Optimal quantization methods with applications to finance.

Abass Sagna

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Abass SAGNA

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Directeur de thèse: Gilles PAGÈS

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Composition du Jury

Jean-Claude	FORT
Jean	JACOD
Antoine	LEJAY
Harald	LUSCHGY
Gilles	PAGÈS
Huyên	PHAM
Jacques	PRINTEMS

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Chapter 1

Introduction et présentation des résultats obtenus

La quantification est apparue pour la première fois en theorie du signal dans les années 1940. Depuis elle s'est élargie dans divers domaines tels que la Physique, l'Informatique, etc, et plus récemment les Probabilités Numériques et leurs applications en Finance. De façon générale, la quantification consiste a représenter un ensemble continu (ou un très grand nombre) d'observables par un ensemble fini (ou un petit nombre) de valeurs.

En traitement du signal par exemple, la quantification et l'échantillonnage sont des étapes permettant le traitement informatique du signal (son, image, etc). Ils autorisent par exemple l'enrègistrement sonore, qui consiste à transformer un son en un signal afin de pouvoir le diffuser ou d'en garder des traces de façon à pouvoir le rediffuser (téléphonie, radiophonie, etc), et la synthèse sonore (génération de signaux sonores).

En Probabilités Numériques la quantification est utilisée pour estimer une espérance mathématique ou une espérance conditionnelle. Ainsi, la quantification optimale est utilisée avec succès pour l'estimation des prix de certaines options financières (européennes comme américaines). Rappelons qu'en Finance les prix (au temps 0) des options s'écrivent sous forme d'espérance (options européennes) ou d'espérances conditionnelles (options américaines). Par ailleurs, l'évaluation du prix des options américaines est souvent plus délicat. Cette difficulté est liée au problème d'arrêt optimal qui fait intervenir des temps d'arrêt. Pour contourner cette difficulté on a souvent recours à des transformations qui font passer d'un suprémum sur des temps d'arrêt à des formules dont l'application numérique est plus simple à mettre en oeuvre: formule de programmation dynamique définie par une récurrence descendante (voir [PPP1]).

Les premières appplications numériques de la quantification optimale remontent à [PAG]. Depuis elle a été utilisée avec succès au pricing d'options américaines multi-sous-jacents, voir [BPP, PPP1], d'options asiatiques (via la quantification fonctionnelle, voir [PAGPRI1]), d'option swing ([BBP, BBP1]), en contrôle optimal stochastique ([CPR, PPP]), en filtrage non linéaire pour l'estimation de volatilité stochastique ([PAGPHA, PRS]).

Du point de vue mathématique, si X est un vecteur aléatoire à valeurs dans $\mathbb{R}^d$ admettant un moment d'ordre $r > 0$ alors quantifier X dans L^r au niveau n consiste à trouver la meilleure approximation possible de X (au sens de l'erreur L^r) par une variable aléatoire $q(X)$ où q est une fonction borélienne de $\mathbb{R}^d$ dans un ensemble fini α de cardinal au plus égal à n. L'erreur de quantification associée est ainsi définie par

$$e_{n,r}(X) = \inf \{\|X - q(X)\|_r, q : \mathbb{R}^d \xrightarrow{\text{Borel}} \mathbb{R}^d, \text{ card}(q(\mathbb{R}^d)) \leq n\}.$$

Au vu de cette définition il est clair que l'erreur de quantification $e_{n,r}(X)$ ne depend que de la loi $P = \mathbb{P}_X$ de X. Ainsi nous la noterons aussi parfois $e_{n,r}(P)$. D'autre part, pour toute fonction borélienne $q : \mathbb{R}^d \to \alpha$ on a

$$|X - q(X)| \geq \min_{a\in\alpha} d(X, a) = d(X, \alpha) \qquad \mathbb{P}\text{- p.s.}$$

de sorte que le problème de quantification se réduit à

$$\begin{aligned} e_{n,r}(X) &= \inf\{\|X - \widehat{X}^\alpha\|_r, \alpha \subset \mathbb{R}^d, \text{ card}(\alpha) \leq n\} \\ &= \inf_{\substack{\alpha\subset\mathbb{R}^d \\ \text{card}(\alpha)\leq n}} \left(\int_{\mathbb{R}^d} d(x, \alpha)^r dP(x)\right)^{1/r} \end{aligned} \tag{1.0.1}$$

où la variable aléatoire

$$\widehat{X}^\alpha := \sum_{a\in\alpha} a\mathbf{1}_{\{X\in C_a(\alpha)\}}$$

(en l'absence d'ambigüité on omettra parfois l'exposant α) est appelée la quantification de X sur la grille α ou un quantifieur de X associé à α. L'ensemble $(C_a(\alpha))_{a\in\alpha}$ est une partition de Voronoï associée à la grille α (par rapport à la norme $|\cdot|$ qui désigne une norme quelconque de $\mathbb{R}^d$), c'est-à-dire une partition borélienne de $\mathbb{R}^d$ vérifiant pour tout $a \in \alpha$,

$$C_a(\alpha) \subset \{x \in \mathbb{R}^d : |x - a| = \min_{b\in\alpha} |x - b|\}.$$

Dans le cas d'une norme euclidienne, on vérifie que les adhérences

$$\bar{C}_a(\alpha) = \{x \in \mathbb{R}^d : |x - a| = \min_{b\in\alpha} |x - b|\}, \quad a \in \alpha$$

sont des convexes fermés obtenus comme intersection de demi-espaces. Si de plus X ne charge aucun hyperplan la partition de Voronoï est $\mathbb{P}_X$- p.s. unique.

Par commodité, on raisonne souvent sur la puissance r-ième de l'erreur de quantification d'ordre r induite par un quantifieur α de taille au plus n. Ce qui conduit à introduire la fonction symétrique de r-distortion définie sur $\left(\mathbb{R}^d\right)^n$ par

$$\forall (a_1, \cdots, a_n) \in \left(\mathbb{R}^d\right)^n, \quad D_{n,r}(a_1, \cdots, a_n) := \mathbb{E}\big(\min_{1\leq i\leq n} |X - a_i|^r\big). \tag{1.0.2}$$

On démontre que la fonction $D_{n,r}$ est continue (en fait $D_{n,r}^{1/r}$ est Lipschitzienne) et atteint son infimum sur $(\mathbb{R}^d)^n$ (voir e.g. [PAG, PAR]). De plus, si la norme considérée est euclidienne, $r > 1$, et si P ne charge par les singletons alors cette fonction est différentiable (voir [GL] pour la démonstration) en tout quantifieur de taille n ($i.e.$ tout n-uplet à cordonnées deux à deux distinctes), de gradient donnée par

$$\nabla D_{n,r}(\alpha) = \left(r \int_{C_a(\alpha)} |x - a|^{r-1}(a - x)dP(x)\right)_{a\in\alpha}.$$

On vérifie alors que lorsque $r = 2$, $\widehat{X}$ est stationnaire (voir définition ci-apès) si et seulement si

$$\mathbb{E}(X|\widehat{X}) = \widehat{X}.$$

Définition. (a) *Un quantifieur de taille n est $L^r(P)$-**stationnaire** si les n-uplets associés sont des points critiques de $D_{n,r}$.*

(b) *Une **suite** de quantifieurs $(\alpha_n)_{n\geq 1}$ est $L^r(P)$-**stationnaire** si pour tout $n \geq 1$, α_n est $L^r(P)$-**stationnaire** de taille n.*

(c) *Par extension si α est $L^r(P)$-stationnaire on dira que $\widehat{X}^\alpha$ est un quantifieur stationnaire de X.*

D'autre part lorsque $n = 1$, $D_{1,r}(\alpha)$ est convexe. Par contre dès que $n \geq 2$, $D_{n,r}$ n'est généralement pas convexe de sorte que $D_{n,r}$ peut a priori admettre plusieurs points critiques qui ne sont pas forcément des minima globaux. Cependant en dimension $d = 1$, lorsque la densité f de X est log-concave $D_{n,r}$ admet un unique point critique qui est alors forcément son minimum global (cf [KIEF, LAMPAG]). Dans ce cas le problème $(1.0.1)$ admet une unique solution.

Dans la suite on s'autorisera l'abus de langage consistant à parler du "quantifieur" $\alpha = \{a_i, i \in \{1, \cdots, n\}\}$ au lieu de "n-uplet" $(a_1, \cdots, a_n)$, sachant que α a alors une taille au plus n.

L'erreur de quantification tend en décroissant vers 0 lorsque n tend vers $+\infty$ et si $\mathrm{card}(\mathrm{supp}(P)) \geq n$ alors le minimum est atteint en un quantifieur α de taille n et $\alpha \in \mathrm{Adh}(\mathrm{conv}(\mathrm{supp}(\mathbb{P}_X)))$. L'un des plus importants résultats en quantification vectorielle est le théorème dit de Zador (dont la preuve est donnée de façon détaillée dans [GL]) qui donne la vitesse de convergence vers zéro de l'erreur de quantification.

Théorème. (Théorème de Zador) *Soit $P = P_a + P_s$ la décomposition de P par rapport à la mesure de Lebesgue λ_d, où P_a est la partie absolument continue et P_s la partie singulière de P. Supposons que $\mathbb{E}|X|^{r+\eta} < +\infty$ pour un $\eta > 0$. Alors*

$$\lim_{n\to+\infty} n^{r/d}(e_{n,r}(P))^r = Q_r(P).$$

avec

$$Q_r(P) = J_{r,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d\right)^{\frac{d+r}{d}} = J_{r,d}\,\|f\|_{\frac{d}{d+r}} \ \in [0, +\infty),$$

où $f = \frac{dP_a}{d\lambda_d}$ et

$$J_{r,d} = \inf_{n\geq 1} n^{r/d} e^r_{n,r}(U([0,1]^d)) \in (0, +\infty),$$

$U([0,1]^d)$ désignant la loi uniforme sur $[0,1]^d$.

La vraie valeur de la constante $J_{r,d}$ est inconnue pour $d \geq 3$ mais on sait que $J_{2,d} \sim \frac{d}{2\pi e}$ lorsque $d \to +\infty$ et que $J_{r,1} = 1/(2^r(r+1))$. Pour n variant de 1 à 900 et pour $r = 1, 2$, la figure 1.1 montre la convergence de la quantité $n^r e^r_{n,r}(\mathcal{N}(0;1))$ vers la constante $Q_r(\mathcal{N}(0;1))$ $\big(Q_1(\mathcal{N}(0;1)) \simeq 1.252$ et $Q_2(\mathcal{N}(0;1)) \simeq 2.714\big)$.

A noter aussi que l'hypothèse de moment $\mathbb{E}|X|^{r+\eta} < +\infty$ est une condition suffisante (mais non nécessaire) pour que la quantité $\|f\|_{\frac{d}{d+r}}$ soit finie (voir [GL], $p.79$).

Lorsque $P_a = 0$ alors $Q_r(P) = 0$. Dans ce cas $e_{n,r}(P) = o(n^{-1/d})$ et la vitesse de convergence est étudiée dans [GL] pour une grande classe de probabilités singulières notamment à support fractal. Du théorème de Zador il apparaît naturel de poser les définitions suivantes.

Définition. • *Lorsque un n-uplet de points $\alpha_n = (\alpha_{n1}, \cdots, \alpha_{nn})$ réalise l'infinimum dans $(1.0.1)$ alors α_n est dit $L^r(P)$-**optimale**.*

• *Une **suite** de quantifieurs $(\alpha_n)_{n\geq 1}$ sera dite $L^r(P)$-**optimale** lorsque, pour tout $n \geq 1$, α_n est $L^r(P)$-optimale.*

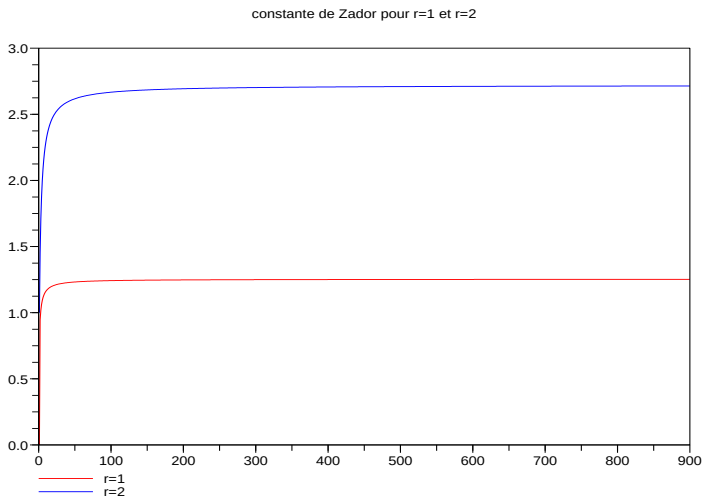

Figure 1.1: Taille de la grille n en fonction de $n^r e^r_{n,r}(\mathcal{N}(0;1))$ pour $n = 1, \cdots, 900, r = 1, 2$.

• *Une **suite** de quantifieurs $(\alpha_n)_{n\geq 1}$ sera dite **asymptotiquement** $L^r(P)$**-optimale** si l'erreur de quantification associée, normalisée à la vitesse optimale: $n^{1/d}e_{n,r}(P)$, converge vers la constante $Q_r(P)^{1/r}$ sans que la suite $(\alpha_n)_{n\geq 1}$ ne soit forcément $L^r(P)$-optimale, c'est-à-dire,*

$$\int_{\mathbb{R}^d} d(x,\alpha_n)^r P(dx) \;=\; e^r_{n,r}(X) + o(e^r_{n,r}(X)) \qquad \text{quand } n \to \infty.$$

Une façon triviale de construire une suite asymptotiquement $L^r(P)$-optimale qui n'est pas forcément $L^r(P)$-optimale est de modifier les $n_0 \geq 1$ premières valeurs de cette dernière.

• *On dira qu'une **suite** de quantifieurs $(\alpha_n)_{n\geq 1}$ est $L^r(P)$**-taux-optimale** si l'erreur de quantification tend vers zero à la vitesse optimale sans que la suite ne soit forcément asymptotiquement $L^r(P)$-optimale. Autrement dit si l'on a simplement*

$$\limsup_{n\to+\infty} n^{1/d}\left(\int_{\mathbb{R}^d} d(x,\alpha_n)^r P(dx)\right)^{1/r} < +\infty.$$

Ainsi si la suite (α_n) est $L^r(P)$-optimale, alors elle est $L^s(P)$-taux-optimale pour tout $s \leq r$.

Après avoir quantifié la variable X une question naturelle est de savoir comment reconstruire la probabilité initiale P à partir de grilles $L^r(P)$-optimales. Une démarche naïve est de considérer la mesure empirique $\frac{1}{n}\sum_{a\in\alpha_n}\delta_a$. Cependant, cela supposerait que les cellules de Voronoi $C_a(\alpha_n)$ sont uniformément distribuées sous P et ont, asymptotiquement au moins, pour poids $1/n$. Ce qui n'est pas le cas sauf pour la loi uniforme sur $[0,1]^d$. Il semble donc naturel de pondérer les mesures de dirac par les poids des cellules de la mosaïque de Voronoï associée (sous P) pour reconstruire la probabilité initiale. On a le résultat suivant (voir [DGLP, PAG]): si (α_n) est une suite de quantifieurs telle que l'erreur de quantification associée tende vers zéro alors,

$$\sum_{a\in\alpha_n} P\big(C_a(\alpha_n)\big)\delta_n \Longrightarrow P. \tag{1.0.3}$$

oú $\Longrightarrow$ désigne la convergence étroite. Néanmoins la mesure empirique converge effectivement mais vers une autre mesure de probabilité. Ce résultat, connu sous le nom de théorème de la mesure empirique, est rappelé ci dessous (voir [DGLP, GL] pour la preuve).

Théorème. (Théorème de la mesure empirique) *Soit $X \sim P$, avec $P_a \neq 0$. Soit $(\alpha_n)_{n\geq 1}$ une suite asymptotiquement $L^r(P)$-optimale. Alors*

$$\frac{1}{n}\sum_{a\in\alpha_n}\delta_a \Longrightarrow P_r \tag{1.0.4}$$

où P_r est définie pour tout borélien A de $\mathbb{R}^d$ par

$$P_r(A) = \frac{1}{C_{f,r}}\int_A f(x)^{\frac{d}{d+r}}d\lambda_d(x), \text{ avec } \quad C_{f,r} = \int_{\mathbb{R}^d} f(x)^{\frac{d}{d+r}}d\lambda_d(x).$$

On dira qu'une suite de quantifieurs d'une loi P vérifie le *théorème de la mesure empirique à l'ordre r* si l'assertion $(1.0.4)$ est vérifiée. Cela signifie simplement que pour tout borélien A de $\mathbb{R}^d$ tel que P ne charge pas la frontière ∂A,

$$\frac{\mathrm{card}\big(A\cap\alpha_n\big)}{n} \longrightarrow P_r(A).$$

Notons que l'hypothèse $P_a \neq 0$ est indispensable pour que P_r soit définie comme une probabilité.

1.1 Applications aux probabilités numériques

Soit X une variable alétoire à valeurs dans $\mathbb{R}^d$ et soit f une fonction borélienne de $\mathbb{R}^d$. On s'intéresse ici à l'estimation de la quantité

$$\mathbb{E}f(X) = \int f(x)\mathbb{P}_X(dx).$$

Lorsque la variable aléatire X est simulable et que $d > 1$ la méthode probabiliste la plus couramment utilisée pour calculer cette intégrale est la méthode de *Monte Carlo*. Elle consiste à simuler un échantillon de taille n c'est-à-dire n réalisations $X_1, \cdots, X_n$ indépendantes de la loi de X et à estimer $\mathbb{E}f(X)$ par la moyenne empirique

$$m_{n,f} := \frac{1}{n}\sum_{i=1}^{n} f(X_i).$$

D'après la loi des grands nombres, l'estimateur sans biais $m_{n,f}$ converge presque sûrement vers $\mathbb{E}f(X)$. Lorsque $f(X)$ est de carré intégrable, la vitesse de convergence est donnée par le théorème central limite:

$$\sqrt{n}\left(\mathbb{E}f(X) - m_{n,f}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0;\sigma_f^2)$$

avec

$$\sigma_f^2 = \mathbb{E}\big(f(X) - \mathbb{E}f(X)\big)^2.$$

La vitesse de convergence est donc de l'ordre de $1/\sqrt{n}$ indépendemment de la dimension du vecteur X.

La méthode de quantification pour le calcul d'intégrales est une alternative à la méthode de Monte Carlo (au moins jusqu'en dimension $d = 4$ comme nous le verrons). Elle consiste, compte tenu de $(1.0.3)$, à estimer $\mathbb{E}f(X)$ par

$$\mathbb{E}f(\widehat{X}^{\alpha_n}) = \sum_{a\in\alpha_n} f(a)\mathbb{P}(\widehat{X}^{\alpha_n} = a) = \sum_{a\in\alpha_n} f(a)\mathbb{P}(X \in C_a(\alpha_n)) \tag{1.1.1}$$

où α_n est une grille si possible $L^r(P)$-optimale. Il s'agit donc d'une formule de "cubature" consistant à pondérer des valeurs de la fonction f en une famille de points de $\mathbb{R}^d$.

Lorsque n tend vers $+\infty$ on voudrait alors pouvoir estimer l'erreur induite par une telle estimation. On rappelle ci-dessous quelques résultats élémentaires liés à cette erreur d'estimation selon la régularité de la fonction f.

• Soit f une fonction Lipschitz de coefficient $[f]_{\mathrm{Lip}}$ et soit (α_n) une suite de quantifieurs. Pour tout $n \geq 1$ on a

$$\begin{aligned} |\mathbb{E}f(X) - \mathbb{E}f(\widehat{X}^{\alpha_n})| &\leq \mathbb{E}|f(X) - f(\widehat{X}^{\alpha_n})| \\ &\leq [f]_{\mathrm{Lip}}\|X - \widehat{X}^{\alpha_n}\|_1 \\ &\leq [f]_{\mathrm{Lip}}\|X - \widehat{X}^{\alpha_n}\|_s \qquad (\text{si } X \in L^s,\ s \geq 1). \end{aligned}$$

Par conséquent lorsque (α_n) est L^s-taux-optimale (voir [GLP, SAG] pour les conditions de taux optimalité) alors

$$\mathbb{E}f(X) - \mathbb{E}f(\widehat{X}^{\alpha_n}) = O(n^{-1/d}). \tag{1.1.2}$$

• Si la dérivée Df de f est Lipschitz alors pour toute suite $L^2(P)$-optimale (α_n) on a pour tout $n \geq 1$,

$$|\mathbb{E}f(X) - \mathbb{E}f(\widehat{X}^{\alpha_n})| \leq [Df]_{\mathrm{Lip}}\|X - \widehat{X}^{\alpha_n}\|_2^2$$

de sorte que si (α_n) est L^2-taux-optimale alors

$$\mathbb{E}f(X) - \mathbb{E}f(\widehat{X}^{\alpha_n}) = O\big(n^{-2/d}\big). \tag{1.1.3}$$

• Lorsque f est convexe et que $\widehat{X}$ est L^2-stationnaire alors par l'inégalité de Jensen on a

$$\mathbb{E}f(\widehat{X}) = \mathbb{E}\big(f(\mathbb{E}(X|\widehat{X}))\big) \leq \mathbb{E}f(X).$$

Ce qui signifie que $\mathbb{E}f(\widehat{X})$ est une borne inférieure de $\mathbb{E}f(X)$ si f est convexe (et une borne supérieure si f est concave).

• Il y a des estimations d'erreurs pour d'autres classes de fonctions comme les fonctions localement Holdériennes vérifiant

$$|f(x) - f(y)| \leq C|x - y|^\theta(1 + |x|^\rho + |y|^\rho)$$

pour $\theta \in]0.1)$ et $\rho \geq 0$.

En effet soit $\rho \geq 0$ et soit $X \in L^{\beta+\varepsilon_0}(\mathbb{P}),\ \varepsilon_0 > 0$. Alors pour toute suite de quantifieurs $(\alpha_n)_{n\geq 1}$ et pour tout $r > 1$, on d'après les inégalités de Hölder et de Minkovski

$$\|f(X) - f(\widehat{X}^{\alpha_n})\|_1 \leq C\,\|X - \widehat{X}^{\alpha_n}\|_{\theta r}(1 + \|X\|_{\rho r'}^\rho + \|\widehat{X}^{\alpha_n}\|_{\rho r'}^\rho)$$

où $r' = r/(r-1)$. Noter que si $\widehat{X}$ est L^2-stationnaire et $\frac{r}{r-1} \geq \frac{1}{\rho}$ alors

$$\|\widehat{X}\|_{\rho r'} \leq \|X\|_{\rho r'}$$

car $u \to u^{\rho r'}$ est convexe.

Pour ce qui concerne l'espérance conditionnelle considérons deux vecteurs aléatoires X et Y à valeurs dans $\mathbb{R}^d$ définies sur un même espace de probabilité et soit f une fonction borélienne de $\mathbb{R}^d$. Alors une idée naturelle est d'estimer $\mathbb{E}(f(X)|Y)$ par $\mathbb{E}(f(\widehat{X})|\widehat{Y})$ où $\widehat{X}$ et $\widehat{Y}$ sont des quantifieurs optimaux de X et de Y, respectivement. L'estimation de l'erreur résultant d'une telle approximation est donnée ci-après (consulter [PAG1] pour plus de détails). Soit $\varphi_f : \mathbb{R}^d \to \mathbb{R}$ une version de l'espérance conditionnelle, c'est-à-dire, une fonction borélienne φ_f vérifiant

$$\varphi_f(Y) := \mathbb{E}(f(X)|Y) \qquad p.s.$$

Si f et φ_f sont Lipschitz continues de coefficients $[f]_{\mathrm{Lip}}$ et $[\varphi_f]_{\mathrm{Lip}}$ alors

$$\|\mathbb{E}(f(\widehat{X})|\widehat{Y}) - \mathbb{E}(f(\widehat{X})|\widehat{Y})\|_2 \leq [f]_{\mathrm{Lip}}\|X - \widehat{X}\|_2 + [\varphi_f]_{\mathrm{Lip}}\|Y - \widehat{Y}\|_2.$$

Le véritable avantage de la méthode de quantification par rapport à la méthode de Monte Carlo est que pour cette première des grilles de quantifieurs optimaux (ou stationnaires) et les poids (mais aussi les erreurs de quantification) associés peuvent être stockées offline de sorte que le calcul de l'espérence dans $(1.1.1)$ se fait de façon instantanée via les formules de cubature. On trouvera ainsi des grilles de quantifieurs quadratiques et leurs poids asssociés pour la loi $\mathcal{N}(0; I_d)$ sur le site internet

`www.quantize.maths-fi.com`

pour $d \in \{1, \cdots, 10\}$ et $n \in \{1, \cdots, 5000\}$.

Par contre en grande dimension (typiquement pour $d > 5$) la méthode de Monte Carlo devient plus compétitive pour n assez grand du fait de la dégradation de la vitesse de convergence (de l'ordre de $n^{-1/d}$ ou au mieux $n^{-2/d}$) dans la méthode de quantification, contrairement à la méthode de Monte Carlo où la vitesse de convergence est de l'orde de $1/\sqrt{n}$ quelque soit $d \geq 1$.

Il faut noter par ailleurs que lorsque $d \geq 3$, l'intégration par cubature via la quantification optimale reste parfois plus efficace que la méthode de Monte Carlo pour tout $n \leq n(d, f)$, où $n(d, f)$ est une valeur critique qui dépend de la dimension d et de l'intégrande f (voir [PAGPRI]).

1.2 Recherche de quantifieurs optimaux

Pour estimer une espérance via la formule de cubature $(1.0.3)$ on doit disposer, à n fixé, de la grille optimale et des poids associés aux cellules de Voronoi. C'est pourquoi l'étape de recherche de quantifieurs optimaux est cruciale pour avoir une bonne estimation de l'espérance. Alors soit on dispose de formules fermées (cas de la loi uniforme) ou de formules semi-fermées (cas de la loi puissance, des lois exponentielle et de Pareto) soit on fait appel à des algorithmes déterministes ou stochastiques basés sur la recherche de zéros de $\nabla D_{n,r}$. Avant de passer en revue ces algorithmes nous rappelons d'abord les formules fermées et semi-fermées disponibles.

• **Loi uniforme** $\mathcal{U}([0, 1])$. Pour tout $n \geq 1$, pour tout $r > 0$, la grille optimale $\alpha_n = (\alpha_{n,1}, \cdots, \alpha_{n,n})$ de la loi uniforme sur $[0, 1]$ est unique et donnée par

$$\alpha_{n,k} = \frac{2k-1}{2n}, \qquad k = 1, \cdots, n.$$

1.2.1 Quelques formules quasi-fermées

On dispose de formules semi-fermées qui permettent d'obtenir les grilles de quantifieurs L^r-optimales pour tout $r > 0$ pour la loi puissance, la loi exponentielle et la loi de Pareto. Ces résultats, parus dans [FP], sont rappelés ci-dessous.

• **Loi exponentielle $\mathscr{E}(1)$.** Pour tout $n \geq 1$, la grille L^r-optimale $\alpha_n^{(r)} = (\alpha_{n1}^{(r)}, \cdots, \alpha_{nn}^{(r)})$ est unique et donnée par

$$\alpha_{nk}^{(r)} = \frac{a_n^{(r)}}{2} + \sum_{i=n+1-k}^{n-1} a_i^{(r)}, \quad 1 \leq k \leq n, \tag{1.2.1}$$

où $(a_k^{(r)})_{k\geq 1}$ est est une suite de réels positifs définie par la formule de récurrence implicite suivante:

$$a_0^{(r)} := +\infty, \quad \phi_r\big(-a_{k+1}^{(r)}\big) := \phi_r\big(a_k^{(r)}\big),\ k \geq 1$$

avec $\phi_r(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)e^{-u}du$ (convention : $0^0 = 1$).

De plus $(a_k^{(r)})_{k\geq 1}$ décroît vers 0 et pour tout $k \geq 1$,

$$a_k^{(r)} = \frac{r+1}{k}\left(1 + \frac{c_r}{k} + \mathrm{O}(\frac{1}{k^2})\right)$$

pour une constante réelle c_r. Noter que les grilles optimales de la loi exponentielle de paramètre $\lambda > 0$ s'en déduisent puisque pour tout $\lambda > 0$, la variable aléatoire $\frac{X}{\lambda}$, $X \sim \mathscr{E}(1)$, suit la loi $\mathscr{E}(\lambda)$.

• **Loi puissance sur $[0,1]$ de paramètre p de densité $px^{p-1}\mathbf{1}_{[0,1]}, p > 0$.** Pour tout $n \geq 1$, la grille L^r-optimale $\alpha_n^{(r)} = (\alpha_{n1}^{(r)}, \cdots, \alpha_{nn}^{(r)})$ est unique et donnée par

$$\alpha_{nk}^{(r)} = \left(1 + \frac{a_n^{(r)}}{2}\right)^{-1} \prod_{i=k}^{n-1} \big(1 + a_i^{(r)}\big)^{-1}, \quad 1 \leq k \leq n, \tag{1.2.2}$$

où $(a_k^{(r)})_{k\geq 1}$ est une suite de réels positifs définie par l'équation de récurrence implicite suivante:

$$a_0^{(r)} := -2, \quad \phi_p\big(a_{k+1}^{(r)}\big) := \phi_p\left(-a_k^{(r)}/(1 + a_k^{(r)})\right),\ k \geq 1$$

avec

$$\phi_p(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)\, p(1+u)^{p-1}du.$$

La suite $(a_k^{(r)})_{k\geq 1}$ décroît vers 0 et il existe une constante réelle c_r telle que pour tout $k \geq 1$,

$$a_k^{(r)} = \frac{r+1}{k(p+r)}\left(1 + \frac{c_r}{k} + \mathrm{O}(\frac{1}{k^2})\right).$$

• **Loi de Pareto de paramètre γ de densité $\gamma x^{-(\gamma+1)}\mathbf{1}_{[1,+\infty)}$, $\gamma > r$.** Pour tout $n \geq 1$, l'unique grille L^r-optimale $\alpha_n^{(r)} = (\alpha_{n1}^{(r)}, \cdots, \alpha_{nn}^{(r)})$ est donnée par

$$\alpha_{nk}^{(r)} = \frac{1}{1 + a_n^{(r)}} \prod_{i=n-k+1}^{n-1} (1 + a_i^{(r)}), \quad 1 \leq k \leq n, \tag{1.2.3}$$

où cette fois-ci $(a_k^{(r)})_{k\geq 1}$ est une suite de réels positifs définie par l'équation de récurrence implicite suivante:

$$a_0^{(r)} := +\infty, \quad \phi_\gamma\left(-\frac{a_{k+1}^{(r)}}{1+a_{k+1}^{(r)}}\right) := \phi_\gamma\left(a_k^{(r)}\right),\ k \geq 1$$

avec

$$\phi_\gamma(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)\ \gamma(1+u)^{-(\gamma+1)}du.$$

La suite $(a_k^{(r)})_{k\geq 1}$ décroît vers 0 et il existe une constante réelle $c > 0$ telle que pour tout $k \geq 1$,

$$a_k^{(r)} = \frac{r+1}{k(\gamma-r)}\left(1+\frac{c}{k}+\mathrm{O}(\frac{1}{k^2})\right).$$

En pratique, lorsqu'on ne dispose pas de formule semi-fermées comme précédemment, les quantifieurs optimaux (ou au moins stationnaires) sont obtenus par la recherche de zéros de $\nabla D_{n,r}(\alpha)$. Rappelons que lorsque $d = 1$ et lorsque la densité de X est log-concave alors il y a unicité des quantifieurs optimaux et donc l'unique zéro de $\nabla D_{n,r}(\alpha)$ est un minimal global. Dans ces cas, la méthode de Newton s'avère très efficace pour trouver les zéros de $\nabla D_{n,r}(\alpha)$. Dès que $d > 2$, la méthode de Newton devient inutilisable en pratique et on fait appel dans ce cas à des algorithme stochastiques tel que l'algorithme de Kohonen (ou du plus proche voisin) ou à l'algorithme de Lloyd.

1.2.2 Algorithme de Newton

Cette méthode consiste à linéaliser la fonction $\nabla D_{n,r}(\alpha)$ et à prendre le point d'annulation de cette linéarisation comme approximation du zéro recherché. Cette procédure réitérée en l'approximation abtenue génère des approximations successives qui convergent en théorie avec une vitesse quadratique, c'est-à-dire, le nombre de décimales correctes double à chaque itération. De façon explicite, pour n fixé (la taille de la grille) et pour M le nombre d'itérations de la procédure, la méthode est décrite comme suit: on choisit un point initial $\alpha_0^{(n)}$ et on définit par récurrence la suite $(\alpha_p^{(n)})_{1\leq p\leq M}$ par

$$\alpha_{p+1}^{(n)} = \alpha_p^{(n)} - \left(\nabla^2 D_{n,r}(\alpha_p^{(n)})\right)^{-1}\nabla D_{n,r}(\alpha_p^{(n)}) \quad p = 1, \cdots, M.$$

où $\nabla^2 D_{n,r}$ est la matrice hessienne de $D_{n,r}$. Remarquons que plutôt que d'inverser la matrice hessienne il est préférable, pour minimiser le temps de résolution, de résoudre en $\alpha_{p+1}^{(n)}$ le système affine

$$\nabla^2 D_{n,r}(\alpha_p^{(n)})\big(\alpha_{p+1}^{(n)} - \alpha_p^{(n)}\big) = -\nabla D_{n,r}(\alpha_p^{(n)}).$$

L'initialisation de la procédure est une étape très importante puisque l'algorithme peut entrer dans une boucle infinie sans produire de meilleures approximations si le point initial est très éloigné du vrai zéro.

Pour la loi $\mathcal{N}(0;1)$ on peut avoir des formules explicites en fonction de la densité et de la fonction de répartition gaussiennes pour le gradient $\nabla D_{n,r}$ et pour la matrice hessienne $\nabla^2 D_{n,r}$ (voir e.g. [PAGPRI] pour $r = 2$). D'autre part les expériences numériques suggèrent de choisir la grille initiale $\alpha_0^{(n)}$ uniformément sur $[-2,2]$ pour $r = 2$. Pour $r \neq 2$, on prend pour point initial (se référer au chapitre 1 pour plus de détails) la grille

$$\alpha_0^{(n)} = \sqrt{\frac{r+1}{3}}\,\alpha_n = \Big\{\sqrt{\frac{r+1}{3}}\,a, a \in \alpha_n\Big\}$$

où α_n est l'approximé de la grille optimale quadratique.

Le même algorithme peut être utilisé pour avoir les grilles de quantifieurs de la loi exponentielle, de Weibull, de Pareto, de la loi puissance, etc.

A noter que pour toutes ces lois, le poinds p_a correspondant à la cellule de Voronoi associée au point $a \in \alpha$ (où α est la grille optimale) est obtenu par $p_a = \mathbb{P}(X \in C_a(\alpha))$. Aussi les erreurs de quantification sont estimées à partir des grilles L^r -optimales à travers des formules pseudo-explicites (pour la gaussienne par exemple) ou explicites (c'est le cas de la loi exponentielle).

1.2.3 Algorithme CLVQ

L'algorithme CLVQ (Competitive Learning Vector Quantization) est aussi appelé algorithme du plus proche voisin ou algorithme de Kohonen à 0 voisin. Pour tout $r > 1$ et pour la taille de la grille n fixée, il consiste à simuler une suite $(\xi_p)_{p\geq 1}$ selon la loi de X et à définir partant de X^0, la suite $(X^p)_{p\geq 1}$ par

$$X^{p+1} = X^p - \gamma_{p+1}\mathbf{1}_{\{\xi^{p+1}\in C_i(X^p)\}}|X_i^p - \xi^{p+1}|^{r-1}(X_i^p - \xi^{p+1}), \quad i = 1, \cdots, n.$$

où $\gamma := (\gamma_p)_p$ est une suite de pas décroissante vers 0 et vérifiant

$$\sum_p \gamma_p = +\infty \quad \text{et} \quad \sum_p \gamma_p^2 < +\infty.$$

L'algorithme de Kohonen se déroule en deux phases. Une phase dite de compétition où l'on sélectionne l'indice $i(p+1)$ (l'indice i à l'étape $p+1$ de la procédure) de la cellule du plus proche voisin et une phase d'apprentissage qui permet de mettre à jour les points de la grille en modifiant le point le "plus proche" de la valeur simulée ξ^{p+1} et en laissant inchangés les autres points, c'est-à-dire,

$$\begin{cases} X_i^{p+1} = X_i^p & \text{si } i \neq i(p+1)\,, \\ X_i^{p+1} = X_i^p - \gamma_p\,|X_i^p - \xi^{p+1}|^{r-1}(X_i^p - \xi^{p+1}) & \text{si } i = i(p+1)\,. \end{cases}$$

La suite $(X_p^{(n)})_{p\geq 1}$ ainsi définie converge presque sûrement vers un point critique de $D_{n,r}$ (voir [PAGPRI]). On se réfère au même article pour plus de détails sur l'initialisation de l'algorithme et sur le choix de (γ_p) pour la loi gaussienne multivariée. Les poids et les erreurs de quantification sont estimés via une simulation de Monte Carlo.

1.2.4 Algorithme de Lloyd I

Lorsque $d \geq 2$, l'algorithme de Lloyd I est largement utilisé en pratique pour obtenir des grilles quadratiques stationnaires du fait de sa mise en œuvre simple. Pour obtenir des grilles L^r-stationnaires, une extension naturelle de l'algorithme de Lloyd's I (qu'on appellera algorithme de Lloyd I généralisé ou étendu) est basée sur l'équation de L^r-stationnarité $\nabla D_{n,r}(\alpha_n) = 0$ avec $\alpha_n = (\alpha_{n1}, \cdots, \alpha_{nn})$. Cette équation s'écrit pour tout $r \geq 1$ (voir [PPP1])

$$\alpha_{ni} = \frac{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X - \alpha_{ni}|^{r-2}X\right)}{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X - \alpha_{ni}|^{r-2}\right)}, \qquad i = 1, \cdots, n. \tag{1.2.4}$$

L'algorithme de Lloyd I (généralisé) est un algorithme du point fixe fondé sur $(1.2.4)$. Partant de la grille initiale $\alpha_n^{(0)}$, on définit de façon récurcive une suite $(\alpha_n^{(l)})_{l=1,\cdots,L}$ de quantifieurs L^r-stationnaires (où L correspond au nombre d'itérations de la procédure) en posant pour tout $l =$

$1, \cdots, L$,

$$\alpha_{ni}^{(l)} = \frac{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}^{(l-1)}|^{r-2}X\right)}{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}^{(l-1)}|^{r-2}\right)}, \qquad i=1,\cdots,n. \tag{1.2.5}$$

Nous "randomisons" alors l'algorithme de Lloyd I en estimant les espérances qui apparaissent dans $(1.2.5)$ par la méthode de Monte Carlo.

Comme en dimension $d \geq 2$ il peut exister une infinité de quantifieurs L^r-stationnaires. L'algorithme converge alors vers l'équilibre le "plus proche" (c'est-à-dire celui qui se trouve dans le basin d'attraction de la grille initiale), la phase d'initialisation est donc crutiale pour avoir de meilleurs quantifieurs. Le choix de la grille initiale pour la gaussienne (pour $r \neq 2$) est discuté dans le chapitre suivant; pour le cas quadratique ($r = 2$) voir par exemple [GG].

Lorsque $d = 1, r = 2$ et lorsque la variable aléatoire X est à densité strictement log-concave il est prouvé dans [KIEF1] que l'algorithme de Lloyd I converge avec une vitesse exponentielle. Par contre la convergence de l'algorithme n'a pas encore été établie de façon rigoureuse dans le cas général.

1.2.5 Exemples de grilles optimales

• *Cas Gaussien.* On considère $P = \mathcal{N}(0;1)$ et une taille de grille $n = 70$. La figure 1.2 représente en abscisse les grilles L^r-optimales obtenues par la méthode de Newton pour $r = 1, 2, 4$ et en ordonnée les poids associés.

Lorsque $P = \mathcal{N}(0; I_2)$, la figure 1.3 représente les grilles L^r-optimales (ou plutôt stationnaires) obtenues par l'algorithme de Lloyd après 10 itérations, pour $r = 2, 4$ et pour une taille de la grille $n = 100$.

• *Cas de la loi exponentielle.* Soit $P = \mathscr{E}(1)$. La figure 1.4 représente en abscisse les grilles $L^r(P)$-optimales de taille $n = 70$ (pour $r = 1, 2, 4$), obtenues par la formule semi-fermée donnée dans $(4.7.3)$ et en ordonnée leurs poids associés.

• *Loi de Weibull de paramètre* $\kappa = 2$. Rappelons que la densité de la loi de Weibull de paramètre κ est donnée par

$$f(x) := \kappa x^{\kappa-1} e^{-x^\kappa} \mathbf{1}_{\{x>0\}}.$$

Pour cet exemple on représente à la figure 1.5 les poids associés à la grille optimale quadratique de taille $n = 70$ de la loi de Weibull de paramètre $\kappa = 2$ en fonction de la grille optimale.

1.3 Principaux résultats obtenus

Lorsque l'on estime $\mathbb{E}f(X)$ par la méthode de quantification on a vu que les erreurs d'estimation associées aux formules de quadratures font intervenir l'erreur de quantification

$$\|X - \widehat{X}^{\alpha_n}\|_s \tag{1.3.1}$$

où (α_n) est une suite de quantifieurs L^r-optimale, avec $r \neq s$. Pour atteindre les vitesses optimales dans $(1.1.2)$ et $(1.1.3)$ il faut que la suite (α_n) soit L^s-taux-optimale. C'est le cas lorsque $s \leq r$ puisque dans ce cas on a par croissance de la norme L^r,

$$\|X - \widehat{X}^{\alpha_n}\|_s \leq \|X - \widehat{X}^{\alpha_n}\|_r.$$

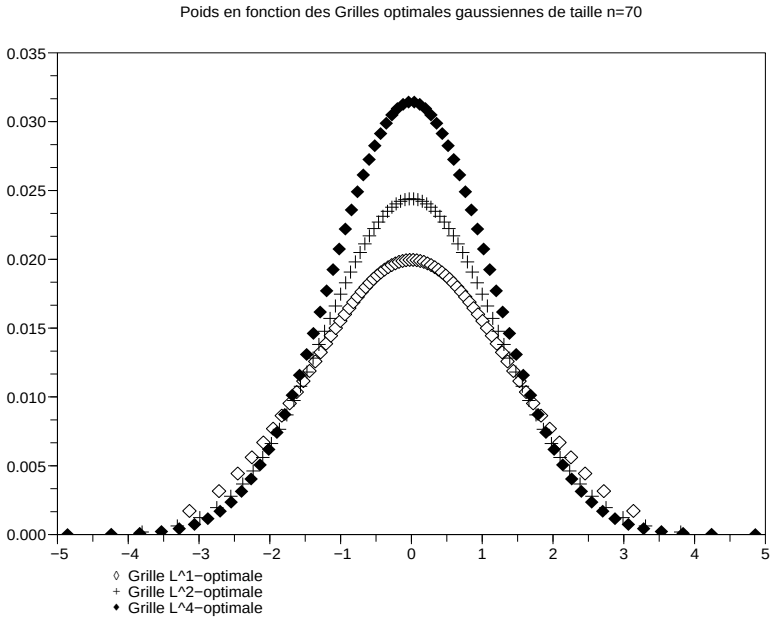

Figure 1.2: Poids en fonction des grilles L^r-optimales pour la $\mathcal{N}(0;1)$, pour $r = 1, 2, 4$.

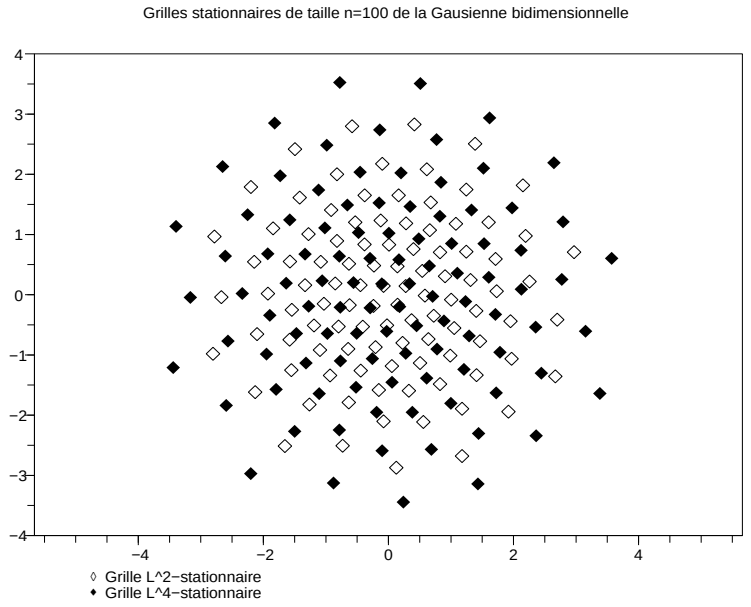

Figure 1.3: Grilles L^r-stationnaires pour la $\mathcal{N}(0; I_2)$, pour $r = 2, 4$.

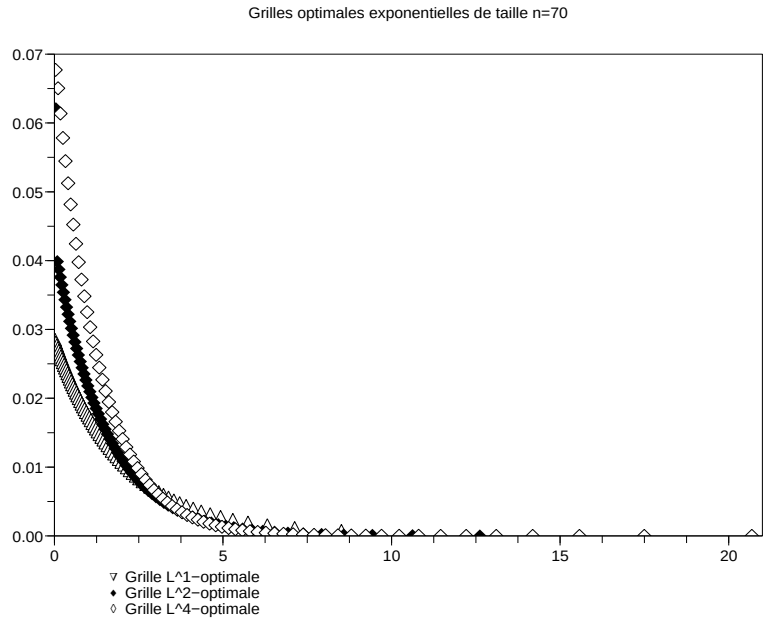

Figure 1.4: Poids en fonction des grilles L^r-optimales pour la loi $\mathscr{E}(1)$, pour $r = 1, 2, 4$.

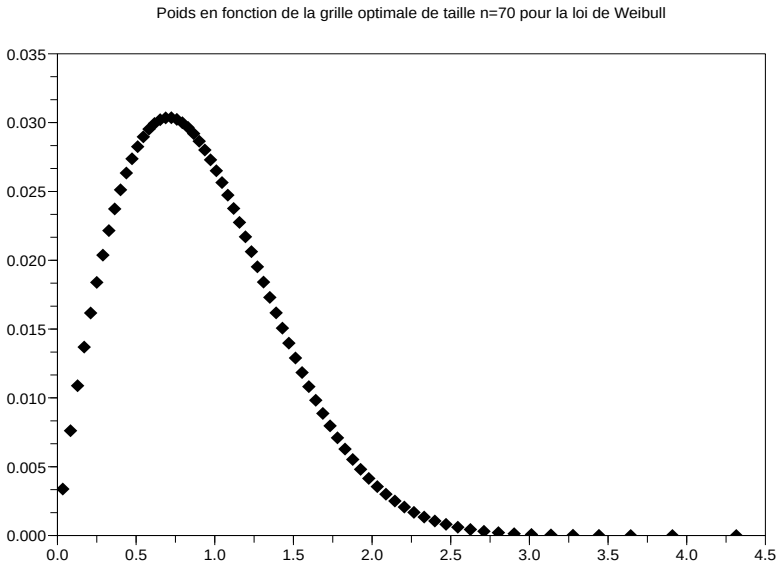

Figure 1.5: Poids en fonction de la grille optimale quadratique de taille $n = 70$ pour la loi de Weibull de paramètre $\kappa = 2$.

Par ailleurs il n'est (a priori) pas sûr que la suite (α_n) reste L^s-taux-optimale pour tout $s > r$. Ceci a amené Graf, Luschgy & Pagès (voir [GLP]) à étudier l'asymptotique de l'erreur de quantification dans L^s pour une suite de quantifieurs L^r-optimale, $s \neq r$. Ils montrent en particulier que si la loi de X admet une partie non singulière de densité f vérifiant $\lambda_d(f > 0) = +\infty$ alors

$$\lim_{n\to+\infty} n^{1/d} \|X - \widehat{X}^{\alpha_n}\|_s = +\infty$$

dès lors que $s > r + d$ et plus généralement que pour tout $s > 0$,

$$\liminf_{n\to+\infty} n^{s/d} \|X - \widehat{X}^{\alpha_n}\|_s^s \geq J_{s,d} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d \right)^{s/d} \int_{\{f>0\}} f^{1-\frac{s}{d+r}} d\lambda_d \tag{1.3.2}$$

où $J_{r,d}$ est la constante intervenant dans le théorème de Zador.

Au chapitre 2 nous montrons qu'une transformée linéaire simple $(\alpha_n^{\theta,\mu})$ definie pour tout $n \geq 1$ par

$$\alpha_n^{\theta,\mu} := \mu + \theta(\alpha_n - \mu) = \{\mu + \theta(a - \mu),\ a \in \alpha_n\}$$

(de paramètre $\theta > 0$ de dilatation ou de condensation, selon que $\theta > 1$ ou que $\theta < 1$, et de paramètre de translation $\mu \in \mathbb{R}^d$) de la suite L^r-optimale (α_n) permet de s'affranchir de ce seuil critique $r + d$ et de construire des suites L^s- rate optimales pour tout $s > 0$, pour une large famille de distributions. Les principaux résultats de ce chapitre sont les suivants. Pour la borne inférieure nous montrons, sous des hypothèses d'existence d'un moment d'ordre $r + \eta$ pour la variable X (avec $\eta > 0$) et de la probabilité non singulière $P_a = f \cdot \lambda_d$ (noter que si $f \equiv 0$ la borne inférieure ci-après est nulle donc semble inutile), que pour toute suite (α_n) asymptotiquement L^r-optimale,

$$\liminf_{n\to+\infty} n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \geq Q_{r,s}^{\text{Inf}}(P, \theta) \tag{1.3.3}$$

pour tout $s > 0$ et pour tout $\theta > 0,\ \mu \in \mathbb{R}^d$; avec $\left(f_{\theta,\mu}(x) = f(\mu + \theta(x - \mu))\right)$

$$Q_{r,s}^{\text{Inf}}(P, \theta) = \theta^{s+d} J_{s,d} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d \right)^{s/d} \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d.$$

Ce qui voudrait dire en particulier que pour tout $s > 0$, si l'erreur de quantification d'ordre s associée à la suite transformée $(\alpha_n^{\theta,\mu})$ converge vers 0, alors elle ne peut converger plus vite que $n^{-1/d}$ que si la loi de X est totalement singulière.

Pour ce qui concerne l'estimation de la borne supérieure nous avons obtenu deux résultats principaux. Le premier se restreint au cas $s < r$. Si X admet pour loi $P = f \cdot \lambda_d$ et si le couple de paramètre (θ, μ) est P-admissible, c'est-à-dire,

$$\{f > 0\} \subset \mu(1 - \theta) + \theta\{f > 0\} \qquad \lambda_d\text{-p.p.}$$

et si la suite (α_n) est asymptotiquement $L^r(P)$-optimale alors

$$\limsup_{n\to+\infty} n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq \theta^{s+d} (Q_r(P))^{s/r} \left(\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d \right)^{1-\frac{s}{r}}; \tag{1.3.4}$$

de sorte que la suite $(\alpha_n^{\theta,\mu})_{n\geq 1}$ est L^s-taux-optimale dès lors que

$$\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d < +\infty.$$

En même temps, ce résultat affirme que l'erreur de quantification d'ordre s associée à la suite transformée converge vers 0 avec une vitesse au moins égale à $n^{-1/d}$ (et donc égal d'après ce qui précède).

Le second résultat est la contrepartie du résultat sur la borne inférieure. En effet soit $X \sim P$ avec $P = f \cdot \lambda_d$. Sous l'hypothèse d'existence d'un moment d'ordre un peu plus égal à r pour X on montre que si $P_{\theta,\mu}$, la loi de $\frac{X-\mu}{\theta} + \mu$, est absolument continue par rapport à P pour un $\theta > 0, \mu \in \mathbb{R}^d$ et si la "fonction b-maximale de (α_n)" définie sur $\mathbb{R}^d$ (pour tout $b > 0$) par

$$\psi_b(x) = \sup_{n \geq 1} \frac{\lambda_d(B(x, bd(x, \alpha_n)))}{P(B(x, bd(x, \alpha_n)))}$$

est intégrable par rapport à $P_{\theta,\mu}$ pour un $b \in (0, 1/2)$, alors pour toute suite $L^r(P)$-optimale (α_n) et pour tout $s > 0$,

$$\limsup_n n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq C(b) \, \theta^{s+d} \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d < +\infty, \tag{1.3.5}$$

pour une constante $C(b) > 0$. Ce résultat nous dit en particulier que la vitesse de convergence de l'erreur de quantification $\|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s$ vers zéro reste optimale $i.e$ de l'ordre de $n^{-1/d}$.

Il faut noter que dans ce dernier résultat la suite (α_n) est (exactement) $L^r(P)$-optimale tandis que dans le premier résultat sur la borne supérieure (lorsque $s < r$) on demande à la suite (α_n) d'être simplement asymptotiquement $L^r(P)$-optimale.

A ce stade, il paraît naturel de chercher parmi les paramètres (θ, μ) qui laissent la suite transformée L^s-taux-optimale, le paramètre minimal $(\theta^\star, \mu^\star$ qui rend la suite $(\alpha_n^{\theta^\star,\mu^\star})$ la plus près de la L^s-optimalité asymptotique. Il s'avère que le paramètre d'intérêt est θ, μ une fois fixé ne jouant qu'un rôle de recentrage indépendant de s et de n (si X est centré $\mu = 0$).

La L^s-optimalité asymptotique ne peut pas être prouvée théoriquement du fait de l'existence d'une constante $C(b)$ non explicite dans $(1.3.5)$ et de la non optimalité de la majoration dans $(1.3.4)$.

Néanmoins on montre l'existence et l'unicité d'un tel paramètre $\theta^\star$ pour la loi Gaussienne, la loi exponentielle, la loi Gamma, et pour les deux premières lois on montre que la suite transformée associée $(\alpha_n^{\theta^\star,\mu})$, vérifie le théorème de la mesure empirique à l'ordre s. Des résultats numériques nous ont mené à conjecturer que cette dernière suite est asymptotiquement L^s-optimale. Elle est aussi numériquement très proche de la L^s-stationnarité et de la L^s-optimalité même si pour ce dernier point l'exemple de la loi exponentielle montre qu'elle ne l'est pas en général.

Pour la loi Gamma on a montré que la suite ainsi "optimalisée" $(\alpha_n^{\theta^\star,\mu})$ ne vérifie pas nécessairement le théorème de la mesure empirique à l'ordre s. Ce qui nous conduit à construire un exemple de suite L^s-taux-optimale qui n'est pas asymptotiquement L^s-optimale.

Au chapitre 3 nous menons à bien une première étude allant dans la direction des propriétés géométriques d'une suite de quantifieurs L^r-optimaux en étudiant le comportement asymptotique de la suite du rayon maximal (ρ_n) associée à une suite de quantifieurs L^r-optimale définie pour tout $n \geq 1$ par

$$\rho_n := \max\{|a|, \ a \in \alpha_n\}$$

où $|\cdot|$ est la norme euclidienne de $\mathbb{R}^d$ (lorsque $d = 1$, pour tenir compte de l'ordre naturel on pose simplement $\rho_n := \max\{a, \ a \in \alpha_n\}$)

On montrera que, dès que $\mathrm{supp}(P)$ est non borné (à droite lorsque $d = 1$), alors

$$\lim_{n \to +\infty} \rho_n = +\infty.$$

Nous donnerons ensuite, pour une grande famille de probabilités, la vitesse à laquelle le rayon maximal converge vers $+\infty$. Cette vitesse de convergence découle de deux résultats principaux. Le premier conduit à la borne supérieure de la suite (ρ_n) ou $(\log \rho_n)$ et s'énonce comme suit. Si X admet pour loi P vérifiant

$$P(dx) \geq \varepsilon_0 \, \mathbf{1}_{\{x \in \bar{B}(x_0, r_0)\}} \lambda_d(dx), \quad \varepsilon_0, \; r_0 > 0, \; x_0 \in \mathbb{R}^d,$$

alors le rayon maximal associé à une suite de quantifieurs L^r-optimale vérifie

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \to +\infty} \left(n^{1+\frac{r}{d}} \bar{F}_r\big(\frac{\rho_n}{2+\varepsilon}\big) \right) \geq C_{r,d,U} \tag{1.3.6}$$

où $\bar{F}_r$ est la fonction de survie généralisée définie pour tout $x \in \mathbb{R}^d$ par $\bar{F}_r(x) = \mathbb{E}(|X|^r \mathbf{1}_{\{|X|>x\}})$ et $C_{r,d,U} > 0$ est une constante dépendant de r, d et de $U \sim \mathcal{U}([0,1]^d)$.

Si $d = 1, r \geq 1$ et si de plus X admet une densité décroissance vers zéro sur un ensemble $[A, +\infty)$ et vérifiant

$$\lim_{y \to +\infty} \int_1^{+\infty} (u-1)^{r-1} \frac{f(uy)}{f(y)} du = 0$$

(hypothèse en particulier vérifiée pour des lois à densité $f(x) \propto |x|^c \, e^{-\vartheta |x|^\kappa}, x \in \mathbb{R}; \vartheta, \kappa > 0; c > -1$) alors on a le résultat suivant (plus précis que le précédent)

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \to +\infty} \left(n^{r+1} \bar{F}_r\big(\frac{\rho_n}{1+\varepsilon}\big) \right) \geq C_{r,1,U}. \tag{1.3.7}$$

Ces résultats donnent entre autre la vitesse de convergence de $\bar{F}_r(\frac{\rho_n}{c_{r,d}+\varepsilon})$ vers 0 pour tout $\varepsilon > 0$, avec $c_{r,d} = 1$ si $d = 1; r \geq 1$ et $c_{r,d} = 2$ sinon. Ils affirment que cette vitesse est d'un ordre au plus égal à $n^{-(1+r/d)}$.

D'autre part, lorsque $\nu > 0$ est choisi de sorte que la suite de quantifieurs (α_n) reste $L^{r+\nu}$-taux-optimale alors,

$$\limsup_{n \to +\infty} \sup_{c>0} \Big(c^{r+\nu} \, n^{\frac{r+\nu}{d}} \bar{F}\big(\rho_n + c\big) \Big) < +\infty \tag{1.3.8}$$

et

$$\limsup_{n \to +\infty} \sup_{u>1} \Big(\big(1 - 1/u\big)^{r+\nu} \, n^{\frac{r+\nu}{d}} \bar{F}_r\big(u\rho_n\big) \Big) < +\infty. \tag{1.3.9}$$

Ces deux derniers résultats qui nous permettent d'avoir la borne inférieure du rayon maximal affirment en particulier que pour tout $u > 1$, $\bar{F}_r(\rho_n u)$ (et $\bar{F}(\rho_n + c), \forall c > 0$) tendent vers 0 lorsque $n \to +\infty$ à une vitesse au moins égale à $n^{-(r+\nu)/d}$ dès que (α_n) est $L^{r+\nu}$-taux-optimale. On verra que le paramètre optimal $\nu^\star$ assurant que la suite (α_n) reste $L^{r+\nu}$-taux-optimale pour tout $\nu \in (0, \nu^\star)$ jouera un rôle crucial pour obtenir une borne inférieure pour la vitesse de convergence du rayon maximal.

Ensuite nous avons montré comment la connaissance du comportement asymptotique de $-\log \bar{F}_r$ permet de déduire les asymptotiques de $\log \rho_n$ ou de ρ_n. Comme exemples importants on peut mentionner deux familles de lois.

$\triangleright$ Les lois à *queue polynomiale* incluant les lois à densité f vérifiant

$$f(x) \propto \frac{(\log |x|)^\beta}{|x|^c} \, \mathbf{1}_{\{|x|>1\}} \qquad x \in \mathbb{R}^d, \; \beta \in \mathbb{R}, \; c > r + d$$

pour lesquelles la vitesse de convergence de la suite $\log \rho_n$ est calculée et donnée pour tout $r > 0$ et pour tout $d \geq 1$ par

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c-r-d}\frac{r+d}{d}.$$

$\triangleright$ Un autre exemple concerne les lois à *queue de distribution exponentielle* incluant les lois à densité f telle que

$$f(x) \ \propto \ |x|^c \, e^{-\vartheta|x|^\kappa} \ x \in \mathbb{R}^d; \ \vartheta, \kappa > 0; \ c > -d$$

pour lesquelles on a pour tout $d \geq 1$ et pour tout $r > 0$,

$$\frac{1}{\vartheta^{1/\kappa}}\left(1+\frac{r}{d}\right)^{1/\kappa} \leq \liminf_{n\to+\infty} \ \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \limsup_{n\to+\infty} \ \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \frac{2}{\vartheta^{1/\kappa}}\left(1+\frac{r}{d}\right)^{1/\kappa}$$

et dans le cas particulier où $d = 1$ et $r \geq 1$ on obtient la vitesse exacte

$$\lim_{n\to+\infty} \ \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = \left(\frac{r+1}{\vartheta}\right)^{1/\kappa}.$$

Notre conjecture générale pour ces lois à queue exponentielle (prouvée pour $d = 1$ et $r \geq 1$) est que la limite inférieure est optimale, c'est-à-dire que

$$\lim_{n\to+\infty} \ \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = \frac{1}{\vartheta^{1/\kappa}}\left(1+\frac{r}{d}\right)^{1/\kappa}.$$

D'autre part, une approche alternative pour la limite inférieure est aussi proposée. Celle-ci est basée sur la quantification aléatoire et relie le rayon maximal ρ_n à $\mathbb{E}(\max_{1\leq k\leq [n^{(r+\nu)/d}]} |X_k|)$ où (X_k) est une suite $i.i.d$ de variable aléatoires distribuées selon la loi de X et $\nu > 0$ est telle que la suite (α_n) est $L^{r+\nu}$-optimale.

Le dernier chapitre est consacré à des applications en Finance plus précisemment aux pricing d'options Lookback and d'options barrières par la méthode de Monte Carlo, par une méthode hybride Monte Carlo-quantification optimale et par une méthode (pure) quantification optimale.

Rappelons qu'une option sur un sous-jacent (stock) est un contrat qui confère à son détenteur le droit et non l'obligation d'acheter (dans le cas d'un call) ou de vendre (dans le cas d'un put) une quantité du sous-jacent pour un montant fixé K (strike) à une date fixée (la maturité). Une option européenne ne peut être exercée qu'à maturité tandis qu'une option américaine peut être exercée à tout instant jusqu'à la maturité. Ces options ont un prix et le prix est déterminé par la valeur courante d'un portefeuille d'autofinancement ayant pour valeur terminale la valeur de l'option à muturité.

Du point de vue mathématique soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé muni de la filtration $\mathcal{F} = \{\mathcal{F}_t, 1 \leq t \leq T\}$. On considère que le prix du sous-jacent est modélisé par l'équation différentielle stochastique (EDS) suivante

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = \zeta \tag{1.3.10}$$

où $(W_t)_{t\in[0,T]}$ est un mouvement brownien de dimension q defini sur $(\Omega, \mathcal{F}, \mathbb{P})$; $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ et $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}(d,q)$ sont des fonctions continues vérifiant

$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq C|x-y| \tag{1.3.11}$$

et

$$|b(t,x)| + \|\sigma(t,x)\| \leq C(1+|x|) \tag{1.3.12}$$

pour tout $t \in [0,T]$ et pour tout $x, y \in \mathbb{R}^d$. La norme $|\cdot|$ est une norme quelconque de $\mathbb{R}^d$ et $\|\cdot\|$ une norme quelconque sur l'espace des matrices $\mathcal{M}(d,q)$. La valeur initiale ζ est une variable aléatoire de carré intégrable, $\mathcal{F}_0$-mesurable, indépendent de W et défini sur $(\Omega, \mathcal{F}, \mathbb{P})$.

Rappelons que sous les hypothèses faites sur les coefficients l'EDS admet une unique solution forte (voir par exemple [OKS]).

On dira que le marché est sans arbitrage s'il existe une probabilité $\widetilde{\mathbb{P}}$ sous laquelle la valeur actualisée du prix du sous-jacent $e^{-rt}S_t$ est une martingale. Il est dit complet si la probabilité $\widetilde{\mathbb{P}}$ est unique. Dans un marché complet, tout actif peut être répliqué avec le stock et un zero-coupon à travers un portefeuille d'autofinancement. De plus sous les hypothèses d'absence d'oportunité d'arbitrage et de complétude du marché le prix des actifs est déterminé de façon unique est correspond à l'espérance de la valeur actualisée du payoff (une fonctionnelle du processus de prix $(X_t)_{t\in[0,T]}$ qui peut dependre de toute la trajectoire du processus) sous la probabilité $\widetilde{\mathbb{P}}$. Si V_t est la valeur de option au temps t et si h dénote le payoff à maturité alors

$$V_t = e^{-r(T-t)}\mathbb{E}(h|\mathcal{F}_t),$$

où $\mathbb{E}$ est l'espérance sous $\widetilde{\mathbb{P}}$ de sortes que le prix au temps 0 est donné par

$$V_0 = e^{-rT}\mathbb{E}(h).$$

Dans ce travail nous nous sommes intéréssé à une classe d'options exotiques dont le payoff dépend de toute la trajectoire du processus sur $[0,T]$. Ces payoff sont de la forme

$$h = F(X_T, \sup_{t\in[0,T]} X_t) \qquad \text{or} \qquad h = F(X_T, \inf_{t\in[0,T]} X_t).$$

Lorsque F est une fonction continue on parle des "options type Lookback " par référence à l'option lookback standard de payoff h défini par

$$h = X_T - \inf_{t\in[0,T]} X_t \qquad (\text{or } h = \sup_{t\in[0,T]} X_t - X_T).$$

Quand le payoff peut être décomposé par

$$h = \varphi(X_T)\mathbf{1}_{\{\sup_{t\in[0.T]} X_t \in I\}} \qquad \text{or } \varphi(X_T)\mathbf{1}_{\{\inf_{t\in[0.T]} X_t \in I\}}$$

où I est un interval non borné de $\mathbb{R}$, on parle d'options barrières.

Noter que dans le modèle de Black-Scholes il existe des formules semi fermées pour les prix des options Lookback et barrières (voir [CV]). Mais dès qu'on quitte le modèle de Black-Scholes il n'existe plus de formules semi fermées de sortes que ces prix doivent être approchés par des méthodes numériques. Ceci nécessite la discrétisation de la trajectoire du processus par des schémas tel que le schema d'Euler. Une fois la trajectoire discrétisée on peut approcher

$$\mathbb{E}F(X_T, \sup_{t\in[0,T]} X_t) \quad (\text{ou } \mathbb{E}F(X_T, \inf_{t\in[0,T]} X_t))$$

par

$$\mathbb{E}F(\bar{X}_T, \max_{0\leq k\leq n} \bar{X}_{t_k}) \quad (\text{ou } \mathbb{E}F(\bar{X}_T, \min_{0\leq k\leq n} \bar{X}_{t_k}))$$

où $(\bar{X}_{t_k})_{0\leq k\leq n}$ est le processus discret issu du schéma d'Euler (continu) et $t_k = \frac{kT}{n}$ les points de discrétisations. Mais on sait d'après [GOB] que l'erreur faible associée à une telle approximation est très faible puisqu'elle est d'un ordre inférieure à $n^{-1/2}$. Ceci pourrait s'expliquer par le fait qu'on perd de l'information entre deux temps de discrétisation puisque la probabilité pour que la barrière soit franchis entre ces deux temps de discrétisation pourrait par exemple être très proche de 1 sans que cette information ne soit prise en compte. A partir de la représentation de l'espérance d'une variable aléatoire par rapport à sa queue de distribution et par conditionnement (en utilisant la méthode du pont brownien) on réecrit le prix de ces options en intégrant probabilté que la barrière soit franchis entre tous les temps de discrétisation. Par exemple le prix du Partial Lookback call de maturité T est estimé par

$$\bar{C}_{PL} := e^{-rT}\mathbb{E}\big(\bar{X}_T - \lambda \min_{t\in[0,T]} \bar{X}_t \vee 0\big)^+ = X_0 - \alpha e^{-rT}\mathbb{E}\left(U^{\alpha-1}\bar{X}_T \prod_{k=1}^{n}\left(1 - F_{\bar{X}_{k-1},\bar{X}_k}\big(\tfrac{\bar{X}_T U^\alpha}{\lambda}\big)\right)\right) \tag{1.3.13}$$

et celui du Partial Lookback put de maturité T est estimé par

$$e^{-rT}\mathbb{E}\big(\lambda \max_{t\in[0,T]} \bar{X}_t - \bar{X}_T\big)^+ = \beta e^{-rT}\mathbb{E}\left(e^V \bar{X}_T\left(1 - \prod_{k=1}^{n} G_{\bar{X}_{k-1},\bar{X}_k}\big(\tfrac{(1+\beta V)\bar{X}_T}{\lambda}\big)\right)\right) \tag{1.3.14}$$

où $\alpha > 0, \beta > 0$ and $U \sim \mathcal{U}([0,1])$ and $V \sim \mathscr{E}(1)$ sont tous deux indépendantes de $(\bar{X}_k)_{k=0,\cdots,n}$ (processus d'Euler continu) et

$$G_{x,y}(u) = \left(1 - e^{-2n\frac{(x-u)(y-u)}{T\sigma^2(x)}}\right)\mathbf{1}_{\{x\leq u; y\leq u\}}$$

and

$$F_{x,y}(u) = \left(1 - e^{-2n\frac{(x-u)(y-u)}{T\sigma^2(x)}}\right)\mathbf{1}_{\{x\geq u; y\geq u\}}.$$

On en déduit alors des formules de prix pour les options Lookback à strike fixé K. Ainsi pour le put de maturité T on a

$$e^{-rT}\,\mathbb{E}(K - \min_{t\in[0,T]} \bar{X}_t)^+ = K\alpha e^{-rT}\mathbb{E}\left[U^{\alpha-1}\left(1 - \left(\prod_{k=1}^{n} F_{\bar{X}_{k-1},\bar{X}_k}\big(KU^\alpha\big)\right)\right)\right]. \tag{1.3.15}$$

et pour le call de maturité T

$$e^{-rT}\,\mathbb{E}(\max_{t\in[0,T]} \bar{X}_t - K)^+ = K\beta e^{-rT}\mathbb{E}\left[e^V\left(1 - \left(\prod_{k=0}^{n-1} G_{\bar{X}_k,\bar{X}_{k+1}}(K+\beta V)\right)\right)\right] \tag{1.3.16}$$

Ces représentations sont censés réduire la complexité du calcul de tels prix par la "methode du Pont Brownien" (voir [PAG2]) qui nécessite le calcul

$$\mathcal{L}(\sup_{[0,T]} \bar{X}_t | \bar{X}_{t_k}, k = 0, \cdots, n).$$

Les résultats numériques obtenus sur l'option Partial Lookback call montre en effet que cette méthode réduit la complexité mais en même temps augmente la variance (par rapport à la methode du Pont Brownien) même après réduction de la variance par des algorithmes stochastiques qui permettent d'estimer les paramètres $\alpha^\star$ et $\beta^\star$ minimisant la variance dans $(1.3.13)$ et $(1.3.14)$.

Ensuite une méthode hybride Monte Carlo-quantification optimale est proposée pour le calcul du prix des options Partial Lookback. Elle consiste à estimer les espérances par rapport à la loi uniforme et la loi exponentielle qui apparaissent dans les équations $(1.3.13)$ et $(1.3.14)$ par la méthode de quantification. Les résultats numériques obtenus sur les Partial Lookback call et put montrent que cette méthode nous permet de retrouver le niveau de variance initiale (par rapport à la méthode du Pont Brownien).

Des formules d'approximation des prix d'options barrières sont aussi obtenus par la même technique de conditionnement. Par exemple l'option up-and-out put de maturité T de strike K et de barrière L peut être estimer par

$$e^{-rT}\mathbb{E}\big((K-\bar{X}_T)^+\mathbf{1}_{\{\sup\limits_{t\in[0,T]}\bar{X}_t\leq L\}}\big)=e^{-rT}\mathbb{E}\left(g(\bar{X}_T)\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}(L)\right). \tag{1.3.17}$$

Pour l'option down-and-out put de maturité T de strike K et de barrière L elle est estimée par

$$e^{-rT}\mathbb{E}\big((\bar{X}_T-K)^+\mathbf{1}_{\{\inf\limits_{t\in[0,T]}\bar{X}_t\geq L\}}\big)=e^{-rT}\mathbb{E}\left(f(\bar{X}_T)\prod_{k=1}^{n}F_{\bar{X}_{k-1},\bar{X}_k}(L)\right). \tag{1.3.18}$$

Ces dernières formules sont classiques (voir par exemple [GLA, PAG2]) mais cette représentation nous permet de pouvoir estimer les prix des options barrières par la méthode de quantification en s'inspirant des méthodes d'estimation de filtres non linéaires par quantification présentée dans [PAGPRI, SEL].

Chapter 2

Universal L^s-rate-optimality of L^r-optimal quantizers by dilatation and contraction

This work will appear in *ESAIM PS* .

We investigate in this paper the properties of some dilatations or contractions of a sequence $(\alpha_n)_{n\geq 1}$ of L^r-optimal quantizers of an $\mathbb{R}^d$-valued random vector $X \in L^r(\mathbb{P})$ defined in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $\mathbb{P}_X = P$. To be precise, we investigate the L^s-quantization rate of sequences $\alpha_n^{\theta,\mu} = \mu + \theta(\alpha_n - \mu) = \{\mu + \theta(a-\mu),\ a \in \alpha_n\}$ when $\theta \in \mathbb{R}_+^\star, \mu \in \mathbb{R}^d, s \in (0, r)$ or $s \in (r, +\infty)$ and $X \in L^s(\mathbb{P})$. We show that for a wide family of distributions, one can always find parameters (θ, μ) such that $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is L^s-rate-optimal. For the Gaussian and the exponential distributions we show the existence of a couple $(\theta^\star, \mu^\star)$ such that $(\alpha_n^{\theta^\star,\mu^\star})_{n\geq 1}$ also satisfies the so-called L^s-empirical measure theorem. Our conjecture, confirmed by numerical experiments, is that such sequences are asymptotically L^s-optimal. In both cases the sequence $(\alpha_n^{\theta^\star,\mu^\star})_{n\geq 1}$ is incredibly close to L^s-optimality. However we show (see Remark $2.5.4$) that this last sequence is not L^s-optimal (e.g when $s = 2,\ r = 1$) for the exponential distribution.

2.1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \mathbb{R}^d$ be a random variable with distribution $\mathbb{P}_X = P$. The $L^r(P)$-optimal quantization problem of size n for P (or X) consists in the study of the best approximation of X by a $\sigma(X)$-measurable random vector taking at most n values. For $X \in L^r(\mathbb{P})$ this leads to the following optimization problem:

$$e_{n,r}(X) = \inf \{\|X - q(X)\|_r, q : \mathbb{R}^d \to \alpha, \alpha \subset \mathbb{R}^d, \text{ card}(\alpha) \leq n\}.$$

Let $\alpha \subset \mathbb{R}^d$ be a subset (a codebook) of size n. A Borel partition $C_a(\alpha)_{a\in\alpha}$ of $\mathbb{R}^d$ satisfying

$$C_a(\alpha) \subset \{x \in \mathbb{R}^d : |x - a| = \min_{b\in\alpha_n} |x - b|\},$$

where $|\cdot|$ denotes a norm on $\mathbb{R}^d$ is called a Voronoi partition of $\mathbb{R}^d$ (with respect to α and $|\cdot|$). The random variable $\widehat{X}^\alpha$ taking values in the codebook α defined by

$$\widehat{X}^\alpha = \sum_{a\in\alpha} a\mathbf{1}_{\{X\in C_a(\alpha)\}}.$$

is called a Voronoi quantization of X. In other words, it is a nearest neighbour projection of X onto the codebook (also called grid) α.

For any Borel function $q : \mathbb{R}^d \to \alpha$,

$$|X - q(X)| \geq \min_{a\in\alpha} d(X, a) = d(X, \alpha) = |X - \widehat{X}^\alpha| \quad \mathbb{P} \text{ a.s}$$

so that

$$\begin{aligned} e_{n,r}(X) &= \inf \{\|X - \widehat{X}^\alpha\|_r, \alpha \subset \mathbb{R}^d, \text{ card}(\alpha) \leq n\} \\ &= \inf_{\substack{\alpha\subset\mathbb{R}^d \\ \text{card}(\alpha)\leq n}} \left(\int_{\mathbb{R}^d} d(x, \alpha)^r dP(x)\right)^{1/r}. \end{aligned} \tag{2.1.1}$$

For all $n \geq 1$, the infimum in $(2.1.1)$ is reached at one (at least) codebook $\alpha^\star$; $\alpha^\star$ is then called a L^r-optimal n-quantizer. In addition, if $\text{card}(\text{supp}(P)) \geq n$ then $\text{card}(\alpha^\star) = n$ (see [GL] or [PAG]). Moreover the quantization error, $e_{n,r}(X)$, decreases to zero as n goes to infinity and the so-called Zador's Theorem gives its convergence rate under a slightly stringent moment assumption on X.

Zador Theorem (see [GL]) : Suppose $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$ and let $P = P_a + P_s$ be the Lebesgue decomposition of P with respect to the Lebesgue measure λ_d, where P_a denotes the absolutely continuous part and P_s the singular part of P. Then

$$\lim_{n\to+\infty} n^{r/d}(e_{n,r}(P))^r = Q_r(P).$$

with

$$Q_r(P) = J_{r,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d\right)^{\frac{d+r}{d}} = J_{r,d}\,\|f\|_{\frac{d}{d+r}} \in [0, +\infty),$$

$$J_{r,d} = \inf_{n\geq 1} n^{r/d} e^r_{n,r}(U([0, 1]^d)) \in (0, +\infty),$$

where $U([0, 1]^d)$ denotes the uniform distribution on the set $[0, 1]^d$ and $f = \frac{dP_a}{d\lambda_d}$. Furthermore, this theorem naturally suggests to set the following definitions.

A sequence of n-quantizers $(\alpha_n)_{n\geq 1}$ is

- $L^r(P)$-**rate-optimal** (or **rate-optimal** for X) if

$$\limsup_{n\to+\infty} n^{1/d}\left(\int_{\mathbb{R}^d} d(x,\alpha_n)^r dP(x)\right)^{1/r} < +\infty,$$

- **asymptotically** $L^r(P)$-**optimal** if

$$\lim_{n\to+\infty} n^{r/d}\int_{\mathbb{R}^d} d(x,\alpha_n)^r dP(x) = Q_r(P),$$

- $L^r(P)$-**optimal** if for all $n \geq 1$,

$$e^r_{n,r}(P) = \int_{\mathbb{R}^d} d(x,\alpha_n)^r dP(x).$$

Optimal quantizers are used in many fields of applications like Signal Processing (discretization of emitted signals) or more recently, Numerical Probability where they provide some simples quadrature formulae for the computation of expectations and conditional expectations. This approach has been extensively developed in Finance for the pricing of American options, swing options (commodities), portfolio management (stochastic control) or stochastic volatility estimation (non linear filtration); we refer for example to [PPP] for applications to stochastic control problems. The errors bounds in these quadrature formulae are always based on the mean quantization error $\|X - \widehat{X}^\alpha\|_s$ where α is an optimal L^r-quantizer, usually with $r \leq s$.

Motivated by this problem, the asymptotic behavior of the L^s-mean quantization error of a sequence of L^r-quantizers has been extensively investigated in [GLP]. A lower bound has been established which shows that if the distribution P is unbounded in the sense that the density function $f = \frac{dP}{d\lambda_d}$ satisfies $\lambda_d(f > 0) = +\infty$ then for any sequence $(\alpha_n)_{n\geq 1}$ of asymptotically L^r-optimal quantizers, $\liminf_n n^{\frac{1}{d}}\|X - \widehat{X}^{\alpha_n}\|_s = +\infty,\ \forall s > r + d$.

On the other hand, under natural assumptions in the tail of the distribution P, it is shown in [GLP] that for any sequence of L^r-optimal quantizers, $\forall s \in (0, r+d),\ \limsup_n n^{\frac{1}{d}}\|X - \widehat{X}^{\alpha_n}\|_s < +\infty\ i.e$ $(\alpha_n)_{n\geq 1}$ remains L^s-rate-optimal as long as $s < r + d$.

The aim of this paper is to show that some simple transformation of the L^r-optimal quantizers, namely some dilatation-translation, makes possible to overcome the critical exponent $r + d$: we will establish that for a wide family of distributions, one can always find $\theta \in \mathbb{R}^\star_+$ and $\mu \in \mathbb{R}$ (depending on r, s and d but not on n) such that $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is L^s-rate-optimal. From a general upper bounds that we establish for such transformed sequences of quantizers we derive an heuristic to specify some explicit optimal (in a sense which will be elucidated later) scaling parameters (θ, μ) for several families of distributions (Gaussian Vector, exponential and gamma distributions). Some numerical computations carried with the Gaussian and the exponential distributions show that the resulting sequence of quantizers is very close to L^s-optimality.

So, one application could be to use these quantizers to initialize the procedures used for L^s-optimal (and local optimal) quantizers search when $s \neq 2$. Indeed, in the quadratic case, $s = 2$, several stochastic procedures like the Competitive Learning Vector Quantization algorithm or the randomized Lloyd's I procedure have been designed. Both rely on the stationary property: $\widehat{X}^\alpha = \mathbb{E}(X|\widehat{X}^\alpha)$, satisfied by optimal (and locally optimal) quadratic quantizers. In one dimension, Newton's method is used to compute the optimal quadratic quantizers. Thus a whole package of optimal n-quantizers

of the $\mathcal{N}(0, I_d)$ distributions are available in the website `www.quantize.maths-fi.com` for $d \in \{1, \cdots, 10\}$ and $n \in \{1, \cdots, 5000\}$. But, when $s \neq 2$, the natural extension of these procedures become more difficult to implement due to some loss of stability. When $s > 2$ the procedures tend to explode more and more often while when $1 \leq s < 2$ the convergence phase becomes chaotic. In particular, the sensibility of the procedure to its initialization increases as s moves away from 2. Thus, initializing theses procedures by the dilated-contracted L^2-optimal (or locally optimal) quantizers would make them more stable and speed up the convergence. This is what we do to carry the L^4-optimal quantizers of the one dimensional Gaussian distribution (used for numerical experiments in Section 5.1.2) by Newton's method. In fact, initializing this procedure to a n-tuple different from the dilated sequence usually makes the hessian matrix of the L^4-quantization error singular (which makes the procedure very unstable), especially when the grid's size becomes large (typically when $n \geq 400$).

The paper is organized as follows. In section 2 we establish a general lower bound for dilated-translated sequences of quantizers. General upper bounds are also established in section 3 for such sequences. In section 4 we provide a necessary and sufficient condition of L^s-rate-optimality for the dilated-translated sequences. Section 5 deals with some examples of distributions for which we give the set of parameters (θ, μ) such that the dilated-translated sequence is L^s-rate-optimal and try to find the couple (if any) which makes the resulted transformed sequence satisfy the L^s-empirical measure theorem. The last section is devoted to some applications.

NOTATIONS : • Let α_n be a set of n points of $\mathbb{R}^d$. For every $\mu \in \mathbb{R}^d$ and every $\theta > 0$ we denote $\alpha_n^{\theta,\mu} = \mu + \theta(\alpha_n - \mu) = \{\mu + \theta(a - \mu),\ a \in \alpha_n\}$.

• Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a Borel function and let $\mu \in \mathbb{R}^d, \theta > 0$. One notes by $f_{\theta,\mu}$ (or f_θ if $\mu = 0$) the function defined by $f_{\theta,\mu}(x) = f(\mu + \theta(x - \mu)),\ x \in \mathbb{R}^d$.

• If $X \sim P$, $P_{\theta,\mu}$ will stand for the probability distribution of the random variable $\frac{X-\mu}{\theta} + \mu,\ \theta > 0, \mu \in \mathbb{R}^d$. In other words, it is the distribution image of P by $x \longmapsto \frac{x-\mu}{\theta} + \mu$. Note that if $P = f \cdot \lambda_d$ then $\frac{dP_{\theta,\mu}}{d\lambda_d} = \theta^d f_{\theta,\mu}$.

• If A is a matrix A' stands for its transpose.

• Set $x = (x_1, \cdots, x_d);\ y = (y_1, \cdots, y_d) \in \mathbb{R}^d$; we denote $[x, y] = [x_1, y_1] \times \cdots \times [x_d, y_d]$.

• Let $|\cdot|$ be a norm on $\mathbb{R}^d$ and let A be a subset of $\mathbb{R}^d$; we denote by $B(x, r)$ the closed ball, centered to x with radius $r > 0$ and by $d(x, A)$ the distance between x and A; both with respect to the norm $|\cdot|$.

2.2 Lower estimate

Let $r, s > 0$. Consider an asymptotically $L^r(P)$-optimal sequence of quantizers $(\alpha_n)_{n\geq 1}$. For every $\mu \in \mathbb{R}^d$ and any $\theta > 0$, we construct the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$ and try to lower bound asymptotically the L^s-quantization error induced by this sequence. This estimation provides a necessary condition of rate-optimality for the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$. In the particular case where $\theta = 1$ and $\mu = 0$ we get the same result as in [GLP].

Theorem 2.2.1. *Let $r, s \in (0, +\infty)$, and let X be a random variable taking values in $\mathbb{R}^d$ with distribution P such that $P_a = f.\lambda_d \not\equiv 0$. Suppose that $\mathbb{E}|X|^{r+\eta} < \infty$ for some $\eta > 0$. Let $(\alpha_n)_{n\geq 1}$ be an asymptotically $L^r(P)$-optimal sequence of quantizers. Then, for every $\theta > 0$ and for every $\mu \in \mathbb{R}^d$,*

$$\liminf_{n\to+\infty} n^{s/d} \left\| X - \widehat{X}^{\alpha_n^{\theta,\mu}} \right\|_s^s \geq Q_{r,s}^{Inf}(P, \theta), \tag{2.2.1}$$

with

$$Q^{Inf}_{r,s}(P,\theta) = \theta^{s+d} J_{s,d} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d \right)^{s/d} \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d.$$

Proof. Let $m \geq 1$ and

$$f_m^{\theta,\mu} = \sum_{k,l=0}^{m2^m-1} \frac{l}{2^m} \mathbf{1}_{E_k^m \cap G_l^m};$$

with

$$E_k^m = \left\{ \frac{k}{2^m} \leq f < \frac{k+1}{2^m} \right\} \cap B(0,m) \text{ and } G_l^m = \left\{ \frac{l}{2^m} \leq f_{\theta,\mu} < \frac{l+1}{2^m} \right\} \cap B(0,m).$$

The sequence $(f_m^{\theta,\mu})_{m\geq 1}$ is non-decreasing and

$$\lim_{m\to+\infty} f_m^{\theta,\mu} = f_{\theta,\mu} \qquad \lambda_d \text{ a.e.}$$

Let

$$I_m = \{(k,l) \in \{0,\cdots,m2^m-1\}^2 : \lambda_d(E_k^m) > 0; \lambda_d(G_l^m) > 0\}.$$

For every $(k,l) \in I_m$ there exists compact sets K_k^m and L_l^m such that :

$$K_k^m \subset E_k^m \text{ , } L_l^m \subset G_l^m \text{ , } \lambda_d(E_k^m \backslash K_k^m) \leq \frac{1}{m^4 2^{2m+1}} \text{ and } \lambda_d(G_l^m \backslash L_l^m) \leq \frac{1}{m^4 2^{2m+1}}.$$

Then

$$\begin{aligned} (E_k^m \cap G_l^m) \backslash (K_k^m \cap L_l^m) &= E_k^m \cap G_l^m \cap ((K_k^m)^c \cup (L_l^m)^c) \\ &\subset (E_k^m \backslash K_k^m) \cup (G_l^m \backslash L_l^m). \end{aligned}$$

Consequently,

$$\begin{aligned} \lambda_d(E_k^m \cap G_l^m \backslash K_k^m \cap L_l^m) &\leq \lambda_d(E_k^m \backslash K_k^m) + \lambda_d(G_l^m \backslash L_l^m) \\ &\leq \frac{1}{m^4 2^{2m+1}} + \frac{1}{m^4 2^{2m+1}} \\ &= \frac{1}{m^4 2^{2m}}. \end{aligned}$$

For every $m \geq 1$ and every $(k,l) \in I_m$, set

$$A_{k,l}^m := K_k^m \cap L_l^m,$$

$$\tilde{f}_m^{\theta,\mu} := \sum_{k,l=0}^{m2^m-1} \frac{l}{2^m} \mathbf{1}_{A_{k,l}^m},$$

and

$$\tilde{f}_m := \sum_{k,l=0}^{m2^m-1} \frac{k}{2^m} \mathbf{1}_{A_{k,l}^m}.$$

We get

$$\{f_m^{\theta,\mu} \neq \tilde{f}_m^{\theta,\mu}\} \subset \bigcup_{k,l\in\{0,\cdots,m2^m-1\}} \left((E_k^m \cap G_l^m)\backslash A_{k,l}^m\right).$$

Therefore, for every $m \geq 1$,

$$\lambda_d(\{f_m^{\theta,\mu} \neq \tilde{f}_m^{\theta,\mu}\}) \leq \sum_{k,l=0}^{m2^m-1} \frac{1}{m^4 2^{2m}} = \frac{1}{m^2}$$

and finally

$$\sum_{m\geq 1} \mathbf{1}_{\{f_m^{\theta,\mu} \neq \tilde{f}_m^{\theta,\mu}\}} < \infty \qquad \lambda_d \text{ a.e.}$$

As a consequence $\lambda_d(dx)$ a.e, $f_m^{\theta,\mu}(x) = \tilde{f}_m^{\theta,\mu}(x)$ for large enough m. Then $\tilde{f}_m^{\theta,\mu} \overset{\lambda_d \text{ a.e.}}{\longrightarrow} f_{\theta,\mu}$ when $m \to +\infty$. Since in addition $A_{k,l}^m \subset E_k^m \cap G_l^m$ we obtain

$$\tilde{f}_m^{\theta,\mu} \leq f_m^{\theta,\mu} \leq f_{\theta,\mu}. \tag{2.2.2}$$

Moreover, for every $n \geq 1$,

$$\begin{aligned} n^{s/d} \left\| X - \widehat{X}^{\alpha_n^{\theta,\mu}} \right\|_s^s &= n^{s/d} \int_{\mathbb{R}^d} d(z, \mu + \theta(\alpha_n - \mu))^s f(z) \lambda_d(dz) \\ &= n^{s/d} \int_{\mathbb{R}^d} \min_{a\in\alpha_n} |z - (\mu + \theta(a-\mu))|^s f(z) \lambda_d(dz) \\ &= \theta^s n^{s/d} \int_{\mathbb{R}^d} \min_{a\in\alpha_n} |(z-\mu)/\theta + \mu - a|^s f(z) \lambda_d(dz). \end{aligned}$$

Making the change of variable $x := (z-\mu)/\theta + \mu$ yields:

$$\begin{aligned} n^{s/d} \left\| X - \widehat{X}^{\alpha_n^{\theta,\mu}} \right\|_s^s &= \theta^{s+d} n^{s/d} \int_{\mathbb{R}^d} d(x, \alpha_n)^s f_{\theta,\mu}(x) \lambda_d(dx) \\ &\geq \theta^{s+d} n^{s/d} \int_{\mathbb{R}^d} d(x, \alpha_n)^s \tilde{f}_m^{\theta,\mu} \lambda_d(dx) \qquad \text{(by (2.2.2))} \\ &= \theta^{s+d} n^{s/d} \sum_{k,l=0}^{m2^m-1} \frac{l}{2^m} \int_{A_{k,l}^m} d(x, \alpha_n)^s \lambda_d(dx). \end{aligned} \tag{2.2.3}$$

Let $m \geq 1$ and $(k,l) \in I_m$. Define the closed sets $\tilde{A}_{k,l}^m$ by $\tilde{A}_{k,l}^m = \emptyset$ if $\lambda_d(\tilde{A}_{k,l}^m) = 0$ and otherwise by

$$\tilde{A}_{k,l}^m = \{x \in \mathbb{R}^d : d(x, A_{k,l}^m) \leq \varepsilon_m\}$$

where $\varepsilon_m \in (0,1]$ is chosen so that

$$\int_{\tilde{A}_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d \leq \left(1 + 1/m\right) \int_{A_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d.$$

Since $\tilde{A}_{k,l}^m$ is compact $\left(\tilde{A}_{k,l}^m \subset B(0, m+1)\ \forall (k,l)\right)$, and

$$A_{k,l}^m \subset (\tilde{A}_{k,l}^m)_{\varepsilon_m/2} := \{x \in \mathbb{R}^d : d(x, A_{k,l}^m) \leq \varepsilon_m/2\} = \{x \in \mathbb{R}^d : d(x, (\tilde{A}_{k,l}^m)^c) > \varepsilon_m/2\},$$

there is (ref. [DGLP], *Lemma 4.3*) a finite " firewall" set $\beta_{k,l}^m$ such that $\forall n \geq 1,\ \forall x \in (\tilde{A}_{k,l}^m)_{\varepsilon_m/2}$,

$$d(x, \alpha_n \cup \beta_{k,l}^m) = d(x, (\alpha_n \cup \beta_{k,l}^m) \cap \tilde{A}_{k,l}^m).$$

This last equality holds in particular for every $x \in A_{k,l}^m$ since $A_{k,l}^m \subset (\tilde{A}_{k,l}^m)_{\varepsilon_m/2}$.

Now set $\beta^m = \bigcup_{k,l} \beta_{k,l}^m$ and $n_{k,l}^m = \text{card}((\alpha_n \cup \beta^m) \cap \tilde{A}_{k,l}^m)$. The empirical measure theorem (see $(2.5.3)$) yields

$$\limsup_n \frac{\text{card}(\alpha_n \cap \tilde{A}_{k,l}^m)}{n} \leq \frac{\int_{\tilde{A}_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d}{\int f^{\frac{d}{d+r}} d\lambda_d}.$$

Moreover

$$\frac{n_{k,l}^m}{n} \sim \frac{\text{card}(\alpha_n \cap \tilde{A}_{k,l}^m)}{n} \qquad \text{when } n \to +\infty$$

then

$$\liminf_{n\to+\infty} \frac{n}{n_{k,l}^m} \geq \frac{\int f^{\frac{d}{d+r}} d\lambda_d}{\int_{\tilde{A}_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d} \geq \frac{m}{m+1} \frac{\int f^{\frac{d}{d+r}} d\lambda_d}{\int_{A_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d}. \tag{2.2.4}$$

On the other hand,

$$\begin{aligned}
\int_{A_{k,l}^m} d(x, \alpha_n)^s \lambda_d(dx) &\geq \int_{A_{k,l}^m} d(x, (\alpha_n \cup \beta_{k,l}^m) \cap \tilde{A}_{k,l}^m)^s \lambda_d(dx) \\
&= \lambda_d(A_{k,l}^m) \int d(x, (\alpha_n \cup \beta_{k,l}^m) \cap \tilde{A}_{k,l}^m)^s \mathbf{1}_{A_{k,l}^m}(x) \frac{\lambda_d(dx)}{\lambda_d(A_{k,l}^m)} \\
&\geq \lambda_d(A_{k,l}^m) e^s_{n_{k,l}^m, s}(U(A_{k,l}^m)),
\end{aligned}$$

where $U(A) = \mathbf{1}_A/\lambda_d(A)$ denotes the uniform distribution in the Borel set A when $\lambda_d(A) \neq 0$. Then we can write for every $(k,l) \in I_m$,

$$\liminf_{n\to+\infty} n^{s/d} \int_{A_{k,l}^m} d(x, \alpha_n)^s \lambda_d(dx) \geq \lambda_d(A_{k,l}^m) \liminf_n \left(\frac{n}{n_{k,l}^m}\right)^{s/d} \liminf_n n^{s/d} e^s_{n,s}(U(A_{k,l}^m)),$$

since

$$\liminf_n n^{s/d} e^s_{n,s}(U(A_{k,l}^m)) \geq J_{s,d} \cdot \lambda_d(A_{k,l}^m)^{s/d}.$$

Owing to Equation $(2.2.4)$, one has

$$\liminf_{n\to+\infty} n^{s/d} \int_{A_{k,l}^m} d(x, \alpha_n)^s \lambda_d(dx) \geq \lambda_d(A_{k,l}^m) \left(\frac{m}{m+1} \frac{\int f^{\frac{d}{d+r}} d\lambda_d}{\int_{A_{k,l}^m} f^{\frac{d}{d+r}} d\lambda_d}\right)^{s/d} J_{s,d} \cdot \lambda_d(A_{k,l}^m)^{s/d}.$$

However, $f < \frac{k+1}{2^m}$ on E_k^m (which still hold on $A_{k,l}^m$ because $A_{k,l}^m \subset E_k^m$). Hence

$$\liminf_{n\to+\infty} n^{s/d} \int_{A_{k,l}^m} d(x, \alpha_n)^s \lambda_d(dx) \geq J_{s,d} \left(\frac{m}{m+1} \int f^{\frac{d}{d+r}} \lambda_d(dx)\right)^{s/d} \left(\frac{k+1}{2^m}\right)^{-\frac{d}{d+r} \cdot \frac{s}{d}} \lambda_d(A_{k,l}^m).$$

It follows from Inequality $(2.2.3)$ and the super-additivity of the liminf that for every $m \geq 1$,

$$\begin{aligned}\liminf_n n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &\geq \theta^{s+d} J_{s,d} \left(\frac{m}{m+1}\int f^{\frac{d}{d+r}} \lambda_d(dx)\right)^{s/d} \sum_{k,l=0}^{m2^m-1} \frac{l}{2^m}\left(\frac{k+1}{2^m}\right)^{-\frac{s}{d+r}} \lambda_d(A_{k,l}^m) \\ &\geq \theta^{s+d} J_{s,d} \left(\frac{m}{m+1}\int f^{\frac{d}{d+r}} \lambda_d(dx)\right)^{s/d} \int_{\{f>0\}} \tilde{f}_m^{\theta,\mu}(\tilde{f}_m + 2^{-m})^{-\frac{s}{d+r}} d\lambda_d.\end{aligned}$$

Finally, applying Fatou's Lemma yields

$$\liminf_{n\to+\infty} n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \geq \theta^{s+d} J_{s,d} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d\right)^{s/d} \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d.$$

□

2.3 Upper estimate

Let $r, s > 0$. Let $(\alpha_n)_{n\geq 1}$ be an (asymptotically) $L^r(P)$ - optimal sequence of quantizers. In this section we will provide some sufficient conditions of $L^s(P)$-rate-optimality for the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$.

Definition 2.3.1. *Let $\theta > 0,\ \mu \in \mathbb{R}^d$ and let P be a probability distribution such that $P = f \cdot \lambda_d$. The couple (θ, μ) is said P-**admissible** if*

$$\{f > 0\} \subset \mu(1-\theta) + \theta\{f > 0\} \qquad \lambda_d\text{-a.e.} \tag{2.3.1}$$

One remarks that when $\text{supp}(P) = \mathbb{R}^d$ then every couple (θ, μ) is P-admissible. Indeed, almost every $x \in \mathbb{R}^d$ can be written $x = \mu(1-\theta) + \theta z$ with $z = \frac{x-\mu(1-\theta)}{\theta}$ and $f(z) > 0$.

Theorem 2.3.1. *Let $r, s \in (0, +\infty), s < r$ and let X be a random variable taking values in $\mathbb{R}^d$ with distribution P such that $P = f \cdot \lambda_d$. Suppose that (θ, μ) is P-admissible for some $\theta > 0; \mu \in \mathbb{R}^d$, and $\mathbb{E}|X|^{r+\eta} < \infty$, for some $\eta > 0$. Let $(\alpha_n)_{n\geq 1}$ be an asymptotically L^r-optimal sequence of n-quantizers. If*

$$\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d < +\infty \tag{2.3.2}$$

then, $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is $L^s(P)$-rate-optimal and

$$\limsup_{n\to+\infty} n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq \theta^{s+d} \left(Q_r(P)\right)^{s/r} \left(\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d\right)^{1-\frac{s}{r}}. \tag{2.3.3}$$

Remark 2.3.1. Note that if $\theta = 1$ and $\mu = 0$ then

$$\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d = \int_{\{f>0\}} f^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d = \int_{\{f>0\}} f d\lambda_d = 1.$$

Which gives the expected result since $\|X - \widehat{X}^{\alpha_n}\|_s \leq \|X - \widehat{X}^{\alpha_n}\|_r$.

Proof. Let P^θ denote the distribution of the random variable θX. P^θ is absolutely continuous with respect to λ_d, with $p.d.f\ g_\theta(x) = \theta^{-d} f(\frac{x}{\theta})$.

For every $n \geq 1$,

$$\begin{aligned} n^{s/d}\, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d} \int_{\mathbb{R}^d} d(x, \alpha_n^{\theta,\mu})^s dP(x) \\ &= n^{s/d} \int_{\{f>0\}} \min_{a\in\alpha_n} |x - \mu(1-\theta) - \theta a|^s f(x) d\lambda_d(x). \end{aligned}$$

Making the change of variable $z := x - \mu(1-\theta)$ gives

$$\begin{aligned} n^{s/d}\, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d} \int_{\{f>0\}-\mu(1-\theta)} d(z, \theta\alpha_n)^s f(z + \mu(1-\theta)) d\lambda_d(z) \\ &\leq n^{s/d} \int_{\theta\{f>0\}} d(z, \theta\alpha_n)^s f(z + \mu(1-\theta)) g_\theta^{-1}(z) dP^\theta(z) \qquad (2.3.4) \\ &\leq n^{s/d} \left(\int_{\mathbb{R}^d} d(z, \theta\alpha_n)^r dP^\theta(z) \right)^{s/r} \left(\int_{\theta\{f>0\}} \left(f(z + \mu(1-\theta)) g_\theta^{-1}(z) \right)^{\frac{r}{r-s}} dP^\theta(z) \right)^{\frac{r-s}{r}} \\ &\leq \left(n^{r/d} \|\theta X - \widehat{\theta X}^{\theta\alpha_n}\|_r^r \right)^{s/r} \left(\int_{\theta\{f>0\}} f(z + \mu(1-\theta))^{\frac{r}{r-s}} g_\theta^{-\frac{s}{r-s}}(z) d\lambda_d(z) \right)^{\frac{r-s}{r}} \end{aligned}$$

where we used the P-admissibility of (θ, μ) in the first inequality. The second inequality is derived from *Hölder* inequality applied with $p = r/s > 1$ and $q = 1 - s/r$.

Moreover

$$\|\theta X - \widehat{\theta X}^{\theta\alpha_n}\|_r^r = \mathbb{E}\big(\min_{a\in\alpha_n} |\theta X - \theta a|^r \big) = \theta^r \|X - \widehat{X}^{\alpha_n}\|_r^r. \qquad (2.3.5)$$

Then

$$n^{s/d}\, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq \theta^s \left(n^{r/d} \|X - \widehat{X}^{\alpha_n}\|_r^r \right)^{s/r} \left(\int_{\theta\{f>0\}} f(z + \mu(1-\theta))^{\frac{r}{r-s}} g_\theta^{-\frac{s}{r-s}}(z) d\lambda_d(z) \right)^{\frac{r-s}{r}}.$$

Owing to the asymptotically $L^r(P)$-optimality of (α_n) and making again the change of variable $x := z/\theta$ yields

$$\begin{aligned} \limsup_{n\to+\infty} n^{s/d}\, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &\leq \theta^s \left(Q_r(P)\right)^{s/r} \left(\theta^{\frac{ds}{r-s}} \int_{\theta\{f>0\}} f(z + \mu(1-\theta)))^{\frac{r}{r-s}} f(z/\theta)^{-\frac{s}{r-s}} d\lambda_d(z) \right)^{\frac{r-s}{r}} \\ &= \theta^s \left(Q_r(P)\right)^{s/r} \left(\theta^{\frac{rd}{r-s}} \int_{\{f>0\}} f_{\theta,\mu}(x)^{\frac{r}{r-s}} f(x)^{-\frac{s}{r-s}} d\lambda_d(x) \right)^{\frac{r-s}{r}} \\ &= \theta^{s+d} \left(Q_r(P)\right)^{s/r} \left(\int_{\{f>0\}} f_{\theta,\mu}(x)^{\frac{r}{r-s}} f(x)^{-\frac{s}{r-s}} d\lambda_d(x) \right)^{\frac{r-s}{r}}. \end{aligned}$$

□

The next theorem provides a less accurate asymptotic upper bound than the previous one since, beyond the restriction on the distribution of X, we need now the sequence (α_n) to be (exactly) $L^r(P)$-optimal. Before giving the theorem, recall first the following result established in [GLP] and related to the maximal function $\psi_b : \mathbb{R}^d \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$\psi_b(x) = \sup_{n\geq 1} \frac{\lambda_d(B(x, bd(x, \alpha_n)))}{P(B(x, bd(x, \alpha_n)))}. \tag{2.3.6}$$

Proposition 2.3.1. *Let $b \in \big(0, 1/2\big), X \sim P$, with $P_a \neq 0$, such that $\mathbb{E}|X|^{r+\eta} < \infty$, for some $\eta > 0$. Let (α_n) be an $L^r(P)$-optimal sequence of quantizers. Then $\forall x \in \mathbb{R}^d$,*

$$\sup_{n\geq 1} n^{1/d} d(x, \alpha_n) \leq C(b)\psi_b(x)^{1/(d+r)} \tag{2.3.7}$$

where $C(b)$ denotes a real constant not depending on n.

Theorem 2.3.2. *Let $r, s \in (0, +\infty)$ and let X be a random variable taking values in $\mathbb{R}^d$ with distribution P such that $P = f \cdot \lambda_d$. Suppose that $\mathbb{E}|X|^{r+\eta} < \infty$ for some $\eta > 0$ and $P_{\theta,\mu} \ll P$ $\big($i.e $P_{\theta,\mu}$ is absolutely continuous with respect to $P\big)$ for some $\theta > 0, \mu \in \mathbb{R}^d$. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers and suppose that the maximal function defined previously satisfies*

$$\psi_b^{s/(d+r)} \in L^1(P_{\theta,\mu}) \text{ for some } b \in (0, 1/2). \tag{2.3.8}$$

Then,

$$\limsup_n n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq C(b)\, \theta^{s+d} \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d < +\infty \tag{2.3.9}$$

where $C(b)$ is a positive real constant not depending on θ, μ and n.

Notice that this theorem does not require (θ, μ) to be P-admissible.

Proof. It follows from the definition of ψ_b that (because f is a limit which is less than the sup)

$$f^{-\frac{s}{d+r}} \leq \psi_b^{\frac{s}{d+r}} \qquad P_{\theta,\mu}\text{-a.s.}$$

Then, under Assumption $(2.3.8)$,

$$\int f^{-\frac{s}{d+r}} dP_{\theta,\mu} < +\infty.$$

For all $n \geq 1$,

$$\begin{aligned} n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d} \int_{\mathbb{R}^d} d(z, \alpha_n^{\theta,\mu})^s f(z) d\lambda_d(z) \\ &= n^{s/d}\theta^s \int_{\mathbb{R}^d} \min_{a\in\alpha_n} |(z-\mu)/\theta + \mu - a|^s f(z) d\lambda_d(z) \end{aligned}$$

We make the change of variable $x := (z - \mu)/\theta + \mu$. Then

$$\begin{aligned} n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s &= n^{s/d}\theta^{s+d} \int_{\mathbb{R}^d} d(x, \alpha_n)^s f(\mu + \theta(x-\mu)) d\lambda_d(x) \\ &= n^{s/d}\theta^s \int_{\mathbb{R}^d} d(x, \alpha_n)^s dP_{\theta,\mu}(x). \end{aligned}$$

Hence, from Inequality $(2.3.7)$ and under Assumption $(2.3.8)$, we can apply the Lebesgue dominated convergence theorem to get

$$\limsup_n \int n^{s/d} d(x,\alpha_n)^s dP_{\theta,\mu}(x) \leq \int \limsup_n n^{s/d} d(x,\alpha_n)^s dP_{\theta,\mu}(x)$$

Now, it is established in [GLP] that

$$\limsup_n n^{s/d} d(\cdot,\alpha_n)^s \leq C(b) f^{-\frac{s}{d+r}}.$$

Then we finally have

$$\begin{aligned}\limsup_n \int n^{s/d} d(x,\alpha_n)^s dP_{\theta,\mu}(x) &\leq C(b) \int f^{-\frac{s}{d+r}} dP_{\theta,\mu}(x).\\ &= \theta^d C(b) \int_{\{f>0\}} f_{\theta,\mu}(x) f^{-\frac{s}{d+r}}(x) d\lambda_d(x).\end{aligned}$$

□

For a given distribution, Assumption $(2.3.8)$ is not easy to verify. But when $s \neq r+d$, the lemma and criterions below provide a sufficient condition so that Assumption $(2.3.8)$ is satisfied. In the rest of this section we extend some of the results obtained in [GLP].

Let $P = f \cdot \lambda_d$ be an absolutely continuous distribution. Let $r, s \in (0, +\infty)$ and (θ, μ) be a P-admissible couple of parameters. We will need the following hypotheses:

(H1) for all $M > 0$,

$$\sup_{z \in B(0,M)} \frac{f(\mu + \theta(z-\mu))}{f(z)} \mathbf{1}_{\{f(z)>0\}} < +\infty. \tag{2.3.10}$$

(H2) There exists $b \in (0, 1/2),\ M \in (0, +\infty)$ such that

$$\int_{B(0,M)^c} \left(\sup_{t \leq 2b|x|} \frac{\lambda_d(B(x,t))}{P(B(x,t))} \right)^{s/(d+r)} dP_{\theta,\mu} < +\infty. \tag{2.3.11}$$

(H3) $\lambda_d(\cdot \cap \mathrm{supp}(P)) \ll P$ and $\mathrm{supp}(P)$ is a finite union of closed convex sets.

Lemma 2.3.1. *Let $P = f \cdot \lambda_d$ and $r > 0$ such that $\int |x|^r P(dx) < +\infty$. Assume $(\alpha_n)_{n\geq 1}$ is a sequence of quantizers such that $\int d(x,\alpha_n)^r dP \to 0$. Let (θ, μ) be a P-admissible couple of parameters for which (**H1**) holds.*

(a) If $p \in (0,1)$ then for every $b > 0$, $\psi_b^p \in L^1_{loc}(P_{\theta,\mu})$.

*(b) If $p \in (1, +\infty]$ and if furthermore **(H3)** holds then for every $b > 0$,*

$$f^{-p} \in L^1_{loc}(P) \Longrightarrow \psi_b^p \in L^1_{loc}(P_{\theta,\mu}).$$

Proof. It follows from the P-admissibility of (θ, μ) that

$$P_{\theta,\mu}(dz) = \theta^d f(\mu + \theta(z-\mu)) \lambda_d(dz) = g_\theta(z) P(dz),$$

where $g_\theta(z) = \theta^d \frac{f(\mu+\theta(z-\mu))}{f(z)} \mathbf{1}_{\{f(z)>0\}}$. Then g_θ is locally bounded by (**H1**).

(a) If $p \in (0,1)$, it follows from Lemma 1 in [GLP] that $\psi_b^p \in L^1_{\mathrm{loc}}(P)$. Hence $\psi_b^p \in L^1_{\mathrm{loc}}(P_{\theta,\mu})$ since g_θ is locally bounded.

(b) If $p \in (1, +\infty)$ it follows from Lemma 2 in [GLP] that if $f^{-p} \in L^1_{\mathrm{loc}}(P)$ then $\psi_b^p \in L^1_{\mathrm{loc}}(P_{\theta,\mu})$ since g_θ is locally bounded. □

Corollary 2.3.1. *(Distributions with unbounded support) Let $r > 0,\ s \in (0, +\infty),\ s \neq r + d$ and let X be a random variable with probability measure $P = f \cdot \lambda_d$ such that $E|X|^{r+\eta} < +\infty$ for some $\eta > 0$. Let (θ, μ) be P-admissible and suppose that **(H1)**, **(H2)** hold.*

(a) If $s \in (0, r + d)$ then Assumption $(2.3.8)$ of Theorem $2.3.2$ holds true.

*(b) If $s \in (r + d, +\infty)$, and if furthermore, **(H3)** holds and $f^{-\frac{s}{r+d}} \in L^1_{loc}(P)$ then Assumption $(2.3.8)$ of Theorem $2.3.2$ holds true.*

Proof. Let $x_0 \in \text{supp}(P)$. We know from [DGLP] that $d(x_0, \alpha_n) \to 0$. Then following the lines of the proof of Corollary 2 in [GLP] one has for $|x| > N = |x_0| + \sup_{n\geq 1} d(x_0, \alpha_n),\ d(x, \alpha_n) \leq 2|x|$ for every $n \geq 1$. Thus for every $b > 0,\ x \in B(0, N)^c$,

$$\psi_b(x) \leq \sup_{t\leq 2b|x|} \frac{\lambda_d(B(x,t))}{P(B(x,t))}.$$

Now, coming back to the core of our proof, it follows from **(H2)** that (for b coming from **(H2)**),

$$\int_{B(0,M\vee N)^c} \psi_b^{s/(d+r)} dP_{\theta,\mu} < +\infty.$$

Since

$$\int \psi_b^{s/(d+r)} dP_{\theta,\mu} = \int_{B(0,M\vee N)} \psi_b^{s/(d+r)} dP_{\theta,\mu} + \int_{B(0,M\vee N)^c} \psi_b^{s/(d+r)} dP_{\theta,\mu},$$

it remains to show that the first term in the right hand side of this last equality is finite.

(a) If $s \in (0, r + d)$ it follows from Lemma $2.3.1,\ (a)$ that the first term in the right hand side of the above equality is finite. As a consequence, $\psi^{\frac{s}{r+d}} \in L^1(P_{\theta,\mu})$.

(b) If $s > r + d$, the first term in the right hand side of the above equality is still finite owing to Lemma $2.3.1,\ (b)$. Consequently, Assumption $(2.3.8)$ of Theorem $2.3.2$ holds true provided **(H3)** holds and $f^{-\frac{s}{r+d}} \in L^1_{loc}(P)$. □

We next give two useful criteria ensuring that Hypothesis **(H2)** holds. The first one is useful for distributions with radial tails and the second one for distributions which do not satisfy this last assumption.

Criterion 2.3.1. *Let $X \sim P$. Suppose that $P = f \cdot \lambda_d$ and $E|X|^{r+\eta} < +\infty$ for some $\eta > 0$.*
(a) Let $r, s > 0$ and $f = h(|\cdot|)$ on $B_{|\cdot|}(0, N)^c$ with $h : (R, +\infty) \to \mathbb{R}_+,\ R \in \mathbb{R}_+$, a decreasing function and $|\cdot|$ any norm on $\mathbb{R}^d$. Suppose that (θ, μ) is a couple of P-admissible parameters such that

$$\int f(cx)^{-\frac{s}{d+r}} dP_{\theta,\mu}(x) < +\infty \tag{2.3.12}$$

*for some $c > 1$. Then **(H2)** holds.*
*(b) Let $r, s > 0$. Suppose supp$(P) \subset [R_0, +\infty)$ for some $R_0 \in \mathbb{R}$ and $f_{|(R_0', +\infty)}$ is decreasing for $R_0' \geq R_0$. Assume furthermore that (θ, μ) is a couple of P-admissible parameters such that $(2.3.12)$ is satisfied for some $c > 1$. Then Hypothesis **(H2)** holds.*

Note that (b) follows from (a) for $d = 1$ and that (a) is simply deduced from the proof of Corollary 3 in [GLP] since it has been shown that for $b \in (0, 1/2),\ M := N/(1-2b)$ one has for every $x \in B(0,M)^c$,

$$\sup_{t \le 2b|x|} \frac{\lambda_d(B(x,t))}{P(B(x,t))} \le \frac{1}{f(x(1+2b))}.$$

Criterion 2.3.2. *Let $r, s > 0,\ P = f \cdot \lambda_d$ and $\int |x|^{r+\eta} P(dx) < +\infty$ for some $\eta > 0$. Let (θ, μ) be a P-admissible couple such that*

$$\sup_{z \neq 0} \frac{f(\mu + \theta(z-\mu))}{f(z)} \mathbf{1}_{\{f(z)>0\}} < +\infty. \tag{2.3.13}$$

Assume furthermore that

$$\inf_{x \in supp(P), \rho > 0} \frac{\lambda_d(supp(P) \cap B(x,\rho))}{\lambda_d(B(x,\rho))} > 0$$

and that f satisfies the local growth control assumption : there exists real numbers $\varepsilon \ge 0,\ \eta \in (0, 1/2),\ M, C > 0$ such that

$$\forall x, y \in supp(P),\ |x| \ge M,\ |y-x| \le 2\eta|x| \implies f(y) \ge C f(x)^{1+\varepsilon}.$$

If

$$\int f(x)^{-\frac{s(1+\varepsilon)}{d+r}} dP(x) < +\infty, \tag{2.3.14}$$

*then **(H2)** holds. If in particular f satisfies the local growth control assumption for $\varepsilon = 0$ or for every $\varepsilon \in (0, \underline{\varepsilon}]$, with $\underline{\varepsilon} > 0$, and if*

$$\int f(x)^{-\frac{s}{d+r}} dP(x) = \int_{\{f>0\}} f(x)^{1-\frac{s}{d+r}} d\lambda_d(x) < +\infty$$

*then Hypothesis **(H2)** holds.*

Notice that Hypothesis $(2.3.13)$ can be relaxed if we suppose that $f(x)^{-\frac{s(1+\varepsilon)}{d+r}} \in L^1(P_{\theta,\mu})$ instead of $(2.3.14)$.

The criterion follows from Corollary 4 in [GLP].

2.4 Toward a necessary and sufficient condition for $L^s(P)$-rate-optimality

Let $X \sim P$. Let us make some comments about inequalities $(2.2.1)$ and $(2.3.9)$. Note first that the moment assumption $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$, ensures that $\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d < +\infty$ (cf [GL]). Consequently, if $\int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d = +\infty$ one derives from inequality $(2.2.1)$ that

$$\lim_{n \to +\infty} n^{s/d} \, \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s = +\infty.$$

Then the sequence $(\alpha_n^{\alpha,\mu})_{n \ge 1}$ is not L^s-rate-optimal.

On the other hand if $\int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d < +\infty$ one derives from Inequality $(2.3.9)$ that $(\alpha_n^{\theta,\mu})_{n \ge 1}$ is L^s-rate-optimal. This leads to a necessary and sufficient condition so that the sequence $(\alpha_n^{\theta,\mu})_{n \ge 1}$ (in particular the sequence $(\alpha_n)_{n \ge 1}$ by taking $\theta = 1$ and $\mu = 0$) is L^s-rate-optimal.

Remark 2.4.1. Let $\mu \in \mathbb{R}^d$, $\theta, r > 0$ and let P be a probability distribution such that $P = f \cdot \lambda_d$. Assume (θ, μ) is P-admissible. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers and suppose that Assumption $(2.3.8)$ of Theorem 2.3.2 holds true. Then for every $s > 0$,

$$(\alpha_n^{\theta,\mu})_{n\geq 1} \text{ is } L^s\text{-rate-optimal} \iff \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d < +\infty. \tag{2.4.1}$$

Remark 2.4.2. If $s < r$ and if $(\alpha_n)_{n\geq 1}$ is asymptotically L^r-optimal, Inequality $(2.3.3)$ provides a sufficient condition so that the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$ is L^s-rate-optimal, which is : $\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d < +\infty$ (always satisfied by $(\alpha_n)_{n\geq 1}$ itself).

However, it follows from Hölder inequality (applied to $p = \frac{r}{r-s} > 1$ and $q = \frac{r}{s}$) that $\forall s < r$,

$$\begin{aligned}\int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d &= \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{r}} f^{\frac{sd}{r(d+r)}} d\lambda_d \\ &\leq \left(\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d\right)^{1-\frac{s}{r}} \left(\int f^{\frac{d}{d+r}} d\lambda_d\right)^{\frac{s}{r}}.\end{aligned}$$

One deduces that

$$\left\{(\theta,\mu) \text{ s. t } \int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d < +\infty\right\} \subset \left\{(\theta,\mu) \text{ s. t } \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d < +\infty\right\}. \tag{2.4.2}$$

As a consequence, if $(\alpha_n)_{n\geq 1}$ is an $L^r(P)$-optimal sequence of quantizers and if assumptions of Theorem 2.3.2 are fulfilled then for every $s < r$ we will rather use Inequality $(2.3.9)$ instead of $(2.3.3)$ to find the couple of parameters (θ, μ) so that the sequence is $L^s(P)$-rate-optimal. But If $(\alpha_n)_{n\geq 1}$ is simply asymptotically $L^r(P)$-optimal, we only have at our disposal Inequality $(2.3.3)$ to find this set of parameters.

Now, for $s \neq r$, is it possible to find a $\theta = \theta^\star$ for which the sequence $(\alpha_n^{\theta^\star,\mu})_{n\geq 1}$ is asymptotically $L^s(P)$-optimal? (when $s < r$ this is the only question of interest since we know that $(\alpha_n)_{n\geq 1}$ is $L^s(P)$-rate-optimal for every $s < r$).

Let $(\alpha_n)_{n\geq 1}$ be an (asymptotically) $L^r(P)$-optimal sequence of quantizers. For a fixed r, b and μ, we can write from inequalities $(2.3.3)$ and $(2.3.9)$:

$$\limsup_n n^{s/d} \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s \leq Q_{r,s}^{\text{Sup}}(P,\theta) \tag{2.4.3}$$

with

$$Q_{r,s}^{\text{Sup}}(P,\theta) = \begin{cases} \theta^{s+d} (Q_r(P))^{s/r} \left(\int_{\{f>0\}} f_{\theta,\mu}^{\frac{r}{r-s}} f^{-\frac{s}{r-s}} d\lambda_d\right)^{1-\frac{s}{r}} & \text{if } s < r \\ \theta^{s+d} C(b) \int_{\{f>0\}} f_{\theta,\mu} f^{-\frac{s}{d+r}} d\lambda_d & \text{if } s > r. \end{cases}$$

One knows that for a given $s > 0$, we have for all $n \geq 1$,

$$e_{n,s}^s(X) \leq \|X - \widehat{X}^{\alpha_n^{\theta,\mu}}\|_s^s.$$

Then for every $\theta > 0$,

$$Q_s(P) \leq Q_{r,s}^{\text{Sup}}(P,\theta).$$

Consequently for a fixed $s > 0$, in order to have the best estimation of Zador's constant in L^s, we must minimize over θ, the quantity $Q^{\text{Sup}}_{r,s}(P,\theta)$. In that way, we may hope to reach the sharp rate of convergence in Zador Theorem and so construct an asymptotically L^s-optimal sequence.

For μ well chosen, the examples below show that, for the Gaussian and the exponential distributions, the minimum $\theta^\star$ exists and the sequence $(\alpha_n^{\theta^\star,\mu})_{n\geq 1}$ satisfies the empirical measure theorem and is suspected to be asymptotically L^s-optimal.

2.5 Examples of distributions

Let $(\alpha_n)_{n\geq 1}$ be an (asymptotically) $L^r(P)$-optimal sequence of quantizers for a given probability distribution P and consider the sequence $(\alpha_n^{\theta,\mu})_{n\geq 1}$. For a fixed μ and s, we try to solve the following minimization problem

$$\theta^\star = \arg\min_{\theta>0} \left\{ Q^{\text{Sup}}_{r,s}(P,\theta),\ (\alpha_n^{\theta,\mu})_{n\geq 1}\ L^s(P)\text{-rate-optimal} \right\}. \tag{2.5.1}$$

In all examples, C will denote a generic real constant (not depending on θ) which may change from line to line. The choice of μ depends on the probability measure and it is not clear how to choose it. But Proposition 2.6.1 morally implies that μ must be chosen so that for every $\theta > 0$, the probability distribution $P_{\theta,\mu}$ lies in the same family of distributions as P so that for the Gamma distribution we will set $\mu = 0$. If for every $\theta > 0, \mu \in \mathbb{R}^d$, $P_{\theta,\mu}$ lies in the same family of distribution as P we will choose μ such that $\theta^\star$ does not depend on μ; which means to put $\mu = \mathbb{E}(X)$ if X is further a symmetric random variable.

2.5.1 The multivariate Gaussian distribution

Optimal dilatation and contraction

Proposition 2.5.1. *Let $r, s > 0$ and let $P = \mathcal{N}(m;\Sigma),\ m \in \mathbb{R}^d, \Sigma \in \mathcal{S}^+(d,\mathbb{R})$ (the set of positive definite real $d \times d$ matrices).*

(a) *If $(\alpha_n)_{n\geq 1}$ is an $L^r(P)$-optimal sequence of quantizers then, for $s \neq r + d$, the sequence $(\alpha_n^{\theta,m})_{n\geq 1}$ is $L^s(P)$-rate-optimal iff $\theta \in \left(\sqrt{s/(d+r)}, +\infty\right)$ and*

$$\theta^\star = \sqrt{(s+d)/(r+d)}$$

is the unique solution of (2.5.1) on the set $\left(\sqrt{s/(d+r)}, +\infty\right)$.

(b) *If $(\alpha_n)_{n\geq 1}$ is an asymptotically $L^r(P)$-optimal sequence of quantizers then, for $s \in (0, r)$, the sequence $(\alpha_n^{\theta,m})_{n\geq 1}$ is $L^s(P)$-rate-optimal if $\theta \in \left(\sqrt{s/r}, +\infty\right)$ and*

$$\theta^\star = \sqrt{(s+d)/(r+d)}\ \in (0,1)$$

is the unique solution of (2.5.1) on the set $\left(\sqrt{s/r}, +\infty\right)$.

Proof. Since the multivariate Gaussian distribution is symmetric and for every $\theta > 0$, $P_{\theta,\mu}$ is also a Gaussian random vector, one sets $\mu = m$. Keep in mind that the probability density function f of P is given for every $x \in \mathbb{R}^d$ by

$$f(x) = \left((2\pi)^d \det \Sigma\right)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-m)'\Sigma^{-1}(x-m)}.$$

Note first that Hypothesis **(H1)** is obviously satisfied from the continuity of $\frac{f(m+\theta(z-m))}{f(z)}\mathbf{1}_{\{f(z)>0\}}$ on every $\bar{B}(0,M),\ M>0$.

(a) Let $s<d+r$. For every $\theta>0, \mu\in\mathbb{R}^d$, the couple (θ,μ) is P-admissible and f is radial since $f(x)=\varphi(|x-m|_\Sigma)$ with $\varphi:(0,+\infty)\longmapsto\mathbb{R}_+$ defined by

$$\varphi(\xi)=\left((2\pi)^d\text{det}\Sigma\right)^{-1/2}\exp(-\frac{1}{2}|\xi|^2),\ \text{ with } |x|_\Sigma=|\Sigma^{-\frac{1}{2}}x|.$$

Let $\theta>\sqrt{s/(r+d)}$. Then Assumption $(3.2.33)$ holds for every $c\in(1,\theta\sqrt{\frac{r+d}{s}})$. Consequently, it follows from Corollary $2.3.1,(a)$ that Assumption $(2.3.8)$ of Theorem 2.3.2 holds.

If $s>d+r$, the required additional hypotheses **(H3)** and $f^{-\frac{s}{r+d}}\in L^1_{loc}(P)$ are clearly satisfied since if $P=f\cdot\lambda_d$ then

$$\lambda_d(\text{supp}(P)\cap\{f=0\})=0\implies\lambda_d(\cdot\cap\text{supp}(P))\ll P \tag{2.5.2}$$

and $f^{-\frac{s}{r+d}}$ is continuous on every $\bar{B}(0,M),\ M>0$. Then it follows from Corollary $2.3.1,(b)$ that Assumption $(2.3.8)$ of Theorem 2.3.2 holds.

On the other hand

$$\begin{aligned}\int_{\mathbb{R}^d}f_{\theta,m}(x)f(x)^{-\frac{s}{d+r}}dx &= \int_{\mathbb{R}^d}f(m+\theta(x-m))f(x)^{-\frac{s}{d+r}}dx\\ &= C\int_{\mathbb{R}^d}e^{-\frac{1}{2}(\theta^2-\frac{s}{d+r})(x-m)'\Sigma^{-1}(x-m)}dx\end{aligned}$$

so that

$$\int_{\mathbb{R}^d}f_{\theta,m}(x)f(x)^{-\frac{s}{d+r}}<+\infty\quad\text{iff}\quad\theta>\sqrt{\frac{s}{d+r}}.$$

Now we are in position to solve the problem $(2.5.1)$. Let $\theta\in\left(\sqrt{s/(d+r)},+\infty\right)$,

$$\begin{aligned}\theta^{s+d}\int_{\mathbb{R}^d}f_{\theta,m}(x)f(x)^{-\frac{s}{d+r}}dx &= \left((2\pi)^d det\ \Sigma\right)^{-\frac{1}{2}(1-\frac{s}{d+r})}\theta^{s+d}\int_{\mathbb{R}^d}e^{-\frac{1}{2}(\theta^2-\frac{s}{d+r})(x-m)'\Sigma^{-1}(x-m)}dx\\ &= \left((2\pi)^d det\ \Sigma\right)^{-\frac{s}{d+r}}\theta^{s+d}\left(\theta^2-\frac{s}{d+r}\right)^{-\frac{d}{2}}.\end{aligned}$$

For $\theta\in\left(\sqrt{s/(d+r)},+\infty\right)$, we want to minimize the function h defined by

$$h(\theta)=\theta^{s+d}\left(\theta^2-\frac{s}{d+r}\right)^{-\frac{d}{2}}.$$

The function h is differentiable on $\left(\sqrt{s/(d+r)},+\infty\right)$ with derivative

$$h'(\theta)=s\theta^{d+s-1}\left(\theta^2-\frac{s}{d+r}\right)^{-1-d/2}\left(\theta^2-\frac{s+d}{r+d}\right).$$

One easily checks that h reaches its unique minimum on $\left(\sqrt{s/(d+r)},+\infty\right)$ at $\theta^\star=\sqrt{(s+d)/(r+d)}$.

(b) Let $s<r$ and consider the inequality $(2.3.3)$. We get

$$\int f_{\theta,m}^{\frac{r}{r-s}}(x)f^{-\frac{s}{r-s}}(x)dx=C\int_{\mathbb{R}^d}e^{-\frac{1}{2}\frac{r}{r-s}(\theta^2-\frac{s}{r})(x-m)'\Sigma^{-1}(x-m)}dx.$$

So if $\theta \in \left(\sqrt{s/r}, +\infty\right)$ then $\int f_{\theta,m}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)dx < +\infty$. This proves the first assertion.

To prove the second assertion, let $\theta \in \left(\sqrt{s/r}, +\infty\right)$. Then

$$\begin{aligned}\theta^{d+s}\left(\int f_{\theta,m}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)dx\right)^{1-\frac{s}{r}} &= C\,\theta^{s+d}\left(\int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{r}{r-s}(\theta^2-\frac{s}{r})(x-m)'\Sigma^{-1}(x-m)}dx\right)^{1-\frac{s}{r}} \\ &= C\,\theta^{s+d}\left(\theta^2-\frac{s}{r}\right)^{-\frac{d}{2r}(r-s)}.\end{aligned}$$

We proceed as before by setting

$$h(\theta) = \theta^{\alpha}\left(\theta^2-\frac{s}{r}\right)^{\beta}, \text{ with } \alpha = d+s \text{ and } \beta = -\frac{d}{2r}(r-s).$$

For all $\theta \in \left(\sqrt{s/r}, +\infty\right)$,

$$h'(\theta) = \theta^{\alpha-1}\left(\theta^2-\frac{s}{r}\right)^{\beta-1}\left((\alpha+2\beta)\theta^2-\frac{\alpha s}{r}\right).$$

The sign of h' depends on the sign of $\left((\alpha+2\beta)\theta^2-\frac{\alpha s}{r}\right)$. Moreover $\alpha+2\beta = \frac{s}{r}(d+r) > 0$ then h' vanishes at $\theta^\star = \sqrt{(s+d)/(r+d)}$, is negative on the set $\left(\sqrt{s/r}, \theta^\star\right)$ and positive on $\left(\theta^\star, +\infty\right)$. Therefore h reaches its minimum on $\left(\sqrt{s/r}, +\infty\right)$ at the unique point $\theta^\star$. □

Definition 2.5.1. *A sequence of quantizers $(\beta_n)_{n\geq 1}$ is called a **dilatation** of the sequence $(\alpha_n)_{n\geq 1}$ with **scaling number** θ and **translating number** μ if, for every $n \geq 1$, $\beta_n = \alpha_n^{\theta,\mu}$, with $\theta > 1$. If $\theta < 1$, one defines likewise the **contraction** of the sequence $(\alpha_n)_{n\geq 1}$ with **scaling number** θ and **translating number** μ.*

From this definition follows the following remark.

Remark 2.5.1. Let $X \sim \mathcal{N}(m;\Sigma)$.
If $s < r$ then $\theta^\star < 1$. Hence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ is a contraction of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$ and translating number m. On the other hand, if $s > r$, then $\theta^\star > 1$. In this case the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ is a dilatation of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$ and translating number m. Also note that $\theta^\star$ does not depend on the covariance matrix Σ.

What we do expect from the resulting sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$? Before giving any answer to this question let us recall first the empirical measure theorem (see [GL]) which gives the asymptotic distribution of the empirical measure induced by an asymptotically L^r-optimal sequence of quantizers.

Theorem 2.5.1. *(Empirical measure theorem) Let $X \sim P$, with $P_a \neq 0$, and let $(\alpha_n)_{n\geq 1}$ be an asymptotically $L^r(P)$-optimal sequence of quantizers. Then*

$$\frac{1}{n}\sum_{a\in\alpha_n}\delta_a \xrightarrow{w} P_r \tag{2.5.3}$$

where $\xrightarrow{w}$ denotes the weak convergence and for every Borel set A of $\mathbb{R}^d$, P_r is defined by

$$P_r(A) = \frac{1}{C_{f,r}}\int_A f(x)^{\frac{d}{d+r}}d\lambda_d(x), \text{ with } \quad C_{f,r} = \int_{\mathbb{R}^d} f(x)^{\frac{d}{d+r}}d\lambda_d(x). \tag{2.5.4}$$

A sequence of quantizers $(\alpha_n)_{n\geq 1}$ will be said to satisfy **the L^r-empirical measure theorem** if $(2.5.3)$ holds. The next proposition shows that the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ satisfies the L^s-empirical measure theorem.

Proposition 2.5.2. *Let $r,s>0$ and let $P=\mathcal{N}(m;\Sigma)$. Assume $(\alpha_n)_{n\geq 1}$ is asymptotically $L^r(P)$-optimal. Then the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ (as defined before with $\theta^\star=\sqrt{(s+d)/(r+d)}$) satisfies the L^s-empirical measure theorem.*

In other words, for every $a,b\ \in\mathbb{R}^d$,

$$\frac{1}{n}\mathit{card}(\{x\in\alpha_n^{\theta^\star,m}\cap[a,b]\})\longrightarrow\frac{1}{C_{f,s}}\int_{[a,b]}f(x)^{\frac{d}{d+s}}dx.$$

Proof. For all $n\geq 1$,

$$\{x\in\alpha_n^{\theta^\star,m}\cap[a,b]\}=\{x\in\alpha_n\cap[(a-m)/\theta^\star+m,(b-m)/\theta^\star]+m\}.$$

Since $(\alpha_n)_{n\geq 1}$ is asymptotically L^r-optimal, applying the empirical measure theorem to the sequence $(\alpha_n)_{n\geq 1}$ yields

$$\frac{1}{n}\mathrm{card}(\{x\in\alpha_n\cap[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]\})\longrightarrow\frac{1}{C_{f,r}}\int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]}f(x)^{\frac{d}{d+r}}dx.$$

It remains to verify that

$$\frac{1}{C_{f,r}}\int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]}f(x)^{\frac{d}{d+r}}dx=\frac{1}{C_{f,s}}\int_{[a,b]}f(x)^{\frac{d}{d+s}}dx.$$

One knows that

$$f(x)=\big((2\pi)^d\det\Sigma\big)^{-\frac{1}{2}}e^{-\frac{1}{2}(x-m)'\Sigma^{-1}(x-m)}$$

and $\big($see $(2.5.4)\big)$

$$C_{f,r}=\int_{\mathbb{R}^d}f(x)^{\frac{d}{d+r}}dx$$

so that for all $r>0$,

$$C_{f,r}=\big((2\pi)^d\det\Sigma\big)^{\frac{r}{2(r+d)}}\left(\frac{d+r}{d}\right)^{\frac{d}{2}}.$$

By making the change of variable $x=m+\theta^\star(z-m)$, one gets :

$$\frac{1}{C_{f,r}}\int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]}f(z)^{\frac{d}{d+r}}dz\ \ =\ \ \frac{1}{C_{f,r}}(\theta^\star)^{-d}\int_{[a,b]}f((x-m)/\theta^\star+m)^{\frac{d}{d+r}}dx.$$

It is easy to check that

$$\big(f((x-m)/\theta^\star+m\big)^{\frac{d}{d+r}}=\big(f(x)\big)^{\frac{d}{d+s}}\big((2\pi)^d\det\Sigma\big)^{-\frac{1}{2}(\frac{d}{d+r}-\frac{d}{d+s})}$$

and that

$$\frac{1}{C_{f,r}}(\theta^\star)^{-d}\big((2\pi)^d\det\Sigma\big)^{-\frac{1}{2}(\frac{d}{d+r}-\frac{d}{d+s})}=\big((2\pi)^d\det\Sigma\big)^{-\frac{s}{2(s+d)}}\left(\frac{d+s}{d}\right)^{-\frac{d}{2}}.$$

The last term is simply equal to $\frac{1}{C_{f,s}}$. We then deduce that

$$\frac{1}{C_{f,r}}\int_{[(a-m)/\theta^\star+m,(b-m)/\theta^\star+m]} f(x)^{\frac{d}{d+r}}dx = \frac{1}{C_{f,s}}\int_{[a,b]} f(x)^{\frac{d}{d+s}}dx.$$

□

We have just built a sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ satisfying the empirical measure theorem. The question of interest is now to know whether or not this sequence is asymptotically L^s-optimal. The following proposition shows that the lower bound in $(2.2.1)$ is in fact reached by considering the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$.

Proposition 2.5.3. *Let $s > 0$ and let $\theta = \theta^\star = \sqrt{(s+d)/(r+d)}$. Then, the constant in the asymptotic lower bound for the L^s-error induced by the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ (see $(2.2.1)$) satisfies :*

$$Q_{r,s}^{Inf}(P,\theta^\star) = Q_s(P). \tag{2.5.5}$$

Proof. Keep in mind that if $P \sim \mathcal{N}(m;\Sigma)$ then for all $r > 0$,

$$\big(Q_r(P)\big)^{1/r} = \big(J_{r,d}\big)^{1/r}\sqrt{2\pi}\left(\frac{d+r}{d}\right)^{\frac{d+r}{2r}}\big(\det\Sigma\big)^{\frac{1}{2d}}.$$

We have on one hand

$$\begin{aligned}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}(x)d(x)\right)^{s/d} &= \left(\big((2\pi)^d\det\Sigma\big)^{-\frac{1}{2}\frac{d}{d+r}}\int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{d}{d+r}(x-m)'\Sigma^{-1}(x-m)}dx\right)^{s/d}\\ &= \left(\big((2\pi)^d\det\Sigma\big)^{\frac{1}{2}\frac{r}{d+r}}\big(\frac{d+r}{d}\big)^{\frac{d}{2}}\right)^{s/d}\end{aligned}$$

and on the other hand

$$\begin{aligned}\int_{\mathbb{R}^d} f_{\theta^\star,\mu}(x)f^{-\frac{s}{d+r}}(x)d(x) &= \big((2\pi)^d\det\Sigma\big)^{-\frac{1}{2}-\frac{s}{d+r}}\int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{d}{d+r}(x-m)'\Sigma^{-1}(x-m)}dx\\ &= \big((2\pi)^d\det\Sigma\big)^{-\frac{s}{d+r}}\big(\frac{d+r}{d}\big)^{\frac{d}{2}}.\end{aligned}$$

Combining these two results yields

$$\begin{aligned}Q_{r,s}^{\text{Inf}}(P,\theta^\star) &= (\theta^\star)^{s+d}J_{s,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}d\lambda_d\right)^{s/d}\int_{\mathbb{R}^d} f_{\theta^\star,\mu}f^{-\frac{s}{d+r}}d\lambda_d\\ &= J_{s,d}\left(\frac{s+d}{r+d}\right)^{\frac{d+s}{2}}\big((2\pi)^d\det\Sigma\big)^{\frac{s}{2d}}\left(\frac{r+d}{d}\right)^{\frac{d+s}{2}}\\ &= J_{s,d}\left(\frac{s+d}{d}\right)^{\frac{d+s}{2}}\big((2\pi)^d\det\Sigma\big)^{\frac{s}{2d}}\\ &= Q_s(P).\end{aligned}$$

□

After some elementary calculations, it follows from the proposition above and inequalities $(2.2.1)$,$(2.4.3)$, the corollary below :

Corollary 2.5.1. *Let $X \sim \mathcal{N}(m;\Sigma)$ and $\theta^\star = \sqrt{(s+d)/(r+d)}$. Then,*

$$Q_s(P)^{1/s} \le \liminf_{n\to\infty} n^{1/d}\,\|X-\widehat{X}^{\alpha_n^{\theta^\star,m}}\|_s \le \limsup_{n\to\infty} n^{1/d}\,\|X-\widehat{X}^{\alpha_n^{\theta^\star,m}}\|_s \le Q_{r,s}^{Sup}(P,\theta^\star)^{1/s} \quad (2.5.6)$$

with

$$Q_{r,s}^{Sup}(P,\theta^\star)^{1/s} = \begin{cases} \left(\frac{s+d}{d}\right)^{\frac{s+d}{2s}} J_{r,d}^{\frac{1}{r}} \left((2\pi)^d det\,\Sigma\right)^{\frac{1}{2d}} & \text{if } s<r \\ \left(\frac{s+d}{d}\right)^{\frac{d}{2}} \sqrt{\frac{s+d}{r+d}}\, C(b)\, \left((2\pi)^d det\,\Sigma\right)^{\frac{1}{2(d+r)}} & \text{if } s>r. \end{cases}$$

Remark 2.5.2. (a) If $s>r$, we cannot prove the asymptotically $L^s(P)$-optimality of $(\alpha_n^{\theta^\star,m})_{n\ge1}$ using $(2.3.9)$ since the constant $C(b)$ is not explicit.

(b) When $s<r$, the corollary above shows that the upper bound in $(2.3.3)$ does not reach the Zador constant. Then our upper estimate does not allow us to show that the sequence $(\alpha_n^{\theta^\star,m})_{n\ge1}$ is asymptotically $L^s(P)$-optimal.

Moreover, using $H\ddot{o}lder$ inequality (with $p=r/(r-s)$ and $q=r/s$), we have for every $\theta>0$,

$$\begin{aligned}\int_{\mathbb{R}^d} f_{\theta,\mu}(x) f^{-\frac{s}{d+r}}(x)d\lambda_d(x) &= \int_{\mathbb{R}^d} f_{\theta,\mu}(x) f^{-s/r}(x) f^{\frac{sd}{r(d+r)}}(x)d\lambda_d(x) \\ &\le \left(\int_{\mathbb{R}^d} f_{\theta,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)d\lambda_d(x)\right)^{\frac{r-s}{r}} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}(x)d\lambda_d\right)^{\frac{s}{r}}.\end{aligned}$$

and (for $\theta=\theta^\star$)

$$\int_{\mathbb{R}^d} f_{\theta^\star,\mu}(x) f^{-\frac{s}{d+r}}(x)d\lambda_d(x) = \left(\int_{\mathbb{R}^d} f_{\theta^\star,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)d\lambda_d(x)\right)^{\frac{r-s}{r}} \left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}}(x)d\lambda_d\right)^{\frac{s}{r}}. \quad (2.5.7)$$

Hence, according to $(2.5.5)$, one gets for every $s<r$,

$$(\theta^\star)^{s+d} J_{s,d} \left(\int_{\mathbb{R}^d} f_{\theta^\star,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)d\lambda_d(x)\right)^{\frac{r-s}{r}} \|f\|_{\frac{d}{d+r}}^{s/r} = Q_s(P). \quad (2.5.8)$$

Thus, to reach the Zador constant in $(2.3.3)$ we must rather have $J_{s,d}$ instead of $J_{r,d}$ (which will be coherent since for all $s<r$, $J_{s,d}^{1/s} \le J_{r,d}^{1/r}$), that is,

$$\limsup_{n\to\infty} n^{1/d}\,\|X-\widehat{X}^{\alpha_n^{\theta,\mu}}\|_s \le \theta^{s+d} J_{s,d} \left(\int_{\mathbb{R}^d} f_{\theta,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x)d\lambda_d(x)\right)^{\frac{r-s}{r}} \|f\|_{\frac{d}{d+r}}^{s/r}.$$

Numerical experiments

For numerical example, supppose that $d=1$ and $r\in\{1,2,4\}$. Let $X\sim\mathcal{N}(0,1)$ and, for a fixed n, let $\alpha_n^{(r)} = \{\alpha_{n1}^{(r)},\cdots,\alpha_{nn}^{(r)}\}$ be the L^r-optimal grid of size n (obtained by a Newton-Raphson zero search). For every $n\in\{20,50,\cdots,900\}$ and for $(s,r)=(1,2)$ and $(4,2)$, we make a linear regression of $\alpha_n^{(r)}$ onto $\alpha_n^{(s)}$:

$$\alpha_{ni}^{(s)} \simeq \hat{a}_{sr}\alpha_{ni}^{(r)} + \hat{b}_{sr}, \quad i=1,\cdots,n.$$

Table 2.1 provides the regression coefficients we obtain for different values of n. We note that when n increases, the coefficients $\hat{a}_{sr}$ tend to the value $\sqrt{(s+1)/(r+1)} = \theta^\star$ whereas the coefficients $\hat{b}_{sr}$ almost vanish. For example, for $n = 900$ and for $(r, s) = (2, 1)$ (resp. $(2, 4)$) we get $\hat{a}_{sr} = 0.8170251$ (resp. 1.2900417). The expected values are $\sqrt{2/3} = 0.8164966$ (resp. $\sqrt{5/3} = 1.2909944$). The absolute errors are then 5.285×10^{-4} $\big($resp. $9.527 \times 10^{-4}\big)$. We remark that the error mainly stems from the tail of the distribution.

n	$\hat{a}_{12}$	$\hat{b}_{12}$	ϵ		$\hat{a}_{42}$	$\hat{b}_{42}$	ϵ
20	0.8250096	1.826E-14	0.0003025		1.2761027	- 3.650E-12	0.0008607
50	0.8211387	- 1.021E-13	0.0006870		1.2828110	3.733E-10	0.0020110
100	0.8193424	8.693E-14	0.0009909		1.2859567	4.059E-09	0.0029445
300	0.8177506	- 1.045E-11	0.0013601		1.2887640	0.0000004	0.0041021
700	0.8171428	- 7.219E-11	0.0015111		1.2898393	- 0.0000089	0.0048006
800	0.8170775	- 6.725E-11	0.0015247		1.2900041	0.0000216	0.0040577
900	0.8170251	4.564E-11	0.0015346		1.2900417	- 0.0000141	0.0048182

Table 2.1: Regression coefficients for the Gaussian.

The previous numerical results, in addition to Equation $(2.5.5)$, strongly suggest that the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ is in fact asymptotically $L^s(P)$-optimal. This leads to the following conjecture.

Conjecture. *Let $P \sim \mathcal{N}(m; \Sigma)$ and let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers. Then, for every $s > 0$, the sequence $(\alpha_n^{\theta^\star,m})_{n\geq 1}$ (with $\theta^\star = \sqrt{(s+d)/(r+d)}$) is asymptotically $L^s(P)$-optimal.*

2.5.2 Exponential distribution

Optimal dilatation and contraction

Proposition 2.5.4. *Let $r, s > 0$ and X be an exponentially distributed random variable with rate parameter $\lambda > 0$. Set $\mu = 0$.*

(a) *If $(\alpha_n)_{n\geq 1}$ is an $L^r(P)$-optimal sequence of quantizers then, for $s \neq r + 1$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal iff $\theta \in \big(s/(r+1), +\infty\big)$ and*

$$\theta^\star = (s+1)/(r+1)$$

is the unique solution of $(2.5.1)$ on the set $\big(s/(r+1), +\infty\big)$.

(b) *If $(\alpha_n)_{n\geq 1}$ is an asymptotically $L^r(P)$-optimal sequence of quantizers then, for $s \in (0, r)$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal for all $\theta \in \big(s/r, +\infty\big)$ and*

$$\theta^\star = (s+1)/(r+1)$$

is the unique solution of $(2.5.1)$ on $\big(s/r, +\infty\big)$.

***Proof*. (a)** Let $s < r+1$. For all $\theta > 0, \mu \in \mathbb{R}^d$, the couple (θ, μ) is P-admissible and the function f is decreasing on $(0, +\infty)$. For $\theta > s/(r+1)$, Assumption $(3.2.33)$ holds true for every $c \in \big(1, \theta(1+r)/s\big)$. Moreover, Hypothesis **(H1)** is clearly satisfied. Consequently, it follows from Corollary $2.3.1, (a)$ that Assumption $(2.3.8)$ holds true.

If $s > r+1$, Assumption $(2.3.8)$ still holds since the additional assumptions **(H3)** and $f^{-\frac{s}{r+1}} \in L^1_{loc}(P)$ required to apply the corollary $2.3.1, (b)$ are satisfied.

On the other hand, one has

$$\int_{\mathbb{R}} f(\theta x) f(x)^{-s/(r+1)} dx = C \int_0^{+\infty} e^{-\lambda(\theta - s/(r+1))x} dx < +\infty \Longleftrightarrow \theta > s/(r+1).$$

Now, let us solve the problem $(2.5.1)$. For all $\theta > s/(r+1)$,

$$\begin{aligned} h(\theta) := \theta^{s+1} \int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{r+1}} dx &= C\, \theta^{s+1} \int_0^{+\infty} e^{-\lambda(\theta - \frac{s}{r+1})x} dx \\ &= C\, \theta^{s+1} \left(\theta - \frac{s}{r+1}\right)^{-1}. \end{aligned}$$

Consequently

$$h'(\theta) = Cs\, \theta^s \left(\theta - \frac{s}{r+1}\right)^{-2} \left(\theta - \frac{s+1}{r+1}\right).$$

Hence, h reaches its unique minimum on $\big(s/(r+1), +\infty\big)$ at $\theta^\star = (s+1)/(r+1)$.

(b) Let $s < r$. We have

$$\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x) dx = C \int_{\mathbb{R}_+} e^{-x\frac{\lambda}{r-s}(r\theta - s)} dx.$$

Then, for all $\theta > s/r$, $\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x) dx < +\infty$. Which gives the first assertion.

For every $\theta > s/r$,

$$\begin{aligned} \theta^{s+1} \left(\int_{\mathbb{R}} f_{\theta,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x) dx\right)^{1-\frac{s}{r}} &= C\, \theta^{s+1} \left(\int_{\mathbb{R}_+} e^{-x\frac{\lambda}{r-s}(r\theta - s)} dx\right)^{\frac{r-s}{r}} \\ &= C\, \theta^{s+1} (r\theta - s)^{\frac{s-r}{r}}. \end{aligned}$$

We easily check that the function $h(\theta) = \theta^{s+1}(r\theta - s)^{\frac{s-r}{r}}$ reaches its minimum on $\big(s/r, +\infty\big)$ at the unique point $\theta^\star = (s+1)/(r+1)$. □

Remark 2.5.3. Let $X \sim \mathcal{E}(\lambda)$. If $s < r$, then $\theta^\star = (s+1)/(r+1) < 1$. As a consequence, the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a contraction of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$. On the other hand, if $s > r$, then $\theta^\star > 1$ and then $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a dilatation of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$. Note that $\theta^\star$ does not depend on the rate parameter λ of the exponential distribution.

One shows below that the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$, with $\theta^\star = (1+s)/(1+r)$, satisfies the L^s-empirical measure theorem.

Proposition 2.5.5. *Let $r, s > 0$ and let X be an exponentially distributed random variable with rate parameter $\lambda > 0$. Assume $(\alpha_n)_{n\geq 1}$ is an asymptotically L^r-optimal sequence of quantizers for X and let $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ be defined as before, with $\theta^\star = (s+1)/(r+1)$. Then, the sequence $(\alpha_n^{\theta^\star,0})$ satisfies the L^s-empirical measure theorem.*

Proof. Since $(\alpha_n^{\theta^\star,0})_{n\geq 1} = (\theta^\star \alpha_n)_{n\geq 1}$ it amounts to show that for every $a, b \in \mathbb{R}_+$

$$\frac{\text{card}(\alpha_n \cap [a/\theta^\star, b/\theta^\star])}{n} \longrightarrow \frac{1}{C_{f,s}} \int_a^b f(x)^{\frac{1}{1+s}} dx$$

i.e that for all $a, b \in \mathbb{R}_+$,

$$\frac{1}{C_{f,r}} \frac{1}{\theta^\star} \int_a^b f(x/\theta^\star)^{\frac{1}{1+r}} dx = \frac{1}{C_{f,s}} \int_a^b f(x)^{\frac{1}{1+s}} dx.$$

Elementary computations show that $\forall\, r > 0$,

$$C_{f,r} = \lambda^{-\frac{r}{1+r}}(1+r).$$

so that

$$\begin{aligned}
\frac{1}{C_{f,r}} \frac{1}{\theta^\star} \int_a^b f(x/\theta^\star)^{\frac{1}{1+r}} dx &= \frac{1}{C_{f,r}} \frac{1+r}{1+s} \int_a^b \left(\lambda e^{-x\lambda \frac{1+r}{1+s}}\right)^{\frac{1}{1+r}} dx \\
&= \frac{1}{C_{f,r}} \frac{1+r}{1+s} \lambda^{\frac{1}{1+r} - \frac{1}{1+s}} \int_a^b \left(\lambda e^{-\lambda x}\right)^{\frac{1}{1+s}} dx \\
&= \frac{1}{C_{f,s}} \int_a^b f(x)^{\frac{1}{1+s}} dx.
\end{aligned}$$

□

Once again, the question of interest is to know if the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is asymptotically L^s-optimal. The remark $2.5.2$ is also valid for the exponential distribution. Our upper bounds in $(2.3.3)$ and $(2.3.9)$ do not allow us to show that $(\theta^\star \alpha_n)$ is asymptotically L^s-optimal because of the corollary below. But the numerical results strongly suggest that it is.

Corollary 2.5.2. *Let $X \sim \mathscr{E}(\lambda)$ and $\theta^\star = (s+1)/(r+1)$. Then,*

$$Q_s(P)^{1/s} \leq \liminf_{n\to\infty} n^{1/d} \|X - \widehat{X}^{\alpha_n^{\theta^\star,0}}\|_s \leq \limsup_{n\to\infty} n^{1/d} \|X - \widehat{X}^{\alpha_n^{\theta^\star,0}}\|_s \leq Q_{r,s}^{\mathit{Sup}}(P, \theta^\star)^{1/s} \quad (2.5.9)$$

with

$$Q_{r,s}^{\mathit{Sup}}(P, \theta^\star)^{1/s} = \begin{cases} \frac{1}{2\lambda}(s+1)^{1+1/s}(r+1)^{-1/r} & \text{if } s < r \\ (s+1)^{1+1/s}\big((r+1)\lambda^{\frac{1}{1+r}}\big)^{-1} C(b)^{1/s} & \text{if } s > r. \end{cases}$$

Proof. We easily prove, like in proposition 2.5.2, that $Q_{r,s}^{\text{Inf}}(P, \theta^\star) = Q_s(P)$. The corollary follows then from $(2.2.1)$ and $(2.4.3)$ $\big($Note that for every $r > 0, \quad J_{r,1} = \frac{1}{(r+1)2^r}\big)$. □

Numerical experiments

We relate first a proposition established in [FP] and used in our context to compute the L^r-optimal quantizers for the exponential distribution.

Proposition 2.5.6. *Let $r > 0$ and let X be an exponentially distributed random variable with scale parameter $\lambda > 0$. Then for every $n \geq 1$, the L^r-optimal quantizer $\alpha_n^{(r)} = (\alpha_{n1}^{(r)}, \cdots, \alpha_{nn}^{(r)})$ is unique and given by*

$$\alpha_{nk}^{(r)} = \frac{a_n^{(r)}}{2} + \sum_{i=n+1-k}^{n-1} a_i^{(r)}, \quad 1 \leq k \leq n, \tag{2.5.10}$$

where $(a_k^{(r)})_{k\geq 1}$ is a $\mathbb{R}_+$-valued sequence defined by the following implicit recursive equation:

$$a_0^{(r)} := +\infty, \quad \phi_r\big(-a_{k+1}^{(r)}\big) := \phi_r\big(a_k^{(r)}\big),\ k \geq 1$$

with $\phi_r(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)e^{-u}du$ (convention : $0^0 = 1$).

Furthermore, the sequence $(a_k^{(r)})_{k\geq 1}$ decreases to zero and for every $k \geq 1$,

$$a_k^{(r)} = \frac{r+1}{k}\big(1 + \frac{c_r}{k} + O(\frac{1}{k^2})\big)$$

for some real constant c_r.

For numerical examples, Table 2.2 gives the regression coefficients we obtain by regressing the L^2 grids onto the grids we get with the L^1 and L^4 norms, for different values of n. The notations are the same as in the previous example. We note that for large enough n, the coefficients $\hat{a}_{sr}$ tend to $(s+1)/(r+1) = \theta^\star$. For example, if $n = 900$, we get $\hat{a}_{12} = 0.6676880;\ \hat{a}_{42} = 1.6640023$ whereas the expected values are respectively $2/3 = 0.66666667$ and $5/3 = 1.6666667$. The absolute errors are in the order of 10^{-3}. Like the Gaussian case, we remark that the error of the estimation results mainly from the tail of the exponential distribution.

n	$\hat{a}_{12}$	$\hat{b}_{12}$	ϵ		$\hat{a}_{42}$	$\hat{b}_{42}$	ϵ
20	0.6765013	- 0.0104881	0.0019489		1.6396807	0.0288348	3.081E-33
50	0.6726145	- 0.0082123	0.0045310		1.6502245	0.0225246	1.149E-28
100	0.6706176	- 0.0062439	0.0070734		1.6556979	0.0172020	1.573E-27
300	0.6686428	- 0.0036234	0.0114628		1.6611520	0.0100523	1.508E-27
700	0.6677864	- 0.0022222	0.0146186		1.6635261	0.0061356	1.222E-25
800	0.6676880	- 0.0020482	0.0150735		1.6638043	0.0057199	2.020E-26
900	0.6676079	- 0.0019043	0.0154634		1.6640023	0.0053173	9.683E-25

Table 2.2: Regression coefficients for exponential distribution.

Conjecture. *Let X be an exponentially distributed random variable with rate parameter λ and let $(\alpha_n)_{n\geq 1}$ be an L^r-optimal sequence of quantizers for X. Then for $s > 0$ and $\theta^\star = (s+1)/(r+1)$ the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is asymptotically L^s-optimal.*

Remark 2.5.4. As a matter of fact, the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is not (exactly) L^s-optimal for every $s \neq r$. Otherwise, if $\alpha_{nk}^{(s)} = \frac{s+1}{r+1}\alpha_{nk}^{(r)}$ for every $k \geq 1$ then if follows by backward induction that

$$\forall k \geq 1, \quad a_k^{(s)} = \frac{s+1}{r+1}a_k^{(r)}.$$

However straightforward calculations show e.g that $a_1^{(2)} = 2$ and $a_1^{(1)} = 2\log(2)$ so that

$$a_1^{(2)} \neq \frac{3}{2} a_1^{(1)}.$$

Moreover, these examples could suggest that a contraction (or a dilatation) parameter $\theta^\star$, solution of the minimization problem $(2.5.1)$, always leads to a sequence of quantizers satisfying the L^s-empirical measure theorem. The following example shows that this can fail.

2.5.3 Gamma distribution

Optimal dilatation and contraction

Proposition 2.5.7. *Let $r, s > 0$ and let P be a Gamma distribution with parameters a and λ : $P = \Gamma(a, \lambda),\ a > 0,\ \lambda > 0$.*

(a) *If $(\alpha_n)_{n\geq 1}$ is an $L^r(P)$-optimal sequence of quantizers then, for $s < r + 1$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal iff $\theta \in \big(s/(r+1), +\infty\big)$ and for all $a > 0$,*

$$\theta^\star = (s + a)/(r + a)$$

is the unique solution of $(2.5.1)$ on the set $\big(s/(r+1), +\infty\big)$.

(b) *Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers then, if $s > r + 1$ and if $a \in \big(0, \frac{s+r+1}{s}\big)$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal for every $\theta \in \big(s/(r+1), +\infty\big)$ and*

$$\theta^\star = (s + a)/(r + a)$$

is the unique solution of $(2.5.1)$ on the set $\big(s/(r+1), +\infty\big)$ (Note that the assumptions imply $a \in (0, 2)$).

(c) *Let $(\alpha_n)_{n\geq 1}$ be an asymptotically $L^r(P)$-optimal sequence of quantizers then, if $s < r$, the sequence $(\alpha_n^{\theta,0})_{n\geq 1}$ is L^s-rate-optimal for every $\theta \in \big(s/r, +\infty\big)$ and for all $a > 0$,*

$$\theta^\star = (s + 1)/(r + 1)$$

is the unique solution of $(2.5.1)$ on the set $\big(s/r, +\infty\big)$.

Proof. We set $\mu = 0$. The density function reads

$$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \mathbf{1}_{\{x>0\}}, \text{ with } \Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx.$$

(a) and **(b)**. Let $s \in (0, r + 1)$ and set $R_0 = \max(0, (a - 1)/\lambda)$. The function f is decreasing on $(R_0, +\infty)$ and for every $\theta > 0, \mu \in \mathbb{R}$, the couple (θ, μ) is P-admissible. For $\theta > s/(r + 1)$, Assumption $(3.2.33)$ holds true for every $c \in \big(1, \theta(1 + r)/s\big)$. Moreover, Hypothesis **(H1)** clearly holds. Consequently, it follows from Corollary $2.3.1, (a)$ that Assumption $(2.3.8)$ of Theorem 2.3.2 holds true.

When $s > r+1$, the additional hypothesis $f^{-\frac{s}{r+1}} \in L^1_{loc}(P)$ holds for $a < \frac{r+1}{s} + 1$. Furthermore, it follows from $(2.5.2)$ that **(H3)** holds. In this case Assumption $(2.3.8)$ of Theorem 2.3.2 holds true.

For all $\theta > 0$,

$$\int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{1+r}} dx = \left(\frac{\lambda^a}{\Gamma(a)}\right)^{1-s/(r+1)} \int_0^{+\infty} x^{(a-1)(1-\frac{s}{r+1})} e^{-(\theta-\frac{s}{r+1})\lambda x} dx,$$

it follows that

$$\int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{r+1}} dx < +\infty \quad \text{iff} \quad \theta > s/(r+1) \text{ and } a(r+1-s)+s > 0.$$

For every $\theta > s/(r+1)$, we have

$$\begin{aligned} h(\theta) &:= \theta^{s+1} \int_{\mathbb{R}} f(\theta x) f(x)^{-\frac{s}{1+r}} dx \\ &= \left(\frac{\lambda^a}{\Gamma(a)}\right)^{1-s/(r+1)} \theta^{s+1}\theta^{a-1} \int_0^{+\infty} x^{(a-1)(1-\frac{s}{1+r})} e^{-(\theta-\frac{s}{1+r})\lambda x} dx \\ &= C\,\theta^{\gamma}\left(\theta - \frac{s}{1+r}\right)^{-\beta} \end{aligned}$$

with

$$\gamma = s + a \text{ and } \beta = (a-1)(1-s/(r+1)) + 1.$$

The function h is differentiable for all $\theta > s/(1+r)$ and

$$h'(\theta) = C\theta^{\gamma-1}\left(\theta - \frac{s}{1+r}\right)^{-\beta-1}\left((\gamma-\beta)\theta - \frac{s\gamma}{1+r}\right).$$

The minimum of h is then unique on $\big(s/(r+1), +\infty\big)$ and is reached at $\theta^\star$.

Notice that the condition required for $f^{-\frac{s}{r+1}}$ to be in $L^1_{loc}(P)$ is $a < \frac{r+1}{s} + 1$ and for every $s > r+1$ one has $1 + \frac{r+1}{s} < \frac{s}{s-(r+1)}$. Combined to the condition $a(r+1-s) > 0$ yields the given constrain on a in (b).

(c) Now let $s < r$ and consider Inequality $(2.3.3)$. One has

$$\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x) dx = \frac{\lambda^a}{\Gamma(a)} \int_0^{+\infty} x^{a-1} e^{-\frac{\lambda x}{r-s}(r\theta - s)} dx.$$

Therefore $\int_{\mathbb{R}} f^{\frac{r}{r-s}}(\theta x) f^{-\frac{s}{r-s}}(x) dx < +\infty$ iff $\theta > s/r$.

On the other hand, for every $\theta > s/r$,

$$\begin{aligned} \theta^{1+s}\left(\int_{\mathbb{R}} f_{\theta,\mu}^{\frac{r}{r-s}}(x) f^{-\frac{s}{r-s}}(x) dx\right)^{1-\frac{s}{r}} &= C\,\theta^{s+a}\left(\int_0^{+\infty} x^{a-1} e^{-\frac{\lambda x}{r-s}(r\theta-s)} dx\right)^{\frac{r-s}{r}} \\ &= C\,\theta^{s+a}\big(r\theta - s\big)^{a\frac{s-r}{r}}. \end{aligned}$$

Considering the function h defined by $h(\theta) = \theta^{s+a}\,(r\theta - s)^{a\frac{s-r}{r}}$ we show that h reaches its minimum on $\big(s/r, +\infty\big)$ at the unique point $\theta^\star = (s+a)/(r+a)$. □

Remark 2.5.5. Let $X \sim \Gamma(a, \lambda)$. If $s < r$, then $\theta^\star = (s+a)/(r+a) < 1$. Then the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a contraction of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$. On the other hand, if $s > r$, then $\theta^\star > 1$ and the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ is a dilatation of $(\alpha_n)_{n\geq 1}$ with scaling number $\theta^\star$.

Moreover there is no constraint on the parameter a as long as $s < r$. In this case when we set $a = 1$ (exponential distribution with parameter λ) we retrieve the result related to the exponential distribution. We notice that $\theta^\star$ does not depend on the parameter λ. That is expected since $\Gamma(1,\lambda) = \mathscr{E}(\lambda)$ and, in the exponential case we know that the scaling number does not depend on λ.

Let $\theta^\star = (s+a)/(r+a)$ and consider now the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ defined as previously. Does this sequence verify the L^s-empirical measure theorem? If $a = 1$ we boil down to the exponential distribution. On the other hand, when $a \neq 1$, one shows below that there exists $a > 1,\ s > 0$ and $r > 0$ such that the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ does not verify the L^s-empirical measure theorem.

Suppose that $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ satisfies the L^s-empirical measure theorem. Then we must have, for all $u \in \mathbb{R}_+$,

$$\frac{1}{C_{f,r}}\frac{1}{\theta^\star}\int_0^u f(x/\theta^\star)^{\frac{1}{1+r}}dx = \frac{1}{C_{f,s}}\int_0^u f(x)^{\frac{1}{1+s}}dx. \tag{2.5.11}$$

with $f(x) = \frac{\lambda^a}{\Gamma(a)}x^{a-1}e^{-\lambda x}\mathbf{1}_{\{x>0\}}$ and $C_{f,r} = \int f(x)^{\frac{1}{1+r}}dx$ for all $r > 0$.

However, we have for any $r > 0$,

$$\begin{aligned}
C_{f,r} &= \lambda^{\frac{a}{1+r}}\Gamma(a)^{-\frac{1}{1+r}}\int_0^{+\infty} x^{(a-1)/(r+1)}e^{-\frac{\lambda}{1+r}x}dx \\
&= \lambda^{\frac{a}{1+r}}\Gamma(a)^{-\frac{1}{1+r}}\int_0^{+\infty} x^{(r+a)/(r+1)-1}e^{-\frac{\lambda}{1+r}x}dx \\
&= \lambda^{\frac{a}{1+r}}\Gamma(a)^{-\frac{1}{1+r}}\Gamma\left(\frac{r+a}{r+1}\right)\lambda^{-\frac{r+a}{r+1}}\big(r+1\big)^{\frac{r+a}{r+1}} \\
&= \Gamma\left(\frac{r+a}{r+1}\right)\Gamma(a)^{-\frac{1}{1+r}}\lambda^{-\frac{r}{r+1}}\big(r+1\big)^{\frac{r+a}{r+1}}.
\end{aligned}$$

Equation $(2.5.11)$ is written down for all $u \in \mathbb{R}_+$,

$$C(r)\left(\frac{r+a}{s+a}\right)^{\frac{r+a}{r+1}}\int_0^u x^{\frac{a-1}{r+1}}e^{-\frac{\lambda(r+a)}{(r+1)(s+a)}x}dx = C(s)\int_0^u x^{\frac{a-1}{s+1}}e^{-\frac{\lambda}{s+1}x}dx$$

with $C(r) = \Gamma\left(\frac{r+a}{r+1}\right)^{-1}\lambda^{\frac{r+a}{r+1}}\big(r+1\big)^{-\frac{r+a}{r+1}}, \qquad \forall\, r > 0.$

Let $m \in \mathbb{N}$ and $\alpha > 0$. We show by induction that, for $u > 0$,

$$\int_0^u x^n e^{-\alpha x}dx = -\left(\frac{1}{\alpha}u^n + \frac{n}{\alpha^2}u^{n-1} + \frac{n(n-1)}{\alpha^3}u^{n-2} + \cdots + \frac{n!}{\alpha^n}u + \frac{n!}{\alpha^{n+1}}\right)e^{-\alpha u} + \frac{n!}{\alpha^{n+1}}.$$

Let us consider $a > 1$ such that $\frac{a-1}{r+1}$ and $\frac{a-1}{s+1}$ are integers and set $n = \frac{a-1}{r+1}$, $m = \frac{a-1}{s+1}$, $\alpha = \frac{\lambda(r+a)}{(r+1)(s+a)}$ and $\beta = \frac{\lambda}{s+1}$. Equation $(2.5.11)$ finally reads

$$\begin{aligned}
C(r)\ &\left(\frac{r+a}{s+a}\right)^{\frac{r+a}{r+1}}\left[\left(\frac{1}{\alpha}u^n + \frac{n}{\alpha^2}u^{n-1} + \frac{n(n-1)}{\alpha^3}u^{n-2} + \cdots + \frac{n!}{\alpha^n}u + \frac{n!}{\alpha^{n+1}}\right)e^{-\alpha u} - \frac{n!}{\alpha^{n+1}}\right] \\
&= C(s)\left[\left(\frac{1}{\beta}u^m + \frac{m}{\beta^2}u^{m-1} + \frac{m(m-1)}{\beta^3}u^{m-2} + \cdots + \frac{m!}{\beta^m}u + \frac{m!}{\beta^{m+1}}\right)e^{-\beta u} - \frac{m!}{\beta^{m+1}}\right].
\end{aligned}$$

Set $a = 7,\ s = 1,\ r = 2,\ \lambda = 1$ and $u = 1$. Then $n = 2, m = 3, \alpha = 3/8, \beta = 1/2$ and we show after some computations that the sequence $(\alpha_n^{\theta^\star,0})_{n\geq 1}$ does not satisfy the L^s-empirical measure theorem since

$$\frac{185}{128}e^{-3/8} - \frac{79}{48}e^{-1/2} \neq -\frac{511}{512}$$

(one side is rational, the other is not). Hence, we have constructed an $L^s(P)$-rate-optimal sequence which does not satisfy the L^s-empirical measure theorem.

2.6 Applications

2.6.1 Application to Lloyd's I algorithm

One of the important issues from a computational point of view is the search of the L^r-optimal quantizers. The quadratic case ($r = 2$) is the commonly implemented for applications and various algorithms like the Competitive Learning Vector Quantization (CLVQ) algorithm (see e.g [PPP1]) and "randomized versions" of the Lloyd's algorithms (I and II) (see e.g. [GG]) are used. In practice, Lloyd's I algorithm is widely used to compute stationary (or optimal) quantizers because it can be easily implemented. We will use the natural extension of Lloyd's I algorithm to compute the L^r-stationary (optimal) quantizers. In a general framework, L^r-stationary quantizers (α_n) (with $\alpha_n = (\alpha_{n1}, \cdots, \alpha_{nn})$) are computed using the L^r-stationary equation $\nabla e_{n,r}(X) = 0$. This equation reads for every $r \geq 1$ (see [PPP1])

$$\alpha_{ni} = \frac{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}|^{r-2}X\right)}{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}|^{r-2}\right)}, \qquad i = 1, \cdots, n. \tag{2.6.1}$$

The Lloyd's I procedure is the fixed point procedure derived from $(2.6.1)$. Starting with an initial quantizer $\alpha_n^{(0)}$ of size n, one defines recursively a sequence $(\alpha_n^{(l)})_{l=1,\cdots,L}$ of L^r-stationary quantizers (where L corresponds to the number of Lloyd's iterations) by setting for every $l = 1, \cdots, L$,

$$\alpha_{ni}^{(l)} = \frac{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}^{(l-1)}|^{r-2}X\right)}{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}^{(l-1)}|^{r-2}\right)}, \qquad i = 1, \cdots, n. \tag{2.6.2}$$

By "randomized version" of the Lloyd's I procedure we mean that both expectations in $(2.6.2)$ are computed using a Monte Carlo simulation of size M. However in higher dimensions there are several L^r-stationary quantizers and the Lloyd's I procedure is somehow a method to compute the "nearest" one (namely the one in the attracting basin of which the procedure has been initialized). This is why the initialization of the procedure at already good grid is a crucial issue to obtain good L^r-quantizers. This leads us to consider the optimally L^r-dilated quadratic quantizers as natural good candidates to initialize the L^r-Lloyd's I procedure. We compare it to a random initialization, which consists on generating a vector of size n distributed as X and multiply it by the same scaled number ($\theta^\star$).

We carried out a numerical test in dimension $d = 2, 3$ with $r = 4$ for the $\mathcal{N}(0; I_d)$ distribution. The Monte Carlo size M is equal to 10^6 and our grid size n moves 10 by 10 from 10 to 100. Numerical results depicted in figure 2.2 (for $d = 2$) show that the dilated L^2-stationary (optimal) grids are already almost L^4-stationary (and likely almost optimal if we suppose that the L^2-stationary quantizers are) since the initial scaled L^2-stationary grids do not move during the successive Lloyd's iterations. This is also confirmed by Figure 2.1 (when $d = 2$) and Figure 2.3 (when $d = 3$) where the logarithm of the L^4 quantization error of the dilated grids remains the same during the successive Lloyd's procedures. The dilated L^2-stationary quantizers initialization seems to be the best choice one can do. The Lloyd's procedure initialization with random grids never leads to lower quantization errors. Moreover it needs several iterations of the procedure, depending to the grid size.

2.6.2 Application of L^r-quantizers to numerical integration

Let $\beta \geq 1$ and let $X \in L^{\beta+\varepsilon_0}(\mathbb{P})$, $\varepsilon_0 > 0$. Let $f \in \mathrm{Lip}_\beta(\mathbb{R}^d) := \{g : \mathbb{R}^d \to \mathbb{R},\ |g(x) - g(y)| \leq C|x-y|(1+|x|^{\beta-1}+|y|^{\beta-1})\}$. For any sequence of quantizers $(\alpha_n)_{n\geq 1}$ and any $r \in [1, +\infty]$ we have

$$\begin{aligned}\|f(X) - f(\widehat{X}^{\alpha_n})\|_1 &\leq C\,\mathbb{E}(|X - \widehat{X}^{\alpha_n}|(1+|X|^{\beta-1}+|\widehat{X}^{\alpha_n}|^{\beta-1})) \\ &\leq C\,\|X - \widehat{X}^{\alpha_n}\|_r(1+\|X\|^{\beta-1}_{(\beta-1)r'} + \|\widehat{X}^{\alpha_n}\|^{\beta-1}_{(\beta-1)r'}) \qquad (2.6.3)\end{aligned}$$

by *Hölder*'s inequality with $r' = r/(r-1)$.

Suppose now that $X \notin L^{\beta+\varepsilon}(\mathbb{P})$, $\forall \varepsilon > \varepsilon_0$. To give a sense to the above inequality as a error bound, we must choose r' such that $(\beta-1)r' \leq \beta + \varepsilon_0$; which in return impose that $r \geq \frac{\beta+\varepsilon_0}{1+\varepsilon_0}$. Now $\frac{\beta+\varepsilon_0}{1+\varepsilon_0} > 2$ as soon as $\beta > 2 + \varepsilon_0$. Furthermore if $\liminf_{|x|\to+\infty} \frac{|f(x)|}{|x|^\beta} > 0$ there is no alternative to these constraints. In this situation it is impossible to use quadrature formulae for numerical integration based on quadratic quantizers. However we can use some dilated L^2-optimal (at least stationary) quantizers $\alpha_n^{\theta^\star,\mu}$, for large enough n. Then

$$\mathbb{E}\big(f(\widehat{X}^{\alpha_n^{\theta^\star,\mu}})\big) = \sum_{i=1}^n f(\alpha_{ni}^{\theta^\star,\mu})\mathbb{P}(X \in C_i(\alpha_n^{\theta^\star,\mu}))$$

and

$$|\mathbb{E}f(X) - \mathbb{E}f\big(\widehat{X}^{\alpha_n^{\theta^\star,\mu}}\big)| \leq \|\mathbb{E}f(X) - \mathbb{E}f\big(\widehat{X}^{\alpha_n^{\theta^\star,\mu}}\big)\|_1$$

so that the error bound $(2.6.3)$ holds true profided f is lipschitz. Such an approach requires to compute the weights $\mathbb{P}(X \in C_i(\alpha_n^{\theta^\star,\mu}))$ associated to the Voronoi cells of $\alpha_n^{\theta^\star,\mu}$. The following easy proposition says how to compute these weights.

Proposition 2.6.1. *Let $X \sim P$ and $P = f \cdot \lambda_d$. Then, for every $n \geq 1$, we have*

$$\forall i \in \{1, \cdots, n\}, \quad P(C_i(\alpha_n^{\theta,\mu})) = P_{\theta,\mu}(C_i(\alpha_n)). \qquad (2.6.4)$$

Proof. Let $n \geq 1$ and $\alpha_n = (\alpha_{n1}, \cdots, \alpha_{nn})$. One has

$$P(C_i(\alpha_n^{\theta,\mu})) = \mathbb{P}(X \in C_i(\alpha_n^{\theta,\mu})) = \int_{\{|x-(\mu+\theta(\alpha_{ni}-\mu))| = \min_{j\neq i}|x-(\mu+\theta(\alpha_{nj}-\mu))|\}} f(x)d\lambda_d(x).$$

Making the change of variable $z = \frac{x-\mu}{\theta} + \mu$ yields

$$P(C_i(\alpha_n^{\theta,\mu})) = \theta^d \int_{\{z\in C_i(\alpha_n)\}} f_{\theta,\mu}(z)d\lambda_d(z) = P_{\theta,\mu}(C_i(\alpha_n)).$$

□

When a closed formula (like for the exponential distribution) is not available for the weights of the dilated cells, these weights can be estimated by the Monte Carlo method using the Nearest-Neighbour algorithm. Fast implementations of this algorithm can be find e.g. in [FBF, McN].

Acknowledgements

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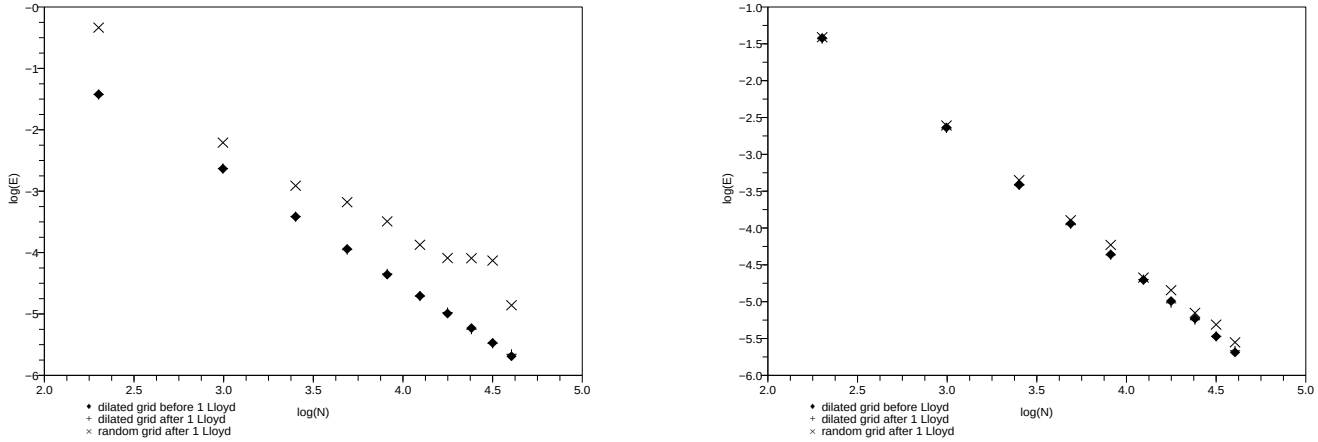

Figure 2.1: Comparison of the log of the L^4-error (power 4) as function of the log of the grid size after 1 (on the left) and 10 (on the right) Lloyd's iterations.

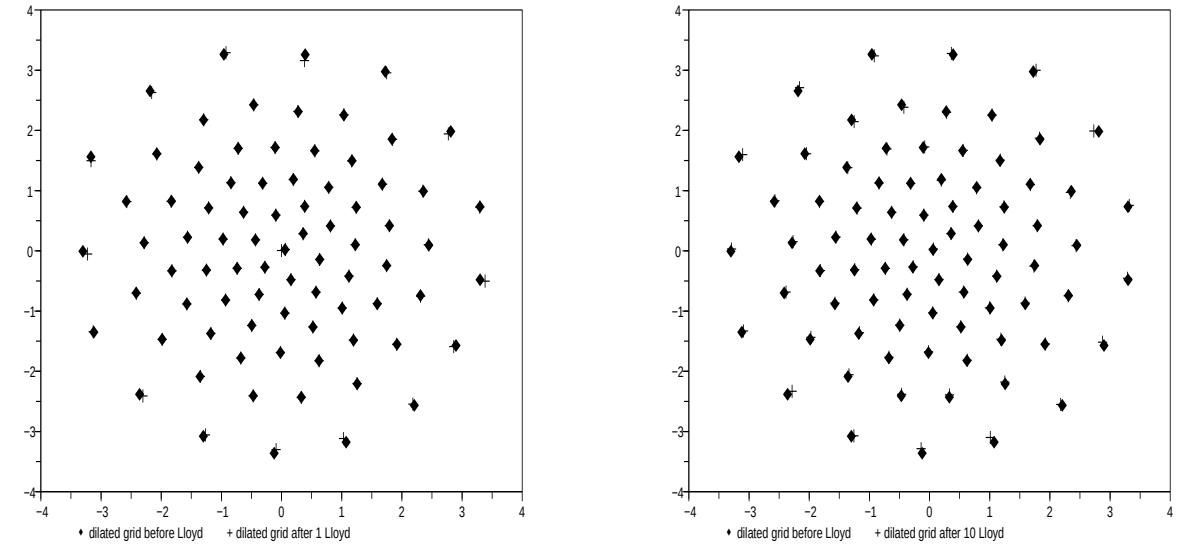

Figure 2.2: The L^2-dilated grid before and after 1 and 10 Lloyd's iterations; the grid size equals 80.

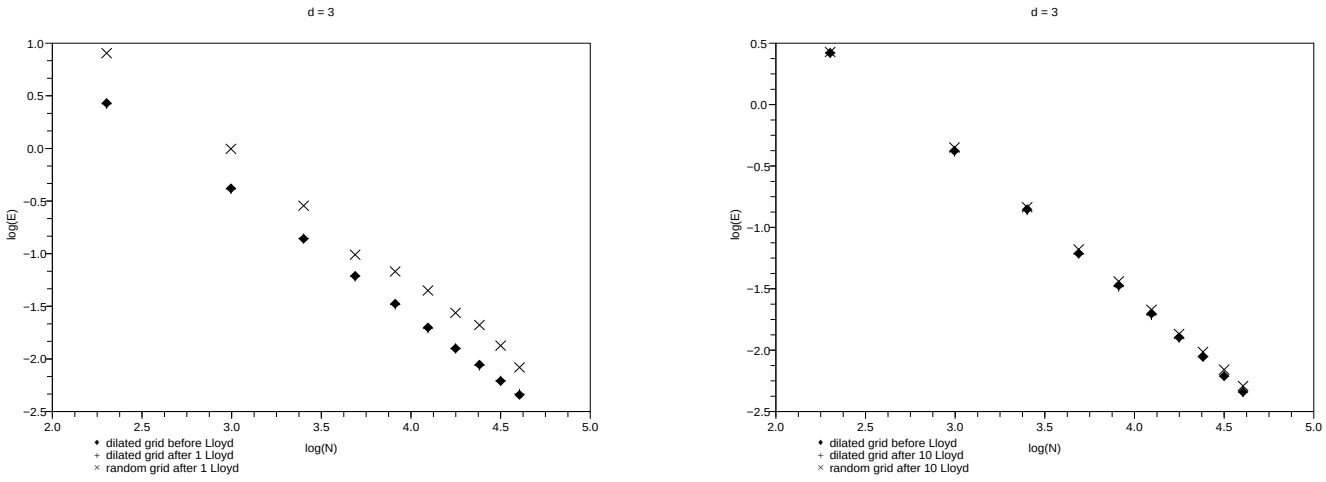

Figure 2.3: Comparison of the log of the L^4-error (power 4) as function of the log of the grid size after 1 (on the left) and 10 (on the right) Lloyd's iterations.

Chapter 3

Asymptotics of the maximal radius of an L^r-optimal sequence of quantizers

Let P be a probability distribution on $\mathbb{R}^d$ (equipped with an Euclidean norm). Let $r, s > 0$ and assume $(\alpha_n)_{n\geq 1}$ is an (asymptotically) $L^r(P)$-optimal sequence of n-quantizers. In this paper we investigate the asymptotic behavior of the maximal radius sequence induced by the sequence $(\alpha_n)_{n\geq 1}$ and defined to be for every $n \geq 1$, $\rho(\alpha_n) = \max\{|a|,\ a \in \alpha_n\}$. We show that if $\mathrm{card}(\mathrm{supp}(P))$ is infinite, the maximal radius sequence goes to $\sup\{|x|, x \in \mathrm{supp}(P)\}$ as n goes to infinity. We then give the rate of convergence for two classes of distributions with unbounded support : distributions with exponential tails and distributions with polynomial tails.

Introduction

The aim of this paper is to provide some precise upper and lower bounds for the radius of an optimal quantizer of an $\mathbb{R}^d$-valued random vector. The main motivation for this comes from the application of optimal quantization in Numerical Probability as detailed below.

Quantization has become an important field of information theory since the early 1940's. Nowadays, it plays an important rule in Digital Signal Processing (DSP), the basis of many areas of technology, from mobile phones to modems and multimedia PCs. In DSP, vector quantization is the process of approximating a continuous range of values or a very large set of discrete values by a relatively small set of discrete values. A common use of quantization is the conversion of a continuous signal into a digital signal. This is performed in analog-to-digital converters with a given quantization level.

Recently, optimal vector quantization has become a promising tool in Numerical Probability: it is an efficient to produce grids optimally fitted to the distribution of a random vector X. This leads to some cubature formulas that may approximate either expectations (see [PAG]) or more significantly some conditional expectations (see [PPP1]). This ability to approximate conditional expectations is the key property called upon in the quantization based numerical schemes used to solve some problems arising in finance: optimal stopping problems (pricing and hedging American style options, see [BP, BPP]), the pricing of swing options (see [BBP, BBP1]), stochastic control problem (see [CPR, PPP]) for portfolio management, nonlinear filtering (see [PAGPHA, PRS]). Other applications like Zakai and McKean-Vlasov equations have also been investigated (see [GPPP]).

In all these procedures the computation of these optimal grids is usually a rather time-consuming step that has to be processed "off line" (with in mind that the resulting algorithm $e.g.$ for american a swing option will be processed to price and hedge a whole book of options with various payoffs functions).

In higher dimension, all algorithms to compute optimal quantization grids rely on simulation procedures which make extensive use of nearest neighbor procedures. Some (exact or approximate) fast procedures have been devised to speed up this phase which confirms that this grid computation phase can be drastically reduced (see [FBF, McN]) . It remains that quantization procedures are not convex and that they may fall in many "trops". So it is a very challenging problem to initialize efficiently and at a low cost the procedures in order to make them converge toward optimal quantization grids.

Several numerical experiments lead with random vector having a radial distribution suggest that a good starting value, rather that a random n-sample of the distribution of X, is to consider a n-sample of the distribution $\rho_n \frac{X}{|X|}$ where ρ_n is the radius of an n-optimal quantizer. To implement this heuristics requires to elucidate the behavior of ρ_n (at least asymptotically). This motivates the extensive study of the asymptotics of the maximal radius sequence (ρ_n). Some of the numerical tests are reproduced further in Section 3.3.

Now let us recall some facts about optimal quantization. The L^r-optimal quantization problem at level n for a $\mathbb{R}^d$-valued random vector X lying in $L^r(\Omega, \mathcal{A}, \mathbb{P})$ consists in finding the best approximation of X by $q(X)$, where q is a Borel function taking at most n values. This reads as the following minimization problem:

$$e_{n,r}(X) = \inf \{\|X - q(X)\|_r, q : \mathbb{R}^d \xrightarrow{\text{Borel}} \mathbb{R}^d, \text{ card}(q(\mathbb{R}^d)) \leq n\}.$$

Note that in fact $e_{n,r}(X)$ only depends on the distribution $P = \mathbb{P}_X$ of X so that we will also use the notation $e_{n,r}(P)$. However, for any Borel function $q : \mathbb{R}^d \to \alpha$ we have

$$|X - q(X)| \geq \min_{a \in \alpha} d(X, a) = d(X, \alpha) = |X - \widehat{X}^{\alpha}| \quad \mathbb{P} \text{ a.s}$$

so that the quantization problem reduces to

$$e_{n,r}(X) = \inf\{\|X - \widehat{X}^\alpha\|_r, \alpha \subset \mathbb{R}^d, \text{ card}(\alpha) \le n\} = \inf_{\substack{\alpha\subset\mathbb{R}^d \\ \text{card}(\alpha)\le n}} \left(\int_{\mathbb{R}^d} d(x,\alpha)^r dP(x)\right)^{1/r}. \tag{3.0.1}$$

where $\widehat{X}^\alpha = \sum_{a\in\alpha} a\mathbf{1}_{\{X\in C_a(\alpha)\}}$ and $(C_a(\alpha))_{a\in\alpha}$ corresponds to a Voronoi partition of $\mathbb{R}^d$ (with respect to a norm $|\cdot|$ on $\mathbb{R}^d$), that is, a Borel partition of $\mathbb{R}^d$ satisfying for every $a \in \alpha$,

$$C_a(\alpha) \subset \{x \in \mathbb{R}^d : |x-a| = \min_{b\in\alpha} |x-b|\}.$$

For every $n \ge 1$, the infimum in $(3.0.1)$ holds as a finite minimum reached (at least) at one grid $\alpha^\star$. In this case $\alpha^\star$ is called an $L^r(P)$**-optimal** (or L^r-optimal for X) and a sequence of n-quantizers $(\alpha_n)_{n\ge1}$ is $L^r(P)$-optimal if for every $n \ge 1$, α_n is $L^r(P)$-optimal. A sequence $(\alpha_n)_{n\ge1}$ is said **asymptotically** $L^r(P)$**-optimal** if

$$\int_{\mathbb{R}^d} d(x,\alpha_n)^r P(dx) \;=\; e^r_{n,r}(X) + o(e^r_{n,r}(X)) \qquad \text{as } n \to \infty.$$

Moreover the L^r-quantization error $e_{n,r}(X)$ decreases to 0 as n goes to infinity and if there is an $(r+\eta)$-moment of X, for $\eta > 0$, the so-called Zador's theorem recalled below rules its rate of convergence to 0.

Zador's Theorem (see [GL]). Let $P = P_a + P_s$ be the Lebesgue decomposition of P with respect to the Lebesgue measure λ_d, where P_a denotes the absolutely continuous part and P_s the singular part of P. Suppose $\mathbb{E}|X|^{r+\eta} < +\infty$ for some $\eta > 0$. Then

$$\lim_{n\to+\infty} n^{r/d}(e_{n,r}(P))^r = Q_r(P).$$

with

$$Q_r(P) = J_{r,d}\left(\int_{\mathbb{R}^d} f^{\frac{d}{d+r}} d\lambda_d\right)^{\frac{d+r}{d}} = J_{r,d}\,\|f\|_{\frac{d}{d+r}} \;\in [0,+\infty),$$

$$J_{r,d} = \inf_{n\ge1} n^{r/d} e^r_{n,r}(U([0,1]^d)) \in (0,+\infty),$$

where $U([0,1]^d)$ denotes the uniform distribution on the set $[0,1]^d$ and $f = \frac{dP_a}{d\lambda_d}$. Note that the moment assumption : $\mathbb{E}|X|^{r+\eta} < +\infty$ ensure that $\|f\|_{\frac{d}{d+r}}$ is finite.

Very little is known about the geometric properties of optimal quantizers. In this paper we address a first problem in this direction: we study the asymptotic behavior of the radii of a sequence $(\alpha_n)_{n\ge1}$ of L^r-optimal quantizers. The maximal radius (or simply radius) $\rho(\alpha)$ of a quantizer $\alpha \subset \mathbb{R}^d$ is defined by

$$\rho(\alpha) = \max\{|a|,\ a \in \alpha\}.$$

In our framework, $|\cdot|$ will be an Euclidean norm on $\mathbb{R}^d$. For the sake of simplicity, we will denote from now on by $(\rho_n)_{n\ge1}$ the sequence $(\rho(\alpha_n))_{n\ge1}$ of radii of a sequence $(\alpha_n)_{n\ge1}$ of optimal quantizers (although it may be not unique).

We will show that, as soon as $\text{supp}(P)$ is unbounded, $\lim_{n\to+\infty} \rho_n = +\infty$. Besides, our key inequalities to get the upper and lower estimates of the maximal radius sequence are provided in Theorem

3.2.1 and Theorem 3.2.2. The first theorem yields amount others the maximal rate of convergence of $\bar{F}_r(\frac{\rho_n}{c_{r,d}+\varepsilon})$ (when $n \to +\infty$) to 0, for every $\varepsilon > 0$, with $c_{r,d} = 1$ if $d = 1; r \geq 1$ and $c_{r,d} = 2$ otherwise. It claims that this rate is at most equals to $n^{-(1+r/d)}$.

Theorem 3.2.2 maintains in particular that for every $u > 1$, $\bar{F}_r(\rho_n u)$ goes to 0, as n goes to infinity, at a rate at least equals to $n^{-\frac{r+\nu}{d}}$, where ν is such that the random vector X has an $(r, r+\nu)$-distribution (see Definition 3.2.1). We will see later on that the index $\nu_X^\star$ ensuring that X has an $(r, r+\nu)$-distribution for every $\nu \in (0, \nu_X^\star)$ will play a crucial role in the lower limit estimates of the maximal radius sequence.

Then we will emphasize how knowing the asymptotic behavior of the function $-\log \bar{F}_r$ allow to derive the asymptotic estimates of ρ_n (or $\log \rho_n$). As an important example we can already mention distributions with density function f satisfying

$$f(x) \ \propto \ \frac{(\log |x|)^\beta}{|x|^c} \, \mathbf{1}_{\{|x|>1\}} \qquad x \in \mathbb{R}^d, \ \beta \in \mathbb{R}, \ c > r + d$$

for which the optimal rate of convergence of $\log \rho_n$ is computed and given by

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c-r-d}\frac{r+d}{d}.$$

Of course, this result is less accurate as giving the rate of convergence of the sequence (ρ_n) itself for which the exact limit can not be computed with our approach because the upper and lower limits make appear no identified constants. Another example concerns distributions with exponential tail for which the upper and lower rates of convergence of the sequence (ρ_n) are provided. This is the case for the normally distributed random vector on $\mathbb{R}^d$ for which we have

$$\sqrt{\frac{2(r+d)}{d}} \leq \liminf_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq \limsup_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq 2\sqrt{\frac{2(r+d)}{d}}.$$

Our general conjecture for such distributions, which is proved when $d = 1$ and $r \geq 1$, is that the liminf bound is sharp, that is,

$$\lim_{n\to+\infty} \frac{\rho_n}{\sqrt{\log(n)}} = \sqrt{\frac{2(r+d)}{d}}.$$

Moreover, an alternative approach is given for the lower limit estimates. This approach is based on random quantization and relies ρ_n to the expectation of an $i.i.d$ sequence of random variables distributed as X.

The paper is organized as follows. In Section 2, upper and lower estimates of the maximal radius sequence are given and the exact limit is provided when the cardinal of the support of P is infinite. This limit corresponds to $\sup\{|x|, \ x \in \mathrm{supp}(P)\}$ and Section 3 is entirely devoted to the convergence rate of the sequence of radii to this limit value.

Notations : Throughout the paper X will denote an $\mathbb{R}^d$-valued random vector defined in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution P having a moment of order $r > 0$ $i.e.$ $\mathbb{E}|X|^r < +\infty$. We define

$$L^{r+}(\mathbb{P}) = \bigcup_{\varepsilon>0} L^{r+\varepsilon}(\mathbb{P}).$$

We will denote by λ_d the Lebesgue measure on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$. We will also denote by $\bar{F}$ the survival function of X, that is, the $(0,1]$-valued function defined on $\mathbb{R}_+$ by

$$\bar{F} : x \mapsto \bar{F}(x) = \mathbb{P}(\{|X| > x\})$$

and for every $r > 0$, we define the generalized survival function of X by

$$\bar{F}_r : x \mapsto \bar{F}_r(x) = \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big).$$

Note that this last function is defined on $\mathbb{R}_+$ and takes values on the set $(0, \mathbb{E}|X|^r]$.

For a given set A, $\overline{A}$ will stand for its closure, ∂A its boundary, $\mathrm{Conv}(A)$ its convex hull and $\mathring{A}$ or $\mathrm{Int}(A)$ its interior. The cardinal of A is denoted by $\mathrm{card}(A)$. For every $x \geq 0$, $[x]$ will denote the integral part of x.

3.1 Asymptotics of the the maximal radius sequence

In this section we give an asymptotic upper bound and a lower bound of the sequence of radii. For distributions supported by a infinite set, the exact limit is provided.

Proposition 3.1.1. *Let $X \in L^r_{\mathbb{R}^d}(\mathbb{P})$. Let $(\alpha_n)_{n\geq 1}$ be a sequence of n-quantizers such that $e_{n,r}(X) \to 0$ as $n \to +\infty$. Then,*

$$\liminf_{n\to+\infty} \rho_n \geq \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{3.1.1}$$

Remark that this result also holds for any norm on $\mathbb{R}^d$.

Proof. Let $x \in \mathrm{supp}(P)$. Suppose that there exists $\varepsilon_0 > 0$ and a subsequence $(\rho_{n_k})_{k\geq 1}$ such that

$$\forall k \geq 1, \qquad \rho_{n_k} < |x| - 2\varepsilon_0. \tag{3.1.2}$$

Thus

$$\exists\, \eta > 0 \text{ such that } \forall k,\ d(B(0,\rho_{n_k}), B(x,\varepsilon_0)) > \eta > 0.$$

Then one has for every $k \geq 1$,

$$\begin{aligned}
e_{n_k,r}(X) &= \|d(X,\alpha_{n_k})\|_r \\
&\geq \|d(X, B(0,\rho_{n_k}))\|_r \qquad\qquad (\text{since } \alpha_{n_k} \subset B(0,\rho_{n_k})) \\
&\geq \|d(X, B(0,\rho_{n_k}))\mathbf{1}_{\{X\in B(x,\varepsilon_0)\}}\|_r \\
&\geq \|d(B(x,\varepsilon_0), B(0,\rho_{n_k}))\mathbf{1}_{\{X\in B(x,\varepsilon_0)\}}\|_r \\
&= d(B(x,\varepsilon_0), B(0,\rho_{n_k}))\mathbb{P}(X \in B(x,\varepsilon_0))^{1/r} \\
&> \eta\, \mathbb{P}(X \in B(x,\varepsilon_0))^{1/r} > 0.
\end{aligned}$$

This is not possible since $e_{n,r}(X) \to 0$. Then, we have shown that

$$\forall x \in \mathrm{supp}(P), \quad \liminf_n \rho_n \geq |x|.$$

Hence $\liminf_n \rho_n \geq \sup\{|x|,\ x \in \mathrm{supp}(P)\}$. □

Among other results, the next proposition provides the limit of the sequence $(\rho_n)_{n\geq 1}$ when the support of P is infinite.

Proposition 3.1.2. ***(a)*** *Let α be an L^r-optimal quantizer at level n. If $\mathrm{card}(\mathrm{supp}(P)) \geq n$ then*

$$\alpha \subset \overline{\mathrm{Conv}(\mathrm{supp}(P))} \quad \textit{and} \quad \rho_n \leq \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{3.1.3}$$

(b) *If $\mathrm{card}(\mathrm{supp}(P)) = +\infty$ then*

$$\lim_{n\to+\infty} \rho_n = \sup_{n\geq 1} \rho_n = \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{3.1.4}$$

for any $L^r(P)$-optimal sequence of quantizers $(\alpha_n)_{n\geq 1}$.

***Proof*. (a)** If α is L^r-optimal at level n then $\mathrm{card}(\alpha) = n$ since $\mathrm{card}(\mathrm{supp}(P)) \geq n$ (see [PAG]).

Now, suppose that $\alpha \not\subset \overline{\mathrm{Conv}(\mathrm{supp}(P))}$. Then let $a \in \alpha \cap \left(\overline{\mathrm{Conv}(\mathrm{supp}(P))}\right)^c$ and set

$$\alpha' = (\alpha \backslash \{a\}) \cup \{\Pi(a)\}$$

where Π denotes the projection on the non empty closed convex set $\overline{\mathrm{Conv}(\mathrm{supp}(P))}$. The projection is 1-Lipschitz and X is $\mathbb{P}$-$a.s$ $\mathrm{supp}(P)$-valued, hence

$$d(X,a) \geq d(\Pi(X), \Pi(a)) \overset{\mathbb{P}\text{-}a.s}{=} d(X, \Pi(a)).$$

It follows that

$$d(X,\alpha) \geq d(X,\alpha') \qquad \mathbb{P}\text{-}a.s.$$

Since α is $L^r(P)$-optimal at level n and $\mathrm{card}(\alpha') \leq \mathrm{card}(\alpha) = n$,

$$\mathbb{E}(d(X,\alpha')^r) = \mathbb{E}(d(X,\alpha)^r)$$

so that the three statements hold:

- $d(X,\alpha') = d(X,\alpha) \quad \mathbb{P}-a.s$
- $\Pi(a) \notin \alpha \backslash \{a\}$ since α' is $L^r(P)$-optimal (which implies that $\mathrm{card}(\alpha') = n$),
- $\mathbb{P}(X \in C_{\Pi(a)}(\alpha')) > 0$ (otherwise $\alpha' \backslash \{\Pi(a)\}$ would be optimal).

On the other hand, $X \in \mathrm{Conv}(\mathrm{supp}(P))$ $\mathbb{P}$-$a.s$ so that

$$\big(a - \Pi(a) | X - \Pi(a)\big) \leq 0 \quad \mathbb{P}\text{-}a.s.$$

Consequently

$$\begin{aligned} |X-a|^2 - |X - \Pi(a)|^2 &= 2(\Pi(a) - a | X - \Pi(a)) + |a - \Pi(a)|^2 \\ &\geq |a - \Pi(a)|^2 > 0 \qquad \text{since } a \notin \overline{\mathrm{Conv}(\mathrm{supp}(P))}. \end{aligned}$$

As a consequence

$$d(X,\alpha') < d(X,\alpha) \qquad \mathbb{P}\text{-a.s on } \{X \in \mathring{C}_{\Pi(a)}(\alpha')\}$$

where $\mathring{C}_{\Pi(a)}(\alpha') = \{\xi \in \mathbb{R}^d,\ d(\xi, \Pi(a)) < d(\xi, \alpha \backslash \{a\})\}$ since the norm is Euclidean.

This implies that $\mathbb{P}(X \in \mathring{C}_{\Pi(a)}(\alpha')) = 0$ and then $\mathbb{P}(X \in \partial C_{\Pi(a)}(\alpha')) > 0$; this is impossible since α' is L^r-optimal (see [GL]). Hence $\alpha \subset \overline{\mathrm{Conv}(\mathrm{supp}(P))}$.

Now, let us prove that $\rho_n \leq \sup\{|x|,\ x \in \mathrm{supp}(P)\}$. Note first that this assertion is obvious if $\mathrm{supp}(P)$ is unbounded.

On the other hand if $\mathrm{supp}(P)$ is bounded then it is compact and so is $\mathrm{Conv}(\mathrm{supp}(P))$. Let $x_0 \in \mathrm{Conv}(\mathrm{supp}(P))$ be such that $|x_0| = \sup\{|x|, x \in \mathrm{Conv}(\mathrm{supp}(P))\}$. Thus

$$x_0 = \lambda_0 \xi_1 + (1-\lambda_0)\xi_2, \qquad \xi_1, \xi_2 \in \mathrm{supp}(P)$$

and $\lambda \longmapsto |\lambda\xi_1 + (1-\lambda)\xi_2|$ is convex so that it reaches its maximum at $\lambda = 0$ or $\lambda = 1$. Consequently $x_0 \in \mathrm{supp}(P)$.

(b) This follows from the assertion about $\rho(\alpha_n)$ in the item (a) and from Proposition 3.1.1. □

Remark 3.1.1. If the norm on $\mathbb{R}^d$ is an arbitrary norm, the assertion (a) of the proposition may fail. An example is given with the l_∞-norm in [GL], p. 25.

3.2 Convergence rate of the maximal radius sequence

We first start by giving two examples of distributions for which the sharp convergence rate of the maximal radius sequence can be computed rather easily. In fact the semi-closed forms established in [FP] for the L^r-optimal quantizers of the exponential and the Pareto distributions and summed up in the following proposition allow to derive some sharp asymptotics for the maximal radius sequence $(\rho_n)_{n\geq 1}$ induced by the unique sequence of L^r-optimal quantizers at level n. These rates will be very useful to validate the asymptotic rates obtained by others approaches.

Proposition 3.2.1. *(see [FP])* ***(a)*** *Let $r > 0$ and let X be an exponentially distributed random variable with scale parameter $\lambda > 0$. Then, for every $n \geq 1$, the L^r-optimal quantizer $\alpha_n = (\alpha_{n,1}, \cdots, \alpha_{n,n})$ is unique and given by*

$$\alpha_{n,k} = \frac{1}{\lambda}\left(\frac{a_n}{2} + \sum_{i=n+1-k}^{n-1} a_i\right), \quad 1 \leq k \leq n, \tag{3.2.1}$$

where $(a_k)_{k\geq 1}$ is an $\mathbb{R}_+$-valued sequence recursively defined by the following implicit equation:

$$a_0 := +\infty, \quad \phi_r(-a_{k+1}) := \phi_r(a_k),\ k \geq 0$$

with $\phi_r(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)e^{-u}du$ (convention : $0^0 = 1$).

Furthermore, the sequence $(a_k)_{k\geq 1}$ decreases to zero and for every $k \geq 1$,

$$a_k = \frac{r+1}{k}\left(1 + \frac{c_r}{k} + O(\frac{1}{k^2})\right)$$

for some positive real constant c_r.

(b) *Let $r > 0$ and let X be a random variable having a Pareto distribution with index $\gamma > r$. Let f be the density function : $f(x) = \gamma x^{-(\gamma+1)}\mathbf{1}_{\{x>1\}}$. Then for every $n \geq 1$, the L^r-optimal quantizer $\alpha_n = (\alpha_{n,1}, \cdots, \alpha_{n,n})$ is unique and given by*

$$\alpha_{n,k} = \frac{1}{1+a_n}\prod_{i=n-k+1}^{n-1}(1+a_i), \quad 1 \leq k \leq n, \tag{3.2.2}$$

where $(a_k)_{k\geq 1}$ is an $\mathbb{R}_+$-valued sequence recursively defined by the following implicit equation:

$$a_0 = +\infty, \quad \phi_\gamma\left(-\frac{a_{k+1}}{1+a_{k+1}}\right) := \phi_\gamma(a_k),\ k \geq 1,$$

with $\phi_\gamma(x) := \int_0^{x/2} \gamma|u|^{r-1}\mathrm{sign}(u)(1+u)^{-(\gamma+1)}du$. The sequence $(a_k)_{k\geq 1}$ decreases to zero and there is some positive real constant c such that for every $k \geq 1$,

$$a_k = \frac{r+1}{(\gamma-r)k}\left(1 + \frac{c}{k} + O(\frac{1}{k^2})\right).$$

Let us give now the sharp asymptotic derived from these semi-closed forms.

Proposition 3.2.2. ***(a)*** *Let $r > 0$ and let X be an exponentially distributed random variable with parameter $\lambda > 0$. Then*

$$\rho_n \quad = \quad \frac{r+1}{\lambda}\log(n) + \frac{C_r}{\lambda} + O(\frac{1}{n}), \tag{3.2.3}$$

where C_r is a real constant depending only on r.

(b) *Let $r > 0$ and let X be a random variable with Pareto distribution of index γ such that $\gamma > r$. Then,*

$$\log(\rho_n) = \frac{r+1}{\gamma - r}\log(n) + C_r + O(\frac{1}{n}), \tag{3.2.4}$$

where C_r is a real constant depending only on r.

Proof. **(a)** It follows from $(3.2.1)$ that

$$\lambda\, \rho_n = \frac{a_n}{2} + \sum_{i=1}^{n-1} a_i$$

where the sequence $(a_n)_{n\geq 1}$ decreases to zero and satisfies for every $n \geq 1$, $a_n = \big((r+1)/n\big)\big(1 + c_r/n + \mathrm{O}(1/n^2)\big)$, for some real constant c_r. Thus,

$$\begin{aligned}\lambda\, \rho_n &= \frac{a_n}{2} + (r+1)\sum_{i=1}^{n-1}\frac{1}{i} + c_r\sum_{i=1}^{n-1}\frac{1}{i^2} + \sum_{i=1}^{n-1}\mathrm{O}(1/i^3)\\ &= (r+1)\log(n) + C_r + \mathrm{O}(\frac{1}{n}).\end{aligned}$$

(b) It follows from $(3.2.2)$ that

$$\rho_n = \frac{1}{1+a_n}\prod_{i=1}^{n-1}(1+a_i)$$

where $(a_n)_{n\geq 1}$ is an $\mathbb{R}_+$-valued sequence, decreasing to zero and satisfying : $\forall n \geq 1$, $a_n = \frac{r+1}{(\gamma-r)n}\big(1 + c_r/n + \mathrm{O}(1/n^2)\big)$, for some real constant c_r.

Then,

$$\begin{aligned}\log(\rho_n) &= -\log(1+a_n) + \sum_{i=1}^{n-1}\big(a_i - \frac{a_i^2}{2} + \mathrm{O}(a_i^3)\big)\\ &= \frac{r+1}{\gamma - r}\log(n) + C_r + \mathrm{O}(\frac{1}{n})\end{aligned}$$

where we used that $\sum_{i=1}^{\infty} a_i^2 < \infty$ and $\sum_{i=1}^{\infty}\mathrm{O}(a_i^3) < \infty$. □

3.2.1 Upper estimate

We investigate in this section the rate of convergence of (ρ_n) to infinity. Let us give first some definitions and some hypotheses which will be useful later on.

Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers at level n. For $n \geq 1$, we define $M(\alpha_n)$ to be

$$M(\alpha_n) = \{a \in \alpha_n \text{ such that } |a| = \max_{b\in\alpha_n}|b|\}.$$

We will need the following assumption on P

$$\textbf{(H)} \equiv \qquad P(dx) \geq \varepsilon_0\, \mathbf{1}_{\{x\in \bar{B}(x_0,r_0)\}}\lambda_d(dx), \quad \varepsilon_0,\ r_0>0,\ x_0\in\mathbb{R}^d.$$

In the one dimensional setting, we will need the following specific assumption depending on $r \in [1,+\infty)$:

$(\mathbf{G_r})$ $P = f\cdot\lambda_d$ where f is non-increasing to 0 on $[A,+\infty)$ for some real constant A and

$$\lim_{y\to+\infty}\int_1^{+\infty}(u-1)^{r-1}\frac{f(uy)}{f(y)}du = 0. \tag{3.2.5}$$

Let us make some brief comments on theses assumptions as well as some simple criterions.

• Note that Assumption **(H)** holds as soon as X has a density f which is bounded away from 0 on a closed ball $\bar{B}(x_0,r_0), r_0>0, x_0\in\mathbb{R}^d$, i.e. $\varepsilon_0 := \min_{x\in\bar{B}(x_0,r_0)} f(x) > 0$. This is a very light assumption satisfied by all usual distributions (Gaussian distribution, the exponential distribution, the Pareto distribution, etc).

• Assumption $(\mathbf{G_r})$ holds for distributions with density functions of the form

$$f(x)\ \propto\ |x|^c\, e^{-\vartheta|x|^\kappa} \qquad x\in\mathbb{R};\ \vartheta,\kappa>0;\ c>-1.$$

Indeed, we have for large enough y, f is non-increasing and

$$\begin{aligned}\int_1^{+\infty}(u-1)^{r-1}\frac{f(uy)}{f(y)}du &= \int_1^{+\infty}(u-1)^{r-1}u^c e^{-\vartheta y^\kappa(u^\kappa-1)}du\\ &\leq \int_1^{+\infty}(u-1)^{r-1}u^c e^{-\vartheta A^\kappa(u^\kappa-1)}du<+\infty.\end{aligned}$$

The existence of the last integral follows from the existence of the moment of every order.

It follows from the Lebesgue convergence theorem that $(3.2.5)$ holds. Then Assumption $(\mathbf{G_r})$ holds in particular for the Gaussian distribution, for the Weibull distribution and for the Gamma distribution. However it fails for example for the Pareto distribution. But, we will see later that we do not need this assumption for distributions with polynomial tails to estimate the sequence $(\log\rho_n)_{n\geq1}$.

Let us recall the L^r-stationary property which will be also useful. Assume $P = f\cdot\lambda_d$. The so-called L^r-distortion function $D^X_{n,r} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+$ is defined by :

$$\alpha=(\alpha_1,\cdots,\alpha_n)\longmapsto\mathbb{E}\left(\min_{i=1,\cdots,n}|X-\alpha_i|^r\right).$$

Then, for every $r\geq1$, $D^X_{n,r}$ is differentiable at any codebook having pairwise distinct components and (see [GLP1] for details)

$$\nabla D^X_{n,r}(\alpha) = r\left(\int_{C_i(\alpha)}(\alpha_i-u)|u-\alpha_i|^{r-2}f(u)du\right)_{1\leq i\leq n}. \tag{3.2.6}$$

An optimal L^r-quantizer at level n $\alpha=\{\alpha_1,\cdots,\alpha_n\}$ for P has full size n, so that,

$$\nabla D^X_{n,r}(\alpha) = 0.$$

α is said to satisfy an L^r-stationary property.

When $d = 1$ then for any (ordered) quantizer $\alpha_n = \{x_1^{(n)}, \cdots, x_n^{(n)}\}$, $x_1^{(n)} < \cdots < x_n^{(n)}$ at level n, its Voronoi partition is given by

$$C_1(\alpha_n) = (-\infty, x_{\frac{1}{2}}^{(n)}],\ C_n(\alpha_n) = (x_{n-\frac{1}{2}}^{(n)}, +\infty),\ C_i(\alpha_n) = (x_{i-\frac{1}{2}}^{(n)}, x_{i+\frac{1}{2}}^{(n)}],\ i = 2, \cdots, n-1,$$

with $x_{i-\frac{1}{2}}^{(n)} = \frac{x_i^{(n)} + x_{i-1}^{(n)}}{2}$ and $x_{i+\frac{1}{2}}^{(n)} = \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2}$.

The main result of this section is the following.

Theorem 3.2.1. *Suppose that X has an unbounded support and that **(H)** holds. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers. Then,*

(a)

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \to +\infty} \left(n^{1+\frac{r}{d}} \bar{F}_r\big(\frac{\rho_n}{2+\varepsilon}\big) \right) \geq C_{r,d,U}. \tag{3.2.7}$$

***(b)** If $d = 1,\ r \geq 1$ and if furthermore $(\mathbf{G_r})$ holds then,*

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \to +\infty} \left(n^{r+1} \bar{F}_r\big(\frac{\rho_n}{1+\varepsilon}\big) \right) \geq C_{r,1,U}. \tag{3.2.8}$$

$C_{r,d,U}$ is a positive real constant depending on r, d and the uniform distribution U on $[0,1]^d$.

The Lemmas below are used to prove this result.

Lemma 3.2.1. *Let X be an $\mathbb{R}^d$ valued random variable with unbounded support and probability distribution P and let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers, $r > 0$. Let $(\rho_n)_{n\geq 1}$ be the maximal radius sequence induced by $(\alpha_n)_{n\geq 1}$. Then,*

***(a)** $\forall\, \varepsilon > 0,\ \exists\, n_\varepsilon$ such that $\forall n \geq n_\varepsilon$,*

$$\forall\, a \in M(\alpha_n),\ \forall\, \xi \in C_a(\alpha_n), \qquad |\xi| \geq \frac{\rho_n}{2+\varepsilon}. \tag{3.2.9}$$

***(b)** If $d = 1,\ r \geq 1$ and if furthermore $(\mathbf{G_r})$ holds then, for large enough n,*

$$\forall\, a \in M(\alpha_n),\ \forall\, \xi \in C_a(\alpha_n), \qquad |\xi| \geq \frac{\rho_n}{1+\varepsilon}. \tag{3.2.10}$$

***Proof*. (a)** Since (α_n) is $L^r(P)$-optimal, $e_{n,r}(X) \to 0$ as $n \to +\infty$. Then, the following asymptotic density property of (α_n) in the support of P holds:

$$\forall \varepsilon > 0,\ \forall x \in \mathrm{supp}(P)\ \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon \quad B(x,\varepsilon) \cap \alpha_n \neq \emptyset. \tag{3.2.11}$$

Otherwise, if there exists $x \in \mathrm{supp}(P), \varepsilon > 0$ and a subsequence $(\alpha_{n_k})_{k\geq 1}$ so that $\forall k \geq 1,\ B(x,\varepsilon) \cap \alpha_{n_k} = \emptyset$, then, for every $k \geq 1$,

$$e_{n_k,r}(X) \geq \|d(X, \alpha_{n_k}) \mathbf{1}_{X \in B(x,\varepsilon/2)}\|_r \geq \frac{\varepsilon}{2} P(B(x,\varepsilon/2))^{1/r} > 0.$$

Which contradicts the fact that $e_{n,r}(X) \to 0$ as $n \to +\infty$.

Now, to prove the result assume first $0 \in \mathrm{supp}(P)$. Let $\varepsilon > 0,\ a \in M(\alpha_n)$. Then, $\exists N_1 \in \mathbb{N}$ such that $B(0,\varepsilon) \cap \alpha_n \neq \emptyset,\ \forall n \geq N_1$. Now $\rho_n \to +\infty$ implies that $B(0,\varepsilon) \cap (\alpha_n \backslash M(\alpha_n)) \neq \emptyset$ for $n \geq N_1'$.

Let $b \in B(0,\varepsilon) \cap (\alpha_n \backslash M(\alpha_n))$. We have for every $\xi \in C_a(\alpha_n)$,

$$|\xi - b|^2 \geq |\xi - a|^2,$$

namely

$$\begin{aligned} 2(\xi|a-b) &\geq |a|^2 - |b|^2 \qquad (\geq 0) \\ &\geq \rho_n^2 - |b|^2. \end{aligned}$$

Now, $|\xi||a-b| \geq (\xi|a-b)$, then,

$$|\xi||a-b| \geq \frac{(\rho_n + |b|)(\rho_n - |b|)}{2}.$$

Moreover, $|a-b| \leq |a| + |b| \leq \rho_n + |b|$. One finally gets

$$|\xi| \geq \frac{\rho_n - |b|}{2} \geq \frac{\rho_n - \varepsilon}{2}.$$

Since $\rho_n \to +\infty$ as $n \to +\infty,\ |\xi| \geq \frac{\rho_n}{2+\varepsilon}$, for large enough n.

If $0 \notin \mathrm{supp}(P)$ we show likewise that $|\xi| \geq \frac{\rho_n - |x_0| - \varepsilon}{2},\ \forall x_0 \in \mathrm{supp}(P)$ which implies the announced result since $\rho_n \to +\infty$.

(b) We will make an abuse of notation by considering that

$$\rho_n = \rho_n^+ := \max\{x,\ x \in \alpha\}.$$

In what follows all results on $\rho_-(\alpha) := \max\{-x,\ x \in \alpha\}$ can be derived by using $-X$ instead of X. Let $\alpha_n = \{x_1^{(n)}, \cdots, x_n^{(n)}\}$ with $x_1^{(n)} < \cdots < x_n^{(n)}$ and suppose that (up to a subsequence)

$$\frac{x_{n-1}^{(n)}}{x_n^{(n)}} \to \rho < 1.$$

Let $\varepsilon > 0$ such that $\rho + \varepsilon < 1$. We have for large enough n,

$$\frac{x_{n-1}^{(n)}}{x_n^{(n)}} < \rho + \varepsilon < 1$$

or equivalently,

$$\frac{x_{n-1}^{(n)} + x_n^{(n)}}{2} < x_n^{(n)} \left(\frac{1 + \rho + \varepsilon}{2} \right). \tag{3.2.12}$$

Let ρ' be such that $0 < \rho' < \frac{1-(\rho+\varepsilon)}{2}$, that is,

$$\left(\frac{1 + \rho + \varepsilon}{2} \right) < 1 - \rho' < 1. \tag{3.2.13}$$

It follows from (3.2.12) and (3.2.12) that

$$\begin{aligned}\int_{\frac{x_{n-1}^{(n)}+x_n^{(n)}}{2}}^{x_n^{(n)}}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du &\geq \int_{\frac{x_n^{(n)}(1+\rho+\varepsilon)}{2}}^{x_n^{(n)}(1-\rho')}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du\\ &\geq (\rho')^{r-1}\int_{\frac{x_n^{(n)}(1+\rho+\varepsilon)}{2}}^{x_n^{(n)}(1-\rho')}f(u)du\\ &\geq \rho'' x_n^{(n)}f(c_n)\end{aligned}$$

with $\rho''=(\rho')^{r-1}(\frac{1}{2}-\rho'-\frac{\rho+\varepsilon}{2})>0$ and $c_n\in(x_n^{(n)}(1+\rho+\varepsilon)/2, x_n^{(n)}(1-\rho'))$.

On the other hand, we have

$$\frac{1}{x_n^{(n)}f(x_n^{(n)})}\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du=\int_1^{+\infty}(u-1)^{r-1}\frac{f(ux_n^{(n)})}{f(x_n^{(n)})}du.$$

It follows from Assumption $(\mathbf{G_r})$ that

$$\limsup_{n\to+\infty}\frac{1}{x_n^{(n)}f(x_n^{(n)})}\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du=0.$$

Consequently we have for large enough n,

$$\frac{1}{x_n^{(n)}f(x_n^{(n)})}\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du<\rho''$$

so that ($A<c_n<x_n^{(n)}$ for large enough n and f is non-increasing in $[A,+\infty)$)

$$\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du<\rho'' x_n^{(n)}f(x_n^{(n)})\leq\rho'' x_n^{(n)}f(c_n)\leq\int_{\frac{x_{n-1}^{(n)}+x_n^{(n)}}{2}}^{x_n^{(n)}}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du$$

which is not possible since the L^r-stationary equation implies

$$\int_{\frac{x_{n-1}^{(n)}+x_n^{(n)}}{2}}^{x_n^{(n)}}\left(1-\frac{u}{x_n^{(n)}}\right)^{r-1}f(u)du=\int_{x_n^{(n)}}^{+\infty}\left(\frac{u}{x_n^{(n)}}-1\right)^{r-1}f(u)du.$$

We have then shown that

$$\lim_{n\to+\infty}\frac{x_n^{(n)}}{x_{n-1}^{(n)}}=1.$$

It follows that $\forall\varepsilon>0,\exists\, n_\varepsilon$ such that $\forall n\geq n_\varepsilon,\ x_n^{(n)}<(1+\varepsilon)x_{n-1}^{(n)}$. Thus,

$$\rho_n=x_n^{(n)}<(1+\varepsilon)x_{n-1}^{(n)}<(1+\varepsilon)\xi\quad\forall\,\xi\in C_a(\alpha_n), a\in M(\alpha_n).$$

This completes the proof. □

Lemma 3.2.2. *Let $(\alpha_n)_{n\geq 1}$ be a sequence of L^r-optimal n-quantizers of the distribution P. Suppose that **(H)** holds. Then for large enough n,*

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq C_{r,d,U}\, n^{-\frac{r+d}{d}}, \tag{3.2.14}$$

where

$$C_{r,d,U} = 2^{-(r+d)} \left(\frac{r}{d+r}\right)^{1/r} \left(\frac{d}{d+r}\right)^{d/r} \frac{\varepsilon_0}{1+\varepsilon_0}\, Q_{d+r}(U(\bar{B}(x_0, \frac{r_0}{2}))). \tag{3.2.15}$$

***Proof*.** **Step** 1. Let $y \in \mathbb{R}^d$. Without loss of generality we temporarily set $\delta_n = d(y, \alpha_n)$. Following the lines of the proof of Theorem 2 in [GLP] we have for every $x \in B(y, \delta_n/2)$ and $a \in \alpha_n$,

$$|x-a| \geq |y-a| - |x-a| \geq \delta_n/2$$

and hence

$$d(x, \alpha_n) \geq \delta_n/2 \geq |x-y|, \quad x \in B(y, \delta_n/2).$$

It follows, by setting $\beta_n = \alpha_n \cup \{y\}$, that

$$d(x, \beta_n) = |x-y|, \quad x \in B(y, \delta_n/2).$$

Consequently for every $b \in (0, 1/2)$,

$$\begin{aligned}
e^r_{n,r}(X) - e^r_{n+1,r}(X) &\geq \int d(x,\alpha_n)^r dP(x) - \int d(x,\beta_n)^r dP(x) \\
&\geq \int_{B(y,\delta_n b)} (d(x,\alpha_n)^r - d(x,\beta_n)^r) dP(x) \\
&= \int_{B(y,\delta_n b)} (d(x,\alpha_n)^r - |x-y|^r) dP(x) \\
&\geq \int_{B(y,\delta_n b)} ((\delta_n/2)^r - (\delta_n b)^r) dP(x) \\
&= (2^{-r} - b^r)\, \delta_n^r\, P(B(y, \delta_n b)).
\end{aligned}$$

Step 2. Now, coming back to the core of our proof let x_0 and r_0 be as in **(H)**. For every $y \in \bar{B}(x_0, \frac{r_0}{2})$ we have

$$\begin{aligned}
e^r_{n,r}(X) - e^r_{n+1,r}(X) &\geq (2^{-r} - b^r)\, \delta_n^r\, P(B(y, (b\,\delta_n) \wedge \frac{r_0}{2})) \\
&\geq (2^{-r} - b^r)\, \delta_n^r \varepsilon_0 \left((b\,\delta_n)^d \wedge (\frac{r_0}{2})^d\right) \mathbf{1}_{\{y \in \bar{B}(x_0, \frac{r_0}{2})\}}.
\end{aligned}$$

One checks that

$$\sup_{y \in \bar{B}(x_0, \frac{r_0}{2})} d(y, \alpha_n) \to 0.$$

Otherwise $\exists y_\infty \in \bar{B}(x_0, \frac{r_0}{2}),\ \eta > 0$ and a subsequence $(\alpha_{\varphi(n)})_{n\geq 1}$ of $(\alpha_n)_{n\geq 1}$ such that for every $n \geq 1,\ d(y_\infty, \alpha_{\varphi(n)}) > \frac{\eta}{2}$. Then

$$\int d(\alpha_{\varphi(n)}, \xi)^r P(d\xi) \geq \int_{\bar{B}(y_\infty, \frac{\eta}{4})} d(\alpha_{\varphi(n)}, \xi)^r P(d\xi).$$

Moreover for every $\xi \in B(y_{\infty,\frac{\eta}{4}})$

$$d(\alpha_{\varphi(n)},\xi) \geq d(y_\infty,\alpha_{\varphi(n)}) - d(y_\infty,\xi) \geq \frac{\eta}{2} - \frac{\eta}{4}$$

so that

$$\int d(\alpha_{\varphi(n)},\xi)^r P(d\xi) \geq (\frac{\eta}{2})^r P(B(y_{\infty,\frac{\eta}{4}})).$$

This contradicts the fact that $e_{n,r}(X) \to 0$ as n goes to infinity. Consequently, for large enough n,

$$\sup_{y\in\bar{B}(x_0,\frac{r_0}{2})} d(y,\alpha_n) \leq \frac{r_0}{2}$$

so that

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq (2^{-r} - b^r)\, b^d d(y,\alpha_n)^{d+r} \varepsilon_0\, \mathbf{1}_{\{y\in\bar{B}(x_0,\frac{r_0}{2})\}}.$$

It follows that

$$\begin{aligned}\int_{\bar{B}(x_0,\frac{r_0}{2})} (e^r_{n,r}(X) - e^r_{n+1,r}(X))\lambda_d(dy) &\geq (2^{-r} - b^r)\, \varepsilon_0\, b^d \int_{\bar{B}(x_0,\frac{r_0}{2})} d(y,\alpha_n)^{d+r}\lambda_d(dy) \\ &\geq (2^{-r} - b^r)\, \varepsilon_0\ \ b^d\, \lambda_d(\bar{B}(x_0,\frac{r_0}{2}))\, e^{r+d}_{n,r+d}(U(\bar{B}(x_0,\frac{r_0}{2}))).\end{aligned}$$

Then,

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq (2^{-r} - b^r)\, \varepsilon_0\, b^d\, e^{r+d}_{n,r+d}(U(\bar{B}(x_0,\frac{r_0}{2}))).$$

Consequently, for large enough n,

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq (2^{-r} - b^r)\, \frac{\varepsilon_0}{1+\varepsilon_0}\, b^d\, Q_{d+r}(U(\bar{B}(x_0,\frac{r_0}{2})))\, n^{-\frac{d+r}{d}}.$$

As a function of b, the right hand side of this last inequality reaches its maximum at $b^\star = \frac{1}{2}\left(\frac{d}{d+r}\right)^{1/r}$. Which completes the proof. □

Proof of Theorem 3.2.1. Let $a \in M(\alpha_n)$ and $\varepsilon > 0$. We have,

$$\mathbb{E}|X - \widehat{X}^{\alpha_{n-1}}|^r \leq \mathbb{E}|X - \widehat{X}^{\alpha_n\setminus\{a\}}|^r$$

and

$$\begin{aligned}\mathbb{E}|X - \widehat{X}^{\alpha_n\setminus\{a\}}|^r &= \mathbb{E}\big(|X - \widehat{X}^{\alpha_n}|^r \mathbf{1}_{\{X\in C^c_a(\alpha_n)\}}\big) + \mathbb{E}\big(\min_{b\in\alpha_n\setminus\{a\}} |X-b|^r \mathbf{1}_{\{X\in C_a(\alpha_n)\}}\big) \\ &\leq \mathbb{E}|X - \widehat{X}^{\alpha_n}|^r + \mathbb{E}\big(\min_{b\in\alpha_n\setminus\{a\}} (|X|+|b|)^r \mathbf{1}_{\{X\in C_a(\alpha_n)\}}\big).\end{aligned}$$

It follows from Lemma 3.2.1 (a) that $\exists\, n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon,\ |X| > \frac{\rho_n}{2+\varepsilon}$ on $\{X \in C_a(\alpha_n)\}$. Consequently, for all $b \in \alpha_n\setminus\{a\},\ |b| \leq |a| = \rho_n < (2+\varepsilon)\,|X|$.

Hence,

$$\mathbb{E}|X - \widehat{X}^{\alpha_{n-1}}|^r - \mathbb{E}|X - \widehat{X}^{\alpha_n}|^r \leq (3+\varepsilon)^r \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>\frac{\rho_n}{2+\varepsilon}\}}\big).$$

Lemma 3.2.2 yields (since $(n-1)^{-\frac{r+d}{d}} \sim n^{-\frac{r+d}{d}}$ as $n \to +\infty$)

$$(1+\varepsilon)^{-1} C_{r,d,U}\, n^{-\frac{r+d}{d}} \leq (3+\varepsilon)^r \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>\frac{\rho_n}{2+\varepsilon}\}}\big)$$

for large enough n, so that for every $\varepsilon > 0$,

$$\liminf_n \left(n^{\frac{r+d}{d}} \bar{F}_r\big(\frac{\rho_n}{2+\varepsilon}\big) \right) \geq \frac{C_{r,d,U}}{(3+\varepsilon)^r(1+\varepsilon)}.$$

Taking the limit as $\varepsilon \to 0$ gives the statement $(3.2.7)$. Assertion $(3.2.8)$ is proved as above by using Lemma 3.2.1 (b) instead of Lemma 3.2.1 (a). □

Recall that $\bar{F}_r(x) = \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big)$. It is clear that this function is non-increasing and goes to 0 as $x \to +\infty$ (provided $\mathbb{E}|X|^r < +\infty$). Consequently, $-\log \bar{F}_r(x)$ is monotone nondecreasing and goes to $+\infty$ as x goes to $+\infty$. Moreover, we know that if a function f defined on $(0, +\infty)$ is increasing to $+\infty$ (at $+\infty$), its generalized inverse function $f^{\leftarrow}$ defined by $\forall x > 0$,

$$f^{\leftarrow}(x) = \inf\{t > 0,\ f(t) \geq x\} \tag{3.2.16}$$

is monotone increasing to $+\infty$. On the other hand, the following result holds (see [BGT]): If furthermore f is regularly varying (at $+\infty$) with index $1/\delta, \delta > 0$, then there exists a function ψ, regularly varying with index δ and satisfying

$$\psi(f(x)) \sim f(\psi(x)) \sim x \quad \text{ as } x \to +\infty. \tag{3.2.17}$$

The function ψ is an asymptotic inverse of f and it is not necessarily increasing neither continuous. Moreover, ψ is unique up to asymptotic equivalence and $f^{\leftarrow}$ is one version of ψ.

We next show that for distributions with exponential tails, specifying either the asymptotic inverse ϕ_r (if any) of the function $-\log \bar{F}_r$ or finding some asymptotic upper bound ψ_r of ϕ_r (having some " nice" properties) leads to an upper estimate of the maximal radius sequence. This estimate is connected to the chosen function ψ_r.

When the distribution has a polynomial tail, we will look for the asymptotic inverse function of $-\log \bar{F}_r(e^x)$ or some asymptotic upper bound ψ_r of it to provide an upper estimate of $(\log \rho_n)_{n\geq 1}$.

Proposition 3.2.3. *Assume that the distribution P of X has an unbounded support and satisfies $(\boldsymbol{H})$. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers.*

(a) *If ψ_r is a measurable nondecreasing function, regularly varying with index δ and*

$$\psi_r(-\log \bar{F}_r(x)) \geq x + o(x) \quad \text{ as } x \to +\infty, \tag{3.2.18}$$

then

$$\limsup_{n\to+\infty} \frac{\rho_n}{\psi_r(\log(n))} \leq 2\left(1 + \frac{r}{d}\right)^{\delta}. \tag{3.2.19}$$

If $d = 1$ and $r \geq 1$ and if $(\mathbf{G_r})$ holds then, one has

$$\limsup_{n\to+\infty} \frac{\rho_n}{\psi_r(\log(n))} \leq (r+1)^{\delta}. \tag{3.2.20}$$

In particular if $-\log \bar{F}_r$ has regular variation with index $1/\delta$ then $(3.2.19)$ holds with $\psi_r = (-\log \bar{F}_r)^{\leftarrow}$.

(b) *If ψ_r is a measurable nondecreasing function, regularly varying with index δ and*

$$\psi_r(-\log \bar{F}_r(e^x)) \geq x + o(x) \quad \text{ as } x \to +\infty, \tag{3.2.21}$$

then

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\psi_r(\log(n))} \leq \left(1 + \frac{r}{d}\right)^{\delta}. \tag{3.2.22}$$

If $-\log \bar{F}_r(e^{\cdot})$ has regular variation of index $1/\delta$ then $(3.2.22)$ holds with $\psi_r = (-\log \bar{F}_r(e^{\cdot}))^{\leftarrow}$.

Prior to the proof, let us make some comment on the proposition. First note that the measurability of ψ_r is necessary to define the regular varying property. On the other hand we have for every $r > 0$ and for every $x > 0$,

$$\bar{F}_r(x) \geq x^r \bar{F}(x).$$

Then

$$-\log \bar{F}_r(x) \leq -\log \bar{F}(x) - r\log(x).$$

According to the nondecreasing hypothesis on ψ we have for every $x > 1$,

$$\psi_r(-\log \bar{F}_r(x)) \leq \psi_r(-\log \bar{F}(x) - r\log(x)) \leq \psi_r(-\log \bar{F}(x)) \tag{3.2.23}$$

since $\log(x) > 0$. Hence if Assumption $(3.2.18)$ holds then

$$\psi_r(-\log \bar{F}(x)) \geq x + o(x).$$

We will see further on that for distributions with exponential tails, the function ψ_r in the statement (a) of the proposition does not depend on r. However in the situation of the item (b) of the proposition, Assumption $(3.2.21)$ implies that

$$\psi_r(-\log \bar{F}(e^x)) \geq (r+1)x + o(x).$$

Consequently, taking $\bar{F}$ instead of $\bar{F}_r$ in Assumption $(3.2.21)$ will induce a loss of precision in the upper estimate of $\log \rho_n$.

Also remark that if $-\log \bar{F}_r$ (resp. $-\log \bar{F}_r(e^\bullet)$) is measurable, locally bounded and regularly varying with index $1/\delta, \delta > 0$ then its generalized inverse function ϕ_r (resp. Φ_r) is measurable increasing to $+\infty$, regularly varying with index δ and, $\phi_r(-\log \bar{F}_r(x)) = x + o(x)$ (resp. $\Phi_r(-\log \bar{F}_r(e^x)) = x + o(x)$). Consequently, both inequalities $(3.2.19)$ and $(3.2.20)$ (resp. claim $(3.2.22)$) hold with ϕ_r (resp. Φ_r) in place of ψ_r. However, ϕ_r (resp. Φ_r) is in general not easy to compute and the examples below show that it is often easier to exhibit directly a function ψ_r satisfying the announced hypotheses without inducing any asymptotic loss of accuracy.

We prove now the proposition.

***Proof*. (a)** It follows from $(3.2.7)$ and $(3.2.8)$ that for every $\varepsilon > 0$, there is a positive real constant $C_{r,d,U,\varepsilon}$ depending on the indexing parameters such that

$$n^{-\frac{d+r}{d}} C_{r,d,U,\varepsilon} \leq \bar{F}_r\big(\frac{\rho_n}{c_{r,d} + \varepsilon}\big)$$

where (from now on) $c_{r,1} = 1$ if $r > 1$; $c_{r,d} = 2$ otherwise. Therefore, one has

$$\frac{r+d}{d}\log(n) - \log(C_{r,d,U,\varepsilon}) \geq -\log \bar{F}_r\big(\frac{\rho_n}{c_{r,d} + \varepsilon}\big).$$

Combining the fact that ψ_r is nondecreasing and Assumption $(3.2.18)$ yields

$$\begin{aligned}\psi_r\big(\frac{r+d}{d}\log(n) - \log(C_{r,d,U,\varepsilon})\big) &\geq \psi_r\big(-\log \bar{F}_r\big(\frac{\rho_n}{c_{r,d}+\varepsilon}\big)\big) \\ &\geq \frac{\rho_n}{c_{r,d}+\varepsilon} + o(\rho_n).\end{aligned}$$

Moreover, dividing by $\psi_r(\log(n))$ (which is positive for large enough n) yields

$$\frac{\rho_n}{\psi_r(\log(n))} \leq (c_{r,d}+\varepsilon)\big(1+\frac{o(\rho_n)}{\rho_n}\big)^{-1}\frac{\psi_r\big(\frac{r+d}{d}\log(n)-\log(C_{r,d,U,\varepsilon})\big)}{\psi_r(\log(n))}.$$

It follows from the regular varying hypothesis on ψ_r and $\lim_n \rho_n = +\infty$ that

$$\limsup_{n\to+\infty}\frac{\rho_n}{\psi_r(\log(n))} \leq (c_{r,d}+\varepsilon)\left(\frac{r+d}{d}\right)^{\delta}, \quad \forall \varepsilon>0.$$

The result follows by letting $\varepsilon \to 0$.

(b) As previously, one derives from $(3.2.7)$ and from Assumption $(3.2.21)$ and the nondecreasing hypothesis on ψ_r that

$$\begin{aligned}\psi_r\big(\frac{r+d}{d}\log(n)-\log(C_{r,d,U,\varepsilon})\big) &\geq \psi_r\big(-\log\bar{F}_r\big(\frac{\rho_n}{c_{r,d}+\varepsilon}\big)\big)\\ &\geq \log\rho_n - \log(c_{r,d}+\varepsilon)+o(\log\rho_n).\end{aligned}$$

It follows that

$$\frac{\log\rho_n}{\psi_r(\log(n))} \leq \left(1-\frac{\log(c_{r,d}+\varepsilon)}{\log\rho_n}+\frac{o(\log\rho_n)}{\log\rho_n}\right)^{-1}\frac{\psi_r\big(\frac{r+d}{d}\log(n)-\log(C_{r,d,U,\varepsilon})\big)}{\psi_r(\log(n))}.$$

Owing to the regular varying hypothesis on ψ_r and the fact that $\lim_n \rho_n = +\infty$, we have

$$\limsup_{n\to+\infty}\frac{\log\rho_n}{\psi_r(\log(n))} \leq \left(\frac{r+d}{d}\right)^{\delta}.$$

□

We next give an explicit asymptotic upper bound for the convergence rate of the maximal radius sequence in the sense that the function ψ_r is made explicit. These bounds are derived on the rate of decay of the generalized survival function $\bar{F}_r$.

Criterion 3.2.1. ***(a)*** *Let X be a random variable with unbounded support. Let $r>0$ and let $(\alpha_n)_{n\geq 1}$ be an L^r-optimal sequence of n-quantizers for X. Let $\kappa>0$ such that $e^{|X|^\kappa}\in L^{0+}(\mathbb{P})$. Set*

$$\theta^\star = \sup\big\{\theta>0,\ \limsup_{x\to+\infty} e^{\theta x^\kappa}\bar{F}_r(x)<+\infty\big\} = \sup\big\{\theta>0,\ \mathbb{E}\,e^{\theta|X|^\kappa}<+\infty\big\}. \tag{3.2.24}$$

Then $\theta^\star\in(0,+\infty]$ and

$$\limsup_{n\to+\infty}\frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq 2\left(\frac{r+d}{d\,\theta^\star}\right)^{1/\kappa}. \tag{3.2.25}$$

When $d=1$ and $r\geq 1$, if $(\mathbf{G_r})$ holds then

$$\limsup_{n\to+\infty}\frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \left(\frac{r+1}{\theta^\star}\right)^{1/\kappa}.$$

(b) *Let $X\in L^{r+}(\mathbb{P})$ be a random variable with unbounded support. Set*

$$\zeta^\star = \sup\big\{\zeta>0,\ \limsup_{x\to+\infty} x^{\zeta-r}\bar{F}_r(x)<+\infty\big\} = \sup\big\{\zeta>r,\ \mathbb{E}|X|^\zeta<+\infty\big\}. \tag{3.2.26}$$

Then $\zeta^\star \in (r, +\infty]$ *and*

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \leq \frac{1}{\zeta^\star - r}\frac{r+d}{d}. \tag{3.2.27}$$

Prior to the proof we can make the following remark.

Remark 3.2.1. If $X \in \bigcap_{r>0} L^r(\mathbb{P})$ then $\zeta^\star = +\infty$ and consequently $\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = 0$.

Proof . **(a)** The equalities in $(3.2.24)$ and $(3.2.26)$ are obvious.

Let $\theta \in (0, \theta^\star)$. We have

$$\mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big) = \mathbb{E}\big(|X|^r \mathbf{1}_{\left\{e^{\theta|X|^\kappa} > e^{\theta x^\kappa}\right\}}\big) \leq e^{-\theta x^\kappa}\mathbb{E}\big(|X|^r e^{\theta|X|^\kappa}\big).$$

Now, the right hand side of this last inequality is finite because if $\theta' \in (\theta, \theta^\star)$,

$$|x|^r e^{\theta|x|^\kappa} \leq 1 + C_{\theta,\theta'} e^{\theta'|x|^\kappa}.$$

As a consequence,

$$-\log \bar{F}_r(x) \geq \theta x^\kappa + C_{\theta,X}, \qquad C_{\theta,X} \in \mathbb{R}.$$

Let $\psi_\theta(y) = \big(\frac{y}{\theta}\big)^{1/\kappa}$. As a function of y, ψ_θ is continuous (then measurable) increasing to $+\infty$, regularly varying with index $\delta = \frac{1}{\kappa}$ and we have

$$\psi_\theta(-\log \bar{F}_r(x)) \geq \left(x^\kappa + \frac{C_X}{\theta}\right)^{1/\kappa} = x + o(x), \quad \text{as } x \to +\infty.$$

It follows from Proposition $3.2.3$ (a) that

$$\limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq c_{r,d}\left(\frac{d+r}{d\,\theta}\right)^{1/\kappa} \qquad \forall \theta \in (0, \theta^\star).$$

Letting $\theta \to \theta^\star$ completes the proof.

(b) Let $\zeta \in (r, \zeta^\star)$. We have

$$\begin{aligned}\mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big) &= \mathbb{E}\big(|X|^r \mathbf{1}_{\{1<x^{-\zeta+r}|X|^{\zeta-r}\}}\big)\\ &\leq x^{-\zeta+r}\mathbb{E}|X|^\zeta.\end{aligned}$$

Then

$$-\log \bar{F}_r(x) \geq (\zeta - r)\log(x) + C$$

so that by setting $\psi_r(x) = \frac{x}{\zeta - r}$, it follows from Proposition $3.2.3$ (b) that

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \leq \frac{1}{\zeta - r}\frac{r+d}{d}.$$

Letting ζ go to $\zeta^\star$ yields the assertion $(3.2.27)$. □

Remark 3.2.2. Note that the choice of the function ψ_r in the statement (a) of Proposition $3.2.3$ does not depend on r as approved in the proof of the item (a) of the previous criterion. But for distributions with polynomial tails the choice of ψ_r clearly depends on r.

We now give more explicit results for specified density functions.

Corollary 3.2.1. ***(a)*** *Suppose that the density f of X satisfies*

$$f(x) \;\propto\; |x|^c\, e^{-\vartheta|x|^\kappa} \qquad x \in \mathbb{R}^d;\ \vartheta, \kappa > 0;\ c > -d. \tag{3.2.28}$$

Then $\theta^\star = \vartheta$ and

$$\limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \le \frac{c_{r,d}}{\vartheta^{1/\kappa}} \Big(1 + \frac{r}{d}\Big)^{1/\kappa}. \tag{3.2.29}$$

(b) *If the density of X reads*

$$f(x) \;\propto\; \frac{(\log|x|)^\beta}{|x|^c}\, \mathbf{1}_{\{|x|>1\}} \qquad x \in \mathbb{R}^d,\ \beta \in \mathbb{R},\ c > r + d \tag{3.2.30}$$

then $\zeta^\star = c - d$ and

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \le \frac{1}{c-d-r} \frac{r+d}{d}. \tag{3.2.31}$$

(c) *The statement $(3.2.29)$ (resp $(3.2.31)$) holds if the density of X is simply equivalent to the specified density in $(3.2.28)$ (resp in $(3.2.30)$).*

Notice that the restriction $c > r + d$ in $(3.2.30)$ ensures that $\mathbb{E}|X|^r < +\infty$.

Proof. **(a)** We have

$$\begin{aligned}\bar{F}_r(x) = \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>x\}}\big) \;&=\; K \int_{\{|u|>x\}} |u|^{r+c}\, e^{-\vartheta|u|^\kappa} d\lambda_d(u) \\ &=\; K\, V_d \int_x^{+\infty} \rho^{r+c+d-1}\, e^{-\vartheta\rho^\kappa} d\rho\end{aligned}$$

where $K^{-1} = \int |x|^c e^{-\vartheta|x|^\kappa} d\lambda_d(x)$ is the normalizing positive real constant in $(3.2.28)$ and V_d denotes the volume of the Euclidean unit ball. Integrating by parts and using usual integral comparison rules yields

$$\bar{F}_r(x) = C_{d,\vartheta,\kappa}\, x^{r+c+d-\kappa}\, e^{-\vartheta x^\kappa} \big(1 + o(1)\big)$$

for a positive real constant $C_{d,\vartheta,\kappa}$. It follows that if $\theta < \vartheta$ then

$$\limsup_{x\to+\infty}\ e^{\theta x^\kappa} \bar{F}_r(x) < +\infty$$

and if $\theta > \vartheta$ then

$$\limsup_{x\to+\infty}\ e^{\theta x^\kappa} \bar{F}_r(x) = +\infty.$$

Which means that $\theta^\star = \vartheta$ and the statement $(3.2.29)$ follows from Criterion $3.2.1\ (a)$.

(b) We have for every $r > 0$, for every $x > 1$,

$$\bar{F}_r(x) = \int_{\{|u|>x\}} \frac{(\log|u|)^\beta}{|u|^{c-r}} d\lambda_d(u) = V_d \int_x^{+\infty} \frac{(\log\rho)^\beta}{\rho^{c'}} d\rho.$$

with $c' = c - r - d + 1$ ($c' > 1$) and V_d defined as previously. Integrating by parts and multiplying by $x^{\zeta - r}$ yields

$$x^{\zeta - r}\bar{F}_r(x) = \frac{K\, V_d}{c - (r+d)} \log(x)^{\beta} x^{\zeta - c + d}(1 + o(1)).$$

Consequently, $\zeta^\star = c - d$. The statement $(3.2.31)$ follows from Criterion 3.2.1 (b).

(c) Obvious from the forgoing. □

We now give some examples for usual distributions.

Example 3.2.1. (a) • If $X \sim \mathcal{N}(0; I_d)$, we have

$$f(x) = (2\pi)^{-d/2} e^{-\frac{1}{2}|x|^2}.$$

It follows from item (a) of the previous corollary (with $\vartheta = 1/2, \kappa = 2, c = 0$) that for every $r > 0$, for every $d \geq 1$,

$$\limsup_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq 2\sqrt{2\left(1 + \frac{r}{d}\right)}.$$

In particular, when $r \geq 1, d = 1$ we have $\limsup_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq \sqrt{2(r+1)}$.

• For a double Gamma distribution in the real line where

$$f(x) = \frac{\lambda^a}{2\Gamma(a)} |x|^{a-1} e^{-\lambda |x|}, \quad x \in \mathbb{R};\ \lambda, a > 0$$

or a Gamma distribution for which

$$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \mathbf{1}_{\{x \geq 0\}}, \quad \lambda, a > 0$$

we have (from Corollary 3.2.1 (a) with $c = a - 1, \vartheta = \lambda, \kappa = 1$) for every $r \in (0, 1)$,

$$\limsup_{n \to +\infty} \frac{\rho_n}{\log(n)} \leq \frac{2(r+1)}{\lambda}$$

and in case $r \geq 1$ we have

$$\limsup_{n \to +\infty} \frac{\rho_n}{\log(n)} \leq \frac{r+1}{\lambda};$$

which coincides with the sharp rate given in $(3.2.3)$.

• When X has a logistic distribution with density $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}$ we have

$$f(x) \ \sim\ e^{-x} \qquad \text{as } x \to +\infty.$$

Then, it follows from Corollary 3.2.1 (c) that $(\rho_n)_{n \geq 1}$ has the same upper asymptotic as the exponential distribution with parameter $\lambda = 1$.

• As concerns the Weibull distribution with shape parameter $\kappa > 0$ with density function

$$f(x) = \kappa x^{\kappa - 1} e^{-x^\kappa} \mathbf{1}_{\{x \geq 0\}}$$

it follows from Corollary 3.2.1, (a) (with $\vartheta = 1$) that for $r \in (0,1)$,

$$\limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \le 2(r+1)^{1/\kappa}$$

and if $r \ge 1$

$$\limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \le (r+1)^{1/\kappa}.$$

(b) Suppose X is a random variable having a Pareto distribution with index $\gamma > r$. The density function reads

$$f(x) = \gamma x^{-(\gamma+1)}\mathbf{1}_{\{x>1\}}.$$

Then we deduce from Corollary 3.2.1 (b) (with $d = 1, c = \gamma+1, \beta = 0$) that

$$\limsup_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \le \frac{r+1}{\gamma - r}.$$

3.2.2 Lower estimate

In this section we investigate the asymptotic lower estimate of the maximal radius sequence $(\rho_n)_{n\ge1}$ induced by an L^r-optimal sequence of n-quantizers. First we introduce the family of the (r,s)-distributions which will play a crucial role to obtain the optimal lower estimate for the rate of the maximal radius sequence.

Let $r > 0, s > r$. Since the L^r-norm is increasing, it is clear that for every $s \le r$ any L^r-optimal sequence of quantizers $(\alpha_n)_{n\ge1}$ is L^s**-rate optimal** $i.e.$

$$\limsup_{n\to+\infty} n^{1/d}\|X - \widehat{X}^{\alpha_n}\|_s < +\infty. \tag{3.2.32}$$

But if $s > r$ (and $X \in L^s(\mathbb{P})$) this asymptotic rate optimality usually fails . So is always the case when $s > r + d$ and X has a probability distribution f satisfying $\lambda_d(f > 0) = +\infty$, see [GLP]. But it is established in [SAG] that some linear transformation of the L^r-optimal quantizers (α_n) makes possible to overcome the critical exponent $r + d$, that is, one can always construct an L^s-rate-optimal sequence of quantizers by a linear transformation of the L^r-optimal sequence of quantizers (α_n). However there are some distributions for which $(3.2.32)$ holds for every $s \in [r, r+d)$. This leads to the following definition:

Definition 3.2.1. *Let $s, r > 0, s > r$. A random vector $X \in L^s(\mathbb{P})$ has an (r,s)-**distribution** if any L^r-optimal sequence $(\alpha_n)_{n\ge1}$ is L^s-rate optimal.*

For $\nu \in (0,d)$, sufficient conditions such that X has an $(r, r+\nu)$-distribution are provided in [GLP]. Let us mention two criteria ensuring that a random vector X has an $(r, r+\nu)$-distribution. The first one deals with distributions with radial tails.

Criterion 3.2.2. ***(a)*** *Let $d \ge 1$. If $f = h(|\cdot|)$ on $B_{|\cdot|}(0,N)^c$ with $h : (R, +\infty) \to \mathbb{R}_+,\ R \in \mathbb{R}_+$, a decreasing function and $|\cdot|$ any norm on $\mathbb{R}^d$. If*

$$\int f(cx)^{-\frac{r+\nu}{r+d}} dP(x) < +\infty \tag{3.2.33}$$

for some $c > 1$. Then X has an $(r, r+\nu)$-distribution.
(b) *If $d = 1$ and if $\mathrm{supp}(P) \subset [R_0, +\infty)$ for some $R_0 \in \mathbb{R}$ and $f_{|(R_0', +\infty)}$ is decreasing for $R_0' \ge R_0$. If further Assumption $(3.2.33)$ holds for some $c > 1$. Then X has an $(r, r+\nu)$-distribution.*

The following criterion works for distributions with non radial tails.

Criterion 3.2.3. *Let $r > 0,\ \nu \in (0,d),\ P = f \cdot \lambda_d$ and $\int |x|^{r+\eta} P(dx) < +\infty$ for some $\eta > 0$. Assume that $\mathrm{supp}(P)$ is convex and that f satisfies the local growth control assumption : there exists real numbers $\varepsilon \geq 0,\ \eta \in (0,1/2),\ M, C > 0$ such that*

$$\forall x, y \in \mathit{supp}(P),\ |x| \geq M,\ |y-x| \leq 2\eta|x| \implies f(y) \geq C f(x)^{1+\varepsilon}.$$

If

$$\int f(x)^{-\frac{(r+\nu)(1+\varepsilon)}{r+d}} dP(x) < +\infty, \tag{3.2.34}$$

then X has an $(r, r+\nu)$-distribution.

Furthermore a necessary condition for X (with density f) to have an $(r, r+\nu)$-distribution is (see [GLP]) :

$$X \text{ has an } (r,r+\nu)\text{-distribution} \implies \int f(x)^{-\frac{(r+\nu)}{d+r}} dP(x) < +\infty. \tag{3.2.35}$$

It follows from $(3.2.33)$ and $(3.2.35)$ that the Gaussian distribution, the Weibull and the Gamma distributions have an $(r, r+\nu)$-distribution if and only if $\nu \in (0,d)$. The Pareto distribution with index $\gamma > r$ has an $(r, r+\nu)$-distribution if and only if $\nu \in (0, \frac{\gamma - r}{\gamma+1})$.

Now, suppose that X has an $(r, r+\nu)$-distribution for some $\nu \in (0,d)$ and set

$$\nu_X^\star := \sup\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\}.$$

Note that

$$\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\} = (0, \nu_X^\star) \text{ or } (0, \nu_X^\star]$$

and that $X \in L^{r+\nu}(\mathbb{P}),\ \forall \nu \in (0, \nu_X^\star)$. When

$$\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\} = \emptyset$$

we set $\nu_X^\star = 0$.

This index $\nu_X^\star$ will play a crucial role to determine the lower bound of the maximal radius sequence. Recall that if X has a density f satisfying $\lambda_d(f > 0) = +\infty$ then a necessary condition for X to have $(r, r+\nu)$-distribution is that $\nu < d$. Which means that $\nu_X^\star \leq d$. However, this inequality may stand strictly as approved by the Pareto distribution with index γ for which $\nu_X^\star = \frac{\gamma - r}{\gamma+1} < 1$.

We present below two different approaches to get the lower bound for the maximal radius sequence. The first one involves the generalized survival functions $\bar{F}_r$ like for upper bounds and is based on tail estimates. The second one is probably more original. It is based on random quantization and provides a close connection between the sequence $(\rho_n)_{n\geq 1}$ and the maximum of the norm of an $i.i.d$ sequence of random variables with distributions P.

Distribution tail approach

The main result of this section is the following theorem.

Theorem 3.2.2. *Let $r > 0$ and let X be a $\mathbb{R}^d$-valued random variable with probability distribution P. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers. For every $\nu \in (0, \nu_X^\star)$, the following statements hold:*

(a)

$$\limsup_{n\to+\infty}\ \sup_{c>0}\Big(c^{r+\nu}\,n^{\frac{r+\nu}{d}}\bar{F}\big(\rho_n+c\big)\Big)<+\infty. \tag{3.2.36}$$

(b)

$$\limsup_{n\to+\infty}\ \sup_{u>1}\Big(\big(1-1/u\big)^{r+\nu}\,n^{\frac{r+\nu}{d}}\bar{F}_r\big(u\rho_n\big)\Big)<+\infty. \tag{3.2.37}$$

Proof. **(a)** Let $c>0$ and let $\nu\in(0,\nu_X^\star)$. Then

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\mathbb{E}\left(\min_{a\in\alpha_n}|X-a|^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\right).$$

On the events $\{|X|>\rho_n+c\}$, we have: $|X|>\rho_n+c>\rho_n\geq|a|,\ \forall a\in\alpha_n$. Then

$$\begin{aligned}\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}&\geq\mathbb{E}\left(\min_{a\in\alpha_n}|X-a|^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\right)\\&\geq\mathbb{E}\left(\min_{a\in\alpha_n}\big(|X|-|a|\big)^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\right)\\&\geq\mathbb{E}\left(\big(|X|-\rho_n\big)^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\right)\\&\geq c^{r+\nu}\,\mathbb{P}\left(\{|X|>\rho_n+c\}\right).\end{aligned} \tag{3.2.38}$$

It follows that

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\sup_{c>0}\left(c^{r+\nu}\,\mathbb{P}\big(\{|X|>\rho_n+c\}\big)\right).$$

Since X has an $(r,r+\nu)$-distribution we have

$$\limsup_n\ n^{\frac{r+\nu}{d}}\|X-\widehat{X}^{\alpha_n}\|_{r+\nu}^{r+\nu}<+\infty.$$

Which completes the proof.

(b) is proved like (a). Inequality $(3.2.38)$ becomes: for every $u>1$,

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\mathbb{E}\left(\big(|X|-\rho_n\big)^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\right)\geq\mathbb{E}\left(|X|^{r+\nu}\,(1-1/u)^{r+\nu}\,\mathbf{1}_{\{|X|>u\rho_n\}}\right).$$

Then,

$$\mathbb{E}|X-\widehat{X}^{\alpha_n}|^{r+\nu}\geq\sup_{u>1}\left[(1-1/u)^{r+\nu}\,\mathbb{E}\left(|X|^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\right)\right].$$

Inequality $(3.2.37)$ follows by noticing that $\limsup_n n^{\frac{r+\nu}{d}}\|X-\widehat{X}^{\alpha_n}\|_{r+\nu}^{r+\nu}<+\infty$. □

Like for the upper estimate, given the asymptotic inverse function ϕ of $-\log\bar{F}$ or given an asymptotic lower bound ψ of ϕ satisfying some standard hypotheses specified below, we provide the asymptotic lower estimate for the maximal radius sequence for distributions with exponential tails.

For distributions with polynomial tails, we will rather look for the asymptotic inverse function $\Phi_{r,\nu}$, $\nu\in(0,\nu_X^\star)$ (if any) of $-\log\bar{F}_{r+\nu}(e^x)$ or some asymptotic lower bound of it to provide a lower estimate of $\log\rho_n$.

Proposition 3.2.4. *Let $r > 0$ and let X be an $\mathbb{R}^d$-valued random variable with distribution P. Suppose that X has an unbounded support. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers.*

(a) *If ψ is a measurable nondecreasing function going to $+\infty$ as $x \to +\infty$, regularly varying with index δ and satisfying*

$$\psi(-\log \bar{F}(x)) \leq x + o(x), \tag{3.2.39}$$

then

$$\liminf_{n\to+\infty} \frac{\rho_n}{\psi(\log(n))} \geq \left(\frac{r+\nu_X^\star}{d}\right)^\delta. \tag{3.2.40}$$

If $-\log \bar{F}$ is regularly varying of index $1/\delta$ then $(3.2.40)$ holds with $\psi = (-\log \bar{F})^{\leftarrow}$.

(b) *Let $\nu \in (0, \nu_X^\star)$. If there is a measurable nondecreasing function $\psi_{r,\nu}(x)$ going to $+\infty$ as $x \to +\infty$, regularly varying with index δ and satisfying*

$$\psi_{r,\nu}(-\log \bar{F}_{r+\nu}(e^x)) \leq x + o(x), \tag{3.2.41}$$

then

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\psi_{r,\nu}(\log(n))} \geq \left(\frac{r+\nu}{d}\right)^\delta. \tag{3.2.42}$$

In particular if $-\log \bar{F}_{r+\nu}(e^x)$ has regular variation with index $1/\delta$ then $(3.2.42)$ holds with $\psi_{r,\nu}(x) = (-\log \bar{F}_{r+\nu}(e^x))^{\leftarrow}$.

Let us provide a few comments on this proposition. We have for every $r > 0$ and for every $x > 0$, $\bar{F}_r(x) > x^r \bar{F}(x)$. Then

$$-\log \bar{F}_r(x) \leq -\log \bar{F}(x) - r\log(x).$$

According to the nondecreasing hypothesis on ψ we have for every $x > 1$

$$\psi(-\log \bar{F}_r(x)) \leq \psi(-\log \bar{F}(x) - r\log(x)) \leq \psi(-\log \bar{F}(x))$$

so that if $(3.2.39)$ holds then for every $r > 0$, for every $\nu \in (0, \nu_X^\star)$,

$$\psi(-\log \bar{F}_{r+\nu}(x)) \leq x + o(x).$$

Reproducing the proof (given below) of Proposition $3.2.4$ (a) by using $(3.2.37)$ instead of $(3.2.36)$ shows that $(3.2.40)$ still holds true. This means that for distributions with exponential tails, the function ψ does not depend on r and ν even if in Assumption $(3.2.39)$ we take the generalized survival function $\bar{F}_{r+\nu}$ in place of the regular survival function $\bar{F}$. However for distributions with polynomial tails like Pareto distribution the function $\psi_{r,\nu}$ in $(3.2.41)$ may depend on r and taking the regular survival function $\bar{F}$ in place of the generalized survival function $\bar{F}_{r+\nu}$ would make lose the dependance upon r and consequently lead to a less accurate result.

We next prove the proposition.

Proof. **(a)** Assume $\nu_X^\star > 0$ and let $\nu \in (0, \nu_X^\star)$. It follows from $(3.2.36)$ that for large enough n,

$$-\log \bar{F}(\rho_n + c) \geq -\log(C_{\nu,c}) + \frac{r+\nu}{d}\log(n)$$

where $C_{\nu,c}$ is a positive real constant depending on the indexing parameters. It follows from the fact that ψ is nondecreasing and goes to $+\infty$ and from Assumption $(3.2.39)$ that

$$\frac{\rho_n}{\psi(\log(n))} \geq \left(1 + \frac{c}{\rho_n} + \frac{o(\rho_n)}{\rho_n}\right)^{-1} \frac{\psi(\frac{r+\nu}{d}\log(n) - \log(C_{\nu,c}))}{\psi(\log(n))}.$$

Since ψ is regularly varying with index δ we have

$$\liminf_{n\to+\infty} \frac{\rho_n}{\psi(\log(n))} \geq \left(\frac{r+\nu}{d}\right)^{\delta}, \qquad \forall \nu \in (0, \nu_X^\star).$$

Letting $\nu \to \nu_X^\star$ give the announced result. If $\nu_X^\star = 0$, one follows the same proof with $\nu = 0$.

(b) This is proved like the statement **(b)** in Proposition 3.2.3 by considering $\bar{F}_{r+\nu}$ instead of $\bar{F}_r$, for $\nu \in (0, \nu_X^\star)$. □

The next criterion is the lower limit counterpart of Criterion 3.2.1.

Criterion 3.2.4. ***(a)*** *Let X be an $\mathbb{R}^d$-valued random variable with unbounded support and suppose that*

$$\theta_\star = \inf\left\{\theta > 0,\ \liminf_{x\to+\infty}\ e^{\theta x^\kappa}\mathbb{P}(|X| > x) > 0\right\} \in (0, +\infty]. \tag{3.2.43}$$

Then

$$\liminf_{n\to+\infty}\ \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \geq \left(\frac{r+\nu_X^\star}{d\,\theta_\star}\right)^{1/\kappa}. \tag{3.2.44}$$

(b) *Let X be a random variable with unbounded support such that $\nu_X^\star > 0$. Set*

$$\zeta_\star = \inf\left\{\zeta > 0,\ \forall \nu \in (0, \nu_X^\star),\ \liminf_{x\to+\infty}\ x^{\zeta-r-\nu}\bar{F}_{r+\nu}(x) > 0\right\} \in [r + \nu_X^\star, +\infty]. \tag{3.2.45}$$

Then

$$\liminf_{n\to+\infty}\ \frac{\log\rho_n}{\log(n)} \geq \frac{1}{\zeta_\star - r - \nu_X^\star}\,\frac{r+\nu_X^\star}{d}. \tag{3.2.46}$$

Proof . **(a)** Let $\theta \in (\theta_\star, +\infty)$. Then

$$\bar{F}(x) \geq C\, e^{-\theta x^\kappa}$$

for large enough x and for a positive real constant C. Therefore

$$-\log\bar{F}(x) \leq \theta x^\kappa(1 - \frac{\log(C)}{x^\kappa})$$

so that by setting $\psi_\theta(x) = (x/\theta)^{1/\kappa}$ we have

$$\psi_\theta(-\log\bar{F}(x)) \leq x + o(x).$$

It follows from Proposition $3.2.4\ (a)$ that

$$\liminf_{n\to+\infty}\ \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \geq \frac{1}{\theta^{1/\kappa}}\left(\frac{r+\nu_X^\star}{d}\right)^{1/\kappa}.$$

We let θ go to $\theta_\star$ to get the announced result.

(b) Let $\zeta \in (\zeta_\star, +\infty)$. We have, for every $\nu \in (0, \nu_X^\star)$,

$$\bar{F}_{r+\nu}(x) \geq C_\nu \, x^{-\zeta+r+\nu}$$

for large enough x and for a positive real constant C_ν. Then, by setting $\psi_{r,\nu}(x) = \frac{x}{\zeta-r-\nu}$ we get $\psi_{r,\nu}(-\log \bar{F}_{r+\nu}(x)) \leq \log(x) + o(\log(x))$. It follows from Proposition $3.2.4$ (b) that

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{1}{\zeta - r - \nu} \frac{r+\nu}{d}.$$

The right hand side of this last inequality is increasing on $(0, \nu_X^\star)$ (as a function of ν) and noincreasing on $(\zeta_\star, +\infty)$ (as a function of ζ) so that by letting ν and ζ go respectively to $\nu_X^\star$ and $\zeta_\star$ we get

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{1}{\zeta_\star - r - \nu_X^\star} \frac{r+\nu_X^\star}{d}.$$

□

Corollary 3.2.2. ***(a)*** *If the density function of X reads*

$$f(x) \;\propto\; |x|^c \, e^{-\vartheta|x|^\kappa} \; x \in \mathbb{R}^d; \; \vartheta, \kappa > 0; \; c > -d \tag{3.2.47}$$

then $\nu_X^\star = d$ and $\theta^\star = \theta_\star = \vartheta$. In this case we have for every $r > 0$, for every $d \geq 1$,

$$\frac{1}{\vartheta^{1/\kappa}} \left(1 + \frac{r}{d}\right)^{1/\kappa} \leq \liminf_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \frac{2}{\vartheta^{1/\kappa}} \left(1 + \frac{r}{d}\right)^{1/\kappa}. \tag{3.2.48}$$

When $d = 1$ and $r \geq 1$ we have

$$\lim_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = \left(\frac{r+1}{\vartheta}\right)^{1/\kappa}. \tag{3.2.49}$$

(b) *If X has a density f satisfying*

$$f(x) \;\propto\; \frac{(\log|x|)^\beta}{|x|^c} \, \mathbf{1}_{\{|x|>1\}} \qquad x \in \mathbb{R}^d, \; \beta \in \mathbb{R}, \; c > r + d \tag{3.2.50}$$

then $\nu_X^\star = d\left(1 - \frac{r+d}{c}\right) \in (0, d)$ and $\zeta_\star = \zeta^\star = c - d$. Furthermore we have for every $r > 0$ and every $d \geq 1$,

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c - r - d} \frac{r+d}{d}. \tag{3.2.51}$$

(c) *The claim $(3.2.48)$ (resp $(3.2.51)$) holds if the density of X is simply equivalent to the specified density in $(3.2.47)$ (resp in $(3.2.50)$).*

Proof . **(a)** It is obvious from Criterion 3.2.2 and $(3.2.35)$ that $\nu_X^\star = d$. Let K be the normalizing positive real constant in $(3.2.47)$. We have for every $x > 0$,

$$\begin{aligned}\mathbb{P}(|X| > x) &= K \int_{\{|u|>x\}} |u|^c \, e^{-\vartheta|u|^\kappa} d\lambda_d(u) \\ &= K_d \int_x^{+\infty} \rho^{c+d-1} \, e^{-\vartheta\rho^\kappa} d\rho \\ &= K_d \, \frac{x^{c+d-\kappa}}{\vartheta} e^{-\vartheta x^\kappa} (1 + o(1))\end{aligned}$$

where we used an integration by parts and usual integral comparison criterions. Consequently, if $\theta > \vartheta$ then

$$\liminf_{x\to+\infty} \; e^{\theta x^\kappa} \mathbb{P}(|X| > x) = +\infty$$

and if $\theta < \vartheta$ then

$$\liminf_{x\to+\infty} \; e^{\theta x^\kappa} \mathbb{P}(|X| > x) = 0.$$

Which means that $\theta_\star = \vartheta$ and the statements $(3.2.48)$ and $(3.2.49)$ follow from Criterion 3.2.1 (a) and Criterion 3.2.4 (a).

(b) Let $\bar{c} > 1$. We have

$$\begin{aligned}\int_{\{f>0\}} f(\bar{c}x)^{-\frac{r+\nu}{r+d}} f(x) d\lambda_d(x) &= K_{\bar{c}} \int_{\{|u|>1\}} \frac{\big(\log \bar{c} + \log|u|\big)^{-\frac{(r+\nu)\beta}{r+d}} (\log|u|)^\beta}{|u|^{c'}} d\lambda_d(u) \\ &= K_{d,\bar{c}} \int_1^{+\infty} \frac{\big(\log \bar{c} + \log\rho\big)^{-\frac{(r+\nu)\beta}{r+d}} (\log\rho)^\beta}{\rho^{c'-d+1}} d\rho \qquad (3.2.52)\end{aligned}$$

with $c' = c(1 - \frac{r+\nu}{r+d})$; $K_{\bar{c}}$ and $K_{d,\bar{c}}$ are some positive real constants. Remark that, as $\rho \to +\infty$, the numerator of the integrand part of the right hand side of Inequality $(3.2.52)$ is equivalent to $(\log\rho)^{\beta'}$, with $\beta' = \beta(d-\nu)/(r+d)$. Then we deduce that if

$$c' > d \iff \nu < d\left(1 - \frac{r+d}{c}\right) \text{ then } \int_{\{f>0\}} f(\bar{c}x)^{-\frac{r+\nu}{r+d}} f(x) d\lambda_d(x) < +\infty$$

and if

$$c' < d \iff \nu > d\left(1 - \frac{r+d}{c}\right) \text{ then } \int_{\{f>0\}} f(x)^{-\frac{r+\nu}{r+d}} f(x) d\lambda_d(x) = +\infty$$

so that (from Criterion 3.2.2 and Statement $(3.2.35)$) $\nu_X^\star = d\left(1 - \frac{r+d}{c}\right)$.

Let us show that $\zeta_\star = c - d$. For every $r > 0$, for every $\nu \in (0, \nu_X^\star)$, for every $x > 1$, integrating by parts and using integral comparison criterions yield

$$\begin{aligned}\bar{F}_{r+\nu}(x) = \mathbb{E}\left(|X|^{r+\nu} \mathbf{1}_{\{|X|>x\}}\right) &= \int_{\{|u|>x\}} \frac{(\log|u|)^\beta}{|u|^{c-r-\nu}} d\lambda_d(u) \\ &= V_d \int_x^{+\infty} \frac{(\log\rho)^\beta}{\rho^{c'}} d\rho \\ &= V_d \, \frac{x^{-c'+1}}{c'-1} (\log x)^\beta (1 + o(1))\end{aligned}$$

with $c' := c - r - \nu - d + 1 > 1$. It follows that

$$x^{\zeta - r - \nu} \bar{F}_{r+\nu}(x) = \frac{V_d}{c' - 1} (\log x)^{\beta} x^{\zeta - c + d} (1 + o(1))$$

so that for every $r > 0$ and for every $\nu \in (0, \nu_X^\star)$, if $\zeta > c - d$ then

$$\liminf_{x \to +\infty} x^{\zeta - r - \nu} \bar{F}_{r+\nu}(x) = +\infty$$

and if $\zeta < c - d$,

$$\liminf_{x \to +\infty} x^{\zeta - r - \nu} \bar{F}_{r+\nu}(x) = 0.$$

Hence $\zeta_\star = c - d$. It follows from Criterion $3.2.1\ (b)$ and Criterion $3.2.4\ (b)$ that

$$\frac{1}{\zeta_\star - r - \nu_X^\star} \frac{r + \nu_X^\star}{d} \leq \liminf_{n \to +\infty} \frac{\log \rho_n}{\log(n)} \leq \limsup_{n \to +\infty} \frac{\log \rho_n}{\log(n)} \leq \frac{1}{c - r - d} \frac{r + d}{d}.$$

Now $\big($recall that $\zeta_\star = c - d$ and $\nu_X^\star = d\left(1 - \frac{r+d}{c}\right)\big)$,

$$\frac{1}{\zeta_\star - r - \nu_X^\star} \frac{r + \nu_X^\star}{d} = \frac{1}{c - r - d} \frac{r + d}{d}$$

so that

$$\lim_{n \to +\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c - r - d} \frac{r + d}{d}.$$

(c) Obvious from what forgoes. □

We deal now with examples.

Example 3.2.2. **(1)** It follows from Corollary $3.2.2\ (a)$ that

• When $X \sim \mathcal{N}(0, I_d)$, for every $r > 0$, for every $d \geq 1$,

$$\sqrt{\frac{2(r+d)}{d}} \leq \liminf_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq \limsup_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} \leq 2\sqrt{\frac{2(r+d)}{d}}.$$

In case $d = 1$ and $r \geq 1$ we have

$$\lim_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} = \sqrt{2(r+1)}.$$

• If $X \sim \Gamma(a, \lambda), a > 0, \lambda > 0$ or if X has a double gamma distribution we have for every $r \geq 1$,

$$\lim_{n \to +\infty} \frac{\rho_n}{\log(n)} = \frac{r+1}{\lambda}$$

(which coincides to the exact rate given in $(3.2.3)$ for the exponential distribution) and for every $r \in (0, 1)$,

$$\frac{r+1}{\lambda} \leq \liminf_{n \to +\infty} \frac{\rho_n}{\log(n)} \leq \limsup_{n \to +\infty} \frac{\rho_n}{\log(n)} \leq \frac{2(r+1)}{\lambda}.$$

• As concern the logistic distribution, the maximal radius sequence has the same asymptotic as the exponential distribution with parameter $\lambda = 1$ following Corollary $3.2.2\ (c)$.

• For a Weibull distribution with shape parameter $\kappa > 0$ we have for every $r \geq 1$,

$$\lim_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = (r+1)^{1/\kappa}.$$

For $r \in (0,1)$, one has

$$(r+1)^{1/\kappa} \leq \liminf_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \limsup_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq 2(r+1)^{1/\kappa}.$$

(2) Suppose X is a random variable having a Pareto distribution with index $\gamma > r$ where the density reads $f(x) = \gamma x^{-(\gamma+1)}\mathbf{1}_{\{x>1\}}$. It follows from Corollary 3.2.2 (b) (with $c = \gamma+1$, $d = 1$)

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{r+1}{\gamma - r}. \tag{3.2.53}$$

We retrieve of course the sharp rate given in $(3.2.4)$.

An alternative approach by random quantization approach

Let $X \sim P$. Random quantization is another tool to compute the lower estimate of the maximal radius sequence. It makes a connection between ρ_n and the maximum of an $i.i.d$ sequence of random variables with distributions P.

Theorem 3.2.3. *Let $r > 0$ and let X be a random variable taking values in $\mathbb{R}^d$ with probability distribution P with $P_a \neq 0$. Assume $(\alpha_n)_{n\geq 1}$ is an $L^r(P)$-optimal sequence of n-quantizers. Let $(X_k)_{k\geq 1}$ be an $i.i.d$ sequence of $\mathbb{R}^d$-valued random variables with probability distribution P. For every $\nu \in (0, \nu_X^\star)$,*

$$\liminf_{n\to+\infty} \big(\rho_n - \mathbb{E}\big(\max_{k\leq [n^{(r+\nu)/d}]} |X_k|\big)\big) \geq -C_\nu \tag{3.2.54}$$

where C_ν is a positive real constant.

Proof. Let $\nu \in (0, \nu_X^\star)$ and let $\widehat{X}_k^{\alpha_n} = \sum_{a\in\alpha_n} a\mathbf{1}_{\{X_k \in C_a(\alpha_n)\}}$. We have,

$$\begin{aligned}
\rho_n &\geq \max_{k\leq m} |\widehat{X}_k^{\alpha_n}| \\
&\geq \sum_{k=1}^m \max_{l\leq m} |\widehat{X}_l^{\alpha_n}| \mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \\
&\geq \sum_{k=1}^m |\widehat{X}_k^{\alpha_n}| \mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \\
&\geq \sum_{k=1}^m \big(|X_k| - |X_k - \widehat{X}_k^{\alpha_n}|\big) \mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \\
&\geq \mathbb{E}\max_{k\leq m} |X_k| - \sum_{k=1}^m \mathbb{E}\left(|X_k - \widehat{X}_k^{\alpha_n}| \mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}}\right).
\end{aligned}$$

Furthermore,

$$\forall k \geq 1, \qquad |X_k - \widehat{X}_k^{\alpha_n}| \mathbf{1}_{\{|X_k| > \max_{i\neq k}|X_i|\}} \stackrel{\mathscr{L}}{=} |X_1 - \widehat{X}_1^{\alpha_n}| \mathbf{1}_{\{|X_1| > \max_{i\neq 1}|X_i|\}}.$$

Hence,

$$\begin{aligned}\rho_n &\geq \mathbb{E}\max_{k\leq m}|X_k| - m\,\mathbb{E}\left(|X_1-\widehat{X}_1^{\alpha_n}|\mathbf{1}_{\{|X_1|>\max_{i\neq 1}|X_i|\}}\right)\\ &\geq \mathbb{E}\max_{k\leq m}|X_k| - m\,\|X_1-\widehat{X}_1^{\alpha_n}\|_{r+\nu}\left(\mathbb{P}\big(|X_1|>\max_{i\neq 1}|X_i|\big)\right)^{1-1/(r+\nu)}.\end{aligned}$$

Since the events

$$\{|X_k|>\max_{i\neq k}|X_i|\},\ k=1,\cdots,m$$

are pairwise disjoint with the same probability we have

$$\mathbb{P}\big(|X_1|>\max_{i\neq 1}|X_i|\big)\leq\frac{1}{m}.$$

Finally,

$$\rho_n\geq\mathbb{E}\max_{k\leq m}|X_k| - m^{\frac{1}{r+\nu}}\,\|X_1-\widehat{X}_1^{\alpha_n}\|_{r+\nu}.$$

It follows, by setting $m=[n^{(r+\nu)/d}]$, that

$$\liminf_{n\to+\infty}\big(\rho_n-\mathbb{E}\big(\max_{k\leq[n^{(r+\nu)/d}]}|X_k|\big)\big)\geq-\limsup_{n\to+\infty}n^{\frac{1}{d}}\|X_1-\widehat{X}_1^{\alpha_n}\|_{r+\nu}.$$

However, since X has an $(r,r+\nu)$-distribution, the upper limit on the right hand side of the equation is finite. □

Example 3.2.3. (Exponential distribution) Let $r>0$ and let X be an exponentially distributed random variable with parameter $\lambda>0$. If $(\alpha_n)_{n\geq1}$ is an L^r-optimal sequence of n-quantizers for X then Theorem 3.2.3 implies

$$\liminf_{n\to+\infty}\frac{\rho_n}{\log(n)}\geq\frac{r+1}{\lambda}. \tag{3.2.55}$$

which is the sharp rate given by (3.2.3).

Indeed, let $\nu\in(0,\nu_X^\star)$ and let $(X_i)_{\{i=1,\cdots,[n^{r+\nu}]\}}$, be an $i.i.d$ exponentially distributed sequence of random variables with parameter λ. We have for every $u\geq0$,

$$\mathbb{P}(\max_{i\leq[n^{r+\nu}]}X_i\geq u)=1-\mathbb{P}(X_1\leq u)^{[n^{r+\nu}]}=1-F(u)^{[n^{r+\nu}]},$$

where F is the distribution function of X (we will denote by f its density function). Then

$$\begin{aligned}\mathbb{E}\big(\max_{i\leq[n^{r+\nu}]}X_i\big) &= \int_0^{+\infty}\mathbb{P}(\max_{i\leq[n^{r+\nu}]}X_i\geq u)du=\int_0^{+\infty}(1-(1-e^{-\lambda u})^{[n^{r+\nu}]})du\\ &= \int_0^{+\infty}\Big(\sum_{i=0}^{[n^{r+\nu}]-1}F(u)^i\Big)e^{-\lambda u}du\\ &= \int_0^{+\infty}\left(1+F(u)+\cdots+F(u)^{[n^{r+\nu}]-1}\right)\frac{f(u)}{\lambda}du\\ &= \frac{1}{\lambda}(1+\frac{1}{2}+\cdots+\frac{1}{[n^{r+\nu}]})\\ &\geq \frac{1}{\lambda}\log(1+[n^{r+\nu}])\ \geq\ \frac{r+\nu}{\lambda}\log(n).\end{aligned}$$

Consequently, it follows from the super-additivity of the liminf that for every $\nu \in (0,1)$,

$$\begin{aligned}\liminf_{n\to+\infty} \frac{\rho_n}{\log(n)} &\geq \liminf_{n\to+\infty} \frac{\rho_n - \mathbb{E}\big(\max_{i\leq[n^{r+\nu}]}\big)}{\log(n)} + \liminf_{n\to+\infty} \frac{\mathbb{E}\big(\max_{i\leq[n^{r+\nu}]}\big)}{\log(n)} \\ &\geq \frac{r+\nu}{\lambda}.\end{aligned}$$

The result follows by letting ν go to $\nu_X^\star = 1$.

Example 3.2.4. (Pareto distribution) Let X be a random variable having a Pareto distribution with index $\gamma > 0$. If $(\alpha_n)_{n\geq 1}$ is an asymptotically L^r-optimal sequence of n-quantizers for X, r is such that $\gamma > r$, then Theorem 3.2.3 yields

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{r+1}{\gamma+1}.$$

Which is not the sharp rate given by $(3.2.4)$.

Notice that if $\gamma > r$ then $X \in L^{r+\eta}(\mathbb{P})$ for $\eta \in \big(0, \gamma - r\big)$. Now, to prove this result, let $\nu \in (0, \nu_X^\star)$ and let $(X_i)_{\{i=1,\cdots,[n^{r+\nu}]\}}$ be an $i.i.d$ sequence of random variables with Pareto distribution with index γ. We have

$$\forall m \geq 1, \forall u \geq 1, \quad \mathbb{P}(\max_{i\leq m} X_i \leq u) = (1-u^{-\gamma})^m.$$

Then, the density function of $\max_{1\leq i\leq m} X_i$ is $m\gamma u^{-(\gamma+1)}(1-u^{-\gamma})^{m-1}$.

Hence

$$\begin{aligned}\mathbb{E}\big(\max_{1\leq i\leq m} X_i\big) &= m\gamma \int_1^{+\infty} x^{-\gamma}(1-x^{-\gamma})^{m-1}dx \\ &= m\int_0^1 u^{-1/\gamma}(1-u)^{m-1}du \qquad (u = x^{-\gamma}) \\ &= m\, B(1-\frac{1}{\gamma}, m) \\ &= \frac{\Gamma(1-\frac{1}{\gamma})\Gamma(m+1)}{\Gamma(m+1-\frac{1}{\gamma})} \\ &\sim \Gamma(1-\frac{1}{\gamma})\, m^{\frac{1}{\gamma}} \qquad \text{as } m \to +\infty\end{aligned}$$

where we used Stirling's formula for the last statement. We finally set $m = [n^{r+\nu}]$ to get

$$\mathbb{E}\big(\max_{1\leq i\leq [n^{r+\nu}]} X_i\big) \sim \Gamma(1-\frac{1}{\gamma})\, n^{\frac{r+\nu}{\gamma}}.$$

It follows from $(3.2.54)$ that for every $\varepsilon \in (0,1)$,

$$\rho_n - (1-\varepsilon)\Gamma(1-\frac{1}{\gamma})\, n^{\frac{r+\nu}{\gamma}} \geq -\varepsilon - \mathrm{C}_\nu.$$

Dividing both side of the inequality by $n^{\frac{r+\nu}{\gamma}}$ and taking the logarithm yields

$$\log \rho_n - \frac{r+\nu}{\gamma}\log(n) \geq \log\left((1-\varepsilon)\Gamma(1-\frac{1}{\gamma}) - (\varepsilon + \mathrm{C}_\nu)n^{-\frac{r+\nu}{\gamma}}\right).$$

Consequently

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{r+\nu}{\gamma}$$

for every $\nu \in (0, \nu_X^\star)$. The announced result follows by letting ν go to $\nu_X^\star = \frac{\gamma-r}{\gamma+1}$.

Comment. Let ϕ be the inverse (if any) function of $-\log \bar{F}$. It can be noticed that in both previous examples we have

$$\mathbb{E}\big(\max_{k\leq[n^{r+\nu_X^\star}]} |X_k|\big) \sim \phi((r+\nu_X^\star)\log(n)) \qquad \text{as } n \to +\infty \tag{3.2.56}$$

which, for distributions with exponential tail leads to the same asymptotic lower bound for the sequence $(\rho_n)_{n\geq 1}$ as in $(3.2.40)$. For Pareto distribution, using the approximation $(3.2.56)$ to compute the asymptotic lower estimate of the maximal radius sequence make us loose the " $-r$" term in the exact asymptotic. To recover this reminding term we have simply to consider the inverse function of $-\log \bar{F}_{r+\nu_X^\star}$ (as done in the previous section) instead of $-\log \bar{F}$, and, the random quantization approach clearly does not allow us to do so.

A conjecture about the sharp rate

The previous results related to distributions with exponential tails strongly suggest the following conjecture: suppose X is a distribution with exponential tail in the sense of claim $(3.2.43)$. Then for every $r > 0$, for every $d \geq 1$,

$$\lim_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = \left(\frac{r+d}{d\,\theta^\star}\right)^{1/\kappa}.$$

This conjecture is proved for $d = 1$ and $r \geq 1$. To be satisfied for higher dimension we need to prove that the geometric statement $(3.2.9)$ of Lemma 3.2.1 (a) holds true with "$1+\varepsilon$" instead of "$2+\varepsilon$" like in 1-dimension. Although this inequality looks quite intuitive in any dimension its proof seems out of reach when $d \geq 2$.

Numerical experiments

We now attempt to focus on numerical experiment of the maximal radius sequence $(\rho_n)_{n\geq 1}$ for the quadratic optimal quantizers of the Gaussian, the Weibull and the exponential distributions. A whole package of quadratic optimal n-quantizers of the $\mathcal{N}(0, I_d)$ distributions are available in the website

`www.quantize.maths-fi.com`

for $d \in \{1, \cdots, 10\}$ and $n \in \{1, \cdots, 5000\}$. When $d = 1$, these L^2-optimal grids are obtained by the Newton method, see e.g. [PAGPRI] for details. For the exponential distribution the quadratic optimal quantizers are computed by using the semi-closed formulae given in Proposition 3.2.1.

As concerns the Weibull distribution with shape parameter $\kappa = 2$, we compute the quadratic optimal quantizers up to 3000 using the Lloyd's I algorithm described in [PAGPRI] (see [GG] for a more itemized description of the algorithm).

In these three cases we depicted the ratio between ρ_n and the expected asymptotic optimal rate. For the exponential distribution we represent the graph of $\frac{\rho_n}{3\log(n)}$ as a function of the grid sizes

(see Figure 3.1). One remarks that the convergence of $\frac{\rho_n}{3\log(n)}$ to 1 as n goes to infinity is almost instantaneous.

However, the cases of the Gaussian and the Weibull distributions are more delicate. Indeed, for the Gaussian distribution the ratio $\frac{\rho_n}{\sqrt{6\log(n)}}$ seems increasing but has not reached yet the value 9 even for a grid size equals 100000, as emphasized by Figure 3.1 (right hand side graph). For the Weibull distribution, $\frac{\rho_n}{\sqrt{3\log(n)}}$ also seems increasing but takes values around 0.927 for a grid size equal to 3000 (see Figure 2). Then for both cases, the convergence to 1 of the ratio between the maximal radius and the expected asymptotic optimal rate seems increasing but very low.

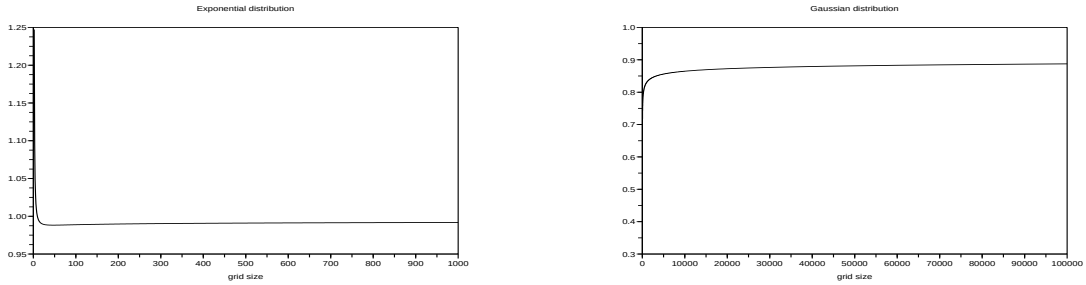

Figure 3.1: Left: $\frac{\rho_n}{3\log(n)}$ as a function of the grid size n for the exponential distribution. Right: $\frac{\rho_n}{\sqrt{6\log(n)}}$ as a function of the grid size n for the normal distribution.

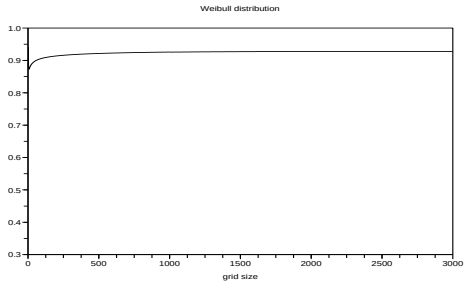

Figure 3.2: $\frac{\rho_n}{\sqrt{3\log(n)}}$ as a function of the grid size for the Weibull distribution with shape parameter $\kappa = 2$.

3.3 Application to the computation of a n-quantizer

One of the important issues from a computational point of view is the search of the L^r-optimal quantizers. The commonly used quantizers for numerical implementations is the quadratic quantizers.Our aim is to use the natural extension of Lloyd's I algorithm to compute the L^r-stationary (optimal) quantizers for the normal distribution, for $r \geq 2$. In a general framework, L^r-stationary quantizers (α_n) (with $\alpha_n = (\alpha_{n1}, \cdots, \alpha_{nn})$) are computed using the L^r-stationary equation $\nabla e_{n,r}(X)^r = 0$. This equation reads for every $r \geq 2$ (see [PPP1, GLP1])

$$\alpha_{ni} = \frac{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}|^{r-2}X\right)}{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}|^{r-2}\right)}, \qquad i = 1, \cdots, n. \tag{3.3.1}$$

Under some additional assumption this formula also holds for $r \in (1,2)$ The so-called Lloyd's I procedure is simply the fixed point procedure attached to Equation $(3.3.1)$. Starting with an initial quantizer $\alpha_n^{(0)}$ of size n, one defines recursively a sequence $(\alpha_n^{(l)})_{l=1,\cdots,L}$ of L^r-stationary quantizers (where L corresponds to the number of Lloyd's iterations) by setting for every $l = 1, \cdots, L$,

$$\alpha_{ni}^{(l)} = \frac{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}^{(l-1)}|^{r-2}X\right)}{\mathbb{E}\left(\mathbf{1}_{X\in C_i(\alpha_n)}|X-\alpha_{ni}^{(l-1)}|^{r-2}\right)}, \qquad i = 1, \cdots, n. \tag{3.3.2}$$

Note that when $d = 1$ and X has an absolutely continuous distribution with a log-concave p.d.f then Equation 3.3.2 has a unique solution as a set (see [KIEF, LAMPAG]) which is the only L^r-optimal n-quantizer. When $d \geq 2$ this uniqueness usually fails.

In practice the above Lloyd's I procedure is randomized $i.e.$ the expectations involved in $(3.3.2)$ are estimated using a regular Monte Carlo (or possibly Quasi-Monte Carlo) simulation.

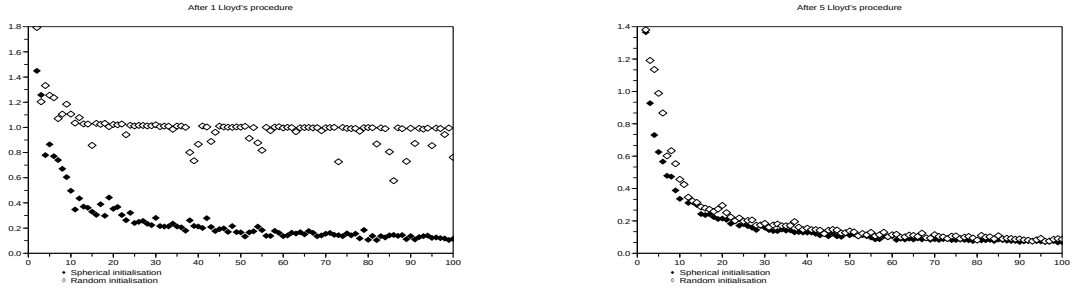

Figure 3.3: Comparison of the distortions for the random initialization method and the initialization over the sphere $S_{d-1}(0,\rho_n)$ after 1 and 5 Lloyd's I procedures.

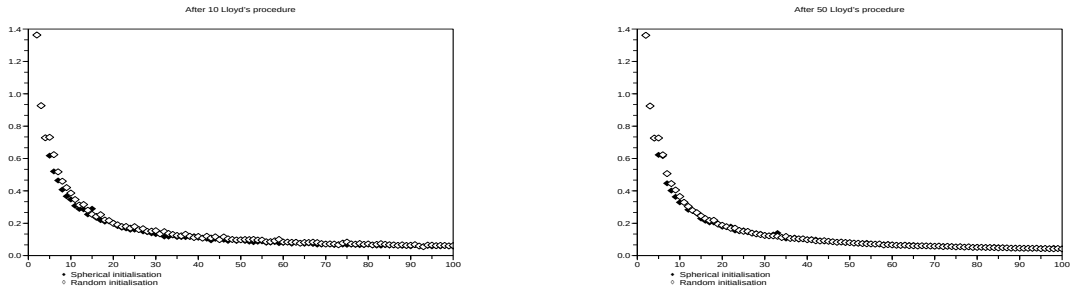

Figure 3.4: Comparison of the distortions for the random initialization method and the initialization over the sphere $S_{d-1}(0,\rho_n)$ after 10 and 50 Lloyd's I procedures.

Owing to our knowledge of the (asymptotic behavior of the) radius ρ_n of a quantizer we propose an alternative way to initialize the Lloyd's I procedure at least when the distribution of X is radial. The starting value $\alpha^{(0)}$ is a sample of the uniform distribution over the sphere $S_{d-1}(0,\rho_n)$, which can easily be obtained from a n-sample $(Z^1, \cdots, Z^n)$ of the $\mathcal{N}(0, I_d)$-distribution by setting

$$\alpha_{ni}^{(0)} = \rho_n \frac{Z^i}{|Z^i|}$$

where the norm $|\cdot|$ denotes the canonical Euclidean distribution. For a grid size varying one by one one from 2 to 100, we compare in figures 3.3 and 3.4 the distortions displayed by the spherical

initialization technique (or Hyperspherical Initialazation Method (HIM) when $d > 2$) and the random initialization method, means, the initial grid (of size n) is set to $\sqrt{(2(r+d)/d)\log(n)}\, Z_i$ where the Z_i's, $i = 1, \cdots, n$ is a sample of size n of the $\mathcal{N}(0; I_d)$ (in this example $d = 2$ and $r = 2$). These graphs show that the spherical initialization method is more competitive than the random initialization method particularly for small values of the number of Lloyd's I procedure.

Chapter 4

Pricing Partial Lookback and Barrier options by Monte Carlo and optimal quantization

This paper is devoted to the pricing of Barrier and Partial Lookback options by Monte Carlo simulations and optimal quantization methods. We represent the premium of these options as an integral of some simpler barrier with respect to the Lebesgue measure either on a bounded or an unbounded interval. Then for Partial Lookback options these integrals have been randomized by a parametrized family of probability density functions in order to reduce the complexity of the Monte Carlo simulations with respect to the regular "Brownian Bridge method " in which the conditional lay of the supremum (or the infimum) of the continuous Euler price process over the time interval from zero to the maturity, given its values in the discretized steps, need to be simulated. Numerical simulations have been carried out for Partial Lookback options by Monte Carlo simulations and show that this randomization technique reduces the complexity but seems to increase the variance with respect to the regular "Brownian Bridge method ". Then an hybrid Monte Carlo-optimal quantization method is used to recover the initial variance level (w.r.t the regular "Brownian Bridge method "). Finally, we show how to price fixed strike Lookback options and barrier options by (pure) optimal quantization from these representation techniques of the premiums.

4.1 Introduction

An *option* on a stock is a contract giving the owner the right but not the obligation to trade a given number of shares of the stock for a fixed price (the *strike*) at a fixed date (the *maturity*).

There are two basic types of options. A *call* option gives the owner the right (but not the obligation) to buy the underlying asset for a certain strike price at a certain maturity. A *put* option allows the owner the right to sell the underlying asset to the writer (seller) of the option for a certain price at a certain time. The option is *European* if it can only be exercised at the maturity; it is *American* if the holder can exercise his right to trade at any time up to maturity.

Note that the writer of the option has the obligation to sell (in the call option) or to purchase (in the put option) the underlying asset if the buyer of the option exercises the option. In exchange for having this option, the buyer pays the writer a fee (the *premium*). The determination of this premium is referred in Mathematical Finance as *Option Pricing*. The fair price for the option is described by the current value of a portfolio that will yield exactly the same return as does the option at the maturity.

When the option is *replicable*, the replicating portfolio can be used to *hedge* the risk inherent in writing the option. For a detailed background on financial derivatives we refer $e.g$ to [CR, HUL].

For the mathematical model consider a time horizon T (which will be typically the maturity) and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space (modelling the randomness of the market) with a filtration $\mathcal{F} = \{\mathcal{F}_t, 1 \leq t \leq T\}$ satisfying the usual requirements. For every $t \in [0, T]$ the filtration describes the information available up to time t. The probability $\mathbb{P}$ is supposed to be the probability in the 'real world' in opposite to the risk neutral probability.

Consider that the stock price process $(X_t)_{t\in[0.T]}$ satisfies the following stochastic differential equation (SDE)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = \zeta \tag{4.1.1}$$

$(W_t)_{t\in[0,T]}$ denotes a q-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathcal{M}(d, q)$ are continuous functions satisfying the global Lipschitz and linear growth conditions:

$$|b(t, x) - b(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leq C|x - y| \tag{4.1.2}$$

and

$$|b(t, x)| + \|\sigma(t, x)\| \leq C(1 + |x|) \tag{4.1.3}$$

for every $t \in [0, T]$ and for every $x, y \in \mathbb{R}^d$.

The norm $|\cdot|$ is any norm on $\mathbb{R}^d$ and $\|\cdot\|$ any norm on the matrix space $\mathcal{M}(d, q)$ (the set of real $d \times q$ matrices). The starting random variable ζ is square integrable, $\mathcal{F}_0$-mesurable, independent from W and defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration considered here is the one generated by ζ and the natural filtration of the brownian motion completed by the $\mathbb{P}$-null sets.

Under the above assumptions on the coefficients of the diffusion one shows (see [**?**, OKS]) that there exists a unique strong solution for the SDE The uniqueness of the solution is ensured by the global Lipschitz assumption $(4.1.2)$ whereas the linear growth assumption $(4.1.3)$ guarantees that this solution does not explode (see [OKS] for more details). Remark that one may always assume that the coefficients b and σ of the SDE only depend on the space variable by considering the transformation $Y_t := (t, X_t)$, see e.g [OKS].

The first workable model for 'rational' market pricing of traded options have been proposed by Black-Scholes in 1973 and extended by Merton in the same year. In the Black-Scholes model the

economics consists of two assets: the stock price with dynamics as the previous SDE with $b(t,x) := \mu x$ and $\sigma(t,x) := \sigma x$, and a zero-coupon bound of constant interest rate r and maturity T.

The market is said to be arbitrage free if there is a probability $\widetilde{\mathbb{P}}$ under which the discounted stock price $e^{-rt}S_t$ is a martingale. It is said to be complete if the probability $\widetilde{\mathbb{P}}$ is unique. In a complete market, any contingent claim (option) may be perfectly replicated with the stock and the zero-coupon using a self financing portfolio. Under arbitrage free and completeness assumptions the price of any contingent claim is uniquely determined and is the discounted expectation of its payoff (a functional of the price process $(X_t)_{t\in[0,T]}$ which may depend on all the trajectory of the process) under $\widetilde{\mathbb{P}}$. If V_t is the value of the option a time t and if h denotes the payoff at the maturity then

$$V_t = e^{-r(T-t)}\mathbb{E}(h|\mathcal{F}_t),$$

where $\mathbb{E}$ is the expectation under $\widetilde{\mathbb{P}}$ so that the price at time 0 is

$$V_0 = e^{-rT}\mathbb{E}(h).$$

Our aim in this work is to estimate such an expectation for a class of path-dependent payoffs: barrier options and (Partial) Lookback options, by optimal quantization method as well as by Monte Carlo method. We will see from the definitions in the next section that the price of Lookback options reads as an expectation of a functional of the supremum (or the infimum) of the stock process over the time interval $[0,T]$, means, $\mathbb{E}F(\sup_{t\in[0,T]} X_t)$ or $\mathbb{E}F(\inf_{t\in[0,T]} X_t)$ and the price of barrier options as $\mathbb{E}F(X_T, \sup_{t\in[0,T]} X_t)$ or $\mathbb{E}F(X_T, \inf_{t\in[0,T]} X_t)$ where F will be specified.

Note that closed formulas are available for these options in the Black-Scholes framework (see [CV]). But this no longer holds when we move out from the Black-Scholes framework so that we are led to estimate the prices by some numerical procedures. But, estimating these prices in a general setting requires first to discretize the stock price trajectories; which leads us to recall in Section 3 some facts about the Euler Scheme and the estimation of the induced discretization errors. Once the stock process is discretized, one can estimate the price by Monte Carlo, by replacing the supremum (or the infimum) of the stock process over $[0,T]$ by the supremum (or infimum) over the time discretization steps. It is established in [GOB] that the weak order of convergence of the supremum (or infimum) of the process over the time discretization steps toward its supremum (or infimum) over $[0,T]$ is very slow since it cannot be greater than $n^{-1/2}$. Using this last approximation method to estimate the price of barrier options for example leads us to neglect the fact that the process can hit the barrier between two discretization steps even if the values taken by the process at these two points are "in-the-money". Hence, to compute efficiently barrier options, one can (as suggested for example in [GLA]) integrate the probability that the barrier is crossed between all the discretization steps. In Section 4 we extend this approach to more general payoffs depending on $\sup_{t\in[0,T]} X_t$ or $\inf_{t\in[0,T]} X_t$ (and X_T) by using a representation of the premium as some integrals of some simpler barrier options with respect to the barriers. As a result this representation appears as an integral with respect to the Lebesque measure either on a bounded or an unbounded interval. Then we simply randomize these integrals

$$\int_I \varphi(z)dz$$

by writing for a given probability density g (positive on I) that

$$\int_I \varphi(z)dz = \mathbb{E}\big(\frac{\varphi}{g}(\xi)\big), \qquad \xi \overset{\mathcal{L}}{\sim} g(z)dz.$$

The final step is to find (among a parametrized family of probability density functions) the one which yields the lowest variance. This is but an importance sampling procedure (with a σ-finite measure as a reference).

The main asset of this approach is that it drastically reduces the complexity of the Monte Carlo simulations with respect to the regular "Brownian Bridge method" in which

$$\mathcal{L}(\sup_{[0,T]} \bar{X}_t | \bar{X}_{t_k}, k = 0, \cdots, n)$$

needs to be simulated. The yield of such a simulation process is quite low (n^{-1}), see Proposition $4.4.1$ below.

On the other hand, this randomization seems to induce an increase of the variance (compared to that obtained in the regular "Brownian Bridge method"). Then an hybrid Monte Carlo-optimal quantization method is used to recover the initial variance level (with respect to the regular "Brownian Bridge method").

Finally, we show how to price some of these options by (pure)optimal quantization techniques.

4.2 Definitions

The European options are sometimes called *vanilla* or *plain vanilla* options and are payoffs depending only on the value of the underlying asset at the maturity. Options whose payoffs depend on the path of the underlying asset are called *exotic* or *path dependent* options. We consider here a class of exotic options whose payoff depend on both the value of the underlying asset at the maturity and its maximum or its minimum over $[0, T]$. That means payoffs h of the form

$$h = F(X_T, \sup_{t\in[0,T]} X_t) \qquad \text{or} \qquad h = F(X_T, \inf_{t\in[0,T]} X_t).$$

When F is a continuous function one speaks of "Lookback like options" refering to the original lookback options where the payoff h is defined by

$$h = X_T - \inf_{t\in[0,T]} X_t \qquad (\text{or } h = \sup_{t\in[0,T]} X_t - X_T).$$

When the payoff can be decomposed as

$$h = \varphi(X_T)\mathbf{1}_{\{\sup_{t\in[0.T]} X_t \in I\}} \qquad \text{or } \varphi(X_T)\mathbf{1}_{\{\inf_{t\in[0.T]} X_t \in I\}}$$

where I is an unbounded interval of $\mathbb{R}$, one speaks about barrier options. This last class is a particular case of payoffs of the form

$$h = \varphi(X_T)\mathbf{1}_{\{\tau_D(X)>T\}}$$

where $\tau_D(X)$ is the exit time of a domain $D \subset \mathbb{R}^d$ by a d-dimensional underlying asset $X = (X^1, \cdots, X^d)$.

Here are some definitions.

Barrier Options. The option is said to be a up-and-out option if it knocks out when the price of its underlying asset crosses a specified value. It is said a down-and-out option if it has barrier below the initial asset price and knocks out if the underlying asset price falls below the barrier.

The payoff of an European up-and-out call expiring at time T, with strike price K and up-and-out barrier L is given by :

$$(X_T - K)^+ \mathbf{1}_{\{\sup_{t\in[0.T]} X_t \leq L\}}$$

and the payoff of a European down-and-out call barrier option with maturity T, strike K and barrier L is given by

$$(X_T - K)^+ \mathbf{1}_{\{\inf_{t\in[0.T]} X_t \geq L\}}.$$

Lookback Options. We will consider fixed and floating strike lookback options. The payoff of a fixed strike K lookback option expiring at time T is the maximum between zero and the difference between the optimal price and the strike. For a call option the payoff is defined by

$$(\max_{t\in[0,T]} X_t - K)^+$$

and for a put option it is defined by

$$(K - \min_{t\in[0,T]} X_t)^+.$$

Partial Lookback Options. The payoff of a partial Lookback call payoff expiring at T is given by

$$(X_T - \lambda \min_{t\in[0,T]} X_t)^+, \qquad \lambda > 1$$

and the put option is defined by its payoff

$$(\lambda \max_{t\in[0,T]} X_t - X_T)^+, \qquad \lambda \in (0,1).$$

When $\lambda = 1$, these payoff are those of classical Lookback options.

To estimate the price of these path dependent options we need first to discretize the paths of the stock process. We next recall one of the paths discretization method, the Euler scheme, and some estimation results of the induced discretization error.

4.3 Euler Scheme

Consider a d-dimensional Brownian diffusion process $(X_t)_{t\in[0,T]}$ solution of the following Stochastic Differential Equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = x \in \mathbb{R}^d \tag{4.3.1}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}(d,q)$ (the set of $d \times q$ matrices) are continuous functions satisfying the usual conditions ensuring the existence and the uniqueness of the SDE and $(W_t)_{t\in[0,T]}$ denotes a q-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let us divide the set $[0,T]$ into n subsets of length T/n and set for every $k = 0, \cdots, n,\ t_k = \frac{kT}{n}$. The stepwise constant Euler scheme is defined by

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(t_k, \bar{X}_{t_k})\,\frac{T}{n} + \sigma(t_k, \bar{X}_{t_k})\sqrt{\frac{T}{n}}Z_{k+1}, \quad \bar{X}_0 = x,\ k = 0, \cdots, n-1 \tag{4.3.2}$$

where $(Z_k)_{1\leq k\leq n}$ is a sequence of $i.i.d$ random variables distributed as $\mathcal{N}(0;1)$ defined by

$$Z_k := \sqrt{\frac{T}{n}}\left(W_{t_k} - W_{t_{k-1}}\right), \quad k = 1, \cdots, n.$$

For every $t \in [0,T]$ the discrete Euler scheme is defined by

$$\tilde{X}_t := \bar{X}_{\underline{t}}$$

with $\underline{t} = t_k$ if $t \in [t_k, t_{k+1}), \quad k = 1, \cdots, n$. A natural extension of the discrete Euler scheme is the continuous Euler scheme defined for every $t \in [0,T]$ by

$$\bar{X}_t = \bar{X}_{\underline{t}} + b(\underline{t}, \bar{X}_{\underline{t}})(t - \underline{t}) + \sigma(\underline{t}, \bar{X}_{\underline{t}})(W_t - W_{\underline{t}}), \quad \bar{X}_0 = x$$

with satisfies the SDE

$$\bar{X}_t = x + \int_0^t b(\underline{s}, \bar{X}_{\underline{s}})ds + \int_0^t \sigma(\underline{s}, \bar{X}_{\underline{s}})dW_s.$$

4.3.1 Strong error rate

Theorem 4.3.1. *Assume b and σ satisfy for every $\alpha \in (0,1)$,*

$$\forall t \in [0,T], \forall y, z \in \mathbb{R}^d, \quad |b(s,y) - b(t,z)| \leq C(|t-s|^\alpha + |y-z|). \tag{4.3.3}$$

Then for every $p > 0$,
(a) for every $n \geq 1$,

$$\| \sup_{t\in[0,T]} |X_t - \bar{X}_t| \|_p \leq C_{b,\sigma,p}\, e^{TC_{b,\sigma,p}}(1+|x|)\left(\frac{T}{n}\right)^{\frac{1}{2}\wedge\alpha}.$$

(b) For every $n \geq 1$,

$$\| \sup_{t\in[0,T]} |X_t - \tilde{X}_t| \|_p \leq C_{b,\sigma,p}\, e^{TC_{b,\sigma,p}}(1+|x|)\sqrt{\frac{\log(n)}{n}}.$$

4.3.2 Weak error

We recall in this section some weak error estimates for path-dependent options (we refer e.g. to [GOB, PAG2] for the proofs). Let

$$\mathbb{D}([0,T],\mathbb{R}^d) := \left\{\xi : [0,T] \to \mathbb{R}^d, \ \text{càdlàg}\right\}.$$

If $F : \mathbb{D}([0,T],\mathbb{R}^d) \to \mathbb{R}$ is a Lipschitz functional for the sup norm, that is,

$$|F(\xi) - F(\xi')| \leq C_F \sup_{t\in[0,T]} |\xi(t) - \xi'(t)|$$

then

$$\left|\mathbb{E}F((X_t)_{t\in[0,T]}) - \mathbb{E}F((\bar{X}_t)_{t\in[0,T]})\right| \leq \frac{C}{\sqrt{n}} \tag{4.3.4}$$

and

$$\left|\mathbb{E}F((X_t)_{t\in[0,T]}) - \mathbb{E}F((\tilde{X}_t)_{t\in[0,T]})\right| \leq C\sqrt{\frac{\log n}{n}}. \tag{4.3.5}$$

In fact the Lookback and Partial Lookback payoffs are continuous with respect to the pathwise L^∞-norm: $\|f\|_{L^\infty_T} := \operatorname{ess\,sup}_{t\in[0,T]} |f(t)|$.

For Barrier options the rate of convergence for the discrete and continuous Euler scheme are given in [GOB] and recalled below.

If a domain D has a smooth enough boundary, $b, \sigma \in \mathcal{C}^3(\mathbb{R}^d)$ and σ uniformly elliptic on D : $\exists \sigma_0,\ \forall x \in \mathbb{R}^d\ \sigma(x)\sigma^\star(x) \geq \sigma_0^2\ I_{\mathbb{R}^d\otimes\mathbb{R}^d}$ then for every bounded measurable function f satisfying $d(\mathrm{supp}(f), \partial D) \geq 2\varepsilon > 0$,

$$\mathbb{E}(f(\bar{X})\mathbf{1}_{\{\tau(\bar{X})>T\}}) - \mathbb{E}(f(X)\mathbf{1}_{\{\tau(X)>T\}}) = Cn^{-1} + o(n^{-1})$$

and

$$\mathbb{E}(f(\tilde{X})\mathbf{1}_{\{\tau(\tilde{X})>T\}}) - \mathbb{E}(f(X)\mathbf{1}_{\{\tau(X)>T\}}) = O(n^{-1/2})$$

where n is the number of discretization steps and $\tau(Y)$ is the exit time of the process Y from the open set D, $i.e$

$$\tau(Y) = \inf\{t \in [0,T],\ Y_t \in D^c\}.$$

Then the convergence rate is of order n^{-1} for the continuous Euler scheme and of order $n^{-1/2}$ for the discrete one.

4.4 Price estimates

According to the convergence rate for the continuous Euler scheme we would like to estimate the price of path-dependent options by replacing the asset price process $(X_t)_{t\in[0,T]}$ by its continuous Euler process $(\bar{X})_{t\in[0,T]}$. The distributions of the maximum and the minimum of the continuous Euler process $(\bar{X}_t)$ over the time interval $[0,T]$ given its values at the discrete time observations t_k are a well known result. We recall it below without giving any proof. The proof can be found e.g in [PAG2].

Proposition 4.4.1. *We have*

$$\mathcal{L}(\max_{t\in[0,T]} \bar{X}_t | \bar{X}_{t_k} = x_k, k = 0, \cdots, n) = \mathcal{L}(\max_{k=0,\cdots,n-1} G^{-1}_{x_k,x_{k+1}}(U_k)) \tag{4.4.1}$$

and

$$\mathcal{L}(\min_{t\in[0,T]} \bar{X}_t | \bar{X}_{t_k} = x_k, k = 0, \cdots, n) = \mathcal{L}(\min_{k=0,\cdots,n-1} F^{-1}_{x_k,x_{k+1}}(U_k)) \tag{4.4.2}$$

where $(U_k)_{k=0,\cdots,n-1}$ are $i.i.d$ random variables uniformly distributed over the unit interval, $G^{-1}_{x,y}$ and $F^{-1}_{x,y}$ are the inverse functions of the conditional distribution functions $G_{x,y}$ and $F_{x,y}$ defined by

$$G_{x,y}(u) = \left(1 - e^{-2n\frac{(x-u)(y-u)}{T\sigma^2(x)}}\right)\mathbf{1}_{\{u\geq\max(x,y)\}}$$

and

$$F_{x,y}(u) = 1 - \left(1 - e^{-2n\frac{(x-u)(y-u)}{T\sigma^2(x)}}\right)\mathbf{1}_{\{u\leq\min(x,y)\}}.$$

From the above results we next deduce general formulas relating the expectation of a functional of both the terminal value $\bar{X}_T$ of the process $(\bar{X}_t)$ and its maximum (or the minimum) over the time interval $[0,T]$. From now on we make the abuse of notation $\bar{X}_k := \bar{X}_{t_k},\ \forall k \in \{0,\cdots,n\}$.

Proposition 4.4.2. (a) *Let f be a real-valued non negative function defined on $\mathbb{R}_+^2$ such that*

$$\sup_{x>0} \mathbb{E} f(x, \max_{t\in[0,T]} \bar{X}_t) < +\infty. \tag{4.4.3}$$

Then

$$\mathbb{E} f(\bar{X}_T, \max_{t\in[0,T]} \bar{X}_t) = \mathbb{E} f(\bar{X}_T, 0) + \mathbb{E}\int_0^{+\infty} \left(1 - \prod_{k=1}^{n} G_{\bar{X}_{k-1},\bar{X}_k}(z)\right) d_z f(\bar{X}_T, z). \tag{4.4.4}$$

Likewise if

$$\sup_{x>0} \mathbb{E} f(x, \min_{t\in[0,T]} \bar{X}_t) < +\infty)$$

then

$$\mathbb{E} f(\bar{X}_T, \min_{t\in[0,T]} \bar{X}_t \vee 0) = \mathbb{E} f(\bar{X}_T, 0) + \mathbb{E}\int_0^{+\infty} \left(\prod_{k=1}^{n} \left(1 - F_{\bar{X}_{k-1},\bar{X}_k}(z)\right)\right) d_z f(\bar{X}_T, z). \tag{4.4.5}$$

(b) *If furthermore $f_\infty(x) := \lim_{y\to+\infty} f(x,y) < +\infty$ for every $x > 0$. Then*

$$\mathbb{E} f(\bar{X}_T, \max_{t\in[0,T]} \bar{X}_t) = \mathbb{E} f_\infty(\bar{X}_T) - \mathbb{E}\int_0^{+\infty} \left(\prod_{k=1}^{n} G_{\bar{X}_{k-1},\bar{X}_k}(z)\right) d_z f(\bar{X}_T, z) \tag{4.4.6}$$

and

$$\mathbb{E} f(\bar{X}_T, \min_{t\in[0,T]} \bar{X}_t \vee 0) = \mathbb{E} f_\infty(\bar{X}_T) - \mathbb{E}\int_0^{+\infty} \left(1 - \prod_{k=1}^{n} \left(1 - F_{\bar{X}_{k-1},\bar{X}_k}(z)\right)\right) d_z f(\bar{X}_T, z). \tag{4.4.7}$$

This proposition follows from the following lemma.

Lemma 4.4.1. (a) *Let Z be a positive random variable and let g be a nonnegative function defined on $\mathbb{R}_+$ with finite variation (on compact sets) such that*

$$\mathbb{E}\left(\int_{]0,Z]} |dg|\right) < +\infty. \tag{4.4.8}$$

Then

$$\mathbb{E} g(Z) = g(0) + \int_{(0,+\infty)} \mathbb{P}(Z \geq z)\, dg(z). \tag{4.4.9}$$

(b) *If furthermore $g_\infty := \lim_{x\to+\infty} g(x) < +\infty$ then*

$$\mathbb{E} g(Z) = g_\infty - \int_{(0,+\infty)} \mathbb{P}(Z < z)\, dg(z). \tag{4.4.10}$$

Proof. (a) We have

$$g(Z) = g(0) + \int_{]0,Z]} dg(u).$$

It follows that

$$\begin{aligned}\mathbb{E}g(Z) &= g(0) + \mathbb{E}\int_{]0,Z]} dg(u)\\ &= g(0) + \int_{(0,+\infty)} \mathbb{P}(Z \geq z)\, dg(z),\end{aligned}$$

the last inequality coming from Fubini's theorem; which can be applied owing to assumption $(4.4.8)$.

(b) We just use the fact that $\mathbb{P}(Z \geq u) = 1 - \mathbb{P}(Z < u)$. □

Now we are in position to prove Proposition $4.4.2$.

Proof of Proposition $4.4.2$. (b) One deduces from $(4.4.1)$ that

$$\mathbb{E}\big(f(x_n, \max_{t\in[0,T]} \bar{X}_t)\big|\bar{X}_k = x_k, k = 0, \cdots, n\big) = \mathbb{E}\big(f(x_n, \max_{0\leq k\leq n-1} G^{-1}_{x_k,x_{k+1}}(U_k))\big)$$

where $G^{-1}_{x,y}$ and the U_k are defined like in $(4.4.1)$. Now it follows from Lemma $4.4.1$ (b) (applied to $g(z) = f(x_n, z)$) that

$$\begin{aligned}\mathbb{E}\big(f(x_n, \max_{0\leq k\leq n-1} G^{-1}_{x_k,x_{k+1}}(U_k))\big) &= f_\infty(x_n) - \int_0^{+\infty} \mathbb{P}(\max_{0\leq k\leq n-1} G^{-1}_{x_k,x_{k+1}}(U_k) \leq z) d_z f(x_n, z)\\ &= f_\infty(x_n) - \int_0^{+\infty} \left(\prod_{k=0}^{n-1} \mathbb{P}(U_k \leq G_{x_k,x_{k+1}}(z))\right) d_z f(x_n, z)\\ &= f_\infty(x_n) - \int_0^{+\infty} \left(\prod_{k=0}^{n-1} G_{x_k,x_{k+1}}(z))\right) d_z f(x_n, z).\end{aligned}$$

Consequently

$$\begin{aligned}\mathbb{E}f(\bar{X}_T, \max_{t\in[0,T]} \bar{X}_t) &= \mathbb{E}\big(\mathbb{E}\big(f(\bar{X}_T, \max_{t\in[0,T]} \bar{X}_t)\big|\bar{X}_k = x_k, k = 0, \cdots, n\big)\big)\\ &= \mathbb{E}f_\infty(\bar{X}_T) - \mathbb{E}\int_0^{+\infty} \left(\prod_{k=1}^{n} G_{\bar{X}_{k-1},\bar{X}_k}(z)\right) d_z f(\bar{X}_T, z).\end{aligned}$$

The formula relative to the minimum is proved likewise by using $(4.4.2)$ in place of $(4.4.1)$.

(a) is proved like (b) by using Lemma $4.4.1$ (a) instead of Lemma $4.4.1$ (b). □

Proposition $4.4.2$ allows us to rewrite the estimates of the premiums of some usual exotic options in a useful form in view of the optimal quantization approximation method as well as of Monte Carlo simulation methods.

Proposition 4.4.3. *Let* $f(x) = (x - K)^+$ *and* $g(x) = (K - x)^+$.

(a) *The price of an up-and-out put option expiring at time T with strike K and up-and-out barrier L is estimated by*

$$\bar{P}_{UB} := e^{-rT}\mathbb{E}\big((K-\bar{X}_T)^+\mathbf{1}_{\{\sup\limits_{t\in[0,T]}\bar{X}_t\le L\}}\big) = e^{-rT}\mathbb{E}\left(g(\bar{X}_T)\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}(L)\right). \quad (4.4.11)$$

(b) *The price of an up-and-out call option expiring at time T with strike K and up-and-out barrier L can be approximated by*

$$\bar{C}_{UB} := e^{-rT}\mathbb{E}\big((\bar{X}_T-K)^+\mathbf{1}_{\{\sup\limits_{t\in[0,T]}\bar{X}_t\le L\}}\big) = e^{-rT}\mathbb{E}\left(f(\bar{X}_T)\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}(L)\right). \quad (4.4.12)$$

(c) *The price of an down-and-out put option expiring at time T with strike K and down-and-out barrier L can be approximated by*

$$\bar{P}_{OB} := e^{-rT}\mathbb{E}\big((K-\bar{X}_T)^+\mathbf{1}_{\{\inf\limits_{t\in[0,T]}\bar{X}_t\ge L\}}\big) = e^{-rT}\mathbb{E}\left(g(\bar{X}_T)\prod_{k=1}^{n}\big(1-F_{\bar{X}_{k-1},\bar{X}_k}(L)\big)\right). \quad (4.4.13)$$

(d) *The price of an down-and-out call option expiring at time T with strike K and up-and-out barrier L is approximated by the following formula :*

$$\bar{C}_{OB} := e^{-rT}\mathbb{E}\big((\bar{X}_T-K)^+\mathbf{1}_{\{\inf\limits_{t\in[0,T]}\bar{X}_t\ge L\}}\big) = e^{-rT}\mathbb{E}\left(f(\bar{X}_T)\prod_{k=1}^{n}\big(1-F_{\bar{X}_{k-1},\bar{X}_k}(L)\big)\right). \quad (4.4.14)$$

(e) *The risk neutral price at time zero of a Partial Lookback call option with maturity T is approximated by the following formula:*

$$\bar{C}_{PL} := e^{-rT}\mathbb{E}\big(\bar{X}_T-\lambda\min_{t\in[0,T]}\bar{X}_t\vee 0\big)^+ = X_0-\alpha e^{-rT}\mathbb{E}\left(U^{\alpha-1}\bar{X}_T\prod_{k=1}^{n}\big(1-F_{\bar{X}_{k-1},\bar{X}_k}\big(\tfrac{\bar{X}_T U^\alpha}{\lambda}\big)\big)\right) \quad (4.4.15)$$

where $U\sim\mathcal{U}([0,1])$ is independent from $(\bar{X}_k)_{k=0,\cdots,n}$ and $\alpha>0$ is a "shape" parameter (to be optimized in order to reduce the variance).

(f) *The risk neutral price at time zero of a Partial Lookback put option with maturity T is approximated by*

$$\bar{P}_{PL} := e^{-rT}\mathbb{E}\big(\lambda\max_{t\in[0,T]}\bar{X}_t-\bar{X}_T\big)^+ = \beta e^{-rT}\mathbb{E}\left(e^V\bar{X}_T\left(1-\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}\big(\tfrac{(1+\beta V)\bar{X}_T}{\lambda}\big)\right)\right) \quad (4.4.16)$$

where $V\sim\mathscr{E}(1)$ independent from $(\bar{X}_k)_{k=0,\cdots,n}$ and $\beta>0$ is a "shape" parameter (to be optimized in order to reduce the variance).

Prior to the proof let us make some remarks about these equations.

Note that the right hand side of Equations $(4.4.11)$, $(4.4.12)$, $(4.4.13)$, $(4.4.14)$ are all obtained by re-conditioning, then it follows from Jensen inequality that this technique reduces the variance. However, the right hand side of formulas $(4.4.15)$ and $(4.4.16)$ are obtained by randomizing the Lebesgue integral of some integrand functions. It then turns out that we add an noise to these formulas;

which a priori increases the variance. Hence, it becomes crucial to reduce the variance using stochastic algorithms which will be briefly recalled in Section 4.5.

Note also that the price at time zero of a Partial Lookback call option with maturity T can also be written as

$$X_0 - \alpha e^{-rT}\mathbb{E}\left(\frac{e^V \bar{X}_T}{(1+\alpha V)^2}\prod_{k=1}^{n}\left(1 - F_{\bar{X}_{k-1},\bar{X}_k}\left(\tfrac{\bar{X}_T}{\lambda(1+\alpha V)}\right)\right)\right) \tag{4.4.17}$$

as well as

$$X_0 - \frac{1}{\alpha}e^{-rT}\mathbb{E}\left(Z^{\frac{\alpha-1}{\alpha}}\bar{X}_T\prod_{k=1}^{n}\left(1 - F_{\bar{X}_{k-1},\bar{X}_k}\left(\tfrac{\bar{X}_T}{\lambda Z^{1/\alpha}}\right)\right)\right) \tag{4.4.18}$$

where $V \sim \mathscr{E}(1)$ and Z has the Pareto distribution with parameter 1 (recall that the density of the Pareto distribution with parameter $a > 0$ reads $f(x) = ax^{-(a+1)}\mathbf{1}_{\{x>1\}}$) both independent from $(\bar{X}_k)_{k=0,\cdots,n}$; $\alpha > 0$ is a shape parameter.

On the other hand the price of a Partial Lookback put option can also be written as

$$\frac{1}{\beta}e^{-rT}\mathbb{E}\left(Z^{\frac{\beta+1}{\beta}}\bar{X}_T\left(1 - \prod_{k=1}^{n}\left(1 - F_{\bar{X}_{k-1},\bar{X}_k}\left(\tfrac{\bar{X}_T Z^{1/\beta}}{\lambda}\right)\right)\right)\right) \tag{4.4.19}$$

where Z has the Pareto distribution with parameter 1.

But for the call option the moment of order 2 of the integrand parts of equations (4.4.17) and (4.4.18) go to zero as α goes to infinity, which, from the stochastic algorithms point of view, is not a suitable behaviour. So we will consider the representation (4.4.15) to reduce the variance (the 2-nd moment of the integrand part goes to zero as α goes to infinity). Note that for the put option the 2-nd moment integrand parts of all equations (4.4.16) and (4.4.19) go to zero as β goes to infinity.

Now let us prove Proposition 4.4.3.

Proof. (a) Let $f(x) = (x-K)^+$, $g(z) = \mathbf{1}_{\{z\le L\}}$ and set $h(x,z) = f(x)g(z)$. Then it follows from (4.4.6) that

$$\mathbb{E}\big((\bar{X}_T - K)^+\mathbf{1}_{\{\sup\limits_{t\in[0,T]}\bar{X}_t\le L\}}\big) = \mathbb{E}h_\infty(\bar{X}_T) - \mathbb{E}\int_0^{+\infty}\left(f(\bar{X}_T)\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}(z)\right)dg(z).$$

Now $\forall x \ge 0,\ h_\infty(x) = 0$ and $dg(z) = -\delta_L(z)$. Then

$$C_{UB} = e^{-rT}\mathbb{E}\left(f(\bar{X}_T)\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}(L)\right).$$

The items (b), (c), (d) are proved in the same way as (a).

(e) We deduce from (4.4.5) $\big($with $f(x,z) = (x-\lambda z)^+\ \big)$ that

$$\begin{aligned}
\mathbb{E}\left(\big(x_n - \lambda\min_{t\in[0,T]}\bar{X}_t \vee 0\big)^+\big|\bar{X}_k = x_k\right) &= x_n - \lambda\int_0^{x_n/\lambda}\left(\prod_{k=1}^{n}\big(1 - F_{x_{k-1},x_k}(z)\big)\right)dz \\
&= x_n - \int_0^1 x_n\left(\prod_{k=1}^{n}\big(1 - F_{x_{k-1},x_k}(\tfrac{zx_n}{\lambda})\big)\right)dz \\
&\overset{u=z^{1/\alpha}}{=} x_n - \alpha\int_0^1 u^{\alpha-1}x_n\left(\prod_{k=1}^{n}\big(1 - F_{x_{k-1},x_k}(\tfrac{x_n u^\alpha}{\lambda})\big)\right)du.
\end{aligned}$$

It follows that

$$\mathbb{E}\big(\bar{X}_T - \lambda \min_{t\in[0,T]} \bar{X}_t \vee 0\big)^+ = \mathbb{E}(\bar{X}_T) - \alpha\, \mathbb{E}\int_0^1 u^{\alpha-1}\bar{X}_T \left(\prod_{k=1}^{n}\big(1 - F_{\bar{X}_{k-1},\bar{X}_k}(\frac{\bar{X}_T u^\alpha}{\lambda})\big)\right) du.$$

Which proves the statement (e). The statement (f) is proved likewise (e).

□

Taking $\lambda = 1$ and $\bar{X}_T \equiv K$ in equations $(4.4.15)$ and $(4.4.16)$ yield the approximations of the price of fixed strike K lookback options.

Proposition 4.4.4. *The risk neutral price at time zero of a fixed strike K lookback put option with maturity T is approximated by*

$$\bar{P}_{LF} := e^{-rT}\,\mathbb{E}(K - \min_{t\in[0,T]} \bar{X}_t \vee 0)^+ = Ke^{-rT} - \alpha e^{-rT}\mathbb{E}\left(U^{\alpha-1}\prod_{k=1}^{n}\big(1 - F_{\bar{X}_{k-1},\bar{X}_k}\big(KU^\alpha\big)\big)\right). \tag{4.4.20}$$

The price at time zero of a fixed strike K lookback call option with maturity T is approximated by

$$\bar{C}_{LF} := e^{-rT}\,\mathbb{E}(\max_{t\in[0,T]} \bar{X}_t - K)^+ = K\beta e^{-rT}\mathbb{E}\left[e^V\left(1 - \left(\prod_{k=0}^{n-1} G_{\bar{X}_k,\bar{X}_{k+1}}(K + \beta V)\right)\right)\right] \tag{4.4.21}$$

where $\alpha > 0, \beta > 0$; $U \sim \mathcal{U}([0,1])$ and $V \sim \mathscr{E}(1)$ both independent from $(\bar{X}_k)_{k=0,\cdots,n}$.

Proof. This is proved as Proposition $4.4.3$. □

4.5 Short background on Stochastic approximation

This section is motivated by the computation of the price of Partial Lookback (or fixed strike Lookback) options from formulas $(4.4.15)$, $(4.4.16)$, $(4.4.21)$ and $(4.4.21)$. In fact any given α or β in these formulas will lead to a estimation of the expectation. The best estimation will be the one with the smallest variance. Computing directly the variance of the corresponding random variables in order to deduce the optimal parameters $\alpha^\star$ and $\beta^\star$ leading to the smallest variance is not possible in general because it does not lead to closed (or semi-closed) formulas.

Stochastic approximations allow to estimate $\alpha^\star$ and $\beta^\star$ using a probabilistic extension of deterministic zero search recursive procedure. Deterministic zero search recursive procedure are displayed as

$$\forall n \geq 1, \qquad y_{n+1} = y_n - \gamma_{n+1}h(y_n) \quad (0 < \gamma_n \leq \gamma_0) \tag{4.5.1}$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function satisfying a sub-linear growth assumption at infinity.

Suppose h has an integral representation with respect to an $\mathbb{R}^q$-valued random variable Z, that is,

$$h(y) = \mathbb{E}(H(y,Z)), \quad H : \mathbb{R}^d \times \mathbb{R}^q \overset{Borel}{\longrightarrow} \mathbb{R}^d, \quad Z \sim \mu,$$

and $\mathbb{E}|H(y,Z)| < +\infty$ for every $y \in \mathbb{R}^d$.

If $H(y,z)$ is computable and if the distribution μ of Z can be simulated one randomizes the zero search procedure $(4.5.1)$ by setting

$$Y_{n+1} = Y_n - \gamma_{n+1}H(Y_n, Z_{n+1}), \quad Z_n\ i.i.d \text{ with distribution } \mu. \tag{4.5.2}$$

where the gain sequence $(\gamma_n)_{n\geq 1}$ satisfies

$$\sum_n \gamma_n = +\infty \quad \text{and} \quad \sum_n \gamma_n^2 < +\infty.$$

Under some appropriate assumptions on H the sequence (Y_n) converge $a.s.$ to a zero $y^\star$ of h. We next recall some convergence results whose proofs are provided for example in [PAG2].

▷ *Robbins-Monro algorithm.* Suppose that

$$(y - y^\star | h(y)) \geq 0$$

and

$$\mathbb{E}|H(y,Z)|_2 \leq C(1+|y|)$$

for every $y \in \mathbb{R}^d$. Suppose furthermore that h is continuous and that

$$\forall y \in \mathbb{R}^d,\ y \neq y^\star, \qquad (y - y^\star | h(y)) > 0.$$

Then

$$Y_n \xrightarrow{a.s.} y^\star$$

and in $L^p,\ p \in [1,2)$.

▷ *Stochastic gradient.* Suppose that h is Lipschitz continuous. Assume that there exists a continuously differentiable function $L : \mathbb{R}^d \to \mathbb{R}_+$ such that

$$\forall y \in \mathbb{R}^d,\ \mathbb{E}|H(y,Z)|_2 \leq C(1+\sqrt{L(y)}).$$

Suppose furthermore that $\lim_{|y|\to\infty} L(y) = +\infty$ and that $\{h=0\} = \{y^\star\}$. Then

$$L(y^\star) = \min\{L(y),\ y \in \mathbb{R}^d\}$$

and

$$Y_n \xrightarrow{a.s.} y^\star.$$

▷ *Pseudo-stochastic gradient.* If $(h|\nabla L) \geq 0$ (with L the function defined previously) and, for every $v \geq 0$, $\{(h|\nabla L)\} \cap \{L = v\}$ is finite on every compact set then $\mathbb{P}$-$a.s.$, Y_n converges toward a point of $\{(h|\nabla L) = 0\}$.

4.6 Estimation of the prices by Monte Carlo simulation

4.6.1 Background

In the previous section we have written the prices as an expectation of some random vector and we would like to estimate these prices by Monte Carlo method. In a general setting, let X be an $\mathbb{R}^d$ random variable defined in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let f be a real values function defined on $\mathbb{R}^d$ and consider the problem of estimating the value

$$\mathbb{E}(f(X)) = \int f(x)\mathbb{P}_X(dx).$$

Suppose that we have a mechanism for simulating n points $X_1, \cdots, X_N$ independently and distributed as X. Evaluating the function f at these n random points and averaging the results produce the Monte Carlo estimate

$$\frac{1}{N}\sum_{i=1}^{N} f(X_i).$$

This method is more suited to calculation by a computer because of its reliance on repeated computation of random or pseudo random numbers. It f is reasonably well behaved, the strong law of large number yields the following convergence result as N goes to infinity:

$$m_{N,f} := \frac{1}{N}\sum_{i=1}^{N} f(X_i) \longrightarrow \mathbb{E}f(X) \qquad \text{a.s.}$$

This convergence result justifies the intuitive interpretation of the expected value as the long-term average when sampling repeatedly. Moreover if $f(X) \in L^2(\mathbb{P})$, the rate of convergence in the strong law of large number is ruled by the central limit theorem,

$$\sqrt{N}\left(\mathbb{E}f(X) - m_{N,f}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0;\sigma_f^2)$$

where

$$\sigma_f^2 = \mathbb{E}\left(f(X) - \mathbb{E}f(X)\right)^2.$$

For the control of the error, one relies on the central limit theorem since the convergence in distribution given above implies that for every real numbers a and b with $a < b$,

$$\begin{aligned}\lim_{N\to+\infty} \mathbb{P}\left(\sqrt{N}\,\frac{m_{N,f} - \mathbb{E}f(X)}{\sigma_f} \in [a,b]\right) &= \mathbb{P}\left(\mathcal{N}(0;1) \in [a,b]\right) \\ &= F(b) - F(a)\end{aligned}$$

where F denotes the cumulative function of the standard normal distribution.

The parameter σ_f is typically unknown in a setting in which $\mathbb{E}f(X)$ is unknown. However it can be estimated by

$$s_{N,f} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(f(X) - m_{N,f})^2}$$

and we have

$$\lim_{N\to+\infty} \sqrt{N}\,\frac{m_{N,f} - \mathbb{E}f(X)}{s_{N,f}} = \lim_{N\to+\infty} \sqrt{N}\,\frac{m_{N,f} - \mathbb{E}f(X)}{\sigma_f}\frac{\sigma_f}{s_{N,f}} \overset{\mathcal{L}}{=} \mathcal{N}(0;1).$$

One derives the confidence interval at level α of the Monte Carlo simulation by

$$I_N = \left[m_{N,f} - a_\alpha\sqrt{s_{N,f}/N}, m_{N,f} + a_\alpha\sqrt{s_{N,f}/N}\right] \tag{4.6.1}$$

which satisfies for large n

$$\mathbb{P}(\mathbb{E}f(X) \in I_n) \approx \mathbb{P}\left(|\mathcal{N}(0;1)| \le a_\alpha\right).$$

Then, from the $f(X_1), \cdots, f(X_N)$, we obtain both an estimation of $\mathbb{E}f(X)$ and a measure of the error estimate.

Note that Monte Carlo method is generally not a competitive method for calculating one dimensional integrals. However when we change the dimension of the random vector X the function f and consequently σ_f will also change but the standard error still have the form $\sigma_f/\sqrt{N}$. Then this method displays $1/\sqrt{N}$ convergence, means, quadrupling the number of sampled points will halve the error estimate, regardless of the number of dimensions.

Finally Equation $(4.6.1)$ emphasizes that the only way to speed up the Monte Carlo is to reduce the variance $i.e.$ finding a $r.v.$ Y such that

$$\mathbb{E}(Y) = \mathbb{E}f(X) \quad \text{and} \quad \mathbb{E}(Y^2) \ll \mathbb{E}(f(X)^2).$$

4.6.2 Numerical implementations

In this scope we deal with numerical illustrations by considering Partial Lookback call options. We consider two models. The Black-Scholes model and a local volatility model inspired by the CEV model already considered in [LEMPAG]. For all options we set the interest rate r equal to 0.15. Furthermore the number of time discretization steps $n = 100$, the maturity $T = 1$. The parameter λ is set to 1.1 for the call and 0.7 for the put. The size of Monte Carlo is set to $M = 10^6$.

▷ *The models.* Recall that in the Black-Scholes framework the stock price process (X_t) is modeled by the following SDE (under the risk neutral probability $\tilde{\mathbb{P}}$)

$$dX_t = rX_t dt + \sigma X_t dW_t \tag{4.6.2}$$

where r is the interest rate, σ the volatility and W a brownian motion under $\tilde{\mathbb{P}}$.

For the pseudo CEV model, the dynamic of the stock price process is ruled by the following SDE (under the risk neutral probability)

$$dX_t = rX_t dt + \vartheta X_t^{\delta} \frac{X_t}{\sqrt{1+X_t^2}} dW_t \tag{4.6.3}$$

for some $\delta \in (0,1)$ and $\vartheta \in (0, \underline{\vartheta}], \underline{\vartheta} > 0$. The parameter r still be the interest rate and $\sigma(x) := \vartheta \frac{x^{\delta}}{\sqrt{1+x^2}}$ corresponds to the local volatility function.

Note that for a fixed $\delta \in (0,1)$, if the initial value of the stock process X_0 is large enough then the pseudo CEV model is very close the time CEV model

$$dX'_t = X'_t(rdt + \vartheta (X'_t)^{\delta-1} dW_t).$$

In particular, for numerical tests we will consider that $\vartheta \approx \sigma X_0^{1-\delta}$ where σ denotes the regular volatility. The only "aim of the really" rough calibration is just to deal with reasonable values.

▷ *Partial Lookback options.* In both models, we estimate the price of the Partial Lookback call option by

$$\alpha^{\star} e^{-rT} \frac{1}{M} \sum_{j=1}^{M} U_j^{\alpha^{\star}-1} \bar{X}_T^j \left(1 - \prod_{k=1}^{n} \left(1 - F_{\bar{X}_{k-1}^j, \bar{X}_k^j} \left(\tfrac{\bar{X}_T^j U_j^{\alpha^{\star}}}{\lambda} \right) \right) \right) \tag{4.6.4}$$

where the $(U_j, (\bar{X}_k^j)_{k=0,\cdots,n})$ for $j = 1, \cdots, M$ are M trials of a random vector with components U_j, the uniform distribution and the trajectories of the process X; $\alpha^{\star}$ is the optimal parameter given by the stochastic algorithm (if any). In practice we only need to know a "good" α which significantly reduces the variance. Note that the estimation in the form $(4.6.4)$ leads to smaller variance than when we isolate the X_0 as in Equation $(4.4.15)$.

Recall that $\alpha^\star$ is defined as the zero of the following function (we make a formal derivative of the moment of order 2 in Equation $(4.4.15)$)

$$h(\alpha) := \mathbb{E}(H(\alpha, X, U))$$

with

$$H(\alpha, X, U) = \bar{X}_T^2 \prod_{k=1}^{n} \Big(1 - F_{\bar{X}_{k-1},\bar{X}_k}(\tfrac{U^\alpha \bar{X}_T}{\lambda})\Big) \Big\{ \Big(2\alpha U^{2(\alpha-1)} + 2\alpha^2 U^{2(\alpha-1)} \log U \Big) \times$$
$$\prod_{k=1}^{n} \Big(1 - F_{\bar{X}_{k-1},\bar{X}_k}(\tfrac{U^\alpha \bar{X}_T}{\lambda})\Big) + \frac{2}{\lambda}\alpha^2 U^{3\alpha-2} \bar{X}_T \log U \sum_{k=1}^{n} \frac{2n}{T\sigma^2(\bar{X}_{k-1})} \big(\bar{X}_{k-1} + \bar{X}_k - 2\tfrac{U^\alpha \bar{X}_T}{\lambda}\big)$$
$$\exp\big(- \tfrac{2n}{T\sigma^2(\bar{X}_{k-1})} \big(\bar{X}_{k-1} - \tfrac{U^\alpha \bar{X}_T}{\lambda}\big)\big(\bar{X}_k - \tfrac{U^\alpha \bar{X}_T}{\lambda}\big)\big) \mathbf{1}_{\{\bar{X}_{k-1};\bar{X}_k \geq \frac{U^\alpha \bar{X}_T}{\lambda}\}}$$
$$\times \prod_{p\neq k} \big(1 - F_{\bar{X}_{p-1},\bar{X}_p}(\tfrac{U^\alpha \bar{X}_T}{\lambda})\big) \Big\}$$

The graphs depicted in Figure 4.1 and Figure 4.2 show some paths of the stochastic algorithm for the Partial Lookback call in both models. Note that the initialization of the algorithm and the choice of the gain sequence $(\gamma_n)_{n\geq 1}$ (both depending on the model parameters) are two crucial steps to make the algorithm converge.

The numerical results corresponding to the previous set of parameters are summarized in Table 4.1 and 4.2. These results are compared to those obtained for the computation of expressions like

$$\mathbb{E}f(\bar{X}_T, \sup_{t\in[0,T]} \bar{X}_t) \quad \text{or} \quad \mathbb{E}f(\bar{X}_T, \inf_{t\in[0,T]} \bar{X}_t)$$

by the regular "Brownian Bridge" method (RBB) based on Proposition $4.4.1$ and consisting (for example for the estimation of $\mathbb{E}f(\bar{X}_T, \inf_{t\in[0,T]} \bar{X}_t)$) in the following steps (see [PAG2]):

Set $S^f = 0$.
`for` $m = 1$ `to` M

- Simulate a path of the discrete time Euler scheme $(\bar{X}^{(m)})$ and set $x_k = \bar{X}^{(m)}_{t_k}, k = 0, \cdots, n$.
- Simulate $\Gamma^{(m)} := \min_{0\leq k\leq n} (F_{x_k,x_{k+1}})^{-1}(U_k^{(m)})$, where $(U_k^{(m)})_{1\leq k\leq m}$ are iid with $\mathcal{U}([0,1])$-distribution.
- compute $f(\bar{X}_T^{(m)}, \Gamma^{(m)})$.
- Compute $S_m^f := f(\bar{X}_T^{(m)}, \Gamma^{(m)}) + S_{m-1}^f$.

`end.` (m)

Eventually, for large enough M,

$$\mathbb{E}f(\bar{X}_T, \inf_{t\in[0,T]} \bar{X}_t) \approx \frac{S_M^f}{M}.$$

Keep in mind that

$$(F_{x,y})^{-1}(u) = \frac{1}{2}\big(x + y - \sqrt{(x-y)^2 - 2T\sigma^2(x)\log(u)/n}\big), \; u \in (0,1)$$

and

$$(G_{x,y})^{-1}(1-u) = \frac{1}{2}\big(x+y+\sqrt{(x-y)^2 - 2T\sigma^2(x)\log(u)/n}\big),\ u \in (0,1).$$

It follows from the numerical results summarized in Table 4.1 and Table 4.2, that in both models, the Randomized Method (RM) increases the variance (it is equal to a little more than the double of the variance in the RBB estimation) even after variance reduction. However, the complexity of the RM method is lower than the one in the RBB method. Then in terms of the risk, the RBB method is clearly better than the RM method. But in term of complexity this is the converse (keep in mind that the code is written in C language). Let V_1 be the variance of the estimator in the RM method (i.e the variance of the payoff as it appears in $4.6.4$) and let T_1 be the CPU time used to simulate one path of this payoff. Let V_2 and T_2 be the corresponding variance and CPU time for the RBB method. The must performing method is the one with the lower T_iV_i since it will provide the lowest variance for a given allocation of CPU time. Numerical tests carried out below with the RM method suggest that the variance increase is not compensated by the complexity reduction. Then the RM method is not an alternative to the RBB method.

σ	True price	$\alpha^\star$	RM price	RM Var.	RM CPU	RBB price	RBB Var.	RBB CPU
0.2	15.12	0.29	15.14	5.398 10^2	37.90	15.14	2.374 10^2	48.84
0.3	21.47	0.43	21.50	1.168 10^3	38.20	21.50	5.503 10^2	48.85
0.4	27.61	0.58	27.68	2.119 10^3	38.64	27.65	1.050 10^3	48.85
0.6	38.93	2.66	38.95	1.368 10^4	40.07	39.05	2.914 10^3	48.92

Table 4.1: B&S model: Partial Lookback call price estimates by the Randomized Method (RM) and the Regular Brownian Bridge Method (RBB), the CPU time is given in second. RM Var. (RBB Var) corresponds to the estimated variance in the RM method (RBB method).

We said that the randomized method increases the variance because we have introduced a noise by randomizing the Lebesgue integral of some integrand function to get formulas $(4.4.15)$ and $(4.4.16)$ and that, unfortunately, the variance reduction technique does not allow us to recover the initial variance level. A solution to this problem can be to compute the expectation with respect to the uniform distribution (resp. the exponential distribution) appearing in Equation $(4.4.15)$ and $(4.4.16)$ by optimal quantization method. It turn out from Section $4.7.2$ that this removes the noise introduced by the RM method. But this increases again the complexity.

σ	$\alpha^\star$	RM price	RM variance	RM CPU	RBB price	RBB variance	RBB CPU
2.1	0.23	14.94	4.68 10^2	60.43	14.97	2.13 10^2	78.82
3.2	0.74	21.85	1.748 10^3	63.93	21.86	4.79 10^2	78.84
4.1	0.22	27.46	1.859 10^3	57.29	27.46	7.88 10^2	78.36

Table 4.2: Pseudo CEV model: Partial Lookback call price estimates by the Randomized Method (RM) and the Regular Brownian Bridge Method (RBB), CPU time given in second.

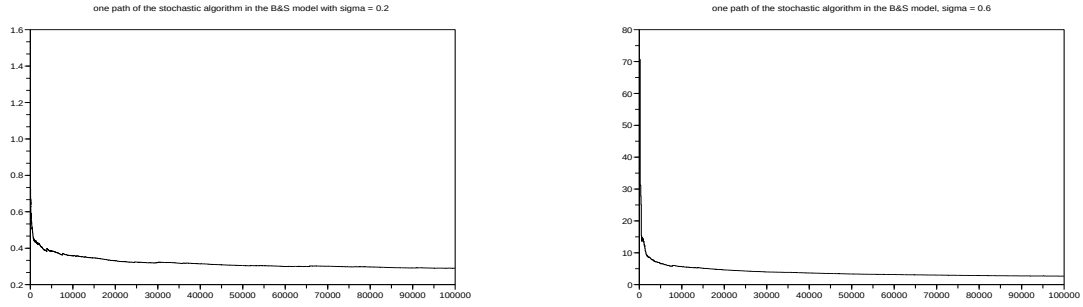

Figure 4.1: Paths of the stochastic algorithm in the B&S model for the Partial Lookback call option with $\sigma = 0.2$ (left) and $\sigma = 0.6$ (right).

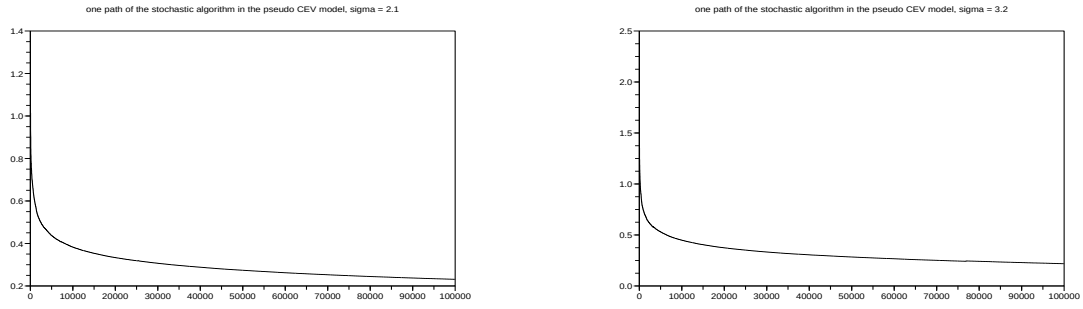

Figure 4.2: Paths of the stochastic algorithm in the pseudo CEV model for the Partial Lookback call option with $\vartheta = 2.1$ (left) and $\vartheta = 4.1$ (right).

4.7 Optimal quantization method

This section is devoted, on one hand, to the premium computation of Partial Lookback options by an hybrid Monte Carlo-optimal quantization and, on the other hand, to the pricing of barrier options and fixed strike Lookback options by a (pure) optimal quantization method. Let us recall first some facts about optimal quantization.

4.7.1 A brief overview on optimal quantization

Consider an $\mathbb{R}^d$ random vector X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with finite r-th moment. Quantizing X on a given grid $\Gamma = \{x^1, \cdots, x^N\}$ consists on projecting X on the grid Γ following the closest neighbour rule. The induced mean L^r-error

$$\|X - \mathrm{Proj}_\Gamma(X)\|_r = \| \min_{1\le i\le N} |X - x^i| \|_r$$

is called the *L^r-mean quantization error* and the projection of X on Γ, $\mathrm{Proj}_\Gamma(X)$, is called *the quantization of X*. As a function of the grid Γ the L^r-mean quantization error is continuous and reaches a minimum over all the grids with size at most N. A grid $\Gamma^\star$ minimizing the L^r-mean quantization error over all the grids with size at most N is called an *L^r-optimal quantizer*.

Moreover the L^r-mean quantization error goes to 0 as the grid size $N \to +\infty$ and the convergence rate is ruled by the so-called Zador theorem:

$$\min_{\Gamma,\ |\Gamma|=N} \|X - \mathrm{Proj}_\Gamma(X)\|_r = Q_r(\mathbb{P}_X) N^{-1/d} + o\big(N^{-1/d}\big)$$

where $Q_r(P) \ge 0$. We shall say no more about the principle of optimal quantization. For a complete background on this field we refer to [GL].

The first application of optimal quantization method to numerical probability appear in [PAG] in 1997. It consists in estimating $\mathbb{E}f(X)$ (it may also be an conditional expectation) by

$$\mathbb{E}f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big) = \sum_{i=1}^N f(x^{\star,i})\, p_i \tag{4.7.1}$$

where $\Gamma^\star = \{x^{\star,1}, \cdots, x^{\star,N}\}$ is an L^r-optimal grid for X and $p_i = \mathbb{P}\big(\mathrm{Proj}_{\Gamma^\star}(X) = x^{\star,i}\big)$.

• If $f : \mathbb{R}^d \mapsto \mathbb{R}$ is Lipschitz continuous then

$$\begin{aligned} |\mathbb{E}f(X) - \mathbb{E}f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big)| &\le \mathbb{E}|f(X) - f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big)| \\ &\le [f]_{\mathrm{Lip}} \|X - \mathrm{Proj}_{\Gamma^\star}(X)\|_1 \\ &\le [f]_{\mathrm{Lip}} \|X - \mathrm{Proj}_{\Gamma^\star}(X)\|_s \qquad (\text{if } X \in L^s,\ s \ge r). \end{aligned}$$

• If the derivative Df of f is Lipschitz then for any optimal grid $\Gamma^\star$ for the quadratic quantization of X we have

$$|\mathbb{E}f(X) - \mathbb{E}f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big)| \le [Df]_{\mathrm{Lip}} \|X - \mathrm{Proj}_{\Gamma^\star}(X)\|_s^2 \qquad (\text{if } X \in L^s,\ s \ge 2).$$

How to compute numerically quadratic optimal quantizers or L^r-optimal (or stationnary) quantizers in general, the associated weights and L^r-mean quantization errors is a important issue from the numerical point of view. Several algorithms are used in practice. In the one dimensional framework,

the L^r-optimal quantizers are unique up to the grid size as soon as the density of X is strictly log-concave. In this case the Newton algorithm is the commonly used algorithm to carry out L^r-optimal quantizers when closed or semi-closed formulas are available for the gradient and the hessian matrix. However, note that semi-closed formulas are available for some scalar distributions: the exponential distribution, the power and the Pareto distributions (see [FP]).

But when the dimension $d \geq 2$, the L^r-optimal grids are not uniquely determined and all L^r-optimal quantizers search algorithms are based to zero search recursive procedures like Lloyd's I algorithms (or generalized Lloyd's I algorithms with are the natural extension of the quadratic case), the Competitive Learning Vector Quantization (CLVQ) algorithm (see [GG]), stochastic algorithms (see [PAG2, PAGPRI]), etc.

4.7.2 An hybrid Monte Carlo-optimal quantization method

The principle

This method consists in computing the expectations with respect to the uniform distribution, the exponential and the Pareto distributions appearing in the price formulas of Partial Lookback options (or fixed strike Lookback options) in Proposition 4.4.3 and the comment following it (or in Corollary 4.4.4) by optimal quantization. In fact, let us consider for example the Partial Lookback call option. Equation (4.4.15) is of the form

$$\mathbb{E}\left(F\big((\bar{X}_{t_k})_{0\leq k\leq n}, U\big)\right), \quad \text{with } U \sim \mathcal{U}([0,1]) \text{ independent from } (\bar{X}_{t_k})_{0\leq k\leq n}.$$

On the other hand, owing to the independence condition, conditioning with respect to the discrete stock process yields

$$\mathbb{E}\left(F\big((\bar{X}_{t_k})_{0\leq k\leq n}, U\big)\big|(\bar{X}_{t_k})_{0\leq k\leq n} = x\right) = \mathbb{E}\left(F\big(x, U\big)\right). \tag{4.7.2}$$

We know that this technique reduces the variance because the conditional expectation is a contraction with respect to the L^2-norm.

The hybrid Monte Carlo-optimal quantization method consists on estimating the expectation appearing in the right hand side of Equation (4.7.2) by optimal quantization for every path of the stock process. The same method can be used for equations (4.4.17), (4.4.18), (4.4.19) and for fixed strike Lookback options. We then need the optimal grids of the associated distributions (the uniform distribution, the exponential and the Pareto distributions). Here is some closed and semi-closed formulas for the optimal grids of the required distributions (see [FP] for more details on the exponential and the Pareto distributions).

▷ **Uniform distribution** $\mathcal{U}([0,1])$. For every $n \geq 1$, for every $r > 0$, the quadratic optimal n-quantizer $\alpha_n = (\alpha_{n,1}, \cdots, \alpha_{n,n})$ of the uniform distribution in the unit interval is uniquely defined by

$$\alpha_{n,k} = \frac{2k-1}{2n}, \qquad k = 1, \cdots, n$$

with associated weight p_k equals to $1/n$ for every $k = 1, \cdots, n$.

▷ **Exponential distribution** $\mathscr{E}(1)$**.** For every $n \geq 1$, the L^r-optimal n-quantizer of the exponential distribution $\alpha_n^{(r)} = (\alpha_{n,1}^{(r)}, \cdots, \alpha_{n,n}^{(r)})$ is uniquely defined and given by

$$\alpha_{n,k}^{(r)} = \frac{a_n^{(r)}}{2} + \sum_{i=n+1-k}^{n-1} a_i^{(r)}, \quad 1 \leq k \leq n, \tag{4.7.3}$$

where $(a_k^{(r)})_{k\geq 1}$ is a positive real sequence defined by the following implicit recursive formula :

$$a_0^{(r)} := +\infty, \quad \phi_r\big(-a_{k+1}^{(r)}\big) := \phi_r\big(a_k^{(r)}\big),\ k \geq 1$$

with $\phi_r(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)e^{-u}du$ (convention : $0^0 = 1$).

The associated weighs are given by

$$p_1 := e^{-\alpha_{n,1+1/2}^{(r)}};\ p_k := e^{-\alpha_{n,k-1/2}^{(r)}} - e^{-\alpha_{n,k+1/2}^{(r)}}; \text{for } k = 2, \cdots, n-1;\ p_n = e^{-\alpha_{n,n-1/2}^{(r)}},$$

with

$$\alpha_{n,k-1/2}^{(r)} = \frac{1}{2}\big(\alpha_{n,k}^{(r)} + \alpha_{n,k-1}^{(r)}\big), \quad k = 2, \cdots, n$$

and

$$\alpha_{n,k+1/2}^{(r)} = \frac{1}{2}\big(\alpha_{n,k}^{(r)} + \alpha_{n,k+1}^{(r)}\big), \quad k = 1, \cdots, n-1.$$

▷ **Pareto distribution with parameter** $\gamma > 0$**.** For every $n \geq 1$, the unique L^r-optimal n-quantizer $\alpha_n^{(r)} = (\alpha_{n,1}^{(r)}, \cdots, \alpha_{n,n}^{(r)})$ is given by

$$\alpha_{n,k}^{(r)} = \frac{1}{1 + a_n^{(r)}} \prod_{i=n-k+1}^{n-1} (1 + a_i^{(r)}), \quad 1 \leq k \leq n, \tag{4.7.4}$$

where $(a_k^{(r)})_{k\geq 1}$ is a positive real sequence defined by

$$a_0^{(r)} := +\infty, \quad \phi_\gamma\left(-\frac{a_{k+1}^{(r)}}{1 + a_{k+1}^{(r)}}\right) := \phi_\gamma\left(a_k^{(r)}\right),\ k \geq 1$$

with

$$\phi_\gamma(x) := \int_0^{x/2} |u|^{r-1}\mathrm{sign}(u)\ \gamma(1+u)^{-(\gamma+1)}du.$$

The associated weights are

$$p_1 := 1 - 1/\alpha_{n,1+1/2}^{(r)};\ p_k := 1/\alpha_{n,k-1/2}^{(r)} - 1/\alpha_{n,k+1/2}^{(r)},\ k = 2, \cdots, n-1;\ p_n = 1/\alpha_{n,n-1/2}^{(r)}.$$

Once we get the optimal (quadratic) quantizers and the associated weights of the previous distributions, we can approximate the premium of the Partial Lookback call by

$$\alpha^\star e^{-rT}\frac{1}{NM}\sum_{j=1}^{M}\sum_{i=1}^{N} U_i^{\alpha^\star - 1}\bar{X}_T^j\left(1 - \prod_{k=1}^{n}\left(1 - F_{\bar{X}_{k-1}^j,\bar{X}_k^j}\big(\tfrac{\bar{X}_T^j U_i^{\alpha^\star}}{\lambda}\big)\right)\right) \tag{4.7.5}$$

where $(U_i)_{i=1,\cdots,N}$ is the optimal N-quantizer of the uniform distribution and $\alpha^\star$ the optimal parameter displayed by the stochastic algorithm. It may also be estimated in the same way from equations $(4.4.17)$ and $(4.4.18)$ by using the optimal quantizers of the exponential distribution or the Pareto distribution.

On the other hand we approximate the price of the Partial Lookback put options by

$$\beta^\star e^{-rT}\frac{1}{M}\sum_{j=1}^{M}\sum_{i=1}^{N} e^{V_i}\bar{X}_T^j\left(1 - \prod_{k=1}^{n} G_{\bar{X}_{k-1}^j,\bar{X}_k^j}\big(\tfrac{(1+\beta^\star V_i)\bar{X}_T^j}{\lambda}\big)\right)p_i. \tag{4.7.6}$$

In this case, $(V_i)_{i=1,\cdots,N}$ is the (quadratic) optimal N-quantizer of the exponential distribution $\mathscr{E}(1)$ and the p_i the associated weights. This premium may also be approximated from Equation $(4.4.19)$ by using optimal quantizers of the Pareto distribution.

Numerical experiments

For the numerical implementations we consider the same models as in Section $4.6.2$.

σ	True price	$\alpha^\star$	HMQ price	HMQ variance	RBB price	RBM variance
0.2	15.12	0.29	15.10	2.30 10^2	15.10	2.31 10^2
0.3	21.47	0.43	21.49	5.31 10^2	21.50	5.35 10^2
0.4	27.61	0.58	27.75	1.01 10^3	27.80	1.02 10^3
0.6	38.93	2.66	39.5	2.76 10^3	39.5	2.77 10^3

Table 4.3: B&S model: Partial Lookback call price estimates by the Hybrid Monte Carlo-Quantization method (HMQ) and the Regular Brownian Bridge Method (RBB).

σ	price	price	variance	variance
0.2	15.10	15.11	2.35 10^2	2.31 10^2
0.3	21.54	21.53	5. 40 10^2	5.34 10^2
0.4	27.80	27.78	1.03 10^2	1.02 10^3
0.6	39.50	39.5	2.79 10^3	2.77 10^3
$(N1, N2)$	$(5, 10)$	$(10, 20)$	$(5, 10)$	$(10, 20)$

Table 4.4: B&S model: Partial Lookback call price estimates by the Hybrid Monte Carlo-Quantization method (HMQ) with Romberg extrapolation.

• *Partial Lookback call.* The numerical results obtained for the Partial Lookback call are summarized in tables 4.3, 4.4 and 4.5. In Table 4.3 we compute the price of the Partial Lookback option in the Black-Scholes model. The grid size is set to $N = 50$ and the number of time discretization steps $n = 20$. All the other parameters are the same as in the previous section.

We observe that the complexity increases with the grid size and the mean computation time is about 90 s for the hybrid Monte Carlo-optimal quantization method (HMQ) whereas it is about 10 s for the RBM method. Hence, in its present form, the HQM method has a slightly "TV" indicator. However, it follows from the numerical results that the hybrid Monte Carlo-optimal quantization method (HMQ) removes the additional noise induced by the RM method. Also, the displayed prices and variances a bit more accurate in the HQM method than in the RBB method.

To reduce the grid size while performing the accuracy we use a classical Richardson Romberg extrapolation (see e.g. [PAG2]) with grid sizes $N1, N2$ such that $N1 + N2 < 50$, of the grid. The numerical results are depicted in Table 4.4. The mean computation time is about 31 s when $(N1, N2) = (5, 10)$ and 56 s when $(N1, N2) = (10, 20)$.

Finally, Table 4.5 displays the estimated prices from the HMQ and the RBB methods in the pseudo CEV model. Here, the number of time discretization steps is set to $n = 100$.

• *Partial Lookback put.* Recall that one computes first the $\beta^\star$ minimizing the variance introduced by the randomized method. This parameter is a zero of (from a formal derivative of the moment of order 2 $(4.4.16)$) the function

$$l(\beta) = \mathbb{E}(L(\beta, X, V))$$

σ	$\alpha^\star$	HMQ price	HMQ variance	RBB price	RBM variance
2.1	0.23	14.96	2.13 10^2	14.97	2.13 10^2
3.2	0.74	21.82	4.77 10^2	21.86	4.79 10^2
4.1	0.22	27.40	7.90 10^2	27.46	7.88 10^2

Table 4.5: Pseudo CEV model: Partial Lookback call price estimates by the Hybrid Monte Carlo-Quantization method (MQM) and the Regular Brownian Bridge Method (RBB).

with

$$L(\beta,X,V)=e^{2V}\bar{X}_T^2\Big(1-\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}(\tfrac{(1+\beta V)\bar{X}_T}{\lambda})\Big)\Big\{2\beta\Big(1-\prod_{k=1}^{n}G_{\bar{X}_{k-1},\bar{X}_k}(\tfrac{(1+\beta V)\bar{X}_T}{\lambda})\Big)$$
$$+\frac{2}{\lambda}\beta^2V\bar{X}_T\sum_{k=1}^{n}\frac{2n}{T\sigma^2(\bar{X}_{k-1})}\Big(\bar{X}_{k-1}+\bar{X}_k-2\tfrac{(1+\beta V)\bar{X}_T}{\lambda}\Big)\times$$
$$\exp\big(-\tfrac{2n}{T\sigma^2(\bar{X}_{k-1})}\big(\bar{X}_{k-1}-\tfrac{(1+\beta V)\bar{X}_T}{\lambda}\big)\big(\bar{X}_k-\tfrac{(1+\beta V)\bar{X}_T}{\lambda}\big)\big)\mathbf{1}_{\{\bar{X}_{k-1};\bar{X}_k\le\frac{(1+\beta V)\bar{X}_T}{\lambda}\}}$$
$$\times\prod_{p\neq k}G_{\bar{X}_{p-1},\bar{X}_p}(\tfrac{(1+\beta V)\bar{X}_T}{\lambda})\Big\}.$$

The expectation is taken with respect to the random vector $(V,(\bar{X}_{t_k})_{0\le k\le n})$ where $V\sim\mathscr{E}(1)$. It turn turn out that in this case, the randomized method introduces a "strong noise" in Equation $(4.4.16)$ so that the regular stochastic algorithm defined by

$$\beta_{n+1}=\beta_n-\gamma_{n+1}L\left(\beta_n,Z_n\right),\quad Z_n\ i.i.d \tag{4.7.7}$$

where $Z=(V,(\bar{X}_{t_k})_{0\le k\le n})$ does not converge. Here we use the same idea of the hybrid Monte Carlo-optimal quantization method by modifying the algorithm as

$$\beta_{n+1}=\beta_n-\gamma_{n+1}\sum_{i=1}^{N}L\left(\beta_n,V_i,Z_n\right)p_i,\quad Z_n\ i.i.d \tag{4.7.8}$$

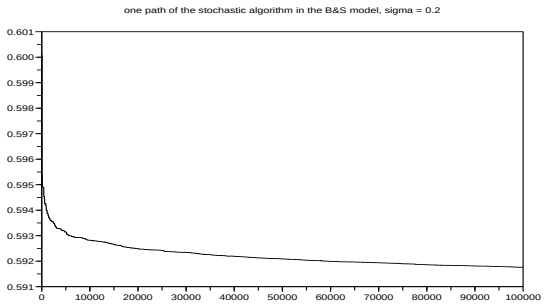

Figure 4.3: Path of the stochastic algorithm in the B&S model for the Partial Lookback put with $\sigma=0.2$.

where in this case $Z=(\bar{X}_{t_k})_{0\le k\le n}$ and $(V_i)_{i=1,\cdots,N}$ is the (quadratic) optimal N-quantizer of the exponential distribution $\mathscr{E}(1)$ and the p_i the associated weights. This method makes the algorithm

σ	True price	$\beta^\star$	HMQ price	HMQ variance	RBB price	RBB variance
0.2	0.08	0.29	0.08	0.58	0.08	0.58
0.3	1.20	0.59	1.20	14.7	1.21	14.7
0.4	4.09	1.36	3.92	67.3	4.10	66.3

Table 4.6: B&S model: Partial Lookback put price estimates by the Hybrid Monte Carlo-Quantization method (HMQ) and the Regular Brownian Bridge Method (RBB).

converge (since it removes the initial noise introduced by the randomized representation). Figure 4.3 shows one path of (β_n) in the Black-Scholes model for $\sigma = 0.2$.

We consider here the Black-Scholes model and numerical results are displayed in Table 4.6. The number of time discretization steps is set to $n = 50$, the grid size $N = 50$ and $\lambda = 0.7$. The other parameters are chosen like in the previous section. One observes that the hybrid Monte Carlo-quantization method allows us to retrieve the initial variances with respect to the regular "Brownian Brigde method" but with a strong complexity. Like for the Partial Lookback call option, one can use a standard Romberg extrapolation to reduce the complexity.

4.7.3 Estimation of the prices by marginal quantization

The aim of this section is to propose an algorithm based exclusively on optimal quantization to compute the path dependent options detailed in Section 4.4, using the RBB quantization based on filtering algorithms and their randomized representation. This is the first step toward an algorithm for the evaluation of credit default in a partial observation model (Callegaro G., Sagna A., An application to credit risk of optimal quantization method for nonlinear filtering, work in progress).

To this end we will approximate by optimal quantizations some expressions of the form

$$V := \mathbb{E}\left(f(\bar{X}_n) \prod_{k=1}^{n} g_k(\bar{X}_{k-1}, \bar{X}_k) \right) \tag{4.7.9}$$

where f is a bounded measurable function on $\mathbb{R}^d$ taking values on $\mathbb{R}$ and $g_k(\cdot,\cdot)$ a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ with can depends on some real parameters like for Barrier options where it depends also on the barrier.

We proceed like in [PAGPRI] where numerical solving of nonlinear filtering with discrete-time observation have been ruled by optimal quantization methods. The only change is that in our case we will drop the dependance on the noisy observations ($i.e$ the Y_k's following the notations in [PAGPRI]) because our problem is not a filtering problem.

We define for any $k = 1, \cdots, n$, the bounded transition kernel H_k by

$$H_k f(x) = \mathbb{E}\left(f(\bar{X}_k) g_k(x, \bar{X}_k) | \bar{X}_{k-1} = x \right) = \int f(y) g_k(x,y) P_k(x, dy) \tag{4.7.10}$$

where $P_k(x,\cdot) = \mathcal{L}(\bar{X}_k = \cdot | \bar{X}_{k-1} = x)$. For convenience, we set

$$H_0 f(x) = \mathbb{E}(f(\bar{X}_0)) = \int f(x)\mu(dx). \tag{4.7.11}$$

Now for any $k \in \{1, \cdots, n\}$ set

$$\pi_k f = \mathbb{E}\big(f(\bar{X}_k) \prod_{i=1}^{k} g_k(\bar{X}_{i-1}, \bar{X}_i) \big).$$

We have

$$\begin{aligned}\pi_k f &= \mathbb{E}\left(\mathbb{E}\big(f(\bar{X}_k)\prod_{i=1}^{k} g_k(\bar{X}_{i-1},\bar{X}_i)|\bar{X}_{k-1}\big)\right)\\ &= \mathbb{E}\left(\mathbb{E}\big(f(\bar{X}_k)g_k(\bar{X}_{k-1},\bar{X}_k)|\bar{X}_{k-1}\big)\prod_{i=1}^{k-1} g_k(\bar{X}_{i-1},\bar{X}_i)\right)\\ &= \mathbb{E}\left(H_k(f(\bar{X}_{k-1}))\prod_{i=1}^{k-1} g_k(\bar{X}_{i-1},\bar{X}_i)\right)\end{aligned}$$

It follows that

$$\pi_k f = \pi_{k-1} H_k f, \quad k = 1,\cdots,n \tag{4.7.12}$$

so that

$$V = \pi_n f = (H_0 \circ H_1 \circ \cdots \circ H_n) f. \tag{4.7.13}$$

Then to estimate V we only need to approximate π_n. The question to ask is then how to estimate π_n. The approximation method proposed in [PAGPRI] consists first in quantizing for every time step t_k the random variable $\bar{X}_k$ by

$$\widehat{X}_k = \mathrm{Proj}_{\Gamma_k}(\bar{X}_k), \quad k = 0,\cdots,n \tag{4.7.14}$$

where for every k, Γ_k is a grid of N_k points $x_k^i,\ i = 1,\cdots,N_k$ to be optimized and Proj_{Γ_k} denotes the closest neighbor projection on the grid Γ_k.

Owing to equation $(4.7.12)$ our aim is to estimate the price using an approximation of the probability transition $P_k(x_k, dx_{k+1})$ of $\bar{X}_{k+1}$ given $\bar{X}_k$. These probability transitions are approximated by the probability transition matrix $\hat{p}_k := (\hat{p}_k^{ij})$ of $\widehat{X}_{k+1}$ given $\widehat{X}_k$:

$$\hat{p}_k^{ij} = \mathbb{P}(\widehat{X}_k = x_k^j | \widehat{X}_{k-1} = x_{k-1}^i),\ i = 1,\cdots,N_{k-1};\ j = 1,\cdots,N_k. \tag{4.7.15}$$

Then, following equation $(4.7.10)$, we estimate the transition kernel matrix H_k by the quantized transition kernel $\widehat{H}_k$ given by

$$\widehat{H}_k = \sum_{j=1}^{N_k} \widehat{H}_k^{ij} \delta_{x_{k-1}^i}, \quad k = 1,\cdots,n$$

where

$$\widehat{H}_k^{ij} = g_k(x_{k-1}^i, x_k^j)\hat{p}_k^{ij}, \quad i = 1,\cdots,N_{k-1};\ j = 1,\cdots,N_k.$$

For $k = 0$, we approximate $\widehat{H}_0$ (owing to $(4.7.11)$) by

$$\widehat{H}_0 = \sum_{i=1}^{N_0} \mathbb{P}(\widehat{X}_0 = x_0^i)\ \delta_{x_0^i}.$$

We finally approximate π_n by

$$\widehat{\pi}_n = \widehat{H}_0 \circ \widehat{H}_1 \circ \cdots \circ \widehat{H}_n; \tag{4.7.16}$$

which in turn can be computed by the forward induction

$$\widehat{\pi}_0 = \widehat{H}_0, \qquad \widehat{\pi}_k = \widehat{\pi}_{k-1}\widehat{H}_k, \qquad k = 1, \cdots, n. \tag{4.7.17}$$

It follows that the price $V = \pi_n f$ can be estimated by summery

$$\widehat{V} := \widehat{\pi}_n f.$$

Now some facts about practical implementation: optimal dispatching of the grid sizes N_k and how to get the optimal grids are discussed in [BP] and [PAGPRI].

This approach leads to the estimation of the following options using Optimal Quantization method. Set in this scope $f(x) := (x - K)^+$ and $g(x) := (K - x)^+$.

▷ *Up-and-out options.* According to the forgoing we estimate the price of an up-and-out put option by

$$\widehat{P}_{UB} := e^{-rT}\widehat{\pi}_n g$$

and the price of up-and-out call option is approximated by

$$\widehat{C}_{UB} := e^{-rT}\widehat{\pi}_n f$$

where $\widehat{\pi}_n$ is defined as in $(4.7.16)$ with the associated transition kernel

$$\widehat{H}_k^{ij} = G_{x_{k-1}^i, x_k^j}(L)\hat{p}_k^{ij}, \quad i = 1, \cdots, N_{k-1};\ j = 1, \cdots, N_k.$$

The advantage here, from a numerical point of view, is that $\widehat{\pi}_n$ is the same for the call and the put options. Then as soon as $\widehat{\pi}_n$ is computed we deduce both the call and the put price approximations. This remark is also valid for Down-and-out options mentioned below.

▷ *Down-and-out options.* The down-and-out put option's price is estimated by

$$\widehat{P}_{OB} := e^{-rT}\widehat{\pi}_n g$$

and the price of down-and-out barrier call option is estimated by

$$\widehat{C}_{OB} := e^{-rT}\widehat{\pi}_n f$$

where for both cases $\widehat{\pi}_n$ is defined as in $(4.7.16)$ with the associated transition kernel

$$\widehat{H}_k^{ij} = F_{x_{k-1}^i, x_k^j}(L)\hat{p}_k^{ij}, \quad i = 1, \cdots, N_{k-1};\ j = 1, \cdots, N_k.$$

▷ *Fixed strike Lookback put option.* To estimate the price of the fixed strike Lookback put option by quantization we need to know the optimal grid $(u_1^{(m)}, \cdots, u_m^{(m)})$ of the uniform distribution in the unit interval for a grid size equals m and the weights of the associated Voronoi tessel. It is easily seen that

$$u_r^{(m)} = \frac{2r-1}{2m}, \qquad r = 1, \cdots, m$$

which Voronoi cell associated to $u_r^{(m)}$ is the interval $[(r-1)/m, r/m]$ whose Lebesgue measure is $1/m$.

Then the price of a fixed strike Lookback option can be estimated by

$$\widehat{P}_{LF} := K\alpha^\star e^{-rT}\frac{1}{m}\sum_{r=1}^{m}\left(u_r^{(m)}\right)^{\alpha^\star-1}\left(1-\widehat{\pi}_n^r\mathbb{1}\right)$$

where $\alpha^\star$ is the optimal parameter displayed by the stochastic algorithm; $\mathbb{1}$ is the unit vector and $\widehat{\pi}_n^r$ satisfies the following forward induction

$$\widehat{\pi}_0^r = \widehat{H}_0^r, \qquad \widehat{\pi}_k^r = \widehat{\pi}_{k-1}^r\widehat{H}_k^r, \qquad k=1,\cdots,n \tag{4.7.18}$$

and

$$\widehat{H}_k^r = \sum_{j=1}^{N_k}(\widehat{H}_k^r)^{ij}\delta_{x_{k-1}^i}, \quad k=1,\cdots,n$$

with

$$(\widehat{H}_k^r)^{ij} = F_{x_{k-1}^i,x_k^j}\left(K\left(u_r^{(m)}\right)^{\alpha^\star}\right)\hat{p}_k^{ij}, \quad i=1,\cdots,N_{k-1};\ j=1,\cdots,N_k.$$

The $\hat{p}_k^{ij}$ are defined like in $(4.7.15)$ and

$$\widehat{H}_0^{(m)} = \widehat{H}_0 = \sum_{i=1}^{N_0}\mathbb{P}(\widehat{X}_0 = x_0^i)\,\delta_{x_0^i}.$$

▷ *Fixed strike Lookback call option.* We need here the optimal grids $v^{(m)} = (v_1^{(m)},\cdots,v_m^{(m)})$ of the exponential distribution with parameter 1. These optimal grids can be computed by a semi-closed formula given in [FP]. Then the price is estimated by

$$\widehat{C}_{LF} := \beta^\star e^{-rT}\sum_{r=1}^{m}e^{v_r^{(m)}}\left(1-\widehat{\pi}_n^r\mathbb{1}\right)\widehat{p}_r$$

where $\widehat{p}_r = \mathbb{P}\left(V\in C_r(v^{(m)})\right)$, $V\sim\mathscr{E}(1)$, $\beta^\star$ is the optimal parameter in $(4.4.16)$ given by the stochastic approximation algorithm and $\widehat{\pi}_n^r$ is defined as in $(4.7.18)$ with transition kernel

$$(\widehat{H}_k^r)^{ij} = G_{x_{k-1}^i,x_k^i}\left(K+\beta^\star v_r^{(m)}\right).$$

4.8 Error analysis

In order to have some upper bound of the quantization error estimate of πf we need the following assumptions **(A1)** and **(A2)** :

(A1) The transition operator $P_k(x,dy)$ of X_k given X_{k-1}, $k=1,\cdots,n$ are Lipschitz.

Recall that a probability transition P on $\mathbb{R}^d$ is C-Lipschitz (with $\mathrm{C}>0$) if for any Lipschitz function f on $\mathbb{R}^d$ with ratio $[f]_{Lip}$, Pf is Lipschitz with ratio $[Pf]_{Lip}\le \mathrm{C}[f]_{Lip}$. Then, one may define the Lipschitz ratio $[P]_{Lip}$ by

$$[P]_{Lip} = \sup\left\{\frac{[Pf]_{Lip}}{[f]_{Lip}}, f \text{ a nonzero Lipschitz function }\right\} < +\infty.$$

Then if the transition operators $P_k(x,dy)$, $k=1,\cdots,n$ are Lipschitz, it follows that

$$[P]_{Lip} := \max_{k=1,\cdots,n}[P_k]_{Lip} < +\infty.$$

(A2) It consists on the following two assumptions.

(i) For every $k = 1, \cdots, n$, the functions $G_{k,L}(\cdot,\cdot)$ are bounded on $\mathbb{R}^d \times \mathbb{R}^d$ and we set

$$\mathrm{K}_G := \max_{k=1,\cdots,n} \|G_{k,L}\|_\infty$$

(ii) For every $k = 1, \cdots, n$, there exist two constants $[G^1_{k,L}]_{Lip}$ and $[G^2_{k,L}]_{Lip}$ so that for every $x, x', \widehat{x}, \widehat{x}' \in \mathbb{R}^d$,

$$|G_{k,L}(x,x') - G_{k,L}(\widehat{x},\widehat{x}')| \le [G^1_{k,L}]_{Lip}\,|x-\widehat{x}| + [G^2_{k,L}]_{Lip}\,|x'-\widehat{x}'|.$$

Theorem 4.8.1. *Under Assumptions **(A1)** and **(A2)** we have for every bounded Lipschitz continuous function f on $\mathbb{R}^d$ and for every $p \ge 1$,*

$$|\pi_n f - \widehat{\pi}_n f| \le \sum_{k=0}^{n} \mathrm{C}^n_k(f,p)\,\|X_k - \widehat{X}_k\|_p \tag{4.8.1}$$

with

$$\mathrm{C}^n_k(f,p) = (2-\delta_{2,p})\,\mathrm{K}^k_G\,[u_k]_{Lip} + \mathrm{K}^{n-1}_G\,\|f\|_\infty([G^1_{k+1,L}]_{Lip} + [G^2_{k+1,L}]_{Lip}).$$

Proof. The proof follows from the proof of Theorem 3.1 in [PAGPRI] by dropping the dependency on the noisy observations $(y_1, \cdots, y_n)$ following the notations of the authors). □

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