Affine cluster algebras

G. Dupont

University of Lyon

november 6th, 2008

Context

- **Introduction :** S. Fomin et A. Zelevinsky (*Cluster Algebras I : Foundations*, J. Amer. Math. Soc. 2001)
- **Motivation :** Framework for a *combinatorial* study of
  - total positivity in algebraic groups,
  - canonical bases in quantum groups.
- **Connections :**
  - Combinatorics,
  - Lie Theory,
  - Poisson Geometry,
  - Representation theory...

Problematics

- **Problem :** *Find* and *compute* bases in cluster algebras.
- **Canonical Bases:**
  - Sherman-Zelevinsky ($\mathbb{A}_2$, $\tilde{\mathbb{A}}_{1,1}$),
  - Cerulli Irelli ($\tilde{\mathbb{A}}_{2,1}$).
- **Bases :**
  - Caldero-Keller (finite type),
  - Geiss-Leclerc-Schröer (general, abstract).
- **Strategy:** Give an unified and explicit method to compute bases in cluster algebras using representation theory.

Contents

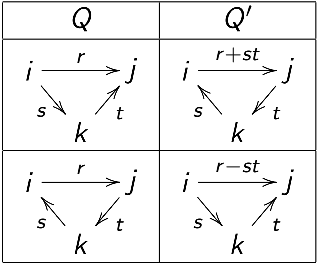
Seeds and clusters

A *seed* is a pair $(Q, \mathbf{x})$ such that:

- $Q = (Q_0, Q_1)$ is a quiver without loops and 2-cycles;
- $\mathbf{x} = (x_i \ : \ i \in Q_0)$ is a Q_0-tuple of indeterminates over $\mathbb{Z}$, called *cluster of the seed* $(Q, \mathbf{x})$.

Mutation of seeds

For every $k \in Q_0$, $\mu_k(Q, \mathbf{x}) = (Q', \mathbf{x}')$ is the new seed given by:



and

$$\mathbf{x}' = \mathbf{x} \setminus \{x_k\} \sqcup \{x'_k\}$$

where

$$x_k x'_k = \prod_{i \longrightarrow k \in Q_1} x_i + \prod_{k \longrightarrow i \in Q_1} x_i.$$

We denote by $(Q, \mathbf{x}) \sim_{\mathrm{mut}} (R, \mathbf{y})$ the generated equivalence relation.

Acyclic cluster algebras

Let $(Q, \mathbf{u})$ be a seed with Q ayclic.

Definition

The cluster algebra $\mathcal{A}(Q)$ with initial seed $(Q, \mathbf{u})$ is

$$\mathcal{A}(Q) = \mathbb{Z}[x | x \in \mathbf{c} \text{ s.t. } (R, \mathbf{c}) \sim_{\mathrm{mut}} (Q, \mathbf{u})] \subset \mathbb{Q}(\mathbf{u})$$

The $\mathbf{c}$ occuring are called *the clusters of* $\mathcal{A}(Q)$,
The $x \in \mathbf{c}$ are called *the cluster variables of* $\mathcal{A}(Q)$.

$$\mathrm{Cl}(Q) = \{\text{cluster variables}\}.$$

The Laurent phenomenon

Theorem, Fomin-Zelevinsky, 2001

$$\mathcal{A}(Q) \subset \mathbb{Z}[\mathbf{u}^{\pm 1}].$$

If $x \in \mathbb{Z}[\mathbf{u}^{\pm 1}]$, the *denominator vector* $\mathrm{den}(x) \in \mathbb{Z}^{Q_0}$ of x is given by

$$x = \frac{P(\mathbf{u})}{\mathbf{u}^{\mathrm{den}(x)}}$$

in its irreducible form.

Cluster monomials

Definition

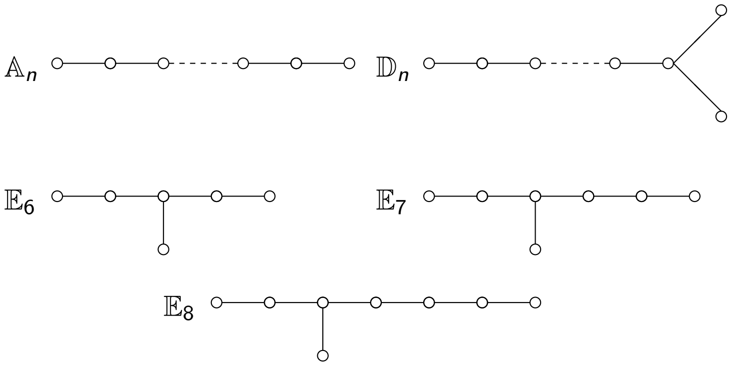
A *cluster monomial* is a monomial in cluster variables belonging to a same cluster.
We set
$$\mathcal{M}(Q) = \{\text{cluster monomials in } \mathcal{A}(Q)\}.$$

Cluster algebras of finite type

Definition

A cluster algebra $\mathcal{A}(Q)$ is said to be of *finite type* if $|\mathrm{Cl}(Q)| < \infty$.

Simply-laced Dynkin diagrams



Finite type classification

Theorem, F.Z., 2002

$\mathcal{A}(Q)$ is of finite type if and only if Q is a Dynkin quiver.
In this case, den induces a 1-1 correspondence

$$\mathrm{den} : \mathrm{Cl}(Q) \longrightarrow \Phi_{>0}(Q) \sqcup (-\Pi(Q)).$$

A $\mathbb{Z}$-basis in finite type

Theorem, Caldero-Keller, 2005

If $\mathcal{A}(Q)$ is of finite type, then $\mathcal{M}(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$.

Fact, Sherman-Zelevinsky

In general, $\mathcal{M}(Q)$ does not span $\mathcal{A}(Q)$.

Conjecture, Zelevinsky

In general, $\mathcal{M}(Q)$ is linearly independent over $\mathbb{Z}$.

The cluster category

Let $k = \mathbb{C}$, Q be an acyclic quiver

$$kQ\text{-mod} \simeq \mathrm{rep}(Q).$$

Definition, BMRRT, 2004

The *cluster category* of Q is the orbit category of the auto-functor $F = \tau^{-1}[1]$ in the bounded derived category $D^b(kQ)$ of kQ-mod.

$$\mathcal{C}_Q = D^b(kQ)/F.$$

Theorem, K, BMRRT, 2004

- $\mathcal{C}_Q$ is a triangulated category;
- $\mathrm{Ext}^1_{\mathcal{C}_Q}(X, Y) \simeq D\mathrm{Ext}^1_{\mathcal{C}_Q}(Y, X)$ (2-Calabi-Yau);
- $\mathrm{ind}(\mathcal{C}_Q) = \mathrm{ind}(kQ\text{-mod}) \sqcup \{P_i[1] \ : \ i \in Q_0\}$.

The quiver grassmannian

- Let M be a kQ-module and $\mathbf{e} \in \mathbb{Z}^{Q_0}$. We write

$$\mathrm{Gr}_{\mathbf{e}}(M) = \{N \subset M \; : \; \mathbf{dim}\, N = \mathbf{e}\}$$

  the quiver grassmannian.
- We denote by χ the Euler-Poincaré characteristic.

Definition, Caldero-Chapoton

The *Caldero-Chapoton* map is the map $X_? : \mathrm{Ob}(\mathcal{C}_Q) \longrightarrow \mathbb{Z}[\mathbf{u}^{\pm 1}]$:

- If M, N are in $\mathrm{Ob}(\mathcal{C}_Q)$, then $X_{M \oplus N} = X_M X_N$;
- If $M \simeq P_i[1]$, then $X_{P_i[1]} = u_i$;
- If M is an indecomposable module, then

$$X_M = \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \prod_{i \in Q_0} u_i^{-\langle \mathbf{e}, \mathbf{dim}\, S_i \rangle - \langle \mathbf{dim}\, S_i, \mathbf{dim}\, M - \mathbf{e} \rangle}. \quad (1)$$

Equality (1) holds for any kQ-module M.

From cluster categories to cluster algebras

Theorem, Caldero-Keller

$X_?$ induces a 1-1 correspondence

$$\{\text{indecomposable rigid objects in } \mathcal{C}_Q\} \xrightarrow{\sim} \mathrm{Cl}(Q).$$

Moreover, the map

$$\left\{ \begin{array}{rcl} \{\text{maximal rigid objects in } \mathcal{C}_Q\} & \xrightarrow{\sim} & \{\text{clusters in } \mathcal{A}(Q)\} \\ T = \bigoplus_{i \in Q_0} T_i & \mapsto & \{X_{T_i} \ : \ i \in Q_0\} \end{array} \right.$$

is a 1-1 correspondence.

Cluster monomials and rigid objects

Corollary

$X_?$ induces a 1-1 correspondence

$$\{\text{ rigid objects in } \mathcal{C}_Q\} \xrightarrow{\sim} \mathcal{M}(Q).$$

Corollary

den induces a 1-1 correspondence

$$\mathrm{Cl}(Q) \xrightarrow{\sim} \Phi^{\mathrm{re,Sc}}(Q) \sqcup (-\Pi(Q))$$

The one-dimensional multiplication formula

Theorem, CK

Let M, N be indecomposable objects in $\mathcal{C}_Q$ such that $\dim \mathrm{Ext}^1_{\mathcal{C}_Q}(M, N) = 1$. Then

$$X_M X_N = X_B + X_{B'}$$

where B and B' are the unique objects such that there exists non-split triangles

$$M \longrightarrow B \longrightarrow N \longrightarrow M[1],$$

$$N \longrightarrow B' \longrightarrow M \longrightarrow N[1]$$

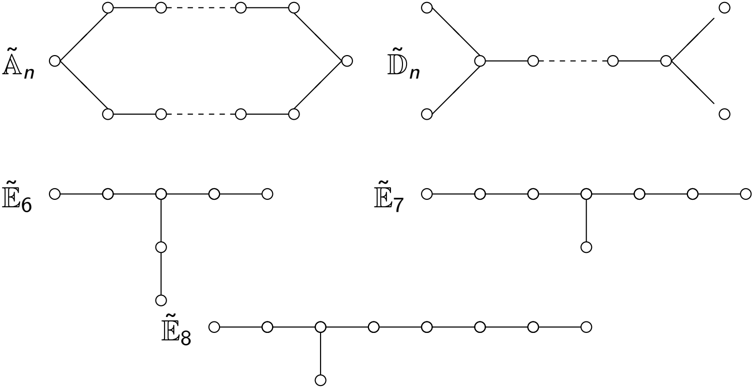
in $\mathcal{C}_Q$.

Contents

Motivation

- Finite-tame-wild classification theorem
- Affine quivers are minimal among representation-infinite quivers
- Representation theory of affine quivers is well-known

Simply laced affine diagrams



Affine cluster algebras

Definition

A quiver Q is called *affine* if it is acyclic and if its underlying diagram is an affine diagram.

Definition

A cluster algebra $\mathcal{A}(Q)$ is called *affine* if Q is an affine quiver.

Affine root systems

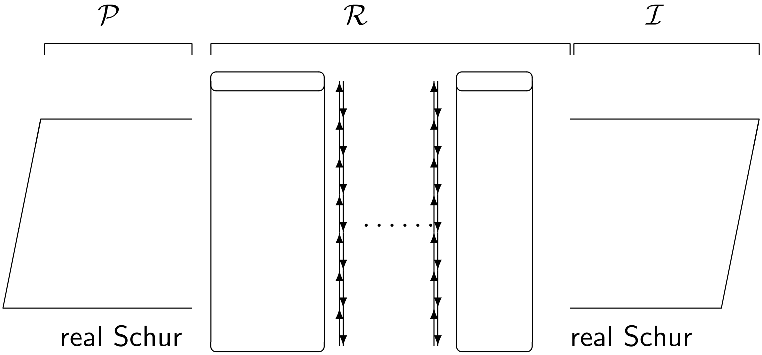
$$\Phi_{>0}(Q) = \Phi^{\mathrm{re}}_{>0}(Q) \sqcup \mathbb{N}^*\delta$$
$$\Phi^{\mathrm{Sc}}(Q) = \Phi^{\mathrm{re,Sc}}(Q) \sqcup \{\delta\}$$

Kac's theorem

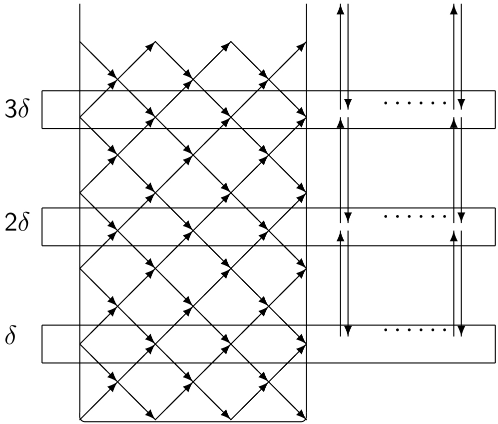
Let $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then

- $\exists M$ indecomposable in $\mathrm{rep}(Q, \mathbf{d})$ iff $\mathbf{d} \in \Phi_{>0}(Q)$;
- $\exists! M$ indecomposable in $\mathrm{rep}(Q, \mathbf{d})$ iff$\mathbf{d} \in \Phi^{\mathrm{re}}_{>0}(Q)$;
- There exists a 1-parameter family of pairwise non-isomorphic indecomposable representations in $\mathrm{rep}(Q, n\delta)$ for every $n \geq 1$.

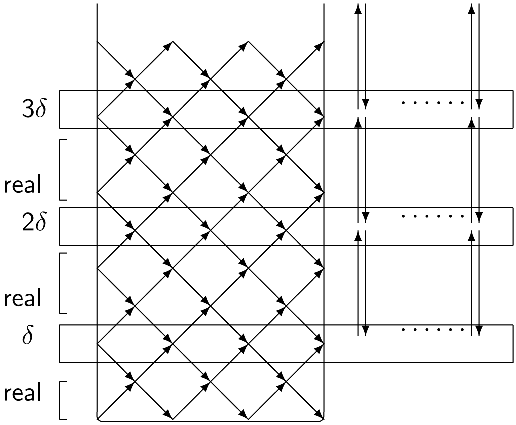
The Auslander-Reiten quiver of kQ-mod



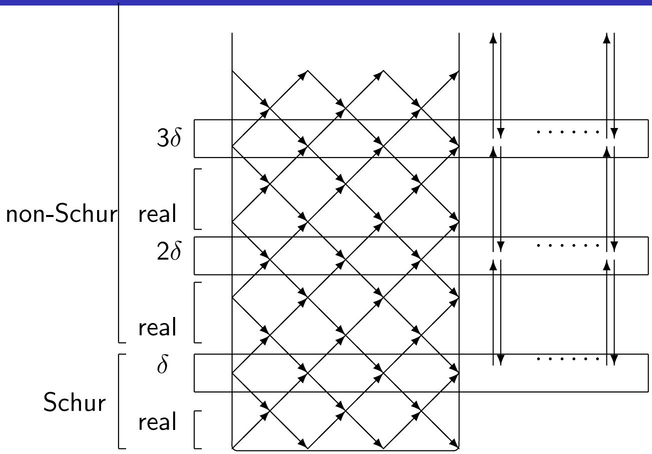
Tubes in $\Gamma(kQ)$



Tubes in $\Gamma(kQ)$



Tubes in $\Gamma(kQ)$



Contents

Motivation

- **Problem:** Understand $X_?$ on regular components.
- **Strategy:** Use the combinatorial description of regular components in order to have a *combinatorial* description of the behaviour of $X_?$.

Generalized Chebyshev polynomials

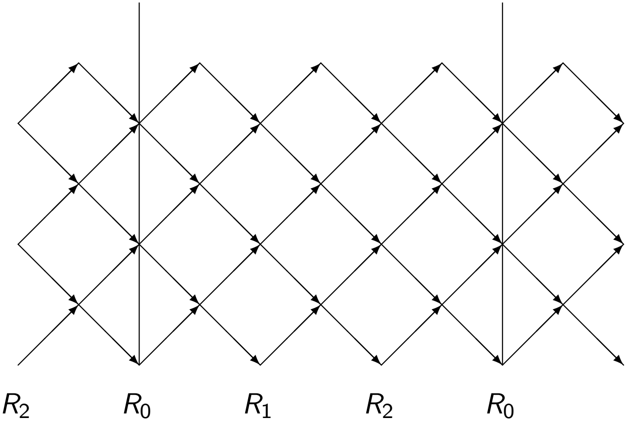
Let $x_i, i \geq 1$ be indeterminates over $\mathbb{Z}$.

Definition

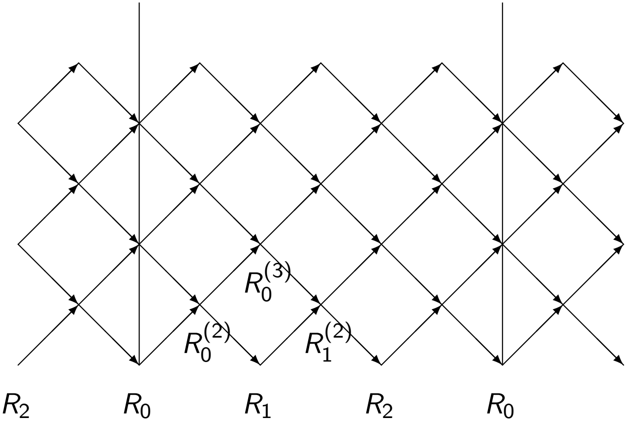
The *n-th generalized Chebyshev polynomial* P_n is given by

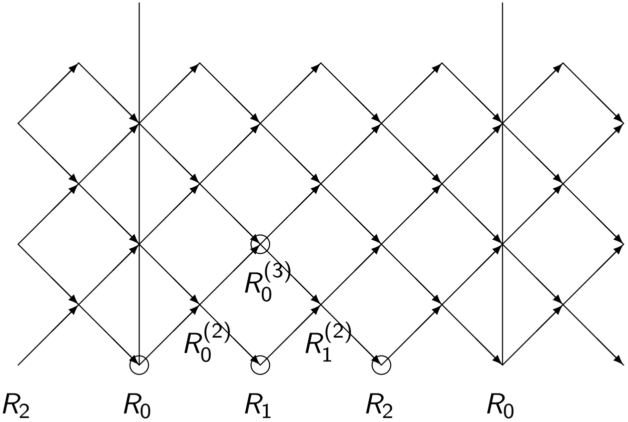
$$P_n(x_1, \ldots, x_n) = \det \begin{bmatrix} x_n & 1 & & (0) \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ (0) & & 1 & x_1 \end{bmatrix} \in \mathbb{Z}[x_1, \ldots, x_n]$$

A tube



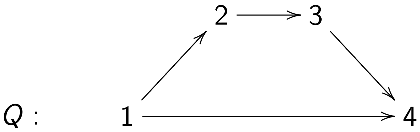
A tube




$R_0^{(3)}$
$R_0^{(2)}$
$R_1^{(2)}$
R_2
R_0
R_1
R_2
R_0

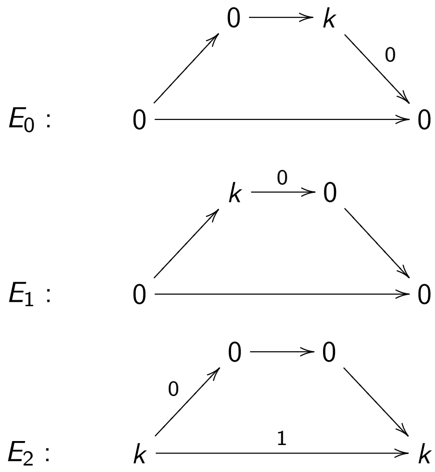
Example in type $\tilde{\mathbb{A}}_{3,1}$

Let Q be an affine quiver of type $\tilde{\mathbb{A}}_{3,1}$.

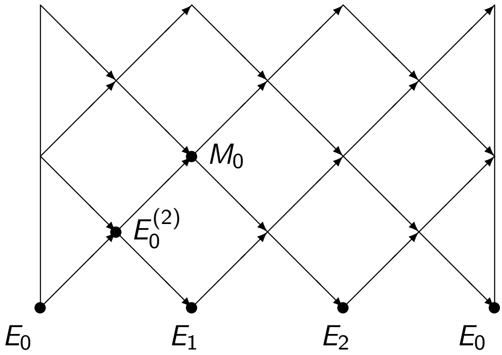


$\Gamma(kQ)$ contains an unique exceptional tube $\mathcal{T}_0$ and $\mathrm{rg}(\mathcal{T}_0) = 3$.
We denote by E_0, E_1, E_2 the quasi-simple modules in $\mathcal{T}_0$.

Example: Quasi-simples in the exceptional tube of $\tilde{\mathbb{A}}_{3,1}$



The exceptional tube of $\tilde{\mathbb{A}}_{3,1}$



Variables in the exceptional tube of $\tilde{\mathbb{A}}_{3,1}$

$$x_0 = X_{E_0} = \frac{u_2 + u_4}{u_3}, \quad x_1 = X_{E_1} = \frac{u_1 + u_3}{u_2},$$

$$x_2 = X_{E_2} = \frac{1 + u_1 u_3 + u_2 u_4}{u_1 u_4}.$$

Variables in the exceptional tube of $\tilde{\mathbb{A}}_{3,1}$

$$x_0 = X_{E_0} = \frac{u_2 + u_4}{u_3}, \quad x_1 = X_{E_1} = \frac{u_1 + u_3}{u_2},$$

$$x_2 = X_{E_2} = \frac{1 + u_1 u_3 + u_2 u_4}{u_1 u_4}.$$

$$X_{E_0^{(2)}} = \frac{u_1 u_2 + u_1 u_4 + u_3 u_4}{u_2 u_3}$$

Variables in the exceptional tube of $\tilde{\mathbb{A}}_{3,1}$

$$x_0 = X_{E_0} = \frac{u_2 + u_4}{u_3}, \quad x_1 = X_{E_1} = \frac{u_1 + u_3}{u_2},$$

$$x_2 = X_{E_2} = \frac{1 + u_1 u_3 + u_2 u_4}{u_1 u_4}.$$

$$X_{E_0^{(2)}} = \frac{u_1 u_2 + u_1 u_4 + u_3 u_4}{u_2 u_3}$$

$$X_{M_0} = \frac{u_1^2 u_3 u_4 + u_1^2 u_2 u_3 + u_1 u_3^2 u_4 + u_1 u_4 + u_1 u_2 + u_3 u_4 + u_2 u_3 u_4^2}{u_1 u_2 u_3 u_4}$$

Contents

Motivations

- "**Generalizing**" cluster monomials,
- **Analogue** of the dual semicanonical basis.

Existence of generic variables

Lemma, D. 2008

Let Q be an acyclic quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then, there exists an open dense subset $U_{\mathbf{d}} \subset \mathrm{rep}(Q, \mathbf{d})$ such that $X_?$ is constant over $U_{\mathbf{d}}$.
We denote by $X_{\mathbf{d}}$ the value of $X_?$ on this open subset.

Definition of generic variables

Definition

Let $\mathbf{d} \in \mathbb{Z}^{Q_0}$. We set

$$X_{\mathbf{d}} = X_{[\mathbf{d}]_+} \prod_{d_i<0} u_i^{-d_i}$$

the *generic variable of dimension* $\mathbf{d}$.

$$\mathcal{B}'(Q) = \left\{ X_{\mathbf{d}} \ : \ \mathbf{d} \in \mathbb{Z}^{Q_0} \right\}$$

Proposition, D. 2008

Let Q be an acyclic quiver. Then

$$\mathcal{M}(Q) \subset \mathcal{B}'(Q).$$

Moreover, if Q is Dynkin, then

$$\mathcal{M}(Q) = \mathcal{B}'(Q).$$

Canonical decomposition and generic variables

Proposition, D. 2008

Let Q be an acyclic quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$ and $\mathbf{d} = \mathbf{d}_1 \oplus \cdots \oplus \mathbf{d}_n$ its canonical decomposition. Then,

$$X_{\mathbf{d}} = \prod_{i=1}^{n} X_{\mathbf{d}_i}.$$

It thus suffices to compute $X_{\mathbf{d}}$ for $\mathbf{d} \in \Phi^{\mathrm{Sc}}$.

Explicitness of generic variables

Proposition, D. 2008

Let Q be an affine quiver and $\mathbf{d} \in \Phi^{\mathrm{Sc}}(Q)$.

- If $\mathbf{d} \in \Phi^{\mathrm{Sc,re}}(Q)$, then $X_{\mathbf{d}} \in \mathcal{M}(Q)$;
- Otherwise, $\mathbf{d} = \delta$ and $X_\delta = X_{M_\lambda}$ for any $\lambda \in \mathbb{P}^1_0$.

Corollary, D. 2008

Let Q be an affine quiver. Then,

$$\mathcal{B}'(Q) = \mathcal{M}(Q) \sqcup \{X_\delta^n X_E \ : \ n \geq 1, E \in \mathcal{E}_R\}$$

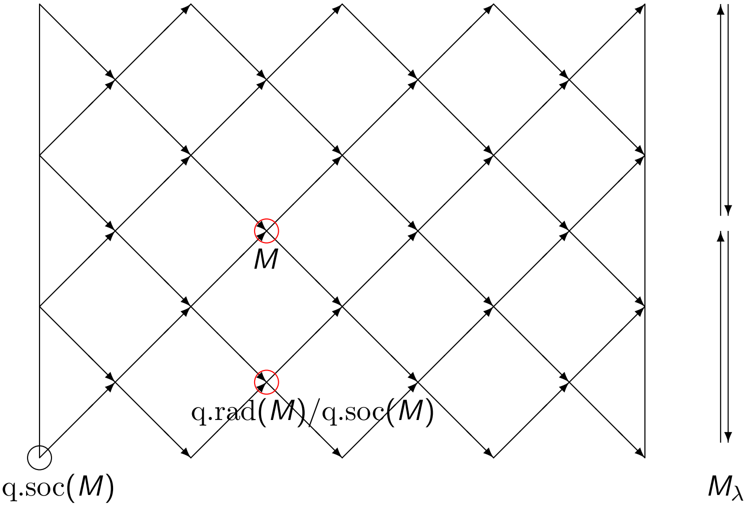
The difference property

Definition

Let Q be an affine quiver. We say that Q *satisfies the difference property* if for every indecomposable kQ-modules M, M_λ in $\mathrm{rep}(Q, \delta)$ belonging respectively to an exceptional and an homogeneous tube, we have:

$$X_{M_\lambda} = X_M - X_{\mathrm{q.rad}M/\mathrm{q.soc}M}.$$

The difference property



The difference property for type $\tilde{\mathbb{A}}$

Theorem, D. 2008

Let Q be an affine quiver of type $\tilde{\mathbb{A}}$. Then Q satisfies the difference property.

Conjecture

Every affine quiver satisfies the difference property.

Generic variables and cluster algebras

Lemma, D. 2008

Let Q be an affine quiver satisfying the difference property. Then,

$$\mathbb{Z}[X_M \ : \ M \in \mathrm{Ob}(\mathcal{C}_Q)] = \mathcal{A}(Q).$$

Corollary, D. 2008

Let Q be an affine quiver satisfying the difference property. Then,

$$\mathcal{B}'(Q) \subset \mathcal{A}(Q).$$

The semicanonical basis

Theorem, D. 2008

Let Q be an affine quiver such that every quiver reflection-equivalent to Q satisfies the difference property. Then, $\mathcal{B}'(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$.

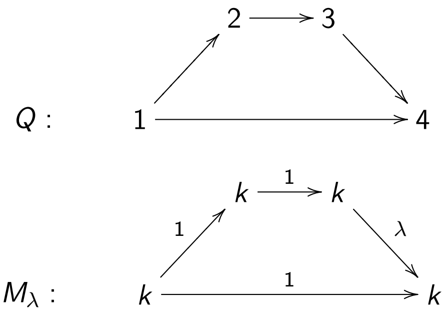
Corollary, D. 2008

Let Q be an affine quiver of type $\tilde{\mathbb{A}}$. Then, $\mathcal{B}'(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$.

Conjecture

Let Q be an affine quiver. Then, $\mathcal{B}'(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$.

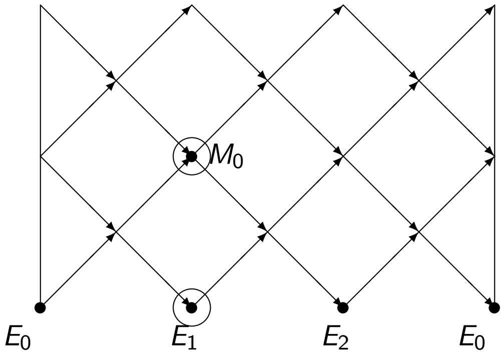
Representations in $\operatorname{rep}(Q, \delta)$ for $\tilde{\mathbb{A}}_{3,1}$



If $\lambda \neq 0$, M_λ is a quasi-simple in an homogeneous tube.
M_0 is in $\mathcal{T}_0$ and

$$\mathrm{q.soc} M_0 \simeq E_0, \quad \mathrm{q.rad} M_0 \simeq E_0^{(2)}.$$

The exceptional tube of $\tilde{\mathbb{A}}_{3,1}$



Example of difference property for type $\tilde{\mathbb{A}}_{3,1}$

$\mathbf{e}$	0	[0001]	[0010]	[0110]	[0011]	[0111]	[1111]
$\mathrm{Gr}_{\mathbf{e}}(M_0)$	0	S_4	E_0	$E_0^{(2)}$	$E_0 \oplus S_4$	$E_0^{(2)} \oplus S_4$	M_0
$\chi(\mathrm{Gr}_{\mathbf{e}}(M_0))$	1	1	1	1	1	1	1
$\mathrm{Gr}_{\mathbf{e}}(M_\lambda)$	0	S_4	$\emptyset$	$\emptyset$	P_3	P_2	M_λ
$\chi(\mathrm{Gr}_{\mathbf{e}}(M_\lambda))$	1	1	0	0	1	1	1

Example of difference property for type $\tilde{\mathbb{A}}_{3,1}$

$\mathbf{e}$	0	[0001]	[0010]	[0110]	[0011]	[0111]	[1111]
$\mathrm{Gr}_{\mathbf{e}}(M_0)$	0	S_4	E_0	$E_0^{(2)}$	$E_0 \oplus S_4$	$E_0^{(2)} \oplus S_4$	M_0
$\chi(\mathrm{Gr}_{\mathbf{e}}(M_0))$	1	1	1	1	1	1	1
$\mathrm{Gr}_{\mathbf{e}}(M_\lambda)$	0	S_4	$\emptyset$	$\emptyset$	P_3	P_2	M_λ
$\chi(\mathrm{Gr}_{\mathbf{e}}(M_\lambda))$	1	1	0	0	1	1	1

$$X_{M_0} = \frac{u_1^2 u_3 u_4 + u_1^2 u_2 u_3 + u_1 u_3^2 u_4 + u_1 u_4 + u_1 u_2 + u_3 u_4 + u_2 u_3 u_4^2}{u_1 u_2 u_3 u_4}$$

$$X_{M_\lambda} = \frac{u_1^2 u_2 u_3 + u_1 u_2 + u_1 u_4 + u_3 u_4 + u_2 u_3 u_4^2}{u_1 u_2 u_3 u_4}$$

$$X_{M_0} = X_{M_\lambda} + \frac{u_2 + u_4}{u_3} = X_{M_\lambda} + X_{E_0}$$

The semicanonical basis of $\mathcal{A}(\tilde{\mathbb{A}}_{3,1})$

$$x_0 = X_{E_0}, x_1 = X_{E_1}, x_2 = X_{E_2},$$

$$y_0 = X_{E_0^{(2)}}, \quad y_1 = X_{E_1^{(2)}}, \quad y_2 = X_{E_2^{(2)}},$$

$$z = X_{M_\lambda}$$

Alors,

$$\mathcal{B}'(Q) = \mathcal{M}(Q) \sqcup \{z^n x_i^r y_i^s \ : \ n > 0, r, s \geq 0, i = 0, 1, 2\}$$

est une $\mathbb{Z}$-base de $\mathcal{A}(Q)$.

Contents

Further directions

- Canonical bases for affine quivers,
- Cluster algebras with coefficients,
- Semicanonical bases for wild quivers,
- Connections with the dual semicanonical basis.