Elliptic problems in unbounded domains: an approach in weighted Sobolev spaces

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Introduction

De nombreux problèmes en mathématique physique et particulièrement en dynamique des fluides sont modélisés par des équations aux dérivées partielles dans des domaines non bornés. L'étude de ces problèmes passe d'abord par la résolution d'équations portant sur des opérateurs différentiels linéaires de base. Le but de cette thèse est d'étudier les opérateurs de Laplace et de Stokes dans différentes géométries non bornées en donnant des résultats d'existence, d'unicité et de régularité.
Nous connaissons les domaines non bornés les plus classiques et les plus étudiés comme, en premier lieu, l'espace $\mathbb{R}^n$ tout entier ou bien un domaine extérieur, *i.e.* le complémentaire d'un compact dans $\mathbb{R}^n$, ou encore le demi espace supérieur $\mathbb{R}^n_+$. Ici nous souhaitons étudier d'autres géométries non bornées, plus particulières, pour lesquelles on ne connaît pas ou peu de résultats, à plus forte raison dans le cadre fonctionnel que nous avons choisi. Parmi elles, une géométrie mixte, *i.e.* un "mélange" de géométries classiques (domaine extérieur et demi-espace) ou encore des géométries originales comme un demi-espace perturbé ou un "aperture domain".

De plus, si nous avons opté pour des conditions aux limites non homogènes assez classiques, à savoir de type Dirichlet ou Neumann, nous regardons également, pour l'opérateur de Laplace, le cas de conditions mixtes, *i.e.* avec une condition de type Dirichlet sur une partie du bord (bornée ou non suivant les cas) et une condition de Neumann sur l'autre partie (bornée ou non), les deux parties étant disjointes.

Le fait d'étudier des domaines non bornés conduit, au contraire du cas borné, à décrire le comportement à l'infini des données et éventuellement des solutions. De plus, dans la plupart des cas où nous nous plaçons, la frontière elle-même n'est pas bornée ce qui implique de devoir définir des espaces de traces permettant, là encore, de décrire le comportement à l'infini de leurs éléments.

Plusieurs cadres fonctionnels ont été proposés pour résoudre ces problèmes en domaine non borné. Il y a par exemple le complété de $\mathcal{D}(\Omega)$ pour la norme L^p du gradient ou bien encore le sous espace des fonctions localement L^p dont le gradient est L^p. Mais ces espaces présentent plusieurs inconvénients. En effet, pour le premier, il existe, dans le cas où $p \geq n$ et $\Omega = \mathbb{R}^n$, des suites de Cauchy dans $\mathcal{D}(\mathbb{R}^n)$ qui ne convergent pas vers des distributions. Ce comportement a été décrit par Deny et Lions en 1954 (voir [20]). Remarquons que dans le cadre fonctionnel que nous choisissons, ces suites seront éliminées car nous équipons nos espaces de la norme complète et pas seulement de la norme du gradient. De plus, cela nous permet d'éviter l'imprécision à l'infini de la seconde approche, imprécision inhérente à la définition de la norme L^p_{loc}.

L'approche que nous choisissons ici est celle des espaces de Sobolev avec poids $W^{m,p}_{\alpha,\beta}(\Omega)$. Ces espaces sont des extensions des espaces de Sobolev classiques, munis de poids qui permettent de contrôler la croissance ou la décroissance des fonctions à l'infini. Ce sont des espaces qui présentent l'avantage de donner des informations sur les fonctions elles-mêmes, en plus de leurs dérivées.

Outre la description à l'infini des fonctions, la raison fondamentale de l'ajout des poids et donc d'une généralisation des espaces de Sobolev classiques provient des inégalités de type Poincaré. En effet, dans les espaces classiques et pour des domaines non bornés, elles ne sont plus vérifiées. Or, ces inégalités de Poincaré jouent un rôle clé dans les méthodes variationnelles permettant de résoudre des problèmes aux limites elliptiques. Il est donc judicieux d'introduire des poids qui nous permettent de les retrouver. Le premier poids que l'on introduit (voir [30]) est :

$$\rho = (1 + |\boldsymbol{x}|^2)^{1/2}.$$

Il apparaît de façon naturelle dans des inégalités de Hardy qui elles-mêmes sont fondamentales pour établir les inégalités de type Poincaré. Notons également que, pour certaines valeurs critiques de n et p, l'introduction du poids ρ est insuffisante pour établir ces inégalités. Il convient donc de rajouter un facteur logarithmique $lg\ \rho$, défini par

$$lg\ \rho = ln(2 + |\boldsymbol{x}|^2)$$

pour lever partiellement ces restrictions. Nous pouvons remarquer ici que l'ajout des constantes dans la définition des poids a pour but de ne pas modifier le comportement à l'origine des espaces de Sobolev avec poids. Leurs propriétés locales sont donc les mêmes que celles des espaces de Sobolev classiques. De nombreux auteurs ont étudié ces espaces sans poids logarithmique. Nous pouvons citer par exemple Hanouzet [30], Kudrjavcev [36], Kufner [37], Kufner et Opic [38] ou Avantaggiati [12]. Par contre, comparativement, peu d'entre eux ont étudié l'espace complet avec poids logarithmique. Nous pouvons néanmoins nommer Lizorkin [43], Leroux [40] et Giroire [27].

Nous voudrions maintenant revenir aux différents problèmes que l'on étudie pour diverses géométries dans le cadre de cette thèse. En premier lieu, nous nous intéressons à l'opérateur de base, à savoir le Laplacien et ce, dans la géométrie d'un domaine extérieur, *i.e.* le complémentaire d'un domaine compact dans $\mathbb{R}^n$. De nombreux auteurs se sont penchés sur ce problème avec des conditions aux bords de type Dirichlet ou bien de type Neumann. Nous pouvons citer Leroux [40] et [41], Giroire et Nedelec [28], Nedelec [47] ou bien Nedelec et Planchard [48] mais aussi Hsiao et Wendland [33] qui ont étudié le problème avec une condition de Dirichlet en dimension 2, Cantor [17] qui a résolu les problèmes de Dirichlet et de Neumann pour certaines valeurs de p et de n ($n \geq 3$ et $p > \dfrac{n}{n-2}$) et pour certains poids ou encore Giroire [27] qui a établit des isomorphismes en dimension 2 et 3 pour un large éventail de poids. Enfin, nous citons les travaux de Amrouche, Girault et Giroire [7] que nous rappelons également dans la section 2.2 et qui fournissent des résultats pour des problèmes de Dirichlet et de Neumann pour toute dimension, là encore avec un large éventail de poids. Dans ce premier travail, nous souhaitons, en nous basant sur [7], considérer le cas où il y a à la fois une condition aux limites de type Dirichlet et une autre de type Neumann. Nous pouvons noter que nous considérons juste un cas particulier de "mixité", en supposant que les surfaces où les conditions de Dirichlet et de Neumann sont données sont disjointes et qu'il pourrait également être intéressant d'étudier le cadre général, quand il y a une frontière commune. Ici, nous définissons donc deux compacts disjoints non vides ω_0 et ω_1 de frontière (lipschitzienne ou de classe $C^{1,1}$ suivant les cas) Γ_0 et Γ_1 et nous voulons établir des résultats d'existence, d'unicité et de régularité pour le problème suivant dans $\Omega = {}^c(\omega_0 \cup \omega_1)$, problème que nous appelons **problème mixte extérieur à deux corps** :

$$(\mathcal{P}) \begin{cases} -\Delta u = f & \text{dans } \Omega, \\ u = g_0 & \text{sur } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{sur } \Gamma_1. \end{cases}$$

Dans un second temps, nous voulons rester dans ce même type d'approche, *i.e.* pour des problèmes avec des conditions de type Dirichlet-Neumann mais cette fois-ci, nous ne voulons plus considérer deux frontières compactes mais une compacte et une autre non compacte, à savoir $\mathbb{R}^{n-1}$. Pour ce type de géométrie, nous regardons d'abord le premier cas, où l'on a sur les deux parties de la frontière des conditions aux limites de type Dirichlet ou bien Neumann puis nous regardons les deux types de mixité, suivant le fait que la condition de type Dirichlet (resp. Neumann) se trouve sur la partie borné ou non bornée de la frontière. Donc, pour ω_0 un compact inclus dans $\mathbb{R}^n_+$, nous voulons établir des résultats d'existence, d'unicité et de régularité dans $\Omega = {}^c\omega_0 \cap \mathbb{R}^n_+$ domaine que l'on appelle **domaine extérieur dans le demi-espace**, pour les problèmes suivants :

$$(\mathcal{P}_D) \ -\Delta u = f \text{ dans } \Omega, \quad u = g_0 \text{ sur } \Gamma_0, \quad u = g_1 \text{ sur } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_N) \ -\Delta u = f \text{ dans } \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_0 \text{ sur } \Gamma_0, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \text{ sur } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_{M_1}) \ -\Delta u = f \text{ dans } \Omega, \quad u = g_0 \text{ sur } \Gamma_0, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \text{ sur } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_{M_2}) \ -\Delta u = f \text{ dans } \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_0 \text{ sur } \Gamma_0, \quad u = g_1 \text{ sur } \mathbb{R}^{n-1}.$$

Une manière de voir cette géométrie est de la considérer comme l'association d'un domaine extérieur classique et du demi-espace. La difficulté vient ici du fait que la frontière n'est pas bornée. Il convient donc de définir des espaces de trace avec poids permettant de décrire le comportement des fonctions à l'infini. Nous pouvons citer à ce sujet les travaux de Hanouzet [30] puis ceux d'Amrouche et Nečasová [8] qui ont étendu cette première définition aux espaces avec poids logarithmiques. Nous renvoyons aux résultats de Boulmezaoud [15], Maz'ya, Plamanevskii et Stupyalis [44], Simader et Sohr [49] et donc Amrouche et Nečasová [8] (présentés dans la section 2.3) pour l'équation de Laplace dans le demi-espace avec condition de Dirichlet, ainsi qu'aux résultats d'Amrouche [4] pour le même problème avec une condition de Neumann.

Après s'être intéressés au Laplacien, nous voulons étudier l'opérateur de Stokes en considérant toujours des géométries non bornées, possédant également des frontières non bornées. Nous nous intéressons en premier lieu au même domaine que précédemment, à savoir un **domaine extérieur dans le demi-espace** et nous étudions le problème suivant, avec des conditons aux limites de type Dirichlet :

$$(\mathcal{S}_{\mathcal{D}}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \\ \boldsymbol{u} = \boldsymbol{g_1} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Ici encore, cette géométrie, mélange de domaine extérieur classique et de demi-espace, est assez nouvelle mais nous pouvons rappeler des résultats précédemment établis pour les deux géométries plus classiques. Tout d'abord, pour un domaine extérieur, citons quelques travaux dans un autre cadre fonctionnel que celui des espaces de Sobolev avec poids, ceux de Kozono et Sohr [34] et [35] et ceux de Galdi et Simader [24] qui établissent des résultats d'existence et d'unicité pour un champ de vitesse dans le complété de $\mathcal{D}(\Omega)$ pour la norme L^p du gradient et pour un champ de pression dans L^p. En ce qui concerne les espaces de Sobolev avec poids, nous renvoyons à Girault et Sequeira [26] (dans les cas $n = 2$ ou $n = 3$, $p = 2$ et $\alpha = 0$), à Specovius et Neugebauer ([52] quand $n \geq 3$ et $\dfrac{n}{p} + \alpha \notin \mathbb{Z}$ pour des solutions fortes et [53] quand $n = 2$ et $\dfrac{2}{p} + \alpha \notin \mathbb{Z}$ pour des solutions faibles) ainsi qu'à Alliot et Amrouche [3]. Pour le problème de Stokes dans le demi-espace, nous pouvons citer Cattabriga [18], Farwig et Sohr [22] et Galdi [23] pour une approche dans les espaces homogènes ou bien Maz'ya, Plamanevskii et Stupyalis [44] (en dimension 3) et Amrouche, Nečasová et Raudin [9] (voir la section 2.5) pour une approche dans les espaces de Sobolev avec poids.

Enfin, dans un dernier temps, nous nous intéressons à d'autres géométries non bornées avec frontière non bornée, pour l'opérateur de Stokes. Nous nous concentrons sur deux domaines particuliers que l'on peut retrouver dans la littérature. Le premier est un demi-espace perturbé *i.e.* un domaine obtenu par une perturbation locale du demi-espace. Le second est un “aperture domain” *i.e.* deux demi-espaces séparés par un mur d'épaisseur $d > 0$ et reliés par un trou. Pour ces domaines, nous voulons donc établir des résultats d'existence, d'unicité et de régularité pour le problème de Stokes suivant :

$$(\mathcal{S}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{dans } \Omega, \\ \text{div } \boldsymbol{u} = h & \text{dans } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{sur } \partial\Omega. \end{cases}$$

Nous remarquons ici, comme l'a montré Heywood ([31] et [32]), l'importance, dans le cas de l'aperture domain, de la conditon de flux

$$\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma,$$

en tant que condition supplémentaire afin d'obtenir l'unicité de la solution. Notons que M est une hypersurface au niveau du trou qui sépare Ω en un domaine supérieur Ω_+ et un domaine inférieur Ω_-. Nous renvoyons également aux travaux de Borchers et Pileckas [14], Galdi [23], Ladyzhenskaya et Solonnikov [39], Solonnikov [50], Solonnikov et Pileckas [51] et Farwig et Sohr [22].

Ce travail de thèse est découpé en sept chapitres. Les chapitres 3, 4 et 5 ont chacun donné lieu à la rédaction d'un article soumis, à paraître ou paru. Un dernier article en cours de rédaction regroupe les résultats des chapitres 6 et 7.

Dans le chapitre 1, nous donnons les notations principales que l'on utilise dans ce mémoire de thèse. Nous y définissons en particulier les espaces de Sobolev avec poids de la manière la plus générale possible, c'est à dire avec un très large éventail d'exposants sur le poids ρ ainsi que sur le poids logarithmique. Ceci étant, dans la pratique nous nous contenterons de donner des résultats dans les espaces à poids les plus basiques (exposant -1, 0 ou 1 sur ρ, exposant 0 sur $lg\ \rho$). Nous pouvons noter ici qu'une extension possible de ce travail pourrait être de donner des résultats similaires dans les espaces plus généraux. Bien sûr, cela entraînera des questions supplémentaires sur les noyaux des problèmes ainsi que sur les conditions de compatibilité. Nous rappelons aussi les résultats connus des inégalités fondamentales de type Poincaré (Theorème 1.2.1) dans différentes géométries. Nous donnons également dans ce chapitre les définitions générales des espaces de traces avec poids. Néanmoins, nous les énonçons ici uniquement dans le cas de l'espace $\mathbb{R}^{n-1}$ et nous renvoyons aux introductions respectives des chapitres 6 et 7 pour leurs définitions dans le cas du demi-espace perturbé et de l'"aperture domain". Enfin, nous donnons un résultat fondamental de relèvement au Lemme 1.3.1.

Dans le second chapitre, nous nous contentons de rappeler des résultats déjà établis dans d'autres géométries plus classiques et dans le cadre des espaces de Sobolev avec poids. Nous rappelons tout d'abord certains isomorphismes pour l'équation de Poisson puis des résultats pour un problème de Laplace en domaine extérieur ainsi que dans le demi-espace avec des conditions aux limites de type Dirichlet ou Neumann. Ensuite, nous énonçons également des résultats pour le problème de Stokes dans $\mathbb{R}^n$ puis dans $\mathbb{R}^n_+$. Nous précisons ici que nous ne rappelons que les résultats sur l'existence et l'unicité de solutions généralisées et nous renvoyons à ces différents travaux pour une étude plus complète de ces problèmes.

La suite du travail est le véritable contenu de cette thèse, à savoir les nouveaux résultats qui sont établis ici. Tout d'abord, dans le chapitre 3, nous nous consacrons à l'étude du problème mixte extérieur à deux corps défini précédemment. Nous étudions tout d'abord le cas hilbertien en démontrant entre autres une inégalité de type Poincaré. Puis nous regardons le cas $p \neq 2$ en résolvant tout d'abord, lorsque $p > 2$, le problème harmonique puis en établissant une condition "inf-sup". Ensuite, nous donnons une caractérisation du noyau et par dualité, nous étudions le cas $p < 2$. Nous établissons également des résultats de régularité et dans la dernière partie, nous abordons brièvement la question des solutions homogènes.

Le chapitre 4 est dévoué à la résolution des quatre problèmes de Laplace cités ci-dessus dans le complémentaire d'un compact dans le demi-espace. Chaque section de ce chapitre est consacrée à l'étude d'un de ces quatre problèmes. Nous y donnons des résultats d'existence et d'unicité de solutions faibles et de solutions fortes.

En se plaçant toujours dans un domaine extérieur dans le demi-espace, nous étudions au chapitre 5 l'opérateur de Stokes avec conditons aux limites de type Dirichlet. Ici encore, nous cherchons à établir des théorèmes d'existence et d'unicité de solutions généralisées. Pour cela nous étudions d'abord le cas $p = 2$ en donnant des lemmes de relèvements, une formulation variationnelle équivalente et un théorème du type De Rham au moyen d'une condition "inf-sup". Puis nous passons au cas $p > 2$ que nous résolvons en considérant tout d'abord des données à support compact de manière à se ramener au cas hilbertien et construire une solution. Ici encore, nous traitons le cas $p < 2$ grâce à un raisonnement par dualité. Enfin, les deux dernières sections de ce chapitre sont vouées à établir des résultats pour des solutions fortes dont un résultat de régularité mais également des théorèmes permettant d'obtenir des solutions très faibles lorsque les données sur le bord sont peu régulières.

Pour terminer, dans les chapitres 6 et 7, consacrés respectivement au demi-espace perturbé et à l'"aperture domain", nous essayons de suivre le même schéma de démonstration qu'au chapitre 5 afin d'obtenir des résultats similaires. Néanmoins, et particulièrement dans le cas de l'"aperture domain", nous devons adapter nos démonstrations à la particularité du domaine et notamment à l'ajout de la condition de flux.

Enfin, pour conclure cette introduction, nous voulons rappeler les choses qui n'ont pas été traitées ici et qui peuvent ouvrir des perspectives pour la suite. Tout d'abord, comme nous l'avons déjà dit, il peut-être intéressant de donner des résultats similaires pour une classe plus large de poids. Ensuite, on peut étudier, pour la géométrie du chapitre 3, le cas où ω_0 et ω_1 ont une frontière commune. En ce qui concerne le problème de Stokes, on pourrait l'étudier dans ces différentes géométries avec d'autres conditions aux limites. Puis, on peut penser au cas où l'épaisseur du "mur" de l'aperture domain est réduite à zéro. Enfin, il conviendrait d'étudier ces problématiques pour des problèmes évolutifs et non linéaires.

Chapter 1

Functional framework

1.1 Notations

Let Ω be an open set of $\mathbb{R}^n$ with $n \geq 2$ and Γ its boundary. In all the sequel, Ω is supposed of class $C^{1,1}$ except in some cases where we will precise that the boundary can be only Lipschitz-continuous. For any real number $p \in \,]1, +\infty[$, we denote by p' the dual exponent of p:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a typical point of $\mathbb{R}^n$ and let $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}$ denote its distance to the origin. We define the upper half-space by

$$\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n,\ x_n > 0\}.$$

We shall use two basic weights:

$$\rho = (1 + r^2)^{1/2} \quad \text{and} \quad lg\ \rho = ln(2 + r^2)$$

For a multi index $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ we set

$$D^\lambda = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \ldots \partial x_n^{\lambda_n}}$$

with $|\lambda| = \lambda_1 + \cdots + \lambda_n$. For any integer q we denote by $\mathcal{P}_q$ the space of polynomials in n variables, smaller than or equal to q, with the convention that $\mathcal{P}_q$ is reduced to $\{0\}$ when q is negative.

For E and F two spaces such that $E \subset F$, we define

$$F' \perp E = \{f \in F',\ \forall x \in E,\ < f, x >_{F',F} = 0\}.$$

For any space E, we denote by $\boldsymbol{E}$ the space E^n.

We will denote by C a positive and real constant which may vary from line to line.

For $R > 0$, we will denote by B_R an open ball of radius R.

We remind here the definition of classical Sobolev spaces: for any nonnegative integers n and m and real numbers $p > 1$ setting:

$$W^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ \forall \lambda \in \mathbb{N}^n : 0 \leq |\lambda| \leq m,\ D^\lambda u \in L^p(\Omega)\}.$$

We equipped this space with its natural norm. When $p = 2$, we note

$$H^m(\Omega) = W^{m,2}(\Omega),$$

then $H^{\frac{1}{2}}(\Gamma)$ is the space of traces of functions in $H^1(\Omega)$ and $H^1_0(\Omega)$ is the subspace of functions in $H^1(\Omega)$ whose the trace is equal to zero on Γ.

Finally, we define the following space:

$$L^p_{loc}(\overline{\Omega}) = \{u,\ \text{for any compact } K \subset \overline{\Omega},\ u \in L^p(K)\}.$$

1.2 Weighted Sobolev spaces

For any nonnegative integers n and m and real numbers $p > 1$, α and β, setting

$$k = k(m,n,p,\alpha) = \begin{cases} -1 & \text{if } \dfrac{n}{p} + \alpha \notin \{1,\dots,m\}, \\ \\ m - \dfrac{n}{p} - \alpha & \text{if } \dfrac{n}{p} + \alpha \in \{1,\dots,m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \{&u \in \mathcal{D}'(\Omega); \\ &\forall \lambda \in \mathbb{N}^n : 0 \leqslant |\lambda| \leqslant k,\ \rho^{\alpha-m+|\lambda|}(lg\ \rho)^{\beta-1} D^\lambda u \in L^p(\Omega); \\ &\forall \lambda \in \mathbb{N}^n : k+1 \leqslant |\lambda| \leqslant m,\ \rho^{\alpha-m+|\lambda|}(lg\ \rho)^{\beta} D^\lambda u \in L^p(\Omega)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm:

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = (&\sum_{0\leqslant|\lambda|\leqslant k} \|\rho^{\alpha-m+|\lambda|}(lg\ \rho)^{\beta-1} D^\lambda u\|^p_{L^p(\Omega)} \\ &+ \sum_{k+1\leqslant|\lambda|\leqslant m} \|\rho^{\alpha-m+|\lambda|}(lg\ \rho)^{\beta} D^\lambda u\|^p_{L^p(\Omega)})^{1/p}. \end{aligned}$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = (\sum_{|\lambda|=m} \|\rho^{\alpha}(lg\ \rho)^{\beta} D^{\lambda} u\|^{p}_{L^{p}(\Omega)})^{1/p}.$$

When $\beta = 0$, we agree to drop the index β and denote simply the space by $W^{m,p}_{\alpha}(\Omega)$.

The constants 1 and 2 in $\rho(r)$ and $lg\ r$ are added so that they do not modify the behaviour of the functions near the origin, in case it belongs to Ω. Thus, the functions of $W^{m,p}_{\alpha,\beta}(\Omega)$ belong to $W^{m,p}(\mathcal{O})$ on all bounded domains $\mathcal{O}$ contained in Ω. Now, we define the space

$$\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\Omega)}},$$

it is characterized by

$$\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\Omega) = \{v \in W^{m,p}_{\alpha,\beta}(\Omega);\ \gamma_0 v = \gamma_1 v = \cdots = \gamma_{m-1} v = 0\},$$

where γ_i is the trace of order i of a function in $W^{m,p}_{\alpha,\beta}(\Omega)$. Now, we define $W^{-m,p'}_{-\alpha,-\beta}(\Omega)$, the dual space of $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\Omega)$, which is a space of distributions.

In the whole space, in the half-space and in an exterior domain in the whole space, we remind that

$$\mathcal{D}(\overline{\Omega}) \text{ is dense in } W^{m,p}_{\alpha,\beta}(\Omega) \tag{1.1}$$

and that we have the following Poincaré-type inequalities:

Theorem 1.2.1. *Let α and β be two real numbers and $m \geq 1$ an integer not satisfying simultaneously:*

$$\frac{n}{p} + \alpha \in \{1, \dots, m\} \quad \textit{and} \quad (\beta - 1)p = -1$$

Let $q' = \min(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\Omega)$. Then:
i) the semi-norm $|\ .\ |_{W^{m,p}_{\alpha,\beta}(\Omega)}$ defined on $W^{m,p}_{\alpha,\beta}(\Omega)/\mathcal{P}_{q'}$ is a norm equivalent to the quotient norm.
ii) the semi-norm $|\ .\ |_{W^{m,p}_{\alpha,\beta}(\Omega)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\Omega)$, which is equivalent to the full norm $\|\ .\ \|_{W^{m,p}_{\alpha,\beta}(\Omega)}$.

We just recall here the definition of the quotient norm:

$$\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)/\mathcal{P}_{q'}} = \inf_{k \in \mathcal{P}_{q'}} \|u + k\|_{W^{m,p}_{\alpha,\beta}(\Omega)}.$$

This theorem is established by Amrouche, Girault an Giroire when $\Omega = \mathbb{R}^n$ (see [6]) or when Ω is an exterior domain in the whole space (see [7]) and by Amrouche and Nečasovà when $\Omega = \mathbb{R}^n_+$ (see [8]). We prove that it is easily extended to an exterior domain in the half-space, a perturbed half-space or an aperture domain (see Sections 1 in Chapters 5, 6 and 7).

Now, we want to recall some properties of the weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\Omega)$. First, we have the algebraic and topological imbeddings:

$$W^{m,p}_{\alpha,\beta}(\Omega) \hookrightarrow W^{m-1,p}_{\alpha-1,\beta}(\Omega) \hookrightarrow \ldots \hookrightarrow W^{0,p}_{\alpha-m,\beta}(\Omega)$$

if $\dfrac{n}{p} + \alpha \notin \{1, \ldots, m\}$ and

$$W^{m,p}_{\alpha,\beta}(\Omega) \hookrightarrow \ldots \hookrightarrow W^{m-j+1,p}_{\alpha-j+1,\beta}(\Omega) \hookrightarrow W^{m-j,p}_{\alpha-j,\beta-1}(\Omega) \hookrightarrow \ldots \hookrightarrow W^{0,p}_{\alpha-m,\beta-1}(\Omega)$$

if $\dfrac{n}{p} + \alpha = j \in \{1, \ldots, m\}$. We notice that in the first case, for any $\gamma \in \mathbb{R}$ such that $\dfrac{n}{p} + \alpha - \gamma \notin \{1, \ldots, m\}$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\Omega) \mapsto \ \rho^\gamma u \in W^{m,p}_{\alpha-\gamma,\beta}(\Omega)$$

is an isomorphism. In both cases and for any multi-index $\lambda \in \mathbb{N}^n$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\Omega) \mapsto \ \partial^\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\Omega)$$

is continuous. Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\Omega)$ is given by

$$q = \begin{cases} m - \dfrac{n}{p} - \alpha - 1 & \text{if} \begin{cases} \dfrac{n}{p} + \alpha \in \{1, \ldots, m\} \quad \text{and} \quad (\beta - 1)p \geq -1, \\ \qquad\qquad \text{or} \\ \dfrac{n}{p} + \alpha \in \{j \in \mathbb{Z}; j \leq 0\} \quad \text{and} \quad \beta p \geq -1, \end{cases} \\ m - \dfrac{n}{p} - \alpha & \text{otherwise.} \end{cases}$$

For any open subset Θ of $\mathbb{R}^n$, we denote the following duality pairing:

$$< .,. >_\Theta = < .,. >_{W^{-1,p}(\Theta) \times \overset{\circ}{W}{}^{1,p'}(\Theta)} \quad \text{if } \Theta \text{ is bounded,}$$

$$< .,. >_\Theta = < .,. >_{W^{-1,p}_0(\Theta) \times \overset{\circ}{W}{}^{1,p'}_0(\Theta)} \quad \text{if } \Theta \text{ is unbounded,}$$

and

$$< .,. >_{\mathbb{R}^n} = < .,. >_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} .$$

1.3 The spaces of traces

Here, we want define the trace of a function in $W^{m,p}_{\alpha,\beta}(\Omega)$. First, when Γ, which is the boundary or a part of the boundary of Ω is bounded, the traces of functions in $W^{m,p}_{\alpha,\beta}(\Omega)$ are in the classical spaces of traces $W^{m-j-\frac{1}{p},p}(\Gamma)$, with $j = 0, \dots, m-1$ and we return to Adams [1] or Nečas [46] for their definition and for the usual trace theorems.

Now, we want to define the traces of functions when $\Omega = \mathbb{R}^n_+$. We will use these spaces in Chapters 4 and 5. For the other types of spaces of traces (in the case of the perturbed half-space (Section 6) and in the case of the aperture domain (Section 7)), we return to the introductions of the corresponding sections.

For any $\sigma \in]0,1[$, we introduce the space

$$W^{\sigma,p}_0(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n), \omega^{-\sigma} u \in L^p(\mathbb{R}^n),\\ \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x} d\boldsymbol{y} < \infty\},$$

where $\omega = \rho$ if $\dfrac{n}{p} \neq \sigma$ and $\omega = \rho\ (lg\rho)^{1/\sigma}$ if $\dfrac{n}{p} = \sigma$. It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W^{\sigma,p}_0(\mathbb{R}^n)} = (\|\frac{u}{\omega^\sigma}\|^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x} d\boldsymbol{y})^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space:

$$W^{\sigma,p}_\alpha(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n), \omega^{\alpha-\sigma} u \in L^p(\mathbb{R}^n),\\ \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\rho^\alpha(\boldsymbol{x})u(\boldsymbol{x}) - \rho^\alpha(\boldsymbol{y})u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x} d\boldsymbol{y} < \infty\},$$

where $\omega = \rho$ if $\dfrac{n}{p} + \alpha \neq \sigma$ and $\omega = \rho\ (lg\rho)^{1/(\sigma-\alpha)}$ if $\dfrac{n}{p} + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$\begin{aligned} W^{s,p}_\alpha(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n);\\ &\forall \lambda \in \mathbb{N}^n : 0 \leqslant |\lambda| \leqslant k,\ \rho^{\alpha-s+|\lambda|}(lg\ \rho)^{-1} D^\lambda u \in L^p(\mathbb{R}^n);\\ &\forall \lambda \in \mathbb{N}^n : k+1 \leqslant |\lambda| \leqslant [s]-1,\ \rho^{\alpha-s+|\lambda|} D^\lambda u \in L^p(\mathbb{R}^n);\\ &\forall \lambda \in \mathbb{N}^n : |\lambda| = [s],\ D^\lambda u \in W^{\sigma,p}_\alpha(\mathbb{R}^n)\}. \end{aligned}$$

where $k = s - \dfrac{n}{p} - \alpha$ if $\dfrac{n}{p} + \alpha \in \{\sigma, \dots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$

otherwise. It is a reflexive Banach space equipped with the norm

$$\|u\|_{W^{s,p}_\alpha(\mathbb{R}^n)} = \Big(\sum_{0\leqslant|\lambda|\leqslant k} \|\rho^{\alpha-s+|\lambda|}(lg\ \rho)^{-1}D^\lambda u\|^p_{L^p(\mathbb{R}^n)} + \sum_{k+1\leqslant|\lambda|\leqslant[s]-1} \|\rho^{\alpha-s+|\lambda|}D^\lambda u\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p} + \sum_{|\lambda|=[s]} \|D^\lambda u\|_{W^{\sigma,p}_\alpha(\mathbb{R}^n)}.$$

We can similarly define, for any real number β, the space:

$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n), (lg\ \rho)^\beta u \in W^{s,p}_\alpha(\mathbb{R}^n)\}.$$

We can prove some properties of the weighted Sobolev spaces $W^{s,p}_{\alpha,\beta}(\mathbb{R}^n)$. We have the algebraic and topological imbeddings in the case where $\dfrac{n}{p} \notin \{\sigma, \dots, \sigma+[s]-1\}$:

$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^n) \hookrightarrow W^{s-1,p}_{\alpha-1,\beta}(\mathbb{R}^n) \hookrightarrow \dots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta}(\mathbb{R}^n),$$

$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^n) \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta}(\mathbb{R}^n) \hookrightarrow \dots \hookrightarrow W^{0,p}_{\alpha-s,\beta}(\mathbb{R}^n).$$

When $\dfrac{n}{p} + \alpha = j \in \{\sigma, \dots, \sigma+[s]-1\}$, then we have:

$$W^{s,p}_{\alpha,\beta} \hookrightarrow \dots \hookrightarrow W^{s-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{s-j,p}_{\alpha-j,\beta-1} \hookrightarrow \dots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta-1},$$

$$W^{s,p}_{\alpha,\beta} \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta} \hookrightarrow \dots \hookrightarrow W^{[s]-j+1,p}_{\alpha-\sigma-j+1,\beta} \hookrightarrow W^{[s]-j,p}_{\alpha-\sigma-j,\beta-1} \hookrightarrow \dots \hookrightarrow W^{0,p}_{\alpha-s,\beta-1}.$$

We denote the trace of order j on $\mathbb{R}^{n-1}$ of a function u by:

$$\forall j \in \mathbb{N},\ \boldsymbol{x'} \in \mathbb{R}^{n-1}, \quad \gamma_j u(\boldsymbol{x'}) = \frac{\partial^j u}{\partial \boldsymbol{n}^j}(\boldsymbol{x'}, 0).$$

We remind the following trace lemma proved by Hanouzet [30] and extended by Amrouche and Nečasová [8] to this class of weighted Sobolev spaces:

Lemma 1.3.1. *For any integer $m \geq 1$ and real number α, the mapping*

$$\gamma : \mathcal{D}(\overline{\mathbb{R}^n_+}) \to (\mathcal{D}(\mathbb{R}^{n-1}))^m$$
$$u \mapsto (\gamma_0 u, \dots, \gamma_{m-1} u)$$

can be extended by continuity to a linear and continuous mapping still denoted by γ from $W^{m,p}_\alpha(\mathbb{R}^n_+)$ to $\prod_{j=0}^{m-1} W_\alpha^{m-j-\frac{1}{p},p}(\mathbb{R}^{n-1})$. Moreover, γ is onto and

$$Ker\ \gamma = \mathring{W}^{m,p}_\alpha(\mathbb{R}^n_+).$$

To finish this section, for any open subset Θ of $\mathbb{R}^n$, we denote the following duality pairing:

$$< .,. >_{\partial\Theta} = < .,. >_{W^{-\frac{1}{p},p}(\partial\Theta)\times W^{1-\frac{1}{p'},p'}(\partial\Theta)} \quad \text{if } \partial\Theta \text{ is bounded},$$

$$< .,. >_{\partial\Theta} = < .,. >_{W_0^{-\frac{1}{p},p}(\partial\Theta)\times W_0^{1-\frac{1}{p'},p'}(\partial\Theta)} \quad \text{if } \partial\Theta \text{ is unbounded},$$

Chapter 2

Known results

In this section, we want to remind some results previously established and usually used in this work. Let us notice that it is not an exhaustive list of all results using here but only a recall of similar results than ours for other geometries. Indeed, here we give results for a Laplace's problem and a Stokes problem in the whole space or the complement of a compact in the whole space or the half-space. In all the sequel, they will be used several times in the proofs of our theorems.

2.1 The Poisson's equation

First, we recall some isomorphisms established by Amrouche, Girault and Giroire in 1994 ([6]) for the Poisson's equation:

Theorem 2.1.1. *The following Laplace operators are isomorphisms:*

$$
\begin{array}{lll}
i) & \Delta: W_0^{1,p}(\mathbb{R}^n)/\mathcal{P}_{[1-n/p]} \;\rightarrow\; W_0^{-1,p}(\mathbb{R}^n)\perp\mathcal{P}_{[1-n/p']}, & \\
ii) & \Delta: W_1^{2,p}(\mathbb{R}^n)/\mathcal{P}_{[1-n/p]} \;\rightarrow\; W_1^{0,p}(\mathbb{R}^n)\perp\mathcal{P}_{[1-n/p']}, & \textit{if } n\neq p' \\
iii) & \Delta: W_1^{2,\frac{n}{n-1}}(\mathbb{R}^n)/\mathcal{P}_{2-n} \;\rightarrow\; (W_1^{0,\frac{n}{n-1}}(\mathbb{R}^n)\cap W_0^{-1,\frac{n}{n-1}}(\mathbb{R}^n))\perp\mathbb{R}, &
\end{array}
$$

Then, we have also this result:

Proposition 2.1.2. *Assume that $p>2$ and $f\in W_0^{-1,p}(\mathbb{R}^n)$ with compact support and satisfying, if $n=2$, the compatibility condition*

$$< f, 1 >_{W_0^{-1,2}(\mathbb{R}^2),W_0^{1,2}(\mathbb{R}^2)} = 0.$$

Then, the problem

$$-\Delta u = f \ \textit{in } \mathbb{R}^n$$

has a solution $u\in W_0^{1,2}(\mathbb{R}^n)\cap\ W_0^{1,p}(\mathbb{R}^n)$, unique up to an additive constant if $n=2$.

2.2 The Laplace's equation in an exterior domain

Here we remind the main theorems for Dirichlet and Neumann problems for the Laplace operator in exterior domains of $\mathbb{R}^n$. This study was done by Amrouche, Girault and Giroire in 1997 ([7]). Let ω_0 a compact region of $\mathbb{R}^n$ and Γ_0 its boundary of class $C^{1,1}$. First, we are interested in a problem with a Dirichlet boundary condition. We give the characterization of the kernel of such a problem. We define

$$\mathcal{A}_0^p({}^c\omega_0) = \{v \in W_0^{1,p}({}^c\omega_0),\ \Delta v = 0 \text{ in } {}^c\omega_0,\ v = 0 \text{ on } \Gamma_0\}.$$

For this, we define the function μ_0 by:

$$\mu_0 = U * (\frac{1}{|\Gamma_0|}\delta_{\Gamma_0})$$

where $U = \dfrac{1}{2\pi}\ln(r)$ is the fundamental solution of the Laplace's equation in $\mathbb{R}^2$ and δ_{Γ_0} is defined by:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2),\ < \delta_{\Gamma_0}, \varphi > = \int_{\Gamma_0} \varphi\, d\sigma.$$

We have the following characterization:

Proposition 2.2.1. *If $p < n$ or $p = n = 2$, then $\mathcal{A}_0^p({}^c\omega_0) = \{0\}$.*

If $p \geq n \geq 3$, then $\mathcal{A}_0^p({}^c\omega_0) = \{c(\lambda - 1), c \in \mathbb{R}\}$, where λ is the unique solution in $W_0^{1,p}({}^c\omega_0) \cap W_0^{1,2}({}^c\omega_0)$ of the problem

$$\Delta\lambda = 0 \text{ in } {}^c\omega_0, \quad \lambda = 1 \text{ on } \Gamma_0.$$

If $p > n = 2$, then $\mathcal{A}_0^p({}^c\omega_0) = \{c(\mu - \mu_0), c \in \mathbb{R}\}$, where μ is the unique solution in $W_0^{1,p}({}^c\omega_0) \cap W_0^{1,2}({}^c\omega_0)$ of the problem

$$\Delta\mu = 0 \text{ in } {}^c\omega_0, \quad \mu = \mu_0 \text{ on } \Gamma_0.$$

Then, we have the following theorem:

Theorem 2.2.2. *If $p \geq 2$, for any $f \in W_0^{-1,p}({}^c\omega_0)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma_0)$, there exists a unique solution $u \in W_0^{1,p}({}^c\omega_0)/\mathcal{A}_0^p({}^c\omega_0)$ of the problem*

$$\begin{cases} -\Delta u = f & \text{in } {}^c\omega_0, \\ u = g & \text{on } \Gamma_0. \end{cases}$$

Moreover, u satisfies

$$\|u\|_{W_0^{1,p}({}^c\omega_0)/\mathcal{A}_0^p({}^c\omega_0)} \leq\ C(\ \|f\|_{W_0^{-1,p}({}^c\omega_0)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}),$$

where C is a real positive constant which depends only on ω_0 and p.

If $p \leq \dfrac{n}{n-1}$ and $p < 2$, for any $f \in W_0^{-1,p}({}^c\omega_0)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma_0)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{A}_0^{p'}({}^c\omega_0), \quad < f, \varphi > = < g, \frac{\partial \varphi}{\partial \boldsymbol{n}} >_{\Gamma_0},$$

this problem has a unique solution $u \in W_0^{1,p}({}^c\omega_0)$.

If $\dfrac{n}{n-1} < p < 2$, for any $f \in W_0^{-1,p}({}^c\omega_0)$ and $g \in W^{1-\frac{1}{p},p}(\Gamma_0)$, this problem has a unique solution $u \in W_0^{1,p}({}^c\omega_0)$. In both cases, u satisfies

$$\|u\|_{W_0^{1,p}({}^c\omega_0)} \leq \ C(\ \|f\|_{W_0^{-1,p}({}^c\omega_0)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}),$$

where C is a real positive constant which depends only on ω_0 and p.

Now, we give results for the Laplace operator with a Neumann boundary condition. We introduce the following partition of unity:

$$\begin{gathered} \psi_1, \ \psi_2 \in C^\infty(\mathbb{R}^n), \ 0 \leq \psi_1, \psi_2 \leq 1, \ \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n, \\ \psi_1 = 1 \text{ in } B_R, \quad \operatorname{supp} \psi_1 \subset B_{R'}, \end{gathered}$$

where $0 < R < R'$ is such that $\omega_0 \subset B_R$ and we have the following theorem:

Theorem 2.2.3. *For any $p > 1$, $f \in W_0^{-1,p}({}^c\omega_0) \cap L^p({}^c\omega_0)$ and $g \in W^{-\frac{1}{p},p}(\Gamma_0)$ satisfying, if $p \leq \dfrac{n}{n-1}$, the compatibility condition*

$$\int_{{}^c\omega_0 \cap B_{R'}} f \psi_1 \ d\boldsymbol{x} + < f, \psi_2 >_{{}^c\omega_0} + < g, 1 >_{\Gamma_0} = 0,$$

there exists a unique solution $u \in W_0^{1,p}({}^c\omega_0)/\mathcal{P}_{[1-n/p]}$ of the problem

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } {}^c\omega_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \Gamma_0. \end{array} \right.$$

Moreover, u satisfies

$$\|u\|_{W_0^{1,p}({}^c\omega_0)/\mathcal{P}_{[1-n/p]}} \leq \ C(\ \|f\|_{W_0^{-1,p}({}^c\omega_0) \cap L^p({}^c\omega_0)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)}),$$

where C is a real positive constant which depends only on ω_0 and p.

2.3 The Laplace's equation in the half-space

Here we remind the main theorems for Dirichlet and Neumann problems for the Laplace operator in the half-space. This study was done by Amrouche and Nečasová in 2001 ([8]) for the Dirichlet boundary condition and by Amrouche in 2002 ([4]) for the Neumann boundary condition.

Theorem 2.3.1. *For any $p > 1$, $f \in W_0^{-1,p}(\mathbb{R}^n_+)$ and $g \in W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique solution $u \in W_0^{1,p}(\mathbb{R}^n_+)$ of the problem*

$$\begin{cases} -\Delta u = f & \text{in } \mathbb{R}^n_+, \\ u = g & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, u satisfies

$$\|u\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq \; C(\; \|f\|_{W_0^{-1,p}(\mathbb{R}^n_+)} + \|g\|_{W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p.

Theorem 2.3.2. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, $f \in W_1^{0,p}(\mathbb{R}^n_+)$ and $g \in W^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ satisfying, if $p \leq \dfrac{n}{n-1}$, the compatibility condition*

$$\int_{\mathbb{R}^n_+} f \; d\boldsymbol{x} =< g, 1 >_{\mathbb{R}^{n-1}},$$

there exists a unique solution $u \in W_0^{1,p}(\mathbb{R}^n_+)/\mathcal{P}_{[1-n/p]}$ of the problem

$$\begin{cases} -\Delta u = f & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, u satisfies

$$\|u\|_{W_0^{1,p}(\mathbb{R}^n_+)\mathcal{P}_{[1-n/p]}} \leq \; C(\; \|f\|_{W_1^{0,p}(\mathbb{R}^n_+)} + \|g\|_{W^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p.

2.4 The Stokes equations in $\mathbb{R}^n$

Here, we give results established by Alliot and Amrouche in 1999 ([2]) for the following Stokes system in the whole space:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \mathbb{R}^n, \\ \text{div } \boldsymbol{u} = h & \text{in } \mathbb{R}^n. \end{cases}$$

We have the following theorem:

Theorem 2.4.1. *For any $p > 1$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)$ and $h \in L^p(\mathbb{R}^n)$ satisfying, if $p \leq \dfrac{n}{n-1}$, the compatibility condition*

$$\forall i = 1, \ldots, n, \quad < f_i, 1 >_{\mathbb{R}^n} = 0,$$

this problem has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$. Moreover, $(\boldsymbol{u}, \pi)$ is unique up to an element of $\boldsymbol{\mathcal{P}}_{[1-n/p]} \times \{0\}$ and satisfies

$$\inf_{\boldsymbol{\lambda} \in \boldsymbol{\mathcal{P}}_{[1-n/p]}} \|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^n)} + \|\pi\|_{L^p(\mathbb{R}^n)} \leq \; C(\; \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)} + \|h\|_{L^p(\mathbb{R}^n)}),$$

where C is a real positive constant which depends only on p.

2.5 The Stokes equations in the half-space

Finally, to finish this section, we give results established by Amrouche, Nečasová and Raudin in 2008 ([9]) for the Stokes system in $\mathbb{R}^n_+$. We have the following theorem:

Theorem 2.5.1. *For any $p > 1$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$, $h \in L^p(\mathbb{R}^n_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ of the problem*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \mathbb{R}^n_+, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+)} + \|\pi\|_{L^p(\mathbb{R}^n_+)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)} + \|h\|_{L^p(\mathbb{R}^n_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p.

Chapter 3

Mixed exterior Laplace's problem

3.1 Introduction and preliminaries

In this chapter, we want to study the mixed exterior Laplace's problem, with both a boundary condition of Dirichlet and a boundary condition of Neumann. Let ω_0 and ω_1 be two compact, disconnected and not empty regions of $\mathbb{R}^n$, $n \geq 2$, let Γ_0 and Γ_1 be their respective boundary, of class $C^{1,1}$ when $p \neq 2$ and Lipschitz-continuous when $p = 2$ and let Ω be the complement of $\omega_0 \cup \omega_1$. We set $\Gamma = \Gamma_0 \cup \Gamma_1 = \partial\Omega$.
This chapter is devoted to solve the following problem:

$$(\mathcal{P}) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \Gamma_1. \end{cases}$$

This chapter is organized as follows. Sections 3.2, 3.3 and 3.4 are devoted to the study of questions of existence, uniqueness and regularity of solutions respectively in cases $p = 2$, $p > 2$ and $p < 2$ and Section 3.5 deals with different behaviours at the infinity of the solution according to the data. The main results are Theorem 3.2.2, Theorem 3.3.6 and Theorem 3.4.3 for generalized solutions and Theorems 3.3.8 and 3.4.4 for regularity results.

Now, we define the space $Y^p(\Omega) = W_0^{-1,p}(\Omega) \cap L^p(\Omega)$ equipped with the following norm:

$$\|u\|_{Y^p(\Omega)} = (\|u\|^p_{W_0^{-1,p}(\Omega)} + \|u\|^p_{L^p(\Omega)})^{1/p}.$$

We easily check that $Y^p(\Omega)$ is complete. We introduce the partition of unity:

$$\psi_1,\ \psi_2 \in C^\infty(\mathbb{R}^n),\ 0 \leq \psi_1, \psi_2 \leq 1,\ \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n,$$
$$\psi_1 = 1 \text{ in } B_R, \quad \text{supp } \psi_1 \subset B_{R+1}.$$

where $R > 0$ is such that $\omega_0 \cup \omega_1 \subset B_R$. For any $v \in W_0^{1,p'}(\Omega)$, we set $v_1 = \psi_1 v$ and $v_2 = \psi_2 v$. We have supp $v_1 \subset \Omega_{R+1} = B_{R+1} \cap \Omega$ and so $v_1 \in W^{1,p'}(\Omega_{R+1})$. Furthermore $v_2 = 0$ on Γ because $\psi_2 = 0$ on $\Omega_R = B_R \cap \Omega$, so $v_2 \in \overset{\circ}{W}{}_0^{1,p'}(\Omega)$. For $f \in Y^p(\Omega)$, we set:

$$\forall v \in W_0^{1,p'}(\Omega),\ T_f(v) = \int_{\Omega_{R+1}} f v_1 \, d\boldsymbol{x} + < f, v_2 >_\Omega .$$

We easily notice that T_f is well defined, linear and we check that:

$$\forall \varphi \in \mathcal{D}(\overline{\Omega}), \quad T_f(\varphi) = \int_\Omega f \varphi \, dx, \tag{3.1}$$

and for any $f \in Y^p(\Omega)$ and $v \in W_0^{1,p'}(\Omega)$,

$$|T_f(v)| \leq C \ \|f\|_{Y^p(\Omega)} \ \|v\|_{W_0^{1,p'}(\Omega)}, \tag{3.2}$$

where $C > 0$ is a constant which does not depend of f and v.

3.2 Case p = 2

We begin to introduce the space

$$D_2 = \{v \in W_0^{1,2}(\Omega),\ v = 0 \ \text{ on } \Gamma_0\}.$$

and to establish a Poincaré type inequality:

Proposition 3.2.1. *There exists a constant $C > 0$ such that:*

$$\forall u \in D_2,\ \|u\|_{W_0^{1,2}(\Omega)} \ \leq \ C \ |u|_{W_0^{1,2}(\Omega)}.$$

Proof- We use an absurd argument; so, assume that

$$\forall n \in \mathbb{N}^*,\ \exists\ w_n \in D_2,\ \|w_n\|_{W_0^{1,2}(\Omega)} \ > n\ |w_n|_{W_0^{1,2}(\Omega)}.$$

Then the sequence defined by $u_n = \dfrac{w_n}{\|w_n\|_{W_0^{1,2}(\Omega)}}$ satisfy

$$\|u_n\|_{W_0^{1,2}(\Omega)} = 1 \quad \text{and} \quad |u_n|_{W_0^{1,2}(\Omega)} < \frac{1}{n}. \tag{3.3}$$

Here, we define another partition of unity:

$$\begin{gathered} \varphi_1,\ \varphi_2 \in C^\infty(\mathbb{R}^n),\ 0 \leq \varphi_1, \varphi_2 \leq 1,\ \varphi_1 + \varphi_2 = 1 \text{ in } \mathbb{R}^n, \\ \varphi_1 = 1 \text{ in } B_{R_1}, \quad \text{supp } \varphi_1 \subset B_{R_1'}, \end{gathered}$$

where $0 < R_1 < R'_1$ are such that $\omega_1 \subset B_{R_1}$, $\omega_0 \cap B_{R'_1} = \emptyset$ and $B_{R'_1} \subset B_R$. We set $u_n^1 = \varphi_1 u_n$ and $u_n^2 = \varphi_2 u_n$, so that $u_n = u_n^1 + u_n^2$. We deduce from (3.3) the existence of $u \in D_2$ such that:

$$u_n \rightharpoonup u \text{ in } W_0^{1,2}(\Omega) \quad \text{and} \quad \nabla u = 0 \text{ in } \Omega.$$

As Ω is connected and $u \in D_2$, then $u = 0$ in Ω and

$$u_n \rightharpoonup 0 \text{ in } W_0^{1,2}(\Omega). \tag{3.4}$$

Thanks to the Rellich's compactness theorem, $u_n \to 0$ in $L^2(\Omega_{R'_1})$ and by (3.3), we easily deduce that $u_n^1 \to 0$ in $W_0^{1,2}(\Omega)$. Now, we prove that $u_n^2 \to 0$ in $W_0^{1,2}(\Omega)$. First, we notice that $u_n^2 \in \mathring{W}_0^{1,2}(\Omega)$. Setting $\Omega' = (\Omega \setminus \overline{B}_{R_1}) \cup \omega_0$, we call again u_n^2 the restriction of u_n^2 to $\Omega \setminus \overline{B}_{R_1}$ and we define:

$$\widetilde{u}_n^2 = u_n^2 \text{ in } \Omega \setminus \overline{B}_{R_1}, \quad \widetilde{u}_n^2 = 0 \text{ in } \omega_0$$

We easily check that $\widetilde{u}_n^2 \in \mathring{W}_0^{1,2}(\Omega')$ with $\|\widetilde{u}_n^2\|_{W_0^{1,2}(\Omega')} = \|u_n^2\|_{W_0^{1,2}(\Omega \setminus \overline{B}_{R_1})}$. Noticing that $\Omega' = {}^c\overline{B}_{R_1}$ and applying a result established by Giroire [27], we have:

$$\|\widetilde{u}_n^2\|_{W_0^{1,2}(\Omega')} \leq C \, |\widetilde{u}_n^2|_{W_0^{1,2}(\Omega')}.$$

We easily show that $|\widetilde{u}_n^2|_{W_0^{1,2}(\Omega')} \to 0$, so in particular $\|u_n^2\|_{W_0^{1,2}(\Omega \setminus \overline{B}_{R_1})} \to 0$. To finish, since $u_n^2 = 0$ on Ω_{R_1}, we have: $\|u_n^2\|_{W_0^{1,2}(\Omega)} = \|u_n^2\|_{W_0^{1,2}(\Omega \setminus \overline{B}_{R_1})} \to 0$. So, $u_n^2 \to 0$ in $W_0^{1,2}(\Omega)$ which implies that $u_n = u_n^1 + u_n^2 \to 0$ in $W_0^{1,2}(\Omega)$, and which contradicts (3.3). In consequence, we have the result searched. $\square$

Remark: For some geometries of compacts ω_0 and ω_1 (for example, when they almost penetrate each other and when they have concave parts), the definition of balls B_{R_1} and $B_{R'_1}$ where $0 < R_1 < R'_1$ are such that $\omega_1 \subset B_{R_1}$ and $\omega_0 \cap B_{R'_1} = \emptyset$ is impossible. Nevertheless, the same reasoning holds taking suitable open neighborhoods instead of open balls.

Theorem 3.2.2. *For any $f \in Y^2(\Omega)$, $g_0 \in H^{\frac{1}{2}}(\Gamma_0)$ and $g_1 \in H^{-\frac{1}{2}}(\Gamma_1)$, there exists a unique $u \in W_0^{1,2}(\Omega)$ solution of the problem $(\mathcal{P})$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,2}(\Omega)} \leq C \, (\|f\|_{Y^2(\Omega)} + \|g_0\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|g_1\|_{H^{-\frac{1}{2}}(\Gamma_1)}),$$

where C is a real positive constant which depends only on Ω.

Proof- First, according to Theorem 2.2.2, there exists a unique $u_0 \in W_0^{1,2}(\Omega_0)$ where $\Omega_0 = \Omega \cup \omega_1$, solution of:

$$\Delta u_0 = 0 \quad \text{in } \Omega_0, \quad u_0 = g_0 \quad \text{on } \Gamma_0,$$

and such that

$$\|u_{0|\Omega}\|_{W_0^{1,2}(\Omega)} \ \leq \ C\ \|g_0\|_{H^{\frac{1}{2}}(\Gamma_0)}.$$

We notice that since $u_{0|\Omega} \in W_0^{1,2}(\Omega)$ and $0 = \Delta u_0 \in L^2(\Omega)$, then $\dfrac{\partial u_0}{\partial \boldsymbol{n}} \in H^{-\frac{1}{2}}(\Gamma_1)$. Moreover, we know that there exists a unique $v \in D_2$ solution of the variational formulation:

$$\forall w \in D_2,\ a(v, w) = L(w),$$

where for $v, w \in D_2$,

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w\ d\boldsymbol{x} \ \text{ and } \ L(w) = T_f(w) + < g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}, w >_{\Gamma_1}.$$

Indeed, this result is a simple consequence of the Lax-Milgram theorem and of Proposition 3.2.1 which shows that the form a is coercive. Then, we check that this solution $v \in W_0^{1,2}(\Omega)$ satisfies

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = g_1 - \dfrac{\partial u_0}{\partial \boldsymbol{n}} & \text{on } \Gamma_1. \end{cases}$$

Indeed, clearly, $v = 0$ on Γ_0 because $v \in D_2$. Now, let show the first and the third relation. Let $w \in \mathcal{D}(\Omega)$, then:

$$\int_\Omega f w\ d\boldsymbol{x} = T_f(w) = \int_\Omega \nabla v \cdot \nabla w\ d\boldsymbol{x} = < -\Delta v, w >_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)},$$

i.e. $f = -\Delta v$ in Ω. Now, let $w \in H^1(\Omega)$ such that supp $w \subset B_{R_1}$. Then

$$\int_\Omega -w\ \Delta v\ d\boldsymbol{x} \ = \int_\Omega f w\ d\boldsymbol{x}.$$

Since $v \in W_0^{1,2}(\Omega)$ and $\Delta v \in L^2(\Omega)$, then $\dfrac{\partial v}{\partial \boldsymbol{n}} \in H^{-\frac{1}{2}}(\partial\Omega_{R_1})$. In other respects, we have

$$\int_\Omega -w\ \Delta v\ d\boldsymbol{x} = \int_{\Omega_{R_1}} -w\ \Delta v\ d\boldsymbol{x} = \int_{\Omega_{R_1}} \nabla v \cdot \nabla w\ d\boldsymbol{x} - < \frac{\partial v}{\partial \boldsymbol{n}}, w >_{\partial\Omega_{R_1}}.$$

But, supp $w \subset B_{R_1}$, so $\displaystyle\int_{\Omega_{R_1}} \nabla v \cdot \nabla w\ d\boldsymbol{x} = \int_\Omega \nabla v \cdot \nabla w\ d\boldsymbol{x}$ and

$$\int_\Omega \nabla v \cdot \nabla w\ d\boldsymbol{x} = \int_\Omega f w\ d\boldsymbol{x} + < \frac{\partial v}{\partial \boldsymbol{n}}, w >_{\Gamma_1}.$$

As v is solution of $(\mathcal{FV})$, we deduce that

$$\int_\Omega fw\ d\boldsymbol{x} + < \frac{\partial v}{\partial \boldsymbol{n}}, w >_{\Gamma_1} = T_f(w) + < g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}, w >_{\Gamma_1} .$$

As the support of w is included in B_{R_1}, then $T_f(w) = \displaystyle\int_\Omega fw\ d\boldsymbol{x}$ and so

$$< \frac{\partial v}{\partial \boldsymbol{n}}, w >_{\Gamma_1} = < g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}, w >_{\Gamma_1} .$$

Now, let $h \in H^{\frac{1}{2}}(\Gamma_1)$; we set

$$h_0 = h \text{ sur } \Gamma_1, \quad h_0 = 0 \text{ sur } \partial B_{R_1}.$$

Since $h_0 \in H^{\frac{1}{2}}(\Gamma_1 \cup \partial B_{R_1})$, there exists $w_{h_0} \in H^1(\Omega_{R_1})$, $w_{h_0} = h_0$ on $\Gamma_1 \cup \partial B_{R_1}$. Let w_h be the extension of w_{h_0} by 0 outside Ω_{R_1}. Then $w_h \in H^1(\Omega)$ and supp $w_h \subset B_{R_1}$; moreover, on Γ_1, $w_h = h$. So, we have, for any $h \in H^{\frac{1}{2}}(\Gamma_1)$:

$$< \frac{\partial v}{\partial \boldsymbol{n}}, h >_{\Gamma_1} = < g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}, h >_{\Gamma_1},$$

i.e. $\dfrac{\partial v}{\partial \boldsymbol{n}} = g_1 - \dfrac{\partial u_0}{\partial \boldsymbol{n}}$ sur Γ_1. Moreover, we easily show that

$$\|v\|_{W_0^{1,2}(\Omega)} \leq\ C\ (\|f\|_{Y^2(\Omega)} + \|g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}\|_{H^{-\frac{1}{2}}(\Gamma_1)}).$$

Finally, the function $u = u_{0|\Omega} + v$ is the solution of $(\mathcal{P})$ and the estimate searched is a consequence of the two previous inequalities. $\square$

3.3 Case p > 2

We propose the following approach: first we solve the harmonic problem, this will enable us to establish an "inf-sup" condition which in turn will solve the full problem by the theorem of Babuška-Brezzi.
In all this section we suppose $p > 2$ (except for the subsection 3.3.5. where we suppose $p \geq 2$) and Γ of class $C^{1,1}$.

3.3.1 Resolution of the harmonic problem

Let g_0 be in $W^{1-\frac{1}{p},p}(\Gamma_0)$ and g_1 be in $W^{-\frac{1}{p},p}(\Gamma_1)$. Here, we consider the problem: find u in $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ solution of

$$(\mathcal{P}_0) \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \Gamma_1. \end{cases}$$

Theorem 3.3.1. *For any $p > 2$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$, there exists a unique $u \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ solution of $(\mathcal{P}_0)$. Moreover, u satisfies:*

$$\|u\|_{W_0^{1,p}(\Omega)} + \|u\|_{W_0^{1,2}(\Omega)} \le C\ (\ \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \ \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}), \quad (3.5)$$

where C is a real positive constant which depends only on p and Ω.

Proof- By Corollary 2.6 in [7], we know there exists $u_0 \in W_0^{1,p}(\Omega_0) \cap W_0^{1,2}(\Omega_0)$ solution of

$$\Delta u_0 = 0 \quad \text{in } \Omega_0, \quad u_0 = g_0 \quad \text{on } \Gamma_0,$$

where we remind that $\Omega_0 = \Omega \cup \omega_1$, with the following estimate:

$$\|u_0\|_{W_0^{1,p}(\Omega_0)} + \|u_0\|_{W_0^{1,2}(\Omega_0)} \ \leqslant\ C\ \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}. \quad (3.6)$$

We notice that since $u_{0|\Omega} \in W_0^{1,p}(\Omega)$ and $0 = \Delta u_0 \in L^p(\Omega)$, we have $\dfrac{\partial u_0}{\partial \boldsymbol{n}} \in W^{-\frac{1}{p},p}(\Gamma_1)$. Moreover

$$\|\frac{\partial u_0}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)} \le C\ \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}$$

Then, we are going to show that there exists a unique v in $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ solution of the following problem:

$$(\mathcal{P}'_0) \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = g_1 - \dfrac{\partial u_0}{\partial \boldsymbol{n}} & \text{on } \Gamma_1. \end{cases}$$

with the estimate:

$$\|v\|_{W_0^{1,p}(\Omega)} + \|v\|_{W_0^{1,2}(\Omega)} \ \leqslant\ C\ (\ \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \ \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}). \quad (3.7)$$

As $p > 2$, $g_1 - \dfrac{\partial u_0}{\partial \boldsymbol{n}} \in H^{-\frac{1}{2}}(\Gamma_1)$ and by Theorem 3.2.2, there exists a unique $v \in W_0^{1,2}(\Omega)$ solution of $(\mathcal{P}'_0)$ and satisfying (3.7) with $p = 2$. It remains to show that $v \in W_0^{1,p}(\Omega)$. For this, we use the partition of unity previously defined:

$$\varphi_1,\ \varphi_2 \in C^\infty(\mathbb{R}^n),\ 0 \le \varphi_1, \varphi_2 \le 1,\ \varphi_1 + \varphi_2 = 1 \text{ in } \mathbb{R}^n,$$
$$\varphi_1 = 1 \text{ in } B_{R_1}, \quad \text{supp } \varphi_1 \subset B_{R'_1},$$

where $0 < R_1 < R'_1$ are such that $\omega_1 \subset B_{R_1}$, $\omega_0 \cap B_{R'_1} = \emptyset$ and $B_{R'_1} \subset B_R$. We set $v_1 = \varphi_1 v$, $v_2 = \varphi_2 v$ and

$$\widetilde{v_2} = v_2 \text{ in } \Omega, \quad \widetilde{v_2} = 0 \text{ in } \omega_1.$$

We remind that $v_2 \in \overset{\circ}{W}{}^{1,2}_0(\Omega)$ and so that $\widetilde{v_2} \in W^{1,2}_0(\Omega_0)$. Moreover, we have:

$$-\Delta v_2 = \Delta v_1 = v\ \Delta\varphi_1 + 2\nabla\varphi_1 \cdot \nabla v := f_1 \text{ in } \Omega.$$

Setting

$$\widetilde{f_1} = f_1 \text{ in } \Omega, \quad \widetilde{f_1} = 0 \text{ in } \omega_1,$$

it is obvious that $\widetilde{f_1} \in L^2(\Omega_0)$ with supp $\widetilde{f_1} \subset \Omega_{R'_1}$ and that $-\Delta\widetilde{v_2} = \widetilde{f_1}$ in Ω_0. Now, we set $s = \widetilde{v_2}$ and we are going to show that $s \in W^{1,p}_0(\Omega_0)$. For this, we define another partition of unity:

$$\xi_1,\ \xi_2 \in C^\infty(\mathbb{R}^n),\ 0 \le \xi_1, \xi_2 \le 1,\ \xi_1 + \xi_2 = 1 \text{ in } \mathbb{R}^n,$$
$$\xi_1 = 1 \text{ in } B_{R_0}, \quad \text{supp } \xi_1 \subset B_{R'_0},$$

where $0 < R_0 < R'_0$ are such that $\omega_0 \subset B_{R_0}$, $\omega_1 \cap B_{R'_0} = \emptyset$ and $B_{R'_0} \subset B_R$. We set $s_1 = \xi_1 s$ and $s_2 = \xi_2 s$, and we notice that:

$$-\Delta s_2 = \widetilde{f_1} + \Delta s_1 = \widetilde{f_1} + s\ \Delta\xi_1 + 2\nabla\xi_1 \cdot \nabla s := F \text{ in } \Omega_0.$$

Finally, we set:

$$\widetilde{s_2} = \begin{cases} s_2 & \text{in } \Omega_0, \\ 0 & \text{in } \omega_0, \end{cases} \quad \text{and} \quad \widetilde{F} = \begin{cases} F & \text{in } \Omega_0, \\ 0 & \text{in } \omega_0. \end{cases}$$

We have $\widetilde{s_2} \in W^{1,2}_0(\mathbb{R}^n)$ because $s_2 \in \overset{\circ}{W}{}^{1,2}_0(\Omega_0)$, $\widetilde{F} \in L^2(\mathbb{R}^n)$ with supp $\widetilde{F} \subset \Omega_{R'_0} \cap \Omega_{R'_1}$, and also $-\Delta\widetilde{s_2} = \widetilde{F}$ in $\mathbb{R}^n$.

i) Case $2 < p \le \dfrac{2n}{n-2}$ and $n \ge 3$ or $p > 2$ and $n = 2$.

Thanks to the injections of Sobolev, , we have $\widetilde{F} \in W^{-1,p}_0(\mathbb{R}^n)$. In consequence, by Theorem 2.1.1 i), (there is no condition of compatibility because $p > 2$), we show that $\widetilde{s_2} \in W^{1,p}_0(\mathbb{R}^n)$ and so $s_2 \in W^{1,p}_0(\Omega_0)$ and we easily check that:

$$\|\widetilde{s_2}\|_{W^{1,p}_0(\mathbb{R}^n)/\mathcal{P}_{[1-\frac{n}{p}]}} \le C\ \|g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}. \tag{3.8}$$

Outside of $B_{R'_0}$, $\xi_1 = 0$ so $s = s_2$ and the trace of s_2 on $\partial B_{R'_0}$ belongs to $W^{1-\frac{1}{p},p}(\partial B_{R'_0})$. So s satisfies:

$$-\Delta s = \widetilde{f_1} \text{ in } \Omega_{R'_0}, \quad s = s_2 \text{ on } \partial B_{R'_0}, \quad s = 0 \text{ on } \Gamma_0.$$

Consequently, (see Lions and Magenes [42]), $s \in W^{1,p}(\Omega_{R_0+1})$ and:

$$\|s\|_{W^{1,p}(\Omega_{R'_0})} \;\leq\; C\; (\|\widetilde{f}_1\|_{W^{-1,p}(\Omega_{R'_0})} \;+\; \|s_2\|_{W^{1-\frac{1}{p},p}(\partial B_{R'_0})}) \tag{3.9}$$

We deduce of this that $s \in W_0^{1,p}(\Omega_0)$; and with (3.8) and (3.9), we have:

$$\|s\|_{W_0^{1,p}(\Omega_0)} \;\leq\; C\; \|g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}. \tag{3.10}$$

ii) Case $n \geq 3$ and $p > \dfrac{2n}{n-2}$.

The argument used above with $p = \dfrac{2n}{n-2}$ shows that $s \in W_0^{1,\frac{2n}{n-2}}(\Omega_0)$ and we use the same proof as **i)** with $s \in W_0^{1,\frac{2n}{n-2}}(\Omega_0)$ instead of $s \in W_0^{1,2}(\Omega_0)$. So, we obtain the result for $n = 3,\; n = 4$ and $n = 5$ if $p < \dfrac{2n}{n-4}$; then we take $\dfrac{2n}{n-4}$ instead of $\dfrac{2n}{n-2}$, and we start again; so we reach for all dimensions, all values of p.

Consequently, we have $s = \tilde{v_2} \in W_0^{1,p}(\Omega_0)$, $v_2 \in W_0^{1,p}(\Omega)$ and

$$\|v_2\|_{W_0^{1,p}(\Omega)} \;\leq\; C\; \|g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}. \tag{3.11}$$

Outside of $B_{R'_1}$, $\varphi_1 = 0$ and $v = v_2$ and the trace of v_2 on $\partial B_{R'_1}$ belongs to $W^{1-\frac{1}{p},p}(\partial B_{R'_1})$. So v satisfies:

$$\Delta v = 0 \text{ in } \Omega_{R'_1}, \quad v = v_2 \text{ on } \partial B_{R'_1}, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}} \text{ on } \Gamma_1.$$

In consequence, (see Lions and Magenes [42]), $v \in W^{1,p}(\Omega_{R'_1})$ and

$$\|v\|_{W^{1,p}(\Omega_{R'_1})} \;\leq\; C\; (\|\; g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)} \;+\; \|v_2\|_{W^{1-\frac{1}{p},p}(\partial B_{R'_1})}). \tag{3.12}$$

We deduce of this that $v \in W_0^{1,p}(\Omega)$ and with (3.11) and (3.12), we have:

$$\|v\|_{W_0^{1,p}(\Omega)} \;\leq\; C\; \|\; g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_1)}.$$

Then, we easily check (3.7). Finally the function $u = u_{0|\Omega} + v \in W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ suits and with (3.6) and (3.7), we have (3.5). $\square$

Remark: In this proof, we can do the same remark as Page 30 when we define balls B_{R_1}, $B_{R'_1}$, B_{R_0} and $B_{R'_0}$.

3.3.2 An "inf-sup" condition

Setting for any $p > 1$,

$$D_p = \{v \in W_0^{1,p}(\Omega),\ v = 0 \text{ on } \Gamma_0\}.$$

we notice that, equipped with the norm $\|\nabla.\|_{\boldsymbol{L}^p(\Omega)}$, D_p is a reflexive Banach space. In this subsection, we are interested in the existence of $\beta > 0$, a constant such that:

$$\inf_{\substack{w\in D_{p'}\\ w\neq 0}} \sup_{\substack{v\in D_p\\ v\neq 0}} \frac{\displaystyle\int_\Omega \nabla v \cdot \nabla w \, d\boldsymbol{x}}{\|\nabla v\|_{\boldsymbol{L}^p(\Omega)} \|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)}} \ \geq\ \beta.$$

We define:

$$\overset{\circ}{H}_p(\Omega) = \{\boldsymbol{z} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega,\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_1\}.$$

Proposition 3.3.2. *For any $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, there exists $\boldsymbol{z} \in \overset{\circ}{H}_p(\Omega)$ and $\varphi \in D_p$, such that:*

$$\boldsymbol{g} = \nabla\varphi + \boldsymbol{z},$$
$$\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)} \leq\ C\ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}$$

where C is a real positive constant which depends only on p and Ω.

Proof- Let $\boldsymbol{g}$ be in $\boldsymbol{L}^p(\Omega)$ and $\widetilde{\boldsymbol{g}}$ the extension by 0 of $\boldsymbol{g}$ in $\mathbb{R}^n$; so we have $\widetilde{\boldsymbol{g}} \in \boldsymbol{L}^p(\mathbb{R}^n)$ and by a result established in [6],
$\operatorname{div} \widetilde{\boldsymbol{g}} \in W_0^{-1,p}(\mathbb{R}^n) \perp \mathcal{P}_{[1-n/p']}$ with

$$\|\operatorname{div} \widetilde{\boldsymbol{g}}\|_{W_0^{-1,p}(\mathbb{R}^n)} \leq\ C\ \|\widetilde{\boldsymbol{g}}\|_{\boldsymbol{L}^p(\mathbb{R}^n)} =\ C\ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}.$$

According to Theorem 2.1.1 i), we know there exists $v \in W_0^{1,p}(\mathbb{R}^n)$ such that $\Delta v = \operatorname{div} \widetilde{\boldsymbol{g}}$ in $\mathbb{R}^n$, and we show that $\|\nabla v\|_{\boldsymbol{L}^p(\mathbb{R}^n)} \leq\ C_1\ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}$. So, we have:

$$\widetilde{\boldsymbol{g}} - \nabla v \in \boldsymbol{L}^p(\mathbb{R}^n) \quad \text{and} \quad \operatorname{div}(\widetilde{\boldsymbol{g}} - \nabla v) = 0 \quad \text{in } \mathbb{R}^n.$$

Consequently, $(\boldsymbol{g} - \nabla v)\cdot\boldsymbol{n} \in W^{-\frac{1}{p},p}(\Gamma_1)$ and $v_{|\Gamma_0} \in W^{1-\frac{1}{p},p}(\Gamma_0)$. Here, we apply the results of Theorem 3.3.1. There exists a unique $w \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ solution of:

$$\Delta w = 0 \text{ in } \Omega, \quad w = -v \text{ on } \Gamma_0, \quad \frac{\partial w}{\partial \boldsymbol{n}} = (\boldsymbol{g} - \nabla v)\cdot\boldsymbol{n} \text{ on } \Gamma_1,$$

and we show that $\|\nabla w\|_{\boldsymbol{L}^p(\Omega)} \leq\ C_2\ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}$.
Finally the functions $\varphi = v_{|\Omega} + w$ and $\boldsymbol{z} = \boldsymbol{g} - \nabla\varphi$ comply with the question.
□

Theorem 3.3.3. *There exists a constant* $\beta > 0$ *such that*

$$\inf_{\substack{w \in D_{p'} \\ w \neq 0}} \sup_{\substack{v \in D_p \\ v \neq 0}} \frac{\displaystyle\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}}{\|\nabla v\|_{\boldsymbol{L}^p(\Omega)} \|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)}} \;\geq\; \beta \tag{3.13}$$

Proof- Let w be in $D_{p'}$ with $w \neq 0$. We notice that $\nabla w \neq 0$ because otherwise w is constant in the connected open region Ω, *i.e.* $w = 0$ in Ω because $w = 0$ on Γ_0. We have

$$\|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)} = \sup_{\substack{\boldsymbol{g} \in \boldsymbol{L}^p(\Omega) \\ \boldsymbol{g} \neq 0}} \frac{\displaystyle\int_\Omega \nabla w \cdot \boldsymbol{g} \; d\boldsymbol{x}}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}}.$$

We suppose that $\boldsymbol{g} \notin \overset{\circ}{H}_p (\Omega)$. By Proposition 3.3.2, there exists $\boldsymbol{z} \in \overset{\circ}{H}_p (\Omega)$ and $\varphi \in D_p$, with $\nabla\varphi \neq 0$ such that $\boldsymbol{g} = \boldsymbol{z} + \nabla\varphi$ and $\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)} \leq C \; \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}$. Thus,

$$\frac{\displaystyle\int_\Omega \nabla w \cdot \boldsymbol{g} \; d\boldsymbol{x}}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \;\leq\; C \; \frac{\displaystyle\int_\Omega \nabla w \cdot \nabla\varphi \; d\boldsymbol{x}}{\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)}} \;\leq\; C \; \sup_{\substack{\varphi \in D_p \\ \varphi \neq 0}} \frac{\displaystyle\int_\Omega \nabla w \cdot \nabla\varphi \; d\boldsymbol{x}}{\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)}}.$$

Noticing that, if $\boldsymbol{g} \in \overset{\circ}{H}_p (\Omega)$, we have $\displaystyle\int_\Omega \nabla w \cdot \boldsymbol{g} \; d\boldsymbol{x} = 0$, this implies that, for any $\boldsymbol{g}$ in $\boldsymbol{L}^p(\Omega)$:

$$\|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)} = \sup_{\substack{\boldsymbol{g} \in \boldsymbol{L}^p(\Omega) \\ \boldsymbol{g} \neq 0}} \frac{\displaystyle\int_\Omega \nabla w \cdot \boldsymbol{g} \; d\boldsymbol{x}}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \;\leq\; C \; \sup_{\substack{\varphi \in D_p \\ \varphi \neq 0}} \frac{\displaystyle\int_\Omega \nabla w \cdot \nabla\varphi \; d\boldsymbol{x}}{\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)}}.$$

We deduce the estimate (3.13) with $\beta = \dfrac{1}{C} > 0$. $\square$

3.3.3 The full problem

We remind here the following result:

Theorem 3.3.4. *Let* X *and* M *be two reflexive Banach spaces and* X' *and* M' *their dual spaces. Let* b *be a bilinear form defined and continuous on* $X \times M$*, let* $B \in \mathcal{L}(X; M')$ *and* $B' \in \mathcal{L}(M, X')$ *be the operators defined by:*

$$\forall v \in X, \; \forall w \in M, \; b(v, w) = < Bv, w > = < v, B'w >$$

The following statements are equivalent:
i) There exists $\beta > 0$*, such that* $\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{b(v,w)}{\|v\|_X \|w\|_M} \geq \beta.$
ii) The operator B *is an isomorphism from* $X/Ker\ B$ *to* M' *and* $\frac{1}{\beta}$ *is the continuity constant of* B^{-1}*.*
iii) The operator B' *is an isomorphism from* M *to* $X' \perp Ker\ B$ *and* $\frac{1}{\beta}$ *is the continuity constant of* B'^{-1}*.*

Here, we apply this theorem with $X = D_p$, $M = D_{p'}$ and:

$$b(v,w) = \int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}.$$

According to (3.13),

$$B \text{ is an isomorphism from } D_p/Ker\ B \text{ to } (D_{p'})'. \tag{3.14}$$

Then, we define for $f \in Y^p(\Omega)$ and $g \in W^{-\frac{1}{p},p}(\Gamma_1)$ the linear form T by

$$\forall w \in D_{p'}, \quad T(w) = T_f(w) + < g, w >_{\Gamma_1} .$$

We check that $T \in (D_{p'})'$ and by (3.14), we deduce the existence of $v \in D_p$, unique up to an element of $Ker\ B$, such that $Bv = T$, *i.e.* solution of the variational formulation:

$$\forall w \in D_{p'}, \quad \int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x} = T_f(w) + < g, w >_{\Gamma_1} .$$

Corollary 3.3.5. *For any* $p > 2$*,* $f \in Y^p(\Omega)$ *and* $g \in W^{-\frac{1}{p},p}(\Gamma_1)$*, there exists a unique* $v \in W_0^{1,p}(\Omega)/Ker\ B$ *solution of*

$$-\Delta v = f \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_1.$$

Moreover, v *satisfies*

$$\|v\|_{W_0^{1,p}(\Omega)/Ker\ B} \leq C \; (\|f\|_{Y^p(\Omega)} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma_1)}) \tag{3.15}$$

where C *is a real positive constant which depends only on* p *and* Ω*.*

Proof- As we have done in Theorem 3.2.2, we show that the solution of the variational formulation is also solution of this problem. □

Theorem 3.3.6. *For any $p > 2$, $f \in Y^p(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$, there exists a unique $u \in W_0^{1,p}(\Omega)/Ker\ B$ solution of $(\mathcal{P})$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)/Ker\ B} \leq C(\ \|f\|_{Y^p(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- First of all, by Theorem 2.2.2, we know there exists a unique $u_0 \in W_0^{1,p}(\Omega_0)$ where $\Omega_0 = \Omega \cup \omega_1$, solution of

$$\Delta u_0 = 0 \quad \text{in } \Omega_0, \quad u_0 = g_0 \quad \text{on } \Gamma_0,$$

and such that

$$\|u_0\|_{W_0^{1,p}(\Omega_0)} \ \leq\ C\|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}. \tag{3.16}$$

By Corollary 3.3.5, we deduce that there exists $v \in W_0^{1,p}(\Omega)$, unique up to an element of $Ker\ B$, solution of

$$-\Delta v = f \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}} \text{ on } \Gamma_1.$$

Finally, the function $u = u_0|_\Omega + v$ is solution of $(\mathcal{P})$ and the estimate comes from (3.15) and (3.16). □

3.3.4 Characterization of the kernel of the operator B

We set:

$$\mathcal{M}_0^p(\Omega) = \{v \in W_0^{1,p}(\Omega);\ \Delta v = 0 \ \text{ in } \Omega,\ v = 0 \ \text{ on } \Gamma_0, \frac{\partial v}{\partial \boldsymbol{n}} = 0 \ \text{ on } \Gamma_1\}.$$

Thanks to the density of $\mathcal{D}(\overline{\Omega})$ in $W_0^{1,p'}(\Omega)$, we easily check that

$$Ker\ B = \mathcal{M}_0^p(\Omega).$$

Now, we characterize $\mathcal{M}_0^p(\Omega)$. For this, first of all, we remind that we define: $\mu_0 = U * (\frac{1}{|\Gamma|}\delta_\Gamma)$, where $U = \frac{1}{2\pi}\ln(r)$ is the fundamental solution of Laplace's equation in $\mathbb{R}^2$ and δ_Γ is the distribution defined by:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2),\ <\delta_\Gamma, \varphi> = \int_\Gamma \varphi\ d\sigma$$

Proposition 3.3.7. *We have the following statements:*
i) If $p < n$, then $\mathcal{M}_0^p(\Omega) = \{0\}$.

ii) If $p \geq n \geq 3$, then $\mathcal{M}_0^p(\Omega) = \{c(\lambda - 1),\ c \in \mathbb{R}\}$ where λ is the only solution in $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ of the following problem $(\mathcal{P}_1)$:

$$\Delta\lambda = 0 \text{ in } \Omega,\ \ \lambda = 1 \text{ on } \Gamma_0,\ \frac{\partial \lambda}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_1.$$

iii) If $p > n = 2$, then $\mathcal{M}_0^p(\Omega) = \{c(\mu - \mu_0),\ c \in \mathbb{R}\}$ where μ is the only solution in $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ of the following problem $(\mathcal{P}_2)$:

$$\Delta\mu = 0 \text{ in } \Omega,\ \ \mu = \mu_0 \text{ on } \Gamma_0,\ \frac{\partial \mu}{\partial \boldsymbol{n}} = \frac{\partial \mu_0}{\partial \boldsymbol{n}} \text{ on } \Gamma_1.$$

Proof- Let $z \in \mathcal{M}_0^p(\Omega)$ and let η be the trace of z on Γ_1. We have $\eta \in W^{-\frac{1}{p},p}(\Gamma_1)$. We know there exists a unique $\xi \in W^{1,p}(\overset{\circ}{\omega}_1)$, where $\overset{\circ}{\omega}_1$ is the interior of the compact ω_1, such that:

$$\Delta\xi = 0 \text{ in } \overset{\circ}{\omega}_1,\ \ \xi = \eta \text{ on } \Gamma_1.$$

Let $\widetilde{z}$ be defined by

$$\widetilde{z} = z \text{ in } \Omega,\ \ \widetilde{z} = \xi \text{ in } \omega_1,\ \ \widetilde{z} = 0 \text{ in } \omega_0.$$

It is obvious that $\widetilde{z} \in W_0^{1,p}(\mathbb{R}^n)$ and $\Delta\widetilde{z} \in W_0^{-1,p}(\mathbb{R}^n)$. Moreover, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have:

$$< \Delta\widetilde{z}, \varphi >_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} = - < \frac{\partial z}{\partial \boldsymbol{n}}, \varphi >_{\Gamma_0} - < \frac{\partial \xi}{\partial \boldsymbol{n}}, \varphi >_{\Gamma_1} .$$

We set $h = \Delta\widetilde{z}$. Then, $h \in W_0^{-1,p}(\mathbb{R}^n)$ and h has a compact support. At this stage, the discussion splits into two parts according to the dimension n:

i) Case $n \geq 3$. By Proposition 2.1.2, we know there exists a unique w such that:

$$w \in W_0^{1,p}(\mathbb{R}^n) \cap W_0^{1,2}(\mathbb{R}^n) \ \text{ and } \ \Delta w = h \text{ in } \mathbb{R}^n.$$

The difference $\widetilde{z} - w$ is in $W_0^{1,p}(\mathbb{R}^n)$ and is harmonic in $\mathbb{R}^n$. If $p < n$, then $w = \widetilde{z}$ in $\mathbb{R}^n$, the restriction of w to Ω is in $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$, and since $z = w$ in Ω, Section 3.2 implies that $w = 0$ in Ω *i.e.* $\mathcal{M}_0^p(\Omega) = \{0\}$.
If $p \geq n \geq 3$, we have $w = \widetilde{z} + c$ in $\mathbb{R}^n$ so $w \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ is the only solution of the problem:

$$\Delta w = 0 \text{ in } \Omega,\ \ w = c \text{ on } \Gamma_0,\ \ \frac{\partial w}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_1.$$

Consequently $\mathcal{M}_0^p(\Omega) = \{c(\lambda - 1),\ c \in \mathbb{R}\}$ where λ is the solution of $(\mathcal{P}_1)$.

ii) Case $n = 2$. The problem

$$\Delta w = h \ \text{ in } \mathbb{R}^2,$$

does not have a solution in $W_0^{1,2}(\mathbb{R}^2)$ unless h satisfies the necessary condition $< h, 1 >_{\mathbb{R}^2} = 0$. In this case, with the arguments above, we obtain $z = c(\lambda - 1)$. However, when $n = 2$, the constant functions are in $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ so $\lambda = 1 \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ is solution of $(\mathcal{P}_1)$. So $z = 0$, which is the trivial case. Thus, we suppose that $< h, 1 >_{\mathbb{R}^2} \neq 0$ and we consider the problem:

$$\Delta w = h \ - < h, 1 >_{\mathbb{R}^2} \Delta \mu_0 \ \text{ in } \mathbb{R}^2. \tag{3.17}$$

We know that $\mu_0 \in W_0^{1,q}(\mathbb{R}^2)$ for any $q > 2$ and moreover

$$\Delta \mu_0 = 0 \ \text{ in } \Omega \cup \overset{\circ}{\omega}_0 \cup \overset{\circ}{\omega}_1 \quad \text{and} \ < \Delta \mu_0, 1 >_{\mathbb{R}^2} = 1.$$

The right-hand side of (3.17) is orthogonal to constants, has compact support and belongs to $W_0^{-1,p}(\mathbb{R}^2)$. So, by Proposition 2.1.2, the problem (3.17) has a solution (unique up to an additive constant) $w \in W_0^{1,2}(\mathbb{R}^2) \cap W_0^{1,p}(\mathbb{R}^2)$. Moreover, the function $w + < h, 1 >_{\mathbb{R}^2} \mu_0 - \widetilde{z}$ is harmonic in $\mathbb{R}^2$. So, there exists $c > 0$ such that $w + < h, 1 >_{\mathbb{R}^2} \mu_0 - \widetilde{z} = c$. The restriction of w to Ω is in $W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ and $w = c + w_1$ where w_1 is the only solution in $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ of the problem:

$$\Delta w_1 = 0 \text{ in } \Omega, \ \ w_1 = - < h, 1 >_{\mathbb{R}^2} \mu_0 \text{ on } \Gamma_0, \ \ \frac{\partial w_1}{\partial \boldsymbol{n}} = - < h, 1 >_{\mathbb{R}^2} \frac{\partial \mu_0}{\partial \boldsymbol{n}} \text{ on } \Gamma_1.$$

The function μ being the only solution in $W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ of the problem $(\mathcal{P}_2)$, we have $\mathcal{M}_0^p(\Omega) = \{c(\mu - \mu_0), \ c \in \mathbb{R}\}$. $\square$

3.3.5 A regularity result

We suppose, in this subsection, that $p \geq 2$. Here, we propose to study the question of the regularity of the solutions when the data are more regular. More precisely, we suppose that:

$$g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0), \ \ g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1) \ \text{ and } \ f \in X_1^{0,p}(\Omega)$$

where

$$X_1^{0,p}(\Omega) = \begin{cases} W_1^{0,p}(\Omega) & \text{if } p \neq \dfrac{n}{n-1}, \\ W_1^{0,p}(\Omega) \cap W_0^{-1,p}(\Omega) & \text{otherwise,} \end{cases}$$

equipped with its natural norm: (we remind that $W_1^{0,p}(\Omega)$ is included in $W_0^{-1,p}(\Omega)$ if and only if $W_0^{1,p'}(\Omega) \subset W_0^{-1,p'}(\Omega)$, this last inclusion taking place if and only if $p \neq \dfrac{n}{n-1}$).

Theorem 3.3.8. *For any $p \geq 2$, $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$, $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$, and $f \in X_1^{0,p}(\Omega)$, there exists a unique $u \in W_1^{2,p}(\Omega)/\mathcal{M}_0^p(\Omega)$ solution of $(\mathcal{P})$. Moreover, u satisfies:*

$$\|u\|_{W_1^{2,p}(\Omega)/\mathcal{M}_0^p(\Omega)} \leq C\ (\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}), \tag{3.18}$$

where C is a real positive constant which depends only on p and Ω.

Proof- For $p \geq 2$ and $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$, we have, by the remark 2.11 in [7] that the solution $u_0 \in W_0^{1,p}(\Omega_0)$ (where $\Omega_0 = \Omega \cup \omega_1$) of the problem

$$\Delta u_0 = 0 \quad \text{in } \Omega_0, \quad u_0 = g_0 \quad \text{on } \Gamma_0,$$

is in $W_1^{2,p}(\Omega_0)$ and it satisfies:

$$\|u_0\|_{W_1^{2,p}(\Omega_0)} \leq C\ \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)}. \tag{3.19}$$

Now, we notice that $\dfrac{\partial u_0}{\partial \boldsymbol{n}} \in W^{1-\frac{1}{p},p}(\Gamma_1)$ because $u_0 \in W_1^{2,p}(\Omega_0)$. Thus, as $f \in X_1^{0,p}(\Omega) \subset Y^p(\Omega)$ and $g_1 - \dfrac{\partial u_0}{\partial \boldsymbol{n}} \in W^{1-\frac{1}{p},p}(\Gamma_1) \subset W^{-\frac{1}{p},p}(\Gamma_1)$, applying Corollary 3.3.5 when $p > 2$ and Theorem 3.2.2 when $p = 2$, there exists, for $p \geq 2$, $v \in W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta v = f \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g_1 - \frac{\partial u_0}{\partial \boldsymbol{n}} \text{ on } \Gamma_1.$$

It remains to show that $v \in W_1^{2,p}(\Omega)$ and that

$$\|v\|_{W_1^{2,p}(\Omega)/\mathcal{M}_0^p(\Omega)} \leq C\ (\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \tag{3.20}$$

For this, we follow the same reasoning as in Theorem 3.3.1 using Theorem 2.1.1 ii) if $p \neq \dfrac{n}{n-1}$ and Theorem 2.1.1 iii) otherwise, and using regularity results in bounded open regions (see Lions and Magenes [42] for instance). Finally, the function $u = u_0|_\Omega + v \in W_1^{2,p}(\Omega)$ is solution of $(\mathcal{P})$ and the estimate (3.18) is a consequence of (3.19) and (3.20). $\square$

3.4 Case p $<$ 2

We are going to proceed in two steps. First of all, by an argument of duality, which allows us to use results of the previous section, we solve the problem in the case where $f = 0$ and $g_1 = 0$. The sum of the solution of this problem and of a solution of a Neumann problem will permit us to solve the general problem $(\mathcal{P})$.

In all the section, we suppose that $p < 2$ and that Γ is of class $C^{1,1}$.

3.4.1 Case where $f = 0$ and $g_1 = 0$.

Let $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ satisfying the condition of compatibility

$$\forall z \in \mathcal{M}_0^{p'}(\Omega) \ , \ < \frac{\partial z}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} = 0. \tag{3.21}$$

With this hypothesis, we consider the problem: find $v \in W_0^{1,p}(\Omega)$ solution of:

$$(\mathcal{Q}) \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma_1. \end{cases}$$

and this other problem $(\mathcal{Q}')$: find $v \in W_0^{1,p}(\Omega)$ such that for any $u \in X_{p'}(\Omega)$ satisfying $u = 0$ on Γ_0 and $\dfrac{\partial u}{\partial \boldsymbol{n}} = 0$ on Γ_1, we have:

$$T_{-\Delta u}(v) = - < \frac{\partial u}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} \tag{3.22}$$

where

$$X_{p'}(\Omega) = \{u \in W_0^{1,p'}(\Omega), \ \Delta u \in L^{p'}(\Omega)\}.$$

Proposition 3.4.1. *The problem $(\mathcal{Q})$ is equivalent to the problem $(\mathcal{Q}')$.*

Proof- Let $v \in W_0^{1,p}(\Omega)$ be a solution of $(\mathcal{Q})$. Then $\Delta v \in Y^p(\Omega)$ and for any $\varphi \in \mathcal{D}(\overline{\Omega})$:

$$0 = \int_\Omega -\varphi \ \Delta v \ d\boldsymbol{x} = \int_\Omega \nabla v \cdot \nabla \varphi \ d\boldsymbol{x} - < \frac{\partial v}{\partial \boldsymbol{n}}, \varphi >_{\Gamma_0},$$

As $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^{1,p'}(\Omega)$, we have for any $u \in W_0^{1,p'}(\Omega)$:

$$\int_\Omega \nabla v \cdot \nabla u \ d\boldsymbol{x} \ = < \frac{\partial v}{\partial \boldsymbol{n}}, u >_{\Gamma_0} .$$

Moreover, if $\Delta u \in L^{p'}(\Omega)$ then:

$$T_{-\Delta u}(v) = \int_\Omega \nabla v \ \cdot \nabla u \ d\boldsymbol{x} - < \frac{\partial u}{\partial \boldsymbol{n}}, v >_{\Gamma} .$$

In particular, for any $u \in X_{p'}(\Omega)$ such that $u = 0$ on Γ_0 and $\dfrac{\partial u}{\partial \boldsymbol{n}} = 0$ on Γ_1, we have:

$$T_{-\Delta u}(v) = - < \frac{\partial u}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0}$$

i.e. v is solution of $(\mathcal{Q}')$.

Conversely, let $v \in W^{1,p}_0(\Omega)$ be a solution of $(\mathcal{Q}')$. For any $u \in \mathcal{D}(\Omega)$, we have:

$$0 = T_{-\Delta u}(v) = < v, -\Delta u >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = < -\Delta v, u >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$$

i.e. $\Delta v = 0$ in Ω. For any $u \in X_{p'}(\Omega)$, we deduce that

$$0 = -\int_\Omega \nabla v \cdot \nabla u \, d\boldsymbol{x} + < \frac{\partial v}{\partial \boldsymbol{n}}, u >_\Gamma = -T_{-\Delta u}(v) - < \frac{\partial u}{\partial \boldsymbol{n}}, v >_\Gamma + < \frac{\partial v}{\partial \boldsymbol{n}}, u >_\Gamma .$$

Moreover, if $u = 0$ on Γ_0 and $\dfrac{\partial u}{\partial \boldsymbol{n}} = 0$ on Γ_1, then

$$< \frac{\partial u}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} - < \frac{\partial u}{\partial \boldsymbol{n}}, v >_{\Gamma_0} + < \frac{\partial v}{\partial \boldsymbol{n}}, u >_{\Gamma_1} = 0.$$

Let $R > 0$ be such that $\omega_0 \cup \omega_1 \subset B_R$ and let μ be in $W^{2-\frac{1}{p'},p'}(\Gamma_1)$. We know there exists a raising up $u \in W^{2,p'}(\Omega_R)$ such that

$$u = \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0 \cup \partial B_R, \quad u = \mu \text{ and } \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_1.$$

Let $\widetilde{u}$ be the extension by 0 outside B_R of the function u. Then $\widetilde{u}$ belongs to $X_{p'}(\Omega)$, $\widetilde{u} = 0$ on Γ_0 and $\dfrac{\partial \widetilde{u}}{\partial \boldsymbol{n}} = 0$ on Γ. So, we have $< \dfrac{\partial v}{\partial \boldsymbol{n}}, \mu >_{\Gamma_1} = 0$ for any $\mu \in W^{2-\frac{1}{p'},p'}(\Gamma_1)$. Thus $\dfrac{\partial v}{\partial \boldsymbol{n}} = 0$ on Γ_1. It remains to show that $v = g_0$ on Γ_0. Let μ be in $W^{1-\frac{1}{p'},p'}(\Gamma_0)$; we know there exists $u \in W^{2,p'}(\Omega_R)$ such that

$$u = 0 \text{ and } \frac{\partial u}{\partial \boldsymbol{n}} = \mu \text{ on } \Gamma_0, \quad u = \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_1 \cup \partial B_R.$$

The extension $\widetilde{u}$ by 0 of the function u outside B_R belongs to $X_{p'}(\Omega)$, $\widetilde{u} = 0$ on Γ and $\dfrac{\partial \widetilde{u}}{\partial \boldsymbol{n}} = 0$ on Γ_1. Thus, we have for any $\mu \in W^{1-\frac{1}{p'},p'}(\Gamma_0)$

$$< \mu, g_0 - v >_{\Gamma_0} = 0.$$

So, $v = g_0$ on Γ_0 and problems $(\mathcal{Q})$ and $(\mathcal{Q}')$ are equivalents. $\square$

Theorem 3.4.2. *For any $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ satisfying the compatibility condition (3.21), there exists a unique $v \in W^{1,p}_0(\Omega)$ solution of the problem $(\mathcal{Q})$. Moreover, v satisfies*

$$\|v\|_{W^{1,p}_0(\Omega)} \leq \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)},$$

where C is a real positive constant which depends only on p and Ω.

Proof- Let f be in $Y^{p'}(\Omega)$. It is obvious that $T_f \in (W_0^{1,p}(\Omega))'$. Since $p' > 2$, by Theorem 3.3.6, we know there exists a unique $u \in W_0^{1,p'}(\Omega)/\mathcal{M}_0^{p'}(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_1,$$

and satisfying

$$\|u\|_{W_0^{1,p'}(\Omega)/\mathcal{M}_0^{p'}(\Omega)} \leq \ C \ \|T_f\|_{(W_0^{1,p}(\Omega))'} \tag{3.23}$$

Let L be the linear form defined on $(W_0^{1,p}(\Omega))'$ by:

$$L(T_f) = \ - < \frac{\partial u}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} .$$

We are going to show that L is continuous. Let z be in $\mathcal{M}_0^{p'}(\Omega)$, then, by (3.21), we have $< \dfrac{\partial u}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} = < \dfrac{\partial (u+z)}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0}$. Now, let θ be an open region of class $C^{1,1}$ such that $\omega_0 \subset \theta \subset B_R$ where R is the radius associated to the partition of unity used in the definition of T_f. We set $\Omega_\theta = \theta \backslash \omega_0$ and let $\varphi \in W^{1,p}(\Omega_\theta)$ be such that $\varphi = 0$ on $\partial\theta$. We have:

$$| < \frac{\partial (u+z)}{\partial \boldsymbol{n}}, \varphi >_{\Gamma_0} | \leq \ \|\nabla(u+z)\|_{L^{p'}(\Omega_\theta)} \|\nabla \varphi\|_{L^p(\Omega_\theta)} + | \int_{\Omega_\theta} \varphi \ \Delta(u+z) \ d\boldsymbol{x}| \ .$$

But,

$$| \int_{\Omega_\theta} \varphi \ \Delta(u+z) \ d\boldsymbol{x}| = | \int_{\Omega_\theta} \varphi \ \Delta u \ d\boldsymbol{x}| = |T_{-\Delta u}(\widetilde{\varphi})|,$$

where $\widetilde{\varphi} \in W_0^{1,p}(\Omega)$ is defined by $\widetilde{\varphi} = \varphi$ in Ω_θ, $\widetilde{\varphi} = 0$ in $\Omega \backslash \theta$. In consequence,

$$| < \frac{\partial (u+z)}{\partial \boldsymbol{n}}, \varphi >_{\Gamma_0} | \leq \ \|\nabla(u+z)\|_{L^{p'}(\Omega_\theta)} \|\nabla \varphi\|_{L^p(\Omega_\theta)} + \|T_f\|_{(W_0^{1,p}(\Omega))'} \ \|\varphi\|_{W^{1,p}(\Omega_\theta)}.$$

Now, for any $\mu \in W^{1-\frac{1}{p},p}(\Gamma_0)$, we know there exists $\varphi \in W^{1,p}(\Omega_\theta)$ such that $\varphi = \mu$ on Γ_0 and $\varphi = 0$ on $\partial\theta$ satisfying

$$\|\varphi\|_{W^{1,p}(\Omega_\theta)} \leq \ C \ \|\mu\|_{W^{1-\frac{1}{p},p}(\Gamma_0)},$$

where $C > 0$ is a constant which depends only on Ω_θ and on μ. So

$$| < \frac{\partial (u+z)}{\partial \boldsymbol{n}}, \mu >_{\Gamma_0} | \leq \ C \ (\ \|\nabla(u+z)\|_{L^{p'}(\Omega_\theta)} + \|T_f\|_{(W_0^{1,p}(\Omega))'} \ ) \ \|\mu\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}.$$

Thus, we deduce of (3.23) that

$$\inf_{z \in \mathcal{M}_0^{p'}(\Omega)} \| \frac{\partial (u+z)}{\partial \boldsymbol{n}} \|_{W^{-1/p',p'}(\Gamma_0)} \leq \ C \ \|T_f\|_{(W_0^{1,p}(\Omega))'}.$$

So

$$|L(T_f)| = | < \frac{\partial u}{\partial \boldsymbol{n}}, g_0 > | \leq \; C \; \|T_f\|_{(W_0^{1,p}(\Omega))'} \; \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)},$$

and the linear form L is continuous on $(W_0^{1,p}(\Omega))'$. Since the space $W_0^{1,p}(\Omega)$ is reflexive, we can identify L to an element of $W_0^{1,p}(\Omega)$, *i.e.* there exists a unique $v \in W_0^{1,p}(\Omega)$ such that:

$$T_f(v) = \; - < \frac{\partial u}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0},$$

and satisfying the estimate searched. In consequence, the function v is solution of the problem $(\mathcal{Q}')$ equivalent to the problem $(\mathcal{Q})$. $\square$

3.4.2 The general problem when p < 2.

Let $f \in Y^p(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$. We remind that we search $u \in W_0^{1,p}(\Omega)$ solution of the problem $(\mathcal{P})$. Assuming that a such solution $u \in W_0^{1,p}(\Omega)$ exists, for any $\varphi \in \mathcal{M}_0^{p'}(\Omega)$, we have by the density of $\mathcal{D}(\overline{\Omega})$ in $W_0^{1,p'}(\Omega)$:

$$T_f(\varphi) = \int_\Omega \nabla u \cdot \nabla \varphi \; d\boldsymbol{x} \; - < g_1, \varphi >_{\Gamma_1} . \tag{3.24}$$

Since $\mathcal{D}(\overline{\Omega})$ is also dense in $W_0^{1,p}(\Omega)$, we have, for any $\varphi \in \mathcal{M}_0^{p'}(\Omega)$:

$$\int_\Omega \nabla \varphi \cdot \nabla u \; d\boldsymbol{x} = < \frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} . \tag{3.25}$$

We deduce from (3.24) and (3.25) that if $u \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P})$, the data must satisfy the following condition of compatibility:

$$\forall \varphi \in \mathcal{M}_0^{p'}(\Omega), \quad T_f(\varphi) = < \frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} - < g_1, \varphi >_{\Gamma_1} . \tag{3.26}$$

Theorem 3.4.3. *For any $p < 2$ and $f \in Y^p(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$ satisfying the condition of compatibility (3.26) if $1 < p \leq \dfrac{n}{n-1}$, there exists a unique $u \in W_0^{1,p}(\Omega)$ solution of the problem $(\mathcal{P})$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)} \leq \; C(\; \|f\|_{Y^p(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- First of all, we notice that the condition (3.26) is always satisfied if $\dfrac{n}{n-1} < p < 2$ because in this case $p' < n$ and $\mathcal{M}_0^{p'}(\Omega) = \{0\}$. Let θ be the constant defined by:

$$T_f(1) + < g_1, 1 >_{\Gamma_1} + < \theta, 1 >_{\Gamma_0} = 0. \tag{3.27}$$

According to Theorem 2.2.3, by (3.27), there exists a unique $w \in W_0^{1,p}(\Omega)$ such that:

$$(\mathcal{Q}_1) \quad -\Delta w = f \text{ in } \Omega,\ \frac{\partial w}{\partial \boldsymbol{n}} = \theta \text{ on } \Gamma_0,\ \frac{\partial w}{\partial \boldsymbol{n}} = g_1 \text{ on } \Gamma_1.$$

Moreover, for any $\varphi \in \mathcal{M}_0^{p'}(\Omega)$, we have, by (3.26), that

$$< \frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0 - w >_{\Gamma_0} = 0. \tag{3.28}$$

So, we can apply Theorem 3.4.2 which assures existence of a unique $v \in W_0^{1,p}(\Omega)$ such that:

$$(\mathcal{Q}_2) \quad \Delta v = 0 \text{ in } \Omega,\ v = g_0 - w \text{ on } \Gamma_0,\ \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_1,$$

satisfying

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \left(\|w\|_{W_0^{1,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} \right). \tag{3.29}$$

Finally, the function $u = v + w \in W_0^{1,p}(\Omega)$ is the solution searched and the inequality of continuous dependance comes from (3.29) and:

$$\|w\|_{W_0^{1,p}(\Omega)} \leq C \left(\|f\|_{Y^p(\Omega)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)} \right). \qquad \square$$

3.4.3 A regularity result

We suppose, in this subsection, that $p < 2$. Here, we study the regularity of solutions when the data are more regular.

Theorem 3.4.4. *For any $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$, $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$, $f \in X_1^{0,p}(\Omega)$ satisfying the condition (3.26) if $1 < p < \dfrac{n}{n-1}$, there exists a unique $u \in W_1^{2,p}(\Omega)$ solution of $(\mathcal{P})$. Moreover u satisfies*

$$\|u\|_{W_1^{2,p}(\Omega)} \leq C \left(\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)} \right), \tag{3.30}$$

where C is a real positive constant which depends only on p and Ω.

Proof- We take back the proof of Theorem 3.4.3. We know, by Proposition 3.12 in [7] that the solution $w \in W_0^{1,p}(\Omega)$ of the problem $(\mathcal{Q}_1)$ is, in this case, in $W_1^{2,p}(\Omega)$ and satisfies:

$$\|w\|_{W_1^{2,p}(\Omega)} \leq \; C\; (\|f\|_{X_1^{0,p}(\Omega)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \tag{3.31}$$

Now, we notice that, on Γ_0, $g_0 - w \in W^{2-\frac{1}{p},p}(\Gamma_0) \subset W^{1-\frac{1}{p},p}(\Gamma_0)$. Thus, applying Theorem 3.4.3, the condition of compatibility (3.26) being satisfied if $1 < p < \dfrac{n}{n-1}$, there exists a unique $v \in W_0^{1,p}(\Omega)$ solution of the problem $(\mathcal{Q}_2)$. It remains to show that $v \in W_1^{2,p}(\Omega)$ and that

$$\|v\|_{W_1^{2,p}(\Omega)} \leq \; C\; (\|f\|_{X_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \tag{3.32}$$

For this, we follow the same reasoning as in Theorem 3.3.1 using Theorem 2.1.1 ii) if $p \neq \dfrac{n}{n-1}$ and Theorem 2.1.1 iii) otherwise and also using regularity results in bounded open regions (see Lions and Magenes [42] for instance).
Finally, $u = w + v \in W_1^{2,p}(\Omega)$ is the solution searched and by (3.31) and (3.32), we have (3.30). $\square$

3.5 Solutions in homogeneous spaces

In all the section we suppose that $p < n$.
Let v_∞ be in $\mathbb{R}$. It is frequent to meet in the literature the following problem: find $v \in \mathcal{D}'(\Omega)$, with $\nabla v \in \boldsymbol{L}^p(\Omega)$ solution of

$$(\mathcal{R}) \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = g_1 & \text{on } \Gamma_1, \\ v \rightharpoonup v_\infty & \text{at infinity.} \end{cases}$$

where the sense of the convergence $v \rightharpoonup v_\infty$ is specified in the following proposition (see [11]).

Proposition 3.5.1. *We suppose that $1 < p < n$ and $z \in \mathcal{D}'(\Omega)$ such that $\nabla z \in \boldsymbol{L}^p(\Omega)$. Then, there exists a unique constant $z_\infty \in \mathbb{R}$ such that $z - z_\infty \in W_0^{1,p}(\Omega)$, where z_∞ is defined by:*

$$z_\infty = \lim_{|\boldsymbol{x}| \to +\infty} \frac{1}{|S_n|} \int_{S_n} z(\sigma|\boldsymbol{x}|)\; d\sigma$$

Moreover, we have the following proprieties:

$$z - z_\infty \in L^{\frac{np}{n-p}}(\Omega),$$
$$\|z - z_\infty\|_{L^{\frac{np}{n-p}}(\Omega)} \leq \; C\|\nabla z\|_{\boldsymbol{L}^p(\Omega)},$$
$$\lim_{|\boldsymbol{x}|\to+\infty} \frac{1}{|S_n|}\int_{S_n} |z(\sigma|\boldsymbol{x}|) - z_\infty| \; d\sigma = \lim_{|\boldsymbol{x}|\to+\infty} \frac{1}{|S_n|}\int_{S_n} |z(\sigma|\boldsymbol{x}|) - z_\infty|^p \; d\sigma = 0$$
$$\int_{S_n} |z(r\sigma) - z_\infty|^p \; d\sigma \leq \; Cr^{p-n}\int_{\{\boldsymbol{x}\in\;\Omega, |\boldsymbol{x}|>r\}} |\nabla z|^p \; d\boldsymbol{x}.$$

Let $z \in \mathcal{D}'(\Omega)$ be such that $\nabla z \in \boldsymbol{L}^p(\Omega)$. So, we say that $z \rightharpoonup z_\infty$ if and only if:

$$\lim_{|\boldsymbol{x}|\to+\infty} \frac{1}{|S_n|}\int_{S_n} (z(\sigma|\boldsymbol{x}|) - z_\infty) \; d\sigma = 0.$$

It is obvious that if $z \in \mathcal{D}'(\Omega)$, $\nabla z \in L^p(\Omega)$, then $z \rightharpoonup z_\infty$ is equivalent to $z - z_\infty \in L^{\frac{np}{n-p}}(\Omega)$ or to $z - z_\infty \in W_0^{1,p}(\Omega)$.

Proposition 3.5.2. *For any $f \in Y^p(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$, $g_1 \in W^{-\frac{1}{p},p}(\Gamma_1)$ and $v_\infty \in \mathbb{R}$, with $1 < p < n$, there exists a unique $v \in \mathcal{D}'(\Omega)$ with $\nabla v \in \boldsymbol{L}^p(\Omega)$ solution of $(\mathcal{R})$. Moreover, we have the following estimate:*

$$\|v - v_\infty\|_{W_0^{1,p}(\Omega)} \leq \; C\; (\; \|f\|_{Y^p(\Omega)} + \|g_0 - v_\infty\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- The previous sections allow us to say that there exists a unique $u \in W_0^{1,p}(\Omega)$ solution of

$$-\Delta u = f \text{ in } \Omega, \; u = g_0 - v_\infty \text{ on } \Gamma_0, \; \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \text{ on } \Gamma_1,$$

and such that

$$\|u\|_{W_0^{1,p}(\Omega)} \leq \; C\; (\; \|f\|_{Y^p(\Omega)} + \|g_0 - v_\infty\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}(\Gamma_1)}). \quad (3.33)$$

It is obvious that $u \rightharpoonup 0$. Thus, the function $v = u + v_\infty$ belongs to $\mathcal{D}'(\Omega)$, satisfies $\nabla v \in \boldsymbol{L}^p(\Omega)$ and $v \rightharpoonup v_\infty$. So, v is solution of $(\mathcal{R})$ and the estimate searched comes from the inequality (3.33). □

Now, let $\boldsymbol{a_\infty}$ be in $\mathbb{R}^n$. We want to solve the following problem: find $v \in \mathcal{D}'(\Omega)$, such that for any $i, j = 1, ..., n$, we have $\dfrac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$, solution of

$$(\mathcal{R}') \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = g_1 & \text{on } \Gamma_1, \\ \nabla v \rightharpoonup \boldsymbol{a_\infty} & \text{at infinity.} \end{cases}$$

Proposition 3.5.3. *For any $f \in L^p(\Omega)$, $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$, $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$ and $\boldsymbol{a_\infty} \in \mathbb{R}^n$, with $1 < p < n$, there exists a unique $v \in \mathcal{D}'(\Omega)$ with, for any $i, j = 1, ..., n$, $\dfrac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$, solution of $(\mathcal{R}')$ and we have the following estimate:*

$$\|v-\boldsymbol{a_\infty}\cdot\boldsymbol{x}\|_{W_0^{2,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)}+\|g_0-\boldsymbol{a_\infty}\cdot\boldsymbol{x}\|_{W^{2-\frac{1}{p},p}(\Gamma_0)}+\|g_1-\boldsymbol{a_\infty}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- First of all, we set $\widetilde{f}$ the extension of f by 0 on $\mathbb{R}^n$. So, we have $\widetilde{f} \in L^p(\mathbb{R}^n)$. Moreover, we know that $\Delta : W_0^{2,p}(\mathbb{R}^n)/\mathcal{P}_{[2-n/p]} \to L^p(\mathbb{R}^n)$ is an isomorphism, so there exists $\widetilde{u} \in W_0^{2,p}(\mathbb{R}^n)$ unique up to an additive constant, such that $-\Delta\widetilde{u} = \widetilde{f}$ in $\mathbb{R}^n$. We have $\widetilde{u}_{|\Omega} \in W_0^{2,p}(\Omega)$ and $-\Delta\widetilde{u}_{|\Omega} = f$ in Ω.
Thanks to the regularity results of the previous sections, we know there exists a unique $z \in W_1^{2,p}(\Omega) \subset W_0^{2,p}(\Omega)$ solution of the problem

$$\Delta z = 0 \text{ in } \Omega, \quad z = g_0 - \boldsymbol{a_\infty}\cdot\boldsymbol{x} - \widetilde{u} \text{ on } \Gamma_0, \quad \frac{\partial z}{\partial n} = g_1 - \boldsymbol{a_\infty}\cdot\boldsymbol{n} - (\frac{\partial\widetilde{u}}{\partial\boldsymbol{n}}) \text{ on } \Gamma_1.$$

So the function $u = \widetilde{u}_{|\Omega} + z$ is in $W_0^{2,p}(\Omega)$ and is solution of the problem

$$-\Delta u = f \text{ in } \Omega, \quad u = g_0 - \boldsymbol{a_\infty}\cdot\boldsymbol{x} \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial\boldsymbol{n}} = g_1 - \boldsymbol{a_\infty}\cdot\boldsymbol{n} \text{ on } \Gamma_1,$$

satisfying the following estimate:

$$\|u\|_{W_0^{2,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)}+\|g_0-\boldsymbol{a_\infty}\cdot\boldsymbol{x}\|_{W^{2-\frac{1}{p},p}(\Gamma_0)}+\|g_1-\boldsymbol{a_\infty}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma_1)}). \tag{3.34}$$

We have also $\nabla u \in W_0^{1,p}(\Omega)$ and $\nabla u \rightharpoonup 0$. Now, we set $v = u + \boldsymbol{a_\infty}\cdot\boldsymbol{x}$. We have $v \in \mathcal{D}'(\Omega)$ and $\forall i, j = 1, ..., n$, $\dfrac{\partial^2 v}{\partial x_i \partial x_j} = \dfrac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$ because $u \in W_0^{2,p}(\Omega)$. Moreover $\nabla v = \nabla u + \boldsymbol{a_\infty}$ so $\nabla v \rightharpoonup \boldsymbol{a_\infty}$. So v is solution of $(\mathcal{R}')$ and the estimate searched comes from (3.34). $\square$

Now, let $\boldsymbol{a_\infty}$ be in $\mathbb{R}^n$ and b_∞ in $\mathbb{R}$. We want to solve the following problem: find $v \in \mathcal{D}'(\Omega)$ with, for any $i, j = 1, ..., n$, $\dfrac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$, solution of

$$(\mathcal{R}'')\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = g_0 & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial\boldsymbol{n}} = g_1 & \text{on } \Gamma_1, \\ v - \boldsymbol{a_\infty}\cdot\boldsymbol{x} - b_\infty \rightharpoonup 0 & \text{at infinity.} \end{cases}$$

Proposition 3.5.4. *For any* $f \in X_1^{0,p}(\Omega)$, $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$, $g_1 \in W^{1-\frac{1}{p},p}(\Gamma_1)$, $\boldsymbol{a_\infty} \in \mathbb{R}^n$ *and* $b_\infty \in \mathbb{R}$, *there exists a unique* $v \in \mathcal{D}'(\Omega)$ *with, for any* $i,j = 1,...,n$, $\dfrac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$, *solution of the problem* $(\mathcal{R}'')$.

Proof- Thanks to regularity results of the previous sections, we know there exists a unique $w \in W_1^{2,p}(\Omega)$ solution of the problem

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = g_0 - \boldsymbol{a_\infty} \cdot \boldsymbol{x} - b_\infty & \text{on } \Gamma_0, \\ \dfrac{\partial w}{\partial \boldsymbol{n}} = g_1 - \boldsymbol{a_\infty} \cdot \boldsymbol{n} & \text{on } \Gamma_1. \end{cases}$$

Since $w \in W_1^{2,p}(\Omega)$, so $w \in W_0^{1,p}(\Omega)$ which implies that $w \rightharpoonup 0$. Now, we set $u = w + \boldsymbol{a_\infty} \cdot \boldsymbol{x}$, $u \in \mathcal{D}'(\Omega)$ with, for any $i,j = 1,...,n$, $\dfrac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$. Moreover, since $w \rightharpoonup 0$, we have $u - \boldsymbol{a_\infty} \cdot \boldsymbol{x} \rightharpoonup 0$ and u is solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_0 - b_\infty & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \Gamma_1. \end{cases}$$

Finally, we set $v = u + b_\infty$. We have $v \in \mathcal{D}'(\Omega)$ with, for any $i,j = 1,...,n$, $\dfrac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega)$ and v is solution of $(\mathcal{R}'')$. $\square$

Chapter 4

Exterior problems in the half-space for the Laplace operator

4.1 Introduction and preliminaries

In this chapter, we want to study the Laplace's equation in an exterior domain in the half-space, *i.e.* the complement of a compact in the half-space. Here, we remind that one of the additional difficulties is that the boundary is not bounded anymore since it contains $\mathbb{R}^{n-1}$. We refer to Section 1.3 for the definitions of spaces of traces with weights. Fortunately, the half-space has a useful symmetric property which allows us to return several times in a problem setted in an exterior domain in the whole space. Moreover, we want to study cases where there is a Dirichlet and/or a Neumann boundary condition.

So, we define ω_0 a compact and non-empty subset of $\mathbb{R}^n_+$, Γ_0 its boundary and we denote by Ω the complement of ω_0 in $\mathbb{R}^n_+$. We easily check that the property (1.1) is satisfied. We want to solve the four following problems:

$$(\mathcal{P}_D)\ \ -\Delta u = f \text{ in } \Omega, \quad u = g_0 \text{ on } \Gamma_0, \quad u = g_1 \text{ on } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_N)\ \ -\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_{M_1})\ \ -\Delta u = f \text{ in } \Omega, \quad u = g_0 \text{ on } \Gamma_0, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

$$(\mathcal{P}_{M_2})\ \ -\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g_0 \text{ on } \Gamma_0, \quad u = g_1 \text{ on } \mathbb{R}^{n-1},$$

We supposed that Ω is connected. We suppose too that Ω is of class $C^{1,1}$, even if, for some values of the exponent p, Ω can be less regular.

Each section of this chapter is devoted to the study of one of these four problems. We will call $(\mathcal{P}_{M_1})$ and $(\mathcal{P}_{M_2})$ the first and the second mixed problem. The main results of this chapter are Theorems 4.2.2, 4.3.3, 4.4.3 and 4.5.4.

We define ω_0' the symmetric region of ω_0 with respect to $\mathbb{R}^{n-1}$, Γ_0' the boundary of ω_0', Ω' the symmetric region of Ω, $\widetilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1}$ and $\widetilde{\Gamma}_0 = \Gamma_0 \cup \Gamma_0'$.

We define too the following functions ℓ^* and ℓ_*. For $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$ and ℓ any function, we set:

$$\ell^*(\boldsymbol{x'}, x_n) = \begin{cases} \ell(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ -\ell(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and

$$\ell_*(\boldsymbol{x'}, x_n) = \begin{cases} \ell(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ \ell(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0. \end{cases}$$

4.2 The problem of Dirichlet

In this section, we want to solve the following problem of Dirichlet:

$$(\mathcal{P}_D) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_0 & \text{on } \Gamma_0, \\ u = g_1 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

First, we characterize the following kernel:

$$\mathcal{D}_0^p(\Omega) = \{z \in W_0^{1,p}(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ z = 0 \text{ on } \Gamma_0,\ z = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

Proposition 4.2.1. *For any $p > 1$, $\mathcal{D}_0^p(\Omega) = \{0\}$.*

Proof- Let z be in $\mathcal{D}_0^p(\Omega)$, we define, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$ the function $z^* \in \overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega})$. For any $\varphi \in \mathcal{D}(\widetilde{\Omega})$, we have:

$$\begin{aligned} < \Delta z^*, \varphi >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} &= < z^*, \Delta\varphi >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} \\ &= \int_\Omega z(\boldsymbol{x'}, x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} - \int_{\Omega'} z(\boldsymbol{x'}, -x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x}. \end{aligned}$$

Moreover

$$\int_\Omega z(\boldsymbol{x'}, x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} = - < \frac{\partial z}{\partial \boldsymbol{n}}, \varphi >_{\mathbb{R}^{n-1}} .$$

Setting $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$, we have $\psi \in \mathcal{D}(\widetilde{\Omega})$ and

$$\begin{aligned} \int_{\Omega'} z(\boldsymbol{x'}, -x_n)\Delta\varphi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} &= \int_\Omega z(\boldsymbol{x'}, x_n)\Delta\psi(\boldsymbol{x'}, x_n)\ d\boldsymbol{x} \\ &= - < \frac{\partial z}{\partial \boldsymbol{n}}, \varphi >_{\mathbb{R}^{n-1}} . \end{aligned}$$

Thus, we deduce that $< \Delta z^*, \varphi >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} = 0$, *i.e.* $\Delta z^* = 0$ in $\widetilde{\Omega}$. So, the function z^* is in the space $\mathcal{A}_0^p(\widetilde{\Omega})$ defined by:

$$\mathcal{A}_0^p(\widetilde{\Omega}) = \{v \in \overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}),\ \Delta v = 0 \text{ in } \widetilde{\Omega},\ v = 0 \text{ on } \widetilde{\Gamma}_0\}.$$

Now, we use the characterization of $\mathcal{A}_0^p(\widetilde{\Omega})$ (see Proposition 2.2.1):

i) If $p < n$ or $p = n = 2$, then $\mathcal{A}_0^p(\widetilde{\Omega}) = \{0\}$ and $z^* = 0$ in $\widetilde{\Omega}$, *i.e.* $z = 0$ in Ω and $\mathcal{D}_0^p(\Omega) = \{0\}$.

ii) If $p \geq n \geq 3$, then we have $z^* = c(\lambda - 1)$, where c is a real constant and λ is the unique solution in $\overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}) \cap \overset{\circ}{W}{}_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta \lambda = 0 \text{ in } \widetilde{\Omega}, \quad \lambda = 1 \text{ on } \widetilde{\Gamma}_0.$$

Thus, on $\mathbb{R}^{n-1}$, $z^* = z = c(\lambda - 1) = 0$. This implies that $c = 0$ because otherwise, λ will be equal to 1 on $\mathbb{R}^{n-1}$, that is not possible because $1 \notin W_0^{\frac{1}{2},2}(\mathbb{R}^{n-1})$. Finally, we deduce that $z = 0$, *i.e.* $\mathcal{D}_0^p(\Omega) = \{0\}$.

iii) If $p > n = 2$, then we have $z^* = c(\mu - \mu_0)$, where c is a real constant and the function μ is the unique solution in $\overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}) \cap \overset{\circ}{W}{}_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta \mu = 0 \text{ in } \widetilde{\Omega}, \quad \mu = \mu_0 \text{ on } \widetilde{\Gamma}_0.$$

Thus, on $\mathbb{R}$, $z = c(\mu - \mu_0) = 0$. This implies again that $c = 0$ because otherwise μ will be equal to μ_0 on $\mathbb{R}$, that is not possible because $\mu_0 \notin W_0^{\frac{1}{2},2}(\mathbb{R})$. Indeed, let $\boldsymbol{x} = (\boldsymbol{x'}, 0)$ be in $\mathbb{R}$, since

$$\mu_0(\boldsymbol{x}) = \frac{1}{2\pi|\widetilde{\Gamma}_0|} \int_{\widetilde{\Gamma}_0} ln(|\boldsymbol{y} - \boldsymbol{x}|)\ d\sigma_{\boldsymbol{y}},$$

then $\mu_0(\boldsymbol{x'}) \geq C\ ln|\boldsymbol{x'}|$ if $|\boldsymbol{x'}| > \alpha$ with α enough big and

$$\int_{|\boldsymbol{x'}|>\alpha} \frac{|\mu_0(\boldsymbol{x'}, 0)|^2}{|\boldsymbol{x'}|\ ln^2(2 + |\boldsymbol{x'}|)}\ d\boldsymbol{x'}\ \geq C \int_{|\boldsymbol{x'}|>\alpha} \frac{d\boldsymbol{x'}}{|\boldsymbol{x'}|} = +\infty$$

that is contradictory with $\mu_0 \in W_0^{\frac{1}{2},2}(\mathbb{R})$. Thus $c = 0$ and we deduce that $z = 0$, *i.e.* $\mathcal{D}_0^p(\Omega) = \{0\}$. $\square$

Theorem 4.2.2. *For any $p > 1$, $f \in W_0^{-1,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_0^{1,p}(\Omega)$ solution of $(\mathcal{P}_D)$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)} \leq\ C(\ \|f\|_{W_0^{-1,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}), \quad (4.1)$$

where C is a real positive constant which depends only on p and ω_0.

Proof- **i)** We begin to show that solving $(\mathcal{P}_D)$ amounts to solve a problem with homogeneous boundary conditions. We know there exists $u_{g_1} \in W_0^{1,p}(\mathbb{R}^n_+)$ such that $u_{g_1} = g_1$ on $\mathbb{R}^{n-1}$ and

$$\|u_{g_1}\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq C \, \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{4.2}$$

We set $u_1 = u_{g_1|\Omega}$. Then $u_1 \in W_0^{1,p}(\Omega)$ and the trace η of u_1 on Γ_0 is in $W^{1-\frac{1}{p},p}(\Gamma_0)$. Setting $z = u - u_1$, the problem $(\mathcal{P}_D)$ is equivalent to the problem:

$$-\Delta z = f + \Delta u_1 \text{ in } \Omega, \quad z = g_0 - \eta \text{ on } \Gamma_0, \quad z = 0 \text{ on } \mathbb{R}^{n-1}.$$

We set $g = g_0 - \eta$ and $R > 0$ such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$. The function h_0 defined by:

$$h_0 = g \text{ on } \Gamma_0, \quad h_0 = 0 \text{ on } \partial B_R,$$

is in $W^{1-\frac{1}{p},p}(\Gamma_0 \cup \partial B_R)$. We know there exists $u_{h_0} \in W^{1,p}(\Omega_R)$, where $\Omega_R = \Omega \cap B_R$, such that $u_{h_0} = h_0$ on $\Gamma_0 \cup \partial B_R$ and satisfying the estimate:

$$\|u_{h_0}\|_{W^{1,p}(\Omega_R)} \leq C \, \|h_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0 \cup \partial B_R)}.$$

We set

$$u_0 = u_{h_0} \text{ in } \Omega_R, \quad u_0 = 0 \text{ in } \Omega \setminus \Omega_R.$$

We have $u_0 \in W^{1,p}(\Omega)$, $u_0 = g$ on Γ_0, $u_0 = 0$ on $\mathbb{R}^{n-1}$ and u_0 satisfies

$$\|u_0\|_{W_0^{1,p}(\Omega)} \leq \; C(\, \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{4.3}$$

Finally, setting $v = z - u_0$, the problem $(\mathcal{P}_D)$ is equivalent to the problem with homogeneous boundary conditions:

$$(\mathcal{P}') \quad -\Delta v = h \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad v = 0 \text{ on } \mathbb{R}^{n-1},$$

where $h = f + \Delta u_1 + \Delta u_0 \in W_0^{-1,p}(\Omega)$.

ii) Now, we want to return to a problem setted in the open region $\widetilde{\Omega}$, problem that we know solving. Let φ be in $\overset{\circ}{W}{}_0^{1,p'}(\widetilde{\Omega})$, we set for almost all $(\boldsymbol{x'}, x_n) \in \Omega$,

$$\pi\varphi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, x_n) - \varphi(\boldsymbol{x'}, -x_n).$$

It is obvious that $\pi\varphi \in \overset{\circ}{W}{}_0^{1,p'}(\Omega)$ and, for any $\varphi \in \overset{\circ}{W}{}_0^{1,p'}(\widetilde{\Omega})$, we define the operator h_π by

$$< h_\pi, \varphi > := < h, \pi\varphi >_{W_0^{-1,p}(\Omega) \times \overset{\circ}{W}{}_0^{1,p'}(\Omega)} .$$

We notice that h_π is in $W_0^{-1,p}(\widetilde{\Omega})$ and satisfies

$$\|h_\pi\|_{W_0^{-1,p}(\widetilde{\Omega})} \leq 2\|h\|_{W_0^{-1,p}(\Omega)}. \tag{4.4}$$

Now, we suppose that $p \geq 2$. By Theorem 2.2.2, we know there exists $w \in W_0^{1,p}(\widetilde{\Omega})$ solution of

$$-\Delta w = h_\pi \text{ in } \widetilde{\Omega}, \quad w = 0 \text{ on } \widetilde{\Gamma}_0,$$

satisfying the estimate

$$\|w\|_{W_0^{1,p}(\widetilde{\Omega})} \leq \; C \; \|h_\pi\|_{W_0^{-1,p}(\widetilde{\Omega})}. \tag{4.5}$$

The function $v = \dfrac{1}{2}\pi w$ belongs to $\overset{\circ}{W}{}_0^{1,p}(\Omega)$ and we have:

$$\|v\|_{W_0^{1,p}(\Omega)} \leq \; 2\|w\|_{W_0^{1,p}(\widetilde{\Omega})}. \tag{4.6}$$

Now, let us show that $-\Delta v = h$ in Ω. Let φ be in $\mathcal{D}(\Omega)$, then:

$$\begin{aligned} 2 < \Delta v, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= 2 < v, \Delta\varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ &= \int_\Omega [w(\boldsymbol{x'}, x_n) - w(\boldsymbol{x'}, -x_n)]\Delta\varphi \; d\boldsymbol{x}. \end{aligned}$$

Moreover, setting $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$, then $\psi \in \mathcal{D}(\Omega')$ and we have the relations

$$\int_\Omega w(\boldsymbol{x'}, x_n)\Delta\varphi \; d\boldsymbol{x} = < \Delta w, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$$

and

$$\int_\Omega w(\boldsymbol{x'}, -x_n)\Delta\varphi \; d\boldsymbol{x} = \int_{\Omega'} w(\boldsymbol{x'}, x_n)\Delta\psi \; d\boldsymbol{x} = < \Delta w, \psi >_{\mathcal{D}'(\Omega'),\mathcal{D}(\Omega')} .$$

Setting $\widetilde{\varphi}$ and $\widetilde{\psi}$ the extensions by 0 in $\widetilde{\Omega}$ of φ and ψ respectively, we deduce that:

$$\begin{aligned} 2 < \Delta v, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= < \Delta w, \widetilde{\varphi} - \widetilde{\psi} >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} \\ &= - < h_\pi, \widetilde{\varphi} - \widetilde{\psi} >_{\mathcal{D}'(\widetilde{\Omega}),\mathcal{D}(\widetilde{\Omega})} \\ &= - < h, \pi\widetilde{\varphi} - \pi\widetilde{\psi} >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ &= -2 < h, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}, \end{aligned}$$

i.e. $-\Delta v = h$ in Ω. So, we have checked that, if $p \geq 2$, the operator

$$\Delta : \overset{\circ}{W}{}_0^{1,p}(\Omega) \mapsto W_0^{-1,p}(\Omega)$$

is a isomorphism, and, by duality, the operator

$$\Delta : \overset{\circ}{W}{}_0^{1,p'}(\Omega) \mapsto W_0^{-1,p'}(\Omega)$$

is an isomorphism too. So, if $p < 2$, the problem with homogeneous boundary conditions has also a unique solution $v \in W_0^{1,p}(\Omega)$. Thus, the problem $(\mathcal{P}_D)$ has a unique solution for $1 < p < \infty$. Finally, by (4.2), (4.3), (4.4), (4.5) and (4.6), we have the estimate (4.1). □

4.3 The problem of Neumann

We remind that in this section and in the following ones, Ω is of class $C^{1,1}$. In this section, we want to solve the following problem:

$$(\mathcal{P}_N)\begin{cases} -\Delta u = f & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_0 & \text{on } \Gamma_0,\\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \mathbb{R}^{n-1}.\end{cases}$$

First, we characterize the following kernel:

$$\mathcal{N}_0^p(\Omega) = \{z \in W_0^{1,p}(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0,\ \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

Proposition 4.3.1. *For any $p > 1$, $\mathcal{N}_0^p(\Omega) = \mathcal{P}_{[1-n/p]}$.*

Proof- First, we notice that $\mathcal{P}_{[1-n/p]} \subset \mathcal{N}_0^p(\Omega)$. Let us show the other inclusion. Let z be in $\mathcal{N}_0^p(\Omega)$ and its associated function z_* which is in $W_0^{1,p}(\widetilde{\Omega})$. Since $\dfrac{\partial z}{\partial \boldsymbol{n}} = 0$ on Γ_0, we have $\dfrac{\partial z_*}{\partial \boldsymbol{n}} = 0$ on $\widetilde{\Gamma}_0$ and we check, like done in the proof of Proposition 4.2.1, that $\Delta z_* = 0$ in $\widetilde{\Omega}$. So, the function z_* belongs to the space $\{v \in W_0^{1,p}(\widetilde{\Omega}),\ \Delta v = 0 \text{ in } \widetilde{\Omega},\ \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \widetilde{\Gamma}_0\}$ which is equal to $\mathcal{P}_{[1-n/p]}$ (see Theorem 2.2.3). Thus, if $p < n$, $z_* = 0$ in $\widetilde{\Omega}$ and $z = 0$ in Ω. If $p \geq n$, z_* is constant in $\widetilde{\Omega}$, so z is constant in Ω. In other words, we have $\mathcal{N}_0^p(\Omega) = \mathcal{P}_{[1-n/p]}$. $\square$

The following theorem allows us to obtain strong solutions of the problem $(\mathcal{P}_N)$.

Theorem 4.3.2. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$ and for any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ satisfying, if $p < \dfrac{n}{n-1}$, the following compatibility condition:*

$$\int_\Omega f\ d\boldsymbol{x} + \int_{\Gamma_0} g_0\ d\sigma + \int_{\mathbb{R}^{n-1}} g_1\ d\boldsymbol{x'} = 0, \tag{4.7}$$

the problem $(\mathcal{P}_N)$ has a unique solution $u \in W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}$. Moreover, u satisfies

$$\|u\|_{W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq C(\ \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}), \tag{4.8}$$

where C is a real positive constant which depends only on p and ω_0.

Proof- First, we notice that, thanks to the hypothesis on the data, any integral of (4.7) has a meaning when $p < \dfrac{n}{n-1}$, the last one being finished because $W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \subset W_{\frac{1}{p}}^{0,p}(\mathbb{R}^{n-1}) \subset L^1(\mathbb{R}^{n-1})$. We know there exists a function $u_{g_1} \in W_1^{2,p}(\mathbb{R}^n_+)$ such that $\dfrac{\partial u_{g_1}}{\partial \boldsymbol{n}} = g_1$ and $u_{g_1} = 0$ on $\mathbb{R}^{n-1}$ satisfying:

$$\|u_{g_1}\|_{W_1^{2,p}(\mathbb{R}^n_+)} \leq C \, \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{4.9}$$

We set u_1 the restriction of u_{g_1} to Ω and η the normal derivative of u_1 on Γ_0. Finally, we set $g = g_0 - \eta \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $h = f + \Delta u_1 \in W_1^{0,p}(\Omega)$. Then, setting $v = u - u_1 \in W_1^{2,p}(\Omega)$, the problem $(\mathcal{P}_N)$ is equivalent to the following problem $(\mathcal{P}')$:

$$(\mathcal{P}') \begin{cases} -\Delta v = h & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = g & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \boldsymbol{n}} = 0 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

We construct the two functions $h_* \in W_1^{0,p}(\widetilde{\Omega})$ and $g_* \in W^{1-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ which satisfy, if $p < \dfrac{n}{n-1}$ and by (4.7), the equality $\displaystyle\int_{\widetilde{\Omega}} h_* \, d\boldsymbol{x} + \int_{\widetilde{\Gamma}_0} g_* \, d\sigma = 0$. By Proposition 3.12 in [7], there exists a function $w \in W_1^{2,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$, solution of

$$-\Delta w = h_* \text{ in } \widetilde{\Omega}, \quad \frac{\partial w}{\partial \boldsymbol{n}} = g_* \text{ on } \widetilde{\Gamma}_0,$$

such that:

$$\|w\|_{W_1^{2,p}(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq C(\, \|h\|_{W_1^{0,p}(\widetilde{\Omega})} + \|g\|_{W^{1-\frac{1}{p},p}(\widetilde{\Gamma}_0)}).$$

Now, let $w_0 \in W_1^{2,p}(\widetilde{\Omega})$ be a solution of the above problem. For almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set $v_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n)$. As h_* is even with respect to x_n, we easily check that we have $-\Delta v_0 = h_*$ in $\widetilde{\Omega}$. Moreover, by the definition of the normal derivative on $\widetilde{\Gamma}_0$, we notice that we have, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Gamma}_0$:

$$\frac{\partial v_0}{\partial \boldsymbol{n}}(\boldsymbol{x'}, x_n) = \frac{\partial w_0}{\partial \boldsymbol{n}}(\boldsymbol{x'}, -x_n).$$

As g_* is even with respect to x_n, we again easily check that we have $\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_*$ on $\widetilde{\Gamma}_0$. So $v_0 \in W_1^{2,p}(\widetilde{\Omega})$ is solution of the same problem that w_0 satisfies. Thus, the difference $v_0 - w_0$ is equal to a constant c which is necessary nil.

So $w_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n)$ and thus $\dfrac{\partial w_0}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$. The restriction v of w_0 to Ω being in $W_1^{2,p}(\Omega)$, is solution of $(\mathcal{P}')$ and satisfies:

$$\|v\|_{W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq \ C(\ \|h\|_{W_1^{0,p}(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}).$$

Finally, from this inequality and (4.9), comes the estimate (4.8). $\square$

Now, we search weak solutions of the problem $(\mathcal{P}_N)$:

Theorem 4.3.3. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, and for any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ satisfying, if $p < \dfrac{n}{n-1}$, the following condition of compatibility:*

$$\int_\Omega f \ d\boldsymbol{x} + < g_0, 1 >_{\Gamma_0} + < g_1, 1 >_{\mathbb{R}^{n-1}} = 0, \tag{4.10}$$

the problem $(\mathcal{P}_N)$ has a unique solution $u \in W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}$. Moreover, u satisfies

$$\|u\|_{W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq \ C(\ \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}) \tag{4.11}$$

where C is a real positive constant which depends only on p and ω_0.

Proof- i) First, we suppose $\dfrac{n}{p'} > 1$.

Theorem 4.3.2 assures the existence of a function $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta s = f \text{ in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

and such that

$$\|s\|_{W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq \|s\|_{W_1^{2,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq \ C \ \|f\|_{W_1^{0,p}(\Omega)}. \tag{4.12}$$

Then, by [4], there exists a function $z \in W_0^{1,p}(\mathbb{R}_+^n)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}_+^n \quad \frac{\partial z}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

satisfying the estimate

$$\|z\|_{W_0^{1,p}(\mathbb{R}_+^n)} \leq \ C \ \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{4.13}$$

We denote again by z the restriction of z to Ω. It is obvious that the normal derivative η of z on Γ_0 is in $W^{-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{-\frac{1}{p},p}(\Gamma_0)$ and we want to solve the following problem:

$$(\mathcal{P}') \quad \Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

Let μ be in $W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$. For almost any $(\boldsymbol{x'}, x_n) \in \Gamma_0$, we set

$$\pi\mu(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, x_n) + \mu(\boldsymbol{x'}, -x_n).$$

We notice that $\pi\mu \in W^{1-\frac{1}{p'},p'}(\Gamma_0)$ and we define

$$< g_\pi, \mu >:= < g, \pi\mu >_{\Gamma_0} .$$

It is obvious that $g_\pi \in W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and that g is the restriction of g_π to Γ_0. Moreover, we easily check that g_π is even with respect to x_n, *i.e.*

$$< g_\pi, \xi >_{\widetilde{\Gamma}_0} = < g_\pi, \mu >_{\widetilde{\Gamma}_0},$$

where $\xi(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, -x_n)$ with $(\boldsymbol{x'}, x_n) \in \widetilde{\Gamma}_0$. By Theorem 2.2.3, there exists a function $w \in W^{1,p}_0(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$ solution of the following problem:

$$\Delta w = 0 \text{ in } \widetilde{\Omega}, \quad \frac{\partial w}{\partial \boldsymbol{n}} = g_\pi \text{ on } \widetilde{\Gamma}_0,$$

and such that:

$$\|w\|_{W^{1,p}_0(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq C \, \|g_\pi\|_{W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)} \leq C \, \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)}.$$

Let w_0 be a solution of the problem and we set for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$:

$$v_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n).$$

The function v_0 is in $W^{1,p}_0(\widetilde{\Omega})$ and since $\Delta w_0 = 0$ on $\widetilde{\Omega}$, we easily check that Δv_0 is nil too. Thus, $\dfrac{\partial v_0}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$. Now, we want to show that $\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_\pi$ on $\widetilde{\Gamma}_0$. Let μ be in $W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$. We know there exists $\varphi \in W^{1,p'}_0(\widetilde{\Omega})$ such that $\varphi = \mu$ on $\widetilde{\Gamma}_0$ and $\|\varphi\|_{W^{1,p'}_0(\widetilde{\Omega})} \leq C \, \|\mu\|_{W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)}$. We have:

$$< \frac{\partial v_0}{\partial \boldsymbol{n}}, \mu >_{\widetilde{\Gamma}_0} = \int_{\widetilde{\Omega}} \nabla v_0 \cdot \nabla \varphi \, d\boldsymbol{x}.$$

For almost any $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$. The function ψ is in $W^{1,p'}_0(\widetilde{\Omega})$ and we set $\xi \in W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$ the trace of ψ on $\widetilde{\Gamma}_0$. We notice that $\xi(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, -x_n)$. Moreover, we show that

$$\int_{\widetilde{\Omega}} \nabla v_0 \cdot \nabla \varphi \, d\boldsymbol{x} = \int_{\widetilde{\Omega}} \nabla w_0 \cdot \nabla \psi \, d\boldsymbol{x}.$$

Thus,

$$< \frac{\partial v_0}{\partial \boldsymbol{n}}, \mu >_{\widetilde{\Gamma}_0} = < \frac{\partial w_0}{\partial \boldsymbol{n}}, \xi >_{\widetilde{\Gamma}_0} = < g_\pi, \xi >_{\widetilde{\Gamma}_0} = < g_\pi, \mu >_{\widetilde{\Gamma}_0} .$$

So $\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_\pi$ on $\widetilde{\Gamma}_0$ and v_0 is solution of the same problem that w_0 satisfies, which implies that $v_0 - w_0$ is a constant, constant which is necessary nil. The restriction of w_0 to Ω, that we denote by v, being in $W_0^{1,p}(\Omega)$, is solution of the problem $(\mathcal{P}')$ and we have the estimate

$$\|v\|_{W_0^{1,p}(\Omega)/\mathcal{P}_{[1-n/p]}} \leq C \, \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)}. \tag{4.14}$$

Finally, the function $u = z + s + v \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_N)$ and by (4.12), (4.13) and (4.14), we have (4.11).

ii) Now, we suppose that $\dfrac{n}{p'} < 1$.

Let α be in $W_1^{0,p}(\Omega)$, β in $W^{1-\frac{1}{p},p}(\Gamma_0)$ and γ in $W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ such that:

$$\int_\Omega \alpha \, d\boldsymbol{x} = \int_{\Gamma_0} \beta \, d\sigma = \int_{\mathbb{R}^{n-1}} \gamma \, d\boldsymbol{x'} = 1.$$

Here, we notice that we have $W^{1-\frac{1}{p},p}(\Gamma_0) \subset W^{-\frac{1}{p},p}(\Gamma_0)$ and since $\dfrac{n}{p'} \neq 1$, $W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \subset W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. We set

$$F = \left(\int_\Omega f \, d\boldsymbol{x}\right) \alpha, \quad G_0 = < g_0, 1 >_{\Gamma_0} \beta \ \text{ and } G_1 = < g_1, 1 >_{\mathbb{R}^{n-1}} \gamma.$$

By Theorem 4.3.2, we know there exists $r \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta r = f - F \text{ in } \Omega, \quad \frac{\partial r}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad \frac{\partial r}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

satisfying by (4.8) (since $\dfrac{n}{p} > 1$):

$$\|r\|_{W_0^{1,p}(\Omega)} \leq \|r\|_{W_1^{2,p}(\Omega)} \leq C \, \|f - F\|_{W_1^{0,p}(\Omega)}.$$

We notice, by the Hölder's inequality and because $\dfrac{n}{p'} < 1$, that

$$\|F\|_{W_1^{0,p}(\Omega)} \leq C \, \|f\|_{L^1(\Omega)} \leq C \, \|f\|_{W_1^{0,p}(\Omega)}.$$

So, we have the estimate

$$\|r\|_{W_0^{1,p}(\Omega)} \leq C \, \|f\|_{W_1^{0,p}(\Omega)}. \tag{4.15}$$

Now, by Theorem 2.3.2, since $< G_1 - g_1, 1 >_{\mathbb{R}^{n-1}} = 0$, there exists a function $z \in W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial z}{\partial \boldsymbol{n}} = g_1 - G_1 \text{ on } \mathbb{R}^{n-1},$$

such that:

$$\|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq C \, \|g_1 - G_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})} \leq C \, \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \qquad (4.16)$$

We denote again by z the restriction of z to Ω. It is obvious that the normal derivative η of z on Γ_0 is in $W^{-\frac{1}{p},p}(\Gamma_0)$ and satisfy the following equality:

$$< \eta, 1 >_{\Gamma_0} = 0.$$

We set $g = g_0 - G_0 - \eta \in W^{-\frac{1}{p},p}(\Gamma_0)$, and we may proceed with the same reasoning as in the point **i)** to show there exists $v \in W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

such that:

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \, \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)} \leq C(\, \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \qquad (4.17)$$

We notice that the compatibility condition on g_π is satisfied because $< g_\pi, 1 >_{\widetilde{\Gamma}_0} = 2 < g, 1 >_{\Gamma_0} = 0$. Finally, noticing that $F \in W_1^{0,p}(\Omega)$, $G_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$, $G_1 \in W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ and that the condition (4.10) is satisfied, thanks to Theorem 4.3.2, there exists a function $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta s = F \text{ in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = G_0 \text{ on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = G_1 \text{ on } \mathbb{R}^{n-1},$$

and satisfying the following estimate:

$$\|s\|_{W_0^{1,p}(\Omega)} \leq C(\, \|F\|_{W_1^{0,p}(\Omega)} + \|G_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|G_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \qquad (4.18)$$

Finally, the function $u = r + z + v + s \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_N)$ and the estimate (4.11) is given by (4.15), (4.16) (4.17) and (4.18). $\square$

Remark: We notice that, when the data are more regular, the weak solution is also more regular; in fact, it is the solution of Theorem 4.3.2.

4.4 The first mixed problem

In this section, we want to solve the following problem:

$$(\mathcal{P}_{M_1})\begin{cases} -\Delta u = f & \text{in } \Omega,\\ u = g_0 & \text{on } \Gamma_0,\\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \mathbb{R}^{n-1}.\end{cases}$$

First, we characterize the following kernel:

$$\mathcal{E}_0^p(\Omega) = \{z \in \overset{\circ}{W}{}_0^{1,p}(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ z = 0 \text{ on } \Gamma_0,\ \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

We have the following result:

Proposition 4.4.1. *i) If $p < n$ or $p = n = 2$, then $\mathcal{E}_0^p(\Omega) = \{0\}$.*
ii) If $p \geq n \geq 3$, then $\mathcal{E}_0^p(\Omega) = \{c(\lambda - 1),\ c \in \mathbb{R}\}$ where λ is the unique solution in $\overset{\circ}{W}{}_0^{1,2}(\Omega) \cap \overset{\circ}{W}{}_0^{1,p}(\Omega)$ of the following problem $(\mathcal{P}_1)$:

$$(\mathcal{P}_1)\ \ \Delta\lambda = 0 \text{ in } \Omega,\quad \lambda = 1 \text{ on } \Gamma_0,\quad \frac{\partial \lambda}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

iii) If $p > n = 2$, then $\mathcal{E}_0^p(\Omega) = \{c(\mu - \mu_0),\ c \in \mathbb{R}\}$ where μ is the unique solution in $\overset{\circ}{W}{}_0^{1,2}(\Omega) \cap \overset{\circ}{W}{}_0^{1,p}(\Omega)$ of the following problem $(\mathcal{P}_2)$:

$$(\mathcal{P}_2)\ \ \Delta\mu = 0 \text{ in } \Omega,\quad \mu = \mu_0 \text{ on } \Gamma_0,\quad \frac{\partial \mu}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}.$$

Proof- Let z be in $\mathcal{E}_0^p(\Omega)$. We define, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$ the function $z_* \in \overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega})$, $z_* = 0$ on $\widetilde{\Gamma}_0$ and we check, like done in the proof of Proposition 2.1 that $\Delta z_* = 0$ in $\widetilde{\Omega}$. So the function z_* is in the space

$$\mathcal{A}_0^p(\widetilde{\Omega}) = \{z \in \overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}),\ \Delta z = 0 \text{ in } \widetilde{\Omega},\ z = 0 \text{ on } \widetilde{\Gamma}_0\}$$

Now, we use the characterization of $\mathcal{A}_0^p(\widetilde{\Omega})$ (see Proposition 2.2.1).

i) If $p < n$ or if $p = n = 2$, then $\mathcal{A}_0^p(\widetilde{\Omega}) = \{0\}$ which implies that $z_* = 0$ in $\widetilde{\Omega}$ and so $z = 0$ in Ω, *i.e.* $\mathcal{E}_0^p(\Omega) = \{0\}$.

ii) If $p \geq n \geq 3$, then $z_* = c(\widetilde{\lambda} - 1)$, where c is a real constant and $\widetilde{\lambda}$ is the unique solution in $\overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}) \cap \overset{\circ}{W}{}_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta\widetilde{\lambda} = 0 \text{ in } \widetilde{\Omega},\quad \widetilde{\lambda} = 1 \text{ on } \widetilde{\Gamma}_0.$$

Now, we set, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, $\beta(\boldsymbol{x'}, x_n) = \widetilde{\lambda}(\boldsymbol{x'}, -x_n)$. We easily check that β, belonging to $\overset{\circ}{W}{}_0^{1,p}(\widetilde{\Omega}) \cap \overset{\circ}{W}{}_0^{1,2}(\widetilde{\Omega})$, is solution of the same problem that $\widetilde{\lambda}$ satisfies, but this solution is unique, so we deduce that

$\beta = \widetilde{\lambda}$ and so on $\mathbb{R}^{n-1}$, $\dfrac{\partial \widetilde{\lambda}}{\partial \boldsymbol{n}} = 0$. Thus, setting λ the restriction of $\widetilde{\lambda}$ to Ω, $\lambda \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ is solution of the problem $(\mathcal{P}_1)$. Moreover, this solution is unique. Indeed, if θ is another solution, θ_* is solution of the same problem that $\widetilde{\lambda}$ satisfies in $\widetilde{\Omega}$, so $\theta_* = \widetilde{\lambda}$ in $\widetilde{\Omega}$ and $\theta = \lambda$ in Ω.

iii) If $p > n = 2$, so, we have $z_* = c(\widetilde{\mu} - \mu_0)$, where c is a real constant and $\widetilde{\mu}$ the unique solution in $W_0^{1,p}(\widetilde{\Omega}) \cap W_0^{1,2}(\widetilde{\Omega})$ of the problem

$$\Delta \widetilde{\mu} = 0 \text{ in } \widetilde{\Omega}, \quad \widetilde{\mu} = \mu_0 \text{ on } \widetilde{\Gamma}_0.$$

But, we notice that μ_0 can also be written

$$\mu_0(\boldsymbol{x}) = \frac{1}{2\pi |\widetilde{\Gamma}_0|} \int_{\widetilde{\Gamma}_0} ln(|\boldsymbol{y} - \boldsymbol{x}|)\, d\sigma_{\boldsymbol{y}}.$$

As $\widetilde{\Gamma}_0$ is symmetric with respect to $\mathbb{R}^{n-1}$, we deduce that μ_0 is symmetric too, and so $\dfrac{\partial \mu_0}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$. Now, for $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set $\xi(\boldsymbol{x'}, x_n) = \widetilde{\mu}(\boldsymbol{x'}, -x_n)$. We check that ξ, belonging to $W_0^{1,p}(\widetilde{\Omega}) \cap W_0^{1,2}(\widetilde{\Omega})$, is solution of the same problem that $\widetilde{\mu}$ satisfies, but this solution being unique, we deduce that $\xi = \widetilde{\mu}$ and so, on $\mathbb{R}^{n-1}$, $\dfrac{\partial \widetilde{\mu}}{\partial \boldsymbol{n}} = 0$. Thus, setting μ the restriction of $\widetilde{\mu}$ to Ω, $\mu \in W_0^{1,p}(\Omega) \cap W_0^{1,2}(\Omega)$ is solution of the problem $(\mathcal{P}_2)$ and we show that this solution is unique like in the point **ii)**. Noticing that we have also $\Delta \mu_0 = 0$ in Ω, the other inclusion becomes obvious. $\square$

Let f be in $W_1^{0,p}(\Omega)$, g_0 in $W^{1-\frac{1}{p},p}(\Gamma_0)$ and g_1 in $W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. We remind that we search $u \in W_0^{1,p}(\Omega)$ solution of the problem $(\mathcal{P}_{M_1})$. We suppose that such a solution $u \in W_0^{1,p}(\Omega)$ exists. Then, for any $v \in W_0^{1,p'}(\Omega)$, we have:

$$\int_\Omega -v \Delta u \, d\boldsymbol{x} \ = \int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} - < \frac{\partial u}{\partial \boldsymbol{n}}, v >_{\Gamma_0 \cup \mathbb{R}^{n-1}} .$$

In particular, for any $\varphi \in \mathcal{E}_0^{p'}(\Omega)$:

$$\int_\Omega f \varphi \, d\boldsymbol{x} \ = \int_\Omega \nabla u \cdot \nabla \varphi \, d\boldsymbol{x} - < g_1, \varphi >_{\mathbb{R}^{n-1}} .$$

We have too:

$$0 = \int_\Omega -u \Delta \varphi \, d\boldsymbol{x} = \int_\Omega \nabla \varphi \cdot \nabla u \, d\boldsymbol{x} - < \frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} .$$

We deduce from this that if $u \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_{M_1})$, the data must satisfy the following compatibility condition:

$$\forall \varphi \in \mathcal{E}_0^{p'}(\Omega), \quad \int_\Omega f\varphi \, d\boldsymbol{x} = < \frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0} - < g_1, \varphi >_{\mathbb{R}^{n-1}} . \qquad (4.19)$$

Now, we are going to search strong solutions for the problem $(\mathcal{P}_{M_1})$.

Theorem 4.4.2. *For any $p > \dfrac{n}{n-1}$, and for any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{2-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_1^{2,p}(\Omega)/\mathcal{E}_0^p(\Omega)$ solution of $(\mathcal{P}_{M_1})$. Moreover, u satisfies*

$$\|u\|_{W_1^{2,p}(\Omega)/\mathcal{E}_0^p(\Omega)} \leq C(\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}), \qquad (4.20)$$

where C is a real positive constant which depends only on p and ω_0.

Proof- We know there exists a function $u_{g_1} \in W_1^{2,p}(\mathbb{R}^n_+)$ such that $u_{g_1} = 0$ and $\dfrac{\partial u_{g_1}}{\partial \boldsymbol{n}} = g_1$ on $\mathbb{R}^{n-1}$ satisfying the estimate:

$$\|u_{g_1}\|_{W_1^{2,p}(\mathbb{R}^n_+)} \leq C \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \qquad (4.21)$$

We set u_1 the restriction of u_{g_1} to Ω and η the trace of u_1 on Γ_0. Then, we set $g = g_0 - \eta \in W^{2-\frac{1}{p},p}(\Gamma_0)$ and $h = f + \Delta u_1 \in W_1^{0,p}(\Omega)$. Now, we must find $v \in W_1^{2,p}(\Omega)$ solution of the following problem $(\mathcal{P}')$:

$$(\mathcal{P}') \quad -\Delta v = h \text{ in } \Omega, \quad v = g \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

For this, we define the functions $h_* \in W_1^{0,p}(\widetilde{\Omega})$ and $g_* \in W^{2-\frac{1}{p},p}(\widetilde{\Gamma}_0)$. By the remark 2.11 in [7], there exists a function $w \in W_1^{2,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{A}_0^p(\widetilde{\Omega})$, solution of

$$-\Delta w = h_* \text{ in } \widetilde{\Omega}, \quad w = g_* \text{ on } \widetilde{\Gamma}_0,$$

and satisfying the estimate:

$$\|w\|_{W_1^{2,p}(\widetilde{\Omega})/\mathcal{A}_0^p(\widetilde{\Omega})} \leq C(\|h\|_{W_1^{0,p}(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma_0)}).$$

Let w_0 be a solution of this problem and for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set:

$$v_0(\boldsymbol{x'}, x_n) = w_0(\boldsymbol{x'}, -x_n).$$

Thanks to the symmetry of h_*, g_*, $\widetilde{\Omega}$ and $\widetilde{\Gamma}_0$ with respect to $\mathbb{R}^{n-1}$, we easily show that v_0 is solution of the same problem that w_0. Thus $v_0 = w_0 + k$

where $k \in \mathcal{A}_0^p(\widetilde{\Omega})$. Moreover, we show that $\dfrac{\partial k}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$ and we deduce that $\dfrac{\partial w_0}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$, so, the function v, restriction of w_0 to Ω, is in $W_1^{2,p}(\Omega)$, is solution of $(\mathcal{P}')$ and satisfies:

$$\|v\|_{W_1^{2,p}(\Omega)} \leq \ C(\ \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{2-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \quad (4.22)$$

Finally, $u = v + u_1 \in W_1^{2,p}(\Omega)$ is solution of $(\mathcal{P}_{M_1})$ and (4.20) comes from (4.21) and (4.22). $\square$

Now we search weak solutions of the problem $(\mathcal{P}_{M_1})$. For this, in the following theorem, we shall introduce a lemma between points **i)** and **ii)**. This lemma, proved thanks to the point **i)**, allows us to obtain an "inf-sup" condition, fundamental condition for the resolution of the point **ii)**.

Theorem 4.4.3. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, and for any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$, satisfying, if $p < \dfrac{n}{n-1}$, the compatibility condition (4.19), there exists a unique $u \in W_0^{1,p}(\Omega)/\mathcal{E}_0^p(\Omega)$ solution of $(\mathcal{P}_{M_1})$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)/\mathcal{E}_0^p(\Omega)} \leq \ C(\ \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}), \quad (4.23)$$

where C is a real positive constant which depends only on p and ω_0.

Proof- i) First, we suppose $\dfrac{n}{p'} > 1$, *i.e.* $p > \dfrac{n}{n-1}$.

By the previous theorem, we begin to show that there exists a function $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta s = f \text{ in } \Omega, \quad s = 0 \text{ on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

and satisfying the estimate:

$$\|s\|_{W_0^{1,p}(\Omega)} \leq \|s\|_{W_1^{2,p}(\Omega)} \leq \ C \ \|f\|_{W_1^{0,p}(\Omega)}. \quad (4.24)$$

Moreover, by Theorem 2.3.2, there exists a function $z \in W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}^n_+, \quad \frac{\partial z}{\partial \boldsymbol{n}} = g_1 \text{ on } \mathbb{R}^{n-1},$$

and such that

$$\|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq \ C \ \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \quad (4.25)$$

We denote again by z the restriction of z to Ω, so $z \in W_0^{1,p}(\Omega)$ and $\Delta z = 0$ in Ω. Now, let η be the trace of z on Γ_0. The function η is in $W^{1-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{1-\frac{1}{p},p}(\Gamma_0)$. Like done in the proof of Theorem 4.4.2 and by Theorem 2.2.2, we show there exists $v \in W_0^{1,p}(\Omega)$ solution of:

$$\Delta v = 0 \text{ in } \Omega, \quad v = g \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

and satisfying the estimate:

$$\|v\|_{W_0^{1,p}(\Omega)} \leq \; C \; \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}. \tag{4.26}$$

Finally, the function $u = s + z + v \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_{M_1})$ and the estimate (4.23) comes from (4.24), (4.25) and (4.26).

Now, we set

$$D_p = \{v \in W_0^{1,p}(\Omega), \; v = 0 \text{ on } \Gamma_0\},$$

and we introduce the following lemma to solve the point **ii)** of the theorem:

Lemma 4.4.4. *Let p be such that $p > \dfrac{n}{n-1}$. There exists a real constant $\beta > 0$ such that*

$$\inf_{\substack{w \in D_{p'} \\ w \neq 0}} \sup_{\substack{v \in D_p \\ v \neq 0}} \frac{\displaystyle\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}}{\|\nabla v\|_{\boldsymbol{L}^p(\Omega)} \|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)}} \; \geq \; \beta,$$

and the operators B from $D_p/Ker\ B$ to $(D_{p'})'$ and B' from $D_{p'}$ to $(D_p)' \perp Ker\ B$ defined by:

$$\forall v \in D_p, \; \forall w \in D_{p'}, \; < Bv, w > = < v, B'w > = \int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}$$

are isomorphisms.

Proof of the Lemma- We must firstly show an equivalent proposition to Proposition 3.3.2, *i.e.* for any $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, there exists $\boldsymbol{z} \in \overset{\circ}{H}_p(\Omega)$ and $\varphi \in D_p$, such that:

$$\begin{aligned} &\boldsymbol{g} = \nabla\varphi + \boldsymbol{z}, \\ &\|\nabla\varphi\|_{\boldsymbol{L}^p(\Omega)} \leq \; C \; \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} \end{aligned}$$

where $C > 0$ is a real constant which depends only on Ω and p and

$$\overset{\circ}{H}_p(\Omega) = \{\boldsymbol{z} \in \boldsymbol{L}^p(\Omega), \; \text{div } \boldsymbol{z} = 0 \text{ in } \Omega, \; \boldsymbol{z} \cdot \boldsymbol{n} = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

The proof takes its inspiration from the proof of Proposition 3.3.2. First, setting

$$\widetilde{\boldsymbol{g}} = \boldsymbol{g} \text{ in } \Omega, \ \widetilde{\boldsymbol{g}} = 0 \text{ in } \omega_0, \ \widetilde{\boldsymbol{g}} = 0 \text{ in } \mathbb{R}^n_-,$$

we remind (see [6]) that there exists $v \in W^{1,p}_0(\mathbb{R}^n)$, unique if $p < n$ and unique up to an additive constant otherwise, solution of

$$\Delta v = \text{div } \widetilde{\boldsymbol{g}} \text{ in } \mathbb{R}^n$$

and satisfying

$$\|\nabla v\|_{\boldsymbol{L}^p(\mathbb{R}^n)} \leq \ C \ \|\text{div } \widetilde{\boldsymbol{g}}\|_{W^{-1,p}_0(\mathbb{R}^n)} \leq \ C \ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}.$$

We denote again by v the restriction of v to Ω. We notice that, by [9], $(\boldsymbol{g} - \nabla v) \cdot \boldsymbol{n} \in W^{-\frac{1}{p},p}_0(\mathbb{R}^{n-1})$ because div $(\boldsymbol{g} - \nabla v) = 0 \in W^{0,p}_1(\Omega)$ and

$$\|(\boldsymbol{g} - \nabla v) \cdot \boldsymbol{n}\|_{W^{-\frac{1}{p},p}_0(\mathbb{R}^{n-1})} \leq \ C \ \|\boldsymbol{g} - \nabla v\|_{L^p(\Omega)}.$$

Moreover, thanks to the point **i)**, there exists a unique $w \in W^{1,p}_0(\Omega)$ solution of:

$$\Delta w = 0 \text{ in } \Omega, \quad w = -v \text{ on } \Gamma_0, \quad \frac{\partial w}{\partial \boldsymbol{n}} = (\boldsymbol{g} - \nabla v) \cdot \boldsymbol{n} \text{ on } \mathbb{R}^{n-1},$$

such that

$$\begin{aligned}\|w\|_{W^{1,p}_0(\Omega)} &\leq \ C \ (\|v\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|(\boldsymbol{g} - \nabla v) \cdot \boldsymbol{n}\|_{W^{-\frac{1}{p},p}_0(\mathbb{R}^{n-1})}) \\ &\leq \ C \ (\|v\|_{W^{1,p}_0(\mathbb{R}^n)} + \|\boldsymbol{g} - \nabla v\|_{L^p(\Omega)}) \leq \ C \ \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}.\end{aligned}$$

Then, setting $\varphi = v + w$ and $\boldsymbol{z} = \boldsymbol{g} - \nabla\varphi$, we have the searched result, and, like done in Section 3.3.2, the "inf-sup" condition. The second part of the lemma comes from the Babuška-Brezzi's theorem (see Theorem 3.3.4). □

End of the proof of Theorem 4.4.3- ii) We suppose $\frac{n}{p'} < 1$, *i.e.* $p < \frac{n}{n-1}$. Thanks to Section 4.2, we know there exists a unique $z \in W^{1,p}_0(\Omega)$ solution of the problem

$$\Delta z = 0 \text{ in } \Omega, \quad z = g_0 \text{ on } \Gamma_0, \quad z = 0 \text{ on } \mathbb{R}^{n-1},$$

and satisfying the estimate:

$$\|z\|_{W^{1,p}_0(\Omega)} \leq \ C \ \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}. \tag{4.27}$$

Since $\Delta z = 0 \in W^{0,p}_1(\Omega)$, $\eta = \dfrac{\partial z}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}_0(\mathbb{R}^{n-1})$. We set $g = g_1 - \eta \in W^{-\frac{1}{p},p}_0(\mathbb{R}^{n-1})$ and we want to solve the following problem $(\mathcal{P}')$:

$$(\mathcal{P}') \quad -\Delta v = f \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \mathbb{R}^{n-1}.$$

For this, for any $w \in D_{p'}$ we define the operator:

$$Tw = \int_\Omega fw \; d\boldsymbol{x} + < g, w >_{\mathbb{R}^{n-1}} .$$

We easily check that $T \in (D_{p'})'$. We define the following problem $(\mathcal{FV})$: find $v \in D_p$ such that for any $w \in D_{p'}$, we have:

$$\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x} = Tw.$$

We notice that if $v \in W_0^{1,p}(\Omega)$ is solution of $(\mathcal{P}')$, it is also solution of $(\mathcal{FV})$. Conversely, let $v \in D_p$ be a solution of $(\mathcal{FV})$ and let φ be in $\mathcal{D}(\Omega) \subset D_{p'}$. So

$$< \Delta v, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = - \int_\Omega \nabla v \cdot \nabla \varphi \; d\boldsymbol{x} = -T\varphi = - < f, \varphi >_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)},$$

i.e. $-\Delta v = f$ in Ω. The function $\Delta v \in W_1^{0,p}(\Omega)$, so $\dfrac{\partial v}{\partial \boldsymbol{n}}$ has a meaning in $W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$. Now, we want to show that we have $\dfrac{\partial v}{\partial \boldsymbol{n}} = g$ on $\mathbb{R}^{n-1}$. We know that, for any $\mu \in W_1^{2-\frac{1}{p'},p'}(\mathbb{R}^{n-1})$, there exists $u_1 \in W_1^{2,p'}(\mathbb{R}^n_+)$ such that

$$u_1 = \mu \;\; \text{and} \;\; \frac{\partial u_1}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

with $\|u_1\|_{W_1^{2,p'}(\mathbb{R}^n_+)} \leq C \, \|\mu\|_{W_1^{2-\frac{1}{p'},p'}(\mathbb{R}^{n-1})}$. We denote again by $u_1 \in W_1^{2,p'}(\Omega)$ the restriction of u_1 to Ω and $\xi \in W^{2-\frac{1}{p'},p'}(\Gamma_0)$ the trace of u_1 on Γ_0. There exists $u_0 \in W^{2,p'}(\Omega_R)$, where $R > 0$ is such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$, B_R is an open ball of radius R and $\Omega_R = \Omega \cap B_R$, satisfying

$$u_0 = \xi \;\; \text{and} \;\; \frac{\partial u_0}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \;\; u_0 = \frac{\partial u_0}{\partial \boldsymbol{n}} = 0 \text{ on } \partial B_R$$

and

$$\|u_0\|_{W^{2,p'}(\Omega_R)} \leq \; C \; \|\xi\|_{W^{2-\frac{1}{p'},p'}(\Gamma_0)}$$

We set $\widetilde{u}_0$ the extension of u_0 by 0 outside B_R. We have $\widetilde{u}_0 \in W_1^{2,p'}(\Omega)$ and

$$\widetilde{u}_0 = \xi \;\; \text{and} \;\; \frac{\partial \widetilde{u}_0}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \;\; \widetilde{u}_0 = \frac{\partial \widetilde{u}_0}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

with $\|\widetilde{u}_0\|_{W_1^{2,p'}(\Omega)} \leq C \, \|u_1\|_{W_1^{2,p'}(\Omega)}$. We set $u = u_1 - \widetilde{u}_0 \in W_1^{2,p'}(\Omega)$, then u satisfies

$$u = 0 \text{ on } \Gamma_0, \;\; u = \mu \;\; \text{and} \;\; \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}$$

and

$$\|u\|_{W_1^{2,p'}(\Omega)} \leq \; C \; \|\mu\|_{W_1^{2-\frac{1}{p'},p'}(\mathbb{R}^{n-1})}$$

Thus, noticing that $u \in D_{p'}$ and $\mu \in W_0^{1-\frac{1}{p'},p'}(\mathbb{R}^{n-1})$ because, for any value of n and p', $W_1^{2,p'}(\Omega) \subset W_0^{1,p'}(\Omega)$, we have

$$\begin{aligned} < \frac{\partial v}{\partial \boldsymbol{n}}, \mu >_{\mathbb{R}^{n-1}} = < \frac{\partial v}{\partial \boldsymbol{n}}, u >_{\Gamma_0 \cup \mathbb{R}^{n-1}} &= \int_\Omega u \Delta v \; d\boldsymbol{x} + \int_\Omega \nabla v \cdot \nabla u \; d\boldsymbol{x} \\ &= - \int_\Omega f u \; d\boldsymbol{x} + T u \\ &= < g, \mu >_{\mathbb{R}^{n-1}}, \end{aligned}$$

i.e. $\dfrac{\partial v}{\partial \boldsymbol{n}} = g$ on $\mathbb{R}^{n-1}$. So, problems $(\mathcal{P}')$ and $(\mathcal{FV})$ are equivalents. Moreover, since $p < \dfrac{n}{n-1}$, $p' > \dfrac{n}{n-1}$ and we apply the previous lemma noticing that we have Ker $B = \mathcal{E}_0^{p'}(\Omega)$. We deduce that

$$B' \text{ is an isomorphism from } D_p \text{ to } (D_{p'})' \perp \mathcal{E}_0^{p'}(\Omega). \tag{4.28}$$

Moreover $T \in (D_{p'})' \perp \mathcal{E}_0^{p'}(\Omega)$. Indeed, for any $\varphi \in \mathcal{E}_0^{p'}(\Omega)$, we have

$$T\varphi = \int_\Omega f \varphi \; d\boldsymbol{x} + < g_1, \varphi >_{\mathbb{R}^{n-1}} - < \eta, \varphi >_{\mathbb{R}^{n-1}}$$

and

$$< \eta, \varphi >_{\mathbb{R}^{n-1}} = < \frac{\partial z}{\partial \boldsymbol{n}}, \varphi >_{\Gamma_0 \cup \mathbb{R}^{n-1}} = \int_\Omega \nabla z \cdot \nabla \varphi \; d\boldsymbol{x} = < \frac{\partial \varphi}{\partial \boldsymbol{n}}, g_0 >_{\Gamma_0},$$

which implies, by the condition (4.19), that $T\varphi = 0$. This allows us to deduce, by (4.28), that there exists a unique $v \in D_p$ such that $B'v = T$, *i.e.* solution of $(\mathcal{FV})$ and consequently of $(\mathcal{P}')$ and we have the following estimate:

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \; (\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{4.29}$$

Finally, $u = z + v \in W_0^{1,p}(\Omega)$ is solution of $(\mathcal{P}_{M_1})$ and we have the estimate (4.23) by (4.27) and (4.29). $\square$

Remark: We notice that when $p > \dfrac{n}{n-1}$ and when the data are more regular, the weak solution is more regular too; it is in fact the solution of Theorem 4.4.2.

4.5 The second mixed problem

In this section, we want to solve the following problem:

$$(\mathcal{P}_{M_2}) \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_0 & \text{on } \Gamma_0, \\ u = g_1 & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

First, we characterize the following kernel:

$$\mathcal{F}_0^p(\Omega) = \{z \in W_0^{1,p}(\Omega),\ \Delta z = 0 \text{ in } \Omega,\ \frac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0,\ z = 0 \text{ on } \mathbb{R}^{n-1}\}.$$

Proposition 4.5.1. *For any $p > 1$, $\mathcal{F}_0^p(\Omega) = \{0\}$.*

Proof- Let z be in $\mathcal{F}_0^p(\Omega)$. We define, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$ the function $z^* \in W_0^{1,p}(\widetilde{\Omega})$. Then $\dfrac{\partial z^*}{\partial \boldsymbol{n}} = 0$ on $\widetilde{\Gamma}_0$ and we check, like done in the proof of Proposition 4.2.1 that $\Delta z^* = 0$ in $\widetilde{\Omega}$. The function z^* is in the space $\{z \in W_0^{1,p}(\widetilde{\Omega}),\ \Delta z = 0 \text{ in } \widetilde{\Omega},\ \dfrac{\partial z}{\partial \boldsymbol{n}} = 0 \text{ on } \widetilde{\Gamma}_0\}$ which is equal to $\mathcal{P}_{[1-n/p]}$ (see Theorem 2.2.3). Thus, if $p < n$, $z^* = 0$ in $\widetilde{\Omega}$ and $z = 0$ in Ω and if $p \geq n$, z^* is a constant in $\widetilde{\Omega}$ so z is constant in Ω, but $z = 0$ on $\mathbb{R}^{n-1}$, so $z = 0$ in Ω and $\mathcal{F}_0^p(\Omega) = \{0\}$. $\square$

The following theorem allows us to obtain strong solutions of the problem $(\mathcal{P}_{M_2})$.

Theorem 4.5.2. *For any $p > \dfrac{n}{n-1}$, and for any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_1^{2,p}(\Omega)$ solution of $(\mathcal{P}_{M_2})$. Moreover, u satisfies*

$$\|u\|_{W_1^{2,p}(\Omega)} \leq C(\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}), \quad (4.30)$$

where C is a real positive constant which depends only on p and ω_0.

Proof- We know there exists a function $u_{g_1} \in W_1^{2,p}(\mathbb{R}_+^n)$ such that $u_{g_1} = g_1$ and $\dfrac{\partial u_{g_1}}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$, satisfying the estimate:

$$\|u_{g_1}\|_{W_1^{2,p}(\mathbb{R}_+^n)} \leq C \|g_1\|_{W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \quad (4.31)$$

We set u_1 the restriction of u_{g_1} to Ω and η the normal derivative of u_1 on Γ_0. Then, we set $g = g_0 - \eta \in W^{1-\frac{1}{p},p}(\Gamma_0)$ and $h = f + \Delta u_1 \in W_1^{0,p}(\Omega)$. Now, we want to find $v \in W_1^{2,p}(\Omega)$ solution of the following problem $(\mathcal{P'})$:

$$(\mathcal{P'}) \quad -\Delta v = h \text{ in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \quad v = 0 \text{ on } \mathbb{R}^{n-1}.$$

We define the functions $h^* \in W_1^{0,p}(\widetilde{\Omega})$ and $g^* \in W^{1-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and, by Proposition 3.12 in [7], there exists a function $w \in W_1^{2,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$, solution of

$$-\Delta w = h^* \text{ in } \widetilde{\Omega}, \quad \frac{\partial w}{\partial \boldsymbol{n}} = g^* \text{ on } \widetilde{\Gamma}_0,$$

and satisfying the estimate:

$$\|w\|_{W_1^{2,p}(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq C(\|h\|_{W_1^{0,p}(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma_0)}).$$

Let w_0 be a solution of this problem and, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, we set:

$$v_0(\boldsymbol{x'}, x_n) = -w_0(\boldsymbol{x'}, -x_n).$$

We easily check that v_0 is solution of the same problem that w_0 satisfies. Thus $v_0 - w_0 \in \mathcal{P}_{[1-n/p]}$.

i) We suppose that $\dfrac{n}{p} > 1$. In this case, $v_0 = w_0$ in $\widetilde{\Omega}$ and we deduce that $w_0 = 0$ on $\mathbb{R}^{n-1}$. So, the function $v \in W_1^{2,p}(\Omega)$, restriction of w_0 to Ω is a solution of $(\mathcal{P}')$.

ii) We suppose that $\dfrac{n}{p} \leq 1$. In this case, $v_0 = w_0 + \alpha$ in $\widetilde{\Omega}$, where α is a real constant, and, setting $c = -\dfrac{1}{2}\alpha$, we deduce that $w_0 = c$ on $\mathbb{R}^{n-1}$. The function $v = w_{0|\Omega} - c$ is an element of $W_1^{2,p}(\Omega)$ and v is solution of $(\mathcal{P}')$.

Moreover, v, solution of $(\mathcal{P}')$, satisfies the estimate:

$$\|v\|_{W_1^{2,p}(\Omega)} \leq C(\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{1-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \quad (4.32)$$

Finally, the function $u = v + u_1 \in W_1^{2,p}(\Omega)$ is solution of $(\mathcal{P}_{M_2})$ and the estimate (4.30) comes from (4.31) and (4.32). $\square$

Now, we search weak solutions of the problem $(\mathcal{P}_{M_2})$. We set

$$N_p = \{v \in W_0^{1,p}(\Omega),\ v = 0 \text{ on } \mathbb{R}^{n-1}\},$$

and we firstly give the following lemma that we demonstrate like to Lemma 4.4.4 reversing only Γ_0 and $\mathbb{R}^{n-1}$ (and so, using in its proof the result of the point **i)** of the following theorem):

Lemma 4.5.3. *Let p be such that $p > \dfrac{n}{n-1}$. There exists a real constant $\beta > 0$ such that*

$$\inf_{\substack{w \in N_{p'} \\ w \neq 0}} \sup_{\substack{v \in N_p \\ v \neq 0}} \frac{\displaystyle\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}}{\|\nabla v\|_{\boldsymbol{L}^p(\Omega)} \|\nabla w\|_{\boldsymbol{L}^{p'}(\Omega)}} \;\geq\; \beta,$$

and the operators B from $N_p/Ker\ B$ to$(N_{p'})'$ and B' from $N_{p'}$ to $(N_p)' \perp Ker\ B$ defined by:

$$\forall v \in N_p,\ \forall w \in N_{p'},\ < Bv, w > = < v, B'w > = \int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x}$$

are isomorphisms.

Theorem 4.5.4. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, and for any $f \in W_1^{0,p}(\Omega)$, $g_0 \in W^{-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $u \in W_0^{1,p}(\Omega)$ solution of $(\mathcal{P}_{M_2})$. Moreover, u satisfies*

$$\|u\|_{W_0^{1,p}(\Omega)} \leq\ C(\ \|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}), \quad (4.33)$$

where C is a real positive constant which depends only on p and ω_0.

Proof- i) We suppose $\dfrac{n}{p'} > 1$, *i.e.* $p > \dfrac{n}{n-1}$.

First, we apply Theorem 4.5.2 to have the existence of $s \in W_1^{2,p}(\Omega) \subset W_0^{1,p}(\Omega)$ solution of the problem

$$-\Delta s = f \text{ in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0, \quad s = 0 \text{ on } \mathbb{R}^{n-1},$$

and satisfying:

$$\|s\|_{W_0^{1,p}(\Omega)} \leq \|s\|_{W_1^{2,p}(\Omega)} \leq\ C\ \|f\|_{W_1^{0,p}(\Omega)}. \quad (4.34)$$

Then, by Theorem 2.3.1, there exists a function $z \in W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$\Delta z = 0 \text{ in } \mathbb{R}^n_+ \quad z = g_1 \text{ on } \mathbb{R}^{n-1},$$

satisfying:

$$\|z\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq\ C\ \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \quad (4.35)$$

We denote again by z the restriction of z to Ω. It is obvious that the normal derivative η of z on Γ_0 is in $W^{-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta \in W^{-\frac{1}{p},p}(\Gamma_0)$ and we want to solve the following problem $(\mathcal{P}')$:

$$(\mathcal{P}')\ \ \Delta v = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \quad v = 0 \text{ on } \mathbb{R}^{n-1}.$$

Let μ be in $W^{1-\frac{1}{p'},p'}(\widetilde{\Gamma}_0)$. For almost any $(\boldsymbol{x'}, x_n) \in \Gamma_0$, we set

$$\pi\mu(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, x_n) - \mu(\boldsymbol{x'}, -x_n).$$

We notice that $\pi\mu \in W^{1-\frac{1}{p'},p'}(\Gamma_0)$, and we define

$$< g_\pi, \mu >:= < g, \pi\mu >_{\Gamma_0} .$$

It is obvious that $g_\pi \in W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and that g is the restriction of g_π to Γ_0. Moreover, we easily check that

$$< g_\pi, \xi >_{\widetilde{\Gamma}_0} = - < g_\pi, \mu >_{\widetilde{\Gamma}_0},$$

where $\xi(\boldsymbol{x'}, x_n) = \mu(\boldsymbol{x'}, -x_n)$ with $(\boldsymbol{x'}, x_n) \in \widetilde{\Gamma}_0$. By Theorem 2.2.3, there exists a function $w \in W_0^{1,p}(\widetilde{\Omega})$, unique up to an element of $\mathcal{P}_{[1-n/p]}$ solution of the following problem:

$$\Delta w = 0 \text{ in } \widetilde{\Omega}, \quad \frac{\partial w}{\partial \boldsymbol{n}} = g_\pi \text{ on } \widetilde{\Gamma}_0,$$

and such that:

$$\|w\|_{W_0^{1,p}(\widetilde{\Omega})/\mathcal{P}_{[1-n/p]}} \leq \; C \; \|g_\pi\|_{W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)}.$$

Let w_0 be a solution of this problem. We set for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$:

$$v_0(\boldsymbol{x'}, x_n) = -w_0(\boldsymbol{x'}, -x_n).$$

The function v_0 is in $W_0^{1,p}(\widetilde{\Omega})$ and since Δw_0 is nil in $\widetilde{\Omega}$, we easily check that Δv_0 is nil too. Thus $\dfrac{\partial v_0}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ and we show, like done in the proof of Theorem 4.3.3 that $\dfrac{\partial v_0}{\partial \boldsymbol{n}} = g_\pi$ on $\widetilde{\Gamma}_0$. So, the function v_0 is solution of the same problem that w_0 satisfies, which implies that $v_0 - w_0 \in \mathcal{P}_{[1-n/p]}$. We conclude like done in the proof of the previous theorem to show the existence of the solution $v \in W_0^{1,p}(\Omega)$ of the problem $(\mathcal{P}')$ satisfying:

$$\|v\|_{W_0^{1,p}(\Omega)} \leq \; C \; \|g\|_{W^{-\frac{1}{p},p}(\Gamma_0)}. \tag{4.36}$$

Finally, the function $u = z + s + v \in W_0^{1,p}(\Omega)$ is solution of the problem $(\mathcal{P}_N)$ and the estimate (4.33) comes from (4.34), (4.35) and (4.36).

ii) We suppose $\dfrac{n}{p'} < 1$, *i.e.* $p < \dfrac{n}{n-1}$.

Thanks to Section 4.2, we know there exists a unique $z \in W_0^{1,p}(\Omega)$ solution of the problem

$$\Delta z = 0 \text{ in } \Omega, \quad z = 0 \text{ on } \Gamma_0, \quad z = g_1 \text{ on } \mathbb{R}^{n-1},$$

satisfying the estimate:

$$\|z\|_{W_0^{1,p}(\Omega)} \leq \; C \; \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}. \tag{4.37}$$

Since $\Delta z = 0 \in L^p(\Omega)$, $\eta = \dfrac{\partial z}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}(\Gamma_0)$. We set $g = g_0 - \eta$ and we want to find $v \in W_0^{1,p}(\Omega)$ solution of the following problem:

$$(\mathcal{P}') \quad -\Delta v = f \text{ in } \Omega, \;\; \frac{\partial v}{\partial \boldsymbol{n}} = g \text{ on } \Gamma_0, \;\; v = 0 \text{ on } \mathbb{R}^{n-1}.$$

For this, we define, for any $w \in N_{p'}$ the operator:

$$Tw = \int_\Omega fw \; d\boldsymbol{x} + < g, w >_{\Gamma_0} .$$

We easily check that $T \in (N_{p'})'$. We define the following problem $(\mathcal{FV})$: find $v \in N_p$ such that for any $w \in N_{p'}$, we have

$$\int_\Omega \nabla v \cdot \nabla w \; d\boldsymbol{x} = Tw.$$

We notice that if $v \in W_0^{1,p}(\Omega)$ is solution of $(\mathcal{P}')$, it is also solution of $(\mathcal{FV})$. Conversely, let $v \in N_p$ be solution of $(\mathcal{FV})$ and let φ be in $\mathcal{D}(\Omega) \subset N_{p'}$. Then

$$< \Delta v, \varphi >_\Omega = -\int_\Omega \nabla v \cdot \nabla \varphi \; d\boldsymbol{x} = -T\varphi = - < f, \varphi >_\Omega,$$

i.e. $-\Delta v = f$ in Ω. The function Δv is in $L^p(\Omega)$, so $\dfrac{\partial v}{\partial \boldsymbol{n}}$ has a meaning in $W^{-\frac{1}{p},p}(\Gamma_0)$. Let us show now that $\dfrac{\partial v}{\partial \boldsymbol{n}} = g$ on Γ_0. We know that for any $\mu \in W^{2-\frac{1}{p'},p'}(\Gamma_0)$, there exists $u \in W^{2,p'}(\Omega_R)$, where $R > 0$ is such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$ and $\Omega_R = \Omega \cap B_R$ satisfying

$$\begin{cases} u = \mu \text{ and } \dfrac{\partial u}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma_0, \\ u = \dfrac{\partial u}{\partial \boldsymbol{n}} = 0 & \text{on } \partial B_R, \end{cases}$$

We denote by $\widetilde{u}$ the extension of u by 0 outside B_R. We have $\widetilde{u} \in W_1^{2,p'}(\Omega)$,

$$\begin{cases} \widetilde{u} = \mu \text{ and } \dfrac{\partial \widetilde{u}}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma_0, \\ \widetilde{u} = \dfrac{\partial \widetilde{u}}{\partial \boldsymbol{n}} = 0 & \text{on } \mathbb{R}^{n-1}, \end{cases}$$

and the following estimate

$$\|\widetilde{u}\|_{W_1^{2,p'}(\Omega)} \leq C \; \|\mu\|_{W_1^{2-\frac{1}{p'},p'}(\Gamma_0)}.$$

Thus

$$< \frac{\partial v}{\partial \boldsymbol{n}}, \mu >_{\Gamma_0} = < \frac{\partial v}{\partial \boldsymbol{n}}, \widetilde{u} >_{\Gamma_0 \cup \mathbb{R}^{n-1}} = \int_\Omega \widetilde{u} \Delta v \; d\boldsymbol{x} + \int_\Omega \nabla v \cdot \nabla \widetilde{u} \; d\boldsymbol{x}$$
$$= - \int_\Omega f \widetilde{u} \; d\boldsymbol{x} + T(\widetilde{u})$$
$$= < g, \mu >_{\Gamma_0},$$

i.e. $\dfrac{\partial v}{\partial \boldsymbol{n}} = g$ on Γ_0 and so, problems $(\mathcal{P}')$ and $(\mathcal{FV})$ are equivalents. Moreover, since $p < \dfrac{n}{n-1}$, then $p' > \dfrac{n}{n-1}$ and we apply the previous lemma noticing that Ker $B = \mathcal{F}_0^{p'}(\Omega) = \{0\}$. We deduce from this that

$$B' \text{ is an isomorphism from } N_p \text{ to } (N_{p'})'. \tag{4.38}$$

Since $T \in (N_{p'})'$, we can deduce from this that there exists a unique $v \in N_p$ such that $B'v = T$, *i.e.* solution of $(\mathcal{FV})$ and so of $(\mathcal{P}')$ and we have the following estimate:

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C \; (\|f\|_{W_1^{0,p}(\Omega)} + \|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}). \tag{4.39}$$

Finally, $u = z + v \in W_0^{1,p}(\Omega)$ is solution of $(\mathcal{P}_{M_2})$ and we have the searched estimate. □

Remark: We notice that when $p > \dfrac{n}{n-1}$ and when the data are more regular, the weak solution is more regular too; it is in fact the solution of Theorem 4.5.2.

Chapter 5

Exterior Stokes problem in the half-space

5.1 Introduction and preliminaries

In this chapter, we consider the same domain as the previous one and here we want to study the Stokes operator with Dirichlet boundary condition. Let ω_0 be a compact region of $\mathbb{R}^n_+$, Γ_0 the boundary of ω_0 and Ω the complement of ω_0 in $\mathbb{R}^n_+$. We remind that the property (1.1) is satisfied in such a domain. This paper is devoted to the resolution of the Stokes system

$$(\mathcal{S}_{\mathcal{D}}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \\ \boldsymbol{u} = \boldsymbol{g_1} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

The chapter is organized as follows. Sections 5.2 and 5.3 are devoted to the case of generalized solutions respectively when $p = 2$ and $p \neq 2$. In Section 5.4, we consider strong solutions and give regularity results according to the data. Finally, in Section 5.5, we find very weak solutions to the homogeneous problem with singular boundary conditions. The main results of this work are Theorems 5.2.5 and 5.3.6 for generalized solutions, Theorems 5.4.2 and 5.4.4 for strong solutions and Theorems 5.5.2 and 5.5.3 for very weak solutions.

In all this article, we suppose that Γ_0 is of class $C^{1,1}$, except when $p = 2$, where Γ_0 can be only considered Lipschitz-continuous.

Corresponding to Theorem 1.2.1, the following Poincaré-type inequalities hold in an exterior domain in the half-space. We give here the theorem and the proof only for the case that we use, the basis space $W^{1,2}_0(\Omega)$. We easily extend to the general case.

Theorem 5.1.1. *i) The semi-norm* $|\cdot|_{W_0^{1,2}(\Omega)}$ *defined on* $W_0^{1,2}(\Omega)/\mathcal{P}_{[1-n/2]}$ *is a norm equivalent to the quotient norm.*
ii) The semi-norm $|\cdot|_{W_0^{1,2}(\Omega)}$ *defined on* $\overset{\circ}{W}{}_0^{1,2}(\Omega)$ *is a norm equivalent to the full norm* $\|\cdot\|_{W_0^{1,2}(\Omega)}$.

Proof- We extend the problem in $\widetilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1}$ with Ω' the symmetric region of Ω with respect to $\mathbb{R}^{n-1}$ and we use results of [7] in an exterior domain. Here, we prove only the case i), the case ii) is similar.

Let u be in $W_0^{1,2}(\Omega)$ and $u_* \in W_0^{1,2}(\widetilde{\Omega})$ its extension in $\widetilde{\Omega}$ defined by

$$u_*(\boldsymbol{x'}, x_n) = \begin{cases} u(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ u(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0. \end{cases}$$

We have

$$\|u_*\|_{W_0^{1,2}(\widetilde{\Omega})} \leq C \; \|u\|_{W_0^{1,2}(\Omega)}.$$

Moreover, by [7]

$$\inf_{k \in \mathcal{P}_{[1-n/2]}} \|u_* + k\|_{W_0^{1,2}(\widetilde{\Omega})} \leq C \; |u_*|_{W_0^{1,2}(\widetilde{\Omega})}.$$

Finally, since for all $k \in \mathcal{P}_{[1-n/2]}$,

$$\|u + k\|_{W_0^{1,2}(\Omega)} \leq \|u_* + k\|_{W_0^{1,2}(\widetilde{\Omega})},$$

we have

$$\inf_{k \in \mathcal{P}_{[1-n/2]}} \|u + k\|_{W_0^{1,2}(\Omega)} \leq C \; |u_*|_{W_0^{1,2}(\widetilde{\Omega})} \leq C \; |u|_{W_0^{1,2}(\Omega)}. \qquad \square$$

5.2 Study of the problem $(\mathcal{S}_{\mathcal{D}})$ when $p = 2$.

First, we notice that it is equivalent to solve the problem with homogeneous boundary conditions. Indeed, the function $\boldsymbol{g_1}$ is in $\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})$, so, by Lemma 1.3.1, there exists $\boldsymbol{u_1} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)$ such that $\boldsymbol{u_1} = \boldsymbol{g_1}$ on $\mathbb{R}^{n-1}$ and

$$\|\boldsymbol{u_1}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} \leq \; C \; \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}.$$

Now, let $\boldsymbol{\eta}$ be the trace of $\boldsymbol{u_1}$ on Γ_0, $\boldsymbol{g} = \boldsymbol{g_0} - \boldsymbol{\eta} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$ and let $R > 0$ be such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$. It is clear that the function $\boldsymbol{h_0}$ defined by

$$\boldsymbol{h_0} = \boldsymbol{g} \text{ on } \Gamma_0, \quad \boldsymbol{h_0} = \boldsymbol{0} \text{ on } \partial B_R,$$

belongs to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0 \cup \partial B_R)$. We know that there exists an extension $\boldsymbol{u_{h_0}} \in \boldsymbol{H}^1(\Omega_R)$, where $\Omega_R = \Omega \cap B_R$, such that $\boldsymbol{u_{h_0}} = \boldsymbol{h_0}$ on $\Gamma_0 \cup \partial B_R$ and such that $\|\boldsymbol{u_{h_0}}\|_{\boldsymbol{H}^1(\Omega_R)} \leq C \, \|\boldsymbol{h_0}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0 \cup \partial B_R)}$. We set

$$\boldsymbol{u_0} = \begin{cases} \boldsymbol{u_{h_0}} & \text{in } \Omega_R, \\ \boldsymbol{0} & \text{in } \Omega \setminus \Omega_R. \end{cases}$$

We have $\boldsymbol{u_0} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u_0} = \boldsymbol{g}$ on Γ_0, $\boldsymbol{u_0} = \boldsymbol{0}$ on $\mathbb{R}^{n-1}$ and

$$\|\boldsymbol{u_0}\|_{\boldsymbol{H}^1(\Omega)} \leq C \, \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)}.$$

Thus the function $\boldsymbol{u_0} + \boldsymbol{u_{1|\Omega}}$ is in $\boldsymbol{W}_0^{1,2}(\Omega)$ and its traces are $\boldsymbol{g_0}$ on Γ_0 and $\boldsymbol{g_1}$ on Γ_1. This allows us to solve only the following problem: let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,2}(\Omega)$ and h be in $L^2(\Omega)$, we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of

$$(\mathcal{S}_0) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Now, we want to establish Lemma 5.2.2 to have a data for the divergence reduced to zero. For this, we use this preliminary lemma:

Lemma 5.2.1. *For any $h \in L^2(\Omega)$, there exists a unique $\varphi \in W_0^{2,2}(\Omega)/\mathbb{R}$ solution of*

$$\Delta \varphi = h \text{ in } \Omega \quad \text{and} \quad \frac{\partial \varphi}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}.$$

Moreover, φ satisfies

$$\|\varphi\|_{W_0^{2,2}(\Omega)/\mathbb{R}} \leq C \, \|h\|_{L^2(\Omega)},$$

where C is a real positive constant which depends only on ω_0.

Proof- First, we define Ω' the symmetric region of Ω with respect to $\mathbb{R}^{n-1}$, $\widetilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1}$ and $\widetilde{\Gamma}_0 = \partial \widetilde{\Omega}$. Let h be in $L^2(\Omega)$ and let the function h_* be defined, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, by

$$h_*(\boldsymbol{x'}, x_n) = \begin{cases} h(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ h(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Then, we set in $\mathbb{R}^n$ the function

$$\widetilde{h} = \begin{cases} h_* & \text{in } \widetilde{\Omega}, \\ 0 & \text{in } \mathbb{R}^n \setminus \widetilde{\Omega}. \end{cases}$$

So, $\widetilde{h} \in L^2(\mathbb{R}^n)$ and, supposing first that $n > 2$, as [6] allows us to say that

$$\Delta : W_0^{2,2}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

is onto, we deduce that there exists $\widetilde{u} \in W_0^{2,2}(\mathbb{R}^n)$ such that $\Delta\widetilde{u} = \widetilde{h}$ in $\mathbb{R}^n$ and $\|\widetilde{u}\|_{W_0^{2,2}(\mathbb{R}^n)} \leq \; C \; \|h\|_{L^2(\Omega)}$. We denote by $u \in W_0^{2,2}(\widetilde{\Omega})$ the restriction of $\widetilde{u}$ to $\widetilde{\Omega}$. We notice that we have $\Delta u = h_*$ in $\widetilde{\Omega}$ and that $\dfrac{\partial u}{\partial \boldsymbol{n}} \in H^{\frac{1}{2}}(\widetilde{\Gamma}_0)$. By Proposition 3.12 in [7], (there is no condition of compatibility because $n > 2$), there exists $z \in W_1^{2,2}(\widetilde{\Omega}) \subset W_0^{2,2}(\widetilde{\Omega})$ such that

$$\Delta z = 0 \text{ in } \widetilde{\Omega} \quad \text{and} \quad \frac{\partial z}{\partial \boldsymbol{n}} = \frac{\partial u}{\partial \boldsymbol{n}} \text{ on } \widetilde{\Gamma}_0,$$

and satisfying

$$\|z\|_{W_0^{2,2}(\widetilde{\Omega})} \leq \; C \; \|u\|_{W_0^{2,2}(\widetilde{\Omega})}.$$

Now, we set $w = u - z$. Then $w \in W_0^{2,2}(\widetilde{\Omega})$ satisfies

$$\Delta w = h_* \text{ in } \widetilde{\Omega} \quad \text{and} \quad \frac{\partial w}{\partial \boldsymbol{n}} = 0 \text{ on } \widetilde{\Gamma}_0, \tag{5.1}$$

and we have

$$\|w\|_{W_0^{2,2}(\widetilde{\Omega})} \leq \; C \; \|h\|_{L^2(\Omega)}.$$

If $n = 2$, we can not apply this reasoning because a condition of compatibility appears when we want to use Proposition 3.12 of [7]. Nevertheless, we can find directly $w \in W_0^{2,2}(\widetilde{\Omega})$, solution of (5.1), without needing the space $W_1^{2,2}(\widetilde{\Omega})$ (see Theorem 7.13 in [27]). Then, we set, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$,

$$v(\boldsymbol{x'}, x_n) = w(\boldsymbol{x'}, -x_n).$$

As h_* is even with respect to x_n, we easily check that v is solution of the same problem that w satisfies. So, noticing that the kernel of this problem is $\mathbb{R}$, we deduce that $v = w + c$ in $\widetilde{\Omega}$, with $c \in \mathbb{R}$, and consequently, $\dfrac{\partial w}{\partial \boldsymbol{n}} = 0$ on $\mathbb{R}^{n-1}$. Thus, the function $w_{|\Omega} \in W_0^{2,2}(\Omega)$ is solution of our problem. Moreover, this solution is unique up to a real constant. Indeed, if $z \in W_0^{2,2}(\Omega)$ is in the kernel of this problem, $z_* \in W_0^{2,2}(\widetilde{\Omega})$ is in $\mathbb{R}$, the kernel of the problem (5.1), so $z \in \mathbb{R}$. $\square$

Lemma 5.2.2. *There exists a real constant $C > 0$ depending only on ω_0 such that for any $h \in L^2(\Omega)$, there exists $\boldsymbol{w} \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega)$ satisfying*

$$\operatorname{div} \boldsymbol{w} = h \text{ in } \Omega \quad \text{and} \quad \|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \; \|h\|_{L^2(\Omega)}.$$

Proof- Let h be in $L^2(\Omega)$. We know, by the previous lemma, that there exists a unique $\varphi \in W_0^{2,2}(\Omega)/\mathbb{R}$ satisfying

$$\Delta\varphi = h \text{ in } \Omega \quad \text{and} \quad \frac{\partial \varphi}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1},$$

with

$$\|\varphi\|_{W_0^{2,2}(\Omega)/\mathbb{R}} \leq \; C \; \|h\|_{L^2(\Omega)}.$$

We set $\boldsymbol{v} = \nabla\varphi \in \boldsymbol{W}_0^{1,2}(\Omega)$. So $\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \; \|h\|_{L^2(\Omega)}$. Moreover, we set $\boldsymbol{g_0} = \boldsymbol{v}_{|\Gamma_0} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$ and $\boldsymbol{g_1} = \boldsymbol{v}_{|\mathbb{R}^{n-1}} \in \boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})$. By Theorem 2.5.1, there exists $(\boldsymbol{z}, \theta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{z} + \nabla\theta = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{z} = \boldsymbol{g_1} \text{ on } \mathbb{R}^{n-1},$$

satisfying

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} \leq \; C \; \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}.$$

We denote again by $\boldsymbol{z}$ the restriction of $\boldsymbol{z}$ to Ω and $\boldsymbol{g} = \boldsymbol{g_0} - \boldsymbol{z}_{|\Gamma_0} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$. We observe that

$$\int_{\Gamma_0} \boldsymbol{g}\cdot\boldsymbol{n}\; d\sigma = \int_{\Gamma_0} \boldsymbol{v}\cdot\boldsymbol{n}\; d\sigma - \int_{\Gamma_0} \boldsymbol{z}\cdot\boldsymbol{n}\; d\sigma = \int_{\Gamma_0} \frac{\partial\varphi}{\partial\boldsymbol{n}}\; d\sigma - \int_{\omega_0} \text{div } \boldsymbol{z}\; d\boldsymbol{x} = 0.$$

Now, let $R > 0$ be such that $\omega_0 \subset B_R \subset \mathbb{R}^n_+$ and $\Omega_R = B_R \cap \Omega$. Then, the previous condition being checked, we have the following result (see Lemma 3.3 in [5]): there exists $\boldsymbol{y} \in \boldsymbol{H}^1(\Omega_R)$ such that

$$\text{div } \boldsymbol{y} = 0 \text{ in } \Omega_R, \quad \boldsymbol{y} = \boldsymbol{g} \text{ on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} \text{ on } \partial B_R,$$

and

$$\|\boldsymbol{y}\|_{\boldsymbol{H}^1(\Omega_R)} \leq \; C_R \; (\|\boldsymbol{g_0}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}).$$

We denote again by $\boldsymbol{y}$ its extension by $\boldsymbol{0}$ in Ω. So $\boldsymbol{y} \in \boldsymbol{W}_0^{1,2}(\Omega)$ and

$$\text{div } \boldsymbol{y} = 0 \text{ in } \Omega, \quad \boldsymbol{y} = \boldsymbol{g} \text{ on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

Finally, we set $\boldsymbol{u} = \boldsymbol{z}_{|\Omega} + \boldsymbol{y} \in \boldsymbol{W}_0^{1,2}(\Omega)$. The function $\boldsymbol{u}$ satisfies

$$\text{div } \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g_0} \text{ on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{g_1} \text{ on } \mathbb{R}^{n-1},$$

and the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \; \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\Omega)}.$$

Finally the function $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ is solution of our problem. $\square$

So to solve $(\mathcal{S}_0)$, it is sufficient to solve the following problem $(\mathcal{S}_{00})$: find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of

$$(\mathcal{S}_{00}) \begin{cases} -\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

For this, as an immediate consequence of the previous lemma, we derive first the following Babuška-Brezzi condition (see [13] and [16]).

Corollary 5.2.3. *There exists a real constant $\beta > 0$, depending only on ω_0, such that*

$$\inf_{h \in L^2(\Omega)} \sup_{\boldsymbol{w} \in \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} \frac{\displaystyle\int_\Omega h \; div \; \boldsymbol{w} \; d\boldsymbol{x}}{\|\boldsymbol{w}\|_{\mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} \|h\|_{L^2(\Omega)}} \; \geq \; \frac{1}{\beta}. \tag{5.2}$$

We introduce the continuous bilinear form defined on $\mathring{\boldsymbol{W}}\,_0^{1,2}(\Omega) \times L^2(\Omega)$ by

$$b(\boldsymbol{w}, q) = -\int_\Omega q \text{ div } \boldsymbol{w} \; d\boldsymbol{x}.$$

Let $B \in \mathcal{L}\ (\mathring{\boldsymbol{W}}\,_0^{1,2}(\Omega), L^2(\Omega))$ be the associated linear operator and let $B' \in \mathcal{L}\ (L^2(\Omega), \boldsymbol{W}_0^{-1,2}(\Omega))$ the dual operator of B, *i.e.*

$$b(\boldsymbol{w}, q) = < B\boldsymbol{w}, q >_{L^2(\Omega)\times L^2(\Omega)} = < \boldsymbol{w}, B'q >_{\mathring{\boldsymbol{W}}_0^{1,2}(\Omega), \boldsymbol{W}_0^{-1,2}(\Omega)} .$$

It is clear that $B = -$ div and that $B' = \nabla$. As a consequence of the "inf-sup" condition (5.2), we know that B is an isomorphism from $\mathring{\boldsymbol{W}}\,_0^{1,2}(\Omega)/\boldsymbol{V}$ onto $L^2(\Omega)$ and B' is an isomorphism from $L^2(\Omega)$ onto $\boldsymbol{V}^\circ$ with

$$\boldsymbol{V} = \{\boldsymbol{v} \in \mathring{\boldsymbol{W}}\,_0^{1,2}(\Omega), \text{ div } \boldsymbol{v} = 0 \text{ in } \Omega\},$$

which is an Hilbert space and

$$\boldsymbol{V}^\circ = \{\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega), \ \forall \boldsymbol{w} \in \boldsymbol{V}, \ < \boldsymbol{f}, \boldsymbol{w} >_{\boldsymbol{W}_0^{-1,2}(\Omega), \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} = 0\}.$$

Thus, we have the following De Rham's theorem:

Corollary 5.2.4. *The operator ∇ is an isomorphism from $L^2(\Omega)$ to $\boldsymbol{V}^\circ$.*

Now, we define the problem: find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \ \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \; d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{v} >_{\boldsymbol{W}_0^{-1,2}(\Omega), \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} .$$

Using the Poincaré-type inequality given in Theorem 5.1.1 ii) and applying Lax-Milgram theorem, we check that the variational formulation has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$ and we notice that it is equivalent to $(\mathcal{S}_{00})$, obtaining the pressure by Corollary 5.2.4. Thus, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of $(\mathcal{S}_{00})$.

In consequence, we have the following theorem:

Theorem 5.2.5. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$, $h \in L^2(\Omega)$, $\boldsymbol{g_0} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of the problem*

$$(\mathcal{S}_\mathcal{D}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{g_1} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \; & C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} \\ & + \|\boldsymbol{g_0}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})}), \end{aligned}$$

where C is a real positive constant which depends only on ω_0.

5.3 Study of the problem $(\mathcal{S}_\mathcal{D})$ when $p \neq 2$.

First, we suppose that $p > 2$ and we want to study the kernel of the Stokes system. We set:

$$\mathcal{D}_0^p(\Omega) = \{(\boldsymbol{z}, \eta) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega), \; -\Delta \boldsymbol{z} + \nabla \eta = \boldsymbol{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega\}.$$

To characterize this space, it is useful to show the following lemma:

Lemma 5.3.1. *Let $p > 2$, $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and h be in $L^p(\mathbb{R}^n_+)$, both with compact support in $\mathbb{R}^n_+$, and $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ the unique solution of*

$$(\mathcal{S}_+) \begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{f} & \text{in } \mathbb{R}^n_+, \\ \text{div } \boldsymbol{v} = h & \text{in } \mathbb{R}^n_+, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Then, we have $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ and $(\boldsymbol{v}, \eta)$ satisfies

$$\begin{aligned} & \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+)} + \|\eta\|_{L^p(\mathbb{R}^n_+)} \\ & \qquad + \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} + \|\eta\|_{L^2(\mathbb{R}^n_+)} \leq \; C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)} + \|h\|_{L^p(\mathbb{R}^n_+)}), \end{aligned}$$

where C is a real positive constant which depends only on p, ω_0 and the support of $\boldsymbol{f}$ and h.

Proof- Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and h in $L^p(\mathbb{R}^n_+)$, both with compact support in $\mathbb{R}^n_+$; we easily check that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\mathbb{R}^n_+)$ and $h \in L^2(\mathbb{R}^n_+)$ because $p > 2$ and let $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ be the solution of $(\mathcal{S}_+)$ satisfying

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} + \|\eta\|_{L^2(\mathbb{R}^n_+)} \leq \; C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^n_+)} + \|h\|_{L^2(\mathbb{R}^n_+)}). \tag{5.3}$$

By Theorem 2.5.1, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^n) \times L^p(\mathbb{R}_+^n)$ solution of $(\mathcal{S}_+)$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}_+^n)} + \|\pi\|_{L^p(\mathbb{R}_+^n)} \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}_+^n)} + \|h\|_{L^p(\mathbb{R}_+^n)}). \tag{5.4}$$

We set $(\boldsymbol{w}, \tau) = (\boldsymbol{u} - \boldsymbol{v}, \pi - \eta)$ which satisfies

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \mathbb{R}_+^n, \quad \operatorname{div} \boldsymbol{w} = 0 \text{ in } \mathbb{R}_+^n, \quad \boldsymbol{w} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

and we want to prove that $(\boldsymbol{w}, \tau) = (\boldsymbol{0}, 0)$. We easily show (see Proposition 4.1 in [9]) that w_n, the nth component of $\boldsymbol{w}$, which is in $W_0^{1,p}(\mathbb{R}_+^n) + W_0^{1,2}(\mathbb{R}_+^n)$, satisfies

$$\Delta^2 w_n = 0 \text{ in } \mathbb{R}_+^n, \quad w_n = 0 \text{ on } \mathbb{R}^{n-1}, \quad \frac{\partial w_n}{\partial x_n} = 0 \text{ on } \mathbb{R}^{n-1}.$$

Here, the discussion splits into three steps: first, if $p \neq n$ and $n \neq 2$, then $w_n \in W_{-1}^{0,p}(\mathbb{R}_+^n) + W_{-1}^{0,2}(\mathbb{R}_+^n)$. For almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$, we set

$$\widetilde{w}_n(\boldsymbol{x'}, x_n) = \begin{cases} w_n(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ (-w_n - 2x_n \dfrac{\partial w_n}{\partial x_n} - x_n^2 \Delta w_n)(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and we check (see [10], [21]) that $\widetilde{w}_n$ is the unique extension of w_n such that $\Delta^2 \widetilde{w}_n = 0$ in $\mathbb{R}^n$. Moreover, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$< \widetilde{w}_n, \varphi >_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = \int_{\mathbb{R}_+^n} w_n [\varphi - 5\psi - 6x_n \frac{\partial \psi}{\partial x_n} - x_n^2 \Delta \psi] \, d\boldsymbol{x}$$

where $\psi \in \mathcal{D}(\mathbb{R}^n)$ is defined by $\psi(\boldsymbol{x'}, x_n) = \varphi(\boldsymbol{x'}, -x_n)$, which allows us to prove that $\widetilde{w}_n$ is in $W_{-3}^{-2,p}(\mathbb{R}^n) + W_{-3}^{-2,2}(\mathbb{R}^n)$. So $\widetilde{w}_n$ is a biharmonic tempered distribution and consequently a biharmonic polynomial. Finally, as the space $W_{-3}^{-2,p}(\mathbb{R}^n) + W_{-3}^{-2,2}(\mathbb{R}^n)$ does not contain polynomial, we deduce from this that $\widetilde{w}_n = 0$ in $\mathbb{R}^n$ and so $w_n = 0$ in $\mathbb{R}_+^n$. Now, if $n = p$, we have $W_0^{1,p}(\mathbb{R}_+^n) \subset W_{-1,-1}^{0,p}(\mathbb{R}_+^n)$, and we may proceed with the same reasoning since the logarithmic factor does not change the proof. When $n = 2$, we have $W_0^{1,2}(\mathbb{R}_+^n) \subset W_{-1,-1}^{0,2}(\mathbb{R}_+^n)$ and get the same result with the same arguments, simply noticing that w_n could be equal to a constant in $\mathbb{R}_+^n$ but that this constant would be necessary equal to zero because $w_n = 0$ on $\mathbb{R}^{n-1}$.

Consequently, in any case, we have $w_n = 0$ in $\mathbb{R}_+^n$. We deduce from this (see Proposition 4.1, [9]) that $\tau \in L^p(\mathbb{R}_+^n) + L^2(\mathbb{R}_+^n)$ satisfies

$$\Delta \tau = 0 \text{ in } \mathbb{R}_+^n, \quad \frac{\partial \tau}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1}.$$

Now, we set for almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$,

$$\tau_*(\boldsymbol{x'}, x_n) = \begin{cases} \tau(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ \tau(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and we easily check that τ_* is a harmonic tempered distribution, so a harmonic polynomial, included in $L^p(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, a space which does not contain polynomial. Thus, we conclude that $\tau = 0$ in $\mathbb{R}^n_+$. Then, we show that $\boldsymbol{w'} = (w_1, \dots w_{n-1}) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^n_+) + \boldsymbol{W}^{1,2}_0(\mathbb{R}^n_+)$ satisfies

$$\Delta \boldsymbol{w'} = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w'} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}.$$

We set for almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$,

$$\boldsymbol{w'}^*(\boldsymbol{x'}, x_n) = \begin{cases} \boldsymbol{w'}(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ -\boldsymbol{w'}(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0, \end{cases}$$

and we easily check that $\boldsymbol{w'}^* \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^n) + \boldsymbol{W}^{1,2}_0(\mathbb{R}^n)$ is a harmonic tempered distribution, so a harmonic polynomial in $\mathbb{R}^n$. Thus, $\boldsymbol{w'}$ is a harmonic polynomial in $\mathbb{R}^n_+$ and $\nabla \boldsymbol{w'}$ is a harmonic polynomial in $\boldsymbol{L}^p(\mathbb{R}^n_+) + \boldsymbol{L}^2(\mathbb{R}^n_+)$, a space which does not contain polynomial. So $\nabla \boldsymbol{w'} = \boldsymbol{0}$ in $\mathbb{R}^n_+$ and since $\boldsymbol{w'} = \boldsymbol{0}$ in $\mathbb{R}^{n-1}$, we have $\boldsymbol{w'} = \boldsymbol{0}$ in $\mathbb{R}^n_+$. Finally, we deduce from this that $(\boldsymbol{w}, \tau) = (\boldsymbol{0}, 0)$. $\square$

Now, we have the following theorem:

Theorem 5.3.2. *The kernel $\mathcal{D}^p_0(\Omega)$ is reduced to $\{(\boldsymbol{0}, 0)\}$ when $p > 2$.*

Proof- Let $(\boldsymbol{z}, \pi)$ be in $\mathcal{D}^p_0(\Omega)$. We denote by $\widetilde{\boldsymbol{z}}$ and $\widetilde{\pi}$ the extensions by 0 of $\boldsymbol{z}$ and π in $\mathbb{R}^n_+$. We have $\widetilde{\boldsymbol{z}} \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^n_+)$ and $\widetilde{\pi} \in L^p(\mathbb{R}^n_+)$. We set $\widetilde{\boldsymbol{h}} = -\Delta \widetilde{\boldsymbol{z}} + \nabla \widetilde{\pi} \in \boldsymbol{W}^{-1,p}_0(\mathbb{R}^n_+)$ and we easily check that $\widetilde{\boldsymbol{h}}$ has a compact support in $\mathbb{R}^n_+$. Thus, we can apply the previous lemma which assures us that there exists a unique $(\boldsymbol{v}, \eta) \in (\boldsymbol{W}^{1,p}_0(\mathbb{R}^n_+) \cap \boldsymbol{W}^{1,2}_0(\mathbb{R}^n_+)) \times (L^p(\mathbb{R}^n_+) \cap L^2(\mathbb{R}^n_+))$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{h}} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}.$$

Noticing that div $\widetilde{\boldsymbol{z}} = 0$ in $\mathbb{R}^n_+$, we see that $(\widetilde{\boldsymbol{z}}, \widetilde{\pi})$ and $(\boldsymbol{v}, \eta)$ are solutions of the same problem, which, by Theorem 2.5.1, has a unique solution in $\boldsymbol{W}^{1,p}_0(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$. So $(\widetilde{\boldsymbol{z}}, \widetilde{\pi}) = (\boldsymbol{v}, \eta)$ in $\mathbb{R}^n_+$ and, setting again $\boldsymbol{v}$ and η the restrictions of $\boldsymbol{v}$ and η to Ω, we deduce that

$$\boldsymbol{v} = \boldsymbol{z}, \quad \eta = \pi \quad \text{in } \Omega.$$

So, $(\boldsymbol{v}, \eta) \in \boldsymbol{W}^{1,p}_0(\Omega) \times L^p(\Omega)$ satisfies

$$-\Delta \boldsymbol{v} + \nabla \eta = 0 \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}.$$

But, $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ and in this space, there is, by Theorem 5.2.5, a unique solution to the above problem, which is $(\boldsymbol{0}, 0)$. Thus, $\mathcal{D}_0^p(\Omega) = \{(\boldsymbol{0}, 0)\}$. $\square$

Now, supposing that $p > 2$, we want to solve the Stokes system with homogeneous boundary conditions, that is to say: let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$, we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of the problem

$$(\mathcal{S}_0) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

First, we establish the following lemma:

Lemma 5.3.3. *For each $p > 2$ and for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$, both with compact support in Ω, there exists a unique $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)) \times (L^p(\Omega) \cap L^2(\Omega))$ solution of $(\mathcal{S}_0)$.*

Proof- Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$, both with compact support in Ω. Then, since $p > 2$, we easily check that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $h \in L^2(\Omega)$ and that

$$\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} \leq \; C \; (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}),$$

where C is a real positive constant which depends only on p, ω_0 and the supports of $\boldsymbol{f}$ and h. We deduce from Theorem 5.2.5 that there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of $(\mathcal{S}_0)$. It remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. We denote by $\widetilde{\boldsymbol{u}} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)$ and $\widetilde{\pi} \in L^2(\mathbb{R}^n_+)$ the extensions by 0 in $\mathbb{R}^n_+$ of $\boldsymbol{u}$ and π and we set

$$\widetilde{\boldsymbol{f}} = -\Delta \widetilde{\boldsymbol{u}} + \nabla \widetilde{\pi} \quad \text{and} \quad \widetilde{h} = \text{ div } \widetilde{\boldsymbol{u}}.$$

Let us show now that $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and $\widetilde{h} \in L^p(\mathbb{R}^n_+)$. We define the function $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ in θ where θ is open bounded subset of Ω such that supp $\boldsymbol{f} \subset \theta$. We denote by $\widetilde{\chi}$ the extension of χ by 0 in $\mathbb{R}^n_+$. For $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)$, we have

$$< \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} >_{\boldsymbol{\mathcal{D}'}(\mathbb{R}^n_+), \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)} = < \widetilde{\boldsymbol{f}}, \widetilde{\chi} \boldsymbol{\varphi} >_{\boldsymbol{\mathcal{D}'}(\mathbb{R}^n_+), \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)} = < \boldsymbol{f}, \chi \boldsymbol{\varphi} >_{\boldsymbol{\mathcal{D}'}(\Omega), \boldsymbol{\mathcal{D}}(\Omega)}$$

and for $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$, we have

$$< \widetilde{h}, \varphi >_{\mathcal{D}'(\mathbb{R}^n_+), \mathcal{D}(\mathbb{R}^n_+)} = \int_\Omega h \varphi \; d\boldsymbol{x}.$$

So, $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+)$ and $\widetilde{h} \in L^p(\mathbb{R}^n_+)$. Finally, we can apply Lemma 5.3.1 to conclude that $(\widetilde{\boldsymbol{u}}, \widetilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$. Thus, by restriction, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. $\square$

Now, we establish the following theorem:

Theorem 5.3.4. *For any $p > 2$, $\boldsymbol{g_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of*

$$(\mathcal{S}')\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{g_1} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C \,(\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- The uniqueness comes from Theorem 5.3.2. Then, thanks to Theorem 2.5.1, there exists a unique $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \operatorname{div} \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \boldsymbol{g_1} \text{ on } \mathbb{R}^{n-1}.$$

We denote again by $\boldsymbol{w}$ and τ the restrictions of $\boldsymbol{w}$ and τ to Ω and we set $\boldsymbol{g} = \boldsymbol{g_0} - \boldsymbol{w}_{|\Gamma_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$. Thus, it remains to show that there exists $(\boldsymbol{y}, \lambda) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}'')$

$$(\mathcal{S}'')\begin{cases} -\Delta \boldsymbol{y} + \nabla \lambda = \boldsymbol{0} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{y} = 0 \quad \text{in } \Omega, \\ \boldsymbol{y} = \boldsymbol{g} & \text{on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

For this, let $R > 0$ be such that $\boldsymbol{w}_0 \subset B_R \subset \mathbb{R}^n_+$, $\Omega_R = B_R \cap \Omega$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$ with support included in Ω_R such that

$$\int_{\Omega_R} \psi(\boldsymbol{x}) \; d\boldsymbol{x} + \int_{\Gamma_0} \boldsymbol{g} \cdot \boldsymbol{n} \; d\boldsymbol{\sigma} = 0.$$

Thanks to results in bounded domains (see [5]), there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}^{1,p}(\Omega_R) \times L^p(\Omega_R)$ such that

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{0} & \text{in } \Omega_R, \quad \operatorname{div} \boldsymbol{v} = \psi \quad \text{in } \Omega_R, \\ \boldsymbol{v} = \boldsymbol{g} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial B_R. \end{cases}$$

Next, we extend $(\boldsymbol{v}, \eta)$ by $(\boldsymbol{0}, 0)$ in Ω and we denote by $(\widetilde{\boldsymbol{v}}, \widetilde{\eta}) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ this extension which satisfies

$$\begin{cases} -\Delta \widetilde{\boldsymbol{v}} + \nabla \widetilde{\eta} = \boldsymbol{\xi} & \text{in } \Omega, \quad \operatorname{div} \widetilde{\boldsymbol{v}} = \psi \quad \text{in } \Omega, \\ \widetilde{\boldsymbol{v}} = \boldsymbol{g} & \text{on } \Gamma_0, \quad \widetilde{\boldsymbol{v}} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

where $\boldsymbol{\xi} \in \boldsymbol{W}_0^{-1,p}(\Omega)$. We notice that $\boldsymbol{\xi}$ and ψ have a compact support in Ω_R, so that by the previous lemma, that there exists $(\boldsymbol{z}, \nu) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{z} + \nabla \nu = -\boldsymbol{\xi} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{z} = -\psi \quad \text{in } \Omega, \\ \boldsymbol{z} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{z} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Finally, $(\boldsymbol{y}, \lambda) = (\widetilde{\boldsymbol{v}} + \boldsymbol{z}, \widetilde{\eta} + \nu) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}'')$, so $(\boldsymbol{u}, \pi) = (\boldsymbol{w} + \boldsymbol{y}, \mu + \lambda) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}')$ and the estimate follows immediately. $\square$

Now, we can solve the problem with homogeneous boundary conditions in the case $p > 2$.

Theorem 5.3.5. *For any $p > 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_0)$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C \; (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- The uniqueness comes from Theorem 5.3.2. Then, there exists, as a consequence of Theorem 5.1.1 ii), a tensor of second order $F \in [L^p(\Omega)]^{n \times n}$ such that div $F = \boldsymbol{f}$. We extend F (respectively h) by 0 in $\mathbb{R}^n$, and we denote by $\widetilde{F}$ (respectively $\widetilde{h}$) this extension. Then, we set $\widetilde{\boldsymbol{f}} = \text{div } \widetilde{F}$ and we notice that $\widetilde{\boldsymbol{f}}_{|\Omega} = \boldsymbol{f}$. We have $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)$ and $\widetilde{h} \in L^p(\mathbb{R}^n)$. By Theorem 2.4.1, there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{f}} \quad \text{and} \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n.$$

We denote again by $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\eta \in L^p(\Omega)$ the restrictions of $\boldsymbol{v}$ and η to Ω. We have $\boldsymbol{v}_{|\Gamma_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{v}_{|\mathbb{R}^{n-1}} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, thus, by Theorem 5.3.4, there exists $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ \boldsymbol{w} = -\boldsymbol{v}_{|\Gamma_0} & \text{on } \Gamma_0, \quad \boldsymbol{w} = -\boldsymbol{v}_{|\mathbb{R}^{n-1}} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{w}, \eta + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}_0)$ and the estimate follows immediately. $\square$

Now, we suppose that $1 < p < 2$. By the previous theorem,

$$\begin{aligned} S : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) &\longrightarrow \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega), \\ (\boldsymbol{u}, \pi) &\longrightarrow (-\Delta \boldsymbol{u} + \nabla \pi, -\text{div } \boldsymbol{u}), \end{aligned}$$

then, S is an isomorphism. So, by duality,

$$S^* : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega) \longrightarrow \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega),$$

is also an isomorphism and, as it is standard to check that $S^*(\boldsymbol{u}, \pi) = (-\Delta \boldsymbol{u} + \nabla \pi, -\text{div } \boldsymbol{u})$, we have Theorem 5.3.5 for any $p < 2$. $\square$

Finally, it remains to return to the general problem with $p \neq 2$ and nonhomogeneous boundary conditions. For this, like for the case $p = 2$, we show that there exists a function $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{w} = \boldsymbol{g_0}$ in Γ_0 and $\boldsymbol{w} = \boldsymbol{g_1}$ in $\mathbb{R}^{n-1}$. Then, we have just seen that there exists a unique $(\boldsymbol{v}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla\pi = \boldsymbol{f} + \Delta \boldsymbol{w} & \text{in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma_0, \\ \text{div } \boldsymbol{v} = h - \text{div } \boldsymbol{w} & \text{in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

In consequence, the function $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is a solution of the problem $(\mathcal{S}_\mathcal{D})$ and we have the following theorem:

Theorem 5.3.6. *For any $p \neq 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g_0} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of the problem $(\mathcal{S}_\mathcal{D})$*

$$(\mathcal{S}_\mathcal{D}) \begin{cases} -\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \quad \boldsymbol{u} = \boldsymbol{g_1} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g_0}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

5.4 Strong solutions and regularity for the Stokes system $(\mathcal{S}_\mathcal{D})$.

In this section, we are interested in the existence of strong solutions of the Stokes system $(\mathcal{S}_\mathcal{D})$, *i.e.* of solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{\ell+1}^{2,p}(\Omega) \times W_{\ell+1}^{1,p}(\Omega)$. Here, we limit ourselves to the two cases $\ell = 0$ and $\ell = -1$.

First, we give results for the case $\ell = 0$. We notice that in this case, we have the continuous injections $\boldsymbol{W}_1^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{1,p}(\Omega)$ and $W_1^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. So, the two theorems which follow show that generalized solutions of Theorems 5.2.5 and 5.3.6, with a stronger hypothesis on the data, are in fact strong solutions.

Theorem 5.4.1. *For any $p > 1$ satisfying $\dfrac{n}{p'} \neq 1$, $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$ and $h \in W_1^{1,p}(\Omega)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ solution of $(\mathcal{S}_0)$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\Omega)} + \|\pi\|_{W_1^{1,p}(\Omega)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\Omega)} + \|h\|_{W_1^{1,p}(\Omega)}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- First, we notice that we have the continuous injections $\boldsymbol{W}_1^{0,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\Omega)$ because $\dfrac{n}{p'} \neq 1$ and $W_1^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. Thus, by Theorems 5.2.5 ($p = 2$) and 5.3.6 ($p \neq 2$), there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_0)$. It remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$. For this, we introduce the following partition of unity:

$$\psi_1,\ \psi_2 \in C^\infty(\mathbb{R}^n),\ 0 \leq \psi_1, \psi_2 \leq 1,\ \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n,$$
$$\psi_1 = 1 \text{ in } B_R, \quad \text{supp } \psi_1 \subset B_{R'},$$

with $0 < R < R' < \infty$ such that $\omega_0 \subset B_R \subset B_{R'} \subset \mathbb{R}^n_+$. We set $\Omega_R = \Omega \cap B_R$, $\Omega_{R'} = \Omega \cap B_{R'}$, $\boldsymbol{u_i} = \psi_i \boldsymbol{u} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega)$ and $\pi_i = \psi_i \pi \in L^p(\Omega)$ for $i = 1$ or 2. We notice that supp $(\boldsymbol{u_1}, \pi_1) \subset \Omega_{R'}$ and we denote by $(\widetilde{\boldsymbol{u}}_{\boldsymbol{1}}, \widetilde{\pi}_1)$ the extension by $(\mathbf{0}, 0)$ of $(\boldsymbol{u_1}, \pi_1)$ in ${}^c\omega_0$. Finally, we set

$$\widetilde{\boldsymbol{f}}_{\boldsymbol{1}} = -\Delta \widetilde{\boldsymbol{u}}_{\boldsymbol{1}} + \nabla \widetilde{\pi}_1, \quad \widetilde{h}_1 = \text{ div } \widetilde{\boldsymbol{u}}_{\boldsymbol{1}}$$

and $(\boldsymbol{f_1}, h_1)$ their restriction to Ω. We have in Ω:

$$\boldsymbol{f_1} = -\Delta \boldsymbol{u_1} + \nabla \pi_1 = \psi_1 \boldsymbol{f} - 2\nabla \psi_1 \cdot \nabla \boldsymbol{u} - \Delta \psi_1 \ \boldsymbol{u} + \pi \nabla \psi_1$$

and

$$h_1 = \text{div } \boldsymbol{u_1} = \psi_1 h + \text{ div } \psi_1 \boldsymbol{u}.$$

As $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and supp $\psi_1 \subset \Omega_{R'}$, then $\boldsymbol{f_1} \in \boldsymbol{W}_1^{0,p}(\Omega)$ and $h_1 \in W_1^{1,p}(\Omega)$. Thus $\widetilde{\boldsymbol{f}}_{\boldsymbol{1}} \in \boldsymbol{W}_1^{0,p}({}^c\omega_0)$, $\widetilde{h}_1 \in W_1^{1,p}({}^c\omega_0)$ and $(\widetilde{\boldsymbol{u}}_{\boldsymbol{1}}, \widetilde{\pi}_1)$ satisfies

$$\begin{cases} -\Delta \widetilde{\boldsymbol{u}}_{\boldsymbol{1}} + \nabla \widetilde{\pi}_1 = \widetilde{\boldsymbol{f}}_{\boldsymbol{1}} & \text{in } {}^c\omega_0, \\ \text{div } \widetilde{\boldsymbol{u}}_{\boldsymbol{1}} = \widetilde{h}_1 & \text{in } {}^c\omega_0, \\ \widetilde{\boldsymbol{u}}_{\boldsymbol{1}} = \mathbf{0} & \text{on } \Gamma_0. \end{cases}$$

So, by regularity results in a ”classical“ exterior domain (see Theorem 3.1 in [3]), we have $(\widetilde{\boldsymbol{u}}_{\boldsymbol{1}}, \widetilde{\pi}_1) \in \boldsymbol{W}_1^{2,p}({}^c\omega_0) \times W_1^{1,p}({}^c\omega_0)$ and consequently $(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$.

Now, we denote by $(\widetilde{\boldsymbol{u}}_{\boldsymbol{2}}, \widetilde{\pi}_2)$ the extension by $(\mathbf{0}, 0)$ of $(\boldsymbol{u_2}, \pi_2)$ in $\mathbb{R}^n_+$ and

$$\widetilde{\boldsymbol{f}}_{\boldsymbol{2}} = -\Delta \widetilde{\boldsymbol{u}}_{\boldsymbol{2}} + \nabla \widetilde{\pi}_2, \quad \widetilde{h}_2 = \text{ div } \widetilde{\boldsymbol{u}}_{\boldsymbol{2}}.$$

As supp $(\widetilde{\boldsymbol{f}}_{\boldsymbol{2}}, \widetilde{h}_2) \subset \Omega$ and as $\widetilde{\boldsymbol{f}}_{\boldsymbol{2}|\Omega} = \boldsymbol{f} - \boldsymbol{f_1} \in \boldsymbol{W}_1^{0,p}(\Omega)$ and $\widetilde{h}_{2|\Omega} = h - h_1 \in W_1^{1,p}(\Omega)$, we have

$$\widetilde{\boldsymbol{f}}_{\boldsymbol{2}} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^n_+), \quad \text{and} \quad \widetilde{h}_2 \in W_1^{1,p}(\mathbb{R}^n_+).$$

Thus, by Theorem 5.2 of [9], we deduce from this that $\widetilde{\boldsymbol{u}}_{\boldsymbol{2}} \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+)$, and $\widetilde{\pi}_2 \in W_1^{1,p}(\mathbb{R}^n_+)$. By restriction, we have $\boldsymbol{u_2} \in \boldsymbol{W}_1^{2,p}(\Omega)$, $\pi_2 \in W_1^{1,p}(\Omega)$ and so $(\boldsymbol{u},\pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$. The estimate follows immediately. $\square$

Now, as at the end of the previous section, we can solve the problem with nonhomogeneous boundary conditions.

Theorem 5.4.2. *For any $p > 1$ such that $\dfrac{n}{p'} \neq 1$, $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$, $h \in W_1^{1,p}(\Omega)$, $\boldsymbol{g_0} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u},\pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ solution of the problem $(\mathcal{S}_{\mathcal{D}})$. Moreover, $(\boldsymbol{u},\pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\Omega)} + \|\pi\|_{W_1^{1,p}(\Omega)} \leq C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\Omega)} + \|h\|_{W_1^{1,p}(\Omega)} + \|\boldsymbol{g_0}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_1^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Now, we examine the basic case $\ell = -1$, corresponding to $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. First, we study the kernel of such a problem. We set

$$\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{z},\pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega), \ -\Delta \boldsymbol{z} + \nabla\pi = \boldsymbol{0} \text{ in } \Omega, \ \text{div } \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \boldsymbol{z} = \boldsymbol{0} \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\}.$$

The characterization of this kernel is given by this proposition:

Proposition 5.4.3. *For each $p > 1$ such that $\dfrac{n}{p'} \neq 1$, we have the following statements: i) If $p < n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{0},0)\}$.*
ii) If $p \geq n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu), \ \ \boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\}, \ \mu \in \mathbb{R}\}$ where $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda})) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ is the unique solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla\eta = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{\lambda} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Proof- Let $(\boldsymbol{z},\pi) \in \mathcal{S}_0^p(\Omega)$. We easily show that there exists $(\widetilde{\boldsymbol{z}},\widetilde{\pi}) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n)$ such that $(\widetilde{\boldsymbol{z}},\widetilde{\pi})_{|\Omega} = (\boldsymbol{z},\pi)$. We set

$$\boldsymbol{\xi} = -\Delta\widetilde{\boldsymbol{z}} + \nabla\widetilde{\pi} \quad \text{and} \quad \sigma = \text{div } \widetilde{\boldsymbol{z}} \quad \text{in } \mathbb{R}_+^n.$$

Then, $\boldsymbol{\xi} \in \boldsymbol{L}^p(\mathbb{R}_+^n)$, $\sigma \in W_0^{1,p}(\mathbb{R}_+^n)$ and $(\widetilde{\boldsymbol{z}},\widetilde{\pi}) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n)$ satisfies

$$(\mathcal{S}_+) \begin{cases} -\Delta\widetilde{\boldsymbol{z}} + \nabla\widetilde{\pi} = \boldsymbol{\xi} & \text{in } \mathbb{R}_+^n, \\ \text{div } \widetilde{\boldsymbol{z}} = \sigma & \text{in } \mathbb{R}_+^n, \\ \widetilde{\boldsymbol{z}} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Moreover, $\boldsymbol{\xi}$ and σ have a compact support, so $\boldsymbol{\xi} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^n)$, $\sigma \in W_1^{1,p}(\mathbb{R}_+^n)$ and by Theorem 5.2 of [9], we know there exists $(\boldsymbol{v},\eta) \in (\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^n) \times$

$W_1^{1,p}(\mathbb{R}^n_+)) \subset (\boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+))$ solution of $(\mathcal{S}_+)$. Thus, (see Theorem 5.6 in [9]), if $p < n$, we deduce from this that

$$\widetilde{\boldsymbol{z}} = \boldsymbol{v} \quad \text{and} \quad \widetilde{\pi} = \eta \quad \text{in } \mathbb{R}^n_+,$$

and if $p \geq n$, there exists $\boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\}$ and $\mu \in \mathbb{R}$ such that

$$\boldsymbol{v} - \widetilde{\boldsymbol{z}} = \boldsymbol{\lambda} \quad \text{and} \quad \eta - \widetilde{\pi} = \mu \quad \text{in } \mathbb{R}^n_+.$$

So, if $p < n$, we have $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and by the uniqueness of the solution of the problem of Theorem 5.4.2, we conclude that $(\boldsymbol{z}, \pi) = \{(\boldsymbol{0}, 0)\}$ and if $p \geq n$, we have the characterization we were looking for. $\square$

We have the following result, corresponding to Theorem 5.4.2:

Theorem 5.4.4. *For any $p > 1$ satisfying $\dfrac{n}{p'} \neq 1$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W_0^{1,p}(\Omega)$, $\boldsymbol{g_0} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a unique $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))/\mathcal{S}_0^p(\Omega)$ solution of the problem $(\mathcal{S}_\mathcal{D})$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\inf_{(\boldsymbol{z},p) \in \mathcal{S}_0^p(\Omega)} (\|\boldsymbol{u} + \boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi + p\|_{W_0^{1,p}(\Omega)}) \leq C \, (\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \|\boldsymbol{g_0}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- We easily show that there exists extensions $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^p(\mathbb{R}^n)$ of $\boldsymbol{f}$ and $\widetilde{h} \in W_0^{1,p}(\mathbb{R}^n)$ of h in $\mathbb{R}^n$ and, by Theorem 3.10 of [3], there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n) \times W_0^{1,p}(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n, \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n.$$

Now, it remains to solve the problem: find $(\boldsymbol{z}, \mu) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta \boldsymbol{z} + \nabla \mu = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{z} = 0 \qquad\quad \text{in } \Omega, \\ \boldsymbol{z} = \boldsymbol{g_0} - \boldsymbol{v}_{|\Gamma_0} & \text{on } \Gamma_0, \quad \boldsymbol{z} = \boldsymbol{g_1} - \boldsymbol{v}_{|\mathbb{R}^{n-1}} \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

By Theorem 5.6 of [9], there exists $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \boldsymbol{g_1} - \boldsymbol{v}_{|\mathbb{R}^{n-1}} \text{ on } \mathbb{R}^{n-1}.$$

Moreover, by Theorem 5.4.2, there exists $(\boldsymbol{y}, p) \in (\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)) \subset (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))$ solution of

$$\begin{cases} -\Delta \boldsymbol{y} + \nabla p = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{y} = 0 \quad \text{in } \Omega, \\ \boldsymbol{y} = \boldsymbol{g_0} - \boldsymbol{v}_{|\Gamma_0} - \boldsymbol{w}_{|\Gamma_0} & \text{on } \Gamma_0, \quad \boldsymbol{y} = \boldsymbol{0} \qquad\quad \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $(\boldsymbol{z}, \mu) = (\boldsymbol{y}+\boldsymbol{w}, p+\tau) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ and $(\boldsymbol{u}, \pi) = (\boldsymbol{v}+\boldsymbol{z}, \eta+\mu)$ is solution to our problem. The estimate follows immediately. $\square$

5.5 Very weak solutions for the homogeneous Stokes system

The aim of this section is to study the system $(\mathcal{S_D})$ with $\boldsymbol{f} = \boldsymbol{0}$, $h = 0$ and singular data on the boundary. For this, we must firstly give a meaning to singular data for this problem. More precisely, we want to show that boundary conditions of the form $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_{\ell-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ are meaningful. Here, we limit ourselves to the two cases $\ell = 0$ and $\ell = 1$. Our work is related to that of Amrouche, Nečasová and Raudin for the half space ([9]) and of Amrouche and Girault for a bounded domain ([5]). We refer to these papers for the ideas of proofs for the first results of this section. Here, we suppose that $\dfrac{n}{p} \neq 1$.

We introduce the space:

$$\boldsymbol{M}_\ell(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega),\ \boldsymbol{u} = \boldsymbol{0} \text{ and div } \boldsymbol{u} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\},$$

and we show that we have the identity

$$\boldsymbol{M}_\ell(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega),\ \boldsymbol{u} = \boldsymbol{0} \text{ and } \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_0 \cup \mathbb{R}^{n-1}\}.$$

Then, we define

$$\boldsymbol{X}_\ell(\Omega) = \{\boldsymbol{v} \in \mathring{\boldsymbol{W}}^{1,p'}_{-\ell}(\Omega),\ \text{div } \boldsymbol{v} \in \mathring{W}^{1,p'}_{-\ell+1}(\Omega)\},$$

which is a reflexive Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{X}_\ell(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}_{-\ell}(\Omega)} + \|\text{div } \boldsymbol{v}\|_{W^{1,p'}_{-\ell+1}(\Omega)}.$$

We check that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{X}_\ell(\Omega)$ and we denote by $\boldsymbol{X}'_\ell(\Omega)$ the dual space of $\boldsymbol{X}_\ell(\Omega)$. Now, we introduce the spaces

$$\boldsymbol{T}_\ell(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega),\ \Delta \boldsymbol{v} \in \boldsymbol{X}'_\ell(\Omega)\},$$

$$\boldsymbol{T}_{\ell,\sigma}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{T}_\ell(\Omega),\ \text{div } \boldsymbol{v} = 0 \text{ dans } \Omega\},$$

which are reflexive Banach spaces for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega)} + \|\Delta \boldsymbol{v}\|_{\boldsymbol{X}'_\ell(\Omega)},$$

where $\| \cdot \|_{\boldsymbol{X}'_\ell(\Omega)}$ denotes the dual norm of the space $\boldsymbol{X}'_\ell(\Omega)$. It can be shown that the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{T}_\ell(\Omega)$ and that the space $\{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}),\ \text{div}\ \boldsymbol{v} = 0\}$ is dense in $\boldsymbol{T}_{\ell,\sigma}(\Omega)$.

Finally, using exactly the same reasoning as in Lemma 6.4 and Remark 6.5 of [9] and Section 4.2 of [5], we conclude that for a function $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$, the trace of $\boldsymbol{u}$ on Γ_0 is in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and the trace of $\boldsymbol{u}$ on $\mathbb{R}^{n-1}$ is in $\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$. Moreover, we have for any $\boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\Omega)$ and for any $\boldsymbol{v} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$

$$
\begin{aligned}
&<\Delta \boldsymbol{v}, \boldsymbol{\varphi}>_{\boldsymbol{X}'_\ell(\Omega),\boldsymbol{X}_\ell(\Omega)} = <\boldsymbol{v}, \Delta\boldsymbol{\varphi}>_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega),\boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} \\
&\quad - <\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}>_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - <\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}}>_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}),\boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})}.
\end{aligned} \tag{5.5}
$$

We remind that, for any $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$, we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ solution of

$$
\begin{cases}
-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} & \text{in } \Omega, \qquad (5.6)\\
\text{div}\ \boldsymbol{u} = 0 & \text{in } \Omega, \qquad (5.7)\\
\boldsymbol{u} = \boldsymbol{g_0} & \text{on } \Gamma_0, \qquad (5.8)\\
\boldsymbol{u} = \boldsymbol{g_1} & \text{on } \mathbb{R}^{n-1}. \quad (5.9)
\end{cases}
$$

First, we remark that if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ satisfies (5.6) and (5.7), then $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$ and thus (5.8) and (5.9) make sense. Indeed, the function $\boldsymbol{u}$ is in $\boldsymbol{W}^{0,p}_{\ell-1}(\Omega)$ and div $\boldsymbol{u} = 0$ in Ω. Moreover, because $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{X}_\ell(\Omega)$, we easily show that $\nabla \pi \in \boldsymbol{X}'_\ell(\Omega)$. Thus, by (5.6), we have $\Delta \boldsymbol{u} \in \boldsymbol{X}'_\ell(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$. So, in this case, we have seen that $\boldsymbol{u}_{|\Gamma_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{u}_{|\mathbb{R}^{n-1}} \in \boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$.

Proposition 5.5.1. *For each $p > 1$ such that $\dfrac{n}{p} \neq 1$, we suppose that the functions $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1})$ satisfy*

$$
\boldsymbol{g_0} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \boldsymbol{g_1} \cdot \boldsymbol{n} = 0 \quad \text{on } \mathbb{R}^{n-1}. \tag{5.10}
$$

Then, problem (5.6)-(5.9) is equivalent to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{M}_\ell(\Omega)$ and for any $\eta \in W^{1,p'}_{-\ell+1}(\Omega)$, we have

$$
\begin{aligned}
(\mathcal{FV}) \quad &<\boldsymbol{u}, -\Delta \boldsymbol{v} + \nabla \eta>_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega),\boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} - <\pi, \text{div}\ \boldsymbol{v}>_{W^{-1,p}_{\ell-1}(\Omega),\mathring{W}^{1,p'}_{-\ell+1}(\Omega)} = \\
&- <\boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}>_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - <\boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}>_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}),\boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})}.
\end{aligned}
$$

Proof- Let $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ be a solution of (5.6)-(5.9). Thanks to the previous remark, we have $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\Omega)$. Let $\boldsymbol{v}$ be in $\boldsymbol{M}_\ell(\Omega)$. We deduce from (5.5) and (5.6) that

$$< \boldsymbol{u}, -\Delta \boldsymbol{v} >_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega), \boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} - < \pi, \operatorname{div}\, \boldsymbol{v} >_{W^{-1,p}_{\ell-1}(\Omega), \overset{\circ}{W}{}^{1,p'}_{-\ell+1}(\Omega)} =$$

$$- < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}), \boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})} .$$

Moreover, since $\frac{n}{p} \neq 1$, the space $\{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}),\ \operatorname{div}\, \boldsymbol{v} = 0\}$ is dense in $\boldsymbol{T}_{\ell,\sigma}(\Omega)$ and using (5.7) and (5.10), we show that for any $\eta \in W^{1,p'}_{-\ell+1}(\Omega)$

$$< \boldsymbol{u}, \nabla \eta >_{\boldsymbol{W}^{0,p}_{\ell-1}(\Omega), \boldsymbol{W}^{0,p'}_{-\ell+1}(\Omega)} = 0.$$

Thus, we conclude that $(\boldsymbol{u},\pi)$ is solution of $(\mathcal{FV})$. Reciprocally, let $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega) \times W^{-1,p}_{\ell-1}(\Omega)$ be a solution of $(\mathcal{FV})$. With $\eta = 0$ and $\boldsymbol{v} \in \mathcal{D}(\Omega)$, we have

$$< -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{v} >_{\boldsymbol{\mathcal{D}}'(\Omega), \boldsymbol{\mathcal{D}}(\Omega)} = 0,$$

and with $\boldsymbol{v} = \boldsymbol{0}$ and $\eta \in \mathcal{D}(\Omega)$, we have

$$< \operatorname{div}\, \boldsymbol{u}, \eta >_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = 0.$$

Thus, (5.6) and (5.7) hold. It remains to show (5.8) and (5.9). Let $\boldsymbol{v} \in \boldsymbol{M}_\ell(\Omega)$. Thanks to Green's formula (5.5) and $(\mathcal{FV})$, we have

$$< \boldsymbol{u}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{u}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}), \boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})} =$$

$$< \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}_{\ell-1}(\mathbb{R}^{n-1}), \boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})} .$$

Now, let $\boldsymbol{\mu}$ be in $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)$. We denote by $\boldsymbol{\mu_\tau}$ the tangential component of $\boldsymbol{\mu}$. It is defined by

$$\boldsymbol{\mu} = \boldsymbol{\mu_\tau} + (\boldsymbol{\mu} \cdot \boldsymbol{n})\boldsymbol{n}.$$

We easily show that there exists $\boldsymbol{w} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega)$ such that

$$\begin{cases} \boldsymbol{w} = \boldsymbol{0} \text{ and } \dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} = \boldsymbol{\mu_\tau} & \text{on } \Gamma_0, \\ \boldsymbol{w} = \dfrac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $\boldsymbol{w} \in \boldsymbol{M}_\ell(\Omega)$ and

$$< \boldsymbol{u}, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} = < \boldsymbol{g_0}, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} .$$

Thus

$$< \boldsymbol{u}, \boldsymbol{\mu_\tau} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} = < \boldsymbol{g_0}, \boldsymbol{\mu_\tau} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} .$$

Finally, since $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_0 and by hypothesis $\boldsymbol{g_0}\cdot\boldsymbol{n} = 0$ on Γ_0, we conclude that

$$< \boldsymbol{u}, \boldsymbol{\mu} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} = < \boldsymbol{g_0}, \boldsymbol{\mu} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)},$$

i.e. $\boldsymbol{u} = \boldsymbol{g_0}$ on Γ_0. Now, let $\boldsymbol{\mu}$ be in $\boldsymbol{W}^{\frac{1}{p},p'}_{-\ell+1}(\mathbb{R}^{n-1})$. We know that there exists $\boldsymbol{s} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)$ such that

$$\boldsymbol{s} = \boldsymbol{0} \quad \text{and} \quad \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{n}} = \boldsymbol{\mu_\tau} \quad \text{on } \mathbb{R}^{n-1}.$$

Moreover, as above, we can find $\boldsymbol{y} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega)$ such that

$$\begin{cases} \boldsymbol{y} = -\boldsymbol{s} \text{ and } \dfrac{\partial \boldsymbol{y}}{\partial \boldsymbol{n}} = -\dfrac{\partial \boldsymbol{s}}{\partial \boldsymbol{n}} & \text{on } \Gamma_0, \\ \boldsymbol{y} = \dfrac{\partial \boldsymbol{y}}{\partial \boldsymbol{n}} = \boldsymbol{0} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

So, $\boldsymbol{z} = \boldsymbol{s}_{|\Omega} + \boldsymbol{y} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\Omega)$ satisfies

$$\begin{cases} \boldsymbol{z} = \dfrac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} = \boldsymbol{0} & \text{on } \Gamma_0, \\ \boldsymbol{z} = \boldsymbol{0} \text{ and } \dfrac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} = \boldsymbol{\mu_\tau} & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

Then, $\boldsymbol{z} \in \boldsymbol{M}_\ell(\Omega)$ and we easily conclude like above that $\boldsymbol{u} = \boldsymbol{g_1}$ on $\mathbb{R}^{n-1}$. Thus, we have the equivalence of the two problems. □

Now, we can solve the homogeneous Stokes system (5.6)-(5.9) with singular boundary conditions. We will give separately the results for $\ell = 0$ and $\ell = 1$. Note that the first theorem (for the case $\ell = 0$) extends Theorems 5.2.5 and 5.3.6 (with $\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$) since $\boldsymbol{W}^{1,p}_0(\Omega) \subset \boldsymbol{W}^{0,p}_{-1}(\Omega)$ if $n \neq p$.

Theorem 5.5.2. *For any $p > 1$ such that $\frac{n}{p} \neq 1$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})$ satisfying (5.10), there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ solution of (5.6)-(5.9). Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + \|\pi\|_{W^{-1,p}_{-1}(\Omega)} \leq C \left(\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}\right),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- In fact, we solve $(\mathcal{FV})$. For this, we argue by duality. Since $\dfrac{n}{p} \neq 1$, by Theorem 5.4.1, we can say that for any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p'}(\Omega)$ and $h \in \mathring{\boldsymbol{W}}\,_1^{1,p'}(\Omega)$, there exists a unique $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_1^{2,p'}(\Omega) \times W_1^{1,p'}(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = h \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

satisfying

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_1^{2,p'}(\Omega)} + \|\eta\|_{W_1^{1,p'}(\Omega)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p'}(\Omega)} + \|h\|_{W_1^{1,p'}(\Omega)}).$$

Then,

$$\begin{aligned} &| < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_1^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} | \\ &\qquad \leq\ C\ (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})})(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p'}(\Omega)} + \|h\|_{W_1^{1,p'}(\Omega)}). \end{aligned}$$

We can deduce from this that the linear mapping T defined by

$$T(\boldsymbol{f}, h) = < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} > + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >, \tag{5.11}$$

on $\boldsymbol{W}_1^{0,p'}(\Omega) \times \mathring{\boldsymbol{W}}\,_1^{1,p'}(\Omega)$ is continuous. So, according to the Riesz representation theorem, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\Omega) \times W_{-1}^{-1,p}(\Omega)$ such that

$$\begin{aligned} &< \boldsymbol{u}, \boldsymbol{f} >_{\boldsymbol{W}_{-1}^{0,p}(\Omega), \boldsymbol{W}_1^{0,p'}(\Omega)} + < \pi, h >_{W_{-1}^{-1,p}(\Omega), W_1^{1,p'}(\Omega)} = \\ &- < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_1^{\frac{1}{p},p'}(\mathbb{R}^{n-1})}. \end{aligned}$$

Thus, noticing that $\boldsymbol{v} \in \boldsymbol{M}_0(\Omega)$, we deduce that $(\boldsymbol{u}, \pi)$ satisfies $(\mathcal{FV})$. □

The next corollary relaxes the constraint (5.10) on the data. In order to establish this corollary, we give the following lemma.

Lemma 5.5.3. *For any $p > 1$, $g_0 \in W^{-\frac{1}{p},p}(\Gamma_0)$ and $g_1 \in W_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$, there exists a function $s \in W_{-1}^{1,p}(\Omega)$ solution of*

$$\Delta s = 0 \quad \text{in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = g_0 \quad \text{on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = g_1 \quad \text{on } \mathbb{R}^{n-1}.$$

Moreover, s satisfies

$$\|s\|_{W_{-1}^{1,p}(\Omega)} \leq\ C\ (\|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W_{-1}^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- Using results in the half space (see Theorem 3.7 in [4]), we know there exists $z \in W^{1,p}_{-1}(\mathbb{R}^n_+)$ solution of

$$-\Delta z = 0 \quad \text{in } \mathbb{R}^n_+, \quad \frac{\partial z}{\partial \boldsymbol{n}} = g_1 \quad \text{on } \mathbb{R}^{n-1},$$

satisfying

$$\|z\|_{W^{1,p}_{-1}(\mathbb{R}^n_+)} \leq \ C \ \|g_1\|_{W^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}.$$

We have $g = g_0 - \dfrac{\partial z}{\partial \boldsymbol{n}} \in W^{-\frac{1}{p},p}(\Gamma_0)$ and it remains to solve the following problem: find $v \in W^{1,p}_{-1}(\Omega)$ solution of

$$\Delta v = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \quad \text{on } \Gamma_0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \quad \text{on } \mathbb{R}^{n-1}. \qquad (5.12)$$

To solve this problem, we solve first the following one: find $y \in W^{1,p}_{-1}(\widetilde{\Omega})$ solution of

$$\Delta y = 0 \quad \text{in } \widetilde{\Omega}, \quad \frac{\partial y}{\partial \boldsymbol{n}} = \widetilde{g} \quad \text{on } \widetilde{\Gamma}_0 \qquad (5.13)$$

such that

$$\|y\|_{W^{1,p}_{-1}(\widetilde{\Omega})} \leq \ C \ (\|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})});$$

here we remind that $\widetilde{\Omega} = \Omega \cup \Omega' \cup \mathbb{R}^{n-1}$ with Ω' the symmetric region of Ω with respect to $\mathbb{R}^{n-1}$ and $\widetilde{\Gamma}_0 = \partial\widetilde{\Omega}$ and that $\widetilde{g}$ is an extension of g in $W^{-\frac{1}{p},p}(\widetilde{\Gamma}_0)$ symmetric with respect to $\mathbb{R}^{n-1}$ (we refer to the proof of Theorem 4.3.3 to construct such an extension). To find a solution y of (5.13), we split the proof into two cases. First, if $\dfrac{n}{p'} > 1$, we apply Theorem 2.2.3 (there is no condition of compatibility), so there exists $y \in W^{1,p}_{0}(\widetilde{\Omega}) \subset W^{1,p}_{-1}(\widetilde{\Omega})$ solution of (5.13) and satisfying the estimate. Next, if $\dfrac{n}{p'} \leq 1$, we set for any $\boldsymbol{x}$ in $\widetilde{\Omega}$

$$w(\boldsymbol{x}) = -\frac{1}{2\pi} \int_{\widetilde{\Gamma}_0} E(\boldsymbol{x} - \boldsymbol{y}) \ d\boldsymbol{x},$$

where E is the fundamental solution of the Laplacian and we easily show that $w \in W^{1,p}_{-1}(\widetilde{\Omega})$ (but $w \notin W^{1,p}_{0}(\widetilde{\Omega})$), that $\Delta w = 0$ in $\widetilde{\Omega}$ and $< \dfrac{\partial w}{\partial \boldsymbol{n}}, 1 >_{\widetilde{\Gamma}_0} \neq 0$. We define λ by

$$\lambda = \frac{< \widetilde{g}, 1 >_{\widetilde{\Gamma}_0}}{< \dfrac{\partial w}{\partial \boldsymbol{n}}, 1 >_{\widetilde{\Gamma}_0}},$$

so that the compatibility condition

$$< \widetilde{g} - \lambda \frac{\partial w}{\partial \boldsymbol{n}}, 1 >_{\widetilde{\Gamma}_0} = 0$$

is satisfied. By Theorem 2.2.3, there exists $u \in W^{1,p}_0(\widetilde{\Omega}) \subset W^{1,p}_{-1}(\widetilde{\Omega})$ solution of

$$\Delta u = 0 \quad \text{in } \widetilde{\Omega}, \quad \frac{\partial u}{\partial \boldsymbol{n}} = \widetilde{g} - \lambda \frac{\partial w}{\partial \boldsymbol{n}} \quad \text{on } \widetilde{\Gamma}_0$$

satisfying

$$\|u\|_{W^{1,p}_0(\widetilde{\Omega})} \leq \; C \; \|\widetilde{g} - \lambda \frac{\partial w}{\partial \boldsymbol{n}}\|_{W^{-\frac{1}{p},p}(\Gamma_0)}.$$

Thus, $y = \lambda w + u$ is solution of (5.13) and satisfies the estimate. Now, let $y_0 \in W^{1,p}_{-1}(\widetilde{\Omega})$ a solution of (5.13) and let $s_0 \in W^{1,p}_{-1}(\widetilde{\Omega})$ be defined, for almost all $(\boldsymbol{x'}, x_n) \in \widetilde{\Omega}$, by

$$s_0(\boldsymbol{x'}, x_n) = y_0(\boldsymbol{x'}, -x_n).$$

Thanks to the symmetry of $\widetilde{\Omega}$ and $\widetilde{g}$ with respect to $\mathbb{R}^{n-1}$, we prove that s_0 is also a solution of (5.13) (here again, for more details, we refer to the proof of Theorem 4.3.3). Then, setting $v = \dfrac{1}{2}(y_0 + s_0)_{|\Omega} \in W^{1,p}_{-1}(\Omega)$, we show that v satisfies (5.12) and we have

$$\|v\|_{W^{1,p}_{-1}(\Omega)} \leq \; C \; (\|g_0\|_{W^{-\frac{1}{p},p}(\Gamma_0)} + \|g_1\|_{W^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}).$$

Finally, the function $s = z + v$ solves the problem and the estimate follows immediately. $\square$

Corollary 5.5.4. *For any $p > 1$ such that $\dfrac{n}{p} \neq 1$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})$ there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ solution of (5.6)-(5.9). Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + \|\pi\|_{W^{-1,p}_{-1}(\Omega)} \leq \; C \; (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- The uniqueness comes from Theorem 5.5.2. For the existence, by the previous lemma, there exists $s \in W^{1,p}_{-1}(\Omega)$ solution of

$$\Delta s = 0 \quad \text{in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_0} \cdot \boldsymbol{n} \quad \text{on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_1} \cdot \boldsymbol{n} \quad \text{on } \mathbb{R}^{n-1}.$$

Now, we define $\boldsymbol{w}$ by $\boldsymbol{w} = \nabla s \in \boldsymbol{W}^{0,p}_{-1}(\Omega)$. Then, $\boldsymbol{w} \in \boldsymbol{T}_{0,\sigma}(\Omega)$ and so its traces on Γ_0 and $\mathbb{R}^{n-1}$ have a sense respectively in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})$. We set $\boldsymbol{g_0}^* = \boldsymbol{g_0} - \boldsymbol{w}_{|\Gamma_0}$ and $\boldsymbol{g_1}^* = \boldsymbol{g_0} - \boldsymbol{w}_{|\mathbb{R}^{n-1}}$ and we notice that the functions $\boldsymbol{g_0}^*$ and $\boldsymbol{g_1}^*$ satisfy (5.10). So we can apply the previous theorem and there exists $(\boldsymbol{v}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{0} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{g_0}^* & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{g_1}^* \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

and satisfying

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + \|\pi\|_{W^{-1,p}_{-1}(\Omega)} \leq \; C\; (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}^{-\frac{1}{p},p}_{-1}(\mathbb{R}^{n-1})}),$$

Finally, $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi)$ is solution of (5.6)-(5.9) and the estimates follows immediately. $\square$

Now, we describe a result for the case $\ell = 1$.

Theorem 5.5.5. *For any $p > 1$ such that $\dfrac{n}{p} \neq 1$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ and $\boldsymbol{g_1} \in \boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$ satisfying (5.10) and the following compatibility condition if $p \leq \dfrac{n}{n-1}$: for each $(\boldsymbol{z}, p) \in \mathcal{S}_0^{p'}(\Omega)$*

$$< \boldsymbol{g_0}, \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} >_{\Gamma_0} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} >_{\mathbb{R}^{n-1}} = 0, \qquad (5.14)$$

there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ solution of (5.6)-(5.9). Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W_0^{-1,p}(\Omega)} \leq \; C\; (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C is a real positive constant which depends only on p and ω_0.

Proof- In fact, here again, we solve $(\mathcal{FV})$. For this, we apply a duality argument. Since $\dfrac{n}{p} \neq 1$, by Theorem 5.4.4, we can say that for any $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ and $h \in \overset{\circ}{W}{}_0^{1,p'}(\Omega)$, there exists a unique $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{2,p'}(\Omega) \times W_0^{1,p'}(\Omega)/\mathcal{S}_0^{p'}(\Omega)$ solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = h \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma_0, \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

satisfying

$$\inf_{(\boldsymbol{z},p)\in\mathcal{S}_0^{p'}(\Omega)} (\|\boldsymbol{v}+\boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p'}(\Omega)} + \|\eta+p\|_{W_0^{1,p'}(\Omega)}) \leq \; C\; (\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|h\|_{W_0^{1,p'}(\Omega)}).$$

Then, for any $(\boldsymbol{z}, p) \in \mathcal{S}_0^{p'}(\Omega)$

$$\begin{aligned} &| < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} + < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_0^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} | \\ &= | < \boldsymbol{g_0}, \frac{\partial}{\partial \boldsymbol{n}}(\boldsymbol{v} + \boldsymbol{z}) >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0), \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} \\ &\qquad\qquad + < \boldsymbol{g_1}, \frac{\partial}{\partial \boldsymbol{n}}(\boldsymbol{v} + \boldsymbol{z}) >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}), \boldsymbol{W}_0^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} | \\ &\leq \; C\; (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})})(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|h\|_{W_0^{1,p'}(\Omega)}). \end{aligned}$$

We can deduce from this that the linear mapping T defined by (5.11) on $\boldsymbol{L}^{p'}(\Omega)\times \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega)$ is continuous. So, according to the Riesz representation theorem, there exists a unique $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ such that

$$< \boldsymbol{u}, \boldsymbol{f} >_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)} + < \pi, h >_{W_0^{-1,p}(\Omega),W_0^{1,p'}(\Omega)} =$$
$$- < \boldsymbol{g_0}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0),\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma_0)} - < \boldsymbol{g_1}, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1}),\boldsymbol{W}_0^{\frac{1}{p},p'}(\mathbb{R}^{n-1})} .$$

Thus, noticing that $\boldsymbol{v} \in \boldsymbol{M}_1(\Omega)$, we deduce that $(\boldsymbol{u},\pi)$ satisfies $(\mathcal{FV})$. □

Here again, with a similar proof to that of Corollary 5.5.4, we want to relax the constraint on the data:

Corollary 5.5.6. *For any* $p > \dfrac{n}{n-1}$, $\boldsymbol{g_0} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)$ *and* $\boldsymbol{g_1} \in \boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})$, *there exists a unique* $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ *solution of (5.1)-(5.4). Moreover,* $(\boldsymbol{u},\pi)$ *satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W_0^{-1,p}(\Omega)} \leq C \, (\|\boldsymbol{g_0}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma_0)} + \|\boldsymbol{g_1}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\mathbb{R}^{n-1})}),$$

where C *is a real positive constant which depends only on* p *and* ω_0.

Proof- When $p > \dfrac{n}{n-1}$, we follow the same reasoning as in Corollary 5.5.4 using Theorem 4.3.3 to find $s \in W_0^{1,p}(\Omega)$ such that

$$\Delta s = 0 \quad \text{in } \Omega, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_0} \cdot \boldsymbol{n} \quad \text{on } \Gamma_0, \quad \frac{\partial s}{\partial \boldsymbol{n}} = \boldsymbol{g_1} \cdot \boldsymbol{n} \quad \text{on } \mathbb{R}^{n-1}.$$

and using the previous theorem. □

Remark: When $1 < p \leq \dfrac{n}{n-1}$, we notice that, because of the compatibility condition (5.14), we can not prove a result similar to Corollary 5.5.4.

Chapter 6

A Stokes problem in a perturbed half-space in dimension $n \geq 3$

6.1 Introduction and preliminaries

In this chapter, we want to study a Stokes system in dimension $n \geq 3$ in a particular unbounded domain with an unbounded boundary, namely a perturbed half-space. Let $\Omega \subset \mathbb{R}^n$ a domain obtained by an arbitrary modification of the half-space

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\}.$$

We are interested in the following problem:

$$(\mathcal{S}_1) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma = \partial\Omega. \end{cases}$$

This chapter is organized as follows. First, we give some definitions and preliminary results. Next, Sections 6.2 and 6.3 are respectively devoted to the resolution of $(\mathcal{S}_1)$ in cases $p = 2$ and $p \neq 2$ and in Section 6.4, we study the case of strong and very weak solutions.

Now, we want to give a precise definition of a perturbed half-space. Let Ω an open and connected domain such that

$$\mathbb{R}^n_+ \subset \Omega \subset \mathbb{R}^n.$$

There exists an open ball $B \subset \mathbb{R}^n$ such that

$$\Omega \cup B = \mathbb{R}^n_+ \cup B.$$

We set $\Gamma = \partial\Omega$ the boundary of Ω that we suppose of class $C^{1,1}$; then, we can choose some bounded subdomain $G \subset \Omega$ with boundary ∂G of class $C^{1,1}$ such that $\Omega \cap B \subset G$. The ball B can be chosen centered on the origin and sufficiently large so that there exists another ball B_0 centered on the origin with closure $\overline{B}_0 \subset B$ such that

$$\Omega \cup B_0 = \mathbb{R}^n_+ \cup B_0.$$

Then, we define the following domains: $\Sigma = \Gamma \cap \mathbb{R}^{n-1}$, $D = \mathbb{R}^{n-1} \setminus \Sigma$ and $S = \Gamma \setminus \Sigma$. Moreover, we define also $\omega = \Omega \setminus \overline{\mathbb{R}^n_+}$ (we notice that $\partial\omega = D \cup S$) and ω' the symmetric region of ω with respect to $\mathbb{R}^{n-1}$. Finally, we choose B_0 and B sufficiently large so that they satisfy $\omega' \subset B_0 \subset B$ and we introduce the following partition of unity

$$\begin{gathered}\psi_1,\ \psi_2 \in C^\infty(\mathbb{R}^n),\ 0 \leq \psi_1, \psi_2 \leq 1,\ \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n,\\ \psi_1 = 1 \text{ in } B_0, \quad \operatorname{supp}\, \psi_1 \subset B.\end{gathered}$$

We easily check that the property (1.1) is satisfied and that, corresponding to Theorem 1.2.1, the following Poincaré-type inequalities hold in a perturbed half-space:

Theorem 6.1.1. *We suppose that* $n \geq 3$.
i) The semi-norm $|\cdot|_{W_0^{1,2}(\Omega)}$ *defined on* $W_0^{1,2}(\Omega)$ *is a norm equivalent to the full norm* $\|\cdot\|_{W_0^{1,2}(\Omega)}$.
ii) The semi-norm $|\cdot|_{W_0^{1,2}(\Omega)}$ *defined on* $\overset{\circ}{W}{}_0^{1,2}(\Omega)$ *is a norm equivalent to the full norm* $\|\cdot\|_{W_0^{1,2}(\Omega)}$.

Here again, we give the theorem only for the case that we use, the basis space $W_0^{1,2}(\Omega)$. We easily prove it thanks to the previous partition of unity using results already known for the Poincaré-type inequalities in the bounded domain G and in the half-space.

The boundary Γ is unbounded. So, like for the domain $\mathbb{R}^n_+$, we define weighted Sobolev traces spaces. For any $\sigma \in \,]0, 1[$, we set

$$\omega_1 = \begin{cases} \rho & \text{if } \dfrac{n}{p} \neq \sigma, \\ \rho(lg\rho)^{1/\sigma} & \text{if } \dfrac{n}{p} = \sigma. \end{cases}$$

We define the space

$$W_0^{\sigma,p}(\Gamma) = \{u, \omega_1^{-\sigma} u \in L^p(\Sigma), u \in L^p(S), \int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x} d\boldsymbol{y} < \infty\}.$$

It is a reflexive Banach space equipped with its natural norm

$$(\|\frac{u}{\omega_1^\sigma}\|^p_{L^p(\Sigma)} + \|u\|^p_{L^p(S)} + \int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{n+\sigma p}}\ d\boldsymbol{x}d\boldsymbol{y})^{1/p}.$$

For any $s \in \mathbb{R}^+$, we set

$$W_0^{s,p}(\Gamma) = \{u \in W^{[s],p}_{[s]-s}(\Gamma),\ \forall|\lambda| = [s],\ D^\lambda u \in W_0^{s-[s],p}(\Gamma)\}.$$

It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W_0^{s,p}(\Gamma)} = \|u\|_{W^{[s],p}_{[s]-s}(\Gamma)} + \sum_{|\lambda|=s} \|D^\lambda u\|_{W_0^{s-[s],p}(\Gamma)}.$$

Then, for any $s \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, we set

$$W_\alpha^{s,p}(\Gamma) = \{u \in W^{[s],p}_{[s]+\alpha-s}(\Gamma),\ \forall|\lambda| = [s],\ \rho^\alpha D^\lambda u \in W_0^{s-[s],p}(\Gamma)\}.$$

Next, we prove that the following traces lemma is satisfied:

Lemma 6.1.2. *For any integer $m \geq 1$ and real number α, we define the mapping*

$$\begin{aligned} \gamma : \mathcal{D}(\overline{\Omega}) \to\ & (\mathcal{D}(\Gamma))^m \\ u \mapsto\ & (\gamma_0 u, \ldots, \gamma_{m-1}u) \end{aligned}$$

where for any $k = 0, \ldots, m-1$, $\gamma_k u = \dfrac{\partial^k u}{\partial \boldsymbol{n}^k}$. Then, γ can be extended by continuity to a linear and continuous mapping still denoted by γ from $W_\alpha^{m,p}(\Omega)$ to $\prod_{j=0}^{m-1} W_\alpha^{m-j-\frac{1}{p},p}(\Gamma)$. Moreover, γ is onto and

$$Ker\ \gamma = \mathring{W}{}_\alpha^{m,p}(\Omega).$$

Proof- We prove this lemma only in the basic case of a function in $W_0^{1,p}(\Omega)$, the generalization being obvious.

First, let u be in $\mathcal{D}(\overline{\Omega})$. We set $u_i = \psi_i u$ for $i = 1, 2$ and we have

$$\|\gamma u_2\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} = \|\gamma u_2\|_{W_0^{1-\frac{1}{p},p}(\Sigma)}$$

because $\gamma u_2 = 0$ on S. Since $\Sigma \subset \mathbb{R}^{n-1}$ and that γ is continuous from $W_0^{1,p}(\mathbb{R}^n_+)$ to $W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ (see [8]), we deduce that

$$\|\gamma u_2\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} \leq\ C\ \|u_2\|_{W_0^{1,p}(\mathbb{R}^n_+)} \leq\ C\ \|u\|_{W_0^{1,p}(\Omega)}.$$

Next, we notice that $\gamma u_1 = 0$ on $\Gamma \cap {}^cB$, so we have

$$\|\gamma u_1\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} \leq C \|\gamma u_1\|_{W^{1-\frac{1}{p},p}(\Gamma\cap B)}.$$

Since $\Gamma \cap B \subset \partial G$ and since, thanks to results in bounded domains, γ is continuous from $W^{1,p}(G)$ to $W^{1-\frac{1}{p},p}(\partial G)$, we have

$$\|\gamma u_1\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} \leq C \|u_1\|_{W^{1,p}(G)} \leq C \|u\|_{W_0^{1,p}(\Omega)}.$$

Finally

$$\|\gamma u\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} \leq C \|u\|_{W_0^{1,p}(\Omega)}.$$

So, by density, γ can be extended by continuity to a linear and continuous mapping from $W_0^{1,p}(\Omega)$ to $W_0^{1-\frac{1}{p},p}(\Gamma)$.

Now, we want to show that γ is onto. Let g be in $W_0^{1-\frac{1}{p},p}(\Gamma)$. We set $g_i = \psi_i g$, $i = 1, 2$ and

$$\widetilde{g_2} = g_2 \text{ on } \Sigma, \quad \widetilde{g_2} = 0 \text{ on } D.$$

We have $\widetilde{g_2} \in W_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ and there exists (see [8]) $v \in W_0^{1,p}(\mathbb{R}^n_+)$ such that $v = \widetilde{g_2}$ on $\mathbb{R}^{n-1}$. We define

$$\widetilde{v} = v \text{ in } \mathbb{R}^n_+, \quad \widetilde{v} = 0 \text{ in } \omega.$$

Then, $\widetilde{v} \in W_0^{1,p}(\Omega)$ and $\widetilde{v} = g_2$ on Γ. We set also

$$\widetilde{g_1} = g_1 \text{ on } \partial G \cap \Gamma, \quad \widetilde{g_1} = 0 \text{ on } \partial G \cap \mathbb{R}^n_+.$$

We have $\widetilde{g_1} \in W^{1-\frac{1}{p},p}(\partial G)$ and there exists, thanks to results in bounded domains, $w \in W^{1,p}(G)$ such that $w = \widetilde{g_1}$ on ∂G. We define

$$\widetilde{w} = w \text{ in } G, \quad \widetilde{w} = 0 \text{ in } \Omega \setminus G.$$

Then, $\widetilde{w} \in W_0^{1,p}(\Omega)$ and $\widetilde{w} = g_1$ sur Γ. Consequently, there exists $u = \widetilde{v} + \widetilde{w} \in W_0^{1,p}(\Omega)$ such that $u = g$ on Γ. So, γ is onto.

Finally, it remains to show that $\operatorname{Ker} \gamma = \mathring{W}{}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|.\|_{W_0^{1,p}(\Omega)}}$. Let u be in $\mathring{W}{}_0^{1,p}(\Omega)$. Then, there exists a sequence $(\varphi_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that $\|u - \varphi_\ell\|_{W_0^{1,p}(\Omega)} \to 0$ when $\ell \to +\infty$. Since γ is continuous, we have

$$\|\gamma(u - \varphi_\ell)\|_{W_0^{1-\frac{1}{p},p}(\Gamma)} \leq C \|u - \varphi_\ell\|_{W_0^{1,p}(\Omega)} \to 0.$$

But $\gamma\varphi_\ell = 0$ for any $\ell \in \mathbb{N}$ because $\varphi_\ell \in \mathcal{D}(\Omega)$, so, $\gamma u = 0$ in $W_0^{1-\frac{1}{p},p}(\Gamma)$ and $u \in \text{Ker } \gamma$. Conversely, let u be in Ker γ, $u_i = \psi_i u$ for $i = 1, 2$. We have $u_2 = 0$ on $\mathbb{R}^{n-1}$ so $u_2 \in \overset{\circ}{W}{}_0^{1,p}(\mathbb{R}^n_+)$ (see [8]). Thus, there exists $(\varphi_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n_+) \subset \mathcal{D}(\Omega)$ such that $\|u_2 - \varphi_\ell\|_{W_0^{1,p}(\mathbb{R}^n_+)} \to 0$. Moreover, since, for any $\ell \in \mathbb{N}$, $u_2 = \varphi_\ell = 0$ in ω, we deduce that

$$\|u_2 - \varphi_\ell\|_{W_0^{1,p}(\Omega)} \to 0,$$

i.e $u_2 \in \overset{\circ}{W}{}_0^{1,p}(\Omega)$. With the same idea, $u_1 = 0$ on ∂G so $u_1 \in \overset{\circ}{W}{}^{1,p}(G)$ (see [19]). Thus, there exists $(\psi_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{D}(G) \subset \mathcal{D}(\Omega)$ such that $\|u_1 - \psi_\ell\|_{W^{1,p}(G)} \to 0$. Moreover, since for any $\ell \in \mathbb{N}$, $u_1 = \varphi_\ell = 0$ in $\Omega \setminus G$, we have

$$\|u_1 - \varphi_\ell\|_{W_0^{1,p}(\Omega)} \to 0,$$

i.e $u_1 \in \overset{\circ}{W}{}_0^{1,p}(\Omega)$. Consequently, $u = u_1 + u_2 \in \overset{\circ}{W}{}_0^{1,p}(\Omega)$ and Ker $\gamma = \overset{\circ}{W}{}_0^{1,p}(\Omega)$. □

For $n \geq 3$, $p \in]1, +\infty[$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$ we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_1)$.

6.2 Case $p = 2$.

Thanks to Lemma 6.1.2, we notice first that there exists $\boldsymbol{u_g} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that $\boldsymbol{u_g} = \boldsymbol{g}$ on Γ and satisfying

$$\|\boldsymbol{u_g}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)}.$$

So, it is equivalent to solve the problem with homogeneous boudary conditions: for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $h \in L^2(\Omega)$, find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ such that

$$(\mathcal{S}_{10}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

Now, we want to establish Lemma 6.2.2 to have a data for the divergence reduced to zero. For this, we use this preliminary lemma:

Lemma 6.2.1. *We suppose that $n > 2$. For any h in $L^2(\Omega)$, there exists $u \in W_0^{2,2}(\Omega)$ solution of*

$$\Delta u = h \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma.$$

Moreover, u satisfies

$$\|u\|_{W_0^{2,2}(\Omega)} \leq C \|h\|_{L^2(\Omega)}, \tag{6.1}$$

where C is a real positive constant which depends only on Ω.

Proof- Let h be in $L^2(\Omega)$. We set $h_1 = \psi_1 h \in L^2(\Omega)$ and $h_2 = \psi_2 h \in L^2(\Omega)$.

i) By Theorem 3.1 in [4], there exists $v \in W_0^{2,2}(\mathbb{R}^n_+)$ such that

$$\Delta v = h_2 \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \frac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \mathbb{R}^{n-1},$$

and satisfying

$$\|v\|_{W_0^{2,2}(\mathbb{R}^n_+)} \leq \; C \; \|h\|_{L^2(\Omega)}.$$

We set for almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$:

$$v_*(\boldsymbol{x'}, x_n) = \begin{cases} v(\boldsymbol{x'}, x_n) & \text{if } x_n > 0, \\ v(\boldsymbol{x'}, -x_n) & \text{if } x_n < 0. \end{cases}$$

It is clear that $v_* \in W_0^{2,2}(\mathbb{R}^n)$ and that

$$\|v_*\|_{W_0^{2,2}(\mathbb{R}^n)} \leq \; C \; \|h\|_{L^2(\Omega)}.$$

and we show that

$$\Delta v_* = h_2 \text{ in } \Omega. \tag{6.2}$$

Indeed, for φ in $\mathcal{D}(\Omega)$, we have

$$\int_\Omega \varphi \Delta v_* \; d\boldsymbol{x} = \int_\Omega v_* \Delta \varphi \; d\boldsymbol{x} = \int_{\mathbb{R}^n_+} v \Delta \varphi \; d\boldsymbol{x} + \int_\omega v_* \Delta \varphi \; d\boldsymbol{x}$$

and next

$$\int_{\mathbb{R}^n_+} v_* \Delta \varphi \; d\boldsymbol{x} = \int_{\mathbb{R}^n_+} h_2 \varphi \; d\boldsymbol{x} \; + \int_D v \frac{\partial \varphi}{\partial \boldsymbol{n}} \; d\sigma$$

and

$$\int_\omega v_* \Delta \varphi \; d\boldsymbol{x} = -\int_D v \frac{\partial \varphi}{\partial \boldsymbol{n}} \; d\sigma.$$

Moreover, we notice that $\dfrac{\partial v_*}{\partial \boldsymbol{n}} \in W_0^{\frac{1}{2},2}(\Gamma)$ and that its support is included in S.

ii) Now, we want to find $w \in W_0^{2,2}(\Omega)$ solution of

$$\Delta w = h_1 \text{ in } \Omega \quad \text{and} \quad \frac{\partial w}{\partial \boldsymbol{n}} = -\frac{\partial v_*}{\partial \boldsymbol{n}} \text{ on } \Gamma, \tag{6.3}$$

satisfying

$$\|w\|_{W_0^{2,2}(\Omega)} \leq \; C \; \|h\|_{L^2(\Omega)}. \tag{6.4}$$

Since $h_1 \in L^2(\Omega)$ and $\dfrac{\partial v_*}{\partial \boldsymbol{n}} \in W_0^{\frac{1}{2},2}(\Gamma)$ have a compact support, we have $h_1 \in W_0^{-1,2}(\Omega)$ and $\dfrac{\partial v_*}{\partial \boldsymbol{n}} \in W_0^{-\frac{1}{2},2}(\Gamma)$ and there is a sense to search first a solution $w \in W_0^{1,2}(\Omega)$. For this, we easily prove that this problem is equivalent to the following variational formulation: find $w \in W_0^{1,2}(\Omega)$ such that for any $z \in W_0^{1,2}(\Omega)$

$$(\mathcal{FV}) \quad \int_\Omega \nabla w \cdot \nabla z \; d\boldsymbol{x} = \int_\Omega h_1 z \; d\boldsymbol{x} - < \frac{\partial v_*}{\partial \boldsymbol{n}}, z >_\Gamma,$$

and that there exists a unique solution of $(\mathcal{FV})$ applying the theorem of Lax-Milgram in $(W_0^{1,2}(\Omega), \| \cdot \|_{W_0^{1,2}(\Omega)})$ (the coercivity is satisfied by the point **i)** of Theorem 6.1.1).

Next, we prove that w is in $W_0^{2,2}(\Omega)$. For this, we set $w_1 = \psi_1 w \in W_0^{1,2}(\Omega)$ and $w_2 = \psi_2 w \in W_0^{1,2}(\Omega)$. Since supp $w_1 \subset G$ and $\Delta w = h_1 \in L^2(\Omega)$, we have

$$\Delta w_1 = w \Delta \psi_1 + 2 \nabla \psi_1 \cdot \nabla w + \psi_1 h_1 \ \in L^2(G),$$

and since $\psi_1 = \dfrac{\partial \psi_1}{\partial \boldsymbol{n}} = 0$ on $\partial G \cap \mathbb{R}^n_+$ and $\dfrac{\partial w}{\partial \boldsymbol{n}} = -\dfrac{\partial v_*}{\partial \boldsymbol{n}} \in H^{\frac{1}{2}}(\partial G \cap \Gamma)$, we have

$$\frac{\partial w_1}{\partial \boldsymbol{n}} = \psi_1 \frac{\partial w}{\partial \boldsymbol{n}} + \frac{\partial \psi_1}{\partial \boldsymbol{n}} w \ \in H^{\frac{1}{2}}(\partial G).$$

Thanks to regularity results in bounded domains (see [42] when the boundary is very smooth and the technique of Grisvard [29] for the extension to a $C^{1,1}$ boundary), we deduce from this that $w_1 \in H^2(G)$ and since supp $w_1 \subset G$, then $w_1 \in W_0^{2,2}(\Omega)$. Thus, in $\mathbb{R}^n_+$ $\Delta w_2 \in L^2(\mathbb{R}^n_+)$ and $\dfrac{\partial w_2}{\partial \boldsymbol{n}} \in W_0^{\frac{1}{2},2}(\mathbb{R}^{n-1})$. We easily conclude thanks to Corollary 3.3 of [4] that $w_2 \in W_0^{2,2}(\mathbb{R}^n_+)$ and since supp $w_2 \subset \mathbb{R}^n_+$, $w_2 \in W_0^{2,2}(\Omega)$. So, $w = w_1 + w_2 \in W_0^{2,2}(\Omega)$, satisfies (6.3) and the estimate (6.4). Finally, from (6.2) and (6.3), $u = v_* + w$ is solution of our problem and the estimate (6.1) follows immediately. □

Lemma 6.2.2. *For any $h \in L^2(\Omega)$, there exists $\boldsymbol{w} \in \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)$ satisfying*

$$\begin{cases} \text{div } \boldsymbol{w} = h \text{ in } \Omega, \\ \|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \; \|h\|_{L^2(\Omega)}, \end{cases}$$

where C is a real positive constant depending only on Ω.

Proof- Let h be in $L^2(\Omega)$. We know, thanks to the previous lemma, that there exists $\varphi \in W_0^{2,2}(\Omega)$ solution of

$$\Delta \varphi = h \text{ in } \Omega \quad \text{and} \quad \frac{\partial \varphi}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma,$$

with

$$\|\varphi\|_{W_0^{2,2}(\Omega)} \leq \; C \; \|h\|_{L^2(\Omega)}.$$

We set $\boldsymbol{v} = \nabla\varphi \in \boldsymbol{W}_0^{1,2}(\Omega)$. So $\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C\ \|h\|_{L^2(\Omega)}$. Moreover, we set $\boldsymbol{g} = \boldsymbol{v}_{|\Gamma}$, $\boldsymbol{g_1} = \psi_1\boldsymbol{g}$, $\boldsymbol{g_2} = \psi_2\boldsymbol{g}$ and we notice that $\boldsymbol{g}$, $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ belong to $\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)$. First, we want to solve the following problem: find $\boldsymbol{t} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that

$$\text{div } \boldsymbol{t} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{t} = \boldsymbol{g_2} \text{ on } \Gamma. \tag{6.5}$$

We define the function $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}}$ by

$$\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{g_2} \text{ on } \Sigma \quad \text{and} \quad \widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{0} \text{ on } D.$$

Noticing that supp $\boldsymbol{g_2} \subset \Sigma$, we have $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \in \boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})$. Moreover, we know, thanks to results in the half-space, that there exists $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)$ such that

$$\text{div } \boldsymbol{u} = 0 \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \boldsymbol{u} = \widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \text{ on } \mathbb{R}^{n-1}$$

and satisfying the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)} \leq\ C\ \|\widetilde{\boldsymbol{g}}_{\boldsymbol{2}}\|_{\boldsymbol{W}_0^{1-\frac{1}{2},2}(\mathbb{R}^{n-1})} \leq\ C\ \|h\|_{L^2(\Omega)}.$$

We define the function $\boldsymbol{t}$ by

$$\boldsymbol{t} = \boldsymbol{u} \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \boldsymbol{t} = \boldsymbol{0} \text{ in } \overline{\omega}.$$

We easily check that $\boldsymbol{t} \in \boldsymbol{W}_0^{1,2}(\Omega)$ and that div $\boldsymbol{t} = 0$ in Ω. Thus, since on Σ, $\boldsymbol{t} = \boldsymbol{g_2}$ and on S, $\boldsymbol{t} = \boldsymbol{g_2} = \boldsymbol{0}$, we have established that $\boldsymbol{t} \in \boldsymbol{W}_0^{1,2}(\Omega)$ is solution of (6.5) and that we have the estimate

$$\|\boldsymbol{t}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)} \leq\ C\ \|h\|_{L^2(\Omega)}.$$

Now, we want to solve the following problem: find $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that

$$\text{div } \boldsymbol{z} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{z} = \boldsymbol{g_1} \text{ on } \Gamma. \tag{6.6}$$

We define the function $\widetilde{\boldsymbol{g}}_{\boldsymbol{1}}$ by

$$\widetilde{\boldsymbol{g}}_{\boldsymbol{1}} = \boldsymbol{g_1} \text{ on } \partial G \cap \Gamma \quad \text{and} \quad \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} = \boldsymbol{0} \text{ on } \partial G \cap \mathbb{R}^n_+.$$

Noticing that supp $\boldsymbol{g_1} \subset \partial G \cap \Gamma$, we have $\widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \in \boldsymbol{H}^{\frac{1}{2}}(G)$ and we notice that

$$\int_{\partial G} \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \cdot \boldsymbol{n}\ d\sigma = \int_{\partial G\cap\Gamma} \boldsymbol{g_1} \cdot \boldsymbol{n}\ d\sigma = \int_{\partial G\cap\Gamma} \psi_1 \frac{\partial\varphi}{\partial\boldsymbol{n}}\ d\sigma = 0, \tag{6.7}$$

because $\dfrac{\partial\varphi}{\partial\boldsymbol{n}} = 0$ on $\partial G \cap \Gamma$. We deduce from (6.7), thanks to results in bounded domain that there exists $\boldsymbol{u_0} \in \boldsymbol{H}^1(G)$ such that

$$\text{div } \boldsymbol{u_0} = 0 \text{ in } G \quad \text{and} \quad \boldsymbol{u_0} = \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \text{ on } \partial G$$

and satisfying the estimate

$$\|\boldsymbol{u_0}\|_{\boldsymbol{H}^1(G)} \leq \; C \; \|\widetilde{\boldsymbol{g}}_{\boldsymbol{1}}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial G)} \leq \; C \; \|h\|_{L^2(\Omega)}.$$

We define the function $\boldsymbol{z}$ by

$$\boldsymbol{z} = \boldsymbol{u_0} \text{ in } G \quad \text{and} \quad \boldsymbol{z} = \boldsymbol{0} \text{ in } \Omega \setminus G.$$

We easily check that $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ and that div $\boldsymbol{z} = 0$ in Ω. Thus, we have established that $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ is solution of (6.6) and that we have the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq \; C \; \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)} \leq \; C \; \|h\|_{L^2(\Omega)}.$$

So, $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{t} - \boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ is solution of our problem and we have the estimate searched. $\square$

So to solve $(\mathcal{S}_{10})$, it is sufficient to solve the following problem $(\mathcal{S}_{100})$: find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of

$$(\mathcal{S}_{100}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

and, the study of this problem is exactly equivalent to the one of the end of Section 5.2. In consequence, we have the following theorem:

Theorem 6.2.3. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$, $h \in L^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)$ there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of (S_1). Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \; C \; (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)}),$$

where C is a real positive constant which depends only on Ω.

6.3 Case $p \neq 2$.

First, we suppose that $p > 2$ and we want to study the kernel of the Stokes system. We set:

$$\mathcal{D}_0^p(\Omega) = \{(\boldsymbol{z}, \pi) \in \mathring{\boldsymbol{W}}\,{}_0^{1,p}(\Omega) \times L^p(\Omega), \; -\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega\}.$$

We have the following result:

Theorem 6.3.1. *For each $p > 2$, the kernel $\mathcal{D}_0^p(\Omega)$ is reduced to $\{(\boldsymbol{0}, 0)\}$.*

Proof- Let $(\boldsymbol{z}, \pi)$ be in $\mathcal{D}_0^p(\Omega)$. We set $(\boldsymbol{z_1}, \pi_1) = (\psi_1 \boldsymbol{z}, \psi_1 \pi) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega)$. Since supp $(\boldsymbol{z_1}, \pi_1)$ is included in G which is bounded, we have

$$(\boldsymbol{z_1}, \pi_1) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$$

and so

$$-\Delta \boldsymbol{z_1} + \nabla \pi_1 \in \boldsymbol{W}_0^{-1,2}(\Omega) \text{ and div } \boldsymbol{z_1} \in L^2(\Omega).$$

Now, we set $(\boldsymbol{z_2}, \pi_2) = (\psi_2 \boldsymbol{z}, \psi_2 \pi) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega)$ and

$$\begin{aligned} &\boldsymbol{f} = -\,\Delta \boldsymbol{z_2} + \nabla \pi_2 = \Delta \boldsymbol{z_1} - \nabla \pi_1 \in \boldsymbol{W}_0^{-1,2}(\Omega) \\ &h = \text{div } \boldsymbol{z_2} = -\text{ div } \boldsymbol{z_1} \in L^2(\Omega). \end{aligned}$$

Thus, $(\boldsymbol{f}, h) \in \boldsymbol{W}_0^{-1,2}(\mathbb{R}_+^n) \times L^2(\mathbb{R}_+^n)$ and by Theorem 2.5.1, there exists $(\boldsymbol{s}, \theta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}_+^n) \times L^2(\mathbb{R}_+^n)$ solution of

$$-\Delta \boldsymbol{s} + \nabla \theta = \boldsymbol{f} \text{ in } \mathbb{R}_+^n, \quad \text{div } \boldsymbol{s} = h \text{ in } \mathbb{R}_+^n, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}. \tag{6.8}$$

But, noticing that on $\mathbb{R}^{n-1}$, $\boldsymbol{z_2} = \boldsymbol{0}$ (because $\boldsymbol{z_2} = \boldsymbol{0}$ on Σ and $\psi_2 = 0$ on D), it is obvious that $(\boldsymbol{z_2}, \pi_2) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^n) \times L^p(\mathbb{R}_+^n)$ is solution of (6.8). So, $(\boldsymbol{w}, \tau) = (\boldsymbol{s} - \boldsymbol{z_2}, \theta - \pi_2)$ satisfy

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \mathbb{R}_+^n, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}_+^n, \quad \boldsymbol{w} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

and we easily deduce that $(\boldsymbol{w}, \tau) = (\boldsymbol{0}, 0)$ in $\mathbb{R}_+^n$ (see Lemma 5.3.1). Thus $(\boldsymbol{z_2}, \pi_2) = (\boldsymbol{s}, \theta) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}_+^n) \times L^2(\mathbb{R}_+^n)$ and since supp $(\boldsymbol{z_2}, \pi_2) \subset \mathbb{R}_+^n$, we deduce that $(\boldsymbol{z_2}, \pi_2) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. Finally, we conclude that $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$, which implies that $(\boldsymbol{z}, \pi) \in \mathcal{D}_0^2(\Omega)$. But, we have seen at Theorem 6.2.3 that when $p = 2$ problem (S_{10}) has a unique solution. Here, $(\boldsymbol{0}, 0)$ is solution, so we have our result. □

Now, supposing that $p > 2$, we want to study the Stokes system with homogeneous boundary conditions, that is to say: let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$, we want to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of the problem

$$(\mathcal{S}_{10}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

First, we establish the following lemma:

Lemma 6.3.2. *For each $p > 2$ and for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$ with a compact support in Ω, there exists a unique $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)) \times (L^2(\Omega) \cap L^p(\Omega))$ solution of (S_{10}).*

Proof- Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$ with a compact support in Ω. Then, since $p > 2$, we easily check that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $h \in L^2(\Omega)$ and we deduce from Theorem 6.2.3 that there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ solution of (S_{10}). It remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. We set $(\boldsymbol{u_1}, \pi_1) = (\psi_1 \boldsymbol{u}, \psi_1 \pi) \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$, it has a compact support included in G. An elementary calculation shows that we have

$$-\Delta \boldsymbol{u_1} + \nabla \pi_1 = \psi_1 \boldsymbol{f} + \boldsymbol{F}_1 \text{ in } G, \quad \text{div } \boldsymbol{u_1} = \psi_1 h + H_1 \text{ in } G, \quad \boldsymbol{u_1} = \boldsymbol{0} \text{ on } \partial G,$$

where

$$\boldsymbol{F}_1 = -(2\nabla \boldsymbol{u} \nabla \psi_1 + \boldsymbol{u} \Delta \psi_1) + \pi \nabla \psi_1 \in \boldsymbol{L}^2(G),$$

and

$$H_1 = \boldsymbol{u} \cdot \nabla \psi_1 \in \boldsymbol{H}^1(G).$$

Thanks to the Sobolev imbeddings, we have

$$(\boldsymbol{F}_1, H_1) \in \boldsymbol{W}^{-1,s}(G) \times L^s(G), \quad \forall 1 < s \leq 2*,$$

where $2* = \dfrac{2n}{n-2}$. So, if $p \leq 2*$, thanks to results in bounded domains (see [5]), we have

$$(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}^{1,p}(G) \times L^p(G) \tag{6.9}$$

and

$$\|\boldsymbol{u_1}\|_{\boldsymbol{W}^{1,p}(G)} + \|\pi_1\|_{L^p(G)} \leq \ C \ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}).$$

Now, if $p > 2*$, we can show that $(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}^{1,2*}(G) \times L^{2*}(G)$ because $(\psi_1 \boldsymbol{f}, \psi_1 h) \in \boldsymbol{W}^{-1,2*}(G) \times L^{2*}(G)$. Thus, we have $(\boldsymbol{F}_1, H_1) \in \boldsymbol{L}^{2*}(G) \times W^{1,2*}(G)$ and we apply the same argument as previously with $2*$ instead of 2. Finally, starting again, we reach any value of $p > 2$. Thus, we have (6.9) in any case and since supp $(\boldsymbol{u_1}, \pi_1) \subset G$, we deduce that

$$(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega) \tag{6.10}$$

Now, we set $(\boldsymbol{u_2}, \pi_2) = (\psi_2 \boldsymbol{u}, \psi_2 \pi) \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$ and

$$\begin{aligned} &\boldsymbol{f_2} = -\,\Delta \boldsymbol{u_2} + \nabla \pi_2 = \boldsymbol{f} - (-\Delta \boldsymbol{u_1} + \nabla \pi_1) \in \boldsymbol{W}_0^{-1,p}(\Omega), \\ &h_2 = \text{div } \boldsymbol{u_2} = h - \text{ div } \boldsymbol{u_1} \in L^p(\Omega). \end{aligned}$$

We have $(\boldsymbol{f_2}, h_2) \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ and by Theorem 2.5.1, there exists $(\boldsymbol{s}, \theta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{s} + \nabla \theta = \boldsymbol{f_2} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{s} = h_2 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

But, noticing that on $\mathbb{R}^{n-1}$, $\boldsymbol{u_2} = \boldsymbol{0}$ (because $\boldsymbol{u_2} = \boldsymbol{0}$ on Σ and $\psi_2 = 0$ on D), it is obvious that in $\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$, $(\boldsymbol{u_2}, \pi_2)$ is solution of the same problem that $(\boldsymbol{s}, \theta)$ satisfies. We use the same reasoning as in Theorem 6.3.1 to conclude that

$$(\boldsymbol{u_2}, \pi_2) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega) \tag{6.11}$$

Finally, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ by (6.10) and (6.11). $\square$

Now, we establish the following theorem:

Theorem 6.3.3. *For any $p > 2$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of*

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma. \tag{6.12}$$

Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \ C \ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)},$$

where C is a real positive constant which depends only on p and Ω.

Proof- The uniqueness comes from Theorem 6.3.1. Now, let $\boldsymbol{g}$ be in $\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$. We set

$$\boldsymbol{g_1} = \psi_1 \boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma) \text{ and } \boldsymbol{g_2} = \psi_2 \boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$$

First, we want to find $(\boldsymbol{v}, \mu) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta \boldsymbol{v} + \nabla \mu = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{g_1} \text{ on } \Gamma. \tag{6.13}$$

We notice that supp $\boldsymbol{g_1} \subset \partial G \cap \Gamma$. We define the function $\widetilde{\boldsymbol{g}}_{\boldsymbol{1}}$ by

$$\widetilde{\boldsymbol{g}}_{\boldsymbol{1}} = \boldsymbol{g_1} \text{ on } \partial G \cap \Gamma \quad \text{and} \quad \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} = \boldsymbol{0} \text{ on } \partial G \cap \mathbb{R}^n_+.$$

We easily check that $\widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\partial G)$. Let ψ be in $\mathcal{D}(\mathbb{R}^n)$ with a compact support in G such that

$$\int_G \psi(\boldsymbol{x}) \, d\boldsymbol{x} = \int_{\partial G} \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \cdot \boldsymbol{n} \, d\sigma.$$

Thanks to this condition and results in bounded domains (see [5]), there exists $(\boldsymbol{z}, \pi) \in \boldsymbol{W}^{1,p}(G) \times L^p(G)$ such that

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ in } G, \quad \text{div } \boldsymbol{z} = \psi \text{ in } G, \quad \boldsymbol{z} = \widetilde{\boldsymbol{g}}_{\boldsymbol{1}} \text{ on } \partial G,$$

We denote again by $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ its extension by $(\boldsymbol{0}, 0)$ in Ω. Thus, $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ satisfies

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{\sigma} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = \psi \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g_1} \text{ on } \Gamma,$$

where we notice that $\boldsymbol{\sigma} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ has a compact support in Ω. As ψ has a compact support too, we deduce from Lemma 6.3.2 that there exists $(\boldsymbol{t}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta \boldsymbol{t} + \nabla \tau = -\boldsymbol{\sigma} \text{ in } \Omega, \quad \text{div } \boldsymbol{t} = -\psi \text{ in } \Omega, \quad \boldsymbol{t} = \boldsymbol{0} \text{ on } \Gamma,$$

Finally, $(\boldsymbol{v}, \mu) = (\boldsymbol{z} + \boldsymbol{t}, \pi + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of (6.13).

Now, we want to find $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \eta = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \Omega, \quad \boldsymbol{w} = \boldsymbol{g_2} \text{ on } \Gamma. \tag{6.14}$$

For this, we notice that supp $\boldsymbol{g_2} \subset \Sigma$. We define the function $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}}$ by

$$\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{g_2} \text{ on } \Sigma \quad \text{and} \quad \widetilde{\boldsymbol{g}}_{\boldsymbol{2}} = \boldsymbol{0} \text{ on } D.$$

We easily check that $\widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$. Thanks to Theorem 2.5.1, there exists $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ such that

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{z} = \widetilde{\boldsymbol{g}}_{\boldsymbol{2}} \text{ on } \mathbb{R}^{n-1}.$$

We denote again by $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ its extension by $(\boldsymbol{0}, 0)$ in Ω. So, $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ satisfies

$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g_2} \text{ on } \Gamma,$$

where $\boldsymbol{\xi}$ has a compact support in Ω. We deduce from Lemma 6.3.2 that there exists $(\boldsymbol{t}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta \boldsymbol{t} + \nabla \tau = -\boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{t} = 0 \text{ in } \Omega, \quad \boldsymbol{t} = \boldsymbol{0} \text{ on } \Gamma.$$

Finally, $(\boldsymbol{w}, \eta) = (\boldsymbol{z} + \boldsymbol{t}, \pi + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of (6.14).

In consequence, $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{w}, p + \eta) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of (6.12) and the estimate follows immediately. $\square$

Now, we can solve the problem with homogeneous boundary conditions in the case $p > 2$.

Theorem 6.3.4. *For any $p > 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of $(\mathcal{S}_{10})$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C \, (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- The uniqueness comes from Theorem 6.3.1. Then, as a consequence of Theorem 6.1.1 ii), we know that there exists a tensor of the second order $F \in [L^p(\Omega)]^{n\times n}$ such that div $F = \boldsymbol{f}$. We extend F (respectively h) by 0 in $\mathbb{R}^n$, and we denote by $\widetilde{F}$ (respectively $\widetilde{h}$) this extension. Then, we set $\widetilde{\boldsymbol{f}} = \text{div } \widetilde{F}$ and we notice that $\widetilde{\boldsymbol{f}}_{|\Omega} = \boldsymbol{f}$. We have $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)$ and $\widetilde{h} \in L^p(\mathbb{R}^n)$. By [2], there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n \quad \text{and} \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n,$$

satisfying the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^n)} + \|\eta\|_{L^p(\mathbb{R}^n)} \leq \ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}).$$

We denote again by $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\eta \in L^p(\Omega)$ the restrictions of $\boldsymbol{v}$ and η to Ω. We have $\boldsymbol{v}_{|\Gamma} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, thus, thanks to Theorem 6.3.3, there exists $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \Omega, \quad \boldsymbol{w} = -\boldsymbol{v}_{|\Gamma} \text{ on } \Gamma,$$

satisfying the estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\tau\|_{L^p(\Omega)} \leq \ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)}).$$

Finally, $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{w}, \eta + \tau) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of $(\mathcal{S}_{10})$ and the estimate follows immediately. $\square$

Now, we suppose that p is such that $p < 2$. Thanks to the previous theorem, if we set

$$S : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \longrightarrow \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega),$$
$$(\boldsymbol{u}, \pi) \longrightarrow (-\Delta \boldsymbol{u} + \nabla \pi, -\text{div } \boldsymbol{u}),$$

then, S is an isomorphism. So, by duality,

$$S^* : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega) \longrightarrow \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega),$$

is an isomorphism too, and, as it is standard to check that $S^*(\boldsymbol{u}, \pi) = (-\Delta \boldsymbol{u} + \nabla \pi, -\text{div } \boldsymbol{u})$, we have Theorem 6.3.4 for any $p < 2$. $\square$

Finally, it remains to return to the general problem with $p \neq 2$ and nonhomogeneous boundary conditions. For this, like for the case $p = 2$, we show that there exists a function $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{w} = \boldsymbol{g}$ in Γ. Then, we have just seen that there exists a unique $(\boldsymbol{v}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{w} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = h - \text{div } \boldsymbol{w} \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma.$$

In consequence, the function $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is solution of the problem $(\mathcal{S}_1)$ and we have the following theorem:

Theorem 6.3.5. *For any* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, *there exists a unique* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ *solution of the problem* $(\mathcal{S}_1)$

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \text{ in } \Omega, \quad \text{div } \boldsymbol{u} = h \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma.$$

Moreover, $(\boldsymbol{u}, \pi)$ *satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)}),$$

where C *is a real positive constant which depends only on* p *and* Ω.

6.4 Regularity, strong solutions and very weak solutions.

First, in this section, we are interested in the existence of strong solutions of the Stokes system $(\mathcal{S}_1)$, *i.e.* of solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{\ell+1}^{2,p}(\Omega) \times W_{\ell+1}^{1,p}(\Omega)$. Here, we limit ourselves to the two cases $\ell = 0$ and $\ell = -1$.

We give at the beginning a regularity result studying the case $\ell = 0$. Indeed, we notice that in this case, we have the continuous injections $\boldsymbol{W}_1^{2,p}(\Omega) \subset \boldsymbol{W}_0^{1,p}(\Omega)$ and $W_1^{1,p}(\Omega) \subset L^p(\Omega)$. So, Theorem which follows shows that the generalized solution of Theorems 6.2.3 and 6.3.5, with a stronger hypothesis on the data, is in fact a strong solution.

Theorem 6.4.1. *For any* $p > 1$ *such that* $\dfrac{n}{p'} \neq 1$, *and for any* $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$, $h \in W_1^{1,p}(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}_1^{2-\frac{1}{p},p}(\Gamma)$, *there exists a unique* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ *solution of the problem* $(\mathcal{S}_1)$. *Moreover,* $(\boldsymbol{u}, \pi)$ *satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\Omega)} + \|\pi\|_{W_1^{1,p}(\Omega)} \leq C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\Omega)} + \|h\|_{W_1^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_1^{2-\frac{1}{p},p}(\Gamma)}),$$

where C *is a real positive constant which depends only on* p *and* Ω.

Proof- First, we want to solve the problem with homogeneous boundary conditions. For this, we notice that we have the continuous injections $\boldsymbol{W}_1^{0,p}(\Omega) \subset \boldsymbol{W}_0^{-1,p}(\Omega)$ because $\dfrac{n}{p'} \neq 1$ and $W_1^{1,p}(\Omega) \subset L^p(\Omega)$. Thus, thanks to Theorems 6.2.3 and 6.3.5, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of (S_{10}). Next, it remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$. We set $(\boldsymbol{u_1}, \pi_1) = (\psi_1 \boldsymbol{u}, \psi_1 \pi) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega)$, it has a compact support included in G. An elementary calculation shows that we have

$$-\Delta \boldsymbol{u_1} + \nabla \pi_1 = \psi_1 \boldsymbol{f} + \boldsymbol{F}_1 \text{ in } G, \quad \text{div } \boldsymbol{u_1} = \psi_1 h + H_1 \text{ in } G, \quad \boldsymbol{u_1} = \boldsymbol{0} \text{ on } \partial G,$$

where

$$\boldsymbol{F}_1 = -(2\nabla \boldsymbol{u} \nabla \psi_1 + \boldsymbol{u} \Delta \psi_1) + \pi \nabla \psi_1 \in \boldsymbol{L}^p(G),$$

and

$$H_1 = \boldsymbol{u} \cdot \nabla \psi_1 \in \boldsymbol{W}^{1,p}(G).$$

Thus, $\psi_1 \boldsymbol{f} + \boldsymbol{F}_1 \in \boldsymbol{L}^p(G)$ and $\psi_1 h + H_1 \in \boldsymbol{W}^{1,p}(G)$. So, using results in bounded domains, we have $(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}^{2,p}(G) \times \boldsymbol{W}^{1,p}(G)$ and since supp $(\boldsymbol{u_1}, \pi_1) \subset G$,

$$(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}_1^{2,p}(\Omega) \times \boldsymbol{W}_1^{1,p}(\Omega). \tag{6.15}$$

Now, we set $(\boldsymbol{u_2}, \pi_2) = (\psi_2 \boldsymbol{u}, \psi_2 \pi) \in \mathring{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p(\Omega)$ and

$$\begin{aligned} \boldsymbol{f_2} &= -\Delta \boldsymbol{u_2} + \nabla \pi_2 = \boldsymbol{f} - (-\Delta \boldsymbol{u_1} + \nabla \pi_1) \in \boldsymbol{W}_1^{0,p}(\Omega), \\ h_2 &= \operatorname{div} \boldsymbol{u_2} = h - \operatorname{div} \boldsymbol{u_1} \in W_1^{1,p}(\Omega). \end{aligned}$$

We have $(\boldsymbol{f_2}, h_2) \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^n_+) \times W_1^{1,p}(\mathbb{R}^n_+)$ and by [9], there exists $(\boldsymbol{s}, \theta) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+) \times W_1^{1,p}(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{s} + \nabla \theta = \boldsymbol{f_2} \text{ in } \mathbb{R}^n_+, \quad \operatorname{div} \boldsymbol{s} = h_2 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}, \tag{6.16}$$

But, noticing that $(\boldsymbol{u_2}, \pi_2)$ is also solution of (6.16) and that, by Theorem 2.5.1, the solution of this problem is unique in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$, we have $(\boldsymbol{u_2}, \pi_2) = (\boldsymbol{s}, \theta) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^n_+) \times W_1^{1,p}(\mathbb{R}^n_+)$. The support of $(\boldsymbol{u_2}, \pi_2)$ being included in $\mathbb{R}^n_+$, we deduce that

$$(\boldsymbol{u_2}, \pi_2) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega) \tag{6.17}$$

So, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ by (6.15) and (6.17). Finally, thanks to Lemma 1.2, we come back to a problem with nonhomogeneous boundary conditions. $\square$

Now, we examine the basic case $\ell = -1$, corresponding to $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and first, we study the kernel of such a problem. We set

$$\begin{aligned} \mathcal{S}_0^p(\Omega) = \{(\boldsymbol{z}, \pi) \in \ \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega),\ -\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ in } \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \boldsymbol{z} = \boldsymbol{0} \text{ on } \Gamma\}. \end{aligned}$$

The characterization of this kernel is given by this proposition:

Proposition 6.4.2. *For each $p > 1$ such that $\dfrac{n}{p'} \neq 1$, we have the following statements: i) If $p < n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{0}, 0)\}$.*
ii) If $p \geq n$, $\mathcal{S}_0^p(\Omega) = \{(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu),\ \ \boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\},\ \mu \in \mathbb{R}\}$ where $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda})) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ is the unique solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \boldsymbol{0} \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{\lambda} \text{ on } \Gamma.$$

Proof- Let $(\boldsymbol{z},\pi) \in \mathcal{S}_0^p(\Omega)$. We set $(\boldsymbol{z_i},\pi_i) = (\psi_i\boldsymbol{z},\psi_i\pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ (for $i = 1$ or 2). Since supp $(\boldsymbol{z_1},\pi_1)$ is bounded, we have $(\boldsymbol{z_1},\pi_1) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$. Now, we set

$$\boldsymbol{f_2} = -\Delta\boldsymbol{z_2} + \nabla\pi_2 = \Delta\boldsymbol{z_1} - \nabla\pi_1, \quad h_2 = \text{ div } \boldsymbol{z_2} = - \text{ div } \boldsymbol{z_1}.$$

We have $(\boldsymbol{f_2}, h_2) \in \boldsymbol{W}_1^{0,p}(\Omega) \times W_1^{1,p}(\Omega)$, so by Theorem 5.2 of [9], there exists $(\boldsymbol{s},\theta) \in (\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^n) \times W_1^{1,p}(\mathbb{R}_+^n)) \subset (\boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n))$ solution of

$$(\mathcal{S}_+) \quad -\Delta\boldsymbol{s} + \nabla\theta = \boldsymbol{f_2} \text{ in } \mathbb{R}_+^n, \quad \text{div } \boldsymbol{s} = h_2 \text{ in } \mathbb{R}_+^n, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1}.$$

Noticing that $(\boldsymbol{z_2},\pi_2) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n)$ is solution of the problem $(\mathcal{S}_+)$, we can deduce, using Theorem 5.6 in [9]), the following results:

i) If $p < n$, the solution of $(\mathcal{S}_+)$ is unique in $\boldsymbol{W}_0^{2,p}(\mathbb{R}_+^n) \times W_0^{1,p}(\mathbb{R}_+^n)$, so $(\boldsymbol{z_2},\pi_2) = (\boldsymbol{s},\theta)$. Thus, as the support of $(\boldsymbol{z_2},\pi_2)$ is included in $\mathbb{R}_+^n$, we have $(\boldsymbol{z_2},\pi_2) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and so $(\boldsymbol{z},\pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$. Thanks to Theorem 6.4.1, we have necessarily $(\boldsymbol{z},\pi) = (\boldsymbol{0},0)$.

ii) If $p \geq n$, there exists $\boldsymbol{\lambda} \in (\mathbb{R}x_n)^{n-1} \times \{0\}$ and $\mu \in \mathbb{R}$ such that

$$\boldsymbol{z_2} = \boldsymbol{s} - \boldsymbol{\lambda} \quad \text{and} \quad \pi_2 = \theta - \mu \quad \text{in } \mathbb{R}_+^n.$$

We define $\boldsymbol{w}$ by $\boldsymbol{w} = \boldsymbol{s}$ in $\mathbb{R}_+^n$, $\boldsymbol{w} = \boldsymbol{\lambda}$ in ω and ξ by $\xi = \theta$ in $\mathbb{R}_+^n$, $\xi = \mu$ in ω. We easliy check that $(\boldsymbol{w},\xi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and that

$$\boldsymbol{z_2} = \boldsymbol{w} - \boldsymbol{\lambda} \quad \text{and} \quad \pi_2 = \xi - \mu \quad \text{in } \Omega.$$

Finally, we set

$$\boldsymbol{v} = \boldsymbol{z} + \boldsymbol{\lambda} \quad \text{and} \quad \eta = \pi + \mu \quad \text{in } \Omega.$$

Then, $(\boldsymbol{v},\eta)$ is in $\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ and is the unique (see Theorem 6.4.1) solution of

$$-\Delta\boldsymbol{v} + \nabla\eta = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{\lambda} \text{ on } \Gamma.$$

In consequence, we have the characterization of the kernel. □

We have the following result, corresponding to Theorem 6.4.1:

Theorem 6.4.3. *For any $p > 1$ such that $\frac{n}{p'} \neq 1$, and for any $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W_0^{1,p}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$, there exists a unique $(\boldsymbol{u},\pi) \in (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))/\mathcal{S}_0^p(\Omega)$ solution of the problem $(\mathcal{S}_1)$. Moreover, $(\boldsymbol{u},\pi)$ satisfies*

$$\inf_{(\boldsymbol{z},\alpha)\in\mathcal{S}_0^p(\Omega)} (\|\boldsymbol{u}+\boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi+\alpha\|_{W_0^{1,p}(\Omega)}) \leq C \, (\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)}),$$

where C is a real positive constant which depends only on p and Ω.

Proof- i) First, we solve the following problem: find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ solution of

$$-\Delta \boldsymbol{u} + \nabla \pi = \mathbf{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma.$$

For this, we set $\boldsymbol{g_i} = \psi_i \boldsymbol{g} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$ (for $i = 1$ or 2) and we define the function $\widetilde{\boldsymbol{g}}_{\mathbf{2}}$ by

$$\widetilde{\boldsymbol{g}}_{\mathbf{2}} = \boldsymbol{g_2} \text{ on } \Sigma, \quad \widetilde{\boldsymbol{g}}_{\mathbf{2}} = \mathbf{0} \text{ on } D.$$

So $\widetilde{\boldsymbol{g}}_{\mathbf{2}} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\mathbb{R}^{n-1})$ and by Theorem 5.6 of [9], there exists $(\boldsymbol{w}, \tau) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n_+) \times W_0^{1,p}(\mathbb{R}^n_+)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \tau = \mathbf{0} \text{ in } \mathbb{R}^n_+, \quad \text{div } \boldsymbol{w} = 0 \text{ in } \mathbb{R}^n_+, \quad \boldsymbol{w} = \widetilde{\boldsymbol{g}}_{\mathbf{2}} \text{ on } \mathbb{R}^{n-1}.$$

Now, we define for almost all $(\boldsymbol{x'}, x_n) \in \mathbb{R}^n$, the following functions $\boldsymbol{w^*}$ and τ_* by

$$\boldsymbol{w^*}(\boldsymbol{x'}, x_n) = \boldsymbol{w}(\boldsymbol{x'}, x_n) \text{ if } x_n > 0, \quad \boldsymbol{w^*}(\boldsymbol{x'}, x_n) = -\boldsymbol{w}(\boldsymbol{x'}, -x_n) \text{ if } x_n < 0$$

and

$$\tau_*(\boldsymbol{x'}, x_n) = \tau(\boldsymbol{x'}, x_n) \text{ if } x_n > 0, \quad \tau_*(\boldsymbol{x'}, x_n) = \tau(\boldsymbol{x'}, -x_n) \text{ if } x_n < 0.$$

Then, we set

$$\widetilde{\boldsymbol{w}} = \boldsymbol{w} \text{ in } \mathbb{R}^n_+, \quad \widetilde{\boldsymbol{w}} = \boldsymbol{w^*} \text{ in } \omega$$

and

$$\widetilde{\tau} = \tau \text{ in } \mathbb{R}^n_+, \quad \widetilde{\tau} = \tau_* \text{ in } \omega.$$

We easily check that $\widetilde{\boldsymbol{w}} \in \boldsymbol{W}_0^{2,p}(\Omega)$ and that $\widetilde{\tau} \in W_0^{1,p}(\Omega)$. Finally, we denote by $\boldsymbol{\mu} \in \boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$ the trace of the function $\widetilde{\boldsymbol{w}}$ and we set

$$-\Delta \widetilde{\boldsymbol{w}} + \nabla \widetilde{\tau} = \boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \widetilde{\boldsymbol{w}} = \sigma \text{ in } \Omega.$$

The functions $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ and $\sigma \in W_0^{1,p}(\Omega)$ have clearly a compact support, so, by Theorem 6.4.1, there exists $(\boldsymbol{t}, \beta) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)$ such that

$$-\Delta \boldsymbol{t} + \nabla \beta = -\boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{t} = -\sigma \text{ in } \Omega, \quad \boldsymbol{t} = \mathbf{0} \text{ on } \Gamma,$$

(indeed, $\xi \in \boldsymbol{W}_1^{0,p}(\Omega)$ and $\sigma \in W_1^{1,p}(\Omega)$). The pair $(\boldsymbol{z}, \eta) = (\widetilde{\boldsymbol{w}} + \boldsymbol{t}, \widetilde{\tau} + \beta) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ satisfy

$$-\Delta \boldsymbol{z} + \nabla \eta = \mathbf{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{\mu} \text{ on } \Gamma.$$

In a last step, noticing that on $\Sigma \cap B_0$, $\boldsymbol{\mu} = \mathbf{0}$ because $\boldsymbol{\mu} = \boldsymbol{g_2}$ on Σ, we can say that the function $\boldsymbol{\gamma}$, defined by $\boldsymbol{\gamma} = -\boldsymbol{\mu}$ on S and $\boldsymbol{\gamma} = \mathbf{0}$ on Σ, belongs to $\boldsymbol{W}_0^{2-\frac{1}{p},p}(\Gamma)$. Moreover, since $\boldsymbol{g_1}$ and $\boldsymbol{\gamma}$ have a compact support, they belong to the space $\boldsymbol{W}_1^{2-\frac{1}{p},p}(\Gamma)$. Thus, applying Theorem 6.4.1, there exists $(\boldsymbol{v}, p) \in (\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)) \subset (\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega))$ such that

$$-\Delta \boldsymbol{v} + \nabla p = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \quad \boldsymbol{v} = \boldsymbol{\gamma} + \boldsymbol{g_1} \text{ on } \Gamma.$$

Noticing that on Γ, $\boldsymbol{\mu} + \boldsymbol{\gamma} + \boldsymbol{g_1} = \boldsymbol{g}$, we conclude that $(\boldsymbol{u}, \pi) = (\boldsymbol{z} + \boldsymbol{v}, \eta + p)$ answers the question.

ii) We easily show that there exists extensions $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^p(\mathbb{R}^n)$ of $\boldsymbol{f}$ and $\widetilde{h} \in W_0^{1,p}(\mathbb{R}^n)$ of h in $\mathbb{R}^n$ and, by Theorem 3.10 of [3], there exists $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^n) \times W_0^{1,p}(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{w} + \nabla \eta = \widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n, \quad \text{div } \boldsymbol{w} = \widetilde{h} \text{ in } \mathbb{R}^n.$$

Moreover, by **i)**, there exists $(\boldsymbol{z}, \mu) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ such that

$$-\Delta \boldsymbol{z} + \nabla \mu = \boldsymbol{0} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g} - \boldsymbol{w}_{|\Gamma} \text{ on } \Gamma.$$

Thus, $(\boldsymbol{u}, p) = (\boldsymbol{z} + \boldsymbol{w}, \mu + \eta)$ is solution of our problem. □

On the other hand, we want to study the case of very weak solutions *i.e.*, we study $(\mathcal{S}_1)$ with $\boldsymbol{f} = 0$, $h = 0$ and singular data on the boundary. We use previous results for strong solutions and we argue by duality. The proofs are exactly the same as Section 5.5. We give the two following theorems:

Theorem 6.4.4. *For each $p > 1$ such that $\dfrac{n}{p} \neq 1$ and for any $\boldsymbol{g} \in \boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\Gamma)$ satisfying*

$$\boldsymbol{g} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{6.18}$$

there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\Omega) \times W_{-1}^{-1,p}(\Omega)$ solution of $(\mathcal{S}_1)$ with $\boldsymbol{f} = 0$, $h = 0$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_{-1}^{0,p}(\Omega)} + \|\pi\|_{W_{-1}^{-1,p}(\Omega)} \leq \; C \; \|\boldsymbol{g}\|_{\boldsymbol{W}_{-1}^{-\frac{1}{p},p}(\Omega)},$$

where C is a real positive constant wich depends only on Ω and p.

Theorem 6.4.5. *For each $p > 1$ such that $\dfrac{n}{p} \neq 1$ and for any $\boldsymbol{g} \in \boldsymbol{W}_0^{-\frac{1}{p},p}(\Gamma)$ satisfying (6.18) and the following condition if $p \leq \dfrac{n}{n-1}$: for any $(\boldsymbol{z}, p) \in \mathcal{S}_0^{p'}(\Omega)$*

$$< \boldsymbol{g}, \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}} >_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\Gamma), \boldsymbol{W}_0^{\frac{1}{p},p'}(\Gamma)} = 0,$$

there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ solution of $(\mathcal{S}_1)$ with $\boldsymbol{f} = 0$, $h = 0$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W_0^{-1,p}(\Omega)} \leq \; C \; \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{-\frac{1}{p},p}(\Gamma)},$$

where C is a real positive constant wich depends only on Ω and p.

Chapter 7

A Stokes problem in an aperture domain in dimension $n \geq 3$

7.1 Introduction and preliminaries

In this chapter, we want to study a Stokes system in dimension $n \geq 3$ in an other particular unbounded domain with an unbounded boundary, namely an aperture domain *i.e.* two half-spaces separated by some wall of thickness $d > 0$ and connected by some hole (aperture). We can remind that the aperture domain became interesting since Heywood ([31], [32], in the Hilbert case) found the important role of the flux condition

$$\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma$$

as an additional boundary condition in order to get uniqueness for the Stokes problem with a Dirichlet condition (see also [14], [39], [50] or [51]). So, we want to find $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^p(\Omega_+)$, $\pi - a_- \in L^p(\Omega_-)$ and

$$(\mathcal{S}_2) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \\ \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha. & \end{cases}$$

This chapter is organized as follows. First, we give some definitions and preliminary results. Next, Sections 7.2 and 7.3 are respectively devoted to the resolution of $(\mathcal{S}_2)$ in cases $p = 2$ and $p \neq 2$ and in Section 6.4, we give a regularity result.

Now, we want to give a precise definition of an aperture domain. For this, we set $d \in \mathbb{R}_*^+$ and $\mathbb{R}^n_{-d} = \{\boldsymbol{x} \in \mathbb{R}^n, x_n < -d\}$. Let Ω an open domain such that

$$\mathbb{R}^n_+ \cup \mathbb{R}^n_{-d} \subset \Omega \subset \mathbb{R}^n.$$

There exists an open ball $B \subset \mathbb{R}^n$ such that

$$\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_{-d} \cup B.$$

We set $\Gamma = \partial\Omega$ the boundary of Ω that we suppose of class $C^{1,1}$; then, we can choose some bounded subdomain $G \subset \Omega$ with boundary ∂G of class $C^{1,1}$ such that $\Omega \cap B \subset G$. The ball B can be chosen centered on $(0, \dots, 0, -\dfrac{d}{2})$ and sufficiently large so that there exists another ball B_0 centered on $(0, \dots, 0, -\dfrac{d}{2})$ with closure $\overline{B}_0 \subset B$ such that

$$\Omega \cup B_0 = \mathbb{R}^n_+ \cup \mathbb{R}^n_{-d} \cup B_0.$$

We define two disjoints subdomains Ω_+ and Ω_- of Ω and an $(n-1)$-dimensional smooth manifold M with the following properties:

$$\Omega = \Omega_+ \cup \Omega_- \cup M, \qquad M = \partial\Omega_+ \cap \partial\Omega_-,$$

and

$$\Omega_+ \cup B = \mathbb{R}^n_+ \cup B, \qquad \Omega_- \cup B = \mathbb{R}^n_{-d} \cup B.$$

We can notice that Ω_+ and Ω_- are perturbed half-spaces. Finally, we define the following partition of unity:

$$\begin{gathered}\psi_1,\ \psi_2 \in C^\infty(\mathbb{R}^n),\ 0 \le \psi_1, \psi_2 \le 1,\ \psi_1 + \psi_2 = 1 \text{ in } \mathbb{R}^n,\\ \psi_1 = 1 \text{ in } B_0, \quad \operatorname{supp} \psi_1 \subset B.\end{gathered}$$

We easily check that the property (1.1) is satisfied and that, exactly like for the perturbed half-space, we have the following Poincaré-type inequalities:

Theorem 7.1.1. *i) The semi-norm* $|\cdot|_{W^{1,2}_0(\Omega)}$ *defined on* $W^{1,2}_0(\Omega)$ *is a norm equivalent to the full norm* $\|\cdot\|_{W^{1,2}_0(\Omega)}$.

ii) The semi-norm $|\cdot|_{W^{1,2}_0(\Omega)}$ *defined on* $\overset{\circ}{W}{}^{1,2}_0(\Omega)$ *is a norm equivalent to the full norm* $\|\cdot\|_{W^{1,2}_0(\Omega)}$.

The boundary Γ is unbounded. So, like for the domain $\mathbb{R}^n_+$ and the perturbed half-space, we define weighted Sobolev traces spaces. For any $\sigma \in]0,1[$, we set

$$\omega_1 = \begin{cases} \rho & \text{if } \dfrac{n}{p} \neq \sigma, \\ \rho(lg\rho)^{1/\sigma} & \text{if } \dfrac{n}{p} = \sigma. \end{cases}$$

We define the space

$$W_0^{\sigma,p}(\Gamma) = \{u, \omega_1^{-\sigma} u \in L^p(\Gamma \cap \ ^cB_0), u \in L^p(\Gamma \cap B_0),$$
$$\int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x}d\boldsymbol{y} < \infty\}.$$

It is a reflexive Banach space equipped with its natural norm

$$(\|\frac{u}{\omega_1^\sigma}\|^p_{L^p(\Gamma\cap\ ^cB_0)} + \|u\|^p_{L^p(\Gamma\cap B_0)} + \int_{\Gamma\times\Gamma} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{n+\sigma p}}\, d\boldsymbol{x}d\boldsymbol{y})^{1/p}.$$

For any $s \in \mathbb{R}^+$, we set

$$W_0^{s,p}(\Gamma) = \{u \in W^{[s],p}_{[s]-s}(\Gamma),\ \forall |\lambda| = [s],\ D^\lambda u \in W_0^{s-[s],p}(\Gamma)\}.$$

It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W_0^{s,p}(\Gamma)} = \|u\|_{W^{[s],p}_{[s]-s}(\Gamma)} + \sum_{|\lambda|=s} \|D^\lambda u\|_{W_0^{s-[s],p}(\Gamma)}.$$

Then, for any $s \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, we set

$$W_\alpha^{s,p}(\Gamma) = \{u \in W^{[s],p}_{[s]+\alpha-s}(\Gamma),\ \forall |\lambda| = [s],\ \rho^\alpha D^\lambda u \in W_0^{s-[s],p}(\Gamma)\}.$$

Next, we prove that the following traces lemma is satisfied:

Lemma 7.1.2. *For any integer $m \geq 1$ and real number α, we define the mapping*

$$\gamma : \mathcal{D}(\overline{\Omega}) \to\ (\mathcal{D}(\Gamma))^m$$
$$u \mapsto (\gamma_0 u, \ldots, \gamma_{m-1} u)$$

where for any $k = 0, \ldots, m-1$, $\gamma_k u = \dfrac{\partial^k u}{\partial \boldsymbol{n}^k}$. Then, γ can be extended by continuity to a linear and continuous mapping still denoted by γ from $W_\alpha^{m,p}(\Omega)$ to $\prod_{j=0}^{m-1} W_\alpha^{m-j-\frac{1}{p},p}(\Gamma)$. Moreover, γ is onto and

$$Ker\ \gamma = \mathring{W}{}_\alpha^{m,p}(\Omega).$$

Proof- We prove this lemma following the same main idea as the perturbed half-space. First, for the continuity, we work like previously, in an unbounded domain and next in a bounded domain. For the surjectivity, we easily prove, using results in half-spaces, that we can find $u \in W_0^{1,p}(\Omega)$ such that $u = g_2$ on Γ and then, we study the bounded part like for the perturbed half-space. For the characterization of the kernel, the bounded part is again the same as the previous case and for the unbounded part, we find two sequences, the first one, C^∞ with compact support in $\mathbb{R}^n_+$ (so in Ω) and the second one in $\mathbb{R}^n_-$ (so in Ω) and we work with the sum of these two sequences. $\square$

Now, we define the following spaces:

$$\boldsymbol{\mathcal{V}}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega),\ \text{div}\ \boldsymbol{v} = 0 \text{ in } \Omega\},$$

$$\boldsymbol{V_p}(\Omega) = \overline{\boldsymbol{\mathcal{V}}(\Omega)}^{\|\cdot\|_{\boldsymbol{W}_0^{1,p}(\Omega)}},$$

$$\hat{\boldsymbol{V}}_{\boldsymbol{p}}(\Omega) = \{\boldsymbol{v} \in \mathring{\boldsymbol{W}}{}_0^{1,p}(\Omega),\ \text{div}\ \boldsymbol{v} = 0 \text{ in } \Omega\}$$

and the polar of $\hat{\boldsymbol{V}}_{\boldsymbol{p}}$:

$$\hat{\boldsymbol{V}}^\circ_{\boldsymbol{p}}(\Omega) = \{\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega),\ \forall \boldsymbol{w} \in \hat{\boldsymbol{V}}_{\boldsymbol{p}}(\Omega),\ < \boldsymbol{f}, \boldsymbol{w} >_\Omega = 0\}.$$

We easily show that:

$$\forall \boldsymbol{u} \in \boldsymbol{V_p}(\Omega),\ \int_M \boldsymbol{u} \cdot \boldsymbol{n}\ d\sigma = 0. \tag{7.1}$$

Moreover, contrary to cases of exterior domains, half-space or perturbed half-space, we notice (see [31]) that, in an aperture domain, we have only the strict inclusion:

$$\boldsymbol{V_p}(\Omega) \subsetneq \hat{\boldsymbol{V}}_{\boldsymbol{p}}(\Omega). \tag{7.2}$$

7.2 Case $p = 2$.

Thanks to Lemma 7.1.2, there exists $\boldsymbol{u_g} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that $\boldsymbol{u_g} = \boldsymbol{g}$ on Γ and satisfying

$$\|\boldsymbol{u_g}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)}.$$

Thus, we show that it is equivalent to study the problem with homogeneous boundary conditions.

Moreover, let h be in $L^2(\Omega)$. We remind that we can consider Ω_+ and Ω_- as perturbed half-spaces, (their boundary is Lipschitz-continuous, which is sufficient when $p = 2$). So, thanks to Lemma 6.2.2, there exists $\boldsymbol{t} \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega_+)$ such that

$$\begin{cases} \text{div}\ \boldsymbol{t} = h \text{ in } \Omega_+, \\ \|\boldsymbol{t}\|_{\boldsymbol{W}_0^{1,2}(\Omega_+)} \leq C\ \|h\|_{L^2(\Omega)}, \end{cases}$$

and there exists $\boldsymbol{z} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega_-)$ such that

$$\begin{cases} \text{div } \boldsymbol{z} = h \text{ in } \Omega_-, \\ \|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\Omega_-)} \leq C \ \|h\|_{L^2(\Omega)}. \end{cases}$$

We define the function $\boldsymbol{w}$ by

$$\boldsymbol{w} = \boldsymbol{t} \text{ in } \Omega_+ \quad \text{and} \quad \boldsymbol{w} = \boldsymbol{z} \text{ in } \Omega_-.$$

We have $\boldsymbol{w} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$ because $\boldsymbol{t} = \boldsymbol{z} = \boldsymbol{0}$ on the join M and we easily show that

$$\text{div } \boldsymbol{w} = h \text{ in } \Omega \quad \text{and} \quad \|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C \ \|h\|_{L^2(\Omega)}.$$

Thus, thanks to this result, it remains only to study the following problem: for $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and $\alpha \in \mathbb{R}$, find $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^2(\Omega_+)$, $\pi - a_- \in L^2(\Omega_-)$ and

$$(\mathcal{S}_{200}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \ d\sigma = \alpha. \end{cases}$$

First, we establish the following lemma:

Lemma 7.2.1. *There exists* $\boldsymbol{b} \in \hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega)$ *such that*

$$\int_M \boldsymbol{b} \cdot \boldsymbol{n} \ d\sigma = 1. \tag{7.3}$$

Proof- By Lemma 11 of Heywood in [31], there exists $\boldsymbol{b} \in \overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\|\nabla \cdot\|_{\boldsymbol{L}^2(\Omega)}}$ such that

$$\text{div } \boldsymbol{b} = \boldsymbol{0} \text{ in } \Omega \quad \text{and} \quad \int_M \boldsymbol{b} \cdot \boldsymbol{n} \ d\sigma = 1.$$

Moreover, thanks to Lemma 2.1 of Farwig and Sohr in [22], we show that, in an aperture domain, $\overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\|\nabla \cdot\|_{\boldsymbol{L}^2(\Omega)}} = \{\boldsymbol{u} \in \boldsymbol{L}^2_{loc}(\overline{\Omega}),\ \nabla \boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma\}$. So, it remains only to show that $\boldsymbol{b} \in \boldsymbol{W}_0^{1,2}(\Omega)$. Let us set, for all $i = 1, \ldots, n$, $\boldsymbol{s_i} = \nabla b_i \in \boldsymbol{L}^2(\Omega)$ and

$$\widetilde{\boldsymbol{s}}_{\boldsymbol{i}} = \boldsymbol{s_i} \text{ in } \Omega \quad \text{and} \quad \widetilde{\boldsymbol{s}}_{\boldsymbol{i}} = \boldsymbol{0} \text{ in } \mathbb{R}^n \setminus \Omega.$$

We have $\widetilde{\boldsymbol{s}}_{\boldsymbol{i}} \in \boldsymbol{L}^2(\mathbb{R}^n)$ and we check that for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \widetilde{\boldsymbol{s}}_{\boldsymbol{i}} \cdot \boldsymbol{\varphi} \ d\boldsymbol{x} = \int_\Omega \boldsymbol{s_i} \cdot \boldsymbol{\varphi} \ d\boldsymbol{x} = 0.$$

Moreover, thanks to Lemma 4.2 of Amrouche, Girault and Giroire in [6], $\boldsymbol{\mathcal{V}}(\mathbb{R}^n)$ is dense in $\boldsymbol{H_2}(\mathbb{R}^n) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathbb{R}^n),\ \text{div } \boldsymbol{v} = 0 \text{ in } \mathbb{R}^n\}$. So, $\widetilde{\boldsymbol{s}}_{\boldsymbol{i}} \in \boldsymbol{L}^2(\mathbb{R}^n) \perp \boldsymbol{H_2}(\mathbb{R}^n)$. Then, thanks to Proposition 9.2 of [6], for each

$i = 1, \ldots, n$, there exists $\widetilde{w}_i \in W_0^{1,2}(\mathbb{R}^n)$ such that $\nabla \widetilde{w}_i = \widetilde{\boldsymbol{s}}_{\boldsymbol{i}}$ in $\mathbb{R}^n$. We set $w_i \in W_0^{1,2}(\Omega)$ the restriction of $\widetilde{w}_i$ in Ω; we have $\nabla w_i = \boldsymbol{s}_i$ *i.e.* $\nabla w_i = \nabla b_i$ in Ω. So, Ω being connected, there exists a real constant $K_i \in \mathbb{R}^n$ such that $w_i = b_i + K_i \in W_0^{1,2}(\Omega)$. Thus, since $b_i = 0$ on Γ, $w_i = K_i$ on Γ. Moreover, we notice that $\nabla \widetilde{w}_i = \mathbf{0}$ in $\mathbb{R}^n \setminus \overline{\Omega}$, so $\widetilde{w}_i$ is constant in each of the two infinite and connected components Θ_j $(j = 1, 2)$ of $\mathbb{R}^n \setminus \overline{\Omega}$. As $\widetilde{w}_i \in W_0^{1,2}(\Theta_j)$ and that constants are not in this space, we deduce that $\widetilde{w}_i = 0$ in $\Theta_1 \cup \Theta_2$. Finally, reminding that $w_i = K_i$ on Γ and that $\widetilde{w}_i \in W_0^{1,2}(\mathbb{R}^n)$, we conclude that $K_i = 0$. Thus, $b_i \in W_0^{1,2}(\Omega)$ for any $i = 1, \ldots, n$. $\square$

Now, for any $\boldsymbol{w} \in \boldsymbol{V_2}(\Omega)$, we define the following bilinear and continuous application $\boldsymbol{T}$ by

$$\boldsymbol{T}\boldsymbol{w} = < \boldsymbol{f}, \boldsymbol{w} >_\Omega - \alpha \int_\Omega \nabla \boldsymbol{b} : \nabla \boldsymbol{w} \, d\boldsymbol{x}.$$

We apply the theorem of Lax-Milgram in $(\boldsymbol{V_2}(\Omega), \| \cdot \|_{\boldsymbol{W}_0^{1,2}(\Omega)})$ to conclude that there exists a unique $\boldsymbol{v} \in \boldsymbol{V_2}(\Omega)$ such that

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} \, d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{w} >_\Omega - \alpha \int_\Omega \nabla \boldsymbol{b} : \nabla \boldsymbol{w} \, d\boldsymbol{x},$$

(we notice that we have the coercivity thanks to the point ii) of Theorem 7.1.1 since $\boldsymbol{V_2}(\Omega) \subset \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$). Then, setting $\boldsymbol{u} = \boldsymbol{v} + \alpha \boldsymbol{b} \in \hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega)$, we have for any $\boldsymbol{w} \in \boldsymbol{V_2}(\Omega)$:

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \, d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{w} >_\Omega$$

and, by (7.1) and (7.3),

$$\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha. \tag{7.4}$$

Then, let Ω' be a connected open bounded subset of Ω. If $\boldsymbol{w} \in \boldsymbol{V_2}(\Omega')$, we easily show that

$$\int_{\Omega'} \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \, d\boldsymbol{x} = < \boldsymbol{f}, \boldsymbol{w} >_{\Omega'} .$$

Now, let $\boldsymbol{w}$ be in $\boldsymbol{H}_0^1(\Omega')$. We define the linear continuous form $\boldsymbol{\mathcal{F}}$ by

$$\boldsymbol{\mathcal{F}}(\boldsymbol{w}) = - \int_{\Omega'} \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \, d\boldsymbol{x} + < \boldsymbol{f}, \boldsymbol{w} >_{\Omega'} .$$

We have $\boldsymbol{\mathcal{F}} \in \boldsymbol{H}^{-1}(\Omega')$, $\boldsymbol{\mathcal{F}}$ is equal to zero on $\boldsymbol{V_2}(\Omega')$ and consequently on $\boldsymbol{\mathcal{V}}(\Omega')$. We apply a result established by Girault and Raviart in bounded domain (see [25]) to deduce from this that there exists $p \in L^2(\Omega')$, unique up to an additive constant, such that

$$\nabla p = \boldsymbol{\mathcal{F}} \text{ in } \Omega'. \tag{7.5}$$

This permits us to prove the following lemma:

Lemma 7.2.2. *There exists $\pi \in L^2_{loc}(\overline{\Omega})$ such that for any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$, we have*

$$\int_\Omega \pi \ div\, \boldsymbol{\psi} \ d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\psi} \ d\boldsymbol{x} \ - < \boldsymbol{f}, \boldsymbol{\psi} >_\Omega \tag{7.6}$$

Proof- Let $(B_m)_{m \in \mathbb{N}^*}$ an increasing sequence of open balls included in $\mathbb{R}^n$. We set, for any $m \geq 1$, $\Omega_m = B_m \cap \Omega$. For any $m \geq 1$, we know, thanks to (7.5), that there exists $p_m \in L^2(\Omega_m)$ such that

$$\nabla p_m = \boldsymbol{\mathcal{F}_m} \text{ in } \Omega_m,$$

where $\boldsymbol{\mathcal{F}_m}$ is defined, for any $\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega_m)$, by

$$\boldsymbol{\mathcal{F}_m}(\boldsymbol{w}) = -\int_{\Omega_m} \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \ d\boldsymbol{x} + < \boldsymbol{f}, \boldsymbol{w} >_{\Omega_m} .$$

Moreover, we easily notice that $\boldsymbol{\mathcal{F}_m} = \boldsymbol{\mathcal{F}_{m+1}}$ in $\boldsymbol{H}^{-1}(\Omega_m)$ which implies that, for any $m \geq 1$, $\nabla p_m = \nabla p_{m+1}$ in $\boldsymbol{H}^{-1}(\Omega_m)$. As each Ω_m is connected, we deduce that each p_m is unique up to an additive constant, constant that we can choose in order to have $p_m = p_{m+1}$ in Ω_m. Thus, starting again, we construct a function π defined by:

$$\forall \ m \geq 1, \ \ \pi = p_m \text{ in } \Omega_m.$$

Because of the definition of the space $L^2_{loc}(\overline{\Omega})$, it becomes obvious that $\pi \in L^2_{loc}(\overline{\Omega})$. Now, let $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$, then, there exists $m \in \mathbb{N}^*$ such that supp $\boldsymbol{\psi} \subset \Omega_m$. Since $\pi = p_m$ in Ω_m and $\boldsymbol{\psi} \in \boldsymbol{H}^1_0(\Omega_m)$, we have

$$\int_{\Omega_m} \pi \text{ div } \boldsymbol{\psi} \ d\boldsymbol{x} = \int_{\Omega_m} \nabla \boldsymbol{u} : \nabla \boldsymbol{\psi} \ d\boldsymbol{x} \ - < \boldsymbol{f}, \boldsymbol{\psi} >_{\Omega_m}$$

and consequently (7.6). □

Thus, we have found $\boldsymbol{u} \in \hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega)$ and $\pi \in L^2_{loc}(\overline{\Omega})$ such that

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \text{ in } \Omega.$$

It remains us to find two real constants a_+ and a_- such that

$$\pi - a_+ \ \in L^2(\Omega_+) \quad \text{and} \quad \pi - a_- \ \in L^2(\Omega_-).$$

Let $\boldsymbol{\varphi}$ be in $\overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega_+)$. We define $\boldsymbol{\mathcal{F}_+} \in \boldsymbol{W}_0^{-1,2}(\Omega_+)$ by

$$\boldsymbol{\mathcal{F}_+}(\boldsymbol{\varphi}) = -\int_{\Omega_+} \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi}\, d\boldsymbol{x} + < \boldsymbol{f}, \boldsymbol{\varphi} >_{\Omega_+} = -\int_{\Omega_+} \pi \text{ div } \boldsymbol{\varphi}\, d\boldsymbol{x}.$$

We notice that $\boldsymbol{\mathcal{F}_+}$ is equal to zero on $\hat{\boldsymbol{V}}_{\boldsymbol{2}}(\Omega_+)$, so $\boldsymbol{\mathcal{F}_+} \in \hat{\boldsymbol{V}}_{\boldsymbol{2}}^{\circ}(\Omega_+)$. Moreover, considering Ω_+ as a perturbed half-space, we establish that there exists $\pi_+ \in L^2(\Omega_+)$ such that $\nabla \pi_+ = \boldsymbol{\mathcal{F}_+}$ (it is the same proof as Corollary 5.2.4). But, in $\Omega_+ \subset \Omega$, we have $\boldsymbol{\mathcal{F}_+} = \nabla \pi$. So, $\nabla \pi = \nabla \pi_+$ and there exists a real constant a_+ such that $\pi - a_+ = \pi_+ \in L^2(\Omega_+)$. Then, we may proceed with the same reasoning for Ω_-.

Finally, we define $\overline{\pi} \in L^2(\Omega)$ by

$$\overline{\pi} = \begin{cases} \pi - a_+ & \in \Omega_+, \\ \pi - a_- & \in \Omega_-, \end{cases}$$

and γ_∞ by

$$\gamma_\infty = a_+ - a_-,$$

and we easily check that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\overline{\pi}\|_{L^2(\Omega)} + |\gamma_\infty| \leq \ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + |\alpha|).$$

Thus, we have solved the problem (S_{200}) and consequently, we have the following theorem:

Theorem 7.2.3. *For any* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$, $h \in L^2(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)$ *and* $\alpha \in \mathbb{R}$, *there exists* $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ *such that* $\pi - a_+ \in L^2(\Omega_+)$, $\pi - a_- \in L^2(\Omega_-)$ *and*

$$(\mathcal{S}_2) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n}\, d\sigma = \alpha. \end{cases}$$

Moreover $\boldsymbol{u}$ *is unique,* π, a_+ *and* a_- *are unique up to an additive and common constant and it holds*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\overline{\pi}\|_{L^2(\Omega)} + |\gamma_\infty| \leq \ C\ (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{\frac{1}{2},2}(\Gamma)} + |\alpha|),$$

where C *is a real positive constant which depends only on* Ω.

Proof- It remains to prove that $\boldsymbol{u}$ is unique and that π, a_+ and a_- are unique up to an additive and common constant. We set:

$$\mathcal{B}_0^2(\Omega) = \{(\boldsymbol{z}, \pi, a_+, a_-) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}, \text{ with } \overline{\pi} \in L^2(\Omega),$$
$$-\Delta \boldsymbol{z} + \nabla \pi = \boldsymbol{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \int_M \boldsymbol{z} \cdot \boldsymbol{n}\, d\sigma = 0\}.$$

Let $(\boldsymbol{u}, \pi, a_+, a_-) \in \mathcal{B}_0^2(\Omega)$. For any $\boldsymbol{v} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$, we define the linear and continuous application $\boldsymbol{\ell} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ by

$$\boldsymbol{\ell}(\boldsymbol{v}) = \int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma.$$

Let $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$. Then, reminding that $\boldsymbol{n}$ is the unit normal vector on M directed to Ω_-, we have

$$\begin{aligned}
&< -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{\varphi} >_\Omega = < -\Delta \boldsymbol{u}, \boldsymbol{\varphi} >_\Omega - \int_\Omega \pi \text{ div } \boldsymbol{\varphi} \, d\boldsymbol{x} \\
&= < -\Delta \boldsymbol{u}, \boldsymbol{\varphi} >_\Omega - \int_\Omega \overline{\pi} \text{ div } \boldsymbol{\varphi} \, d\boldsymbol{x} - \ a_+ \int_{\Omega_+} \text{div } \boldsymbol{\varphi} \, d\boldsymbol{x} - \ a_- \int_{\Omega_-} \text{div } \boldsymbol{\varphi} \, d\boldsymbol{x} \\
&=< -\Delta \boldsymbol{u} + \nabla \overline{\pi}, \boldsymbol{\varphi} >_\Omega - a_+ \int_{\partial\Omega_+} \boldsymbol{\varphi} \cdot \boldsymbol{n} \, d\boldsymbol{x} + \ a_- \int_{\partial\Omega_-} \boldsymbol{\varphi} \cdot \boldsymbol{n} \, d\boldsymbol{x} \\
&= < -\Delta \boldsymbol{u} + \nabla \overline{\pi}, \boldsymbol{\varphi} >_\Omega - (a_+ - a_-) \int_M \boldsymbol{\varphi} \cdot \boldsymbol{n} \, d\boldsymbol{x} \\
&= < -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, \boldsymbol{\varphi} >_\Omega,
\end{aligned}$$

i.e. $\mathbf{0} = -\Delta \boldsymbol{u} + \nabla \pi = -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}$ in Ω. Now, let $\boldsymbol{v}$ be in $\overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$, we have $< -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, \boldsymbol{v} >_\Omega = 0$ and so

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\boldsymbol{x} - \int_\Omega \overline{\pi} \text{ div } \boldsymbol{v} \, d\boldsymbol{x} - \gamma_\infty \int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = 0.$$

But $\boldsymbol{u} \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,2}(\Omega)$, div $\boldsymbol{u} = 0$ in Ω and $\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = 0$. Thus, $\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} = 0$ which implies that $\boldsymbol{u}$ is a constant vector which is equal to zero because $\boldsymbol{u} = \mathbf{0}$ on Γ. Consequently, π is constant in Ω. So $\pi - a_+ \in L^2(\Omega_+)$ is constant and since Ω_+ is not bounded, $\pi = a_+$ on Ω_+. With the same reasoning, we establish that $\pi = a_-$ on Ω_- and since π is constant in Ω, we have $\pi = a_+ = a_-$ in Ω and our result. □

7.3 Case $p \neq 2$.

First, we suppose that $p > 2$ and we study the kernel of the Stokes system. We set:

$$\begin{aligned}
\mathcal{B}_0^p(\Omega) = \{(\boldsymbol{z}, \pi, a_+, a_-) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}, \text{ with } \overline{\pi} \in L^p(\Omega), \\
- \Delta \boldsymbol{z} + \nabla \pi = \mathbf{0} \text{ and div } \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \int_M \boldsymbol{z} \cdot \boldsymbol{n} \, d\sigma = 0\}.
\end{aligned}$$

Theorem 7.3.1. *We have* $\mathcal{B}_0^p(\Omega) = \{\lambda(\mathbf{0}, 1, 1, 1), \lambda \in \mathbb{R}\}$.

Proof- Let $(\boldsymbol{z},\pi,a_+,a_-)$ be in $\mathcal{B}_0^p(\Omega)$ and $(\boldsymbol{z_1},\overline{\pi}_1)=(\psi_1\boldsymbol{z},\psi_1\overline{\pi})\in\overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega)\times L^p(\Omega)$. Since supp $(\boldsymbol{z_1},\overline{\pi}_1)\subset G$ which is bounded, we have

$$(\boldsymbol{z_1},\overline{\pi}_1)\in\boldsymbol{W}_0^{1,2}(\Omega)\times L^2(\Omega)$$

and so

$$-\Delta\boldsymbol{z_1}+\nabla\overline{\pi}_1\in\boldsymbol{W}_0^{-1,2}(\Omega)\text{ and div }\boldsymbol{z_1}\in L^2(\Omega).$$

In $\mathbb{R}^n_+$, we notice that $\overline{\pi}=\pi-a_+$, so $\nabla\overline{\pi}=\nabla\pi$ which implies that $-\Delta\boldsymbol{z}+\nabla\overline{\pi}=\boldsymbol{0}$ in $\mathbb{R}^n_+$. We set $(\boldsymbol{z_2},\overline{\pi}_2)=(\psi_2\boldsymbol{z},\psi_2\overline{\pi})\in\overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega)\times L^p(\Omega)$ and

$$\begin{aligned}\boldsymbol{f}_+&=-\,\Delta\boldsymbol{z_2}+\nabla\overline{\pi}_2=\Delta\boldsymbol{z_1}-\nabla\overline{\pi}_1)\in\boldsymbol{W}_0^{-1,2}(\mathbb{R}^n_+),\\ h_+&=\text{div }\boldsymbol{z_2}=-\text{ div }\boldsymbol{z_1}\in L^2(\mathbb{R}^n_+).\end{aligned}$$

We may proceed with the same reasoning as in Theorem 6.3.1 to obtain that $(\boldsymbol{z_2},\overline{\pi}_2)\in\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_+)\times L^2(\mathbb{R}^n_+)$. Now, working on $\mathbb{R}^n_-$ instead of $\mathbb{R}^n_+$ (even if we move the origin to a distance equal to d), we obtain too that $(\boldsymbol{z_2},\overline{\pi}_2)\in\boldsymbol{W}_0^{1,2}(\mathbb{R}^n_{-d})\times L^2(\mathbb{R}^n_{-d})$. Finally, since supp $(\boldsymbol{z_2},\overline{\pi}_2)\subset\mathbb{R}^n_+\cup\mathbb{R}^n_{-d}$, we conclude that $(\boldsymbol{z_2},\overline{\pi}_2)\in\boldsymbol{W}_0^{1,2}(\Omega)\times L^2(\Omega)$ and so $(\boldsymbol{z},\overline{\pi})\in\boldsymbol{W}_0^{1,2}(\Omega)\times L^2(\Omega)$ too. Moreover, it is obvious that $\pi\in L^2_{loc}(\overline{\Omega})$ because $\pi\in L^p_{loc}(\overline{\Omega})$. Thus, we conclude that $(\boldsymbol{z},\pi,a_+,a_-)\in\mathcal{B}_0^2(\Omega)$. $\square$

Now, supposing again that $p>2$, we want to study the Stokes system with homogeneous boundary conditions, that is to say: for $\boldsymbol{f}\in\boldsymbol{W}_0^{-1,p}(\Omega)$, $h\in L^p(\Omega)$ and $\alpha\in\mathbb{R}$, we want to find $(\boldsymbol{u},\pi,a_+,a_-)\in\boldsymbol{W}_0^{1,p}(\Omega)\times L^p_{loc}(\overline{\Omega})\times\mathbb{R}\times\mathbb{R}$ such that $\pi-a_+\in L^p(\Omega_+)$, $\pi-a_-\in L^p(\Omega_-)$ and

$$(\mathcal{S}_{20})\begin{cases}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f} & \text{in }\Omega,\quad\text{div }\boldsymbol{u}=h\quad\text{in }\Omega,\\ \boldsymbol{u}=\boldsymbol{0} & \text{on }\Gamma,\quad\displaystyle\int_M\boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma=\alpha.\end{cases}$$

First, we establish the following lemma:

Lemma 7.3.2. *For each $p>2$, for any $\boldsymbol{f}\in\boldsymbol{W}_0^{-1,p}(\Omega)$ and $h\in L^p(\Omega)$ with a compact support in Ω and for any $\alpha\in\mathbb{R}$, there exists $(\boldsymbol{u},\pi,a_+,a_-)\in(\boldsymbol{W}_0^{1,p}(\Omega)\cap W_0^{1,2}(\Omega))\times L^p_{loc}(\overline{\Omega})\times\mathbb{R}\times\mathbb{R}$ solution of (S_{20}) with*

$$\pi-a_+\in L^p(\Omega_+)\cap L^2(\Omega_+)\quad\text{and}\quad\pi-a_-\in L^p(\Omega_-)\cap L^2(\Omega_-).$$

Moreover $\boldsymbol{u}$ is unique and π, a_+ and a_- are unique up to an additive and common constant

Proof- Let $\boldsymbol{f}$ be in $\boldsymbol{W}_0^{-1,p}(\Omega)$ and h be in $L^p(\Omega)$ with a compact support in Ω and $\alpha\in\mathbb{R}$. Then, since $p>2$, we easily check that $\boldsymbol{f}\in\boldsymbol{W}_0^{-1,2}(\Omega)$ and $h\in L^2(\Omega)$ and we deduce from Theorem 7.2.3 that there exists $(\boldsymbol{u},\pi,a_+,a_-)\in\boldsymbol{W}_0^{1,2}(\Omega)\times L^2_{loc}(\overline{\Omega})\times\mathbb{R}\times\mathbb{R}$ solution of (S_{20}) where $\boldsymbol{u}$ is unique and π, a_+ and a_- are unique up to an additive and common constant. It remains to show that $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega})$. For any $\boldsymbol{v} \in \mathring{\boldsymbol{W}}{}_0^{1,p'}(\Omega)$, we define the linear and continuous application $\boldsymbol{\ell} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ by

$$\boldsymbol{\ell}(\boldsymbol{v}) = \int_M \boldsymbol{v}\cdot\boldsymbol{n}\; d\sigma,$$

and we recall that

$$-\Delta \boldsymbol{u} + \nabla\pi = -\Delta\boldsymbol{u} + \nabla\overline{\pi} - \gamma_\infty \boldsymbol{\ell} \text{ in } \Omega.$$

We set, for $i = 1,2$, $(\boldsymbol{u_i},\overline{\pi}_i) = (\psi_i\boldsymbol{u},\psi_i\overline{\pi}) \in \mathring{\boldsymbol{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$ and we notice that $(u_1,\overline{\pi}_1)$ has a compact support included in G. An elementary calculation shows that we have

$$\begin{cases} -\Delta\boldsymbol{u_1} + \nabla\overline{\pi}_1 = \psi_1\boldsymbol{f} + \boldsymbol{F}_1 + \psi_1\gamma_\infty\boldsymbol{\ell} & \text{in } G,\\ \operatorname{div}\, \boldsymbol{u_1} = \psi_1 h + H_1 & \text{in } G,\\ \boldsymbol{u_1} = \boldsymbol{0} & \text{on } \partial G,\end{cases}$$

where

$$\boldsymbol{F}_1 = -(2\nabla\boldsymbol{u}\nabla\psi_1 + \boldsymbol{u}\Delta\psi_1) + \overline{\pi}\nabla\psi_1 \in \boldsymbol{L}^2(G),$$

and

$$H_1 = \boldsymbol{u}\cdot\nabla\psi_1 \in \boldsymbol{H}^1(G).$$

Noticing that the support of $\boldsymbol{\ell}$, subset of M, is compact and using the same reasoning as Lemma 6.3.2 for the perturbed half-space, we conclude that we have $(\boldsymbol{u_1},\overline{\pi}_1) \in \boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ and here again, we do like Lemma 6.3.2 to obtain that $(\boldsymbol{u_2},\overline{\pi}_2) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. Thus $(\boldsymbol{u},\overline{\pi}) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and the estimate follows immediately. Finally, we easily deduce from this, since $\overline{\pi} \in L^p(\Omega)$, that $\pi \in L^p_{loc}(\overline{\Omega})$ and we have our result. □

Now, we establish the following theorem:

Theorem 7.3.3. *For any $p > 2$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, there exists a unique $(\boldsymbol{u},\pi,a_+,a_-) \in (\boldsymbol{W}_0^{1,p}(\Omega)\times L^p_{loc}(\overline{\Omega})\times\mathbb{R}\times\mathbb{R})/\mathcal{B}_0^p(\Omega)$ with $\pi - a_+ \in L^p(\Omega_+)$ and $\pi - a_- \in L^p(\Omega_-)$ solution of*

$$\begin{cases} -\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{0} & \text{in } \Omega, & \operatorname{div}\,\boldsymbol{u} = 0 \quad \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, & \displaystyle\int_M \boldsymbol{u}\cdot\boldsymbol{n}\; d\sigma = \alpha,\end{cases}$$

and satisfying

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\overline{\pi}\|_{L^p(\Omega)} + |\gamma_\infty| \leq\; C\; (\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)} + |\alpha|),$$

where C is a real positive constant which depends only on p and Ω.

Proof- The idea is the same as Theorem 6.3.3. First, using results in bounded domains, we can deduce that there exists $(\boldsymbol{z}, \theta) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{z} + \nabla \theta = \boldsymbol{\sigma} \text{ in } \Omega, \quad \text{div } \boldsymbol{z} = \psi \text{ in } \Omega, \quad \boldsymbol{z} = \boldsymbol{g_1} \text{ on } \Gamma,$$

where $\boldsymbol{\sigma} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $\psi \in L^p(\Omega)$ have a compact support. Using Lemma 7.3.2, then there exists $(\boldsymbol{t}, \tau, a'_+, a'_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} -\Delta \boldsymbol{t} + \nabla \tau = -\boldsymbol{\sigma} & \text{in } \Omega, \quad \text{div } \boldsymbol{t} = -\psi \quad \text{in } \Omega, \\ \boldsymbol{t} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{t} \cdot \boldsymbol{n} \, d\sigma = \frac{1}{2}\alpha - \int_M \boldsymbol{z} \cdot \boldsymbol{n} \, d\sigma, \end{cases}$$

and

$$\tau - a'_+ \in L^p(\Omega_+) \quad \text{and} \quad \tau - a'_- \in L^p(\Omega_-).$$

Noticing that $\theta \in L^p(\Omega) \subset L^p_{loc}(\overline{\Omega})$, we deduce that $(\boldsymbol{v} = \boldsymbol{z} + \boldsymbol{t}, \mu = \theta + \tau, a'_+, a'_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \mu = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{g_1} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma = \frac{1}{2}\alpha, \end{cases}$$

and

$$\mu - a'_+ \in L^p(\Omega_+) \quad \text{and} \quad \mu - a'_- \in L^p(\Omega_-).$$

Next, we follow again the same ideas as Theorem 6.3.3. First, we use results in $\mathbb{R}^n_+$ and in $\mathbb{R}^n_{-d}$ and we extend by $(\boldsymbol{0}, 0)$ in Ω. Then, summing the two found pairs, we construct $(\boldsymbol{r}, \alpha) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$-\Delta \boldsymbol{r} + \nabla \alpha = \boldsymbol{\xi} \text{ in } \Omega, \quad \text{div } \boldsymbol{r} = 0 \text{ in } \Omega, \quad \boldsymbol{r} = \boldsymbol{g_2} \text{ on } \Gamma.$$

Then, using like previously Lemma 7.3.2, we are able to find $(\boldsymbol{w}, \eta, a''_+, a''_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \eta = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{g_2} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{w} \cdot \boldsymbol{n} \, d\sigma = \frac{1}{2}\alpha, \end{cases}$$

and

$$\eta - a''_+ \in L^p(\Omega_+) \quad \text{and} \quad \eta - a''_- \in L^p(\Omega_-).$$

Finally $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi = \mu + \eta, a_+ = a'_+ + a''_+, a_- = a'_- + a''_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of our problem and the estimate follows immediately. □

Theorem 7.3.4. *For any $p > 2$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of $(\mathcal{S}_{20})$. Moreover, $\boldsymbol{u}$*

is unique, π , a_+ and a_- are unique up to an additive and common constant and we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\overline{\pi}\|_{L^p(\Omega)} + |\gamma_\infty| \leq \; C \; (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + |\alpha|),$$

where C is a real positive constant which depends only on p and Ω.

Proof- The uniqueness comes from Theorem 7.3.1. Then, there exists, as a consequence of Theorem 7.1.1 ii), a tensor of the second order $F \in [L^p(\Omega)]^{n\times n}$ such that div $F = \boldsymbol{f}$. We extend F (respectively h) by 0 in $\mathbb{R}^n$, and we denote by $\widetilde{F}$ (respectively $\widetilde{h}$) this extension. Then, we set $\widetilde{\boldsymbol{f}} =$ div $\widetilde{F}$ and we notice that $\widetilde{\boldsymbol{f}}_{|\Omega} = \boldsymbol{f}$. We have $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^n)$ and $\widetilde{h} \in L^p(\mathbb{R}^n)$. By [2], there exists $(\boldsymbol{v}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ solution of

$$-\Delta \boldsymbol{v} + \nabla \eta = \widetilde{\boldsymbol{f}} \text{ in } \mathbb{R}^n \quad \text{and} \quad \text{div } \boldsymbol{v} = \widetilde{h} \text{ in } \mathbb{R}^n.$$

We denote again by $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\eta \in L^p(\Omega) \subset L^p_{loc}(\overline{\Omega})$ the restrictions of $\boldsymbol{v}$ and η to Ω. We have $\boldsymbol{v}_{|\Gamma} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$, thus, thanks to Theorem 7.3.3, there exists $(\boldsymbol{w}, \tau, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{0} & \text{in } \Omega, \quad \text{div } \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ \boldsymbol{w} = -\boldsymbol{v}_{|\Gamma} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{w}\cdot\boldsymbol{n}\, d\sigma = \alpha - \int_M \boldsymbol{v}\cdot\boldsymbol{n}\, d\sigma, \end{cases}$$

and

$$\tau - a_+ \in L^p(\Omega_+) \quad \text{and} \quad \tau - a_- \in L^p(\Omega_-).$$

Finally, $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \pi = \eta + \tau, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of $(\mathcal{S}_{20})$ and the estimate follows immediately. □

Now, we suppose that p is such that $p < 2$ and we want to solve $(\mathcal{S}_{20})$. Since $p < 2$, its dual exponent p' satisfies $p' > 2$. So, if $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p'}(\Omega)$, $h \in L^{p'}(\Omega)$ and $\alpha \in \mathbb{R}$, there exists, thanks to Theorem 7.3.4, $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p'}(\Omega) \times L^{p'}_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u}\cdot\boldsymbol{n}\, d\sigma = \alpha, \end{cases}$$

and

$$\pi - a_+ \in L^{p'}(\Omega_+) \quad \text{and} \quad \pi - a_- \in L^{p'}(\Omega_-).$$

We easily notice that it is equivalent to say that, for any $(\boldsymbol{f}, h, \alpha) \in \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R}$, the following problem $(\overline{\mathcal{S}}_{20})$

$$(\overline{\mathcal{S}}_{20}) \begin{cases} -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell} = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = h & \text{in } \Omega, \\ \boldsymbol{\ell}(\boldsymbol{u}) = \alpha, \end{cases}$$

possesses a unique solution $(\boldsymbol{u}, \overline{\pi}, \gamma_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R}$ (the uniqueness comes from the definition of $\mathcal{B}_0^{p'}(\Omega)$). So, the mapping

$$S : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R} \to \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R}$$
$$(\boldsymbol{u}, \overline{\pi}, \gamma_\infty) \mapsto (-\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, -\text{div }\boldsymbol{u}, -\boldsymbol{\ell}(\boldsymbol{u}))$$

is an isomorphism. Furthermore, we have, for any $(\boldsymbol{u}, \overline{\pi}, \gamma_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p'}(\Omega) \times L^{p'}(\Omega) \times \mathbb{R}$ and $(\boldsymbol{v}, \overline{\eta}, \theta_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^{p}(\Omega) \times \mathbb{R}$,

$$\begin{aligned} &< -\Delta \boldsymbol{u} + \nabla \overline{\pi} - \gamma_\infty \boldsymbol{\ell}, \boldsymbol{v} >_\Omega - \int_\Omega \text{div }\boldsymbol{u}\ \overline{\eta}\ d\boldsymbol{x}\ - \theta_\infty \boldsymbol{\ell}(\boldsymbol{u}) \\ = &< -\Delta \boldsymbol{v} + \nabla \overline{\eta} - \theta_\infty \boldsymbol{\ell}, \boldsymbol{u} >_\Omega - \int_\Omega \text{div }\boldsymbol{v}\ \overline{\pi}\ d\boldsymbol{x}\ - \gamma_\infty \boldsymbol{\ell}(\boldsymbol{v}). \end{aligned}$$

Thus, by duality

$$S^* : \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^{p}(\Omega) \times \mathbb{R} \to \boldsymbol{W}_0^{-1,p}(\Omega) \times L^{p}(\Omega) \times \mathbb{R}$$
$$(\boldsymbol{v}, \overline{\eta}, \theta_\infty) \mapsto (-\Delta \boldsymbol{v} + \nabla \overline{\eta} - \theta_\infty \boldsymbol{\ell}, -\text{div }\boldsymbol{v}, -\boldsymbol{\ell}(\boldsymbol{v}))$$

is also an isomorphism, *i.e.*, when $p < 2$, there exists a unique $(\boldsymbol{v}, \overline{\eta}, \theta_\infty) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^{p}(\Omega) \times \mathbb{R}$ solution of $(\overline{\mathcal{S}}_{20})$. Finally, it remains to return to the problem $(\mathcal{S}_{20})$. For this, let c_+ and c_- be two constants such that $\theta_\infty = c_+ - c_-$. We set

$$\eta = \begin{cases} \overline{\eta} + c_+ & \text{in } \Omega_+, \\ \overline{\eta} + c_- & \text{in } \Omega_-, \end{cases}$$

and we easily show that

$$-\Delta \boldsymbol{v} + \nabla \eta = -\Delta \boldsymbol{v} + \nabla \overline{\eta} - \theta_\infty \boldsymbol{\ell} \text{ in } \Omega.$$

Thus, $(\boldsymbol{v}, \eta, c_+, c_-) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of $(\mathcal{S}_{20})$. For the kernel, let $(\boldsymbol{w}, \mu, k_+, k_-) \in \overset{\circ}{\boldsymbol{W}}{}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ be an other solution of $(\mathcal{S}_{20})$. So, setting

$$\overline{\mu} = \begin{cases} \mu - k_+ & \text{in } \Omega_+, \\ \mu - k_- & \text{in } \Omega_-, \end{cases}$$

and $\alpha_\infty = k_+ - k_-$, we easily check that $(\boldsymbol{w}, \overline{\mu}, \alpha_\infty)$ is solution of $(\overline{\mathcal{S}}_{20})$, problem which admits a unique solution. Consequently, $\boldsymbol{w} = \boldsymbol{v}$, $\overline{\mu} = \overline{\eta}$ and $\alpha_\infty = \theta_\infty$ and so, there exists $\lambda \in \mathbb{R}$ such that

$$k_+ = c_+ + \lambda, \quad \text{and} \quad k_- = c_- + \lambda.$$

We easily deduce from this that $\eta = \mu - \lambda$ and thus, the kernel of the problem when $p < 2$ is again $\mathcal{B}_0^p(\Omega)$.

Finally, it remains to return to the general problem when $p \neq 2$. Thanks to Lemma 7.1.2, there exists $\boldsymbol{u_g} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{u_g} = \boldsymbol{g}$ on Γ and satisfying

$$\|\boldsymbol{u_g}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq \; C \; \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)},$$

and thanks to previous results, we have seen that there exists a unique $(\boldsymbol{v}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^p(\Omega_+)$, $\pi - a_- \in L^p(\Omega_-)$ and

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{f} - \Delta \boldsymbol{u_g} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{v} = h - \operatorname{div} \boldsymbol{u_g} \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{v} \cdot \boldsymbol{n} \; d\sigma = \alpha - \int_M \boldsymbol{u_g} \cdot \boldsymbol{n} \; d\sigma. \end{cases}$$

Finally, the function $(\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u_g}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ is solution of the problem $(\mathcal{S}_2)$ and the estimate follows immediately. In consequence, we have the following theorem:

Theorem 7.3.5. *For any $p > 1$, $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)$ and $\alpha \in \mathbb{R}$, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ such that $\pi - a_+ \in L^p(\Omega_+)$, $\pi - a_- \in L^p(\Omega_-)$ and*

$$(\mathcal{S}_2) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \; d\sigma = \alpha. \end{cases}$$

Moreover $\boldsymbol{u}$ is unique, π, a_+ and a_- are unique up to an additive and common constant and it holds

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\overline{\pi}\|_{L^p(\Omega)} + |\gamma_\infty| \leq \; C \; (\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-\frac{1}{p},p}(\Gamma)} + |\alpha|),$$

where C is a real positive constant which depends only on p and Ω.

7.4 Regularity result

Here, we want to give a regularity result for the problem (S_2). For this, we define the space:

$$D_1^p(\Omega) = \{\pi \in L^p_{loc}(\overline{\Omega}), \rho \nabla \pi \in L^p(\Omega)\}.$$

We have the following regularity theorem:

Theorem 7.4.1. *For any $p > 1$ satisfying $\dfrac{n}{p'} \neq 1$ and for any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$, $h \in W_1^{1,p}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}_1^{2-\frac{1}{p},p}(\Gamma)$ and $\alpha \in \mathbb{R}$, there exists a unique $(\boldsymbol{u}, \pi, a_+, a_-) \in$*

$(\boldsymbol{W}_1^{2,p}(\Omega) \times D_1^p(\Omega) \times \mathbb{R} \times \mathbb{R})/\mathcal{B}_0^p(\Omega)$ *such that* $\pi - a_+ \in W_1^{1,p}(\Omega_+)$, $\pi - a_- \in W_1^{1,p}(\Omega_-)$ *and*

$$(\mathcal{S}_2) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \quad \text{div } \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \quad \displaystyle\int_M \boldsymbol{u} \cdot \boldsymbol{n} \, d\sigma = \alpha. \end{cases}$$

Moreover $\boldsymbol{u}$ *is unique and* π, a_+ *and* a_- *are unique up to an additive and common constant.*

Proof- Thanks to Lemma 7.1.2, it is easily to show that it is sufficient to solve the problem with $\boldsymbol{g} = \boldsymbol{0}$. Now, we notice that we have the continuous injections $\boldsymbol{W}_1^{0,p}(\Omega) \subset \boldsymbol{W}_0^{-1,p}(\Omega)$ because $\dfrac{n}{p'} \neq 1$ and $W_1^{1,p}(\Omega) \subset L^p(\Omega)$. Thus, thanks to Theorems 7.2.3 and 7.3.5, there exists $(\boldsymbol{u}, \pi, a_+, a_-) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p_{loc}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$ solution of $(\mathcal{S}_{20})$ and such that

$$\pi - a_+ \in L^p(\Omega_+), \quad \pi - a_- \in L^p(\Omega_-).$$

Then, it remains to show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times D_1^p(\Omega)$ and that

$$\pi - a_+ \in W_1^{1,p}(\Omega_+), \quad \pi - a_- \in W_1^{1,p}(\Omega_-).$$

We set $(\boldsymbol{u_i}, \pi_i) = (\psi_i \boldsymbol{u}, \psi_i \pi)$, for $i = 1, 2$. An elementary calculation shows that we have

$$\begin{cases} -\Delta \boldsymbol{u_1} + \nabla \pi_1 = \psi_1 \boldsymbol{f} + \boldsymbol{F}_1 & \text{in } G, \\ \text{div } \boldsymbol{u_1} = \psi_1 h + H_1 & \text{in } G, \\ \boldsymbol{u_1} = \boldsymbol{0} & \text{on } \partial G, \end{cases}$$

where

$$\boldsymbol{F}_1 = -(2\nabla \boldsymbol{u} \nabla \psi_1 + \boldsymbol{u} \Delta \psi_1) + \pi \nabla \psi_1 \in \boldsymbol{L}^p(G),$$

and

$$H_1 = \boldsymbol{u} \cdot \nabla \psi_1 \in \boldsymbol{W}^{1,p}(G).$$

Thanks to results in bounded domains and since supp $(\boldsymbol{u_1}, \pi_1)$ is included in G, we have

$$(\boldsymbol{u_1}, \pi_1) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega).$$

Now, we define the function $\overline{\pi_2}$ by

$$\overline{\pi_2} = \pi_2 - a_+ \text{ in } \Omega_+, \quad \overline{\pi_2} = \pi_2 - a_- \text{ in } \Omega_-,$$

and we easily check that $\overline{\pi_2} \in L^p(\Omega)$. Now, considering the half-space $\mathbb{R}^n_+$ and noticing that $\nabla \overline{\pi_2} = \nabla \pi_2$ in $\mathbb{R}^n_+$, we define

$$\begin{aligned} &\boldsymbol{f_2} = -\Delta \boldsymbol{u_2} + \nabla \overline{\pi_2} = \boldsymbol{f} + \Delta \boldsymbol{u_1} - \nabla \pi_1 \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^n_+), \\ &h_2 = \text{div } \boldsymbol{u_2} = h - \text{ div } \boldsymbol{u_1} \in W_1^{1,p}(\mathbb{R}^n_+). \end{aligned}$$

By [9], there exists $(\boldsymbol{s}, \theta) \in (\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^n) \times W_1^{1,p}(\mathbb{R}_+^n)) \subset (\boldsymbol{W}_0^{1,p}(\mathbb{R}_+^n) \times L^p(\mathbb{R}_+^n))$ solution of

$$-\Delta \boldsymbol{s} + \nabla\theta = \boldsymbol{f_2} \text{ in } \mathbb{R}_+^n, \quad \text{div } \boldsymbol{s} = h_2 \text{ in } \mathbb{R}_+^n, \quad \boldsymbol{s} = \boldsymbol{0} \text{ on } \mathbb{R}^{n-1},$$

and since in $\boldsymbol{W}_0^{1,p}(\mathbb{R}_+^n) \times L^p(\mathbb{R}_+^n)$ there is a unique solution of this problem, we have

$$(\boldsymbol{u_2}, \overline{\pi_2}) = (\boldsymbol{s}, \theta) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^n) \times W_1^{1,p}(\mathbb{R}_+^n).$$

We use the same reasoning in $\mathbb{R}_{-d}^n$ and since supp $(\boldsymbol{u_2}, \overline{\pi_2})$ is included in $\mathbb{R}_+^n \cup \mathbb{R}_{-d}^n$, we have

$$(\boldsymbol{u_2}, \overline{\pi_2}) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega).$$

Finally, we deduce from this that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times D_1^p(\Omega)$ and that

$$\pi - a_+ \ \in W_1^{1,p}(\Omega_+), \quad \pi - a_- \ \in W_1^{1,p}(\Omega_-). \qquad \square$$

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Résumé

L'objet de cette thèse est la résolution de problèmes elliptiques dans différents domaines non bornés. Dans un premier temps, nous étudions l'opérateur de Laplace dans un domaine extérieur avec des conditions aux limites non homogènes mêlées, puis dans un domaine extérieur dans le demi-espace avec des conditions de type Dirichlet, Neumann et mêlées. Nous considérons ensuite le problème de Stokes dans trois géométries non bornées : un domaine extérieur dans le demi-espace, un demi-espace perturbé et un domaine avec ouverture. Nous donnons pour chacun de ces problèmes des résultats fondamentaux d'existence et d'unicité en théorie L^p avec $1 < p < \infty$ dans le cadre fonctionnel des espaces de Sobolev avec poids. De plus, nous nous intéressons également aux cas des solutions fortes (avec en particulier des résultats de régularité) et aux cas des solutions très faibles.

Mots clés : Espaces de Sobolev avec poids ; Problème de Laplace ; Problème de Stokes ; Demi-espace ; Domaine extérieur ; Demi-espace perturbé ; Domaine avec ouverture.

Abstract

The aim of this PhD thesis is the resolution of elliptic problems in several unbounded domains. First, we study the Laplace operator in an exterior domain with nonhomogeneous and mixed boundary conditions and next in an exterior domain in the half-space with Dirichlet, Neumann and mixed boundary conditions. Then, we consider the Stokes problem in three different unbounded geometries : an exterior domain in the half-space, a perturbed half-space and an aperture domain. We give, for these problems, existence and uniqueness fundamental results in L^p's theory with $1 < p < \infty$ in the functional framework of weighted Sobolev spaces. Moreover, we are also interested in strong solutions (particularly with regularity results) and in very weak solutions.

Keywords : Weighted Sobolev spaces ; Laplace's problem ; Stokes problem ; Half-space ; Exterior domain ; Perturbed half-space ; Aperture domain.