Two-phase flows in heterogeneous porous media: modeling and analysis of the flows of the effects involved by the discontinuities of the capillary pressure.

Clément Cancès

UNIVERSITÉ DE PROVENCE
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Clément CANCÈS

sous la direction du Pr. Thierry GALLOUËT

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ÉCOULEMENTS DIPHASIQUES EN MILIEUX POREUX HÉTÉROGÈNES : MODÉLISATION ET ANALYSE DES EFFETS LIÉS AUX DISCONTINUITÉS DE LA PRESSION CAPILLAIRE

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JURY

M. Robert EYMARD	Université de Marne-la-Vallée	*Président*
M. Roland MASSON	IFP	*Rapporteur*
M. Michel PIERRE	ENS Cachan	*Rapporteur*
M. Thierry GALLOUËT	Université de Provence	*Directeur*
M. Franck BOYER	Université Paul Cézanne	*Examinateur*
Mme Florence HUBERT	Université de Provence	*Examinateur*

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Résumé

On s'intéresse à l'écoulement d'un mélange d'eau et d'huile dans une matrice poreuse supposée hétérogène, et plus particulièrement apposition de différentes sous-matrices poreuses supposées homogènes. Si la modélisation et l'analyse des écoulements diphasiques dans des milieux poreux homogènes a fait l'objet de nombreuses études préalables, ce travail s'intéresse aux phénomènes liés aux forces provenant de la pression capillaire au niveau des interfaces entre des milieux différents.

Dans un premier temps, on suppose que l'on peut connecter les pressions au niveau des interfaces. Cela nécessite des hypothèses sur les profils de pression capillaire, afin que les raccords soient possibles. On démontre l'existence d'une solution faible du problème parabolique dégénéré obtenu par convergence d'une famille de solutions approchées obtenues à l'aide d'un schéma Volumes Finis. L'unicité est garantie, sous hypothèse sur les dégénérescence, par une méthode de dédoublement de variable aboutissant à un principe de contraction L^1.

La modélisation ne garantit pas forcément que le raccord des pressions capillaires aux interfaces soit possible. Dans le chapitre 3, on donne une condition de raccord graphique des pressions capillaires aux interfaces qui permet de traiter des cas beaucoup plus généraux. On montre que de le problème avec raccords graphiques admet une solution. Un résultat d'unicité et de contraction L^1 est donné dans le cas unidimensionnel.

Dans le chapitre 4, on montre la convergence d'une approximation Volumes Finis vers l'unique solution du problème unidimensionnel. Ce résultat utilise une borne uniforme sur les flux discrets, analogie discrète de la preuve dans le cas continue faite au chapitre précédent.

On étudie dans les chapitres 5 et 6 la limite des solutions lorsque la dépendance de la pression capillaire par rapport à l'inconnue saturation devient très faible, et que la pression capillaire ne dépend plus que du sous milieux poreux homogène. Il apparaît alors des phénomènes différents selon l'orientation des forces de gravité et de capillarité. Soit la solution du problème est la solution entropique d'une équation hyperbolique à flux discontinus, soit une solution faible, entropique à l'intérieur des sous-domaines homogènes, et laissant apparaître un choc non classique à l'interface.

Mots-clefs :
Équation parabolique, milieux poreux, capillarités discontinues, méthode Volumes Finis.

Summary

We consider the flow of a fluid made of oil and water in a heterogeneous porous medium, which is supposed to be a apposition of several homogeneous isotropic porous media. We are interested in the phenomena occurring at the interfaces between the different sub-media, and particularly if the capillary pressure is discontinuous w.r.t. the space.

We first deal with the case where some compatibility relations ensure that the capillary pressure connect in a strong sense. We prove the existence and the uniqueness of the weak solution. The existence result is obtained by proving the convergence of a Finite Volume scheme, while the uniqueness proof is based on the doubling variable technique.

In the following chapter, we deal with the case where no compatibility relation holds, so a new frame for the connection of the pressures is defined, using monotonous graphs. The existence of a weak solution is proven in the multidimensional case, while an uniqueness result, based on a uniform bound on the flux, is given for the one dimensional frame.

The results of the previous chapter are then adapted to numerical methods, that is Finite Volume schemes. It is shown that the discrete solution admits discrete bounded fluxes, and thus that it converge toward the unique solution involving bounded fluxes.

In the chapter 5 and 6, we study the case where the capillary pressure does not depend on the saturation any-more, but only on the porous medium. It is shown that is the orientation of the gravity forces and the capillary forces is the same, some entropy criterion are fulfilled by the solution at the interface. Reversly, if the forces are oriented in opposite senses, then non-classical shock occur at the interface, leading to the entrapment of the hydrocarbons.

keywords :
parabolic equations, porous media, discontinuous capillarity fields, finite volume methods.

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Chapitre 1

Introduction

1.1 Modélisation des bassins sédimentaires

Cette partie est fortement inspirée du manuscrit de thèse de Guillaume Enchéry [Enc04, chapitre 2], et du remarquable travail de synthèse qui y a été opéré.

1.1.1 Bassins sédimentaires et hydrocarbures

L'étude et la simulation de modèles de bassins cherche à prédire les mouvements naturels des hydrocarbures dans le sous-sol, contrairement aux simulations de réservoir qui se concentre sur la phase de récupération des hydrocarbures. Cependant, la physique sous-jacente à ces problèmes est bien évidemment assez proche.

Les hydrocarbures sont des molécules constituées (principalement) d'atomes de carbone et d'hydrogène, de la forme C_nH_m. Par exemple, le méthane CH_4 est constitué d'un seul atome de carbone pour 4 atomes d'hydrogène, et l'octane C_8H_{18} contient 8 atomes de carbone. La famille des hydrocarbures est très grande, et très variée : par exemple, le méthane est très léger et aura tendance à se trouver sous forme gazeuse, alors que des chaînes carbonées plus longues auront tendance à rester sous forme liquide. On pourra consulter [EMMQ87] pour avoir une composition réaliste pour un champ pétrolier. Ces grandes différences entre espèces chimiques se retrouvent sur les propriétés physiques, comme par exemple la masse volumique et la viscosité. Dans la suite de ce manuscrit, cette large variété d'espèces chimique sera négligée.

Si les hydrocarbures circulent dans le sous-sol, de l'eau est aussi présente. Les hydrocarbures sous forme liquide sont non miscibles dans l'eau et donc deux phases liquides sont en présence : une aqueuse, et une huileuse. Uns phase gazeuse doit en plus être considérée, et celle-ci est miscible avec les autres phases. Cependant, dans la suite de ce manuscrit, elle sera négligés, ce qui revient à supposer que la pression est suffisamment élevée pour empêcher les hydrocarbures de se gazéifier.

Le modèle simplifié d'écoulement ne prenant en compte que deux phases liquides, et négligeant la variété des constituants de chaque phase est appelé modèle *dead-oil.* Grâce à l'absence de phase gazeuse, on suppose de plus que chaque phase est incompressible. Si ce modèle simplifié ne saurait rendre compte des situations complexes venant des gisements réels, il permet déjà de laisser apparaître de nombreux phénomènes physiques, et est donc largement étudié (cf. [GMT96],[Che01]).

1.1.2 Approche microscopique

Les roches dites poreuses comportent de l'espace disponible pour un écoulement, par exemple d'eau concernant les nappes phréatiques, ou d'un mélange d'eau et d'hydrocarbures.

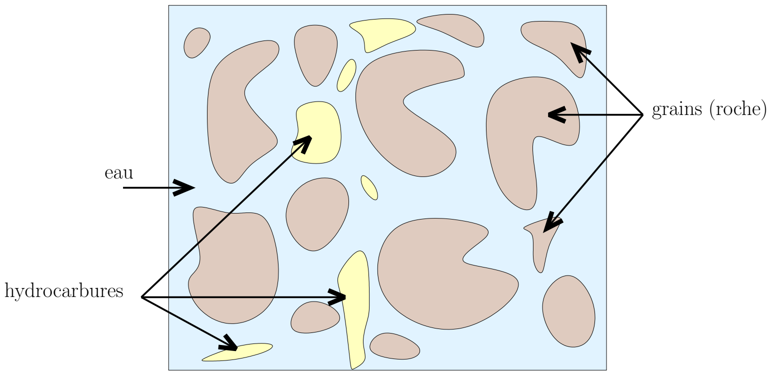

FIG. 1.1 – Exemple de matrice poreuse : la roche constituée de grains laisse de la place pour l'écoulement d'un fluide constitué d'eau et d'hydrocarbure.

Chaque phase a sa pression propre, et à l'interface entre les phases, l'équilibre statique impose la relation suivante entre les pressions : si p_o désigne la pression de la phase huileuse (hydrocarbures) et p_w celle de la phase aqueuse, alors

$$p_o - p_w = \frac{2\gamma_{o,w}}{R} = \frac{2\gamma_{o,w}\cos(\theta)}{r_{pore}}, \tag{1.1}$$

où

- $\gamma_{o,w}$ est la tension superficielle,
- R est le rayon de courbure de l'interface eau huile,
- θ est l'angle de mouillage,
- r_{pore} est le rayon du pore.

Une approche plus complète du problème, prenant en compte aussi des termes dynamiques dans l'équilibre peut-être faite [Pav89].

Les forces en présence sont :

- la poussée d'Archimède liée à la différence de masse volumique entre l'eau et les hydrocarbures, qui sont généralement plus légers,
- la capillarité, exprimée par (1.1),
- les forces de pression au sein du fluide environnant.

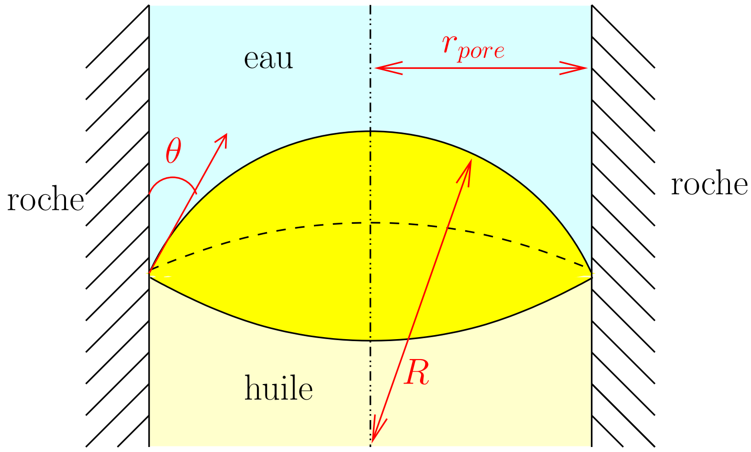

Fig. 1.2 – Interface eau huile dans un pore.

1.1.3 Approche macroscopique

Nous ne sommes pas en mesure d'étudier le problème à l'échelle microscopique, entre autre parce que nous ne savons pas où se trouvent exactement les pores. Un changement d'échelle est alors nécessaire. On considère donc des grandeurs moyennes :

- la porosité ϕ, qui est le ratio volumique de fluide dans la roche.

$$\phi = \frac{\text{Volume de fluide}}{\text{Volume total}}.$$

  La porosité est supposée ne pas dépendre du temps, et on a bien évidemment $\phi(x) \in (0,1)$ pour tout x.
- la saturation en huile u, qui est la ratio volumique d'huile dans le fluide.

$$u = \frac{\text{Volume d'hydrocarbure}}{\text{Volume de fluide}}.$$

  On a bien évidemment aussi $u(x,t) \in [0,1]$ pour tout (x,t).

Il en découle alors que la saturation en eau est $(1-u) \in [0,1]$, que le ratio volumique d'huile dans la roche est ϕu, et le ratio volumique d'eau est $\phi(1-u)$.

Comme chaque phase est supposée incompressible, et comme les deux phases sont supposées non miscibles, alors on peut écrire la conservation volumique de chacune. En désignant par $\vec{v}_\beta$ la vitesse de la phase $\beta = o, w$,

$$\phi\partial_t u + div\,(\phi u \vec{v}_o) = 0, \tag{1.2}$$

$$\phi\partial_t (1-u) + div\,(\phi(1-u)\vec{v}_w) = 0. \tag{1.3}$$

Loi de Darcy monophasique

Cette loi est à l'origine une loi expérimentale [Dar56] qui établit que le débit d'eau Q dans une colonne de sable de hauteur L et de section A est relié à la perte de charge Δh entre le sommet et la base de la colonne, suivant la relation

$$Q = \Lambda A \frac{\Delta h}{L}$$

où Λ est la mobilité du milieu. D'après [dM81], les pertes de charge, pour un écoulement monophasique d'eau sont de la forme

$$\phi \vec{v}_w = \frac{K}{\mu_w} \left(\vec{\nabla} p_w - \rho_w \vec{g} \right)$$

où K désigne le tenseur de perméabilité intrinsèque du milieu, μ_w la viscosité dynamique de l'eau, p_w sa pression, ρ_w sa masse volumique, et $\vec{g}$ est le vecteur gravité. Dans la suite, les milieux poreux seront toujours supposés isotropes, et donc K sera un réel strictement positif.

Loi de Darcy diphasique

Si la loi de Darcy monophasique peut se déduire des équations de Stokes dans un milieu poreux idéalisé (voir [dM81]), on ne sait pas adapter cette preuve pour un fluide diphasique, et pour un milieu poreux général. Une généralisation empirique de la loi de Darcy pour les écoulements diphasiques est alors :

$$\phi u \vec{v}_o = \frac{K k_{r,o}(u)}{\mu_o} \left(\vec{\nabla} p_o - \rho_o \vec{g} \right), \tag{1.4}$$

$$\phi (1-u) \vec{v}_w = \frac{K k_{r,w}(u)}{\mu_w} \left(\vec{\nabla} p_w - \rho_w \vec{g} \right). \tag{1.5}$$

Dans (1.4) et (1.5), les termes correctifs $k_{r,\beta}$ ont été ajouté pour prendre en compte le fait que les deux fluides présents dans le même réseau se gênent mutuellement, l'espace poreux disponible étant réduit. Ces fonctions sont supposées lipschitziennes, positives, monotones, et vérifiant

$$k_{r,o}(u) \leq u, \qquad k_{r,w}(u) \leq (1-u).$$

En utilisant les équations (1.4) et (1.5) dans (1.2) et (1.3), on obtient :

$$\phi \partial_t u + div \left(\frac{K k_{r,o}(u)}{\mu_o} \left(\vec{\nabla} p_o - \rho_o \vec{g} \right) \right) = 0, \tag{1.6}$$

$$-\phi \partial_t u + div \left(\frac{K k_{r,w}(u)}{\mu_w} \left(\vec{\nabla} p_w - \rho_w \vec{g} \right) \right) = 0. \tag{1.7}$$

Loi de pression capillaire

Le système (1.6)-(1.7) admet trois inconnues : u, p_o et p_w pour seulement deux équations. Un relation supplémentaire est nécessaire pour fermer le problème. Cette relation vient de l'équivalent macroscopique de (1.1) :

$$p_o - p_w = \pi(u), \tag{1.8}$$

où π est une fonction croissante supposée lipschitzienne dans la suite. Comme pour (1.1), la relation (1.8) ne prend en compte que les effets statiques. Il est montré dans [Pav89] que la prise en compte de termes dynamiques à l'échelle microscopique conduit à la relation macroscopique

$$p_o - p_w = \pi(u) + \tau\partial_t u.$$

Cette modélisation plus complète a donné lieu à un travail récent [vDPP07].

1.1.4 Réduction du problème à une équation parabolique

Le problème mathématique sortant de la modélisation simplifiée du problème physique dans un milieu homogène peut donc se mettre sous la forme :

$$\phi\partial_t u - div\left(\eta_o(u)(\vec{\nabla}p_o - \rho_o\vec{g})\right) = 0, \tag{1.9}$$

$$-div\left(\sum_{\beta=o,w}\eta_\beta(u)(\vec{\nabla}p_\beta - \rho_\beta\vec{g})\right) = 0, \tag{1.10}$$

$$p_o - p_w = \pi(u). \tag{1.11}$$

Il est clair que l'équation (1.11) permet d'éliminer une pression p_o ou p_w dans l'équation (1.10). Un choix pratique de formulation des pressions est de prendre une sorte de moyenne pondérée entre les deux pressions de phase pour remplacer les deux pressions. On introduit donc la pression globale $\overline{P}$ (cf. [CJ86], [AKM90]).

$$\overline{P} = p_w + \int_0^u \frac{\eta_o(s)}{\eta_o(s)+\eta_w(s)}\pi'(s)ds \tag{1.12}$$

qui permet de réécrire le système (1.9)-(1.10)-(1.11) sous la forme

$$\phi\partial_t u - div\left(\eta_o(u)(\vec{\nabla}\overline{P} - \rho_o\vec{g}) + \frac{\eta_o(u)\eta_w(u)}{\eta_o(u)+\eta_w(u)}\vec{\nabla}\pi(u)\right) = 0, \tag{1.13}$$

$$-div\left(\sum_{\beta=o,w}\eta_\beta(u)(\vec{\nabla}\overline{P} - \rho_\beta\vec{g})\right) = 0. \tag{1.14}$$

Dans la suite de ce manuscrit, nous négligeons les difficultés provenant du couplage de l'équation en pression (1.14) et de l'équation en saturation (1.13), en supposant le profil de pression globale connu, du moins à une constante près. Ce couplage peut néanmoins être traité, au moins dans le cas de domaines homogènes [GMT96] ou aux hétérogénéités régulières [Che01]. Supposons que l'on connaisse le flux volumique total

$$\vec{q} = -\sum_{\beta=o,w}\eta_\beta(u)(\vec{\nabla}\overline{P} - \rho_\beta\vec{g}), \tag{1.15}$$

de manière à ce que l'équation (1.14) se réduise uniquement à $div(\vec{q}) = 0$, alors l'équation en saturation (1.13) peut s'écrire

$$\phi\partial_t u + div\left(\frac{\eta_o(u)}{\eta_o(u)+\eta_w(u)}\vec{q} - \frac{\eta_o(u)\eta_w(u)}{\eta_o(u)+\eta_w(u)}\left(\vec{\nabla}\pi(u) - (\rho_o-\rho_w)\vec{g}\right)\right) = 0. \tag{1.16}$$

Il est intéressant de remarquer que cette réduction ne résulte d'aucune hypothèse dans le cas unidimensionnel : en effet, l'équation (1.14) devient $\partial_x q = 0$, et le débit total q est donc constant en espace, et son comportement temporel est donné par les conditions aux limites.

On se concentrera donc par la suite sur l'étude d'équations quasi-linéaires paraboliques dégénérées de la forme

$$\phi\partial_t u + div\left(\vec{q}f(u) - \lambda(u)\left(\vec{\nabla}\pi(u) - (\rho_o - \rho_w)\vec{g}\right)\right) = 0, \tag{1.17}$$

où $\vec{q}$ est un champ de vecteur régulier à divergence nulle, f est une fonction croissante et lipschitzienne vérifiant $f(0) = 0$, $f(1) = 1$, λ est une fonction lipschitzienne et positive vérifiant $\lambda(0) = \lambda(1) = 0$, $\lambda(s) > 0$ si $s \in]0,1[$, et π est une fonction strictement croissante et lipschitzienne.

1.1.5 Hétérogénéités

La géologie des bassins sédimentaires peut être complexe, différentes couches lithologiques peuvent cohabiter, et des variations très brutales des propriétés physiques de la matrice poreuse par rapport à la variable d'espace doivent être prises en considération dans les modèles.

Si le fait d'introduire des irrégularités sur ϕ ne pose pas de problèmes majeurs tant que l'on suppose qu'il existe $\eta > 0$ tel que $\phi \geq \eta$, par contre, les variations brutales des autres données K, $k_{r,\beta}$ et π induisent des difficultés dans l'analyse du problème, ainsi que dans sa modélisation.

Lois de conservation hyperboliques scalaires à flux discontinus

Dans le cadre de la simulation de réservoirs, les forces liées à la capillarité sont faibles par rapport à celles liées à la convection. Elles sont donc souvent négligées, et on considère alors les solutions faibles entropiques pour des lois de conservation scalaires hyperboliques (cf. [Kru70]), ou paraboliques fortement dégénérées (cf. [Car99]).

L'adaptation de la notion de solution entropique pour des problèmes dont les données sont discontinues en espace a été l'objet de nombreux travaux récents [Tow00], [Tow01], [KRT02a], [KRT02b], [KRT03], [AG03], [SV03], [AJVG04], [Bac04], [AMG05], [Bac05], [BV06], [Jim07]. On renvoie à la définition 5.1 pour le détail de la notion de solution entropique pour un problème à données discontinues.

Piégeage des hydrocarbures

Les hydrocarbures étant en général plus légers que l'eau, la gravité les pousse à remonter vers la surface. Cependant, ils peuvent être arrêtés dans leur progression par des changements de type de roche, et se retrouver piégés par la géométrie de la lithologie (cf. figure 1.3).

Il semble que la pression capillaire joue un rôle très important dans le piégeage des hydrocarbures, et en particulier les discontinuités spatiales de la pression capillaire, mais la compréhension des mécanismes générant ce phénomène de piégeage des hydrocarbures est encore très partielle.

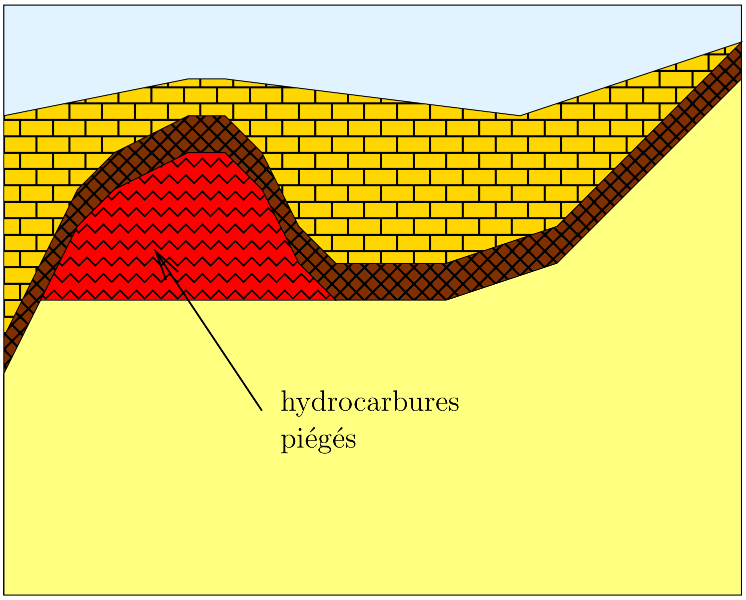

FIG. 1.3 – Un exemple de piège à hydrocarbures.

Si quelques travaux pointent le rôle de la pression capillaire (cf. [vDMdN95], [BDPvD03], [EEM06]), aucune formulation satisfaisante n'a pu être donnée jusqu'à présent. En particulier, les formulation données dans [BDPvD03] et [EEM06] ne permettent pas de traiter des profils de pression capillaire quelconques. Ce manuscrit apporte quelques réponses concernant la modélisation ainsi que l'analyse de ces problèmes de piégeages des hydrocarbures en introduisant des barrières capillaires au niveau des interfaces entres des roches différentes.

1.2 Organisation du manuscrit

La suite du manuscrit est constituée de chapitres correspondant chacun à des travaux publiés, soumis, ou bien en fin de préparation.

1.2.1 Chapitre 2 : Nonlinear Parabolic Equations with Spatial Discontinuities

Dans ce chapitre, on considère un milieux poreux hétérogène Ω constitué d'une apposition de différents sous milieux poreux homogènes isotropes et polygonaux Ω_i ($i \in \{1, \dots, N\}$) qui représentent des lithologies différentes, chacune possédant ses caractéristiques physiques propres (cf. figure 1.4). On considère alors un écoulement diphasique, immiscible et incompressible.

Cherchant à comprendre les mécanismes liés à la pression capillaire, tous les termes convectifs sont négligés, c'est à dire $\vec{q} = 0$ et $\rho_w = \rho_o$. L'équation gouvernant le mouvement de l'huile dans chaque Ω_i est donc de la forme :

$$\phi_i \partial_t u - div\left(\lambda_i(u)\vec{\nabla}\pi_i(u)\right) = \phi_i \partial_t u - \Delta\varphi_i(u) = 0, \qquad (1.18)$$

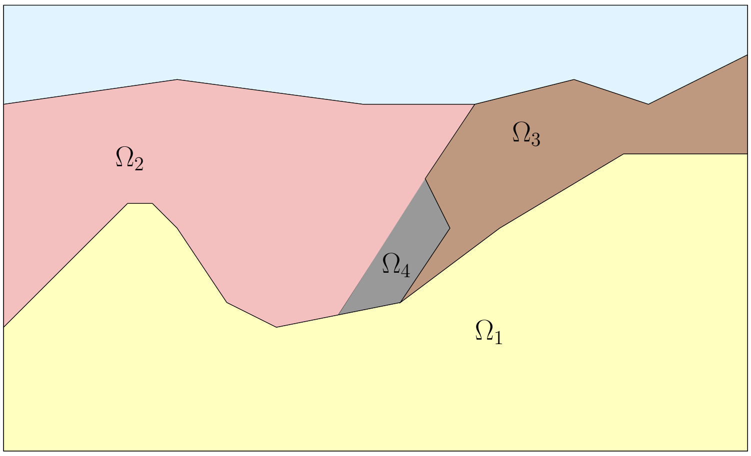

FIG. 1.4 – Un exemple de géométrie pour le domaine Ω.

pour $\varphi_i(u) = \displaystyle\int_0^u \lambda_i(s)\pi_i'(s)ds$.

On note $\Gamma_{i,j}$ l'interface entre Ω_i et Ω_j, et $\vec{n}_i$ la normale sortante à Ω_i, alors la conservation de la masse au niveau de l'interface impose

$$\vec{\nabla}\varphi_i(u)\cdot\vec{n}_i + \vec{\nabla}\varphi_j(u)\cdot\vec{n}_j = 0 \quad \text{sur } \Gamma_{i,j}.$$

On impose aussi le raccord des pressions de chaque phase, et donc celui des pressions capillaires :

$$\pi_i(u_i) = \pi_j(u_j), \tag{1.19}$$

mais cette condition doit être accompagnée de la condition de compatibilité

$$\pi_i(0) = \pi_j(0), \qquad \pi_i(1) = \pi_j(1). \tag{1.20}$$

Il faut signaler que cette condition (1.20) n'est pas forcément vérifiée dans les modèles utilisés par les ingénieurs. Au cours du chapitre 3, on s'affranchira de cette hypothèse.

Le problème à résoudre s'écrit alors

$$\left\{\begin{array}{ll} \phi_i\partial_t u - \Delta\varphi_i(u) = 0 & \text{dans } \Omega_i\times(0,T),\\ \pi_i(u_i) = \pi_j(u_j) & \text{sur } \Gamma_{i,j}\times(0,T),\\ \vec{\nabla}\varphi_i(u)\cdot\vec{n}_i + \vec{\nabla}\varphi_j(u)\cdot\vec{n}_j = 0 & \text{sur } \Gamma_{i,j}\times(0,T),\\ \vec{\nabla}\varphi_i(u)\cdot\vec{n}_i = 0 & \text{sur } (\partial\Omega_i\cap\partial\Omega)\times(0,T),\\ u(\cdot,0) = u_0 & \text{dans } \Omega. \end{array}\right. \tag{1.21}$$

Existence d'une solution faible : convergence d'un schéma volumes finis Afin de prouver l'existence d'une solution faible au problème (1.21), on introduit une approximation volumes finis $u_{\mathcal{D}}$ de la solution u. L'équation (1.18) étant quasi-linéaire, et le problème étant isotrope, on peut utiliser une discrétisation admissible de Ω au sens de la définition 2.3. Avec les notations de cette définition, qui sont aussi essentiellement celles de [EGH00], si K et L sont deux volumes de contrôle voisins, alors la droite liant les centres de maille x_K et x_L est orthogonale à l'arête $K|L$ entre K et L (cf. figure 1.5). De plus, on suppose que chaque volume de contrôle K est inclus dans un sous domaine Ω_i. Enfin, on considère une discrétisation uniforme $(t^n)_n = (n\delta t)_n$ de l'intervalle $(0, T)$.

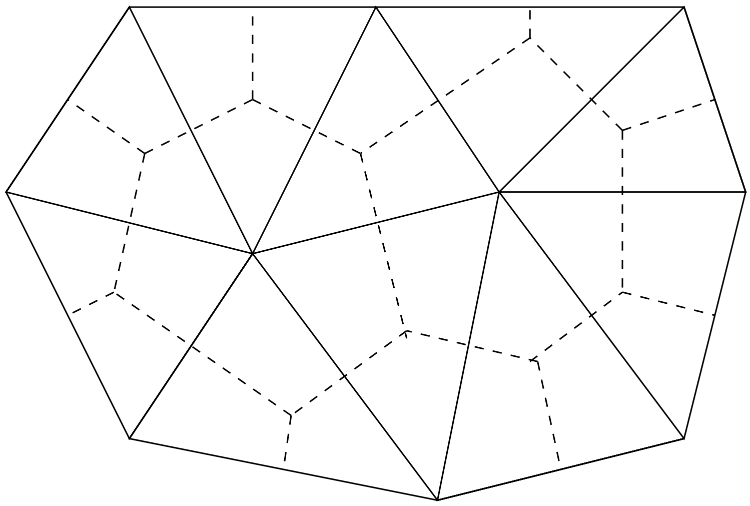

FIG. 1.5 – Un exemple de maillage admissible : triangulation de Delaunay, les centres des mailles sont alors les centres des cercles circonscrits des triangles.

Les inconnues u_K^n sont des approximations des moyennes de u sur la maille K au temps t^n. On choisit de considérer un schéma implicite de la forme :

$$\phi_i \frac{u_K^{n+1} - u_K^n}{\delta t} m(K) + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = 0, \tag{1.22}$$

où les $F_{K,\sigma}^{n+1}$ sont des approximations consistantes des flux sortants de K à travers les arêtes $\sigma \subset \partial K$ au temps t^{n+1} :

$$F_{K,\sigma}^{n+1} \sim \int_\sigma \vec{\nabla} \varphi_i(u_{\mathcal{D}})(x, t^{n+1}) \cdot \vec{n}_{K,\sigma} dx.$$

Les arêtes σ sont soit incluses dans $\partial\Omega$, soit incluses dans un Ω_i, soit incluses dans une interface $\Gamma_{i,j}$.

– $\sigma \subset \partial\Omega$, alors
$$F_{K,\sigma}^n = 0; \tag{1.23}$$

– $\sigma = K|L \subset \Omega_i$, alors
$$F_{K,\sigma}^n = -F_{L,\sigma}^n = m(K|L) \frac{\varphi_i(u_K^n) - \varphi_i(u_L^n)}{d(x_K, x_L)}; \tag{1.24}$$

– $\sigma = K|L$, $K \in \Omega_i$, $L \in \Omega_j$, alors on introduit deux nouvelles inconnues intermédiaires

$u^n_{K,\sigma}$ et $u^n_{L,\sigma}$ au niveau de l'arête σ telles que :

$$\pi_i(u^n_{K,\sigma}) = \pi_j(u^n_{L,s}), \tag{1.25}$$

$$m(K|L)\frac{\varphi_i(u^n_K) - \varphi_i(u^n_{K,\sigma})}{d(x_K,\sigma)} + m(K|L)\frac{\varphi_j(u^n_L) - \varphi_j(u^n_{L,\sigma})}{d(x_L,\sigma)} = 0. \tag{1.26}$$

En choisissant

$$u^0_K = \frac{1}{m(K)}\int_K u_0(x)dx,$$

alors le schéma monotone donné par (1.22), (1.23), (1.24), (1.25) et (1.26) admet une unique solution discrète $u_{\mathcal{D}} \in L^\infty(\Omega \times (0,T))$, avec

$$0 \le u_{\mathcal{D}} \le 1.$$

Des estimations d'énergie sont prouvées, et permettent de prouver la convergence su schéma numérique par des techniques inspirées de [EGH00], et donc l'existence d'une solution faible u au sens de la définition 2.2. En particulier, on démontre deux estimations très importantes pour la suite :

- une estimation $L^2((0,T);H^1(\Omega_i))$ sur $\varphi_i(u)$,

$$\int_0^T \sum_{i=1}^N \int_{\Omega_i} \left(\vec{\nabla}\varphi_i(u)\right)^2 dxdt \le C \max_{i\in\{1,\dots,N\}} \int_0^1 |\pi_i(s)|ds, \tag{1.27}$$

  où C dépend uniquement de λ_i et de Ω,
- L'existence de fonctions Π_i strictement croissantes telle que la fonction $w : (x,t) \mapsto \Pi_i(u)(x,t)$ si $x \in \Omega_i$ appartient à $L^2((0,T);H^1(\Omega))$.

Continuité temporelle d'une solution

Pour traiter la donnée initiale dans la preuve d'unicité de la solution qui suit, on a besoin de prouver l'existence d'une solution faible u telle que

$$\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^1(\Omega)} = 0.$$

Pour cela, on prouve que si u_0 est régulière par morceaux, et sous une hypothèse sur la discrétisation $\mathcal{D}$ de $\Omega \times (0,T)$ de type

$$\frac{\delta t}{\max_K (diam(K)^2)} \le C,$$

alors la solution approchée $u_{\mathcal{D}}$ converge pour la topologie $L^\infty((0,T);L^p(\Omega))$ vers une solution faible $u \in \mathcal{C}([0,T],L^p(\Omega))$ pour $1 \le p < \infty$. Cette démonstration, qui utilise le théorème d'Ascoli sur la solution discrète, fait l'objet de toute la partie 2.2.5. Une autre démonstration de la continuité temporelle est donnée en annexe A, montrant que toute solution faible du problème appartient à $\mathcal{C}([0,T],L^p(\Omega))$.

Unicité de la solution faible

Enfin, l'unicité de la solution faible du problème est prouvée en utilisant un dédoublement de variable en temps (cf. [AL83], [Ott96b]). Cependant, une hypothèse technique apparaît au cours de la démonstration :

$$\forall i \in \{1, \dots, N\}, \qquad \left(\Pi_i \circ \varphi_i^{-1}\right)' \text{ est une fonction lipschitzienne.} \tag{1.28}$$

On utilise durant la preuve de l'unicité deux ingrédients clefs.

1. Si u et v sont deux solutions faibles associées à des données initiales u_0 et v_0, les fonction $w_u : (x,t) \mapsto \Pi_i(u)(x,t)$ si $x \in \Omega_i$ et $w_v : (x,t) \mapsto \Pi_i(ux)(x,t)$ si $x \in \Omega_i$ appartiennent à $L^2((0,T); H^1(\Omega))$. Comme $\partial_t u, \partial_t v \in L^2((0,T); \left(H^1(\Omega)\right)')$, on peut prendre $S_n(w_u - w_v)$ comme fonction test, où S_n est une approximation lipschitzienne de la fonction sign.
2. Les fonctions Π_i sont strictement croissantes, et donc
$$\mathrm{sign}(\Pi_i(u) - \Pi_i(v)) = \mathrm{sign}(u - v).$$

Le résultat principal de ce chapitre peut donc s'énoncer comme suit (voir aussi le théorème 2.1.1) :

Théorème 1.2.1 *Supposons que (1.28) soit vérifiée. Soit $\mathcal{D}$ une discrétisation admissible de $\Omega \times (0,T)$, alors l'unique solution discrète $u_{\mathcal{D}}$ au schéma (1.22), (1.23), (1.24), (1.25) et (1.26) converge dans $L^p(\Omega \times (0,T))$ vers l'unique solution faible u du problème (1.21) pour tout $1 \le p < \infty$.De plus, $u \in \mathcal{C}([0,T]; L^p(\Omega))$. Si u et v sont deux solutions faibles, associées à des données initiales u_0 et v_0 appartenant à $L^\infty(\Omega)$, $0 \le u_0, v_0 \le 1$, alors on a le principe de comparaison et contraction L^1 suivant : $\forall t \in [0,T]$,*

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^{\pm} dx \le \sum_{i=1}^{N} \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^{\pm} dx,$$

$(\cdot)^{\pm}$ désignant les parties positives ou négatives.

Ce travail a été accepté pour publication dans *Nonlinear Differential Equations and Applications (NoDEA).*

1.2.2 Chapitre 3 : Two-phase flows involving capillary barriers in heterogeneous porous media

L'objectif de ce chapitre est de donner un sens aux conditions de transmission de pression aux niveau des interfaces lorsque la condition de compatibilité (1.20) n'est pas vérifiée. On ne peut pas dans ce cas demander le raccord des pressions au sens $\pi_i(u_i) = \pi_j(u_j)$ sur les interfaces et une nouvelle condition doit être donnée. Des premiers résultat sur ce problème ont été donné par Bertsch, Dal Passo et Van Duijn [BDPvD03], puis par Enchéry, Eymard et Michel [EEM06]. Des discontinuités de pression capillaire apparaissent au niveau des interfaces, et il semblerait que ces discontinuités de pression capillaire puissent être à l'origine du piégeage des hydrocarbures.

Dans une première partie de chapitre, on approche les fonctions π_i $(i = 1, \dots, N)$ par des fonctions $\pi_{i,n}$ se raccordant en 0 et 1, tendant vers les fonctions π_i au sens $L^1(0,1)$, ainsi que uniformément sur tout compact de $(0,1)$. Les mobilités globales λ_i sont aussi légèrement modifiées en $\lambda_{i,n}$, afin de s'assurer que $\varphi_{i,n} = \left(s \mapsto \int_0^s \lambda_{i,n}(a)\pi'_{i,n}(a)da\right)$ tende vers φ_i pour la topologie $W^{1,\infty}(0,1)$, et que $\varphi_i([0,1]) = \varphi_{i,n}([0,1])$. D'après le chapitre

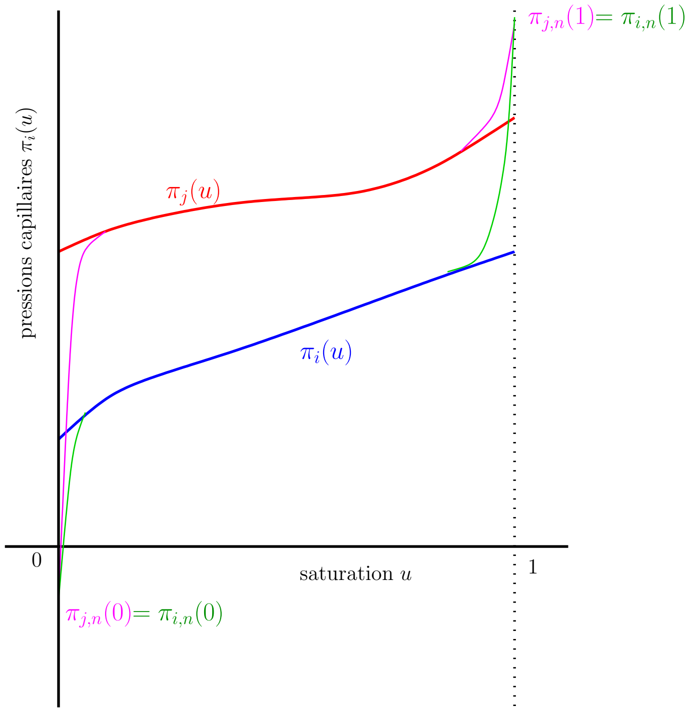

FIG. 1.6 – Courbes de pressions capillaires approchées : les fonctions $\pi_{i,n}$ et $\pi_{j,n}$ tendent dans $L^1(0,1)$ et uniformément sur tout compact de (0,1) vers π_i et π_j, sont régulières, et se raccordent en 0 et en 1.

précédent, le problème

$$\begin{cases} \phi_i\partial_t u_n - \Delta\varphi_{i,n}(u_n) = 0 & \text{dans } \Omega_i \times (0,T), \\ \pi_{i,n}(u_{i,n}) = \pi_{j,n}(u_{j,n}) & \text{sur } \Gamma_{i,j} \times (0,T), \\ \vec{\nabla}\varphi_{i,n}(u_n)\cdot\vec{n}_i + \vec{\nabla}\varphi_{j,n}(u_n)\cdot\vec{n}_j = 0 & \text{sur } \Gamma_{i,j} \times (0,T), \\ \vec{\nabla}\varphi_{i,n}(u_n)\cdot\vec{n}_i = 0 & \text{sur } (\partial\Omega_i \cap \partial\Omega) \times (0,T), \\ u_n(\cdot,0) = u_0 & \text{dans } \Omega \end{cases} \tag{1.29}$$

admet une unique solution faible.

Cette solution faible vérifie

$$0 \le u_n \le 1, \tag{1.30}$$

et d'après (1.27) :

$$\int_0^T \sum_{i=1}^N \int_{\Omega_i} \left(\vec{\nabla}\varphi_{i,n}(u_n)\right)^2 dxdt \le C \max_{i\in\{1,\dots,N\}} \int_0^1 |\pi_{i,n}(s)|ds \le C', \tag{1.31}$$

où C, C' sont indépendants de n. Il découle alors de (1.30) qu'il existe $u \in L^\infty(\Omega \times (0,T))$, $0 \le u \le 1$ tel que

$$u_n \to u \quad \text{pour la topologie } L^\infty(\Omega \times (0,T)) \text{ faible-}\star$$

L'estimation uniforme (1.31) sur les semi-normes $L^2((0,T); H^1(\Omega_i))$ des $\varphi_{i,n}(u_n)$, permet de montrer par une estimation sur les translatés en espace et en temps qu'il existe $f_i \in L^2((0,T); H^1(\Omega_i))$ telle que $\varphi_{i,n}(u_n) \to f_i$ fortement dans $L^2(\Omega_i \times (0,T))$, et faiblement dans $L^2((0,T); H^1(\Omega_i))$. On peut alors montrer, par exemple en utilisant l'astuce de Minty, que

$$f_i = \varphi_i(u) \in L^2((0,T), H^1(\Omega_i)),$$

et comme φ_i^{-1} est continue, u_n converge presque partout vers u quand $n \to \infty$.

La condition de raccord des pressions apparaissant alors à l'interface comme limite de la relation $\pi_{i,n}(u_{i,n}) = \pi_{j,n}(u_{j,n})$ est alors

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset, \tag{1.32}$$

où $\tilde{\pi}_i$ est le graphe monotone donné par

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{pour } s \in (0,1), \\ (-\infty, \pi_i(0)] & \text{pour } s = 0, \\ [\pi_i(1), +\infty) & \text{pour } s = 1. \end{cases}$$

Nous avons donc prouvé l'existence d'une solution faible du problème :

$$\begin{cases} \phi_i \partial_t u - \Delta\varphi_i(u) = 0 & \text{dans } \Omega_i \times (0,T), \\ \tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset & \text{sur } \Gamma_{i,j} \times (0,T), \\ \vec{\nabla}\varphi_i(u) \cdot \vec{n}_i + \vec{\nabla}\varphi_j(u) \cdot \vec{n}_j = 0 & \text{sur } \Gamma_{i,j} \times (0,T), \\ \vec{\nabla}\varphi_i(u) \cdot \vec{n}_i = 0 & \text{sur } (\partial\Omega_i \cap \partial\Omega) \times (0,T), \\ u(\cdot, 0) = u_0 & \text{dans } \Omega \end{cases} \tag{1.33}$$

On vérifie que la relation (1.32) est en fait une extension des relations de raccord de pression basées sur des troncatures introduite dans [BDPvD03] et [EEM06], permettant de supprimer toute relation de compatibilité de type (1.20).

Si la question de l'unicité de la solution faible du problème (1.33) dans le cas multidimensionnel est encore ouverte à notre connaissance, nous donnons une preuve dans le cas unidimensionnel. Le problème étant une loi de conservation, on peut écrire dans $\Omega_i \times (0,T)$,

$$\phi_i \partial_t u + \partial_x F_i = 0,$$

avec

$$F_i = \partial_x \varphi_i(u) = \lambda_i(u) \partial_x \pi_i(u).$$

Formellement, en dérivant F_i par rapport au temps et en utilisant sa définition, on obtient

$$\begin{aligned} \partial_t F_i &= \partial_x \partial_t \varphi_i(u) \\ &= \partial_x \varphi_i'(u) \partial_t u \\ &= \partial_x \left(\frac{1}{\phi_i} \varphi_i'(u) \partial_x F_i \right). \end{aligned}$$

La solution à une telle équation est susceptible de vérifier le principe du maximum pour des conditions aux limites et initiales convenablement choisies. En régularisant le problème, on peut voir que la discontinuité des coefficients $\frac{1}{\phi_i}\varphi_i'(u)$ au niveau des interfaces ne pose pas de grosses difficultés, si ce n'est des régularisations successives sur les données du problème, et les conditions de transmission vont permettre à un tel principe du maximum d'être satisfait par le problème limite, pour peu que le flux initial $F_i(x,0)$ soit borné. On obtient donc deux conditions sur la donnée initiale pour obtenir l'existence d'une solution impliquant des flux bornés :

$$\partial_x\varphi_i(u_0) \in L^\infty(\Omega_i), \tag{1.34}$$
$$\tilde{\pi}_i(u_{0,i}) \cap \tilde{\pi}_j(u_{0,j}) \neq \emptyset. \tag{1.35}$$

Sous ces deux hypothèses, il existe une solution au problème vérifiant

$$\partial_x\varphi_i(u) \in L^\infty(\Omega_i \times (0,T)). \tag{1.36}$$

Cette solution à flux bornés est limite de solutions u_n au problème (1.29).

Nous prouvons ensuite que pour une donnée initiale vérifiant les critères (1.34) et (1.35) ci-dessus, la solution vérifiant (1.36) est unique. En effet, on peut montrer en utilisant la technique du dédoublement de variables que si u et v sont deux solutions à flux bornés associées à des données initiales u_0 et v_0 : pour tout $\psi \in \mathcal{D}^+(\overline{\Omega})$, avec $\psi(x) = 0$ pour $x \in \Gamma_{i,j}$, pour tout $t \in [0,T)$,

$$\begin{aligned}\sum_{i=1}^N \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^\pm \psi(x)dx \leq \sum_{i=1}^N \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^\pm \psi(x)dx \\ - \sum_{i=1}^N \int_0^t \int_{\Omega_i} \partial_x \left(\varphi_i(u)(x,s) - \varphi_i(v)(x,s)\right)^\pm \partial_x\psi(x)dxds.\end{aligned} \tag{1.37}$$

On choisit alors $\psi_\varepsilon(x) = \min\left(1, \frac{d(x,\bigcup\Gamma_{i,j})}{\varepsilon}\right)$ comme fonction test dans (1.37), et montre en utilisant la régularité de la solution (1.36) que pour tout $t \in [0,T)$,

$$\limsup_{\varepsilon\to 0} \sum_{i=1}^N \int_0^t \int_{\Omega_i} \partial_x \left(\varphi_i(u)(x,s) - \varphi_i(v)(x,s)\right)^\pm \partial_x\psi_\varepsilon(x)dxds \geq 0.$$

On en déduit alors le principe de comparaison suivant : Si u et v sont deux solutions à flux bornés correspondant à des données initiales u_0 et v_0, alors :

$$\sum_{i=1}^N \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^\pm dx \leq \sum_{i=1}^N \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^\pm dx. \tag{1.38}$$

On en déduit que pour toute donnée initiale u_0 vérifiant (1.34) et (1.35), on a existence et unicité de la solution à flux borné. Comme l'ensemble des telles conditions initiales est dense dans $L^\infty(\Omega)$ pour la topologie de $L^1(\Omega)$, alors ce résultat d'existence et unicité est étendu aux données initiales $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq 1$ avec une notion de « SOLA »(Solution Obtained as Limit of Approximation).

Ce travail a été fait en collaboration avec Thierry Gallouët et Alessio Porretta de l'université Tor Vergata de Rome, et a été accepté pour publication dans "Interfaces and free boundaries".

1.2.3 Chapitre 4 : Finite volume scheme for two-phase flow in heterogeneous porous media

Dans ce chapitre, on étudie le problème unidimensionnel du chapitre précédent pour deux domaines $\Omega_1 = (-1,0)$ et $\Omega_2 = (0,1)$, $\Gamma = \{0\}$, en rajoutant la convection. On rappelle qu'en dimension 1, le débit total q dans le milieu poreux ne dépend pas de x, et on supposera dans ce chapitre que $q \geq 0$, $q \in BV(0,T)$. Des conditions au limite non homogènes sont prises en compte : condition de Neumann $g \in L^\infty(0,T)$, $0 \leq g \leq q$ en $x = -1$, et $\partial_x \varphi_2(u)(1,\cdot) = 0$. Le problème induit est alors le suivant :

$$\left\{\begin{array}{ll} \phi_i \partial_t u + \partial_x F_i = 0 & \text{dans } \Omega_i \times (0,T), \\ F_i = q f_i(u) - \partial_x \varphi_i(u) & \text{dans } \Omega_i \times (0,T), \\ \tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset & \text{sur } (0,T), \\ F_1(0,\cdot) = F_2(0,\cdot) & \text{sur } (0,T), \\ F_1(-1,\cdot) = g & \text{sur } (0,T), \\ F_2(1,\cdot) = q f_2(u)(1,\cdot) & \text{sur } (0,T). \end{array}\right. \tag{1.39}$$

Il est démontré au chapitre 3 que le cadre des solutions à flux bornés est un bon cadre pour la résolution des problèmes du type (1.39), et que l'existence de telles solutions pouvait être démontré en régularisant tous les paramètres du problème. Les approximation Volumes Finis par des schéma monotones peuvent être vues comme des régularisation visqueuses, et l'idée du chapitre 4 est donc de montrer la convergence de la solution obtenue par le schéma numérique vers l'unique SOLA du problème (1.39) lorsque le pas de discrétisation tend vers 0. Les conditions aux limites choisies permettent de montrer que la solution est bornée entre 0 et 1, et d'obtenir une borne L^∞ sur les flux entrants et sortants.

Schéma Volumes Finis monotone

Afin de simplifier l'étude du problème, on choisit de considérer une discrétisation uniforme de Ω.

– Pour $N \in \mathbb{N}$, on définit les arêtes

$$(x_j)_{-N \leq j \leq N} = (j\delta x)_{-N \leq j \leq N}, \qquad \text{avec } \delta x = 1/N.$$

– Les centres de maille sont alors donnés par

$$\left(x_{j+1/2}\right)_{-N \leq j \leq N-1} = ((j+1/2)\delta x)_{-N \leq j \leq N}.$$

L'interface est alors en $x_0 = 0$. On choisit aussi une discrétisation uniforme de $(0,T)$,

$$t^n = n\delta t, \qquad \text{avec } \delta t = T/M.$$

Les inconnues du problème sont les saturations au centre des mailles $u^n_{i+1/2}$, et les flux à travers les arêtes F^n_i. En définissant

$$q^{n+1} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} q(t)dt, \qquad q^0 = q(0), \qquad g^{n+1} = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} g(t)dt,$$

et

$$u^0_{j+1/2} = \frac{1}{\delta x}\int_{x_i}^{x_{j+1}} u_0(x)dx,$$

les inconnues $\left(u^{n+1}_{j+1/2}\right)$ et $\left(F^{n+1}_j\right)$ sont reliées par les relations :

– $\forall j \in \{-N, \dots, N-1\}, \forall n \in \{0, \dots, M-1\}$,

$$\phi_i \frac{u^{n+1}_{j+1/2} - u^n_{j+1/2}}{\delta t}\delta x + F^{n+1}_{j+1} - F^{n+1}_j = 0 \tag{1.40}$$

où i est choisi tel que $x_{j+1/2} \in \Omega_i$,

– $\forall j \in \{-N+1, -1\} \cup \{1, N-1\}, \forall n \in \{0, \dots, M-1\}$,

$$F^{n+1}_j = q^{n+1} f_i(u^{n+1}_{j-1/2}) - \frac{\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(u^{n+1}_{j-1/2})}{\delta x}, \tag{1.41}$$

– $\forall n \in \{0, \dots, M-1\}$,

$$F^{n+1}_{-N} = g^{n+1}, \qquad F^{n+1}_N = f_2(u^{n+1}_{N-1/2}). \tag{1.42}$$

Comme dans le chapitre 2, on introduit des inconnues supplémentaires au niveau de l'interface, à savoir, $u^n_{0,1}$ et $u^n_{0,2}$ vérifiant : $\forall n \in \{0, \dots, M\}$,

$$\tilde\pi_1(u^n_{0,1}) \cap \tilde\pi_2(u^n_{0,2}) \neq \emptyset, \tag{1.43}$$

$$F^n_0 = q^n f_1(u^n_{-1/2}) - \frac{\varphi_1(u^n_{0,1}) - \varphi_1(u^n_{-1/2})}{\delta x} \tag{1.44}$$

$$= q^n f_2(u^n_{0,2}) - \frac{\varphi_2(u^n_{1/2}) - \varphi_2(u^n_{0,2})}{\delta x}. \tag{1.45}$$

Le système (1.43), (1.44), (1.45) admet une unique solution $(u^n_{0,1}, u^n_{0,2})$, et les fonctions

$$(u^n_{-1/2}, u^n_{1/2}) \mapsto u^n_{0,1}, \qquad (u^n_{-1/2}, u^n_{1/2}) \mapsto u^n_{0,2}$$

sont croissantes.

Le schéma donné par (1.40), (1.41), (1.42), (1.43), (1.44), (1.45) est alors monotone, admet une unique solution discrète $u_\mathcal{D} \in L^\infty(\Omega \times (0,T))$, vérifiant

$$0 \le u_\mathcal{D} \le 1.$$

De plus, si u_0 et v_0 sont deux données initiales auxquelles correspondent les solutions discrètes $u_\mathcal{D}$ et $v_\mathcal{D}$, on a grâce à la monotonie du schéma : $\forall t \in [0,T]$,

$$\sum_{i=1,2}\int_{\Omega_i} \phi_i |u_\mathcal{D}(x,t) - v_\mathcal{D}(x,t)| dx \le \sum_{i=1,2}\int_{\Omega_i} \phi_i |u_0(x) - v_0(x)| dx. \tag{1.46}$$

On a aussi une estimation d'énergie de la forme

$$\sum_{n=1}^{M} \delta t \sum_{j=-N+1}^{N-1} \delta x \left(F^n_j\right)^2 \le C,$$

ce qui permet de prouver la proposition suivante.

Proposition 1.2.2 *Soit $(\mathcal{D}_m)_m$ une suite de discrétisation de $\Omega \times (0,T)$ dont le pas tend vers 0, et soit $(u_{\mathcal{D}_m})_m$ la suite de solutions discrètes correspondantes. Alors à une sous-suite près, $u_{\mathcal{D}_m}$ tend dans $L^p(\Omega \times (0,T))$ $(1 \le p < \infty)$ vers une solution faible u du problème (1.39).*

Convergence vers une solution à flux bornés

Nous ne sommes pas capable de démontrer l'unicité de solutions faibles pour des flux seulement dans $L^2(\Omega \times (0,T))$, et on a vu au chapitre précédent que l'on avait besoin que les flux soient dans $L^\infty(\Omega \times (0,T))$. Ceci n'est possible bien sûr que pour des données initiales assez régulières, à savoir vérifiant

$$\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i \times (0,T)), \tag{1.47}$$

$$\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset. \tag{1.48}$$

où $u_{0,i}$ désigne la trace de $(u_0)_{|\Omega_i}$ sur l'interface $\{x = 0\}$.

Dans ce cas, on peut montrer que les flux discrets initiaux sont bornées, c'est à dire :

$$\max_{-N+1 \le j \le N-1} F_j^0 \le C,$$

où C ne dépend pas du pas de la discrétisation.

En calculant $F_j^{n+1} - F_j^n$, et en utilisant la monotonie du schéma, et en particulier celle des conditions de raccord au niveau des interfaces, on a : $\forall j \in \{-N+1, \ldots, N-1\}, \forall n \in \{0, \ldots, M-1\}$,

$$F_j^{n+1} - F_j^n \le a_{j+1/2}^{n+1/2}\left(F_{j+1}^{n+1} - F_j^{n+1}\right) + b_{j-1/2}^{n+1/2}\left(F_{j-1}^{n+1} - F_j^{n+1}\right) + TV(q) \tag{1.49}$$

$$F_j^{n+1} - F_j^n \ge c_{j+1/2}^{n+1/2}\left(F_{j+1}^{n+1} - F_j^{n+1}\right) + d_{j-1/2}^{n+1/2}\left(F_{j-1}^{n+1} - F_j^{n+1}\right) - TV(q) \tag{1.50}$$

où les coefficient $a_{j+1/2}^{n+1/2}$, $b_{j-1/2}^{n+1/2}$, $c_{j+1/2}^{n+1/2}$ et $d_{j-1/2}^{n+1/2}$ sont positifs et uniformément bornés.

Les conditions aux limites discrètes (1.42) assurent que

$$\left|F_{-N}^{n+1}\right| \le \|q\|_{L^\infty(0,T)}, \qquad \left|F_N^{n+1}\right| \le \|q\|_{L^\infty(0,T)}. \tag{1.51}$$

En combinant (1.49), (1.50) et (1.51), on obtient alors

$$\max_{0 \le n \le M}\left(\max_{-N+1 \le j \le N-1} F_j^n\right) \le C,$$

ce qui permet de prouver la convergence du schéma numérique vers une solution à flux bornés, c'est à dire vérifiant :

$$F_i \in L^\infty(\Omega_i \times (0,T)), \qquad \forall i \in 1, 2.$$

Unicité de la solution à flux bornés

Comme au chapitre précédent, le cadre des solutions à flux bornés est un cadre où l'on a unicité de la solution, et existence sous les hypothèses (1.47) et (1.48). La présence de termes de convection dans le problème (1.39) ne pose pas de problème particulier au cours de la preuve d'unicité, et si u_0 et v_0 sont deux données initiales vérifiant (1.47) et (1.48), les solutions à flux bornés correspondantes vérifient

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^{\pm} dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^{\pm} dx. \tag{1.52}$$

On peut alors affirmer que l'intégralité de la suite des solutions discrètes $(u_{\mathcal{D}_m})$ converge vers l'unique solution à flux bornés u du problème.

Convergence de la solution discrète vers l'unique SOLA
Dans le cas où u_0 n'appartient qu'à $L^\infty(\Omega)$, et ne vérifie donc pas (1.47) et (1.48), il n'existe pas de solution à flux bornés u correspondante, mais le résultat d'unicité peut être étendu en utilisant la notion de SOLA introduite au chapitre précédent. En effet, si $(u_{0,\nu})_\nu$ est une suite de données initiales régulières tendant vers u_0 dans $L^1(\Omega)$, alors l'estimation (1.52) nous assure que la suite des solutions à flux bornés associées $(u_\nu)_\nu$ est de Cauchy dans $\mathcal{C}([0,T];L^1(\Omega))$, et donc converge vers une unique SOLA u.

Si u et v sont deux SOLA associées à des données initiales $u_0, v_0 \in L^\infty(\Omega)$, alors on conserve le principe de comparaison

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^{\pm} dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^{\pm} dx. \tag{1.53}$$

Il est alors facile de vérifier en utilisant (1.46) et (1.53) que la solution discrète $u_{\mathcal{D}}$ converge vers l'unique SOLA u lorsque le pas de maillage tend vers 0

Théorème 1.2.3 *Soit $u_0 \in L^\infty$, et $(\mathcal{D}_m)_m$ une suite de discrétisation de $\Omega \times (0,T)$ dont le pas tend vers 0, et soit $(u_{\mathcal{D}_m})_m$ la suite de solutions discrètes données par le schéma (1.40), (1.41), (1.42), (1.43), (1.44), (1.45), alors*

$$\lim_{m\to\infty} u_{\mathcal{D}_m}(x,t) = u(x,t) \qquad \textit{p.p. dans } \Omega \times (0,T),$$

où u est l'unique SOLA du problème (1.39).

Ce travail a été soumis pour publication, et a donné lieu à un acte pour la conférence Finite Volumes for Complex Applications V [Can08b].

1.2.4 Chapitre 5 : Hyperbolic limit for two-phase flow in heterogeneous porous media with discontinuous capillary forces : convergence toward the entropy solution

Dans ce chapitre, on suppose que le domaine Ω est non borné, $\Omega_1 = \mathbb{R}_-^\star$, $\Omega_2 = \mathbb{R}_+^\star$ afin d'éviter les difficultés liées aux conditions aux limites.

La formulation « graphique » de connexion des pressions capillaires au niveau des interfaces (1.32) permet de traiter une très large classe de fonctions π_i, en particulier toutes les fonctions continues et strictement croissantes. On cherche dans ce chapitre, ainsi que dans le chapitre 6 à comprendre ce qu'il se produit lorsque le problème dégénère, c'est à dire lorsque la pression capillaire ne dépend plus de la saturation :

$$\pi_i(u) = P_i.$$

Soit $\varepsilon > 0$, on introduit les pressions capillaires approchées

$$\pi_i^\varepsilon(u) = P_i + \varepsilon u.$$

Ce choix de fonction π_i^ε a été opéré par simplicité, mais tout autre choix vérifiant π_i^ε strictement croissante, convergeant uniformément vers P_i et telle que $u \mapsto \int_0^u \lambda_i(s)\,(\pi_i^\varepsilon)'(s)ds$ converge uniformément vers 0 convient. On définit $\varphi_i(u) = \int_0^u \lambda_i(s)ds$, si bien que le problème approché considéré peut s'écrire :

$$\begin{cases} \partial_t u^\varepsilon + \partial_x F_i^\varepsilon = 0 & \text{dans } \Omega_i \times (0,T), \\ F_i^\varepsilon = f_i(u) - \varepsilon\partial_x\varphi_i(u) & \text{dans } \Omega_i \times (0,T), \\ \tilde{\pi}_i^\varepsilon(u_i^\varepsilon) \cap \tilde{\pi}_j^\varepsilon(u_j^\varepsilon) \neq \emptyset & \text{sur } (0,T), \\ F_1^\varepsilon(0,\cdot) = F_2^\varepsilon(0,\cdot) & \text{sur } (0,T), \\ u^\varepsilon(\cdot,0) = u_0^\varepsilon & \text{dans } \Omega, \end{cases} \tag{1.54}$$

où u_0^ε est une régularisation bien choisie de la donnée initiale, et f_i sont des fonctions lipschitziennes se raccordant en 0 et en 1 :

$$f_1(0) = f_2(0), \qquad f_1(1) = f_2(1).$$

En utilisant les résultats des deux chapitres précédents on montre que le problème (1.54) admet une unique solution à flux bornés, et que la borne uniforme sur les flux ne dépend pas de ε :

$$\|F_i^\varepsilon\|_{L^\infty(\Omega_i\times(0,T))} \leq C. \tag{1.55}$$

L'objectif de ce chapitre est de montrer que sous une hypothèse $(\mathcal{H})$ page 117 sur les f_i et P_i, qui pourrait se traduire physiquement par

> *« les forces de capillarité et de gravité sont orientées dans le même sens, ou la convection globale est très forte par rapport à celle engendrée par la gravité, »*

alors l'unique solution à flux bornés u^ε au problème (1.54) converge lorsque ε tend vers 0 vers l'unique solution entropique au problème

$$\begin{cases} \partial_t u + \partial_x f_i(u) = 0 & \text{dans } \Omega_i \times (0,T), \\ f_1(u_1) = f_2(u_2) & \text{sur } (0,T), \\ u(\cdot,0) = u_0 & \text{dans } \Omega, \end{cases} \tag{1.56}$$

définie ci dessous.

Définition 1.1 (Solution entropique de (1.56)) *Soit* $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, *alors* $u \in L^\infty(\mathbb{R} \times (0,T))$ *est solution entropique de (1.56) si :* $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times [0,T))$, $\forall \kappa \in [0,1]$,

$$\int_0^T \int_{\mathbb{R}} |u(x,t) - \kappa| \partial_t \psi(x,t) dxdt + \int_{\mathbb{R}} |u_0(x) - \kappa| \psi(x,0) dx$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \operatorname{sign}(u(x,t) - \kappa) \left(f_i(u)(x,t) - f_i(\kappa) \right) \partial_x \psi(x,t) dxdt$$
$$+ \int_0^T |f_1(\kappa) - f_2(\kappa)| \psi(0,t) dt \ge 0. \tag{1.57}$$

L'étude de telles solutions entropique a donné lieu à de nombreux travaux. On mentionnera parmi eux le travail de John D. Towers [Tow00], [Tow01] (voir aussi [KRT02a], [KRT02b], [KRT03]), de Seguin et Vovelle [SV03], de Adimurthi et Veerappa Gowda [AG03], (voir aussi [AJVG04], [AMG05], [AMG07]), de Florence Bachmann durant sa thèse [Bac05], [Bac04], [BV06], ou encore de Julien Jimenez [Jim07].

Une estimation $L^2((0,T); H^1_{loc}(\overline{\Omega}_i))$

Pour démontrer la convergence de la solution approchée u^ε vers une solution entropique u, il faut prouver que

$$\varepsilon \partial_x \varphi_i(u^\varepsilon) \to 0 \qquad \text{p.p. sur } \Omega_i \times (0,T). \tag{1.58}$$

Pour ce faire, une méthode différente de celle des chapitres précédents doit être utilisée pour obtenir une estimation $L^2((0,T); H^1_{loc}(\overline{\Omega}_i))$, et en utilisant l'estimation uniforme sur les flux (1.55), on a : pour tout compact K_i inclus dans $\overline{\Omega}_i$,

$$\varepsilon \int_0^T \int_{K_i} \left(\partial_x \varphi_i(u^\varepsilon) \right)^2 dxdt \le C. \tag{1.59}$$

Le cadre BV, et la convergence presque partout

Afin de prouver la convergence de u^ε vers une solution faible u (qui sera même entropique) de (1.56), il est nécessaire de montrer la convergence presque partout de u^ε vers une fonction u à sous suite près quand ε tend vers 0. Pour cela, on utilise le cadre des solutions à variations bornées introduit par Vol'pert [Vol67], et on montre alors la proposition suivante.

Proposition 1.2.4 *Si* u_0 *appartient à* $BV(\mathbb{R})$, *et* u_0^ε *est une approximation régulière de* u_0, *alors il existe une fonction* $u \in L^\infty(\mathbb{R} \times (0,T))$, $0 \le u \le 1$ *telle que, à une sous-suite près,*

$$u^\varepsilon \to u \qquad \textit{p.p. dans } \mathbb{R} \times (0,T).$$

De plus, il existe des fonctions u_1, u_2 *dans* $L^\infty(0,T)$ *telles que*

$$\lim_{\eta \to 0} \frac{1}{\eta} \int_0^T \int_{-\eta}^0 |u(x,t) - u_1(t)| dxdt = 0.$$

$$\lim_{\eta \to 0} \frac{1}{\eta} \int_0^T \int_0^\eta |u(x,t) - u_2(t)| dxdt = 0.$$

On peut alors vérifier grâce à (1.58) que la limite u des u^ε est solution faible du problème (1.56), et donc on a une condition de Rankine-Hugoniot liant les fonctions u_1 et u_2. La discontinuité étant stationnaire, on a

$$f_1(u_1) = f_2(u_2). \tag{1.60}$$

Convergence vers une solution entropique
Comme u^ε est solution faible d'un problème parabolique dans chaque sous-domaine Ω_i, on a d'après [Car99] : $\forall \kappa \in [0,1]$, $\forall \psi \in \mathcal{D}^+(\Omega_i \times [0,T))$,

$$\begin{aligned}\int_0^T \int_{\Omega_i} |u^\varepsilon(x,t) - \kappa| \partial_t \psi(x,t) dxdt + \int_{\Omega_i} |u_0^\varepsilon(x) - \kappa| \psi(x,0) dx \\ + \int_0^T \int_{\Omega_i} \operatorname{sign}(u^\varepsilon(x) - \kappa)(f_i(u^\varepsilon)(x,t) - f_i(\kappa)) \partial_x \psi(x,t) dxdt \\ -\varepsilon \int_0^T \int_{\Omega_i} \partial_x(\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa)) \partial_x \psi(x,t) dxdt \geq 0.\end{aligned} \tag{1.61}$$

Laissant alors tendre ε vers 0, on obtient grâce à (1.58) et à la proposition 1.2.4

$$\begin{aligned}\int_0^T \int_{\Omega_i} |u(x,t) - \kappa| \partial_t \psi(x,t) dxdt + \int_{\Omega_i} |u_0(x) - \kappa| \psi(x,0) dx \\ + \int_0^T \int_{\Omega_i} \operatorname{sign}(u(x,t) - \kappa)(f_i(u)(x,t) - f_i(\kappa)) \partial_x \psi(x,t) dxdt \geq 0.\end{aligned}$$

Cette inégalité n'est alors valable que pour des fonctions test ψ nulle sur l'interface $\{x = 0\}$. Cependant, au cours des preuves d'unicité de la solutions à flux bornés u^ε présentée aux chapitres 2 et 4, on prouve que des inégalités ressemblant à (1.61) pour des fonctions test ψ non nulles sur l'interface. Si κ^ε est un régime permanent de (1.54), c'est à dire si

$$f_i(\kappa^\varepsilon) - \varepsilon \partial_x \varphi_i(\kappa^\varepsilon) = M,$$

$$\tilde{\pi}_1^\varepsilon(\kappa_1^\varepsilon) \cap \tilde{\pi}_2^\varepsilon(\kappa^\varepsilon) \neq \emptyset,$$

où κ_i^ε est la trace en $x = 0$ de κ^ε du coté Ω_i, alors : $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times [0,T))$,

$$\begin{aligned}\int_0^T \int_{\Omega_i} |u^\varepsilon(x,t) - \kappa^\varepsilon(x)| \partial_t \psi(x,t) dxdt + \int_{\Omega_i} |u_0^\varepsilon(x) - \kappa^\varepsilon(x)| \psi(x,0) dx \\ + \int_0^T \int_{\Omega_i} \operatorname{sign}(u^\varepsilon(x,t) - \kappa^\varepsilon(x))(f_i(u^\varepsilon)(x,t) - f_i(\kappa^\varepsilon)(x)) \partial_x \psi(x,t) dxdt \\ -\varepsilon \int_0^T \int_{\Omega_i} \partial_x(\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa^\varepsilon)(x)) \partial_x \psi(x,t) dxdt \geq 0.\end{aligned} \tag{1.62}$$

On considère l'ensemble Λ des valeurs d'adhérence $\tilde{\kappa}$ pour ε tendant vers 0 des régimes permanents κ^ε, et en passant à la limite dans (1.62), on obtient alors : $\forall \tilde{\kappa} \in \Lambda$, $\forall \psi \in$

$\mathcal{D}^+(\mathbb{R}\times[0,T))$,

$$\int_0^T \sum_{i=1,2}\int_{\Omega_i} |u(x,t)-\tilde{\kappa}(x)|\partial_t\psi(x,t)dxdt + \sum_{i=1,2}\int_{\Omega_i}|u_0(x)-\tilde{\kappa}(x)|\psi(x,0)dx$$
$$+\int_0^T\sum_{i=1,2}\int_{\Omega_i}\mathrm{sign}(u(x,t)-\tilde{\kappa}(x))(f_i(u^\varepsilon)(x,t)-f_i(\tilde{\kappa})(x))\partial_x\psi(x,t)dxdt \geq 0.$$

On prouve alors que cette dernière formulation est équivalente à (1.57) sous l'hypothèse $(\mathcal{H})$ page 117.

Le résultat est ensuite étendu par densité aux données initiales u_0 appartenant à $L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ au théorème 5.3.4.

Pour finir, on montre des résultats numériques confortant la preuve de convergence sous l'hypothèse $(\mathcal{H})$ page 117, ainsi que des exemples où cette dernière est mise en défaut, et où la limite n'est clairement pas entropique.

1.2.5 Chapitre 6 : Occurrence of non classical shocks in modeling of oil-trapping

Nous avons vu au chapitre 5 que la solution u^ε du problème (1.54) tend vers l'unique solution entropique du problème (1.56) lorsque ε tend vers 0 si les forces de capillarité à l'interface sont orientées dans le même sens que la force de gravité. L'objectif de ce chapitre est de démontrer que dans le cas contraire, des chocs non classiques peuvent apparaître à l'interface, c'est à dire des discontinuités ne vérifiant pas les critères d'entropie usuels.

Pour simplifier l'étude, on suppose que le débit total est nul, et donc que

$$f_i(0)=f_i(1)=0.$$

On suppose que les f_i sont positives, c'est à dire que les forces de gravité sont orientée dans le sens des x croissants. On suppose de plus que les f_i sont localement inversibles près de 0 et de 1, et que les inverses locaux sont localement höldériens. On suppose que les forces dues à la capillarité au niveau de l'interface sont orientées dans le sens des x décroissants, ce qui se traduit au niveau de la relation de raccord des pressions, si $\varepsilon>0$ est suffisamment petit, par

$$u_1^\varepsilon = 1 \quad \text{ou} \quad u_2^\varepsilon = 0.$$

Comme la fonction $u^\varepsilon=1$ sur $\mathbb{R}_-$ et $u^\varepsilon=0$ sur $\mathbb{R}_+$ est solution à flux bornés pour tout ε assez petit, il est clair que le problème entretient des chocs non classiques au niveau de l'interface. En fait, nous allons prouver que la discontinuité de la pression capillaire génère de tels chocs pour toute donnée initiale non triviale.

Pour cela, on commence par approcher une donnée initiale quelconque par une famille de données laissant apparaître le choc non classique à l'interface.

Lemma 1.2.5 (condition initiale préparée) *Soit $u_0\in L^1(\mathbb{R})$, $0\leq u_0\leq 1$, pour tout $\eta>0$, il existe $u_{0,\eta}$ tel que*

i) $u_{0,\eta}\in C^\infty(\mathbb{R}^\star)$ à support compact dans $\mathbb{R}$, $0\leq u_{0,\eta}\leq 1$,

ii) $u_{0,\eta}(x)=1$ sur $(-\eta,0)$, et $u_{0,\eta}(x)=0$ sur $(0,\eta)$,

iii) $\lim_{\eta\to 0} u_{0,\eta}=u_0(x)$ dans $L^1(\mathbb{R})$.

Une condition initiale $u_{0,\eta}$ satisfaisant i) et ii) est dite η-préparée. Une condition initiale est dite préparée si elle est η-préparée pour un $\eta > 0$.

On montre alors que pour toute donnée initiale $u_{0,\eta}$ η-préparée, si ε est assez petit,

$$\begin{cases} u^\varepsilon = 1 & \text{sur} \quad (-\eta/2, 0) \times [0, \infty), \\ u^\varepsilon = 0 & \text{sur} \quad (0, \eta/2) \times [0, \infty), \end{cases} \tag{1.63}$$

où u^ε est l'unique solution à flux bornés du problème (1.54) pour la donnée initiale $u_{0,\eta}$. La discontinuité est donc entretenue, et perdure pour tout temps.

Comme au chapitre précédent, on a l'estimation $L^2((0,T); H^1_{loc}(\overline{\Omega}_i))$ (1.59), qui assure la disparition des termes de diffusion :

$$\varepsilon \partial_x \varphi_i(u^\varepsilon) \to 0 \qquad \text{dans } L^2((0,T); L^2_{loc}(\overline{\Omega}_i)). \tag{1.64}$$

L'estimation $0 \leq u^\varepsilon \leq 1$ reste valable, et assure que u^ε converge vers une limite u au sens $L^\infty(\mathbb{R} \times (0, \infty))$ faible-$\star$. Ceci n'est toutefois pas suffisant pour passer à la limite dans les non-linéarités du problème. On utilise alors la notion de solution processus introduite dans [EGGH98], équivalente à la notion de *measure-valued solution* introduite pas DiPerna [DiP85] (voir aussi [Sze91]). Une telle solution processus dépend d'une variable supplémentaire α provenant d'une mesure de Young, et est basée sur le résultat de compacité suivant.

Théorème 1.2.6 (Convergence non-linéaire faible-$\star$) *Soit $\mathcal{Q}$ une borélien de $\mathbb{R}^k$, et soit (u_n) une suite bornée dans $L^\infty(\mathcal{Q})$. Alors il existe une fonction $u \in L^\infty(\mathcal{Q} \times (0,1))$, telle que, à une sous-suite près, u_n tend vers u « au sens non-linéaire faible-$\star$ » quand $n \to \infty$, c'est à dire : $\forall g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$,*

$$g(u_n) \to \int_0^1 g(u(\cdot, \alpha)) d\alpha \qquad \textit{pour la topologie } L^\infty(\mathcal{Q}) \textit{ faible} - \star.$$

Néanmoins, il découle de (1.63) que $u(x,t,\alpha) = 1$ sur $(-\eta/2, 0) \times [0, \infty) \times (0,1)$, et $u(x,t,\alpha) = 0$ sur $(0, \eta/2) \times [0,\infty) \times (0,1)$. En particulier, $u(x,\alpha)$ ne dépend pas de α pour x proche de l'interface, et admet donc une trace $\overline{u}_i$ forte indépendante de α de chaque coté de l'interface.

u^ε est solution d'une équation parabolique dans chaque sous-domaine Ω_i, et comme au chapitre précédent, on a les inégalités 1.61. Si (1.64) nous permet de passer à la limite sur les termes de diffusion, on n'a pas prouvé la convergence presque partout de la suite u^ε, et le passage à la limite sur les nonlinéarités se fait grâce au théorème 1.2.6 ce qui mène à : $\forall \psi \in \mathcal{D}^+(\Omega_i \times [0,T))$, $\forall \kappa \in [0,1]$,

$$\int_0^T \int_{\Omega_i} \int_0^1 (u(x,t,\alpha) - \kappa)^\pm \partial_t \psi(x,t) d\alpha dx dt + \int_{\Omega_i} (u_{0,\eta}(x) - \kappa)^\pm \psi(x,0) dx$$
$$+ \int_0^T \int_{\Omega_i} \int_0^1 \operatorname{sign}_\pm(u(x,t,\alpha) - \kappa)(f_i(u(x,t,\alpha)) - f_i(\kappa)) \partial_x \psi(x,t) d\alpha dx dt \geq 0.$$

Comme u admet des traces $\overline{u}_i$ de part et d'autre de l'interface, on peut alors utiliser des fonctions test non nulles en $x = 0$, et donc : $\forall i = 1, 2$, $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0, T))$, $\forall \kappa \in [0, 1]$,

$$\int_0^T \int_{\Omega_i} \int_0^1 (u(x,t,\alpha) - \kappa)^{\pm} \partial_t \psi(x,t) d\alpha dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^{\pm} \psi(x,0) dx$$
$$+ \int_0^T \int_{\Omega_i} \int_0^1 \mathrm{sign}_{\pm}(u(x,t,\alpha) - \kappa) \left(f_i(u)(x,t,\alpha) - f_i(\kappa)\right) \partial_x \psi(x,t) d\alpha dx dt$$
$$+ M_i \int_0^T \left(u(x,t) - \overline{u}_i\right)^{\pm} \psi(0,t) dt \geq 0, \tag{1.65}$$

où $\overline{u}_1 = 1$, $\overline{u}_2 = 0$, et M_i désigne n'importe quelle constante de Lipschitz de f_i. Une telle fonction u est unique d'après une résultat inspiré de la thèse de Felix Otto [Ott96a], [MNRR96], ainsi que dans celle de Julien Vovelle [Vov02]. De plus, si u et v sont deux solutions de (1.67) associées à des données initiales préparées u_0 et v_0, on a : $\forall t \in [0, \infty)$,

$$\int_{-R}^{R} \int_0^1 \int_0^1 (u(x,t,\alpha) - v(x,t,\beta))^{\pm} d\alpha d\beta dx \leq \int_{-R-M_1T}^{R+M_2T} (u_0(x) - v_0(x))^{\pm} dx. \tag{1.66}$$

En prenant $u_0 = v_0$ dans (1.66), on obtient alors que l'unique solution processus $u(x,t,\alpha)$ ne dépend pas de α, et est donc une solution du problème au sens de la définition suivante.

Définition 1.2 *Soit $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, on dit que u est solution du problème si $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, et : $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0,T))$, $\forall \kappa \in [0,1]$,*

$$\int_0^T \int_{\Omega_i} (u(x,t) - \kappa)^{\pm} \partial_t \psi(x,t) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^{\pm} \psi(x,0) dx$$
$$+ \int_0^T \int_{\Omega_i} \mathrm{sign}_{\pm}(u(x,t) - \kappa) \left(f_i(u)(x,t) - f_i(\kappa)\right) \partial_x \psi(x,t) dx dt$$
$$+ M_i \int_0^T \left(u(x,t) - \overline{u}_i\right)^{\pm} \psi(0,t) dt \geq 0. \tag{1.67}$$

On obtient donc au final le résultat suivant, un argument classique de densité permettant de traiter toute donnée initiale $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$.

Théorème 1.2.7 *Soit $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, soit u^ε l'unique SOLA au problème approché (1.54) associée à la donnée initiale u_0, alors u^ε converge dans $L^p((0,T); L^p_{loc}(\mathbb{R}))$ $(1 \leq p < \infty)$ vers l'unique solution du problème limite u lorsque $\varepsilon \to 0$.*

La solution limite u est donc entropique à l'intérieur de chaque Ω_i, et laisse apparaître une discontinuité stationnaire à l'interface qui ne vérifie pas de critère d'entropie.

Chapitre 2

Nonlinear Parabolic Equations with Spatial Discontinuities

2.1 Presentation of the problem and main results

2.1.1 Presentation of the problem

In this paper, we are interested by the parabolic equation obtained by modeling a two phase flow in a heterogeneous porous media. Let Ω be an open polygonal subset of $\mathbb{R}^d$ ($d \leq 3$), let $T > 0$. One assumes that there exists a finite number of polygonal open subsets $\Omega_i \subset \Omega$, $i \in \{1, ..., N\}$ such that :

$$\bigcup_i \overline{\Omega_i} = \overline{\Omega} \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \;\text{ if }\; i \neq j. \tag{2.1}$$

For all $(i, j) \in \{1, ..., N\}^2$, $i \neq j$, one defines $\Gamma_{i,j}$ the subset of Ω given by :

$$\overline{\Gamma}_{i,j} = \overline{\Omega}_i \cap \overline{\Omega}_j.$$

Each Ω_i will represent a porous media with its own physical characteristics, u will represent the saturation of the oily phase. We aim to solve the problem :

$$\left\{\begin{array}{rcll} \partial_t u - div(\lambda_i(u)\nabla\overline{\pi}_i(u)) & = & 0 & \text{in } \Omega_i \times (0,T), \\ \overline{\pi}_i(u) & = & \overline{\pi}_j(u) & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_i(u)\nabla\overline{\pi}_i(u)\cdot\mathbf{n_i} + \lambda_j(u)\nabla\overline{\pi}_j(u)\cdot\mathbf{n_j} & = & 0 & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_i(u)\nabla\overline{\pi}_i(u)\cdot\mathbf{n_i} & = & 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) & = & u_0(x) & \text{in } \Omega, \end{array}\right. \tag{2.2}$$

where $\overline{\pi}_i$ is an increasing Lipschitz continuous function associated to Ω_i, λ_i is a non negative continuous function with $\lambda_{i|]0,1[} > 0$, the initial data $u_0 \in L^\infty(\Omega)$ with $0 \leq u_0 \leq 1$ a.e. in Ω.

Remark 2.1.1 *We choose to consider only homogeneous Neumann boundary conditions on $\partial\Omega$, but this work can be easily generalized for non-homogeneous Dirichlet conditions, the same way as in [EGHM02, MV03].*

2.1.2 Mathematical definition of the problem

In this part, one defines all the functions necessary to explicit the problem (2.2) and the notion of weak solution.

Particularly, one can associate to each Ω_i two functions, the capillary pressure $\overline{\pi_i}$ and the global mobility λ_i, on which we do the following assumptions :

Assumption 2.1

1. *For all $i \in \{1, ..., N\}$, the function $\overline{\pi}_i : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies :*
   - $\forall s \leq 0, \quad \overline{\pi}_i(s) = \overline{\pi}_i(0)$
   - $\forall s \geq 1, \quad \overline{\pi}_i(s) = \overline{\pi}_i(1)$
   - $\overline{\pi}_{i|[0,1]} \in C^1([0,1], \mathbb{R})$ *is an increasing function*
   - $\forall (i,j) \in \{1, ..., N\}^2, \quad \overline{\pi}_i(0) = \overline{\pi}_j(0)$
   - $\forall (i,j) \in \{1, ..., N\}^2, \quad \overline{\pi}_i(1) = \overline{\pi}_j(1).$

   Let $m_0 \geq 1$. For all $i \in \{1, ..., N\}$, for all $s \in [0,1]$, we might choose $\overline{\pi}_i(s) = \beta_i s^{m_0} + (1-\beta_i) s^{m_i}$ for any $\beta_i \in]0,1]$ and any $m_i > m_0$.
2. *For all $i \in \{1, ...N\}$, the function $\lambda_i : \mathbb{R} \to \mathbb{R}^+$ is continuous and satisfies :*
   - $\forall s \leq 0, \quad \lambda_i(s) = \lambda_i(0)$
   - $\forall s \geq 1, \quad \lambda_i(s) = \lambda_i(1)$
   - $\forall s \in]0,1[, \quad \lambda_i(s) > 0.$

   One can now define a function $\lambda : \bigcup_i \Omega_i \times \mathbb{R} \to \mathbb{R}^+$ by $(x,s) \mapsto \lambda_i(s)$, for all $x \in \Omega_i$, for all $s \in \mathbb{R}$. One denotes $C_\lambda = \max_i(\sup_{s \in \mathbb{R}}(|\lambda_i(s)|))$.

   A classical choice for λ_i is : $\forall i \in \{1, ..., N\}, \forall s \in [0,1], \lambda_i(s) = \alpha_i s(1-s)$ with $\alpha_i > 0$.

We can now define the functions φ_i and Π_i, $i \in \{1, ..., N\}$.

Definition 2.1 *Under Assumptions 2.1, for all $i \in \{1, ..., N\}$, we define :*

$$\varphi_i \; : \; \left\{ \begin{array}{lcl} \mathbb{R} & \to & \mathbb{R}^+ \\ s & \mapsto & \int_0^s \lambda_i(a) \overline{\pi}_i'(a) da \end{array} \right. \tag{2.3}$$

- $\forall i \in \{1, ..., N\}, \; \varphi_{i|[0,1]}$ *is a derivable increasing function*
- $\forall s \leq 0, \; \varphi_i(s) = \varphi_i(0) = 0$
- $\forall s \geq 1, \; \varphi_i(s) = \varphi_i(1).$

We denote by L_φ a Lipschitz constant for all φ_i, $i \in \{1, ..., N\}$.

We also define the function Π_i :

$$\Pi_i \; : \; \left\{ \begin{array}{lcl} \mathbb{R} & \to & \mathbb{R}^+ \\ s & \mapsto & \displaystyle\int_{\overline{\pi}_i(0)}^{\overline{\pi}_i(s)} \sqrt{\min_{j \in \{1,...,N\}} (\lambda_j \circ \overline{\pi}_j^{(-1)}(a)) da} \end{array} \right. \tag{2.4}$$

- $\forall i \in \{1, ..., N\}, \; \Pi_{i|[0,1]}$ *is a derivable increasing function*
- $\forall s \leq 0, \; \Pi_i(s) = \Pi_i(0) = 0$
- $\forall s \geq 1, \; \Pi_i(s) = \Pi_i(1)$
- $\forall (i,j) \in \{1, ..., N\}, \; \forall (s_i, s_j) \in \mathbb{R}, \quad \overline{\pi}_i(s_i) = \overline{\pi}_j(s_j) \Leftrightarrow \Pi_i(s_i) = \Pi_j(s_j).$

We denote by $\Pi(s,x) = \Pi_i(s)$ and $\varphi(s,x) = \varphi_i(s)$ if $x \in \Omega_i$.

Remark 2.1.2 *The last point seen in the previous definition allows us to connect* Π_i *and* Π_j *instead of* $\overline{\pi}_i$ *and* $\overline{\pi}_j$ *on* $\Gamma_{i,j}$.

The definition of φ_i implies that $\partial_t u - \Delta\varphi_i(u) = 0$ in Ω_i, and we can rewrite the transmission conditions on $\Gamma_{i,j}$: $\overline{\pi}_i(u) = \overline{\pi}_j(u)$ and $\nabla\varphi_i(u)\cdot\mathbf{n}_i + \nabla\varphi_j(u)\cdot\mathbf{n}_j = 0$, where $\mathbf{n}_i$ represents the outward normal to Ω_i. We can now define the notion of weak solution for the problem (2.2) :

Definition 2.2 (weak solution) *Under assumptions 2.1, a function u is said to be a weak solution to problem (2.2) if it satisfies :*

1. $u \in L^\infty(\Omega\times(0,T))$, $0 \le u \le 1$ *a.e. in* $\Omega\times(0,T)$,
2. $\forall i \in \{1,...,N\}$, $\varphi_i(u) \in L^2(0,T;H^1(\Omega_i))$,
3. $\Pi(u,\cdot) \in L^2(0,T;H^1(\Omega))$,
4. *for all* $\psi \in \mathcal{D}(\Omega\times[0,T))$,

$$\int_\Omega\int_0^T u(x,t)\partial_t\psi(x,t)dxdt + \int_\Omega u_0(x)\psi(x,0)dx \\ -\sum_{i=1}^N \int_{\Omega_i}\int_0^T \nabla\varphi_i(u(x,t))\cdot\nabla\psi(x,t)dxdt \qquad = 0. \tag{2.5}$$

Remark 2.1.3 *The transmission condition* $\overline{\pi}_i(u) = \overline{\pi}_j(u)$ *on the interface* $\Gamma_{i,j}$ *is now replaced by the point 3 in the previous definition. Because of the lack of regularity on the solution, we cannot write* $\nabla\varphi_i(u)\cdot\mathbf{n}_i + \nabla\varphi_j(u)\cdot\mathbf{n}_j = 0$ *on* $\Gamma_{i,j}$ *in a strong sense. But it is easy to check that, if u is a regular enough weak solution of (2.2), this condition is imposed by point 4 of the previous definition.*

2.1.3 Finite volume approximation and main convergence result

Let us first define space and time discretization of $\Omega\times(0,T)$.

Definition 2.3 *(***Admissible mesh of** Ω*) An admissible mesh of* Ω *is given by a set* $\mathcal{T}$ *of open bounded convex subsets of* Ω *called control volumes, a family* $\mathcal{E}$ *of subsets of* $\overline{\Omega}$ *contained in hyperplanes of* $\mathbb{R}^d$ *with strictly positive measure, and a family of points* $(x_K)_{K\in\mathcal{T}}$ *(the "centers" of control volumes) satisfying the following properties :*

1. $\exists i \in \{1,...,N\}$, $K \subset \Omega_i$. *We denote by* $\mathcal{T}_i = \{K\in\mathcal{T}/K\subset\Omega_i\}$.
2. $\overline{\bigcup_{K\in\mathcal{T}_i} K} = \overline{\Omega_i}$. *Thus,* $\overline{\bigcup_{K\in\mathcal{T}} K} = \overline{\Omega}$.
3. *For any* $K\in\mathcal{T}$, *there exists a subset* $\mathcal{E}_K$ *of* $\mathcal{E}$ *such that* $\partial K = \overline{K}\backslash = \bigcup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. *Furthermore,* $\mathcal{E} = \bigcup_{K\in\mathcal{T}}\mathcal{E}_K$.
4. *For any* $(K,L)\in\mathcal{T}^2$ *with* $K \neq L$, *either the "length" (i.e. the* $(d-1)$ *Lebesgue measure) of* $\overline{K}\cap\overline{L}$ *is 0 or* $\overline{K}\cap\overline{L} = \overline{\sigma}$ *for some* $\sigma\in\mathcal{E}$. *In the latter case, we shall write* $\sigma = K|L$, *and*
   - $\mathcal{E}_i = \{\sigma\in\mathcal{E},\ \exists(K,L)\in\mathcal{T}_i^2,\ \sigma = K|L\}$
   - $\mathcal{E}_{ext} = \{\sigma\in\mathcal{E},\ \sigma\subset\partial\Omega\}$, $\mathcal{E}_{ext,i} = \{\sigma\in\mathcal{E},\ \sigma\subset\partial\Omega_i\cap\partial\Omega\}$
   - $\mathcal{E}_{\Gamma_{i,j}} = \{\sigma\in\mathcal{E},\ \exists(K,L)\in\mathcal{T}_i\times\mathcal{T}_j,\ \sigma = K|L\}$

– $\mathcal{F} = \bigcup_{i,j} \mathcal{E}_{\Gamma_{i,j}}$.
For any $i \in \{1, ..., N\}$, *for any* $K \in \mathcal{T}_i$, *we shall denote by :*
– $\mathcal{N}_K = \{L \in \mathcal{T},\ \exists \sigma \in \mathcal{E},\ \sigma = K|L\}$
– $\mathcal{N}_{K,i} = \{L \in \mathcal{T}_i,\ \exists \sigma \in \mathcal{E}_i,\ \sigma = K|L\}$
– $\mathcal{F}_K = \{L \in \mathcal{T},\ \exists j \neq i,\ \exists \sigma \in \mathcal{E}_{\Gamma_{i,j}},\ \sigma = K|L\}$
– $\mathcal{E}_{K,i} = \mathcal{E}_K \cap \mathcal{E}_i$
– $\mathcal{E}_{K,ext} = \mathcal{E}_K \cap \mathcal{E}_{ext}$.

5. *The family of points* $(x_K)_{K\in\mathcal{T}}$ *is such that* $x_K \in K$ *(for all* $K \in \mathcal{T}$*) and, if* $\sigma = K|L$, *it is assumed that the straight line* (x_K, x_L) *is orthogonal to* σ.

For a control volume $K \in \mathcal{T}$, *we will denote by* $m(K)$ *its measure and* $\mathcal{E}_{ext,K} = \mathcal{E}_K \cap \partial\Omega$. *For all* $\sigma \in \mathcal{E}$,*we denote by* $m(\sigma)$ *the* $(d-1)$*-Lebesgue measure of* σ. *If* $\sigma \in \mathcal{E}_K$, *we denote by* $\tau_{K,\sigma}$ *the transmissibility of* K *through* σ, *defined by* $\tau_{K,\sigma} = \frac{m(\sigma)}{d(x_K,\sigma)}$. *We also define* $\tau_{K|L} = \frac{m(\sigma)}{d(x_K,x_L)}$. *The size of the mesh is defined by :*

$$size(\mathcal{T}) = \max_{K\in\mathcal{T}} diam(K),$$

and a geometrical factor, linked with the regularity of the mesh, is defined by

$$reg(\mathcal{T}) = \max_{K\in\mathcal{T}}(card(\mathcal{E}_K), \max_{\sigma\in\mathcal{E}_K} \frac{diam(K)}{d(x_K,\sigma)}).$$

Remark 2.1.4 *For all* $\sigma \in \mathcal{E}$, $\dfrac{1}{\tau_{K|L}} = \dfrac{1}{\tau_{K,\sigma}} + \dfrac{1}{\tau_{L,\sigma}}$.

Definition 2.4 (Uniform time discretization of $(0,T)$) *A uniform time discretization of* $(0,T)$ *is given by an integer value* M *and a sequence of non-negative real values* $(t^n)_{n=0,...,M+1}$. *We define* $\delta t = \frac{T}{M+1}$ *and,* $\forall n \in \{0, ..., M\},\ t^n = n\delta t$. *Thus we have* $t^0 = 0$ *and* $t^{M+1} = T$.

Remark 2.1.5 *We can easily prove all the results of this paper for general time discretizations, but for the sake of simplicity, we choose to consider only uniform time discretizations.*

Definition 2.5 (Space-time discretization of $\Omega\times(0,T)$) *A finite volume discretization* $\mathcal{D}$ *of* $\Omega \times (0,T)$ *is the family*

$$\mathcal{D} = (\mathcal{T}, \mathcal{E}, (x_K)_{K\in\mathcal{T}}, N, (t^n)_{n\in\{0,...,M\}}),$$

where $(\mathcal{T}, \mathcal{E}, (x_K)_{K\in\mathcal{T}})$ *is an admissible mesh of* Ω *in the sense of Definition 2.3 and* $(N, (t^n)_{n\in\{0,...,M\}})$ *is a discretization of* $(0,T)$ *in the sense of Definition 2.4. For a given mesh* $\mathcal{D}$, *one defines :*

$$size(\mathcal{D}) = max(size(\mathcal{T}), \delta t), \qquad and\ reg(\mathcal{D}) = reg(\mathcal{T}).$$

We will now define the finite volume discretization of problem (2.2). Let $\mathcal{D}$ be a finite volume discretization of $\Omega \times (0,T)$ in the sense of Definition 2.5. The initial condition is discretized by :

$$U_K^0 = \frac{1}{m(K)} \int_K u_0(x)dx, \quad \forall K \in \mathcal{T}. \tag{2.6}$$

An **implicit finite volume scheme** for the discretization of problem (2.2) is given by the following set of nonlinear equations, whose discrete unknowns are $U = (U_K^{n+1})_{K\in\mathcal{T},n\in\{0,...,M\}}$: $\forall K \in \mathcal{T}, \forall n \in \{0,...,M\}$

$$m(K)\frac{U_K^{n+1} - U_K^n}{\delta t} + \sum_{\sigma\in\mathcal{E}_{K,i}} \tau_{K,\sigma}(\varphi(U_K^{n+1}, x_K) - \varphi(U_{K,\sigma}^{n+1}, x_K))$$
$$+ \sum_{\sigma\in\mathcal{F}_K} \tau_{K,\sigma}(\varphi(U_K^{n+1}, x_K) - \varphi(U_{K,\sigma}^{n+1}, x_K)) = 0. \tag{2.7}$$

where $\forall L \in \mathcal{N}_K,\ U_{K,K|L}^{n+1}, U_{L,K|L}^{n+1}$ are the only values in $[0,1]$ that satisfy the transmission conditions :

$$\begin{cases} \tau_{K,\sigma}(\varphi(U_K^{n+1}, x_K) - \varphi(U_{K,\sigma}^{n+1}, x_K)) + \tau_{L,\sigma}(\varphi(U_L^{n+1}, x_L) - \varphi(U_{L,\sigma}^{n+1}, x_L)) = 0 \\ \Pi(U_{K,\sigma}^{n+1}, x_K) = \Pi(U_{L,\sigma}^{n+1}, x_L). \end{cases} \tag{2.8}$$

Definition 2.6 *Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0,T)$ in the sense of Definition 2.5. The approximate solution of problem (2.2) associated to the discretization $\mathcal{D}$ is defined almost everywhere in $\Omega \times (0,T)$ by :*

$$\forall x \in K,\ \forall t \in (t^n, t^{n+1}),\ \forall K \in \mathcal{T},\ \forall n \in \{0,...,M\},$$
$$u_{\mathcal{D}}(x,t) = U_K^{n+1}, \tag{2.9}$$

where $(U_K^{n+1})_{K\in\mathcal{T},n\in\{0,...,M\}}$ is the unique solution to (2.7).

We will now state an assumption which will be useful to prove the uniqueness of the weak-solution in section 2.3.

Assumption 2.2 *For all $i \in \{1,...,N\}$, $(\varphi_i \circ \Pi_i^{(-1)})'$ is a Lipschitz continuous function on $[0,1]$.*

We will now state our main result :

Theorem 2.1.1 (Convergence to the weak solution) *Let $\xi \in \mathbb{R}$, consider a family of admissible discretizations of $\Omega\times(0,T)$ in the sense of Definition 2.5 such that, for all $\mathcal{D}$ in the family, one has $\xi \geq reg(\mathcal{D})$. For a given admissible discretization $\mathcal{D}$ of this family, let $u_{\mathcal{D}}$ denote the associated approximate solution as defined in Definition 2.6. Then, under assumptions 2.1-2.2 :*

$$u_{\mathcal{D}} \to u \in L^p(\Omega \times (0,T)) \text{ as } size(\mathcal{D}) \to 0,\ \forall p \in [1, +\infty)$$

where u is the unique weak solution to problem (2.2) in the sense of Definition 2.2. Furthermore, we have the following regularity result :

$$u \in C^0([0,T], L^p(\Omega)), \quad \forall p \in [1, +\infty)$$

All this paper will be devoted to the proof of Theorem 2.1.1. First, we will use the work of [Enc04] to prove the existence of a weak-solution. Then, in subsection 2.2.5, we will prove the existence of a time continuous solution, applying Ascoli theorem to a family of approximate solutions for a regular enough initial data. The uniqueness of the weak solution will be proven in the section 2.3 by using a doubling variable method inspired from [Kru70, AL83, Ott96b, Car99].

2.2 Existence of a weak solution

The main work of this section has already been done in [Enc04]. We only need to get enough results to prove the time-continuity in section 2.2.5. This proof will need estimates obtained by working on the scheme, so we prefer to give the whole proof of convergence. In this whole part, any sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of admissible discretizations of $\Omega\times(0,T)$ in the sense of Definition 2.5 will be supposed to have a bounded regularity.

$$\exists\zeta\in\mathbb{R},\ \forall m\in\mathbb{N},\quad \zeta\geq reg(\mathcal{D}_m). \tag{2.10}$$

2.2.1 Existence, uniqueness of the approximate solution

We state here the properties and estimates which are satisfied by the scheme (2.7) which we introduced in the previous section and prove existence and uniqueness of the solution to this scheme. First, we will take some notations for the convenience of the reader :

Notations 1 *for all $K\in\mathcal{T}$, for all $\sigma\in\mathcal{E}_K\cap\mathcal{F}_K$, for all $n\in\{0,...,M+1\}$,*

$$\left\{\begin{array}{rcl}\varphi_K^n &=& \varphi(U_K^n,x_K),\\ \varphi_{K,\sigma}^n &=& \varphi(U_{K,\sigma}^n,x_K),\\ \Pi_K^n &=& \Pi(U_K^n,x_K),\\ \Pi_{K,\sigma}^n &=& \Pi(U_{K,\sigma}^n,x_K),\\ \overline{\pi}_K^n &=& \overline{\pi}(U_K^n,x_K),\\ \overline{\pi}_{K,\sigma}^n &=& \overline{\pi}(U_{K,\sigma}^n,x_K).\end{array}\right.$$

We will need the following lemma :

Lemma 2.2.1 *For all $K\in\mathcal{T}$, for all $L\in\mathcal{N}_K$, for all $n\in\{0,...,M\}$, for all $(U_K^{n+1},U_L^{n+1})\in\mathbb{R}^2$, there exists an unique $(U_{K,\sigma}^{n+1},U_{L,\sigma}^{n+1})\in[0,1]^2$ solution of (2.8).*

Proof
Suppose that there exists $i\in\{1,...,N\}$ such that $(x_K,x_L)\in\Omega_i^2$. Then (2.8) can be written :

$$\left\{\begin{array}{l}\tau_{K,\sigma}\Big(\varphi_K^{n+1}-\varphi_{K,\sigma}^{n+1}\Big)+\tau_{L,\sigma}\Big(\varphi_L^{n+1}-\varphi_{L,\sigma}^{n+1}\Big)=0\\ U_{K,\sigma}^{n+1}=U_{L,\sigma}^{n+1}\end{array}\right.$$

which leads to

$$\left\{\begin{array}{l}U_{K,\sigma}^{n+1}=U_{L,\sigma}^{n+1}\\ \varphi_{K,\sigma}^{n+1}=\frac{1}{\tau_{K,\sigma}+\tau_{L,\sigma}}\Big(\tau_{K,\sigma}\varphi_K^{n+1}+\tau_{L,\sigma}\varphi_L^{n+1}\Big)\end{array}\right.$$

and $\frac{1}{\tau_{K,\sigma}+\tau_{L,\sigma}}\Big(\tau_{K,\sigma}\varphi_K^{n+1}+\tau_{L,\sigma}\varphi_L^{n+1}\Big)$ admits an unique antecedent through φ_i in $[0,1]$.

Let us now suppose that $(x_K,x_L)\in\Omega_i\times\Omega_j$ with $j\neq i$

$$(2.8)\Leftrightarrow\left\{\begin{array}{l}U_{K,\sigma}^{n+1}=\overline{\pi}_i^{-1}(\overline{\pi}_{L,\sigma}^{n+1})\\ \tau_{K,\sigma}\Big(\varphi_K^{n+1}-\varphi_i(\overline{\pi}_i^{-1}(\overline{\pi}_{L,\sigma}^{n+1}))\Big)+\tau_{L,\sigma}\Big(\varphi_L^{n+1}-\varphi_{L,\sigma}^{n+1}\Big)=0\end{array}\right.$$

then

$$\begin{cases} U_{K,\sigma}^{n+1} = \overline{\pi}_i^{-1}(\overline{\pi}_{L,\sigma}^{n+1}) \\ \tau_{K,\sigma}\varphi_i(\overline{\pi}_i^{-1}(\overline{\pi}_{L,\sigma}^{n+1})) + \tau_{L,\sigma}\varphi_{L,\sigma}^{n+1} = \tau_{K,\sigma}\varphi_K^{n+1} + \tau_{L,\sigma}\varphi_L^{n+1}. \end{cases}$$

Since $\overline{\pi}_i(0) = \overline{\pi}_j(0)$ and $\overline{\pi}_i(1) = \overline{\pi}_j(1)$ (Assumption 2.1), then :

$$\forall U_{L,\sigma}^{n+1} \in [0,1], \; U_{K,\sigma}^{n+1} \in [0,1].$$

The application $\theta : z \mapsto \tau_{K,\sigma}\varphi_i(\overline{\pi}_i^{-1}(\overline{\pi}_j(z))) + \tau_{L,\sigma}\varphi_j(z)$ is increasing on $[0,1]$, ensuring this way the uniqueness of the solution of (2.8). Furthermore, it satisfies :

$$\begin{cases} \theta(0) = 0 \\ \theta(1) = \tau_{K,\sigma}\varphi_i(1) + \tau_{L,\sigma}\varphi_j(1). \end{cases}$$

We conclude the proof by obtaining the existence of the solution by using the intermediate value theorem. □

L^∞-stability of the scheme

One assumes $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq 1$ a.e in Ω, then for all $K \in \mathcal{T}$, $U_K^0 \in [0,1]$ where U_K^0 is given by (2.6).

Proposition 2.2.2 *Let $\mathcal{D}$ an admissible discretization of $\Omega \times (0,T)$ in the sense of Definition 2.5, let $(U_K^{n+1})_{K\in\mathcal{T}, n\in\{0,...,M\}}$ be a solution of the scheme (2.7), then for all $K \in \mathcal{T}$, for all $n \in \{0,...,M\}$:*

$$0 \leq U_K^{n+1} \leq 1. \tag{2.11}$$

Proof

Let us first remark that the scheme can be written :

$$\begin{aligned} \frac{U_K^{n+1} - U_K^n}{\delta t} m(K) \quad &+ \sum_{L\in\mathcal{N}_{K,i}} \tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1}) \\ &+ \sum_{\sigma\in\mathcal{F}_K} \tau_{K,\sigma}(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1}) = 0, \end{aligned}$$

with $\tau_{K|L} = \frac{m(K|L)}{d(x_K,x_L)}$. Let us now rewrite it once again. $\forall K \in \mathcal{T}_i$, $\forall n \in \{0,...,M\}$:

$$U_K^{n+1} = H_K\Big(U_K^n, (U_L^{n+1})_{L\in\mathcal{T}}\Big)$$

with :

$$H_K\Big(a, (a_L)_{L\in\mathcal{T}}\Big) = \frac{a + \lambda_K a_K + \frac{\delta t}{m(K)} \begin{bmatrix} \sum_{L\in\mathcal{N}_{K,i}} \tau_{K|L}(\varphi_i(a_L) - \varphi_i(a_K)) \\ + \sum_{\sigma\in\mathcal{F}_K} \tau_{K,\sigma}(\varphi_i(a_{K,\sigma}) - \varphi_i(a_K)) \end{bmatrix}}{1 + \lambda_K}$$

where λ_K is given by :

$$\lambda_K = \frac{\delta t L_\varphi}{m(K)}\Big(\sum_{L\in\mathcal{N}_{K,i}} \tau_{K|L} + \sum_{\sigma\in\mathcal{F}_K} \tau_{K,\sigma}\Big)$$

and L_φ is a Lipschitz constant for all φ_i. However, $a_{K,\sigma} = g(a_K, a_L)$ where g is a nondecreasing function. We deduce from it that H_K is a nondecreasing function of all its arguments.

Let $n \in \{0, ..., M\}$, let us assume $0 \leq U_K^n \leq 1$, $\forall K \in \mathcal{T}$. Let us assume that there exists $K_{max} \in \mathcal{T}$ such that :

$$U_{K_{max}}^{n+1} = \max_{K \in \mathcal{T}} (U_K^{n+1}) > 1,$$

then :

$$1 < U_{K_{max}}^{n+1} \leq H_K\Big(1, (U_{K_{max}}^{n+1})_{L \in \mathcal{T}}\Big) = \frac{1 + \lambda_K U_{K_{max}}^{n+1}}{1 + \lambda_K} < U_{K_{max}}^{n+1},$$

a contradiction. Therefore :

$$U_{K_{max}}^{n+1} \leq 1.$$

We can prove exactly in the same way that

$$U_{K_{min}}^{n+1} = \min_{K \in \mathcal{T}} (U_K^{n+1}) \geq 0.$$

□

Proposition 2.2.3 *Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 2.5. There exists a unique solution to the scheme (2.7).*

Proof

Existence of the discrete solution

Let $n \in \{0, ..., M\}$ Since $(U_{K,\sigma}^{n+1})_{\sigma \in \mathcal{E}}$ in a function of $(U_L^{n+1})_{L \in \mathcal{T}}$, we shall see the scheme as a non linear system of equations only depending on $(U_L^{n+1})_{L \in \mathcal{T}}$.

Let us consider the application $\Psi : \begin{cases} \mathbb{R}^{\sharp \mathcal{T}} \times [0,1] & \rightarrow \mathbb{R}^{\sharp \mathcal{T}} \\ \Big((U_K^{n+1})_{K \in \mathcal{T}}, \lambda\Big) & \mapsto (V_K)_{K \in \mathcal{T}} \end{cases}$ where :

$$\begin{aligned} V_K \quad = \quad & \frac{U_K^{n+1} - \lambda U_K^n}{\delta t} m(K) + \lambda \sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L} (\varphi_K^{n+1} - \varphi_L^{n+1}) \\ & + \lambda \sum_{\sigma \in \mathcal{F}_K} \tau_{K,\sigma} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1}) \end{aligned}$$

The linear system of equation $\Psi\Big((U_K^{n+1})_{K \in \mathcal{T}}, 0\Big) = (0)_{K \in \mathcal{T}}$ admits a unique trivial solution. The continuity of $\lambda \mapsto \Psi(\cdot, \lambda)$ and the L^∞-estimate (2.11) allows us to use a topological degree argument, insuring the existence of a solution for $\lambda = 1$.

Uniqueness of the discrete solution

Assume that, for a given value of n, there exist two solutions to the scheme (2.7), $(U_K^{n+1})_{K \in \mathcal{T}}$ and $(V_K^{n+1})_{K \in \mathcal{T}}$. Then, for all $K \in \mathcal{T}$, using the monotony of H_K :

$$\max(U_K^{n+1}, V_K^{n+1}) \leq H_K(U_K^n, (\max(U_L^{n+1}, V_L^{n+1}))_{L \in \mathcal{T}}), \tag{2.12}$$

$$\min(U_K^{n+1}, V_K^{n+1}) \leq H_K(U_K^n, (\min(U_L^{n+1}, V_L^{n+1}))_{L \in \mathcal{T}}). \tag{2.13}$$

One multiplies (2.12) and (2.13) by $(1+\lambda_K)m(K)$, substracts (2.13) to (2.12) and sum on $K \in \mathcal{T}$. Remarking that all the exchange terms between neighboring control volumes disappear, we get :

$$\sum_{K\in\mathcal{T}} |U_K^{n+1} - V_K^{n+1}| m(K) \leq 0.$$

□

2.2.2 Discrete $L^2(0,T;H^1(\Omega))$ estimates

In this section, we will prove some estimates on the approximate solution. We first have to define the space $\mathcal{X}(\mathcal{D})$ the solution belongs to.

Definition 2.7 *Let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of the Definition 2.5. We denote by $\mathcal{X}(\mathcal{D})$ the functional space :*

$$\mathcal{X}(\mathcal{D}) = \left\{ \begin{array}{c} v \in L^\infty(\Omega\times(0,T)), \forall K \in \mathcal{T},\ \forall n \in \{0,...,M\}, \\ \exists V_K^{n+1}, v(x,t) = V_K^{n+1} \text{ a.e. in } K\times(t^n,t^{n+1}) \end{array} \right\}.$$

Definition 2.8 (Discrete $L^2(0,T;H^1(\Omega_i))$ semi-norm) *We define the discrete $L^2(0,T;H^1(\Omega_i))$ semi norm on $\mathcal{X}(\mathcal{D})$ by :*

$$|v|^2_{1,\mathcal{D},i} = \sum_{n=0}^{M} \delta t \sum_{K|L\in\mathcal{E}_i} \tau_{K,\sigma}(V_K^{n+1} - V_L^{n+1})^2.$$

We will need the following lemma :

Lemma 2.2.4 *Let $\mathcal{D}$ be an admissible discretization of $\Omega\times(0,T)$ in the sense of the Definition 2.5. Let $u_\mathcal{D}$ be the discrete solution to (2.7). Let $\sigma \in \Gamma_{i,j}$ for some $i,j \in \{1,...,N\}^2$, $\sigma = K|L, K \in \mathcal{T}_i, L \in \mathcal{T}_j$. Then :*

$$0 \leq (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1}) \leq (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_L^{n+1}), \tag{2.14}$$

$$0 \leq (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\Pi_K^{n+1} - \Pi_{K,\sigma}^{n+1}) \leq (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\Pi_K^{n+1} - \Pi_L^{n+1}). \tag{2.15}$$

Proof
$\forall K \in \mathcal{T},\ \varphi(\cdot,x_K)$, and $\overline{\pi}(\cdot,x_K)$ are increasing functions on $[0,1]$, thus

$$(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1}) \geq 0.$$

Furthermore, $(\overline{\pi}_K^{n+1} - \overline{\pi}_L^{n+1}) = (\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1} + \overline{\pi}_{L,\sigma}^{n+1} - \overline{\pi}_L^{n+1})$ then :

$$\begin{aligned} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_L^{n+1}) &= (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1}) \\ &\quad + (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_L^{n+1} - \overline{\pi}_{L,\sigma}^{n+1}) \\ &= (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1}) \\ &\quad + \frac{\tau_{L,\sigma}}{\tau_{K,\sigma}}(\varphi_L^{n+1} - \varphi_{L,\sigma}^{n+1})(\overline{\pi}_L^{n+1} - \overline{\pi}_{L,\sigma}^{n+1}) \\ &\geq (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1}). \end{aligned}$$

This proof is also true with Π instead of $\overline{\pi}$. □

One introduces the function

$$\eta_i : s \mapsto \int_0^s \sqrt{\lambda_i(a)}\overline{\pi}_i'(a)da,$$

which fulfills thanks to Cauchy-Schwarz inequality, for all $(a,b) \in [0,1]^2$, for all $i \in [\![1,N]\!]$,

$$(\eta_i(a) - \eta_i(b))^2 \le (\varphi_i(a) - \varphi_i(b))(\overline{\pi}_i(a) - \overline{\pi}_i(b)). \tag{2.16}$$

Proposition 2.2.5 (Discrete $L^2(0,T;H^1(\Omega_i))$ estimate) *Under Assumption 2.1, let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0,T)$ in the sense of Definition 2.5, let $u_{\mathcal{D}}$ be the solution of the scheme (2.7). Then there exists C only depending on $\overline{\pi}_i$, Ω_i, $i \in \{1,...,N\}$ such that :*

$$\sum_{i=1}^{N} |\eta_i(u_{\mathcal{D}})|^2_{1,\mathcal{D},i} \le C. \tag{2.17}$$

$$0 \le \sum_{n=0}^{M} \delta t \sum_{\substack{\sigma \in \mathcal{F} \\ \sigma = K|L}} \tau_{K,\sigma}(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_L^{n+1}) \le C \tag{2.18}$$

Proof

Accumulation term :

Let us multiply the equations (2.7) by $\delta t \overline{\pi}_K^{n+1}$ and sum on $K \in \mathcal{T}, n \in \{0,...,M\}$. We get :

$$\sum_{n=0}^{M} \sum_{K \in \mathcal{T}} \begin{pmatrix} \Big(m(K)(U_K^{n+1} - U_K^n) + \delta t \sum_{L \in \mathcal{N}_{K,i}} \tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1}) + \\ \sum_{s \in \mathcal{F}_K} \tau_{K,\sigma}(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})\Big)\overline{\pi}_K^{n+1} \end{pmatrix} = 0.$$

Let i belong to $\{1,...,N\}$, let $K \in \mathcal{T}_i$. Since $\overline{\pi}_i$ is an increasing function, $g_i : s \mapsto \int_0^s \overline{\pi}_i(a)da$ is a convex function. Then :

$$(U_K^{n+1} - U_K^n)\overline{\pi}_K^{n+1} \ge g_i(U_K^{n+1}) - g_i(U_K^n).$$

Thus :

$$\begin{aligned} \sum_{n=0}^{M} \sum_{K \in \mathcal{T}} m(K)(U_K^{n+1} - U_K^n)\overline{\pi}_K^{n+1} &\ge \sum_{K \in \mathcal{T}} m(K)(g_i(U_K^{M+1}) - g_i(U_K^0)) \\ &\ge -m(\Omega) \int_0^1 \max_{i \in \{1,...,N\}} |\overline{\pi}_i(a)| da. \end{aligned}$$

Diffusion term :

One gets, thanks to (2.16)

$$\begin{aligned} &\sum_{n=0}^{M} \delta t \sum_{i=1}^{N} \sum_{K|L \in \mathcal{E}_i} \left[\tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_L^{n+1})\right] \\ &\quad \ge \sum_{n=0}^{M} \delta t \sum_{i=1}^{N} \sum_{K|L \in \mathcal{E}_i} \tau_{K|L}(\eta_K^{n+1} - \eta_L^{n+1})^2, \end{aligned}$$

where η_K^{n+1} denotes $\eta_i(U_K^{n+1})$ if $K \subset \Omega_i$. Furthermore, for all $\sigma = K|L \in \mathcal{F}$, Lemma 2.2.4 implies :

$$(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_L^{n+1}) \geq 0,$$

then, we have the following estimates :

$$\sum_{i=1}^{N} |\eta_i(u_{\mathcal{D}})|^2_{1,\mathcal{D},i} \leq m(\Omega) \int_0^1 \max_{i\in\{1,\dots,N\}} |\overline{\pi}_i(a)| da,$$

$$0 \leq \sum_{n=0}^{M} \delta t \sum_{\sigma=K|L\in\mathcal{F}} (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_L^{n+1}) \leq m(\Omega) \int_0^1 \max_{i\in\{1,\dots,N\}} |\overline{\pi}_i(a)| da.$$

□

Definition 2.9 (Discrete $L^2(0,T;H^1(\Omega))$ semi-norm) *Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0,T)$ in the sense of the Definition 2.5. One defines the discrete $L^2(0,T;H^1(\Omega))$ semi norm of $v \in \mathcal{X}(\mathcal{D})$ by :*

$$|v|^2_{1,\mathcal{D}} \;=\; \sum_{i=1}^{N} |v|^2_{1,\mathcal{D},i} + \sum_{\sigma=K|L\in\mathcal{F}} \Big[\tau_{K|L}(v(x_K,t^{n+1}) - v(x_L,t^{n+1}))^2\Big].$$

We will need the following lemma :

Lemma 2.2.6 *Under assumptions 2.1, for all i in $\{1,...,N\}$, the function $\Pi_i \circ \eta_i^{(-1)}$ admits 1 as Lipschitz constant.*

Proof
Let $i \in \{1,...,N\}$, let $a \in]0,\varphi_i(1)[$, let $b \in]0,\eta_i(1)[$ with $b \neq a$.
We set $A = \eta_i^{(-1)}(a)$ and $B = \eta_i^{(-1)}(b)$. One has :

$$\frac{\overline{\pi}_i \circ \eta_i^{(-1)}(b) - \overline{\pi}_i \circ \eta_i^{(-1)}(a)}{b-a} = \frac{\overline{\pi}_i(B) - \overline{\pi}_i(A)}{\eta_i(B) - \eta_i(A)}.$$

One denote by $I(A,B)$ the interval $[A,B]$ if $B \geq A$, and $[B,A]$ if $A \geq B$. The definition of the function η_i implies :

$$\min_{C\in I(A,B)} \sqrt{\lambda_i(C)}(\overline{\pi}_i(B) - \overline{\pi}_i(A)) \leq \eta_i(B) - \eta_i(A) \leq \max_{C\in I(A,B)} \sqrt{\lambda_i(C)}(\overline{\pi}_i(B) - \overline{\pi}_i(A)).$$

Then, there exists $C \in I(A,B)$ such that :

$$\eta_i(B) - \eta_i(A) = \sqrt{\lambda_i(C)}(\overline{\pi}_i(B) - \overline{\pi}_i(A)).$$

So one gets :

$$\frac{\overline{\pi}_i \circ \eta_i^{(-1)}(b) - \overline{\pi}_i \circ \eta_i^{(-1)}(a)}{b-a} = \frac{1}{\sqrt{\lambda_i(C)}}.$$

Letting b tend to a, we get, using the continuity of $\eta_i^{(-1)}$:

$$(\overline{\pi}_i \circ \eta_i^{(-1)})'(a) = \frac{1}{\sqrt{\lambda_i \circ \eta_i^{(-1)}(a)}}.$$

Remarking that $\Pi_i = \Psi \circ \overline{\pi}_i$ with

$$\Psi \ : \left\{ \begin{array}{rcl} [\overline{\pi}_i(0), \overline{\pi}_i(1)] & \rightarrow & \mathbb{R}^+ \\ p & \mapsto & \int_{\overline{\pi}_i(0)}^{p} \min_{j\in\{1,\dots,N\}} \left(\sqrt{\lambda_j \circ \overline{\pi}_j^{(-1)}(a)} \right) da \end{array} \right.$$

we obtain :

$$(\Pi_i \circ \eta_i^{(-1)})'(a) = \Psi'(\overline{\pi}_i \circ \eta_i^{(-1)}(a))(\overline{\pi}_i \circ \eta_i^{(-1)})'(a) = \frac{\Psi'(\overline{\pi}_i \circ \eta_i^{(-1)}(a))}{\sqrt{\lambda_i \circ \eta_i^{(-1)}(a)}}.$$

Remarking that the definition of Ψ implies $\Psi'(\overline{\pi}_i(y)) \leq \sqrt{\lambda_i(y)}$, we get that

$$(\Pi_i \circ \eta_i^{(-1)})'(a) \leq 1.$$

□

Proposition 2.2.7 (Discrete $L^2(0,T;H^1(\Omega))$ estimate) *Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0,T)$ in the sense of Definition 2.5, let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be the approximate solution given by the scheme (2.7). There exists a constant C only depending on $\Omega, \overline{\pi}_i, \forall i \in \{1, ..., N\}$ such that :*

$$|\Pi(u_{\mathcal{D}}, \cdot)|_{1,\mathcal{D}}^2 \leq C.$$

Proof
Using inequality (2.17) proven in Proposition 2.2.5 and Lemma 2.2.6, we immediately get that :

$$\sum_{i=1}^{N} |\Pi_i(u_{\mathcal{D}})|_{1,\mathcal{D},i}^2 \leq C.$$

Let us now consider the case $\sigma = K|L \in \mathcal{F}$. Using $\overline{\pi}_{K,\sigma}^{n+1} = \overline{\pi}_{L,\sigma}^{n+1}$, inequality (2.18) together with (2.8) leads to :

$$\sum_{n=0}^{M} \delta t \sum_{\sigma=K|L\in\mathcal{F}} \left[\begin{array}{c} \tau_{K,\sigma}(\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1}) + \\ \tau_{L,\sigma}(\varphi_L^{n+1} - \varphi_{L,\sigma}^{n+1})(\overline{\pi}_L^{n+1} - \overline{\pi}_{L,\sigma}^{n+1}) \end{array} \right] \leq C. \tag{2.19}$$

For all $\sigma \in \mathcal{F}$, for all K such that $\sigma \in \mathcal{E}_K$, we have thanks to (2.16) :

$$(\eta_K^{n+1} - \eta_{K,\sigma}^{n+1})^2 \leq (\varphi_K^{n+1} - \varphi_{K,\sigma}^{n+1})(\overline{\pi}_K^{n+1} - \overline{\pi}_{K,\sigma}^{n+1}).$$

Lemma 2.2.6 implies :

$$(\Pi_K^{n+1} - \Pi_{K,\sigma}^{n+1})^2 \leq (\eta_K^{n+1} - \eta_{K,\sigma}^{n+1})^2$$

and

$$(\Pi_L^{n+1} - \Pi_{L,\sigma}^{n+1})^2 \le (\eta_L^{n+1} - \eta_{L,\sigma}^{n+1})^2,$$

thus (2.19) leads to

$$\sum_{n=0}^{M} \delta t \sum_{\sigma=K|L\in\mathcal{F}} \left[\tau_{K,\sigma}(\Pi_K^{n+1} - \Pi_{K,\sigma}^{n+1})^2 + \tau_{L,\sigma}(\Pi_L^{n+1} - \Pi_{L,\sigma}^{n+1})^2\right] \le C. \tag{2.20}$$

The convexity of the function $x \mapsto x^2$ together with the relation
$\frac{1}{\tau_{K|L}} = \frac{1}{\tau_{K,\sigma}} + \frac{1}{\tau_{L,\sigma}}$ leads to :

$$\sum_{n=0}^{M} \delta t \sum_{\sigma=K|L\in\mathcal{F}} \tau_{K|L}(\Pi_K^{n+1} - \Pi_L^{n+1})^2 \le C.$$

□

2.2.3 Some compactness results

We aim in this section to get enough compactness results to be able to let $size(\mathcal{D})$ tend to 0. Proposition 2.2.2 insures that, up to a subsequence, $\exists u \in L^\infty(\Omega \times (0,T))$ such that $u_\mathcal{D} \rightharpoonup u$ in the $L^\infty(\Omega \times (0,T))$-weak $\star$ topology.

Let us now turn to the Kolmogorov compactness criterion (see e.g. [Bré83]) which will allow us to pass to the limit in the nonlinear second order terms.

Theorem 2.2.8 (Kolmogorov) *Let $\mathcal{Q}$ be an open bounded subset of $\mathbb{R}^k$, and $(v_n)_n$ be a bounded sequence in $L^2(\mathbb{R}^k)$ such that :*

$$\lim_{\delta\to 0}[\sup_{n\in\mathbb{N}} \|v_n(\cdot+\delta) - v_n(\cdot)] = 0,$$

then there exists $v \in L^2(\mathcal{Q})$ such that, up to a subsequence,

$$v_n \to v \text{ in } L^2(\mathcal{Q}) \text{ as } n \to \infty.$$

Let us now show that we are in position to apply the Kolmogorov compactness criterion to $(\eta_i(u_{\mathcal{D}_m}))_{m\in\mathbb{N}}$ where $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is a sequence of admissible discretization of $\Omega \times (0,T)$, with $\lim_{m\to\infty} size(\mathcal{D}_m) = 0$.

Space translates estimates

We will now state a proposition proven in [Enc04], which is a consequence of Proposition 2.2.5.

Proposition 2.2.9 *Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0,T)$ in the sense of Definition 2.5, let $u_\mathcal{D} \in \mathcal{X}(\mathcal{D})$ be the approximate solution given by the scheme (2.7), let $i \in \{1,...,N\}$, let $\xi \in \mathbb{R}^d$, and $\Omega_{i,\xi}$ the open subset of Ω_i defined by :*

$$\Omega_{i,\xi} = \{x \in \Omega_i \ / \ [x, x+\xi] \subset \Omega_i\}.$$

Then there exists C only depending on Ω_i such that :

$$\int_0^T \int_{\Omega_{i,\xi}} |\eta_i(u_\mathcal{D}(x+\xi,t) - \eta_i(u_\mathcal{D}(x,t)|^2 dxdt \le |\xi|(|\xi| + Csize(\mathcal{D}))|\eta_i(u_\mathcal{D})|_{1,\mathcal{D},i}. \tag{2.21}$$

Time translates estimates

We state here a first result on the time translates of $\varphi_i(u)$, already proven in [Enc04].

Proposition 2.2.10 *Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0,T)$. Let $u_{\mathcal{D}} \in \mathcal{X}(\mathcal{D})$ be the approximate solution obtained with the scheme (2.7). Let $\omega_i \subset \overline{\omega}_i \subset \Omega_i$ be an open subset of Ω_i. We assume that size($\mathcal{T}$) is small enough to ensure that :*

$$\omega_i \subset \Omega_{i,size(\mathcal{T})} = \{x \in \Omega_i \ / \ \overline{B(x, size(\mathcal{T}))} \subset \Omega_i\}.$$

We set :

$$\eta_{\mathcal{D},\omega_i} = \begin{cases} \eta_i(u_{\mathcal{D}}) & on \quad \omega_i \times (0,T) \\ 0 & on \quad \mathbb{R}^{d+1} \backslash (\omega_i \times (0,T)) \end{cases}$$

then, for all $\tau \in \mathbb{R}$, we get the following inequality :

$$\|\eta_{\mathcal{D},\omega_i}(\cdot, \cdot + \tau) - \eta_{\mathcal{D},\omega_i}\|^2 \leq C\tau, \tag{2.22}$$

where C is a constant which only depends on $T, \Omega, d, \varphi_i, \overline{\pi}_i, \Theta_i$.

Thus we can apply the Kolmogorov compactness criterion, and claim that there exists a function f such that, up to a subsequence, $\eta(u_{\mathcal{D}}, \cdot)$ converges to f in $L^2(\Omega \times (0,T))$ as $size(\mathcal{D})$ tends to 0. Furthermore, letting $size(\mathcal{D})$ tend to 0 in estimates (2.21) ensures that $f \in L^2(0,T; H^1(\Omega_i))$ for all $i \in \{1, ..., N\}$. The same way, we can prove that $\Pi(u_{\mathcal{D}}, \cdot)$ converges to some $g \in L^2(0,T; H^1(\Omega))$ in the $L^2(\Omega \times (0,T))$-topology.

Remark 2.2.1 *Since the functions $\eta_i \circ {\varphi_i}^{(-1)}$ are Lipschitz continuous, estimates (2.21) and (2.22) still hold with φ_i instead of η_i. Then $\varphi(u_{\mathcal{D}}, \cdot)$ also converges to a function h in $L^2(\Omega \times (0,T))$.*

2.2.4 Convergence to a weak solution

In this section, we aim to prove that, for an admissible sequence $(\mathcal{D}_m)$ of discetization of $\Omega \times (0,T)$ in the sense of the Definition 2.5, with $\lim_{m \to +\infty} size(\mathcal{D}_m) = 0$ the solution of the scheme $u_{\mathcal{D}_m}$ tends to a weak solution of the problem (2.2).

We have already proven that $\eta(u_{\mathcal{D}_m}, \cdot)$ (resp. $\Pi(u_{\mathcal{D}_m}, \cdot)$, $\varphi(u_{\mathcal{D}_m}, \cdot)$) converges in $L^2(\Omega \times (0,T))$ to a function f (resp. g, h). Proposition 2.2.2 allows us to assume that $u_{\mathcal{D}_m}$ converges to u in $L^\infty(\Omega \times (0,T))$ for the weak $\star$ topology. Using Minty Lemma, stated below, let us show that $f = \eta(u, \cdot)$, $g = \Pi(u, \cdot)$ and $h = \varphi(u, \cdot)$.

Lemma 2.2.11 (Minty lemma) *Let Ω be an open subset of $\mathbb{R}^k$, let $\Psi : \mathbb{R} \to \mathbb{R}$ be a monotonous continuous function. Let $(u_n)_{n \in \mathbb{N}}$ such that :*

$$\begin{cases} u_n \rightharpoonup u & in \quad L^\infty(\Omega) \ weak - \star \\ \Psi(u_n) \to f & in \quad L^1(\Omega) \end{cases}$$

then

$$\Psi(u) = f.$$

From the Lemma 4.2.8, we can deduce that, for all $i \in \{1, ..., N\}$:

$$\eta_i(u_{\mathcal{D}}) \to \eta_i(u) \qquad \text{in } L^1(\Omega_i \times (0,T)),$$

thus :

$$\eta(u_{\mathcal{D}}, \cdot) \to \eta(u, \cdot) \qquad \text{in } L^1(\Omega \times (0,T)).$$

The same way, $\varphi(u_{\mathcal{D}}, \cdot)$ and $\Pi(u_{\mathcal{D}}, \cdot)$ converge in $L^2(\Omega \times (0,T))$ to $\varphi(u, \cdot)$ and $\Pi(u, \cdot)$, respectively. Estimate (2.21) insures that, for all $i \in \{1, ..., N\}$, $\eta_i(u)$ belongs to $L^2(0,T; H^1(\Omega_i))$, and thus $\varphi_i(u)$ too. A straightforward adaptation of Proposition 2.2.9 with $\Pi(u_{\mathcal{D}}, \cdot)$ instead of $\eta_i(u_{\mathcal{D}})$ allows us to claim that $\Pi(u, \cdot)$ belongs to $L^2(0,T; H^1(\Omega))$.

Proposition 2.2.12 (Convergence to a weak solution) *Under assumption 2.1, let $(\mathcal{D}_m)$ be a sequence of admissible discretizations in the sense of Definition 2.5 which fulfill the assumption (2.10) and such that $size(\mathcal{D}_m) \to 0$. Let $(u_{\mathcal{D}_m})$ be the sequence of approximate solutions given by the scheme (2.7). Then there exists a subsequence of approximate solutions still denoted $(u_{\mathcal{D}_m})$ which converges to u in $L^q(\Omega \times (0,T))$ for all $q \in [1, +\infty)$. Furthermore, u is a weak solution to the problem (2.2) in the sense of Definition 2.2.*

The proof of this proposition is a straightforward adaptation of the proof stated in [Enc04].

2.2.5 Time continuity of the approximation limit

The aim of this section is to prove the existence of a time continuous solution. This result will be fundamental to prove the uniqueness of the weak solution to the problem (2.2).

In order to prove the continuity of a time continuous solution to the problem (2.2), we will apply the Ascoli theorem on a family of approximate solutions obtained through the scheme (2.7).

We need a classical CFL assumption on the family of space-time discretizations.

Assumption 2.3 *Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be an admissible space-time discretization of $\Omega \times (0,T)$ in the sense of Definition 2.5. In all this subsection, we furthermore assume that there exists $S_1 > 0$ which does not depend on m such that :*

$$\max_m \frac{\delta t_m}{(size(\mathcal{T}_m))^2} \leq S_1.$$

Remark 2.2.2 *Assumption 5.82 and (2.10) ensure us that the quantity $\max_{K \in \mathcal{T}}(\max_{L \in \mathcal{N}_{K,i}}(\frac{\delta t \tau_{K|L}}{m(K)}))$ stays bounded as m tends to $+\infty$. The assumption 5.82 is not hard to fulfill in the theorical framework. One just has to choose a convenient time step. Nevertheless, this assumption is very demanding in the numerical framework, but we will be able to relax it in the sequel of this work.*

We also need to make an assumption on the regularity of the initial data, but once again this assumption will be relaxed in the sequel, thanks to a density argument.

Assumption 2.4 *The initial data u_0 belongs to $L^\infty(\Omega)$, $0 \leq u_0 \leq 1$, and furthermore fulfills : $\varphi(u_0, \cdot)$ is a piecewise Lipschitz function.*

We will first need the following lemmas :

Lemma 2.2.13 (Discrete $H^1(0,T;L^2(\omega_i))$ estimate) *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of $\Omega\times(0,T)$ fulfilling assumption 5.82. Let $(u_{\mathcal{D}_m})$ the sequence of approximate solutions given by the scheme (2.7). Let O_i be an open subset of Ω_i such that $\varphi_i(u_0)_{|O_i}$ is a Lipschitz continuous function. Let ω_i be an open subset of O_i, with $\overline{\omega}_i\subset O_i$. Then there exists C only depending on $O_i,\omega_i,\lambda_i,\overline{\pi}_i,u_0,S_1,\zeta$ such that :*

$$\sum_{n=0}^{M}\sum_{\substack{K\in\mathcal{T}\\K\subset\omega_i}} m(K)(\varphi_i(u_{\mathcal{D}_m}(x_K,t^{n+1}))-\varphi_i(u_{\mathcal{D}_m}(x_K,t^n)))^2\le C\delta t.$$

Proof

We use the following notations :

$$\mathcal{E}_{O_i}=\{\sigma\in\mathcal{E}_i,\ \sigma=K|L/\ K\subset O_i,\ L\subset O_i\},$$

$$\mathcal{E}_{\omega_i}=\{\sigma\in\mathcal{E}_i,\ \sigma=K|L/\ K\subset \omega_i,\ L\subset \omega_i\}.$$

Let $\Theta_i\in C_c^\infty(\Omega)$ such that $supp(\Theta_i)\subset O_i,\ \Theta_{i|\omega_i}=1,\ 1\ge\Theta_i\ge0$. For all $K\in\mathcal{T}$, we denote $\Theta_{i,K}=\Theta_i(x_K)$. We multiply the scheme (2.7) by $(\varphi_K^{n+1}-\varphi_K^n){\Theta_{i,K}}^2\delta t$:

$$\frac{U_K^{n+1}-U_K^n}{\delta t}m(K)(\varphi_K^{n+1}-\varphi_K^n){\Theta_{i,K}}^2+\sum_{L\in\mathcal{N}_{K,i}}\left[\begin{array}{c}\tau_{K|L}(\varphi_K^{n+1}-\varphi_L^{n+1})\\(\varphi_K^{n+1}-\varphi_K^n){\Theta_{i,K}}^2\end{array}\right]=0,$$

thus :

$$m(K)(\varphi_K^{n+1}-\varphi_K^n)^2{\Theta_{i,K}}^2\le L_{\varphi_i}\delta t\sum_{L\in\mathcal{N}_{K,i}}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})(\varphi_K^{n+1}-\varphi_K^n){\Theta_{i,K}}^2.$$

Let $M_1\in\{0,...,M\}$. We sum on $K\in\mathcal{T}$ and on $n\in\{0,M_1\}$:

$$\sum_{n=0}^{M_1}\sum_{K\in\mathcal{T}}m(K)(\varphi_K^{n+1}-\varphi_K^n)^2{\Theta_{i,K}}^2$$

$$\le L_{\varphi_i}\sum_{n=0}^{M_1}\delta t\sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})\\((\varphi_K^{n+1}-\varphi_K^n){\Theta_{i,K}}^2-(\varphi_L^{n+1}-\varphi_L^n){\Theta_{i,L}}^2)\end{array}\right]$$

$$\le\left(\begin{array}{c}L_{\varphi_i}\displaystyle\sum_{n=0}^{M_1}\delta t\sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})\left(\frac{{\Theta_{i,K}}^2+{\Theta_{i,L}}^2}{2}\right)\\((\varphi_L^n-\varphi_K^n)-(\varphi_L^{n+1}-\varphi_K^{n+1}))\end{array}\right]\\+\dfrac{L_{\varphi_i}}{2}\displaystyle\sum_{n=0}^{M_1}\delta t\sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})({\Theta_{i,L}}^2-{\Theta_{i,K}}^2)\\((\varphi_K^{n+1}-\varphi_K^n)+(\varphi_L^{n+1}-\varphi_L^n))\end{array}\right]\end{array}\right)$$

$$\sum_{n=0}^{M_1}\sum_{K\in\mathcal{T}}m(K)(\varphi_K^{n+1}-\varphi_K^n)^2{\Theta_{i,K}}^2\le A_1+A_2,\qquad(2.23)$$

with

$$\left\{\begin{array}{l} A_1 = L_{\varphi_i}\sum_{n=0}^{M_1}\delta t \sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})\left(\frac{\Theta_{i,K}{}^2+\Theta_{i,L}{}^2}{2}\right)\\ ((\varphi_L^n-\varphi_K^n)-(\varphi_L^{n+1}-\varphi_K^{n+1}))\end{array}\right]\\ A_2 = \frac{L_{\varphi_i}}{2}\sum_{n=0}^{M_1}\delta t \sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})(\Theta_{i,L}{}^2-\Theta_{i,K}{}^2)\\ ((\varphi_K^{n+1}-\varphi_K^n)+(\varphi_L^{n+1}-\varphi_L^n))\end{array}\right].\end{array}\right.$$

Using $xy \leq \frac{1}{2}(x^2+y^2)$ in A_1 leads to :

$$\begin{aligned} A_1 &\leq \frac{L_{\varphi_i}\delta t}{2}\sum_{n=0}^{M_1}\sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}\frac{\Theta_{i,K}{}^2+\Theta_{i,L}{}^2}{2}\\ ((\varphi_K^n-\varphi_L^n)^2-(\varphi_K^{n+1}-\varphi_L^{n+1})^2)\end{array}\right] &(2.24)\\ &\leq \frac{L_{\varphi_i}\delta t}{2}\sum_{K|L\in\mathcal{E}_{O_i}}\tau_{K|L}(\varphi_K^0-\varphi_L^0)^2. &(2.25)\end{aligned}$$

Since $\varphi_i(u_0)$ is a continuous Lipschitz function on O_i, there exists C_{O_i,u_0}, which does not depend on the mesh such that :

$$|\varphi_K^0-\varphi_L^0| \leq C_{O_i,u_0} d(x_K,x_L). \tag{2.26}$$

Thus, using (2.26) in inequality (2.25) leads to :

$$A_1 \leq \frac{C_{O_i,u_0} d m(O_i) L_{\varphi_i}}{2}\delta t. \tag{2.27}$$

We will now prove a similar estimate for A_2 :

$$A_2 = \frac{\delta t L_{\varphi_i}}{2}\sum_{n=0}^{M_1}\sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})(\Theta_{i,K}+\Theta_{i,L})(\Theta_{i,L}-\Theta_{i,K})\\ \left((\varphi_K^{n+1}+\varphi_L^{n+1})-(\varphi_K^n+\varphi_L^n)\right)\end{array}\right].$$

For all $\alpha > 0$, we deduce from Young inequality that :

$$\begin{aligned} A_2 &\leq \alpha\delta t L_{\varphi_i}\sum_{n=0}^{M_1}\sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\Theta_{i,K}+\Theta_{i,L})^2\\ \left((\varphi_K^{n+1}+\varphi_L^{n+1})-(\varphi_K^n+\varphi_L^n)\right)^2\end{array}\right]\\ &\quad+\frac{L_{\varphi_i}\delta t}{4\alpha}\sum_{n=0}^{M_1}\sum_{K|L\in\mathcal{E}_i}\tau_{K|L}(\varphi_L^{n+1}-\varphi_K^{n+1})^2(\Theta_{i,L}-\Theta_{i,K})^2 = A_{21}+A_{22}.\end{aligned}$$

Since for all $(x,y)\in\mathbb{R}^2$, $(x+y)^2 \leq 2x^2+2y^2$, we can write :

$$\begin{aligned} A_{21} &\leq 2\alpha\delta t L_{\varphi_i}\sum_{n=0}^{M_1}\sum_{K|L\in\mathcal{E}_i}\left[\begin{array}{c}\tau_{K|L}(\Theta_{i,K}+\Theta_{i,L})^2\\ \left((\varphi_K^{n+1}-\varphi_K^n)^2+(\varphi_L^{n+1}-\varphi_L^n)^2\right)\end{array}\right]\\ &\leq 2\alpha\delta t L_{\varphi_i}\sum_{n=0}^{M_1}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_{K,i}}\tau_{K|L}(\Theta_{i,K}+\Theta_{i,L})^2(\varphi_K^{n+1}-\varphi_K^n)^2\\ &\leq 2\alpha L_{\varphi_i}\sum_{n=0}^{M_1}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_{K,i}}\left[\begin{array}{c}\frac{\tau_{K|L}\delta t}{m(K)}m(K)(\varphi_K^{n+1}-\varphi_K^n)^2\\ (2\Theta_{i,K}+(\Theta_{i,L}-\Theta_{i,K}))^2\end{array}\right].\end{aligned}$$

The Remark 2.2.2 allows us to take a constant S_2 not depending on m such that, for all $K \in \mathcal{T}$, for all $L \in \mathcal{N}_{K,i}$:

$$\frac{\tau_{K|L}\delta t}{m(K)} \leq S_2,$$

thus, using once again $(x+y)^2 \leq 2x^2 + 2y^2$, and that the number of edges of K is not bigger than $reg(\mathcal{T})$:

$$\begin{aligned} A_{21} \quad \leq \quad & 16\alpha L_{\varphi_i} reg(\mathcal{T}) S_2 \sum_{n=0}^{M_1} \sum_{K\in\mathcal{T}} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 {\Theta_{i,K}}^2 \\ & + 4\alpha L_{\varphi_i} \sum_{n=0}^{M_1} \sum_{K|L\in\mathcal{E}_i} \tau_{K|L}(\Theta_{i,L} - \Theta_{i,K})^2 (\varphi_K^{n+1} - \varphi_K^n)^2. \end{aligned} \tag{2.28}$$

One applies exactly the same method with a regular function Ψ_i instead of Θ_i, with $supp(\Psi_i) \subset O_i$, $\exists \varepsilon > 0$, $\Psi_{i|supp(\Theta_i)+\varepsilon} = 1$, $\Psi_i \geq 0$, where :

$$supp(\Theta_i) + \varepsilon = \{x \in \Omega_i / \ d(x, supp(\Theta_i)) < \varepsilon\}$$

Then, for $size(\mathcal{T})$ small enough, we obtain that there exists $H > 0$ such that :

$$\sum_{n=0}^{M_1} \sum_{K\subset(supp(\Theta_i)+\varepsilon)} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 \leq \sum_{n=0}^{M_1} \sum_{K\in\mathcal{T}} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 \Psi_K^2 \leq H. \tag{2.29}$$

Denoting by C_{Θ_i} the Lipschitz constant of the function Θ_i, using (2.29) in (2.28) and Remark 2.2.2, one gets :

$$\begin{aligned} A_{21} \quad \leq \quad & 16\alpha L_{\varphi_i} reg(\mathcal{T}) S_2 \sum_{n=0}^{M_1} \sum_{K\in\mathcal{T}} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 {\Theta_{i,K}}^2 \\ & + 4\alpha L_{\varphi_i} (reg(\mathcal{D}))^2 C_{\Theta_i}^2 H S_2 (size(\mathcal{T}))^2. \end{aligned} \tag{2.30}$$

A similar estimate on A_{22} is obvious.

$$A_{22} \leq \left(\frac{L_{\varphi_i} C_{\Theta_i}^2}{4\alpha} |\varphi_i(u_{\mathcal{D}})|_{1,\mathcal{D},i} \right) (reg(\mathcal{D}))^2 (size(\mathcal{T}))^2.$$

Assumption 5.82 ensures that there exist constants C_1, C', C_{21}, C_{22} such that :

$$\begin{cases} A_1 \leq C_1 \delta t, \\ A_{21} \leq \alpha C' \displaystyle\sum_{n=0}^{M_1} \sum_{K\in\mathcal{T}} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 {\Theta_{i,K}}^2 + \alpha C_{21} \delta t, \\ A_{22} \leq \dfrac{C_{22}}{\alpha} \delta t. \end{cases} \tag{2.31}$$

We can now choose $\alpha = \frac{1}{2C'}$ and claim that inequalities (2.31) together with (2.23) lead to the existence of a constant C such that

$$\sum_{n=0}^{M_1} \sum_{\substack{K\in\mathcal{T} \\ K\subset\omega_i}} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 \leq \sum_{n=0}^{M_1} \sum_{K\in\mathcal{T}} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 {\Theta_{i,K}}^2 \leq C\delta t. \tag{2.32}$$

□

Lemma 2.2.14 (Discrete $L^\infty(0,T,H^1(\omega_i))$ estimate) *With the same assumptions and notations as in Lemma 2.2.13, there exists C such that :*

$$\sup_{m\in\mathbb{N}}(\sup_{t\in[0,T]} (|\varphi_i(u_{\mathcal{D}_m})(\cdot,t)|_{1,\mathcal{T}_m,\omega_i})) \leq C,$$

where :

$$|\varphi_i(u_{\mathcal{D}_m})(\cdot,t)|^2_{1,\mathcal{T}_m,\omega_i} = \sum_{K|L\subset\omega_i} \tau_{K|L} \left(\varphi_i(u_{\mathcal{D}_m})(x_K,t) - \varphi_i(u_{\mathcal{D}_m})(x_L,t)\right)^2.$$

Proof

Keeping the notations of the previous proof, inequality (2.24) leads to :

$$A_1 \leq \frac{L_{\varphi_i}\delta t}{2}\left[\sum_{K|L\in\mathcal{E}_{O_i}} \tau_{K|L}(\varphi_K^0 - \varphi_L^0)^2 - \sum_{K|L\in\mathcal{E}_{\omega_i}} \tau_{K|L}(\varphi_K^{M_1+1} - \varphi_L^{M_1+1})^2\right].$$

So we can deduce from (2.32) the following estimate :

$$\sum_{n=0}^{M_1} \sum_{\substack{K\in\mathcal{T}\\ K\subset\omega_i}} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 + \frac{L_{\varphi_i}\delta t}{2}\sum_{K|L\in\mathcal{E}_{\omega_i}} \tau_{K|L}(\varphi_K^{M_1+1} - \varphi_L^{M_1+1})^2 \leq C\delta t. \quad (2.33)$$

Dividing by δt leads to :

$$\frac{L_{\varphi_i}}{2}\sum_{K|L\in\mathcal{E}_{\omega_i}} \tau_{K|L}(\varphi_K^{M_1+1} - \varphi_L^{M_1+1})^2 \leq C.$$

This estimates holds for any $M_1 \in \{0,..,M\}$ and also for $M_1 = -1$ because of Assumption 2.4. □

Proposition 2.2.15 (Time continuity of a weak solution) *One supposes that assumptions 2.1 and 2.4 is fulfilled. Then there exists a weak solution to the problem (2.2) in the sense of Definition 2.2 satisfying :*

$$\forall p \in [1,+\infty), \quad u \in C([0,T],L^p(\Omega)).$$

Proof

Let $(\mathcal{D}_m)$ be a sequence of admissible discretizations of $\Omega\times(0,T)$ in the sense of Definition 2.5 fulfilling assumption 5.82. We will apply Ascoli theorem to the family of approximate solutions obtained through the scheme (2.7). We will first build another sequence of approximate solutions $(v_m)_m$, whose terms will be continuous with respect to the time variable. We denote by v_m the function defined almost everywhere in Ω for all $t\in[0,T]$ by :

$$v_m(x,t) = \frac{t^{n+1}-t}{\delta t}(\varphi_K^n - \varphi_K^{n+1}) + \varphi_K^{n+1} \qquad \text{if } (x,t)\in K\times[t^n,t^{n+1}].$$

Let $i\in\{1,...,N\}$. Let O_i be an open subset of Ω_i such that $\varphi_i(u_0)$ is a Lipschitz continuous function on O_i. Let U_i be an open subset of Ω_i such that $\overline{U}_i \subset O_i$. Let ω_i be an open subset of Ω_i such that $\overline{\omega}_i \subset U_i$. Let $\Theta_i \in \mathcal{D}(U_i)$ such that $\Theta_{i\omega_i} = 1$ and $0\leq\Theta_i\leq 1$. We suppose that m is large enough to ensure that :

– $size(\mathcal{T}_m) < d(U_i, \partial O_i)$,
– $size(\mathcal{T}_m) < d(supp(\Theta_i), \partial U_i)$.
We denote by $\mathcal{E}_{U_i} = \{\sigma \in \mathcal{E}, \sigma = K|L \ / \ K \subset U_i, L \subset U_i\}$.
Then, for all $t \in [0,T]$:

$$\begin{aligned}
&\|v_m(\cdot,t) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|^2_{L^2(\omega_i)} \\
&\leq \int_{O_i} \Theta_i(x)(v_m(x,t) - \varphi_i(u_{\mathcal{D}_m}(x,t)))^2 dx \\
&\leq \frac{(t^{n+1}-t)^2}{\delta t^2} \sum_{K\in\mathcal{T}} \Theta_{i,K} m(K)(\varphi_K^{n+1} - \varphi_K^n)^2 \\
&\leq \frac{(t^{n+1}-t)^2}{\delta t^2} L_{\varphi_i} \sum_{K\in\mathcal{T}} \Theta_{i,K} m(K)(\varphi_K^{n+1} - \varphi_K^n)(U_K^{n+1} - U_K^n) \\
&\leq \frac{(t^{n+1}-t)^2}{\delta t} L_{\varphi_i} \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}_{K,i}} \Theta_{i,K}(\varphi_K^{n+1} - \varphi_K^n)\tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1}) \\
&\leq \frac{(t^{n+1}-t)^2}{\delta t} L_{\varphi_i} \sum_{K|L\in\mathcal{E}_{U_i}} \left[\begin{array}{c} (\Theta_{i,K}(\varphi_K^{n+1} - \varphi_K^n) - \Theta_{i,L}(\varphi_L^{n+1} - \varphi_L^n)) \\ \tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1}) \end{array} \right],
\end{aligned}$$

then we have :

$$\|v_m(\cdot,t) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|^2_{L^2(\omega_i)} \leq A_1(t) + A_0(t),$$

with

$$\left\{ \begin{array}{l} A_0(t) = \dfrac{(t^{n+1}-t)^2}{2\delta t} L_{\varphi_i} \displaystyle\sum_{K|L\in\mathcal{E}_{U_i}} \left[\begin{array}{c} \tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1})(\Theta_{i,K} + \Theta_{i,L}) \\ (\varphi_K^{n+1} - \varphi_K^n - \varphi_L^{n+1} + \varphi_L^n) \end{array} \right], \\ A_1(t) = \dfrac{(t^{n+1}-t)^2}{2\delta t} L_{\varphi_i} \displaystyle\sum_{K|L\in\mathcal{E}_{U_i}} \left[\begin{array}{c} \tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1})(\Theta_{i,K} - \Theta_{i,L}) \\ (\varphi_K^{n+1} - \varphi_K^n + \varphi_L^{n+1} - \varphi_L^n) \end{array} \right]. \end{array} \right.$$

We apply Cauchy-Schwarz on A_1, so we get :

$$|A_1(t)| \leq \delta t \varphi_i(1) L_{\varphi_i} \left[\begin{array}{c} \left(\sum_{K|L\in\mathcal{E}_{U_i}} \tau_{K|L}(\Theta_{i,K} - \Theta_{i,L})^2\right)^{1/2} \\ \left(\sum_{K|L\in\mathcal{E}_{U_i}} \tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1})^2\right)^{1/2} \end{array} \right].$$

Using Lemma 2.2.14, we can claim that there exists C_1 only depending on the data and on the regularity of the mesh such that, for all $t \in [0,T]$,

$$|A_1(t)| \leq C_1 \delta t.$$

Let us now have a look on A_0.

$$\begin{aligned}
|A_0(t)| &\leq L_{\varphi_i} \delta t \sum_{K|L\in\mathcal{E}_{U_i}} \tau_{K|L} |\varphi_K^{n+1} - \varphi_L^{n+1}| (|\varphi_K^{n+1} - \varphi_K^n| + |\varphi_L^{n+1} - \varphi_L^n|) \\
&\leq \delta t \sqrt{\sum_{K|L\in\mathcal{E}_{U_i}} \tau_{K|L}(\varphi_K^{n+1} - \varphi_L^{n+1})^2} \left[\begin{array}{c} \sqrt{\displaystyle\sum_{K|L\in\mathcal{E}_{U_i}} \tau_{K|L}(\varphi_K^{n+1} - \varphi_K^n)^2} \\ +\sqrt{\displaystyle\sum_{K|L\in\mathcal{E}_{U_i}} \tau_{K|L}(\varphi_L^{n+1} - \varphi_L^n)^2} \end{array} \right].
\end{aligned}$$

Using Remark 2.2.2 and lemmas 2.2.13 and 2.2.14, we can find C_0, not depending on m such that :

$$|A_0(t)| \leq C_0 \delta t.$$

So, we have shown that there exists C, only depending on the data and the regularity of the mesh, such that :

$$\forall m \in \mathbb{N},\ \forall t \in [0,T], \qquad \|v_m(\cdot,t) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|^2_{L^2(\omega_i)} \leq C\delta t. \tag{2.34}$$

We are now able to prove the relative compactness of the family $(v_m)_{m\in\mathbb{N}}$ in $C([0,T];L^2(\omega_i))$.

Uniform equicontinuity : Let $\varepsilon > 0$, let $m \in \mathbb{N}$, let $t \in [0,T)$, let $\tau \in (0, T-t)$. We denote $N_1 = \lceil \frac{t}{\delta t} \rceil$, $N_2 = \lceil \frac{t+\tau}{\delta t} \rceil$.

$$\begin{aligned}\|v_m(\cdot,t+\tau) - v_m(\cdot,t)\|_{L^2(\omega_i)} \leq\ & \|v_m(\cdot,t+\tau) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t+\tau))\|_{L^2(\omega_i)} \\ & + \|v_m(\cdot,t) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|_{L^2(\omega_i)} \\ & + \|\varphi_i(u_{\mathcal{D}_m}(\cdot,t+\tau)) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|_{L^2(\omega_i)}.\end{aligned}$$

Using estimate (2.34), we can choose $m_1 \in \mathbb{N}$ large enough so that :

$$\forall m \geq m_1,\ \forall t \in [0,T), \qquad \|v_m(\cdot,t) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|_{L^2(\omega_i)} \leq \varepsilon/3. \tag{2.35}$$

There exists $m_2 \in \mathbb{N}$ such that, for all $m \geq m_2$, $size(\mathcal{D}_m) \leq d(\omega_i, \partial U_i)$. Then, for all $m \geq m_2$, one has :

$$\begin{aligned}\|\varphi_i(u_{\mathcal{D}_m}(\cdot,t+\tau)) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|^2_{L^2(\omega_i)} &\leq \sum_{K\subset U_i} m(K)(\varphi_K^{N_2} - \varphi_K^{N_1})^2 \\ &\leq \sum_{K\subset U_i} m(K) \left(\sum_{n=N_1}^{N_2-1} (\varphi_K^{n+1} - \varphi_K^n) \right)^2 \\ &\leq (N_2 - N_1) \sum_{K\subset U_i} m(K) \sum_{n=N_1}^{N_2-1} (\varphi_K^{n+1} - \varphi_K^n)^2.\end{aligned}$$

Lemma 2.2.13 ensures that there exists C, not depending on m, such that :

$$\sum_{K\subset U_i} m(K) \sum_{n=N_1}^{N_2-1} (\varphi_K^{n+1} - \varphi_K^n)^2 \leq C\delta t,$$

thus

$$\|\varphi_i(u_{\mathcal{D}_m}(\cdot,t+\tau)) - \varphi_i(u_{\mathcal{D}_m}(\cdot,t))\|^2_{L^2(\omega_i)} \leq C(N_2 - N_1)\delta t.$$

The definition of N_1 and N_2 implies that $(N_2 - N_1)\delta t \leq \tau + dt$. So we can claim that : $\forall \varepsilon > 0,\ \exists m_3 \in \mathbb{N},\ \exists \alpha > 0,\ \forall t \in [0, T-\alpha],\ \forall \tau \in (0,\alpha),\ \forall m \geq m_3,$

$$\|v_m(\cdot,t+\tau) - v_m(\cdot,t)\|_{L^2(\omega_i)} \leq \varepsilon.$$

Local relative compactness : We state the following lemma which is a straightforward generalization of Lemma 3.3 of [EGH00] together with Lemma 2.2.14.

Lemma 2.2.16 *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of $\Omega\times(0,T)$ fulfilling assumption 5.82. Let $(u_{\mathcal{D}_m})$ the sequence of approximate solutions given by the scheme (2.7). Let O_i be an open subset of Ω_i such that $\varphi_i(u_0)_{|O_i}$ is a Lipschitz continuous function. Let ω_i be an open subset of O_i, with $\overline{\omega}_i\subset O_i$. Then there exist C_1, C_2 and an integer m_0 such that, for all $m\geq m_0$, for all $t\in[0,T[$, for all $\eta\in\mathbb{R}^d$ such that $|\eta|\leq\frac{1}{2}d(\omega_i,\partial O_i)$:*

$$\|\varphi_i(u_{\mathcal{D}_m}(x+\eta,t))-\varphi_i(u_{\mathcal{D}_m}(x,t))\|_{L^2(\omega_i)}\leq C_1|\eta|(|\eta|+C_2 size(\mathcal{T}))$$

where C_1 only depends on $O_i,\omega_i,u_0,\zeta,S_1,\lambda_i,\overline{\pi}_i$ for all $i\in\{1,...,N\}$, and C_2 only depends on Ω.

It is easy to check that, for all $t>0$:

$$\|v_m(x+\eta,t)-v_m(x,t)\|_{L^2(\omega_i)}\leq\|\varphi_i(u_{\mathcal{D}_m}(x+\eta,t))-\varphi_i(u_{\mathcal{D}_m}(x,t))\|_{L^2(\omega_i)}$$
$$+\|\varphi_i(u_{\mathcal{D}_m}(x+\eta,t-\delta t))-\varphi_i(u_{\mathcal{D}_m}(x,t-\delta t))\|_{L^2(\omega_i)}$$

with the convention $\varphi_i(u_{\mathcal{D}_m}(x,t))=\varphi_i(u_{\mathcal{D}_m}(x,0))$ if $t<0$. Then using lemma 2.2.16

$$\|v_m(x+\eta,t)-v_m(x,t)\|_{L^2(\omega_i)}\leq 2C_1|\eta|(|\eta|+C_2 size(\mathcal{T})).$$

Then we can apply Theorem 2.2.8 to state that, for all $t\geq0$, $(v_m(\cdot,t))_m$ is relatively compact in $L^2(\omega_i)$.

Ascoli theorem implies that the sequence $(v_m)_m$ is relatively compact in $C([0,T],L^2(\omega_i))$, so, up to a subsequence, it converges to $v\in C([0,T],L^2(\omega_i))$. It is now obvious that $\varphi_i(u)_{|\omega_i\times[0,T]}=v$. Thus $\varphi_i(u)\in C([0,T];L^1(\omega_i))$ for all $\omega_i\subset\overline{\omega}_i\subset O_i$, then $\varphi_i(u)$ belongs to $C([0,T];L^1(O_i))$.

Assumption 2.4 implies that, for all $i\in\{1,...,N\}$, there exists a family $(O_{i,j})_j$ of open subsets of Ω_i such that $\overline{\Omega}_i=\bigcup_j\overline{O}_{i,j}$. So $\varphi_i(u)\in C([0,T];L^1(\Omega_i))$, and :

$$\varphi_i(u)\in C([0,T];L^1(\Omega)). \tag{2.36}$$

We deduce from (2.36) that, for almost every $x\in\Omega$, for all $t\in[0,T]$:

$$\lim_{\tau\to0}\varphi(u(x,t+\tau),x)=\varphi(u(x,t),x).$$

The continuity of ${\varphi_i}^{-1}$ for all $i\in\{1,...,N\}$ leads to : for all $t\in[0,T]$, for almost every $x\in\Omega$,

$$\lim_{\tau\to0}|u(x,t+\tau)-u(x,t)|=0.$$

Let $p\in[1,+\infty)$. The continuity of $s\mapsto s^p$ leads to : for all $t\in[0,T]$, for almost every $x\in\Omega$,

$$\lim_{\tau\to0}|u(x,t+\tau)-u(x,t)|^p=0.$$

Since $|u(x,t+\tau)-u(x,t)|^p\leq1$ a.e. $x\in\Omega$, $\forall t\in[0,T]$, the dominated convergence theorem leads to :

$$u\in C([0,T];L^p(\Omega)).$$

□

2.3 Uniqueness of the weak solution

In this section, we aim to prove the following L^1-contraction principle, which directly implies the uniqueness of the weak solution to the problem (2.2) under Assumption 2.2. The method is inspired from [AL83, Ott96b].

Theorem 2.3.1 *Let u_0, v_0 belong to $L^\infty(\Omega)$, $0 \le u_0, v_0 \le 1$, and let u, v be weak solutions associated to the initial data u_0, v_0. Then under assumptions 2.1 and 2.2, u and v belong to $C([0,T], L^p(\Omega))$ for all $p \in [1, +\infty[$. Furthermore, for all $t \in [0,T]$,*

$$\int_\Omega (u(x,t) - v(x,t))^\pm dxdt \le \int_\Omega (u_0(x) - v_0(x))^\pm dx$$

where $(\cdot)^+$ (resp. $(\cdot)^-$) denotes the positive (resp. negative) part.

Proof
Let u be a weak solution to the problem (2.2) in the sense of Definition 2.2. It is easy to check that $\partial_t u \in L^2(0,T;(H^1(\Omega))')$, and for any $\theta \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T \Big\langle \partial_t u(\cdot,t), \theta(\cdot,t)\Big\rangle dt = -\sum_{i=1}^N \int_0^T \int_{\Omega_i} \nabla\varphi_i(u(x,t)) \cdot \nabla\theta(x,t) dxdt. \tag{2.37}$$

Let $S_n^\pm$ be a Lipschitz continuous non-decreasing function fulfilling :

$$S_n^+(a) = \left\{ \begin{array}{ll} 0 & \text{if} \quad a < 0, \\ 1 & \text{if} \quad a > \frac{1}{n}, \end{array}\right. \qquad S_n^-(a) = -S_n^+(-a) \tag{2.38}$$

Let $\kappa(x)$ such that $\Pi(\kappa(x),x) \in H^1(\Omega)$, then for almost every $x \in \Omega$, the functions $a \mapsto S_n^\pm(\Pi(a,x) - \Pi(\kappa(x),x))$ is non-decreasing, and so $\mu_{n,x}^\pm : a \mapsto \int_0^a S_n^\pm(\Pi(s,x) - \Pi(\kappa(x),x))ds$ is convex. One defines for $u(x,t) = u_0(x)$ if $t < 0$, then for almost every $(x,t) \in \Omega \times (0,T)$, for almost every $\tau > 0$,

$$\mu_{n,x}^\pm(u(x,t)) - \mu_{n,x}^\pm(u(x,t-\tau)) \le S_n^\pm(\Pi(u(x,t),x) - \Pi(\kappa(x),x))(u(x,t) - u(x,t-\tau))$$

One multiplies the previous inequality by $\psi(x,t) \ge 0$, with $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0,T))$, one divides by τ, on integrates on $\Omega_i \times (0,T)$ and sums for $i \in [\![1,N]\!]$, so one gets :

$$\begin{aligned} &\frac{1}{\tau}\sum_{i=1}^N \int_{T-\tau}^T \int_{\Omega_i} \mu_{n,x}^\pm(u(x,t))\psi(x,t)dxdt \\ &-\frac{1}{\tau}\sum_{i=1}^N \int_0^\tau \int_{\Omega_i} \mu_{n,x}^\pm(u_0(x))\psi(x,t)dxdt \\ &+\frac{1}{\tau}\sum_{i=1}^N \int_0^T \int_{\Omega_i} \mu_{n,x}^\pm(u(x,t-\tau))(\psi(x,t-\tau) - \psi(x,t))dxdt \\ &\le \frac{1}{\tau}\sum_{i=1}^N \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^\pm(\Pi_i(u(x,t)) - \Pi_i(\kappa(x))) \\ (u(x,t) - u(x,t-\tau)) \end{array} \right] \psi(x,t)dxdt. \end{aligned} \tag{2.39}$$

One can let τ tend to 0. Since the function $(x,t) \mapsto S_n^{\pm}(\Pi(u(x,t),x) - \Pi(\kappa(x),x))$ belongs to $L^2(0,T;H^1(\Omega))$, one gets :

$$\sum_{i=1}^{N} \int_{\Omega_i} \mu_{n,x}^{\pm}(u_0(x))\psi(x,0)dx + \sum_{i=1}^{N} \int_0^T \int_{\Omega_i} \mu_{n,x}^{\pm}(u(x,t))\partial_t\psi(x,t)dxdt$$
$$\geq -\int_0^T \left\langle \partial_t u(\cdot,t), S_n^{\pm}(\Pi(u(x,t),x) - \Pi(\kappa(x),x))\psi(x,t)\right\rangle dt. \quad (2.40)$$

Thus using (2.37) in (2.40) leads to :

$$\sum_{i=1}^{N} \int_{\Omega_i} \mu_{n,x}^{\pm}(u_0(x))\psi(x,0)dx + \sum_{i=1}^{N} \int_0^T \int_{\Omega_i} \mu_{n,x}^{\pm}(u(x,t))\partial_t\psi(x,t)dxdt$$
$$-\sum_{i=1}^{N} \int_0^T \int_{\Omega_i} S_n^{\pm}(\Pi_i(u(x,t)) - \Pi_i(\kappa(x)))\nabla\varphi_i(u(x,t)) \cdot \nabla\psi(x,t)dxdt$$
$$-\sum_{i=1}^{N} \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(\kappa(x))) \\ \nabla\varphi_i(u(x,t)) \cdot \nabla(\Pi_i(u(x,t)) - \Pi_i(\kappa(x))) \end{array} \right] \psi(x,t)dxdt \geq 0. \quad (2.41)$$

Let ξ belong to $\mathcal{D}^+(\overline{\Omega}\times[0,T)\times(0,T))$. Let v be a weak solution for the problem (2.2) for an initial data v_0 regular enough to insure $v \in C([0,T];L^1(\Omega))$ (e.g. fulfilling Assumption 2.4, thanks to Proposition 2.2.15). For almost every $s \in (0,T)$, one has $\Pi(v(x,s),x) \in H^1(\Omega)$, and so we can substitute $v(x,s)$ to $\kappa(x)$ in (2.41), and integrate for $s \in (0,T)$.

$$\int_0^T \sum_{i=1}^{N} \int_{\Omega_i} \left(\int_0^{u_0(x)} S_n^{\pm}(\Pi_i(a) - \Pi_i(v(x,s)))da \right) \xi(x,0,s)dxds$$
$$+\sum_{i=1}^{N} \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{u(x,t)} S_n^{\pm}(\Pi_i(a) - \Pi_i(v(x,s)))da \right) \partial_t\xi(x,t,s)dxdtds$$
$$-\sum_{i=1}^{N} \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla\varphi_i(u(x,t)) \cdot \nabla\xi(x,t,s) \end{array} \right] dxdtds$$
$$-\sum_{i=1}^{N} \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\Pi_i(u(x,t)) - \Pi_i(v(x,s)))\cdot \\ \nabla\varphi_i(u(x,t))\xi(x,t,s) \end{array} \right] dxdtds \geq 0. \quad (2.42)$$

Inverting the roles of $u(x,t)$ and $v(x,s)$, using $\xi(\cdot,\cdot,0) = 0$, one gets :

$$\sum_{i=1}^{N} \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{v(x,s)} S_n^{\mp}(\Pi_i(a) - \Pi_i(u(x,t)))da \right) \partial_s\xi(x,t,s)dxdtds$$
$$-\sum_{i=1}^{N} \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\mp}(\Pi_i(v(x,s)) - \Pi_i(u(x,t))) \\ \nabla\varphi_i(v(x,s)) \cdot \nabla\xi(x,t,s) \end{array} \right] dxdtds$$
$$-\sum_{i=1}^{N} \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\mp\prime}(\Pi_i(v(x,s)) - \Pi_i(u(x,t))) \\ \nabla(\Pi_i(v(x,s)) - \Pi_i(u(x,t)))\cdot \\ \nabla\varphi_i(v(x,s))\xi(x,t,s) \end{array} \right] dxdtds \geq 0. \quad (2.43)$$

Adding (2.42) and (2.43), and using (2.38), we get :

$$\begin{aligned}
&\int_0^T \sum_{i=1}^N \int_{\Omega_i} \left(\int_0^{u_0(x)} S_n^{\pm}(\Pi_i(a) - \Pi_i(v(x,s)))da \right) \xi(x,0,s)dxds \\
&+ \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{u(x,t)} S_n^{\pm}(\Pi_i(a) - \Pi_i(v(x,s)))da \right) \partial_t \xi(x,t,s)dxdtds \\
+ \quad & \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{v(x,s)} S_n^{\mp}(\Pi_i(a) - \Pi_i(u(x,t)))da \right) \partial_s \xi(x,t,s)dxdtds \\
&- \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,s))) \end{array} \right] \cdot \nabla \xi(x,t,s)dxdtds \\
&- \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,s))) \cdot \\ \nabla(\Pi_i(v(x,s)) - \Pi_i(u(x,t))) \end{array} \right] \xi(x,t,s)dxdtds \geq 0.
\end{aligned}$$

Let us rewrite it $A_1^n + A_2^n + A_3^n + A_4^n \geq 0$, with :

$$A_1^n = \int_0^T \sum_{i=1}^N \int_{\Omega_i} \left(\int_0^{u_0(x)} S_n^{\pm}(\Pi_i(a) - \Pi_i(v(x,s)))da \right) \xi(x,0,s)dxds,$$

$$\begin{aligned}
A_2^n \;=\; & \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{u(x,t)} S_n^{\pm}(\Pi_i(a) - \Pi_i(v(x,s)))da \right) \partial_t \xi(x,t,s)dxdtds \\
&+ \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left(\int_0^{v(x,s)} S_n^{\mp}(\Pi_i(a) - \Pi_i(u(x,t)))da \right) \partial_s \xi(x,t,s)dxdtds,
\end{aligned}$$

$$A_3^n = -\sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,s))) \end{array} \right] \cdot \nabla \xi(x,t,s)dxdtds,$$

$$A_4^n = -\sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \left[\begin{array}{c} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,s))) \cdot \\ \nabla(\Pi_i(v(x,s)) - \Pi_i(u(x,t))) \end{array} \right] \xi(x,t,s)dxdtds.$$

Now, we let n tend to $+\infty$, then, using the dominated convergence theorem and the fact that Π_i is strictly increasing,

$$\lim_{n\to+\infty} A_1^n = \int_0^T \sum_{i=1}^N \int_{\Omega_i} (u_0(x) - v(x,s))^{\pm} \xi(x,0,s)dxds. \tag{2.44}$$

The same way, remarking that

$$Sign^+(\Pi_i(u) - \Pi_i(v)) = Sign^+(u-v) = -Sign^-(v-u),$$

$$\lim_{n\to+\infty} A_2^n = \int_0^T \int_0^T \sum_{i=1}^N \int_{\Omega_i} (u(x,t) - v(x,s))^{\pm} (\partial_t \xi(x,t,s) + \partial_s \xi(x,t,s))dxds, \tag{2.45}$$

$$\lim_{n\to+\infty} A_3^n = -\sum_{i=1}^{N}\int_0^T\int_0^T\int_{\Omega_i} \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,s)))^{\pm}\cdot\nabla\xi(x,t,s)dxdtds. \quad (2.46)$$

$$\begin{aligned}
A_4^n &= -\sum_{i=1}^{N}\int_0^T\int_0^T\int_{\Omega_i}\begin{bmatrix} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \nabla(\varphi_i\circ\Pi_i^{-1}(\Pi_i(u(x,t))) - \varphi_i\circ\Pi_i^{-1}(\Pi_i(v(x,s))))\cdot \\ \nabla(\Pi_i(v(x,s)) - \Pi_i(u(x,t)))\xi(x,t,s)\end{bmatrix}dxdtds \\
&= -\sum_{i=1}^{N}\int_0^T\int_0^T\int_{\Omega_i}\begin{bmatrix} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ (\varphi_i\circ\Pi_i^{-1})'(\Pi_i(u(x,t))) \\ |\nabla(\Pi_i(v(x,s)) - \Pi_i(u(x,t)))|^2\xi(x,t,s)\end{bmatrix}dxdtds \\
&\quad +\sum_{i=1}^{N}\int_0^T\int_0^T\int_{\Omega_i}\begin{bmatrix} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s)))\xi(x,t,s) \\ \nabla(\Pi_i(u(x,t)) - \Pi_i(v(x,s)))\cdot\nabla\Pi_i(v(x,s)) \\ ((\varphi_i\circ\Pi_i^{-1})'(\Pi_i(v(x,s))) - (\varphi_i\circ\Pi_i^{-1})'(\Pi_i(u(x,t))))\end{bmatrix}dxdtds \\
&\le \sum_{i=1}^{N}\int_0^T\int_0^T\int_{\Omega_i}\begin{bmatrix} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ \Big((\varphi_i\circ\Pi_i^{-1})'(\Pi_i(v(x,s))) - (\varphi_i\circ\Pi_i^{-1})'(\Pi_i(u(x,t))\Big) \\ \nabla(\Pi_i(u(x,t)) - \Pi_i(v(x,s)))\cdot\nabla\Pi_i(v(x,s))\xi(x,t,s)\end{bmatrix}dxdtds \\
&\le \sum_{i=1}^{N}\int_0^T\int_0^T\int_{\Omega_i}\begin{bmatrix} S_n^{\pm\prime}(\Pi_i(u(x,t)) - \Pi_i(v(x,s))) \\ M_i|\Pi_i(u(x,t)) - \Pi_i(v(x,s))||\nabla\Pi_i(v(x,s))|\cdot \\ |\nabla(\Pi_i(u(x,t)) - \Pi_i(v(x,s)))|\xi(x,t,s)\end{bmatrix}dxdtds,
\end{aligned}$$

where M_i denotes a Lipschitz constant for the function $(\varphi_i\circ\Pi_i^{-1})'$. Such a constant exists thanks to Assumption 2.2. Let us now define a partition of $\Omega_i\times(0,T)\times(0,T)$:

- $E1 = \{(x,t,s)\in\Omega\times(0,T)\times(0,T), \Pi_i(u)(x,t) = \Pi_i(v)(x,s)\}$,
- $E2 = \{(x,t,s)\in\Omega\times(0,T)\times(0,T), \Pi_i(u)(x,t) \neq \Pi_i(v)(x,s)\}$.

Then

$$\text{for almost every } (x,t,s)\in E1, \qquad \nabla(\Pi_i(u) - \Pi_i(v)) = 0,$$

$$\text{for all } (x,t,s)\in E2, \qquad \lim_{n\to\infty} S_n^{\pm\prime}(\Pi_i(u) - \Pi_i(v)) = 0.$$

Thus for almost every $(x,t,s)\in\Omega\times(0,T)\times(0,T)$:

$$\lim_{n\to\infty} S_n^{\pm\prime}(\Pi_i(u) - \Pi_i(v))\nabla(\Pi_i(u) - \Pi_i(v)) = 0.$$

Furthermore, $(x,t,s)\mapsto S_n^{\pm\prime}(\Pi_i(u) - \Pi_i(v))M_i|\Pi_i(u) - \Pi_i(v)||\nabla\Pi_i(v)||\nabla(\Pi_i(u) - \Pi_i(v))|$ is integrable on $\Omega_i\times(0,T)\times(0,T)$, thus, using the dominated convergence theorem, we can claim that :

$$\liminf_{n\to\infty} A_4^n \le 0. \quad (2.47)$$

So (2.44)-(2.45)-(2.46)-(2.47) implies :

$$\int_0^T \sum_{i=1}^N \int_{\Omega_i} (u_0(x) - v(x,s))^{\pm} \xi(x,0,s) dx ds$$
$$+ \int_0^T \int_0^T \sum_{i=1}^N \int_{\Omega_i} (u(x,t) - v(x,s))^{\pm} (\partial_t \xi(x,t,s) + \partial_s \xi(x,t,s)) dx ds$$
$$- \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,s)))^{\pm} \cdot \nabla \xi(x,t,s) dx dt ds \geq 0. \quad (2.48)$$

Let $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0,T))$, let ρ belong to $\mathcal{D}^+(\mathbb{R})$, with $supp(\rho) \subset [-1,0]$ and $\int_{\mathbb{R}} \rho(s) ds = 1$. For $n \geq 1$, one sets $\rho_n(s) = n\rho(ns)$. One sets $\xi(x,t,s) = \psi(x,t)\rho_n(t-s)$, so that ξ belongs to $\mathcal{D}^+(\overline{\Omega} \times [0,T) \times (0,T))$. One has :

$$\partial_t \xi(x,t,s) + \partial_s \xi(x,t,s) = \partial_t \psi(x,t) \rho_n(t-s),$$

and then inequality (2.48) can be rewritten :

$$\int_0^T \sum_{i=1}^N \int_{\Omega_i} (u_0(x) - v(x,s))^{\pm} \psi(x,0) \rho_n(-s) dx ds$$
$$+ \int_0^T \int_0^T \sum_{i=1}^N \int_{\Omega_i} (u(x,t) - v(x,s))^{\pm} \partial_t \psi(x,t) \rho_n(t-s) dx ds$$
$$- \sum_{i=1}^N \int_0^T \int_0^T \int_{\Omega_i} \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,s)))^{\pm} \cdot \nabla \psi(x,t) \rho_n(t-s) dx dt ds \geq 0. \quad (2.49)$$

The weak solution v has been chosen in $C([0,T], L^1(\Omega))$, (such a solution exists for regular enough initial data v_0, i.e. $\varphi(v_0, \cdot) \in W^{1\infty}_{pw}(\Omega)$, as exposed in Proposition 2.2.15). We can apply the theorem of continuity in mean to let tend n to $+\infty$ in inequality (2.49), thus we get, for all $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0,T))$,

$$\int_0^T \int_\Omega (u(x,t) - v(x,t))^{\pm} \partial_t \psi(x,t) dx dt + \int_\Omega (u_0(x) - v_0(x))^{\pm} \psi(x,0) dx$$
$$- \sum_{i=1}^N \int_0^T \int_{\Omega_i} \nabla(\varphi_i(u(x,t)) - \varphi_i(v(x,t)))^{\pm} \nabla \psi(x,t) dx dt \geq 0. \quad (2.50)$$

The inequality (2.50) still holds for any $\psi \in W^{1,1}(0,T; L^1(\Omega)) \cap L^2(0,T; H^1(\Omega))$ with $\psi(\cdot, T) = 0$, and so for $\psi(x,t) = (T-t)$. In this case, we get the following comparison principle :

$$\int_0^T \int_\Omega (u(x,t) - v(x,t))^{\pm} dx dt \leq T \int_\Omega (u_0(x) - v_0(x))^{\pm} dx. \quad (2.51)$$

This particularly insures the uniqueness of the solution, and its time-continuity, if $\varphi_i(u_0)$ belongs to $W^{1,\infty}(\Omega_i)$. Moreover, we can state the following L^1-contraction principle in this case : $\forall t \in [0,T]$,

$$\int_\Omega (u(x,t) - v(x,t))^{\pm} dx dt \leq \int_\Omega (u_0(x) - v_0(x))^{\pm} dx. \quad (2.52)$$

If u_0 belongs to $L^\infty(\Omega)$, $0 \le u_0 \le 1$, there exists a sequence $(u_{0,n})_{n\ge 1}$ of approximated initial data fulfilling :

– $\forall n \ge 1$, $\varphi_i(u_{0,n}) \in W^{1,\infty}(\Omega_i)$,

– $\|u_{0,n} - u_0\|_{L^1(\Omega)} \to 0$ as $n \to +\infty$.

Let u_n be the unique solution associated to initial data $u_{0,n}$. Then for all $t \in [0,T]$, the sequence $(u_n(\cdot,t))_n$ is a Cauchy sequence thanks to (2.52), and so it converges to $u(\cdot,t)$ thanks to (2.51).

Let $v_0 \in L^\infty(\Omega)$. Let $(v_{0,n})_n$ such that, for all $n \in \mathbb{N}$, $\varphi_i(v_{0,n}) \in W^{1,\infty}(\Omega_i)$ and $v_{0,n} \to v_0$ in the $L^1(\Omega)$-topology. Then, thanks to (2.52),

$$\int_\Omega (u_n(x,t) - v_n(x,t))^\pm dxdt \le \int_\Omega (u_{0,n}(x) - v_{0,n}(x))^\pm dx \tag{2.53}$$

where v_n is the unique weak solution associated to the initial data $v_{0,n}$. We can now let tend n to $+\infty$. Thanks to the short discussion stated above, for all $t \in [0,T]$, $v_n(\cdot,t)$ tends to $v(\cdot,t)$ in the $L^1(\Omega)$-topology. Then we deduce from (2.53) that for any $u_0, v_0 \in L^\infty(\Omega)$, $0 \le u_0, v_0 \le 1$, for any $t \in [0,T]$, one has the following L^1-contraction principle :

$$\int_\Omega (u(x,t) - v(x,t))^\pm dxdt \le \int_\Omega (u_0(x) - v_0(x))^\pm dx. \tag{2.54}$$

Let $t \in [0,T]$, $\tau \in]0, T-t[$. For all $n \ge 1$, one has, using (2.54)

$$\begin{aligned}\int_\Omega (u(x,t+\tau) - u(x,t))^\pm dx &\le \int_\Omega \left[\begin{array}{cc} & (u(x,t+\tau) - u_n(x,t+\tau))^\pm \\ + & (u_n(x,t+\tau) - u_n(x,t))^\pm \\ + & (u(x,t) - u_n(x,t))^\pm \end{array}\right] dx \\ &\le \int_\Omega \left[\begin{array}{cc} & 2(u_0(x) - u_{0,n}(x))^\pm \\ + & (u_n(x,t+\tau) - u_n(x,t))^\pm \end{array}\right] dx\end{aligned}$$

Since $u_n \in C([0,T]; L^1(\Omega))$, one gets, for all $t \in [0,T]$, for all $n \ge 1$,

$$\lim_{\tau \to 0} \int_\Omega (u(x,t+\tau) - u(x,t))^\pm dx \le 2 \int_\Omega (u_0(x) - u_{0,n}(x))^\pm dx.$$

Letting n tend to $+\infty$ gives the time-continuity of u.

□

The proof of Theorem 2.3.1 completes the proof of Theorem 2.1.1.

Chapitre 3

Two-phase flows involving capillary barriers in heterogeneous porous media.

3.1 Presentation of the problem

The models of immiscible two-phase flows are widely used in petroleum engineering, particularly in basin modeling, whose aim can be the prediction of the migration of hydrocarbon components at geological time scale in a sedimentary basin.

The heterogeneousness of the porous medium leads to the phenomena of oil-trapping and oil-expulsion, which is modeled with discontinuous capillary pressures between the different geological layers.

The physical principles models and the mathematical models can be found in [AS79, Bea72, CJ86, vDMdN95, Enc04]. The phenomenon of capillary trapping has been completed only in simplified cases (see [BDPvD03]), and several numerical methods have been developed (see e.g. [EEN98, EEM06]).

The aim of this paper is to introduce a new notion of weak solution, which allows us to deal with more general cases than those treated in [EEM06], while it is equivalent to the notion of weak solution introduced in [EEM06] on the already treated cases. We will consider a simplified model $(\mathcal{P})$ defined page 76, in which the convection is neglected,

We then give a uniqueness result in the one dimensional case which is inspired from the result in [BDPvD03] and extends this latter one to more general situations, by requiring weaker assumptions on the solutions and applying to a larger class of initial data.

We have to make some assumptions on the heterogeneous porous medium :

Assumptions 3.1 (Geometrical assumptions)

1. *The heterogeneous porous medium is represented by a polygonal bounded connected domain $\Omega \subset \mathbb{R}^d$ with $meas_{\mathbb{R}^d}(\Omega) > 0$, where $meas_{\mathbb{R}^n}$ is the Lebesgue's measure of $\mathbb{R}^n$.*
2. *There exists a finite number N of polygonal connected subdomains $(\Omega_i)_{1\le i\le N}$ of Ω such that :*
   (a) for all $i \in [\![1, \mathrm{N}]\!]$, $meas_{\mathbb{R}^d}(\Omega_i) > 0$,

(b) $\bigcup_{i=1}^{N} \overline{\Omega}_i = \overline{\Omega}$,

(c) for $(i,j) \in [\![1,\mathrm{N}]\!]^2$ *with* $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$.

Each Ω_i *represents an homogeneous porous medium. One denotes, for all* $(i,j) \in [\![1,\mathrm{N}]\!]^2$, $\Gamma_{i,j} \subset \Omega$ *the interface between the geological layers* Ω_i *and* Ω_j*, defined by* $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$.

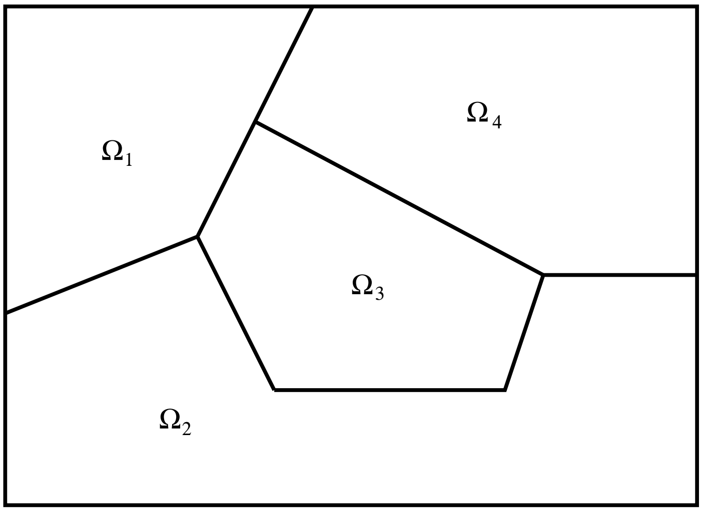

Fig. 3.1 – An example for the domain Ω

We consider an incompressible and immiscible oil-water flow through Ω, and thus through each Ω_i. Using Darcy's law, the conservation of oil and water phases is given for all $(x,t) \in \Omega_i \times (0,T)$,

$$\left\{ \begin{array}{l} \phi_i \partial_t u_i(x,t) - \nabla \cdot \big(\eta_{o,i}(u_i(x,t))(\nabla p_{o,i}(x,t) - \rho_o \mathbf{g})\big) = 0, \\ -\phi_i \partial_t u_i(x,t) - \nabla \cdot \big(\eta_{w,i}(u_i(x,t))(\nabla p_{w,i}(x,t) - \rho_w \mathbf{g})\big) = 0, \\ p_{o,i}(x,t) - p_{w,i}(x,t) = \pi_i(u_i(x,t)), \end{array} \right. \tag{3.1}$$

where $u_i \in [0,1]$ is the oil saturation in Ω_i (and therefore $1 - u_i$ the water saturation), $\phi_i \in]0,1[$ is the porosity of Ω_i, which is supposed to be constant in each Ω_i for the sake of simplicity, $\pi_i(u_i(x,t))$ is the capillary pressure, and $\mathbf{g}$ is the gravity acceleration. The indices o and w respectively stand for the oil and the water phase. Thus, for $\sigma = o, w$, $p_{\sigma,i}$ is the pressure of the phase σ, $\eta_{\sigma,i}$ is the mobility of the phase σ, and ρ_σ is the density of the phase σ.

We have now to make assumptions on the data to explicit the transmission conditions through the interfaces $\Gamma_{i,j}$:

Assumptions 3.2 (Assumptions on the data)

1. *for all* $i \in [\![1,\mathrm{N}]\!]$, $\pi_i \in C^1([0,1],\mathbb{R})$, *with* $\pi_i'(x) > 0$ *for* $x \in]0,1[$,
2. *for all* $i \in [\![1,\mathrm{N}]\!]$, $\eta_{o,i} \in C^0([0,1],\mathbb{R}_+)$ *is an increasing function fulfilling* $\eta_{o,i}(0) = 0$,
3. *for all* $i \in [\![1,\mathrm{N}]\!]$, $\eta_{w,i} \in C^0([0,1],\mathbb{R}_+)$ *is a decreasing function fulfilling* $\eta_{w,i}(1) = 0$,

4. the initial data u_0 *belongs to* $L^\infty(\Omega)$, $0 \le u_0 \le 1$.

One denotes $\alpha_i = \lim_{s\to 0} \pi_i(s)$ and $\beta_i = \lim_{s\to 1} \pi_i(s)$. We can now define the monotonous graphs $\tilde{\pi}_i$ by :

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0,1[, \\ ]-\infty, \alpha_i] & \text{if } s = 0, \\ [\beta_i, +\infty[& \text{if } s = 1. \end{cases} \quad (3.2)$$

As it is exposed in [EEM06], the following conditions must be satisfied on the traces of u_i,

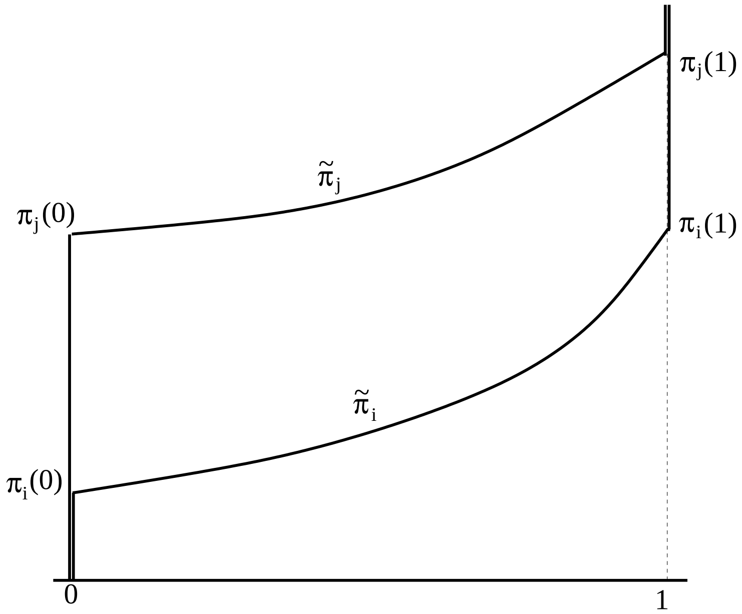

FIG. 3.2 – Graphs for the capillary pressures

$p_{\sigma,i}$ and $\nabla p_{\sigma,i}$ on $\Gamma_{i,j} \times (0,T)$, still denoted respectively u_i, $p_{\sigma,i}$ and $\nabla p_{\sigma,i}$ (see [Bea72]) :

1. for any $\sigma = o, w$, $(i,j) \in [\![1,\mathrm{N}]\!]^2$ such that $\Gamma_{i,j} \neq \emptyset$, the flux of the phase σ through $\Gamma_{i,j}$ must be continuous :

$$\eta_{\sigma,i}(u_i)(\nabla p_{\sigma,i} - \rho_\sigma \mathbf{g}) \cdot \mathbf{n}_i + \eta_{\sigma,j}(u_j)(\nabla p_{\sigma,j} - \rho_\sigma \mathbf{g}) \cdot \mathbf{n}_j = 0, \quad (3.3)$$

   where $\mathbf{n}_i$ denotes the outward normal of $\Gamma_{i,j}$ to Ω_i ;

2. for any $\sigma = o, w$, $(i,j) \in [\![1,\mathrm{N}]\!]^2$ such that $\Gamma_{i,j} \neq \emptyset$, either p_σ is continuous or $\eta_\sigma = 0$. Since the saturation is itself discontinuous across $\Gamma_{i,j}$, one must express the mobility at the upstream side of the interface. This gives

$$\eta_{\sigma,i}(u_i)(p_{\sigma,i} - p_{\sigma,j})^+ - \eta_{\sigma,j}(u_j)(p_{\sigma,j} - p_{\sigma,i})^+ = 0. \quad (3.4)$$

The conditions (4.8) have direct consequences on the behaviour of the capillary pressures on both side of $\Gamma_{i,j}$. Indeed, if $0 < u_i, u_j < 1$, then the partial pressures p_o and p_w have both to be continuous, and so we have the connection of the capillary pressures $\pi_i(u_i) = \pi_j(u_j)$. If $u_i = 0$ and $0 < u_j < 1$, then $p_{o,i} \ge p_{o,j}$ and $p_{w,i} = p_{w,j}$, thus $\pi_j(u_j) \le \pi_i(0)$. The same way, $u_i = 1$ and $0 < u_j < 1$ implies $\pi_j(u_j) \ge \pi_i(1)$. If $u_i = 0$,

$u_j = 1$, then $p_{o,i} \geq p_{o,j}$ and $p_{w,i} \leq p_{w,j}$, so $\pi_i(0) \geq \pi_j(1)$. Checking that the definition of the graphs $\tilde{\pi}_i$ and $\tilde{\pi}_j$ implies $\tilde{\pi}_i(0) \cap \tilde{\pi}_j(0) \neq \emptyset$, $\tilde{\pi}_i(1) \cap \tilde{\pi}_j(1) \neq \emptyset$, we can claim that (4.8) leads to :

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset. \tag{3.5}$$

We introduce the global pressure in Ω_i

$$\overline{p}_i(x,t) = p_{w,i}(x,t) + \int_0^{u_i(x,t)} \frac{\eta_{o,i}(a)}{\eta_{o,i}(a) + \eta_{w,i}(a)} \pi_i'(a)\mathrm{d}a \tag{3.6}$$

(see e.g. [AKM90] or [CJ86]), and the global mobility in Ω_i

$$\lambda_i(u_i(x,t)) = \frac{\eta_{o,i}(u_i(x,t))\eta_{w,i}(u_i(x,t))}{\eta_{o,i}(u_i(x,t)) + \eta_{w,i}(u_i(x,t))} \tag{3.7}$$

which verifies $\lambda_i(0) = \lambda_i(1) = 0$, and $\lambda_i(s) > 0$ for $0 < s < 1$. Taking into account (3.6) and (3.7) in (3.1), and adding the conservation laws leads to, for $(x,t) \in \Omega_i \times (0,T)$:

$$\left\{ \begin{array}{l} \phi_i \partial_t u_i(x,t) - \nabla \cdot \big(\eta_{o,i}(u_i(x,t))(\nabla \overline{p}_i(x,t) - \rho_o \mathbf{g}) - \lambda_i(u_i(x,t)) \nabla \pi_i(u_i(x,t))\big) = 0, \\ -\nabla \cdot \left(\displaystyle\sum_{\sigma = o,w} \eta_{\sigma,i}(u_i(x,t))(\nabla \overline{p}_i(x,t) - \rho_\sigma \mathbf{g}) \right) = 0. \end{array} \right. \tag{3.8}$$

We neglect the convective effects, so that we focus on the mathematical modeling of flows with discontinuous capillary pressures, which seem to necessary to explain the phenomena of oil trapping. This simplification will allow us to neglect the coupling with the second equation of (3.8), and we get the simple degenerated parabolic equation in $\Omega_i \times (0,T)$:

$$\phi_i \partial_t u_i(x,t) - \nabla \cdot (\lambda_i(u_i(x,t)) \nabla \pi_i(u_i(x,t))) = 0 \quad \text{in } \Omega_i \times (0,T). \tag{3.9}$$

In this simplified framework, the transmission condition (3.3) on the fluxes through $\Gamma_{i,j}$ can be rewritten :

$$\lambda_i(u_i(x,t)) \nabla(\pi_i(u_i(x,t))) \cdot \mathbf{n}_i + \lambda_j(u_j(x,t)) \nabla(\pi_j(u_j(x,t))) \cdot \mathbf{n}_j = 0 \quad \text{on } \Gamma_{i,j} \times (0,T). \tag{3.10}$$

We suppose furthermore that $u_i(x,0) = u_0(x)$ for $x \in \Omega_i$. In the remainder of this paper, we suppose to take a homogeneous Neumann boundary condition, The existence of a weak solution proven in section 3.3 can be extended to the case of non-homogeneous Dirichlet conditions. Nevertheless, homogeneous Neumann boundary conditions are needed to prove the theorem 3.4.1, and thus to prove the conclusion theorem 3.5.4

Taking into account the equations (4.9), (3.9), (3.10), the boundary condition, and the initial condition, we can write the problem we aim to solve this way : for all $i \in [\![1, \mathrm{N}]\!]$, for all $j \in [\![1, \mathrm{N}]\!]$ such that $\Gamma_{i,j} \neq \emptyset$,

$$\left\{ \begin{array}{ll} \phi_i \partial_t u_i - \nabla \cdot (\lambda_i(u_i) \nabla \pi_i(u_i)) = 0 & \text{in } \Omega_i \times (0,T), \\ \tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_i(u_i) \nabla(\pi_i(u_i)) \cdot \mathbf{n}_i + \lambda_j(u_j) \nabla(\pi_j(u_j)) \cdot \mathbf{n}_j = 0 & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_i(u_i) \nabla(\pi_i(u_i)) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times (0,T), \\ u_i(\cdot, 0) = u_0(x) & \text{in } \Omega_i. \end{array} \right. \tag{$\mathcal{P}$}$$

Remark 3.1.1 *All the results presented in this paper still hold if one not neglects the effect of the gravity and if one assumes that the global pressure is known, that is for problems of the type :*

$$\begin{cases} \phi_i \partial_t u_i + \nabla \cdot (\mathbf{q} f_i(u_i) + \lambda_i(u_i)(\rho_o - \rho_w)\mathbf{g} - \lambda_i(u_i)\nabla \pi_i(u_i)) = 0 & \text{in } \Omega_i \times (0,T), \\ \tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset & \text{on } \Gamma_{i,j} \times (0,T), \\ \displaystyle\sum_{k=i,j} (\mathbf{q} f_k(u_k) + \lambda_k(u_k)(\rho_o - \rho_w)\mathbf{g} - \lambda_k(u_k)\nabla \pi_k(u_k)) \cdot \mathbf{n}_k = 0 & \text{on } \Gamma_{i,j} \times (0,T), \\ (\mathbf{q} f_i(u_i) + \lambda_i(u_i)(\rho_o - \rho_w)\mathbf{g} - \lambda_i(u_i)\nabla \pi_i(u_i)) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times (0,T), \\ u_i(\cdot, 0) = u_0(x) & \text{in } \Omega_i, \end{cases}$$

where f_i is supposed to be a $C^1([0,1],\mathbb{R})$-increasing function, λ_i is also supposed to belong to $C^1([0,1],\mathbb{R}_+)$ and $\mathbf{q}$ satisfies

- $\forall i$, $\mathbf{q} \in \left(C^1(\overline{\Omega}_i \times [0,T])\right)^d$,
- $\nabla \cdot \mathbf{q} = 0$ *in* $\Omega_i \times (0,T)$,
- $\mathbf{q}_{|\Omega_i} \cdot \mathbf{n}_i + \mathbf{q}_{|\Omega_j} \cdot \mathbf{n}_j = 0$ *on* $\Gamma_{i,j} \times (0,T)$,
- $\mathbf{q} \cdot \mathbf{n} = 0$.

In order to ensure the uniqueness result stated in theorem 3.5.1, the technical condition (see [AL83] or [Ott96b]) :

$$\forall i, \qquad f_i \circ \varphi_i^{-1}, \lambda_i \circ \varphi_i^{-1} \in C^{0,1/2}([0,\varphi_i(1)],\mathbb{R}).$$

Remark 3.1.2 *In the modeling of two-phase flows, irreducible saturations are often taken into account. One can suppose that there exists s_i and S_i ($0 < s_i < S_i < 1$) such that $\lambda_i(s) = 0$ if $s \notin (s_i, S_i)$. In such a case, the problem $(\mathcal{P})$ becomes strongly degenerated, but a convenient scaling eliminates this difficulty (at least if $s_i \leq u_0 \leq S_i$ a.e. in Ω_i). Moreover, the dependance of the capillary pressure with regard to the saturation can be weak, at least for saturations not too close to 0 or 1. Thus the effects of the capillarity are often neglected for the study of flows in homogeneous porous media, leading to the Buckley-Leverett equation (see e.g. [GMT96]). Looking for degeneracy of $u \mapsto \pi_i(u)$ is a more complex problem, particularly if the convection is not neglected as above. Suppose for example that $\pi_i(u) = \varepsilon u + P_i$, where P_i are constants, and let ε tend 0. Non-classical shocks can appear at the level of the interfaces $\Gamma_{i,j}$ (see [Canb]). Thus the notion of entropy solution used by Adimurthi, J. Jaffré, and G.D. Veerappa Gowda [AJVG04] is not sufficient to deal with this problem. This difficulty has to be overcome to consider degenerate parabolic problem. But it seems clear that the notion of entropy solution developed by K.H. Karlsen, N.H. Risebro, J.D. Towers [KRT02a, KRT02b, KRT03] is not adapted to our problem.*

3.2 The notion of weak solution

In this section, we introduce the notion of weak solution to the problem $(\mathcal{P})$, which is more general than the notion of weak solution given in [Enc04, EEM06]. Indeed, we are able to define such a solution even in the case of an arbitrary finite number of different homogeneous porous media. Furthermore, the notion of weak solution introduced in this paper is still available in cases where the one defined in [EEM06] has no more sense. We finally show that the two notions of solution are equivalent in the case where the notion of weak solution in the sense of [EEM06] is well defined. The existence of a weak solution to problem $(\mathcal{P})$ in a wider case is the aim of the section 3.3.

One denotes by φ_i the $C^1([0,1],\mathbb{R}_+)$ function which naturally appears in the problem $(\mathcal{P})$ and which is defined by : $\forall s \in [0,1]$,

$$\varphi_i(s) = \int_0^s \lambda_i(a)\pi_i'(a)da. \tag{3.11}$$

Remark 3.2.1 *The assumptions on the data insure that $\varphi_i' > 0$ on $]0,1[$, and so we can define an increasing continuous function $\varphi_i^{-1} : [0, \varphi_i(1)] \to [0,1]$.*

We are now able to define the notion of weak solution to the problem $(\mathcal{P})$.

Definition 3.1 (weak solution to the problem $(\mathcal{P})$) *Under assumptions 3.1 and 3.2, a function u is said to be a weak solution to the problem $(\mathcal{P})$ if it verifies :*

1. *$u \in L^\infty(\Omega \times (0,T))), 0 \le u \le 1$ a.e. in $\Omega \times (0,T)$,*
2. *$\forall i \in [\![1,\mathrm{N}]\!]$, $\varphi_i(u_i) \in L^2(0,T;H^1(\Omega_i))$, where u_i denotes the restriction of u to $\Omega_i \times (0,T)$,*
3. *$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times (0,T)$,*
4. *for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T))$,*

$$\begin{aligned}\sum_{i=1}^N \int_{\Omega_i}\int_0^T \phi_i u_i(x,t)\partial_t\psi(x,t)dxdt + \sum_{i=1}^N \int_{\Omega_i} \phi_i u_0(x)\psi(x,0)dx \\ - \sum_{i=1}^N \int_{\Omega_i}\int_0^T \nabla\varphi_i(u_i(x,t)) \cdot \nabla\psi(x,t)dxdt = 0.\end{aligned} \tag{3.12}$$

The third point of the previous definition, which insures the connection in the graph sense of the capillary pressures on the interfaces between several porous media, is well defined. Indeed, since $\varphi_i(u_i)$ belongs to $L^2(0,T;H^1(\Omega_i))$, it admits a trace still denoted $\varphi_i(u_i)$ on $\Gamma_{i,j} \times (0,T)$. Thanks to the remark 3.2.1, we can define the trace of u_i on $\Gamma_{i,j} \times (0,T)$.

Remark 3.2.2 *One can equivalently substitute the condition :*

$$3bis. \quad \check{\pi}_i(u_i) \cap \check{\pi}_j(u_j) \neq \emptyset \text{ a.e. on } \Gamma_{i,j} \times (0,T),$$

to the third point of the definition 3.1, where $\check{\pi}_i$ is the monotonous graph given by :

$$\check{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0,1[, \\ [\min_j(\alpha_j), \alpha_i] & \text{if } s = 0, \\ [\beta_i, \max_j(\beta_j)] & \text{if } s = 1. \end{cases} \tag{3.13}$$

We will now quickly show the equivalence between the notion of weak solution to the problem $(\mathcal{P})$ and the notion of weak solution given in [EEM06], in the case where this one is well defined, i.e. $N = 2$ and $\max(\alpha_1, \alpha_2) = \alpha < \beta = \min(\beta_1, \beta_2)$. We denote as in

[EEM06] the truncated capillary pressures by $\hat\pi_1 = \max(\alpha, \pi_1)$, $\hat\pi_2 = \min(\beta, \pi_2)$, and we introduce the problem $(\widetilde{\mathcal{P}})$, which is treated in [EEM06].

$$\left\{\begin{array}{ll} \phi_i \partial_t u_i - \nabla\cdot(\lambda_i(u_i)\nabla\pi_i(u_i)) = 0 & \text{in } \Omega_i \times (0,T), \\ \hat\pi_1(u_1) = \hat\pi_2(u_2) & \text{on } \Gamma_{i,j}\times(0,T), \\ \lambda_1(u_1)\nabla(\pi_1(u_1))\cdot\mathbf{n}_1 + \lambda_2(u_2)\nabla(\pi_2(u_2))\cdot\mathbf{n}_2 = 0 & \text{on } \Gamma_{i,j}\times(0,T), \\ \lambda_i(u_i)\nabla(\pi_i(u_i))\cdot\mathbf{n}_i = 0 & \text{on } \partial\Omega_i\cap\partial\Omega\times(0,T), \\ u_i(\cdot,0) = u_0(x) & \text{in } \Omega_i. \end{array}\right. \qquad (\widetilde{\mathcal{P}})$$

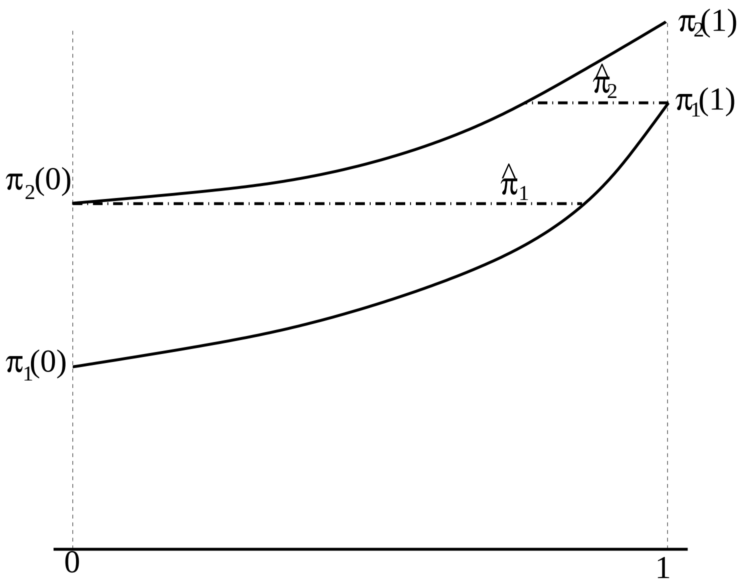

FIG. 3.3 – Truncated capillary pressures

Then it is easy to check that : $\forall (s_1, s_2) \in [0,1]^2$,

$$\hat\pi_1(s_1) = \hat\pi_2(s_2) \Leftrightarrow \tilde\pi_1(s_1)\cap\tilde\pi_2(s_2) \neq \emptyset \Leftrightarrow \breve\pi_1(s_1)\cap\breve\pi_2(s_2)\neq\emptyset. \qquad (3.14)$$

In order to recall the definition of weak solution, we have to introduce the function

$$\Psi : \left\{\begin{array}{l} [\alpha,\beta] \to \mathbb{R} \\ p \mapsto \displaystyle\int_\alpha^p \min_{j=1,2}(\lambda_j \circ \pi_j^{-1}(a))\mathrm{d}a. \end{array}\right.$$

Ψ is increasing, and for $i = 1, 2$, $\Psi\circ\hat\pi_i\circ\varphi_i^{-1}$ is a Lipschitz continuous function.

Definition 3.2 (weak solution to the problem $(\widetilde{\mathcal{P}})$) *A function u is said to be a weak solution to the problem $(\widetilde{\mathcal{P}})$ if it verifies :*

1. $u \in L^\infty(\Omega\times(0,T))), 0 \le u \le 1$ *a.e. in* $\Omega\times(0,T)$,
2. $\forall i \in \{1,2\}, \varphi_i(u_i) \in L^2(0,T;H^1(\Omega_i))$,
3. $w : \Omega\times(0,T)\to\mathbb{R}$, *defined for* $(x,t)\in\Omega_i\times(0,T)$ *by* $w(x,t) = \Psi\circ\hat\pi_i(u_i)(x,t)$ *belongs to* $L^2(0,T;H^1(\Omega))$,

4. for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T))$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \phi_i u_i(x,t) \partial_t \psi(x,t) dx dt + \sum_{i=1}^{N} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx$$
$$- \sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x,t)) \cdot \nabla \psi(x,t) dx dt = 0.$$

Remark 3.2.3 *The notion of weak solution to the problem* $(\widetilde{\mathcal{P}})$ *can be adapted in the case where there are* $N > 2$ *homogeneous domains, but we have to keep conditions of compatibility on* $(\alpha_i)_{1 \le i \le N}$ *and* $(\beta_i)_{1 \le i \le N}$.

Proof of the equivalence of the weak solutions

On the one hand, if u is a weak solution to the problem $(\widetilde{\mathcal{P}})$ in the sense of definition 3.2, then for a.e. $t \in (0,T)$, $w(\cdot,t) \in H^1(\Omega)$,and particularly $w(\cdot,t)$ admits a trace on $\Gamma_{i,j}$, whose value is in the same time $\Psi(\hat{\pi}_i(u_i(\cdot,t)))$ and $\Psi(\hat{\pi}_j(u_j(\cdot,t)))$. Since Ψ in increasing, for a.e $(x,t) \in \Gamma_{i,j} \times (0,T)$, $\hat{\pi}_i(u_i(x,t)) = \hat{\pi}_j(u_j(x,t))$. Using (3.14), we conclude that any weak solution to the problem $(\widetilde{\mathcal{P}})$ is a weak solution to the problem $(\mathcal{P})$ in the sense of definition 3.1.

On the other hand, if u is a weak solution to the problem $(\mathcal{P})$ in the sense of definition 3.1, then thanks to (3.14), for almost every $(x,t) \in \Gamma_{i,j} \times (0,T)$,

$$\hat{\pi}_i(u_i(x,t)) = \hat{\pi}_j(u_j(x,t)) \Leftrightarrow \Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}(\varphi_i(u_i(x,t))) = \Psi \circ \hat{\pi}_j \circ \varphi_j^{-1}(\varphi_j(u_j(x,t))). \quad (3.15)$$

Since $\Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}$ is a Lipschitz continuous function, the second point in definition 3.1 insures us that $\Psi \circ \hat{\pi}_i(u_i)$ belongs to $L^2(0,T,H^1(\Omega_i))$ for $i = 1,2$, and (3.15) insures the connection of the traces on $\Gamma_{i,j} \times (0,T)$, then the third point of definition 3.2 is fulfilled and u is a weak solution to the problem $(\widetilde{\mathcal{P}})$. □

Remark 3.2.4 *We can define a* function $\tilde{\pi}_i^{-1}, i \in [\![1,\mathrm{N}]\!]$, *which verifies* $\tilde{\pi}_i^{-1} \circ \tilde{\pi}_i(s) = s$ *for any* $s \in [0,1]$. *Using the function defined on* $\mathbb{R}$ *by* $\tilde{\Psi}(p) = \int_{-\infty}^{p} \min_{j=1,2}(\lambda_j \circ \tilde{\pi}_j^{-1}(a)) da$, *it is easy to check that we can equivalently substitute the function* $\tilde{\Psi} \circ \pi_i(u_i)$ *to* $\Psi \circ \hat{\pi}_i(u_i)$ *in the third point of definition 3.2. This function is still defined if* $\alpha \ge \beta$, *but it becomes identically* 0, *so the notion of weak solution to the problem* $(\widetilde{\mathcal{P}})$ *is weaker than the notion of weak solution to the problem* $(\mathcal{P})$. *Indeed, in such a case,* $u(x,t) = u_0(x) = a \in]0,1[$ *for any* $(x,t) \in \Omega \times (0,T)$ *is a weak solution to the problem* $(\widetilde{\mathcal{P}})$, *but it does not fulfill the third point in definition 3.1.*

3.3 Existence of a weak solution

The aim of this section is to prove the following theorem, which claims the existence of a weak solution to the problem $(\mathcal{P})$. This result has already been proven in section 3.2 in the case $N = 2$ and $\alpha > \beta$, for which the notion of weak solution in the sense of definition 3.1 is equivalent to the notion of weak solution in the sense of definition 3.2.

Theorem 3.3.1 (Existence of a weak solution) *Under assumptions 3.1 and 3.2, there exists a weak solution to problem* $(\mathcal{P})$ *in the sense of definition 3.1.*

Proof

In order to prove the existence of a weak solution to the problem $(\mathcal{P})$ in the sense of the definition 3.1, we build a sequence of solutions to approximated problems (3.16), which converges, up to a subsequence, toward a weak solution to the problem $(\mathcal{P})$. The approximated problems do not involve capillary barriers, so existence and uniqueness of such approximated solutions is given in [Can08a]. We let the proof of the following technical lemma to the reader.

Lemma 3.3.2 *There exists sequences $(\lambda_{i,n})_n$, $(\pi_{i,n})_n$ belonging to $(C^\infty([0,1],\mathbb{R}))^{\mathrm{N}}$ such that, for $i \in [\![1,\mathrm{N}]\!]$, and for n large enough :*

- $\lambda_{i,n|[0,1/n]\cup[1-1/n,1]} = \dfrac{1}{n^2}$, $\lambda_{i,n}(s) > \dfrac{1}{2n^2}$, *for all* $s \in [0,1]$, $\lambda_{i,n} \to \lambda_i$ *uniformly on* $[0,1]$,
- $\pi_{i,n}(0) = \pi_{j,n}(0) \to -\infty$, $\pi_{i,n}(1) = \pi_{j,n}(1) \to +\infty$, $Kn^{\frac{3}{2}} > \pi'_{i,n} \geq \dfrac{1}{n}$, $\pi_{i,n} \to \pi_i$ *in* $L^1(0,1)$, $\pi_{i,n} \to \pi_i$ *and* $\pi'_{i,n} \to \pi'_i$ *uniformly on any compact set of* $]0,1[$,
- *the function* $\varphi_{i,n} : s \mapsto \int_0^s \lambda_{i,n}(a)\pi'_{i,n}(a)da$ *furthermore fulfills* $\varphi_{i,n}([0,1]) = \varphi_i([0,1])$ *and* $\varphi_{i,n} \to \varphi_i$ *in* $W^{1,\infty}(0,1)$.

We also define the increasing functions :

$$\Psi_n : \begin{cases} [a_n, b_n] \to \mathbb{R} \\ p \mapsto \displaystyle\int_{a_n}^{p} \min_{j\in[\![1,\mathrm{N}]\!]} (\lambda_{j,n} \circ \pi_{j,n}^{-1}(a))\mathrm{d}a. \end{cases}$$

The conditions on the functions on the intervals $[0,\frac{1}{n}] \cup [1-\frac{1}{n},1]$ insures that for any fixed large n, the functions $(\varphi_{i,n} \circ \pi_{i,n}^{-1} \circ \Psi_n^{-1})'$ are Lipschitz continuous. Then thanks to [Can08a], for all n, the approximated problems :

$$\begin{cases} \phi_i \partial_t u_{i,n} - \nabla\cdot(\lambda_{i,n}(u_{i,n})\nabla\pi_{i,n}(u_{i,n})) = 0 & \text{in } \Omega_i \times (0,T), \\ \pi_{i,n}(u_{i,n}) = \pi_{j,n}(u_{j,n}) & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_{i,n}(u_{i,n})\nabla(\pi_{i,n}(u_{i,n}))\cdot\mathbf{n}_i + \lambda_{j,n}(u_{j,n})\nabla(\pi_{j,n}(u_{j,n}))\cdot\mathbf{n}_j = 0 & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_{i,n}(u_{i,n})\nabla(\pi_{i,n}(u_{i,n}))\cdot\mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times (0,T), \\ u_{i,n}(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.16}$$

admit a unique weak solution in the sense of definition 3.3 given below, and this solution belongs to $C([0,T], L^p(\Omega))$ for $1 \leq p < +\infty$.

Definition 3.3 (Weak solutions for approximated problems)
A function u_n is said to be a weak solution to the problem (3.16) if it verifies :

1. $u_n \in L^\infty(\Omega \times (0,T))), 0 \leq u_n \leq 1$ *a.e. in* $\Omega \times (0,T)$,
2. $\forall i \in \{1,2\}, \varphi_{i,n}(u_{i,n}) \in L^2(0,T;H^1(\Omega_i))$,
3. $w_n : \Omega \times (0,T) \to \mathbb{R}$, *defined on* $\Omega_i \times (0,T)$ *by* $w_n = \Psi_n \circ \pi_{i,n}(u_{i,n})$ *belongs to* $L^2(0,T;H^1(\Omega))$,

4. for all $\psi \in \mathcal{D}(\Omega \times [0,T))$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \phi_i u_{i,n}(x,t) \partial_t \psi(x,t) dx dt + \sum_{i=1}^{N} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx \\ - \sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \nabla \varphi_{i,n}(u_{i,n}(x,t)) \cdot \nabla \psi(x,t) dx dt = 0. \tag{3.17}$$

The proof of existence of a weak solution given in [Can08a], shows that for all $i \in [\![1, \mathrm{N}]\!]$, for all n, there exists $C_1 > 0$ not depending on n such that, for all $i \in [\![1, \mathrm{N}]\!]$:

$$\|\varphi_{i,n}(u_{i,n})\|^2_{L^2(0,T;H^1(\Omega_i))} \leq C_1 \|\pi_{i,n}\|_{L^1(0,1)}, \tag{3.18}$$

thus $(\varphi_{i,n}(u_{i,n}))_n$ is a bounded sequence of $L^2(0,T;H^1(\Omega_i))$ using lemma 3.3.2. A study of the proof of the time translate estimate used in [Can08a, EEM06], and detailed in [EGH00, lemma 4.6] leads to the existence of C_2 not depending on n such that :

$$\|\varphi_{i,n}(u_{i,n}(\cdot, \cdot + \tau)) - \varphi_{i,n}(u_{i,n}(\cdot,\cdot))\|^2_{L^2(\Omega_i \times (0,T-\tau))} \leq \tau C_2 \|\pi_{i,n}\|_{L^1(0,1)} \|\varphi'_{i,n}\|_{L^\infty(0,1)}. \tag{3.19}$$

Using lemma 3.3.2 once again, estimates (3.18), (4.56) allow us to apply Kolmogorov's compactness criterion (see e.g. [Bré83]), thus we can claim the relative compactness of the sequence $(\varphi_{i,n}(u_{i,n}))_n$ in $L^2(\Omega_i \times (0,T))$. There exists $f_i \in L^2(0,T;H^1(\Omega_i))$ such that

$$\varphi_{i,n}(u_{i,n}) \to f_i \text{ in } L^2(\Omega_i \times (0,T)),$$
$$\varphi_{i,n}(u_{i,n}) \to f_i \text{ weakly in } L^2(0,T;H^1(\Omega_i)).$$

Let us now recall a very useful lemma, classically called Minty trick, and introduced in this framework by Leray and Lions in the famous paper [LL65].

Lemma 3.3.3 (Minty trick) *Let* $(\phi_n)_n$ *be a sequence of non-decreasing functions with for all* n, $\phi_n : \mathbb{R} \to \mathbb{R}$*, and let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a non-decreasing continuous function such that :*
- $\phi_n \to \phi$ *pointwise,*
- *there exists* $g \in L^1_{loc}(\mathbb{R})$ *such that* $|\phi_n| \leq g$.

Let $\mathcal{O}$ *be an open subset of* $\mathbb{R}^k$, $k \geq 1$. *Let* $(u_n)_n \in (L^\infty(\mathcal{O}))^{\mathbb{N}}$, *let* $u \in L^\infty(\mathcal{O})$ *and let* $f \in L^1(\mathcal{O})$ *such that :*
- $u_n \to u$ *in the* $L^\infty(\mathcal{O})$*-weak-*$\star$ *sense,*
- $\phi_n(u_n) \to f$ *in* $L^1(\mathcal{O})$.

Then

$$f = \phi(u).$$

Since $0 \leq u_{i,n} \leq 1$, $(u_{i,n})_n$ converges up to a subsequence to u_i in the $L^\infty(\Omega_i \times (0,T))$-weak-$\star$ sense. $(\varphi_{i,n})_n$ converges uniformly toward φ_i on $[0,1]$, and we can easily check, using Minty trick, that $f_i = \varphi_i(u_i) \in L^2(0,T;H^1(\Omega_i))$. Thus we can pass to the limit in the formulation (3.17) to obtain the wanted weak formulation :

$$\sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \phi_i u_i(x,t) \partial_t \psi(x,t) dx dt + \sum_{i=1}^{N} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx \\ - \sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x,t)) \cdot \nabla \psi(x,t) dx dt = 0.$$

The last point needed to achieve the proof of theorem 3.3.1 is the convergence of the traces of the approximate solutions $(u_{i,n})_n$ on $\Gamma_{i,j} \times (0,T)$ toward the trace of u_i, and to verify that $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times (0,T)$.

Since Ω_i has a Lipschitz boundary, there exists an operator P, continuous from $H^1(\Omega_i)$ into $H^1(\mathbb{R}^d)$, and also from $L^2(\Omega_i)$ into $L^2(\mathbb{R}^d)$, such that $Pv_{|\Omega_i} = v$ for all $v \in L^2(\Omega_i)$. Then P is continuous from $H^s(\Omega_i)$ into $H^s(\mathbb{R}^d)$ for all $s \in [0,1]$. One has, for all $v \in H^s(\Omega_i)$,

$$\|v\|_{H^s(\Omega_i)} \leq \|Pv\|_{H^s(\mathbb{R}^d)} \leq \|Pv\|^s_{H^1(\mathbb{R}^d)} \|Pv\|^{1-s}_{L^2(\mathbb{R}^d)} \leq C\|v\|^s_{H^1(\Omega_i)} \|v\|^{1-s}_{L^2(\Omega_i)}.$$

One deduces from the previous inequality and from (4.56) that for all $s \in]0,1[$, for all $\tau \in]0,T[$, there exists C_3 not depending on n, τ such that

$$\|\varphi_{i,n}(u_{i,n}(\cdot, \cdot + \tau)) - \varphi_{i,n}(u_{i,n}(\cdot,\cdot))\|^2_{L^2(0,T-\tau;H^s(\Omega_i))} \leq \tau^{1-s} C_3 \tag{3.20}$$

For $s_1 > s_2$, H^{s_1} is compactly imbedded in H^{s_2}, and then estimate (3.20) allows us to claim that the sequence $(\varphi_{i,n}(u_{i,n}))_n$ is relatively compact in $L^2(0,T;H^s(\Omega_i))$ for all $s \in]0,1[$. Particularly, one can extract a subsequence converging toward $\varphi_i(u_i)$ in $L^2(0,T;H^s(\Omega_i))$. We can claim, using once again Minty trick, that the traces of $(\varphi_{i,n}(u_{i,n}))_n$ on $\Gamma_{i,j}$ also converge toward the trace of $\varphi_i(u_i)$, still denoted $\varphi_i(u_i)$ in $L^2(0,T;H^{s-1/2}(\Gamma_{i,j}))$, and particularly for almost every $(x,t) \in \Gamma_{i,j} \times (0,T)$. Since φ_i is increasing, $(u_{i,n}(x,t))_n$ converges almost everywhere on $\Gamma_{i,j} \times (0,T)$ toward $u_i(x,t)$.

Let us now check that $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times (0,T)$. For almost every $(x,t) \in \Gamma_{i,j} \times (0,T)$ the sequence $(\pi_{i,n}(u_{i,n}(x,t)))_n$ converges (up to a new extraction) toward $\gamma_i(x,t) \in \overline{\mathbb{R}}$. Since for all n, $\pi_{i,n}(u_{i,n}(x,t)) = \pi_{j,n}(u_{j,n}(x,t))$, one has :

$$\gamma_i(x,t) = \gamma_j(x,t) \text{ a.e. on } \Gamma_{i,j} \times (0,T). \tag{3.21}$$

If $u_i(x,t) \in]0,1[$, then $\gamma_i(x,t) = \pi_i(u_i(x,t))$. If $u_i(x,t) = 0$, $\gamma_i(x,t) \leq \alpha_i$, and $\gamma_i(x,t) \in \tilde{\pi}_i(0)$. In the same way, if $u_i(x,t) = 1$, $\gamma_i(x,t) \in \tilde{\pi}_i(1)$.

This achieves the proof of theorem 3.3.1, because relation (3.21) insures the connection of the traces in the sense of :

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset \text{ a.e. on } \Gamma_{i,j} \times (0,T).$$

□

3.4 A regularity result

In this section and in section 3.5, we show the existence and the uniqueness of a solution with bounded flux to the problem $(\mathcal{P})$ in the one-dimensional case. We make the proofs in the case where there are only two sub-domains $\Omega_1 =]-1,0[$ and $\Omega_2 =]0,1[$, but a straightforward adaptation of them gives the same result for an arbitrary finite number of Ω_i, each one with an arbitrary finite measure. We now state the main result of this section, which claims the existence of a solution with bounded spatial derivatives on $\mathcal{Q}_i$, where $\mathcal{Q}_i = \Omega_i \times (0,T)$. We also set $\mathcal{Q} =]-1,1[\times]0,T[$ and $\Gamma = \{x = 0\}$.

Theorem 3.4.1 (Existence of a bounded flux solution) *Let $u_0 \in L^\infty(-1,1)$, $0 \leq u_0 \leq 1$ such that :*

– $\varphi_i(u_0) \in W^{1,\infty}(\Omega_i)$,
– $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$ *on* Γ.
Then there exists a weak solution u *to the problem* $(\mathcal{P})$ *such that* $\partial_x \varphi_i(u_i) \in L^\infty(\mathcal{Q}_i)$.

All the section will be devoted to the proof of the theorem 3.4.1. As in section 3.3, we will get this existence result by taking the limit of a sequence of solutions to approximate problems (3.16) involving no capillary barriers, whose data fulfill the properties stated in lemma 3.3.2.

Proof
We will now build a sequence of approximate initial data $(u_{0,n})$ adapted to the sequence of approximate problems.

Lemma 3.4.2 *Let* u_0 *be chosen as in theorem 3.4.1, then there exists* $(u_{0,n})_n$ *such that, for all* n*,*
– $0 \le u_{0,n} \le 1$,
– $\pi_{1,n}(u_{0,n,1}) = \pi_{2,n}(u_{0,n,2})$ *on* Γ.
The sequence $(u_{0,n})_n$ *furthermore fulfills :*

$$\lim_{n\to\infty} \|u_{0,n} - u_0\|_\infty = 0, \qquad \|\partial_x \varphi_{i,n}(u_{0,n})\|_{L^\infty(\Omega_i)} \le \|\partial_x \varphi_i(u_0)\|_{L^\infty(\Omega_i)}. \tag{3.22}$$

Proof
Since $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, then there exists $(a_{1,n}, a_{2,n}) \in [0,1]^2$ such that one has $\pi_{1,n}(a_{1,n}) = \pi_{2,n}(a_{2,n})$ and $|a_{1,n} - u_{0,1}| + |a_{2,n} - u_{0,2}| \to 0$. One sets, for $x \in \Omega_i$:

$$u_{0,n}(x) = \varphi_{i,n}^{-1}\left(T_{\varphi_i}\left[\varphi_i(u_0) + \varphi_{i,n}(a_{i,n}) - \varphi_i(u_{0,i})\right]\right)$$

where

$$T_{\varphi_i}(s) = \begin{cases} s & \text{if} \quad s \in [0, \varphi_i(1)] = [0, \varphi_{i,n}(1)], \\ \varphi_{i,n}(1) & \text{if} \quad s > \varphi_i(1), \\ 0 & \text{if} \quad s < 0. \end{cases}$$

Then the sequence $(u_{0,n})$ converges uniformly toward u_0. For all n, $0 \le u_{0,n} \le 1$ and either $\partial_x \varphi_{i,n}(u_{0,n}) = \partial_x \varphi_i(u_0)$, or $\partial_x \varphi_{i,n}(u_{0,n}) = 0$. □

The approximate problem (3.16) admits a unique solution u_n thanks to [Can08a], which belongs to $C([0,T], L^1(\Omega))$. Now, in order to get a $L^\infty(\mathcal{Q}_i)$-estimate on the sequence $(\partial_x \varphi_{i,n}(u_n))_n$, we introduce a new family of approximate problems (3.23) for which the spatial dependence of the data is smooth.

Let $\theta \in C^\infty(\mathbb{R}), 0 \le \theta \le 1$, with $\theta(x) = 0$ if $x < -1$, and $\theta(x) = 1$ if $x > 1$. Let $k \in \mathbb{N}^\star$, one sets :
– $\phi^k(x) = (1 - \theta(kx))\phi_1 + \theta(kx)\phi_2$,
– $\lambda_{n,k}(s,x) = (1 - \theta(kx))\lambda_{1,n}(s) + \theta(kx)\lambda_{2,n}(s)$,
– $\pi_{n,k}(s,x) = (1 - \theta(kx))\pi_{1,n}(s) + \theta(kx)\pi_{2,n}(s)$.

We will now take a new approximation of the initial data.

$$u_{0,n,k}(x) = \begin{cases} u_{0,n}\left(\frac{k}{k-1}\left(x + \frac{1}{k}\right)\right) & \text{if} \quad x < -1/k, \\ u_{0,n}\left(\frac{k}{k-1}\left(x - \frac{1}{k}\right)\right) & \text{if} \quad x > 1/k. \end{cases}$$

In the layer $[-1/k, 1/k]$, $u_{0,n,k}$ is defined by the relation

$$(1-\theta(kx))\pi_{1,n}(u_{0,n,k}(x)) + \theta(kx)\pi_{2,n}(u_{0,n,k}(x)) = \pi_{1,n}(a_{1,n}) = \pi_{2,n}(a_{2,n}),$$

so that the approximate capillary pressure $\pi_{n,k}(u_{0,n,k}, \cdot)$ is constant through the layer.

Moreover one has either

$$\lambda_{n,k}(u_{0,n,k}, x)\partial_x(\pi_{n,k}(u_{0,n,k}, x)) = \frac{k}{k-1}\partial_x\varphi_{i,n}(u_{0,n}) \qquad \text{if } |x| > \frac{1}{k},$$

or

$$\partial_x(\pi_{n,k}(u_{0,n,k}, x)) = 0 \qquad \text{if } |x| < \frac{1}{k}.$$

So we directly deduce from the definition of $u_{0,n,k}$ the following lemma :

Lemma 3.4.3 *Let $n \geq 1$, $0 \leq u_{0,n} \leq 1$ with $\varphi_{i,n}(u_{0,n}) \in W^{1,\infty}(\Omega_i)$ and $\pi_{1,n}(u_{0,n,1}) = \pi_{2,n}(u_{0,n,2})$, then there exists a sequence $(u_{0,n,k})_k$ satisfying, for all $k \geq 2$, that $0 \leq u_{0,n,k} \leq 1$ and*

$$\|\lambda_{n,k}(u_{0,n,k}, \cdot)\partial_x(\pi_{n,k}(u_{0,n,k}, \cdot))\|_\infty \leq 2\max_{i=1,2}(\|\partial_x\varphi_{i,n}(u_{0,n})\|_\infty),$$

$$u_{0,n,k} \to u_{0,n} \text{ in } L^1(\Omega) \text{ as } k \to +\infty.$$

For any fixed $k \geq 2$ and n large enough, we can now introduce the smooth non-degenerate parabolic problem (3.23) :

$$\begin{cases} \phi^k(x)\partial_t u_{n,k} - \partial_x(\lambda_{n,k}(u_{n,k}, x)\partial_x\pi_{n,k}(u_{n,k}, x)) = 0, \\ \partial_x u_{n,k}(-1, t) = \partial_x u_{n,k}(1, t) = 0, \\ u_{n,k}(x, 0) = u_{0,n,k}(x). \end{cases} \tag{3.23}$$

Moreover, one can furthermore suppose, up to a new regularization, that $u_{0,n,k} \in C^\infty([-1, 1])$. Then (3.23) admits a unique strong solution $u_{n,k} \in C^\infty([0, T] \times [-1, 1])$ (see for instance [Fri64, LSU67]).

Now one sets $f_{n,k}(x, t) = \lambda_{n,k}(u_{n,k}, x)\partial_x\pi_{n,k}(u_{n,k}, x)$, so the main equation of (3.23) can be rewritten :

$$\phi^k\partial_t u_{n,k} = \partial_x f_{n,k}.$$

A short calculation shows that $f_{n,k}(x, t)$ is the solution of the problem :

$$\begin{cases} \partial_t f_{n,k} = a_{n,k}\partial^2_{xx} f_{n,k} + b_{n,k}\partial_x f_{n,k}, \\ f_{n,k}(-1, t) = f_{n,k}(1, t) = 0, \\ f_{n,k}(x, 0) = \lambda_{n,k}(u_{0,n,k}, \cdot)\partial_x(\pi_{n,k}(u_{0,n,k}, \cdot)), \end{cases} \tag{3.24}$$

where $a_{n,k}, b_{n,k}$ are the regular functions defined below.

$$a_{n,k} = \lambda_{n,k}(u_{n,k}, x)\frac{(\pi_{n,k})'(u_{n,k}, x)}{\phi^k(x)} > 0,$$

$$b_{n,k} = (\lambda_{n,k})'(u_{n,k}, x)\frac{\partial_x[\pi_{n,k}(u_{n,k}, x)]}{\phi^k(x)} + \lambda_{n,k}(u_{n,k}, x)\partial_x\left[\frac{(\pi_{n,k})'(u_{n,k}, x)}{\phi^k(x)}\right].$$

The fact that $u_{0,n,k}$ is supposed to be regular allows us to write the problem (3.24) in a strong sense (this is necessary, because this problem can not be written in a conservative form). In particular, $f_{n,k}$ satisfies the maximum principle, and thus

$$\|f_{n,k}\|_{L^\infty((-1,1)\times(0,T))} \leq \|\lambda_{n,k}(u_{0,n,k},\cdot)\partial_x(\pi_{n,k}(u_{0,n,k},\cdot))\|_{L^\infty(-1,1)}.$$

Thanks to the lemmas 3.4.3 and 3.4.2, we have a uniform bound on $(f_{n,k})$:

$$\|f_{n,k}\|_{L^\infty((-1,1)\times(0,T))} \leq 2\max_{i=1,2}(\|\partial_x\varphi_i(u_0)\|_\infty). \tag{3.25}$$

Since the problem (3.23) is fully non degenerated (recall that $\lambda_{i,n} > \frac{1}{2n^2}$ and $\pi'_{i,n} \geq \dfrac{1}{n}$) it follows that $\partial_x u_{n,k}$ and $\partial_t u_{n,k}$ are uniformly bounded respectively in $L^\infty(\mathcal{Q}_i)$ and in $L^2(0,T : H^{-1}(\Omega_i))$ with respect to k, then the sequence $(u_{n,k})_k$ converges toward u_n in $L^2(\mathcal{Q}_i)$, and the limit u_n fulfills, thank to estimate (3.25) :

$$\|\partial_x\varphi_{i,n}(u_n)\|_{L^\infty(\mathcal{Q}_i)} \leq 2\max_{i=1,2}(\|\partial_x\varphi_i(u_0)\|_\infty). \tag{3.26}$$

One has for all $\psi \in \mathcal{D}([-1,1]\times[0,T[)$,

$$\int_0^T\int_{-1}^1 \phi^k u_{n,k}\partial_t\psi + \int_{-1}^1 \phi^k u_{0,n}^k\psi_0 - \int_0^T\int_{-1}^1 f_{n,k}\partial_x\psi = 0. \tag{3.27}$$

Thanks to (3.25),

$$\lim_{k\to+\infty}\int_0^T\int_{-\frac{1}{k}}^{\frac{1}{k}} f_{n,k}\partial_x\psi = 0.$$

One has $u_{n,k} \to u_n$ in the $L^\infty(\mathcal{Q})$-weak- $\star$ and $L^2(\mathcal{Q})$ senses, $u_{0,n,k} \to u_{0,n}$ in $L^1(-1,1)$ thanks to lemma 3.4.3. Moreover, thanks to estimate (3.25), $\partial_x\pi_{i,n,k}(u_{n,k}) \to \partial_x\pi_{i,n}(u_{n,k})$ in the $L^\infty(\mathcal{Q})$-weak- $\star$ sense. Thus we can let k tend toward $+\infty$ in (3.27) to get

$$\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i u_n\partial_t\psi + \sum_{i=1,2}\int_{\Omega_i}\phi_i u_{0,n}\psi_0 - \int_0^T\sum_{i=1,2}\int_{\Omega_i}\lambda_{i,n}(u_n)\partial_x\pi_{i,n}(u_n)\partial_x\psi = 0. \tag{3.28}$$

Furthermore, using the fact that $\pi_{n,k}(u_{n,k},x)$ belongs to $L^2(0,T;H^1(\Omega))$ and, even more, that $\partial_x(\pi_{n,k}(u_{n,k},x))$ is bounded uniformly in k, we can claim that $\pi_{1,n}(u_{1,n}) = \pi_{2,n}(u_{2,n})$, and so u_n is the unique weak solution to the approximate problem (3.16) for $u_{0,n}$ as initial data.

When n tends toward $+\infty$, the sequence $(u_n)_n$ converges, up to a subsequence toward a weak solution to the problem $(\mathcal{P})$, as seen in section 3.3, but the estimate (3.26) insures that

$$\partial_x\varphi_i(u) \in L^\infty(\mathcal{Q}_i).$$

This achieves the proof of theorem 3.4.1. □

3.5 A uniqueness result

In this section, we give a uniqueness result in the one dimensional case in a framework where the existence results are stronger than the general existence result stated in theorem 3.3.1. Under a regularity assumption on the initial data u_0, we proved in section 3.4 the existence of a solution having bounded flux, for which we give a uniqueness result in theorem 3.5.1 and corollary 3.5.2. The bound on the flux will be necessary to prove that the contraction property is also available in the neighborhood of the interface $\{x = 0\}$. Then we show in theorem 3.5.4 the existence and uniqueness of the weak solution which is the limit of bounded flux solutions for any initial data u_0 with $0 \leq u_0 \leq 1$. Indeed, the set of initial data giving a bounded flux solution is dense in $L^\infty(\Omega)$ for the $L^1(\Omega)$ topology, and theorem 3.5.1 has for consequence that the contraction property can be extended to a larger class of solution, defined for all initial data in $L^\infty(\Omega)$. We unfortunately are not able to characterize them differently than by a limit of bounded flux solutions, and we can not either exhibit a weak solution which is not the limit of bounded flux solutions.

Theorem 3.5.1 (L^1-contraction principle for bounded flux solutions) *Let u, v be two weak solutions to the problem ($\mathcal{P}$) for the initial data u_0, v_0. Then, if $\partial_x \varphi_i(u_i)$ and $\partial_x \varphi_i(v_i)$ belong to $L^\infty(\mathcal{Q}_i)$, we have the following L^1-contraction principle :* $\forall t \in [0, T]$,

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^{\pm} dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^{\pm} dx. \tag{3.29}$$

The first part of this section is devoted to the proof of the theorem 3.5.1 which, with theorem 3.4.1, admits the following straightforward consequence :

Corollary 3.5.2 (Uniqueness of the bounded flux solution) *For all $u_0 \in L^\infty(-1,1)$ with $0 \leq u_0 \leq 1$, such that, for $i = 1, 2$, $\varphi_i(u_0) \in W^{1,\infty}(\Omega_i)$, and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, there exists a unique weak solution to the problem ($\mathcal{P}$) in the sense of definition 3.1 and such that $\partial_x \varphi_i(u) \in L^\infty(\mathcal{Q}_i)$; moreover $u \in C([0,T], L^p(\Omega))$ for all $1 \leq p < +\infty$.*

Proof

The proof of the theorem 3.5.1 is based on entropy inequalities, obtained through the method of doubling variables, first introduced by S. Kružkov [Kru70] for first order equations, and then adapted by J. Carrillo [Car99] for degenerate parabolic problems. Note that in the present setting, we only need doubling with respect to the time–variable, as it is done, for instance by F. Otto [Ott96b] for elliptic–parabolic problems (or in [BP05] for Stefan–type problems).

In the sequel of the proof, we will only give the comparison

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u(x,t) - v(x,t)\right)^{+} dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i \left(u_0(x) - v_0(x)\right)^{+} dx.$$

The comparison with $(\cdot)^-$ instead of $(\cdot)^+$ can be proven exactly the same way.

Let u be a bounded flux solution to the one-dimensional problem, i.e $\partial_x \varphi_i(u) \in L^\infty(\mathcal{Q}_i)$, $i = 1, 2$. The weak formulation of definition 3.1 adapted to the one-dimensional framework of the section can be rewritten, for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T[)$,

$$\begin{aligned}\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u(x,t) \partial_t \psi(x,t) dxdt + \sum_{i=1,2} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx \\ - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x \varphi_i(u)(x,t) \partial_x \psi(x,t) dxdt = 0\end{aligned} \tag{3.30}$$

This formulation clearly implies, for $i = 1,2$, for all $\psi \in C_c^\infty(\overline{\Omega}_i \times [0,T[)$ with $\psi(0,t) = 0$,

$$\begin{aligned}\int_0^T \int_{\Omega_i} \phi_i u(x,t) \partial_t \psi(x,t) dxdt + \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx \\ - \int_0^T \int_{\Omega_i} \partial_x \varphi_i(u)(x,t) \partial_x \psi(x,t) dxdt = 0\end{aligned} \tag{3.31}$$

Classical computations (see e.g. [BP05, Car99, Ott96b]) on equation (3.31) lead to the following entropy inequalities : for all weak solutions u, v, for initial data u_0, v_0, for all $\xi \in \mathcal{D}^+(\overline{\Omega}_i \times [0,T[\times[0,T[)$ such that $\xi(0,t,s) = 0$,

$$\begin{aligned}\int_0^T \int_0^T \int_{\Omega_i} \phi_i (u(x,t) - v(x,s))^+ (\partial_t \xi(x,t,s) + \partial_s \xi(x,t,s)) dxdtds \\ + \int_0^T \int_{\Omega_i} \phi_i (u_0(x) - v(x,s))^+ \xi(x,0,s) dxds \\ + \int_0^T \int_{\Omega_i} \phi_i (u(x,t) - v_0(x))^+ \xi(x,t,0) dxdt \\ - \int_0^T \int_0^T \int_{\Omega_i} \partial_x (\varphi_i(u)(x,t) - \varphi_i(v)(x,s))^+ \partial_x \xi(x,t,s) dxdtds \geq 0.\end{aligned} \tag{3.32}$$

Let us note here an important consequence of the entropy inequality (4.105) (and of the corresponding one for $(u-v)^-$), namely that u can be proved to satisfy

$$ess\,-\lim_{t\to 0} \int_{\Omega_i} |u(x,t) - u_0(x)| dx = 0\,. \tag{3.33}$$

Indeed, this follows by taking v as a constant in (4.105) and using an approximation argument, see e.g. Lemma 7.41 in [MNRR96]. We deduce the time continuity at $t = 0$ for any solution and in particular for both u and v taken above.

Now, let $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$ with $supp(\rho) \subset [-1,1]$ and $\int_{\mathbb{R}} \rho(t) dt = 1$. One denotes $\rho_m(t) = m\rho(mt)$. Let $\psi \in \mathcal{D}^+([-1,1] \times [0,T[)$ with $\psi(0,\cdot) = 0$. For m large enough, $\xi(x,t,s) = \psi(x,t)\rho_m(t-s)$ belongs to $\mathcal{D}^+([-1,1] \times [0,T[\times[0,T[)$, and we can take it as test function in (4.105). Then summing on $i = 1,2$ leads to

$$\begin{aligned}\int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v(x,s))^+ \partial_t \psi(x,t) \rho_m(t-s) dxdtds \\ + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u_0(x) - v(x,s))^+ \psi(x,0) \rho_m(-s) dxds \\ + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v_0(x))^+ \psi(x,t) \rho_m(t) dxdt \\ - \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x (\varphi_i(u)(x,t) - \varphi_i(v)(x,s))^+ \partial_x \psi(x,t) \rho_m(t-s) dxdtds \geq 0.\end{aligned} \tag{3.34}$$

We can now let m tend toward $+\infty$ in (3.34), and using (3.33) for u and v, and the theorem of continuity in mean, we get : for all $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0,T[)$ such that $\psi(0,t) = 0$,

$$\begin{aligned} \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t \psi(x,t) dxdt \\ + \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ \psi(x,0) dx \\ - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x (\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x \psi(x,t) dxdt \geq 0. \end{aligned} \tag{3.35}$$

We aim now to extend the inequality (3.35) in the case where $\psi(0,t) \neq 0$, and particularly in the case $\psi(x,t) = \theta(t)$, so that the third term disappears in (3.35).

To this purpose, let us set here $u_i(t) = u_i(0,t)$ to denote the trace of u_i at the interface Γ (and correspondingly, $v_i(t) = v_i(0,t)$). We introduce the subsets of $(0,T)$:

– $E_{u>v} = \{t \in [0,T] \mid u_1(t) > v_1(t) \text{ or } u_2(t) > v_2(t)\}$,

– $E_{u\leq v} = \{t \in [0,T] \mid u_1(t) \leq v_1(t) \text{ and } u_2(t) \leq v_2(t)\}$,

so that $E_{u\leq v}$ is the complement of $E_{u>v}$ in $[0,T]$.

For all $\varepsilon > 0$, one defines $\psi_\varepsilon(x) = \max\left(1 - \frac{|x|}{\varepsilon}, 0\right)$. For all $\theta \in \mathcal{D}^+([0,T[)$, we take $(x,t) \mapsto \theta(t)(1 - \psi_\varepsilon(x))$ instead of $\psi(x,t)$ as test-function in (3.35), thus we get :

$$\begin{aligned} \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t \theta(t)(1 - \psi_\varepsilon(x)) dxdt \\ + \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ (1 - \psi_\varepsilon)(x) \theta(0) dx \\ - \int_0^T \frac{\theta(t)}{\varepsilon} \begin{pmatrix} (\varphi_1(u)(-\varepsilon,t) - \varphi_1(v)(-\varepsilon,t))^+ - (\varphi_1(u_1)(t) - \varphi_1(v_1)(t))^+ \\ +(\varphi_2(u)(\varepsilon,t) - \varphi_2(v)(\varepsilon,t))^+ - (\varphi_2(u_2)(t) - \varphi_2(v_2)(t))^+ \end{pmatrix} dt \geq 0. \end{aligned}$$

For almost every $t \in E_{u\leq v}$, the function $(\varphi_i(u) - \varphi_i(v))^+(\cdot,t)$ admits a nil trace on $\{x = 0\}$, thus the third term in the previous inequality can be reduced to the set $E_{u>v}$ obtaining

$$\begin{aligned} \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t \theta(t)(1 - \psi_\varepsilon(x)) dxdt \\ + \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ (1 - \psi_\varepsilon)(x) \theta(0) dx \\ + \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x (\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x \psi_\varepsilon(x) dxdt \geq 0. \end{aligned} \tag{3.36}$$

We show now the crucial point of the uniqueness proof, which is the subject of the following lemma.

Lemma 3.5.3 *For all $\theta \in \mathcal{D}^+([0,T[)$, if u, v are both bounded flux solutions, i.e. if one has $\partial_x \varphi_i(u), \partial_x \varphi_i(v) \in L^\infty(\mathcal{Q}_i)$ one has,*

$$\limsup_{\varepsilon \to 0} \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x (\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x \psi_\varepsilon(x) dxdt \leq 0.$$

Using the weak formulation (3.30), we can claim that for any regular function $\vartheta \in \mathcal{D}([0,T[)$,

$$\lim_{\varepsilon\to 0}\int_0^T \vartheta(t)\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)-\varphi_i(v))\partial_x\psi_\varepsilon(x)dxdt = 0. \tag{3.37}$$

Since for $i=1,2$, $\partial_x(\varphi_i(u)-\varphi_i(v))$ belongs to $L^\infty(\Omega_i\times(0,T))$, one has

$$\left|\int_0^T \vartheta(t)\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)-\varphi_i(v))\partial_x\psi_\varepsilon(x)dxdt\right| \le C\|\vartheta\|_{L^1(0,T)},$$

then a density argument allows us to claim that (3.37) still holds for any $\vartheta \in L^1(0,T)$, and particularly for $\vartheta(t)=\theta(t)1\!\!1_{E_{u>v}}(t)$. Thus there exists $A(\varepsilon)$ tending to 0 as ε tends to 0 such that

$$\int_{E_{u>v}}\theta(t)\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)(x,t)-\varphi_i(v)(x,t))\partial_x\psi_\varepsilon(x)dxdt = A(\varepsilon). \tag{3.38}$$

Splitting up the positive and negative parts of $(\varphi_i(u)(x,t)-\varphi_i(v)(x,t))$, (3.38) becomes :

$$\begin{aligned}&\int_{E_{u>v}}\theta(t)\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)(x,t)-\varphi_i(v)(x,t))^+\partial_x\psi_\varepsilon(x)dxdt\\ =&\int_{E_{u>v}}\theta(t)\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)(x,t)-\varphi_i(v)(x,t))^-\partial_x\psi_\varepsilon(x)dxdt + A(\varepsilon).\end{aligned} \tag{3.39}$$

It is at this point that we actually use the monotony of the transmission condition, i.e. condition 3 in Definition 3.1. Indeed, the conditions $\tilde\pi_1(u_1(t))\cap\tilde\pi_2(u_2(t))\neq\emptyset$ and $\tilde\pi_1(v_1(t))\cap\tilde\pi_2(v_2(t))\neq\emptyset$ insure that :

$$u_1>v_1 \implies u_2\ge v_2 \qquad \text{and} \qquad u_1<v_1 \implies u_2\le v_2\,. \tag{3.40}$$

Therefore, recalling the definition of the set $E_{u>v}$ and of ψ_ε, the first term in the right member of (3.39) is non-positive, and then we conclude

$$\limsup_{\varepsilon\to 0}\int_{E_{u>v}}\theta(t)\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)(x,t)-\varphi_i(v)(x,t))^+\partial_x\psi_\varepsilon(x)dxdt \le 0.$$

This achieves the proof of lemma 3.5.3, and allows us to take the limit in inequality (3.36) for $\varepsilon\to 0$. Then for all $\psi\in\mathcal{D}^+([0,T[)$, one gets

$$-\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i(u(x,t)-v(x,t))^+\partial_t\psi(t)dxdt \le \sum_{i=1,2}\int_{\Omega_i}\phi_i(u_0(x)-v_0(x))^+\psi(0)dx. \tag{3.41}$$

One can also prove exactly the same way that

$$-\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i(u(x,t)-v(x,t))^-\partial_t\psi(t)dxdt \le \sum_{i=1,2}\int_{\Omega_i}\phi_i(u_0(x)-v_0(x))^-\psi(0)dx. \tag{3.42}$$

These inequalities still hold for $\psi = (T-t)$, and then if $u_0 = v_0$, one has $u = v$ almost everywhere in $\mathcal{Q}$. Moreover we can take $\psi(t) = 1\!\!1_{[0,s]}(t)$ as test function in (5.51) to get the L^1-contraction principle (3.29) stated in theorem 3.5.1. □

In the sequel, we prove that for any u_0 in $L^\infty(-1;1)$, $0 \le u_0 \le 1$, there exists a unique weak solution of problem $(\mathcal{P})$ which is the limit of a sequence of bounded flux solutions $(u_n)_n$, i.e. for all $n \ge 1$, $\partial_x \varphi_i(u_n) \in L^\infty(\mathcal{Q}_i)$.

Theorem 3.5.4 (Existence and uniqueness of the SOLA) *Let $u_0 \in L^\infty(-1,1)$, $0 \le u_0 \le 1$, and let $(u_{0,n})_{n\ge 1}$ be a sequence of bounded flux initial data, i.e. for all $n \ge 1$,*

– $0 \le u_{0,n} \le 1$,
– $\varphi_i(u_{0,n}) \in W^{1,\infty}(\Omega_i)$,
– $\tilde{\pi}_1(u_{0,n,1}) \cap \tilde{\pi}_2(u_{0,n,2}) \ne \emptyset$,

such that

$$\lim_{n\to+\infty} \|u_{0,n} - u_0\|_{L^1(\Omega)} = 0.$$

Let $(u_n)_{n\ge1}$ be the sequence of the bounded flux solutions to the problem $(\mathcal{P})$ for $u_{0,n}$ as initial data. Then the sequence $(u_n)_{n\ge1}$ converges toward u in $C(]0,T[, L^p(-1,1))$, $1 \le p < +\infty$, where u is a solution to the problem $(\mathcal{P})$, called Solution Obtained as Limit of Approximation (SOLA). Furthermore, if u, v are two SOLAs, for initial data u_0, v_0, one has the following L^1-contraction principle : $\forall t \in [0,T]$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i (u(x,t) - v(x,t))^{\pm} dx \le \sum_{i=1}^{N} \int_{\Omega_i} \phi_i (u_0(x) - v_0(x))^{\pm} dx. \tag{3.43}$$

This particularly leads to the uniqueness of the SOLA.

Proof

Let $(u_{0,n})$ be a regular sequence of initial data converging toward u_0 in $L^1(-1,1)$ - one take e.g. $u_{0,n} \in C_c^\infty(]-1,0[\cup]0,1[)$. Then $(u_{0,n})$ is a Cauchy sequence, and thanks to (3.29), for all $t \in [0,T]$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_n(x,t) - u_m(x,t)| dx \le \sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_{0,n}(x) - u_{0,m}(x)| dx.$$

Thus $(u_n)_n$ is a Cauchy sequence in $C([0,T]; L^1(\Omega))$ and converges to a function u in $C([0,T]; L^1(\Omega))$. Since $(u_n)_n$ is bounded in $L^\infty(\mathcal{Q})$, one has $u_n \to u$ in $C([0,T]; L^p(-1,1))$.

We now have to check that u is a weak solution to the problem $(\mathcal{P})$. It is easy to check, using to the L^∞-bound of u_n, that $\varphi_i(u_n)$ tends toward $\varphi_i(u)$ in $L^p(\Omega_i \times (0,T))$, for all $p \in [1,+\infty[$. Thanks to (3.18), the sequence $(\varphi_i(u_n))_n$ is bounded in $L^2(0,T; H^1(\Omega_i))$, and thus $\varphi_i(u_n) \to \varphi_i(u)$ weakly in $L^2(0,T; H^1(\Omega_i))$, and $\varphi_i(u_n)$ converges in $L^2(0,T; H^s(\Omega_i))$, for all $s \in]0,1[$, still toward $\varphi_i(u)$. Particularly, $u_{n,i}(t)$ tends toward $u_i(t)$. Since the set $\{(a,b) \in [0,1]^2 \mid \tilde{\pi}_1(a) \cap \tilde{\pi}_2(b) \ne \emptyset\}$ is closed, we can claim that

$$\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \ne \emptyset \quad \text{for a.e. } t \in [0,T].$$

We can also pass to the limit in the weak formulation in order to conclude that u is a weak solution to the problem $(\mathcal{P})$, achieving this way the existence of a SOLA u.

Let now v be another SOLA, obtained through a sequence $(v_{0,n})_n$ of regular initial data converging toward v_0. Thanks to (3.29), one has,

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_n(x,t) - v_n(x,t)| dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_{0,n}(x) - v_{0,n}(x)| dx,$$

whose limit as n tends toward $+\infty$ gives the attempted L^1-contraction principle :

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u(x,t) - v(x,t)| dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_0(x) - v_0(x)| dx,$$

and so the uniqueness of the SOLA, completing the proof of theorem 3.5.4. □

Chapitre 4

Finite volume scheme for two-phase flow in heterogeneous porous media

Introduction

The models of immiscible two-phase flows in porous media are widely used in petroleum engineering to understand the moves of hydrocarbon in subsoil, because they are quite light but they let important phenomena appear.

We will focus on the influence of strong heterogeneities in the porous media, which will represent changes of rock type. The discontinuity of the physical properties of the porous media can lead to the phenomena of oil-trapping, which can be explained by discontinuities of the capillary pressures on the interfaces between the different rocks.

In the sequel, we restrict our frame to one-dimensional problems. We will give in this paper a finite volume scheme to approach the solution of such two phase flows. We prove some convergence results, and some uniqueness results. The main results of this paper are summarized in theorem 4.7.1 in the end of this paper.

The section 4.1 is devoted to understand the physical model, especially in the case where there is a pressure discontinuity. We will keep the approach introduced in [CGP] to connect the capillary pressures on the interfaces, which allows to deal with a larger class of problem than the one introduced in [EEM06] (or in [BDPvD03] in a particular case). Some monotonous transmission conditions through the interfaces appear, and this monotony will be crucial all along this paper, as it will be stressed in the conclusion of this paper. We then get a spatial coupling of degenerated parabolic equations, leading to a notion of weak solution. We refer to [AS79], [Bea72] for more complete explanations on the physical models than those given in section 4.1.

The implicit finite volume scheme is introduced in section 4.2, where we make a classical convergence study in the way of [EGH00]. All this study can be generalised in multidimensional case for unstructured admissible meshes.We use the monotony of the scheme to get a L^∞-estimate on the discrete solution, and also to prove the uniqueness of the discrete solution. The existence of the discrete solution is got by a topological degree argument. The convergence (up to subsequences) of the discrete solution toward a weak solution to the problem as the step of the discretization tends to 0 follows from compactness results on the set of discrete solutions.

Under regularity assumptions on the data, we prove in section 4.3 that the discrete fluxes are uniformly bounded with regard to time and space. Thus the limit of the discrete solution as the discretization step tends to zero (which exists thanks to section 4.2) admits also bounded fluxes. So we extend the result of regularity stated in theorem 4.1 in [CGP] in the case where there is a convective term, and we show a way to approximate this bounded flux solution.

The uniqueness of such a bounded flux solution is proven in section 4.4, using a classical doubling variable method (see e.g. [AL83], [Ott96b] or [Car99]). The way to deal with the interfaces is inspired from [CGP] with the add of convective terms. This particularly uses the fact that the flux stays bounded, and the monotony of the transmission conditions.

The previous uniqueness result can be extended using density arguments to a larger class of solution.We claim in section 4.5 that for non regular data, there exists a unique weak solution which is limit of weak solutions associated to regular initial data. It is shown that if the total flow-rate belongs to $BV(0,T)$, we can approach this particular weak solution using the finite volume scheme.

Some numerical simulations are given in section 4.6, giving a clear vision of the effects of capillary barriers.

4.1 The Physical model

We consider a heterogeneous porous medium, which is an apposition of homogeneous porous media, representing the different geological layers, so that the physical properties of the medium only depend of the rock type and are piece-wise constant. We restrict our study to the one-dimensional case, even if all the results stated in section 4.2 can be quite easily adapted to larger dimensions (see [EEM06]). For the sake of simplicity, we only deal with two geological layers with same size, because a generalisation to a arbitrary finite number of geological layers would only lead to notation difficulties. In the sequel, one denotes $\Omega =]-1,1[$ the heterogeneous porous medium, and $\Omega_1 =]-1,0[$, $\Omega_2 =]0,1[$ the two homogeneous layers. The interface between the layers is thus $\{x = 0\}$. T is a positive integer.

We consider an incompressible an immiscible oil-water flow through Ω. Writing the conservation of each phase, and using Darcy's law leads to : for all $(x,t) \in \Omega_i \times (0,T)$,

$$\phi_i \partial_t u - \partial_x \left[\mu_{w,i}(u) \left(\partial_x P_{w,i} - \rho_w \mathbf{g}\right)\right] = 0, \tag{4.1}$$

$$-\phi_i \partial_t u - \partial_x \left[\mu_{o,i}(u) \left(\partial_x P_{o,i} - \rho_o \mathbf{g}\right)\right] = 0, \tag{4.2}$$

where $\phi_i \in]0,1[$ is the porosity of the porous media Ω_i, u is the water saturation (and then $(1-u)$ is the oil saturation), $\mu_{\beta,i}$ is the mobility of the phase $\beta = w, o$, where w stands for water, and o for oil. $P_{\beta,i}$ denotes the pressure of the phase β, ρ_β its density, and $\mathbf{g}$ the gravity.

Adding (4.1) and (4.2) shows that :

$$\partial_x q = 0,$$

where

$$q = -\mu_{w,i}(u)\left(\partial_x P_{w,i} - \rho_w \mathbf{g}\right) - \mu_{o,i}(u)\left(\partial_x P_{o,i} - \rho_o \mathbf{g}\right) \tag{4.3}$$

is the total flow-rate, only depending on time. Assume that $\rho_w = \rho_o = \rho$, then using (4.3) in (4.1) and (4.2) yields :

$$\phi_i \partial_t u + \partial_x \left(\frac{\mu_{w,i}(u)}{\mu_{w,i}(u) + \mu_{o,i}(u)} q - \frac{\mu_{w,i}(u)\mu_{o,i}(u)}{\mu_{w,i}(u) + \mu_{o,i}(u)} \partial_x (P_{w,i} - P_{o,i}) \right) = 0. \tag{4.4}$$

One assumes that the capillary pressure $(P_{w,i} - P_{o,i})$ depends only of the saturation and of the rock type. Thus $(P_{w,i} - P_{o,i}) = \pi_i(u)$, where $\pi_i(u)$ is supposed to be an increasing function. One denotes $\lambda_i(u) = \frac{\mu_{w,i}(u)\mu_{o,i}(u)}{\mu_{w,i}(u)+\mu_{o,i}(u)}$, and $f_i(u) = \frac{\mu_{w,i}(u)}{\mu_{w,i}(u)+\mu_{o,i}(u)}$, then (4.4) becomes

$$\phi_i \partial_t u + \partial_x \left(q f_i(u) - \lambda_i(u) \partial_x \pi_i(u) \right) = 0. \tag{4.5}$$

Assumptions 4.1 *For $i = 1, 2$, one has :*

1. *π_i is an increasing $C^1([0,1])$ function,*
2. *$\mu_{w,i}$ is a continuous increasing function on $[0,1]$, with $\mu_{w,i}(0) = 0$,*
3. *$\mu_{o,i}$ is a continuous decreasing function on $[0,1]$, with $\mu_{o,i}(1) = 0$,*
4. *$f_i = \dfrac{\mu_{w,i}}{\mu_{w,i} + \mu_{o,i}}$ is an increasing continuous function on $[0,1]$, with $f(0) = 0$ and $f(1) = 1$.*

We deduce from assumptions 4.1 that (4.5) is a degenerated nonlinear parabolic equation. Denoting

$$\varphi_i(s) = \int_0^s \lambda_i(a) \pi_i'(a) da,$$

the equation (4.5) can be rewritten

$$\phi_i \partial_t u + \partial_x \left(q f_i(u) - \partial_x \varphi_i(u) \right) = 0, \tag{4.6}$$

where $\varphi_i \in C^1([0,1])$ is an increasing function.

Let us now focus on the transmission conditions through the interface $\{x = 0\}$. One denotes $\alpha_i = \lim_{s \to 0} \pi_i(s)$ and $\beta_i = \lim_{s \to 1} \pi_i(s)$. We can now define the monotonous graphs $\tilde{\pi}_i$ by :

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0,1[, \\ ]-\infty, \alpha_i] & \text{if } s = 0, \\ [\beta_i, +\infty[& \text{if } s = 1. \end{cases} \tag{4.7}$$

Let u_i denote the trace of $u_{|\Omega_i}$ on $\{x = 0\}$. The trace on $\{x = 0\}$ from Ω_i of the pressure $P_{\beta,i}$ of the phase β is still denoted $P_{\beta,i}$. As it is exposed in [CGP], the pressure of the phase β can be discontinuous through the interface $\{x = 0\}$ in the case where it is missing in the upstream side. This can be written

$$\mu_{\beta,1}(u_1)(P_{\beta,1} - P_{\beta,2})^+ - \mu_{\beta,2}(u_2)(P_{\beta,2} - P_{\beta,1})^+ = 0. \tag{4.8}$$

The conditions (4.8) have direct consequences on the connection of the capillary pressures through $\{x = 0\}$. Indeed, if $0 < u_1, u_2 < 1$, then the partial pressures P_o and P_w

have both to be continuous, and so we have the connection of the capillary pressures $\pi_1(u_1) = \pi_2(u_2)$. If $u_1 = 0$ and $0 < u_2 < 1$, then $P_{w,1} \geq P_{w,2}$ and $P_{o,1} = P_{o,2}$, thus $\pi_2(u_2) \leq \pi_1(0)$. The same way, $u_1 = 1$ and $0 < u_2 < 1$ implies $\pi_2(u_2) \geq \pi_1(1)$. If $u_1 = 0$, $u_2 = 1$, then $P_{w,1} \geq P_{w,2}$ and $P_{o,1} \leq P_{o,2}$, so $\pi_1(0) \geq \pi_2(1)$. Checking that the definition of the graphs $\tilde{\pi}_1$ and $\tilde{\pi}_1$ implies $\tilde{\pi}_1(0) \cap \tilde{\pi}_2(0) \neq \emptyset$, $\tilde{\pi}_1(1) \cap \tilde{\pi}_2(1) \neq \emptyset$, we can claim that (4.8) leads to :

$$\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset. \tag{4.9}$$

The conservation of each phase leads to the connection of the fluxes on $\{x = 0\}$:

$$q(t)f_1(u)(0^-, t) - \partial_x\varphi_1(u)(0^-, t) = q(t)f_2(u)(0^+, t) - \partial_x\varphi_2(u)(0^+, t). \tag{4.10}$$

We suppose that $q \geq 0$, i.e. that the fluids moves from the $x < 0$ to $x > 0$. We have to choose some boundary conditions on $\{x = -1\}$ and $\{x = 1\}$. Let g such that $0 \leq g(t) \leq q(t)$, we set :

$$q(t)f_1(u)(-1, t) - \partial_x\varphi_1(u)(-1, t) = g(t). \tag{4.11}$$

This can be understood as the injection of a fluid with $g(t)/q(t)$ as water saturation with $q(t)$ as flow-rate.

Assumption 4.2 *We suppose that* $0 \leq g \leq q \in L^1(0, T)$.

We choose to take

$$\partial_x\varphi_2(u)(1, t) = 0. \tag{4.12}$$

We take an initial data $u_0 \in L^\infty(\Omega)$, with $0 \leq u_0 \leq 1$, so we can write the initial-boundary value problem :

$$\left\{\begin{array}{ll}
\phi_i\partial_t u + \partial_x\left[q(t)f_i(u) - \partial_x\varphi_i(u)\right] = 0 & \text{in } \Omega_i \times (0, T), \\
q(t)f_1(u)(0^-, t) - \partial_x\varphi_1(u)(0^-, t) = q(t)f_2(u)(0^+, t) - \partial_x\varphi_2(u)(0^+, t) & \text{on } (0, T), \\
\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset & \text{on } (0, T), \\
u(t = 0) = u_0 & \text{in } \Omega, \\
q(t)f_1(u)(-1, t) - \partial_x\varphi_1(u)(-1, t) = g(t) & \text{on } (0, T), \\
\partial_x\varphi_2(u)(1, t) = 0 & \text{on } (0, T).
\end{array}\right. \tag{$\mathcal{P}$}$$

We now define the notion a weak-solution

Definition 4.1 *A function u is said to be a weak solution to the problem ($\mathcal{P}$) if it fulfills :*

1. $u \in L^\infty(\Omega \times (0, T))$, *with* $0 \leq u \leq 1$,
2. *for* $i = 1, 2$, $\varphi_i(u) \in L^2(0, T; H^1(\Omega_i))$,
3. *for a.e.* $t \in (0, T)$, $\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset$, *where u_i denotes the trace of $u_{|\Omega_i}$ on* $\{x = 0\}$,

4. for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T[)$,

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u(x,t) \partial_t \psi(x,t) dx dt + \sum_{i=1,2} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left[q(t) f_i(u)(x,t) - \partial_x \varphi_i(u)(x,t) \right] \partial_x \psi(x,t) dx dt$$
$$+ \int_0^T g(t)\psi(-1,t) dt - \int_0^T q(t) f_2(u)(1,t) \psi(1,t) dt = 0. \tag{4.13}$$

4.2 A finite volume scheme

In this section, we introduce an implicit finite volume scheme to approach a solution of $(\mathcal{P})$. We deduce from this scheme some estimates on the discrete solutions to the scheme, which will lead to some compactness results on the family of approximate solutions Letting the discretization step tend to 0, we check that the discrete solution obtained via the finite volume scheme converges to a weak solution to the problem. An existence result follows.

4.2.1 The finite volume approximation

We first need to discretize all the data, so that we can define an approximate problem through the finite volume scheme.

Discretization of Ω : for the sake of simplicity, we will only deal with uniform spatial discretizations. Let $N \in \mathbb{N}^\star$, one defines :

$$\begin{cases} x_j = j/N, & \forall j \in [\![-\mathrm{N}, \mathrm{N}]\!], \\ x_{j+1/2} = \dfrac{j+1/2}{N}, & \forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]. \end{cases}$$

One denotes $\delta x = 1/N$.

Discretization of $(0,T)$: once again, we will only deal with uniform discretizations. Let $M \in \mathbb{N}^\star$, one defines : for all $n \in [\![0, \mathrm{M}]\!]$, $t_n = nT/M$. One denotes $\delta t = T/M$.

Discretization of u_0 : $\forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$,

$$u_{0,\mathcal{D}}(x_{j+1/2}) = u^0_{j+1/2} = \frac{1}{\delta x} \int_{x_j}^{x_{j+1}} u_0(x) dx \tag{4.14}$$

The first equation of $(\mathcal{P})$ can be rewritten :

$$\phi_i \partial_t u + \partial_x F(x,t) = 0$$

with $F(x,t) = q(t) f(u) - \partial_x \varphi_i(u)$. We consider the following implicit scheme : $\forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$\phi_i \frac{u^{n+1}_{j+1/2} - u^n_{j+1/2}}{\delta t} \delta x + F^{n+1}_{j+1} - F^{n+1}_j = 0 \tag{4.15}$$

where F_j^{n+1} is an approximation of the mean flux through x_j on $]t_n, t_{n+1}[$, and i is chosen such that $]x_j, x_{j+1}[\subset \Omega_i$. We choose an implicit upwind discretization of the fluxes, so that we get : $\forall j \in [\![-\mathrm{N}+1, -1]\!] \cup [\![1, \mathrm{N}-1]\!]$, $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$F_j^{n+1} = q^{n+1} f_i(u_{j-1/2}^{n+1}) - \frac{\varphi_i(u_{j+1/2}^{n+1}) - \varphi_i(u_{j-1/2}^{n+1})}{\delta x} \tag{4.16}$$

where $q^{n+1} = \dfrac{1}{\delta t}\displaystyle\int_{t^n}^{t^{n+1}} q(t)dt$. We also define $g^{n+1} = \dfrac{1}{\delta t}\displaystyle\int_{t^n}^{t^{n+1}} g(t)dt$.

Discretization of the boundary conditions : $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$F_{-N}^{n+1} = g^{n+1}, \tag{4.17}$$

$$F_N^{n+1} = q^{n+1} f_2(u_{N-1/2}^{n+1}). \tag{4.18}$$

Discretization of the transmission conditions : we first write the connection of the discrete fluxes through the interface $\{x = 0\}$. $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$F_0^{n+1} = q^{n+1} f_1(u_{-1/2}^{n+1}) - \frac{2(\varphi_1(u_{0,1}^{n+1}) - \varphi_1(u_{-1/2}^{n+1}))}{\delta x} \tag{4.19}$$

$$= q^{n+1} f_2(u_{0,2}^{n+1}) - \frac{2(\varphi_2(u_{1/2}^{n+1}) - \varphi_2(u_{0,2}^{n+1}))}{\delta x}. \tag{4.20}$$

We also discretize the condition on the connection of the traces : $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$\tilde{\pi}_1(u_{0,1}^{n+1}) \cap \tilde{\pi}_2(u_{0,2}^{n+1}) \neq \emptyset. \tag{4.21}$$

To justify these discrete transmission conditions, we now state the following lemma :

Lemma 4.2.1 *Let (a, b) belong to R^2, let $n \in [\![0, \mathrm{M}-1]\!]$, then there exists a unique couple $(c, d) \in [0, 1]^2$ such that :*

$$\begin{cases} q^{n+1} f_1(a) - \dfrac{2(\varphi_1(c) - \varphi_1(a))}{\delta x} = q^{n+1} f_2(d) - \dfrac{2(\varphi_2(b) - \varphi_2(d))}{\delta x}, \\ \tilde{\pi}_1(c) \cap \tilde{\pi}_2(d) \neq \emptyset, \end{cases} \tag{4.22}$$

where for all $s \leq 0$, $f_i(s) = 0$, $\varphi_i(s) = 0$, and for all $s \geq 1$, $f_i(s) = 1$, $\varphi_i(s) = \varphi_i(1)$.

Furthermore, $(a, b) \mapsto c$ and $(a, b) \mapsto d$ are continuous and nondecreasing w.r.t. each one of their arguments.

Proof

For $i = 1, 2$, the functions $\varphi_i \circ \tilde{\pi}_i^{-1}$ and $f_i \circ \tilde{\pi}_i^{-1}$ are nondecreasing continuous functions, and then,

$$G : \begin{cases} [\underline{p}, \overline{p}] & \rightarrow \mathbb{R} \\ p & \mapsto q^{n+1} f_2 \circ \tilde{\pi}_2^{-1}(p) + \dfrac{2}{\delta x}\left(\varphi_1 \circ \tilde{\pi}_1^{-1}(p) + \varphi_2 \circ \tilde{\pi}_2^{-1}(p)\right) \end{cases}$$

is a continuous non decreasing function, whose restricition to $[\underline{p}, \overline{p}] \setminus] \min_{i=1,2} \overline{p_i}, \max_{i=1,2} \underline{p_i}]$ (with the convention $]a, b] = \emptyset$ if $a \geq b$) is an increasing function realising a bijection on

$[G(\underline{p}), G(\overline{p})]$.
One defines the increasing function

$$\Psi : \left\{ \begin{array}{ccl} \mathbb{R}^2 & \to & \mathbb{R} \\ (a,b) & \mapsto & q^{n+1} f_1(a) + \dfrac{2}{\delta x} \left(\varphi_1(a) + \varphi_2(b) \right). \end{array} \right.$$

Then, if (c,d) is a solution to the system (4.22), for all $p \in \tilde{\pi}_1(c) \cap \tilde{\pi}_2(d)$,

$$G(p) = \Psi(a,b).$$

First, we suppose that $]\min_{i=1,2} \overline{p_i}, \max_{i=1,2} \underline{p_i}] = \emptyset$. Then G is a continuous increasing function from $[\underline{p}, \overline{p}]$ onto $[G(\underline{p}), G(\overline{p})]$, and we can define its inverse function G^{-1}.
Let $p = G^{-1} \circ \Psi(a,b)$, and $(c,d) = (\tilde{\pi}_1^{-1}(p), \tilde{\pi}_2^{-1}(p))$. Then (c,d) is a solution to (4.22). The uniqueness is a direct consequence of the fact that G is increasing.

Now if $]\min_{i=1,2} \overline{p_i}, \max_{i=1,2} \underline{p_i}] \neq \emptyset$, we can also define an inverse for G, but in a graph sense. One suppose, for the sake of simplicity, that

$$\underline{p} = \underline{p_1} < \overline{p_1} < \underline{p_2} < \overline{p_2} = \overline{p},$$

the other case where $\underline{p_1} > \overline{p_2}$ can be treated exactly the same way. G is an increasing function from $[\underline{p_1}, \overline{p_1}] \cup]\underline{p_2}, \overline{p_2}]$ onto $[G(\underline{p}), G(\overline{p})]$, and $G^{-1}(G(\overline{p_1})) = [\overline{p_1}, \underline{p_2}]$. Particularly, there is not a univocal p such that $G(p) = \Psi(a,b)$ if $\Psi(a,b) = G(\overline{p_1})$. Nevertheless, for all $p \in G^{-1}(G(\overline{p_1}))$, we have $\tilde{\pi}_1^{-1}(p) = 1$ and $\tilde{\pi}_2^{-1}(p) = 0$, and then one gets the uniqueness of the solution to the system (4.22). □

4.2.2 The discrete solution

We will now work on the implicit finite volume scheme given by (4.14)-(4.21) to show that this approximate problem is well-posed.

Definition 4.2 (Discrete solution) *Let N, M be two positive integers and $\mathcal{D}$ be the associated discretization of $\Omega \times (0,T)$. One defines :*

$$\mathcal{X}_{\mathcal{D},i} = \left\{ \begin{array}{c} z \in L^\infty(\Omega_i \times (0,T)) \ / \ \forall]x_j, x_{j+1}[\subset \Omega_i, \forall n \in [\![0, \mathrm{M}-1]\!], \\ z_{|]x_j,x_{j+1}[\times]t^n,t^{n+1}]} \text{ is a constant} \end{array} \right\},$$

and

$$\mathcal{X}_{\mathcal{D}} = \left\{ z \in L^\infty(\Omega \times (0,T)) \ / \ \forall i = 1,2, \ z_{|\Omega_i \times (0,T)} \in \mathcal{X}_{\mathcal{D},i} \right\}.$$

One defines $u_{\mathcal{D}}(x,t) \in \mathcal{X}_{\mathcal{D}}$, called discrete solution, defined almost everywhere in $]-1,1[\times]0,T[$ by : for all $j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, for all $n \in [\![0, \mathrm{M}-1]\!]$,

$$\left\{ \begin{array}{l} u_{\mathcal{D}}(x,t) = u_{j+1/2}^{n+1} \text{ if } (x,t) \in]x_j, x_{j+1}[\times]t_n, t_{n+1}], \\ u_{\mathcal{D}}(x,0) = u_{0,\mathcal{D}}(x), \end{array} \right.$$

We now matter about the existence and the uniqueness of the discrete solution to the scheme. In order to prove the existence of a discrete solution, we first need an a priori estimate on it.

Lemma 4.2.2 **(L^∞-estimate)** *Let $N, M > 0$, let $(u^n_{j+1/2})_{j\in[\![-\mathrm{N},\mathrm{N}-1]\!],\mathrm{n}\in[\![0,\mathrm{M}]\!]}$ be a solution to the scheme (4.14)-(4.21), then*

$$0 \leq u^n_{j+1/2} \leq 1, \qquad \forall j \in [\![-\mathrm{N},\mathrm{N}-1]\!], \forall \mathrm{n} \in [\![0,\mathrm{M}]\!].$$

Proof

For all $0 \leq n \leq M-1$, for all $j \in [\![-\mathrm{N},\mathrm{N}-1]\!]$, one has thanks to (4.15) :

$$u^{n+1}_{j+1/2} = u^n_{j+1/2} + \frac{\delta t}{\phi_i \delta x}\left[F^{n+1}_j - F^{n+1}_{j+1}\right].$$

So for $j \in [\![-\mathrm{N}+1,-2]\!] \cup [\![1,\mathrm{N}-2]\!]$, thanks to (4.16), one has :

$$\begin{aligned} u^{n+1}_{j+1/2} &= u^n_{j+1/2} + \frac{\delta t}{\phi_i \delta x}\left[q^{n+1}(f_i(u^{n+1}_{j-1/2}) - f_i(u^{n+1}_{j+1/2}))\right] \\ &+ \frac{\delta t}{\phi_i \delta x^2}\left[\varphi_i(u^{n+1}_{j-1/2}) + \varphi_i(u^{n+1}_{j+3/2}) - 2\varphi_i(u^{n+1}_{j+1/2})\right], \end{aligned} \tag{4.23}$$

For $j = -N$, one has thanks to (4.17), and using $f(1) = 1$,

$$\begin{aligned} u^{n+1}_{-N+1/2} &= u^n_{-N+1/2} + \frac{\delta t}{\phi_1 \delta x}\left[g^{n+1} - q^{n+1} f_1(u^{n+1}_{-N+1/2})\right] \\ &+ \frac{\delta t}{\phi_1 \delta x^2}\left[\varphi_1(u^{n+1}_{-N+3/2}) - \varphi_1(u^{n+1}_{-N+1/2})\right]. \end{aligned} \tag{4.24}$$

For $j = N-1$, one has thanks to (4.18),

$$\begin{aligned} u^{n+1}_{N-1/2} &= u^n_{N-1/2} + \frac{\delta t}{\phi_2 \delta x}\left[q^{n+1}(f_2(u^{n+1}_{N-3/2}) - f_2(u^{n+1}_{N-1/2}))\right] \\ &+ \frac{\delta t}{\phi_2 \delta x^2}\left[\varphi_2(u^{n+1}_{N-3/2}) - \varphi_2(u^{n+1}_{N-1/2})\right]. \end{aligned} \tag{4.25}$$

For $j = -1$, one has thanks to (4.19),

$$\begin{aligned} u^{n+1}_{-1/2} &= u^n_{-1/2} + \frac{\delta t}{\phi_1 \delta x}\left[q^{n+1}(f_1(u^{n+1}_{-3/2}) - f_1(u^{n+1}_{-1/2}))\right] \\ &+ \frac{\delta t}{\phi_1 \delta x^2}\left[2\varphi_1(u^{n+1}_{0,1}) + \varphi_1(u^{n+1}_{-3/2}) - 3\varphi_1(u^{n+1}_{-1/2})\right]. \end{aligned} \tag{4.26}$$

For $j = 0$, one has thanks to (4.20),

$$\begin{aligned} u^{n+1}_{1/2} &= u^n_{1/2} + \frac{\delta t}{\phi_2 \delta x}\left[q^{n+1}(f_2(u^{n+1}_{0,2}) - f_2(u^{n+1}_{1/2}))\right] \\ &+ \frac{\delta t}{\phi_2 \delta x^2}\left[2\varphi_2(u^{n+1}_{0,2}) + \varphi_2(u^{n+1}_{3/2}) - 3\varphi_2(u^{n+1}_{1/2})\right]. \end{aligned} \tag{4.27}$$

Let $j_M \in [\![-\mathrm{N},\mathrm{N}-1]\!]$, $n_M \in [\![0,\mathrm{M}]\!]$, such that

$$u^{n_M}_{j_M+1/2} = \max_{\substack{-N \leq j \leq N-1, \\ 0 \leq n \leq M}} (u^n_{j+1/2}). \tag{4.28}$$

First, it is clear thanks to (4.14) that for all $j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $0 \le u^0_{j+1/2} \le 1$, then it ends the proof in the case $n_M = 0$.
We suppose now that $n_M \ge 1$, and that $u^{n_M}_{j_M+1/2} > 1$. Thanks to lemma 4.2.1, which insures that we can suppose $u^{n_M}_{0,1} \le 1$, $u^{n_M}_{0,2} \le 1$, and thanks to the monotony of f_i, φ_i for $i = 1, 2$, and to the positivity of q^{n_M}, we can claim using (4.23)-(4.27) that $u^{n_M}_{j_M+1/2} < u^{n_M-1}_{j_M+1/2}$. This is in contradiction with the definition (4.28). We can thus claim that

$$u^n_{j+1/2} \le 1.$$

One can show in the same way that for any $j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $n \in [\![0, \mathrm{M}]\!]$,

$$u^n_{j+1/2} \ge 0.$$

□

We can now prove the following lemma, that insures the well-posedness of the scheme, that is the system made of equations numbered from (4.14) to (4.21) admits a unique solution.

Lemma 4.2.3 *The implicit finite volume scheme (4.14)-(4.21) admits a unique discrete solution*

Proof
Uniqueness of the discrete solution :
The proof of uniqueness uses a monotony method inspired from [EGHM02] : suppose that $u_\mathcal{D}$ and $v_\mathcal{D}$ are two discrete solutions associated to the discrete initial data $u_{0,\mathcal{D}}$ and $v_{0,\mathcal{D}}$. The equations (4.23)-(4.27) can be rewritten, for all $j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, for all $n \in [\![0, \mathrm{M}-1]\!]$,

$$B_j\left(u^{n+1}_{j+1/2}, u^n_{j+1/2}, (u^{n+1}_{k+1/2})_{k\neq j}, (u^{n+1}_{0,i})_{i=1,2}\right) = 0, \tag{4.29}$$

where B_j is non-decreasing w.r.t. $u^{n+1}_{j+1/2}$, and non-increasing w.r.t. each others. One thus has

$$B_j\left(u^{n+1}_{j+1/2}, u^n_{j+1/2}\top v^n_{j+1/2}, (u^{n+1}_{k+1/2}\top v^{n+1}_{k+1/2})_{k\neq j}, (u^{n+1}_{0,i}\top v^{n+1}_{0,i})_{i=1,2}\right) \le 0,$$

with the notation $a\top b = \max(a, b)$. Inverting the roles of $u_\mathcal{D}$ and $v_\mathcal{D}$ leads to

$$B_j\left(v^{n+1}_{j+1/2}, u^n_{j+1/2}\top v^n_{j+1/2}, (u^{n+1}_{k+1/2}\top v^{n+1}_{k+1/2})_{k\neq j}, (u^{n+1}_{0,i}\top v^{n+1}_{0,i})_{i=1,2}\right) \le 0.$$

Since $u^{n+1}_{j+1/2}\top v^{n+1}_{j+1/2}$ is either equal to $u^{n+1}_{j+1/2}$ or $v^{n+1}_{j+1/2}$, one gets :

$$B_j\left(u^{n+1}_{j+1/2}\top v^{n+1}_{j+1/2}, u^n_{j+1/2}\top v^n_{j+1/2}, (u^{n+1}_{k+1/2}\top v^{n+1}_{k+1/2})_{k\neq j}, (u^{n+1}_{0,i}\top v^{n+1}_{0,i})_{i=1,2}\right) \le 0. \tag{4.30}$$

One get the same way, with the convention $a\bot b = \min(a, b)$:

$$B_j\left(u^{n+1}_{j+1/2}\bot v^{n+1}_{j+1/2}, u^n_{j+1/2}\bot v^n_{j+1/2}, (u^{n+1}_{k+1/2}\bot v^{n+1}_{k+1/2})_{k\neq j}, (u^{n+1}_{0,i}\bot v^{n+1}_{0,i})_{i=1,2}\right) \ge 0. \tag{4.31}$$

Subtracting (4.31) to (4.30), gives using the expression of B_j :

– for $j \in [\![-\mathrm{N}+1,-2]\!] \cup [\![1,\mathrm{N}-2]\!]$, using (4.23),

$$rcl \qquad \phi_i \frac{|u^{n+1}_{j+1/2} - v^{n+1}_{j+1/2}| - |u^n_{j+1/2} - v^n_{j+1/2}|}{\delta t}\delta x$$
$$+ \quad q^{n+1}\left(|f_i(u^{n+1}_{j+1/2}) - f_i(v^{n+1}_{j+1/2})| - |f_i(u^{n+1}_{j-1/2}) - f_i(v^{n+1}_{j-1/2})|\right)$$
$$+ \quad \frac{|\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(v^{n+1}_{j+1/2})| - |\varphi_i(u^{n+1}_{j-1/2}) - \varphi_i(v^{n+1}_{j-1/2})|}{\delta x}$$
$$+ \quad \frac{|\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(v^{n+1}_{j+1/2})| - |\varphi_i(u^{n+1}_{j+3/2}) - \varphi_i(v^{n+1}_{j+3/2})|}{\delta x} \leq 0; \quad (4.32)$$

– for $j = -N$, using (4.24),

$$\phi_1 \frac{|u^{n+1}_{-N+1/2} - v^{n+1}_{-N+1/2}| - |u^n_{-N+1/2} - v^n_{-N+1/2}|}{\delta t}\delta x$$
$$+ \quad q^{n+1}\left(|f_1(u^{n+1}_{-N+1/2}) - f_1(v^{n+1}_{-N+1/2})|\right)$$
$$+ \quad \frac{|\varphi_1(u^{n+1}_{-N+1/2}) - \varphi_1(v^{n+1}_{-N+1/2})| - |\varphi_1(u^{n+1}_{-N+3/2}) - \varphi_1(v^{n+1}_{-N+3/2})|}{\delta x} \leq 0; \quad (4.33)$$

– for $j = N-1$, using (4.25),

$$\phi_2 \frac{|u^{n+1}_{N-1/2} - v^{n+1}_{N-1/2}| - |u^n_{N-1/2} - v^n_{N-1/2}|}{\delta t}\delta x$$
$$+ \quad q^{n+1}\left(|f_2(u^{n+1}_{N-1/2}) - f_2(v^{n+1}_{N-1/2})| - |f_2(u^{n+1}_{N-3/2}) - f_2(v^{n+1}_{N-3/2})|\right)$$
$$+ \quad \frac{|\varphi_2(u^{n+1}_{N-1/2}) - \varphi_2(v^{n+1}_{N-1/2})| - |\varphi_2(u^{n+1}_{N-3/2}) - \varphi_2(v^{n+1}_{N-3/2})|}{\delta x} \leq 0; \quad (4.34)$$

– for $j = -1$, using (4.26),

$$\phi_1 \frac{|u^{n+1}_{-1/2} - v^{n+1}_{-1/2}| - |u^n_{-1/2} - v^n_{-1/2}|}{\delta t}\delta x$$
$$+ \quad q^{n+1}\left(|f_1(u^{n+1}_{-1/2}) - f_1(v^{n+1}_{-1/2})| - |f_1(u^{n+1}_{-3/2}) - f_1(v^{n+1}_{-3/2})|\right)$$
$$+ \quad \frac{|\varphi_1(u^{n+1}_{-1/2}) - \varphi_1(v^{n+1}_{-1/2})| - |\varphi_1(u^{n+1}_{-3/2}) - \varphi_1(v^{n+1}_{-3/2})|}{\delta x}$$
$$+ \quad 2\frac{|\varphi_1(u^{n+1}_{-1/2}) - \varphi_1(v^{n+1}_{-1/2})| - |\varphi_1(u^{n+1}_{0,1}) - \varphi_1(v^{n+1}_{0,1})|}{\delta x} \leq 0; \quad (4.35)$$

– for $j = 0$, using (4.27),

$$\phi_2 \frac{|u^{n+1}_{1/2} - v^{n+1}_{1/2}| - |u^n_{1/2} - v^n_{1/2}|}{\delta t}\delta x$$
$$+ \quad q^{n+1}\left(|f_2(u^{n+1}_{1/2}) - f_2(v^{n+1}_{1/2})| - |f_2(u^{n+1}_{0,2}) - f_2(v^{n+1}_{0,2})|\right)$$
$$+ \quad \frac{|\varphi_2(u^{n+1}_{1/2}) - \varphi_2(v^{n+1}_{1/2})| - |\varphi_2(u^{n+1}_{3/2}) - \varphi_2(v^{n+1}_{3/2})|}{\delta x}$$
$$+ \quad 2\frac{|\varphi_2(u^{n+1}_{1/2}) - \varphi_2(v^{n+1}_{1/2})| - |\varphi_2(u^{n+1}_{0,2}) - \varphi_2(v^{n+1}_{0,2})|}{\delta x} \leq 0. \quad (4.36)$$

We now sum those inequalities for $j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, and we remark that the internal exchange terms vanishes. So this sum can be written :

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}}(x, t_{n+1}) - v_{\mathcal{D}}(x, t_{n+1})| dx - \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}}(x, t_n) - v_{\mathcal{D}}(x, t_n)| dx$$
$$\leq -\delta t q^{n+1} |f_2(u^{n+1}_{N-1/2}) - f_2(v^{n+1}_{N-1/2})| + \delta t A^{n+1} \leq \delta t A^{n+1}, \tag{4.37}$$

with

$$\begin{aligned} A^{n+1} \;=\;& q^{n+1} \left(|f_2(u^{n+1}_{0,2}) - f_2(v^{n+1}_{0,2})| - |f_1(u^{n+1}_{-1/2}) - f_1(v^{n+1}_{-1/2})| \right) \\ &+2\frac{|\varphi_1(u^{n+1}_{0,1}) - \varphi_1(v^{n+1}_{0,1})| + |\varphi_2(u^{n+1}_{0,2}) - \varphi_2(v^{n+1}_{0,2})|}{\delta x} \\ &-2\frac{|\varphi_1(u^{n+1}_{-1/2}) - \varphi_1(v^{n+1}_{-1/2})| + |\varphi_2(u^{n+1}_{1/2}) - \varphi_2(v^{n+1}_{1/2})|}{\delta x}. \end{aligned} \tag{4.38}$$

First, it is clear that if $u^{n+1}_{0,1} = v^{n+1}_{0,1}$ and $u^{n+1}_{0,2} = v^{n+1}_{0,2}$, then $A^{n+1} \leq 0$.
Suppose now that $u^{n+1}_{0,1} > v^{n+1}_{0,1}$, then using the monotony inducted by the relation (4.21), we get that $u^{n+1}_{0,2} \geq v^{n+1}_{0,2}$. Thanks to the relations (4.19),(4.20), one has :

$$\begin{aligned} & q^{n+1} \left(f_2(u^{n+1}_{0,2}) - f_2(v^{n+1}_{0,2}) \right) + 2\frac{\left(\varphi_1(u^{n+1}_{0,1}) - \varphi_1(v^{n+1}_{0,1})\right) + \left(\varphi_2(u^{n+1}_{0,2}) - \varphi_2(v^{n+1}_{0,2})\right)}{\delta x} \\ = \; & q^{n+1} \left(f_1(u^{n+1}_{-1/2}) - f_1(v^{n+1}_{-1/2}) \right) + 2\frac{\left(\varphi_1(u^{n+1}_{-1/2}) - \varphi_1(v^{n+1}_{-1/2})\right) + \left(\varphi_2(u^{n+1}_{1/2}) - \varphi_2(v^{n+1}_{1/2})\right)}{\delta x}. \end{aligned} \tag{4.39}$$

Using (4.39) in (4.38), one gets $A^{n+1} \leq 0$. One can treat the cases $u^{n+1}_{0,1} < v^{n+1}_{0,1}$, $u^{n+1}_{0,2} > v^{n+1}_{0,2}$ and $u^{n+1}_{0,2} < v^{n+1}_{0,2}$ exactly the same way. Thus we deduce from (4.37) that for all $n \in [\![0, \mathrm{M}-1]\!]$,

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}}(x, t_{n+1}) - v_{\mathcal{D}}(x, t_{n+1})| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}}(x, t_n) - v_{\mathcal{D}}(x, t_n)| dx.$$

This particularly implies the following L^1-contraction principle for the discrete solution : $\forall t \in [0, T]$,

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}}(x, t) - v_{\mathcal{D}}(x, t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{0,\mathcal{D}}(x) - v_{0,\mathcal{D}}(x)| dx. \tag{4.40}$$

The uniqueness of the discrete solution to the scheme (4.23)-(4.27) is a direct consequence of (4.40).

<u>**Existence of the discrete solution :**</u>
The proof of existence, based on a argument of topological degree (see e.g. [Cro64]), is already stated in a similar case in [EEM06]. One can easily rewrite the numerical scheme (4.23)-(4.27) under the form : $\forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$u^{n+1}_{j+1/2} = u^n_{j+1/2} + H_j \left(\left(u^{n+1}_{k+1/2} \right)_{-N \leq k \leq N-1} \right), \tag{4.41}$$

where H_j is continuous w.r.t. each one of its arguments (we use in particular the fact that $(u^{n+1}_{-1/2}, u^{n+1}_{1/2}) \mapsto (u^{n+1}_{0,1}, u^{n+1}_{0,2})$ is continuous, as stated in lemma 4.2.1). Let $\lambda \in [0,1]$, let $n \in [\![0, \mathrm{M}-1]\!]$ such that $\left(u^n_{k+1/2}\right)_{-N\le k\le N-1}$ exists, one defines (if it exists) for all $j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$:

$$u^{n+1}_{j+1/2,\lambda} + \lambda H_j\left(\left(u^{n+1}_{k+1/2,\lambda}\right)_{-N\le k\le N-1}\right) = u^n_{j+1/2}. \tag{4.42}$$

Since the proof of lemma 4.2.2 can be extended to this case as soon as $\lambda \ge 0$, we have the a priori estimate : $\forall \lambda \in [0,1]$, $\forall j \in [\![-\mathrm{N}, \mathrm{N}-1]\!]$, $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$0 \le u^{n+1}_{j+1/2,\lambda} \le 1. \tag{4.43}$$

One defines the compact set $K = \{x \in \mathbb{R}^{2N} / \max_j |x_j| \le 2\}$.
For $\lambda = 0$, the equation (4.42) becomes trivial, and then the topological degree $d(\lambda = 0) = 1$. The a priori estimate (4.43) insures that for all $\lambda \ge 0$, $u^{n+1}_{j+1/2,\lambda} \notin \partial K$, and then $d(\lambda = 1) = 1$. So the equation (4.41) admits at least one solution. This concludes the proof of lemma 4.2.3. □

In order to prove lemma 4.2.5, we will need the following technical lemma, which is a consequence of the monotony of the discrete transmission conditions (4.19)-(4.21).

Lemma 4.2.4 *Let $(a,b) \in [0,1]^2$, and let $(c,d) \in [0,1]^2$ be the unique solution to the system (4.22), as stated in lemma 4.2.1, then the following inequality holds :*

$$(\pi_1(c) - \pi_2(d))\left(\frac{\varphi_1(a) - \varphi_1(c)}{\delta x}\right) \ge 0.$$

Proof
In this proof, we suppose that $\pi_1(0) \ge \pi_2(0)$ and $\pi_1(1) \ge \pi_2(1)$, the other cases do not bring any other difficulties. One has $\tilde\pi_1(c) \cap \tilde\pi_2(d) \neq \emptyset$, so there are three different cases :

- $\underline{\pi_1(c) = \pi_2(d)}$: in this case, one has directly :

$$(\pi_1(c) - \pi_2(d))\left(\frac{\varphi_1(a) - \varphi_1(c)}{\delta x}\right) = 0.$$

- $\underline{\pi_2(d) < \pi_1(0)}$: the relation $\tilde\pi_1(c) \cap \tilde\pi_2(d) \neq \emptyset$ insures that $c = 0$, and thus $\varphi_1(a) \ge \varphi_1(c)$. This gives :

$$(\pi_1(c) - \pi_2(d))\left(\frac{\varphi_1(a) - \varphi_1(c)}{\delta x}\right) \ge 0.$$

- $\underline{\pi_1(c) > \pi_2(1)}$: this implies $d = 1$. From (4.22), we deduce that :

$$\left(\frac{\varphi_1(a) - \varphi_1(c)}{\delta x}\right) = \left(\frac{\varphi_2(d) - \varphi_2(b)}{\delta x}\right) + \frac{q^{n+1}}{2}\left(f_2(d) - f_1(a)\right).$$

  Since $d = 1$, $\varphi_2(d) \ge \varphi_2(b)$ and $f_2(d) = 1 \ge f_1(a)$, and then, since $q^{n+1} \ge 0$, one has also $\varphi_1(a) \ge \varphi_1(c)$. This insures :

$$(\pi_1(c) - \pi_2(d))\left(\frac{\varphi_1(a) - \varphi_1(c)}{\delta x}\right) \ge 0.$$

This ends the proof of lemma 4.2.4. □

Definition 4.3 (discrete $L^2(0,T;H^1(\Omega_i))$ semi-norms) *Let $i = 1, 2$, one defines the discrete $L^2(0,T;H^1(\Omega_i))$ semi-norms $|\cdot|_{1,\mathcal{D},i}$ on $\mathcal{X}_{\mathcal{D},i}$ by : $\forall z \in \mathcal{X}_{\mathcal{D},i}$,*

$$|z|^2_{1,\mathcal{D},i} = \sum_{n=0}^{M-1} \delta t \sum_{j \in J_{int,i}} \delta x \left(\frac{z(x_{j+1/2}, t^{n+1}) - z(x_{j-1/2}, t^{n+1})}{\delta x} \right)^2,$$

where $J_{int,1} = [\![-\mathrm{N}+1, -1]\!]$ and $J_{int,2} = [\![1, \mathrm{N}-1]\!]$.

Lemma 4.2.5 (discrete $L^2(0,T;H^1(\Omega_i))$ estimates) *For $i = 1, 2$, one defines the Lipschitz continuous increasing function $\xi_i : s \mapsto \int_0^s \sqrt{\lambda_i(a)}\pi_i'(a)da$. There exists $C > 0$ only depending on q, π_i, ϕ_i, T such that :*

$$\sum_{i=1,2} |\xi_i(u_{\mathcal{D}})|^2_{1,\mathcal{D},i} \le C.$$

Proof
One multiplies the equation (4.15) by $\delta t \pi_i(u^{n+1}_{j+1/2})$, and sum on $j = -N, N-1$:

$$\sum_{j=-N}^{N-1} \phi_i \pi_i(u^{n+1}_{j+1/2}) \left(u^{n+1}_{j+1/2} - u^n_{j+1/2} \right) \delta x + \sum_{j=-N}^{N-1} \delta t \pi_i(u^{n+1}_{j+1/2}) \left(F^{n+1}_{j+1} - F^{n+1}_j \right) = 0. \tag{4.44}$$

Taking the definitions of F^{n+1}_j, $j \in [\![-\mathrm{N}, \mathrm{N}]\!]$, given by equations (4.23)-(4.27) in (4.44), it follows :

$$A^{n+1} + B^{n+1} + C^{n+1} = 0, \tag{4.45}$$

with, denoting $J_{\mathrm{int}} = [\![-\mathrm{N}+1, -1]\!] \cup [\![1, \mathrm{N}-1]\!]$,

$$A^{n+1} = \sum_{j=-N}^{N-1} \phi_i \pi_i(u^{n+1}_{j+1/2}) \left(u^{n+1}_{j+1/2} - u^n_{j+1/2} \right) \delta x, \tag{4.46}$$

$$\begin{aligned} B^{n+1} = & \sum_{j \in J_{\mathrm{int}}} q^{n+1} \delta t \pi_i(u^{n+1}_{j+1/2}) \left(f_i(u^{n+1}_{j+1/2}) - f_i(u^{n+1}_{j-1/2}) \right) \\ & + q^{n+1} \delta t \pi_2(u^{n+1}_{1/2}) \left(f_2(u^{n+1}_{1/2}) - f_2(u^{n+1}_{0,2}) \right) \\ & + \delta t \pi_1(u^{n+1}_{-N+1/2}) \left(q^{n+1} f_1(u^{n+1}_{-N+1/2}) - g^{n+1} \right), \end{aligned} \tag{4.47}$$

$$\begin{aligned} C^{n+1} = & \sum_{j \in J_{\mathrm{int}}} \delta t \left(\pi_i(u^{n+1}_{j+1/2}) - \pi_i(u^{n+1}_{j-1/2}) \right) \frac{\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(u^{n+1}_{j-1/2})}{\delta x} \\ & + \delta t \left(\pi_1(u^{n+1}_{0,1}) - \pi_1(u^{n+1}_{-1/2}) \right) \frac{2(\varphi_1(u^{n+1}_{0,1}) - \varphi_1(u^{n+1}_{-1/2}))}{\delta x} \\ & + \delta t \left(\pi_2(u^{n+1}_{1/2}) - \pi_2(u^{n+1}_{0,2}) \right) \frac{2(\varphi_2(u^{n+1}_{1/2}) - \varphi_2(u^{n+1}_{0,2}))}{\delta x} \\ & + \delta t \left(\pi_1(u^{n+1}_{0,1}) - \pi_2(u^{n+1}_{0,2}) \right) \frac{2(\varphi_1(u^{n+1}_{-1/2}) - \varphi_1(u^{n+1}_{0,1}))}{\delta x}. \end{aligned} \tag{4.48}$$

Thanks to assumption 4.2, one has $0 \leq g^{n+1} \leq q^{n+1}$. Since f_1 is a continuous increasing function fulfilling $f_1(0) = 0$ and $f_1(1) = 1$, there exists u_{-N}^{n+1} such that $g^{n+1} = q^{n+1} f_1(u_{-N}^{n+1})$. So we deduce from (4.47) that :

$$\begin{aligned} B^{n+1} &= \sum_{j \in J_{\text{int}}} q^{n+1} \delta t \pi_i(u_{j+1/2}^{n+1}) \left(f_i(u_{j+1/2}^{n+1}) - f_i(u_{j-1/2}^{n+1}) \right) \\ &\quad + q^{n+1} \delta t \pi_2(u_{1/2}^{n+1}) \left(f_2(u_{1/2}^{n+1}) - f_2(u_{0,2}^{n+1}) \right) \\ &\quad + \delta t q^{n+1} \pi_1(u_{-N+1/2}^{n+1}) \left(f_1(u_{-N+1/2}^{n+1}) - f_1(u_{-N}^{n+1}) \right), \end{aligned} \tag{4.49}$$

Since π_i is a non-decreasing function, $\mathcal{G}_i : s \mapsto \int_0^s \phi_i \pi_i(a) da$ is convex, and we deduce from (4.46) that : $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$A^{n+1} \geq \sum_{j=-N}^{N-1} \left(\mathcal{G}_i(u_{j+1/2}^{n+1}) - \mathcal{G}_i(u_{j+1/2}^{n}) \right) \delta x. \tag{4.50}$$

The function $\pi_i \circ f_i^{-1}$ is increasing, and then $\mathcal{H}_i : s \mapsto \int_0^s \pi_i \circ f_i^{-1}(a) da$ is convex, and we deduce from (4.49) that : $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$\begin{aligned} B^{n+1} &\geq \delta t q^{n+1} \begin{pmatrix} \sum_{j \in J_{\text{int}}} \left(\mathcal{H}_i(f_i(u_{j+1/2}^{n+1})) - \mathcal{H}_i(f_i(u_{j-1/2}^{n+1})) \right) \\ + \mathcal{H}_2(f_2(u_{1/2}^{n+1})) - \mathcal{H}_2(f_2(u_{0,2}^{n+1})) \\ + \mathcal{H}_1(f_1(u_{-N+1/2}^{n+1})) - \mathcal{H}_1(f_1(u_{-N}^{n+1})) \end{pmatrix} \\ &\geq \delta t q^{n+1} \left(\mathcal{H}_1(f_1(u_{-1/2}^{n+1})) - \mathcal{H}_1(f_1(u_{-N}^{n+1})) + \mathcal{H}_2(f_2(u_{N-1/2}^{n+1})) - \mathcal{H}_2(f_2(u_{0,2}^{n+1})) \right), \end{aligned}$$

and thus, $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$B^{n+1} \geq -\delta t q^{n+1} \left(\sum_{i=1,2} \int_0^1 |\pi_i(a)| da \right). \tag{4.51}$$

Let $\xi_i : s \mapsto \int_0^s \sqrt{\lambda_i(a)} \pi_i'(a) da$, Cauchy-Schwarz inequality yields : $\forall (a,b) \in [0,1]^2$,

$$(\pi_i(a) - \pi_i(b))(\varphi_i(a) - \varphi_i(b)) \geq (\xi_i(a) - \xi_i(b))^2.$$

Thanks to lemma 4.2.4, one has :

$$\delta t \left(\pi_1(u_{0,1}^{n+1}) - \pi_2(u_{0,2}^{n+1}) \right) \frac{2(\varphi_1(u_{-1/2}^{n+1}) - \varphi_1(u_{0,1}^{n+1}))}{\delta x} \geq 0,$$

and then (4.48) yields :

$$\begin{aligned} C^{n+1} &\geq \sum_{j \in J_{\text{int}}} \delta t \delta x \left(\frac{\xi_i(u_{j+1/2}^{n+1}) - \xi_i(u_{j-1/2}^{n+1})}{\delta x} \right)^2 \\ &\quad + \delta t (\delta x/2) \left(\frac{\xi_1(u_{0,1}^{n+1}) - \xi_1(u_{-1/2}^{n+1})}{\delta x/2} \right)^2 \\ &\quad + \delta t (\delta x/2) \left(\frac{\xi_2(u_{0,2}^{n+1}) - \xi_2(u_{1/2}^{n+1})}{\delta x/2} \right)^2. \end{aligned} \tag{4.52}$$

This particularly insures that : $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$C^{n+1} \geq \sum_{j \in J_{\mathrm{int}}} \delta t \delta x \left(\frac{\xi_i(u_{j+1/2}^{n+1}) - \xi_i(u_{j-1/2}^{n+1})}{\delta x} \right)^2 \tag{4.53}$$

Using (4.50),(4.51) and (4.53) in (4.45), and summing on $n \in [\![0, \mathrm{M}-1]\!]$ yields :

$$\sum_{i=1,2} |\xi_i(u_{\mathcal{D}})|^2_{1,\mathcal{D},i} \leq \sum_{i=1,2} \left(\int_0^1 |\pi_i(a)| da \right) \left(\phi_i + \int_0^T q(t) dt \right).$$

□

Remark 4.2.1 *Under the additional assumption : $f_i \circ \pi_i^{-1}$ is a Lipschitz function, one can adapt the proof of proposition 3.1 in [EGHM02] to get a weak BV estimate, which seems to be necessary to prove the convergence for unstructured meshes of the upstream scheme to the entropy-weak solution to a hyperbolic equation (see [CGH93] or chapter 5 in [EGH00]). Nevertheless such an estimate will not be useful in the sequel, even to adapt this work to non-uniform discretization of Ω. Indeed, the parabolic regularization (due to the fact that φ_i is increasing) will be enough to get the convergence to a weak solution, and we will not need to let the numerical diffusion appear, so we can avoid this weak BV estimate.*

4.2.3 Compactness of a family of approximate solutions

Let $(M_p)_{p\in\mathbb{N}}, (N_p)_{p\in\mathbb{N}}$ be two sequences of positive integers tending to $+\infty$. We denote $\mathcal{D}_p$ the discretization of $\Omega \times (0,T)$ associated to M_p, and N_p. The L^∞-estimate stated in lemma 4.2.2 shows that there exists $u \in L^\infty(\Omega \times (0,T))$, $0 \leq u \leq 1$, such that, up to a subsequence, $u_{\mathcal{D}_p} \to u$ in the $L^\infty(\Omega \times (0,T))$ weak-$\star$ sense as $p \to +\infty$.

We just need to prove that $u_{\mathcal{D}_p} \to u$ almost everywhere in $\Omega \times (0,T)$ to get the convergence of $(u_{\mathcal{D}_p})$ toward u in $L^r(\Omega \times (0,T))$ for any $1 \leq r < +\infty$. This will come from the Kolmogorov compactness criterion stated below and for example in [Bré83].

Theorem 4.2.6 (Kolmogorov compactness criterion)*Let $\mathcal{Q}$ be an open bounded subset of $\mathbb{R}^k$, and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^2(\mathbb{R}^k)$ such that :*

$$\lim_{|\delta|\to 0} \left(\sup_{n\in\mathbb{N}} \| \ u_n(\cdot+\delta) - u_n(\cdot) \ \|_{L^2(\mathcal{Q})} \right) = 0,$$

then there exists $u \in L^2(\mathcal{Q})$ such that, up to a subsequence,

$$u_n \to u \text{ in } L^2(\mathcal{Q}) \text{ as } n \to +\infty.$$

To apply Kolmogorov criterion we need some estimates on the space and time translates of $\xi_i(u_{\mathcal{D}})$.

Lemma 4.2.7 (space and time translates estimate) *For all $\eta \in \mathbb{R}$, for $i = 1, 2$, one denotes $\Omega_{i,\eta} = \{x \in \Omega_i \ / \ (x+\eta) \in \Omega_i\}$, then the following estimate holds :*

$$\|\xi_i(u_{\mathcal{D}})(\cdot+\eta,\cdot) - \xi_i(u_{\mathcal{D}})(\cdot,\cdot)\|_{L^2(\Omega_{i,\eta}\times(0,T))} \leq |\xi_i(u_{\mathcal{D}})|_{1,\mathcal{D},i} |\eta| (|\eta| + 2\delta x). \tag{4.54}$$

One denotes $w_{i,\mathcal{D}}$ the function defined almost everywhere by :

$$w_{i,\mathcal{D}}(x,t) = \begin{cases} \xi_i(u_{\mathcal{D}})(x,t) & in \quad \Omega_i \times (0,T), \\ 0 & in \quad \mathbb{R}^2 \backslash (\Omega_i \times (0,T)). \end{cases}$$

There exists C_1 depending only on q, π_i, ϕ_i, T and C_2 only depending on such that : $q, \pi_i, \phi_i, T, \varphi_i$

$$\forall \eta \in \mathbb{R}, \qquad \|w_{i,\mathcal{D}}(\cdot + \eta, \cdot) - w_{i,\mathcal{D}}(\cdot, \cdot)\|_{L^2(\mathbb{R}^2)} \le C_1 \eta, \tag{4.55}$$

$$\forall \tau \in (0,T), \qquad \|w_{i,\mathcal{D}}(\cdot, \cdot + \tau) - w_{i,\mathcal{D}}(\cdot, \cdot)\|_{L^2(\Omega_i \times (0,T-\tau))} \le C_2 \tau. \tag{4.56}$$

The previous lemma is in fact a compilation of lemmata 4.2, 4.3 and 4.6 of [EGH00] adapted to our framework. The estimates (4.55) and (4.56) allows us to use the Kolmogorov compactness criterion on the sequence $(w_{i,\mathcal{D}_p})_{p\in\mathbb{N}}$, and thus, there exists $w_i \in L^2(\Omega_i \times (0,T))$ such that for almost every $(x,t) \in (\Omega_i \times (0,T))$, $\xi_i(u_{\mathcal{D}_p})(x,t) \to w_i(x,t)$, and then thanks to the L^∞-estimate $0 \le u_{\mathcal{D}_p}(x,t) \le 1$, one can claim that $\xi_i(u_{\mathcal{D}_p}) \to w_i$ in $L^r(\Omega_i \times (0,T))$, for all $r \in [1, +\infty[$. Letting p tend to $+\infty$ in (4.54) insures that w_i belongs to $L^2(0,T;H^1(\Omega_i))$.

In order to identify w_i and $\xi_i(u)$, we use the following lemma, called Minty trick. Since ξ_i^{-1} is a continuous function, this lemma could be replaced by a pointwise argument. Nevertheless, we can not refrain to use this beautiful trick, based on the monotony (in a large sense) of ξ_i.

Lemma 4.2.8 (Minty trick) *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^k$, $k \ge 1$, let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\mathcal{O})$. Let $u \in L^\infty(\mathcal{O})$ such that u_n tends to u in the $L^\infty(\mathcal{O})$ weak-$\star$ sense. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous non-decreasing function, such that $\psi(u_n)$ tends to f in $L^1(\mathcal{O})$, then :*

$$\psi(u) = f.$$

This allows us to say that $\xi_i(u) \in L^2(0,T;H^1(\Omega_i))$, and since $\varphi_i \circ \xi_i^{-1}$ is a Lipschitz function, there exists C depending only on $T, \|q\|_{L^1(0,T)}, \pi_i, \phi_i$ such that :

$$\|\varphi_i(u)\|_{L^2(0,T;H^1(\Omega_i))} \le C, \tag{4.57}$$

and that $\xi_i(u_{\mathcal{D}_p}) \to \xi_i(u)$, up to a subsequence, in $L^r(\Omega_i \times (0,T))$ as $p \to +\infty$ for any $r \in [1, +\infty[$. Since ξ_i, $i = 1, 2$, is an increasing function, one can claim that $u_{\mathcal{D}_p}$ converges a.e. in $\Omega \times (0,T)$ toward u, and then :

$$u_{\mathcal{D}_p} \to u \qquad \text{in the } L^\infty(\Omega \times (0,T))\text{-weak-} \star \text{ sense}, \tag{4.58}$$

$$u_{\mathcal{D}_p} \to u \qquad \text{in } L^r(\Omega_i \times (0,T)),\ \forall r \in [1, +\infty[. \tag{4.59}$$

4.2.4 Convergence of the scheme

Let $(M_p)_{p\in\mathbb{N}}, (N_p)_{p\in\mathbb{N}}$ be two sequences of positive integers tending to $+\infty$, and $(\mathcal{D}_p)_{p\in\mathbb{N}}$ the associated sequence of discretizations of $\Omega \times (0,T)$. As seen in (4.58) and (4.59) , one can already claim the convergence of the sequence $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ toward a function u, with $0 \le u \le 1$, and $\varphi_i(u) \in L^2(0,T;H^1(\Omega_i))$, $i = 1, 2$ (and even $\xi_i(u) \in L^2(0,T;H^1(\Omega_i))$). We now aim to verify that u is a weak solution to the problem $(\mathcal{P})$ in the sense of definition 4.1. We need the convergence of the traces of $u_{\mathcal{D}_p}$ toward the traces of u :

Lemma 4.2.9 (convergence of the traces) *Let $i = 1,2$, and let $\alpha \in \partial\Omega_i$. One denotes $\overline{u}_{\alpha,\mathcal{D}_p}$ the trace of $(u_{\mathcal{D}_p})_{|\Omega_i}$ on $\{x=\alpha\}$. Then, one has : for all $r \in [1,\infty[$,*

$$\overline{u}_{\alpha,\mathcal{D}_p}(t) \to u_{|\Omega_i}(\alpha,t) \text{ in } L^r(0,T) \text{ as } p \to +\infty.$$

Proof
Let $h>0$, $h \notin \mathbb{Q}$, and $B_{i,\alpha,h} = \{x \in \Omega_i \ / \ |x-\alpha| < h\}$. One supposes in the sequel of the proof that $\alpha = \inf \Omega_i$ (the proof can easily be adapted in the other case). Since $\xi_i(u)$ belongs to $L^2(0,T;H^1(\Omega_i))$, $\xi_i(u)(\cdot,t)$ is continuous for a.e. $t \in (0,T)$, and then,

$$\lim_{h\to 0} \int_0^T \frac{1}{h}\int_\alpha^{\alpha+h} |\xi_i(u)(x,t) - \xi_i(u)(\alpha,t)| dxdt = 0. \tag{4.60}$$

Moreover, since $\xi_i(u_{\mathcal{D}_p})$ converges toward $\xi_i(u)$ in $L^1(\Omega_i \times (0,T))$, one has :

$$\lim_{p\to+\infty} \int_0^T \frac{1}{h}\int_\alpha^{\alpha+h} |\xi_i(u_{\mathcal{D}_p})(x,t) - \xi_i(u)(x,t)| dxdt = 0. \tag{4.61}$$

Let $p \in \mathbb{N}$, there exists $(j_0,j_1) \in [\![-\mathrm{N},\mathrm{N}-1]\!]^2$ and $n \in [\![0,\mathrm{M}-1]\!]$ such that

$$\overline{u}_{\alpha,\mathcal{D}_p}(t) = u^{n+1}_{j_0+1/2}, \quad u_{\mathcal{D}_p}(\alpha+h,t) = u^{n+1}_{j_1+1/2}.$$

Then for a.e. $x \in]\alpha,\alpha+h[$, there exists $j_2 \in [\![\mathrm{j}_0,\mathrm{j}_1]\!]$ such that $\xi_i(u_{\mathcal{D}_p})(x,t) = \xi_i(u^{n+1}_{j_2+1/2})$, and so :

$$\int_0^T |\xi_i(u_{\mathcal{D}_p})(x,t) - \xi_i(\overline{u}_{\alpha,\mathcal{D}_p}(t))|dt \le \sum_{n=0}^{M-1}\delta t \sum_{j=j_0+1}^{j_2} |\xi_i(u^{n+1}_{j+1/2}) - \xi_i(u^{n+1}_{j-1/2})|$$

$$\le \left(\sum_{n=0}^{M-1}\delta t \sum_{j=j_0+1}^{j_2} \frac{\left(\xi_i(u^{n+1}_{j+1/2}) - \xi_i(u^{n+1}_{j-1/2})\right)^2}{\delta x}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{M-1}\delta t \sum_{j=j_0+1}^{j_2} \delta x\right)^{\frac{1}{2}},$$

and so

$$\int_0^T |\xi_i(u_{\mathcal{D}_p})(x,t) - \xi_i(\overline{u}_{\alpha,\mathcal{D}_p}(t))|dt \le |\xi_i(u_{\mathcal{D}_p})|_{1,\mathcal{D},i}\sqrt{T(h+\delta x)}.$$

This estimate is uniform w.r.t. x, and then, using lemma 4.2.5, there exists C depending neither on p, nor on h such that

$$\int_0^T \frac{1}{h}\int_\alpha^{\alpha+h} |\xi_i(u_{\mathcal{D}_p})(x,t) - \xi_i(\overline{u}_{\alpha,\mathcal{D}_p}(t))|dxdt \le C\sqrt{h+\delta x}. \tag{4.62}$$

For all $h \in]0,1[\setminus\mathbb{Q}$, for all $p \in \mathbb{N}$,

$$\begin{aligned}\int_0^T |\xi_i(u)(\alpha,t) - \xi_i(\overline{u}_{\alpha,\mathcal{D}_p}(t))|dt \le\ & \int_0^T \frac{1}{h}\int_\alpha^{\alpha+h} |\xi_i(u)(\alpha,t) - \xi_i(u(x,t))|dxdt \\ &+ \int_0^T \frac{1}{h}\int_\alpha^{\alpha+h} |\xi_i(u(x,t)) - \xi_i(u_{\mathcal{D}_p}(x,t))|dxdt \\ &+ \int_0^T \frac{1}{h}\int_\alpha^{\alpha+h} |\xi_i(u_{\mathcal{D}_p}(x,t)) - \xi_i(\overline{u}_{\alpha,\mathcal{D}_p}(t))|dxdt.\end{aligned}$$

Using the estimates (4.61) and (4.62), one get : for all $h \in]0,1[\backslash\mathbb{Q}$,

$$\lim_{p\to+\infty}\int_0^T |\xi_i(u)(\alpha,t)-\xi_i(\overline{u}_{\alpha,\mathcal{D}_p}(t))|dt \le \int_0^T \frac{1}{h}\int_\alpha^{\alpha+h} |\xi_i(u)(\alpha,t)-\xi_i(u(x,t))|dxdt + C\sqrt{h}. \tag{4.63}$$

The right hand side member of (4.63) tends to 0 thanks to (4.60) as h tends to 0. Thus for a.e. $t \in (0,T)$, $\lim_p \xi_i(\overline{u}_{\alpha,\mathcal{D}_p}(t)) = \xi_i(u)(\alpha,t)$, and since ξ_i is a continuous increasing function, one get :

$$\text{for a.e. } t \in (0,T),\ \lim_{p\to+\infty}\overline{u}_{\alpha,\mathcal{D}_p}(t) = u(\alpha,t),$$

and the uniform bound $0 \le \overline{u}_{\alpha,\mathcal{D}_p} \le 1$ gives directly the convergence in $L^r(0,T)$, $r \in [1,+\infty[$. This achieves the proof of lemma 4.2.9. □

In order to simplify the proof of convergence of the scheme toward a weak solution, we will use a density result in the same spirit than those stated in [Dro02].

Lemma 4.2.10 *Let $a,b \in \mathbb{R}$, $a<b$, then : $\{\psi \in C_c^\infty([a,b])/\psi' \in C_c^\infty(]a,b[)\}$ is dense in $W^{1,q}(a,b)$, $q \in [1,+\infty[$.*

This lemma particularly allows us, thanks to a straightforward generalisation, to restrict the set of test function for the weak formulation (4.13) to $\{\psi \in \mathcal{D}(\overline{\Omega}\times[0,T[)/\ \partial_x\psi \in \mathcal{D}((\cup_{i=1,2}\Omega_i)\times[0,T[)\}$.

Proof

Let $\psi \in C_c^\infty([a,b])$, let n be a large enough integer, let $\chi_n \in C_c^\infty(]a,b[)$, with $0 \le \chi_n \le 1$, and $\chi_n(x) = 1$ for $x \in [a+1/n, b-1/n]$. One defines :

$$\psi_n(x) = \psi(a) + \int_a^x \psi'(t)\chi_n(t)dt. \tag{4.64}$$

ψ_n' clearly converges in $L^q(a,b)$ toward ψ'. Let $x \in]a,b[$, and let n be large enough to insure that $x \in [a+1/n, b-1/n]$. An integration by parts in (4.64) yields

$$\psi_n(x) = \psi(x) + \psi(a) - \int_a^{a+1/n}\psi(t)\chi_n'(t)dt.$$

Thus, for all $x \in]a,b[$,

$$\lim_{n\to+\infty}\psi_n(x) = \psi(x).$$

Moreover,

$$|\psi_n(x)| \le |\psi(a)| + \int_a^b |\psi'(t)|dt,$$

thus we can apply the dominated convergence theorem to claim that ψ_n tends to ψ in $W^{1,q}(a,b)$, $q \in [1,+\infty[$. The density of $C_c^\infty([a,b])$ in $W^{1,q}(a,b)$ concludes the proof. □

Proposition 4.2.11 (convergence to a weak solution) *Let u be obtained as limit of the sequence of discrete solution as in section 4.2.3. Then u is a weak solution to the problem $(\mathcal{P})$ in the sense of definition 4.1. This particularly insures the existence of a weak solution.*

Proof
One already has the wanted regularity on the limit u to fulfil the points 1 and 2 of definition 4.1. So we can concentrate on the points 3 and 4 of the definition.

Let us first verify the third point, i.e. that for a.e. $t \in (0,T)$, $\tilde{\pi}_i(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset$. Let $u_{\mathcal{D}}$ be a discrete solution obtained via the scheme (4.23)-(4.27). For $i = 1,2$, one denotes $\gamma_{i,\mathcal{D}}$ the function defined for a.e. t in $(0,T)$ by $\gamma_{i,\mathcal{D}}(t) = u_{0,i}^{n+1}$ if $t \in (t^n, t^{n+1})$, so that, thanks to (4.21), for a.e. $t \in (0,T)$,

$$\tilde{\pi}_1(\gamma_{1,\mathcal{D}}(t)) \cap \tilde{\pi}_2(\gamma_{2,\mathcal{D}}(t)) \neq \emptyset. \tag{4.65}$$

Let $(\mathcal{D}_p)_{p\in\mathbb{N}}$ be a sequence of discretizations of $\Omega \times (0,T)$, Let t chosen such that (4.65) holds for any $p \in \mathbb{N}$ (in fact, any $t > 0$ works). Thanks to lemma 4.2.1, one can claim that $0 \le \gamma_{i,\mathcal{D}_p}(t) \le 1$, and thus there exists $\gamma_i(t) \in [0,1]$ such that, up to a subsequence,

$$\lim_{p\to+\infty} \gamma_{i,\mathcal{D}_p}(t) = \gamma_i(t).$$

Since $\{(a,b) \in [0,1]^2 \ / \ \tilde{\pi}_1(a) \cap \tilde{\pi}_2(b) \neq \emptyset\}$ is closed in $[0,1]^2$, one has, thanks to (4.65) : for a.e. $t \in (0,T)$,

$$\tilde{\pi}_1(\gamma_1(t)) \cap \tilde{\pi}_2(\gamma_2(t)) \neq \emptyset. \tag{4.66}$$

We now need to verify that the sequence $(\gamma_{i,\mathcal{D}_p})_{p\in\mathbb{N}}$ tends to u_i a.e. on $(0,T)$. Thanks to (4.52), one can claim that for all $t > 0$,

$$\lim_{p\to+\infty} |u_{i,\mathcal{D}_p}(t) - \gamma_{i,\mathcal{D}_p}(t)| = 0, \tag{4.67}$$

where $u_{i,\mathcal{D}_p}$ is the trace of $u_{\mathcal{D}_p}$ from Ω_i on $\{x = 0\}$. Using the lemma 4.2.9 yields :

$$\text{for a.e. } t \in (0,T), \quad \gamma_i(t) = u_i(t),$$

and thus (4.66) becomes :

$$\text{for a.e. } t \in (0,T), \quad \tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset. \tag{4.68}$$

Thanks to lemma 4.2.10, it suffices to show that the weak formulation (4.13) holds for any ψ in $\mathcal{D}(\overline{\Omega} \times [0,T[)$ with $\partial_x\psi \in \mathcal{D}((\cup_{i=1,2}\Omega_i) \times [0,T[)$. Let $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T[)$ with $\partial_x\psi \in \mathcal{D}((\cup_{i=1,2}\Omega_i) \times [0,T[)$, for $j \in [\![-\mathrm{N_p}, \mathrm{N_p}-1]\!]$, $n \in [\![0, \mathrm{M_p}-1]\!]$, one denotes $\psi_{j+1/2}^n = \psi(x_{j+1/2}, t^n)$. Assume that p is large enough to insure :

$$\psi_{-5/2}^n = \psi_{-3/2}^n = \psi_{-1/2}^n = \psi_{1/2}^n = \psi_{3/2}^n = \psi_{5/2}^n, \qquad \forall n \in [\![0, \mathrm{M_p}-1]\!], \tag{4.69}$$

$$\forall n \in [\![0, \mathrm{M_p}-1]\!], \qquad \begin{cases} \psi_{-N+1/2}^n = \psi_{-N+3/2}^n = \psi_{-N+5/2}^n = \psi(-1, t^n), \\ \psi_{N-1/2}^n = \psi_{N-3/2}^n = \psi_{N-5/2}^n = \psi(1, t^n). \end{cases} \tag{4.70}$$

One has also

$$\psi_{j+1/2}^{M_p} = 0, \qquad \forall j \in [\![-\mathrm{N_p}, \mathrm{N_p}-1]\!]. \tag{4.71}$$

For $j \in [\![-\mathrm{N_p}, \mathrm{N_p}-1]\!]$, $n \in [\![0, \mathrm{M_p}-1]\!]$, let us multiply equation (4.15) by $\psi_{j+1/2}^n \delta t$, and sum on $j \in [\![-\mathrm{N_p}, \mathrm{N_p}-1]\!]$, $n \in [\![0, \mathrm{M_p}-1]\!]$, we get :

$$\sum_{n=0}^{M_p-1} \sum_{j=-N_p}^{N_p-1} \phi_i (u_{j+1/2}^{n+1} - u_{j+1/2}^n) \psi_{j+1/2}^n \delta x + \sum_{n=0}^{M_p-1} \delta t \sum_{j=-N_p}^{N_p-1} (F_{j+1}^{n+1} - F_j^{n+1}) \psi_{j+1/2}^n = 0,$$

which can be rewritten thanks to (4.71) :

$$\sum_{n=0}^{M_p-1}\sum_{j=-N_p}^{N_p-1}\phi_i u_{j+1/2}^{n+1}(\psi_{j+1/2}^{n}-\psi_{j+1/2}^{n+1})\delta x-\sum_{j=-N_p}^{N_p-1}\phi_i u_{j+1/2}^{0}\psi_{j+1/2}^{0}$$
$$+\sum_{n=0}^{M_p-1}\delta t\sum_{j=-N_p+1}^{N_p-1}F_j^{n+1}(\psi_{j-1/2}^{n}-\psi_{j+1/2}^{n})$$
$$-\sum_{n=0}^{M_p-1}\delta t F_{-N}^{n+1}\psi_{-N+1/2}^{n}+\sum_{n=0}^{M_p-1}\delta t F_{N}^{n+1}\psi_{N-1/2}^{n}=0. \tag{4.72}$$

Thanks to (4.69), for all $n\in [\![0,\mathrm{M_p}-1]\!]$, $F_0^{n+1}(\psi_{-1/2}^{n}-\psi_{1/2}^{n})=0$, and then, using also (4.70)

$$\sum_{n=0}^{M_p-1}\sum_{j=-N_p}^{N_p-1}\phi_i u_{j+1/2}^{n+1}(\psi_{j+1/2}^{n+1}-\psi_{j+1/2}^{n})\delta x+\sum_{j=-N_p}^{N_p-1}\phi_i u_{j+1/2}^{0}\psi_{j+1/2}^{0}$$
$$+\sum_{n=0}^{M_p-1}\delta t\sum_{j\in J^\star}F_j^{n+1}(\psi_{j+1/2}^{n}-\psi_{j-1/2}^{n})$$
$$+\sum_{n=0}^{M_p-1}\delta t F_{-N}^{n+1}\psi_{-N+1/2}^{n}-\sum_{n=0}^{M_p-1}\delta t F_{N}^{n+1}\psi_{N-1/2}^{n}=0, \tag{4.73}$$

where $J^\star=[\![-\mathrm{N}+2,2]\!]\cup[\![2,\mathrm{N}-2]\!]$.
Taking into account the definitions of the F_j^{n+1} given by (4.16), (4.17) and (4.18) in (4.73), one has :

$$A_{\mathcal{D}_p}+B_{\mathcal{D}_p}+C_{\mathcal{D}_p}+D_{\mathcal{D}_p}+E_{\mathcal{D}_p}=0, \tag{4.74}$$

where

$$A_{\mathcal{D}_p}=\sum_{n=0}^{M_p-1}\sum_{j=-N_p}^{N_p-1}\phi_i u_{j+1/2}^{n+1}(\psi_{j+1/2}^{n+1}-\psi_{j+1/2}^{n})\delta x,$$

$$B_{\mathcal{D}_p}=\sum_{j=-N_p}^{N_p-1}\phi_i u_{j+1/2}^{0}\psi_{j+1/2}^{0},$$

$$C_{\mathcal{D}_p}=\sum_{n=0}^{M_p-1}\delta t\sum_{j\in J^\star}q^{n+1}f_i(u_{j-1/2}^{n+1})(\psi_{j+1/2}^{n}-\psi_{j-1/2}^{n}),$$

$$D_{\mathcal{D}_p}=-\sum_{n=0}^{M_p-1}\delta t\sum_{j\in J^\star}\frac{\varphi_i(u_{j+1/2}^{n+1})-\varphi_i(u_{j-1/2}^{n+1})}{\delta x}(\psi_{j+1/2}^{n}-\psi_{j-1/2}^{n}),$$

and

$$E_{\mathcal{D}_p}=\sum_{n=0}^{M_p-1}\delta t g^{n+1}\psi_{-N+1/2}^{n}-\sum_{n=0}^{M_p-1}\delta t q^{n+1}f_2(u_{N-1/2}^{n+1})\psi_{N-1/2}^{n}.$$

Let $\mathfrak{g}_p : \Omega \times (0,T) \to \mathbb{R}$ defined on $]x_i, x_i+1[\times]t^n, t^{n+1}]$ by $\mathfrak{g}_p(x,t) = (\psi^{n+1}_{j+1/2} - \psi^n_{j+1/2})/\delta t$, then

$$A_{\mathcal{D}_p} = \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u_{\mathcal{D}_p}(x,t)\mathfrak{g}_p(x,t)dxdt.$$

$\mathfrak{g}_p$ converges toward $\partial_t\psi$ in $L^1(\Omega \times (0,T))$ as p tends to $+\infty$, and thanks to (4.58), one has

$$\lim_{p\to+\infty} A_{\mathcal{D}_p} = \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u(x,t)\partial_t\psi(x,t)dxdt. \tag{4.75}$$

One denotes $\psi_{\mathcal{D}_p}(x,t) = \psi^n_{j+1/2}$ if $x \in]x_i, x_{i+1}[\times[t^n, t^{n+1}[$. It is easy to check, using (4.14), that $u_{\mathcal{D}_p}(\cdot,0)$ converges in $L^1(\Omega)$ toward u_0, and $\psi_{\mathcal{D}_p}(\cdot,0)$ converges uniformly toward $\psi(\cdot,0)$, thus we get

$$\lim_{p\to+\infty} B_{\mathcal{D}_p} = \sum_{i=1,2} \int_{\Omega_i} \phi_i u_0(x)\psi(x,0)dx. \tag{4.76}$$

Thanks to (4.70), denoting $q_{\mathcal{D}_p}(t) = q^{n+1}$ and $g_{\mathcal{D}_p}(t) = g^{n+1}$ if $t \in]t^n, t^n+1]$, $E_{\mathcal{D}_p}$ can be written :

$$E_{\mathcal{D}_p} = \int_0^T g_{\mathcal{D}_p}(t)\psi_{\mathcal{D}_p}(-1,t)dt - \int_0^T q_{\mathcal{D}_p}(t)f_2(u_{\mathcal{D}_p})(1,t)\psi_{\mathcal{D}_p}(1,t)dt.$$

The theorem of continuity in mean insures that $q_{\mathcal{D}_p}$ (resp. $g_{\mathcal{D}_p}$) tends to q (resp. g) in $L^1(0,T)$. Thanks to lemma 4.2.9 and to the continuity of f_2, $t \mapsto f_2(u_{\mathcal{D}_p})(1,t)$ converges a.e. on $(0,T)$ toward $f_2(u)(1,t)$, and is bounded in $L^\infty(0,T)$, thus, since $\psi_{\mathcal{D}_p}$ converges uniformly toward ψ,

$$\lim_{p\to+\infty} E_{\mathcal{D}_p} = \int_0^T g(t)\psi(-1,t)dt - \int_0^T q(t)f_2(u)(1,t)\psi(1,t)dt. \tag{4.77}$$

Let $\mathfrak{h}_p$ be the function defined almost everywhere on $\mathbb{R} \times (0,T)$ by :

$$\mathfrak{h}_p(x,t) = \begin{cases} \dfrac{\psi^n_{j+1/2} - \psi^n_{j-1/2}}{\delta x} & \text{if } (x,t) \in]x_{j-1/2}, x_{j+1/2}[\times[t^n, t^{n+1}[,\ j \in J^\star \\ 0 & \text{elsewhere ,} \end{cases}$$

so that $C_{\mathcal{D}_p}$ can be written :

$$C_{\mathcal{D}_p} = \int_0^T \sum_{i=1,2} \int_{\Omega_i} q_{\mathcal{D}_p}(t)f_i(u_{\mathcal{D}_p})(x,t)\mathfrak{h}_p(x+\delta x,t)dxdt. \tag{4.78}$$

$\mathfrak{h}_p$ converges a.e. on $\Omega \times (0,T)$ to $\partial_x\psi$, and is uniformly bounded. The same way, $f_i(u_{\mathcal{D}_p})$ converges a.e. on $\Omega \times (0,T)$ to $f_i(u)$ thanks to (4.59) and the continuity of f_i, and is uniformly bounded thanks to lemma 4.2.2. Since $q_{\mathcal{D}_p}$ converges to q in $L^1(0,T)$, one has :

$$\lim_{p\to+\infty} C_{\mathcal{D}_p} = \int_0^T \sum_{i=1,2} \int_{\Omega_i} q(t)f_i(u)(x,t)\partial_x\psi(x,t)dxdt. \tag{4.79}$$

Remark 4.2.2 *We strongly use in this proof the fact that the spatial discretization of Ω is uniform. Indeed, in the case of a non-uniform mesh, an additional term would appear in (4.78). In the case of a purely hyperbolic equation, an additional estimate called weak BV about which we already discussed in remark 4.2.1 allows us to claim that this additional term converges to* 0. *Nevertheless, this can be avoided for the problem ($\mathcal{P}$) because of the parabolic term.*

Let us now turn to the term $D_{\mathcal{D}_p}$. Thanks to (4.69) and (4.70), one can rewrite it

$$D_{\mathcal{D}_p} = \sum_{n=0}^{M_p-1} \delta t \sum_{j\in J^\star} \varphi_i(u^{n+1}_{j+1/2}) \frac{\psi^n_{j+3/2} - 2\psi^n_{j+1/2} + \psi^n_{j-1/2}}{\delta x}.$$

We introduce the function $\mathfrak{m}_p$ defined almost everywhere on $\mathbb{R}\times(0,T)$ by :

$$\mathfrak{m}_p(x,t) = \begin{cases} \dfrac{\psi^n_{j+3/2} - 2\psi^n_{j+1/2} + \psi^n_{j-1/2}}{\delta x} & \text{if } (x,t) \in]x_j, x_{j+1}[\times[t^n, t^{n+1}[,\ j \in J^\star \\ 0 & \text{elsewhere,} \end{cases}$$

so that we can rewrite

$$D_{\mathcal{D}_p} = \int_0^T \sum_{i=1,2} \int_{\Omega_i} \varphi_i(u_{\mathcal{D}_p})(x,t)\mathfrak{m}_p(x,t)dxdt.$$

One uses once again the regularity of ψ and the convergence results (4.59) to conclude that :

$$\lim_{p\to+\infty} D_{\mathcal{D}_p} = \int_0^T \sum_{i=1,2} \int_{\Omega_i} \varphi_i(u)(x,t)\partial_{xx}\psi(x,t)dxdt.$$

Since $\varphi_i(u)$ belongs to $L^2(0,T;H^1(\Omega_i))$, one can integrate by parts, and using

$$\partial_x\psi(1,t) = \partial_x\psi(-1,t) = \partial_x\psi(0,t) = 0,$$

one gets

$$\lim_{p\to+\infty} D_{\mathcal{D}_p} = -\int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x\varphi_i(u)(x,t)\partial_x\psi(x,t)dxdt. \tag{4.80}$$

Using (4.75),(4.76),(4.77),(4.79),(4.80) in (4.74) achieves the proof of proposition 4.2.11 □

4.3 Uniform bound on the fluxes

In this section, we show that, under some regularity assumptions on the data, there exists a solution with bounded fluxes. This existence result is the consequence of some new estimates on the discrete solution, and will be necessary to get the uniqueness result of theorem 4.4.1.

Definition 4.4 (bounded-flux solution) *A function u is said to be a bounded-flux solution to the problem ($\mathcal{P}$) if :*

1. *u is a weak solution to the problem ($\mathcal{P}$) in the sense of definition 4.1,*

2. *$\partial_x \varphi_i(u)$ belongs to $L^\infty(\Omega_i \times (0,T))$.*

In order to get an existence result, we need some more regularity on the data, as stated below.

Assumptions 4.3 (additional regularity on the data) *We assume that :*

1. *$\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$, $0 \le u_0 \le 1$,*
2. *$\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, where $u_{0,i}$ is the trace of $u_{0|\Omega_i}$ on $\{x = 0\}$,*
3. *$q \in BV(0,T)$, $q \ge g \ge 0$ (thus $g \in L^\infty(0,T)$).*

Theorem 4.3.1 (existence of a bounded-flux solution) *Suppose that the data are regular, that is assumptions 4.3 are fulfilled. Let $(M_p)_{p\in\mathbb{N}}$, $(N_p)_{p\in\mathbb{N}}$ two sequences of positive integers tending to $+\infty$. Let $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ be the sequence of the associated discrete solutions obtained via the finite volume scheme (4.23)-(4.27), and let u be an adherence value of the sequence $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$. Then u is a bounded flux solution to the problem $(\mathcal{P})$ in the sense of definition 4.4. This particularly insures the existence of such a bounded-flux solution.*

All the section 4.3 will be devoted to the proof of theorem 4.3.1. We only need to verify the second point in definition 4.4, because we have already proven in proposition 4.2.11 that u is a weak solution. So the aim of this section is to get the uniform bound on the fluxes. Such an estimate (in the case where $q = 0$) can be found in [CGP], and is obtained using a thin regular transition layer between Ω_1 and Ω_2, and a regularization of the initial data u_0. This technique was also used in [BDPvD03] to get a strong BV estimate on the fluxes in the case of a non-bounded domain Ω, and for particular values of the data (which are supposed to be more regular). In this paper, we only deal with the discrete solution, which is in a way, a regularization of the solution.

One extends the definitions of the discrete fluxes (4.16)-(4.20) to the case $n = -1$, i.e. in the time $t = 0$. For all $j \in [\![-\mathrm{N}+1, \mathrm{N}-1]\!]$, $j \neq 0$, $]x_j, x_j + 1[\subset \Omega_i$,

$$F_j^0 = q^0 f_i(u_{j-1/2}^0) - \frac{\varphi_i(u_{j+1/2}^0) - \varphi_i(u_{j-1/2}^0)}{\delta x} \tag{4.81}$$

where $q^0 = q^1 = \int_0^{\delta t} q(t)dt$. One also denotes $g^0 = g^1 = \int_0^{\delta t} g(t)dt$.

$$\begin{aligned} F_{-N}^0 &= g^0, && (4.82)\\ F_N^0 &= q^0 f_2(u_{N-1/2}^0), && (4.83)\\ F_0^0 &= q^0 f_1(u_{-1/2}^0) - \frac{2(\varphi_1(u_{0,1}^0) - \varphi_1(u_{-1/2}^0))}{\delta x}, && (4.84)\\ &= q^0 f_2(u_{0,2}^0) - \frac{2(\varphi_2(u_{1/2}^0) - \varphi_2(u_{0,2}^0))}{\delta x}. && (4.85) \end{aligned}$$

We also discretize the condition on the connection of the traces :

$$\tilde{\pi}_1(u_{0,1}^0) \cap \tilde{\pi}_2(u_{0,2}^0) \neq \emptyset. \tag{4.86}$$

$u_{0,1}^0$ and $u_{0,2}^0$ are given by lemma 4.2.1, and so they are different of $u_{0,1}$ and $u_{0,2}$.

Lemma 4.3.2 (uniform bound on the initial flux) *There exists $C > 0$ depending only on u_0, φ_i, q such that*

$$\max_{j=-N,\dots,N} |F_j^0| \le C.$$

Proof

Since $\varphi_i(u_0)$ is a Lipschitz continuous function, and φ_i is continuous and increasing, $u_{0|\Omega_i}$ is a continuous function, and there exists $y_{j+1/2} \in]x_j, x_{j+1}[$ such that $u^0_{j+1/2} = u_0(y_{j+1/2})$. The monotony of the transmission conditions $\tilde{\pi}_1(u^0_{0,1}) \cap \tilde{\pi}_2(u^0_{0,2}) \neq \emptyset$ and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$ implies that either $u^0_{0,1} \ge u_{0,1}$ and $u^0_{0,2} \ge u_{0,2}$, or $u^0_{0,1} \le u_{0,1}$ and $u^0_{0,2} \le u_{0,2}$.

- Assume that $u^0_{0,1} \ge u_{0,1}$ and $u^0_{0,2} \ge u_{0,2}$, then one deduce from (4.84) and (4.85) that :

$$-\frac{2(\varphi_2(u_0(y_{1/2})) - \varphi_2(u_0(y_{-1/2})))}{\delta x} \le F_0^0 \le q^0 - \frac{2(\varphi_1(u_{0,1}) - \varphi_1(u_0(y_{-1/2})))}{\delta x},$$

  and so since $\varphi_i(u_0)$ is a Lipschitz continuous function,

$$|F_0^0| \le q^0 + 2\max_{i=1,2} \left(\|\partial_x \varphi_i(u_0)\|_{L^\infty(\Omega_i)} \right).$$

- If $u^0_{0,1} \ge u_{0,1}$ and $u^0_{0,2} \ge u_{0,2}$, similar computations lead to the same estimate :

$$|F_0^0| \le q^0 + 2\max_{i=1,2} \left(\|\partial_x \varphi_i(u_0)\|_{L^\infty(\Omega_i)} \right).$$

For all $j \in [\![-\mathrm{N}+1, \mathrm{N}-1]\!]$, $j \neq 0$, one has, from (4.81) :

$$|F_j^0| \le q^0 + 2\|\varphi_i(u_0)\|_{L^\infty(\Omega_i)}.$$

One has also, from (4.82) and (4.83) :

$$|F_N^0| \le |F_{-N}^0| = q^0.$$

□

Proposition 4.3.3 (uniform bound on the discrete fluxes) *There exists $C > 0$ depending only on u_0, φ_i, q, g and T such that*

$$\max_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(\max_{n\in[\![0,\mathrm{M}]\!]} |F_j^n| \right) \le C.$$

Proof

For all $j \in J_{\text{int}}$, for all $n \in [\![0, \mathrm{M}-1]\!]$,

$$\begin{aligned} F_j^{n+1} - F_j^n &= q^{n+1} f_i(u^{n+1}_{j-1/2}) - q^n f_i(u^n_{j-1/2}) + \left(\frac{\varphi_i(u^{n+1}_{j-1/2}) - \varphi_i(u^n_{j-1/2})}{\delta x} \right) \\ &\quad - \left(\frac{\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(u^n_{j+1/2})}{\delta x} \right) \\ &= q^{n+1} \left(f_i(u^{n+1}_{j-1/2}) - f_i(u^n_{j-1/2}) \right) + f_i(u^n_{j-1/2}) \left(q^{n+1} - q^n \right) \\ &\quad + \left(\frac{\varphi_i(u^{n+1}_{j-1/2}) - \varphi_i(u^n_{j-1/2})}{\delta x} \right) - \left(\frac{\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(u^n_{j+1/2})}{\delta x} \right). \end{aligned} \tag{4.87}$$

Thus, using (4.15) in (4.87) yields

$$\begin{aligned} F_j^{n+1} - F_j^n &= q^{n+1}\frac{\delta t}{\phi_i \delta x}\left(\frac{f_i(u_{j-1/2}^{n+1}) - f_i(u_{j-1/2}^n)}{u_{j-1/2}^{n+1} - u_{j-1/2}^n}\right)\left(F_{j-1}^{n+1} - F_j^{n+1}\right) \\ &+\frac{\delta t}{\phi_i \delta x^2}\left(\frac{\varphi_i(u_{j-1/2}^{n+1}) - \varphi_i(u_{j-1/2}^n)}{u_{j-1/2}^{n+1} - u_{j-1/2}^n}\right)\left(F_{j-1}^{n+1} - F_j^{n+1}\right) \\ &+\frac{\delta t}{\phi_i \delta x^2}\left(\frac{\varphi_i(u_{j+1/2}^{n+1}) - \varphi_i(u_{j+1/2}^n)}{u_{j+1/2}^{n+1} - u_{j+1/2}^n}\right)\left(F_{j+1}^{n+1} - F_j^{n+1}\right) \\ &+f_i(u_{j-1/2}^n)\left(q^{n+1} - q^n\right), \end{aligned}$$

Notice that if $u_{j-1/2}^{n+1} = u_{j-1/2}^n$ (resp. $u_{j+1/2}^{n+1} = u_{j+1/2}^n$), then $F_{j-1}^{n+1} = F_j^{n+1}$ (resp. $F_j^{n+1} = F_{j+1}^{n+1}$), and so the corresponding terms in the left-hand side member are equal to 0. So, since f_i and φ_i are increasing functions, for all $j \in J_{\text{int}}$, for all $n \in [\![0, \mathrm{M}-1]\!]$, there exist $a_{j,j-1}^{n+1} \geq 0$ and $a_{j,j+1}^{n+1} \geq 0$ such that

$$\left(1 + a_{j,j-1}^{n+1} + a_{j,j+1}^{n+1}\right) F_j^{n+1} - a_{j,j-1}^{n+1} F_{j-1}^{n+1} - a_{j,j+1}^{n+1} F_{j+1}^{n+1} \leq F_j^n + \left(q^{n+1} - q^n\right)^+. \quad (4.88)$$

Let $n \in [\![0, \mathrm{M}-1]\!]$ and let $j_{\max}^{n+1} \in [\![-\mathrm{N}, \mathrm{N}]\!]$ such that $F_{j_{\max}^{n+1}}^{n+1} = \max_j \left(F_j^{n+1}\right)$.

- Suppose that $j_{\max}^{n+1} \in J_{\text{int}}$, then (4.88) yields :

$$\max_j \left(F_j^{n+1}\right) \leq \max_j \left(F_j^n\right) + \left(q^{n+1} - q^n\right)^+. \quad (4.89)$$

- Suppose now that $j_{\max}^{n+1} = 0$, i.e. $F_0^{n+1} \geq F_j^{n+1}$, $j \in [\![-\mathrm{N}, \mathrm{N}]\!]$. Using (4.19) and (4.20), one gets for all $n \in [\![0, \mathrm{M}-1]\!]$,

$$\begin{aligned} F_0^{n+1} - F_0^n &= q^{n+1}\left(f_1(u_{-1/2}^{n+1}) - f_1(u_{-1/2}^n)\right) + f_1(u_{-1/2}^n)\left(q^{n+1} - q^n\right) \\ &+\left(\frac{\varphi_1(u_{-1/2}^{n+1}) - \varphi_1(u_{-1/2}^n)}{\delta x/2}\right) - \left(\frac{\varphi_1(u_{0,1}^{n+1}) - \varphi_1(u_{0,1}^n)}{\delta x/2}\right) \quad (4.90) \\ &= q^{n+1}\left(f_2(u_{0,2}^{n+1}) - f_2(u_{0,2}^n)\right) + f_2(u_{0,2}^n)\left(q^{n+1} - q^n\right) \\ &+\left(\frac{\varphi_2(u_{0,2}^{n+1}) - \varphi_2(u_{0,2}^n)}{\delta x/2}\right) - \left(\frac{\varphi_2(u_{1/2}^{n+1}) - \varphi_2(u_{1/2}^n)}{\delta x/2}\right). \quad (4.91) \end{aligned}$$

Using again (4.15) in (4.90) and (4.91), one gets :

$$\begin{aligned} F_0^{n+1} - F_0^n &= q^{n+1}\frac{\delta t}{\phi_1 \delta x}\left(\frac{f_1(u_{-1/2}^{n+1}) - f_1(u_{-1/2}^n)}{u_{-1/2}^{n+1} - u_{-1/2}^n}\right)\left(F_{-1}^{n+1} - F_0^{n+1}\right) \\ &+\frac{2\delta t}{\phi_1 \delta x^2}\left(\frac{\varphi_1(u_{-1/2}^{n+1}) - \varphi_1(u_{-1/2}^n)}{u_{-1/2}^{n+1} - u_{-1/2}^n}\right)\left(F_{-1}^{n+1} - F_0^{n+1}\right) \\ &-\left(\frac{\varphi_1(u_{0,1}^{n+1}) - \varphi_1(u_{0,1}^n)}{\delta x/2}\right) + f_1(u_{-1/2}^n)\left(q^{n+1} - q^n\right), \quad (4.92) \end{aligned}$$

$$
\begin{aligned}
F_0^{n+1} - F_0^n &= q^{n+1}\left(f_2(u_{0,2}^{n+1}) - f_2(u_{0,2}^n)\right) + f_2(u_{0,2}^n)\left(q^{n+1} - q^n\right) \\
&+ \frac{2\delta t}{\phi_2 \delta x^2}\left(\frac{\varphi_2(u_{1/2}^{n+1}) - \varphi_2(u_{1/2}^n)}{u_{1/2}^{n+1} - u_{1/2}^n}\right)\left(F_1^{n+1} - F_0^{n+1}\right) \\
&+ \left(\frac{\varphi_2(u_{0,2}^{n+1}) - \varphi_2(u_{0,2}^n)}{\delta x/2}\right). \qquad (4.93)
\end{aligned}
$$

Thanks to the monotony of the conditions (4.21) and (4.86), one has either

$$u_{0,1}^{n+1} \geq u_{0,1}^n \text{ or } u_{0,2}^{n+1} \leq u_{0,2}^n.$$

Assume that $u_{0,1}^{n+1} \geq u_{0,1}^n$, since f_1 and φ_1 are increasing, we deduce from (4.92) that there exists $a_{0,-1}^{n+1} \geq 0$ such that :

$$\left(1 + a_{0,-1}^{n+1}\right) F_0^{n+1} - a_{0,-1}^{n+1} F_{-1}^{n+1} \leq F_0^n + f_1(u_{-1/2}^n)(q^{n+1} - q^n). \qquad (4.94)$$

Assume now that $u_{0,2}^{n+1} \leq u_{0,2}^n$, since f_2 and φ_2 are increasing, we deduce from (4.93) that there exists $a_{0,1}^{n+1} \geq 0$ such that :

$$\left(1 + a_{0,1}^{n+1}\right) F_0^{n+1} - a_{0,1}^{n+1} F_1^{n+1} \leq F_0^n + f_2(u_{0,2}^n)(q^{n+1} - q^n). \qquad (4.95)$$

Since $F_0^{n+1} \geq F_j^{n+1}$, $j \in [\![-\mathrm{N}, \mathrm{N}]\!]$ and $0 \leq f_i(u_{-1/2}^n) \leq 1$, one deduces from (4.94) and (4.95) that :

$$F_0^{n+1} = \max_{j \in [\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^{n+1}\right) \leq \max_{j \in [\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^n\right) + \left(q^{n+1} - q^n\right)^+. \qquad (4.96)$$

- Suppose now that $j_{\max}^{n+1} = -N$, i.e. $F_{-N}^{n+1} \geq F_j^{n+1}$, $j \in [\![-\mathrm{N}, \mathrm{N}]\!]$. From (4.17) and assumption 4.3, one has :

$$F_{-N}^{n+1} = g^{n+1} \leq \|g\|_{L^\infty(0,T)} \qquad (4.97)$$

- Suppose now that $j_{\max}^{n+1} = N$, i.e. $F_N^{n+1} \geq F_j^{n+1}$, $j \in [\![-\mathrm{N}, \mathrm{N}]\!]$. From (4.18), one has :

$$
\begin{aligned}
F_N^{n+1} - F_{-N}^n &= q^{n+1}\left(f(u_{N-1/2}^{n+1}) - f(u_{N-1/2}^n)\right) + f(u_{N-1/2}^n)\left(q^{n+1} - q^n\right) \\
&\leq q^{n+1}\frac{\delta t}{\phi_2 \delta x}\left(\frac{f(u_{N-1/2}^{n+1}) - f(u_{N-1/2}^n)}{u_{N-1/2}^{n+1} - u_{N-1/2}^n}\right)\left(F_N^{n+1} - F_{N-1}^{n+1}\right) \\
&+ \left(q^{n+1} - q^n\right)^+. \qquad (4.98)
\end{aligned}
$$

We can use the monotony of f_2 once again, and the same way than in the previous cases, one can claim :

$$F_N^{n+1} \leq \max_{j \in [\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^n\right) + \left(q^{n+1} - q^n\right)^+. \qquad (4.99)$$

The estimates (4.89), (4.96), (4.97), (4.99) insure that : $\forall n \in [\![0, \mathrm{M}-1]\!]$,

$$\max_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^{n+1}\right) \leq \max\left(\|g\|_{L^\infty(0,T)}, \max_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^n\right) + \left(q^{n+1} - q^n\right)^+\right), \tag{4.100}$$

and thus

$$\max_{n\in[\![0,\mathrm{M}]\!]} \left(\max_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^n\right)\right) \leq \max\left(\|g\|_{L^\infty(0,T)}, \max_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^0\right)\right) + \sum_{n=0}^{M-1} \left(q^{n+1} - q^n\right)^+ .$$

Since $q \in BV(0,T)$, one has, thanks to lemma 5.2.5 stated below :

$$\max_{n\in[\![0,\mathrm{M}]\!]} \left(\max_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^n\right)\right) \leq \max\left(\|g\|_{L^\infty(0,T)}, \max_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^0\right)\right) + 2\left(TV(q)\right). \tag{4.101}$$

One can also prove, following the same sketch, and using $g \geq 0$, that :

$$\min_{n\in[\![0,\mathrm{M}]\!]} \left(\min_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^n\right)\right) \geq \min_{j\in[\![-\mathrm{N},\mathrm{N}]\!]} \left(F_j^0\right) - 2\left(TV(q)\right). \tag{4.102}$$

One achieves the proof of proposition 4.3.3 using lemma 4.3.2 in (4.101) and (4.102). □

Lemma 4.3.4 *If $q \in BV(0,T)$, for all $M \in \mathbb{N}^\star$, one has :* $\sum_{n=0}^{M-1} \left|q^{n+1} - q^n\right| \leq 2TV(q)$.

Proof
Assume first that q belongs to $C^1([0,T])$. Since q is continuous, for all $n \in [\![0, \mathrm{M}-1]\!]$, there exists $t^{n+1/2} \in]t^n, t^{n+1}[$ such that $q^{n+1} = q(t^{n+1/2})$ (recall that $q^0 = q^1 = q(t^{1/2})$), and so

$$\begin{aligned}
\sum_{n=0}^{M-1} \left|q^{n+1} - q^n\right| &= \sum_{n=1}^{M-1} \left|q(t^{n+1/2}) - q(t^{n-1/2})\right| \\
&\leq \sum_{n=1}^{M-1} \int_{t^{n-1/2}}^{t^{n+1/2}} |q'(t)| dt \\
&\leq \sum_{n=1}^{M-1} \int_{t^{n-1}}^{t^{n+1}} |q'(t)| dt,
\end{aligned}$$

and so

$$\sum_{n=0}^{M-1} \left|q^{n+1} - q^n\right| \leq \|q'\|_{L^1(0,T)} = 2TV(q). \tag{4.103}$$

A density argument allows us to extend estimate (4.103) to all q in $BV(0T)$. □

Conclusion of proof of theorem 4.3.1
Let $(N_p)_{p\in\mathbb{N}}$, $(M_p)_{p\in\mathbb{N}}$ be two sequences of positive integers tending to $+\infty$, and let $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ the sequences of associated discrete solutions. It has been seen in proposition 4.2.11 that $(u_{\mathcal{D}_p})_p$ tends to a weak solution u in $L^r(\Omega \times (0,T))$, for all $r \in [1, +\infty[$.

Let $i = 1, 2$, let $(x, y) \in \Omega_i$, let $t \in]0, T]$. For p large enough,there exists $j_0, j_1 \in J_{\text{int}}$ such that $x_{j_0} \le x \le x_{j_0+1}$ and $x_{j_1} \le y \le x_{j_1+1}$, and there exists n such that $t \in]t^n, t^{n+1}]$.

$$\begin{aligned} \left|\varphi_i(u_{\mathcal{D}_p})(x,t) - \varphi_i(u_{\mathcal{D}_p})(y,t)\right| &= \left|\varphi_i(u^{n+1}_{j_0+1/2}) - \varphi_i(u^{n+1}_{j_1+1/2})\right| \\ &= \left|\sum_{j=j_1+1}^{j_0} \varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(u^{n+1}_{j-1/2})\right| \\ &\le \sum_{j=j_1+1}^{j_0} \left|\varphi_i(u^{n+1}_{j+1/2}) - \varphi_i(u^{n+1}_{j-1/2})\right|. \end{aligned}$$

Using the definition of the discrete flux (4.16) :

$$\left|\varphi_i(u_{\mathcal{D}_p})(x,t) - \varphi_i(u_{\mathcal{D}_p})(y,t)\right| \le \sum_{j=j_1+1}^{j_0} \delta x \left|F^{n+1}_j - q^{n+1} f_i(u^{n+1}_{j-1/2})\right|.$$

q is assumed to belong to $BV(0,T)$, thus $q \in L^\infty(0,T)$, and one deduce from proposition 4.3.3 that there exists $C > 0$, depending only on u_0, φ_i, q, g, T such that :

$$\left|\varphi_i(u_{\mathcal{D}_p})(x,t) - \varphi_i(u_{\mathcal{D}_p})(y,t)\right| \le \sum_{j=j_1+1}^{j_0} \delta x C \le C(|x-y| + 2\delta x).$$

Letting p tends toward $+\infty$, i.e. δx and δt toward 0 gives

$$|\varphi_i(u)(x,t) - \varphi_i(u)(y,t)| \le C|x-y|. \tag{4.104}$$

So we deduce from (4.104) that $\partial_x \varphi_i(u) \in L^\infty(\Omega_i \times (0,T))$. □

4.4 Uniqueness of the bounded-flux solution

In this part, we will state a result which is an generalization in the case where there is a non-zero flux of the one stated in [CGP]. As in [CGP], we will use the monotony of the transmission conditions and the uniform bound on the fluxes obtained in section 4.3.

Theorem 4.4.1 (L^1-contraction principle) *If u,v are bounded-flux solutions in the sense of definition 4.4 associated to the initial data u_0, v_0, then for all $p \in [1, +\infty[$, u and v belong to $C([0,T]; L^p(\Omega))$, and the following L^1-contraction principle holds : $\forall t \in [0,T]$,*

$$\int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^{\pm} dx \le \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^{\pm} dx.$$

This particularly implies the uniqueness of the bounded flux solution to the problem $(\mathcal{P})$ for a given initial data u_0.

Proof
Let u be a weak solution with bounded flux, then for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T[)$,

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u(x,t) \partial_t \psi(x,t) dxdt + \sum_{i=1,2} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} [q(t) f_i(u)(x,t) - \partial_x \varphi_i(u)(x,t)] \, \partial_x \psi(x,t) dxdt$$
$$+ \int_0^T g(t) \psi(-1,t) dt - \int_0^T q(t) f_2(u)(1,t) dt = 0.$$

Some classical computations (see e.g. [AL83, Ott96b, Car99] and many others...) shows that, if u and v are two weak solution respectively for initial data u_0 and v_0, one has the following inequalities : for any $\xi \in \mathcal{D}^+(\overline{\Omega} \times [0,T[\times[0,T[)$ such that $\xi(x=0,t,s) = 0$,

$$\int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v(x,s))^+ (\partial_t \xi(x,t,s) + \partial_s \xi(x,t,s)) dxdtds$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u_0(x) - v(x,s))^+ \xi(x,0,s) dxds$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v_0(x))^+ \xi(x,t,0) dxdt$$
$$+ \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,s))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,s))^+ \end{array} \right] \partial_x \xi(x,t,s) dxdtds$$
$$+ \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} Sg^+(u(x,t) - v(x,s)) f_i(v)(x,s)(q(t) - q(s)) \partial_x \xi(x,t,s) dxdtds$$
$$- \int_0^T \int_0^T Sg^+(u(1,t) - v(1,s)) \, [q(t) f_2(u)(1,t) - q(s) f_2(v)(1,s)] \, \xi(1,t,s) dtds$$
$$+ \int_0^T \int_0^T Sg^+(u(-1,t) - v(-1,s))(g(t) - g(s)) \xi(-1,t,s) dtds \geq 0. \qquad (4.105)$$

Let $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$ with $supp(\rho) \subset [-1,1]$ and $\int_{\mathbb{R}} \rho(t) dt = 1$. One denotes $\rho_m(t) = m\rho(mt)$. Let $\psi \in \mathcal{D}^+([-1,1] \times [0,T[)$ with $\psi(0,\cdot) = 0$. For m large enough, $\xi(x,t,s) = \psi(x,t)\rho_m(t-s)$ belongs to $\mathcal{D}^+([-1,1] \times [0,T[\times[0,T[)$, and we can take it as test function in (4.105), them it leads to

$$A_m + B_m + C_m + D_m \geq 0 \qquad (4.106)$$

with :

$$A_m = \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v(x,s))^+ \partial_t \psi(x,t) \rho_m(t-s) dxdtds,$$

$$B_m = \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u_0(x) - v(x,s))^+ \psi(x,0) \rho_m(-s) dxds$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v_0(x))^+ \psi(x,t) \rho_m(t) dxdt,$$

$$C_m = \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,s))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,s))^+ \end{array} \right] \partial_x \psi(x,t) \rho_m(t-s) dxdtds$$
$$+ \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} Sg^+(u(x,t) - v(x,s)) f_i(v)(x,s) \\ (q(t) - q(s)) \partial_x \psi(x,t) \rho_m(t-s) \end{array} \right] dxdtds,$$

$$D_m = -\int_0^T \int_0^T \left[\begin{array}{c} Sg^+(u(1,t) - v(1,s)) \\ [q(t) f_2(u)(1,t) - q(s) f_2(v)(1,s)] \end{array} \right] \psi(1,t) \rho_m(t-s) dtds$$
$$+ \int_0^T \int_0^T Sg^+(u(-1,t) - v(-1,s))(g(t) - g(s)) \psi(-1,t) \rho_m(t-s) dtds.$$

One will now let tend m to $+\infty$. The theorem of continuity in mean gives

$$\lim_{m\to+\infty} A_m = \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v(x,t))^+ \partial_t \psi(x,t) dxdt, \tag{4.107}$$

$$\lim_{m\to+\infty} C_m = \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \end{array} \right] \partial_x \psi(x,t) dxdt, \tag{4.108}$$

$$\lim_{m\to+\infty} D_m = -\int_0^T q(t)(f_2(u)(x,t) - f_2(v)(x,t))^+ \psi(1,t) dt \le 0. \tag{4.109}$$

In order to let m tend to $+\infty$ in B_m, we need to quote a result of time continuity available in [CG] : let u be a weak solution for initial data u_0, then :

$$\exists \tilde{u} \in C([0,T], L^1(\Omega)) \ s.t. \ u(\cdot,t) = \tilde{u}(\cdot,t) \text{ a.e. for a.e. } t \in [0,T]. \tag{4.110}$$

In the sequel, we suppose that u belong to $C([0,T]; L^1(\Omega))$. This particularly implies

$$\lim_{m\to+\infty} B_m = 0. \tag{4.111}$$

Using (4.107), (4.108), (4.109), (4.111) in (4.106) leads to : for all $\psi \in \mathcal{D}^+([-1,1] \times [0,T[)$ with $\psi(0,\cdot) = 0$,

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i (u(x,t) - v(x,t))^+ \partial_t \psi(x,t) dxdt$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \end{array} \right] \partial_x \psi(x,t) dxdt \ge 0. \tag{4.112}$$

We now want to show that inequality (4.112) is still true if $\psi(0,t) \neq 0$, and particularly for $\psi(x,t) = \theta(t)$, with $\theta \in \mathcal{D}^+([0,T[)$.

For all $\varepsilon > 0$, one defines $\psi_\varepsilon(x) = \max(1 - \frac{|x|}{\varepsilon}, 1)$. Let $\theta \in \mathcal{D}^+([0,T[)$, one takes $(x,t) \mapsto \psi(x,t) = \theta(t)\psi_\varepsilon(x)$ as test function in (4.112), so that we get :

$$\begin{aligned}\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t\theta(t)(1-\psi_\varepsilon(x))dxdt \\ + \sum_{i=1,2}\int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+(1-\psi_\varepsilon)(x)\theta(0)dx \\ - \int_0^T \theta(t) \sum_{i=1,2}\int_{\Omega_i} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \partial_x\psi_\varepsilon(x)dxdt \\ + \int_0^T \theta(t)\sum_{i=1,2}\int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+\partial_x\psi_\varepsilon(x)dxdt \geq 0. \end{aligned} \quad (4.113)$$

One denotes :

$$E_{u>v} = \{t \in (0,T) \ / \ u_1 > v_1 \text{ or } u_2 > v_2\},$$

where u_i denotes the trace seen from Ω_i of u on $\{x = 0\}$. One also denotes

$$E_{u\leq v} = \{t \in (0,T) \ / \ u_1 \leq v_1 \text{ and } u_2 \leq v_2\} = E^c_{u>v}.$$

For almost every t in $E_{u\leq v}$, one has

$$\sum_{i=1,2}\int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+\partial_x\psi_\varepsilon(x)dx \leq 0,$$

thus for all $\theta \in \mathcal{D}^+([0,T[)$,

$$\int_{E_{u\leq v}} \theta(t)\sum_{i=1,2}\int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+\partial_x\psi_\varepsilon(x)dx \leq 0.$$

The same way, for almost every t in $E_{u\leq v}$, $x \mapsto (f_i(u)(x,t) - f_i(v)(x,t))^+$ is a continuous function with a zero trace on $\{x = 0\}$, and so

$$\lim_{\varepsilon\to 0}\int_{E_{u\leq v}} \theta(t)q(t)\sum_{i=1,2}\int_{\Omega_i}(f_i(u)(x,t) - f_i(v)(x,t))^+\partial_x\psi_\varepsilon(x)dxdt = 0.$$

and then

$$\limsup_{\varepsilon\to 0}\int_{E_{u\leq v}} \theta(t)\sum_{i=1,2}\int_{\Omega_i}\left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+\end{array}\right]\partial_x\psi_\varepsilon(x)dxdt \leq 0 \quad (4.114)$$

We let ε tend to 0 in (4.113), and using (4.114), one gets :

$$\begin{aligned}\int_0^T \sum_{i=1,2}\int_{\Omega_i}\phi_i(u(x,t) - v(x,t))^+\partial_t\theta(t)dxdt + \sum_{i=1,2}\int_{\Omega_i}\phi_i(u_0(x) - v_0(x))^+\theta(0)dx \\ - \liminf_{\varepsilon\to 0}\int_{E_{u>v}}\theta(t)\sum_{i=1,2}\int_{\Omega_i}\left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+\end{array}\right]\partial_x\psi_\varepsilon(x)dxdt \geq 0. \end{aligned} \quad (4.115)$$

Now, we give a straightforward generalization of a lemma stated in [CGP], which is the heart of the proof of uniqueness.

Lemma 4.4.2 *Let u, v be two bounded flux solutions for initial data u_0, v_0. For all $\theta \in \mathcal{D}^+([0,T[)$, one has :*

$$\liminf_{\varepsilon\to 0} \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \end{array} \right] \partial_x \psi_\varepsilon(x) dx dt \geq 0.$$

Proof of lemma 4.4.2 Subtracting the weak formulation obtained for v to the one obtained for u leads to : for all $\vartheta \in \mathcal{D}([0,T[)$,

$$\lim_{\varepsilon\to 0} \int_0^T \vartheta(t) \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t)) \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t)) \end{array} \right] \partial_x \psi_\varepsilon(x) dx dt = 0 \qquad (4.116)$$

Since u and v are bounded flux solution, for $i = 1, 2$,

$$(x,t) \mapsto q(t)(f_i(u)(x,t) - f_i(v)(x,t)) - \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)) \in L^\infty(\Omega_i \times (0,T)),$$

and then for all $\varepsilon > 0$, one has :

$$\left| \int_0^T \vartheta(t) \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f(u)(x,t) - f(v)(x,t)) \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t)) \end{array} \right] \partial_x \psi_\varepsilon(x) dx dt \right| \leq C \|\vartheta\|_{L^1(0,T)}.$$

A density argument allows us to say that (4.116) still holds for any $\vartheta \in L^1(0,T)$, and particularly for $\vartheta(t) = \theta(t) \mathbb{1}_{E_{u>v}}(t)$. Thus there exists a quantity $A(\varepsilon)$ tending to 0 as ε tends to 0 such that : for all $\varepsilon > 0$, for all $\theta \in \mathcal{D}([0,T[)$,

$$\int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t)) \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t)) \end{array} \right] \partial_x \psi_\varepsilon(x) dx dt = A(\varepsilon) \qquad (4.117)$$

Spitting the positive and negative parts in (4.117) leads to : for all $\varepsilon > 0$, $\theta \in \mathcal{D}([0,T[)$,

$$\begin{aligned} &\int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^+ \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \end{array} \right] \partial_x \psi_\varepsilon(x) dx dt \\ = &\int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \left[\begin{array}{c} q(t)(f_i(u)(x,t) - f_i(v)(x,t))^- \\ -\partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^- \end{array} \right] \partial_x \psi_\varepsilon(x) dx dt + A(\varepsilon) \end{aligned} \qquad (4.118)$$

Here, we will strongly use the monotony of the transmission condition $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset$, which implies that, up to a negligible set,

$$E_{u>v} \subset E_{u \geq v} = \{t \in (0,T) \ / \ u_1 \geq v_1 \text{ and } u_2 \geq v_2\}.$$

So we can make the same computation than those made to get (4.114) to claim that, for all $\theta \in \mathcal{D}^+([0,T[)$, the right-hand side member in (4.118) has a non negative $\liminf_{\varepsilon\to 0}$. This achieves the proof of lemma 4.4.2. □

We have proven that : for all $\theta \in \mathcal{D}^+([0,T[)$,

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t \theta(t) dx dt \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ \theta(0) dx, \quad (4.119)$$

but we can exactly the prove that

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t)-v(x,t))^- \partial_t \theta(t) dx dt \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x)-v_0(x))^- \theta(0) dx. \quad (4.120)$$

So one gets the uniqueness the bounded flux solution to the problem $(\mathcal{P})$ for any given initial data u_0 chosen such that such a bounded flux solution exists.

One recalls that u is supposed to belong to $C([0,T];L^1(\Omega))$, thanks to (4.110). This allows us to use inequality (4.119) for any $\theta \in BV(0,T)$, and particularly for

$$s \mapsto \theta(s) = \begin{cases} 1 \text{ if } s \leq t \\ 0 \text{ if } s > t. \end{cases}$$

We get the following L^1-contraction principle : for all $t \in [0,T[$,

$$\int_{\Omega_i} \phi_i(u(x,t)-v(x,t))^+ dx \leq \int_{\Omega_i} \phi_i(u_0(x)-v_0(x))^+ dx, \quad (4.121)$$

and of course the same with $(\cdot)^-$ instead of $(\cdot)^+$. □

4.5 The SOLA approach

We aim in this section to extend the existence-uniqueness result obtained in theorems 4.3.1 and 4.4.1 to a larger class of data, i.e. in the case where assumptions 4.3 do not hold. We are unfortunately not able to prove the uniqueness of the weak solution to the problem $(\mathcal{P})$ in such a case, but we are able to prove the existence and the uniqueness of the solution obtained as limit of approximation by bounded flux solution. Moreover, this limit is the weak solution obtained via the scheme (4.23)-(4.27).

Definition 4.5 (SOLA to the problem $(\mathcal{P})$) *A function u is said to be a SOLA to the problem $(\mathcal{P})$ if it fulfils :*
– *u is a weak solution to the problem $(\mathcal{P})$,*
– *there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of bounded flux solutions such that*

$$u_n \to u \text{ in } C([0,T];L^r(\Omega)), r \in [1,+\infty[, \text{ as } n \to +\infty.$$

Theorem 4.5.1 (Existence and uniqueness of the SOLA) *Let $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq 1$, let $q \in BV(0,T)$, $g \in L^\infty(0,T)$, $q \geq g \geq 0$, there exists a unique SOLA u to the problem $(\mathcal{P})$ in the sense of definition 4.5.*
Furthermore, if $(M_p)_{p\in\mathbb{N}}$, $(N_p)_{p\in\mathbb{N}}$ are to sequences of positive integers tending to $+\infty$, and if $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ is the corresponding sequence of discrete solutions, then $u_{\mathcal{D}_p} \to u$ in $L^r(\Omega\times(0,T))$, $r \in [1,+\infty[$.

Proof
Let $u_0 \in L^\infty(\Omega)$ and $q \in BV(0,T)$, $g \in L^\infty(0,T)$, $q \geq g \geq 0$, and let $(u_{0,\nu})_{\nu\in\mathbb{N}}$ fulfilling assumptions 4.3 such that :

$$\|u_0 - u_{0,\nu}\|_{L^1(\Omega)} \to 0 \qquad (\nu \to +\infty). \quad (4.122)$$

Such a approximation of u_0 exists, one can take e.g. $u_{0,\nu} \in C_c^\infty(]-1,0[\cup]0,1[)$. Then $(u_{0,\nu})$ is a Cauchy sequence, and thanks to theorem 4.4.1, for all $t \in [0,T]$, for all $(\nu,\mu) \in \mathbb{N}^2$

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_\nu(x,t) - u_\mu(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{0,\nu}(x) - u_{0,\mu}(x)| dx.$$

Thus $(u_\nu)_\nu$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$ and so it converges toward a function u in $C([0,T];L^1(\Omega))$. Since $(u_\nu)_\nu$ is bounded in $L^\infty(\mathcal{Q})$, one has $u_\nu \to u$ in $C([0,T];L^r(-1,1))$, $r \in [1,+\infty[$.

We now have to check that u is a weak solution to the problem $(\mathcal{P})$. Since $u_\nu \to u$ a.e. in $\Omega \times (0,T)$ as $\nu \to +\infty$, then $f_i(u_\nu) \to f_i(u)$ a.e. on $\Omega_i \times (0,T)$, with $0 \leq f_i(u_\nu) \leq 1$, thus

$$qf_i(u_\nu) \to qf_i(u) \text{ in } L^1(\Omega_i \times (0,T)) \text{ as } n \to +\infty.$$

It is easy to check, using to the L^∞-bound of u_ν, that $\varphi_i(u_\nu)$ tends toward $\varphi_i(u)$ in $L^r(\Omega_i \times (0,T))$, for all $r \in [1,+\infty[$. Thanks to (4.57), the sequence $(\varphi_i(u_\nu))_n$ is bounded in $L^2(0,T;H^1(\Omega_i))$, and thus $\varphi_i(u_\nu) \to \varphi_i(u)$ weakly in $L^2(0,T;H^1(\Omega_i))$, and $\varphi_i(u_\nu)$ converges in $L^2(0,T;H^s(\Omega_i))$, for all $s \in]0,1[$, still toward $\varphi_i(u)$. Particularly, $u_{\nu,i}(t)$ tends toward $u_i(t)$. Since the set $\{(a,b) \in [0,1]^2 \mid \tilde{\pi}_1(a) \cap \tilde{\pi}_2(b) \neq \emptyset\}$ is closed, we can claim that

$$\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset \quad \text{ for a.e. } t \in [0,T].$$

We can also pass to the limit in the weak formulation in order to conclude that u is a weak solution to the problem $(\mathcal{P})$, achieving this way the existence of a SOLA u.

Let now v be another SOLA, obtained through a sequence $(v_{0,\nu})_\nu$ of regular initial data converging toward v_0. Thanks to theorem 4.4.1, one has,

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_\nu(x,t) - v_\nu(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{0,n}(x) - v_{0,n}(x)| dx,$$

whose limit as ν tends toward $+\infty$ gives the attempted L^1-contraction principle :

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u(x,t) - v(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_0(x) - v_0(x)| dx,$$

and so the uniqueness of the SOLA.

Let $(M_p)_{p\in\mathbb{N}}$, $(N_p)_{p\in\mathbb{N}}$ be two sequences of positive integers tending to $+\infty$. Let $p \in \mathbb{N}$, let $u_{\mathcal{D}_p}$ the discrete solution corresponding to M_p, N_p, u_0, q and g, and let $u_{\nu,\mathcal{D}_p}$ the discrete solution corresponding to M_p, N_p, $u_{0,\nu}$, q and g. Using (4.40) yields

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{\mathcal{D}_p}(x,t) - u_{\nu,\mathcal{D}_p}(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_{0,\mathcal{D}_p}(x) - u_{0,\nu,\mathcal{D}_p}(x)| dx. \tag{4.123}$$

Thanks to proposition 4.2.11, one can claim that there exists two weak solution u, u_ν associated to initial data $u_0, u_{0,\nu}$ such that : for a.e. $t \in (0,T)$,

$$\lim_{p\to+\infty} \|u_{\mathcal{D}_p}(x,t) - u(x,t)\|_{L^1(\Omega)} = 0, \qquad \lim_{p\to+\infty} \|u_{\nu,\mathcal{D}_p}(x,t) - u_\nu(x,t)\|_{L^1(\Omega)} = 0.$$

One can also claim thanks to theorems 4.3.1 and 4.4.1 that u_ν is the unique bounded flux solution associated to initial data $u_{0,\nu}$.
Letting p tend to $+\infty$ in (4.123) leads to : for a.e. $t \in (0,T)$,

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i |u(x,t) - u_\nu(x,t)| dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i |u_0(x) - u_{0,\nu}(x)| dx. \tag{4.124}$$

It is now obvious using (4.122) in (4.124) that $u_\nu \to u$ in $C([0,T]; L^1(\Omega))$, and thus in $C([0,T]; L^r(\Omega))$ for all $r \in [1, +\infty[$. The limit u as p tends to $+\infty$ of the discrete solution $u_{\mathcal{D}_p}$ is thus the limit as ν tends to $+\infty$ of u_ν, and then u is a SOLA to the problem $(\mathcal{P})$ in the sense of definition 4.5. This conclude the proof of theorem 4.5.1. □

4.6 Numerical Result

In order to illustrate this difficult model, we give here a simple numerical example, which will be considered for several values of the parameters. All the computations have been made with MATLAB, using an *explicit* finite volume scheme. The study convergence for such a scheme follows the same steps than those presented in this paper, but always under a convenient stability condition between the time and space steps, leading to heavy notations.

$\phi_1 = \phi_2 = 1$, $\pi_1(u) = 1 + \frac{1}{2}u$, $\pi_1(u) = 2 + \frac{1}{2}u$, $\mu_{o,1}(u) = \mu_{o,2}(u) = u^2$, $\mu_{w,1}(u) = \mu_{w,2}(u) = (1-u)^2$, so that

$$f_1(u) = f_2(u) = \frac{u^2}{1 + 2u(u-1)},$$

$$\varphi_1(u) = \varphi_2(u) = \frac{1}{4}\left(\frac{u^3}{3} - \frac{u^2}{2} - \frac{u}{2} + \frac{1}{2}\tan^{-1}(2u-1)\right).$$

One choose $q = g = 1$, and $u_0 = 0$.

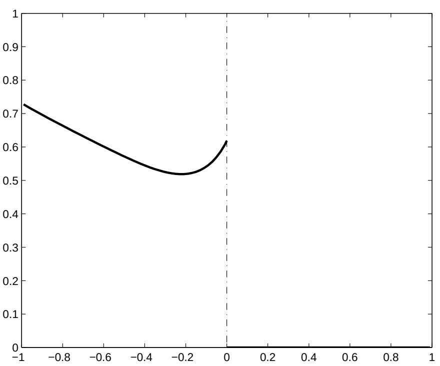

FIG. 4.1 – saturation $u(x, 0.6)$

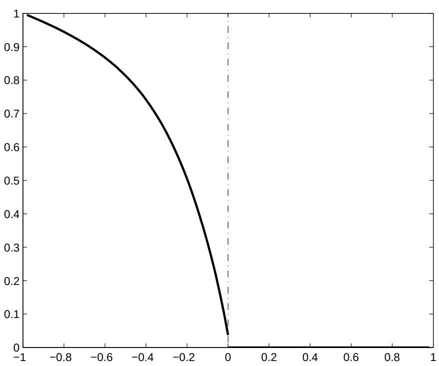

FIG. 4.2 – flux $F(x, 0.6)$

Since $u_1(0, 0.6) < 1$, the flow can not traverse the interface $\{x = 0\}$, as it clearly appears on figures 4.1 and 4.2. One can see on figures 4.3 and 4.4 that the flow can pass through the interface, thanks to the fact that the saturation $u_1(0, 0.8) = 1$. Since q does not depend on time, the flux F fulfills a true maximum principle, due to inequalities (4.101) and (4.102). This can be checked on figures 4.2 and 4.4, where F takes all its values between 0 and 1.

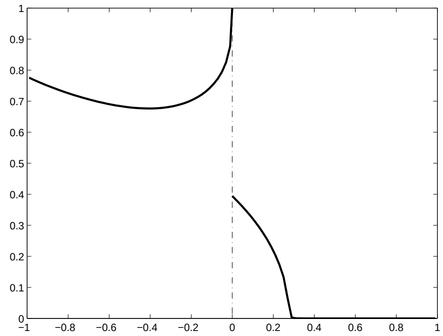

Fig. 4.3 – saturation $u(x, 0.8)$

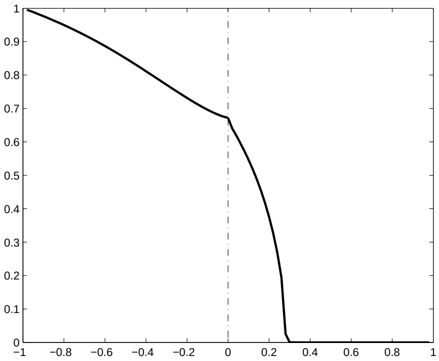

Fig. 4.4 – flux $F(x, 0.8)$

4.7 Conclusion

We will now make a synthesis of the results stated in this paper, to conclude by a theorem summarizing all the results.

We exposed in section 4.1 the physical model to represent the two-phase flow, and we exhibit how to connect the capillary pressures through the interfaces between different homogeneous porous media. This transmission conditions are monotonous, and this plays a crucial role all along the paper. We also set a definition of weak solution to the problem $(\mathcal{P})$.

We built in section 4.2 a finite volume scheme to approach solutions to the problem $(\mathcal{P})$. It has been proven in proposition 4.2.11 that the discrete solution obtained via this scheme converges to a weak solution to the problem $(\mathcal{P})$ for any $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq 1$, and for all $q, g \in L^1(0, T)$, $q \geq g \geq 0$. The proof of uniqueness of the discrete solution and the $L^\infty(\Omega \times (0, T))$ estimate, which is necessary to get the existence of the discrete solution, widely use the monotony of the transmission conditions.

It has been seen in section 4.3 that under regularity assumption on q, g and on u_0 stated in assumptions 4.3 page 95, the numerical solution has bounded-discrete fluxes, as stated in proposition 4.3.3, and thus it converges toward a weak solution fulfilling $\partial_x \varphi_i(u) \in L^\infty(\Omega_i \times (0, T))$. Such a weak solution can also be obtained as in [CGP] via a regularization of the graphs of capillary pressure and of the heterogeneity of the porous media. Once again, the monotony of the transmission conditions will appear to prove proposition 4.3.3, particularly to get the estimate (4.96) on the flux through the interface $\{x = 0\}$.

Such a solution is unique, as stated in section 4.4. We strongly use here the monotony of the transmission conditions (4.9) and (4.10) through the interface $\{x = 0\}$, particularly to prove the key-lemma 4.4.2.

The section 4.5 extends the results of uniqueness to a larger class of solutions than the bounded-flux solutions. For $u_0 \in L^\infty(\Omega)$, $0 \le u_0 \le 1$, for all $q \in BV(0,T)$, $g \in L^\infty(0,T)$, $q \ge g \ge 0$, there exists a unique solution obtained as limit of bounded flux solutions in the sense of definition 4.5. We also proved that the discrete solution converges to this unique SOLA as the discretization step tends to 0, and so, if u_0 is such that $\partial_x \varphi_i(u_0)$ belongs to $L^\infty(\Omega_i)$ and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, then the discrete solution converges to the unique bounded flux solution.

We summarize the discussion above in the following theorem :

Theorem 4.7.1 (conclusion theorem) *Let $u_0 \in L^\infty(\Omega)$, $0 \le u_0 \le 1$, let $q, g \in L^1(0,T)$, $q \ge g \ge 0$. For $(M,N) \in (\mathbb{N}^\star)^2$, there exists a unique discrete solution given by the scheme (4.23)-(4.27).*

Let $(M_p)_{p\in\mathbb{N}}$ and $(N_p)_{p\in\mathbb{N}}$ two sequences of positive integers tending to $+\infty$ as p tends to $+\infty$ and let $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ be the sequence of associated discrete solutions. One has the following convergence results :

- *Suppose that $u_0 \in L^\infty(\Omega)$, $0 \le u_0 \le 1$, and that $q, g \in L^1(0,T)$, $q \ge g \ge 0$. Then $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ converges up to a subsequence to a weak solution to the problem $(\mathcal{P})$ in the sense of definition 4.1 in $L^r(\Omega \times (0,T))$ for all $r \in [1, +\infty[$.*
- *Suppose moreover that assumptions 4.3 page 95 are fulfilled. Then the whole sequence $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ converges to the unique bounded flux solution to the problem $(\mathcal{P})$.*
- *Suppose that $u_0 \in L^\infty(\Omega)$, $0 \le u_0 \le 1$, and that $q \in BV(0,T)$, $g \in L^\infty(0,T)$, $q \ge g \ge 0$. Then the whole sequence $(u_{\mathcal{D}_p})_{p\in\mathbb{N}}$ converges to the unique SOLA to the problem $(\mathcal{P})$.*

Chapitre 5

Hyperbolic limit for two-phase flow in heterogeneous porous media with discontinuous capillary forces : convergence toward the entropy solution

5.1 Presentation of the problem

The resolution of multi-phase flows in porous media are widely used in oil engineering to predict the moves of oil in subsoil. Their mathematical study is however difficult, so that some physical assumptions have to be done, in order to get simpler problems (see e.g. [AS79],[Bea72],[GMT96]). A classical simplified model is called *dead-oil* approximation, and consists in assuming that there is no gas, so that the fluid is composed of two immiscible and incompressible phases and in neglecting all the different chemical species, the oil-phase and the water-phase are then both made of only one component.

5.1.1 the dead-oil problem in the one dimensional case

Suppose that $\mathbb{R}$ represents a one dimensional homogeneous porous medium, with porosity ϕ (which is supposed to be constant for the sake of simplicity). If u denotes the saturation of the water phase, and so $(1-u)$ the saturation of the oil phase thanks to the dead-oil approximation, writing the volume conservation of each phase leads to :

$$\phi\partial_t u + \partial_x V_w = 0, \tag{5.1}$$

$$-\phi\partial_t u + \partial_x V_o = 0, \tag{5.2}$$

where V_o (resp. V_w) is the filtration speed of the oil phase (resp. water phase). Using the empirical diphasic Darcy law, we claim that

$$V_\beta = -K\frac{k_{r,\beta}(u)}{\mu_\beta}\left(\partial_x P_\beta - \rho_\beta \mathbf{g}\right), \qquad \beta = o, w, \tag{5.3}$$

where K is the global permeability, only depending on the porous media, $\mu_\beta, P_\beta, \rho_\beta$ are respectively the dynamical viscosity, the pressure and the density of the phase β, $\mathbf{g}$ represents the effect of gravity, $k_{r,\beta}$ denotes the relative permeability of the phase β. This last term comes from the interference of the two phases in the porous media.

There exists $s_\star \in [0,1)$ such that the function $k_{r,w}$ is non-decreasing, with $k_{r,w}(u) = 0$ if $0 \leq u \leq s_\star < 1$, and $k_{r,w}$ is increasing on $[s_\star, 1]$. The function $k_{r,o}$ is supposed to be non-increasing, with $k_{r,o}(1) = 0$. We suppose that there exists $s^\star \in (s_\star, 1]$ such that $k_{r,o}(s) = 0$ for $s \in [s^\star, 1)$, and $k_{r,o}$ is decreasing on $[0, s^\star)$.

The pressures are supposed to be linked by the relation

$$P_{cap}(u) = P_o - P_w, \tag{5.4}$$

where P_{cap} is a smooth non-increasing function called *capillary pressure.*

Adding (5.1) and (5.2), and using (5.3) and (5.4) yields

$$-\partial_x \left(\sum_{\beta=o,w} K \frac{k_{r,\beta}(u)}{\mu_\beta} \left(\partial_x P_\beta - \rho_\beta \mathbf{g}\right) \right) = 0,$$

and thus there exists q, called total flow-rate, only depending on time, such that

$$-\sum_{\beta=o,w} K \frac{k_{r,\beta}(u)}{\mu_\beta} \left(\partial_x P_\beta - \rho_\beta \mathbf{g}\right) = q. \tag{5.5}$$

Using (5.3), (5.4) and (5.5), (5.1) can be rewritten

$$\begin{aligned} &\phi \partial_t u + \partial_x \left(\frac{q k_{r,w}(u)}{k_{r,w}(u) + \frac{\mu_w}{\mu_o} k_{r,o}(u)} \right) \\ + \quad &K \partial_x \left(\frac{k_{r,w}(u) k_{r,o}(u)}{\mu_o k_{r,w}(u) + \mu_w k_{r,o}(u)} \left(\partial_x P_{cap}(u) - (\rho_o - \rho_w)\mathbf{g}\right) \right) = 0. \end{aligned} \tag{5.6}$$

Supposing that the total flow rate q does not depend on times, and after a convenient rescaling, equation (5.6) becomes

$$\partial_t u + \partial_x (f(u) - \lambda(u) \partial_x \pi(u)) = 0, \tag{5.7}$$

where f is a Lipschitz continuous function, fulfilling $f(0) = 0$, $f(1) = q$, λ is a nonnegative Lipschitz continuous functions, with $\lambda(0) = \lambda(1) = 0$, and π is a non-decreasing function, also called capillary pressure. The effects of capillarity are often neglected, particularly in the case of reservoir simulation, and so (5.7) turns to a nonlinear hyperbolic equation called Buckley-Leverett equation, and we have to consider the initial-value problem

$$\left\{ \begin{array}{l} \partial_t u + \partial_x f(u) = 0, \\ u(0) = u_0. \end{array} \right. \tag{$\mathcal{BL}$}$$

5.1.2 discontinuous flux functions

We now consider heterogeneous one dimensional porous media, that is an apposition of several homogeneous porous media with different physical properties. This leads to discontinuous functions with respect to the spatial variable. For the sake of simplicity, we assume that the heterogeneous porous medium is made of only two homogeneous porous media represented by the open subsets $\Omega_1 = \mathbb{R}^\star_-$ and $\Omega_2 = \mathbb{R}^\star_+$. Keeping the notations of (5.6), $\phi, K, k_{r,\beta}(u,\cdot)$ and $\pi(u,\cdot)$ are now discontinuous functions, that is piecewise constant functions, denoted $\phi_i, K_i, k_{r,\beta,i}$ and π_i in Ω_i. So the problem turns to

$$\left\{\begin{array}{l} \partial_t u + \partial_x (f_i(u) - \lambda_i(u)\partial_x \pi_i(u)) = 0, \\ u(0) = u_0, \\ + \text{ transmission condition at } x = 0, \end{array}\right. \tag{5.8}$$

where f_i are Lipschitz continuous functions on $[0,1]$, and can be decomposed in the following way :

$$f_i(u) = q r_i(u) + \lambda_i(u)(\rho_w - \rho_o)\mathbf{g}, \tag{5.9}$$

where r_i is a non-decreasing Lipschitz continuous function fulfilling $r_i(0) = 0$, $r_i(1) = 1$, and λ_i is a non-negative Lipschitz continuous function fulfilling $\lambda_i(0) = 0$, $\lambda_i(1) = 0$. We stress here the fact that q and $(\rho_w - \rho_o)\mathbf{g}$ neither depend on the subdomain i nor on time.

We now have to give more details on this transmission conditions at $x = 0$. First neglect the effects of capillarity, so that (5.8) becomes the apposition of two Buckley-Leverett equations, linked by a transmission condition.

$$\left\{\begin{array}{l} \partial_t u + \partial_x f_i(u) = 0, \\ u(0) = u_0, \\ + \text{ transmission condition at } x = 0, \end{array}\right. \tag{5.10}$$

We ask the conservation of mass at the interface between the two porous media, then we have to connect the flux. Denoting u_i the trace (if it exists) of $u_{|\Omega_i}$ on $\{x = 0\}$, this means

$$f_1(u_1) = f_2(u_2). \tag{5.11}$$

The problem (5.10)-(5.11) has been widely studied recently (see e.g. [Tow00], [Tow01], [KRT02a], [KRT02b], [KRT03], [SV03], [AJVG04], [Bac04], [Bac06], [BV06], [AMG07]).

It is particularly proven in [Bac05] that the problem admits a unique entropy solution, defined below.

In the sequel, sign is the function defined by

$$\operatorname{sign}(s) = \left\{\begin{array}{lll} 0 & \text{if} & s = 0, \\ 1 & \text{if} & s > 0, \\ -1 & \text{if} & s < 0. \end{array}\right.$$

Let $\mathcal{U}$ be an open set, we denote by $\mathcal{D}(\mathcal{U})$ the set of the smooth functions ψ compactly supported in $\mathcal{U}$ that is $\mathcal{C}^\infty_c(\mathcal{U})$. The function ψ is supposed to belong to $\mathcal{D}^+(\mathcal{U})$ if it furthermore fulfills $\psi \geq 0$.

Definition 5.1 (entropy solution to (5.10)-(5.11)) *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$, and let $T > 0$. A function u is said to be an entropy solution to (5.10)-(5.11) if*

1. $u \in L^\infty(\mathbb{R}\times(0,T))$, $0 \le u \le 1$ *a.e.,*
2. *for all* $\psi \in \mathcal{D}^+(\mathbb{R}\times[0,T))$, *for all* $\kappa \in [0,1]$,

$$\int_0^T \int_{\mathbb{R}} |u(x,t)-\kappa|\partial_t\psi(x,t)dxdt + \int_{\mathbb{R}} |u_0(x)-\kappa|\psi(x,0)dx$$
$$+\sum_{i=1,2}\int_0^T\int_{\Omega_i} \operatorname{sign}(u(x,t)-\kappa)(f_i(u)(x,t)-f_i(\kappa))\partial_x\psi(x,t)dxdt$$
$$+|f_2(\kappa)-f_1(\kappa)|\int_0^T \psi(0,t)dt \ge 0. \tag{5.12}$$

Theorem 5.1.1 (Existence and uniqueness of entropy solution) *Let* $u_0 \in L^\infty(\mathbb{R})$ *with* $0 \le u_0 \le 1$*, and let* $T > 0$*, then there exists a unique entropy solution to (5.10)-(5.11) in the sense of definition 5.1.*
Furthermore, the function u *can be supposed to belong to* $C([0,T];L^1_{loc}(\mathbb{R}))$*, and if* u, v *are two entropy solutions associated to initial data* u_0, v_0*, then, for all* $R > 0$*, the following comparison principle holds :* $\forall t \in [0,T]$,

$$\int_{-R}^{R}(u(x,t)-v(x,t))^{\pm}dx \le \int_{-R-Ct}^{R+Ct}(u_0(x)-v_0(x))^{\pm}dx,$$

where $C = \max_i(Lip(f_i))$*, with* $Lip(f_i) = \sup_{s\in(0,1)}|f_i'(s)|$.

The existence of an entropy solution can be proven by the convergence of a finite volume approximation toward an entropy process solution [Bac05, chapter 5], which is a kind of measure valued solution in the way of Di Perna [DiP85]. The comparison principle is proven in [Bac05, chapter 4], using the concept of kinetic solutions [Per98], and leads to an uniqueness result. The time continuity is proven in appendix A.

5.1.3 heterogeneities involving discontinuous capillarities

Let us now come back to problem (5.8). Suppose for the sake of simplicity that the functions π_i are smooth and increasing on $[0,1]$, and that $\lambda_i(u) > 0$ if $0 < u < 1$. The problem is then a coupling of two parabolic problems, and we will need to ask two transmission conditions : one for the trace, and one for the flux. Concerning the latter, the conservation of mass yields a relation analogous to (5.11), which can be written with rough notations :

$$f_1(u_1) - \lambda_1(u_1)\partial_x\pi_1(u_1) = f_2(u_2) - \lambda_2(u_2)\partial_x\pi_2(u_2). \tag{5.13}$$

Let us now focus on the trace transmission. It has been shown in [CGP] (but see also [BDPvD03] and [EEM06]) that the connection of the capillary pressures $\pi_i(u_i)$ has to be done in a graphical sense, so that phenomena like oil trapping can appear. Thus we have to define the monotonous graphs $\tilde{\pi}_i$.

$$\tilde{\pi}_i(u) = \begin{cases} \pi_i(u) & \text{if } 0 < u < 1, \\ (-\infty, \pi_i(0)] & \text{if } u = 0, \\ [\pi_i(1), +\infty) & \text{if } u = 1. \end{cases}$$

It is shown in [CGP] and [Cana] that a natural way to connect the capillary pressures on the interface consists in asking :

$$\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset. \tag{5.14}$$

In order to state a convenient definition for the solution of (5.8)-(5.13)-(5.14), we need to introduce $\varphi_i(u) = \int_0^u \lambda_i(s)\pi_i'(s)ds$.

Definition 5.2 (bounded flux solution) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, and let $T > 0$. A function u is said to be a bounded flux solution to (5.8)-(5.13)-(5.14) if it fulfills :*

1. $u \in L^\infty(\mathbb{R} \times (0,T))$, $0 \leq u \leq 1$ *a.e.,*
2. $\partial_x \varphi_i(u) \in L^\infty(\Omega_i \times (0,T))$,
3. $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset$ *for a.e.* $t \in (0,T)$,
4. $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0,T))$,

$$\int_0^T \int_{\mathbb{R}} u(x,t)\partial_t \psi(x,t)dxdt + \int_{\mathbb{R}} u_0(x)\psi(x,0)dx$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \left(f_i(u)(x,t) - \partial_x \varphi_i(u)(x,t)\right) \partial_x \psi(x,t)dxdt = 0. \tag{5.15}$$

The bounded flux solution are so called since the point 2 of definition 5.2 insures that the flux $f_i(u) - \partial_x \varphi_i(u)$ remains uniformly bounded. Such a condition will require assumptions on the initial data u_0, as it will be stated in theorem 5.1.2.

Theorem 5.1.2 (existence of a bounded flux solution) *Let f_1, f_2 be Lipschitz continuous functions, and φ_1, φ_2 be increasing Lipschitz continuous functions.*

Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$ fulfilling $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$, and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, where $u_{0,i}$ denotes the trace on $\{x = 0\}$ of $u_{0|\Omega_i}$. Then there exists a bounded flux solution.

If u_0 furthermore belongs to $L^1(\mathbb{R})$, there exists a bounded flux solution $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}))$

The first part of this theorem is a straightforward adaptation to the case of unbounded domains and non-monotonous f_i of a result from [Cana] and [CGP]. This is based on a maximum principle on the fluxes $(f_i(u) - \partial_x \varphi_i(u))$. This particularly yields :

$$\|f_i(u) - \partial_x \varphi_i(u)\|_{L^\infty(\Omega_i \times (0,T))} \leq \max_{j=1,2} \left(\|f_j(u_0) - \partial_x \varphi_j(u_0)\|_{L^\infty(\Omega_j)} \right). \tag{5.16}$$

If $u_0 \in L^1(\Omega)$, then choosing $\psi = \min(1, (1, R - |x|)^+)$ and letting R tend to ∞ gives $u \in L^\infty((0,T); L^1(\mathbb{R}))$. Moreover, thanks to theorem A.1.2 in appendix A, u can be supposed to belong to $\mathcal{C}([0,T]; L^1_{loc}(\mathbb{R}))$. Then u belongs to $\mathcal{C}([0,T]; L^1(\mathbb{R}))$.

The choice of bounded flux solutions instead of more classical weak solution with $\partial_x \varphi_i(u)$ only belonging to $L^2((0,T); L^2_{loc}(\overline{\Omega}_i))$ has been motivated by the fact that it provides a comparison principle if we furthermore suppose that $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}))$.

Proposition 5.1.3 *Let u, v be two bounded flux solutions in the sense of definition 5.2 associated to initial data u_0, v_0. Then, for all $\psi \in \mathcal{D}^+(\mathbb{R} \times [0,T))$,*

$$\int_0^T \int_{\mathbb{R}} |u(x,t) - v(x,t)| \partial_t \psi(x,t) dx dt + \int_{\mathbb{R}} |u_0(x) - v_0(x)| \psi(x,0) dx$$
$$+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \operatorname{sign}(u(x,t) - v(x,t)) \left(f_i(u)(x,t) - f_i(v)(x,t) \right) \partial_x \psi(x,t) dx dt$$
$$- \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x |\varphi_i(u)(x,t) - \varphi_i(v)(x,t)| \partial_x \psi(x,t) dx dt \geq 0. \tag{5.17}$$

This proposition is not sufficient to claim the uniqueness, but it will be very useful in the sequel. In order to obtain a uniqueness result, we have to ask furthermore that the initial data belongs to $L^1(\mathbb{R})$.

Theorem 5.1.4 (uniqueness of bounded flux solution) *Let $u_0 \in L^1(\mathbb{R})$, $0 \leq u_0 \leq 1$ a.e., with $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$ and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$. Then there exists a unique bounded flux solution $u \in \mathcal{C}([0,T]; L^1(\mathbb{R}))$ in the sense of definition 5.2.*

This theorem is a straightforward consequence proposition 5.1.3. Indeed, choosing $\psi = \min(1, (1, R - |x|)^+)$ in (5.17), and letting $\mathbb{R}$ tend to $+\infty$ gives the comparison principle : $\forall t \in [0,T]$,

$$\int_{\mathbb{R}} (u(x,t) - v(x,t))^{\pm} dx \leq \int_{\mathbb{R}} (u_0(x) - v_0(x))^{\pm} dx. \tag{5.18}$$

The uniqueness result follows.

5.1.4 capillary pressure independent of the saturation

The dependance of the capillary pressure π_i with respect to the saturation seems to be weak, and some numerical simulation consider capillary pressures only depending on the porous medium, but not on the saturation. More precisely, we aim to consider graphs of capillary pressure on the form

$$\tilde{\pi}_i(u) = \begin{cases} P_i & \text{if } 0 < u < 1, \\ (-\infty, P_i] & \text{if } u = 0, \\ [P_i, +\infty) & \text{if } u = 1, \end{cases} \tag{5.19}$$

so that the capillary pressure would roughly speaking not depend on u.

If one considers an interface $\{x = 0\}$ between two Ω_i, where the $\tilde{\pi}_i$ are on the form (5.19), we can give an orientation to the interface : the interface is said to be positively oriented if $P_1 > P_2$, and negatively oriented if $P_1 < P_2$. A positively oriented interface involve positive capillary forces, and a negatively oriented involve positive capillary forces. The gravity effects are also oriented by the sign of $(\rho_w - \rho_o)\mathbf{g}$ in (5.9). We have to make the assumption that

« either the gravity effects and the interface are oriented in the same way, or the convective effects are larger than the gravity effects. »

We also suppose that each f_i has a simple dynamic, that is it admits at most one extremum in $(0,1)$.

We suppose in the sequel that $P_1 < P2$. The assumption on the orientation of the forces and on the dynamic of f_i can be stated as follow :

$$\exists b_i \in [0,1) \text{ s.t. } f_i \text{ is decreasing on } [0,b_i] \text{ and increasing on } [b_i,1]. \qquad (\mathcal{H})$$

With this assumption $(\mathcal{H})$, we particularly insure that

$$f_i(1) = \max_{s\in[0,1]}(f_i(s)).$$

The assumption on the dynamic on f_i is often fulfilled by the physical models, as it is stressed in [AJVG04] (see also [EGV03]).

In order to simplify the study, we also suppose that $q \geq 0$, but we can also deal with $q < 0$.

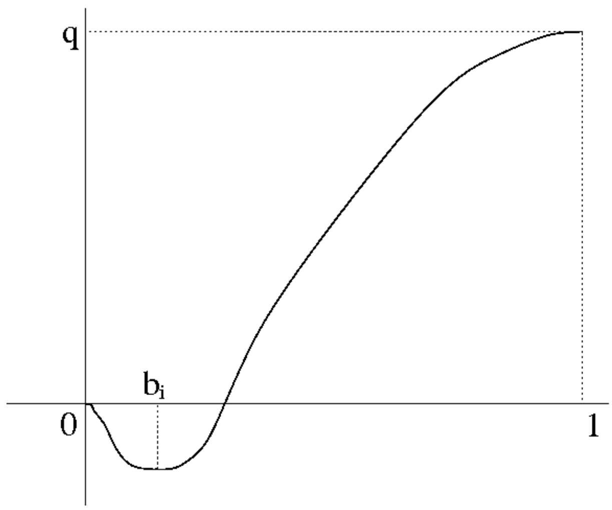

FIG. 5.1 – example of f_i fulfilling $(\mathcal{H})$

We build a family of approximate problems $(\mathcal{P}^\varepsilon)$ taking into account the capillary pressure : one suppose that $\pi_i^\varepsilon(u) = P_i + \varepsilon u$, where P_i is a constant depending only on the homogeneous subdomain Ω_i. In fact, any π_i^ε converging uniformly to P_i on $[0,1]$ and such that $u \mapsto \int_0^u \lambda_i(s) \left(\pi_i^\varepsilon\right)'(s)ds$ converges uniformly toward 0 would fit.

Up to a smoothing of the initial data, we obtain a resulting sequence $(u^\varepsilon)_\varepsilon$ of bounded flux solutions for a problem of type (5.8)-(5.13)-(5.14). We will show that under assumption $(\mathcal{H})$, this sequence tends almost everywhere to the unique entropy solution to (5.10)-(5.11).

5.1.5 organization of the paper

The paper is organized as follow : section 5.2 is devoted to the study of the approximate problem $(\mathcal{P}^\varepsilon)$. We first smooth the initial data in a convenient way, and then we give a $L^2((0,T);H^1(\Omega_i))$-estimate on the approximate solutions, and a family of BV-estimates. We also study the steady states which be useful to derive the entropy inequalities.

The convergence of the approximate solution u^ε toward the unique entropy solution u of (5.10)-(5.11) is proven in section 5.3. We first prove using the BV-estimates that up to a subsequence, the family of bounded flux solutions $(u^\varepsilon)_\varepsilon$ converges almost everywhere, toward a weak solution u. We show that the limit is an entropy solution using the Rankine-Hugoniot relation, and the steady solution.

We give a numerical evidence of this convergence in section 5.4 of the convergence toward an entropy solution under assumption $(\mathcal{H})$. In the case where f_i is not fulfilled, for example in the case where the gravity forces and the capillary forces are oriented in inverse senses at the interface, the convergence toward an entropy solution fails, and non classical shocks can occur at the interface, as it is proven in chapter 6. We give also a numerical evidence of this behavior.

5.2 The approximate problems

In this section we will define the approximate problem $(\mathcal{P}^\varepsilon)$, and its solution u^ε. We will state a $L^2((0,T);H^1_{loc}(\overline{\Omega}_i))$-estimate and a family of BV-estimates, which will be the key points of the proof of convergence of u^ε toward a weak solution of the problem $(\mathcal{P}^\varepsilon)$.

In order to recover a family of entropy inequalities, we will build some steady solutions κ^ε to the problem $(\mathcal{P}^\varepsilon)$, and study their limit as $\varepsilon \to 0$. This last point will require strongly the assumption $(\mathcal{H})$.

5.2.1 smoothing the initial data

As it has already been stressed in theorem 5.1.2, we need to assume some regularity on the initial data to insure the existence of a bounded flux solution to problems of the type (5.8)-(5.13)-(5.14).
Let u_0 belong to $L^\infty(\mathbb{R})$, with $0 \le u_0 \le 1$, we will build a family $(u_0^\varepsilon)_\varepsilon$ of convenient approximate initial data.

Lemma 5.2.1 *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, then there exists a family $(u_0^\varepsilon)_\varepsilon$ of approximate initial data such that :*

- $u_0^\varepsilon \in C_c^\infty(\mathbb{R}^\star)$, $0 \le u_0^\varepsilon \le 1$,
- $u_0^\varepsilon \to u_0$ *a.e. in* $\mathbb{R}$, $\|\varepsilon\partial_x u_0^\varepsilon\|_\infty \to 0$ *as* $\varepsilon \to 0$, *and* $\|\varepsilon\partial_x u_0^\varepsilon\|_\infty \le 1$ *for all* $\varepsilon > 0$,
- *If* $u_0 \in BV(\mathbb{R})$, *then* $\|\partial_x u_0^\varepsilon\|_{L^1(\mathbb{R})} \le TV(u_0) + 4$.

Proof
Let $\alpha > 0$, and let ρ_α be a mollifier with support in $(-\alpha,\alpha)$. Let $v^\alpha = \left(u_0 \circ \chi_{\alpha<|x|<1/\alpha}\right)\star\rho_\alpha$, then it is clear that $v^\alpha \in \mathcal{C}_c^\infty(\mathbb{R}^\star)$, and that $v^\alpha \to u_0$ a.e. in $\mathbb{R}$ as $\alpha \to 0$. Choosing $\varepsilon = \min\left(\alpha, \frac{\min(1,\sqrt{\alpha})}{\|\partial_x v^\alpha\|_\infty}\right)$, and $u_0^\varepsilon = v^\alpha$ ends the proof of lemma 5.2.1. □

5.2.2 the problem $(\mathcal{P}^\varepsilon)$

Let $P_1, P_2 \in \mathbb{R}$, we define the functions π_i^ε by $\pi_i^\varepsilon(u) = P_i + \varepsilon u$, and

$$\tilde{\pi}_i^\varepsilon(u) = \begin{cases} P_i + \varepsilon u & \text{if } 0 < u < 1, \\ (-\infty, P_i] & \text{if } u = 0, \\ [P_i + \varepsilon, +\infty) & \text{if } u = 1. \end{cases}$$

If ε is small, the intersection of the ranges of the functions π_1^ε and π_2^ε is empty, and

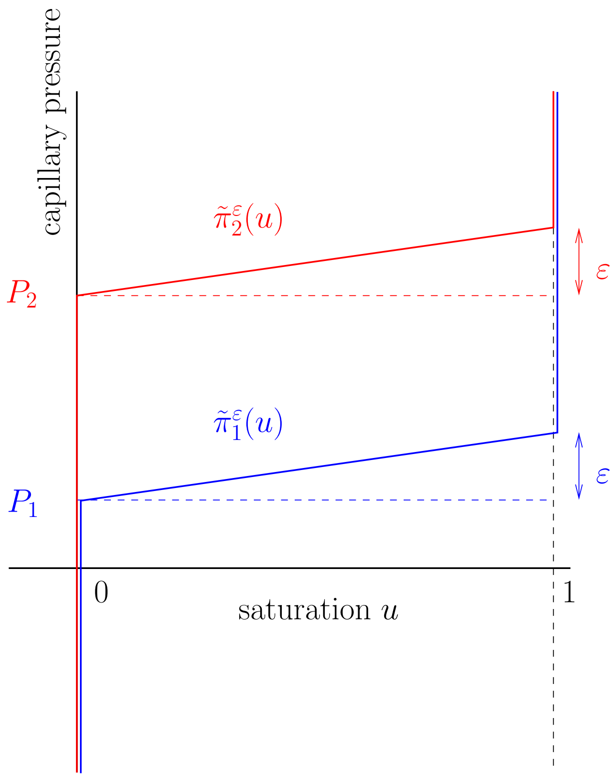

FIG. 5.2 – capillary pressures graphs $\tilde{\pi}_i^\varepsilon$.

then the graphical relation $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset$ connecting the capillary pressures at the interface becomes

$$(1 - u_1)u_2 = 0.$$

Let $0 \leq u_0 \leq 1$, and let $(u_0^\varepsilon)_\varepsilon$ be built as in lemma 5.2.1, let $\varphi_i(u) = \int_0^u \lambda_i(s)ds$. The approximate problem is : find u^ε s.t.

$$\left\{\begin{array}{ll} \partial_t u^\varepsilon + \partial_x \left(f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)\right) = 0 & \text{in } \Omega_i \times (0,T), \\ \tilde{\pi}_1^\varepsilon(u^\varepsilon)(0^-,t) \cap \tilde{\pi}_2^\varepsilon(u^\varepsilon)(0^+,t) \neq \emptyset & \text{in } (0,T), \\ f_1(u^\varepsilon)(0^-,t) - \varepsilon \partial_x \varphi_1(u^\varepsilon)(0^-,t) = f_2(u_2^\varepsilon)(0^+,t) - \varepsilon \partial_x \varphi_2(u^\varepsilon)(0^+,t) & \text{in } (0,T), \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in } \mathbb{R}. \end{array}\right. \quad (\mathcal{P}^\varepsilon)$$

This problems is of type (5.8)-(5.13)-(5.14), and the notion of bounded flux solution is a good frame to solve it

Definition 5.3 (solution to $(\mathcal{P}^\varepsilon)$) *A function u^ε is said to be a (bounded flux) solution to $(\mathcal{P}^\varepsilon)$ if it fulfills*

1. $u^\varepsilon \in L^\infty(\mathbb{R} \times (0,T))$, $0 \leq u^\varepsilon \leq 1$ *a.e.,*
2. $\partial_x \varphi_i(u^\varepsilon) \in L^\infty(\Omega_i \times (0,T))$,

3. $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0,T))$,

$$\int_0^T \int_{\mathbb{R}} u^\varepsilon(x,t)\partial_t\psi(x,t)dxdt + \int_{\mathbb{R}} u_0^\varepsilon(x)\psi(x,0)dx$$
$$+\int_0^T \sum_{i=1,2} \int_{\Omega_i} \left(f_i(u^\varepsilon)(x,t) - \varepsilon\partial_x\varphi_i(u^\varepsilon)(x,t)\right)\partial_x\psi(x,t)dxdt = 0. \quad (5.20)$$

We can use theorem 5.1.2 to claim that there exists a family $(u^\varepsilon)_\varepsilon$ of bounded flux solution to $(\mathcal{P}^\varepsilon)$ in the sense of definition 5.3. Moreover, this family of solution fulfills, thanks to (5.16) and lemma 5.2.1 : for all $\varepsilon > 0$,

$$\|\varepsilon\partial_x\varphi_i(u^\varepsilon)\|_\infty \le (\max_i(Lip(\varphi_i)) + \max_i \|f_i\|_{L^\infty(0,1)}). \quad (5.21)$$

Since u_0^ε belongs to $L^1(\mathbb{R})$, the solution u^ε is furthermore unique in $\mathcal{C}([0,T], L^1(\mathbb{R}))$ thanks to theorem 5.1.4.

5.2.3 a $L^2((0,T); H^1_{loc}(\overline{\Omega}_i))$-estimate

All this subsection is devoted to prove the following estimate.

Proposition 5.2.2 *Let K be a compact subset of $\overline{\Omega}_i$, and let u^ε be a solution of $(\mathcal{P}^\varepsilon)$ in the sense of definition 5.3, then there exists C depending only on f_i, K, T (and not on ε) such that*

$$\sqrt{\varepsilon}\|\varphi_i(u^\varepsilon)\|_{L^2((0,T);H^1(K))} \le C.$$

Particularly, this implies that $\varepsilon\partial_x\varphi_i(u^\varepsilon) \to 0$ a.e. in $\Omega_i \times (0,T)$ as $\varepsilon \to 0$.

Proof
We fix $\varepsilon > 0$. Since the functions φ_i^{-1} are not Lipschitz continuous, the problem $(\mathcal{P}^\varepsilon)$ is not strictly parabolic, and the function u^ε is not a strong solution. In order to get more regularity on the approximate solution, we regularize the problem by adding an additional viscosity $1/n$ ($n \ge 1$), so that the so built approximate solution u_n^ε is regular enough to perform the calculation below.

Let $n > 1$, and let $\varphi_{i,n}(u) = \varphi_i(u) + u/n$, and let u_n^ε be a bounded flux solution of $(\mathcal{P}^\varepsilon)$ with $\varphi_{i,n}$ instead of φ_i. From (5.21), we know that $\partial_x\varphi_{i,n}(u_n^\varepsilon)$ is uniformly bounded in $L^\infty(\Omega_i \times (0,T))$, and since $\varphi_{i,n}^{-1}$ is a Lipschitz continuous function, one has $\partial_x u_n^\varepsilon \in L^\infty(\Omega_i \times (0,T))$. The following weak formulation holds : $\forall \psi \in \mathcal{D}(\Omega \times [0,T))$,

$$\int_0^T \int_{\mathbb{R}} u_n^\varepsilon(x,t)\partial_t\psi(x,t)dxdt + \int_{\mathbb{R}} u_0^\varepsilon(x)\psi(x,0)dx$$
$$+\int_0^T \sum_{i=1,2} \int_{\Omega_i} \left(f_i(u_n^\varepsilon)(x,t) - \varepsilon\partial_x\varphi_{i,n}(u_n^\varepsilon)(x,t)\right)\partial_x\psi(x,t)dxdt = 0. \quad (5.22)$$

Let $(a,b) \subset \Omega_i$. Let $\zeta \in \mathcal{D}^+((a,b))$, we deduce from (5.22) that

$$\left\langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon\zeta^2 \right\rangle = \int_0^T\int_a^b f_i(u_n^\varepsilon)\partial_x(u_n^\varepsilon\zeta^2)dxdt - \varepsilon\int_0^T\int_a^b \partial_x\varphi_{i,n}(u_n^\varepsilon)\partial_x(u_n^\varepsilon\zeta^2)dxdt, \quad (5.23)$$

where $\left\langle \cdot \mid \cdot \right\rangle$ is the duality bracket between $L^2((0,T);H^{-1}(a,b))$ and $L^2((0,T);H^1_0(a,b))$. Since $\varphi_{i,n}$ is a Lipschitz continuous function with $(\|\lambda_i\|_\infty + 1/n)$ as Lipschitz constant, one has

$$\int_0^T\int_a^b \partial_x\varphi_{i,n}(u_n^\varepsilon)\partial_x(u_n^\varepsilon)\zeta^2 dxdt \geq \frac{1}{\|\lambda_i\|_\infty + 1/n}\int_0^T\int_a^b \left(\partial_x\varphi_{i,n}(u_n^\varepsilon)\right)^2\zeta^2 dxdt. \tag{5.24}$$

Let Φ_i be a primitive of f_i, then :

$$\begin{aligned}\int_0^T\int_a^b f_i(u_n^\varepsilon)\partial_x(u_n^\varepsilon\zeta^2)dxdt &= \int_0^T\int_U \partial_x\Phi_i(u_n^\varepsilon)\zeta^2 dxdt + \int_0^T\int_a^b f_i(u_n^\varepsilon)u_n^\varepsilon\partial_x\zeta^2 dxdt \\ &= \int_0^T\int_a^b \left[f_i(u_n^\varepsilon)u_n^\varepsilon - \Phi_i(u_n^\varepsilon)\right]\partial_x\zeta^2 dxdt. \end{aligned}\tag{5.25}$$

We now need the following technical lemma, proven below.

Lemma 5.2.3 *Let u_n^ε be an approximate solution of $(\mathcal{P}^\varepsilon)$ with $\varphi_{i,n}$ instead of φ_i, and let $\psi \in \mathcal{D}^+((a,b))$, then*

$$\left\langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon\zeta^2\right\rangle \geq -\frac{1}{2}\|\zeta\|_\infty^2|b-a|.$$

Admit for the moment lemma 5.2.3. We deduce from (5.23), (5.24), (5.25) and lemma 5.2.3 that

$$\begin{aligned}&\frac{\varepsilon}{\|\lambda_i\|_\infty + 1}\int_0^T\int_a^b \left(\partial_x\varphi_{i,n}(u_n^\varepsilon)\right)^2\zeta^2 dxdt \\ &\leq \frac{1}{2}\|\zeta\|_\infty^2|b-a| + \int_0^T\int_a^b \left|u_n^\varepsilon\left(f_i(u_n^\varepsilon) - \varepsilon\partial_x\varphi_{i,n}(u_n^\varepsilon)\right) - \Phi_i(u_n^\varepsilon)\right| |\partial_x\zeta^2| dxdt.\end{aligned}$$

Using now the fact that u_n^ε is a bounded flux solution, we deduce from (5.16) that $\left[u_n^\varepsilon(f_i(u_n^\varepsilon) - \varepsilon\partial_x\varphi_{i,n}(u_n^\varepsilon)) - \Phi_i(u_n^\varepsilon)\right]$ is uniformly bounded independently of ε and n, and so there exists C only depending on f_i, $|b-a|$ and λ_i such that :

$$\varepsilon\int_0^T\int_a^b \left(\partial_x\varphi_{i,n}(u_n^\varepsilon)\right)^2\zeta^2 dxdt \leq C\left(\|\partial_x\zeta^2\|_{L^1((0,T);\mathcal{M}(\mathbb{R}))} + \|\zeta\|_\infty^2\right).$$

This estimate still holds for $\zeta(x,t) = \chi_{(a,b)}(x)$, for all $(a,b) \in \overline{\Omega_i}^2$, so we obtain

$$\varepsilon\int_0^T\int_a^b \left(\partial_x\varphi_{i,n}(u_n^\varepsilon)\right)^2 dxdt \leq C(2T+1). \tag{5.26}$$

Classical compactness arguments provide the convergence, up to a subsequence, of $(u_n^\varepsilon)_n$ to a solution u^ε of $(\mathcal{P}^\varepsilon)$ in $L^p(\mathbb{R}\times(0,T))$, $1 \leq p < +\infty$. This insures particularly that, up to a subsequence,

$$\lim_{n\to+\infty} u_n^\varepsilon = u^\varepsilon \text{ a.e. in } \mathbb{R}\times(0,T).$$

Taking the limit w.r.t. n in (5.26) yields

$$\varepsilon\int_0^T\int_a^b \left(\partial_x\varphi_i(u^\varepsilon)\right)^2 dxdt \leq C(2T+1).$$

This ends the proof of proposition 5.2.2. □

Proof of lemma 5.2.3
Since $(u_n^\varepsilon \zeta) \in L^2((0,T); H_0^1(a,b))$, and $\partial_t (u_n^\varepsilon \zeta) \in L^2((0,T); H^{-1}(a,b))$, and so, up to a negligible set, $(u_n^\varepsilon \zeta) \in C([0,T]; L^2(a,b))$, and

$$\begin{aligned}\left\langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon \zeta^2 \right\rangle &= \left\langle \partial_t u_n^\varepsilon \zeta \mid u_n^\varepsilon \zeta \right\rangle \\ &= \frac{1}{2}\int_a^b u_n^\varepsilon(x,T)^2 \zeta^2(x)dx - \frac{1}{2}\int_a^b u_0(x)^2\zeta^2(x)dx \\ &\geq -\frac{1}{2}\|\zeta\|_\infty^2 |b-a|.\end{aligned}$$

□

5.2.4 the BV-estimates

In this section we suppose that $u_0 \in BV(\mathbb{R})$. We will prove that the $BV(\Omega_i \times (0,T))$-semi-norm of $f_i(u^\varepsilon)$ remains uniformly bounded with respect to ε. In order to avoid heavy notations which would not lead to a good comprehension of the problem, the following proof will be formal. To establish the following estimates in a rigorous frame, one can introduce of a thin layer $(-\eta, \eta)$ on which the pressure variates smoothly to replace the interface, and add some additional viscosity would lead to smooth strong solutions to the problem. This regularization of the problem has been performed in [CGP] and in [BDPvD03], where a BV-estimate is also derived.

For $a, b \in [0,1]$, we denote by

$$F_i(a,b) = sign(a-b)(f_i(a) - f_i(b)).$$

Lemma 5.2.4 *There exists C depending only on f_i, T, u_0 such that*

$$|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0,T))} \leq C.$$

Proof
Suppose in the sequel that u^ε is a strong solution, that is

$$\partial_t u^\varepsilon + \partial_x \left[f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)\right] = 0$$

holds point-wise in $\Omega_i \times (0,T)$. Let $h > 0$, and let $t \in (0, T-h)$. Comparing $u^\varepsilon(\cdot, \cdot + h)$ and u^ε with (5.18) yieds

$$\int_{\mathbb{R}} |u^\varepsilon(x,t+h) - u^\varepsilon(x,t)|dx \leq \int_{\mathbb{R}} |u^\varepsilon(x,h) - u_0^\varepsilon(x)|dx.$$

Dividing by h and letting h tend to 0, one can claim using the fact that u^ε is supposed to be a strong solution

$$\begin{aligned}\int_{\mathbb{R}} |\partial_t u^\varepsilon(x,t)|dx &= \sum_{i=1,2}\int_{\Omega_i} \left|\partial_x \left[f_i(u^\varepsilon)(x,t) - \varepsilon\partial_x\varphi_i(u^\varepsilon)(x,t)\right]\right|dx \\ &\leq \int_{\mathbb{R}} |\partial_t u_0^\varepsilon(x)|dx \\ &\leq \sum_{i=1,2}\int_{\Omega_i} \left|\partial_x \left[f_i(u_0^\varepsilon)(x) - \varepsilon\partial_x\varphi_i(u_0^\varepsilon)(x)\right]\right|dx\end{aligned}$$

Lemma 5.2.1 then insures that there exists C not depending on ε such that

$$\int_0^T \int_{\mathbb{R}} |\partial_t u^\varepsilon(x,t)| dxdt = \int_0^T \sum_{i=1,2} \int_{\Omega_i} |\partial_x [f_i(u^\varepsilon)(x,t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x,t)]| dxdt \leq C. \quad (5.27)$$

Thanks to the regularity of f_i, this particularly ensures that, if we denote by $\mathcal{M}_b(\Omega_i \times (0,T))$ the set of the bounded Radon measure on $\Omega_i \times (0,T)$, that is the dual space of $\mathcal{C}_c(\overline{\Omega}_i \times [0,T), \mathbb{R})$ with the uniform norm, we obtain : $\forall \kappa \in [0,1]$,

$$|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0,T))} \leq C \|f_i'\|_\infty.$$

□

Lemma 5.2.5 *There exists C depending only on u_0, f_i and T such that*

$$\left| \partial_x \Big(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \Big) \right|_{\mathcal{M}_b(\Omega_i \times (0,T))} \leq C.$$

Proof
It follows from the work of Carillo (see e.g. [Car99]) that for all $\kappa \in [0,1]$, for all $\psi \in \mathcal{D}^+(\Omega_i \times [0,T))$,

$$\begin{aligned} \int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| \partial_t \psi dxdt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) dx \\ \int_0^T \int_{\Omega_i} \Big(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \Big) \partial_x \psi dxdt \geq 0. \end{aligned} \quad (5.28)$$

Let $\eta > 0$, we denote by $\omega_\eta(x) = (1 - |x|/\eta)^+$, and suppose now that ψ belongs to $\mathcal{D}^+(\overline{\Omega}_i \times [0,T))$, that is ψ does not vanish on the interface $\{x = 0\}$. Estimate (5.28) still holds when we consider $\psi_\eta = \psi(1 - \omega_\eta)$ as test function.

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| (1 - \omega_\eta) \partial_t \psi dxdt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0)(1 - \omega_\eta) dx \\ & \quad + \int_0^T \int_{\Omega_i} \Big(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \Big) (1 - \omega_\eta) \partial_x \psi dxdt \\ \geq \quad & \int_0^T \int_{\Omega_i} \Big(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \Big) \psi \partial_x \omega_\eta dxdt. \end{aligned} \quad (5.29)$$

The fact that the flux inducted by u^ε is uniformly bounded w.r.t. ε thanks to (5.21) implies that there exists C depending only on u_0, f_i such that

$$\|F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)|\|_{L^\infty(\Omega_i \times (0,T))} \leq C. \quad (5.30)$$

Then we obtain the following estimate on the right-hand-side in (5.29) :

$$\left| \int_0^T \int_{\Omega_i} \Big(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \Big) \psi \partial_x \omega_\eta dxdt \right| \leq CT \|\psi\|_\infty.$$

Letting η tend to 0 in inequality (5.29) gives : $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0,T))$, $\forall \kappa \in [0,1]$,

$$\int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| \partial_t \psi dxdt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) dx$$
$$+ \int_0^T \int_{\Omega_i} \Big(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \Big) \partial_x \psi dxdt \geq -CT\|\psi\|_\infty \tag{5.31}$$

We introduce now a monotonous function $\chi_\psi \in \mathcal{D}^+(\overline{\Omega}_i)$ equal to 1 on the support of $\psi(\cdot,t)$ for all $t \in [0,T]$ (so that $\|\partial_x \chi_\psi\|_{L^1(\Omega_i)} = 1$), then $\|\psi\|_\infty \chi_\psi \geq \psi$. Choosing $\|\psi\|_\infty \chi_\psi - \psi$ and $\|\psi\|_\infty \chi_\psi + \psi$ as test function in (5.31) yields, using (5.30) once again

$$\Big| \partial_t |u^\varepsilon - \kappa| - \partial_x \Big(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \Big) \Big|_{\mathcal{M}_b(\Omega_i \times (0,T))} \leq 2CT. \tag{5.32}$$

Thanks to (5.27), there exists C' depending only on u_0, f_i and T such that

$$|\partial_t |u^\varepsilon - \kappa||_{\mathcal{M}_b(\Omega_i \times (0,T))} \leq C'. \tag{5.33}$$

The lemma 5.2.5 is so a consequence of (5.32) and (5.33). $\square$

Proposition 5.2.6 *Let $u_0 \in BV(\mathbb{R})$ and let $K = [a,b] \subset \overline{\Omega}_i$. We introduce $z^\varepsilon_K(x,t)$ defined on the whole space $\mathbb{R}^2$ given by :*

$$z^\varepsilon_{K,\kappa}(x,t) = \begin{cases} F_i(u^\varepsilon, \kappa)(x,t) & \textit{if } (x,t) \in K \times (0,T), \\ 0 & \textit{otherwise.} \end{cases}$$

There exists C depending only on u_0, f_i, T, K and a uniformly bounded function $r_{K,\kappa}$, with $r_{K,\kappa}(\varepsilon)$ tends uniformly to 0 with respect to κ as $\varepsilon \to 0$, such that, for all $(\xi, h) \in \mathbb{R}^2$,

$$\iint_{\mathbb{R}^2} \left| z^\varepsilon_{K,\kappa}(x+\xi, t+h) - z^\varepsilon_{K,\kappa}(x,t) \right| dxdt \leq C(|\xi| + |h|) + r_{K,\kappa}(\varepsilon).$$

Proof
Let $h \in \mathbb{R}$, one has

$$\iint_{\mathbb{R}^2} \left| z^\varepsilon_{K,\kappa}(x, t+h) - z^\varepsilon_{K,\kappa}(x,t) \right| dxdt \leq \left| \partial_t z^\varepsilon_{K,\kappa} \right|_{\mathcal{M}_b(\mathbb{R}^2)} |h|$$
$$\leq \quad \Big(|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0,T))} + 2\|F_i(u^\varepsilon, \kappa)\|_{L^\infty}(T + b - a) \Big) |h|$$

It follows from lemma 5.2.4 that there exists C_1 depending only on $u_0, f_i, K, T, \varphi_i$ such that

$$\iint_{\mathbb{R}^2} \left| z^\varepsilon_{K,\kappa}(x, t+h) - z^\varepsilon_{K,\kappa}(x,t) \right| dxdt \leq C_1 |h|. \tag{5.34}$$

We also define

$$q^\varepsilon_{K,\kappa}(x,t) = \begin{cases} \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa)| & \text{if } (x,t) \in K \times (0,T), \\ 0 & \text{otherwise.} \end{cases}$$

The proposition 5.2.2 insures that $q^\varepsilon_{K,\kappa}(x,t)$ converges to 0 almost everywhere in $\mathbb{R}^2$, and the estimate (5.30) insures us that $q^\varepsilon_{K,\kappa}(x,t)$ stays uniformly bounded in $L^\infty(\mathbb{R}^2)$ with respect to ε.
Let $\xi \in \mathbb{R}$, then we have :

$$\begin{aligned}&\iint_{\mathbb{R}^2} \left| z^\varepsilon_{K,\kappa}(x+\xi,t) - q^\varepsilon_{K,\kappa}(x+\xi,t) - \left(z^\varepsilon_{K,\kappa}(x,t) - q^\varepsilon_{K,\kappa}(x,t)\right)\right| dxdt \\ \leq & \left(\begin{array}{c} |\partial_x \left(F_i(u^\varepsilon,\kappa) - \varepsilon\partial_x|\varphi_i(u^\varepsilon)(x,t)-\varphi_i(\kappa)|\right)|_{\mathcal{M}_b(\Omega_i\times(0,T))} \\ +2(T+b-a)\left(\|F_i(u^\varepsilon,\kappa) - \varepsilon\partial_x|\varphi_i(u^\varepsilon)(x,t)-\varphi_i(\kappa)|\|_{L^\infty}\right)\end{array}\right)|\xi|. \end{aligned} \tag{5.35}$$

Using (5.30) and lemma 5.2.5 in (5.35) yields that there exists C_2 depending only on $u_0, f_i, K, T, \varphi_i$ such that

$$\iint_{\mathbb{R}^2} \left| z^\varepsilon_{K,\kappa}(x+\xi,t) - q^\varepsilon_{K,\kappa}(x+\xi,t) - \left(z^\varepsilon_{K,\kappa}(x,t) - q^\varepsilon_{K,\kappa}(x,t)\right)\right| dxdt \leq C_2|\xi|.$$

This particularly insures that :

$$\iint_{\mathbb{R}^2} \left| z^\varepsilon_{K,\kappa}(x+\xi,t) - z^\varepsilon_{K,\kappa}(x,t)\right| dxdt \leq C_2|\xi| + 2\|q^\varepsilon_{K,\kappa}(x,t)\|_{L^1(\mathbb{R}^2)}. \tag{5.36}$$

By choosing $C = \max(C_1, C_2)$, one we deduce from (5.34) and (5.36) that

$$\iint_{\mathbb{R}^2} \left| z^\varepsilon_{K,\kappa}(x+\xi,t+h) - z^\varepsilon_{K,\kappa}(x,t)\right| dxdt \leq C(|h|+|\xi|) + 2\|q^\varepsilon_{K,\kappa}(x,t)\|_{L^1(\mathbb{R}^2)}.$$

We conclude the proof of proposition 5.2.6 by checking that $\|q^\varepsilon_{K,\kappa}(x,t)\|_{L^1(\mathbb{R}^2)}$ converges uniformly to 0 with respect to κ as ε tends to 0. □

5.2.5 some steady solutions

The spatial discontinuities of the saturation and capillary pressure allow us to consider Kružkov entropies $|u-\kappa|$ only for non-negative test functions *vanishing on the interface*, that is in $\mathcal{D}^+(\mathbb{R}^\star \times [0,T))$. This is not enough to obtain the convergence of u^ε toward an entropy solution u in the sense of definition 5.1, and a relation has also to be derived at the level of the interface.

In order to deal with general test functions belonging to $\mathcal{D}^+(\mathbb{R}\times[0,T))$, we will so have to introduce some approximate Kružkov entropies $|u-\kappa^\varepsilon(x)|$, where κ^ε are steady solutions of $(\mathcal{P}^\varepsilon)$. Letting $\varepsilon \to 0$, those steady solutions converges to piecewise constant function $\tilde\kappa^j$ defined below. The limit u of approximate solutions u^ε will then be compared to this limit $\tilde\kappa^j$ to prove that u is an entropy solution.

The building of convenient κ^ε strongly uses the assumption $(\mathcal{H})$. It will be shown in chapter 6 that if $(\mathcal{H})$ fails, non classical shocks can occur at the interface, and the limit $\tilde u$ of the approximate solutions u^ε is thus not an entropy solution.

Recall that $q \geq 0$, $P_1 < P_2$, and suppose that $0 < \varepsilon < P_2 - P_1$. Some simple adaptations can be done to cover the case $q < 0$. The transmission condition (5.14) can be summarized as follow : either $u_1 = 1$, or $u_2 = 0$.

We have to introduce the following sets :

$$\mathcal{E}_1 = \{\kappa_1 \ / \exists \kappa_2 \text{ with } f_1(\kappa_1) = f_2(\kappa_2)\},$$

$$\mathcal{E}_2 = \{\kappa_2 \ / \exists \kappa_1 \text{ with } f_1(\kappa_1) = f_2(\kappa_2)\}.$$

It follows from assumption $(\mathcal{H})$ that either $\mathcal{E}_1 = [0,1]$, or $\mathcal{E}_2 = [0,1]$, and so we are insured that $\kappa \in [0,1]$ belongs either to $\mathcal{E}_1$, or to $\mathcal{E}_2$ (or of course to both $\mathcal{E}_1$ and $\mathcal{E}_2$).

Check also that for all $z \in (0,q]$, it follows from $(\mathcal{H})$ that there exists a unique $\kappa_i(z)$ such that $f_i(\kappa_i(z)) = z$. On the contrary, if $z \in \mathcal{E}_i$ with $z \le 0$, z has a priori not a unique antecedent through f_i. Let κ belong to $\mathcal{E}_j$, one denotes

$$\overline{\kappa}_i^j = \max_{\nu}\{f_i(\nu) = f_j(\kappa)\} \tag{5.37}$$

and

$$\underline{\kappa}_i^j = \min_{\nu}\{f_i(\nu) = f_j(\kappa)\}. \tag{5.38}$$

This section is devoted to establish the following proposition.

Proposition 5.2.7 *For all $\kappa \in \mathcal{E}_j$, there exists a family of steady solutions $(\kappa^\varepsilon)_\varepsilon$ to the problem $(\mathcal{P}^\varepsilon)$ such that*

$$\kappa^\varepsilon \to \tilde{\kappa}^j \ \textit{a.e. in } \Omega_i, \tag{5.39}$$

where $\tilde{\kappa}^j(x)$ can be chosen between :

i) $\tilde{\kappa}^j(x) = \begin{cases} \overline{\kappa}_1^j & \text{if } x < 0, \\ \underline{\kappa}_2^j & \text{if } x > 0, \end{cases}$

ii) $\tilde{\kappa}^j(x) = \begin{cases} \overline{\kappa}_1^j & \text{if } x < 0, \\ \overline{\kappa}_2^j & \text{if } x > 0, \end{cases}$

iii) $\tilde{\kappa}^j(x) = \begin{cases} \underline{\kappa}_1^j & \text{if } x < 0, \\ \underline{\kappa}_2^j & \text{if } x > 0, \end{cases}$

Proof

Let $\kappa \in \mathcal{E}_j$.

– If $\kappa = 1$, the three limits $\tilde{\kappa}^j$ are identically equal to 1, which is a steady solution fulfilling proposition 5.2.7.

– We suppose now that $\kappa < 1$, and $f_j(\kappa) > 0$. Even in this case, the three reachable limit are the same. Thus we only have to build one sequence of converging steady solutions. Let y be a solution of :

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\varphi_1(y) = f_j(\kappa) - f_1(y), & \text{for } x > 0, \\ y(0) = 1. \end{cases} \tag{5.40}$$

The solution $y(x)$ converges to $\overline{\kappa}_1^j$ as $x \to +\infty$. The family $(\kappa^\varepsilon)_\varepsilon$ defined by : $\forall \varepsilon$

$$\kappa^\varepsilon(x) = \begin{cases} \overline{\kappa}_2^j \text{ or } \underline{\kappa}_2^j & \text{if } x > 0, \\ y(-x/\varepsilon) & \text{if } x < 0. \end{cases} \tag{5.41}$$

fulfills so the conclusion of proposition 5.2.7.

– Suppose now $f_j(\kappa) = 0$. The solutions $\kappa^\varepsilon(x)$ built with (5.40)-(5.41) converges toward the two reachable states $i)$ and $ii)$. One can also choose $\kappa^\varepsilon(x) = 0$, which is of course a steady solution.
– It remains the case $f_j(\kappa) < 0$. The solutions $\kappa^\varepsilon(x)$ built with (5.40)-(5.41) still converges toward the two reachable states $i)$ and $ii)$.
  Let w be a solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\varphi_2(w) = f_2(w) - f_j(\kappa), & \text{for } x > 0, \\ w(0) = 0. \end{cases}$$

  and

$$\kappa^\varepsilon(x) = \begin{cases} \underline{\kappa}_1^j & \text{if } x < 0, \\ w(x/\varepsilon) & \text{if } x > 0, \end{cases}$$

  then κ^ε converges toward the third reachable state.

□

5.3 Convergence toward the entropy solution

5.3.1 almost everywhere convergence

We state now a lemma which is an adaptation of Helly's selection theorem criterion, which states that if $(v^\varepsilon)_\varepsilon$ is a family of measurable functions on an open subset $\mathcal{U}$ of $\mathbb{R}^k$ ($k \geq 1$), uniformly bounded in $L^\infty_{loc}(\mathcal{U})$ and $BV_{loc}(\mathcal{U})$, one can extract a subfamily still denoted by $(v^\varepsilon)_\varepsilon$ that converges almost everywhere in $\mathcal{U}$, and the limit belongs to $BV_{loc}(\mathcal{U})$.

Lemma 5.3.1 *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^k$ ($k \geq 1$). Let $(v^\varepsilon)_\varepsilon$ be a family of functions uniformly bounded in $L^\infty(\mathcal{U})$ with respect to ε. Let K be a compact subset of $\mathcal{U}$. For $\zeta \in \mathbb{R}^k$, we denote by*

$$K_\zeta = \{\ x \in K \mid x + \zeta \in K\ \}.$$

One assumes that there exists $C > 0$ depending only on K (and thus not on ε) and a function r fulfilling $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$, such that for all $\zeta \in \mathbb{R}^k$,

$$\int_{K_\zeta} |v^\varepsilon(x+\zeta) - v^\varepsilon(x)|dx \leq C|\zeta| + r(\varepsilon). \tag{5.42}$$

Then, there exists a sequence $(\varepsilon_n)_n$ tending to 0, and $v \in BV_{loc}(\mathcal{U})$ such that

$$\lim_{n\to+\infty} v^{\varepsilon_n} = v \qquad \text{a.e. in } \mathcal{U}.$$

Proof
Let K be a compact subset of $\mathcal{U}$. Estimate (5.42) says, roughly speaking, that v^ε is almost a BV-function on K, that is as close to $BV(K)$ as wanted, provided that ε is supposed to be small enough. So we will build a family $(w^\varepsilon)_\varepsilon$ of BV-functions, which will be close to the family $(v^\varepsilon)_\varepsilon$, at least for small ε, and we will show that $(w^\varepsilon)_\varepsilon$ admits an adherence value v in $BV(K)$ for the $L^1(K)$-topology, and that this v is also an adherence value of $(v^\varepsilon)_\varepsilon$.

A shorter proof for the a.e. convergence toward a function v can be derived directly from Kolmogorov compactness criterion (see e.g. [Bré83]). But the advantage of the following method is that it provides directly some regularity on the limit $v \in BV_{loc}(\mathcal{U})$.

Let $(\rho^\varepsilon)_\varepsilon$ be a sequence of mollifiers, that is smooth, non negative and compactly supported functions with support included in the ball of center 0 and radius ε, and fulfilling $\|\rho^\varepsilon\|_{L^1(\mathbb{R}^k)} = 1$ for all $\varepsilon > 0$. We define the smooth functions

$$w^\varepsilon = \tilde{v}^\varepsilon \star \rho^\varepsilon,$$

where $\tilde{v}^\varepsilon(x) = v^\varepsilon(x)$ if $x \in \mathcal{U}$ and $\tilde{v}^\varepsilon(x) = 0$ if $x \in \mathcal{U}^c$.

Thanks to the regularity of w^ε,

$$\int_{K_\zeta} |w^\varepsilon(x+\zeta) - w^\varepsilon(x)|dx \le \|\nabla w^\varepsilon\|_{(L^1(K))^k}|\zeta|. \tag{5.43}$$

Suppose that $\varepsilon < d(K, \partial\mathcal{U})$ (with the convention $d(K, \emptyset) = \infty$). Thanks to (5.42), we have also

$$\int_{K_\zeta} |w^\varepsilon(x+\zeta) - w^\varepsilon(x)|dx \le \int_{\{K_\zeta+\varepsilon\}} |v^\varepsilon(x+\zeta) - v^\varepsilon(x)|dx \le C|\zeta| + r(\varepsilon), \tag{5.44}$$

where

$$\{K_\zeta + \varepsilon\} = \{\ x \in \mathcal{U} \mid d(x, K) \le \varepsilon\}.$$

Since $r(\varepsilon)$ tends to 0 as $\varepsilon \to 0$, this particularly insures

$$\limsup_{\varepsilon \to 0} \|\nabla w^\varepsilon\|_{\left(L^1(\mathbb{R}^k)\right)^k} \le C.$$

The family $(w^\varepsilon)_\varepsilon$ is thus bounded in $BV(K_\zeta)$ in the neighborhood of $\varepsilon = 0$, and thus, thanks to Helly's selection criterion, there exist $v \in BV(K_\zeta)$, and $(\varepsilon_n)_n$ tending to 0 such that

$$w^{\varepsilon_n} \to v \quad \text{a.e. in } K \text{ as } n \to \infty.$$

Furthermore, for all $n \in \mathbb{N}^\star$,

$$\begin{aligned} \|w^{\varepsilon_n} - v^{\varepsilon_n}\|_{L^1(K)} &\le \int_K \int_{B(0,\varepsilon_n)} |v^{\varepsilon_n}(x-y) - v^{\varepsilon_n}(x)|\rho^{\varepsilon_n}(y)dydx \\ &\le C\varepsilon_n + r(\varepsilon_n). \end{aligned}$$

This insures that v^{ε_n} tends also almost everywhere toward v as n tends to $+\infty$. □

The lemma 5.3.1 will be used to prove the following convergence assertion.

Proposition 5.3.2 *Suppose that $u_0 \in BV(\mathbb{R})$, and let u^ε be a solution to $(\mathcal{P}^\varepsilon)$. Up to an extraction, there exists $u \in L^\infty(\mathbb{R} \times (0,T))$, $0 \le u \le 1$ a.e. such that*

$$u^\varepsilon \to u \text{ a.e. in } \mathbb{R} \times (0,T).$$

Furthermore, there exists $u_1, u_2 \in L^\infty(0,T)$, such that

$$\lim_{\eta\to 0} \frac{1}{\eta} \int_0^T \int_{-\eta}^0 |u(x,t) - u_1(t)| dxdt = 0,$$

$$\lim_{\eta\to 0} \frac{1}{\eta} \int_0^T \int_0^\eta |u(x,t) - u_2(t)| dxdt = 0.$$

Proof

Let K be a compact subset of $\overline{\Omega}_i$.
We define the function $H_i : [0,1] \mapsto \mathbb{R}$ by

$$H_i(u) = \int_0^1 (F_i(u,\sigma) - f_i(\sigma))\, d\sigma, \tag{5.45}$$

so that, thanks to proposition 5.2.6, there exists C depending on u_0, f_i, T, K, and a function r tending to 0 as ε tends to 0 such that for all $\xi \in \mathbb{R}$, $h \in (0,T)$,

$$\int_0^{T-h} \int_K |H_i(u^\varepsilon)(x+\xi, t+h) - H_i(u^\varepsilon)(x,t)|\, dxdt \le C(|\xi| + |h|) + r(\varepsilon). \tag{5.46}$$

An integration by parts in (5.45) yields : $\forall u \in [0,1]$

$$\begin{aligned} H_i(u) &= -\int_0^1 (\sigma - b_i)\partial_\sigma(F_i(u,\sigma) - f_i(\sigma))d\sigma + (2b_i - 1)f_i(u) \\ &= 2\int_0^u (\sigma - b_i) f_i'(\sigma) d\sigma + (2b_i - 1) f_i(u) \end{aligned} \tag{5.47}$$

where, thanks to $(\mathcal{H})$, f_i is decreasing on $[0, b_i]$ and increasing on $[b_i, 1]$.

Using proposition 5.2.6 with $\kappa = 0$,

$$\int_0^{T-h} \int_K |f_i(u^\varepsilon)(x+\xi, t+h) - f_i(u^\varepsilon)(x,t)|\, dxdt \le C(|\xi| + |h|) + r(\varepsilon), \tag{5.48}$$

where C and r have been updated. Denoting by $A_i(u) = \displaystyle\int_0^u (\sigma - b_i) f_i'(\sigma) d\sigma$, we obtain from (5.47) and (5.48)

$$\int_0^{T-h} \int_K |A_i(u^\varepsilon)(x+\xi, t+h) - A_i(u^\varepsilon)(x,t)|\, dxdt \le C(|\xi| + |h|) + r(\varepsilon), \tag{5.49}$$

with a new update for C and r. Thus we deduce from proposition 5.3.1 that, up to an extraction, $A_i(u^\varepsilon)$ converges almost everywhere toward $\overline{A}_i \in BV(K \times (0,T))$. It follows from $(\mathcal{H})$ that A_i is an increasing function, and so we obtain the convergence almost everywhere in $K \times (0,T)$ of u^ε toward a measurable function v.

Since for all $\varepsilon > 0$, $0 \le u^\varepsilon \le 1$, there exists $u \in L^\infty(\mathbb{R} \times (0,T))$, $0 \le u \le 1$ such that u^ε converges to u in the $L^\infty(\mathbb{R} \times (0,T))$-$\star$-weak sense, we thus have, up to an extraction,

$$u^\varepsilon \to u \text{ a.e. in } K \times (0,T). \tag{5.50}$$

Since (5.50) holds for any compact subset K of Ω_i, for $i = 1, 2$, we can claim that, up to an extraction,

$$u^\varepsilon \to u \text{ a.e. in } \mathbb{R} \times (0, T).$$

Moreover, since $A_i(u)$ belongs to $BV(\mathbb{R} \times (0, T))$, we can claim that $A_i(u)$ admits a strong trace on $\{x = 0\} \times (0, T)$ (see for instance [ABM06]). Using once again the fact that A_i^{-1} is a continuous function, we can claim that u admits also a strong trace on each side of the interface. □

5.3.2 convergence toward an entropy solution

Proposition 5.3.3 *Let $u_0 \in BV(\mathbb{R})$, $0 \le u_0 \le 1$ a.e., and let $(u_0^\varepsilon)_\varepsilon$ a family of approximation of u_0 given by lemma 5.2.1, and let $(u^\varepsilon)_\varepsilon$ be the inducted sequence of bounded flux solution of (5.8)-(5.13)-(5.14). Then under assumption ($\mathcal{H}$)*

$$\lim_{\varepsilon \to 0} u^\varepsilon = u \text{ in } L^p_{loc}(\mathbb{R} \times [0, T]), \quad \forall p \in [1, \infty)$$

where u is the unique entropy solution to (5.10)-(5.11) associated to initial data u_0.

Proof
As already stated in section 5.2.5, either $\mathcal{E}_1 = [0, 1]$, or $\mathcal{E}_2 = [0, 1]$. We suppose that $\mathcal{E}_j = [0, 1]$, a straightforward adaptation of the following proof would allow to treat the other case.

Let $\kappa \in [0, 1]$. One can let ε tend to 0 in (5.28), and it follows from propositions 5.2.2 and 5.3.2 and lemma 5.2.1 that for all $\psi \in \mathcal{D}^+(\mathbb{R}^\star \times [0, T))$,

$$\int_0^T \int_{\Omega_i} |u(x,t) - \kappa| \partial_t \psi(x,t) dx dt + \int_{\Omega_i} |u_0(x) - \kappa| \psi(0, x) dx$$
$$\int_0^T \int_{\Omega_i} F_i(u, \kappa)(x,t) \partial_x \psi(x,t) dx dt \ge 0. \tag{5.51}$$

Let κ^ε be a steady solution as built in proposition 5.2.7. Since κ^ε is a bounded flux solution, one can compare u^ε and κ^ε thanks to theorem 5.1.3. For all $\kappa \in [0, 1]$, for all $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}} |u^\varepsilon(x,t) - \kappa^\varepsilon(x)| \partial_t \psi(x,t) dx dt + \int_{\mathbb{R}_-} |u_0^\varepsilon(x) - \kappa^\varepsilon(x)| \psi(x, 0) dx$$
$$+ \sum_{i=1,2} \int_0^T \int_{\Omega_i} \text{sign}(u^\varepsilon(x,t) - \kappa^\varepsilon(x))(f_i(u^\varepsilon)(x,t) - f_i(\kappa^\varepsilon(x)))\partial_x \psi(x,t) dx dt$$
$$- \sum_{i=1,2} \int_0^T \int_{\Omega_i} \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x,t) - \varphi_(\kappa^\varepsilon)(x)| \partial_x \psi(x,t) dx dt \ge 0. \tag{5.52}$$

We let ε tend to 0 in (5.52). Thanks to proposition 5.2.7, we can suppose that $\kappa^\varepsilon(x)$ tends to the reachable steady state $\tilde{\kappa}^j(x)$. Letting ε tend to 0 in (5.52) yields : for all

$\psi \in \mathcal{D}^+(\mathbb{R} \times [0,T))$,

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} |u(x,t) - \tilde{\kappa}_i^j| \partial_t \psi(x,t) dx dt + \sum_{i=1,2} \int_{\Omega_i} |u_0(x) - \tilde{\kappa}_i^j| \psi(0,x) dx$$
$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} F_i(u, \tilde{\kappa}_i^j)(x,t) \partial_x \psi(x,t) dx dt \geq 0. \tag{5.53}$$

Since $\tilde{\kappa}^j = 0$ and $\tilde{\kappa}^j = 1$ are reachable states, we deduce from (5.53) that u is a weak solution to (5.10)-(5.11), that is : $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0,T))$,

$$\int_0^T \int_{\Omega_i} u(x,t) \partial_t \psi(x,t) dx dt + \int_{\Omega_i} u_0(x) \psi(0,x) dx$$
$$\int_0^T \int_{\Omega_i} f_i(u)(x,t) \partial_x \psi(x,t) dx dt \geq 0. \tag{5.54}$$

Since u is a weak solution, u_1 and u_2 fulfill the Rankine-Hugoniot condition :

$$f_1(u_1) = f_2(u_2) \text{ a.e. in } (0,T). \tag{5.55}$$

Let $\eta > 0$, we define $\omega_\eta(x) = (1 - |x|/\eta)^+$. Let $\psi \in \mathcal{D}^+(\mathbb{R} \times [0,T))$ taking $\psi(1 - \omega_\eta)$ as test function in (5.51), and letting η tend to 0 yields

$$\int_0^T \int_{\Omega_i} |u(x,t) - \kappa| \partial_t \psi(x,t) dx dt + \int_{\Omega_i} |u_0(x) - \kappa| \psi(0,x) dx$$
$$\int_0^T \int_{\Omega_i} F_i(u,\kappa)(x,t) \partial_x \psi(x,t) dx dt$$
$$+ \int_0^T \left[F_2(u_2,\kappa)(t) - F_1(u_1,\kappa)(t) \right] \psi(0,t) dt \geq 0. \tag{5.56}$$

We will conclude the proof of the proposition 5.3.3 by the following inequality : $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times [0,T))$, $\forall \kappa \in [0,1]$,

$$F_2(u_2,\kappa) - F_1(u_1,\kappa) \leq |f_2(\kappa) - f_1(\kappa)| \text{ a.e. in } (0,T). \tag{5.57}$$

Indeed, assume that (5.57) holds, then using (5.57) in (5.56) gives the entropy formulation (5.12), and u is an entropy solution in the sense of definition 5.1. The convergence of the whole sequence is just a consequence of the uniqueness of the entropy solution, as stated in theorem 5.1.1. □

We focus now on the last point to achieve the proof of the proposition 5.3.3, that is the proof of (5.57). This relation insures that the limit u of the solutions u^ε of the approximate problem $(\mathcal{P}^\varepsilon)$ fulfills a entropy criterion at the level interface. As it was already stressed before, the relation (5.57) strongly uses the assumption $(\mathcal{H})$. Indeed, the forthcoming proof uses the reachable steady states $\tilde{\kappa}^j$, whose building is based on $(\mathcal{H})$.

Proof of (5.57)
Choosing $\psi\omega_\eta$ as test function in (5.53), and letting η tend to 0 yields

$$\int_0^T \left[F_1(u_1,\tilde{\kappa}_1^j)(t) - F_2(u_2,\tilde{\kappa}_2^j)(t)\right]\psi(0,t)dt \geq 0 \tag{5.58}$$

where $\tilde{\kappa}_i^j$ is the value of $\tilde{\kappa}^j$ in Ω_i. Since this inequality holds for any non negative ψ and since $F_i(u_i,\tilde{\kappa}_i^j)$ is uniformly bounded w.r.t. $t \in (0,T)$, a classical density argument provides

$$F_1(u_1,\tilde{\kappa}_1^j) - F_2(u_2,\tilde{\kappa}_2^j) \ \geq 0 \text{ a.e. in } (0,T). \tag{5.59}$$

This insures that

$$F_2(u_2,\kappa) - F_1(u_1,\kappa) \leq A_1(u_1,\kappa) - A_2(u_2,\kappa) \tag{5.60}$$

with

$$\begin{aligned} A_1(u_1,\kappa) &= F_1(u_1,\tilde{\kappa}_1^j) - F_1(u_1,\kappa), \\ A_2(u_2,\kappa) &= F_2(u_2,\tilde{\kappa}_2^j) - F_2(u_2,\kappa), \end{aligned}$$

and the aim is now to show

$$A_1(u_1,\kappa) - A_2(u_2,\kappa) \leq |f_1(\kappa) - f_2(\kappa)|. \tag{5.61}$$

We will perform a case by case study to prove (5.61), and we will suppose for the sake of simplicity that $j=1$, that is

$$f_i(\tilde{\kappa}_i^1) = f_1(\kappa), \qquad i = 1,2.$$

The proof in the case $j=2$ can be easily derived from the case $j=1$. We also suppose in the sequel that $q \geq 0$. The case $q<0$ can also be derived, comparing a convenient choice of steady states $\tilde{\kappa}^j$ with the solution u.

- <u>If $u_1 \geq \overline{\kappa}_1^1$:</u>
  since κ is equal either to $\underline{\kappa}_1^1$ or to $\overline{\kappa}_1^1$ (that are defined by (5.37) and (5.38)), one gets

  $$u_1 \geq \overline{\kappa}_1^1 \geq \kappa \geq \underline{\kappa}_1^1$$

  and $f_1(\kappa) = f_1(\tilde{\kappa}_1^1)$. This insures

  $$A_1(u_1,\kappa) = 0. \tag{5.62}$$

  One can furthermore claim thanks to assumption $(\mathcal{H})$ that $f_1(u_1) \geq f_1(\kappa)$. Using the definition of $\tilde{\kappa}_2^1$, and (5.55), we obtain $f_2(u_2) \geq f_2(\tilde{\kappa}_2^1)$, and so either $u_2 \geq \overline{\kappa}_2^1$, or $u_2 \leq \underline{\kappa}_2^1$ and $0 \geq f_2(u_2) \geq f_2(\underline{\kappa}_2^1) = f_1(\kappa)$.
  – If $u_2 \geq \max(\kappa, \tilde{\kappa}_2^1)$,

  $$A_2(u_2,\kappa) = f_2(\tilde{\kappa}_2^1) - f_2(\kappa) = f_1(\kappa) - f_2(\kappa). \tag{5.63}$$

– If $u_2 \leq \min(\kappa, \tilde{\kappa}_2^1)$,

$$A_2(u_2,\kappa) = f_2(\kappa) - f_2(\tilde{\kappa}_2^1) = f_2(\kappa) - f_1(\kappa). \tag{5.64}$$

– If $\overline{\kappa}_2^1 \leq u_2 \leq \kappa$,

$$A_2(u_2,\kappa) = 2f_2(u_2) - f_2(\kappa) - f_2(\overline{\kappa}_2^1).$$

Since one has $f_2(u_2) \geq f_2(\tilde{\kappa}_2^1) = f_1(\kappa)$, we obtain

$$A_2(u_2,\kappa) \geq f_1(\kappa) - f_2(\kappa). \tag{5.65}$$

– If $\kappa \leq u_2 \leq \underline{\kappa}_2^1$,

$$A_2(u_2,\kappa) = -2f_2(u_2) + f_2(\kappa) + f_2(\overline{\kappa}_2^1) \geq f_2(\kappa) - f_1(\kappa). \tag{5.66}$$

Estimate (5.61) follows from (5.62)-(5.63)-(5.64)-(5.65)-(5.66).

- <u>If $u_1 \leq \underline{\kappa}_1^1$:</u>
  then (5.62) still holds. In this case, $(\mathcal{H})$ provides the fact that either $0 \geq f_1(u_1) \geq f_1(\underline{\kappa}_1^1)$, or $f_1(u_1) \leq f_1(\underline{\kappa}_1^1)$. Here again, the cases $u_2 \geq \max(\kappa, \tilde{\kappa}_2^1)$ and $u_2 \leq \min(\kappa, \tilde{\kappa}_2^1)$ will not lead to any difficulty.
  – Suppose $f_1(u_1) \leq f_1(\underline{\kappa}_1^1)$, so that $f_2(u_2) \leq f_2(\underline{\kappa}_2^1)$, and so either $u_2 \geq \overline{\kappa}_2^1$, or $u_2 \leq \underline{\kappa}_2^1$ and $0 \geq f_2(u_2) \geq f_2(\underline{\kappa}_2^1) = f_1(\kappa)$. We can reproduce the discussion already done to obtain (5.65) and (5.66).
  – Suppose now that $0 \geq f_1(u_1) \geq f_1(\underline{\kappa}_1^1)$, then $0 \geq f_2(u_2) \geq f_2(\underline{\kappa}_2^1)$ and so u_2 does not belong to $(\underline{\kappa}_2^1, \overline{\kappa}_2^1)$. Estimate (5.65) or (5.66) holds, and thus (5.61) too.

- <u>If $u_1 \in (\underline{\kappa}_1^1, \overline{\kappa}_1^1)$:</u>
  then it follows from $(\mathcal{H})$ that $0 \geq f_1(\kappa) \geq f_1(u_1)$, and then $0 \geq f_2(\tilde{\kappa}_2^1) \geq f_2(u_2)$. We can thus claim that $\underline{\kappa}_2^1 \leq u_2 \leq \overline{\kappa}_2^1$.
  – If $\kappa = \overline{\kappa}_1^1$, choosing the reachable steady state $\tilde{\kappa}^1(x) = \overline{\kappa}_1^1$ if $x < 0$ and $\tilde{\kappa}(x) = \overline{\kappa}_2^1$ if $x > 0$ leads to

$$A_1(u_1,\kappa) = 0,$$

  The case $\kappa \geq u_2$ can be treated as in (5.64), and so we focus on the case $\kappa \leq u_2 \leq \overline{\kappa}_2^1$, then

$$A_2(u_2,\kappa) = -2f_2(u_2) + f_2(\kappa) + f_2(\overline{\kappa}_2^1) \geq f_2(\kappa) - f_1(\kappa).$$

  – If $\kappa = \underline{\kappa}_1$, choosing once again $\tilde{\kappa}_1^1 = \overline{\kappa}_1^1$ and $\tilde{\kappa}_2^1 = \overline{\kappa}_2^1$, we get

$$\begin{aligned} A_1(u_1,\kappa) &= 2\left(f_1(\kappa) - f_1(u_1)\right), \\ A_2(u_2,\kappa) &= -2f_2(u_2) + f_2(\kappa) + f_2(\overline{\kappa}_2^1), \end{aligned}$$

  and so (5.55) provides

$$A_1(u_1,\kappa) - A_2(u_2,\kappa) = f_1(\kappa) - f_2(\kappa).$$

This discussion ends the proof of (5.57), and so the one of the proposition 5.3.3. □

We state now the main result, which is in fact the extension of the proposition 5.3.3 to a larger class of initial data.

Theorem 5.3.4 (main result) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, and let $(u_0^\varepsilon)_\varepsilon$ a family of approximation of u_0 given by lemma 5.2.1, and let $(u^\varepsilon)_\varepsilon$ be the inducted sequence of bounded flux solution of $(\mathcal{P}^\varepsilon)$. Then under assumption $(\mathcal{H})$,*

$$\lim_{\varepsilon\to 0} u^\varepsilon = u \text{ in } L^p_{loc}(\mathbb{R}\times[0,T]), \quad \forall p \in [1,\infty)$$

where u is the unique entropy solution to (5.10)-(5.11) associated to initial data u_0.

Proof
Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, and let $\nu > 0$. There exists $u_{0,\nu}$ in $BV(\mathbb{R})$, $0 \le u_{0,\nu} \le 1$ such that for all $R > 0$,

$$\|u_{0,\nu} - u_0\|_{L^1(-R,R)} \le C(R)\nu. \tag{5.67}$$

If one regularizes $u_{0,\nu}$ into $u_{0,\nu}^\varepsilon$ using the lemma 5.2.1, and if one denotes by u_ν^ε the associated unique bounded flux solution, we have seen that u_ν^ε converges almost everywhere to u_ν as ε tends to 0. As previously, we denote by u_0^ε the regularization of u_0 obtained via the lemma 5.2.1, and u^ε the unique associated bounded flux solution.

Let $R > 0$,

$$\begin{aligned}\int_0^T\int_{-R}^R |u^\varepsilon - u|dxdt \quad &\le \quad \int_0^T\int_{-R}^R |u^\varepsilon - u_\nu^\varepsilon|dxdt + \int_0^T\int_{-R}^R |u_\nu^\varepsilon - u_\nu|dxdt \\ &\quad + \int_0^T\int_{-R}^R |u_\nu - u|dxdt.\end{aligned} \tag{5.68}$$

The contraction principle stated in theorem 5.1.1 yields

$$\int_0^T\int_{-R}^R |u_\nu(x,t) - u(x,t)|dxdt \le T\int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)|dx \tag{5.69}$$

where $M \ge \max_i Lip(f_i)$.

We denote by $\zeta(x,t) = \min(1, (R+1+M(T-t)-|x|)^+)$. One has $\zeta = 1$ on $(-R,R)$, $\zeta \ge 0$ and $\zeta \in L^1(\mathbb{R})$ thus

$$\int_0^T\int_{-R}^R |u^\varepsilon - u_\nu^\varepsilon|dxdt \quad \le \quad \int_0^T\int_{\mathbb{R}} |u^\varepsilon - u_\nu^\varepsilon|\zeta dxdt. \tag{5.70}$$

It follows from proposition 5.1.3 that for all $\psi \in W^{1,1}(\mathbb{R}\times(0,T))$ with $\psi \ge 0$ a.e. and $\psi(\cdot,T) = 0$, one has

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}} |u^\varepsilon(x,t) - u_\nu^\varepsilon(x,t)|\partial_t\psi(x,t)dxdt + \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)|\psi(x,0)dx \\ &+ \int_0^T \sum_{i=1,2}\int_{\Omega_i} \operatorname{sign}(u^\varepsilon(x,t) - u_\nu^\varepsilon(x,t))\left(f_i(u^\varepsilon)(x,t) - f_i(u_\nu^\varepsilon)(x,t)\right)\partial_x\psi(x,t)dxdt \\ &\quad - \int_0^T \sum_{i=1,2}\int_{\Omega_i} \varepsilon\partial_x|\varphi_i(u^\varepsilon)(x,t) - \varphi_i(u_\nu^\varepsilon)(x,t)|\partial_x\psi(x,t)dxdt \ge 0.\end{aligned} \tag{5.71}$$

We denote by

$$\Lambda_1(t) = [-R-1-M(T-t), -R-M(T-t)],$$

$$\Lambda_2(t) = [R+M(T-t), R+1+M(T-t)],$$

$$\vartheta(x) = \begin{cases} 1 & \text{if} \quad x<0, \\ -1 & \text{if} \quad x>0, \end{cases}$$

so that

$$\partial_x \zeta(x,t) = \sum_{i=1,2} \vartheta(x) \chi_{\Lambda_i(t)}(x),$$

$$\partial_t \zeta(x,t) = -M \sum_{i=1,2} \chi_{\Lambda_i(t)}(x).$$

Taking $\psi(x,t) = (T-t)\zeta(x,t)$ in (5.71) yields :

$$\begin{aligned}
&-\int_0^T \int_{\mathbb{R}} |u^\varepsilon(x,t) - u^\varepsilon_\nu(x,t)| \zeta(x,t) dx dt \\
&-M \int_0^T (T-t)) \sum_{i=1,2} \int_{\Lambda_i(t)} |u^\varepsilon(x,t) - u^\varepsilon_\nu(x,t)| dx dt \\
&+T \int_{\mathbb{R}} |u^\varepsilon_0(x) - u^\varepsilon_{0,\nu}(x)| \zeta(x,0) dx \\
&+\int_0^T (T-t) \sum_{i=1,2} \int_{\Lambda_i(t)} \operatorname{sign}(u^\varepsilon(x,t) - u^\varepsilon_\nu(x,t)) \left(f_i(u^\varepsilon)(x,t) - f_i(u^\varepsilon_\nu)(x,t) \right) dx dt \\
&-\int_0^T (T-t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x,t) - \varphi_i(u^\varepsilon_\nu(x,t))| dx dt \geq 0.
\end{aligned}$$

Since $M \geq \max_i Lip(f_i)$, one has

$$\begin{aligned}
&\int_0^T (T-t) \sum_{i=1,2} \int_{\Lambda_i(t)} \operatorname{sign}(u^\varepsilon(x,t) - u^\varepsilon_\nu(x,t)) \left(f_i(u^\varepsilon)(x,t) - f_i(u^\varepsilon_\nu)(x,t) \right) dx dt \\
&\leq M \int_0^T (T-t)) \sum_{i=1,2} \int_{\Lambda_i(t)} |u^\varepsilon(x,t) - u^\varepsilon_\nu(x,t)| dx dt
\end{aligned}$$

and thus

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} |u^\varepsilon(x,t) - u^\varepsilon_\nu(x,t)| \zeta(x,t) dx dt &\leq T \int_{\mathbb{R}} |u^\varepsilon_0(x) - u^\varepsilon_{0,\nu}(x)| \zeta(x,0) dx \\
&- \int_0^T (T-t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x,t) - \varphi_i(u^\varepsilon_\nu(x,t))| dx dt. \qquad (5.72)
\end{aligned}$$

We deduce, using (5.69) (5.70) and (5.72) in (5.68), that

$$\begin{aligned}\int_0^T \int_{-R}^R |u^\varepsilon(x,t) - u(x,t)| dxdt \leq T \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x,0) dx \\ - \int_0^T (T-t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x,t) - \varphi_i(u_\nu^\varepsilon(x,t))| dxdt \\ + T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx + \int_0^T \int_{-R}^R |u_\nu^\varepsilon(x,t) - u_\nu(x,t)| dxdt. \qquad (5.73)\end{aligned}$$

We can now let ε tend to 0. Thanks to the proposition 5.2.2, we can claim that

$$\lim_{\varepsilon \to 0} \int_0^T (T-t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x,t) - \varphi_i(u_\nu^\varepsilon(x,t))| dxdt = 0.$$

We also deduce from the proposition 5.3.3 that

$$\lim_{\varepsilon \to 0} \int_0^T \int_{-R}^R |u_\nu^\varepsilon(x,t) - u_\nu(x,t)| dxdt = 0.$$

Since $\zeta(x,0)$ is compactly supported, it follows from lemma 5.2.1 and the dominated convergence theorem that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x,0) dx = \int_{\mathbb{R}} |u_0(x) - u_{0,\nu}(x)| \zeta(x,0) dx.$$

Thus (5.73) becomes :

$$\begin{aligned}\limsup_{\varepsilon \to 0} \int_0^T \int_{-R}^R |u^\varepsilon(x,t) - u(x,t)| dxdt \quad \leq \quad & T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx \\ & + \int_{\mathbb{R}} |u_0(x) - u_{0,\nu}(x)| \zeta(x,0) dx. \qquad (5.74)\end{aligned}$$

The inequality (5.74) holds for any $\nu > 0$, and letting ν tend to 0 leads to

$$\lim_{\varepsilon \to 0} \int_0^T \int_{-R}^R |u^\varepsilon(x,t) - u(x,t)| dxdt = 0,$$

then u^ε tends to u in $L^1_{loc}(\mathbb{R} \times [0,T])$ as $\varepsilon \to 0$. Since $0 \leq u^\varepsilon \leq 1$ a.e., Hölder inequality gives the convergence in $L^p_{loc}(\mathbb{R} \times [0,T])$ for all finite p. □

5.4 Numerical examples

5.4.1 the test cases

We do the most simple choice for the functions $k_{r,\beta}$ in (5.3), that is $k_{r,w} = u$, and $k_{r,o} = (1-u)$. We furthermore suppose that $(\rho_w - \rho_o)\mathbf{g} = -1$, then the functions f_i are given by :

$$f_i(u) = qu - K_i u(1-u) = qu + g_i(u) \qquad (5.75)$$

where g_i is the non positive function taking into account the global permeability of the porous medium Ω_i and the gravity term $(\rho_w - \rho_o)\mathbf{g}$, and admits K_i as a Lipschitz constant.

We choose $P_1 = 1$, and $P_2 = 2$, and $\varepsilon < 1$. The functions φ_i are so given by :

$$\varphi_i(u) = K_i(u^2/2 - u^3/3), \tag{5.76}$$

The problem will be solved on $(-1,1)$, with $\Omega_1 = (-1,0)$, and $\Omega_2 = (0,1)$. We thus have to choose some boundary condition u_L at $x = -1$ and u_R at $x = 1$. To minimize the effects of the bounded domains, we choose to approximate $\partial_x u(\pm 1) = 0$.

5.4.2 a modified Godunov scheme

We choose, for the sake of simplicity, a uniform discretization w.r.t. the space and time variable. If one denotes by $\delta t = T/M$ and $\delta x = 1/N$ respectively the time and space step. We denote by $(x_i = i\delta x)_{-N\le i\le N}$ the discrete edges, and $\big(x_{i+1/2} = (i+1/2)\delta x\big)_{-N\le i\le N-1}$ the cell centers. Check that the interface is in x_0. The initial data is discretized as follow :

$$u^0_{i+1/2} = \frac{1}{\delta x}\int_{x_i}^{x_{i+1}} u_0(x)dx.$$

One introduces : $\forall a, b \in [0,1]$

$$G_i(a,b) = \begin{cases} \min_{s\in[a,b]} f_i(s) & \text{if } a \ge b, \\ \max_{s\in[b,a]} f_i(s) & \text{if } a \le b. \end{cases}$$

We denote by F_i^n the numerical approximation of the flux at $t = n\delta t$ and $x = x_i$, and $u^n_{i+1/2}$ the discrete approximation of u at $t = n\delta t$ and $x = x_{i+1/2}$. The inner flux F_i^n, $i \notin \{-N, 0, N\}$ is given by, if $x_i \in \Omega_j$,

$$F_i^n = G_j(u^n_{i-1/2}, u^n_{i+1/2}) - \frac{\varepsilon}{\delta x}\Big(\varphi_j(u^n_{i+1/2}) - \varphi_j(u^n_{i-1/2})\Big) \tag{5.77}$$

where the hyperbolic case is obtained by taking $\varepsilon = 0$.

Let us now focus on the flux at the interface. For the sake of simplicity, we do not use Godunov approximation at the interface, but a Rusanov approximation. We have to introduce the discrete traces on the interface $u^n_{0,1}, u^n_{0,2}$, which are the unique solutions of the system (5.78)-(5.79), where (5.78) is a discrete way to connect the flux, and (5.79) is a simple way to write in our case $\tilde\pi_1(u^n_{0,1}) = \tilde\pi_2(u^n_{0,2})$

$$\begin{aligned} F_0^n &= qu^n_{-1/2} - K_1 u^n_{-1/2}(1 - u^n_{-1/2}) + K_1(u^n_{-1/2} - u^n_{0,1}) \\ &\quad -\frac{2\varepsilon}{\delta x}\Big(\varphi_1(u^n_{0,1}) - \varphi_1(u^n_{-1/2})\Big) \\ &= qu^n_{0,2} - K_2 u^n_{1/2}(1 - u^n_{1/2}) + K_2(u^n_{0,2} - u^n_{1/2}) \\ &\quad -\frac{2\varepsilon}{\delta x}\Big(\varphi_2(u^n_{1/2}) - \varphi_2(u^n_{0,2})\Big) \end{aligned} \tag{5.78}$$

and

$$(1 - u^n_{0,1})u^n_{0,2} = 0. \tag{5.79}$$

It is important to check that F_0^n is a non-deacreasing function w.r.t. $u^n_{-1/2}$ and $u^n_{0,2}$, and a non-increasing function w.r.t. $u^n_{0,1}$ and $u^n_{1/2}$. This monotony result is the key of the proof of existence and uniqueness of $u^n_{0,1}$ and $u^n_{0,2}$. Since the monotony still hold for $\varepsilon = 0$, the system (5.78)-(5.79) admits a unique solution even in the case $\varepsilon = 0$.

The solution obtained via the scheme (5.77), (5.78), (5.79) is then compared to the solution to a purely Godunov scheme, given in [AJVG04]. In this paper, the explicit resolution of the Riemann problem is performed under assumption ($\mathcal{H}$). A very simple formula is derived for the flux at the interface.

$$F^n_0 = \max\left(f_1(\max(u^n_{-1/2}), b_1), f_2(\min(b_2, u^n_{1/2}))\right). \tag{5.80}$$

We refer to [Bac06] for such an explicit resolution in more complex cases.

We also define the discrete boundary conditions

$$F^n_{-N} = f_1(u^n_{-N+1/2}), \qquad F^n_N = f_2(u^n_{N-1/2}).$$

The scheme is then given by : $\forall n = 0, M-1$, $\forall i = -N, N-1$,

$$\frac{u^{n+1}_{i+1/2} - u^n_{i+1/2}}{\delta t} + F^n_{i+1} - F^n_i = 0. \tag{5.81}$$

5.4.3 convergence (or not) of the numerical approximations

The numerical approximation u^ε_{disc} for the parabolic problem converges almost everywhere to the unique solution u^ε , as it can be shown with an easy adaptation of the convergence results stated in chapter 4 his convergence occur under the following CFL relation, which insure the monotony of the schemes : $\exists\alpha \in (0, 1)$ s.t.

$$\frac{\delta t}{C(\delta x + \varepsilon\delta x^2)} \leq \alpha, \tag{5.82}$$

where C depends on the data. Once again, we find a classical CFL condition for hyperbolic equations by letting ε tend to 0.

Concerning the approximation u_{disc} obtained using $\varepsilon = 0$ in (5.77) for the inner edges, and (5.80) for the computation of the fluxes at interface, it has been proven in [AJVG04] that the convergence occurs under a classical CFL condition

$$\frac{\delta t}{\delta x} \leq C.$$

Test case 1 : $u_0 = 0.2$, $q = 0$, $K_1 = 2$, $K_2 = 3$, $\delta x = 0.02$, $\delta t = 10^{-4}$.

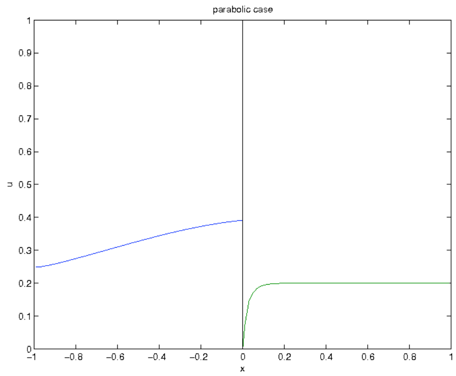

FIG. 5.3 – $t = 0.8$, $\varepsilon = 0.1$, F_0^n given by (5.78)-(5.79).

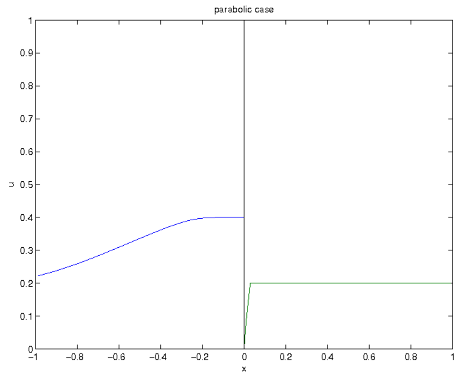

FIG. 5.4 – $t = 0.8$, $\varepsilon = 0$, F_0^n given by (5.78)-(5.79).

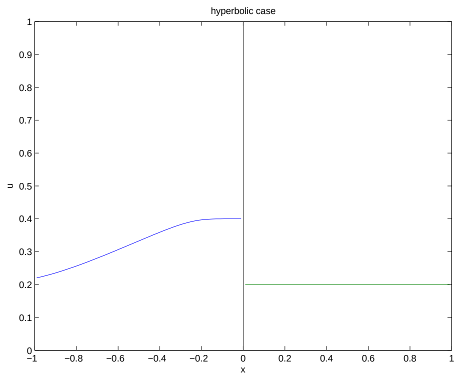

FIG. 5.5 – $t = 0.8$, $\varepsilon = 0$, F_0^n given by (5.80).

We can see that a rarefaction occur at the left of the interface for the purely hyperbolic case with F_0^n given by (5.80) (Figure 5.5). This rarefaction wave is very good mimicked by the numerical solution given where the F_0^n are computed using (5.78)-(5.79) (Figure 5.4), but the precision naturally decreases as ε grows (Figure 5.3). At the right of the interface, the transmission conditions (5.78)-(5.79) generate a thin boundary layer (Figure 5.3), which is reduced to one point in the case $\varepsilon = 0$ (Figure 5.4). Check that as expected, no irrelevant phenomena take place at the boundaries $x = -1$ and $x = 1$.

Test case 2 : $u_0 = 0.2$, $q = 1$, $K_1 = 2$, $K_2 = 3$, $\delta x = 0.02$, $\delta t = 10^{-4}$.

The approximate solution u_{disc} computed with (5.80) let a rarefaction wave appear at the left hand side of the interface, and a shock wave at the right hand side (Figures 5.6 and 5.8). The solution u_{disc}^0 computed with (5.78)-(5.79) for $\varepsilon = 0$ (Figures 5.7 and 5.9) lets also a shock wave appear at the right hand side of the interface, but this shock wave occur with a little delay, due to fact that the limit layer involved by the transmission conditions (5.78)-(5.79) traps some mass at the left hand side of the interface.

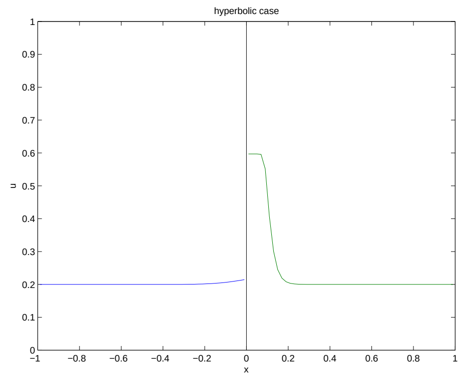

Fig. 5.6 – $t = 0.3$, $\varepsilon = 0$, F_0^n given by (5.80).

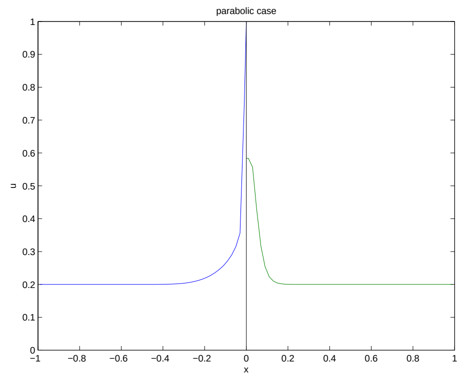

Fig. 5.7 – $t = 0.3$, $\varepsilon = 0$, F_0^n given by (5.78)-(5.79).

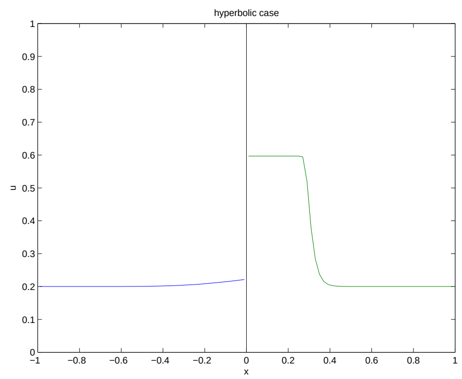

Fig. 5.8 – $t = 0.8$, $\varepsilon = 0$, F_0^n given by (5.80).

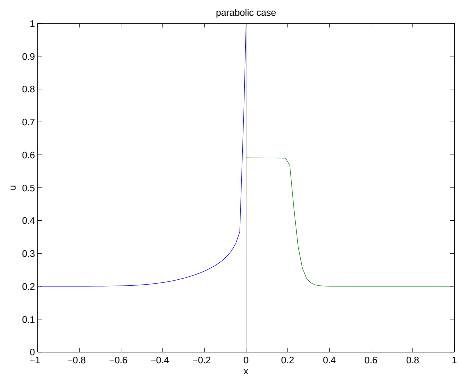

Fig. 5.9 – $t = 0.8$, $\varepsilon = 0$, F_0^n given by (5.78)-(5.79).

Test case 3 : $P_2 < P_1$, $u_0 = 0.2$, $q = 0$, $K_1 = 2$, $K_2 = 3$, $\delta x = 0.02$, $\delta t = 10^{-4}$.

This test case is chosen to give a numerical evidence that assumption $(\mathcal{H})$ is not only technical. We consider a case where the gravity force and the capillary force are oriented in different directions, and so we choose $P_2 < P_1$. In this case, (5.79) has to be replaced by

$$u_{0,1}^n(1 - u_{0,2}^n) = 0 \tag{5.83}$$

to take into account the inversion of the pressures.

Since the solution of the Godunov/VFRoe scheme u_{disc} does not depend on P_1, P_2, it is the same as in test case 1, that is a rarefaction wave at the left hand side of the interface, and a constant equal to u_0 at the right hand side (Figure 5.5). It is clear that the solutions $u_{disc}^{0.1}$ and u_{disc}^{0} given by the scheme respectively for $\varepsilon = 0.1$ and $\varepsilon = 0$ is very different (Figures 5.10, 5.11, 5.12 and 5.13). Indeed, the condition $u_{disc}^{\varepsilon}(0^-) = 0$ and $u_{disc}^{\varepsilon}(0^+) = 1$ are effective, and we see a non entropic steady shock take place at the interface.

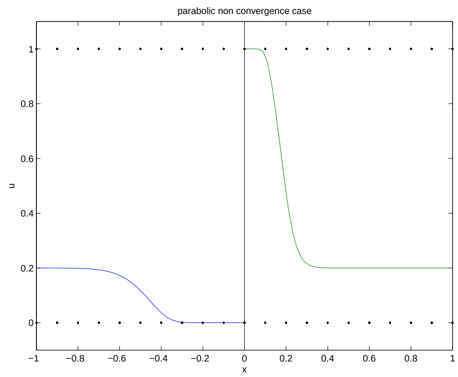

FIG. 5.10 – $t = 0.3$, $\varepsilon = 0.1$, F_0^n given by(5.78)-(5.79).

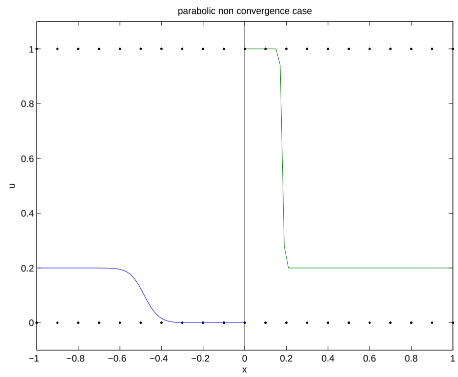

FIG. 5.11 – $t = 0.3$, $\varepsilon = 0$, F_0^n given by (5.78)-(5.79).

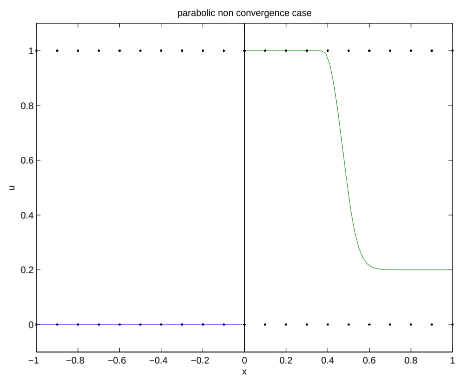

FIG. 5.12 – $t = 0.8$, $\varepsilon = 0.1$, F_0^n given by (5.78)-(5.79).

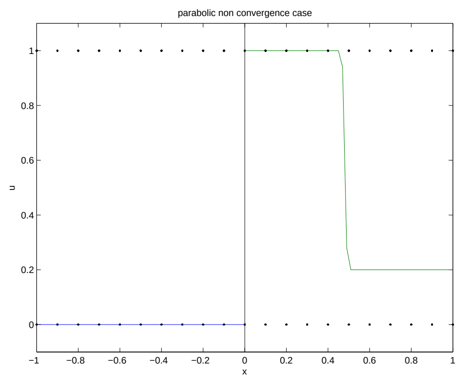

FIG. 5.13 – $t = 0.8$, $\varepsilon = 0$, F_0^n given by (5.78)-(5.79).

The effect of $\varepsilon > 0$ is once again of course the smoothing of the solution u^ε_{disc}, since the function $\varphi(u^\varepsilon_{disc})(\cdot, t)$ is a Lipschitz function for $t > 0$. Indeed, if $x_i \in \Omega_j$

$$|\varphi_j(u^n_{i+1/2}) - \varphi_j(u^n_{i-1/2})| \leq \delta x \frac{(\|f_j\|_\infty + |F^n_i|)}{\varepsilon}. \tag{5.84}$$

Thanks to the uniform estimate on the discrete fluxes available in [Cana]

$$\max_i(F_i^{n+1}) \leq \max_i(F_i^n), \qquad \min_i(F_i^{n+1}) \geq \min_i(F_i^n). \tag{5.85}$$

This estimate, which is a kind of discrete maximum principle for the fluxes, provides a Lipschitz constant for $\varphi_j(u^\varepsilon_{disc})(\cdot, t)$ uniform with respect to $t \geq n\delta t$ which depends on $\max_i |F_i^n|$.

Chapitre 6

Occurrence of non classical shocks in modeling of oil-trapping

6.1 Introduction

Let $\Omega_1 = \mathbb{R}_-$ and $\Omega_2 = \mathbb{R}_+$. In this chapter we will still deal with the solution to the model of two phase flow introduced in chapter 3, computed in chapter 4, but in the case where the convection terms are only involved by the effects of the gravity. We suppose that the gravity works in the sense of increasing x.

Let $\pi(u,x)$ be the capillary pressure, then it has been shown in the previous chapters (but see also [BDPvD03, EEM06]) that the equation governing the two phase flow can be written under the form

$$\partial_t u + \partial_x\Big(g(u,x)\Big(1 - C(x)\partial_x\pi(u,x)\Big)\Big) = 0, \tag{6.1}$$

where $g(u,x) = g_i(u)$ if $x \in \Omega_i$, $C(x) = C_i$ if $x \in \Omega_i$, and $\pi(u,x) = \pi_i(u)$ if $x \in \Omega_i$. Physical experiments let think that the dependence of π_i with respect to u is weak, at least for u not too close to 0 or 1. So we want to choose $\pi_1(u) = P_1$, and $\pi_2(u) = P_2$. The equation (6.1) becomes for $x \neq 0$

$$\partial_t u + \partial_x g(u,x) = 0.$$

We suppose that for $i = 1, 2$, g_i fulfills the following assumptions.

Assumptions 6.1 *For $i = 1, 2$,*

(H1) g_i is a Lipschitz continuous function,

(H2) $g_i(0) = g_i(1) = 0$, and $g_i(s) > 0$ if $s \in (0,1)$,

(H3) There exists neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of 0 and 1 in $[0,1]$ such that g_i is invertible on $\mathcal{U}_0, \mathcal{U}_1$, and $\Big((g_i)_{|\mathcal{U}_z}\Big)^{-1}$ is a Hölder-continuous function.

This assumptions are fulfilled by some models used by the engineers, for which a classical choice of g_i is

$$g_i(u) = K_i \frac{u^{\alpha_i}(1-u)^{\beta_i}}{u^{\alpha_i} + C(1-u)^{\beta_i}},$$

where $\alpha_i, \beta_i \geq 1$.

The goal of this chapter is to show that if the capillary forces at the interface $\{x = 0\}$ are oriented in the inverse sense with respect to the gravity forces (in our case $P_1 < P_2$), then a stationary non-classical shock occurs at the interface. It has been (partially) shown in chapter 5 that if the capillary forces and the gravity forces are oriented in the same sense, the good notion of solution is the one of entropy solution, studied for example in [Tow00], [Tow01], [AG03], [SV03], [AJVG04], [Bac04], [AMG05], [Bac05], [BV06] or [Jim07]. If assumptions 6.2.4 are fulfilled, and if $P_1 < P_2$, we will show that the limit is not the entropy solution, but a solution to the problem

$$\begin{cases} \partial_t u + \partial_x g_i(u) = 0, \\ u(x = 0^-) = 1 \text{ and } u(x = 0^+) = 0, \\ u(t = 0) = u_0. \end{cases} \tag{$\mathcal{P}_{\lim}$}$$

In the sequel, we denote by a^+ (resp. a^-) the positive (resp. negative) part of a, that is $\max(0, a)$ (resp. $\max(0, -a)$), and for $i = 1, 2$, for $u, \kappa \in [0, 1]$, one denotes by

$$G_{i+}(u, \kappa) = \begin{cases} g_i(u) - g_i(\kappa) & \text{if } u \geq \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_{i-}(u, \kappa) = \begin{cases} g_i(\kappa) - g_i(u) & \text{if } u \leq \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G_i(u, \kappa) = G_{i+}(u, \kappa) + G_{i-}(u, \kappa) = g_i(\max(u, \kappa)) - g_i(\min(u, \kappa)).$$

We can now define the notion of solution to ($\mathcal{P}_{\lim}$), which is in fact an entropy solution in each subdomain Ω_i, with an internal boundary condition at the interface.

Definition 6.1 (solution to ($\mathcal{P}_{\lim}$)) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, A function u is said to be a solution of ($\mathcal{P}_{\lim}$) if it belongs to $L^\infty(\mathbb{R} \times (0, T))$, $0 \leq u \leq 1$, and for $i = 1, 2$, for all $\psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0, T))$, for all $\kappa \in [0, 1]$,*

$$\int_0^T \int_{\Omega_i} (u(x,t) - \kappa)^\pm \partial_t \psi dxdt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0)dx$$
$$+ \int_0^T \int_{\Omega_i} G_{i\pm}(u(x,t), \kappa)\partial_x \psi(x,t)dxdt + M_{g_i} \int_0^T (\overline{u}_i - \kappa)^\pm \psi(0,t)dt \geq 0, \tag{6.2}$$

where M_{g_i} is a Lipschitz constant of g_i, and $\overline{u}_1 = 1$, $\overline{u}_2 = 0$.

For a given u_0 in $L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$ there exists a unique solution u to ($\mathcal{P}_{\lim}$) in the sense of definition 6.1, which is in fact made on an apposition of two entropy solutions in $\mathbb{R}_\pm \times (0, T)$. The assumptions on g_i will insure us that the conditions at the interface $\lim_{x \nearrow 0} u(x, t) = 1$ and $\lim_{x \searrow 0} u(x, t) = 0$ are fulfilled for almost every $t \in (0, T)$.

6.1.1 non classical shock at the interface

The study of entropy solutions of hyperbolic scalar conservation laws with discontinuous flux functions has been performed recently, by John D. Towers [Tow00, Tow01], Nicolas Seguin and Julien Vovelle [SV03], Adimurthi and Veerappa Gowda [AG03, AJVG04, AMG05, AMG07], in the work of Florence Bachmann [Bac04, BV06, Bac05] and by Julien Jimenez [Jim07]. They introduced the entropy formulation for this type of problems. $u \in L^\infty(\mathbb{R} \times (0,T))$ is said to be an entropy solution if $0 \le u \le 1$ and $\forall \kappa \in [0,1], \forall \psi \in \mathcal{D}^+(\mathbb{R} \times (0,T))$,

$$\int_0^T \int_{\mathbb{R}} |u(x,t) - \kappa| \partial_t \psi(x,t) dx dt + \int_{\mathbb{R}} |u_0(x) - \kappa| \psi(x,0) dx$$
$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} G_i(u,\kappa) \partial_x \psi(x,t) dx dt + |g_1(\kappa) - g_2(\kappa)| \int_0^T \psi(0,t) dt \ge 0. \tag{6.3}$$

Suppose that an entropy solution u admits strong traces u_i on the interface $\{x = 0\}$, as it is for example the case if g_i, $i = 1,2$ are genuinely non linear [Bac04] from a result of Alexis Vasseur [Vas01]. Then choosing a test function $\psi_\eta(x,t) \ge 0$ such that $\psi_\eta(0,t) = 1$, and $\psi(x,t) = 0$ if $|x| \ge \eta$, and letting η tend to 0 leads to

$$G_1(u_1,\kappa) - G_2(u_2,\kappa) + |g_1(\kappa) - g_2(\kappa)| \ge 0. \tag{6.4}$$

We will show that the solution to $(\mathcal{P}_{\lim})$ is not an entropy solution, and particularly that the conditions at the interface involved by the formulation (6.2) do not fulfill inequality (6.4).

Suppose for the moment that the traces $u_{|\Omega_i}(0) = \overline{u}_i$ of the solution to $(\mathcal{P}_{\lim})$ are fulfilled in a strong sense as it will be the case in the sequel. Since $g_1(\overline{u}_1) = g_2(\overline{u}_2)$, the Rankine-Hugoniot condition is fulfilled at the interface, and it is then easy to check that a u solution to $(\mathcal{P}_{\lim})$ is a weak solution, that is : $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0,T))$,

$$\int_0^T \int_{\mathbb{R}} u(x,t) \partial_t \psi(x,t) dx dt + \int_0^T u_0(x) \psi(x,0) dx$$
$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} g_i(u)(x,t) \partial_x \psi(x,t) dx dt = 0.$$

It follows also from (6.2), taking test functions compactly supported in $\Omega_i \times [0,T)$ that u satisfies an entropy condition in Ω_i, and the lack of entropy can only come from the discontinuity between $\overline{u}_1$ and $\overline{u}_2$ at the interface. Since $\overline{u}_1 = 1$ and $\overline{u}_2 = 0$, we have for all $\kappa \in [0,1]$, $G_1(\overline{u}_1,\kappa) = -g_1(\kappa)$, and $G_2(\overline{u}_2,\kappa) = g_2(\kappa)$. Using the fact that $g_i(\kappa) > 0$ if $\kappa \in (0,1)$, this implies that

$$G_1(\overline{u}_1,\kappa) - G_2(\overline{u}_2,\kappa) + |g_1(\kappa) - g_2(\kappa)| < 0, \qquad \forall \kappa \in (0,1). \tag{6.5}$$

The inequality (6.5) insures that the solution to $(\mathcal{P}_{\lim})$ given by $u(x) = 1$ if $x < 0$ and $u(x) = 0$ if $x > 0$ is not an entropy solution, and so the stationary discontinuity at $\{x = 0\}$ is not entropic.

6.1.2 organization of the chapter

We will introduce a family of approximate problems $(\mathcal{P}^\varepsilon)$ in section 6.2, which takes into account the capillarity, with a small dependance ε of the capillary pressure with respect to the saturation u^ε . We use the transmission conditions introduced in the previous chapters (particularly chapter 3). Then we show that if the initial data is prepared, that is if $u_0(x) = \overline{u}_i$ for $x \in \Omega_i$, $|x| < \eta$ for some $\eta > 0$, then, if the dependance of the capillary pressure is small enough, one has $u^\varepsilon(x,t) = \overline{u}_i$ for $x \in \Omega_i$, $|x| < \eta/2$. Since this equality holds for any ε small enough, we can claim that

$$\forall x \in \Omega_i, |x| < \eta/2, \qquad \lim_{\varepsilon \to 0} u^\varepsilon(x) = \overline{u}_i.$$

We also derive a $L^2((0,T); H^1_{loc}(\Omega_i))$-estimate, which insures that the diffusive effects due to the capillary pressure vanish when we let ε tend to 0.

In section 6.3, we let this parameter ε tend to 0. The only estimate not depending on ε we have on the approximate solution is a $L^\infty(\Omega_i \times (0,T))$ estimate, that is $0 \le u^\varepsilon \le 1$. This insures that there exists $u \in L^\infty(\mathbb{R} \times (0,T))$ such that u^ε tends to u for the $L^\infty(\mathbb{R} \times (0,T))$ weak star topology. This is of course not sufficient to pass to the limit in the nonlinearities. To avoid this difficulty, we use the notion of process solution, which is equivalent to the notion of measure valued solutions of DiPerna [DiP85]. We show that the approximate solution u^ε tends to a process solution u as ε tends to 0. We use then the uniqueness of the process solution to claim that u^ε tends almost everywhere to the unique solution to $(\mathcal{P}_{\lim})$.

6.2 The approximate problem

In this section, we take into account the effects of the capillarity, supposing that they are small We will so build an approximate problem $(\mathcal{P}^\varepsilon)$, whose unknown u^ε will depend on a small parameter ε representing the dependance of the capillary pressure with respect to the saturation. We assume that the capillary pressure in Ω_i is given by :

$$\pi_i^\varepsilon(u^\varepsilon) = P_i + \varepsilon u^\varepsilon. \tag{6.6}$$

It has been seen in the previous chapters that a good way to connect the capillary pressures at the level of the interface is to ask

$$\tilde{\pi}_1^\varepsilon(u_1^\varepsilon) \cap \tilde{\pi}_2^\varepsilon(u_2^\varepsilon) \neq \emptyset, \tag{6.7}$$

where u_1^ε and u_2^ε are the traces of u^ε on the interface, and where $\tilde{\pi}_i^\varepsilon$ is the monotonous graph given by

$$\tilde{\pi}_i^\varepsilon(s) = \begin{cases} \pi_i^\varepsilon(s) & \text{if } s \in (0,1), \\ (-\infty, P_i] & \text{if } s = 0, \\ [P_i + \varepsilon, \infty) & \text{if } s = 1. \end{cases}$$

We suppose that the capillary forces are oriented in the sense of decreasing x, i.e. $P_1 < P_2$ (the capillary forces go from the high pressure to the low pressure). Since ε is assumed to be a small parameter, we can suppose that $0 < \varepsilon < P_2 - P_1$, so that the relation (6.7) becomes :

$$u_1^\varepsilon = 1 \text{ or } \ u_2^\varepsilon = 0. \tag{6.8}$$

We denote by $\varphi_i(s) = C_i \displaystyle\int_0^s g_i(a)da$, where C_i is equal to $C(x)$ introduced in equation (6.1) for $x \in \Omega_i$. The flux function in Ω_i is then given by :

$$F_i^\varepsilon(x,t) = g_i(u^\varepsilon)(x,t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x,t).$$

Thanks to the conservation of mass, we have the continuity of the flux functions at the interface, and the approximate problem becomes :

$$\begin{cases} \partial_t u^\varepsilon + \partial_x F_i^\varepsilon = 0, \\ u^\varepsilon(x=0^-) = 1 \text{ or } u^\varepsilon(x=0^+) = 0, \\ F_1^\varepsilon(0^-) = F_2^\varepsilon(0^+), \\ u(t=0) = u_0. \end{cases} \qquad (\mathcal{P}^\varepsilon)$$

We are not able to prove the uniqueness of a weak solution of $(\mathcal{P}_{\lim})$ if the flux F_i^ε "only" belongs to $L^2(\Omega_i \times (0,T))$, and we will define the notion of prepared initial data, so that the flux belongs to $L^\infty(\Omega_i \times (0,T))$, and the uniqueness holds.

6.2.1 prepared initial data

The following lemma state that one can approach the initial data u_0 by a function $u_{0,\eta}$ for which the expected discontinuity for the non classical shock already occurs.

Lemma 6.2.1 (prepared initial data) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, then there exists $u_{0,\eta}$ such that*
i) $u_{0,\eta} \in C^\infty(\mathbb{R}^\star)$ with a compact support in $\mathbb{R}$, $0 \le u_{0,\eta} \le 1$,
ii) $u_{0,\eta}(x) = 1$ on $(-\eta, 0)$, and $u_{0,\eta}(x) = 0$ on $(0,\eta)$,
iii) $\lim_{\eta\to 0} u_{0,\eta}(x) = u_0(x)$ in $L^1_{loc}(\mathbb{R})$.
We furthermore suppose that if $u_0 \in L^1(\mathbb{R})$, then $\lim_{\eta\to 0} u_{0,\eta} = u_0$ in $L^1(\mathbb{R})$.

A function $u_{0,\eta}$ fulfilling i) and ii) is said to be a η-prepared initial data. An initial data is said to be prepared if it is η-prepared for some $\eta > 0$.

This lemma can be proven exactly in the same way as lemma 5.2.1.

6.2.2 bounded flux solutions

We define now the notion of bounded flux solution, that was already introduced in chapters 3, 4 and 5.

Definition 6.2 (bounded flux solution to $(\mathcal{P}^\varepsilon)$) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, a function u^ε is said to be a bounded flux solution if*
1. *$u^\varepsilon \in L^\infty(\mathbb{R} \times (0,T))$, $0 \le u \le 1$,*
2. *$\partial_x \varphi_i(u^\varepsilon) \in L^\infty(\mathbb{R} \times (0,T))$,*
3. *$\forall \psi \in \mathcal{D}(\mathbb{R} \times [0,T))$,*

$$\int_0^T \int_{\mathbb{R}} u^\varepsilon(x,t)\partial_t\psi(x,t)dxdt + \int_{\mathbb{R}} u_0(x)\psi(x,0)dx + \int_0^T \int_{\mathbb{R}} \left[g_i(u^\varepsilon) - \varepsilon\partial_x\varphi_i(u^\varepsilon)\right]\partial_x\psi(x,t)dxdt = 0. \qquad (6.9)$$

We state now a theorem which is a straightforward generalization to theorem 4.7.1. The fact that g_i is not monotonous can be compensated by the choice of a monotonous scheme to discretize the convection, like for example a Godunov scheme, in order to build a finite volume approximation fulfilling a maximum principle on the flux.

Theorem 6.2.2 (existence and uniqueness of bounded flux solutions) *Let u_0 be a prepared initial data, then there exists a unique bounded flux solution u^ε to the problem $(\mathcal{P}^\varepsilon)$ in the sense of definition 6.2. Furthermore, if $u^\varepsilon, v^\varepsilon$ are two bounded flux solutions associated to initial data u_0, v_0, then, for a.e. $t \in (0,T)$,*

$$\int_{\mathbb{R}} (u^\varepsilon(x,t) - v^\varepsilon(x,t))^{\pm} dx \leq \int_{\mathbb{R}} (u_0(x) - v_0(x))^{\pm} dx. \tag{6.10}$$

Moreover, we can suppose that $u^\varepsilon \in \mathcal{C}([0,T]; L^1(\mathbb{R}))$.

Obviously, the existence of a bounded flux solution can not be extended to any initial data in $L^\infty(\mathbb{R})$. They at least have to involve bounded initial fluxes, that is $\partial_x \varphi_i(u_0) \in L^\infty(\mathbb{R})$. An additional natural assumption is needed to insure the existence of such a bounded flux solution : the connection in the graphical sense of the capillary pressures at the interface.

We need to suppose that u_0 belongs to $L^1(\mathbb{R})$ to insure the comparison principle, and so the uniqueness of solution. Since a prepared initial data is supposed to be compactly supported and uniformly bounded, the property $u_0 \in L^1(\mathbb{R})$ is fulfilled. Then, the associated bounded flux solution u^ε belongs to $L^\infty((0,T); L^1(\mathbb{R}))$ (and even in $\mathcal{C}([0,T]; L^1(\mathbb{R}))$ thanks to appendix A).

We have restricted the frame of theorem 6.2.2 to prepared initial data, but the existence and uniqueness result can be extended to the following larger frame.

Theorem 6.2.3 (existence and uniqueness of the SOLA) *Let $u_0 \in L^1(\mathbb{R})$, $0 \leq u_0 \leq 1$ a.e., there exists a unique $u^\varepsilon \in \mathcal{C}([0,T]; L^1(\Omega))$ such that, for any family $(u_{0,\eta})$ of prepared initial data tending to u_0 as $\eta \to 0$, the corresponding solution u^ε_η tends to u^ε in $\mathcal{C}([0,T]; L^1(\Omega))$. Such a u^ε is called solution obtained as limit of approximations (SOLA) associated to u_0.*

This theorem, also contained in theorem 4.7.1 up to the choice of a non increasing g_i, is in fact a straightforward consequence of the continuous dependance of u^ε_η with regard to $u_{0,\eta}$ stated in (6.10). Indeed, if $(\eta_n)_n$ is a sequence tending to 0, and let $(u_{0,\eta_n})_n$ be a sequence of prepared initial data tending to u_0. Then it follows from (6.10) that for all $t \in (0,T)$, $\left(u^\varepsilon_{\eta_n}(\cdot,t)\right)_n$ is a Cauchy sequence in $L^1(\mathbb{R})$, and so there exists $u^\varepsilon(\cdot,t) \in L^1(\mathbb{R})$ such that

$$u^\varepsilon(\cdot,t) = \lim_{\eta\to 0} u^\varepsilon_\eta(\cdot,t) \quad \text{ in } L^1(\mathbb{R}).$$

Furthermore, we can check that : $\forall t \in [0,T)$,

$$\|u^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}. \tag{6.11}$$

6.2.3 particular sub- and super-solutions

Thanks to lemma 6.2.1, we can approach any initial data $u_0 \in L^\infty$ by a prepared initial data $u_{0,\eta}$ which is smooth in subdomains $\mathbb{R}_-$ and $\mathbb{R}_+$, and which is constant on a small interval on each side of the interface. For the sake of simplicity, we will remove the η in the notation, and so we suppose that u_0 is a prepared initial data. Let u^ε be the unique solution to the approximate problem $(\mathcal{P}^\varepsilon)$. We compare u^ε and particular suband super-solutions, and so we deduce in proposition 6.2.6 that for ε small enough, u^ε is constant on a small interval on each side of the interface.

First, we state the following technical lemma which is just a consequence of the assumptions 6.2.4 and to the fact that $\varphi_i(s) = C_i \int_0^s g_i(a)da$.

Lemma 6.2.4 *Let $i = 1, 2$, and let g_i satisfy the assumptions , there exist $m \in (0,1)$, $R > 0$, and $\alpha \in (0, \varphi_i(1))$ such that :*

$$\begin{aligned} g_i \circ \varphi_i^{-1}(s) \geq Rs^m \quad & \textit{if} \quad s \leq \alpha, & (6.12)\\ g_i \circ \varphi_i^{-1}(s) \geq R(\varphi_i(1) - s)^m \quad & \textit{if} \quad s \geq \varphi_i(1) - \alpha, & (6.13)\\ g_i \circ \varphi_i^{-1} \textit{ is locally Lipschitz continuous} \quad & \textit{on} \quad (0, \varphi_i(1)). & (6.14) \end{aligned}$$

Proof

Let $\alpha \in (0, 1/2)$, then since $g_i > 0$ on $(0,1)$, there exists $\beta > 0$ such that $g_i(s) \geq \beta$ if $s \in [\alpha, 1-\alpha]$. So $\varphi_i'(s)$ is strictly positive on $[\alpha, 1-\alpha]$, and φ_i^{-1} is a Lipschitz continuous function on $[\varphi_i(\alpha), \varphi_i(1-\alpha)]$. Since g_i is also Lipschtiz continuous on $[\alpha, 1-\alpha]$, we deduce that $g_i \circ \varphi_i^{-1}$ is a Lipschitz continuous function on $[\alpha, \varphi_i(1-\alpha)]$, and thus (6.14) holds.

The inequalities (6.12) and (6.13) are similar, indeed they consist in showing that $\varphi_i \circ \left((g_i)_{|\mathcal{U}_z}\right)^{-1}$ are Hölder continuous functions, for $z = 0$ in (6.12), and $z = 1$ in (6.13). This is of course the case, since $\left((g_i)_{|\mathcal{U}_z}\right)^{-1}$ and φ_i are Hölder continuous functions (φ_i are even Lipschitz continuous functions). □

We will introduce now particular solutions of the ordinary differential equation

$$y_i' = g_i \circ \varphi_i^{-1}(y_i). \tag{6.15}$$

Lemma 6.2.5 *There exists a solution y_i to (6.15) on $\mathbb{R}$ and $C(g_i, \varphi_i) > 0$ such that $y_i(x) = 0$ if $x \leq 0$ and $y_i(x) = \varphi_i(1)$ if $x \geq C(g_i, \varphi_i)$.*

Proof

The ordinary differential equation $w' = Rw^m$ with initial condition $w(0) = 0$ admits multiple solutions, and $w(x) = (R(1-m)x)^{\frac{1}{1-m}}$ if $x > 0$ and $w(x) = 0$ if $x \leq 0$ is a particular solution. It follows from inequality (6.12) that w is a subsolution to (6.15) on a neighborhood of $\{x = 0\}$, so there exists a solution y_i to (6.15) such that $y_i \geq w$ and $y_i = 0$ on $\mathbb{R}_-$ (see e.g. [HS99]). This insures the existence of $\eta_0 > 0$ such that $y_i(\eta_0) = \alpha$.
We consider now the initial value problem $z' = R(\varphi_i(1) - z)^m$, with $z(0) = \varphi_i(1)$. It admits of course also multiple solutions, since $z(x) = \varphi_i(1)$ if $x \geq 0$ and $z(x) = \varphi_i(1) - (R(1-m)(-x))^{\frac{1}{1-m}}$ if $x \leq 0$ is a particular solution. Then, thanks to (6.13), there exists $\tilde{y}_i$ solution to (6.15) with $\tilde{y}_i = \varphi_i(1)$ if $x \geq 0$ and $\tilde{y}_i \leq z$ if $x \leq 0$. This ensures the existence of $\eta_1 < 0$ such that $\tilde{y}_i(\eta_1) = \varphi_i(1) - \alpha$.
Since $g_i \circ \varphi_i^{-1}$ is Lipschitz continuous on $[\alpha, \varphi_i(1) - \alpha]$, and $g_i \circ \varphi_i^{-1}(s) \geq \beta > 0$ if $s \in [\alpha, \varphi_i(1) - \alpha]$, then there exists a unique increasing solution y to (6.15) with $y(\eta_0) = \alpha$. This solution is greater than $\hat{y}_i(x) = \min((\varphi_i(1) - \alpha), \max(\alpha, \alpha \exp(\beta x - \eta_0)))$, thus in particular, there exists $\gamma > 0$ such that $y_i(\gamma) = (\varphi_i(1) - \alpha)$.

The function y_i can be extended by $y_i(x) = \tilde{y}_i(x - \gamma + \eta_1)$ if $x \geq \gamma$, and so this function reaches the value $\varphi_i(1)$ in finite x. □

Proposition 6.2.6 *Let $u_0$be a η-prepared initial data, then there exists ε_0, such that for all $\varepsilon \in (0, \varepsilon_0)$, $u^\varepsilon(x) = 1$ on $(-\eta/2, 0)$ and $u^\varepsilon(x) = 0$ on $(0, \eta/2)$, where u^ε is the unique bounded flux solution to ($\mathcal{P}^\varepsilon$).*

Proof
Let $C = C(g_i, \phi_i)$ be a positive real value chosen as in lemma 6.2.5, that is chosen such that a solution y_i to (6.15) needs less space than C to pass from the state 0 to the state $\varphi_i(1)$. We define $\varepsilon_0 = \frac{\eta}{2C}$, and let $\varepsilon \in (0, \varepsilon_0)$

$$\underline{w}(x) = \begin{cases} \varphi_1^{-1}\left(y_1\left(\dfrac{x+\eta}{\varepsilon}\right)\right) & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x > 0, \end{cases} \tag{6.16}$$

$$\overline{w}(x) = \begin{cases} 1 & \text{if} \quad x < 0, \\ \varphi_2\left(y_2\left(\dfrac{x-\eta/2}{\varepsilon}\right)\right) & \text{if} \quad x > 0. \end{cases} \tag{6.17}$$

It follows from lemma 6.2.5 that $\underline{w}(x) = 0$ if $x \notin (-\eta, 0)$, $\overline{w}(x) = 1$ if $x \notin (0, \eta)$, and that $\underline{w}(x) = 1$ if $x \in (-\eta/2, 0)$, $\overline{w}(x) = 1$ if $x \in (0, \eta/2)$.

This functions have been built so that $\underline{w}$ is a sub-solution and $\overline{w}$ is a super-solution to the problem ($\mathcal{P}^\varepsilon$). The comparison principle insures that for every $t \in [0, T]$,

$$\underline{w} \leq u^\varepsilon(\cdot, t) \leq \overline{w}.$$

This particularly insures that $u^\varepsilon(x, \cdot) = 1$ for a.e. $x \in (-\eta/2, 0)$ and $u^\varepsilon(x, \cdot) = 0$ for a.e. $x \in (0, \eta/2)$. □

6.2.4 a $L^2((0, T); H^1_{loc}(\overline{\Omega}_i))$ estimate

Our goal is now to derive an estimate which insures that the effects of capillarity vanish almost everywhere in $\Omega_i \times (0, T)$ as ε tends to 0.

Proposition 6.2.7 *Let K be a compact subset of Ω_i, then there exists C depending only on $u_0, g_i, \varphi_i, T, K$ such that*

$$\sqrt{\varepsilon}\|\varphi_i(u^\varepsilon)\|_{L^2((0,T);H_1(K))} \leq C. \tag{6.18}$$

This particularly ensures that

$$\varepsilon\partial_x\varphi_i(u^\varepsilon) \to 0 \quad \text{ in } L^2((0, T); L^2_{loc}(\overline{\Omega}_i)) \text{ as } \varepsilon \to 0. \tag{6.19}$$

The idea of the proof of proposition 6.2.7 is formally to choose $u^\varepsilon\psi$ as test function in (6.9) for a function $x \mapsto \psi(x)$ compactly supported in K. Using the fact that the flux F_i^ε is uniformly bounded in $L^\infty(\Omega_i \times (0, T))$, we can let ψ tend toward χ_K, with $\chi_K(x) = 1$ if $x \in K$ and 0 otherwise, and the estimate (6.18) follows. To obtain (6.19), it suffices to multiply (6.18) by $\sqrt{\varepsilon}$.

We refer to proposition 5.2.2 for a rigorous proof of proposition 6.2.7.

6.3 Convergence

6.3.1 A compactness result

Since $(u^\varepsilon)_\varepsilon$ is uniformly bounded between 0 and 1, there exists $u \in L^\infty(\mathbb{R} \times (0,T))$ such that $u^\varepsilon \to u$ is the L^∞ weak-$\star$ sense. This is of course insufficient to pass in the limit in the nonlinear terms. Either better estimates are needed, like for example a BV-estimate introduced in the work of Vol′pert [Vol67] and in chapter 5, or we have to use a compactness result allowing to deal with non-linearities. This idea motivates the introduction of Young measures as in the papers of DiPerna [DiP85] and Szepessy [Sze91], or equivalently the notion of nonlinear weak star convergence, introduced in [EGGH98] and [EGH00], which will lead to the notion of process solution given in definition 6.3.

Theorem 6.3.1 (Nonlinear weak-$\star$ convergence) *Let $\mathcal{Q}$ be a Borelian subset of $\mathbb{R}^k$, and (u_n) be a bounded sequence in $L^\infty(\mathcal{Q})$. Then there exists $u \in L^\infty(\mathcal{Q} \times (0,1))$, such that up to a subsequence, u_n tends to u "in the non linear weak star sense" as $n \to \infty$, that is : $\forall g \in C(\mathbb{R},\mathbb{R})$,*

$$g(u_n) \to \int_0^1 g(u(\cdot,\alpha))d\alpha \text{ for the weak-}\star\text{ topology of } L^\infty(\mathcal{Q}) \text{ as } n \to \infty.$$

We refer to [DiP85] and [EGGH98] for the proof of theorem 6.3.1.

6.3.2 convergence toward an process solution

Definition 6.3 (process solution to ($\mathcal{P}_{\lim}$)) *A function $u \in L^\infty(\mathbb{R} \times (0,T) \times (0,1))$ is said to be a process solution to ($\mathcal{P}_{\lim}$) if $0 \le u \le 1$ and for $i = 1,2$, $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times (0,T))$, $\forall \kappa \in [0,1]$,*

$$\int_0^T \int_{\Omega_i} \int_0^1 (u(x,t,\alpha) - \kappa)^\pm \partial_t \psi(x,t) d\alpha dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0) dx$$
$$+ \int_0^T \int_{\Omega_i} \int_0^1 G_{i\pm}(u(x,t,\alpha),\kappa) \partial_x \psi(x,t) d\alpha dx dt + M_{g_i} \int_0^T (\overline{u}_i - \kappa)^\pm \psi(0,t) dt \ge 0,$$

where M_{g_i} is any Lipschitz constant of g_i, $\overline{u}_1 = 1$ and $\overline{u}_2 = 0$.

Proposition 6.3.2 (convergence toward a process solution) *Let u_0 be a η-prepared initial data, and let $(u^\varepsilon)_\varepsilon$ be the corresponding family of approximate solutions obtained via ($\mathcal{P}^\varepsilon$). Then, up to an extraction, u^ε converges in the nonlinear weak-$\star$ sense toward a process solution u to the problem ($\mathcal{P}_{\lim}$).*

Proof
Since u^ε is a weak solution of ($\mathcal{P}^\varepsilon$), which is a non-fully degenerate parabolic problem, that is φ_i^{-1} is continuous, it follows from the work of Carrillo [Car99] that u^ε is an entropy weak solution, that is : $\forall \psi \in \mathcal{D}^+(\Omega_i \times [0,T))$, $\forall \kappa \in [0,1]$,

$$\int_0^T \int_{\Omega_i} (u^\varepsilon(x,t) - \kappa)^\pm \partial_t \psi(x,t) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0) dx$$
$$+ \int_0^T \int_{\Omega_i} \left[G_{i\pm}(u^\varepsilon(x,t),\kappa) - \varepsilon \partial_x (\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa))^\pm \right] \partial_x \psi(x,t) dx dt \ge 0. \quad (6.20)$$

This family of inequalities is only available for non-negative functions ψ compactly supported in Ω_i, and so vanishing on the interface $\{x = 0\}$. To overpass this difficulty, we use cut-off functions $\chi_{i,\delta}$.

Let $\delta > 0$, we denote by $\chi_{i,\delta}$ a smooth non-negative function, with $\chi_{i,\delta}(x) = 0$ if $x \notin \Omega_i$, and $\chi_{i,\delta}(x) = 1$ if $x \in \Omega_i$, $|x| \geq \delta$. Let $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0,T))$, then $\psi\chi_{i,\delta} \in \mathcal{D}^+(\Omega_i \times [0,T))$ can be used as test function in (6.20). This yields

$$\int_0^T \int_{\Omega_i} (u^\varepsilon(x,t) - \kappa)^\pm \partial_t\psi(x,t)\chi_{i,\delta}(x)dxdt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0)\chi_{i,\delta}(x)dx$$
$$+ \int_0^T \int_{\Omega_i} \left[G_{i\pm}(u^\varepsilon(x,t),\kappa) - \varepsilon\partial_x(\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa))^\pm\right] \partial_x\psi(x,t)\chi_{i,\delta}(x)dxdt$$
$$+\int_0^T\int_{\Omega_i} \left[G_{i\pm}(u^\varepsilon(x,t),\kappa) - \varepsilon\partial_x(\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa))^\pm\right] \psi(x,t)\partial_x\chi_{i,\delta}(x)dxdt \geq 0 \quad (6.21)$$

Since u_0 is supposed to be a η-prepared initial data, we can claim thanks to proposition 6.2.6 that, if ε is small enough, u is constant in Ω_i on a neighborhood of $\{x = 0\}$, equal to $\overline{u}_i$, that is 1 is $i = 1$ and 0 if $i = 2$. It follows that for δ small enough, the support of $\partial_x\chi_{i,\delta}$ is included in the set where $u^\varepsilon = \overline{u}_i$, and so the last term in inequality (6.21) becomes can be rewritten in a very simple way.

$$\int_0^T \int_{\Omega_i} (u^\varepsilon(x,t) - \kappa)^\pm \partial_t\psi(x,t)\chi_{i,\delta}(x)dxdt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0)\chi_{i,\delta}(x)dx$$
$$+ \int_0^T \int_{\Omega_i} \left[G_{i\pm}(u^\varepsilon(x,t),\kappa) - \varepsilon\partial_x(\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa))^\pm\right] \partial_x\psi(x,t)\chi_{i,\delta}(x)dxdt$$
$$+ \int_0^T \int_{\Omega_i} G_{i\pm}(\overline{u}_i,\kappa)\psi(x,t)\partial_x\chi_{i,\delta}(x)dxdt \geq 0. \quad (6.22)$$

We denote by $n(i)$ the inward normal to Ω_i, that is $n(1) = -1$ and $n(2) = +1$. Then letting δ tend to 0 in (6.22) yields : $\forall\psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0,T))$, $\forall\kappa \in [0,1]$,

$$\int_0^T \int_{\Omega_i} (u^\varepsilon(x,t) - \kappa)^\pm \partial_t\psi(x,t)dxdt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0)dx$$
$$+ \int_0^T \int_{\Omega_i} \left[G_{i\pm}(u^\varepsilon(x,t),\kappa) - \varepsilon\partial_x(\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa))^\pm\right] \partial_x\psi(x,t)dxdt$$
$$+n(i) \int_0^T G_{i\pm}(\overline{u}_i,\kappa)\psi(0,t)dt \geq 0. \quad (6.23)$$

Let M_{g_i} be a Lipschitz constant of g_i, then

$$|G_{i\pm}(\overline{u}_i,\kappa)| \leq M_{g_i} (\overline{u}_i - \kappa)^\pm ,$$

and it follows from (6.23) that

$$\int_0^T \int_{\Omega_i} (u^\varepsilon(x,t) - \kappa)^\pm \partial_t\psi(x,t)dxdt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0)dx$$
$$+ \int_0^T \int_{\Omega_i} \left[G_{i\pm}(u^\varepsilon(x,t),\kappa) - \varepsilon\partial_x(\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa))^\pm\right] \partial_x\psi(x,t)dxdt$$
$$+M_{g_i} \int_0^T \int_{\Omega_i} (\overline{u}_i - \kappa)^\pm \psi(0,t)dt \geq 0. \quad (6.24)$$

We can now let ε tend to 0. We deduce from proposition 6.2.7 that, up to an extraction, for all $\kappa \in [0,1]$,

$$\varepsilon\partial_x(\varphi_i(u^\varepsilon) - \varphi_i(\kappa))^\pm \text{ tends to } 0 \quad \text{a.e. in } \Omega_i \times (0,T) \text{ as } \varepsilon \to 0, \tag{6.25}$$

and using $0 \le u^\varepsilon \le 1$, theorem 6.3.1 insures the existence of $u \in L^\infty(\mathbb{R} \times (0,T) \times (0,1))$ such that

$$(u^\varepsilon - \kappa)^\pm \to \int_0^1 (u(\cdot,\cdot,\alpha) - \kappa)^\pm d\alpha \quad \text{as } \varepsilon \to 0, \tag{6.26}$$

$$G_{i\pm}(u^\varepsilon, \kappa) \to \int_0^1 G_{i\pm}(u(\cdot,\cdot,\alpha), \kappa) d\alpha \quad \text{as } \varepsilon \to 0. \tag{6.27}$$

Letting ε tend to 0 in (6.24), using (6.25), (6.26) and (6.27) yields : $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times (0,T))$, $\forall \kappa \in [0,1]$,

$$\int_0^T \int_{\Omega_i} \int_0^1 (u(x,t,\alpha) - \kappa)^\pm \partial_t \psi(x,t) d\alpha dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0) dx$$
$$+ \int_0^T \int_{\Omega_i} \int_0^1 G_{i\pm}(u(x,t,\alpha),\kappa) \partial_x \psi(x,t) d\alpha dx dt + M_{g_i} \int_0^T \int_{\Omega_i} (\overline{u}_i - \kappa)^\pm \psi(0,t) dt \ge 0.$$

u is thus a process solution in the sense of definition 6.3. □

6.3.3 uniqueness of the (process) solution

It is clear that the notion of process solution is weaker than the one of solution given in definition 6.1. We state here a theorem which claims the equivalence of the two notions, that is any process solution is a solution in the sense of definition 6.1. Furthermore, such a solution is unique, and a L^1-contraction principle can be proven.

Theorem 6.3.3 (uniqueness of the (process) solutions) *There exists a unique process solution u to the problem $(\mathcal{P}_{\lim})$, and furthermore this solution does not depend on α, that is u is a solution to the problem $(\mathcal{P}_{\lim})$ in the sense of definition 6.1. Furthermore, if u_0, v_0 are two initial data in $L^\infty(\mathbb{R})$, and let u and v be two solutions associated to those initial data, then for almost every $t \in [0,T)$, for all $R > 0$,*

$$\int_{-R}^{R} (u(x,t) - v(x,t))^\pm dx \le \int_{-R-Mt}^{R+Mt} (u_0(x) - v_0(x))^\pm dx, \tag{6.28}$$

where $M \ge \max_i(M_{g_i})$.

The proof of this theorem can be done using the doubling variable method. The presence of the process variable α does not lead to any difficulties all along the proof. At the end, if u and $\tilde{u}$ are two process solutions associated to the same initial data u_0, we obtain a L^1-contraction principle of the following form : for a.e. $t \in [0,T)$,

$$\int_{\mathbb{R}} \int_0^1 \int_0^1 (u(x,t,\alpha) - \tilde{u}(x,t,\beta))^\pm d\alpha d\beta dx \le 0.$$

Thus $u = \tilde{u}$, and u does not depend on α. In the doubling variable method, the treatment of the boundary conditions has been performed by Felix Otto in his PhD Thesis, summarized in [Ott96a], and explained in [MNRR96]. We refer to this later reference and to [Vov02] for a complete proof of theorem 6.3.3.

Proposition 6.3.4 *Let $u_0 \in L^1(\mathbb{R})$ be a prepared initial data, and let u^ε be the corresponding solution to the approximate problem ($\mathcal{P}^\varepsilon$). Then u^ε converges in $L^p(\mathbb{R}\times(0,T))$, $1 \le p < \infty$ to the unique solution u to ($\mathcal{P}_{\lim}$) associated to initial data u_0.*

Proof

We have seen in proposition 6.3.2 that u^ε converges up to extraction to a process solution. Thanks to theorem 6.3.3, the process solution is unique, and is a solution to ($\mathcal{P}_{\lim}$) in the sense of definition 6.1. The family $(u^\varepsilon)_\varepsilon$ admits then a unique adherence value, which is a solution thanks to theorem 6.3.3, thus all the family converges toward this unique limit u in the non-linear L^∞-weak-$\star$ sense. One has

$$\int_0^T\int_{\mathbb{R}}(u^\varepsilon-u)^2dxdt = \int_0^T\int_{\mathbb{R}}(u^\varepsilon)^2dxdt - 2\int_0^T\int_{\mathbb{R}}u^\varepsilon u dxdt + \int_0^T\int_{\mathbb{R}}u^2dxdt.$$

Thanks to the theorem 6.3.1, since u^ε tends to u in the non-linear L^∞-weak-$\star$ sense, and since u does not depend on α, one has

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}}(u^\varepsilon)^2dxdt = \int_0^T\int_{\mathbb{R}}u^2dxdt,$$

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}}u^\varepsilon u dxdt = \int_0^T\int_{\mathbb{R}}u^2dxdt.$$

Thus we obtain

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}}(u^\varepsilon-u)^2dxdt = 0. \tag{6.29}$$

This particularly ensures that

$$u^\varepsilon \to u \quad \text{in } L^1_{loc}(\mathbb{R}\times[0,T]) \text{ as } \varepsilon\to 0. \tag{6.30}$$

Moreover, using $\kappa = 0$ and $\kappa = 1$ in (6.2), we can check that u is a weak solution to ($\mathcal{P}_{\lim}$), that is : $\forall \psi \in \mathcal{D}(\mathbb{R}\times[0,T))$,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}u(x,t)\partial_t\psi(x,t)dxdt + \int_{\mathbb{R}}u_0(x)\psi(x,0)dx \\ + \int_0^T\sum_{i=1,2}\int_{\Omega_i}g_i(u)(x,t)\partial_x\psi(x,t)dxdt = 0.\end{aligned} \tag{6.31}$$

Since g_i is a Lipschitz continuous function, $g_i(u) \in L^\infty((0,T);L^1(\Omega_i))$, and thus

$$\lim_{R\to\infty}\int_0^T\int_R^{R+1}g_i(u)(x,t)dxdt = 0.$$

Then, choosing $\psi(x,t) = \chi_{[0,t)}\min(1,(R+1-|x|)^+)$ in (6.31) yields, using (6.11) : $\forall t \in [0,T]$

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})} = \|u^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})}. \tag{6.32}$$

Let $R > 0$, we denote by B_R the interval $(-R,R)$, then from (6.30)

$$\int_0^T\int_{B_R}u^\varepsilon(x,t)dxdt \to \int_0^T\int_{B_R}u(x,t)dxdt \qquad \text{as } \varepsilon\to 0. \tag{6.33}$$

We deduce from (6.32) and (6.33) that

$$\int_0^T \int_{(B_R)^c} u^\varepsilon(x,t)dxdt \to \int_0^T \int_{(B_R)^c} u(x,t)dxdt \qquad \text{as } \varepsilon \to 0. \tag{6.34}$$

Let $\delta > 0$. Since $u \in L^1(\mathbb{R} \times (0,T))$, we can choose R large enough to ensure

$$\int_0^T \int_{B_R^c} u(x,t)dxdt \le \delta. \tag{6.35}$$

One has

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} |u^\varepsilon - u|dxdt \quad &\le \quad \int_0^T \int_{B_R} |u^\varepsilon - u|dxdt + \int_0^T \int_{(B_R)^c} u^\varepsilon dxdt \\ &\quad + \int_0^T \int_{(B_R)^c} u \; dxdt. \end{aligned} \tag{6.36}$$

We deduce from (6.34) and (6.35) that

$$\lim_{\varepsilon \to 0} \int_0^T \int_{(B_R)^c} u^\varepsilon dxdt \le \delta,$$

then letting ε tend to 0 in (6.36) provides, thanks to (6.30) :

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}} |u^\varepsilon - u|dxdt \le 2\delta.$$

This inequality holds for any $\delta > 0$, then we can claim

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}} |u^\varepsilon - u|dxdt = 0.$$

Since $0 \le u^\varepsilon, u \le 1$ a.e., one has $|u^\varepsilon - u|^p \le |u^\varepsilon - u|$ for all $p \in [1, \infty)$, and then

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}} |u^\varepsilon - u|^p dxdt = 0.$$

□

6.3.4 initial data u_0 in $L^\infty(\mathbb{R})$

The notion of bounded flux solution is too restrictive to provide existence for any u_0 in $L^\infty(\mathbb{R})$, but the existence and uniqueness frame can be extended to general initial data using density arguments.

Theorem 6.3.5 *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, and let u^ε be the unique SOLA corresponding to u_0. Then u^ε tends in $L^p((0,T); L^p_{loc}(\mathbb{R}))$ toward the unique solution u to ($\mathcal{P}_{\lim}$) as ε tends to* 0.

Proof
The extension using a density argument to initial data $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$ a.e., leads to non difficult but quite long calculations performed in the proof of theorem 5.3.4 in the previous chapter. This is based on the fact that the propagation speed for the limit problem is finite, because g_i, $i = 1, 2$, are Lipschitz continuous functions.

We will now give a much shorter proof, which only holds for integrable initial data $u_0 \in L^1(\mathbb{R})$. In such a case, we have global L^1-contraction principles (6.10) and (6.28).

Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, there exists thanks to lemma 6.2.1 a sequence $(u_{0,\eta})_\eta$ of prepared initial data tending to u_0. Let $(u^\varepsilon_\eta)_\eta$ be the associated family of solutions to the problem $(\mathcal{P}^\varepsilon)$. Thanks to (6.10), one has for almost every $t \in (0, T)$, for all $\eta, \delta > 0$,

$$\int_{\mathbb{R}} (u^\varepsilon_\eta(x,t) - u^\varepsilon_\delta(x,t))^\pm dx \le \int_{\mathbb{R}} (u_{0,\eta} - u_{0,\delta})^\pm dx.$$

We can thus build a Cauchy sequence in $L^\infty((0,T); L^1(\mathbb{R}))$, then there exists a unique u^ε limit of u^ε_η for $\eta \to 0$, and

$$\int_{\mathbb{R}} (u^\varepsilon_\eta(x,t) - u^\varepsilon(x,t))^\pm dx \le \int_{\mathbb{R}} (u_{0,\eta} - u_0)^\pm dx. \tag{6.37}$$

This u^ε fulfills the problem $(\mathcal{P}^\varepsilon)$ in a weaker sense, as it is explained in chapters 3 and 4. The function u^ε is furthermore the limit of the finite volume approximation introduced in chapter 4.

Let u (resp. u_η) be the solution to $(\mathcal{P}_{\lim})$ associated to u_0 (resp. $u_{0,\eta}$). We now aim to show that u^ε tends to u as ε tends to 0. For a.e. $t \in (0, T)$,

$$\begin{aligned}\int_{\mathbb{R}} |u(x,t) - u^\varepsilon(x,t)| dx \quad &\le \quad \int_{\mathbb{R}} |u(x,t) - u_\eta(x,t)| dx \\ &\quad + \int_{\mathbb{R}} |u_\eta(x,t) - u^\varepsilon_\eta(x,t)| dx \\ &\quad + \int_{\mathbb{R}} |u^\varepsilon_\eta(x,t) - u^\varepsilon(x,t)| dx.\end{aligned} \tag{6.38}$$

Using (6.28) and (6.37) in (6.38) yields

$$\begin{aligned}\int_{\mathbb{R}} |u(x,t) - u^\varepsilon(x,t)| dx \quad &\le \quad \int_{\mathbb{R}} |u_\eta(x,t) - u^\varepsilon_\eta(x,t)| dx \\ &\quad +2 \int_{\mathbb{R}} |u_{0,\eta}(x,t) - u_0(x,t)| dx.\end{aligned} \tag{6.39}$$

Letting ε tend to 0 in (6.39), it follows from proposition 6.3.4 that

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}} |u(x,t) - u^\varepsilon(x,t)| dx \le +2 \int_{\mathbb{R}} |u_{0,\eta}(x,t) - u_0(x,t)| dx. \tag{6.40}$$

Since (6.40) holds for any η, we can let η tend to 0 and we obtain

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} |u(x,t) - u^\varepsilon(x,t)| dx = 0.$$

□

6.4 Conclusion

We have proven in chapter 5 that under convenient assumptions, which can be mainly summarized as follow :

« either the capillary forces and the gravity are oriented in the same sense, or the global convection is larger than the one inducted by the gravity term, »

the solution to the approximate problem $(\mathcal{P}^\varepsilon)$ taking into account the discontinuous capillary forces at the interface converges toward the unique entropy solution to the problem.

In this chapter, we have set the assumptions 6.2.4, which particularly insure that the gravity forces and the capillary forces are oriented in the reverse sense at the interface. Under this assumptions, we have shown in theorem 6.3.5 that the solution u^ε to the approximate problem $(\mathcal{P}^\varepsilon)$ converges toward a weak solution u, which is entropic in each subdomain $\mathbb{R}^\star_\pm$, but which admits a non classical shock at the interface. Indeed, as we stressed it in section 6.1.1, the function $u = 1$ for $x < 0$ and $u = 0$ for $x > 0$ is not an entropy solution in the sense of definition 5.1, but it is clearly a steady solution to the problem $(\mathcal{P}_{\text{lim}})$.

It is also interesting to check that the limit u is unique, and that it can be easily computed using a Godunov scheme inside of each $\mathbb{R}^\star_\pm$, and the very simple Rusanov approximation introduced in chapter 5 at the interface.

The lack of estimates on the approximate solutions impeach us to extend this result to larger dimension. We would particularly loose the uniform bound on the flux, and so the comparison principle (6.10), and the $L^2((0,T);H^1_{loc}(\overline{\Omega}_i))$-estimate stated in proposition 6.2.7, which are key-points of the proof of theorem 6.3.5.

Chapitre 7

Conclusion et Perspectives

7.1 Un nouveau modèle de raccord aux interfaces

Dans ce manuscrit, nous avons étudié les écoulements en milieux poreux hétérogènes, et en particulier dans des milieux « discontinus », c'est à dire dont les caractéristiques physiques varient de manière discontinue par rapport à la variable d'espace. Ceci correspond au fait que différents types de roches, aux caractéristiques physiques très différentes, cohabitent au sein d'un même bassin sédimentaire. Nous n'avons pas considéré de variations « progressives »des grandeurs physiques, mais cela pourrait être rajouté en s'inspirant par exemple du travail de Chen [Che01] sans apport de difficultés majeures. Les caractéristiques physiques sont donc constantes par morceaux, ce qui revient à considérer le milieux poreux hétérogène Ω comme une apposition de différents milieux poreux homogènes Ω_i, $i = 1, \dots, N$.

A sein des sous-matrices poreuses circule un mélange non miscible d'eau et d'huile, et on suppose qu'il n'y a pas de gaz, si bien que les phases sont supposées incompressibles. Chacune des deux phases possède sa propre pression, p_o pour la phase huileuse et p_w pour la phase aqueuse. La pression capillaire, c'est à dire la différence entre les pressions de phase, est supposée ne dépendre que de la saturation en huile u et de la matrice poreuse Ω_i :

$$p_o - p_w = \pi_i(u),$$

où π_i est supposée lipschitzienne et strictement croissante.

Au niveau des interfaces entre les différents sous-domaines, la conservation de la masse impose le raccord des flux. S'il semble naturel aussi d'imposer le raccord des pressions de chaque phase, et donc des pressions capillaires, ceci n'est pas toujours possible du fait que la pression capillaire est une fonction donnée de la saturation $\pi_i(u)$ dans chaque sous-domaine Ω_i. Raccorder les pressions capillaires, au sens fort, à l'interface impose donc une relation de compatibilité sur les courbes de pression capillaire :

$$\pi_i(0) = \pi_j(0), \qquad \pi_i(1) = \pi_j(1).$$

Le cas où les conditions de compatibilité sont vérifiées est traité dans le premier chapitre en négligeant tous les termes de convection. On y a prouvé l'existence et l'unicité de la solution faible, moyennant une hypothèse sur les différents sous-milieux poreux, exigeant qu'ils aient des caractéristiques pas trop différentes qualitativement.

Supposons que u désigne la saturation de la phase huileuse, dire alors que u augmente et tend vers 1 revient à dire que l'huile envahit tous les pores du réseau poreux, en particulier des pores ayant des diamètres de plus en plus petits. La relation (1.1) liant pression capillaire et diamètre de pores incite à penser que la pression capillaire augmente, éventuellement beaucoup, lorsque u tend vers 1 pour permettre l'invasion des pores de plus en plus étroit. La limite $\lim_{u\to 1}\pi_i(u) = +\infty$ apparaît alors naturellement. Il semble aussi assez naturel que la pression capillaire ne soit pas univoque lorsque l'on se trouve avec une seule phase en présence, en raison de l'incompressibilité du fluide. On considère donc des graphes de pression capillaire $\tilde{\pi}_i$ définis par

$$\tilde{\pi}_i(u) = \begin{cases} \pi_i(u) & \text{if} \quad 0 < u < 1, \\ (-\infty, \pi_i(0)] & \text{if} \quad u = 0, \\ [\pi_i(1), +\infty) & \text{if} \quad u = 1, \end{cases}$$

et le raccord des pressions capillaires à l'interface $\Gamma_{i,j}$ entre Ω_i et Ω_j devient :

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset. \tag{7.1}$$

Cette manière de connecter les pressions capillaires enlève toute relation de compatibilité entre les fonctions π_i.

On a montré au chapitre 3 l'existence d'une solution faible pour un écoulement de type *dead-oil* (i.e. mélange eau huile, immiscible et incompressible) dans un tel milieu poreux hétérogène pour des conditions de type graphique. N'étant pas capable de démontrer l'unicité d'une telle solution faible dans le cadre multidimensionnel, nous avons cependant prouvé un résultat d'unicité dans le cas unidimensionnel. Celui-ci utilise la notion de solution à flux bornés, qu'il semble difficile d'adapter aux dimensions supérieures.

Si ce résultat d'existence d'une solution faible peut s'étendre sans difficulté dans le cas où on a une équation parabolique dégénérée dans chaque sous-domaine Ω_i

$$\phi_i \partial_t u + div\left(\vec{q} f_i(u) + (\rho_o - \rho_w)\vec{g} - \lambda_i(u)\vec{\nabla}\pi_i(u)\right) = 0.$$

avec les notations de l'introduction (1.17), et raccord des flux, et des pressions capillaires au sens graphique au niveau des interfaces, la prise en compte du problème complet semble plus délicate. En effet, on a alors un couplage entre une équation parabolique (1.13) et une équation elliptique (1.14)

$$\phi_i \partial_t u - div\left(\eta_{o,i}(u)(\vec{\nabla}\overline{P} - \rho_o\vec{g}) + \frac{\eta_{o,i}(u)\eta_{w,i}(u)}{\eta_{o,i}(u) + \eta_{w,i}(u)}\vec{\nabla}\pi_i(u)\right) = 0, \tag{7.2}$$

$$-div\left(\sum_{\beta=o,w}\eta_{\beta,i}(u)(\vec{\nabla}\overline{P} - \rho_\beta\vec{g})\right) = 0. \tag{7.3}$$

Les conditions de raccord associées à l'inconnue $\overline{P}$ à imposer au niveau des interfaces ne sont pas claires : si un raccord des flux induits par la deuxième équation à travers l'interface

$$\left(\sum_{\beta=o,w}\eta_{\beta,i}(u)(\vec{\nabla}\overline{P}-\rho_\beta\vec{g})\right)\cdot\vec{n}_i+\left(\sum_{\beta=o,w}\eta_{\beta,j}(u)(\vec{\nabla}\overline{P}-\rho_\beta\vec{g})\right)\cdot\vec{n}_j=0$$

est nécessaire , une condition supplémentaire de type « raccord des traces » doit être rajoutée.

7.2 Schémas numériques

Dans le chapitre 4, nous avons montré la convergence d'un schéma numérique de type Volumes Finis vers l'unique solution à flux bornés au problème unidimensionnel, voire l'unique SOLA pour des données initiales non régulières. La première partie de la démonstration de convergence, à savoir toute la partie 4.2, peut s'adapter très facilement au cadre multidimensionnel, toujours en considérant le débit total connu afin de négliger le couplage. Cette généralisation se fait de manière directe en utilisant des maillages admissibles (voir définition 2.3), mais il serait intéressant de la généraliser pour des maillages non structurés, les maillages industriels étant souvent directement inspirés de la géologie, et ne respectant pas les conditions d'admissibilité requises.

Une fois que les conditions de raccord sur la pression globale $\overline{P}$ seront trouvées, Il est probable que la convergence d'un schéma numérique pour le cas multidimensionnel ne posera pas de problème. Une autre idée serait de trouver les conditions de transmission phase par phase, afin d'utiliser le schéma amont des pétroliers [BJ91], [Mic01], [Enc04].

7.3 Pression capillaire indépendante de la saturation

La condition de raccord graphique des pressions capillaires aux interfaces (7.1) permet de considérer n'importe quelles fonctions π_i strictement croissantes et lipschitziennes, sans aucune condition de compatibilité. Comme les forces de capillarité sont souvent négligeables en comparaison aux forces de gravité, nous avons étudié la limite

$$\pi(u,x)\to P_c(x).$$

Ceci n'a été fait que dans le cas unidimensionnel, et dans des cas particuliers dans les chapitres 5 et 6.

L'influence de l'orientation des discontinuités de pression capillaire a alors une influence très importante sur le profil de solution, sélectionnant soit la solution entropique de la loi de conservation scalaire à coefficients discontinus (cf. [Tow00], [Tow01], [AG03], [SV03], [Bac05]) lorsque l'orientation des forces de capillarité à l'interface est la même que celle de la gravité, comme cela a été montré au chapitre 5, soit une solution faible, entropique partout à l'intérieur des sous-domaines Ω_i, et laissant apparaître un choc non classic au niveau de l'interface. Ce dernier cas donne des profils de saturation faisant très clairement penser au phénomène de piégeage des hydrocarbures.

Le contexte des solutions à flux bornés a été utilisé pour démontrer la convergence de la saturation vers une solution entropique ou non-entropique, et la preuve donnée ne peut donc pas être adaptée directement aux dimensions supérieures. Cependant, il semble possible (mais délicat) d'étudier un schéma Volumes Finis pour ce type de problèmes, et il serait très intéressant d'en comparer les résultats avec ceux donnés par les algorithmes de percolation des hydrocarbures prédisant la position des pièges à hydrocarbures, ainsi qu'avec des résultats provenant du modèle *Temis*, qui est un modèle beaucoup plus complet de résolution d'écoulements triphasiques en milieux anisotropes, hétérogènes.

Si l'étude de la convergence d'un schéma Volumes Finis est envisageable avec des techniques classiques pour la loi de conservation scalaire hyperbolique seule, le couplage ajoute une nouvelle difficulté due à la dépendance par rapport à u en général de

$$u \mapsto \sum_{\beta=o,w} \eta_{\beta,i}(u),$$

et la convergence ne pourra être prouvée (cf. [EG03]), une fois les difficultés liées au couplage d'une équation parabolique avec une équation elliptique surmontées, que dans des cas particuliers, déjà intéressants et étudiés par exemple dans [EG93] et [Vig96], où

$$\sum_{\beta=o,w} \eta_{\beta,i}(u) = K_i.$$

Annexe A

Annexe : On the time continuity of entropy solutions

A.1 The problem, and main result

Convection diffusion equations appear in a large class of problems, and have been widely studied. We consider in the sequel only equations under conservative form :

$$\partial_t u + \nabla \cdot F(u) - \Delta\phi(u) = b, \tag{A.1}$$

so that we can give some sense to (A.1) in the distributional sense. In this paper, we consider entropy solutions of (A.1) that do not take into account any boundary condition, or condition for $|x| \to +\infty$.

The proof does not use a L^1-contraction principle (see e.g. Alt & Luckaus [AL83] or Otto [Ott96b]), so that it can be applied in case where uniqueness is not insured, like for example complex spatial coupling of different conservation laws as in [CGP], or for cases where uniqueness fails because of boundary conditions or conditions at $|x| = +\infty$, as it will be stressed in the sequel.

Let us now state the required assumptions on the data. Let Ω be an open subset of $\mathbb{R}^d$ ($d \geq 1$), and let T be a positive real value or $+\infty$.

$$F \text{ is a continuous function,} \tag{H1}$$

$$\phi \text{ is a nondecreasing Lipschitz function,} \tag{H2}$$

$$u_0 \in L^1_{Loc}(\Omega). \tag{H3}$$

One has to make the following assumption on the source term :

$$b \in L^2_{Loc}([0,T); H^{-1}(\Omega)) \cap L^1_{Loc}(\Omega \times [0,T)). \tag{H4}$$

In the sequel, $v \top w$ (resp. $v \bot w$) denotes $\max(v,w)$ (resp. $\min(v,w)$), and sign is the function defined by

$$\operatorname{sign}(s) = \begin{cases} 0 & \text{if} \quad s = 0, \\ 1 & \text{if} \quad s > 0, \\ -1 & \text{if} \quad s < 0. \end{cases}$$

We consider entropy weak solutions of (A.1), as in the famous work of Kružkov [Kru70] for hyperbolic equations. This notion can be extended to degenerated parabolic equations, as noticed by Carrillo [Car99]. This leads to the following definition of entropy weak solution :

Definition A.1 *A function u is said to be an entropy weak solution if :*

1. $u \in L^1_{Loc}(\Omega \times [0,T))$,
2. $F(u) \in \left(L^2_{Loc}(\Omega \times [0,T))\right)^d$,
3. $\phi(u) \in L^2_{Loc}([0,T); H^1_{Loc}(\Omega))$,
4. $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$, $\forall \kappa \in \mathbb{R}$,

$$\int_0^T \int_\Omega |u-\kappa|\partial_t \psi dxdt + \int_\Omega |u_0 - \kappa|\psi(0)dx$$
$$+ \int_0^T \int_\Omega (F(u\top\kappa) - F(u\bot\kappa) - \nabla|\phi(u) - \phi(\kappa)|) \cdot \nabla\psi dxdt$$
$$+ \int_0^T \int_\Omega sign(u-\kappa) b\psi dxdt \geq 0. \tag{A.2}$$

Proposition A.1.1 *Any entropy weak solution is a weak solution, that is it fulfills the three first points in definition A.1, and : $\forall \psi \in \mathcal{D}(\Omega \times [0,T))$,*

$$\int_0^T \int_\Omega u\partial_t \psi dxdt + \int_\Omega u_0 \psi(0)dx$$
$$+ \int_0^T \int_\Omega (F(u) - \nabla\phi(u)) \cdot \nabla\psi dxdt + \int_0^T \int_\Omega b\psi dxdt = 0. \tag{A.3}$$

Reciprocally, if ϕ^{-1} is a continuous function, the any weak solution is an entropy solution.

Proof

Suppose first that ϕ^{-1} is a continuous function, then the fact that any weak solution u is an entropy weak solution is just based on a convexity inequality, and on the fact that $\text{sign}(\phi(a) - \phi(b)) = \text{sign}(a-b)$ for all $(a,b) \in \mathbb{R}^2$. More details are available in [Car99] (see also [GMT94]).

The fact that an entropy weak solution u is a weak solution is obvious if u belongs to $L^\infty_{Loc}(\Omega \times [0,T))$ (consider $\kappa = \pm\|u\|_{L^\infty(\text{supp}(\psi))}$).

Suppose now that u only belongs to $L^1_{Loc}(\Omega \times [0,T))$. Let $\kappa \in \mathbb{R}$, then for all $\psi \in \mathcal{D}(\Omega \times [0,T))$, one has

$$\int_0^T \int_\Omega \kappa\partial_t \psi dxdt + \int_\Omega \kappa\psi(0)dx = 0, \tag{A.4}$$

which added to (A.2) yields : $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega (|u-\kappa| + \kappa)\, \partial_t \psi dxdt + \int_\Omega (|u_0 - \kappa| + \kappa)\, \psi(0)dx$$
$$+ \int_0^T \int_\Omega \text{sign}(u-\kappa)\,(F(u) - \nabla\phi(u)) \cdot \nabla\psi dxdt$$
$$+ \int_0^T \int_\Omega \text{sign}(u-\kappa) b\psi dxdt \geq 0. \tag{A.5}$$

One will now let κ tend to $-\infty$ in (A.5). Suppose that $\kappa < 0$, then

$$\big||u-\kappa|+\kappa\big| \leq |u| \text{ and } \big||u-\kappa|+\kappa\big| \to u \text{ a.e. in supp}(\psi),$$

and the dominated convergence theorem gives : $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega u\partial_t\psi dxdt + \int_\Omega u_0\psi(0)dx + \int_0^T \int_\Omega (F(u) - \nabla\phi(u)) \cdot \nabla\psi dxdt + \int_0^T \int_\Omega b\psi dxdt \geq 0.$$

The same way, one has : $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega (|u-\kappa| - \kappa)\, \partial_t\psi dxdt + \int_\Omega (|u_0-\kappa| - \kappa)\, \psi(0)dx + \int_0^T \int_\Omega \text{sign}(u-\kappa)\, (F(u) - \nabla\phi(u)) \cdot \nabla\psi dxdt + \int_0^T \int_\Omega \text{sign}(u-\kappa) b\psi dxdt \geq 0.$$

Letting κ tend to $+\infty$, one gets : $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega u\partial_t\psi dxdt + \int_\Omega u_0\psi(0)dx + \int_0^T \int_\Omega (F(u) - \nabla\phi(u)) \cdot \nabla\psi dxdt + \int_0^T \int_\Omega b\psi dxdt \leq 0.$$

This insures that : $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega u\partial_t\psi dxdt + \int_\Omega u_0\psi(0)dx + \int_0^T \int_\Omega (F(u) - \nabla\phi(u)) \cdot \nabla\psi dxdt + \int_0^T \int_\Omega b\psi dxdt = 0. \tag{A.6}$$

It is now easy to check that (A.6) still holds for $\psi \in \mathcal{D}(\Omega \times [0,T))$, and so this achieves the proof of propostion A.1.1 □

Remark A.1.1 *In the case where $\phi \equiv 0$, the point 2 of definition A.1 can be replaced by*

$$F(u) \in \left(L^1_{Loc}(\Omega \times [0,T))\right)^d,$$

and one can remove the assumption $b \in L^2_{Loc}([0,T); H^{-1}(\Omega))$ in (H4). Actually, in such a case, Kružkov entropies $|\cdot - \kappa|$ are sufficient to obtain the time continuity. The assumptions $F(u) \in \left(L^2_{Loc}(\Omega \times [0,T))\right)^d$ and $b \in L^2_{Loc}([0,T); H^{-1}(\Omega))$ will only be useful to insure $\partial_t u$ belongs to $L^2_{Loc}([0,T); H^{-1}(\Omega))$ in order to recover the regular convex entropies, which are necessary to treat the parabolic case, as it was shown in the work of Carrillo [Car99].

The definition A.1 does not take into account any boundary condition, or condition at $|x| \to +\infty$. This lack of regularity can lead to non-uniqueness cases, as the one shown in the book of Friedman [Fri64] (also available in the one of Smoller [Smo94]) : the very simple problem

$$\begin{cases} \partial_t u - \partial_{xx}^2 u = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(\cdot, 0) = 0 & \text{in } \mathbb{R} \end{cases} \tag{A.7}$$

admits multiple classical solutions if one does not ask some condition for large x like e.g. $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$. Indeed, it is easy to check that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{2k!} x^{2k} \frac{d^k}{dt^k} e^{-1/t^2}$$

is a classical solution of (A.7). So u is a weak solution of (A.7), and thus an entropy weak solution thanks to proposition A.1.1. It also belongs to $C([0,T], L^1_{Loc}(\mathbb{R}))$, thanks to its regularity.

In the following theorem, we claim that any entropy solution is time continuous with respect with the time variable, at least locally with respect to the space variable.

Theorem A.1.2 *Let u be a entropy solution in the sense of definition A.1, then there exists $\overline{u}$ such that $u = \overline{u}$ a.e. on $\Omega \times [0,T)$ and fulfilling*

$$\overline{u} \in C([0,T); L^1_{Loc}(\Omega)).$$

Furthermore, if there exists $p > 1$ and a neighborhood $\mathcal{U}$ of $\partial\Omega$ in Ω such that

$$u_0 \in L^p_{Loc}(\mathcal{U}), \qquad u \in L^\infty_{Loc}([0,T); L^p_{Loc}(\mathcal{U})),$$

then we have :

$$\overline{u} \in C([0,T); L^1_{Loc}(\overline{\Omega})).$$

A.2 Essential continuity for $t = 0$

In this section, we give a simple way to prove the classical result stated in proposition A.2.1.

Definition A.2 *One says that $t \in [0,T)$ is a right-Lebesgue point if there exists $\overline{u}(t)$ in $L^1_{Loc}(\Omega)$ such that for all compact subset K of Ω,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \|u(s) - \overline{u}(t)\|_{L^1(K)} ds = 0.$$

We denote by $\mathcal{L}$ the set of right-Lebesgue points.

It is well known that $meas\,((0,T) \setminus \mathcal{L}) = 0$ and that $u = \overline{u}$ (in the $L^1_{Loc}(\Omega)$-sense) a.e. in $(0,T)$. In the sequel, we will prove that $\mathcal{L} = [0,T)$, and that $\overline{u}$ belongs to $C([0,T); L^1_{Loc}(\Omega))$. We begin by considering the essential continuity for the initial time $t = 0$.

Proposition A.2.1 *For all $\zeta \in \mathcal{D}^+(\Omega)$, one has :*

$$\lim_{\substack{t \to 0 \\ t \in \mathcal{L}}} \int_\Omega |\overline{u}(x,t) - u_0(x)|\zeta(x)dx = 0.$$

Particularly, this ensures that $0 \in \mathcal{L}$.

The limit as t tends to 0, $t \in \mathcal{L}$ can be seen as an essential limit, as it is done in lemma 7.41 in the book of Màlek et al. [MNRR96] in the case of a purely hyperbolic problem, or by Otto [Ott96b] in the case of a non strongly degenerated parabolic equation. See also the paper of Blanchard and Porretta [BP05] for the case of renormalized solutions for degenerate parabolic equations.

Proof

First, notice that for all $t \in \mathcal{L}$, and for all $\kappa \in \mathbb{R}$, t is also a right-hand side Lebesgue point of $|u-\kappa|$. Indeed, if K denotes a compact subset of $\overline{\Omega}$, one has for a.e $(x,s) \in \Omega \cap K \times (0,T)$

$$\big||u(x,s) - \kappa| - |u(x,t) - \kappa|\big| \leq |u(x,s) - u(x,t)|,$$

and so, for all $t \in \mathcal{L}$,

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_t^{t+\alpha} \int_{\Omega \cap K} \big||u(x,s) - \kappa| - |u(x,t) - \kappa|\big| dxds = 0. \tag{A.8}$$

Let $\alpha > 0$, and $t^\star \in \mathcal{L}$, one denotes

$$\chi^\alpha_{[0,t^\star[}(t) = \begin{cases} 1 & \text{if } t \leq t^\star \\ 0 & \text{if } t \geq t^\star + \alpha \\ \frac{t^\star + \alpha - t}{\alpha} & \text{if } t^\star < t < t^\star + \alpha. \end{cases}$$

Let $\zeta \in \mathcal{D}(\Omega)$, and let $\varepsilon > 0$ be such that $d(supp(\zeta), \partial\Omega) > \varepsilon$. Let $\rho \in \mathcal{D}^+(\mathbb{R}^d)$, with $supp(\rho) \subset B(0,1)$ and $\int_{\mathbb{R}^d} \rho(z)dz = 1$. One denotes $\rho_\varepsilon(z) = \frac{1}{\varepsilon^d}\rho(\frac{z}{\varepsilon})$. The function $y \mapsto \zeta(x)\rho_\varepsilon(x-y)$ belongs to $\mathcal{D}^+(\Omega)$.

Taking $\kappa = u_0(y)$ and $\psi(x,y,t) = \zeta(x)\rho_\varepsilon(x-y)\chi^\alpha_{[0,t^\star[}(t)$ in (A.2), an integrating with respect to $y \in \Omega$ yields :

$$\begin{aligned}
&\int_0^T \int_\Omega \int_\Omega |u(x,t) - u_0(y)|\zeta(x)\rho_\varepsilon(x-y)\partial_t \chi^\alpha_{[0,t^\star[}(t)dxdydt \\
&\quad + \int_\Omega \int_\Omega |u_0(x) - u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy \\
&+ \int_0^T \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega \left[\begin{array}{c} (F(u(x,t) \top u_0(y)) - F(u(x,t) \bot u_0(y))) \\ \cdot \nabla\left(\zeta(x)\rho_\varepsilon(x-y)\right) \end{array} \right] dxdydt \\
&- \int_0^T \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega \nabla|\phi(u(x,t)) - \phi(u_0(y))| \cdot \nabla\left(\zeta(x)\rho_\varepsilon(x-y)\right) dxdydt \\
&+ \int_0^T \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega \left[\begin{array}{c} \mathrm{sign}(u(x,t) - u_0(y))b(x,t) \\ \zeta(x)\rho_\varepsilon(x-y) \end{array} \right] dxdydt \geq 0,
\end{aligned} \tag{A.9}$$

where all the gradient are considered with respect to x, and not y.

One has

$$|u(x,t)-u_0(y)| = |u(x,t)-u_0(x)| + |u(x,t)-u_0(y)| - |u(x,t)-u_0(x)|,$$

then, since $\int_{\mathbb{R}^d} \rho_\varepsilon(x-y)dy = 1$ for all x in supp(ζ), using

$$|u_0(x)-u_0(y)| \geq \big||u(x,t)-u_0(y)| - |u(x,t)-u_0(x)|\big|,$$

we obtain

$$\begin{aligned} &\int_0^T \partial_t \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega |u(x,t)-u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdydt \\ &\quad\leq \int_0^T \partial_t \chi^\alpha_{[0,t^\star[}(t) \int_\Omega |u(x,t)-u_0(x)|\zeta(x)dxdt \\ &+\|\partial_t \chi^\alpha_{[0,t^\star[}\|_{L^1(0,T)} \int_\Omega \int_\Omega |u_0(x)-u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy. \end{aligned} \tag{A.10}$$

For all $\alpha \in]0, T-t^\star]$,

$$\|\partial_t \chi^\alpha_{[0,t^\star[}\|_{L^1(0,T)} = 1,$$

and then, one can let α tend to 0 in (A.10), so that (A.9) implies :

$$\begin{aligned} &-\int_\Omega \int_\Omega |\overline{u}(x,t^\star)-u_0(x)|\zeta(x)dxdy \\ &+2\int_\Omega \int_\Omega |u_0(x)-u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy + \int_0^{t^\star} \mathcal{R}_\varepsilon(t)dt \geq 0, \end{aligned} \tag{A.11}$$

where $\mathcal{R}_\varepsilon$ belongs to $L^1(0,T)$ for all $\varepsilon > 0$. Since $\mathcal{L}$ is dense in $[0,T]$, one can let in a first step $t^\star$ tend to 0, so that $\int_0^{t^\star} \mathcal{R}_\varepsilon(t)dt$ vanishes :

$$\begin{aligned} &\limsup_{\substack{t^\star \to 0 \\ t^\star \in \mathcal{L}}} \int_\Omega \int_\Omega |\overline{u}(x,t^\star)-u_0(x)|\zeta(x)dxdy \\ &\leq 2\int_\Omega \int_\Omega |u_0(x)-u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy. \end{aligned} \tag{A.12}$$

One can now let ε tend to 0, and using the fact that u_0 belongs to $L^1_{Loc}(\Omega)$, and that ζ is compactly supported in Ω, one gets :

$$\lim_{\substack{t^\star \to 0 \\ t^\star \in \mathcal{L}}} \int_\Omega \int_\Omega |\overline{u}(x,t^\star)-u_0(x)|\zeta(x)dxdy = 0.$$

This achieves the proof of proposition A.2.1. □

A.3 Time continuity for any $t \geq 0$

In this section, we want to prove the following proposition :

Proposition A.3.1 *Let u be a entropy solution in the sense of definition A.1, then there exists $\overline{u}$ such that $u = \overline{u}$ a.e. on $\Omega \times (0,T)$ and fulfilling*

$$\overline{u} \in C([0,T); L^1_{Loc}(\Omega)).$$

In the sequel, we still denote by $\overline{u}$ the representative defined using the right Lebesgue points introduced in definition A.2. Proving the essential continuity for every $t^\star \in \mathcal{L}$ is easy. Indeed, if one replaces $\psi(x,t)$ by $(1-\chi^\alpha_{[0,t^\star[})(t)\psi(x,t)$ in (A.2), and then if one lets α tend to 0, one gets :

$$\begin{aligned} \int_{t^\star}^T \int_\Omega |u-\kappa|\partial_t\psi dxdt + \int_\Omega |\overline{u}(t^\star)-\kappa|\psi(t^\star)dx \\ + \int_{t^\star}^T \int_\Omega (F(u\top\kappa) - F(u\bot\kappa) - \nabla|\phi(u)-\phi(\kappa)|)\cdot\nabla\psi dxdt \\ + \int_{t^\star}^T \int_\Omega \mathrm{sign}(u-\kappa)b\psi dxdt \geq 0. \end{aligned} \tag{A.13}$$

One can thus apply the proposition A.2.1 with $t^\star$ instead of 0, and $\overline{u}(t^\star)$ instead of u_0 : $\forall\zeta \in \mathcal{D}^+(\Omega)$,

$$\lim_{\substack{s^\star \to t^\star \\ s^\star \in \mathcal{L}}} \int_\Omega\int_\Omega |\overline{u}(x,s^\star) - \overline{u}(x,t^\star)|\zeta(x)dxdy = 0.$$

We will prove the uniform continuity of $t \mapsto \overline{u}(t)$ from $\mathcal{L}\cap[0,T-\gamma]$ to $L^1_{Loc}(\Omega)$ for all $\gamma \in (0,T)$. This will give as a direct consequence that $\mathcal{L} = [0,T)$ and $\overline{u} \in C([0,T); L^1_{Loc}(\Omega))$. This uniform continuity will come from theorem 13 in the paper of Carrillo [Car99], which, adapted to our case, can be stated as follow :

Theorem A.3.2 *Suppose that (H1), (H2) hold. Let u_0, v_0 belong to $L^1_{Loc}(\Omega)$, let b_u, b_v belong to $L^2((0,T);H^{-1}(\Omega)) \cap L^1((0,T);L^1_{Loc}(\Omega))$, and let u, v be two entropy solutions associated to the choice of $b = b_u$ and initial data u_0 for u and $b = b_v$ and initial data v_0 for v in definition A.1. Then $\forall\psi \in \mathcal{D}^+(\Omega\times[0,T[)$,*

$$\begin{aligned} \int_0^T \int_\Omega |u-v|\partial_t\psi dxdt + \int_\Omega |u_0-v_0|\psi(0)dx \\ \int_0^T \int_\Omega (F(u\top v) - F(u\bot v) - \nabla|\phi(u)-\phi(v)|)\cdot\nabla\psi dxdt \\ + \int_0^T \int_\Omega sign(u-v)(b_u-b_v)\psi dxdt \geq 0. \end{aligned} \tag{A.14}$$

We now have all the tools for the proof of proposition A.3.1.

Proof of proposition A.3.1

Let $\gamma > 0$, let $t^\star \in \mathcal{L}_\gamma = \mathcal{L}\cap[0,T-\gamma]$, and $h \in \mathcal{L}_\gamma$ such that $t^\star + h \in \mathcal{L}_\gamma$ (this is the case of almost every $h \in (0, T-t^\star-\gamma)$). Let $\zeta \in \mathcal{D}^+(\Omega)$, let $\alpha \in]0, T-t^\star-\gamma-h[$.

Taking $\psi(x,t) = \zeta(x)\chi^{\alpha}_{[0,t^\star[}(t)$, $v_0(x) = u(x,h)$, $v(x,t) = v(x,t+h)$ in (A.14), and letting α tend to 0 yields :

$$\begin{aligned}
&-\int_\Omega |\overline{u}(x,t^\star) - \overline{u}(x,t^\star+h)|\zeta(x)dx + \int_\Omega |u_0(x) - \overline{u}(x,h)|\zeta(x)dx \\
&\int_0^{t^\star}\int_\Omega \left[\begin{array}{c} F(u(x,t)\top u(x,t+h)) - F(u(x,t)\bot u(x,t+h)) \\ -\nabla|\phi(u(x,t)) - \phi(u(x,t+h))| \end{array}\right]\cdot\nabla\zeta(x)dxdt \\
&\quad + \int_0^{t^\star}\int_\Omega \left[\begin{array}{c} \operatorname{sign}(u(x,t) - u(x,t+h)) \\ (b(x,t) - b(x,t+h)) \end{array}\right]\zeta(x)dxdt \ge 0.
\end{aligned} \tag{A.15}$$

We deduce from (A.15) that

$$\begin{aligned}
&\int_\Omega |\overline{u}(x,t^\star) - \overline{u}(x,t^\star+h)|\zeta(x)dx \le \int_\Omega |u_0(x) - \overline{u}(x,h)|\zeta(x)dx \\
&+\int_0^{T-\gamma-h}\int_\Omega |F(u(x,t)\top u(x,t+h)) - F(u(x,t)\bot u(x,t+h))|\,|\nabla\zeta(x)|dxdt \\
&\quad +\int_0^{T-\gamma-h}\int_\Omega |\nabla\phi(u)(x,t+h) - \nabla\phi(u)(x,t)|\,|\nabla\zeta(x)|dxdt \\
&\quad\quad +\int_0^{T-\gamma-h}\int_\Omega |b(x,t+h) - b(x,t)|\,\zeta(x)dxdt,
\end{aligned}$$

and since $F(u), \nabla\phi(u)$ and b belong to $L^1_{Loc}(\Omega\times(0,T))$, one can claim that :

$$\begin{aligned}
&\forall\varepsilon>0, \forall t^\star\in\mathcal{L}_\gamma, \exists\eta>0 \ s.t.\ \forall h\in\mathcal{L}\cap[0,T-\gamma-t^\star], h\le\eta \Rightarrow \\
&\int_\Omega |\overline{u}(x,t^\star) - \overline{u}(x,t^\star+h)|\zeta(x)dx \le \int_\Omega |u_0(x) - \overline{u}(x,h)|\zeta(x)dx + \varepsilon.
\end{aligned} \tag{A.16}$$

One can now use proposition A.2.1 in (A.16), so that we get that

$$t\mapsto\overline{u}(x,t) \text{ is uniformly continuous from } \mathcal{L} \text{ to } L^1(\Omega,\zeta),$$

which is the L^1-space for measure of density ζ w.r.t. Lebesgue measure. We deduce that, for all $\gamma\in(0,T)$, $t\mapsto\overline{u}$ is uniformly continuous from $\mathcal{L}_\gamma$ to $L^1_{Loc}(\Omega)$, and this insures that $\mathcal{L}_\gamma = [0, T-\gamma]$. This holds for any $\gamma\in(0,T)$, and so we can claim that $\overline{u}\in C([0,T);L^1_{Loc}(\Omega))$. ■

It remains to prove the last part of theorem A.1.2 by considering some test functions $\zeta\in\mathcal{D}^+(\overline{\Omega})$ instead of $\zeta\in\mathcal{D}^+(\Omega)$. We will need some additional regularity on the solution :

$$\left\{\begin{array}{c} \text{There exists an open neighborhood } \mathcal{U} \text{ of } \partial\Omega \text{ in } \overline{\Omega} \text{ s.t.} \\ u_0\in L^p_{Loc}(\mathcal{U}), \qquad u\in L^\infty_{Loc}([0,T);L^p_{Loc}(\mathcal{U})). \end{array}\right\} \tag{H5}$$

(H5) gives the uniform (w.r.t. t) local equiintegrability of u (and so of $\overline{u}$) on a neighborhood of $\mathcal{U}$. We deduce, using $\overline{u}\in C([0,T);L^1_{Loc}(\Omega))$ that $\overline{u}\in C([0,T);L^1_{Loc}(\overline{\Omega}))$.

End of the proof of theorem A.1.2

Suppose that (H1),(H2),(H3),(H4) hold, then thanks to proposition A.3.1, there exists a weak solution $\overline{u} \in C([0,T), L^1_{Loc}(\Omega))$.
For $\varepsilon > 0$, $\gamma \in (0,T)$, $\zeta \in \mathcal{D}^+(\Omega)$, there exists $\eta > 0$ such that : $\forall t \in [0, T-\gamma]$, $\forall h \in [0, \min(\eta, T-t-\gamma)]$,

$$\int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)dx \leq \varepsilon.$$

Let K be a compact subset of $\overline{\Omega}$. Then there exists $\zeta \in \mathcal{D}^+(\overline{\Omega})$ such that $0 \leq \zeta(x) \leq 1$ for all $x \in \mathbb{R}^d$, and $\zeta(x) = 1$ if $x \in \overline{\Omega}$. Let $\alpha > 0$ and let $\beta_\alpha \in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that :

$$\begin{array}{ll} 0 \leq \beta_\alpha(x) \leq 1 & \text{for all } x \in \mathbb{R}^d, \\ \beta_\alpha(x) = 1 & \text{if } d(x,\partial\Omega) \leq \alpha/2, \\ \beta_\alpha(x) = 0 & \text{if } d(x,\partial\Omega) \geq \alpha. \end{array}$$

Suppose that (H5) holds. For α small enough, one has $supp(\zeta\beta_\alpha) \subset \mathcal{U}$ and then, for all $t \in [0, T-\gamma]$, for all $h \in [0, T-t-\gamma]$,

$$\int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)\beta_\alpha dx \leq 2\|u\|_{L^\infty((0,T-\gamma);L^p(\mathcal{U}_\zeta))}\|\beta_\alpha\|_{L^{p'}(\mathcal{U}_\zeta)},$$

where $\mathcal{U}_\zeta$ denotes $\mathcal{U} \cap supp(\zeta)$, and $p' = \frac{p}{p-1} < +\infty$. Since $\|\beta_\alpha\|_{L^{p'}(\mathcal{U}_\zeta)}$ tends to 0 as α tends to 0, there exists $\delta > 0$ such that :

$$\alpha \leq \delta \Rightarrow \int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)\beta_\alpha dx \leq \varepsilon. \tag{A.17}$$

Suppose now that α has been chosen such that (A.17) holds. The function $\zeta(1-\beta_\alpha)$ belongs to $\mathcal{D}^+(\Omega)$, and then there exists η such that :$\forall t \in [0, T-\gamma]$, $\forall h \in [0, \min(\eta, T-\gamma-t)]$,

$$\int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)(1-\beta_\alpha(x))dx \leq \varepsilon. \tag{A.18}$$

Adding (A.17) and (A.18) shows that for all t in $[0, T-\gamma-\eta]$, for all $h \in [0,\eta]$,

$$\int_K |\overline{u}(x,t+h) - \overline{u}(x,t)|dx \leq 2\varepsilon. \tag{A.19}$$

So $\overline{u}$ is uniformly continuous from $[0, T-\gamma]$ to $L^1(K)$, and then

$$\overline{u} \in C([0,T); L^1_{Loc}(\overline{\Omega})).$$

■

To conclude this paper, let us give a counter-example to the time continuity in the case where the entropy criterion is not fulfilled for t=0. Consider the Burgers equation, in the one dimensional case, leading to the following initial value problem.

$$\begin{cases} \partial_t u - \partial_x \left(u^2\right) = 0, & (x,t) \in (\mathbb{R} \times \mathbb{R}_+), \\ u(\cdot,0) = u_0 = 0. \end{cases} \tag{A.20}$$

Problem (A.20) admits $u = 0$ as unique entropy solution in the sense of definition A.1.

We define

$$\tilde{u}(x,t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ 0 & \text{if} \quad |x| > \sqrt(t), \\ \dfrac{x}{2t} & \text{if} \quad |x| < \sqrt(t). \end{cases}$$

Then it is easy to check that :

- $\tilde{u} \in L^1_{Loc}(\mathbb{R} \times \mathbb{R}_+)$,
- $\tilde{u}^2 \in L^1_{Loc}(\mathbb{R} \times \mathbb{R}_+)$,
- $\forall \psi \in \mathcal{D}(\Omega \times \mathbb{R}_+)$,

$$\int_0^{+\infty} \int_{\mathbb{R}} \tilde{u}(x,t) \partial_t \psi(x,t) dx dt + \int_0^{+\infty} \int_{\mathbb{R}} \tilde{u}^2(x,t) \partial_x \psi(x,t) dx dt = 0, \tag{A.21}$$

- $\forall \psi \in \mathcal{D}^+(\Omega \times \mathbb{R}_+^\star)$, $\forall \kappa \in \mathbb{R}$,

$$\int_0^{+\infty} \int_{\mathbb{R}} |\tilde{u} - \kappa|(x,t) \partial_t \psi(x,t) dx dt + \int_0^{+\infty} \int_{\mathbb{R}} \text{sign}(\tilde{u} - \kappa) \tilde{u}^2(x,t) \partial_x \psi(x,t) dx dt = 0. \tag{A.22}$$

Thanks to (A.21), $\tilde{u}$ is a weak solution of (A.20), and an entropy criterion (A.22) is fulfilled only for $t > 0$. The fact that the entropy criterion fails for $t > 0$, and that the flux $\tilde{u}^2$ is not bounded (see [CR00]) allows the function $\tilde{u}$ to be discontinuous at $t = 0$. Indeed, for all $t > 0$,

$$\|\tilde{u}(\cdot,t)\|_{L^1(\mathbb{R})} = \frac{1}{2} \neq \|u_0\|_{L^1(\mathbb{R})} = 0.$$

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