{1/3}\|f\|_2). \tag{4.35}$$

The result follows. □

Equivalence of norms

Proposition 4.6. *Let $f \in H^2(\Omega)$ satisfying the mixed boundary conditions $f|_{\Gamma_l} = 0$ and $\nabla f \cdot \boldsymbol{n}|_{\Gamma_0} = 0$. Then the L^2-norm of the Laplacian is equivalent to the H^2-norm:*

$$\|f\|_2 \leq \bar{C}|\Delta f|_2.$$

Proof. This result is proved in [BdV74]. □

Let us state the following corollary, which results immediately from Proposition 4.6, and which is the formulation that will be used in the sequel.

Corollary 4.7. *Let $\varphi_l \in H^{5/2}(\Gamma_l)$, and let $\varphi, \hat{\varphi}_l \in \Phi_l^2$. We have*

$$\|\varphi\|_2 \leq \bar{C}|\Delta\varphi|_2 + \|\hat{\varphi}_l\|_2. \tag{4.36}$$

Moreover, we can combine this result with Proposition 4.5:

Corollary 4.8. *Let $\Omega \subset \mathbb{R}^2$. Let $\varphi_l \in H^{5/2}(\Gamma_l)$, and let $\varphi, \hat{\varphi}_l \in \Phi_l^2$. Let $R = \min(h_m, L)$. The following inequality applies:*

$$|\varphi|_\infty \leq \bar{C}(R^{-2/3}|\Omega|^{5/6} + R^{1/3}|\Omega|^{1/3})\,(|\Delta\varphi|_2 + \|\hat{\varphi}_l\|_2). \tag{4.37}$$

4.3.2 Boundary conditions and lift operator

In order to treat the boundary terms, it is a classical approach for the velocity $\boldsymbol{u}$ to introduce a lift operator of the boundary values by means of a divergence-free function.

Lemma 4.9. *Let $(s, q) \in \mathbb{R}^2$. There exists a constant $C > 0$ such that for any $\tau > 0$, there exists a vector field on $\bar{\Omega}$, denoted by $\mathbf{g}^\tau = (g_1^\tau, g_2^\tau)$, satisfying the following conditions:*

i) $\mathbf{g}^\tau \in H^1(\Omega)^2$,

ii) $\operatorname{div} \mathbf{g}^\tau = 0$ *in* Ω,

iii) $\mathbf{g}^\tau$ *satisfies* $\mathbf{g}^\tau|_\Gamma = \boldsymbol{u}|_\Gamma$, *which corresponds to the following conditions:*

$$\mathbf{g}^\tau(x, 0) = (s, 0), \quad \mathbf{g}^\tau(x, h(x)) = (0, 0), \quad \int_0^{h(0)} \mathbf{g}^\tau|_{x=0} \cdot \boldsymbol{n} = q,$$

iv) $|\mathbf{g}^\tau|_4 \leq C\tau$.

Proof. Let us recall that the boundary value of $\boldsymbol{u}$ is not specified yet on the lateral part of the boundary, and only the input flow q is given (see condition (4.24)). Therefore we can build in an obvious way a function $J \in H^{1/2}(\Gamma)$ such that

$$\int_0^{h(0)} J|_{x=0} = q, \qquad \text{and} \qquad J|_{z=0} = s, \qquad J|_{z=h(x)} = 0,$$

so that the boundary conditions on $\boldsymbol{u}$ read:

$$\boldsymbol{u}|_\Gamma = J.$$

Then it suffices to prove the existence of $\mathbf{g}^\tau$ satisfying *i)*, *ii)*, *iv)*, and the following Dirichlet boundary condition:

$$\mathbf{g}^\tau|_\Gamma = \boldsymbol{u}|_\Gamma.$$

This result is proved in [Chu03]. □

As far as the order parameter φ is concerned, the Dirichlet boundary conditions imposed on Γ_l lead us to introduce also a lifting of the boundary values. However, the way to define it is different from the one for the velocity, since we do not need any divergence-free or smallness condition. We only impose some particular boundary conditions. Let us make the following assumption on φ_l, which allows us to define the lifting $\hat{\varphi}_l$ of the boundary values without any restrictive assumption on the geometry of Ω:

Assumption 4.10. *Let $\varphi \in H^{5/2}(\Gamma_l)$. Suppose that*

$$\exists(\phi_1, \phi_2) \in \mathbb{R}^2,\ \exists r > 0,\ \textit{such that } \varphi_l|_{[0,r]} = \phi_1 \textit{ and } \varphi_l|_{[h(0)-r,h(0)]} = \phi_2. \tag{4.38}$$

This condition is realistic from a physical point of view, since it means that there exists two small zones $[0, r]$ and $[h(0) - r, h(0)]$ of Γ_l along which the injection fluid remains the same (and in general $\phi_{1,2} \in \{-1, 1\}$).

Lemma 4.11. *Let $\varphi_l \in H^{5/2}(\Gamma_l)$, satisfying hypothesis (4.38). There exists a function $\hat{\varphi}_l \in H^3(\Omega)$, such that*

$$\hat{\varphi}_l|_{\Gamma_l} = \varphi_l, \tag{4.39}$$

$$\nabla\hat{\varphi}_l \cdot \boldsymbol{n}|_{\Gamma_0} = 0, \tag{4.40}$$

$$\Big(\Delta\hat{\varphi}_l\Big)|_{\Gamma_l} = \frac{1}{\alpha^2}F'(\varphi_l), \quad i.e. \quad \partial_x^2\hat{\varphi}_l = -\partial_z^2\varphi_l + \frac{1}{\alpha^2}F'(\varphi_l), \tag{4.41}$$

$$\nabla\Delta\hat{\varphi}_l \cdot \boldsymbol{n}|_{\Gamma_0} = 0. \tag{4.42}$$

Proof. Since the claim is not a classical result, let us construct this function explicitly. Recall that r is a parameter defined in hypothesis (4.38) on φ_l. Since h is a continuous function, it is possible to define $0 < r' < \frac{L}{4}$ sufficiently small such that $h(x) > h(r) - \frac{r}{2} > h(0) - r$ for all $0 \le x \le r'$. Let us define the following parts of the domain (see Figure 4.5):

$$\Omega_{1c} = \Big\{(x,z) \in \Omega\,, r' \le x < L - r'\,, h(x) - \frac{r}{2} \le z \le h(x)\Big\}\,,$$
$$\Omega_{1l} = \Big\{(x,z) \in \Omega\,, 0 \le x \le r'\,, \min_{0\le x\le r'}(h(x) - \frac{r}{2}) \le z \le h(x)\Big\}\,,$$
$$\Omega_{1r} = \Big\{(x,z) \in \Omega\,, L - r' \le x \le L\,, \min_{L-r'\le x\le L}(h(x) - \frac{r}{2}) \le z \le h(x)\Big\}\,,$$

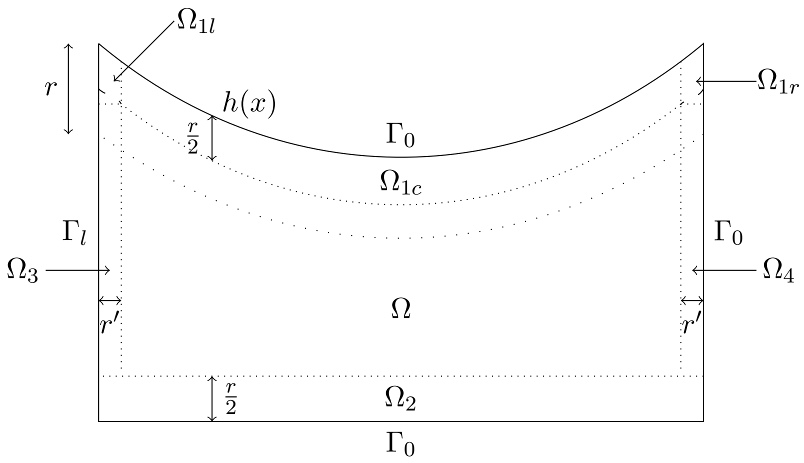

Figure 4.5: Domain Ω and partitions Ω_i

$$\Omega_1 = \Omega_{1l} \cup \Omega_{1c} \cup \Omega_{1r},$$
$$\Omega_2 = \left\{(x,z) \in \Omega\,, x \in (0,L)\,, 0 \leq z \leq \min_{0 \leq x \leq r'} (h(x) - \frac{r}{2})\right\},$$
$$\Omega_3 = \left\{(x,z) \in \Omega\,, 0 \leq x \leq r', \frac{r}{2} < z < h(x) - \frac{r}{2}\right\},$$
$$\Omega_4 = \left\{(x,z) \in \Omega\,, L - r' \leq x \leq L, \frac{r}{2} < z < \min_{L-r' \leq x \leq L} (h(x) - \frac{r}{2})\right\}.$$

Let us define φ_i on Ω_i for $1 \leq i \leq 4$ by:

$$\varphi_1 \equiv \phi_1, \qquad \varphi_2 \equiv \phi_2,$$
$$\forall (x,z) \in \Omega_3, \qquad \varphi_3(x,z) = \left(\frac{1}{\alpha^2} F'(\varphi_l) - \partial_z^2 \varphi_l(z)\right) \frac{x^2}{2} + \varphi_l(z),$$
$$\forall (x,z) \in \Omega_4, \qquad \varphi_4(x,z) = P(z),$$

where ϕ_1 and ϕ_2 are the parameters mentioned in hypothesis (4.38), and $P \in \mathbb{R}[X]$ with $\deg(P) = 7$, satisfying the following conditions:

$$P(h(x) - r) = \phi_2, \qquad P(r) = \phi_1,$$
$$P'(X) = \beta_r (X - r)^3 (X - z_r)^3,$$

where $z_r = \min\{h(L - r') - \frac{r}{2}, h(x) - \frac{r}{2}\}$, and $\beta_r \in R$ is a constant determined by the two first conditions.

Let φ^* defined on $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ by $\varphi^*|_{\Omega_i} = \varphi_i$, for $1 \leq i \leq 4$. Let us check that it satisfies the boundary conditions we claimed (4.39), (4.40), (4.41), (4.42):

- On Ω_1 (resp. Ω_2), $\varphi^* \equiv \phi_1$ (resp. $\equiv \phi_2$), thus $\nabla\varphi^* = (0,0)$, and on Γ_t (resp. Γ_b), the boundary conditions $\nabla\varphi^* \cdot \boldsymbol{n}|_{\Gamma_t} = \nabla\varphi^* \cdot \boldsymbol{n}|_{\Gamma_b} = 0$ and $\nabla\Delta\varphi^* \cdot \boldsymbol{n}|_{\Gamma_t} = \nabla\Delta\varphi^* \cdot \boldsymbol{n}|_{\Gamma_b} = 0$ are satisfied. The same computation holds true on $\Gamma_r \cap \partial\Omega_1$ and $\Gamma_r \cap \partial\Omega_2$.

- On $\Gamma_l \cap \partial\Omega_1$, it follows from hypothesis (4.38) that $\varphi^*|_{\Gamma_l \cap \partial\Omega_1} = \phi_1 = \varphi_l|_{\Gamma_l \cap \partial\Omega_1}$. The same computation holds true on $\Gamma_l \cap \partial\Omega_2$ with ϕ_2.

- On Ω_3, $\varphi^* = \left(\frac{1}{\alpha^2}F'(\varphi_l) - \partial_z^2\varphi_l\right)\frac{x^2}{2} + \varphi_l$. Therefore, on $\Gamma_l \cap \partial\Omega_3$, since $x = 0$, $\varphi^*|_{\Gamma_l \cap \partial\Omega_3} = \varphi_l$.

- On Γ_l, by construction, $\partial_x^2\varphi^*|_{\Gamma_l} = -\partial_z^2\varphi_l$.

- On Ω_4, $\varphi^* = P(z)$, thus φ^* is independent of x. This yields that on $\Gamma_r \cap \partial\Omega_4$, we have $\nabla\varphi^* \cdot \boldsymbol{n}|_{\Gamma_r \cap \partial\Omega_4} = \partial_x P(z) = 0$, and the same argument is used in order to prove that $\nabla\Delta\varphi^* \cdot \boldsymbol{n}|_{\Gamma_r \cap \partial\Omega_4} = 0$.

Moreover, we check that $\varphi^* \in \mathcal{C}^3(\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4)$:

- On $\partial\Omega_1 \cap \partial\Omega_3$, we have to check that $\partial_z^j\varphi^*$ is continuous for $0 \le j \le 3$. By hypothesis (4.38), and by construction of Ω_1 and Ω_3, we have $\partial_z\varphi_l|_{\partial\Omega_1 \cap \partial\Omega_3} = \partial_z\phi_1 = 0$. Thus, for $j \ge 1$,

$$\partial_z^j\varphi_1|_{\partial\Omega_1 \cap \partial\Omega_3} = 0 = \partial_z^j\varphi_3|_{\partial\Omega_1 \cap \partial\Omega_3} = -\frac{x^2}{2}\partial_z^{j+2}\varphi_l|_{\partial\Omega_1 \cap \partial\Omega_3} + \partial_z^j\varphi_l|_{\partial\Omega_1 \cap \partial\Omega_3}.$$

  As far as the continuity of the function is concerned,

$$\varphi_1|_{\partial\Omega_1 \cap \partial\Omega_3} = \phi_1 = -\frac{x^2}{2}\underbrace{\partial_z^2\varphi_l}_{=0} + \phi_1.$$

- On $\partial\Omega_2 \cap \partial\Omega_3$, the same argument with ϕ_2 allows us to conclude that $\varphi \in \mathcal{C}^3(\partial\Omega_2 \cap \partial\Omega_3)$.

- On $\partial\Omega_1 \cap \partial\Omega_4$, by construction of P, it follows that $P(z_r) = \phi_1$, $\partial_z^j P(z_r) = 0$, for $1 \le j \le 3$, which proves the continuity.

- The same reasoning proves the $\mathcal{C}^3$-regularity on $\partial\Omega_2 \cap \partial\Omega_4$.

In order to end the proof, we extend the function φ^* in the whole domain ω by the Calderón extension theorem (see e.g. [Ada75, Th. 4.3]) to $\hat{\varphi}_l \in H^3(\Omega)$, which satisfies $\hat{\varphi}_l|_{\Omega_i} = \varphi^*|_{\Omega_i}$, and thus satisfies the boundary conditions claimed. □

4.3.3 Anisotropic trace estimates

Proposition 4.12. *For $f \in H^1(\Omega)$, the following applies:*

$$|f|^2_{L^2(\Gamma_l)} \leq \bar{C}\left(L|\partial_x f|_2^2 + \Big(\frac{1}{L} + Lh'_M\Big)|f|_2^2\right).$$

Proof. Introduce the auxiliary function $\xi(x) = \frac{1}{2}(x-L)^2$. This function satisfies for all $x \in \mathbb{R}$, $\xi''(x) = 1$. Integration by parts gives

$$\int_0^L \int_0^{h(x)} f^2 = \int_0^L \int_0^{h(x)} f^2 \xi'' = \left[\int_0^{h(x)} \xi' f^2\right]_{x=0}^{x=L} - \int_0^L \int_0^{h(x)} 2f\partial_x f \xi' - \int_0^L h'(x) \int_0^{h(x)} \xi' f^2.$$

Since $\xi'(L) = 0$, and $\xi'(0) = -L$, we get

$$L \int_0^{h(0)} f^2|_{x=0} = |f|_2^2 + \int_0^L \int_0^{h(x)} 2f\partial_x f \xi' + \int_0^L h'(x) \int_0^{h(x)} \xi' f^2.$$

Moreover $|\xi'(x)| \leq L$ for $x \in [0, L]$, $|h'|_\infty \leq h'_M$, and the Cauchy-Schwarz inequality and Young's inequality imply

$$|f|^2_{L^2(\Gamma_l)} \leq \frac{1}{L}|f|_2^2 + 2L|f|_2|\partial_x f|_2 + Lh'_M|f|_2^2 \leq \bar{C}\left(\Big(\frac{1}{L} + Lh'_M\Big)|f|_2^2 + L|\partial_x f|_2^2\right).$$

□

Proposition 4.13. *If $f \in H^1(\Omega)$ and if $\bar{x} \in [0, L]$, then*

$$|f(\bar{x}, \cdot)|^2_{L^2(0,h(\bar{x}))} \leq \bar{C}\left(L|\partial_x f|_2^2 + \Big(\frac{1}{L} + Lh'_M\Big)|f|_2^2\right).$$

Proof. We adapt the proof of Proposition 4.12 for $\bar{x} \in (0, L)$. Let ξ be a function satisfying for all $x \in \mathbb{R}$, $\xi''(x) = 1$ and $\xi'(\bar{x}) = L$. Then it follows that $|\xi'(0)| \leq 2L$. Integration by parts yields

$$\left[\int_0^h \xi' f^2\right]_{x=0}^{x=\bar{x}} = \int_0^{\bar{x}} \int_0^h f^2 + \int_0^{\bar{x}} \int_0^h 2f\partial_x f \xi' + \int_0^{\bar{x}} h'(x) \int_0^{h(x)} \xi' f^2,$$

i.e., using the hypotheses on ξ'

$$L \int_0^{h(\bar{x})} f^2|_{x=\bar{x}} \leq 2L \int_0^{h(0)} f^2|_{x=0} + \int_0^L \int_0^h f^2 + \int_0^L \int_0^h 2|f\partial_x f \xi'| + \int_0^L |h'(x)| \int_0^{h(x)} |\xi'| f^2.$$

Applying the previous proposition to the term $\int_0^{h(0)} f^2|_{x=0}$, and using that $|\xi'(x)| \leq 3L$

for any $x \in [0, L]$, it follows that

$$L|f(\bar{x}, \cdot)|^2_{L^2(0,h(\bar{x}))} \leq \bar{C}\left(L^2|\partial_x f|_2^2 + |f|_2^2\right) + |f|_2^2 + 6L|f|_2|\partial_x f|_2 + 3Lh'_M|f|_2^2,$$

and thus

$$|f(\bar{x}, \cdot)|^2_{L^2(0,h(\bar{x}))} \leq \bar{C}\left(L|\partial_x f|_2^2 + \left(\frac{1}{L} + Lh'_M\right)|f|_2^2\right).$$

□

Corollary 4.14. *If $f \in H^1(\Omega)$, then*

$$\left|f|_{y=0}\right|^2_{L^2(0,L)} \leq \bar{C}\left(h_M|\partial_y f|_2^2 + \frac{1}{h_M}|f|_2^2\right).$$

Proof. The adaptation of the proof of Proposition 4.13 is straightforward. □

Remark 4.15. *We can apply the previous result to φ and μ, leading to the following estimates for $\varphi \in \Phi^1_l$, $\mu \in \Phi^1$:*

$$\begin{aligned}|\varphi|^2_{L^2(\Gamma_l)}, |\varphi|^2_{L^2(\Gamma_r)} &\leq \bar{C}\left(L|\partial_x\varphi|_2^2 + \left(\frac{1}{L} + Lh'_M\right)|\varphi|_2^2\right),\\ |\mu|^2_{L^2(\Gamma_l)}, |\mu|^2_{L^2(\Gamma_r)} &\leq \bar{C}\left(L|\partial_x\mu|_2^2 + \left(\frac{1}{L} + Lh'_M\right)|\mu|_2^2\right).\end{aligned} \tag{4.43}$$

For $\varphi \in \Phi^2_l$, we can also apply this proposition to $\partial_x\varphi$. Since $(\partial_x\varphi)|_{\Gamma_r} = 0$, we can apply the Poincaré inequality: $|\partial_x\varphi|_2^2 \leq L^2|\partial_x^2\varphi|_2^2$. Thus,

$$|\partial_x\varphi|^2_{L^2(\Gamma_l)} \leq \bar{C}L(1 + L^2h'_M)|\partial_x^2\varphi|_2^2. \tag{4.44}$$

4.3.4 Specific Poincaré inequalities

The Poincaré inequalities stated in this section are specific to the functions φ and μ satisfying the boundary conditions (4.26) and such that μ is given as a function of φ by (4.8).

Proposition 4.16 (A Poincaré inequality for φ)**.** *Let $\varphi \in \Phi^1_l$. Let $L_h^2 = L^2(1+h_M^2+{h'_M}^2)$. We have*

$$|\varphi|_2^2 \leq \bar{C}\left(L^2(1 + h_M^2 + {h'_M}^2)|\nabla\varphi|_2^2 + L|\varphi_l|^2_{L^2(\Gamma_l)}\right) = \bar{C}\left(L_h^2|\nabla\varphi|_2^2 + L|\varphi_l|^2_{L^2(\Gamma_l)}\right). \tag{4.45}$$

Proof. This is a consequence of the usual Poincaré inequality with $\varphi|_{x=0} = \varphi_l$ (see for example [Tem97, § II.1.4]). Let $(x, \tilde{z}) \in (0, L) \times (0, 1)$, and define $\tilde{\varphi}(x, \tilde{z})$ such that

$\tilde{\varphi}(x,\tilde{z}) = \varphi(x,z)$, with $z = h(x)\tilde{z}$. For $\tilde{\varphi}$, it is well-known that

$$|\tilde{\varphi}|_2^2 \leq \bar{C}\left(L^2|\partial_x\tilde{\varphi}|_2^2 + L\|\varphi_l\|^2_{L^2(\Gamma_l)}\right).$$

Since $\partial_x\tilde{\varphi} = \partial_x\varphi + z\frac{h'}{h}\partial_z\varphi$ and $\partial_{\tilde{z}}\tilde{\varphi} = h\partial_z\varphi$, we deduce from the fact that $z/h(x) \leq 1$ that

$$|\varphi|_2^2 \leq \bar{C}\left(L^2\left(|\partial_x\varphi|_2^2 + h_M^2|\partial_z\varphi|_2^2 + {h'_M}^2|\partial_z\varphi|_2^2\right) + L|\varphi_l|^2_{L^2(\Gamma_l)}\right), \tag{4.46}$$

which proves the inequality claimed. □

Proposition 4.17 (A Poincaré inequality for μ)**.** *Let $\mu \in \Phi^1$. Let $L_h^2 = L^2(1+h_M^2+{h'_M}^2)$. We have*

$$|\mu|_2^2 \leq \bar{C}L_h^2|\nabla\mu|_2^2. \tag{4.47}$$

Proof. Since $\mu|_{\Gamma_l} = 0$, this result follows directly from the Poincaré inequality for φ. □

4.4 About the Reynolds equation

The Reynolds equation (4.18) governing the behavior of the pressure p as a function of φ can be solved explicitly, if the coefficients are regular enough. In this section, we prove that as soon as the order parameter φ belongs to $H^1(\Omega)$ (this hypothesis is included in the results obtained in (4.114)), then the Reynolds equation (first equation in (4.22)) admits indeed a solution.

Proposition 4.18. *Let $\varphi \in H^1(\Omega)$. Then the Reynolds equation (4.22).i) equipped with the boundary conditions (4.25) admits a unique solution which satisfies*

$$\partial_x p \in H^1(0,L).$$

The velocity field (u,v) given as a function of p by (4.19) satisfies

$$u \in H^1(\Omega)\cap L^\infty(\Omega) \quad \textit{and} \quad v \in L^2(\Omega), \textit{ with } \partial_z v \in L^2(\Omega).$$

Moreover, we have the following estimates

$$|u|_\infty \leq \bar{C}(1+h_M^2) \qquad \textit{and} \quad |v|_2 \leq \bar{C}(1+h_M^2)\|\varphi\|_1. \tag{4.48}$$

Proof. Let us sketch the main steps of the proof of Proposition 4.18:

- The Reynolds equation can be solved explicitly, so that p is given as a function of the coefficients $\widetilde{d}$ and $\widetilde{e}$ (given as functions of φ by (4.15)): recalling definition (4.25) of w_{in}, we can integrate the Reynolds equation once and obtain

$$\widetilde{d}\,\partial_x p = s\,\widetilde{e} + \widetilde{d_l}\,w_{\text{in}} - s\,\widetilde{e}_l, \tag{4.49}$$

where $\widetilde{d_l}$ and $\widetilde{e}_l$ denote respectively $\widetilde{d}|_{x=0}$ and $\widetilde{e}|_{x=0}$. These quantities only depend on φ_l and are thus known. If $\widetilde{d}$ does not vanish, we compute formally $\partial_x p$, and then p using the boundary condition (4.25): $p(L) = 0$. In order to obtain estimates on the pressure, we have to prove that the coefficients $\widetilde{d}$ and $\widetilde{e}$ are regular enough (see Lemma 4.19), and that $\widetilde{d}(\varphi)$ is greater than a strictly positive constant (i.e. the operator $\partial_x(d\,\partial_x\cdot)$ must be coercive, see Lemma 4.20).

- As far as the velocity is concerned,

$$u = f\partial_x p + g,$$

where the coefficients are given by $f = \left(b - \frac{\widetilde{b}}{\widetilde{a}}a\right)$ and $g = \left(1 - \frac{a}{\widetilde{a}}\right)s$ (and a, b, $\widetilde{a}$, $\widetilde{b}$ are defined in (4.14)). It is enough to prove the regularity of f and g in order to deduce the needed estimate on u from the estimate on $\partial_x p$ (see Lemma 4.21).

- The velocity v is given by

$$v(x,z) = -\int_0^z \partial_x u(x,Z)\,dZ,$$

and the regularity of v follows from the regularity of u (see Lemma 4.22).

4.4.1 Regularity of the coefficients

Let us introduce the following function space:

$$X(\Omega) = \{f \in H^1(\Omega) \cap L^\infty(\Omega)\}. \tag{4.50}$$

Lemma 4.19. *If $\varphi \in H^1(\Omega)$, the following regularity holds true for the coefficients:*

$$a,\ b,\ c\ \in X(\Omega),$$

$$\widetilde{a},\ \widetilde{b},\ \widetilde{c},\ \widetilde{d},\ \widetilde{e} \in H^1(0,L).$$

Proof. Let $\varphi \in H^1(\Omega)$. Observe that the terms a, b, c are of the form $\displaystyle\int_0^z Z^i/\eta(\varphi(x,Z))\,dZ$, for $i = 0, 1, 2$ (see definition (4.14) of a, b, c). Therefore we will present the details of the proof that the mapping f defined by $f(x,z) = \int_0^z Z/\eta(\varphi(x,Z))\,dZ$ satisfies $f \in X(\Omega)$. The regularity of the coefficients follows immediately.

- First we prove by Cauchy-Schwarz inequality that $f \in L^2(\Omega)$: for any $(x,z) \in \Omega$, we have

$$f(x,z)^2 = \Big(\int_0^z \frac{Z}{\eta(\varphi(x,Z))}\,dZ\Big)^2 \leq \Big(\frac{1}{\eta_m}\int_0^z Z dZ\Big)^2 \leq \frac{z^2}{4\eta_m^2}.$$

  After integrating with respect to z and x, we get

$$\int_0^L \int_0^{h(x)} f(x,z)^2 dz\,dx \leq \frac{h_M^5 L}{20\eta_m^2} < \infty.$$

- By integration of f with respect to x and z, we show that $f \in H^1(\Omega)$ and $\partial_z f \in H^1(\Omega)$:

  - On one hand,

$$\partial_x f(x,z) = \int_0^z \frac{Z\eta'(\varphi(x,Z))}{\eta(\varphi(x,Z))^2}\,\partial_x\varphi(x,Z)\,dZ,$$

    with $\partial_x\varphi \in L^2(\Omega)$. Let $(x,z) \in \Omega$. Since η and η' are bounded from above and below, we compute

$$\begin{aligned}\partial_x f(x,y)^2 &= \left(\int_0^z \frac{Z\eta'(\varphi)}{\eta(\varphi)}\partial_x\varphi(x,Z)dZ\right)^2\\ &\leq \frac{{\eta'_M}^2}{\eta_m^4}\int_0^z Z^2 dZ \int_0^z \partial_x\varphi(x,Z)^2 dZ \leq \frac{{\eta'_M}^2 z^3}{3\eta_m^4}\int_0^{h(x)} \partial_x\varphi(x,Z)^2 dZ.\end{aligned}$$

    After integrating with respect to z, we get

$$\int_0^{h(x)} \partial_x f(x,y)^2 dy \leq \frac{{\eta'_M}^2 h_M^4}{36\eta_m^4}\int_0^{h(x)} \partial_x\varphi(x,Z)^2 dZ,$$

    and after integrating with respect to x

$$|\partial_x f|_2^2 = \int_0^L \int_0^{h(x)} \partial_x f(x,y)^2 dy\,dx \leq \frac{{\eta'_M}^2 h_M^4}{36\eta_m^4}|\partial_x\varphi|_2^2 < \infty.$$

    It follows that $\partial_x f \in L^2(\Omega)$.

  - On the other hand, $\partial_z f(x,z) = z/\eta(\varphi(x,z)) \in H^1(\Omega)$, since $\varphi \in H^1(\Omega)$ and

$\eta \in \mathcal{C}^1(\mathbb{R})$.

- Next we show that $f \in L^\infty(\Omega)$: since $\partial_z f \in L^2(\Omega)$, we can write

$$f(x,z) = f(x,0) + \int_0^z \partial_Z f(x,Z)\,dZ.$$

By definition of f, we know that $f(x,0) = 0$, $\forall x \in [0,L]$. Therefore, the usual trace theorem for the Sobolev space $H^1(\Omega)$ implies that

$$\begin{aligned}|f(x,z)|^2 &\le z\int_0^z (\partial_Z f(x,Z))^2 dZ \le h_M \int_0^{h(x)} (\partial_Z f(x,Z))^2 dZ = h_M |\partial_z f|^2_{L^2(0,h(x))} \\ &\le C\|\partial_z f\|^2_{H^{1/2}(0,h(x))} \le C\|\partial_z f\|^2_{H^1(\Omega)},\end{aligned}$$

thus

$$|f|^2_\infty \le C\|\partial_z f\|^2_1 < \infty.$$

This proves that $f \in X(\Omega)$. As stated at the beginning of the proof, it follows that $a, b, c \in X(\Omega)$. It remains to prove the regularity of $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$, $\widetilde{e}$.

- For the coefficients of the form $\widetilde{a}(x) = a(x, h(x))$, the regularity in $H^1(0,L)$ is obtained similarly to the regularity of a.

- The key point of the proof is to observe that $X(\Omega)$ and $H^1(0,L)$ (which is embedded in $L^\infty(0,L)$) are algebras:

$$(f,g) \in X(\Omega)^2 \Rightarrow fg \in X(\Omega), \quad \text{and} \quad (f,g) \in H^1(0,L)^2 \Rightarrow fg \in H^1(0,L).$$

Recalling the definitions $\widetilde{d} = \left(\widetilde{c} - \dfrac{\widetilde{b}^2}{\widetilde{a}}\right)$ and $\widetilde{e} = \dfrac{\widetilde{b}}{\widetilde{a}}$, and using the fact that $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ belong to $H^1(0,L)$, we need to show that $1/\widetilde{a}$ remains bounded. Since $\eta \le \eta_M$, we have

$$\widetilde{a}(x) = \int_0^{h(x)} \frac{1}{\eta(\varphi(x,Z))} dZ \ge \frac{h_m}{\eta_M} \quad \text{i.e.} \quad \frac{1}{\widetilde{a}} \le \frac{\eta_M}{h_m}. \tag{4.51}$$

From the regularity of $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, from the algebra structure and from (4.51), we deduce that

$$\widetilde{d} \in X(0,L), \quad \widetilde{e} \in X(0,L).$$

□

4.4.2 Coercivity of the operator

Lemma 4.20. *Let $\widetilde{d}$ be defined by (4.15). There exists $\delta > 0$ such that*

$$\forall x \in (0,L), \quad \widetilde{d}(x) \geq \delta > 0.$$

Moreover, $\delta = h_m^3/4\eta_M$.

Proof. By definition (4.15), $\widetilde{d}(x)$ can be written in the form:

$$\widetilde{d}(x) = \widetilde{c}(x) - \frac{\widetilde{b}(x)^2}{\widetilde{a}(x)} = \int_0^{h(x)} \frac{z^2}{\eta(x,z)} dz - \frac{\left(\displaystyle\int_0^{h(x)} \frac{z}{\eta(x,z)} dz\right)^2}{\displaystyle\int_0^{h(x)} \frac{1}{\eta(x,z)} dz}.$$

In order to prove the assertion, it suffices to prove that there exists $\delta > 0$ such that

$$\left(\int_0^h \frac{z^2}{\eta} dz\right)\left(\int_0^h \frac{1}{\eta} dz\right) - \left(\int_0^h \frac{z}{\eta} dz\right)^2 \geq \delta \left(\int_0^h \frac{1}{\eta} dz\right).$$

Denote by P the following polynomial

$$\begin{aligned} P : \lambda \mapsto & \int_0^{h(x)} \left(\frac{z}{\sqrt{\eta(\varphi(x,z))}} + \frac{\lambda}{\sqrt{\eta(\varphi(x,z))}}\right)^2 dz \\ &= \int_0^{h(x)} \frac{z^2}{\eta(\varphi(x,z))} + \frac{\lambda^2}{\eta(\varphi(x,z))} + \frac{2z\lambda}{\eta(\varphi(x,z))} dz. \end{aligned}$$

Since $\forall (x,z) \in \Omega,\ \eta(x,z) \leq \eta_M$, we get $\forall \lambda \in \mathbb{R}$

$$P(\lambda) \geq \frac{1}{\eta_M} \int_0^{h(x)} z^2 + 2z\lambda + \lambda^2 dz = \frac{1}{3\eta_M}(h(x)^3 + 3h(x)^2\lambda + 3h(x)\lambda^2).$$

A simple study of the right-hand side polynomial proves that

$$\forall \lambda \in \mathbb{R},\ \forall x \in (0,L), \quad h(x)^2 + 3h(x)\lambda + 3\lambda^2 \geq \frac{h(x)^2}{4},$$

thus

$$P(\lambda) \geq \frac{h(x)^3}{12\eta_M}, \quad \text{i.e.} \quad P(\lambda) - \frac{h(x)^3}{12\eta_M} \geq 0,$$

therefore the discriminant of the polynomial

$$P(\lambda) - \frac{h(x)^3}{12\eta_M} = \lambda^2 \int_0^h \frac{1}{\eta} + 2\lambda \int_0^h \frac{z}{\eta} + \int_0^h \frac{z^2}{\eta} - \frac{h^3}{12\eta_M}$$

is negative:

$$4\left(\int_0^{h(x)}\frac{zdz}{\eta(\varphi(x,z))}\right)^2-4\left(\int_0^{h(x)}\frac{dz}{\eta(\varphi(x,z))}\right)\left[\left(\int_0^{h(x)}\frac{z^2dz}{\eta(\varphi(x,z))}\right)-\frac{h(x)^3}{12\eta_M}\right]\leq 0,$$

that is to say

$$\left(\int_0^h\frac{z^2}{\eta}dz\right)\left(\int_0^h\frac{1}{\eta}dz\right)-\left(\int_0^h\frac{z}{\eta}dz\right)^2\geq\frac{h_m^3}{12\eta_M}\left(\int_0^h\frac{1}{\eta}dz\right),\quad \text{i.e.}\quad \widetilde{d}\geq\frac{h_m^3}{4\eta_M}>0.$$

□

The two previous lemmas 4.19 (regularity of the coefficients) and 4.20 (coercivity of the operator) with the formula (4.49) imply that $\partial_x p\in H^1(0,L)$, thus $p\in H^2(0,L)$. The regularity of $\boldsymbol{u}$ follows from the second and third equations of (4.22), and the regularity of the coefficients (Lemma 4.19).

$$u=(b-\frac{a\widetilde{b}}{\widetilde{a}})\partial_x p+s(1-\frac{a}{\widetilde{a}})\in X(\Omega),$$
$$v(x,z)=-\int_0^z\partial_x u(x,Z)dZ\in L^2(\Omega),\quad \partial_z v=-\partial_x u\in L^2(\Omega).$$

4.4.3 Estimates of $|u|_\infty$ and $|v|_2$

Lemma 4.21. *Let $\varphi\in H^1(\Omega)$. The horizontal velocity u given by the two first equations of (4.22) satisfies*

$$|u|_\infty\leq\bar{C}(1+h_M^2),$$

where $\bar{C}$ denotes a constant depending on Ω only through the ratio $\sigma=h_M/h_m$.

Proof. By definition (4.22)ii) of u, we know that u is a combination of coefficients of the form $\int_0^z Z/\eta(\varphi)dZ$. Indeed

$$|u|_\infty\leq\left(|b|_\infty+\frac{|a|_\infty|\widetilde{b}|_\infty}{\min\limits_{x\in(0,L)}\widetilde{a}(x)}\right)|\partial_x p|_\infty+s\left(1+\frac{|a|_\infty}{\min\limits_{x\in(0,L)}\widetilde{a}(x)}\right),\tag{4.52}$$

and $\partial_x p$ is given by (4.49), thus:

$$|\partial_x p|_\infty\leq\frac{1}{\min\limits_{x\in(0,L)}\widetilde{d}(x)}\left(s|e|_\infty+|\widetilde{d}_l|_\infty|w_{\text{in}}|+s|\widetilde{e}_l|_\infty\right).\tag{4.53}$$

Let us obtain estimates for these coefficients.

▷ Using the boundedness hypothesis on η, and applying the Cauchy-Schwarz inequality and the fact that $\forall x \in (0, L)$, $h(x) \le h_M$, we can write for all $(x, z) \in \Omega$

$$a(x,z) = \int_0^z \frac{dZ}{\eta(\varphi(x,Z))} \le \frac{h_M}{\eta_m}.$$

We thus obtain

$$|a|_\infty \le \bar{C} h_M, \quad \text{and} \quad |\widetilde{a}|_\infty \le \bar{C} h_M. \tag{4.54}$$

▷ Similar computations for b, c and $\widetilde{b}$, $\widetilde{c}$ give

$$|b|_\infty, |\widetilde{b}|_\infty \le \bar{C} h_M^2, \quad |c|_\infty, |\widetilde{c}|_\infty \le \bar{C} h_M^3. \tag{4.55}$$

▷ It has already been proved in (4.51) that

$$\widetilde{a} \ge \frac{h_m}{\eta_M}. \tag{4.56}$$

▷ Recalling definition (4.15) of $\widetilde{e}$, and using (4.56), it follows from (4.55):

$$|\widetilde{e}|_\infty = \frac{|b|_\infty}{\min\limits_{x \in (0,L)} \widetilde{a}(x)} \le \frac{\bar{C} h_M^2}{h_m} \le \bar{C} \sigma h_M = \bar{C} h_M. \tag{4.57}$$

We recall that we denote by $\bar{C}$ any constant independent of Ω or depending on Ω only through the rate $\sigma = \dfrac{h_M}{h_m}$.

▷ The coercivity lemma 4.20 implies that

$$\widetilde{d} \ge \delta = \frac{h_m^3}{4\eta_M}. \tag{4.58}$$

▷ Moreover, the same computations as for estimates (4.54), (4.55) lead to

$$|\widetilde{a}_l|_\infty \le \bar{C} h_M, \quad |\widetilde{b}_l|_\infty \le \bar{C} h_M^2, \quad |\widetilde{c}_l|_\infty \le \bar{C} h_M^3.$$

We get (since $h_M \ge h_m$)

$$\begin{aligned} |\widetilde{d}_l|_\infty &\le |\widetilde{c}_l|_\infty + |\widetilde{b}_l|_\infty^2 \frac{h_m}{\eta_M} \le \bar{C} h_M^3, \\ |\widetilde{e}_l|_\infty &\le |\widetilde{b}_l|_\infty \frac{h_m}{\eta_M} \le \bar{C} h_M. \end{aligned} \tag{4.59}$$

Thus, using (4.58), (4.57), (4.59) in (4.53), it follows

$$|\partial_x p|_\infty \leq \bar{C}(1 + \frac{1}{h_m^2}). \tag{4.60}$$

Now, using (4.54), (4.55), (4.56) and (4.60) in (4.52), we obtain the following estimate:

$$|u|_\infty \leq \bar{C}h_M^2 \left(1 + \frac{1}{h_m^2}\right) \leq \bar{C}(1 + h_M^2), \tag{4.61}$$

which ends the proof. □

Lemma 4.22. *Let $\varphi \in H^1(\Omega)$. The vertical velocity v given by (4.19) satisfies*

$$|v|_2 \leq \bar{C}(1 + h_M^2)\|\varphi\|_1,$$

where $\bar{C}$ denotes a constant depending on Ω only through the ratio $\sigma = h_M/h_m$.

Proof. It is clear from definition (4.19) of v and the Cauchy-Schwarz inequality that

$$|v|_2 \leq h_M |\partial_x u|_2. \tag{4.62}$$

Let us introduce the coefficients $f = b - \frac{a\widetilde{b}}{\widetilde{a}}$ and $g = 1 - \frac{a}{\widetilde{a}}$, so that $u = f\partial_x p + sg$. Therefore

$$|\partial_x u|_2 \leq |\partial_x f|_2 |\partial_x p|_\infty + |f|_\infty |\partial_x^2 p|_2 + s|\partial_x g|_2, \tag{4.63}$$

and $\partial_x^2 p$ is given by taking the derivative of (4.49) with respect to x:

$$|\partial_x^2 p|_2 \leq \frac{1}{\min\limits_{x\in(0,L)} \widetilde{d}(x)} \left(s|\partial_x \widetilde{e}|_2 - |\partial_x \widetilde{d}|_2 |\partial_x p|_\infty\right). \tag{4.64}$$

Let us obtain estimates for each coefficient separately:

▷ We have

$$|f|_\infty \leq |\widetilde{b}|_\infty + \frac{\bar{C}}{h_m} |a|_\infty |\widetilde{b}|_\infty. \tag{4.65}$$

▷ It remains to obtain estimates of the derivatives of the coefficients with respect to x.

We can compute $\partial_x a = \displaystyle\int_0^y \frac{\eta'(\varphi)}{\eta(\varphi)^2}\partial_x\varphi$, and Cauchy-Schwarz inequality yields

$$|\partial_x a|_2^2 \leq \frac{{\eta'_M}^2}{\eta_m^4} \int_\Omega \left(\int_0^y \partial_x \varphi(x,z)\, dz \right)^2$$
$$\leq \bar{C} h_M \int_\Omega \int_0^y |\partial_y \varphi|^2 \leq \bar{C} h_M^2 |\partial_x \varphi|_2^2 \leq \bar{C} h_M^2 \|\varphi\|_1^2,$$

and similar estimates for all the other coefficients:

$$|\partial_x a|_2, |\partial_x \widetilde{a}|_2 \leq \bar{C} h_M \|\varphi\|_1, \qquad |\partial_x b|_2, |\partial_x \widetilde{b}|_2 \leq \bar{C} h_M^2 \|\varphi\|_1,$$
$$|\partial_x c|_2, |\partial_x \widetilde{c}|_2 \leq \bar{C} h_M^3 \|\varphi\|_1. \tag{4.66}$$

▷ Let us write

$$\partial_x \left(\frac{a}{\widetilde{a}} \right) = \frac{\partial_x a\, \widetilde{a} - a\, \partial_x \widetilde{a}}{\widetilde{a}^2}.$$

From (4.56), we know that $\widetilde{a} \geq \dfrac{h_m}{\eta_M}$. This estimate combined with (4.66) suffices to prove that

$$\left| \partial_x \left(\frac{a}{\widetilde{a}} \right) \right|_2 \leq \bar{C} \|\varphi\|_1, \tag{4.67}$$

and

$$\left| \partial_x \left(\frac{\widetilde{b}}{\widetilde{a}} \right) \right|_2 \leq \bar{C} h_M \|\varphi\|_1. \tag{4.68}$$

▷ Since

$$\partial_x d = \partial_x c - \partial_x \widetilde{b} \frac{\widetilde{b}}{\widetilde{a}} - \widetilde{b} \partial_x \left(\frac{\widetilde{b}}{\widetilde{a}} \right), \qquad \partial_x e = \partial_x \left(\frac{\widetilde{b}}{\widetilde{a}} \right),$$
$$\partial_x f = \partial_x b - \partial_x a \frac{\widetilde{b}}{\widetilde{a}} - a \partial_x \left(\frac{\widetilde{b}}{\widetilde{a}} \right), \qquad \partial_x g = \partial_x \left(\frac{a}{\widetilde{a}} \right), \tag{4.69}$$

it follows, using (4.66), (4.67), (4.68) in (4.69), that

$$|\partial_x \widetilde{d}|_2 \leq \bar{C} h_M^3 \|\varphi\|_1, \qquad |\partial_x \widetilde{e}|_2 \leq \bar{C} h_M \|\varphi\|_1,$$
$$|\partial_x f|_2 \leq \bar{C} h_M^2 \|\varphi\|_1, \qquad |\partial_x g|_2 \leq \bar{C} \|\varphi\|_1. \tag{4.70}$$

Putting (4.58), (4.70), (4.60) in (4.64) and (4.63), we deduce an estimate for each of the three terms in (4.63):

▶ The first term is estimated by:

$$|\partial_x f|_2 |\partial_x p|_\infty \leq \bar{C} h_M^2 \|\varphi\|_1 \Big(1 + \frac{1}{h_m^2}\Big) \leq \bar{C}(1 + h_M^2)\|\varphi\|_1.$$

▶ For the second term, we have:

$$\begin{aligned}\frac{|f|_\infty}{\delta}\Big(s|\partial_x \widetilde{e}|_2 + |\partial_x \widetilde{d}|_2 |\partial_x p|_\infty\Big) &\leq \frac{1}{h_m^3} h_M^2 \bigg(h_M \|\varphi\|_1 + h_M^3 \|\varphi\|_1 \Big(1 + \frac{1}{h_m^2}\Big)\bigg) \\ &\leq \bar{C}(1 + h_M^2)\|\varphi\|_1.\end{aligned}$$

▶ The third term follows directly from (4.70):

$$|\partial_x g|_2 \leq \bar{C}\|\varphi\|_1.$$

Therefore, using (4.62) and these three estimates for $|\partial_x u|_2$, we obtain:

$$|v|_2 \leq h_M |\partial_x u|_2 \leq \bar{C}(1 + h_M^2)\|\varphi\|_1,$$

which proves the lemma. □

Remark 4.23. *Observe that it is not straightforward to prove that $v \in L^\infty(\Omega)$ if $\varphi \in H^1(\Omega)$. Computing $|v|_\infty$, it is bounded by $|\partial_x u|_\infty$, and thus by $|\partial_x f|_\infty$ for example, i.e. by $|\partial_x a|_\infty$. But writing $|\partial_x a|_\infty \leq C|\partial_x \varphi|_\infty$, the regularity of φ does not allow to conclude.*

4.5 About the Cahn-Hilliard equation

4.5.1 Galerkin approximations

Before explaining the construction of the Galerkin approximations, let us state an additional condition on φ_l.

Assumption 4.24. *For any $y \in \Gamma_l$, let $\varphi_l(y) \in \{-1, 0, 1\}$. Since $\varphi_l \in H^{5/2}(\Gamma_l)$, this condition is equivalent to saying:*

$$\varphi_l \equiv \pm 1, \qquad \textit{or} \qquad \varphi_l \equiv 0.$$

This condition is quite restrictive, since it imposes that on the whole injection zone, a pure fluid or an homogeneous mixture (i.e. with the same proportion of each fluid) is injected. The three values 1, −1, and 0 correspond to the values that cancel F'. Thus, this assumption allows to construct a Galerkin approximation for μ which is compatible with the boundary conditions. However, as a perspective of the work, it would be of interest to remove this assumption.

As in the earlier works on Cahn-Hilliard equation (e.g. [Boy99]), we apply the Galerkin method in order to prove the existence of a solution to the system (4.22). This process consists in building approximate solutions in finite dimension, for which the existence follows from the Cauchy-Lipschitz theorem.

Since Φ^1 is a separable Hilbert space, there exists an Hilbertian basis $(\psi_i)_{i\geq 1}$ of Φ^1. The functions ψ_i can be chosen to be eigenfunctions of the Laplacian $-\Delta$ with domain Φ^1: $-\Delta\psi_i = \lambda_i\psi_i$, for any $i \in \mathbb{N}^*$. We define $\Psi_n = \mathrm{Span}(\psi_1, \cdots, \psi_n)$, and $\mathbb{P}_{\Psi_n}$ the orthogonal projector on Ψ_n in $L^2(\Omega)$. As a projector, $\mathbb{P}_{\Psi_n}$ satisfies:

$$(\mathbb{P}_{\Psi_n} f, g) = (f, \mathbb{P}_{\Psi_n} g), \qquad \forall (f,g) \in L^2(\Omega)^2, \tag{4.71}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

Recalling that we denoted $\hat{\varphi}_l \in \Phi_l^2$ a lifting of the boundary conditions (4.26) for φ which is independent of t (section 4.3.2), we consider the following approximation of φ:

$$\varphi_n(t) = \sum_{i=1}^{n} \beta_i(t)\psi_i + \hat{\varphi}_l,$$

where β_i are $\mathcal{C}^1$-functions, satisfying a system of ordinary differential equations. Indeed, the weak form of the system (4.7)-(4.8) reads, when using integration by parts:

$$\int_\Omega \partial_t\varphi_n\psi + \int_\Omega \frac{1}{\mathcal{P}e}\mathcal{B}(\varphi_n)\nabla\mu_n\,\nabla\psi - \cancel{\int_\Gamma \mathcal{B}(\varphi_n)\nabla\mu_n\cdot\boldsymbol{n}\,\psi} + \int_\Omega \boldsymbol{u}(\varphi_n)\cdot\nabla\varphi_n\psi = 0, \qquad \forall\psi \in \Phi^1, \tag{4.72}$$

$$\mu_n = -\alpha^2\Delta\varphi_n + \mathbb{P}_{\Psi_n} F'(\varphi_n), \tag{4.73}$$

where $\boldsymbol{u}(\varphi_n)$ is defined as a function of φ_n by the formulas (4.19) and (4.18). We recall the boundary conditions:

$$\mu|_{\Gamma_l} = 0, \quad \varphi|_{\Gamma_l} = \varphi_l, \quad \nabla\mu\cdot\boldsymbol{n}|_{\Gamma_0} = \nabla\varphi\cdot\boldsymbol{n}|_{\Gamma_0} = 0. \tag{4.74}$$

Let us explain why the boundary term $\displaystyle\int_\Gamma B(\varphi_n)\nabla\mu_n\cdot\boldsymbol{n}\,\psi$ is zero:

- On Γ_0, we can compute $\nabla\mu_n\cdot\boldsymbol{n}|_{\Gamma_0}$, using that the functions ψ_i are eigenfunctions of $-\Delta$:

$$\nabla \mu_n \cdot \boldsymbol{n}|_{\Gamma_0} = -\alpha^2 \nabla \Delta \varphi_n \cdot \boldsymbol{n}|_{\Gamma_0} + \underbrace{\nabla F'(\varphi_n) \cdot \boldsymbol{n}|_{\Gamma_0}}_{=0, \text{ since } \mathbb{P}_{\Psi_n} F'(\varphi_n) \in \Psi_n}$$
$$= -\alpha^2 \nabla \left(\sum_{i=1}^{n} \beta_i \lambda_i \psi_i \right) \cdot \boldsymbol{n}|_{\Gamma_0}$$

Since $\psi_i \in \Psi_n$ for any $i \leq n$, we have $\nabla \psi_i \cdot \boldsymbol{n}|_{\Gamma_0} = 0$, and thus $\nabla \mu_n \cdot \boldsymbol{n}|_{\Gamma_0} = 0$.

- On Γ_l, the boundary term is also equal to zero, since $\psi \in \Phi^1$, and thus vanishes on Γ_l.

Observe that the weak formulation (4.72)-(4.73) is well-defined since $\psi_i \in H^1(\Omega)$ implies that $\mu_n \in H^1(\Omega)$. Indeed, the functions ψ_i are eigenfunctions of $-\Delta$, thus the regularity follows from definition (4.73). Replacing φ_n by its expression as a function of β_i, it becomes:

$$\sum_{i=1}^{n} \beta_i'(t) \int_\Omega \psi_i \, \psi + \int_\Omega \frac{1}{\mathcal{P}e} \mathcal{B} \left(\sum_{i=1}^{n} \beta_i(t) \psi_i + \hat{\varphi}_l \right) \nabla \mu_n \, \nabla \psi$$
$$+ \sum_{i=1}^{n} \beta_i(t) \int_\Omega \boldsymbol{u} \left(\sum_{i=1}^{n} \beta_i(t) \psi_i + \hat{\varphi}_l \right) \cdot \nabla \psi_i \psi = 0, \qquad \forall \psi \in \Phi^1,$$
$$\mu_n = -\alpha^2 \sum_{i=1}^{n} \beta_i(t) \lambda_i \psi_i + \mathbb{P}_{\Psi_n} F' \left(\sum_{i=1}^{n} \beta_i(t) \psi_i + \hat{\varphi}_l \right).$$

As previously stated, this formulation is an ordinary differential equation on $(\beta_i)_{1 \leq i \leq n}$. The functions $\mathcal{B}$ and F' are of class $\mathcal{C}^1$ on $\mathbb{R}$. Moreover, the function $\boldsymbol{u}$ as a function of φ_n given by (4.19) and (4.18) is also of class $\mathcal{C}^1$ on $\mathbb{R}$: indeed, $u(\varphi_n)$ is given as a combination of coefficients of the form $\displaystyle\int_0^z Z/\eta(\varphi_n(x, Z))dZ$, and the function η is of class $\mathcal{C}^\infty$. The second component of the velocity v is given as a function of u, and is also of class $\mathcal{C}^1$.

Therefore, the Cauchy-Lipschitz theorem ensures the existence of a unique solution $(\beta_i)_{1 \leq i \leq n}$ on a time interval $[0, t_n)$. The proof of the main theorem consists in showing that $t_n = +\infty$ for any $n \geq 1$, and that φ_n is bounded in appropriate function spaces. In the sequel, we drop the subscripts $_n$ for readability.

With this formulation, we can define weak solutions to (4.22).

Definition 4.25. *Let $T > 0$, $\varphi_0 \in \Phi^1_l$. We say that $(p, \boldsymbol{u}, \varphi, \mu)$ is a weak solution of (4.22) on $[0, T)$ if*

- *For any* $\psi \in \Phi^1$,

$$\int_\Omega \partial_t \varphi \psi + \int_\Omega \frac{1}{\mathcal{P}e} \mathcal{B}(\varphi) \nabla \mu \nabla \psi + \int_\Omega \boldsymbol{u}(\varphi) \cdot \nabla \varphi \psi = 0,$$

  with

$$\mu = -\alpha^2 \Delta \varphi + F'(\varphi),$$

  and the boundary conditions (4.74)*. The velocity field* $\mathbf{u}(\varphi) = (u(\varphi), v(\varphi))$ *is given as a function of* φ *by the three first equations of* (4.22)*, equipped with the boundary conditions* (4.23)*,* (4.24)*,* (4.25)*.*

- *The initial condition is satisfied* $\varphi|_{t=0} = \varphi_0$*.*

- *The following regularity is satisfied:*

$$p \in L^\infty(0, \infty; H^2(0, L)), \qquad u \in L^\infty(0, \infty; X(\Omega)), \qquad v \in L^\infty(0, \infty; L^2(\Omega)),$$
$$\varphi \in L^\infty(0, \infty; \Phi^1_l) \cap L^2_{loc}(0, \infty; \Phi^3_l), \qquad \mu \in L^2_{loc}(0, \infty; \Phi^1),$$

  where $X(\Omega)$ *is defined by* (4.50)*.*

4.5.2 Equation on φ

The estimates obtained in this section are similar to the ones in the papers on the Cahn-Hilliard equation by F. Boyer [Boy99] and L. Chupin [Chu04]. Nevertheless, the case considered here differs from these works by the new boundary conditions on φ (and therefore μ): the fluid injection on the left-hand side of the domain is modeled by a non-homogeneous Dirichlet condition, instead of the homogeneous Neumann condition considered previously. In this case, the estimates are of different type, since the conservation of the quantity of each fluid is not satisfied anymore (in the sense that the mean value $m(\varphi)$ of φ is not constant).
Moreover, since $m(\varphi)$ is not constant, we cannot apply classical inequalities on $\varphi - m(\varphi)$, such as the Poincaré inequality. We have to work with the boundary value of φ given on the left-hand side of the domain.

On the other hand, the boundary conditions of $\boldsymbol{u}$ on the lateral sides of the domain also differ from the previous works. In [Boy99], periodical boundary conditions are considered. In [Chu04], Dirichlet boundary conditions $\boldsymbol{u}|_\Gamma = h$ are chosen on the velocity, but such that $h \cdot \boldsymbol{n} = 0$. Here the input flow $w_{\text{in}} = \int_{\Gamma_l} \boldsymbol{u} \cdot \boldsymbol{n} = \int_{\Gamma_l} u$ is given, so that $\boldsymbol{u} \cdot \boldsymbol{n} \neq 0$, and new terms have to be treated.

The new boundary terms that must be estimated have to be treated with care, since we “loose” some regularity because of the trace theorems. Indeed, in order to control these terms with the ones on the left-hand side of the estimate, we have to introduce the adequate norms and to choose in a suitable way the coefficients in front of each term, and this requires smallness conditions on some parameters. The smallness condition on the data (on $|\Omega|$) stated in Theorem 4.32 in order to obtain a global existence result are similar to the hypothesis in [Boy99] or [Chu04].

Lemma 4.26. *For φ and μ solutions of (4.72)-(4.73) with the boundary conditions (4.74), the following estimate holds true:*

$$\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + \left(\frac{3\mathcal{B}_m}{4\mathcal{P}e} - L_0\right)|\nabla\mu|_2^2 \leq L_1(\boldsymbol{u})|\Delta\varphi|_2^2 + L_3(\boldsymbol{u})|\nabla\varphi|_2^2 + L_4(\boldsymbol{u})\|\hat{\varphi}_l\|_2^2, \tag{4.75}$$

where the terms L_i for $0 \leq i \leq 4$ are given by

$$L_0 = \frac{\bar{C}}{\beta}\left(\frac{L_h^2}{L} + LL_h^2 h'_M + L\right),$$

$$L_1(\boldsymbol{u}) = \bar{C}\left(\frac{\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m} + \beta L^3(1 + L^2h'_M)(1 + h_M^2 + {h'_M}^2)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\right),$$

$$L_3(\boldsymbol{u}) = \frac{\bar{C}\mathcal{P}eL_h^2|u|_\infty^2}{\mathcal{B}_m}$$

$$L_4(\boldsymbol{u}) = \bar{C}\left(\frac{\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m} + \frac{\mathcal{P}eL|u|_\infty^2}{\mathcal{B}_m} + \beta L(1 + L^2h'_M)(L(h_M^2 + {h'_M}^2) + h_M)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\right),$$

for any $\beta > 0$ and where p is defined in hypothesis (4.10) on F.

Proof. Let us take $\psi = \mu \in \Phi^1$ in the weak formulation (4.72). Using definition (4.73) for μ, we get

$$\underbrace{\int_\Omega \partial_t\varphi(-\alpha^2\Delta\varphi + \mathbb{P}_{\Psi_n}F'(\varphi))}_{=:A} + \underbrace{\frac{1}{\mathcal{P}e}\int_\Omega \mathcal{B}(\varphi)|\nabla\mu|^2}_{=:B} = \underbrace{-\int_\Omega \boldsymbol{u}\cdot\nabla\varphi\mu}_{=:D}. \tag{4.76}$$

Let us obtain estimates for each term A, B, D:

▷ The A-term is composed of two parts:

$$A = \underbrace{-\alpha^2\int_\Omega \partial_t\varphi\Delta\varphi}_{=:A_1} + \underbrace{\int_\Omega \partial_t\,\varphi\mathbb{P}_{\Psi_n}F'(\varphi))}_{=:A_2}.$$

⋆ For A_1, we use integration by parts:

$$A_1 = -\alpha^2 \int_\Omega \partial_t\varphi \Delta\varphi = \frac{\alpha^2}{2}\frac{d}{dt}|\nabla\varphi|_2^2 - \alpha^2 \int_\Gamma \partial_t\varphi \, \nabla\varphi \cdot \boldsymbol{n}$$

The boundary condition $\nabla\psi_i \cdot \boldsymbol{n}|_{\Gamma_0} = 0$, and the fact that φ_l is independent of t allow us to treat the boundary term:

$$-\alpha^2 \int_\Gamma \underbrace{\partial_t\varphi}_{=0 \text{ on } \Gamma_l} \underbrace{\nabla\varphi \cdot \boldsymbol{n}}_{=0 \text{ on } \Gamma_0} = 0,$$

thus

$$A_1 = \frac{\alpha^2}{2}\frac{d}{dt}|\nabla\varphi|_2^2. \tag{4.77}$$

⋆ For the second term A_2, we use the property (4.71):

$$A_2 = (\partial_t\varphi, \mathbb{P}_{\Psi_n} F'(\varphi)) = (\mathbb{P}_{\Psi_n}\partial_t \varphi, F'(\varphi)) = (\partial_t \varphi, F'(\varphi)).$$

Indeed, $\hat{\varphi}_l$ does not depend on time, and thus

$$\mathbb{P}_{\Psi_n}\partial_t \varphi = \mathbb{P}_{\Psi_n}\left(\sum_{i=1}^n \beta_i'(t)\psi_i\right) = \sum_{i=1}^n \beta_i'(t)\psi_i = \partial_t \varphi,$$

since $\psi_i \in \Psi_n$.

Thus, A_2 can be expressed as a time derivative:

$$A_2 = \int_\Omega \partial_t \varphi F'(\varphi) = \frac{d}{dt}\int_\Omega F(\varphi). \tag{4.78}$$

▷ The B-term is trivially estimated using $\mathcal{B}(\varphi) \geq \mathcal{B}_m$:

$$B = \frac{1}{\mathcal{P}e}\int_\Omega \mathcal{B}(\varphi)|\nabla\mu|^2 \geq \frac{\mathcal{B}_m}{\mathcal{P}e}|\nabla\mu|_2^2. \tag{4.79}$$

▷ For the D-term, we use integration by parts, the fact that $\operatorname{div}\boldsymbol{u} = 0$ and that $\boldsymbol{u}|_\Gamma = \mathbf{g}|_\Gamma$ (where $\mathbf{g}$ is a lifting of the boundary conditions on $\boldsymbol{u}$ defined by Lemma 4.9, e.g. for $\tau = 1$):

$$D = -\int_\Omega \boldsymbol{u} \cdot \nabla\varphi \, \mu = \int_\Omega \varphi \, \boldsymbol{u} \cdot \nabla\mu \underbrace{- \int_\Gamma \mathbf{g} \cdot \boldsymbol{n} \, \varphi \, \mu}_{=:D_3}.$$

Decomposing

$$\int_\Omega \varphi\, \boldsymbol{u}\cdot\nabla\mu = \underbrace{\int_\Omega \varphi\, u\, \partial_x\mu}_{=:D_1} + \underbrace{\int_\Omega \varphi\, v\, \partial_z\mu}_{=:D_2},$$

we observe that both terms must be handled separately, since $v \notin L^\infty(\Omega)$.

⋆ By Young's inequality, we have for D_1:

$$D_1 = \int_\Omega \varphi\, u\, \partial_x\mu \le |\varphi|_2|u|_\infty|\partial_x\mu|_2 \le \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_x\mu|_2^2 + \frac{\mathcal{P}e}{\mathcal{B}_m}|u|_\infty^2|\varphi|_2^2.$$

Using the Poincaré inequality (4.45) for $|\varphi|_2$, and using the fact that by definition of the trace space $H^{1/2}(\Gamma_l)$, the following estimate holds true

$$|\varphi_l|_{L^2(\Gamma_l)} \le \|\varphi_l\|_{H^{1/2}(\Gamma_l)} \le \|\hat{\varphi}_l\|_1, \tag{4.80}$$

we conclude

$$D_1 \le \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_x\mu|_2^2 + \frac{\bar{C}\mathcal{P}eL_h^2}{\mathcal{B}_m}|u|_\infty^2|\nabla\varphi|_2^2 + + \frac{\bar{C}\mathcal{P}eL}{\mathcal{B}_m}|u|_\infty^2\|\hat{\varphi}_l\|_1^2. \tag{4.81}$$

⋆ For D_2, we get

$$D_2 = \int_\Omega \varphi\, v\, \partial_z\mu \le |\varphi|_\infty|v|_2|\partial_z\mu|_2 \le \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_z\mu|_2^2 + \frac{\mathcal{P}e}{\mathcal{B}_m}|v|_2^2|\varphi|_\infty^2.$$

We recall that by (4.37), $|\varphi|_\infty^2 \le C_\infty^2(|\Delta\varphi|_2^2 + \|\hat{\varphi}_l\|_2^2)$, so we obtain

$$D_2 \le \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_z\mu|_2^2 + \frac{C_\infty^2\mathcal{P}e}{\mathcal{B}_m}|v|_2^2|\Delta\varphi|_2^2 + \frac{C_\infty^2\mathcal{P}e}{\mathcal{B}_m}|v|_2^2\|\hat{\varphi}_l\|_2^2. \tag{4.82}$$

⋆ For the boundary term D_3, we make use of the boundary conditions on $\mathbf{g}$:

$$D_3 = \int_\Gamma \mathbf{g}\cdot\boldsymbol{n}\,\varphi\,\mu = \int_{\Gamma_l\cup\Gamma_r} g_1\,\varphi\,\mu.$$

We apply Young's inequality (with $\beta > 0$), and combine it with the trace estimate (4.43) for $|\mu|_{L^2(\Gamma_l\cup\Gamma_r)}$ and $|\varphi|_{L^2(\Gamma_l\cup\Gamma_r)}$:

$$\begin{aligned} D_3 &\le \frac{1}{4\beta}|\mu|_{L^2(\Gamma_l\cup\Gamma_r)}^2 + \beta|g_1|_{L^\infty(\Gamma_l\cup\Gamma_r)}^2|\varphi|_{L^2(\Gamma_l\cup\Gamma_r)}^2 \\ &\le \frac{\bar{C}}{\beta}\left(\left(\frac{1}{L} + Lh_M'\right)|\mu|_2^2 + L|\partial_x\mu|_2^2\right) + \bar{C}\beta|g_1|_{L^\infty(\Gamma_l\cup\Gamma_r)}^2\left(\left(\frac{1}{L} + Lh_M'\right)|\varphi|_2^2 + L|\partial_x\varphi|_2^2\right). \end{aligned}$$

Now, with the Poincaré inequalities (4.46) and (4.47) it follows

$$\begin{aligned}D_3 \leq& \frac{\bar{C}}{\beta}(\frac{L_h^2}{L} + LL_h^2 h'_M + L)|\nabla\mu|_2^2\\&+\bar{C}\beta|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\Big(L(1+L^2h'_M)|\partial_x\varphi|_2^2 + L(1+L^2h'_M)(h_M^2+{h'_M}^2)|\partial_z\varphi|_2^2\\&+(1+L^2h'_M)|\varphi_l|^2_{L^2(\Gamma_l)} + L|\partial_x\varphi|_2^2\Big).\end{aligned}$$

Let us write

$$\begin{aligned}D'_3 :=& \bar{C}\beta|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\Big(\underbrace{L(1+L^2h'_M)|\partial_x\varphi|_2^2}_{=:D'_{31}}\\&+\underbrace{L(1+L^2h'_M)(h_M^2+{h'_M}^2)|\partial_z\varphi|_2^2}_{=:D'_{32}}+\underbrace{(1+L^2h'_M)|\varphi_l|^2_{L^2(\Gamma_l)}}_{=:D'_{33}}\Big).\end{aligned}\tag{4.83}$$

- The Poincaré inequality applied to $\partial_x\varphi$ implies, since $(\partial_x\varphi)|_{\Gamma_l}=0$:

$$D'_{31} = L(1+L^2h'_M)|\partial_x\varphi|_2^2 \leq \bar{C}L^3(1+L^2h'_M)|\partial_x^2\varphi|_2^2.\tag{4.84}$$

- The Poincaré inequality applied to $\partial_z\varphi$ (since $(\partial_z\varphi)|_{\Gamma_l} = \partial_z\varphi_l$) and (4.36) yield:

$$\begin{aligned}D'_{32} =& L(1+L^2h'_M)(h_M^2+{h'_M}^2)|\partial_z\varphi|_2^2\\\leq& \bar{C}L(1+L^2h'_M)(h_M^2+{h'_M}^2)(L^2|\partial^2_{xz}\varphi|_2^2 + L|\partial_z\varphi_l|^2_{L^2(\Gamma_l)})\\\leq& \bar{C}L^2(1+L^2h'_M)(h_M^2+{h'_M}^2)(L|\Delta\varphi|_2^2 + |\partial_z\varphi_l|^2_{L^2(\Gamma_l)}).\end{aligned}\tag{4.85}$$

- By (4.80) combined with (4.33), it follows

$$D'_{33} \leq (1+L^2h'_M)\|\hat{\varphi}_l\|_1^2 = (1+L^2h'_M)(|\hat{\varphi}_l|_2^2+|\nabla\hat{\varphi}_l|_2^2) \leq |\Omega|(1+L^2h'_M)\|\hat{\varphi}_l\|_2^2.\tag{4.86}$$

We conclude from (4.84), (4.85) combined with (4.80), (4.86) and from the fact that $|\Omega| \leq Lh_M$ that (4.83) becomes

$$\begin{aligned}D'_3 \leq& \bar{C}\beta|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\Big((L^3(1+L^2h'_M)(1+h_M^2+{h'_M}^2)|\Delta\varphi|_2^2\\&+(1+L^2h'_M)(L^2(h_M^2+{h'_M}^2)+Lh_M)\|\hat{\varphi}_l\|_2^2\Big).\end{aligned}\tag{4.87}$$

Hence we obtain the following estimate on D_3, after rearranging terms:

$$\begin{aligned} D_3 \leq & \frac{\bar{C}}{\beta}(\frac{L_h^2}{L} + LL_h^2 h'_M + L)|\nabla\mu|_2^2 \\ & + \bar{C}\beta L^3(1 + L^2 h'_M)(1 + h_M^2 + {h'_M}^2)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}|\Delta\varphi|_2^2 \\ & + \bar{C}\beta L(1 + L^2 h'_M)(L(h_M^2 + {h'_M}^2) + h_M)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\|\hat{\varphi}_l\|_2^2 \end{aligned} \tag{4.88}$$

Putting (4.77), (4.78), (4.79), (4.81), (4.82), (4.88) into (4.76), and rearranging terms, we get

$$\begin{aligned} & \frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + \frac{3\mathcal{B}_m}{4\mathcal{P}e}|\nabla\mu|_2^2 \\ & \leq \frac{\bar{C}}{\beta}(\frac{L_h^2}{L} + LL_h^2 h'_M + L)|\nabla\mu|_2^2 \\ & + \bar{C}\Big(\frac{\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m} + \beta L^3(1 + L^2 h'_M)(1 + h_M^2 + {h'_M}^2)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\Big)|\Delta\varphi|_2^2 \\ & + \frac{\bar{C}\mathcal{P}eL_h^2|u|_\infty^2}{\mathcal{B}_m}|\nabla\varphi|_2^2 + \frac{\bar{C}\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m}\|\hat{\varphi}_l\|_2^2 + \frac{\bar{C}\mathcal{P}eL|u|_\infty^2}{\mathcal{B}_m}\|\hat{\varphi}_l\|_2^2 \\ & + \bar{C}\beta L(1 + L^2 h'_M)(L(h_M^2 + {h'_M}^2) + h_M)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\|\hat{\varphi}_l\|_2^2 \end{aligned} \tag{4.89}$$

This proves the inequality (4.75). □

4.5.3 Equation on μ

It is possible to obtain some "information" on $|\nabla\varphi|_2$ and $|\Delta\varphi|_2$ by using equation (4.73). Again, the main difference with previous works consists in the original boundary conditions, and the various boundary terms that are induced.

Lemma 4.27. *For φ and μ solutions of (4.72)-(4.73) with the boundary conditions (4.74), the following estimate holds true:*

$$\begin{aligned} & \alpha^2|\nabla\varphi|_2^2 + F_3(0)\int_\Omega F(\varphi) \\ & \qquad \leq M_0|\nabla\mu|_2^2 + M_1|\Delta\varphi|_2^2 + M_2|\nabla\varphi|_2^{2p} + M_3|\nabla\varphi|_2^2 + M_4\|\hat{\varphi}_l\|_2^2 + M_5, \end{aligned} \tag{4.90}$$

where

$$M_0 = \bar{C}\gamma L_h^2, \quad M_1 = \frac{\bar{C}\alpha^2 L(1 + L^2 h'_M)}{4\lambda}, \quad M_2 = \bar{C}|\Omega|^{1/2}F_1^2(1 + L_h^{2p}),$$
$$M_3 = \frac{\bar{C}L_h^2}{4\gamma}, \quad M_4 = \bar{C}|\Omega|^{1/2}F_1^2 L^p\|\hat{\varphi}_l\|_2^{2(p-1)} + \frac{\bar{C}L}{4\gamma} + \bar{C}|\Omega|^{1/2} + \alpha^2\lambda,$$

$$M_5 = |\Omega| F_4(0) + \bar{C} F_2^2 |\Omega|^{3/2},$$

for $\gamma > 0$, $\lambda > 0$ arbitrary constants and p defined in hypothesis (4.10) on F.

Proof. If we multiply (4.73) by φ, we get

$$\underbrace{(\mu, \varphi)}_{=:A} = \underbrace{-\alpha^2 (\Delta\varphi, \varphi)}_{=:B} + \underbrace{(\mathbb{P}_{\Psi_n} F'(\varphi), \varphi)}_{=:D}. \tag{4.91}$$

Let us treat as before each term separately.

▷ For B, we use integration by parts, and obtain:

$$B = \alpha^2 |\nabla\varphi|_2^2 \underbrace{- \alpha^2 \int_\Gamma \varphi \nabla\varphi \cdot \boldsymbol{n}}_{=:B_1} \tag{4.92}$$

Observe that since $\nabla\varphi \cdot \boldsymbol{n}|_{\Gamma_0} = 0$, the boundary term B_1 is zero on $\Gamma \backslash \Gamma_l$. Using Young's inequality with $\lambda > 0$ and (4.44) for the term $|\partial_x \varphi|_{L^2(\Gamma_l)}$, it follows:

$$\begin{aligned} |B_1| &= \alpha^2 \left| \int_{\Gamma_l} \varphi_l \, \partial_x \varphi \right| \leq \alpha^2 |\varphi_l|_{L^2(\Gamma_l)} |\partial_x \varphi|_{L^2(\Gamma_l)} \leq \frac{\alpha^2}{4\lambda} |\partial_x \varphi|^2_{L^2(\Gamma_l)} + \alpha^2 \lambda |\varphi_l|^2_{L^2(\Gamma_l)} \\ &\leq \alpha^2 \bar{C} \Big(\frac{L(1 + L^2 h'_M)}{4\lambda} |\partial_x^2 \varphi|_2^2 + \lambda |\varphi_l|^2_{L^2(\Gamma_l)} \Big) \leq \alpha^2 \bar{C} \Big(\frac{L(1 + L^2 h'_M)}{4\lambda} |\partial_x^2 \varphi|_2^2 + \lambda \|\hat{\varphi}_l\|_2^2 \Big), \end{aligned} \tag{4.93}$$

making use of (4.80) in the last estimate.

▷ For the D-term, let us use the projector property (4.71) and the fact that $\varphi - \hat{\varphi}_l = \sum_{i=1}^n \beta_i \psi_i \in \Psi_n$:

$$\begin{aligned} D &= (\mathbb{P}_{\Psi_n} F'(\varphi), \varphi) = (F'(\varphi), \mathbb{P}_{\Psi_n} \varphi) = (F'(\varphi), \mathbb{P}_{\Psi_n} (\varphi - \hat{\varphi}_l) + \mathbb{P}_{\Psi_n} \hat{\varphi}_l) \\ &= \underbrace{(F'(\varphi), \varphi)}_{=:D_1} \underbrace{-(F'(\varphi), (\mathrm{Id} - \mathbb{P}_{\Psi_n}) \hat{\varphi}_l)}_{=:D_2}. \end{aligned}$$

Hypothesis (4.11) with $\gamma = 0$ yields

$$D_1 = \int_\Omega F'(\varphi) \, \varphi \geq \int_\Omega F_3(0) F(\varphi) - F_4(0) |\Omega|. \tag{4.94}$$

As far as D_2 is concerned, we use the fact that $\mathrm{Id} - \mathbb{P}_{\Psi_n}$ is a projector, thus its norm is equal to 1, and the property (4.10) for $|F'(\varphi)|$ and (4.33) for $|\varphi|_{2p}^p$:

$$|D_2| = |(F'(\varphi), (\mathrm{Id} - \mathbb{P}_{\Psi_n})\hat{\varphi}_l)| \leq |\hat{\varphi}_l|_2 |F'(\varphi)|_2 \leq |\hat{\varphi}_l|_2 (F_1 |\varphi|_{2p}^p + F_2|\Omega|)$$
$$\leq \bar{C} |\hat{\varphi}_l|_2 (F_1 |\Omega|^{1/2} \|\varphi\|_1^p + F_2|\Omega|).$$

Last, we use the Poincaré inequality (4.45) by rewriting $\|\varphi\|_1^p$ in terms of $|\varphi|_2^p$ and $|\nabla\varphi|_2^p$, and we obtain

$$|D_2| \leq \bar{C}|\hat{\varphi}_l|_2 \left(F_1 |\Omega|^{1/2} \Big((1 + L_h^p)|\nabla\varphi|_2^p + L^{p/2} |\varphi_l|_{L^2(\Gamma_l)}^p \Big) + F_2|\Omega| \right)$$
$$= \bar{C}|\Omega|^{1/4}|\hat{\varphi}_l|_2 \left(F_1 |\Omega|^{1/4} \Big((1 + L_h^p)|\nabla\varphi|_2^p + L^{p/2} |\varphi_l|_{L^2(\Gamma_l)}^p \Big) + F_2|\Omega|^{3/4} \right)$$

and by Young's inequality

$$|D_2| \leq \bar{C} F_1^2 |\Omega|^{1/2} \Big((1 + L_h^{2p})|\nabla\varphi|_2^{2p} + L^p |\varphi_l|_{L^2(\Gamma_l)}^{2p} \Big) + \bar{C} F_2^2 |\Omega|^{3/2} + \bar{C}|\Omega|^{1/2}|\hat{\varphi}_l|_2^2.$$

When combining this estimate with (4.80), it follows

$$\begin{aligned} |D_2| \leq &\bar{C} F_1^2 |\Omega|^{1/2} (1 + L^{2p}) |\nabla\varphi|_2^{2p} + \bar{C} F_1^2 |\Omega|^{1/2} L^p \|\hat{\varphi}_l\|_2^{2p} \\ &+ \bar{C}|\Omega|^{1/2} \|\hat{\varphi}_l\|_2^2 + \bar{C} F_2^2 |\Omega|^{3/2}. \end{aligned} \tag{4.95}$$

▷ For the A-term, Cauchy-Schwarz inequality and Young's inequality with $\gamma > 0$ imply:

$$A = \int_\Omega \mu\, \varphi \leq |\mu|_2 |\varphi|_2 \leq \bar{C} \left(\gamma |\mu|_2^2 + \frac{1}{4\gamma} |\varphi|_2^2 \right).$$

The last step consists in using the two Poincaré inequalities (4.45) for $|\varphi|_2$ and (4.47) for $|\mu|_2$, and combining them with (4.80):

$$A \leq \bar{C}\gamma L_h^2 |\nabla\mu|_2^2 + \frac{\bar{C}}{4\gamma} (L_h^2 |\nabla\varphi|_2^2 + L\|\hat{\varphi}_l\|_2^2) \tag{4.96}$$

Putting (4.93), (4.94), (4.95) and (4.96) in (4.91), and rearranging terms, it follows:

$$\begin{aligned} \alpha^2 |\nabla\varphi|_2^2 + F_3(0) \int_\Omega F(\varphi) \leq &\bar{C}\gamma L_h^2 |\nabla\mu|_2^2 + \frac{\bar{C}\alpha^2 L (1 + L^2 h'_M)}{4\lambda} |\Delta\varphi|_2^2 \\ &+ \bar{C}|\Omega|^{1/2} F_1^2 (1 + L_h^{2p}) |\nabla\varphi|_2^{2p} + \frac{\bar{C} L_h^2}{4\gamma} |\nabla\varphi|_2^2 + \bar{C}|\Omega|^{1/2} F_1^2 L^p \|\hat{\varphi}_l\|_2^{2p} \\ &+ \Big(\frac{\bar{C} L}{4\gamma} + \bar{C}|\Omega|^{1/2} + \alpha^2\lambda \Big) \|\hat{\varphi}_l\|_2^2 + |\Omega| F_4(0) + \bar{C} F_2^2 |\Omega|^{3/2}. \end{aligned}$$

which is the inequality (4.90) we claimed. □

Lemma 4.28. *For φ and μ solutions of (4.72)-(4.73) with the boundary conditions (4.74), the following estimate holds true:*

$$\alpha^2|\Delta\varphi|_2^2 \leq N_0|\nabla\mu|_2^2 + N_1|\Delta\varphi|_2^2 + N_2|\nabla\varphi|_2^{2p} + N_2'|\nabla\varphi|_2^4 + N_3|\nabla\varphi|_2^2 + N_4\|\hat{\varphi}_l\|_2^2 + N_5, \quad (4.97)$$

with

$$N_0 = \bar{C}\Big(\frac{L_h^2}{4\zeta L} + \frac{LL_h^2 h_M'}{4\zeta} + \frac{L}{4\zeta} + \frac{1}{4\delta}\Big), \quad N_1 = \bar{C}(1 + L^2 h_M')\Big(\zeta L + \frac{L}{4\theta}\Big),$$
$$N_2 = \bar{C}|\Omega|^{1/2}F_1^2(1 + L^{2p}), \quad N_2' = \frac{\bar{C}}{\nu}, \quad N_3 = \delta,$$
$$N_4 = \bar{C}|\Omega|^{1/2}F_1^2 L^p\|\hat{\varphi}_l\|_2^{2(p-1)} + \bar{C}|\Omega|^{1/2},$$
$$N_5 = \bar{C}\theta|F'(\varphi_l)|_{L^2(\Gamma_l)}^2 + \bar{C}\nu F_5^2 + \bar{C}F_2^2|\Omega|^{3/2},$$

for $\delta > 0$, $\zeta > 0$, $\theta > 0$, $\nu > 0$ arbitrary constants, and p defined in hypothesis (4.10) on F.

Proof. If we multiply (4.8) by $-\Delta\varphi$ and integrate by parts, we get

$$\alpha^2|\Delta\varphi|_2^2 = \underbrace{-(\mu, \Delta\varphi)}_{=:A} + \underbrace{\int_\Omega \mathbb{P}_{\Psi_n}F'(\varphi)\,\Delta\varphi}_{=:B} \quad (4.98)$$

▷ For the B-term, we use that the functions ψ_i are chosen to be eigenfunctions of $-\Delta$, and we proceed as previously to obtain the following relation:

$$B = (\mathbb{P}_{\Psi_n}F'(\varphi), \Delta\varphi) = (F'(\varphi), \mathbb{P}_{\Psi_n}\Delta\varphi) = \underbrace{(F'(\varphi), \Delta\varphi)}_{=:B_1} - \underbrace{(F'(\varphi), (\mathrm{Id} - \mathbb{P}_{\Psi_n})\Delta\hat{\varphi}_l)}_{=:B_2},$$

since $\Delta\varphi - \Delta\hat{\varphi}_l \in \Psi_n$.

⋆ We can compute the B_1-term by integration by parts:

$$B_1 = \underbrace{-\int_\Omega F''(\varphi)|\nabla\varphi|^2}_{=:B_{11}} + \underbrace{\int_\Gamma F'(\varphi)\nabla\varphi\cdot\boldsymbol{n}}_{=:B_{12}}. \quad (4.99)$$

· We use hypothesis (4.9) on F'' and Young's inequality with $\nu > 0$ in order to obtain

$$B_{11} = -\int_\Omega F''(\varphi)|\nabla\varphi|^2 \leq F_5|\nabla\varphi|_2^2 \leq \bar{C}\left(\nu F_5^2 + \frac{1}{\nu}|\nabla\varphi|_2^4\right). \quad (4.100)$$

· For the boundary term B_{12}, let us observe that it is non-zero on Γ_l only, since $\nabla\varphi \cdot \boldsymbol{n}|_{\Gamma_0} = 0$. Moreover, Young's inequality with $\theta > 0$ and (4.44) yield

$$
\begin{aligned}
B_{12} &= \int_{\Gamma_l} F'(\varphi)\nabla\varphi \cdot \boldsymbol{n} \le |F'(\varphi_l)|_{L^2(\Gamma_l)} |\partial_x \varphi|_{L^2(\Gamma_l)} \\
&\le \bar{C}\Big(\theta |F'(\varphi_l)|^2_{L^2(\Gamma_l)} + \frac{1}{4\theta}|\partial_x\varphi|^2_{L^2(\Gamma_l)}\Big) \\
&\le \bar{C}\left(\theta |F'(\varphi_l)|^2_{L^2(\Gamma_l)} + \frac{L(1+L^2 h'_M)}{4\theta}|\partial_x^2\varphi|_2^2\right)
\end{aligned}
\tag{4.101}
$$

⋆ The term B_2 is estimated in a similar way as for (4.95):

$$
\begin{aligned}
|B_2| \le& \bar{C}F_1^2|\Omega|^{1/2}(1+L^{2p})|\nabla\varphi|_2^{2p} + \bar{C}F_1^2|\Omega|^{1/2}L^p\|\hat{\varphi}_l\|_2^{2p} \\
&+ \bar{C}|\Omega|^{1/2}\|\hat{\varphi}_l\|_2^2 + \bar{C}F_2^2|\Omega|^{3/2}.
\end{aligned}
\tag{4.102}
$$

▷ As far as the A-term is concerned, it is computed by integration by parts:

$$
A = -(\mu, \Delta\varphi) = \underbrace{\int_\Omega \nabla\mu\,\nabla\varphi}_{=:A_1} - \underbrace{\int_\Gamma \mu\nabla\varphi\cdot\boldsymbol{n}}_{=:A_2}. \tag{4.103}
$$

⋆ The term A_1 is easily bounded thanks to Young's inequality with $\delta > 0$:

$$
A_1 = -(\nabla\mu, \nabla\varphi) \le \frac{1}{4\delta}|\nabla\mu|_2^2 + \delta|\nabla\varphi|_2^2. \tag{4.104}
$$

⋆ With the same argument as for (4.101), the boundary term A_2 is non-zero on Γ_l only. It is treated with the help of Young's inequality with $\zeta > 0$, the trace estimates (4.43), (4.44) and the Poincaré inequality (4.47):

$$
\begin{aligned}
A_2 &= \int_{\Gamma_l} \mu\,\nabla\varphi\cdot\boldsymbol{n} \le |\mu|_{L^2(\Gamma_l)}|\partial_x\varphi|_{L^2(\Gamma_l)} \qquad (4.105)\\
&\le \frac{\bar{C}}{4\zeta}\left(\Big(\frac{1}{L} + Lh'_M\Big)|\mu|_2^2 + L|\partial_x\mu|_2^2\right) + \bar{C}\zeta L(1+L^2h'_M)|\partial_x^2\varphi|_2^2 \\
&\le \frac{\bar{C}}{4\zeta}\Big(\Big(\frac{L_h^2}{L} + LL_h^2h'_M\Big)|\nabla\mu|_2^2 + L|\partial_x\mu|_2^2\Big) + \bar{C}\zeta L(1+L^2h'_M)|\Delta\varphi|_2^2 \qquad (4.106)
\end{aligned}
$$

Finally, we combine (4.100) and (4.101) in (4.99), (4.102), and (4.104) and (4.105) in (4.103), and use these estimates in (4.98) to obtain

$$
\alpha^2|\Delta\varphi|_2^2 \le \bar{C}\Big(\frac{L_h^2}{4\zeta L} + \frac{LL_h^2h'_M}{4\zeta} + \frac{L}{4\zeta} + \frac{1}{4\delta}\Big)|\nabla\mu|_2^2 + \bar{C}(1+L^2h'_M)\Big(\zeta L + \frac{L}{4\theta}\Big)|\Delta\varphi|_2^2
$$

$$+ \bar{C}|\Omega|^{1/2}F_1^2(1+L^{2p})|\nabla\varphi|_2^{2p} + \frac{\bar{C}}{\nu}|\nabla\varphi|_2^4 + \delta|\nabla\varphi|_2^2$$
$$+ \bar{C}|\Omega|^{1/2}F_1^2L^p\|\hat{\varphi}_l\|_2^{2p} + \bar{C}|\Omega|^{1/2}\|\hat{\varphi}_l\|_2^2$$
$$+ \bar{C}\theta|F'(\varphi_l)|^2_{L^2(\Gamma_l)} + \bar{C}\nu F_5^2 + \bar{C}F_2^2|\Omega|^{3/2}.$$

This concludes the proof. □

4.6 Existence result without surface tension

4.6.1 *A priori* estimates

Let us sum (4.75), $c_1\times$ (4.90) and $c_2\times$ (4.97), where c_1 and c_2 are two positive constants that will be determined in the sequel. We obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + \Big(\frac{3\mathcal{B}_m}{4\mathcal{P}e} - L_0 - c_1M_0 - c_2N_0\Big)|\nabla\mu|_2^2 + c_1\alpha^2|\nabla\varphi|_2^2 \\
&+c_2\alpha^2|\Delta\varphi|_2^2 + c_1F_3(0)\int_\Omega F(\varphi) \\
&\quad\leq \Big(L_1(\boldsymbol{u}) + c_1M_1 + c_2N_1\Big)|\Delta\varphi|_2^2 + \Big(c_1M_2 + c_2N_2\Big)|\nabla\varphi|_2^{2p} \\
&\quad + c_2N_2'|\nabla\varphi|_2^4 + \Big(L_3(\boldsymbol{u}) + c_1M_3 + c_2N_3\Big)|\nabla\varphi|_2^2 + \Big(L_4(\boldsymbol{u}) + c_1M_4 + c_2N_4\Big)\|\hat{\varphi}_l\|_2^2 \\
&\quad + \Big(L_5 + c_1M_5 + c_2N_5\Big).
\end{aligned} \tag{4.107}$$

We define for all $t \geq 0$,

$$\mathcal{Y}(t) = \frac{\alpha^2}{2}|\nabla\varphi(t)|_2^2 + \int_\Omega F(\varphi(t)),$$
$$\mathcal{Z}(t) = \frac{\alpha^2}{2}|\nabla\varphi(t)|_2^2 + |\nabla\mu(t)|_2^2 + |\Delta\varphi(t)|_2^2 + \int_\Omega F(\varphi(t)),$$

so that $0 \leq \mathcal{Y}(t) \leq \mathcal{Z}(t)$, since $F > 0$ (by assumption in the subsection 4.2.2).

We wish to write the *a priori* estimate (4.107) in the following form:

$$\mathcal{Y}'(t) + C_1\mathcal{Z}(t) \leq f(\mathcal{Y}(t))\mathcal{Z}(t) + C_2\mathcal{Z}(t) + C_3, \tag{4.108}$$

where $C_1\mathcal{Z}$ contains the terms on the left-hand side of (4.107), $C_2\mathcal{Z}$ contains some of the terms on the right-hand side of (4.107), and where $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$f(0) = 0$. Let us first choose the constant C_1:

$$C_1 = \min\left\{\left(\frac{3\mathcal{B}_m}{4\mathcal{P}e} - L_0 - c_1 M_0 - c_2 N_0\right), 2c_1, c_2\alpha^2, c_1 F_3(0)\right\},$$

so that the left-hand side of (4.107) is greater than $\mathcal{Y}'(t) + C_1\mathcal{Z}(t)$.

In order to put (4.107) in the form (4.108), we have to set apart the constant terms, the linear terms (with respect to $\mathcal{Z}$) and the nonlinear terms (which will appear in $f(\mathcal{Y})\mathcal{Z}$). Let us recall that all coefficients L_i, M_i, N_i are functions of φ and μ, except for $L_1(\boldsymbol{u})$, $L_3(\boldsymbol{u})$, $L_4(\boldsymbol{u})$, in which the terms $|u|_\infty$ and $|v|_2$ appear. For these terms, we proved in (4.48) that

$$|u|_\infty \le \bar{C}(1 + h_M^2), \qquad |v|_2 \le \bar{C}(1 + h_M^2)\|\varphi\|_1.$$

We apply the Poincaré inequality (4.45) to $\|\varphi\|_1$ combined with the fact that $|\varphi_l|_{L^2(\Gamma_l)} \le \|\hat{\varphi}_l\|_2$, and we can write

$$|u|_\infty^2 \le \bar{C}(1 + h_M^2)^2, \qquad |v|_2^2 \le \bar{C}(1 + h_M^2)^2\Big((1 + L_h^2)|\nabla\varphi|_2^2 + L\|\hat{\varphi}_l\|_2^2\Big). \tag{4.109}$$

Let us explain how the terms on the right hand side of (4.108) can be obtained.

i) It is easy to determine the contributions to the *constant part* C_3:

- ⋆ $C_{31} := (c_1 M_4 + c_2 N_4)\|\hat{\varphi}_l\|_2^2$;
- ⋆ $C_{32} := (c_1 M_5 + c_2 N_5)$;
- ⋆ the constant part of $L_4(\boldsymbol{u})\|\hat{\varphi}_l\|_2^2$, when using (4.109):

$$\begin{aligned} C_{33} :=& \bar{C}\Big(\frac{\mathcal{P}e C_\infty^2 (1 + h_M^2)^2 L \|\hat{\varphi}_l\|_2^2}{\mathcal{B}_m} \\ &+ \frac{\mathcal{P}e L (1 + h_M^2)^2}{\mathcal{B}_m} + \beta L(1 + L^2 h'_M)(L(h_M^2 + {h'_M}^2) + h_M)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\Big)\|\hat{\varphi}_l\|_2^2. \end{aligned}$$

Thus, we have:

$$C_3 = C_{31} + C_{32} + C_{33}. \tag{4.110}$$

ii) The *linear terms* come from:

- ⋆ $C_{21}|\Delta\varphi|_2^2 := (c_1 M_1 + c_2 N_1)|\Delta\varphi|_2^2$;
- ⋆ if $p = 1$, $C_{22}|\nabla\varphi|_2^{2p} := (c_1 M_2 + c_2 N_2)|\nabla\varphi|_2^{2p}$;
- ⋆ $C_{23}|\nabla\varphi|_2^2 := (c_1 M_3 + c_2 N_3)|\nabla\varphi|_2^2$;

⋆ the terms $L_1(\boldsymbol{u})|\Delta\varphi|_2^2$ and $L_3(\boldsymbol{u})|\nabla\varphi|_2^2$ lead to the following contributions:

$$C_{24}|\Delta\varphi|_2^2 := \bar{C}\Big(\frac{\mathcal{P}eC_\infty^2(1+h_M^2)^2L\|\hat{\varphi}_l\|_2^2}{\mathcal{B}_m} + \beta L^3(1+L^2h'_M)(1+h_M^2+{h'_M}^2)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\Big)|\Delta\varphi|_2^2,$$

$$C_{25}|\nabla\varphi|_2^2 := \frac{\bar{C}\mathcal{P}eL_h^2(1+h_M^2)}{\mathcal{B}_m}|\nabla\varphi|_2^2;$$

⋆ in $L_4(\boldsymbol{u})\|\hat{\varphi}_l\|_2^2$, the product $|v|_2^2\|\hat{\varphi}_l\|_2^2$ contains the terms

$$C_{26}|\nabla\varphi|_2^2 := \frac{\bar{C}\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)}{\mathcal{B}_m}|\nabla\varphi|_2^2\|\hat{\varphi}_l\|_2^2,$$

which is a linear term with respect to $|\nabla\varphi|_2^2$.

Therefore, since all the terms are positive, we can bound these linear terms by $C_2\mathcal{Z}$, with

$$C_2 = C_{21} + C_{22} + C_{23} + C_{24} + C_{25} + C_{26}. \tag{4.111}$$

iii) As far as the *nonlinear terms* are concerned, there are also several contributions:

⋆ the term $c_2N_2'|\nabla\varphi|_2^4$;

⋆ if $p > 1$, the term $(c_1M_2 + c_2N_2)|\nabla\varphi|_2^{2p}$;

⋆ in $L_1(\boldsymbol{u})|\Delta\varphi|_2^2$, the term $\dfrac{\bar{C}\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)|\nabla\varphi|_2^2}{\mathcal{B}_m}|\Delta\varphi|_2^2$ is a nonlinear term.

Since all nonlinear terms are positive, we can bound them by $f(\mathcal{Y})\mathcal{Z}$, with the following expression of the function f defined in $\mathbb{R}^+$: for all $\xi \in \mathbb{R}^+$,

$$f(\xi) = c_2N_2'\xi + \underbrace{\Big(c_1M_2 + c_2N_2\Big)\xi^{p-1}}_{\text{if } p>1} + \frac{\bar{C}\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)\xi}{\mathcal{B}_m}. \tag{4.112}$$

This allows us to write (4.107) in the form (4.108). With the estimate (4.108), we will be able to prove that φ and μ are bounded in adequate function spaces for any time $T > 0$, if we ensure that C_1 is positive and that C_2 and C_3 are sufficiently small, with the help of the following proposition.

Proposition 4.29. *Let $T > 0$. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two functions in $\mathcal{C}^1([0,T])$, such that*

$$\mathcal{Y}' + C_1\mathcal{Z} \le f(\mathcal{Y})\mathcal{Z} + C_2\mathcal{Z} + C_3, \quad 0 \le \mathcal{Y} \le \mathcal{Z}. \tag{4.113}$$

Suppose that f is an increasing continuous function such that $f(0) = 0$, that the constant

C_1 is positive, and that $C_2 < \dfrac{C_1}{2}$. Let $M > 0$ such that $f(M) + C_2 < \dfrac{C_1}{2}$. Then, if $C_3 < \dfrac{MC_1}{2}$, we have the following implication

$$\mathcal{Y}(0) < M \Longrightarrow \mathcal{Y}(t) < M \quad \textit{for } t \in [0, T].$$

This means that if $\mathcal{Y}(0) < M$, then there exists a constant C such that for any $T > 0$,

$$\|\mathcal{Y}(t)\|_{L^\infty(0,T)} \leq M.$$

Moreover, we have

$$\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq CT + C.$$

Proof. Suppose that there exists $0 < T^* < T$, such that $\mathcal{Y}(T^*) = M$ and $\mathcal{Y}'(T^*) > 0$. Then, evaluating (4.113) at T^*, and using the hypothesis on C_2 and C_3, we get

$$0 < \mathcal{Y}'(T^*) \leq \mathcal{Z}(T^*)(f(M) - C_1 + C_2) + C_3 \leq -\frac{C_1}{2}\mathcal{Z}(T^*) + C_3 \leq \frac{C_1}{2}(M - \mathcal{Z}(T^*)).$$

But since $M = \mathcal{Y}(T^*) \leq \mathcal{Z}(T^*)$, we have $M - \mathcal{Z}(T^*) \leq 0$, which leads to a contradiction. The regularity of $\mathcal{Z}$ follows by integrating (4.113) over $(0, T)$, and using the regularity of $\mathcal{Y}$:

$$\frac{C_1}{2}\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq \mathcal{Y}(T) + \frac{C_1}{2}\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq \mathcal{Y}(0) + C_3 T \leq M + C_3 T,$$

which writes $\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq CT + C$. □

In order to apply this proposition, let us give explicitly the constants C_1, C_2, C_3, using the expressions of L_i, M_i, N_i given in Lemmas 4.26, 4.27 and 4.28:

$$C_1 = \min\Bigg\{\frac{3\mathcal{B}_m}{4\mathcal{P}e} - \bar{C}\Big(\frac{L_h^2}{\beta L} + \frac{LL_h^2 h'_M}{\beta} + \frac{L}{\beta} + c_1\gamma L_h^2 + \frac{c_2 L_h^2}{4\zeta L} + \frac{c_2 L L_h^2 h'_M}{4\zeta} + \frac{c_2 L}{4\zeta} + \frac{c_2}{4\delta}\Big), \\ 2c_1, c_2\alpha^2, c_1 F_3(0)\Bigg\},$$

$$C_2 = \bar{C}\Bigg(\frac{\mathcal{P}e C_\infty^2 L(1+h_M^2)^2\|\hat{\varphi}_l\|_2^2}{\mathcal{B}_m} + \bar{C}\beta L^3(1 + L^2 h'_M)(1 + h_M^2 + {h'_M}^2)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)} \\ + \frac{c_1\alpha^2 L(1 + L^2 h'_M)}{4\lambda} + c_2(1 + L^2 h'_M)\Big(\zeta L + \frac{L}{4\theta}\Big)\Bigg) \\ + \frac{2\bar{C}}{\alpha^2}\Bigg(\frac{\mathcal{P}e L_h^2(1 + h_M^2)}{\mathcal{B}_m} + c_1\frac{L_h^2}{4\gamma} + c_2\delta\Bigg)$$

$$
\begin{aligned}
&+ \frac{2\bar{C}}{\alpha^2}\frac{\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)}{\mathcal{B}_m}\|\hat{\varphi}_l\|_2^2 + C_2', \\
C_3 =& \bar{C}F_1^2L^p|\Omega|^{1/2}(c_1+c_2)\|\hat{\varphi}_l\|_2^{2p} \\
&+ \bar{C}\Big(\frac{\mathcal{P}eC_\infty^2L(1+h_M^2)^2\|\hat{\varphi}_l\|_2^2}{\mathcal{B}_m} + \frac{\mathcal{P}eL(1+h_M^2)^2}{\mathcal{B}_m} \\
&+ \bar{C}\beta L(1+L^2h_M')(L(h_M^2+{h_M'}^2)+h_M)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)} \\
&+ c_1\big(\frac{L}{4\gamma}+|\Omega|^{1/2}+\alpha^2\lambda\big) + c_2|\Omega|^{1/2}\Big)\|\hat{\varphi}_l\|_2^2 \\
&+ c_1\Big(F_2^2|\Omega|^{3/2}+|\Omega|F_4(0)\Big) + c_2\bar{C}\Big(\theta|F'(\varphi_l)|^2_{L^2(\Gamma_l)} + \nu F_5^2 + F_2^2|\Omega|^{3/2}\Big).
\end{aligned}
$$

where C_2' is given by

$$
C_2' = \begin{cases} \bar{C}|\Omega|^{1/2}F_1^2(1+L^{2p})(c_1+c_2), & \text{if } p=1, \\ 0, & \text{if } p>1. \end{cases}
$$

Let us prove that there exists β_1^*, β_2^*, γ^*, δ^*, ζ^*, θ^*, λ^*, c_1^*, c_2^*, ν^*, L^* such that for any $\gamma<\gamma^*$, $\delta<\delta^*$, $\theta<\theta^*$, $\lambda<\lambda^*$, $c_1>c_1^*$, $c_2<c_2^*$, $\nu<\nu^*$, $L<L^*$, and for $\beta=\beta^*$, $\zeta=\zeta^*$, the conditions of Proposition 4.29 are satisfied:

- $C_1>0$;
- $C_2<C_1/2$;
- there exists $M>0$ such that $f(M)+C_2<C_1/2$;
- $C_3<MC_1/2$.

To do this, we will prove that there exists $c_2^*>0$ such that for all $c_2<c_2^*$, we have

$$
C_1 = c_2\alpha^2 > 0, \qquad C_2 < C_1/2 = c_2\alpha^2/2.
$$

Since f is a continuous increasing function satisfying $f(0)=0$, it is possible to define $M>0$ such that

$$
f(M)+C_2<C_1/2.
$$

Then we will also prove that

$$
C_3<MC_1/2.
$$

Remark 4.30. *Let us explain in a few words the main idea of the proof: the constants C_i can be written as functions of $X=(\zeta,\beta,\delta,\gamma,\theta,\lambda,\nu,c_2,c_1,L)$. The idea consists in observing that $C_i(X=0)$ satisfy the conditions claimed, and thus that, by continuity of C_i with respect to X, the same is true for $C_i(X)$ for X small enough.*

However, this is not entirely true, since there are some terms involving the inverse of $\zeta, \beta, \delta, \gamma, \theta, \lambda, L$, and thus cannot be evaluated at zero. Therefore, we have to proceed carefully in several steps, choosing the constants small in the "right order" in order to ensure the claimed result.

Let us introduce the following quantities $\bar{\zeta} = \zeta L$ and $\bar{\beta} = \beta L$. Thus the corresponding terms in C_1, C_2, C_3 can be rewritten with these new variables.

- Let $\delta^* > 0$ such that
$$\frac{2\bar{C}}{\alpha^2}\delta^* < \frac{\alpha^2}{2}.$$
This is possible for δ^* small enough.
- Then let $c_2^* > 0$ small enough such that
$$c_2^*\bar{C}\left(\frac{1}{\delta^*} + \alpha^2\right) \leq \frac{3\mathcal{B}_m}{4\mathcal{P}e}, \quad \text{i.e.} \quad \frac{3\mathcal{B}_m}{4\mathcal{P}e} - \frac{c_2^*\bar{C}}{\delta^*} \geq c_2^*\alpha^2.$$
Moreover, take
$$c_1^* \geq \max\{c_2^*\alpha^2, 1/2, 1/F_3(0)\}.$$
At this point, we thus have, for any $\delta < \delta^*$, $c_1 > c_1^*$, $c_2 < c_2^*$:
$$\min\left\{\frac{3\mathcal{B}_m}{4\mathcal{P}e} - \bar{C}\frac{c_2}{4\delta}, 2c_1, c_2\alpha^2\right\} = c_2\alpha^2 > 0.$$
- By continuity, there exists $\bar{\beta}^* > 0$, $\bar{\zeta}^* > 0$, $\gamma^* > 0$, $\theta^* > 0$, $\lambda^* > 0$, $\nu^* > 0$ such that for any $\bar{\beta} < \bar{\beta}^*$, $\bar{\zeta} < \bar{\zeta}^*$, $\gamma < \gamma^*$, $\theta < \theta^*$, $\lambda < \lambda^*$, $\nu < \nu^*$, $\delta < \delta^*$, $\bar{\zeta} \leq \bar{\zeta}^*$, $c_1 > c_1^*$, $c_2 < c_2^*$, we have:
$$\begin{aligned}
&\min\left\{\frac{3\mathcal{B}_m}{4\mathcal{P}e} - \bar{C}\Big(c_1\gamma L_h^2 + \frac{c_2}{4\delta}\Big), 2c_1, c_2\alpha^2\right\} = c_2\alpha^2 > 0,\\
&c_2\bar{\zeta} + \frac{2\bar{C}}{\alpha^2}c_2\delta < \frac{c_2\alpha^2}{2},\\
&\bar{C}\Big(\bar{\beta}h_M|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)} + \alpha^2\lambda)\|\hat{\varphi}_l\|_2^2 + \bar{\beta}{h'_M}^2|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\\
&\qquad + \theta|F'(\varphi_l)|^2_{L^2(\Gamma_l)} + \bar{\zeta}{h'_M}^2 + \nu F_5^2\Big) < \frac{c_2\alpha^2 M}{2}.
\end{aligned}$$
- At last, by continuity also, there exists $L^* > 0$ such that for any $L \leq L^*$, $\bar{\beta} < \bar{\beta}^*$, $\gamma < \gamma^*$, $\theta < \theta^*$, $\lambda < \lambda^*$, $\delta < \delta^*$, $\bar{\zeta} \leq \bar{\zeta}^*$, $c_1 > c_1^*$, $c_2 < c_2^*$, $F_5 < F_5^*$, it follows:
$$C_1 = \min\left\{\frac{3\mathcal{B}_m}{4\mathcal{P}e} - \bar{C}\Big(\frac{L_h^2}{\bar{\beta}} + \frac{L^2L_h^2h'_M}{\bar{\beta}} + \frac{L^2}{\bar{\beta}} + c_1\gamma L_h^2 + \frac{c_2L_h^2}{4\bar{\zeta}} + \frac{c_2L^2L_h^2h'_M}{4\bar{\zeta}} + \frac{c_2L^2}{4\bar{\zeta}} + \frac{c_2}{4\delta}\Big),\right.$$

$$2c_1, c_2\alpha^2, c_1F_3(0)\Big\} = c_2\alpha^2 > 0,$$

$$C_2 = \bar{C}\Bigg(\frac{\mathcal{P}eC_\infty^2 L(1+h_M^2)^2\|\hat{\varphi}_l\|_2^2}{\mathcal{B}_m} + \bar{\beta}L^2(1+L^2h'_M)(1+h_M^2+{h'_M}^2)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}$$
$$+\frac{c_1\alpha^2L(1+L^2h'_M)}{4\lambda}) + c_2(1+L^2h'_M)\Big(\bar{\zeta}+\frac{L}{4\theta}\Big)\Bigg) + \frac{2\bar{C}}{\alpha^2}\Big(\frac{\mathcal{P}eL_h^2(1+h_M^2)^2}{\mathcal{B}_m} + \frac{c_1L_h^2}{4\gamma} + c_2\delta\Big)$$
$$+\frac{2\bar{C}}{\alpha^2}\frac{\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)}{\mathcal{B}_m}\|\hat{\varphi}_l\|_2^2 + C'_2 < \frac{c_2\alpha^2}{2} = \frac{C_1}{2},$$

$$C_3 = \bar{C}F_1^2L^p|\Omega|^{1/2}(c_1+c_2)\|\hat{\varphi}_l\|_2^{2p}$$
$$+\bar{C}\Bigg(\frac{\mathcal{P}eC_\infty^2L(1+h_M^2)^2\|\hat{\varphi}_l\|_2^2}{\mathcal{B}_m} + \frac{\mathcal{P}eL(1+h_M^2)^2}{\mathcal{B}_m}$$
$$+\bar{C}\bar{\beta}(1+L^2h'_M)(L(h_M^2+{h'_M}^2)+h_M)|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}$$
$$+c_1\Big(\frac{L}{4\gamma}+|\Omega|^{1/2}+\alpha^2\lambda\Big)+c_2|\Omega|^{1/2}\Bigg)\|\hat{\varphi}_l\|_2^2 + c_1\Big(F_2^2|\Omega|^{3/2}+|\Omega|F_4(0)\Big)$$
$$+c_2\bar{C}\Big(\theta|F'(\varphi_l)|^2_{L^2(\Gamma_l)}+\nu F_5^2+F_2^2|\Omega|^{3/2}\Big) < \frac{c_2\alpha^2M}{2} = \frac{MC_3}{2}.$$

This is true since all the terms added at this step are of the form L^sC, with $s>0$ and C which remains bounded as $L\to 0$. In particular, all the terms of the form $C|\Omega|$ are of this form, since $|\Omega|\le Lh_M$, and C_∞ is also of this form (see Proposition 4.5).

- Thus, for $\zeta^* = \dfrac{\bar{\zeta}^*}{L^*}$ and $\beta^* = \dfrac{\bar{\beta}^*}{L^*}$, the claimed assertion is proved.

First convergence results

In this paragraph, let us come back to the notation with the subscripts $_n$ introduced in section 4.5.1, denoting the Galerkin approximations. Proposition 4.29 implies that for any $T>0$, $\mathcal{Y}_n \in L^\infty(0,T)$ with a bound independent of T, and $\mathcal{Z}_n\in L^1(0,T)$ with a bound depending on T. From this, we deduce several results on φ_n, μ_n:

- The quantity $\nabla\varphi_n$ is bounded in $L^\infty(0,\infty;L^2(\Omega))$.

- The quantities $\nabla\mu_n$, $\nabla\varphi_n$ and $\Delta\varphi_n$ are bounded in $L^2_{\text{loc}}(0,\infty;L^2(\Omega))$.

- Furthermore, applying the Poincaré inequality (4.45) to φ_n allows us to control the whole $H^1(\Omega)$-norm by the L^2-norm of the gradient.

- As far as the H^2-norm of φ_n is concerned, we know by Proposition 4.6 that it is equivalent to the L^2-norm of the Laplacian, and thus controlling $|\Delta\varphi_n|_2$ is enough to control the whole $H^2(\Omega)$-norm.

- For μ_n, the Poincaré inequality (4.47) also allows to control the H^1-norm by the L^2-norm of the gradient.

From these arguments, we conclude that there exists $C > 0$ such that for any $T > 0$,

$$\|\varphi_n\|_{L^\infty(\mathbb{R}^+;H^1)} \leq C, \qquad \|\varphi_n\|_{L^2(0,T;H^2)} \leq CT, \qquad \|\mu_n\|_{L^2(0,T;H^1)} \leq CT. \tag{4.114}$$

Let us observe that the first estimate is enough to show that the time interval on which the functions φ_n exist is $(0, +\infty)$ (i.e. $t_n = +\infty$).

The estimate (4.114) is not enough to conclude for the convergence of the nonlinear terms and of the initial condition $\varphi_n(0)$. Therefore, some more regularity on φ_n and its time derivative will be proved in the next subsection.

4.6.2 Convergence result of the nonlinear terms and the initial condition

In this section, we will obtain more regularity on φ_n for the convergence of the nonlinear terms and the initial condition. Indeed, in order to apply Proposition 4.2, we will first prove that φ_n is bounded in $H^3(\Omega)$, and next that the time derivative of φ_n is bounded in an adequate function space.

Lemma 4.31. *For φ_n and μ_n solutions of (4.72)-(4.73) with the boundary conditions (4.74), there exists $C > 0$ such that for any $T > 0$:*

$$\|\varphi_n\|_{L^2(0,T;\Phi_l^3)} \leq CT + C, \qquad \left\|\frac{d\varphi_n}{dt}\right\|_{L^2(0,T;{\Phi_l^1}^*)} \leq CT + C, \tag{4.115}$$

where ${\Phi_l^1}^*$ is the dual space of Φ_l^1.

H^3-estimate for φ

In order to treat the H^3-norm of φ_n, we take the gradient of (4.73), we obtain

$$\alpha^2 \nabla\Delta\varphi_n = \underbrace{\nabla\mathbb{P}_{\Psi_n} F'(\varphi_n)}_{=:A} - \nabla\mu_n. \tag{4.116}$$

▷ We can compute the A-term:

$$A = \nabla\mathbb{P}_{\Psi_n} F'(\varphi_n) = \nabla\mathbb{P}_{\Psi_n}(F'(\varphi_n) - F'(\hat{\varphi}_l)) + \nabla\mathbb{P}_{\Psi_n} F'(\hat{\varphi}_l).$$

Let us prove that $F'(\varphi_n) - F'(\hat{\varphi}_l) \in \Psi_n$:

– $\nabla(F'(\varphi_n) - F'(\hat{\varphi}_l)) \cdot \boldsymbol{n}|_{\Gamma_0} = F''(\varphi_n)\underbrace{\nabla\varphi_n \cdot \boldsymbol{n}|_{\Gamma_0}}_{=0} - F''(\hat{\varphi}_l)\underbrace{\nabla\hat{\varphi}_l \cdot \boldsymbol{n}|_{\Gamma_0}}_{=0 \text{ by } (4.40)}.$

– $(F'(\varphi_n) - F'(\hat{\varphi}_l))|_{\Gamma_l} = F'(\varphi_l) - F'(\varphi_l) = 0.$

Thus $\mathbb{P}_{\Psi_n}(F'(\varphi_n) - F'(\hat{\varphi}_l)) = (F'(\varphi_n) - F'(\hat{\varphi}_l))$, and therefore

$$A = \nabla F'(\varphi_n) - \nabla(\mathrm{Id} - \mathbb{P}_{\Psi_n})F'(\hat{\varphi}_l).$$

Taking the L^2-norm, it follows

$$\begin{aligned}|A|_2 = |\nabla\mathbb{P}_{\Psi_n}F'(\varphi_n)|_2 &\leq |\nabla F'(\varphi_n)|_2 + |\nabla(\mathrm{Id} - \mathbb{P}_{\Psi_n})F'(\hat{\varphi}_l)|_2 \\ &\leq \underbrace{|\nabla F'(\varphi_n)|_2}_{=:A_1} + \|(\mathrm{Id} - \mathbb{P}_{\Psi_n})F'(\hat{\varphi}_l)\|_1 \leq |\nabla F'(\varphi_n)|_2 + \|F'(\hat{\varphi}_l)\|_1.\end{aligned}$$

since $\mathrm{Id} - \mathbb{P}_{\Psi_n}$ is a projector, which means that its operator norm is equal to 1.

▷ Let us consider the term $A_1 = |\nabla F'(\varphi_n)|_2$. It follows from hypothesis (4.10) on F:

$$A_1 = |\nabla F'(\varphi_n)|_2^2 \leq \int_\Omega (F_1|\varphi_n|^{p-1} + F_2)^2|\nabla\varphi_n|^2 \leq \bar{C}(|\nabla\varphi_n|_2^2 + |\varphi_n^{p-1}\nabla\varphi_n|_2^2),$$

where $\bar{C}$ is a constant depending on F_1 and F_2. Let us distinguish two cases:

- If $p = 1$, then $\varphi_n^{p-1}\nabla\varphi_n = \nabla\varphi_n$, and the estimate (4.117) is obvious.
- If $p > 1$, the Hölder inequality implies

$$\begin{aligned}|\nabla F'(\varphi_n)|_2^2 &\leq \bar{C}(|\nabla\varphi_n|_2^2 + \left(\int_\Omega |\varphi_n^{2(p-1)}|^q\right)^{1/q}\left(\int_\Omega |\nabla\varphi_n|^{2q'}\right)^{1/q'}) \\ &= \bar{C}(|\nabla\varphi_n|_2^2 + |\varphi_n|_{2(p-1)q}^{2(p-1)}|\nabla\varphi_n|_{2q'}^2),\end{aligned}$$

with $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, for any $q > 1$. Let $q = \dfrac{1}{p-1}$. Then $2(p-1)q \geq 2$, thus $H^1(\Omega) \hookrightarrow L^{2(p-1)q}(\Omega)$ and $2q' \geq 2$, thus $H^1(\Omega) \hookrightarrow L^{2q'}(\Omega)$. We finally obtain

$$A_1 \leq \bar{C}(|\nabla\varphi_n|_2^2 + \|\varphi_n\|_1^{p-1}\|\varphi_n\|_2^2), \tag{4.117}$$

▷ At last, taking the L^2-norm of (4.116), it follows that using (4.117),

$$\alpha^2|\nabla\Delta\varphi_n|_2^2 \leq \bar{C}(|\nabla\mu_n|_2^2 + |\nabla\varphi_n|_2^2 + \|\varphi_n\|_1^{p-1}\|\varphi_n\|_2^2 + \|F'(\hat{\varphi}_l)\|_1),$$

This estimate combined with (4.114) allows us to conclude that there exists $C > 0$ such that for any $T > 0$,

$$\|\varphi_n\|_{L^2(0,T;\Phi_l^3)} \leq CT + C. \tag{4.118}$$

Time derivative estimate for φ

Let us now estimate the time derivative of φ_n. To this end, we introduce the dual operator $P^*_{\Psi_n}$ of P_{Ψ_n}. Equation (4.72) can be rewritten in the following form:

$$(\partial_t \varphi_n, \mathbb{P}_{\Psi_n}\chi) + (\boldsymbol{u}(\varphi_n) \cdot \nabla\varphi_n, \mathbb{P}_{\Psi_n}\chi) + (\operatorname{div}(\mathcal{B}(\varphi_n)\nabla\mu_n), \mathbb{P}_{\Psi_n}\chi) = 0, \qquad \forall\chi \in \Phi_l^1,$$

which becomes

$$\frac{d\varphi_n}{dt} = -P^*_{\Psi_n}\Big(u(\varphi_n)\,\partial_x\varphi_n + v(\varphi_n)\,\partial_z\varphi_n + \operatorname{div}(\mathcal{B}(\varphi_n)\nabla\mu_n))\Big).$$

Let us treat each term separately:

▷ By Proposition 4.18 and estimate (4.114), we have $u(\varphi_n) \in L^\infty(0,T;H^1)$, and $v(\varphi_n) \in L^\infty(0,T;L^2)$. Moreover, previous estimate (4.118) implies that $\varphi_n \in L^2(0,T;\Phi_l^3)$. By Proposition 4.3, we deduce that $u(\varphi_n)\,\partial_x\varphi_n \in L^2(0,T;H^1(\Omega))$ and $v(\varphi_n)\partial_z\varphi_n \in L^2(0,T;L^2(\Omega))$, with

$$\begin{aligned}\|u(\varphi_n)\,\partial_x\varphi_n\|_{L^2(0,T;H^1)} &+ \|v(\varphi_n)\,\partial_z\varphi_n\|_{L^2(0,T;L^2)} \\ &\leq C\left(\|u(\varphi_n)\|_{L^\infty(0,T;H^1)} + \|v(\varphi_n)\|_{L^2(0,T;L^2)} + \|\varphi_n\|_{L^2(0,T;H^3)}\right).\end{aligned}$$

▷ Furthermore, since $\mathcal{B} \leq \mathcal{B}_m$:

$$\|\operatorname{div}(\mathcal{B}(\varphi_n)\nabla\mu_n)\|_{H^{-1}} \leq \mathcal{B}_m|\nabla\mu_n|_2.$$

▷ Moreover, since P_{Ψ_n} is a projector, its operator norm is $\|P_{\Psi_n}\| = \|P^*_{\Psi_n}\| = 1$.

Using the previous estimates (4.114) and (4.48), it follows

$$\left\|\frac{d\varphi_n}{dt}\right\|_{L^2(0,T;\Phi_l^{1*})} \leq CT + C, \tag{4.119}$$

where Φ_l^{1*} is the dual space of Φ_l^1.

4.6.3 Main theorem by convergence results

From the previous estimates (4.114), (4.118), (4.119), we obtain the following convergence results (up to a subsequence):

$$\begin{aligned}
\varphi_n &\rightharpoonup \varphi && \text{in } L^\infty(\mathbb{R}^+;\Phi_l^1) \quad *\text{-weak},\\
\varphi_n &\rightharpoonup \varphi && \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+;\Phi_l^3) \quad \text{weak},\\
\mu_n &\rightharpoonup \mu && \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+;\Phi^1) \quad \text{weak},\\
\frac{d\varphi_n}{dt} &\rightharpoonup \frac{d\varphi}{dt} && \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+;\Phi_l^{1*}) \quad \text{weak}.
\end{aligned}$$

Moreover, Proposition 4.18 combined with the previous global convergence result on φ implies the following convergence results (up to a subsequence):

$$\begin{aligned}
u_n &\rightharpoonup u && \text{in } L^\infty(\mathbb{R}^+;X(\Omega)) \quad *\text{-weak},\\
v_n &\rightharpoonup v && \text{in } L^\infty(\mathbb{R}^+;L^2(\Omega)) \quad *\text{-weak},\\
p_n &\rightharpoonup p && \text{in } L^\infty(\mathbb{R}^+;H^2(0,L)) \quad *\text{-weak}.
\end{aligned}$$

Furthermore, by Proposition 4.2, we deduce from (4.115) that for any $T > 0$

$$\begin{aligned}
\varphi_n &\to \varphi && \text{in } L^2(0,T;H^2(\Omega)) \quad \text{strong},\\
\varphi_n &\to \varphi && \text{in } \mathcal{C}(0,T;L^2(\Omega)) \quad \text{strong},\\
\varphi_n &\rightharpoonup \varphi && \text{in } \mathcal{C}^0(0,T;\Phi_l^1) \quad \text{weak}.
\end{aligned}$$

Therefore, we can conclude that the nonlinear terms $B(\varphi_n)$ and $F'(\varphi_n)$ converge strongly in $\mathcal{C}(0,T;L^2(\Omega))$, since φ_n converges strongly in $\mathcal{C}(0,T;L^2(\Omega))$. As far as the convection term $\boldsymbol{u}(\varphi_n)\cdot\nabla\varphi_n$ is concerned, we know from Lemmas 4.21 and 4.22 that $\boldsymbol{u}(\varphi_n)$ is bounded in $L^\infty(0,T;L^2(\Omega))$. The strong convergence of $\nabla\varphi_n$ in $L^2(0,T;L^2(\Omega))$ allows to conclude for the convergence of this nonlinear term. Lastly, we deduce from the last convergence result that $\varphi(0)$ converges weakly to $\varphi(0)$ in $H^1(\Omega)$, and thus $\varphi(0) = \varphi_0$ because P_{Ψ_n} converges to the identity for the strong topology of operators.

It remains to prove that the functions $\boldsymbol{u}$, φ and μ satisfy (4.72), (4.73). Let $\rho \in \mathcal{D}'(\mathbb{R}^+)$, and let $N > 1$. For any $n \geq N$, φ_n satisfies (4.72) with $\psi = \mu_N$. We multiply this equation by ρ and integrate by parts. From the convergence results stated above, we can pass to the limit in this equation. The limit equation obtained is fulfilled for any $N > 1$, and any $\rho \in \mathcal{D}'(\mathbb{R}^+)$, thus we conclude that $\boldsymbol{u}$, φ and μ satisfy (4.72). At last, since P_{Ψ_n} converges to the identity for the strong topology of operators, the dominated convergence theorem allows to conclude that φ and μ also satisfy (4.73). This proves the following main theorem.

Theorem 4.32. *Let* $\Omega = \{(x,z) \in \mathbb{R}^2, x \in (0,L), z \in (0,h(x)\}$ *for* $h \in \mathcal{C}^1(\mathbb{R})$*. Let* $\varphi_0 \in \Phi^1_l$*, and let* F *satisfy the assumptions stated in Section 4.2.2. Under a smallness assumption on* L*, there exists a weak solution* $(p, \boldsymbol{u}, \varphi, \mu)$ *of* (4.22) *(in the sense of Definition 4.25) such that*

$$p \in L^\infty(0,\infty; H^2(0,L)), \qquad u \in L^\infty(0,\infty; X(\Omega)), \qquad v \in L^\infty(0,\infty; L^2(\Omega))$$
$$\varphi \in L^\infty(0,\infty;\Phi^1_l) \cap L^2_{loc}(0,\infty;\Phi^3_l), \qquad \mu \in L^2_{loc}(0,\infty;\Phi^1),$$

where $X(\Omega)$ *is defined by* (4.50)*.*

4.7 Existence result with surface tension

In this section, we consider the system (4.32). The main difference with the previous section is that the term $\boldsymbol{u}\cdot\nabla\varphi$ in the Cahn-Hilliard equation cancels with the surface tension term $\kappa\mu\nabla\varphi$, when multiplying the Cahn-Hilliard equation by μ and the Navier-Stokes equation by $\boldsymbol{u}$, integrating over Ω and summing the two equations (as it has been done in [Boy99] for example). Therefore, it is of interest to work with the Navier-Stokes equation instead of the Reynolds equation (which uncouples the pressure and the velocity, and thus "looses" the term $\kappa\mu\nabla\varphi$).

However, in order to prove the convergence of the Reynolds equation, we study its regularity in a similar way to the previous case (without surface tension).

Let us recall the system (4.32) in the case when the surface tension is taken into account.

$$\begin{cases} \partial_x(\widetilde{d}\,\partial_x p_\kappa) = s\,\partial_x\widetilde{e} + \kappa\,\partial_x\left(\left(h - \dfrac{\widetilde{b}}{\widetilde{\widetilde{a}}}\right)\widetilde{\mathcal{K}} - \displaystyle\int_0^h \mathcal{K}(\cdot,Z)\,dZ\right) \\ \partial_z p_\kappa = 0, \\ u = \left(b - \dfrac{a\widetilde{b}}{\widetilde{\widetilde{a}}}\right)\partial_x p_\kappa + s\left(1 - \dfrac{a}{\widetilde{\widetilde{a}}}\right) + \kappa\left(\dfrac{a}{\widetilde{\widetilde{a}}}\widetilde{\mathcal{K}} - \mathcal{K}\right) \\ v(\cdot,z) = -\displaystyle\int_0^z \partial_x u(\cdot,Z)dZ \\ \partial_t\,\varphi + u\,\partial_x\varphi + v\,\partial_z\varphi - \dfrac{1}{\mathcal{P}e}\,\mathrm{div}\,(\mathcal{B}(\varphi)\nabla\mu) = 0 \\ \mu = -\,\mathrm{div}\,(\alpha^2\nabla\varphi) + F'(\varphi). \end{cases} \tag{4.120}$$

The coefficients a, b, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{d}$, $\widetilde{e}$, $\mathcal{K}$, $\widetilde{\mathcal{K}}$ are explicit functions of φ (given by (4.14), (4.15), (4.29)), as well as the functions $\mathcal{B}$, F. We remind that $\mathcal{K}^*$ is given in terms of k by

$$k[\varphi] = \mu\,\partial_x\,\varphi - \partial_x\left(\int_0^z \mu\,\partial_Z\varphi\;dZ\right), \tag{4.121}$$

$$\mathcal{K}(x,z) = \int_0^z \frac{1}{\eta(\varphi(x,Z))} \int_0^Z k[\varphi](x,\xi)\, d\xi\, dZ, \qquad \widetilde{\mathcal{K}}(x) = \mathcal{K}(x,h(x)). \tag{4.122}$$

The quantities s, κ, $\mathcal{P}e$, α are physical constants. The boundary conditions are given by (4.23), (4.24), (4.25), (4.26). Recall that the two first equations of (4.120) are obtained from the following relation coupling p and u:

$$-\partial_z(\eta(\varphi)\,\partial_z u) + \partial_x p_\kappa = \kappa\left(\mu\,\partial_x\,\varphi - \partial_x\left(\int_0^z \mu\,\partial_Z\varphi\; dZ\right)\right). \tag{4.123}$$

We can define weak solutions to (4.120) as in the case without surface tension.

Definition 4.33. *Let $T > 0$, $\varphi_0 \in \Phi^1_l$. We say that $(p, \boldsymbol{u}, \varphi, \mu)$ is a weak solution of (4.120) on $[0,T)$ if*

- *For any $\psi \in \Phi^1$,*

$$\int_\Omega \partial_t\varphi\psi + \int_\Omega \frac{1}{\mathcal{P}e}\mathcal{B}(\varphi)\nabla\mu\,\nabla\psi + \int_\Omega \boldsymbol{u}(\varphi)\cdot\nabla\varphi\psi = 0,$$

  with

$$\mu = -\alpha^2\Delta\varphi + F'(\varphi),$$

  and the boundary conditions (4.74).

- *For any $w \in H^1_0(\Omega)$,*

$$\int_\Omega \eta(\varphi)\partial_x u\partial_x w + \int_\Omega \partial_x p_\kappa w = \kappa\int_\Omega \mu\partial_x\varphi w - \kappa\int_\Omega \partial_x\Big(\int_0^z \mu\partial_Z\varphi dZ\Big)w,$$

$$\int_\Omega u\partial_x w + \int_\Omega v\partial_z w = 0,$$

  and p is given by

$$p = p_\kappa + \kappa\int_0^z \mu\partial_Z\varphi dZ,$$

  with the boundary conditions (4.23), (4.24), (4.25).

- *The initial condition is satisfied $\varphi|_{t=0} = \varphi_0$.*

- *The following regularity is satisfied:*

$$p \in L^1_{loc}(0,\infty; H^1(0,L)),$$
$$u \in L^1_{loc}(0,\infty; L^2(\Omega)), \qquad \partial_z u \in L^1_{loc}(0,\infty; L^2(\Omega)), \qquad v \in L^1_{loc}(0,\infty; H^{-1}(\Omega))$$
$$\varphi \in L^\infty(0,\infty; \Phi^1_l) \cap L^2_{loc}(0,\infty; \Phi^3_l), \qquad \mu \in L^2_{loc}(0,\infty; \Phi^1).$$

Let us state the main result.

Theorem 4.34. *Let* $\Omega = \{(x,z) \in \mathbb{R}^2, x \in (0,L), z \in (0,h(x)\}$ *for* $h \in \mathcal{C}^1(\mathbb{R})$*. Let* $\varphi_0 \in \Phi_l^1$*, and* F *satisfying the assumptions stated in Section 4.2.2. Under a smallness assumption on* L*, there exists a solution* $(p, \boldsymbol{u}, \varphi, \mu)$ *of* (4.120) *(in the sense of Definition 4.33).*

4.7.1 About the asymptotic Navier-Stokes equation

Lemma 4.35. *Let* $(p, \boldsymbol{u}, \varphi, \mu)$ *be solution of* (4.120) *(in the sense of Definition 4.33). Then the following estimate holds true:*

$$\begin{aligned}\frac{\eta_m}{2}|\partial_z u|_2^2 \leq &\frac{\eta_M^2}{2\eta_m}|\partial_z g_1^\tau|_2^2 + \frac{\kappa\mathcal{B}_m}{2\mathcal{P}e}|\nabla\mu|_2^2 + \kappa\int_\Omega \mu\nabla\varphi\cdot\boldsymbol{u} \\ &+ L_1'|\Delta\varphi|_2^2 + L_3'|\nabla\varphi|_2^2 + L_4'\|\hat{\varphi}_l\|_2^2,\end{aligned} \tag{4.124}$$

where the coefficients L_i' *are given by:*

$$L_1' = L_3' = L_4' = \bar{C}\frac{\tau^2 h_M^2 \mathcal{P}e L_h^2}{2\kappa\mathcal{B}_m}|\Omega|^{1/2},$$

and $\mathbf{g}^\tau$ *is defined as the lifting of the boundary conditions for* $\boldsymbol{u}$ *in Lemma 4.9.*

Proof. Multiplying (4.123) by $u - g_1^\tau$, and integrating over Ω, it follows with integration by parts:

$$\begin{aligned}\int_\Omega \eta\,|\partial_z u|^2 - \int_\Omega \eta\,\partial_z u\,\partial_z g_1^\tau - \int_{\Gamma_h}\eta\underbrace{(u-g_1^\tau)}_{=0 \text{ on } \Gamma}\partial_z u + \int_\Omega \partial_x p_\kappa\,(u-g_1^\tau) \\ = \kappa\left(\int_\Omega \mu\,\partial_x\varphi\,(u-g_1^\tau) - \int_\Omega \partial_x\left(\int_0^z \mu\,\partial_Z\varphi\; dZ\right)(u-g_1^\tau)\right).\end{aligned}$$

Using that $\eta \geq \eta_m$, it becomes:

$$\begin{aligned}\eta_m|\partial_z u|_2^2 \leq &\underbrace{\int_\Omega \eta\,\partial_z u\,\partial_z g_1^\tau}_{=:A} + \underbrace{\kappa\int_\Omega \mu\,\partial_x\varphi\,(u-g_1^\tau)}_{=:B} \\ &\underbrace{-\int_\Omega \partial_x p_\kappa\,(u-g_1^\tau)}_{=:D}\ \underbrace{-\kappa\int_\Omega \partial_x\left(\int_0^z \mu\,\partial_Z\varphi\; dZ\right)(u-g_1^\tau)}_{=:E}\end{aligned} \tag{4.125}$$

▷ We use Young inequality for the A-term:

$$A = \int_\Omega \eta\, \partial_z u\, \partial_z g_1^\tau \leq \frac{\eta_m}{2} |\partial_z u|_2^2 + \frac{\eta_M^2}{2\eta_m} |\partial_z g_1^\tau|_2^2, \tag{4.126}$$

▷ We prove that the pressure term D is zero using $\operatorname{div} \boldsymbol{u} = \operatorname{div} \mathbf{g}^\tau = 0$:

$$\begin{aligned} -D &= -\int_\Omega p_\kappa \partial_x (u - g_1^\tau) + \int_{\Gamma_l \cup \Gamma_r} p_\kappa \underbrace{(u - g_1^\tau)}_{=0 \text{ on } \Gamma} \\ &= \int_\Omega p_\kappa \partial_z (v - g_2^\tau) = -\int_\Omega \underbrace{\partial_z p_\kappa}_{=0} (v - g_2^\tau) + \int_{\Gamma_h} p_\kappa \underbrace{(v - g_2^\tau)}_{=0 \text{ on } \Gamma} = 0, \end{aligned} \tag{4.127}$$

▷ The E-term is treated similarly:

$$\begin{aligned} E &= \int_\Omega \left(\int_0^z \mu\, \partial_Z \varphi\, dZ \right) \partial_x (u - g_1^\tau) - \int_{\Gamma_l \cup \Gamma_r} \left(\int_0^z \mu\, \partial_Z \varphi\, dZ \right) \underbrace{(u - g_1^\tau)}_{=0 \text{ on } \Gamma} \\ &= -\int_\Omega \left(\int_0^z \mu\, \partial_Z \varphi\, dZ \right) \partial_z (v - g_2^\tau) \\ &= \int_\Omega \mu\, \partial_z \varphi\, (v - g_2^\tau) - \int_{\Gamma_h} \left(\int_0^z \mu\, \partial_Z \varphi\, dZ \right) \underbrace{(v - g_2^\tau)}_{=0 \text{ on } \Gamma} \end{aligned}$$

▷ Moreover, observe that

$$B + E = \int_\Omega \mu \nabla \varphi \cdot \boldsymbol{u} - \underbrace{\int_\Omega \mu \nabla \varphi \cdot \mathbf{g}^\tau}_{=:G}. \tag{4.128}$$

▷ For the G-term, we proceed using hypothesis *iv)* of Lemma 4.9 on $\mathbf{g}$, (4.47) for $|\mu|_2$ and (4.33) combined with (4.36) for $|\nabla\varphi|_2$:

$$\begin{aligned} |G| &\leq |\mu|_2 |\nabla \varphi|_4 |\mathbf{g}^\tau|_4 \\ &\leq \tau \bar{C} L_h |\nabla \mu|_2^2 h_M |\Omega|^{1/4} \left(|\nabla \varphi|_2 + |\Delta \varphi|_2 + \|\hat{\varphi}_l\|_2 \right). \end{aligned}$$

Young's inequality yields

$$|G| \leq \frac{\kappa \mathcal{B}_m}{2\mathcal{P}e} |\nabla \mu|_2^2 + \bar{C} \frac{\tau^2 h_M^2 \mathcal{P}e L_h^2}{2\kappa \mathcal{B}_m} |\Omega|^{1/2} \left(|\nabla \varphi|_2^2 + |\Delta \varphi|_2^2 + \|\hat{\varphi}_l\|_2^2 \right) \tag{4.129}$$

Finally, using (4.126), (4.127), (4.128) with (4.129) in (4.125), we have the following estimate on $|\partial_y u|_2$:

$$\frac{\eta_m}{2}\,|\partial_z u|_2^2 \leq \frac{\eta_M^2}{2\eta_m}\,|\partial_z g_1^\tau|_2^2 + \frac{\kappa B_m}{2\mathcal{P}e}|\nabla\mu|_2^2 + \kappa\int_\Omega \mu\,\nabla\varphi\cdot\boldsymbol{u}$$
$$+\,\bar{C}\frac{\tau^2 h_M^2\mathcal{P}eL_h^2}{2\kappa\mathcal{B}_m}|\Omega|^{1/2}\left(|\nabla\varphi|_2^2+|\Delta\varphi|_2^2+\|\hat{\varphi}_l\|_2^2\right), \tag{4.130}$$

which corresponds to the estimate (4.124) when rearranging terms. □

4.7.2 About the Cahn-Hilliard equation

The weak formulation (4.72)-(4.73) is still valid in this case, since the Cahn-Hilliard equations remain unchanged.

Equation on φ

As before (section 4.5.2), we can choose $\psi=\mu$ as a test function in (4.72). The estimate (4.76) is still valid. Multiplying this estimate by κ, we obtain:

$$\frac{d}{dt}\left(\frac{\kappa\alpha^2}{2}|\nabla\varphi|_2^2+\int_\Omega \kappa F(\varphi)\right)+\frac{\kappa\mathcal{B}_m}{\mathcal{P}e}|\nabla\mu|_2^2 \leq -\kappa\int_\Omega \mu\nabla\varphi\cdot\boldsymbol{u} \tag{4.131}$$

The term $\int\limits_\Omega \mu\nabla\varphi\cdot\boldsymbol{u}$ does not need to be estimated, since it cancels with the same term in (4.124).

Equation on μ

Let us observe that the estimates (4.90) and (4.97) are still valid, since they only depend on the Cahn-Hilliard equation, which has not been changed.

A priori estimates

Summing (4.124), (4.131), $c_1\times$(4.90) and $c_2\times$(4.97), we obtain

$$\frac{d}{dt}\left(\frac{\kappa\alpha^2}{2}|\nabla\varphi|_2^2+\int_\Omega \kappa F(\varphi)\right)+\left(\frac{\kappa\mathcal{B}_m}{2\mathcal{P}e}-c_1M_0-c_2N_0\right)|\nabla\mu|_2^2+c_1\alpha^2|\nabla\varphi|_2^2$$
$$+\,c_2\alpha^2|\Delta\varphi|_2^2+\frac{\eta_m}{2}\,|\partial_z u|_2^2+c_1F_3(0)\int\limits_\Omega F(\varphi)\leq\frac{\eta_M^2}{2\eta_m}\,|\partial_z g_1^\tau|_2^2 \tag{4.132}$$
$$+\,(L_1'+c_1M_1+c_2N_1)|\Delta\varphi|_2^2+(c_1M_2+c_2N_2)|\nabla\varphi|_2^{2p}+c_2N_2'|\nabla\varphi|_2^4$$
$$+\,(L_3'+c_1M_3+c_2N_3)|\nabla\varphi|_2^2+(L_4'+c_1M_4+c_2N_4)\|\hat{\varphi}_l\|_2^2+c_1M_5+c_2N_5,$$

where we defined the modified coefficients L'_i are defined in Lemma 4.35, and the coefficients M_i, N_i are unchanged (defined in Lemmas 4.27 and 4.28). Let us point out that in comparison to the previous section, the terms $|\partial_z u|_2$ and $|\partial_z g_1^\tau|_2$ are added, and the coefficients L_i are modified (in particular they do not depend on $\boldsymbol{u}$ anymore). As we announced it before, the terms $\int_\Omega \mu \nabla\varphi \cdot \boldsymbol{u}$ cancel.

4.7.3 Obtaining the convergence results

The point is to use as before the modified Gronwall's inequality of Lemma 4.29. To this end, we define again the following quantities:

$$\begin{aligned}
\mathcal{Y} =& \frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi), \\
\mathcal{Z} =& c_1\alpha^2|\nabla\varphi|_2^2 + \left(\frac{\kappa\mathcal{B}_m}{2\mathcal{P}e} - L'_0 - c_1 M_0 - c_2 N_0\right)|\nabla\mu|_2^2 \\
&+ c_2\alpha^2|\Delta\varphi|_2^2 + c_1 F_3(0)\int_\Omega F(\varphi) + \frac{\eta_m}{2}|\partial_z u|_2^2,
\end{aligned}$$

and the constants C_1, C_2, C_3 by:

$$\begin{aligned}
C_1 &= \min\left\{\frac{\kappa\mathcal{B}_m}{2\mathcal{P}e} - \bar{C}\Big(c_1\gamma L_h^2 + \frac{c_2 L_h^2}{4\zeta L} + \frac{c_2 L L_h^2 h'_M}{4\zeta} + \frac{c_2 L}{4\zeta} + \frac{c_2}{4\delta}\Big), c_1\alpha^2, c_2\alpha^2\right\} > 0, \\
C_2 &= \bar{C}\left(\frac{\tau^2 h_M^2 \mathcal{P}e L_h^2}{2\kappa\mathcal{B}_m}|\Omega|^{1/2} + \frac{c_1\alpha^2 L(1+L^2 h'_M)}{4\lambda} + c_2(1+L^2h'_M)\Big(\zeta L + \frac{L}{4\theta}\Big)\right) \\
&\quad + \frac{2\bar{C}}{\alpha^2}\Big(\frac{\tau^2 h_M^2 \mathcal{P}e L_h^2}{2\kappa\mathcal{B}_m}|\Omega|^{1/2} + \frac{c_1 L_h^2}{4\gamma} + c_2\delta\Big) + C'_2, \\
C_3 &= \frac{\eta_M^2}{2\eta_m}|\partial_y g_1^\tau|_2^2 + \bar{C}F_1^2 L^p(c_1+c_2)\|\hat{\varphi}_l\|_2^{2p} \\
&\quad + \bar{C}\left(\frac{\tau^2 h_M^2 \mathcal{P}e L_h^2}{2\kappa\mathcal{B}_m}|\Omega|^{1/2} + c_1\Big(\frac{L}{4\gamma} + |\Omega|^{1/2} + \alpha^2\lambda\Big) + c_2|\Omega|^{1/2}\right)\|\hat{\varphi}_l\|_2^2 \\
&\quad + c_1\bar{C}\Big(F_2^2|\Omega|^{3/2} + |\Omega|F_4(0)\Big) + c_2\bar{C}\Big(\theta|F'(\varphi_l)|_{L^2(\Gamma_l)}^2 + \nu F_5^2 + F_2^2|\Omega|^{3/2}\Big),
\end{aligned}$$

where C'_2 is given by

$$C'_2 = \begin{cases} \bar{C}|\Omega|^{1/2}F_1^2(1+L^{2p})(c_1+c_2), & \text{if } p = 1, \\ 0, & \text{if } p > 1. \end{cases}$$

With these constants, the *a priori* estimate (4.132) reads:

$$\mathcal{Y}' + C_1\mathcal{Z} \le f(\mathcal{Y})\mathcal{Z} + C_2\mathcal{Z} + C_3.$$

The only additional terms are $\frac{\eta_M^2}{2\eta_m}|\partial_z g_1^\tau|_2^2$ in C_3 and the terms in factor of τ^2 in C_2 and C_3.

- ⋆ By hypothesis *iv)* of Lemma 4.9, the terms in factor if τ^2 can be chosen arbitrarily small for τ small enough.
- ⋆ As far as the term $\frac{\eta_M^2}{2\eta_m}|\partial_z g_1^\tau|_2^2$ is concerned, it is treated with the help of the term $c_1F_3(0)\int_\Omega F(\varphi)$ in $\mathcal{Y}$ on the left-hand side: since $F \ge F_0$, we have

  $$c_1F_3(0)\int_\Omega F(\varphi) \ge c_1F_3(0)|\Omega|F_0.$$

  We can choose c_1 big enough such that

  $$c_1F_3(0)\int_\Omega F(\varphi) \ge c_1F_3(0)|\Omega|F_0 \ge \frac{\eta_M^2}{2\eta_m}|\partial_z g_1^\tau|_2^2.$$

The same reasoning as in Section 4.6 shows that there exists τ^*, γ^*, δ^*, ζ^*, θ^*, λ^*, c_1^*, c_2^*, L^* such that for any τ^*, $\gamma < \gamma^*$, $\delta < \delta^*$, $\theta < \theta^*$, $\lambda < \lambda^*$, $c_1 > c_1^*$, $c_2 < c_2^*$, $L < L^*$, and for $\zeta = \zeta^*$, the conditions of Proposition 4.29 are satisfied.

Let us come back to the notations with the subscripts $_n$ introduced in 4.5.1 denoting the Galerkin approximation. Proposition 4.29 yields that for any $T > 0$, $\mathcal{Y}_n \in L^\infty(0,T)$ and $\mathcal{Z}_n \in L^1(0,T)$, which means that there exists a constant C such that for any $T > 0$

$$\|\varphi_n\|_{L^\infty(\mathbb{R}^+;H^1(\Omega))} \le C, \qquad \|\varphi_n\|_{L^2(0,T;H^2(\Omega))} \le CT + C,$$
$$\|\mu_n\|_{L^2(0,T;H^1(\Omega))} \le CT + C, \qquad \|\partial_z u_n\|_{L^2(0,T;L^2(\Omega)} \le CT + C.$$

The other convergence results are obtained as before, and (4.118) and (4.119) still hold. Therefore

$$\begin{aligned}
\varphi_n &\rightharpoonup \varphi \quad \text{in } L^\infty(\mathbb{R}^+;\Phi_l^1) \quad *\text{-weak},\\
\varphi_n &\rightharpoonup \varphi \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^+;\Phi_l^3) \quad \text{weak},\\
\mu_n &\rightharpoonup \mu \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^+;\Phi^1) \quad \text{weak},\\
\frac{d\varphi_n}{dt} &\rightharpoonup \frac{d\varphi}{dt} \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^+;\Phi_l^{1*}) \quad \text{weak},
\end{aligned}$$

$$\begin{aligned} \varphi_n &\rightharpoonup \varphi \qquad \text{in } \mathcal{C}^0(0,T;\Phi_l^1) \quad \text{weak}, \\ \partial_z u_n &\rightharpoonup \partial_z u \qquad \text{in } L^2_{\text{loc}}(\mathbb{R}^+;L^2(\Omega)) \quad \text{weak}. \end{aligned}$$

From estimate (4.132), it only follows regularity on $\partial_z u$. In order to prove some more regularity on $\boldsymbol{u}$ and p, we will proceed in the next section as in Section 4.4 for Reynolds equation with surface tension.

4.7.4 About the Reynolds equation

Lemma 4.36. *Let $\varphi \in \Phi_l^3(\Omega)$ and $\mu \in \Phi^1(\Omega)$. Suppose that there exists a constant $C > 0$ such that φ and μ satisfy*

$$\|\varphi\|_{L^2(0,T;H^3(\Omega))} \leq C, \qquad \|\mu\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

Let $(p, \boldsymbol{u})$ be the solution of the Reynolds equation (the three first equations of (4.120)). Then there exists another constant $C' > 0$ such that

$$\|\partial_x p\|_{L^1(0,T;L^2(0,L))} \leq C', \qquad \|u\|_{L^1(0,T;L^2(\Omega))} \leq C', \qquad \|v\|_{L^1(0,T;H^{-1}(\Omega))} \leq C'. \quad (4.133)$$

Proof. As we did in the section 4.4, we can express $\partial_x p_\kappa$ as a function of φ and μ by

$$\begin{aligned} \widetilde{d}(\varphi)\partial_x p_\kappa &= s\widetilde{e}(\varphi) + \widetilde{d}(\varphi_l)w_{\text{in}} - s\widetilde{e}(\varphi_l) \\ &+ \kappa\left(\Big(h(x) - \frac{\widetilde{b}}{\widetilde{a}}\Big)\widetilde{\mathcal{K}}(x) - \Big(h(0) - \frac{\widetilde{b}(\varphi_l)}{\widetilde{a}(\varphi_l)}\Big)\widetilde{\mathcal{K}}(0) - \int_0^{h(x)} \mathcal{K}(x,Z)dZ + \int_0^{h(0)} \mathcal{K}(0,Z)dZ\right). \end{aligned}$$

▷ It has already been proved in the case without surface tension in Section 4.4 that for $\varphi \in H^1(\Omega)$, the coefficients a, b, a belong to $X(\Omega)$ and that $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}, \widetilde{e} \in H^1(0,L)$. Thus it remains to prove the regularity of $\mathcal{K}$. Recalling definition (4.122)-(4.121) of $\mathcal{K}$, we have

$$\widetilde{\mathcal{K}}(x) = \int_0^{h(x)} \frac{1}{\eta(x,z)} \int_0^z \left(\mu(x,Z)\partial_x\varphi(x,Z) - \partial_x \int_0^Z \mu(x,\xi)\partial_\xi\varphi(x,\xi)d\xi\right) dz\, dZ.$$

Since μ is bounded in $L^2(0,T;H^1(\Omega))$ and φ in $L^2(0,T;H^3(\Omega))$, we deduce from Proposition 4.3 that $\mu\partial_x\varphi$ is bounded in $L^1(0,T;H^1(\Omega))$, and $\mu\partial_z\varphi$ in $L^1(0,T;H^1(\Omega))$. Thus there exists $C > 0$ such that

$$\|k[\varphi]\|_{L^1(0,T;L^2(\Omega))} \leq C, \qquad \|\widetilde{\mathcal{K}}\|_{L^1(0,T;L^2(0,L))} \leq C, \qquad \|\mathcal{K}\|_{L^1(0,T;L^2(\Omega))} \leq C.$$

▷ For the trace $\mathcal{K}(0,y)$, we have $\mathcal{K}(0,y) = \int_0^{h(0)} \frac{1}{\eta(\varphi_l(y))} \int_0^y K_0$, where

$$K_0 = \mu(0,z)\partial_x\varphi(0,z) - \int_0^z \underbrace{\partial_x\mu(0,\xi)}_{=0 \text{ on } \Gamma_l} \partial_\xi\varphi(0,\xi)d\xi - \int_0^z \mu(0,\xi)\partial_x\partial_\xi\varphi(0,\xi)d\xi.$$

Since $\mu(0,\cdot) \in L^2(0,T;H^{1/2}(\Gamma_l))$ and $\partial_x\partial_\xi\varphi(0,\cdot) \in L^2(0,T;H^{1/2}(\Gamma_l))$, we apply again Proposition 4.3 to conclude that there exists $C > 0$ such that

$$\|K_0\|_{L^1(0,T;L^2(\Gamma_l))} \leq C, \qquad \text{and thus } \|\mathcal{K}(0,\cdot)\|_{L^1(0,T;L^2(\Gamma_l))} \leq C.$$

▷ Finally, we conclude from equation (4.134) that there exists $C > 0$ such that

$$\|\partial_x p_\kappa\|_{L^1(0,T;L^2(0,L)} \leq C.$$

Moreover, since

$$p = p_\kappa + \kappa \int_0^z \mu\, \partial_Z\varphi\, dZ,$$

it follows that

$$\partial_x p = \partial_x p_\kappa + \kappa \int_0^z \Big(\partial_x\mu\, \partial_Z\varphi + \mu\, \partial_x\partial_Z\varphi\, dZ\Big).$$

Again, Proposition 4.3 and the boundedness of $\varphi \in L^2(0,T;H^3(\Omega))$ and $\mu \in L^2(0,T;H^1(\Omega))$ allow to conclude that $\partial_x p$ is bounded in $L^1(0,T;L^2(0,L))$.

▷ At last, it follows from (4.31) and the regularity of the coefficients and of $\partial_x p^*$ that there exists $C > 0$ such that $\|u\|_{L^1(0,T;L^2(\Omega))} \leq C$, and $\|v\|_{L^1(0,T;H^{-1}(\Omega))} \leq C$.

□

We thus obtained similar convergence results for the order parameter φ and the chemical potential μ as in the previous case without surface tension. However, let us point out that the regularity of $\boldsymbol{u}$, p is much weaker than previously, since the Reynolds equation is only satisfied in a weak sense.

Bibliography

[ABC94] A. Assemien, G. Bayada, and M. Chambat. Inertial effects in the asymptotic behavior of a thin film flow. *Asymptotic Anal.*, 9(3):177–208, 1994.

[Ada75] R. A. Adams. *Sobolev spaces.* Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[BC86] G. Bayada and M. Chambat. The transition between the Stokes equations and the Reynolds equation: a mathematical proof. *Appl. Math. Optim.*, 14(1):73–93, 1986.

[BCC99] G. Bayada, M. Chambat, and I. Ciuperca. Asymptotic Navier-Stokes equations in a thin moving boundary domain. *Asymptot. Anal.*, 21(2):117–132, 1999.

[BdV74] H. Beirão da Veiga. On the $W^{2,p}$-regularity for solutions of mixed problems. *J. Math. Pures Appl. (9)*, 53:279–290, 1974.

[BMV06] G. Bayada, S. Martin, and C. Vázquez. About a generalized Buckley-Leverett equation and lubrication multifluid flow. *European J. Appl. Math.*, 17(5):491–524, 2006.

[Boy99] F. Boyer. Mathematical study of multi-phase flow under shear through order parameter formulation. *Asymptot. Anal.*, 20(2):175–212, 1999.

[Boy01] F. Boyer. Nonhomogeneous Cahn-Hilliard fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(2):225–259, 2001.

[Chu03] L. Chupin. Existence result for a mixture of non Newtonian flows with stress diffusion using the Cahn-Hilliard formulation. *Discrete Contin. Dyn. Syst. Ser. B*, 3(1):45–68, 2003.

[Chu04] L. Chupin. Some theoretical results concerning diphasic viscoelastic flows of the Oldroyd kind. *Adv. Differential Equations*, 9(9-10):1039–1078, 2004.

[Doi97] M. Doi. Dynamics of domains and textures. in *Theoretical Challenges in the Dynamics of Complex Fluids*, pages 293–314. T.C.B. McLeish, 1997.

[GSMS83] J. D. Guton, M. San Miguel, and P. S. Sahni. *Phase transitions and critical phenomena*, volume 8. Academic, London, 1983.

[Hör97] L Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications*. Springer-Verlag, Berlin, 1997.

[MPS05] E. Marušić-Paloka and M. Starčević. Rigorous justification of the Reynolds equations for gas lubrication. *C. R. Mécanique*, 33(7):534–541, 2005.

[Pao03] L. Paoli. Asymptotic behavior of a two fluid flow in a thin domain: from Stokes equations to Buckley-Leverett equation and Reynolds law. *Asymptot. Anal.*, 34(2):93–120, 2003.

[Sim87] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[Tem79] R. Temam. *Navier-Stokes equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, revised edition, 1979.

[Tem97] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1997.

5

Numerical study of diphasic fluids in thin films

Abstract *In this work, diphasic flows in thin films are simulated, for which the diffusion effects at the interface between the two fluids and the capillarity are taken into account. A numerical scheme implemented by Boyer for the hydrodynamical Cahn-Hilliard equation [Boy02] is used and coupled with a resolution of the Reynolds equation. Afterwards, we present numerical simulations for several applications. We introduce some test cases, which allows us to validate the program. Next, we are interested in modelling the cavitation phenomena, and some new aspects can be put forward, since the presence of several layers of fluid is handled in the model.*

5.1 Introduction

In this chapter, we are concerned with the numerical study of diphasic fluids in thin flows. The framework of thin films and the different approaches for diphasic flows have already been introduced in the previous chapter. The model chosen here is a coupled system between a modified Reynolds equation (in which the coefficients depend on the viscosity, which is a function of the order parameter φ) and an hydrodynamical Cahn-Hilliard equation (which controls the behavior of the order parameter φ), as introduced by Boyer [Boy99].

From a numerical point of view, diphasic problems have been widely studied, with several methods in order to follow the position of the interface. We mentioned in the previous chapter that there are two main approaches, the sharp-interface approach and the diffuse-interface approach. As far as the sharp interface models are concerned, many numerical works are available, e.g. [CFBT99], [GGL$^+$98], [TB01]... For more details, we refer for example to [Vig07]. For the diffuse-interface models, they are also several methods for tracking the interface. Let us mention the second gradient method (e.g. [JLCD01]), or works in the compressible case (e.g. [SA99]). We are interested here in "phase-field" models, which introduce a free energy characterizing the equilibrium of the two phases. A classical example of such models is the Cahn-Hilliard model, which has been studied from a numerical point of view by Jacqmin [Jac99], Boyer [Boy02], Kim, Kang and Lowengrub [KKL04]. The scheme presented in the Cahn-Hilliard part of this work is based on the work of Boyer [Boy02].

One of the possible application of such models in thin films is to simulate lubrication phenomena, for example the apparition of cavitation. A first study has been carried out by Bayada, Martin, Vázquez [BMV06] with a sharp-interface model. The limitation of this model consists in the fact that the interface between the two fluids is supposed to be the graph of a function, which hinders many physical features, while the model developed here overcomes these difficulties.

This chapter is organized as follows. In Section 5.2, we recall the system Reynolds/Cahn-Hilliard which had been introduced in Chapter 4. Section 5.3 is dedicated to the description of the numerical scheme. In Section 5.4, we introduce several test cases, which allows us to validate the behavior of the program. Lastly, in Section 5.5, we present an application modelling the cavitation.

5.2 The mathematical model

We are interested in simulating the flow of a diphasic fluid in a thin domain $\Omega^\varepsilon = \{(x, y) \in \mathbb{R}^2, x \in (0, L), y \in (0, \varepsilon h(x))\}$. In this specific geometry, the flow is well represented by the Reynolds equation, which is an asymptotic limit of the Navier-Stokes equations when ε tends to zero. However, since the viscosities of the two fluids are different, a modified Reynolds equation has to be considered, with a variable viscosity. Moreover, since we are interested in a diffuse interface model, we introduce an order parameter φ, representing the volumic fraction of one fluid in the mixture, which satisfies the hydrodynamic Cahn-Hilliard equation. For each point, the viscosity η of the mixture can then be written as a function of φ and the viscosities η_1 and η_2 of the two fluids:

$$\frac{1}{\eta(\varphi)} = \begin{cases} \dfrac{1+\varphi}{2\eta_1} + \dfrac{1-\varphi}{2\eta_2} & \text{if } \varphi \in [-1, 1], \\ 1/\eta_1 & \text{if } \varphi > 1, \\ 1/\eta_2 & \text{if } \varphi < -1, \end{cases} \tag{5.1}$$

The whole system on the velocity $\boldsymbol{u} = (u, v)$, the pressure p, the order parameter φ reads, in $\Omega = \{(x, y) \in \mathbb{R}^2, x \in (0, L), y \in (0, h(x))\}$:

$$\partial_x(\widetilde{d}(\varphi)\partial_x p) = s\partial_x \widetilde{e}(\varphi), \tag{5.2a}$$

$$u = \left(b(\varphi) - \frac{a(\varphi)\widetilde{b}(\varphi)}{\widetilde{a}(\varphi)}\right)\partial_x p + s\left(1 - \frac{a(\varphi)}{\widetilde{a}(\varphi)}\right), \tag{5.2b}$$

$$v(x, y) = -\int_0^y \partial_x u(x, z)dz, \tag{5.2c}$$

$$\partial_t \varphi + \boldsymbol{u} \cdot \nabla\varphi - \frac{1}{\mathcal{P}e}\operatorname{div}(B(\varphi)\nabla\mu) = 0, \tag{5.2d}$$

$$\mu = -\alpha^2\Delta\varphi + F'(\varphi), \tag{5.2e}$$

equipped with the following boundary conditions on $\partial\Omega = \Gamma = \Gamma_l \cup \Gamma_0$, where $\Gamma_l = \Gamma \cap \{x \in \mathbb{R}, x = 0\}$:

$$u|_{y=0} = s, \quad u|_{y=h(x)} = 0, \quad v|_{y=0} = v|_{y=h(x)} = 0, \tag{5.3a}$$

$$\int_0^{h(0)} u|_{x=0} = q, \quad \partial_x p(0) = w_{\text{in}}, \quad p(L) = 0, \tag{5.3b}$$

$$\nabla\varphi \cdot \boldsymbol{n}|_{\Gamma_0} = \nabla\mu \cdot \boldsymbol{n}|_{\Gamma_0} = 0, \quad \varphi|_{\Gamma_l} = \varphi_l, \quad \mu|_{\Gamma_l} = -\alpha^2\Delta\varphi_l + F'(\varphi_l), \tag{5.3c}$$

where s is the shear velocity, $\mathcal{P}e$ the Péclet number, $B(\varphi)$ the mobility, μ the chemical potential, α a parameter measuring the thickness of the interface, $F(\varphi)$ the Cahn-Hilliard potential, q the entrance flow, w_{in} is related to q, s, h and the viscosity η through the following formula:

$$w_{\text{in}} = \frac{q - s\left(h(0) - 1/\widetilde{a}_0 \int_0^{h(0)} a(0, z)dz\right)}{\int_0^{h(0)} b(0, z)dz - \widetilde{b}_0/\widetilde{a}_0 \int_0^{h(0)} a(0, z)dz},$$

and the coefficients $a(\varphi)$, $\widetilde{a}(\varphi)$, $b(\varphi)$, $\widetilde{b}(\varphi)$, $\widetilde{d}(\varphi)$, $\widetilde{e}(\varphi)$, $\widetilde{a}_0$, $\widetilde{b}_0$ are given by:

$$a(\varphi)(x, y) = \int_0^y \frac{dz}{\eta(\varphi(x, z))}, \quad b(\varphi)(x, y) = \int_0^y \frac{zdz}{\eta(\varphi(x, z))}, \quad \widetilde{c}(\varphi)(x) = \int_0^{h(x)} \frac{z^2dz}{\eta(\varphi(x, z))},$$

$$\widetilde{a}(\varphi)(x) = a(\varphi)(x, h(x)), \quad \widetilde{b}(\varphi)(x) = b(\varphi)(x, h(x)),$$

$$\widetilde{d}(\varphi)(x) = \widetilde{c}(\varphi)(x) - \frac{\widetilde{b}(\varphi)(x)^2}{\widetilde{a}(\varphi)(x)}, \quad \widetilde{e}(\varphi)(x) = \frac{\widetilde{b}(\varphi)(x)}{\widetilde{a}(\varphi)(x)}, \quad \widetilde{a}_0 = \widetilde{a}(\varphi)(0), \quad \widetilde{b}_0 = \widetilde{b}(\varphi)(0).$$

Moreover, the initial condition on φ reads $\varphi|_{t=0} = \varphi_0$.

Remark 5.1. *The boundary conditions on μ stated in (5.3c) are adequate in order to* *study the system from a theoretical point of view. However, it is also interesting to consider boundary conditions of the following form:*

$$\nabla\mu \cdot \boldsymbol{n}|_\Gamma = 0, \tag{5.4}$$

meaning that there is no diffusion through the boundary. Therefore, in the numerical simulations, the two cases are taken into account, and different simulations are carried out for each boundary condition on μ.

Taking the surface tension into account

In the case when the surface tension is taken into account, the Reynolds equation is modified when introducing a new pressure $p^* = p - \kappa \int_0^y \mu(x,z)\partial_z\varphi(x,z)dz$, where κ is the capillarity coefficient. This new pressure satisfies also $\partial_y p^* = 0$. Let us define:

$$\begin{aligned} k(\varphi)(x,y) &= \mu(x,y)\partial_x\varphi(x,y) - \partial_x\left(\int_0^y \mu(x,z)\partial_z\varphi(x,z)dz\right), \\ \mathcal{K}(x,y) &= \int_0^y \frac{1}{\eta(\varphi(x,z))}\int_0^z k(\varphi)(x,\xi),d\xi\,dz, \quad \widetilde{\mathcal{K}}(x) = \mathcal{K}(x,h(x)). \end{aligned} \tag{5.5}$$

With these quantities, the modified Reynolds equation becomes:

$$\partial_x(\widetilde{d}(x)\partial_x p^*) = s\partial_x(\widetilde{e}(x)) + \kappa\partial_x\left(\Big(h(x) - \frac{\widetilde{b}(x)}{\widetilde{a}(x)}\Big)\widetilde{\mathcal{K}}(x) - \int_0^{h(x)} \mathcal{K}(x,z)dy\right). \tag{5.6}$$

Similarly to the case without surface tension, the velocity is deduced from the pressure by:

$$\begin{aligned} u(x,y) =& \Big(b(x,y) - \frac{a(x,y)\widetilde{b}(x)}{\widetilde{a}(x)}\Big)\partial_x p^* + s\Big(1 - \frac{a(x,y)}{\widetilde{a}(x)}\Big) \\ &+ \kappa\Big(\frac{a(x,y)}{\widetilde{a}(x)}\widetilde{\mathcal{K}}(x) - \mathcal{K}(x,y)\Big), \end{aligned} \tag{5.7a}$$

$$v(x,y) = -\int_0^y \partial_x u(x,z)dz. \tag{5.7b}$$

Thus, in the case when the surface tension is taken into account, the equations on p and u are only modified by an additional term depending on φ and μ.

5.3 Numerical scheme

Let us describe in a few words the main steps of the numerical computations. First, for a given φ, the pressure is computed by (5.2a), and the velocity field is deduced by (5.2b) and (5.2c). To this end, the coefficients $\widetilde{d}$, $\widetilde{e}$, a, b, $\widetilde{a}$, $\widetilde{b}$ have to be computed. The second step consists in computing φ, μ by (5.2d), (5.2e).

5.3.1 Rescaling the domain

The domain considered here is not rectangular, but since a simple change of variables allows us to work in a rectangular domain, we work with finite differences, thus with a rectangular mesh of uniform cells, and the equations are rewritten in a rescaled rectangular domain. For the sake of clarity, we will keep the same notations for the variables in the rescaled domain, so that the domain now writes $\Omega = \{(x,z) \in \mathbb{R}^2, x \in (0,L), z \in (0,1)\}$. Therefore, the integrals with respect to y change into integrals with respect to z, and are thus multiplied by $h(x)$. With this observation, the system for *the pressure and the velocity field* reads:

$$\partial_x(h^3(x)\widetilde{d}(\varphi)\partial_x p) = s\partial_x\left(h(x)\widetilde{e}(\varphi)\right), \tag{5.8a}$$

$$u = h^2(x)\left(b(\varphi) - \frac{a(\varphi)\widetilde{b}(\varphi)}{\widetilde{a}(\varphi)}\right)\partial_x p + s\left(1 - \frac{a(\varphi)}{\widetilde{a}(\varphi)}\right), \tag{5.8b}$$

$$v(x,z) = -h(x)\int_0^z \partial_x u(x,Z)dZ + h'(x)\int_0^z Z\partial_z u(x,Z)dZ, \tag{5.8c}$$

where the coefficients are now given by:

$$\begin{aligned}
&a(\varphi)(x,z) = \int_0^z \frac{dZ}{\eta(\varphi(x,Z))}, \quad b(\varphi)(x,z) = \int_0^z \frac{ZdZ}{\eta(\varphi(x,Z))}, \quad \widetilde{c}(\varphi)(x) = \int_0^1 \frac{Z^2dZ}{\eta(\varphi(x,Z))},\\
&\widetilde{a}(\varphi)(x) = a(\varphi)(x,1), \quad \widetilde{b}(\varphi)(x) = b(\varphi)(x,1),\\
&\widetilde{d}(\varphi)(x) = \widetilde{c}(\varphi)(x) - \frac{\widetilde{b}(\varphi)(x)^2}{\widetilde{a}(\varphi)(x)}, \quad \widetilde{e}(\varphi)(x) = \frac{\widetilde{b}(\varphi)(x)}{\widetilde{a}(\varphi)(x)}.
\end{aligned} \tag{5.9}$$

Let us observe that due to the change of variables, we have the following correspondences:

$$\begin{aligned}
\partial_x \cdot &\longleftrightarrow \partial_x \cdot - z\frac{h'(x)}{h(x)}\partial_z \cdot,\\
\partial_y \cdot &\longleftrightarrow \frac{1}{h(x)}\partial_z \cdot\,.
\end{aligned} \tag{5.10}$$

This explains in particular why the expression of v contains two terms.It is thus possible to rewrite also *the Cahn-Hilliard equation*:

$$\partial_t \varphi + u\left(\partial_x\varphi - z\frac{h'}{h}\partial_z\varphi\right) + \frac{1}{h}v\partial_z\varphi - \frac{1}{\mathcal{P}e}\left(\partial_x - z\frac{h'}{h}\partial_z\right)\left(B(\varphi)\left(\partial_x\mu - z\frac{h'}{h}\partial_z\mu\right)\right)$$
$$- \frac{1}{\mathcal{P}e}\frac{1}{h^2}\partial_z\Big(B(\varphi)\partial_z\mu\Big) = 0, \tag{5.11a}$$
$$\mu = -\alpha^2\left(\partial_x^2\varphi - z\frac{h'}{h}\partial_x\partial_z\varphi - z\partial_x\left(\frac{h'}{h}\partial_z\varphi\right) + z\left(\frac{h'}{h}\right)^2\partial_z(z\partial_z\varphi)\right.$$
$$\left. + \frac{1}{h^2}\partial_z^2\varphi\right) + F'(\varphi). \tag{5.11b}$$

As far as *the surface tension terms* are concerned, the same change of variables in (5.5) implies, for $z \in (0,1)$:

$$k(x,z) = \mu(x,z)\partial_x\varphi(x,z) - \cancel{z\frac{h'(x)}{h(x)}\mu(x,z)\partial_z\varphi(x,z)}$$
$$- \partial_x\left(\int_0^z \mu(x,\xi)\partial_\xi\varphi(x,\xi)d\xi\right) + \cancel{z\frac{h'(x)}{h(x)}\mu(x,z)\partial_z\varphi(x,z)}$$

Then we compute the integral arising in $\mathcal{K}$ to which we apply the change of variables:

$$h(x)\int_0^z k(x,\xi)d\xi = h(x)\int_0^z (\mu\partial_x\varphi)(x,\xi)d\xi - h(x)\partial_x \underbrace{\int_0^z\int_0^\xi (\mu\partial_\omega\varphi)(x,\omega)d\omega d\xi}_{=\int_0^z (z-\xi)(\mu\partial_\xi\varphi)(x,\xi)d\xi},$$

thus it follows

$$\mathcal{K}(\cdot,z) = h^2\int_0^z \frac{1}{\eta(\varphi(\cdot,Z))}\left[\int_0^Z (\mu\partial_x\varphi)(\cdot,\xi)d\xi - \partial_x\int_0^Z (Z-\xi)(\mu\partial_\xi\varphi)(\cdot,\xi)d\xi\right]dZ, \tag{5.12}$$

and $\widetilde{\mathcal{K}}(x) = \mathcal{K}(x,1)$.

5.3.2 Discretization of the Reynolds equation

The Reynolds equation on the pressure (5.2a) is an elliptic equation, and it has been proved in Chapter 4 that the operator $\partial_x(\widetilde{d}(\varphi)\partial_x\cdot)$ is coercive, thus $\partial_x p$ can be computed explicitly. Using the boundary conditions on p, it follows:

$$h^3(x)\widetilde{d}(\varphi)\partial_x p = sh(x)\widetilde{e}(\varphi) + h(0)^3\widetilde{d}_0 w_{\text{in}} - sh(0)\widetilde{e}_0, \tag{5.13}$$

where $\widetilde{d}_0 = \widetilde{d}(\varphi_l)$, $\widetilde{e}_0 = \widetilde{e}(\varphi_l)$. Thus $\partial_x p$ is computed explicitly through this formula, and since u depends on p through $\partial_x p$, the velocity is also computed. If the pressure is needed, it is computed using the second boundary condition on p:

$$p(x) = p(L) - \int_x^L \partial_X p(X) dX = -\int_x^L \partial_X p(X) dX. \tag{5.14}$$

Space discretization

The mesh is constituted by a grid of $N \times M$ uniform cells. Let us define the two space steps $\delta x = L/N$, $\delta z = 1/M$. The unknowns are either sought at the center of the cells (which is the case of the pressure) or on the boundary of the cells (the two velocity components). More precisely, for a cell (i,j), the value of p is defined at (i,j), the value of u at $(i+1/2, j)$ and the value of v at $(i, j+1/2)$. Moreover, the order parameter φ and the chemical potential μ are also taken at point (i,j) (see Figure 5.1). In this manner,

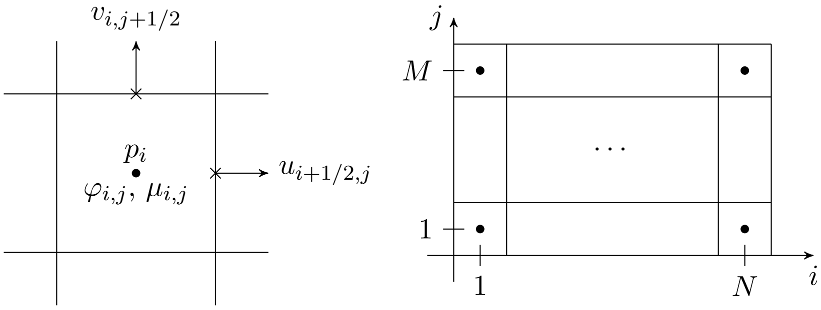

Figure 5.1: Positions of the unknowns for the cell (i,j) and numbering of the cells

the value of u is defined at the same point that $\partial_x p$, which is consistent with formula (5.8b). For the sake of readability, we introduce the following notations for all $1 \le i \le N$, $1 \le j \le M$:

$$h_i = h(x_i), \quad \widetilde{d}_i = \widetilde{d}(\varphi_i), \quad \widetilde{e}_i = \widetilde{e}(\varphi_i), \quad a_{i,j} = a(\varphi_{i,j}), \quad b_{i,j} = b(\varphi_{i,j}).$$

The discretization for $\partial_x p$ writes:

$$(\partial_x p)_{i+1/2} = \frac{2\Big(sh_{i+1/2}\frac{\widetilde{e}_i + \widetilde{e}_{i+1}}{2} + h(0)^3 \widetilde{d}_0 w_{\text{in}} - sh(0)\widetilde{e}_0\Big)}{h_{i+1/2}^3\Big(\widetilde{d}_i + \widetilde{d}_{i+1}\Big)}. \tag{5.15}$$

The coefficients are averaged to be taken at the point $(i+1/2, j)$. Note that this discretization is valid since the coefficients $\widetilde{d}$ and $\widetilde{e}$ are of the form $1/\eta(\varphi)$, and are thus linear with respect to φ by (5.1). Therefore, averaging φ at the point $(i+1/2, j)$ is equivalent to averaging the coefficients. For u, the discretization reads:

$$\begin{aligned} u_{i+1/2,j} = \\ \frac{1}{4}(h_{i+1}+h_i)^2 \left(b_{i+1,j} + b_{i,j} + \frac{(a_{i+1,j}+a_{i,j})(\widetilde{b}_{i+1,j}+\widetilde{b}_{i,j})}{\widetilde{a}_{i+1,j}+\widetilde{a}_{i,j}} \right) (\partial_x p)_{i+1/2} \\ + s\left(1 - \frac{a_{i+1,j}+a_{i,j}}{\widetilde{a}_{i+1,j}+\widetilde{a}_{i,j}}\right). \end{aligned} \tag{5.16}$$

As far as the value of v is concerned, it is defined as the integral with respect to z of $\partial_x u$ and $\partial_z u$ (see (5.8c)). Since the two terms are not naturally expressed at the same point, we average the term $\partial_z u$:

$$\begin{aligned} v_{i,j+1/2} = -h_i \int_0^{y_{j+1/2}} \frac{u_{i+1/2,J} - u_{i-1/2,J}}{\delta x} dJ \\ + h_i' \int_0^{y_{j+1/2}} z \frac{u_{i+1/2,J+1} + u_{i-1/2,J+1} - u_{i+1/2,J} - u_{i-1/2,J}}{2\delta z} dJ. \end{aligned} \tag{5.17}$$

The notation "dJ" means that J is the index corresponding to the variable of integration.

Remark 5.2. *For the Cahn-Hilliard equation, we will define the discretization of the equation on φ later. However, observe that p depends on φ through $\eta(\varphi)$. It is thus convenient to define p and φ at the same point. For μ, it is convenient to define it at the same point as φ because of equation (5.2e). Thus p, φ and μ are computed at the same point (i, j).*

For *the surface tension terms*, since they occur in the equations on $\partial_x p^*$ and u, they are computed at the point $(i + 1/2, j)$. To obtain the value at this point, the contributions μ and $\dfrac{1}{\eta(\varphi)}$ are averaged. Recalling the expression of $\mathcal{K}$ (5.12), we have the following discretization:

$$\begin{aligned} \mathcal{K}_{i+1/2,j} = \int_0^{z_j} \Big(\frac{1}{2\eta(\varphi_{i,J})} + \frac{1}{2\eta(\varphi_{i+1,J})} \Big) \Bigg[\int_0^{\xi_J} \frac{1}{2}(\mu_{i,k} + \mu_{i+1,k}) \frac{\varphi_{i+1,k} - \varphi_{i,k}}{\delta x} dk \\ - \frac{1}{\delta z} \Big(\int_0^{\xi_J} (z - \xi_k) \mu_{i+1,k} \frac{\varphi_{i+1,k+1/2} - \varphi_{i+1,k-1/2}}{\delta z} dk \\ - \int_0^{\xi_J} (z - \xi_k) \mu_{i,k} \frac{\varphi_{i,k+1/2} - \varphi_{i,k-1/2}}{\delta z} dk \Big) \Bigg] dJ. \end{aligned} \tag{5.18}$$

Let us emphasize as before that this is true because the function $\dfrac{1}{\eta(\varphi)}$ is linear with respect to φ, by (5.1). Moreover, as in (5.17), the notations "dk" and "dJ" mean that k and J are the indices corresponding to the variables of integration.

Computation of the coefficients

Recalling that the coefficients a, b, c are of the form $\displaystyle\int_0^z \frac{Z^i dZ}{\eta(\varphi(x,Z))}$, we have to compute several integral terms. To this end, the trapezoidal method is used.

- For the coefficient $a = \displaystyle\int_0^z \frac{dZ}{\eta(\varphi(x,Z))} =: \int_0^z f_a(x,Z)dZ$:
  - $\dfrac{\partial f_a}{\partial Z}(x,0) = 0$, since on the boundary $\{z=0\}$, φ satisfies $\nabla\varphi \cdot \boldsymbol{n}|_{z=0} = 0$, and f_a is a composite function of φ.
  - For the term $\widetilde{a}$, we also need the value for $Z = 1$. Since we do not know any easy property of $f_a(x,1)$, we extend the function f_a by continuity.

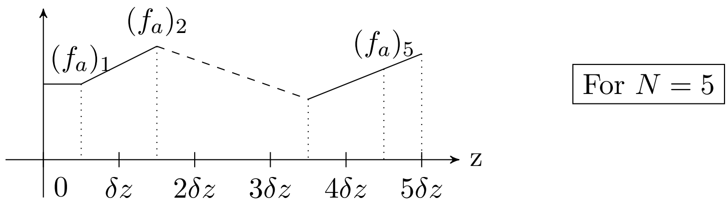

Figure 5.2: Trapezoidal method for the integration for the coefficients a and $\widetilde{a}$

  - Defining A the matrix of all values $(a_{i,j})_{\substack{1\le i\le N\\1\le j\le M}}$, and F_a the matrix of the values $((f_a)_{i,j})_{\substack{1\le i\le N\\1\le j\le M}}$ of f_a at the points (i,j), A satisfies $A = M_a F_a$, with

$$M_a = \delta z \begin{pmatrix} \frac{1}{2} & 1 & \dots & 1 \\ & \frac{1}{2} & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \frac{1}{2} \end{pmatrix}.$$

  - We have also $\tilde{A} = V_a F_a$, where $V_a = \delta z \underbrace{(1,\dots,1)}_{M \text{ components}}$.

- With a similar reasoning, we treat the coefficient

$$b = \int_0^z f_b(x,Z)dZ, \qquad \text{with } f_b(x,z) = \frac{z}{\eta(\varphi(x,z))}:$$

  - It follows directly from the definition of f_b that $f_b(x,0) = 0$.
  - For the term $\widetilde{b}$, we also need the value for $Z = 1$: since we do not know any easy property of $f_b(x,1)$, we extend the function f_b by continuity.

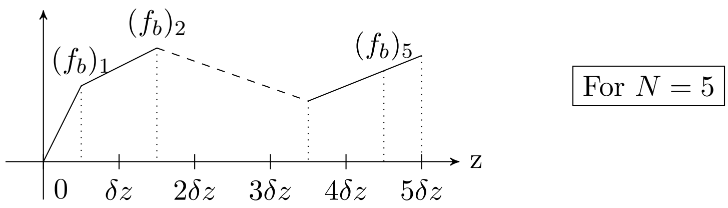

Figure 5.3: Trapezoidal method for the integration for the coefficients b and $\widetilde{b}$

- $B = M_b F_b$, with analogous notations as before, with

$$M_b = \delta z \begin{pmatrix} \frac{1}{2} & 1 & \dots & 1 & \frac{3}{4} \\ & \frac{1}{2} & \ddots & \vdots & \vdots \\ & & \ddots & 1 & \vdots \\ & & & \frac{1}{2} & \frac{3}{4} \\ & & & & \frac{1}{4} \end{pmatrix}.$$

- We have also $\tilde{B} = V_b F_b$, where $V_b = \delta z \, \underbrace{(\frac{9}{8}, \frac{7}{8}, 1, \dots, 1, \frac{3}{4})}_{M \text{ components}}$.

- For the coefficient $\widetilde{c}$, we use that for $f_c(x, z) = \dfrac{z^2}{\eta(\varphi(x, z))}$:
  - $f_c(x, 0) = 0$.
  - For the value of $f_c(x, 1)$, we extend the function f_c by continuity.
  - $\tilde{C} = V_c F_c$, with the same vector $V_c = V_b$, since the "boundary conditions" for f_c are the same as for f_b.
- The coefficients $\widetilde{d}$ and $\widetilde{e}$ are easily deduced from the previous coefficients by (5.9).
- As far as the values at $x = 0$ of the coefficients are concerned, they are simply computed from the value φ_l of φ at the boundary $\{x = 0\}$.

From (5.15), the derivative of the pressure $\partial_x p$ is computed, as a function of these coefficients. The pressure is computed with the following discretization:

$$p_i = \int_{x_i}^{L} (\partial_x p)_I dI. \tag{5.19}$$

For the computation of this integral, we use the same notations as before: we introduce the vector P of all components p_i of p, the function $f_p = \partial_x p$, and F the vector of all components $(f_p)_i$. Since we do not know the value of $\partial_x p$ at $x = L$, the function f_p is extended by continuity at $x = L$. Then $P = M_p F_p$, with

$$M_p = -\delta x \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\ 1 & \frac{3}{4} & \frac{1}{4} & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 1 & \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{7}{8} & \dots & \dots & \dots & \frac{7}{8} & \frac{5}{8} & \frac{1}{8} \\ \frac{5}{8} & \dots & \dots & \dots & \dots & \frac{5}{8} & \frac{3}{8} \end{pmatrix}.$$

Computation of the velocity field

From the derivative of the pressure computed by (5.13), using the coefficients computed as presented in the previous subsection, we deduce the velocity u by (5.16). For the vertical component of the velocity v by (5.17), some integrations are needed. For the $\partial_x u$ term, we use the fact that $\partial_x u|_{z=0} = 0$ (because $u(x,0) \equiv s$). For the $\partial_z u$-term, we have also that $(z\partial_z u)|_{z=0} = 0$. It follows that for $f_v(x,z) = \partial_x u(x,z) + z\partial_z u(x,z)$, the values of v satisfy $V = M_v F_v$, with

$$M_v = \delta z \begin{pmatrix} \frac{1}{8} & \frac{7}{8} & 1 & \dots & 1 & \frac{3}{4} \\ & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & \ddots & 1 & \vdots \\ & & & \ddots & \frac{7}{8} & \frac{3}{4} \\ & & & & \frac{1}{8} & \frac{5}{8} \end{pmatrix}.$$

Observe that this matrix is not a $M \times N$-matrix, since V is a $(M-1) \times N$-matrix. Indeed, it follows from the boundary conditions that $\forall 1 \leq i \leq N$, $v_{i,M} = 0$.

The surface tension terms

The capillarity terms are defined as functions of φ and μ, although they appear in the equations on p and u. From (5.6), applying the change of variables as in (5.8a) and using (5.12), we obtain an equation on $\partial_x(h^3 \widetilde{d} \partial_x p^*)$ which can be integrated, as in (5.13). It follows:

$$\begin{aligned}
&h^3(x)\widetilde{d}(x)\partial_x p^* = sh(x)\widetilde{e}(x) + h^3(0)\widetilde{d}_0 w_{\text{in}} - sh(0)\widetilde{e}_0 \\
&+\kappa\Bigg(h^3(x)\big(1-\frac{\widetilde{b}(x)}{\widetilde{a}(x)}\big)\int_0^1 \frac{1}{\eta(\varphi(x,Z))}\left[\int_0^Z (\mu\partial_x\varphi)(x,\xi)d\xi - \partial_x\int_0^Z (Z-\xi)(\mu\partial_\xi\varphi)(x,\xi)d\xi\right]dZ \\
&- h^3(x)\underbrace{\int_0^1\int_0^z \frac{1}{\eta(\varphi(x,Z))}\left[\int_0^Z (\mu\partial_x\varphi)(x,\xi)d\xi - \partial_x\int_0^Z (Z-\xi)(\mu\partial_\xi\varphi)(x,\xi)d\xi\right]dZdz}_{=\int_0^1 \frac{1-z}{\eta(\varphi(x,z))}\left[\int_0^z (\mu\partial_x\varphi)(x,\xi)d\xi - \partial_x\int_0^z (z-\xi)(\mu\partial_\xi\varphi)(x,\xi)d\xi\right]dz} \\
&- h^3(0)\big(1-\frac{\widetilde{b}_0}{\widetilde{a}_0}\big)\int_0^1 \frac{1}{\eta(\varphi_l(Z))}\left[\int_0^Z (\mu\partial_x\varphi)(0,\xi)d\xi - \partial_x\int_0^Z (Z-\xi)(\mu\partial_\xi\varphi)(0,\xi)d\xi\right]dZ \\
&+ h^3(0)\int_0^1 \frac{1-z}{\eta(\varphi_l(z))}\left[\int_0^z (\mu\partial_x\varphi)(0,\xi)d\xi - \partial_x\int_0^z (z-\xi)(\mu\partial_\xi\varphi)(0,\xi)d\xi\right]dz\Bigg).
\end{aligned}$$

The discretization is then straightforward, using finite differences to discretize the derivatives. In order to compute the integrals and the derivatives, the boundary conditions on φ and μ are needed. They are given in more detail in the next section on the Cahn-Hilliard equation. The value of the real pressure p can be computed, since

$$p = p^* + \kappa\int_0^z (\mu\partial_Z\varphi)(x,Z)dZ.$$

The value of u follows from (5.7a) and the value of $\partial_x p^*$.

5.3.3 Discretization of the Cahn-Hilliard equation

The time discretization is inspired from the works of Boyer [Boy02] and Boyer, Chupin, Fabrie [BCF04]. It is done with a variable time step δt, using a fractional step method. The two main steps consist in the two main equations of the system: first the Reynolds part is solved, as described in the previous subsection, then the Cahn-Hilliard equation is treated. In this part, each time step is decomposed into two steps, in order to treat the convection terms.

Time discretization of the diffusion part

Knowing the values of φ^n, μ^n at time t_n, the first step is to compute the solution $\varphi^{n+1/2}$, $\mu^{n+1/2}$ of the Cahn-Hilliard part of the equation, by the following θ-scheme:

$$\frac{\varphi^{n+1/2}-\varphi^n}{\delta t} - \frac{1}{\mathcal{P}e}\,\text{div}\Big(B(\varphi^n)\nabla(\theta\mu^{n+1/2} + (1-\theta)\mu^n)\Big) = 0, \tag{5.20a}$$

$$\theta\mu^{n+1/2} + (1-\theta)\mu^n + \alpha^2\Delta\left(\theta\varphi^{n+1/2} + (1-\theta)\varphi^n\right) = F'(\theta\varphi^{n+1/2} + (1-\theta)\varphi^n). \quad (5.20b)$$

It is well-known that this kind of schemes are unconditionally stable for $\frac{1}{2} \leq \theta \leq 1$, and of order 2 for $\theta = \frac{1}{2}$. In order to stay away from the stability limits, we choose θ slightly greater than $\frac{1}{2}$, but close enough to $\frac{1}{2}$ so as to keep the method precise. The numerical simulations are carried out with $\theta = 0.6$. This non-linear system is solved by a fixed point method. Defining $\Phi_0^{n+1/2} = \varphi^n$ and $\mathcal{M}_0^{n+1/2} = \mu^n$, let us look for $\Phi_k^{n+1/2}$ and $\mathcal{M}_k^{n+1/2}$ as the solutions of

$$\begin{aligned}
&\Phi_{k+1}^{n+1/2} - \frac{\theta\delta t}{\mathcal{P}e}\,\mathrm{div}\left(B(\varphi^n)\nabla\mathcal{M}_{k+1}^{n+1/2}\right) = \varphi^n,\\
&\mathcal{M}_{k+1}^{n+1/2} + \alpha^2\Delta\Phi_{k+1}^{n+1/2} = F'(\Phi_k^{n+1/2}).
\end{aligned}$$

In this way, if they are convergent, the two sequences $\left(\Phi_k^{n+1/2}\right)_k$ and $\left(\mathcal{M}_k^{n+1/2}\right)_k$ converge respectively to:

$$\begin{aligned}
\Phi_k^{n+1/2} &\longrightarrow \theta\varphi^{n+1/2} + (1-\theta)\varphi^n,\\
\mathcal{M}_k^{n+1/2} &\longrightarrow \theta\mu^{n+1/2} + (1-\theta)\mu^n.
\end{aligned}$$

It is then easy to deduce the values of $\varphi^{n+1/2}$ and $\mu^{n+1/2}$. From a practical point of view, a few iterations are needed for the method to converge.

Space discretization of the diffusion part

The diffusion part of the Cahn-Hilliard equation is discretized with finite differences in a usual way, from the formula (5.11a):

$$\begin{aligned}
&\partial_t\,\varphi_{i,j} - \frac{1}{\mathcal{P}e}\Bigg[\frac{1}{2\delta x^2}\Big((B_{i+1,j}+B_{i,j})(\mu_{i+1,j}-\mu_{i,j}) - (B_{i,j}+B_{i-1,j})(\mu_{i,j}-\mu_{i-1,j})\Big)\\
&- z_j\frac{h_i'}{h_i}\frac{1}{4\delta x\delta z}\Big(B_{i,j+1}(\mu_{i+1,j+1}-\mu_{i-1,j+1}) + B_{i,j-1}(\mu_{i+1,j-1}-\mu_{i-1,j-1})\Big)\\
&- z_j\frac{1}{4\delta x\delta z}\left(\frac{h_{i+1}'}{h_{i+1}}B_{i+1,j}(\mu_{i+1,j+1}-\mu_{i+1,j-1}) - \frac{h_{i-1}'}{h_{i-1}}B_{i-1,j}(\mu_{i-1,j+1}-\mu_{i-1,j-1})\right)\\
&+ z_j\frac{h_i'}{h_i}\frac{1}{2\delta z^2}\Big(z_{j+1/2}(B_{i,j+1}+B_{i,j})(\mu_{i,j+1}-\mu_{i,j}) - z_{j-1/2}(B_{i,j}+B_{i,j-1})(\mu_{i,j}-\mu_{i,j-1})\Big)\\
&+ \frac{1}{h_i^2}\frac{1}{2\delta z^2}\Big((B_{i,j+1}+B_{i,j})(\mu_{i,j+1}-\mu_{i,j}) - (B_{i,j}+B_{i,j-1})(\mu_{i,j}-\mu_{i,j-1})\Big)\Bigg] = 0,
\end{aligned}$$

where $B_{i,j}$ denotes $B(\varphi_{i,j})$. The time discretization is done following (5.20a).

The equation on the chemical potential reads by (5.11b):

$$
\begin{aligned}
\mu_{i,j} = -\alpha^2 \Bigg[& \frac{1}{\delta x^2}\Big(\varphi_{i+1,j} - \varphi_{i,j} - \varphi_{i,j} + \varphi_{i-1,j}\Big) \\
& - z_j \frac{h_i'}{h_i}\frac{1}{4\delta x \delta z}\Big(\varphi_{i+1,j+1} - \varphi_{i-1,j+1} + \varphi_{i+1,j-1} - \varphi_{i-1,j-1}\Big) \\
& - z_j \frac{1}{4\delta x \delta z}\left(\frac{h_{i+1}'}{h_{i+1}}(\varphi_{i+1,j+1} - \varphi_{i+1,j-1}) - \frac{h_{i-1}'}{h_{i-1}}(\varphi_{i-1,j+1} - \varphi_{i-1,j-1})\right) \\
& + z_j \frac{h_i'}{h_i}\frac{1}{\delta z^2}\Big(z_{j+1/2}(\varphi_{i,j+1} - \varphi_{i,j}) - z_{j-1/2}(\varphi_{i,j} - \varphi_{i,j-1})\Big) \\
& + \frac{1}{h_i^2}\frac{1}{\delta z^2}\Big(\varphi_{i,j+1} - \varphi_{i,j} - \varphi_{i,j} + \varphi_{i,j-1}\Big)\Bigg] + F'(\varphi_{i,j}).
\end{aligned}
$$

Time discretization of the convection part

The second step consists in solving taking the convection term into account. Such a term has to be correctly discretized in order to avoid numerical diffusion. We implement a Runge-Kutta scheme of order three in time (as in [BCF04]), which is modified in space in order to avoid the diffusion. The spacial part is explained in more details in the next subsection. We define the convection operator $K(f) = \boldsymbol{u} \cdot \nabla f$, and the scheme reads:

$$\varphi^{n+1} - \varphi^{n+1/2} = -\delta t K(\varphi^{n+1/2}) + \frac{1}{2}\delta t^2 K^2(\varphi^{n+1/2}) - \frac{1}{6}\delta t^3 K^3(\varphi^{n+1/2}),$$

where K^i denotes the composition $\underbrace{K \circ \cdots \circ K}_{i \text{ times}}$.

Space discretization of the convection part

Let us describe now how the convection part (i.e. the operator K) is discretized. We use a centered discretization, with some limiters in order to ensure the L^∞-stability of the scheme, as proposed for example in [GR91], and applied to the Cahn-Hilliard equation by Boyer, Chupin, Fabrie in [BCF04]. Indeed, since values of φ outside the interval $[-1;1]$ do not have any physical meaning, it is crucial that the numerical scheme ensures that φ remains in this interval.

Recall that the explicit Runge-Kutta scheme writes:

$$\varphi^{n+1} = \varphi^{n+1/2} - \delta t K(\varphi^{n+1/2}) + \frac{1}{2}\delta t^2 K^2(\varphi^{n+1/2}) - \frac{1}{6}\delta t^3 K^3(\varphi^{n+1/2}),$$

which becomes, when introducing $L = Id - \delta t K$

$$\varphi^{n+1} = \frac{1}{3}\varphi^{n+1/2} + \frac{1}{2}L(\varphi^{n+1/2}) + \frac{1}{6}L^3(\varphi^{n+1/2}), \tag{5.22}$$

which is a convex combination of $\varphi^{n+1/2}$, $L(\varphi^{n+1/2})$ and $L^3(\varphi^{n+1/2})$. With this form it is easy to verify the positivity of the discretization.

Let us define the following notations: $u = u^+ - u^-$, with $u^\pm \geq 0$. Moreover, let Δ denote in this paragraph the discrete difference operator: $\Delta\varphi_{i+1/2} = \varphi_{i+1} - \varphi_i$. At last, let $\lambda = \delta t/\delta x$. As in [BCF04], we define first the operator for a one-dimensional transport, with a centered discretization:

$$\begin{aligned}(L(\varphi))_i^{\text{centered}} = \varphi_i &- \frac{\lambda}{2}\left(u^+_{i+1/2}\Delta\varphi_{i+1/2} - u^+_{i-1/2}\Delta\varphi_{i-1/2}\right)\\ &- \frac{\lambda}{2}\left(u^-_{i+1/2}\Delta\varphi_{i+1/2} - u^-_{i-1/2}\Delta\varphi_{i-1/2}\right)\\ &- \lambda\left(u^+_{i-1/2}\Delta\varphi_{i-1/2} - u^-_{i+1/2}\Delta\varphi_{i+1/2}\right).\end{aligned}$$

Since this discretization is not L^∞-stable, we use the following modified scheme instead:

$$\begin{aligned}(L(\varphi))_i = \varphi_i &- \frac{\lambda}{2}\left(\theta^+_{i+1/2}u^+_{i+1/2}\Delta\varphi_{i+1/2} - \theta^+_{i-1/2}u^+_{i-1/2}\Delta\varphi_{i-1/2}\right)\\ &- \frac{\lambda}{2}\left(\theta^-_{i+1/2}u^-_{i+1/2}\Delta\varphi_{i+1/2} - \theta^-_{i-1/2}u^-_{i-1/2}\Delta\varphi_{i-1/2}\right)\\ &- \lambda\left(u^+_{i-1/2}\Delta\varphi_{i-1/2} - u^-_{i+1/2}\Delta\varphi_{i+1/2}\right).\end{aligned}$$

The quantities $\theta^+_{i+1/2}$ and $\theta^-_{i+1/2}$ are defined as the ratio of two consecutive gradients denoted by $p_{i\pm1/2}$ and $q_{i\pm1/2}$:

$$\begin{aligned}\theta^+_{i+1/2} &= \theta(p_{i+1/2}), & p_{i+1/2} &= \Delta\varphi_{i-1/2}/\Delta\varphi_{i+1/2},\\ \theta^+_{i-1/2} &= \theta(p_{i-1/2}), & p_{i-1/2} &= \Delta\varphi_{i-3/2}/\Delta\varphi_{i-1/2},\\ \theta^-_{i-1/2} &= \theta(q_{i-1/2}), & q_{i-1/2} &= \Delta\varphi_{i+1/2}/\Delta\varphi_{i-1/2},\\ \theta^-_{i+1/2} &= \theta(q_{i+1/2}), & q_{i+1/2} &= \Delta\varphi_{i+3/2}/\Delta\varphi_{i+1/2}.\end{aligned}$$

There are many choices for the limiter θ such that this scheme is positive (see [GR91]). In the simulations, we use the following limiter[1] (see Figure 5.4):

$$\theta(\xi) = 1 - \min\left(|1-\xi|, \frac{1}{|1-\xi|}\right)(1-\xi).$$

Now we can generalize this scheme for the two-dimensional case (i.e. for the velocity

[1]This limiter is second-order TVD (total variation diminishing).

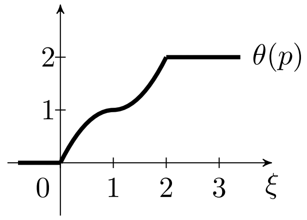

Figure 5.4: Shape of the limiter θ

field $\boldsymbol{u} = (u, v)$):

$$
\begin{aligned}
(L(\varphi))_{i,j} = \varphi_{i,j} &- \frac{\lambda_x}{2}\left(\theta^+_{i+1/2,j}u^+_{i+1/2,j}\Delta\varphi_{i+1/2,j} - \theta^+_{i-1/2,j}u^+_{i-1/2,j}\Delta\varphi_{i-1/2,j}\right)\\
&- \frac{\lambda_x}{2}\left(\theta^-_{i+1/2,j}u^-_{i+1/2,j}\Delta\varphi_{i+1/2,j} - \theta^-_{i-1/2,j}u^-_{i-1/2,j}\Delta\varphi_{i-1/2,j}\right)\\
&- \lambda_x\left(u^+_{i-1/2,j}\Delta\varphi_{i-1/2,j} - u^-_{i+1/2,j}\Delta\varphi_{i+1/2,j}\right)\\
&- \frac{\lambda_z}{2}\left(\theta^+_{i,j+1/2}v^+_{i,j+1/2}\Delta\varphi_{i,j+1/2} - \theta^+_{i,j-1/2}v^+_{i,j-1/2}\Delta\varphi_{i,j-1/2}\right)\\
&- \frac{\lambda_z}{2}\left(\theta^-_{i,j+1/2}v^-_{i,j+1/2}\Delta\varphi_{i,j+1/2} - \theta^-_{i,j-1/2}v^-_{i,j-1/2}\Delta\varphi_{i,j-1/2}\right)\\
&- \lambda_z\left(v^+_{i,j-1/2}\Delta\varphi_{i,j-1/2} - v^-_{i,j+1/2}\Delta\varphi_{i,j+1/2}\right),
\end{aligned}
\tag{5.23}
$$

where $\lambda_x = \delta t/\delta x$, $\lambda_z = \delta t/\delta z$, and the other notations extend naturally the one-dimensional notations.

The following proposition is proved in [BCF04]:

Proposition 5.3. *Under the C.F.L. (Courant-Friedrich-Levy) condition:*

$$\frac{\delta t}{\delta x}\max_{i,j}\left(|u_{i+1/2,j}| + |u_{i-1/2,j}|\right) + \frac{\delta t}{\delta z}\max_{i,j}\left(|v_{i,j+1/2}| + |v_{i,j-1/2}|\right) \leq 1,$$

the scheme defined by (5.22) *and* (5.23) *is a positive scheme and discretizes the equation* $\partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi = 0$.

It remains to apply this scheme for equation (5.11a): it suffices to define the modified velocity field $\boldsymbol{v} = \left(u, \dfrac{v}{h} - \dfrac{zh'u}{h}\right)$, and apply the previous scheme with this velocity field.

Treatment of the boundary conditions

In order to treat the mixed boundary conditions (Dirichlet and Neumann boundary conditions), we introduce artificial unknowns around the physical domain, i.e. we define $\varphi_{0,j}$, $\varphi_{i,0}$, $\varphi_{N+1,j}$, $\varphi_{i,M+1}$ for $1 \leq i \leq N$, $0 \leq j \leq M+1$, and the same for μ (see Figure 5.5).

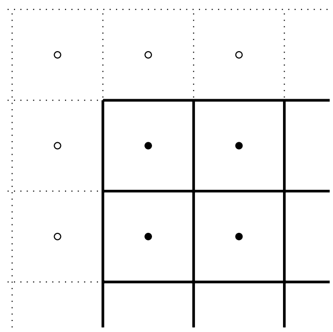

Figure 5.5: Position of the real and fictive unknowns for φ and μ

Furthermore, since we make the computations in a rescaled domain, the boundary conditions (5.3c) also have to be rescaled. They become:

- On $\{z = 0\}$: $\partial_z \varphi = 0$, i.e.

$$\frac{\varphi_{i,1} - \varphi_{i,0}}{\delta z} = 0, \quad \forall\ 1 \leq i \leq N+1.$$

- On $\{x = L\}$: $\left(\partial_x - z\frac{h'}{h} z\partial_z\right)\varphi = 0$, i.e.

$$\frac{\varphi_{N+1,j} - \varphi_{N,j}}{\delta x} - z_j \frac{h'(L)}{h(L)} \frac{\varphi_{N,j+1} - \varphi_{N,j-1} + \varphi_{N+1,j+1} - \varphi_{N+1,j-1}}{4\delta z} = 0,\ \forall 1 \leq j \leq M.$$

- On $\{x = 0\}$: $\varphi = \varphi_l$, i.e.

$$\frac{1}{2}(\varphi_{0,j} + \varphi_{1,j}) = (\varphi_l)_j, \qquad \forall\ 1 \leq j \leq M.$$

Let us observe that this condition is also written at $z = 0$ and $z = 1$, which implies two conditions on $\varphi_{0,0}$ and $\varphi_{0,M+1}$:

$$\begin{aligned}
&\frac{1}{4}(\varphi_{0,0} + \varphi_{1,0} + \varphi_{0,1} + \varphi_{1,1}) = \varphi_l|_{z=0}, \\
&\frac{1}{4}(\varphi_{0,M} + \varphi_{1,M} + \varphi_{0,M+1} + \varphi_{1,M+1}) = \varphi_l|_{z=1}.
\end{aligned}$$

- On $\{z = 1\}$: $-h'\partial_x\varphi + \underbrace{z}_{=1}\frac{h'^2}{h}\partial_z\varphi + \frac{1}{h}\partial_z\varphi = 0$, i.e.

$$-h_i' \frac{\varphi_{i+1,M} - \varphi_{i-1,M} + \varphi_{i+1,M+1} - \varphi_{i-1,M+1}}{4\delta x} + \frac{{h_i'}^2 + 1}{h_i} \frac{\varphi_{i,M+1} - \varphi_{i,M}}{\delta z} = 0,$$
$$\forall 1 \leq i \leq N+1.$$

For the last relation, we use that $\nabla\varphi \cdot \boldsymbol{n}|_{z=0} = 0$ means $-h'\partial_x\varphi + \partial_y\varphi = 0$. Furthermore, using (5.10), the relation becomes $-h'\partial_x\varphi + z\frac{h'^2}{h}\partial_z\varphi + \frac{1}{h}\partial_z\varphi = 0$. In this case, we considered the corners on the right-hand side of the domain to be part of the upper and the lower boundaries (i.e. $\varphi_{N+1,0}$ and $\varphi_{N+1,M+1}$ are determined by the boundary conditions respectively on $\{z = 0\}$ and $\{z = 1\}$), whereas the corners on the left-hand side of the domain are considered to belong to Γ_l (i.e. $\varphi_{0,0}$ and $\varphi_{0,M+1}$ are determined by the boundary conditions on $\{x = 0\}$).

For μ, the corners on the right-hand side are treated as for φ, and the ones on the left-hand side are supposed to be also part of the upper and lower boundaries. Therefore, the same conditions as for φ hold on $\{z = 0\}$ and $\{z = 1\}$ for all $0 \leq i \leq N+1$. The condition on $\Gamma_l = \{x = 0\}$ is $\left(\partial_x - z\frac{h'}{h}\partial_z\right)\varphi = 0$, i.e.

$$\frac{\varphi_{1,j} - \varphi_{0,j}}{\delta x} - z_j\frac{h'(0)}{h(0)}\frac{\varphi_{0,j+1} - \varphi_{0,j-1} + \varphi_{1,j+1} - \varphi_{1,j-1}}{4\delta z} = 0, \ \forall 1 \leq j \leq M.$$

Remark 5.4. *The boundary condition $\varphi|_{\Gamma_l} = \varphi_l$ corresponds to injection boundary conditions. In the algorithm, it is used in two different steps, the diffusion step and the convection step. However, as far as the transport is concerned, since the scheme is not a centered one, this condition is not used anymore when $\boldsymbol{u} \cdot \boldsymbol{n}|_{\Gamma_l} \geq 0$. The diffusion part still uses this boundary condition.*

5.4 Validation of the program

In order to check the validity of the program, we present some tests for the different features of the program.

5.4.1 Lubrication applications

In order to test the Reynolds part of the program, we consider the case of one fluid in a convergent-divergent geometry, which is the geometry used in lubrication applications. For example, this is the case when modelling the flow of a lubricant in the space between bearing rings and rolling elements. In this case, we chose a shear velocity $s = 1$, and the same fluid in all the domain $\varphi \equiv 1$. The input flow corresponds to a Neuman boundary condition on p, i.e. it determines the slope of the pressure curve at $x = 0$. On the figures, this slope remains constant, however, due to the change of scale for the three different figures, it is not clearly visible. We tested three different forms of the domain, for the input flow $q = 0.28$.

✗ For $h(x) = \frac{1}{3}\left((2x-1)^2 + 2\right)$, i.e. $\frac{h_M}{h_m} = \frac{3}{2}$. For such a small ratio, the pressure is always increasing, and we obtain a situation similar to the case of a rectangular domain. Indeed, for $h \equiv 1$, the pressure is linear. Since the value at $x = L$ is fixed ($p(L) = 0$), we obtain a straight line with the given slope at $x = 0$.

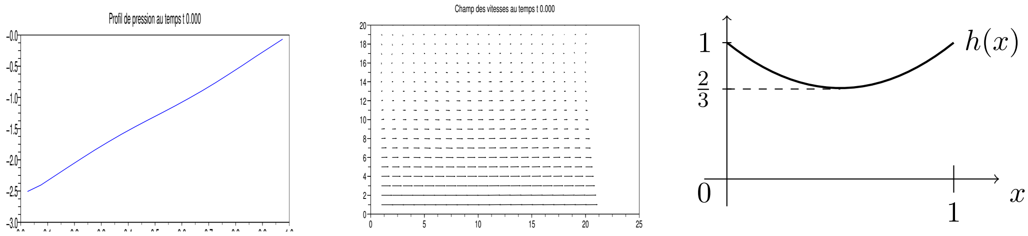

Figure 5.6: Pressure, velocity field and form of the domain for a small ratio $\frac{h_M}{h_m}$

✗ For $h(x) = \frac{2}{3}\left((2x-1)^2 + \frac{1}{2}\right)$, i.e. $\frac{h_M}{h_m} = 3$. In this case, the form of the domain influences the results, and we observe a pressure curve significantly different.

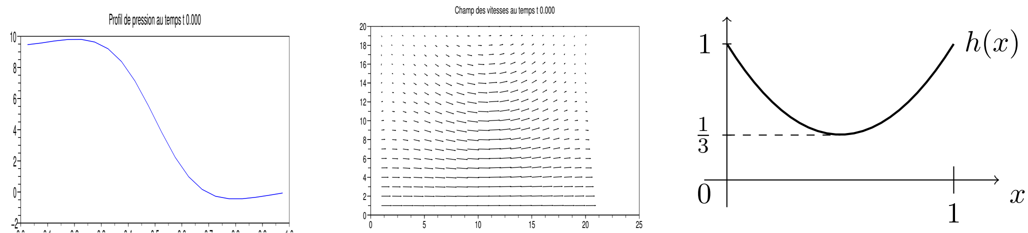

Figure 5.7: Pressure, velocity field and form of the domain for a medium ratio $\frac{h_M}{h_m}$

✗ For $h(x) = \frac{7}{8}\left((2x-1)^2 + \frac{1}{7}\right)$, i.e. $\frac{h_M}{h_m} = 8$. For this test, the pressure has the same form as in the "medium ratio" case, but the pressure values are much higher, which is due to the sharper form of the domain.

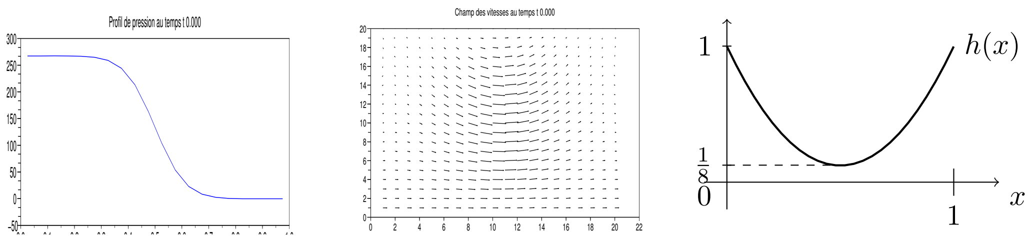

Figure 5.8: Pressure, velocity field and form of the domain for a big ratio $\frac{h_M}{h_m}$

We can also study the influence of the input flow. We consider the same geometry as in Figure 5.7, and we consider different values of q.

✗ For $q = 0.34$, the results are presented in Figure 5.9. The shape of the pressure curve is similar to the one obtained in Figure 5.7, but the pressure values are higher, which is consistent with the fact that the input velocity of the fluid is higher.

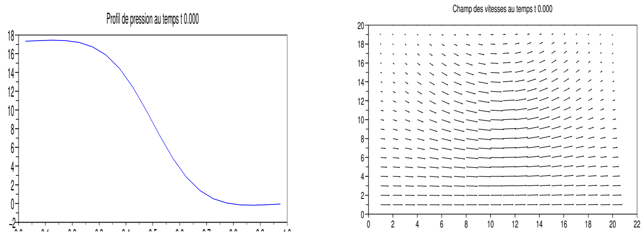

Figure 5.9: Pressure and velocity field for an input flow $q = 0.34$

✗ For $q = 0.22$, the results are given in Figure 5.10. We observe that the shape of the pressure curve is significantly different, and we obtain a classical result in lubrication applications: the pressure is higher in the convergent part of the domain.

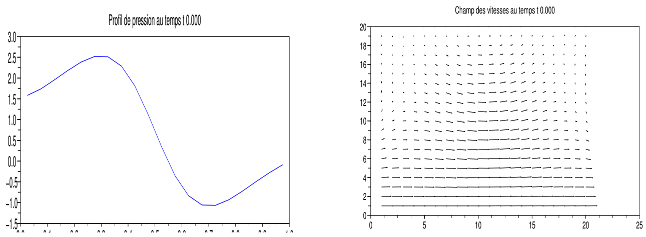

Figure 5.10: Pressure and velocity field for an input flow $q = 0.22$

5.4.2 Influence of the different viscosities

Viscosity is a measure of the resistance of a fluid which is being deformed by either shear stress or extensional stress. This parameter is widely used for characterization of the fluids, and allows us to model different types of behavior for the fluids, even for Newtonian ones (which is the framework of this study). The viscosity values of many materials are well known, since experimental manipulations exist in order to determine the viscosity of a fluid. We give some usual values in Table 5.1.

It is of interest to compare the results obtained in both scenarios, when a drop of a less viscous fluid is immersed in a more viscous one, or when a drop of a more viscous fluid is immersed in a less viscous one. Indeed, the results can vary in a qualitative way.

In order to focus on the influence of the viscosity, we use a simple domain of constant thickness $h \equiv 1$, and we neglect the shear effects by choosing the shear velocity $s = 0$. The surface tension effects are not taken into account, and we choose $\kappa = 0$. The test cases are carried out with the parameter α related to the thickness of the interface chosen

Fluid	Viscosity ($\times 10^{-3}$ Pa·s, at 20 °C)
air	$\sim 1.78\ 10^{-3}$
water	~ 1
blood	~ 1.37
oil	~ 81
honey	~ 5000

Table 5.1: Viscosity values of usual fluids

equal to $\alpha = 0.015$, with an input flow $q = 0.5$. The mesh density is chosen such that $M = N = 65$. The time step δt is adapted from the C.F.L. condition, with $\delta t \leq 0.01$. Thus, we model a situation in which the flow "pushes" the drop in the other fluid, from the left hand side to the right. The two fluids are chosen of viscosities η_2 (in black on the figures), η_1 (in white).

✗ If we want to model for example a drop of oil in water, we choose $\eta_2/\eta_1 = 80$. We obtain the results presented in Figure 5.11. We observe that a viscous drop is not really deformed when immersed in a less viscous fluid.

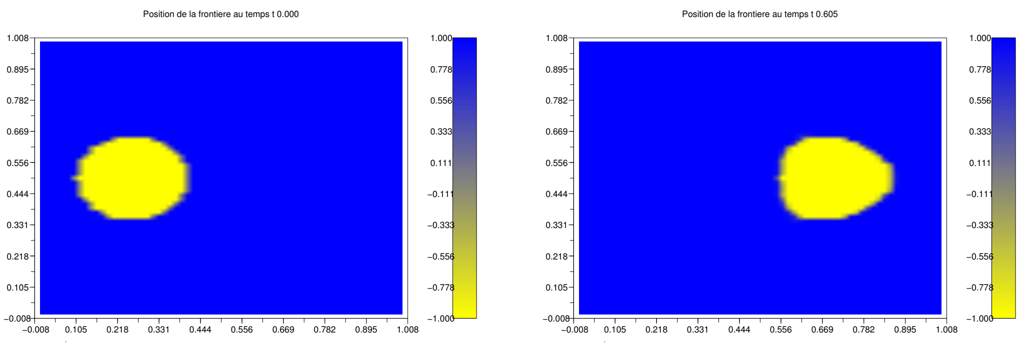

Figure 5.11: A drop of oil (in yellow) in water (in dark blue)

The velocity field is given by Figure 5.12. It is hardly perturbed by the presence of the drop.

✗ On the other hand, choosing $\eta_2/\eta_1 = 1/80$, we model a drop of water in oil. The results are given in 5.13. On the contrary to the previous case, the drop is strongly deformed, independently of the surface tension effects.

The velocity field in figure 5.14 is more perturbed, since the drop is much more deformed.

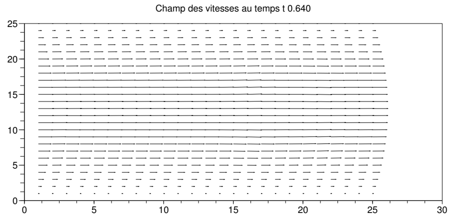

Figure 5.12: Velocity field for a drop of oil in water

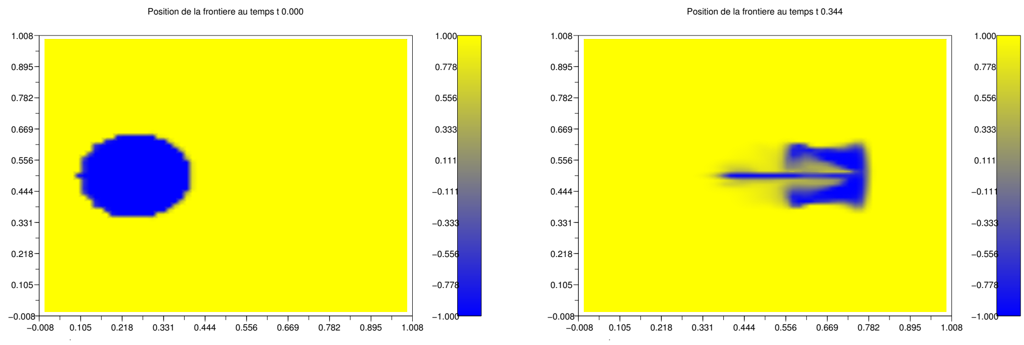

Figure 5.13: A drop of oil (in yellow) in water (in dark blue)

5.4.3 Diffusion part - Anisotropy of the coefficients

Let us recall (see Section 4.2 in Chapter 4) that the asymptotic system (5.2d)-(5.2e) is obtain after a suitable choice of order of magnitude for the coefficients $\mathcal{B}(\varphi)$ (which is the mobility coefficient) and α (which is related to the thickness of the interface). From a mathematical point of view, it is crucial to keep all the derivatives of φ and μ in order to be able to prove some regularity on these quantities. In this case, the equation reads:

$$\partial_t \varphi + \boldsymbol{u} \cdot \nabla\varphi + \partial_x(\mathcal{B}(\varphi)\partial_x\mu) + \partial_z(\mathcal{B}(\varphi)\partial_z\mu) = 0,$$
$$\mu = -\alpha^2 \left(\partial_x^2\varphi + \partial_z^2\varphi\right) + F'(\varphi).$$

However, it seems appropriate from a physical point of view to introduce anisotropy effects in the direction of the thickness of the domain. Therefore, we compare in this section (at least numerically) the results obtained when choosing other orders of magnitude.

- As far as the coefficient $\mathcal{B}(\varphi)$ is concerned, it is related to the friction between the two fluids, and can depend on the geometry of the domain (and thus of the fluid layers).
- As far as the parameter α is concerned, it is related to the thickness of the interface, and can also be related to geometry.

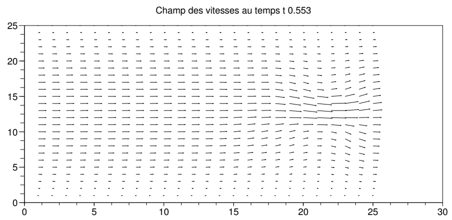

Figure 5.14: Velocity field for a drop of water in oil

We define the two parameters $(a, b) \in [0; 1]^2$, which occur respectively in front of the ∂_x and the ∂_z derivatives in the diffusion terms, so that the system becomes:

$$\begin{aligned} &\partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi + a^2 \partial_x (\mathcal{B}(\varphi) \partial_x \mu) + b^2 \partial_z (\mathcal{B}(\varphi) \partial_z \mu) = 0, \\ &\mu = -\alpha^2 \left(a^2 \partial_x^2 \varphi + b^2 \partial_z^2 \varphi \right) + F'(\varphi). \end{aligned} \tag{5.24}$$

As in the previous subsection, in order to put forward the anisotropy effects, we consider the case of a domain of constant thickness $h \equiv 1$, we neglect the shear effects $s = 0$, and the surface tension effects $\kappa = 0$. Moreover, the thickness of the interface is regulated by $\alpha = 0.015$, and the input flow by $q = 0.1$. The mesh density is chosen such that $M = N = 50$. The time step δt is adapted from the C.F.L. condition, with $\delta t \leq 0.01$. The two fluids are chosen of viscosities $\eta_2 = 1$ (in black on the figures), $\eta_1 = 10$ (in white). First let us point out that numerically, we obtain a solution for any combination of a, b.

✗ With all derivatives (i.e. for $a = 1$, $b = 1$), we obtain the figures presented in Figure 5.15. We observe diffusion effects around the drop, which are of the same order for all directions (and thus on every side of the drop).

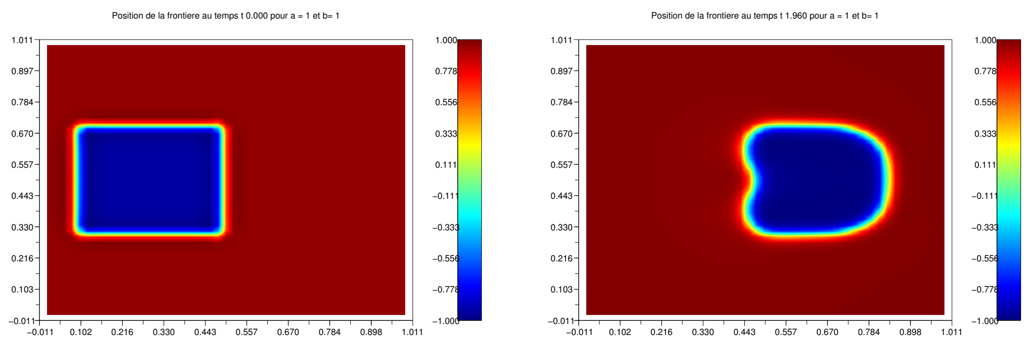

Figure 5.15: Repartition of the fluids for $a = 1$, $b = 1$

✗ When $a = 0$, $b = 1$ (i.e. when keeping only the ∂_z-derivatives in (5.24)), we obtain Figure 5.16. We observe that the lack of diffusion in the horizontal direction leads to the apparition of sharper profiles, due to the influence of the transport part.

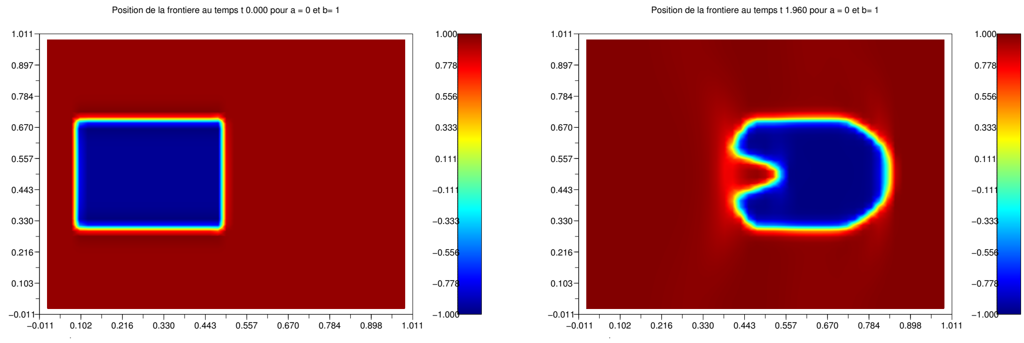

Figure 5.16: Repartition of the fluids for $a = 0$, $b = 1$

✗ When $b = 0$, $a = 1$, the results are presented in Figure 5.17. This case corresponds to keeping only the ∂_x-derivatives, which lacks physical meaning. However, this allows to test if the diffusion in the z-direction disappears. Indeed, the lack of diffusion in the vertical diffusion is clearly visible, since there is no transport in this direction, and thus no diffusion corresponding to the transport scheme.

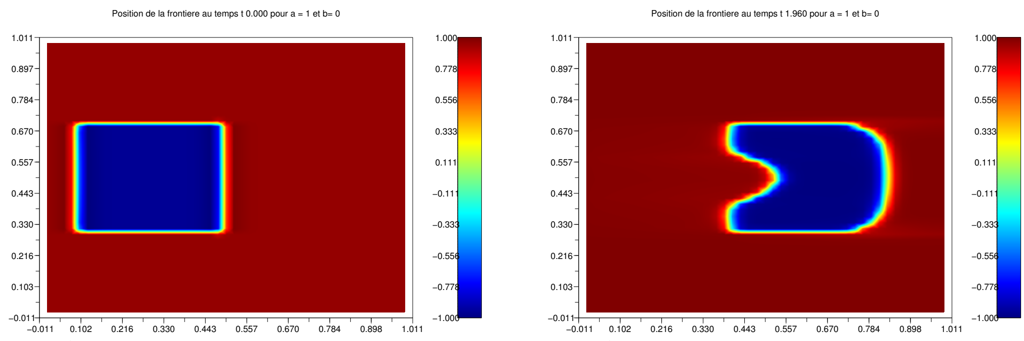

Figure 5.17: Repartition of the fluids for $a = 1$, $b = 0$

5.4.4 Drop transport applications

Another example which allows to validate the program corresponds to the observation of recirculations inside a drop. Indeed, numerical and experimental works [CCG$^+$07], [SLP$^+$06] have showed that due to the blending dynamics, recirculations are observed.

If we compute the relative velocity, we observe recirculations inside the drops, as in Figure 5.18. To this end, we define a mean value of the velocity $\bar{\boldsymbol{u}}$, for example the value on Γ_l (outside the drop), and we compute $\boldsymbol{u} - \bar{\boldsymbol{u}}$, which is represented in the figure. This

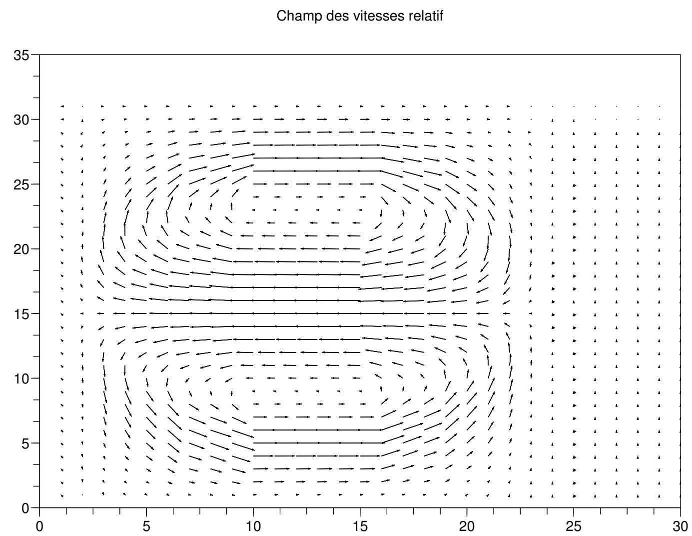

Figure 5.18: Recirculations in a drop

is done with a "big" drop as showed in Figure 5.19 in order to highlight the recirculations.

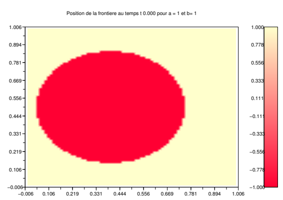

Figure 5.19: Shape of the drop

It is of interest to note that this asymptotic model, which is in fact a very simple one when comparing to the whole Navier-Stokes system coupled with the Cahn-Hilliard equation, allows us nevertheless to observe very fine phenomena, such as recirculations inside a drop.

5.5 Modelling the cavitation

We consider the case of a convergent-divergent geometry, given by

$$h(x) = \frac{2}{3}\left((2x-1)^2 + \frac{1}{2}\right),$$

which corresponds to the geometry used in lubrication applications. The whole system (5.2a)-(5.2e) with the boundary conditions is solved. In order to compare the results to those obtained in [BMV06], we choose the following parameters:

$$q = 0.28, \qquad s = 1,$$

and an injection height equals to $0.45h(0)$. The mesh density is chosen such that $M = N = 35$. The time step δt is adapted from the C.F.L. condition, with $\delta t \leq 0.01$. The viscosity ratio is chosen equal to 1000 ($\eta_1 = 1$, $\eta_2 = 1000$), which corresponds to the ratio for air and water. Let us remark that the pressure is not put in an nondimensional form, since the viscosity remains in the equations. Therefore, the values of the pressure obtained in Figure 5.22 are to relate to the viscosity values. We consider here the case treated in the previous chapter, i.e. the diffusion exists in both x and z directions.

The numerical results are presented in Figure 5.20. The computations are not carried out until a steady state, since the only resolution method implemented for solving the diffusion part is a fixed point method, which is not satisfactory for large times. As a perspective, it would be of interest to implement a second method, e.g. a Newton's algorithm as in [Boy02] to carry out the computations until a steady state.

We observe several features of the the flow. First, we point out that the program makes it possible to have more than two layers of fluids, since there is no hypothesis on the interface to be the graph of a function. Indeed, we observe such a situation. This result was to be expected, but could not be obtained by the previous sharp-interface models [BMV06]. In fact, the velocity field is "negative" in the area where the fluid "returns" (see Figure 5.21).

We observe in Figure 5.22 that in comparison with the values of the pressure observed for large time, the initial pressure is quite constante, and there are two zones of "constant" pressures where there is no saturation. Let us point out that these simulations can capture this physical effect, whereas it was not possible in [BMV06]. This observation is in agreement with physical simulations [Bay], and will be investigated later.

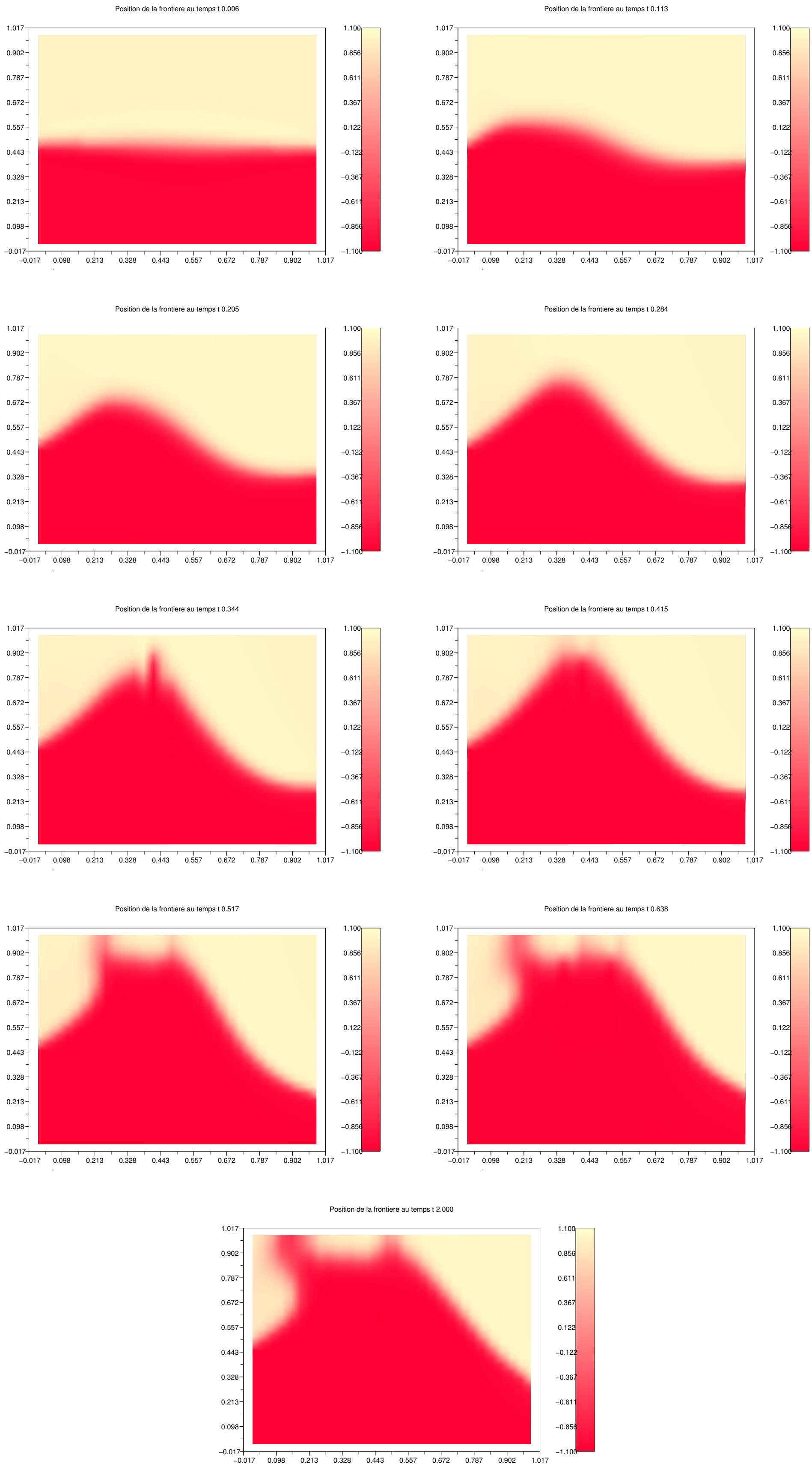

Figure 5.20: Repartition of the two fluids for different times

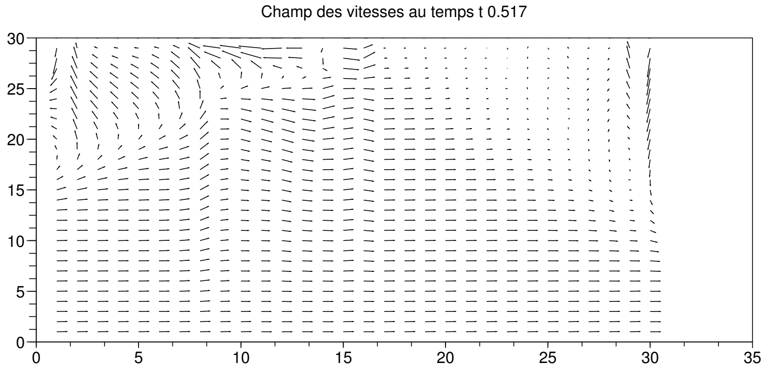

Figure 5.21: Velocity field during the process

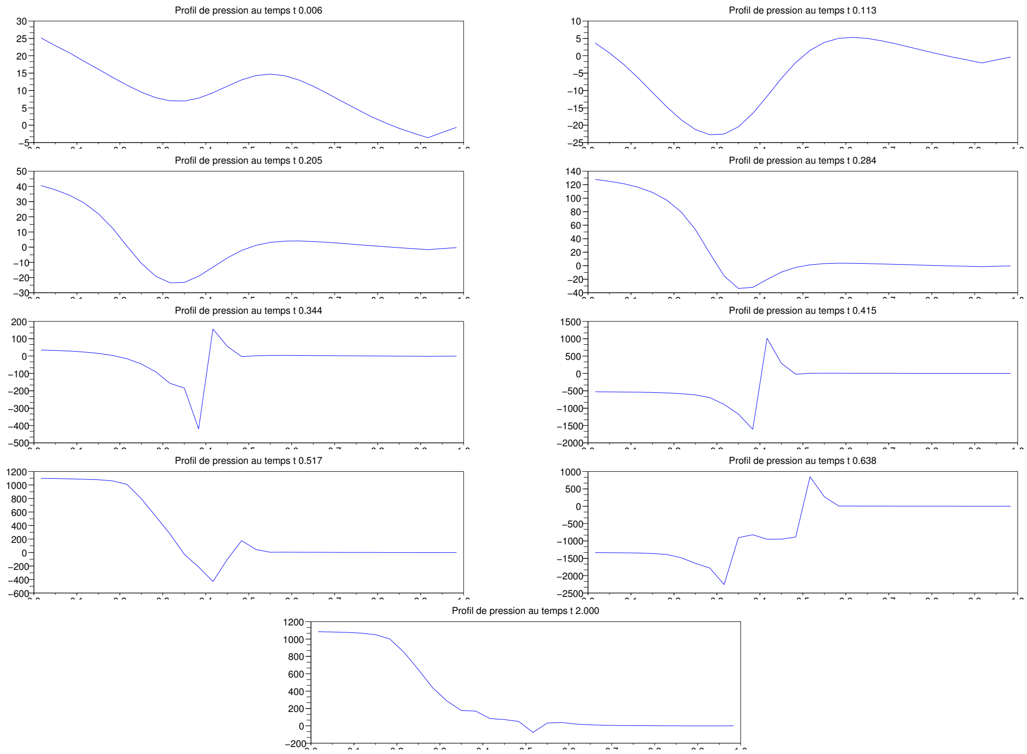

Figure 5.22: Pressure for different times

Bibliography

[Bay] G. Bayada. Communication personnelle.

[BCF04] F. Boyer, L. Chupin, and P. Fabrie. Numerical study of viscoelastic mixtures through a Cahn-Hilliard flow model. *Eur. J. Mech. B Fluids*, 23(5):759–780, 2004.

[BMV06] G. Bayada, S. Martin, and C. Vázquez. About a generalized Buckley-Leverett equation and lubrication multifluid flow. *European J. Appl. Math.*, 17(5):491–524, 2006.

[Boy99] F. Boyer. Mathematical study of multi-phase flow under shear through order parameter formulation. *Asymptot. Anal.*, 20(2):175–212, 1999.

[Boy02] F. Boyer. A theoretical and numerical model for the study of incompressible mixture flows. *Computers and Fluids*, 31(1):41–68, 2002.

[CCG+07] T. Colin, G. Cristobal, C. Galusinski, K. Khadra, and P. Vigneaux. Ecoulement de gouttes dans des microcanaux: simulations numériques et expériences. in *Proceedings du 18e congrès français de mécanique, Grenoble).* 2007.

[CFBT99] J. A. Cuminato, A. C. Filho, M. Boaventura, and M. F. Tomé. Simulation of free surface flows in a distributed memory environment. *J. Comput. Appl. Math.*, 103(1):77–92, 1999. Applied and computational topics in partial differential equations (Gramado, 1997).

[GGL+98] J. Glimm, J. W. Grove, X. L. Li, K.-M. Shyue, Y. Zeng, and Q. Zhang. Three-dimensional front tracking. *SIAM J. Sci. Comput.*, 19(3):703–727 (electronic), 1998.

[GR91] E. Godlewski and P.-A. Raviart. *Hyperbolic systems of conservation laws*, volume 3/4 of *Mathématiques & Applications (Paris) [Mathematics and Applications]*. Ellipses, Paris, 1991.

[Jac99] D. Jacqmin. Calculation of two-phase Navier-Stokes flows using phase-field modeling. *Journal of Computational Physics*, 155:96–127, 1999.

[JLCD01] D. Jamet, O. Lebaigue, N. Coutris, and J. M. Delhaye. The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change. *J. Comput. Phys.*, 169(2):624–651, 2001.

[KKL04] J. Kim, K. Kang, and J. Lowengrub. Conservative multigrid methods for Cahn-Hilliard fluids. *J. Comput. Phys.*, 193(2):511–543, 2004.

[SA99] R. Saurel and R. Abgrall. A multiphase Godunov method for compressible multifluid and multiphase flows. *J. Comput. Phys.*, 150(2):425–467, 1999.

[SLP+06] F. Sarrazin, K. Loubière, L. Prat, C. Gourdon, T. Bonometti, and J. Magnaudet. Experimental and numerical study of droplets hydrodynamics in microchannels. *AIChE Journal*, 52(12):4061–4070, 2006.

[TB01] G. Tryggvason and B. Bunner. Direct numerical simulations of multiphase flows. in *Parallel computational fluid dynamics (Trondheim, 2000)*, pages 77–84. North-Holland, Amsterdam, 2001.

[Vig07] P. Vigneaux. *Méthodes level-set pour des problèmes d'interface en microfluidique*. PhD Thesis, Université Bordeaux I, 2007.

6

Conclusion et Perspectives

Conclusion

Au cours de ce travail, nous nous sommes intéressés à différents problèmes intervenant dans l'étude de fluides complexes en domaines minces, et nous avons obtenu des résultats aussi bien théoriques que numériques pour de tels écoulements.

Dans une première partie, nous avons étudié la pertinence mathématique d'un modèle de fluides visco-élastiques en film mince. En effet, si de nombreux modèles de fluides newtoniens ou non ont été étudiés en film mince, les effets de l'élasticité n'avaient pas été pris en compte. Afin de modéliser le comportement visco-élastique des fluides, nous avons utilisé la loi d'Oldroyd, qui est une loi différentielle sur le tenseur des contraintes. De manière heuristique, un système "limite" correspondant au comportement asymptotique du système Navier-Stokes/Oldroyd lorsque l'épaisseur du domaine tend vers zéro est obtenu. Ce système couple à la fois la vitesse et la pression, à la différence du cas newtonien ou quasi-newtonien, où une équation sur la pression uniquement de type Reynolds est obtenue. Nous avons donc été amenés à étudier la régularité de la solution de ce système. Par ailleurs, nous avons montré la convergence mathématique du système Navier-Stokes/Oldroyd vers l'équation obtenue heuristiquement.

Dans une seconde partie, nous avons introduit dans la modélisation deux autres phénomènes qui interviennent par exemple en lubrification : d'une part, les hautes valeurs de pression observées dans les mécanismes lubrifiés induisent une déformation des surfaces entourant le fluide (aspect élastohydrodynamique), ainsi qu'une variation de la viscosité en fonction de la pression (piezoviscosité). Ces deux caractéristiques ont été prises en compte, en introduisant un couplage fluide-structure par l'intermédiaire de la loi de Hertz ainsi qu'une loi de viscosité variable (par exemple loi de Barus). Enfin, dans de telles conditions de fonctionnement, le phénomène de cavitation doit être pris en compte. La rupture du film de lubrifiant et l'apparition d'une zone de mélange lubrifiant/air sont décrits par le modèle d'Elrod-Adams. L'ajout de ces différentes particularités à l'équation de Reynolds donne un système dont nous avons montré l'existence d'une solution sans l'hypothèse de petitesse sur les données imposée dans des travaux précédents (en particulier, sur la vitesse de cisaillement du fluide, ce qui n'était pas réaliste).

Une troisième partie est dédiée à un autre modèle permettant de prendre en compte l'aspect diphasique d'un écoulement en film mince. Afin de s'affranchir des restrictions liées aux modèles à interface ponctuelle, et pour prendre en compte les effets diffusifs entre deux phases à l'interface, nous avons choisi le modèle de Cahn-Hilliard (avec terme hydrodynamique). Celui-ci fait intervenir un paramètre d'ordre correspondant à la composition du mélange en tout point, par exemple la fraction volumique d'une phase dans le mélange. De manière heuristique, nous avons obtenu un modèle asymptotique pour le système Navier-Stokes/Cahn-Hilliard, qui s'écrit sous la forme d'une équation de Reynolds généralisée (prenant en compte la variation de la viscosité en fonction du paramètre d'ordre) couplée avec l'équation de Cahn-Hilliard. Selon les choix d'adimensionnement considérés, les terme de tension de surface sont ou non conservés dans le modèle limite. Dans les deux cas, nous montrons l'existence d'une solution au système limite. Soulignons qu'afin de décrire des phénomènes d'injection, nous avons choisi des conditions limites originales par rapport aux travaux déjà existants sur le modèle de Cahn-Hilliard, qui modifient le traitement théorique de cette équation.

Enfin, nous avons utilisé le schéma numérique développé par Boyer[1] pour l'équation de Cahn-Hilliard pour le modèle d'écoulements diphasiques en film mince décrit précédemment. Nous avons adapté ce schéma aux conditions limites d'injection, et l'avons couplé avec une discrétisation de la partie Reynolds du système. Nous avons présenté différentes simulations numériques, permettant à la fois de valider le programme et d'observer certaines caractéristiques des écoulements considérés. En particulier, le modèle est utilisé pour simuler le phénomène de cavitation, et permet d'obtenir des profils de répartition du lubrifiant dans l'interstice ne pouvant pas être reproduits avec les modèles de type "interface ponctuelle" utilisés dans des travaux précédents.

Perspectives

Nous proposons quelques développements qui s'inscrivent dans la continuité des travaux présentés dans ce mémoire. Comme nous l'avons signalé en introduction, les problèmes mathématiques survenant dans l'étude d'écoulements en domaines minces sont de deux

[1] F. Boyer, A theoretical and numerical model for the study of incompressible mixture flows, *Computers and Fluids*, 31(1):41–68, 2002.

types :

- la justification rigoureuse de la convergence des équations initiales vers le modèle limite, obtenu généralement heuristiquement ;

- d'autre part, l'étude de ces équations limites, d'un point de vue théorique et numérique.

Nous présentons quelques perspectives dans ces deux directions.

Justification mathématique de modèles limites en films minces

Fluides non-newtoniens

Comme nous l'avons vu dans l'introduction, l'étude en film mince de fluides non-newtoniens "simples" (par exemple quasi-newtoniens) permet d'obtenir un modèle limite qui s'écrit sous la forme d'une équation de Reynolds généralisée. En revanche, nous avons étudié dans le chapitre 2 des écoulements de fluides visco-élastiques en film mince, et mis en évidence le fait que les équations limites obtenues à partir du système Navier-Stokes/Oldroyd ne s'écrivent pas comme une équation sur la pression uniquement (de type Reynolds). Il pourrait être intéressant d'étudier le cas d'autres modèles non-newtoniens, par exemple d'autres lois visco-élastiques de type Phan-Thien-Tanner ou Giesekus. A plus long terme, des modèles plus complexes prenant en compte des effets non seulement macroscopiques mais aussi miscroscopiques pourraient être considérés en film mince, par exemple le modèle FENE, dans la lignée de travaux récents[2].

D'autre part, ce travail a été effectué dans le cadre où le paramètre a intervenant dans l'équation d'Oldroyd est égal à zéro :

$$\lambda(\partial_t \boldsymbol{\sigma} + \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} - W(\boldsymbol{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot W(\boldsymbol{u}) + a(D(\boldsymbol{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot D(\boldsymbol{u})) + \boldsymbol{\sigma} = 2r\eta D(\boldsymbol{u}).$$

Dans le cas où ce paramètre est non nul, l'existence d'une solution globale en temps pour le système Navier-Stokes/Oldroyd n'est pas prouvée. Il serait intéressant de voir si l'étude en film mince ne permet pas d'aborder ce problème d'un point de vue différent. En effet, nous avons introduit les développements asymptotiques suivants :

$$\boldsymbol{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \boldsymbol{u}^* + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \boldsymbol{v}, \qquad p = \frac{1}{\varepsilon^2} p^* + \frac{1}{\varepsilon^2} q, \qquad \boldsymbol{\sigma} = \frac{1}{\varepsilon} \boldsymbol{\sigma}^* + \frac{1}{\varepsilon} \boldsymbol{\tau}.$$

Comme l'existence d'une solution $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$ est connue, il suffit de prouver l'existence

[2] L. Chupin, The FENE model for viscoelastic thin film flows: Justification of new models and applications, soumis.

de $(\boldsymbol{v}, q, \boldsymbol{\tau})$ pour en déduire un résultat d'existence pour $(\boldsymbol{u}, p, \boldsymbol{\sigma})$. Des travaux[3] ont montré l'existence d'une solution locale en temps au système Navier-Stokes/Oldroyd, ou de manière équivalente globale en temps sous l'hypothèse de données petites, pour toute valeur de a. Ces résultats d'existence pourraient être appliqués au problème sur $(\boldsymbol{v}, q, \boldsymbol{\tau})$ pour le cas de données petites (ce qui revient à supposer que le problème en $(\boldsymbol{u}, p, \boldsymbol{\sigma})$ est muni de conditions "bien préparées", c'est-à-dire suffisamment proches de $(\boldsymbol{u}^*, p^*, \boldsymbol{\sigma}^*)$). En effet, le système vérifié par $(\boldsymbol{v}, q, \boldsymbol{\tau})$ a une structure similaire au système Navier-Stokes/Oldroyd, avec de nombreux termes supplémentaires au second membre.

Etude de mélanges

Dans l'étude présentée au chapitre 4, nous avons obtenu de manière heuristique un modèle limite permettant de représenter le comportement de mélanges diphasiques en domaines minces. Cependant, la justification de la convergence du système Navier-Stokes/Cahn-Hilliard n'a pas été effectuée, et serait intéressante d'un point de vue mathématique. De manière similaire à ce qui a été proposé dans le chapitre 2 pour les fluides visco-élastiques (ou dans des travaux antérieurs pour d'autres types de fluides), l'introduction de développements asymptotiques permettrait de se ramener à l'étude d'un système sur les restes, dont il faudrait montrer la convergence vers zéro en des normes adéquates (suffisamment fortes pour conclure de la convergence des termes non-linéaires).

Etude théorique et numérique de modèles limites pour des fluides complexes en film mince

Etude théorique de mélanges diphasiques en film mince

Nous nous sommes intéressés dans le chapitre 4 à des écoulements diphasiques en domaine mince, et plus particulièrement à l'étude du système couplé entre une équation de Reynolds généralisée (prenant en compte la variation de la viscosité en fonction de la composition du mélange) et l'équation de Cahn-Hilliard hydrodynamique. Cette étude a été effectuée dans le cadre de certaines hypothèses, et l'on pourrait considérer des cas plus généraux :

- Le choix des ordres de grandeur dans le processus d'adimensionalisation de l'équation de Cahn-Hilliard peut prêter à controverse. D'un point de vue mathématique, il est en effet utile de conserver les différentes dérivées dans le gradient afin de pouvoir montrer un résultat de régularité satisfaisant. Cependant, la signification physique de ces

[3] E. Fernández-Cara, F. Guillén, et R. R. Ortega, Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):1–29, 1998.

paramètres ne justifie pas complètement ces choix. Il serait intéressant de considérer les équations de Cahn-Hilliard dégénérées, et d'en étudier l'existence d'une solution.

- Nous avons pris en compte le phénomène d'injection dans le choix des conditions limites, en imposant une condition de Dirichlet sur le paramètre d'ordre (c'est-à-dire sur la composition du mélange) sur un des bords du domaine : $\varphi = \varphi_l$ sur Γ_l. Nous avons supposé que cette valeur φ_l devait annuler le potentiel de Cahn-Hilliard (et donc prendre les valeurs $\{-1, 0, 1\}$) ainsi que vérifier une certaine régularité $\varphi_l \in H^{5/2}(\Gamma_l)$. Il serait intéressant de lever cette hypothèse relativement restrictives. Cependant, la définition d'une approximation de Galerkin pour le potentiel chimique μ dans l'étude de l'équation de Cahn-Hilliard est délicate dans le cas général.

- De manière plus générale, cette étude se restreint aux fluides de même densité. Des travaux ont été réalisés[4] dans le cadre du couplage Navier-Stokes/Cahn-Hilliard pour des fluides de densité différentes (mais proches). Cette approche pourrait être appliquée au cas du couplage Reynolds/Cahn-Hilliard. De même, l'hypothèse sur la mobilité imposée dans ce travail (de non-dégénérescence) n'est pas vérifiée d'un point de vue physique, et a été levée dans certains travaux. Des difficultés mathématiques sont engendrées, mais on observe que d'un point de vue théorique, il est possible de montrer que le paramètre d'ordre φ reste dans l'intervalle $[-1, 1]$. A nouveau, il serait intéressant d'aborder cet aspect dans le cadre des écoulements en domaines minces.

Etude numérique de mélanges diphasiques en film mince

L'étude numérique présentée dans le chapitre 5 débouche naturellement sur de nombreuses perspectives :

- Le schéma développé pour la partie diffusive de l'équation de Cahn-Hilliard s'appuie sur un schéma de point fixe, qui converge généralement en peu d'itérations. Cependant, pour les cas où celui-ci ne converge pas rapidement, il serait utile d'implémenter, comme cela avait été fait précédemment[5], un algorithme de Newton pour améliorer l'efficacité de la résolution.

- Il est bien connu que la prise en compte numérique de la tension de surface est un aspect délicat. Une étude de convergence de la méthode proposée, ou l'introduction d'un schéma adapté serait très enrichissant, et permettrait de réaliser des simulations

[4] F. Boyer, Nonhomogeneous Cahn-Hilliard fluids, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(2):225–269, 2001.

[5] F. Boyer, A theoretical and numerical model for the study of incompressible mixture flows, *Computers and Fluids*, 31(1):41–68, 2002.

numériques plus réalistes, par exemple l'étude de la formation de bulles à partir d'un mélange homogène.

- Enfin, le couplage d'un modèle de mélanges tel que le modèle de Cahn-Hilliard avec un modèle non-newtonien de type Oldroyd permettrait de simuler des phénomènes encore mal compris d'un point de vue industriel, par exemple la fabrication de plaques de polymères bicouches (ou multicouches) par l'injection de différents fluides viscoélastiques. Par ailleurs, dans le cadre de telles applications, on pourrait être amené à considérer le cas triphasique, pour lequel des travaux ont été effectués ces dernières années. Bien sûr, les simulations sont actuellement effectuées en deux dimensions, et il serait plus réaliste de travailler dans le cas tridimensionnel.