Partially oriented Markov models. Geometrical approach of Probabilistic Cellular Automata

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THÈSE

en vue de l'obtention du titre de

Docteur de l'Université de Rouen

présentée par

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Discipline	:	Mathématiques Appliquées
Spécialité	:	Probabilités

Modèles markoviens partiellement orientés.
Approche géométrique des Automates cellulaires probabilistes.

Date de soutenance : 14 mai 2008

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Remerciements :

Je voudrais tout d'abord remercier chaleureusement mes deux directeurs de thèse, Roberto Fernández et Thierry de la Rue. Leur vision très complémentaire des mathématiques et leur optimisme constant m'ont beaucoup apporté. Ils sont restés à mon écoute et ont toujours eu confiance en moi.

Je tiens aussi à remercier sincèrement Élise Janvresse et Yvan Velenik pour leur présence lors de mes exposés "hebdomadaires" avec mes encadrants. C'est aussi grâce à leurs questions que mes recherches ont pu progresser.

Merci à Thierry Bodineau, Claude Dellacherie, Olivier Garet, Arnaud le Ny et Dimitri Petritis pour avoir accepté de faire partie de mon jury.

Je souhaite aussi remercier tous les membres du laboratoire Raphaël Salem et notamment Olivier Guibé, Gérard Grancher, Marguerite Losada, Edwige Auvray, Isabelle Lamitte et Marc Jolly.

Je salue également tous mes collègues doctorants, en particulier Nicolas Bruyère, Olivier Bertoncini, Olivier Durieu et plus spécialement ceux avec qui j'ai partagé mon bureau : Nicolas Lanchier, Lamia Belhadji et Nicolas Pétrélis.

L'Atelier des Doctorants est organisé par Claude Dellacherie, Élise Janvresse et Thierry de la Rue. Il m'a permis d'enrichir ma culture mathématique, de profiter de nombreux conseils pour mes présentations orales et de m'initier à la gestion d'emploi du temps.

Enfin une pensée spéciale pour ma famille. Merci à mes parents de m'avoir laissé le choix de la longueur de mes études. Merci à Annabelle pour son abnégation et son courage de m'avoir supporté pendant ces longues années de thèse. Merci à mes enfants, Louisa et Gédéon, pour tous ces petits bonheurs qu'ils m'ont donnés, me donnent et me donneront...

Abstract.

The global subject of this thesis is probabilistic cellular automata (PCA). It is divided into two parts.

In the first part, we define the notion of partially ordered chains (POC) that generalise PCA. They are defined thought partially ordered specification (POS) in analogy with the statistical mechanics notion of Gibbs measure. We obtain the analogous of Gibbs measure phase space properties: characterization of extremal measures, construction/reconstruction starting from single site kernels, criterion of uniqueness. These results are applied to some well-known PCA.

The second part is essentially devoted to 1-dimensional PCA with two neighbours and two states. We show two decompositions of space-time configurations in flow of information. Those flows have a geometrical meaning that induce two uniqueness criteria.

In appendix, we give a version of the proof of phase transition of the NEC Toom's PCA. The whole thesis is punctuated by simulations.

Keywords:

probability measures, probabilistic cellular automata, statistical mechanics, specifications, percolation, simulations.

Résumé.

Le sujet global de cette thèse est l'étude d'automates cellulaires probabilistes. Elle est divisée en deux grandes parties.

Au cours de la première, nous définissons la notion de chaîne partiellement ordonnée qui généralise celle d'automate cellulaire probabiliste. Cette définition se fait par l'intermédiaire de spécification partiellement ordonnée de la même façon que les mesures de Gibbs sont définies à l'aide de spécifications. Nous obtenons des résultats analogues sur l'espace des phases : caractérisation des mesures extrêmes, construction/reconstruction en partant des noyaux sur un seul site, critères d'unicité. Les résultats sont appliqués tout au long du texte à des automates déjà connus.

La deuxième partie est essentiellement vouée à l'étude d'automates cellulaires unidimensionnels à deux voisins et deux états. Nous donnons deux décompositions des configurations spatio-temporelles en flot d'information. Ces flots ont une signification géométrique. De cela nous tirons deux critères d'unicité.

En annexe, nous donnons une démonstration de transition de phase d'un automate cellulaire défini par A. Toom, le modèle NEC. Tout au long du texte, des simulations sont présentées.

Mots clés :

probabilités, automates cellulaires probabilistes, mécanique statistique, spécifications, percolation, simulations.

Contents

Chapitre 1

Introduction

L'idée qui soutend toute cette thèse est la synthèse de textures. Ce sont des images très particulières. En effet, elles présentent deux échelles avec des comportements radicalement différents. D'une part, leur aspect global semble déterministe avec, la plupart du temps des formes géométriques simples : il s'agit de rayures, de cercles, de quadrillages,... Pourtant, si l'on regarde à l'échelle du pixel, c'est-à-dire à l'échelle du plus petit composant de l'image, il ne semble y avoir aucune règle rigide. L'image paraît chaotique.

Ce genre de comportement, désordonné à très petite échelle et quasi-déterministe à grande échelle est une des caractéristiques des modèles probabilistes ergodiques. Il est donc bien naturel de modéliser ce genre d'images à l'aide des probabilités. De plus, beaucoup de textures présentent au moins une direction particulière (le sens des rayures, du quadrillage,...). Il n'est donc pas insensé de voir ces images comme le résultat d'un processus "temporel" dont le temps emprunte cette direction privilégiée. C'est ce qu'ont fait, entre autres, N. Cressie et J. Davidson ([CD98], [CDH99]). Ces deux statisticiens se basent sur un modèle qu'ils appellent POMM (partially ordered Markov models) pour bâtir une analyse et un moyen de synthèse de textures. Leurs travaux portent essentiellement sur l'établissement d'estimateurs pour trouver de manière empirique les paramètres sous-jacents qui dirigent la texture. Ceci fait, il leur est facile de générer des images similaires à l'originale en simulant le modèle obtenu. Cette thèse a pour but d'explorer les probabilités qui se cachent derrière ce genre de modèles.

1.1 Les modèles abordés

Dans cet paragraphe, nous présentons de manière succincte les différents modèles que nous étudions dans cette thèse.

Commençons par les POMM. Donnons-nous S un ensemble dénombrable (l'ensemble des pixels) muni d'une relation d'ordre $\leqslant$. Si x est un élément de S, le ***passé de*** x, noté x_-, est constitué des y plus petits que x :

$$x_- := \{y \in S,\ y < x\}$$

De la même manière, le ***futur de*** x (x_+) est

$$x_+ := \{y \in S,\ y > x\}$$

Enfin les points qui ne sont pas en relation avec x constituent le ***temps extérieur à*** x. Leur ensemble est noté x^*. Nous avons aussi besoin de définir la notion de voisinage du passé proche $\underline{\partial} x$: ce sont les points maximaux (pour $\leqslant$) de x_-.

Les POMM sont alors définis à l'aide de cette structure géométrique. Chaque point x de S possède une couleur aléatoire σ_x qui vérifie

$$P\big(\sigma_x | \sigma_{x^*_-}\big) = P\big(\sigma_x | \sigma_{\underline{\partial} x}\big)$$

où $x^*_- = x_- \cup x^*$. En d'autres termes, sachant l'état de son passé et du temps extérieur, la couleur d'un point x de S ne dépend que de l'état de son voisinage du passé proche. C'est bien l'idée que l'on souhaitait mettre en oeuvre sur les textures.

Un modèle voisin de celui-ci (c'est en fait un cas particulier des POMM) est celui des automates cellulaires probabilistes (PCA en anglais). L'approche ici est assez différente.

Prenons U un ensemble dénombrable ($\mathbb{Z}^n$ la plupart du temps). Cet ensemble sera l'ensemble des sites. Chaque point x de U possède une couleur et un voisinage. Mais à la différence des POMM, les PCA sont dynamiques : les couleurs évoluent au cours de temps. Cette évolution se passe en temps discret. La caractéristique des PCA est que tous les sites évoluent simultanément et indépendemment les uns des autres en fonction de l'état de leur voisinage au temps précédent.

Cela paraît éloigné des POMM mais ce n'est qu'une apparence. En effet, si nous regardons, non pas les configurations du système à un instant donné mais les configurations spatio-temporelles (c'est-à-dire sur $U \times \mathbb{Z}$) alors nous retrouvons un POMM. En effet, il nous est possible de définir un passé pour chaque point (ceux qui interagissent avec lui) et un ordre (partiel) sur $U \times \mathbb{Z}$.

1.2 Les question posées

Une fois les fondements établis, venons en à la question centrale de cette thèse : la simulation de POMM. est la suivante : Pour la synthèse de textures, est-il indispensable de partir de configurations initiales fixées ? La synthèse de textures signifiera pour nous une simulation de notre automate cellulaire ou de notre POMM. On peut aussi reformuler la question ainsi : Est-ce qu'en partant de n'importe quelle configuration initiale notre image sera la même ? Par "la même", il faut bien sûr entendre globalement la même. Il est bien évident que deux simulations d'un objet probabiliste ne pourront donner des résultats strictement identiques. Cette question est en relation étroite avec la notion de transition de phase des modèles considérés. Pour illustrer ce propos, prenons un exemple parmi les champs de Gibbs. Le modèle d'Ising est l'un des modèles les plus simples et les plus étudiés de ces champs. Il est bien connu que, pour des paramètres bien choisis (champs magnétique nul et température suffisamment basse), ce modèle présente une transition de phase : si l'on part d'une configuration "tout plus" au bord d'une grande boîte, notre champs se comportera comme un océan de plus avec quelques îlots de moins. Au contraire, si nous imposons "tout moins", il sera un océan de moins avec des îlots de plus. Ceci nous montre que le choix de la configuration initialement choisie peut être très importante et donner des images complètement différentes en fonction de ce choix. Ce phénomène apparaît-il pour les POMM ? La réponse pour les PCA, est oui. En dimension superieure à 2, elle est connu depuis le début des années 80 avec notamment les travaux de André Toom ([Too80]). Pour la dimension 1, il a fallu attendre 2000 et un article de 220 pages de Peter Gács ([Gác01], [Gra01]). Le modèle de ce dernier est d'ailleur assez exotique. Il ne lui faut pas moins de 2^{100} couleurs et des probabilités de l'ordre de 2^{-50} pour parvenir à obtenir cette fameuse transition de phase. C'est à ce jour le seul modèle en dimension un qui possède cette propriété. Tout ceci est bien hors de porté d'un ordinateur de bureau mais le fait que ce modèle existe nous montre que le problème de la condition initiale n'est pas à négliger.

1.3 Cadre et résultats de nos travaux

Remarquons tout d'abord que si l'ordre sur S est total, nos POMM sont des chaînes de Markov. D'un autre côté, si l'ordre est extrêmement partiel (et à la limite n'existe plus), les POMM se rapprochent sensiblement des champs gibbsiens.

Cette remarque nous montre que les POMM sont en quelque sorte à mi-chemin entre les chaînes de Markov et les champs gibbsiens. Il n'est donc pas étonnant de retrouver des propriétés des uns et des autres satisfaites par les POMM.

Nous avons choisi de travailler avec des outils de physique mathématique. En nous inspirant des spécifications pour les champs gibbsiens, nous avons défini (Définition 3.7 et 3.13) des spécifications partiellement ordonnées (POS en anglais) sur un ensemble de "bonnes boîtes". Ceci nous a alors conduit à explorer les possibilités offertes par la mécanique statistique.

Le premier résultat obtenu (Théorème 4.10) est qu'il n'est pas nécessaire de définir les POS sur toutes les (bonnes) boîtes. Tout comme les chaînes de Markov, le comportement des singletons suffit à définir complètement le modèle. Ce n'est pas le cas pour les champs gibbsiens : il faut une hypothèse supplémentaire pour y arriver (par exemple la positivité des noyaux des singletons).

Ensuite, nous nous sommes attachés à caractériser certaines mesures compatibles avec nos POS. Comme pour les spécifications, l'ensemble des mesures compatibles est convexe et les mesures “visible localement” sont les points extrémaux de cet ensemble (Théorème 6.3). De plus, ces mesures sont mélangeantes (Théorème 6.6) et atteignables par un “simple” passage à la limite (Théorème 6.7).

Parmi toute la panoplie d'outils disponibles pour les champs gibbsiens, nous avons adapté les inégalités FKG à nos POS (Théorème 7.5). Elles sont en effet très pratiques à mettre en oeuvre et, quand elles sont vraies, permettent de réduire considérablement le travail pour la recherche d'unicité des mesures compatibles, recherche qui est la principale préoccupation de cette thèse.

Nous nous sommes enfin intéressés à des critères simples à manipuler pour décider de l'unicité. Nous en avons trouvé deux. Le premier est dit de Dobrushin car il dérive d'un de ses critères pour les champs gibbsiens (Théorème 8.5). Le second s'appuie sur la percolation orientée (Théorème 8.16).

Le reste de nos travaux porte sur les textures bicolores. Nous les avons modélisées par des trajectoires espace-temps d'automates cellulaires probabilistes à deux états dans $\mathbb{Z}$.

Plutôt que de determiner la couleur d'un site en utilisant une probablité qui dépend de la couleur de ses voisins, nous tirons au hasard d'où vient l'information puis nous en déduisons la couleur du site en question. Avec cette façon de procéder, nous avons décomposé les trajectoires espace-temps des ces automates de deux façons différentes. Au lieu de tirer au hasard ce qui se passe temporellement, nous tirons globalement comment se diffuse l'information, puis nous tirons au hasard les configurations compatibles avec cette diffusion. Nous faisons donc intervenir deux mesures. La première peut poser problème car elle est signée (mais de masse unitaire). La deuxième est une probabilité.

La différence des deux décompositions se situe au niveau de la deuxième mesure : le choix des configurations espace-temps dépend de la signification du terme “compatible avec la diffusion d'information”. La première signification que l'on donne est reliée à une fonction de majorité (Proposition 10.4 et Équation (10.6)). Nous tirons alors de cette décomposition un critère d'unicité simple et très facile d'utilisation (Théorème 10.16).

La deuxième signification est reliée à un produit (Équation (11.1)). Elle permet également d'obtenir un critère d'unicité (Théorème 11.7).

Ces deux critères sont complémentaires mais ne permettent pas de couvrir la totalité des PCA à 2 couleurs.

En annexe, nous donnons la preuve de transition de phase du modèle de A. Toom ([Too80]). C'est, à peu de chose près, la démonstration originale.

Enfin, nous terminons avec les programmes qui nous ont servi à faire les simulations qui jalonnent cette thèse. Ils ont été fait en C et sont sous licence GPL.

Part I

General theory of Partially Ordered Markov Models

Chapter 2

Framework

This part is devoted to the description of partially ordered Markov models (POMMs). This name appeared first in two articles from statistics: [CD98] and [CDH99]. The authors describe POMMs features and apply them for white-and-black textures. They are able to analyse a texture with a well chosen POMM and simulate the latter to obtain an image similar to the original one. unfortunately, this seems to be efficient only for textures coming from POMMs.

Nevertheless, this theory is attracting and we worked on the probability side of this model: we define partially oriented specifications (POS). It is an adaptation of Gibbs specifications. It generalizes left interval specifications (see [FM05]) that describe time processes with statistical mechanics tools. POMMs are also a generalisation of probabilistic cellular automata (PCA). Some work has already been done for PCA in statistical mechanics (see [LMS90a], [LMS90b], [GKLM89]). These papers have two drawbacks. First, the main work is not done into the PCA theory. They transform a PCA into a Gibbs field, get results on this Gibbs field and then try to go back to the original PCA. This construction is complicated and the Gibbs fields obtained are not so easy to manipulate. Moreover, in the process to go back they use a powerful but restrictive tool: they use a variational principle. In particular, this implies that the considered PCA must be translation invariant in space and in time.

On the contrary, we do not suppose such a property. Our results are valid for a wider set of PCA. Additionally, we stay in the POMM theory and adapt tools from statistical mechanics to our purpose.

We first present the model: its geometry and the probability context.

We get general results on POS such as construction/reconstruction by singleton kernels. This property is very important because it allows to define a POS by giving only what happens on single sites.

We then obtain the analogous of Gibbs phase space properties: characterization of extremal measures in terms of mixing, triviality on $\mathcal{F}_{-\infty}$ and mutual singularity.

An FKG inequality is also proved. It is a powerful tool that can simplify the research for extremal measures.

At last, we give two criteria of uniqueness for consistent measures. Theses criteria are applied to two examples that are recurrent along this part.

Chapter 3

Preliminaries

3.1 Geometrical aspects

In this chapter, we define the geometry of our model and establish various properties. They will be useful in the following chapters.

Let $(S, \leqslant)$ be a countable (partially) ordered set.

Definition 3.1.
Let $\Upsilon \subset S$. We define

$$\begin{aligned} \max(\Upsilon) &:= \{x \in \Upsilon;\ \forall y > x,\ y \notin \Upsilon\} \\ \min(\Upsilon) &:= \{x \in \Upsilon;\ \forall y < x,\ y \notin \Upsilon\} \end{aligned} \tag{3.1}$$

Definition 3.2.
Let $\Upsilon \subset S$. The ***past*** of Υ is defined by

$$\Upsilon_- := \{x \in S, x \notin \Upsilon; \exists y \in \Upsilon, x < y\} \tag{3.2}$$

Similarly the ***future*** of Υ is

$$\Upsilon_+ := \{x \in S, x \notin \Upsilon; \exists y \in \Upsilon, x > y\} \tag{3.3}$$

We also define the ***outer time*** of Υ:

$$\Upsilon^* := \{x \in S; \forall y \in \Upsilon, x \text{ is unrelated to } y\} \tag{3.4}$$

In the whole thesis, we assume that for all $x \in S$, $\max(\{x\}_-)$ and $\min(\{x\}_+)$ are nonempty finite sets such that

$$\begin{aligned} \forall y < x,\ \exists y_0 \in \max(\{x\}_-),\ y \leqslant y_0 < x \\ \forall z > x,\ \exists z_0 \in \min(\{x\}_+),\ z \geqslant z_0 > x \end{aligned}$$

This property implies that S is totally discontinuous. Moreover, we will suppose that S does not contain minimal points: $\min(S) = \varnothing$. Such an S is called a ***site space*** and a point of S is a ***site***. The last hypothesis is not really crucial. It is useful to avoid some problems. In particular, this permits to have no site in the infinite past (this notion is described later).

Definition 3.3.
Let Λ be a finite part of S. We say that Λ is a ***time box*** if $\Lambda_- \cap \Lambda_+ = \varnothing$, and a ***bad box*** otherwise. The set of time boxes is denoted by $\mathcal{T}_b$.

In the whole thesis, we will use the notation Λ, Δ or Γ for time boxes and Υ for subsets of S without any assumption.

Let Λ be a time box. We define ***the complementary of the past*** as $\Lambda_+^c := S \backslash \Lambda_+$. For brevity, we denote $\Lambda_-^* := \Lambda_- \cup \Lambda^*$, and for $x \in S$, $x_- := \{x\}_-$ and $x_+ := \{x\}_+$.

Remark 3.4.
1. $\mathcal{T}_b$ is not an empty set. Indeed, for every single site $x \in S$, $\{x\}$ is a time box.
2. For any time box Λ, $\{\Lambda,\ \Lambda_+,\ \Lambda_-,\ \Lambda^*\}$ is a partition of S. Moreover, $\Lambda_-^* = \Lambda_+^c \backslash \Lambda$ and $\Lambda_- \subset \Lambda_-^* \subset \Lambda_+^c$.

Intuitively, there are no holes in a time box. We want to define a process that depends on past, without looking at the future. That is why we define time boxes. The process will be defined only on such boxes. To clarify all those definitions, here are some examples.

Example 3.5.
Let us consider $\mathbb{Z}$ with its natural (total) order. Since it is a total order, there is no outer time. $\mathcal{T}_b$ is exactly the set of finite intervals. The theory of this thesis is reduced to Left Interval Specifications (see [FM05], [Mai03]) in this case.

Example 3.6.
In $\mathbb{Z}^2$, define the natural partial order

$$(x_1, y_1) \leqslant (x_2, y_2) \iff x_1 \leqslant x_2 \text{ and } y_1 \leqslant y_2 \tag{3.5}$$

Figure 3.1 shows examples of time boxes and bad boxes.

When it is not specified, the drawings of the whole thesis will be according to the preceding example: in $\mathbb{Z}^2$, with its natural order. Moreover, the vertical axis will be oriented from up to down. This is the usual habit in computer science. We used this convention because it was easier for simulations.

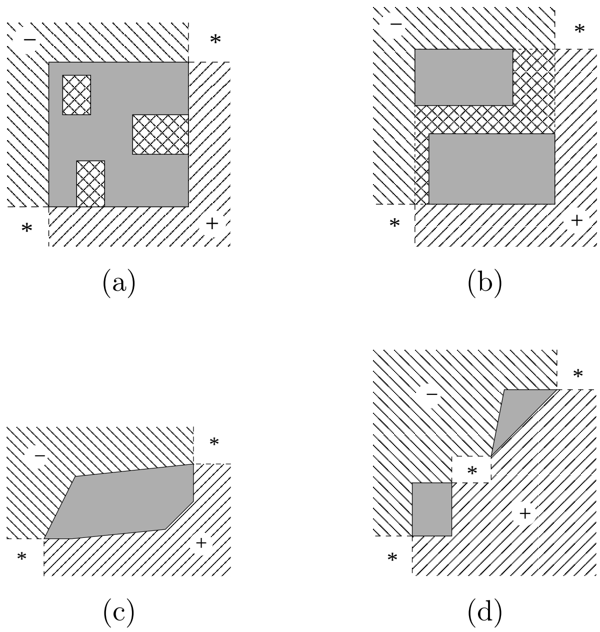


Figure 3.1 – $\mathbb{Z}^2$ is endowed with its natural partial order (3.5). The orientation is from the left-up corner to the right-down one, according to computer science conventions. If we denote Λ the grey box, the symbol * stands for Λ^*, - for Λ_- and + for Λ_+. (a), (b): bad boxes; (c), (d): time boxes.

3.2 Probabilistic notions

Let $(E, \mathcal{E})$ be a measurable set called ***color space***. It is supposed neither finite nor countable for the moment. The ***set of configurations*** is the product space $(\Omega, \mathcal{F}) = (E^S, \mathcal{E}^S)$. If $\Upsilon \subset S$, $\mathcal{F}_\Upsilon$ denotes the sub-σ-algebra of $\mathcal{F}$ generated by the cylinders with base in E^Υ and a configuration on Υ will be denoted by ω_Υ.

Definition 3.7.
Let Λ be a time box. A ***proper oriented kernel*** γ_Λ on Λ is a function $\gamma_\Lambda : \mathcal{F}_{\Lambda_+^c} \times \Omega \longrightarrow [0, 1]$ satisfying the following properties:

i. For each $\omega \in \Omega$, $\gamma_\Lambda(\cdot, \omega)$ is a probability measure,

ii. For each $A \in \mathcal{F}_{\Lambda_+^c}$, $\gamma_\Lambda(A, \cdot)$ is $\mathcal{F}_{\Lambda_-^*}$-measurable,

iii. For each $A \in \mathcal{F}_\Lambda$, $\gamma_\Lambda(A, \cdot)$ is $\mathcal{F}_{\Lambda_-}$-measurable,

iv. For each $B \in \mathcal{F}_{\Lambda_-^*}$, and $\omega \in \Omega$, $\gamma_\Lambda(B, \omega) = \mathbb{1}_B(\omega)$.

The general notion of kernel only refers to the two first properties. (iv) expresses the fact that the past and the outer time of Λ are frozen: The only quantities which may vary are those in Λ. Kernels satisfying (iv) are often called proper kernels. (iii.) means that γ_Λ depends only on the past of Λ. This is the only property that takes into account the structure of partial order of the site space S.

We will use indistinctly the two following notations: $\gamma_\Lambda(\cdot, \cdot)$ or $\gamma_\Lambda(\cdot|\cdot)$ for oriented kernels.

Proposition 3.8.
Let $\Lambda \in \mathcal{T}_b$ and γ_Λ be a proper oriented kernel. For all $A \in \mathcal{F}_\Lambda$ and $B \in \mathcal{F}_{\Lambda_-^*}$,

$$\gamma_\Lambda(A \cap B|\omega) = \mathbb{1}_B(\omega)\gamma_\Lambda(A|\omega)$$

Proof.
Fix $B \in \mathcal{F}_{\Lambda_-^*}$. For $\omega \notin B$, $\gamma_\Lambda(A \cap B|\omega) = 0$ and for $\omega \in B$, $\gamma_\Lambda(A \cap B|\omega) \leqslant \gamma_\Lambda(A|\omega)$. Then for all $A \in \mathcal{F}_\Lambda$, we can write $\gamma_\Lambda(A \cap B|\omega) \leqslant \gamma_\Lambda(A|\omega)\ \mathbb{1}_B(\omega)$. Now,

$$0 = \Big(\gamma_\Lambda(A \cap B|\omega) - \gamma_\Lambda(A|\omega)\,\mathbb{1}_B(\omega)\Big) + \Big(\gamma_\Lambda(A^c \cap B|\omega) - \gamma_\Lambda(A^c|\omega)\,\mathbb{1}_B(\omega)\Big)$$

But the two terms are non-positive. ✠

Remark 3.9.
This proposition shows that $\gamma_\Lambda(d\sigma|\omega) = \bar{\gamma}_\Lambda(d\sigma_\Lambda|\omega)\ \mathbb{1}_{\omega_{\Lambda_-^*}}(\sigma_{\Lambda_-^*})$ where $\bar{\gamma}_\Lambda$ is a kernel on Λ. In the sequel, we shall use this property without introducing $\bar{\gamma}$. This property reinforce the idea of a frozen past: the varying quantities in γ_Λ are only those in Λ.

Proposition 3.10.
Let $\Lambda \in \mathcal{T}_b$, $\Upsilon \subset \Lambda_+^c$, f be an $\mathcal{F}_\Upsilon$-measurable function and γ_Λ be a proper oriented kernel. Then $\gamma_\Lambda(f) := \gamma_\Lambda(f, \cdot)$ is $\mathcal{F}_{\Lambda_- \cup (\Upsilon \backslash \Lambda)}$-measurable.

Proof.
Let $B = \Lambda_- \cup (\Upsilon \backslash \Lambda)$ and $\xi, \eta \in \Omega$ such that $\xi_B = \eta_B$. Let us define $f_\eta : \Omega \mapsto \mathbb{R}$ by $f_\eta(\omega) := f(\omega_\Lambda \eta_{\Lambda_-^*})$. f_η is $\mathcal{F}_\Lambda$-measurable so $\gamma_\Lambda(f_\eta)$ is $\mathcal{F}_{\Lambda_-}$-measurable. This implies that $\gamma_\Lambda(f_\eta, \xi) = \gamma_\Lambda(f_\eta, \eta)$. Moreover, Proposition 3.8 gives $\gamma_\Lambda(f) = \gamma_\Lambda(f_\eta)$. So we have

$$\begin{aligned}
\gamma_\Lambda(f, \xi) - \gamma_\Lambda(f, \eta) &= \gamma_\Lambda(f_\xi, \xi) - \gamma_\Lambda(f_\eta, \eta) \\
&= \gamma_\Lambda(f_\xi, \xi) - \gamma_\Lambda(f_\eta, \xi) \\
&= \gamma_\Lambda(f_\xi - f_\eta, \xi) \\
&= 0
\end{aligned}$$

The last line comes from the fact that f is $\mathcal{F}_\Upsilon$-measurable so $f_\eta = f_\xi$. ✠

Remark 3.11.
In particular, if f is $\mathcal{F}_{\Lambda \cup \Lambda_-}$-measurable, $\gamma_\Lambda(f)$ is $\mathcal{F}_{\Lambda_-}$-measurable. In other words, no dependency of Λ^* is added by applying γ_Λ on f. What happens in Λ^* does not influence what happens in Λ.

Proposition 3.12.
Let $\Lambda \in \mathcal{T}_b$ and γ_Λ, $\bar{\gamma}_\Lambda$ be two proper oriented kernels on Λ such that for all $\mathcal{F}_\Lambda$-measurable functions f,

$$\gamma_\Lambda(f) = \bar{\gamma}_\Lambda(f)$$

then $\gamma_\Lambda = \bar{\gamma}_\Lambda$.

Proof.
Let g be an $\mathcal{F}_{\Lambda_+^c}$-measurable function and η be a configuration. We have to prove that $\gamma_\Lambda(g)(\eta) = \bar{\gamma}_\Lambda(g)(\eta)$. As in the proof of Proposition 3.10, we denote g_η the function defined by $g_\eta(\xi) := g(\xi_\Lambda \eta_{\Lambda_-^*})$. The function g_η is $\mathcal{F}_\Lambda$-measurable, thus

$$\gamma_\Lambda(g, \eta) = \gamma_\Lambda(g_\eta, \eta) = \bar{\gamma}_\Lambda(g_\eta, \eta) = \bar{\gamma}_\Lambda(g, \eta)$$

✠

Definition 3.13.
A ***partially oriented specification*** (***POS***) γ on $(\Omega, \mathcal{F})$ is a family of proper oriented kernels $\{\gamma_\Lambda\}_{\Lambda \in \mathcal{T}_b}$ such that

v. For all $\Lambda, \Delta \in \mathcal{T}_b$ such that $\Lambda \subset \Delta$, $\quad \gamma_\Delta \gamma_\Lambda = \gamma_\Delta \quad$ on $\mathcal{F}_{\Lambda_+^c}$.

This property is usually labeled ***consistency***. Explicitly, this means that for each $\mathcal{F}_{\Lambda_+^c}$-measurable function h and each configuration $\omega \in \Omega$,

$$\iint h(\xi)\ \gamma_\Lambda(d\xi,\sigma)\ \gamma_\Delta(d\sigma,\omega) = \int h(\sigma)\ \gamma_\Delta(d\sigma,\omega)$$

Concretely, this means that, integrating a function on Λ and then integrating the result on Δ is exactly the same as integrating the function directly on Δ. This property is fundamental if we want the family $\{\gamma_\Lambda\}$ to be a family of conditional expectations with respect to the same measure. A POS has all the properties required for this. One of the natural question is then the existence of such a measure and how many they are. We have essentially worked on their uniqueness.

Definition 3.14.
A probability measure μ on $(\Omega, \mathcal{F})$ is said to be ***consistent with a POS*** γ if for each $\Lambda \in \mathcal{T}_b$, $\mu\gamma_\Lambda = \mu \quad$ on $\mathcal{F}_{\Lambda_+^c}$.
Such a measure μ is called a $\gamma-$**partially oriented chain** or a $\gamma-$**POC**. The set of $\gamma-$POCs will be denoted by $\mathcal{G}(\gamma)$.

Definition 3.15.
Let $x \in S$ and $\Lambda \in \mathcal{T}_b$. Denote the ***nearest past*** of x and Λ by

$$\begin{aligned} \underline{\partial} x &:= \max(x_-) \\ \underline{\partial}\Lambda &:= \left(\bigcup_{x\in\Lambda} \underline{\partial} x \right) \setminus \Lambda \end{aligned} \tag{3.6}$$

Definition 3.16.
We will say that a POS γ is ***local*** if for all $\Lambda \in \mathcal{T}_b$ and all $A \in \mathcal{F}_\Lambda$, $\gamma_\Lambda(A|\cdot)$ is $\mathcal{F}_{\underline{\partial}\Lambda}$-measurable. We will also use the term of ***partially ordered Markov model*** or ***POMM***. This assumption replace (iii) in the definition 3.7.

A POMM depends only on the nearest past. This is a generalization of Markov chains for partially ordered "time". The set of dependence of a local POS can be extended to some finite subset of the past. This will generally not affect our results. We made the hypothesis of dependence only on the nearest past for convenience and simplicity.

Definition 3.17.
We define the notion of ***quasilocal*** POS as a limit of local (with the extended meaning) POS: γ is quasilocal if for all $\epsilon > 0$, there exists a local POS $\tilde{\gamma}$ such that for all $\Lambda \in \mathcal{T}_b$, for all $A \in \mathcal{F}_{\Lambda_+^c}$ and for all $\omega \in \Omega$

$$\Big| \gamma_\Lambda(A,\omega) - \tilde{\gamma}_\Lambda(A,\omega) \Big| < \epsilon$$

Definition 3.18.
A POS will be said to be ***continuous*** if for all $\Lambda \in \mathcal{T}_b$ and all continuous local $\mathcal{F}_{\Lambda_+^c}$-measurable functions f, $\gamma_\Lambda(f)$ is continuous.

In general, continuity and quasilocality are distinct notions. They are equivalent if the color space E is finite. In this case, the compactness of Ω induces the compactness of the set of probability measures on Ω endowed with the weak convergence. In particular, this implies existence of POC for the model. Since nearly all the work done here is for E finite, the question of existence of POCs has not been studied.

3.3 Examples

We present here two models which will be recurrent through this thesis.

We will see in the following chapter that it is not necessary to define the whole specification. A family of singleton kernels determines completely a POS. We shall use (before proving it) this property here. Nevertheless, we have decided to introduce now these examples.

For those models, we have done simulations. Programs are available in Appendix B.1 concerning simulations.

3.3.1 The POMM-Ising Model

The model is derived from the 2-dimensional Ising model in Gibbs Theory.

It has the same color space and the interactions are exactly the interaction of the Ising model but in the context of a partially ordered site space.

Let S be equal to $\mathbb{Z}^2$. We define on S the natural partial order defined by Equation (3.5). The color space, as in the Ising model, will consists in only two elements: $E = \{-1, +1\}$. Let $h \in \mathbb{R}$ and $\beta > 0$ be two parameters. In Ising model, h represents the magnetisation and β the inverse of the temperature. Here, those coefficients have no physical interpretation.

We first define the two spatial operators:

$$N : \mathbb{Z}^2 \to \mathbb{Z}^2 : x = (x_1, x_2) \mapsto Nx = (x_1, x_2 - 1)$$
$$W : \mathbb{Z}^2 \to \mathbb{Z}^2 : x = (x_1, x_2) \mapsto Wx = (x_1 - 1, x_2)$$

N stands for North and W for West. It is consistent with the convention used for our drawings.

We define the single site kernel on $x \in S$ by:

$$\forall \xi \in \Omega,\ \forall \sigma \in E, \quad \mathcal{I}_{\{x\}}(\sigma, \xi) := \frac{1}{Z_\xi} \exp\Big(\beta\sigma\big(\xi_{Nx} + \xi_{Wx} + h\big)\Big) \tag{3.7}$$

where Z_ξ is the normalizing coefficient:

$$Z_\xi := \exp\left(-\beta(\xi_{Nx} + \xi_{Wx} + h)\right) + \exp\left(\beta(\xi_{Nx} + \xi_{Wx} + h)\right)$$

This defines a family of single site kernels. We will see in the following chapter (Theorem 4.10) that this is sufficient to completely define a POS.

Note that if the external field is null ($h = 0$), the model is a voter model:

$$\mathcal{J}_{\{x\}}(\sigma, \xi) = \begin{cases} 1 - \epsilon & \text{if } \xi_{Nx} = \xi_{Wx} = \sigma \\ 1/2 & \text{if } \xi_{Nx} \neq \xi_{Wx} \end{cases} \tag{3.8}$$

where $\epsilon = \dfrac{e^{-2\beta}}{e^{2\beta} + e^{-2\beta}} \in \left(0, \dfrac{1}{2}\right)$.

Simulations are shown in Figure 3.2. They have been done with the program given in appendix. We see that for small β (high temperature for the original model), disorder is predominant. At the opposite, for high β, the general behaviour of the model is to stay with the same color. At last, the influence of h is very important. As soon as it is no more nonnul, the model is highly perturbed.

$(\beta, h) = (0.5,\ 0)$ $(\beta, h) = (2,\ 0)$ $(\beta, h) = (2,\ 0.05)$

Figure 3.2 – Simulations of the Pomm-Ising model. color 1 is black and -1 is white.

We will show in 8.4.1 and 12.2 that this model has no phase transition: for fixed parameters β and h there exists only one $\mathcal{J}$-POC.

3.3.2 The Stavskaya's Model

Here is a motivation for looking at this model, and why it is interesting.

One of the main question in the percolation theory is the existence (or not) of an infinite cluster. Usually, one takes a Bernoulli field on the lattice, constructs clusters and finally looks for infinite clusters. Here, we present another method. We directly search for infinite clusters.

Let us expose this on the simple case where $S = \mathbb{Z}^2$ and with the same order than previously (i.e. defined by Equation (3.5)).

By translation invariance, oriented percolation is realized if $P\big((0,0) \in \text{infinite cluster}\big) > 0$. An infinite oriented cluster can be viewed as going from $(0,0)$ to $+\infty$ as well as going from $-\infty$ to $(0,0)$. The second description is considered here.

We use the same operators N and W as in the previous example.

The color space is $E = \{0,1\}$. Let p be a parameter in the unit interval $[0,1]$ and $\xi \in \Omega$. We define the model on $x \in S$ by

$$\mathcal{S}_{\{x\}}(\sigma = 1, \xi) := \begin{cases} p & \text{if } \xi_{Nx} + \xi_{Wx} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Like for the POMM-Ising model, we have defined the POS only on single site boxes. Theorem 4.10 shows that it is sufficient to define the whole POS.

We interpret $\sigma_x = 1$ as the site x is in an oriented cluster. Indeed, a 1 can appear at the site x only if there is at least one 1 in $\underline{\partial} x$. We have then the following interpretation: Let $\Lambda \in \mathcal{T}_b$ and $\xi \in \Omega$. The $1-$valued sites in $\underline{\partial}\Lambda$ can be viewed as seeds of clusters. These clusters "grow" in the box according to the same process as in the classical oriented Bernoulli percolation.

In order to explain a bit more the correspondence between the two models, let us introduce another POS on the same site space but with the color space $\bar{E} = \{-1, 0, 1\}$: Let $x \in S$ and $\xi \in \Omega$.

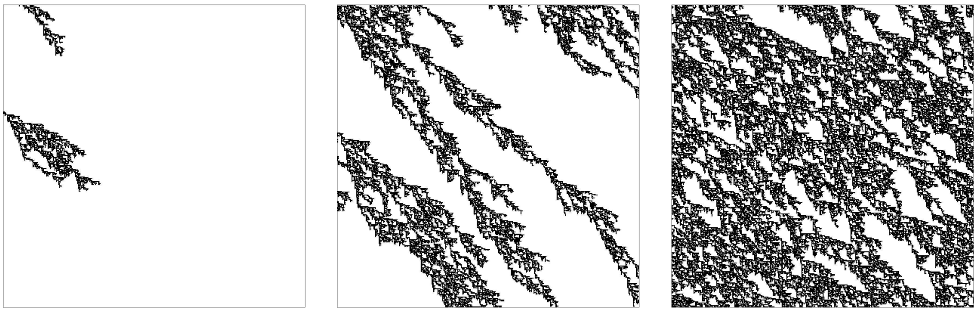
$$\bar{\mathcal{S}}_{\{x\}}(\sigma, \xi) = \begin{cases} p\, \mathbb{1}_{\xi_{Nx} \neq 1 \text{ and } \xi_{Wx} \neq 1} & \text{if } \sigma = -1 \\ 1 - p & \text{if } \sigma = 0 \\ p\, \mathbb{1}_{\xi_{Nx} = 1 \text{ or } \xi_{Wx} = 1} & \text{if } \sigma = 1 \end{cases} \tag{3.10}$$

Turning all -1's to 1 leads to a field of independent Bernoulli random variables. Changing all -1's to 0 leads to our model $\mathcal{S}$. Now, if we have an independent Bernoulli field, we can reconstruct $\bar{\mathcal{S}}$ in a time box by marking some oriented clusters touching the past frontier to $+1$ and fix all the others to -1. Since a $+1$ can not appear at x if there is no 1 in $\underline{\partial} x$, the marked clusters can not appear from nowhere. They must come from the past frontier. Thus at the limit (when the box grows to $\mathbb{Z}^2$), they are infinite.

Let us go back to the description of $\mathcal{S}$, our first POS of this subsection. Clearly, the Dirac measure $\delta_{\underline{0}}$ on the "all 0"-configuration is consistent with this model and $\delta_{\underline{0}}(\sigma = 1) = 0$. Thus, under this measure, no infinite cluster can exist. Let ν be another $\mathcal{S}$-POC (if such a measure exists). This time, $\nu(\sigma_0 = 1) > 0$. So under the measure ν, oriented percolation can be realized. Thus, oriented percolation is realized if and only if there is a phase transition for $\mathcal{S}$.

Our goal will be to determine if there exists such a phenomenon or not. The answer is contained in sub section 8.4.2 and Proposition 12.6: there exists a critical parameter p_c^+ such that for $p < p_c^+$, there exists a unique $\mathcal{S}$-POC and for $p > p_c^+$, a phase transition appears.

As for the POMM-Ising, we show simulations (Figure 3.3). The phase transition can be seen on them. Indeed, for small ϵ, nothing appears; the image is blank. For ϵ near 0.7, some filaments are present and for larger ϵ, there is a real net of 1-valued sites. The transition appears to be sharp.



$\epsilon = 0.7$ $\epsilon = 0.702$ $\epsilon = 0.72$

Figure 3.3 – Simulations of the Stavskaya's model. color 0 is white and 1 is black.

Chapter 4

Construction and reconstruction

This chapter has two aims. The first one is to reconstruct a POS from its singletons. The second one is more important: To show that any family of singleton kernels defines completely a POS. This allows us to only specify the singleton kernels of a POS.

In the whole chapter, we will suppose that E is countable.

4.1 Reconstruction

We begin this chapter by propositions on time boxes. We shall apply them to boxes reduced to singleton.

Proposition 4.1.
Let $\Lambda, \Delta \in \mathfrak{T}_b$ such that $\Delta \cap \Lambda = \varnothing$, $\Delta \cap \Lambda_+ = \varnothing$ and $\Lambda_- \cap \Delta_+ = \varnothing$. Let γ_Λ and γ_Δ be two proper oriented kernels on Λ (resp. Δ). Denote $\Gamma := \Lambda \cup \Delta$. Then,

1. *$\Gamma \in \mathfrak{T}_b$, and $\Gamma_+ = \Lambda_+ \cup (\Delta_+ \backslash \Lambda)$, $\Gamma_- = \Delta_- \cup (\Lambda_- \backslash \Delta)$,*
2. *$\gamma_\Gamma := \gamma_\Delta \gamma_\Lambda$ is well defined and is a proper oriented kernel on Γ.*

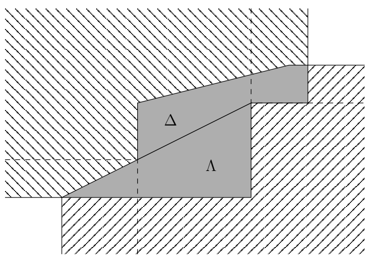


Figure 4.1 – Typical use of Proposition 4.1

Proof.
First of all, let us show that $\Lambda \cap \Delta_- = \varnothing$: Suppose that there exists $x \in \Delta_- \cap \Lambda$. There exists $y \in \Delta$ such that $y > x$. But then $y \in \Lambda_+ \cap \Delta = \varnothing$. We can then affirm that $\Lambda_+ = \Lambda_+ \backslash \Delta$ and $\Delta_- = \Delta_- \backslash \Lambda$.
Let us denote $U_\Delta := \bigcup_{t \in \Delta} t_-$. Similarly, we define U_Λ and U_Γ. Remark that $\Delta_- = U_\Delta \backslash \Delta$ (sim. for Λ_- and Γ_-).

$$\begin{aligned} \Delta_- \cup (\Lambda_- \backslash \Delta) &= (\Delta_- \backslash \Lambda) \cup (\Lambda_- \backslash \Delta) \\ &= \big(U_\Delta \backslash \Gamma\big) \cup \big(U_\Lambda \backslash \Gamma\big) \\ &= U_\Gamma \backslash \Gamma \\ &= \Gamma_- \end{aligned}$$

The same computation gives $\Lambda_+ \cup \big(\Delta_+ \backslash \Lambda\big) = \Gamma_+$. Thus,

$$\begin{aligned} \Gamma_+ \cap \Gamma_- &= \Big(\Lambda_+ \cup \big(\Delta_+ \backslash \Lambda\big)\Big) \cap \Big(\Delta_- \cup (\Lambda_- \backslash \Delta)\Big) \\ &= (\Lambda_+ \cap \Delta_-) \cup \Big(\Lambda_+ \cap (\Lambda_- \backslash \Delta)\Big) \cup \Big((\Delta_+ \backslash \Lambda) \cap \Delta_-\Big) \cup \Big((\Delta_+ \backslash \Lambda) \cap (\Lambda_- \backslash \Delta)\Big) \\ &= \Lambda_+ \cap \Delta_- \end{aligned}$$

Suppose that there exists $x \in \Lambda_+ \cap \Delta_-$. There exists $y \in \Delta$ such that $x < y$. Hence $y \in \Delta \cap \Lambda_+ = \varnothing$. We thus have $\Gamma_+ \cap \Gamma_- = \varnothing$, that is $\Gamma \in \mathfrak{T}_b$.

Since $\Gamma_+ = \Lambda_+ \cup (\Delta_+ \backslash \Lambda)$, we have $\Gamma_+^c \subset \Lambda_+^c$, so we can apply γ_Λ on any $A \in \mathfrak{F}_{\Gamma_+^c}$. Proposition 3.10 gives that $\gamma_\Lambda(A)$ is $\mathfrak{F}_\Upsilon$-measurable, where $\Upsilon = \Lambda_- \cup [\Gamma_+^c \backslash \Lambda] = \Lambda_- \cup \Gamma_-^* \cup \Delta$. Moreover, $\Upsilon \cap \Delta_+ = (\Lambda_- \cap \Delta_+) \cup (\Gamma_-^* \cap \Delta_+) = \varnothing$, so it is possible to apply γ_Δ on $\gamma_\Lambda(A)$. The function $\gamma_\Gamma(A)$ is then well defined and is $\mathfrak{F}_{\Upsilon'}$-measurable, where $\Upsilon' = \Delta_- \cup [\Upsilon \backslash \Delta] = \Gamma_- \cup \Gamma_-^* = \Gamma_-^*$. We have proved that $\gamma_\Gamma(A)$ is $\mathfrak{F}_{\Gamma_-^*}$-measurable for each $A \in \mathfrak{F}_{\Gamma_+^c}$.
Let $A \in \mathfrak{F}_\Gamma$. $\gamma_\Gamma(A)$ is $\mathfrak{F}_{\Upsilon_1}$ where $\Upsilon_1 = \Delta_- \cup [[\Lambda_- \cup (\Gamma \backslash \Lambda)] \backslash \Delta] = \Delta_- \cup [(\Lambda_- \cup \Delta) \backslash \Delta] = \Delta_- \cup (\Lambda_- \backslash \Delta) = \Gamma_-$. The kernel γ_Γ is then oriented.
Let $B \in \mathfrak{F}_{\Gamma_-^*}$. Since

$$\begin{aligned} \Gamma_-^* \cap (\Lambda_+ \cup \Lambda) &= \Gamma_-^* \cap \Lambda_+ \\ &\subset \Gamma_-^* \cap (\Lambda_+ \cup (\Delta_+ \backslash \Lambda)) \\ &\subset \Gamma_-^* \cap \Gamma_+ \\ &\subset \varnothing \end{aligned}$$

we have $\Gamma_-^* \subset \Lambda_-^*$ so $\gamma_\Lambda(B) = \mathbb{1}_B$ and $\gamma_\Lambda(B)$ is $\mathfrak{F}_{\Gamma_-^*}$-measurable. Finally,

$$\begin{aligned} \Gamma_-^* \cap (\Delta_+ \cup \Delta) &= \Gamma_-^* \cap \Delta_+ \\ &= \Gamma_-^* \cap (\Delta_+ \backslash \Lambda \\ &\subset \Gamma_-^* \cap (\Lambda_+ \cup (\Delta_+ \backslash \Lambda)) \\ &\subset \Gamma_-^* \cap \ \Gamma_+ \\ &\subset \varnothing \end{aligned}$$

so $\gamma_\Gamma(B) = \gamma_\Lambda(B) = \mathbb{1}_B$.
Thus γ_Γ is proper. ✠

Corollary 4.2.
Let $\Delta, \Lambda \in \mathfrak{T}_b$ such that $\Delta \subset \Lambda^*$ and denote $\Gamma := \Delta \cup \Lambda$. Let γ_Δ and γ_Λ be two proper oriented kernels on Δ (resp. Λ). Then,

1. $\Gamma \in \mathfrak{T}_b$, *and* $\Gamma_+ = \Delta_+ \cup \Lambda_+$, $\Gamma_- = \Delta_- \cup \Lambda_-$,
2. $\gamma_\Gamma := \gamma_\Delta\gamma_\Lambda$ *is well defined and is a proper oriented kernel on* Γ,
3. $\gamma_\Gamma = \gamma_\Delta\gamma_\Lambda = \gamma_\Lambda\gamma_\Delta$.

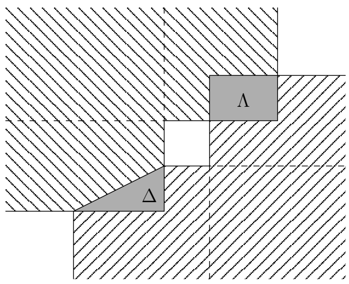


Figure 4.2 – Typical use of Corollary 4.2

Proof.
Δ and Λ satisfy the preceding proposition hypothesis so the two first points are proved. Note that we can exchange the role of Δ and Λ. $\bar{\gamma}_\Gamma := \gamma_\Lambda\gamma_\Delta$ is then a well-defined oriented kernel.

For f $\mathcal{F}_\Delta$-measurable and g $\mathcal{F}_\Lambda$-measurable functions, we have

$$\gamma_\Gamma(fg) = \gamma_\Delta(f)\gamma_\Lambda(g) = \bar{\gamma}_\Gamma(fg)$$

Now, for any $\mathcal{F}_\Gamma$-measurable function h, we can write the following countable sum:

$$h = \sum_{\substack{\sigma_\Delta \in E^\Delta \\ \sigma_\Lambda \in E^\Lambda}} h(\sigma_\Delta\sigma_\Lambda)\,\mathbb{1}_{\sigma_\Delta}\,\mathbb{1}_{\sigma_\Lambda}$$

so $\gamma_\Gamma(h) = \bar{\gamma}_\Gamma(h)$ and we conclude by using Proposition 3.12. ✠

These two results give an easy way to construct proper oriented kernels on unrelated and ordered sets. We shall see in the sequel that this is sufficient to achieve our goal: to construct a proper oriented kernel on each time box given kernels on singletons.

Before this, we introduce the main tool of this chapter: slices.

Definition 4.3.
Let Λ be a finite part of S. It is said to be a ***slice*** if all points of Λ are pairwise unrelated.

Proposition 4.4.

1. *A slice is a time box.*
2. *for each finite subset Υ of S, $\max(\Upsilon)$ and $\min(\Upsilon)$ are slices.*
3. *Each finite subset of S is contained in the past of a time box.*

Proof.
(i). Let Δ be a slice and assume there exists $x \in \Delta_- \cap \Delta_+$. Then, there exist $y, z \in \Delta$ such that $y < x$ and $x < z$. So $y \leqslant z$, which is absurd by definition of Δ.
(ii). This is a simple consequence of the definition of max (resp. min) and slice.
(iii). Let Υ be a finite subset of S. For each $x \in \max(\Upsilon)$, choose some $y > x$. Denote Δ the collection of chosen y. Remark that it is possible that $\Delta \cap \Upsilon \neq \varnothing$. But $\max(\Delta)$ corresponds to what we look for. ✠

Definition 4.5.
Let $\Delta \in \mathcal{T}_b$ and define the following sequence of slices:

$$\begin{aligned}
\Delta_1 &:= \min(\Delta) \\
\Delta_2 &:= \min\left(\Delta \backslash \Delta_1\right) \\
&\vdots \\
\Delta_k &:= \min\left(\Delta \backslash (\Delta_1 \cup \cdots \cup \Delta_{k-1})\right) \\
&\vdots
\end{aligned} \tag{4.1}$$

Let n be the greatest integer such that $\Delta_n \neq \varnothing$. We call ***slices of*** Δ the finite sequence of non empty sets $\Delta_1, \cdots, \Delta_n$.

Remark 4.6.
In general, Δ_n is not equal to $\max(\Delta)$ but it is a subset of it. Figure 4.3 shows an example of such a case.

Now we are ready to achieve the first aim of this chapter: reconstruction of a POS from its singleton kernels.

Theorem 4.7 (Reconstruction Theorem)**.**
Let γ be a POS and $\Delta \in \mathcal{T}_b$. There exists a sequence $x_1, \cdots, x_n$ of the points of Δ such that

$$\gamma_\Delta = \gamma_{x_1} \cdots \gamma_{x_n}$$

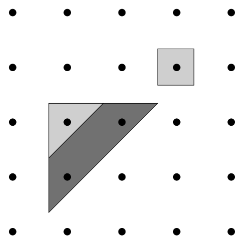


Figure 4.3 – Δ_1 is in light grey, Δ_2 in dark grey. $\Delta_2 \neq \max(\Delta)$.

Proof.
The proof is divided into two parts. First, we split the box into slices, then we split each slice into points.

Denote $\Delta_1, \cdots, \Delta_n$ the slices of Δ. Let

$$\bar{\gamma}_\Delta = \gamma_{\Delta_1}\gamma_{\Delta_2}\cdots\gamma_{\Delta_n} \tag{4.2}$$

We just have to check that the hypothesis of Proposition 4.1 are satisfied. Let $\Lambda_k = \bigcup_{i=k+1}^{n} \Delta_i$ for $1 \leqslant k \leqslant n-1$. By Definition 4.5 of slices, $\Lambda_k \subset (\Delta_k)_+$ and $(\Delta_k)_+ \subset \Lambda_k \cup \Delta_+$ so $(\Lambda_k)_+ \cap \Delta_k = \varnothing$ and $(\Lambda_k)_- \cap (\Delta_k)_+ = \varnothing$. Proposition 4.1 gives that $\bar{\gamma}_\Delta$ define a proper oriented kernel on Δ_+^c.

Now, by applying n times the consistency property, we have that for all $\mathcal{F}_{\Gamma_+^c}$-measurable functions f

$$\begin{aligned}
\gamma_\Delta(f) &= \gamma_\Delta\left(\gamma_{\Delta_n}(f)\right) \\
&= \cdots \\
&= \gamma_\Delta\left(\gamma_{\Delta_1}\cdots\gamma_{\Delta_n}(f)\right) \\
&= \gamma_\Delta\left(\bar{\gamma}_\Delta(f)\right) \\
&= \gamma_\Delta(1)\bar{\gamma}_\Delta(f) \\
&= \bar{\gamma}_\Delta(f)
\end{aligned}$$

So the first step is proved.

Now, let $\Lambda \in \mathcal{T}_b$ be a slice. Denote its elements by $y_1, \cdots, y_m$. Then Corollary 4.2 gives that $\gamma_{y_1}\cdots\gamma_{y_m}$ is a proper oriented kernel on Λ. Moreover, the consistency implies $\gamma_\Lambda = \gamma_\Lambda\big(\gamma_{y_1}\cdots\gamma_{y_m}\big) = \gamma_{y_1}\cdots\gamma_{y_m}$. Thus we have the desired result. ✠

Remark 4.8.
Note that the order of the terms for a slice is not important: another order will give the same kernel. It is obviously not the case if Δ is not a slice: we have to apply kernels according to the order given on S.

Corollary 4.9.
Let γ be a POS. Then for all $\Delta \in \mathcal{T}_b$ and $\eta \in \Omega$,

$$\gamma_\Delta(\sigma|\eta) = \prod_{x \in \Delta} \gamma_x(\sigma_x|\sigma_\Delta \eta_{\Delta_-^*})$$

Proof.
We just have to remark that, since E is finite or countable, $\gamma_x(\sigma|\eta)$ is a real number in the interval $[0, 1]$. The Reconstruction Theorem 4.7 gives then the result. ✠

4.2 Construction from a singleton-kernel family

In the whole first part of this chapter, we had a POS and we tried to reconstruct the kernel of a time box with singletons kernels. Now, we are interested into another problem: we only have a family of singleton kernels and we try to define a POS compatible with it. The result of this section is that no assumption is necessary to construct a POS from a family of single site kernels.

Theorem 4.10.
Let $(\gamma_x)_{x\in S}$ be a family of singleton proper oriented kernels $\gamma_x : \mathcal{F}_{x_+^c} \times \Omega \to [0, 1]$. There exists a unique POS $\gamma = (\gamma_\Delta)_{\Delta\in\mathcal{T}_b}$ such that $\gamma_{\{x\}} = \gamma_x$ for all $x \in S$. Furthermore,

$$\mathcal{G}(\gamma) = \Big\{\mu : \mu\gamma_x = \mu, \quad \textit{for all } x \in S\Big\} \tag{4.3}$$

Proof.
Let Δ be in $\mathcal{T}_b$. If it is a slice, define

$$\gamma_\Delta := \gamma_{y_1} \cdots \gamma_{y_m}$$

where $\Delta = \{y_1, \cdots, y_m\}$. This is well defined according to Corollary 4.2. Otherwise, let $\Delta_1, \cdots, \Delta_n$ be the slices of Δ and define

$$\gamma_\Delta := \gamma_{\Delta_1} \cdots \gamma_{\Delta_n}$$

Proposition 4.1 gives that γ_Δ is a well defined proper oriented kernel. To finish the proof, we just have to prove consistency.
The rest of the proof relies on the following observation, valid for any measure μ on $\mathcal{F}$ and any $\Delta \in \mathcal{T}_b$:

$$\Big[\forall x \in \Delta,\ \mu\gamma_x = \mu\Big] \quad \Longrightarrow \quad \mu\gamma_\Delta = \mu \tag{4.4}$$

This is proved by induction on the cardinality of Δ and comes from our definition of γ_Δ. Let $\Lambda \in \mathcal{T}_b$ such that $\Lambda \subset \Delta$. Let $y \in \Lambda$, h be a $\mathcal{F}_{y_+^c}$-measurable function and denote $f := \gamma_y(h)$. Let p be such that $y \in \Delta_p$. $\forall x \in \Delta_{p+1} \cup \cdots \cup \Delta_n$, $y_-^* \subset x_-^*$ so

$$\gamma_\Delta \gamma_y(h) = \gamma_\Delta(f) = \gamma_{\Delta_1} \cdots \gamma_{\Delta_n}(f) = \gamma_{\Delta_1} \cdots \gamma_{\Delta_p}(f)$$

Let us denote $\Delta_p := \{y_1, \cdots, y_m\}$. We can suppose that $y_m = t$. Thus

$$\gamma_{\Delta_p}(f) = \gamma_{y_1} \cdots \gamma_{y_m} \gamma_y(h) = \gamma_{\Delta_p}(h)$$

So

$$\gamma_\Delta \gamma_y(h) = \gamma_{\Delta_1} \cdots \gamma_{\Delta_p}(h) = \gamma_\Delta(h)$$

We have just proved that for all $y \in \Lambda$, $\gamma_\Delta \gamma_y = \gamma_\Delta$. The observation (4.4) yields to $\gamma_\Delta \gamma_\Lambda = \gamma_\Delta$. $(\gamma_\Delta)_{\Delta \in \mathcal{T}_b}$ is then a well defined POS.

Now (4.4) shows that

$$\mathcal{G}(\gamma) = \Big\{ \mu \ : \ \mu \gamma_x = \mu, \quad \text{for all } x \in S \Big\}$$

Furthermore, it yields uniqueness. Indeed, consider a POS $(\bar{\gamma}_\Lambda)_{\Lambda \in \mathcal{T}_b}$ consistent with the family $(\gamma_x)_{x \in S}$. It follows immediately that $\bar{\gamma}_\Lambda$ must be consistent with γ_Λ for each $\Lambda \in \mathcal{T}_b$. But then, if f is $\mathcal{F}_{\Lambda_+^c}$-measurable

$$\bar{\gamma}_\Lambda(f) = \bar{\gamma}_\Lambda\big(\gamma_\Lambda(f)\big) = \gamma_\Lambda(f)\bar{\gamma}_\Lambda(1) = \gamma_\Lambda(f)$$

and the theorem in proved. ✠

This theorem is very powerful. It is possible to define a POS only by defining the singleton kernels and no assumption on them must be made. It extends the result on LIS in [FM05].

There exists a similar result in the Gibbs theory but in order to construct a specification from singleton kernels, it is necessary to add some additional hypothesis on singleton kernels (positivity) to ensure existence (see [FM06]). These conditions are not necessary in our model.

Thus, the POMM-Ising and Stavskaya models defined in section 3.3 are now well defined.

Chapter 5

POMM versus PCA

We present here the most general notion of probabilistic cellular automata (see [Too01]). We will see that it is a particular case of POMM.

Let U be a finite or countable set. For each $i \in U$, let V_i be a finite subset of U containing i. This V_i is called ***the neighborhood*** of i. Let E be a finite set and $\tilde{\Omega} = E^U$. In most articles that deal with PCA, $\tilde{\Omega}$ is called the *set of configurations*. We already have used this name but it is not the same notion as for PCA. This fact will be stressed in the proof of Theorem 5.1. Our definition corresponds to space-time configurations of PCA.

For each $i \in U$, we define a transition probability θ_i from $\tilde{\Omega}$ to $\tilde{\Omega}$. $\theta_i(x_i|y_{V_i})$ is the probability of having x_i in i given the configuration y_{V_i}. Then we define a transition probability on any finite subset A of U by

$$P(x_A|y) = \prod_{i \in A} \theta_i(x_i|y_{V_i}) \tag{5.1}$$

This formula means that conditionally on y, what happens at $i \in U$ is independent of what happens at $j \in U$, for $j \neq i$. We define then consistent measures with P like in Definition 3.14: μ is said to be consistent with P if $\mu P = \mu$.

We define now the notion of time. Let $y \in \tilde{\Omega}$ be distributed according to μ, a consistent measure. This y is said to be the *original configuration* or the *configuration at time* 0. The configuration at time 1 is chosen according to P with (5.1). Then we determine the configuration at time 2 by the same principle, using the configuration at time 1. And so on. This defines a time process with time on $\mathbb{N}$ and with a parallel updating mechanism.

To define a process with time on $\mathbb{Z}$, we shift the time 0 to time $-n$ (where $n \in \mathbb{N}$) and let n tend to infinity. Since we have chosen the original configuration according to an invariant measure μ, this definition is correct and it defines a time-invariant process.

Theorem 5.1.
Let $S := U \times \mathbb{Z}$. Define a partial order on S by

$$(i,t) \leqslant (j,s) \quad \Longleftrightarrow \quad \begin{cases} (i,t) = (j,s) \\ \quad or \\ \exists (k_n)_{0 \leqslant n \leqslant s-t} \in U^{s-t+1} \; k_0 = j,\; k_{s-t} = i,\; k_n \in V_{k_{n+1}} \end{cases}$$

Define the POMM γ on S by

$$\gamma_{(i,t)}(\sigma|\eta) := \theta_i(\sigma_{(i,t)}|\eta_{V_i,t-1})$$

for every $i \in U$ and $t \in \mathbb{Z}$. Then for all $A \subset U$, $t \in \mathbb{Z}$ and $y \in \Omega$,

$$\gamma_{(A,t)}(x_A|y) = P(x_A|y_{t-1})$$

Proof.
Let $A \subset U$ and $t \in \mathbb{Z}$. The set $(A,t) := \{(i,t) \in S;\; i \in A\}$ is clearly a slice so by Corollary 4.9

$$\begin{aligned} P(x_A|y_{t-1}) &= \prod_{i \in A} \theta_i(x_i|y_{V_i,t-1}) \\ &= \prod_{i \in A} \gamma_{(i,t)}(x_i|y) \\ &= \gamma_{(A,t)}(x_A|y) \end{aligned}$$

✠

Remark 5.2.
1. Figure 5.1 gives an idea of the construction.

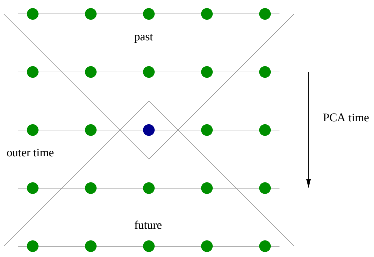


Figure 5.1 – PCA seen as a POMM: here $U = \mathbb{Z}$ and $V_i = \{i-1,\; i,\; i+1\}$. The blue site depends on the three sites above.

2. This theorem can be summarized by the sentence: Every probabilistic cellular automata is a partially ordered Markov model.

We found out that a PCA was a POMM. The contrary is false. Indeed, the geometry of PCA is simple: it is a product space with an order compatible with the product. Each copy of U is interpreted to be at a different time.

We present here counter-example where the geometry can not be summarized to a product space.

Example 5.3.
Let $S := \mathbb{Z} \times \{0\} \cup \{0\} \times \mathbb{N}$. We define a partial order on S by:

$$(x,i) \leqslant (y,j) \iff \begin{cases} i = j = 0,\ x \leqslant y \\ \quad \text{or} \\ i = y = 0,\ x \leqslant j \\ \quad \text{or} \\ x = j = 0,\ i \leqslant y \end{cases}$$

The geometry of S is shown in Figure 5.2.

This geometry can not be written as a product of space because of the shortcuts created by $\{0\} \times \mathbb{N}$. Thus, any POMM on that space can not be a PCA.

In general, the fact that a POMM is not a PCA is only due to its geometry.

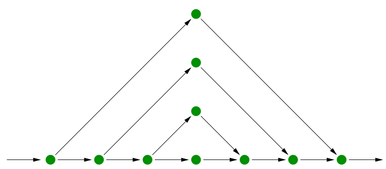


Figure 5.2 – An example of geometry that shows that a POMM can not be a PCA. Arrows indicates the partial order: they go from small sites to big sites.

Chapter 6

Theorems linked with extremality

We are naturally interested in extremal points of $\mathcal{G}$. Indeed, we will see in this chapter that they are the only measures that can be "locally seen". We link here extremality with triviality on $\mathcal{F}_{-\infty}$. We then establish some consequences such as mutual singularity on $\mathcal{F}_{-\infty}$ and mixing property.

Let us begin by defining properly what $\mathcal{F}_{-\infty}$ is. Denote

$$\mathcal{F}_{-\infty} := \bigcap_{\Lambda \in \mathcal{T}_b} \mathcal{F}_{\Lambda^*_-} \tag{6.1}$$

Remark 6.1.
1. One may be tempted to define $\mathcal{F}_{-\infty}$ as $\bigcap_{\Lambda \in \mathcal{T}_b} \mathcal{F}_{\Lambda_-}$. This is not a good definition. Indeed, it may happen that there exist $x, y \in S$ such that $x_- \cap y_- = \varnothing$ (see Figure 6.1 for an example), which implies $\bigcap_{\Lambda \in \mathcal{T}_b} \mathcal{F}_{\Lambda_-} = \{\Omega, \varnothing\}$.
2. On the other hand, according to Lemma 6.2 bellow, $\mathcal{F}_{-\infty}$ as defined above is not a trivial σ-algebra i.e. $\mathcal{F}_{-\infty} \neq \{\Omega, \varnothing\}$.

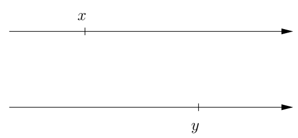


Figure 6.1 – In the context of two independent chains on $\mathbb{Z}$, we can find x, y such that $x_- \cap y_- = \varnothing$

Lemma 6.2.
For all $x, y \in S$, $x^*_- \cap y^*_- \neq \varnothing$.

Proof.
By contradition, if $x_-^* \cap y_-^* = \varnothing$, then $x_-^* \subset S \backslash y_-^* = y_+ \cup \{y\}$. So, by picking $z \in x_- \subset y_+ \cup \{y\}$, we have that $z \leqslant x$ and $y \leqslant z$ so $y \leqslant x$. But this implies that $y_- \subset x_-$. This contradicts $x_-^* \cap y_-^* = \varnothing$. ✠

The theorem below establishes the main properties of extremal measures of $\mathcal{G}$.

Theorem 6.3.
Let γ be a POS on $(\Omega, \mathcal{F})$. The following properties hold:

(a) *$\mathcal{G}(\gamma)$ is a convex set.*

(b) *A measure μ is extremal in $\mathcal{G}(\gamma)$ if and only if μ is trivial on $\mathcal{F}_{-\infty}$.*

(c) *Let $\mu \in \mathcal{G}(\gamma)$ and ν be a measure on $\mathcal{F}$ such that $\nu \ll \mu$. Then $\nu \in \mathcal{G}(\gamma)$ if and only if there exists a nonnegative $\mathcal{F}_{-\infty}$-measurable function h such that $\nu = h\mu$.*

(d) *Each $\mu \in \mathcal{G}(\gamma)$ is uniquely determined (within $\mathcal{G}(\gamma)$) by its restriction to $\mathcal{F}_{-\infty}$.*

(e) *Two distinct extremal elements of $\mathcal{G}(\gamma)$ are mutually singular on $\mathcal{F}_{-\infty}$.*

The proof of this theorem is based on the following lemmas from Georgii ([Geo88, pages 115-117]):

Lemma 6.4.
Let $(\Omega, \mathcal{B})$ be a measurable space, π a probability kernel from $\mathcal{B}$ to $\mathcal{B}$ and μ a measure on $\mathcal{B}$ such that $\mu\pi = \mu$. Denote

$$\mathcal{A}_\pi^{\mathcal{B}}(\mu) := \{A \in \mathcal{B}, \pi(A, \cdot) = \mathbb{1}_A(\cdot) \ \mu\text{-a.s.}\}$$

Then, $\mathcal{A}_\pi^{\mathcal{B}}(\mu)$ is a σ-algebra and for every $\mathcal{B}$-measurable nonnegative function h,

$$\Big((h\mu)\pi = h\mu\Big) \Longleftrightarrow \Big(h \text{ is } \mathcal{A}_\pi^{\mathcal{B}}(\mu)\text{-measurable}\Big).$$

Lemma 6.5.
Let $(\Omega, \mathcal{B})$ be a measurable space and Π a non-empty set of kernels such that, for all $\pi \in \Pi$, π is a probability kernel from $\mathcal{B}_\pi$ to $\mathcal{B}$, where $\mathcal{B}_\pi$ is a sub-σ-algebra of $\mathcal{B}$. Denote

$$\mathcal{G}(\Pi) := \{\mu \in \mathcal{P}(\Omega, \mathcal{B}) : \mu\pi = \mu \ \ \forall \pi \in \Pi\}$$

the convex set of Π-invariant probability measures and for $\mu \in \mathcal{G}(\Pi)$, $\mathcal{A}_\Pi(\mu) := \bigcap_{\pi \in \Pi} \mathcal{A}_\pi^{\mathcal{B}_\pi}(\mu)$. Then

$$\Big(\mu \text{ is extremal in } \mathcal{G}(\Pi)\Big) \Longleftrightarrow \Big(\mu \text{ is trivial on } \mathcal{A}_\Pi(\mu)\Big)$$

proof of the theorem.
(a) It is immediate.
(b) Denote $\mathcal{F}^{\mu}_{-\infty}$ the μ-completion of $\mathcal{F}_{-\infty}$. We only have to prove that $\mathcal{F}^{\mu}_{-\infty} = \mathcal{A}_{\gamma}(\mu)$, because μ is trivial on $\mathcal{F}_{-\infty}$ if and only if μ is trivial on $\mathcal{F}^{\mu}_{-\infty}$.
Let $A \in \mathcal{A}_{\gamma}(\mu)$. For each time box Λ, $A \in \mathcal{A}_{\gamma_\Lambda}^{\mathcal{F}_{\Lambda^c_+}}$ so $\gamma_\Lambda(A,\cdot) = \mathbb{1}_A(\cdot)$ μ-a.s.. Then, μ-a.s., $\mathbb{1}_A$ is $\mathcal{F}_{\Lambda^*_-}$-measurable, i.e. $A \in \mathcal{F}_{\Lambda^*_-}$ μ-a.s., so $A \in \mathcal{F}^{\mu}_{-\infty}$.
Conversely let $A \in \mathcal{F}^{\mu}_{-\infty}$ and Λ be a time box. There exists a set $B \in \mathcal{F}_{-\infty}$ such that $A = B$ μ-almost surely. Since $B \in \mathcal{F}_{\Lambda^*_-}$ for all $\Lambda \in \mathcal{T}_b$, $\gamma_\Lambda(B,\cdot) = \mathbb{1}_B(\cdot)$, thus $B \in \mathcal{A}_{\gamma_\Lambda}^{\mathcal{F}_{\Lambda^c_+}}$ and $A \in \mathcal{A}_{\gamma_\Lambda}^{\mathcal{F}_{\Lambda^c_+}}(\mu)$. Thus, we have proved that $A \in \mathcal{A}_{\gamma}(\mu)$.
(c) $\nu \ll \mu$ implies that there exists an $\mathcal{F}$-measurable non-negative function f such that $\nu = f\mu$.

$$\begin{aligned}
\nu \in \mathcal{G}(\gamma) &\iff \forall \Lambda \in \mathcal{T}_b,\ \nu\gamma_\Lambda = \nu \\
&\iff \forall \Lambda \in \mathcal{T}_b,\ (f\mu)\gamma_\Lambda = f\mu \\
&\iff \forall \Lambda \in \mathcal{T}_b,\ f \text{ is } \mathcal{A}_{\gamma_\Lambda}^{\mathcal{F}_{\Lambda^c_+}}(\mu)\text{-measurable} \\
&\iff f \text{ is } \mathcal{A}_{\gamma}(\mu)\text{-measurable} \\
&\iff f \text{ is } \mathcal{F}^{\mu}_{-\infty}\text{-measurable} \\
&\iff \exists h\ \mathcal{F}_{-\infty}\text{-measurable such that } h = f\ \mu\text{-a.s.} \\
&\iff \exists h\ \mathcal{F}_{-\infty}\text{-measurable such that } \nu = h\mu
\end{aligned}$$

(d) Let $\mu, \nu \in \mathcal{G}(\gamma)$ such that their restrictions to $\mathcal{F}_{-\infty}$ coincide and define $\overline{\mu} := \dfrac{\mu+\nu}{2} \in \mathcal{G}(\gamma)$. $\mu \ll \overline{\mu}$ so there exists a $\mathcal{F}_{-\infty}$-measurable function f such that $\mu = f\overline{\mu}$. Since $\mu = \overline{\mu}$ on $\mathcal{F}_{-\infty}$, then $f = 1$ $\overline{\mu}$-a.s. so $\mu = \overline{\mu}$. Analogously $\nu = \overline{\mu}$.
(e) immediate consequence of (b) and (d). ✠

The following theorem characterizes extremal measures in terms of mixing.

Theorem 6.6.
For each probability measure μ on $(\Omega, \mathcal{F})$, the following statements are equivalent:

(a) *μ is trivial on $\mathcal{F}_{-\infty}$.*

(b) *For all cylinder sets $A \in \mathcal{F}$,*

$$\lim_{\Lambda \uparrow S} \sup_{B \in \mathcal{F}_{\Lambda^*_-}} \left|\mu(A \cap B) - \mu(A)\mu(B)\right| = 0 \tag{6.2}$$

(c) *For all $A \in \mathcal{F}$,*

$$\lim_{\Lambda \uparrow S} \sup_{B \in \mathcal{F}_{\Lambda^*_-}} \left|\mu(A \cap B) - \mu(A)\mu(B)\right| = 0 \tag{6.3}$$

Proof.
(c)⇒(b) is immediate.
(b)⇒(a) Let $B \in \mathcal{F}_{-\infty}$. Let $\mathcal{D} := \{A \in \mathcal{F} : \mu(A \cap B) = \mu(A)\mu(B)\}$. $\mathcal{D}$ satisfies
- $\Omega \in \mathcal{D}$,
- $A_1, A_2 \in \mathcal{D},\ A_1 \subset A_2$ implies $A_2 \backslash A_1 \in \mathcal{D}$ and
- if $(A_n)_{n>0}$ is a sequence of disjoint sets of $\mathcal{D}$ then, $\bigcup_{n>0} A_n \in \mathcal{D}$.

This makes $\mathcal{D}$ a Dynkin system. Thus $\mathcal{D}$ is a sub-σ-algebra of $\mathcal{F}$. Moreover, the hypothesis ensures that all cylinders are in $\mathcal{D}$, so that $\mathcal{D} = \mathcal{F}$. In particular, $B \in \mathcal{F}$ so $\mu(B) = \mu(B)^2$ i.e. $\mu(B) \in \{0, 1\}$.
(a)⇒(c) Let $A \in \mathcal{F}$ and $(\Lambda_n)_{n>0}$ be an increasing sequence of time boxes which converges to S. The reverse martingale theorem yields $\mu\big(A|\mathcal{F}_{(\Lambda_n)^*_-}\big) \xrightarrow{L^1} \mu\big(A|\mathcal{F}_{-\infty}\big)$. Since μ is trivial on $\mathcal{F}_{-\infty}$, $\mu\big(A|\mathcal{F}_{-\infty}\big) = \mu(A)$ μ-a.s.. We deduce that

$$\forall \varepsilon > 0, \quad \exists \Delta \in \mathcal{T}_b, \mu\Big(\big|\mu(A|\mathcal{F}_{\Delta^*_-}) - \mu(A)\big|\Big) < \varepsilon$$

Hence, for all $\Lambda \in \mathcal{T}_b : \Lambda \supset \Delta$,

$$\begin{aligned}
\sup_{B \in \mathcal{F}_{\Lambda^c_+}} \big|\mu(A \cap B) - \mu(A)\mu(B)\big| &\leqslant \sup_{B \in \mathcal{F}_{\Delta^c_+}} \big|\mu(A \cap B) - \mu(A)\mu(B)\big| \\
&\leqslant \sup_{B \in \mathcal{F}_{\Delta^c_+}} \Big|\mu\Big(\mathbb{1}_B \left[\mu(A|\mathcal{F}_{\Delta^*_-}) - \mu(A)\right]\Big)\Big| \\
&\leqslant \mu\Big(\big|\mu(A|\mathcal{F}_{\Delta^*_-}) - \mu(A)\big|\Big) \\
&\leqslant \varepsilon
\end{aligned}$$

✠

Theorem 6.7.
Let γ be a POS, μ an extremal point of $\mathcal{G}(\gamma)$ and $(\Lambda_n)_{n\in\mathbb{N}}$ a sequence of time boxes such that $\Lambda_n \uparrow S$. Then

1. *$\gamma_{\Lambda_n}(h) \to \mu(h)$ μ-a.s. for each bounded local function h on Ω.*
2. *If Ω is a compact metric space, then for μ-almost all $\omega \in \Omega$, $\gamma_{\Lambda_n}(h, \omega) \to \mu(h)$ for all continuous local functions h on Ω.*

Proof.
(i) Since μ is consistent with γ, $\gamma_{\Lambda_n}(h) = \mu\big(h|\mathcal{F}_{(\Lambda_n)^c_+}\big)$. The reverse martingale theorem thus yields

$$\gamma_{\Lambda_n}(h) \xrightarrow[\mu\text{-a.s.}]{L^1} \mu\big(h|\mathcal{F}_{-\infty}\big) = \mu(h)$$

(ii) The set of continuous local functions contains a countable dense set (for the sup-norm). So the result follows from (i). ✠

Suppose E is finite. Ω is then a compact metric set. Let $\mu \in \mathcal{G}(\gamma)$ be extremal. If we choose a typical $\omega \in \Omega$ for μ then

$$\gamma_{\Lambda_n}(h|\omega) \xrightarrow[n\to\infty]{} \mu(h)$$

So we can "locally see" μ that is, the comportment of $\gamma_\Lambda(\cdot|\omega)$ in a big but finite box is nearly the comportment of μ. The same holds for every extremal measure.

In particular, if we can found a local function h and two configurations $\omega_1, \omega_2 \in \Omega$ such that

$$\lim_{n\to\infty} \gamma_{\Lambda_n}(h|\omega_1) \neq \lim_{n\to\infty} \gamma_{\Lambda_n}(h|\omega_2)$$

then, we can deduce that there is a phase transition for γ.

Chapter 7

Inequalities linked with order

In this chapter, we determine some useful inequalities related to the order on the color space E.

7.1 FKG inequalities

In the whole chapter, we will suppose that E is totally ordered. This induces a partial order on Ω: for all $\omega, \eta \in \Omega$,

$$\omega \leqslant \eta \qquad \Longleftrightarrow \qquad \forall x \in S,\ \omega_x \leqslant \eta_x$$

We will use the same symbol for the (partial) order on S, on Ω and on E. The context will determine which one applies.

We first introduce a comparison tool.

Definition 7.1.
Let μ and ν be two measures on the same (partially) ordered space X. We say that μ is ***stochastically dominated*** by ν and we denote $\mu \preccurlyeq \nu$ if for all non-decreasing functions f, we have $\mu(f) \leqslant \nu(f)$.

Definition 7.2.
A coupling P between two measures μ and ν on Ω is a measure P on Ω^2 satisfying

$$\begin{aligned} \sum_{\sigma \in \Omega} P(\sigma, \sigma') = \nu(\sigma') \quad \text{for all } \sigma' \in \Omega \\ \sum_{\sigma' \in \Omega} P(\sigma, \sigma') = \mu(\sigma) \quad \text{for all } \sigma \in \Omega \end{aligned} \tag{7.1}$$

Proposition 7.3 (Strassen theorem)**.**
For any two probability measures μ and ν on Ω, the following statements are equivalent:

1. $\mu \preccurlyeq \nu$

2. *There exists a coupling* P *of* μ *and* ν *such that* $P(\sigma \leqslant \sigma') = 1$.

See [GHM01] for a proof.

Theorem 7.4.
Let γ *and* γ' *be two POS on the same color space* E, $\Lambda \in \mathfrak{T}_b$ *and* η, η' *be two configurations on* Λ_-^*. *If for all* $x \in \Lambda$, $e \in E$ *and* $\xi, \xi' \in \Omega_{\Lambda \cap x_-}$ *satisfying* $\xi \leqslant \xi'$ *we have*

$$\gamma_x\big(\sigma \geqslant e|\eta\xi\big) \leqslant \gamma'_x\big(\sigma \geqslant e|\eta'\xi'\big) \tag{7.2}$$

then $\gamma_\Lambda(\cdot|\eta) \preccurlyeq \gamma'_\Lambda(\cdot|\eta')$.

Theorem 7.4 is the counterpart in POS theory of a theorem proved by Holley (see [GHM01]) in Gibbs fields. We therefor refer to this theorem by ***POS-Holley Theorem***.

Proof.
Let x be in Λ. If we have ξ, ξ' such that $\xi \leqslant \xi'$, we can construct a coupling P_x at the site x preserving the inequality. We will then combine all P_x to construct a coupling satisfying the Strassen Theorem. Let U be a realization of a uniform probability on $[0, 1]$ and define

$$\begin{aligned}
\xi_x &:= \max\Big\{e \in E: \ \gamma_x\big(\sigma_x \geqslant e|\eta\xi\big) \geqslant U\Big\} \\
\xi'_x &:= \max\Big\{e \in E: \ \gamma'_x\big(\sigma_x \geqslant e|\eta'\xi'\big) \geqslant U\Big\}
\end{aligned}$$

Clearly, $\xi_x \leqslant \xi'_x$ and it defines a coupling of the desired probabilities:

$$P_x\big(\sigma_x = e, \sigma'_x \in E\big) = \gamma_x\big(\sigma_x = e|\eta\xi\big)$$

Note that P_x depends on η but we have not specified it into the notation for brevity.

Now, in order to construct the coupling P_Λ, we will use the Reconstruction Theorem 4.7. Let $y_1, \cdots, y_n$ be a sequence such that $\gamma_\Lambda = \gamma_{y_1} \cdots \gamma_{y_n}$.

First, we construct (ξ_{y_n}, ξ'_{y_n}) by using P_{y_n}. Then we obtain $(\xi_{y_{n-1}}, \xi'_{y_{n-1}})$ from $P_{y_{n-1}}$ given (ξ_{y_n}, ξ'_{y_n}) and so on until (ξ_{y_1}, ξ'_{y_1}).

$$\begin{aligned}
P_\Lambda\big(\sigma_\Lambda = \xi, \sigma'_\Lambda \in E^\Lambda\big) &= P_{y_1,\cdots,y_n}\big(\sigma_{y_1,\cdots,y_{n-1}} = \xi, \sigma'_{y_1,\cdots,y_{n-1}} \in E^\Lambda\big) \\
&= P_{y_n}\big(\sigma_{y_n} = \xi_{y_n}, \sigma'_{y_n} \in E\big) \\
&\qquad P_{y_1,\cdots,y_{n-1}}\big(\sigma_{y_1,\cdots,y_{n-1}} = \xi, \sigma'_{y_1,\cdots,y_{n-1}} \in E^\Lambda|(\xi_{y_n}, \xi'_{y_n})\big) \\
&= \gamma_{y_1} \cdots \gamma_{y_n}\big(\xi_\Lambda|\eta\big) \\
&= \gamma_\Lambda\big(\xi_\Lambda|\eta\big)
\end{aligned}$$

So P_Λ is a coupling of γ_Λ and γ'_Λ, and $\xi \leqslant \xi'$. We conclude by applying Strassen Theorem. ✠

Theorem 7.5 (FKG).
Let γ be a POS verifying For all $y \in S$, $a \in E$, $\omega, \eta \in \Omega$ such that $\omega \leqslant \eta$,

$$\gamma_y\Big(\sigma_y \geqslant a \;\Big|\; \omega\Big) \leqslant \gamma_y\Big(\sigma_y \geqslant a \;\Big|\; \eta\Big)$$

Then it satisfies the following FKG inequalities:
For all local increasing functions f, g, for all $\Delta \in \mathcal{T}_b$ such that Supp(f) *and* Supp(g) *are in Δ_+^c and for all $\omega \in \Omega$,*

$$\gamma_\Delta\Big(fg \;\Big|\; \omega\Big) \geqslant \gamma_\Delta\Big(f \;\Big|\; \omega\Big)\gamma_\Delta\Big(g \;\Big|\; \omega\Big) \tag{7.3}$$

and for all extremal measures μ in $\mathcal{G}(\gamma)$ and all increasing functions f, g,

$$\mu(fg) \geqslant \mu(f)\mu(g) \tag{7.4}$$

It is obvious that the FKG theorem will give the same result for both f and g decreasing.

FKG stands for Fortuin, Kasteleyn and Ginibre. They have proved such inequalities for Gibbs fields. This is a very powerful tool. An application of this FKG inequality is shown in section 7.2.

Proof.
First, we show (7.3) for f, g depending only on one site y.
Fix g and $\omega \in \Omega$ and remark that for $m \in \mathbb{R}$ and $\alpha > 0$, if $\gamma_y(fg) \geqslant \gamma_y(f)\gamma_y(g)$, then $\gamma_y(f.(g+m)) \geqslant \gamma_y(f)\gamma_y(g+m)$ and $\gamma_y(f.(\alpha g)) \geqslant \gamma_y(f)\gamma_y(\alpha g)$. So that we can suppose, without loss of generality, that g is positive and $\gamma_y(g|\omega) = 1$. Then, $\pi(.) = \gamma_y(.g|\omega)$ is a measure on $\mathcal{F}_{\Omega_y}$.

For brevity, let us denote

$$\begin{aligned} q(a) &= \gamma_y(\sigma_y \geqslant a|\omega) \\ q'(a) &= \pi(\sigma_y \geqslant a) = \sum_{e \geqslant a} g(e)\gamma_y(\sigma_y = e|\omega) \end{aligned}$$

then we have

$$\begin{aligned} \frac{q'(a)}{1-q'(a)} &= \frac{\displaystyle\sum_{e \geqslant a} g(e)\gamma_y(\sigma_y = e|\omega)}{\displaystyle\sum_{e<a} g(e)\gamma_y(\sigma_y = e|\omega)} \\ &\geqslant \frac{g(a)\gamma_y(\sigma_y \geqslant a|\omega)}{g(a)\gamma_y(\sigma_y < a|\omega)} \\ &= \frac{q(a)}{1-q(a)} \end{aligned}$$

But the function $x \mapsto \dfrac{x}{1-x}$ is increasing so $q(a) \leqslant q'(a)$ that is $\gamma_y(\sigma_y \geqslant a|\omega) \leqslant \pi(\sigma_y \geqslant a)$.

The POS-Holley theorem 7.4 gives that for all increasing $\mathcal{F}_{\Omega_y}$-measurable functions f, $\gamma_y(f|\omega) \leqslant \pi(f)$. But,

$$\begin{aligned}\pi(f) &= \sum_{e \in E} f(e)g(e)\gamma_y(\sigma_y = e|\omega) \\ &= \gamma_y(fg|\omega)\end{aligned}$$

So, we have proved that for all increasing functions f, g depending on the single site y the inequality (7.3) is satisfied.

Now if f, g are $\mathcal{F}_{\underline{y}^*}$-measurable, we use the same trick as in the proof of Proposition 3.12: let us denote $f_\omega(\eta) := f(\eta_y \omega_{S\setminus\{y\}})$ (resp. g).

$$\begin{aligned}\gamma_y(fg|\omega) &= \gamma_y(f_\omega g_\omega|\omega) \\ &\geqslant \gamma_y(f_\omega|\omega)\gamma_y(g_\omega|\omega) \\ &\geqslant \gamma_y(f|\omega)\gamma_y(g|\omega)\end{aligned}$$

Let us finish the proof of (7.3) by induction on the number of sites in the time box Δ: Suppose (7.3) is true for all timeboxes with n sites and let Δ be a time box with $n+1$ sites. We can write the kernel γ_Δ as $\gamma_\Delta = \gamma_{y_1} \cdots \gamma_{y_{n+1}}$ according to the Reconstruction Theorem 4.7. Denote $\Lambda := \{y_1, \cdots, y_n\}$ and $x := y_{n+1}$. Then,

$$\begin{aligned}\gamma_\Delta(fg|\omega) &= \gamma_\Lambda(\gamma_x(fg|\cdot)|\omega) \\ &\geqslant \gamma_\Lambda(\gamma_x(f|\cdot)\gamma_x(g|\cdot)|\omega) \\ &\geqslant \gamma_\Lambda(\gamma_x(f|\cdot)|\omega)\gamma_\Lambda(\gamma_x(g|\cdot)|\omega) \\ &\geqslant \gamma_\Delta(f|\omega)\gamma_\Delta(g|\omega)\end{aligned}$$

The second inequality comes from the fact that $\omega \mapsto \gamma_x(f|\omega)$ is an increasing function by the POS-Holley theorem.

The proof of the inequality (7.4) is just an application of Theorem 6.7. ✠

7.2 Application to examples

We will study the properties of the POMM-Ising model introduced in section 3.3. The same is true for the Stavskaya's model.

Proposition 7.6.
The POMM-Ising model satisfies the FKG inequality.

Proof.
Let ξ, η be two configurations and $x \in S$.

$$\begin{aligned}\xi \leqslant \eta &\Longleftrightarrow \xi_{Nx} + \xi_{Wx} \leqslant \eta_{Nx} + \eta_{Wx} \\ &\Longleftrightarrow \frac{1}{1 + \exp\big(-\beta(\xi_{Nx} + \xi_{Wx} + h)\big)} \leqslant \frac{1}{1 + \exp\big(-\beta(\eta_{Nx} + \eta_{Wx} + h)\big)} \\ &\Longleftrightarrow \mathcal{I}_x(\sigma \geqslant 1|\xi) \leqslant \mathcal{I}_x(\sigma \geqslant 1|\eta)\end{aligned}$$

Since the model has only two possible colors, this is sufficient to apply Theorem 7.5. ✠

Proposition 7.7.
Let Λ, Δ be time boxes such that $\Lambda \subset \Delta$ and $\Lambda_+ \cap \Delta = \varnothing$. Denote $\oplus$ the "all plus"-configuration (for all $x \in S$, $\oplus_x = +1$) and $\mathcal{I}_\Lambda^\oplus := \mathcal{I}_\Lambda(\cdot, \oplus)$. Then for all increasing $\mathcal{F}_\Lambda$-measurable functions f,

$$\mathcal{I}_\Delta^\oplus(f) \leqslant \mathcal{I}_\Lambda^\oplus(f)$$

Proof.
Let $A := \{\eta \in \Omega_\Delta;\ \forall x \in \Delta\backslash\Lambda,\ \eta_x = +1\}$. We have $\mathcal{I}_\Delta^\oplus(\cdot|A) = \mathcal{I}_\Lambda^\oplus(\cdot)$. Moreover, it is clear that $\mathbb{1}_A$ is an increasing function. So the FKG inequality yields to

$$\mathcal{I}_\Delta^\oplus(f\,\mathbb{1}_A) \geqslant \mathcal{I}_\Delta^\oplus(f)\ \mathcal{I}_\Delta^\oplus(A)$$

Thus

$$\begin{aligned}\mathcal{I}_\Lambda^\oplus(f) &= \mathcal{I}_\Delta^\oplus(f|A) \\ &= \frac{\mathcal{I}_\Delta^\oplus(f\,\mathbb{1}_A)}{\mathcal{I}_\Delta^\oplus(A)} \\ &\geqslant \mathcal{I}_\Delta^\oplus(f)\end{aligned}$$

✠

Proposition 7.8.

$$\mathcal{I}^\oplus := \lim_{\Delta \uparrow S} \mathcal{I}_\Delta^\oplus$$

exists and is consistent with the POMM-Ising model.

Proof.
We will construct $\mathcal{I}^\oplus$ step by step. First, suppose that f is a local bounded increasing function. Let $\Delta \in \mathcal{T}_b$ such that $\mathrm{Supp}(f) \subset \Delta$. Choose $(\Lambda_n)_{n\in\mathbb{N}}$ an increasing sequence of time boxes such that

- $\Lambda_0 = \Delta$,
- for all $n \in \mathbb{N}$, $(\Lambda_n)_+^* \cap \Lambda_{n+1} = \varnothing$,

• $\lim_{n\to\infty} \Lambda_n = \Delta \cup \Delta_-$.
The preceding proposition gives that $\left(\mathcal{I}^{\oplus}_{\Lambda_n}(f)\right)_{n\in\mathbb{N}}$ is a decreasing sequence. Since f is bounded, this sequence converges. Considering two such sequences, we can construct a third one satisfying the same properties. Thus the limit obtained is independent of the chosen sequence. Let us denote it $\mathcal{I}^{\oplus}(f)$.

Now, if g is a bounded local function, there exist well-defined real numbers $(\alpha_\Upsilon)_{\Upsilon\subset \mathrm{Supp}(g)}$ such that

$$g(\omega) = \sum_{\Upsilon\subset \mathrm{Supp}(g)} \alpha_\Upsilon \ \mathbb{1}_{\omega_\Upsilon=+1}$$

But for all $\Upsilon \subset \mathrm{Supp}(g)$, $\mathbb{1}_{\omega_\Upsilon=+1}$ is increasing, so that we can define $\mathcal{I}^{\oplus}(g)$ by

$$\mathcal{I}^{\oplus}(g) := \sum_{\Upsilon\subset \mathrm{Supp}(g)} \alpha_\Upsilon \ \mathcal{I}^{\oplus}(\mathbb{1}_{\omega_\Upsilon=+1})$$

At last, we can define $\mathcal{I}^{\oplus}(h)$ for any function h by a limit process. It is obvious that $\mathcal{I}^{\oplus}$ define a probability measure and that it is consistent with the model. ✠

$\mathcal{I}^{\ominus}$ can be defined with the same process. Another application of FKG inequality gives that for all $\omega \in \Omega$ and all increasing $\mathcal{F}_{\Lambda_+^c}$-measurable functions f,

$$\mathcal{I}^{\ominus}_{\Lambda}(f) \leqslant \mathcal{I}^{\omega}_{\Lambda}(f) \leqslant \mathcal{I}^{\oplus}_{\Lambda}(f).$$

Thus, we can conclude that $\mathcal{I}^{\oplus}$ and $\mathcal{I}^{\ominus}$ are the only extremal $\mathcal{I}$-POCs.

Proposition 7.9.

$$\mathcal{I}^{\oplus} = \mathcal{I}^{\ominus} \quad \Longleftrightarrow \quad \mathcal{I}^{\oplus}(\sigma_{(0,0)}) = \mathcal{I}^{\ominus}(\sigma_{(0,0)})$$

The proof is left to the reader (see [LML72]).

The last proposition is very useful. To prove uniqueness of consistent measures, we only have to show that $\lim_{n\to\infty} \mathcal{I}^{\oplus}_{\Lambda_n}(\sigma_{(0,0)}) = \lim_{n\to\infty} \mathcal{I}^{\ominus}_{\Lambda_n}(\sigma_{(0,0)})$.

The same can be done for the Stavskaya's model: There exist at most two extremal measures and they are distinguishable by looking at the site $(0, 0)$. We already know one of them: δ_0, the Dirac measure concentrated in the "all zero" configuration. Then if we prove that there exists a consistent measure ν such that $\nu(\sigma_{(0,0)}) > 0$, we prove the phase transition.

In general, this can be applied as soon as the considered POMM is FKG and has only two possible colors ($|E| = 2$). That is why the FKG inequality is so important.

Chapter 8

Uniqueness results

One of the most important question in the POMM theory is: how many consistent measures are there for a given POS? Since $\mathcal{G}$ is convex, there are only three possibilities: $\mathcal{G}$ is infinite, empty or it has only one element. In the majority of cases, the color space E is finite. This implies that $\mathcal{G}$ is nonempty. The question of uniqueness is therefore essential.

In this chapter, we shall give a partial answer. We present three criteria to decide uniqueness.

In Gibbsian theory or in PCA, the dimension of the space plays an important role. This is not the case for POS. Indeed, we can map every POS into $\mathbb{Z}$: let S be as general as possible in POS theory. There exists $\varphi : S \to \mathbb{Z}$ a bijection from S to $\mathbb{Z}$. S induces, via φ, a partial order on $\mathbb{Z}$ and the POS can be transfered too. Of course in the general case, this partial order do not respect the natural topology of $\mathbb{Z}$. This is why it does not appear in PCA theory.

What really plays a role is the number of past neighbours. The more they are, the easier it is to find a phase transition.

8.1 Bounded uniformity

Theorem 8.1.
Let γ be a POS for which there exists a constant $c > 0$ such that for all cylinders A there exists a time box Λ such that $A \in \mathcal{F}_{\Lambda_+^c}$ and

$$\forall \omega, \xi \in \Omega \quad \gamma_\Lambda(A, \omega) \geqslant c \; \gamma_\Lambda(A, \xi) \tag{8.1}$$

Then $\left|\mathcal{G}(\gamma)\right| \leqslant 1$.

Proof.
We will prove that each measure in $\mathcal{G}(\gamma)$ is extremal. This is the case only if $\mathcal{G}(\gamma)$ contains at most one element.

Let μ be in $\mathcal{G}(\gamma)$ and $B \in \mathcal{F}_{-\infty}$ such that $\mu(B) > 0$. Let us denote $\nu(\cdot) := \mu(\cdot|B) = \mathbb{1}_B(\cdot)\dfrac{\mu(\cdot)}{\mu(B)}$. Since $\nu \ll \mu$ and $\omega \mapsto \dfrac{\mathbb{1}_B(\omega)}{\mu(B)}$ is $\mathcal{F}_{-\infty}$-measurable, $\nu \in \mathcal{G}(\gamma)$. Moreover, for all cylinders A,

$$\begin{aligned}
\nu(A) &= \nu\gamma_\Lambda(A) \\
&= \mu\big(\nu\gamma_\Lambda(A)\big) \\
&= \iint \gamma_\Lambda(A,\omega)d\nu(\omega)d\mu(\xi) \\
&\geqslant \iint c\ \gamma_\Lambda(A,\xi)d\nu(\omega)d\mu(\xi) \\
&\geqslant c\ \nu\big(\mu\gamma_\Lambda(A)\big) \\
&\geqslant c\ \mu(A)
\end{aligned}$$

where Λ satisfies the hypothesis of the theorem. So, $\nu(A) \geqslant c\ \mu(A)$ for all cylinder sets A. Hence $\nu \geqslant c\ \mu$. In particular $0 = \nu(\Omega\backslash B) \geqslant c\ \mu(\Omega\backslash B)$ i.e. $\mu(B) = 1$. ✠

This is not a very useful criterion. It can only be applied when the specification is local and when the number of nearest past points of a time box remains constant when the box grows.

8.2 Dobrushin criterion

In order to state this criterion, we have to suppose that the color space E is finite. Moreover, we must introduce some more definitions.

For $\xi, \eta \in \Omega$ and $x \in S$, let us write $\xi \overset{\neq x}{=} \eta$ if for all $y \in S\backslash\{x\}$, $\xi_y = \eta_y$.

Definition 8.2.
Let $f : \Omega \mapsto \mathbb{R}$ be a measurable function. The **oscillation** of f with respect to the site x is

$$\delta_x(f) := \sup_{\xi \overset{\neq x}{=} \eta} \big|f(\xi) - f(\eta)\big| \tag{8.2}$$

The **total oscillation** of f is

$$\Delta(f) := \sum_{x\in S} \delta_x(f) \tag{8.3}$$

We are interested in functions with bounded total oscillation. Note that

$$\sup(f) - \inf(f) \leqslant \Delta(f) < \infty \tag{8.4}$$

for any local bounded measurable function f.

Definition 8.3.
A **cleaning-rate matrix** $(\alpha_{y,x})_{x \leqslant y}$ is a matrix of nonnegative real numbers such that for all $y \in S$, $\alpha_{y,y} = 0$ and for all $x \in y_-$ and all $\mathcal{F}_{\{y\}}$-measurable functions f,

$$\delta_x(\gamma_y f) \leqslant \delta_y(f)\ \alpha_{y,x}. \tag{8.5}$$

There is no need to define $\alpha_{y,x}$ for $x \in y_+^*$ according to Proposition 3.10. Alternatively, we can set $\alpha_{y,x} := 0$ if $x \in y_+^*$.

The term "cleaning-rate matrix" comes from an interpretation due to Aizenman: Imagine that S is a tiling of an infinite room. The oscillation of a local function f at $y \in S$ can be seen as dust on this site. The application of the kernel γ_y on f can be viewed as cleaning y with a broom. As everybody knows, a broom is not perfect, and a part of the dust is thrown on y_-. $\alpha_{y,x}$ represents the maximal rate of the dust sent from y to x.

The following criterion says that if the broom catches a piece of dust at each step then, by pushing little by little the dust to infinity we can clean the entire room. Doing so removes all the dust, including at the "infinite sites". The resulting function is constant and does not depend on configurations "at infinity". Thus, there can only exist one measure in $\mathcal{G}(\gamma)$.

When the color space has only two elements, the computation of the cleaning-rate matrix is quite simple.

Proposition 8.4.
Let γ be a POS with two possible colors: $E = \{a, b\}$. for $\eta \in \Omega$, $x \in S$ and $e \in E$, denote $\eta^{x,e}$ the configuration defined by

$$\forall z \in S, \qquad (\eta^{x,e})_z = \begin{cases} \eta_z & \text{if } z \neq x \\ e & \text{if } z = x \end{cases}$$

Then the matrix α defined for all $y \in S$ and all x such that $x < y$ by

$$\alpha_{y,x} := \sup_{\eta \in \Omega} \left| \gamma_y(a, \eta^{x,a}) - \gamma_y(a, \eta^{x,b}) \right|$$

is the best cleaning-rate matrix associated with γ.

The best means that all the coefficients of the matrix are as small as possible.

Proof.
Fix $y \in S$ and $x < y$. Let f be a $\mathcal{F}_y$-measurable function. By definition, $\delta_y(f) = |f(a) - f(b)|$ and $\gamma_y(f, \eta) = f(a)\gamma_y(a, \eta) + f(b)\gamma_y(b, \eta)$. Thus for all $\eta \in \Omega$

$$\left| \gamma_y(f,\ \eta^{x,a}) - \gamma_y(f,\ \eta^{x,b}) \right| = \left| f(a) - f(b) \right| \cdot \left| \gamma_y(a, \eta^{x,a}) - \gamma_y(a, \eta^{x,b}) \right|$$

The smallest possible coefficient is then

$$\alpha_{y,x} := \sup_{\eta\in\Omega} \left|\gamma_y(a,\eta^{x,a}) - \gamma_y(a,\eta^{x,b})\right|$$

✠

Here is the criterion. We will refer to it as the ***Dobrushin criterion*** because it comes from a criterion originally proved by Dobrushin.

Theorem 8.5.
Let γ be a quasi-local POS.
If there exists a cleaning-rate matrix α such that

$$\Gamma = \sup_{y\in S} \sum_{x\leqslant y} \alpha_{y,x} < 1 \tag{8.6}$$

then there exists only one γ-POC.

We first prove two lemmas.

Lemma 8.6 (Multisite dusting lemma)**.**
Let $y \in S$, $x \in y_+^c$ and f be a $\mathcal{F}_{y_+^c}$-measurable function. Then,

$$\delta_x(\gamma_y f) \begin{cases} = 0 & \text{if } x = y \\ \leqslant \delta_x(f) & \text{if } x \in y^* \\ \leqslant \delta_x(f) + \delta_y(f)\alpha_{y,x} & \text{if } x \in y_- \text{ (i.e. } x < y) \end{cases} \tag{8.7}$$

proof of lemma.
The case $x = y$ is evident.
Denote $f_\xi : E \mapsto \mathbb{R}$ the function defined by $f_\xi(\eta_y) := f(\eta_y \xi_{S\setminus\{y\}})$. Let $x \in y_-$ and $\xi \overset{\neq x}{=} \eta$. Then we have

$$\begin{aligned} \left|\gamma_y f(\xi) - \gamma_y f(\eta)\right| &= \left|\gamma_y(f_\xi|\xi) - \gamma_y(f_\eta|\eta)\right| \\ &\leqslant \left|\gamma_y(f_\xi|\xi) - \gamma_y(f_\eta|\xi)\right| + \left|\gamma_y(f_\eta|\xi) - \gamma_y(f_\eta|\eta)\right| \\ &\leqslant \gamma_y(\delta_x(f)|\xi) + \delta_x(\gamma_y f) \\ &\leqslant \delta_x(f) + \delta_y(f)\alpha_{y,x} \end{aligned}$$

For $x \in y^*$, the computation is almost the same: the right hand side in the second line is equal to $\left|\gamma_y(f_\xi|\xi) - \gamma_y(f_\eta|\xi)\right|$ because $\gamma_y f_\eta$ is $\mathcal{F}_{y_-}$-measurable. ✠

Lemma 8.7.
Let Λ be a time box. There exists a one-to-one sequence $(x_n)_{n\geqslant 1}$ of sites such that

1. $\bigcup_{n\geqslant 1}\{x_n\} = \Lambda \cup \Lambda_-$

2. $\forall k \geqslant 1, \bigcup_{n\leqslant k}\{x_n\} \in \mathcal{T}_b$

Proof.
Let us denote

$$\{x_1, \cdots, x_{r_1}\} := \max(\Lambda) = \max(\Lambda \cup \Lambda_-)$$

This finite sequence satisfies (2) because it is a slice (see Definition 4.3 and Proposition 4.4). Now let us construct

$$\{x_{r_1+1}, \cdots, x_{r_2}\} := \max(\{x_1, \cdots, x_{r_1}\}_-) = \max(\Lambda \cup \Lambda_- \backslash \{x_1, \cdots, x_{r_1}\})$$

and by repeating this operation,

$$\{x_{r_k+1}, \cdots, x_{r_{k+1}}\} := \max(\{x_1, \cdots, x_{r_k}\}_-) = \max(\Lambda \cup \Lambda_- \backslash \{x_1, \cdots, x_{r_k}\})$$

Since S is countable, the infinite sequence $(x_n)_{n\geqslant 1}$ satisfies (1). To prove (2), it is sufficient to show that if $\Delta \in \mathcal{T}_b$ and $x \in \max(\Delta_-)$, then $\Delta \cup \{x\} \in \mathcal{T}_b$.
Since $x \in \Delta_-$, $x_- \subset \Delta_-$ and $x_- \cap \Delta_+ = \varnothing$. Moreover $x \in \max(\Delta_-)$ implies that $x_+ \cap \Delta_- = \varnothing$. In conclusion,

$$\begin{aligned}(\Delta \cup \{x\})_+ \bigcap (\Delta \cup \{x\})_- &= \Big[(\Delta_+ \cup x_+) \bigcap (\Delta_- \cup x_-)\Big] \backslash (\Delta \cup \{x\}) \\ &= \Big[(\Delta_- \cap x_+) \bigcup (\Delta_+ \cap x_-)\Big] \backslash (\Delta \cup \{x\}) \\ &= \varnothing\end{aligned}$$

so $\Delta \cup \{x\} \in \mathcal{T}_b$. ✠

proof of the Dobrushin Criterion.
Let f be a local bounded function. Choose a time box Λ such that $\mathrm{Supp}(f) \subset \Lambda \cup \Lambda_-$. There exists a sequence $(y_n)_{n\in\mathbb{N}^*}$ verifying Lemma 8.7 for Λ. The kernel $T_n := \gamma_{y_n} \cdots \gamma_{y_2}\gamma_{y_1}$ is then well defined. The Multisite dusting lemma gives us the inequality

$$\Delta(T_1 f) = \Delta(\gamma_{y_1} f) \leqslant \sum_{x\neq y_1} \big(\delta_x(f) + \delta_{y_1}(f)\alpha_{y_1,x}\big) \leqslant \sum_{x\neq y_1} \delta_x(f) + \Gamma\delta_{y_1}(f)$$

so, by induction, for $n \geqslant 1$ we have

$$\begin{aligned}
\Delta(T_n f) &= \Delta\big(\gamma_{y_n} \cdots \gamma_{y_2}(\gamma_{y_1} f)\big) \\
&\leqslant \sum_{x \neq y_2, \cdots, y_n} \delta_x(\gamma_{y_1} f) + \Gamma \sum_{k=2}^{n} \delta_{y_k}(\gamma_{y_1} f) \\
&\leqslant \sum_{x \neq y_1, \cdots, y_n} \delta_x(f) + \Gamma \sum_{k=2}^{n} \delta_{y_k}(f) + \delta_{y_1}(f) \left(\sum_{x \neq y_1, \cdots, y_n} \alpha_{y_1, x} + \Gamma \sum_{k=2}^{n} \alpha_{y_1 y_k} \right) \\
&\leqslant \sum_{x \neq y_1, \cdots, y_n} \delta_x(f) + \Gamma \sum_{k=1}^{n} \delta_{y_k}(f)
\end{aligned}$$

The last line comes from the fact that

$$\sum_{x \neq y_1, \cdots, y_n} \alpha_{y_1, x} + \Gamma \sum_{k=2}^{n} \alpha_{y_1, y_k} \leqslant \sum_{x \neq y_1} \alpha_{y_1, x} \leqslant \Gamma$$

Let $\mu, \nu \in \mathcal{G}(\gamma)$. To prove the criterion, it is sufficient to prove that for all local bounded functions f we have $\mu(f) = \nu(f)$. Let $n \in \mathbb{N}$ be large. γ is quasi-local so $T_n f$ is continuous. We then have

$$\begin{aligned}
\big|\nu(f) - \mu(f)\big| &= \big|\nu(T_n f) - \mu(T_n f)\big| \\
&\leqslant \Delta(T_n f) \\
&\leqslant \sum_{k > n} \delta_{y_k}(f) + \Gamma \sum_{k \leqslant n} \delta_{y_k}(f)
\end{aligned}$$

By letting n go to infinity, we have $\big|\nu(f) - \mu(f)\big| \leqslant \Gamma \Delta(f)$. Now, we can apply this to $T_n f$ instead of f:

$$\begin{aligned}
\big|\nu(f) - \mu(f)\big| &= \big|\nu(T_n f) - \mu(T_n f)\big| \\
&\leqslant \Gamma \Delta(T_n f) \\
&\leqslant \Gamma\big(\sum_{k > n} \delta_{y_k}(f) + \Gamma \sum_{k \leqslant n} \delta_{y_k}(f)\big)
\end{aligned}$$

and pushing n to infinity leads to $\big|\nu(f) - \mu(f)\big| \leqslant \Gamma^2 \Delta(f)$. By induction, $\big|\nu(f) - \mu(f)\big| \leqslant \Gamma^m \Delta(f)$, for all $m \in \mathbb{N}$. Since $\Gamma < 1$, the limit $m \to \infty$ yields to $\nu(f) = \mu(f)$. ✠

8.3 Oriented disagreement percolation

This criterion can only be applied on POMM.

It is based on an optimal coupling between the POMM starting from two different configurations. We then look at disagreement percolation of the realisation.

The criterion in the sequel can be generalized in the case where the POMM depends on a larger (but finite) part of the past by changing the notion of percolation.

We begin this section by recalling the notion of oriented percolation.

Definition 8.8.
Let $\mathbf{p} = (p_x)_{x \in S}$ be a collection of parameters in the interval $[0, 1]$. Let us denote $\psi_{\mathbf{p},\Lambda}$ the independent Bernoulli distribution on Λ with parameters $\mathbf{p}$, that is, for all $x \in \Lambda$, the random variable of the field at x is 1 with probability p_x and 0 with probability $1 - p_x$. Let q be a real number in the interval $[0, 1]$. $\psi_{q,\Lambda}$ will be the Bernoulli distribution where all parameters are equal to q.
We denote $\psi_{\mathbf{p}}$ and ψ_q the Bernoulli field on the whole lattice S.

By a consequence of our model, we are interested in oriented percolation. It consists in looking for infinite oriented 1-clusters.

There is a difference between 1-clusters and oriented 1-clusters. In the first case, we take into account all 1-valued sites. In the second one, only 1-valued sites that can be linked with a decreasing path are taken into account (see Figure 8.1).

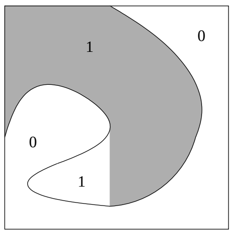


Figure 8.1 – oriented 1-cluster (in grey) versus non-oriented 1-cluster in $\mathbb{Z}^2$ with our drawing conventions.

Here is the definition of 1-clusters and critical parameter of the Bernoulli field. We recall that $\underline{\partial} x$ is the nearest past of the site x, as defined in Definition 3.15.

Definition 8.9.
Let $x, y \in S$. We denote $(x \overset{+1}{\rightsquigarrow} y)$ the event $\{X \in \{0,1\}^S : \exists (y_k)_{1 \leqslant k \leqslant n} : y_1 = x,\ y_n = y,\ y_k \in \underline{\partial} y_{k+1},\ X_{y_k} = 1\}$.
For $\Delta, \Lambda \subset S$, $(\Delta \overset{+1}{\rightsquigarrow} \Lambda)$ stands for $\{\exists x \in \Delta,\ \exists y \in \Lambda :\ (x \overset{+1}{\rightsquigarrow} y)\}$.
By words, it means that there exists a path from x to y (or to Δ to Λ) such that each site of the path is 1-valued.

We say that $x \in S$ belongs to an ***infinite oriented 1-cluster***, denoted by $(\infty \overset{+1}{\rightsquigarrow} x)$ if there exists an infinite decreasing sequence $(y_n)_{n \in \mathbb{N}}$ such that $(y_n \overset{+1}{\rightsquigarrow} x)$ for all $n \in \mathbb{N}$.

If $q < q' \in [0,1]$, the POS-Holley Theorem gives that $\psi_q \preccurlyeq \psi_{q'}$. This induces the existence of a critical parameter, denoted $p_c^+(S)$ such that:

$$\begin{aligned} &\forall q < p_c^+(S),\ \psi_q(\infty \overset{+1}{\rightsquigarrow} x) = 0 \\ &\forall q > p_c^+(S),\ \psi_q(\infty \overset{+1}{\rightsquigarrow} x) > 0 \end{aligned} \tag{8.8}$$

$p_c^+(S)$ is called the ***critical parameter of oriented percolation*** of the lattice S. In general, $p_c^+(S) \in [0,1]$. In general, the bounds 0 and 1 can not be excluded. Moreover, what happens at the critical value depends on the lattice S and is either unknown or very difficult to obtain.

Let us go back to our POMM γ.

Definition 8.10.
For two probability measures μ and ν on a time box Δ, let us denote the ***variational norm*** of $\mu - \nu$ in Δ by

$$\|\mu - \nu\|_\Delta := \sup_{A \in \mathcal{F}_\Delta} |\mu(A) - \nu(A)| \tag{8.9}$$

Since it is easier to manipulate a sum than a sup, we give the following lemma:

Lemma 8.11.
Assume that E is countable. Let Δ be a time box. For μ and ν any probability measures on Δ,

$$\|\mu - \nu\|_\Delta = \frac{1}{2} \sum_{\omega \in \Omega_\Delta} |\mu(\omega) - \nu(\omega)| \tag{8.10}$$

Proof.
Let us denote ϱ_1 the lhs of (8.10) and ϱ_2 the rhs. Define

$$B := \{\omega \in \Omega_\Delta;\ \mu(\omega) > \nu(\omega)\}$$

and let $B^c := \Omega_\Delta \backslash B$. Since $\mu(B^c) - \nu(B^c) = \nu(B) - \mu(B)$,

$$\begin{aligned} 2\varrho_2 &= \sum_{\omega \in B} (\mu(\omega) - \nu(\omega)) + \sum_{\omega \in B^c} (\nu(\omega) - \mu(\omega)) \\ &= (\mu(B) - \nu(B)) - (\nu(B^c) - \mu(B^c)) \\ &= 2|\mu(B) - \nu(B)| \\ &\leqslant 2\varrho_1 \end{aligned}$$

To prove the other inequality, remark that for all a, b two nonnegative numbers, $|a - b| \leqslant \max(a, b)$. Then for any $A \subset \Omega_\Delta$,

$$\begin{aligned} |\mu(A) - \nu(A)| &= \Big| \big(\mu(A \cap B) - \nu(A \cap B)\big) + \big(\mu(A \cap B^c) - \nu(A \cap B^c)\big) \Big| \\ &\leqslant \max\big(|\mu(A \cap B) - \nu(A \cap B)|, |\mu(A \cap B^c) - \nu(A \cap B^c)| \big) \\ &\leqslant \max\big(|\mu(B) - \nu(B)|, |\mu(B^c) - \nu(B^c)| \big) \\ &\leqslant \varrho_2 \end{aligned}$$

✠

One of the main tool of the criterion below is a coupling inequality. For a coupling P of μ and ν, we have

$$\|\mu - \nu\|_\Delta \leqslant P_\Delta(\sigma \neq \sigma') \tag{8.11}$$

and there exists a coupling satisfying the equality. Such a coupling is called optimal coupling (see [FG] for an explicit construction). Optimal couplings are not just a detail for a proof. They are very useful.

Definition 8.12.
Let γ be a POMM. The collection $\mathbf{p} = (p_x)_{x\in S}$ defined by

$$\forall x \in S, \qquad p_x = \sup_{\eta,\xi\in\Omega} \|\gamma_x(\cdot,\eta) - \gamma_x(\cdot,\xi)\|_x \tag{8.12}$$

is called the ***maximal percolation parameters*** of γ.

Proposition 8.13.
Let γ be a POS with two possible colors ($E = \{a, b\}$). Then the maximal percolation parameter at the site $x \in S$ for γ is given by

$$p_x = \sup_{\eta,\xi\in\Omega} \big|\gamma_x(a|\eta) - \gamma_x(a|\xi)\big|$$

Proof.
It come from a simple computation, using (8.12) and (8.10). ✠

Definition 8.14.
Let Λ be a time box and $x \in \Lambda$. We denote $(\underline{\partial}\Lambda \overset{\neq}{\rightsquigarrow} x)$ the event in Ω^2 $\big\{(\sigma, \sigma') \in \Omega^2 : \exists (y_k)_{1\leqslant k\leqslant n} :\ y_1 = x,\ y_n \in \underline{\partial}\Lambda,\ y_k \in \underline{\partial} y_{k+1},\ \sigma_{y_k} \neq \sigma'_{y_k}\big\}$. It means that there exists a path of disagreement from $\underline{\partial}\Lambda$ to x.

Proposition 8.15.
Let γ be a POMM, Λ be a time box and η, η' be two configurations. Denote $\mathbf{p}$ the maximal percolation parameters of γ. There exists a coupling $P_\Lambda = P_{\Lambda,\eta,\eta'}$ of $\gamma_\Lambda(\cdot,\eta)$ and $\gamma_\Lambda(\cdot,\eta')$ such that:

1. $\forall x \in \Lambda,\ \{\sigma_x \neq \sigma'_x\} = (\underline{\partial}\Lambda \overset{\neq}{\rightsquigarrow} x)$ *P_Λ-a.s.,*
2. *the law of $\big(\mathbb{1}_{\{\sigma_x\neq\sigma'_x\}}\big)_{x\in\Lambda}$, denoted by $P^{\neq}_\Lambda$, is such that $P^{\neq}_\Lambda \preccurlyeq \psi_{\mathbf{p},\Lambda}$.*

Proof.
We will construct a coupling (σ, σ') of $\gamma_\Lambda(\cdot,\eta)$ and $\gamma_\Lambda(\cdot,\eta')$ by the following algorithm. In a preparatory step, we set $\Delta = \Lambda$, and define $\sigma_x = \eta_x$, $\sigma'_x = \eta'_x$ for $x \in \Delta^*_-$.

For fixing the main iteration step, suppose that (σ, σ') is already defined on Δ_- for a non-empty set $\Delta \subset \Lambda$ and is realized as a pair (ξ, ξ') on Δ_-^*, where $(\xi, \xi') = (\eta, \eta')$ on Λ_-^*.

Pick $x \in \min \Delta$ such that there exists some $y \in \underline{\partial} x \subset \Delta_-$ satisfying $\xi_y \neq \xi'_y$.
If such an x does not exist, we have $\gamma_\Delta(\cdot|\xi) = \gamma_\Delta(\cdot|\xi')$ on $\mathcal{F}_\Delta$, so that we can take the obvious optimal coupling for which $\sigma = \sigma'$ on Δ.
If such an x exists, we consider the single site distribution γ_x: let (ξ_x, ξ'_x) be distributed according to an optimal coupling of $\gamma_x(\cdot|\xi)$ and $\gamma_x(\cdot|\xi')$. The coupling (ξ, ξ') is then defined on the set $\{x\} \cup \Delta_-^*$, so that we can replace Δ by $\Delta \backslash \{x\}$ and repeat the preceding iteration step.

It is clear that the algorithm above stops after finitely many iterations. Theorem 4.7 gives that our construction is a coupling of $\gamma_\Lambda(\cdot, \eta)$ and $\gamma_\Lambda(\cdot, \eta')$. Property (i) is evident from the construction, since disagreement at a site is only possible if a path of disagreement leads from this site to the boundary $\underline{\partial}\Lambda$.
For (ii), note that in every site of Λ, we have chosen an optimal coupling hence the total coupling is still optimal. Let $x \in \Lambda$, $\eta, \eta' \in \Omega$ and $\xi, \xi' \in \Omega_{\Lambda \cap x_-^*}$.

$$\begin{aligned} P_\Lambda\Big(\sigma_x \neq \sigma'_x \mid (\sigma, \sigma') = (\xi\eta, \xi'\eta') \text{ on } x_-^* \cup \Lambda_-^*\Big) &= P_x\Big(\sigma_x \neq \sigma'_x \mid (\sigma, \sigma') = (\xi\eta, \xi'\eta') \text{ on } x_-^* \cup \Lambda_-^*\Big) \\ &\leqslant \big\|\gamma_x(\cdot, \xi\eta) - \gamma_x(\cdot, \xi'\eta')\big\|_x \\ &\leqslant p_x \end{aligned}$$

The first inequality comes from the fact that we have constructed an optimal coupling.

Since we can interpret $P_\Lambda^{\neq}$ and $\psi_{\mathbf{p},\Lambda}$ as partially oriented kernels, the POS-Holley Theorem yields to $P_\Lambda^{\neq} \preccurlyeq \psi_{\mathbf{p},\Lambda}$. ✠

Theorem 8.16.
Let γ be a POMM and $\mathbf{p}$ be its maximal percolation parameters. If

$$\sup_{x \in S} p_x < p_c^+(S) \tag{8.13}$$

then there exists exactly one γ-POC.

Proof.
We will use the coupling created in the last proposition. Let $\mu, \nu \in \mathcal{G}(\gamma)$, and Δ, Λ be two

time boxes such that $\Delta \subset \Lambda$. Using the coupling inequality (8.11), we have

$$\begin{aligned}
\|\mu - \nu\|_\Delta &\leqslant \sup_{\eta,\eta' \in \Omega_{\Lambda^*_-}} \left\|\mu(\cdot|\eta) - \nu(\cdot|\eta')\right\|_\Delta \\
&\leqslant \sup_{\eta,\eta' \in \Omega} \left\|\gamma_\Lambda(\cdot,\eta) - \gamma_\Lambda(\cdot,\eta')\right\|_\Delta \\
&\leqslant \sup_{\eta,\eta' \in \Omega} P_{\Lambda,\eta,\eta'}\left(\sigma \neq \sigma' \ in \ \Delta\right) \\
&\leqslant \sup_{\eta,\eta' \in \Omega} P_{\Lambda,\eta,\eta'}\left(\exists x \in \Delta,\ \sigma_x \neq \sigma'_x\right) \\
&\leqslant \sup_{\eta,\eta' \in \Omega} P_{\Lambda,\eta,\eta'}\left(\underline{\partial}\Lambda \overset{\neq}{\rightsquigarrow} \Delta\right) \\
&\leqslant \psi_{\mathbf{p},\Lambda}\left(\underline{\partial}\Lambda \rightsquigarrow \Delta\right)
\end{aligned}$$

By letting Λ tend to S, we get

$$\|\mu - \nu\|_\Delta \leqslant \psi_{\mathbf{p}}\left(\infty \rightsquigarrow \Delta\right)$$

Since there exists q such that $\sup_{s\in S} p_s < q < p_c^+(S)$ we have $\psi_{\mathbf{p}} \preccurlyeq \psi_q$. But $(\infty \rightsquigarrow \Delta)$ is clearly an increasing event so $\psi_{\mathbf{p}}\left(\infty \rightsquigarrow \Delta\right) = 0$. Then μ and ν coincide on all time boxes. ✠

Remark 8.17.
In general, oriented percolation is more difficult to obtain than non-oriented percolation: $p_c^+(S) \geqslant p_c(S)$. The preceding theorem can then be applied with the non-oriented percolation critical parameter. It yields to a less precise but easier to use condition. Indeed, in general $p_c(S)$ is better known than $p_c^+(S)$.

8.4 Application

We will apply Dobrushin and oriented disagreement percolation criteria to the examples given through this part.

We begin with the easiest: the geometry introduced in Example 5.3. In this geometry, the critical parameter of oriented percolation is 1. Indeed, let $0 \leqslant q < 1$ be the parameter of a Bernoulli field. It is impossible to find any infinite path of 1-valued sites. This is due to the fact that the geometry is nearly the geometry of $\mathbb{Z}$ endowed with its natural (total) order. Thus, the oriented disagreement percolation criterion leads to the following conclusion:

Proposition 8.18.
Let γ be a POS with the geometry given by Example 5.3. If for all $x \in S$, all $\omega \in \Omega$ and all $e \in E$,

$$\gamma_x(e|\omega) > 0$$

Then there is only one γ-POC.

Proof.
The condition imposed on γ implies that for all $x \in S$, the maximal percolation parameter at x is strictly less than one. ✠

Thus, this geometry is not very exciting: it has the same comportment as Markov Chains.

8.4.1 The POMM-Ising model

First, we look at the Dobrushin criterion.

Let x be in $\mathbb{Z}^2$. $\mathfrak{I}_x$ depends only on the sites Nx and Wx so $\alpha_{x,z} = 0$ for all $z \in S \backslash \{Nx, Wx\}$. Since Nx and Wx can be exchanged, $\alpha_{x,Wx} = \alpha_{x,Nx}$. We shall only compute $\alpha_{x,Nx}$. Proposition 8.4 gives that the smallest cleaning-rate coefficient at x is

$$\begin{aligned}\alpha_{x,Nx} &= \sup_{\eta \in \Omega} \left| \mathfrak{I}_x(1, \eta^{Nx,1}) - \mathfrak{I}_x(1, \eta^{Nx,-1}) \right| \\ &= \frac{\sinh(2\beta)}{\cosh(2\beta) + \cosh(2\beta(|h|-1))}\end{aligned}$$

The Dobrushin criterion gives uniqueness of the $\mathfrak{I}$-POC for

$$\sum_{x \leqslant y} \alpha_{x,y} = 2\alpha_{x,Nx} = \frac{2\sinh(2\beta)}{\cosh(2\beta) + \cosh(2\beta(|h|-1))} < 1 \tag{8.14}$$

Remark 8.19.
If the external field h is equal to zero, the Dobrushin criterion gives uniqueness for all $\beta > 0$. In conclusion, there is no phase transition for the Voter model in $\mathbb{Z}^2$.

We shall see in section 12.2 that the same conclusion holds even if $h \neq 0$.

Now, we look at the oriented disagreement percolation criterion for the POMM-Ising model. As in the preceding criterion, we just have to compute p_x. Proposition 8.13 gives

$$\begin{aligned}p_x &= \sup_{\eta, \xi \in \Omega} \left| \mathfrak{I}_x(1, \eta) - \mathfrak{I}_x(1, \xi) \right| \\ &= \frac{1}{2} \tanh(\beta(|h|+2)) - \tanh(\beta(|h|-2))\end{aligned}$$

It is well known that $p_c^+(\mathbb{Z}^2) > \frac{1}{2}$ so the disagreement-percolation criterion yields to uniqueness when

$$\tanh(\beta(|h|+2)) - \tanh(\beta(|h|-2)) < 1 \tag{8.15}$$

Figure 8.2 summarizes the two criteria on the same phase diagram. We see that the Dobrushin criterion is more efficient than the disagreement percolation if we use $1/2$ as lower bound of p_c^+. On the contrary, if we use the real value of p_c^+, we see that no criterion is better than the other: there exists a zone where only one criterion can be applied. Unfortunately, p_c^+ is unknown. In [BR06], a Monte Carlo method gives that $p_c^+ \approx 0.64450$. In addition, we see that our criteria are not sufficient: there exists (β, h) such that none of our criteria give information.

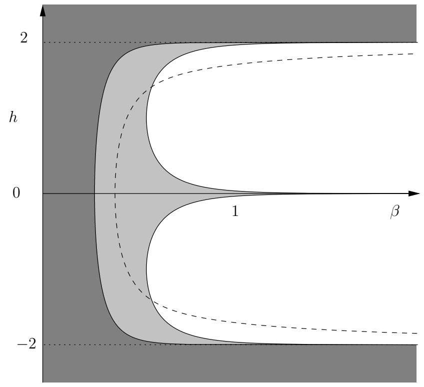


Figure 8.2 – The dark grey region corresponds to the parameters for which both Dobrushin and oriented disagreement percolation criteria (with $1/2$ instead of p_c^+) can be applied. In the light grey zone, only Dobrushin criterion is valid. The dashed line corresponds to the frontier of the region for the real parameter p_c^+. In the white zone, none of the criteria can be applied.

8.4.2 The Stavskaya's model

For the Dobrushin criterion, we use Proposition 8.4 to deduce the cleaning-rate matrix:

$$\alpha_{x,Nx} = \sup_{\eta_{Wx} \in \{0,1\}} \left| p\,\mathbb{1}_{\eta_{Wx}>0} - p\,\mathbb{1}_{1+\eta_{Wx}>0} \right| = p \tag{8.16}$$

so the criterion gives uniqueness for $2p < 1$.

For disagreement percolation, we have

$$p_x = \sup_{\eta_{Nx},\eta_{Wx},\xi_{Nx},\xi_{Wx} \in \{0,1\}} \left| p\,\mathbb{1}_{\eta_{Nx}+\eta_{Wx}>0} - p\,\mathbb{1}_{\xi_{Nx}+\xi_{Wx}>0} \right| = p \tag{8.17}$$

The criterion is then $p < p_c^+$.

Remark 8.20.
1. These criteria lead to the well known lower bound: $p_c^+ \geqslant 1/2$.
2. In this example, the disagreement percolation is a criterion more efficient than the Dobrushin one, even if we don't really know the exact value of p_c^+.
3. This example can easily be generalized to any lattice with finite (oriented) neighbourhood. If such a lattice has n past neighbours per site then $p_c^+(S) \leqslant 1/n$. This is not a good bound because it do not take into account the geometry of the lattice. For example, the same bound

holds for $\mathbb{Z}^2$ with its natural partial order and for the infinite binary tree whereas the geometry of those lattices is very different.

For this model, disagreement percolation is optimal that is, for $p > p_c^+$ there are at least two $\mathcal{S}$-POC. This will be proved in Proposition 12.6, in the next part.

Part II

Geometrical approach of Probabilistic Cellular Automata

Chapter 9

Preliminaries

Our work is aimed at image analysis and synthesis. We are specifically interested into textures. Indeed, this type of image is characterized by a global homogeneity but a local disorder. PCA seems to be well adapted to this. We shall work on the most simple textures: binary textures. Theses images have only two colors: black and white.

Dealing with synthesis, one of the main question is the choice of initial conditions. Indeed, it is not impossible that two different initial conditions lead to very different images. This is a problem for the implementation of the model on computers. This phenomenon is highly linked with the phase transition of PCA. If there is no phase transition, whatever the initial condition, the PCA converges to the same equilibrium measure. Thus the images coming from different initial configurations are governed by the same measure. This implies that they are very similar.

This part tries to answer to the question: *Is there a phase transition for a given 2-color PCA?* Our approach is to separate the information from the colors. Indeed, with the notation $P(\sigma|\eta)$, the color of the neighbours determines which probability to choose for σ. We have decomposed this probability to first (randomly) determine the origin of the information independently of the colors then we fix (deterministically) σ. In other terms, we look at where comes the information available for a single site. This leads to the notion of flow of information. This information is divided into two parts: the real information, coming from the initial conditions and a wrong information. The last one is due to the fact that the PCA is a stochastic model. If we had access to the whole space-time configuration, it is possible to determine the nature of each information. But a single site has only information from its nearest neighbours. It is a very truncated part of it.

That is why we can hope that every positive PCA (that is PCA with positive transition probabilities) has no phase transition. Unfortunately, we have not answer completely to the question. But we have established two criteria very simple to use. Those are the main results of this part.

In the rest of this chapter, we define the context of Part II. As in the first part, we begin be defining the geometrical environment then the notion of PCA. Our point of view is slightly different from the usual one used in the literature on PCA but it is better adapted to our analysis. In fact, we will work on the space-time configurations of the PCA. We will build the notion of flow of information on those configurations.

Let n be a nonnegative integer. It represents the dimension in which the PCA evolves. Since the time can be considered as one more dimension, the ***space-time*** is $\mathbb{Z}^{n+1}$.

Even if our results will essentially be applied for PCA in $\mathbb{Z}$, we define the main notions in $\mathbb{Z}^n$.

Definition 9.1.
For $x = (x_1, \cdots, x_{n+1}) \in \mathbb{Z}^{n+1}$, we define the ***time of the site*** x by

$$t_x := \sum_{i=1}^{n+1} x_i$$

If $t \in \mathbb{Z}$, $T_t := \{x \in \mathbb{Z}^{n+1} : \ t_x = t\}$ defines the ***layer of time*** t and $T_x := T_{t_x}$ is ***the set of points at the same time as*** x.

Definition 9.2.
Let $x = (x_1, \cdots, x_{n+1}) \in \mathbb{Z}^{n+1}$. We define the ***nearest past*** of the site x by

$$\underline{\partial} x = \left\{ y \in T_{t_x - 1} : \ \sum_{k=0}^{n+1} |x_k - y_k| = 1 \right\}$$

In words, $\underline{\partial} x$ is composed by the nearest neighbors of x that lie on the preceding layer of time.

Let $A \subset \mathbb{Z}^{n+1}$. the nearest past of A is defined by

$$\underline{\partial} A = \left(\bigcup_{x \in A} \underline{\partial} x \right) \backslash A$$

Let $E = \{-1, +1\}$ be the ***color space***. The set of ***space-time configurations*** is the product space $\Omega = E^{\mathbb{Z}^{n+1}}$. We endow E with its natural product σ-algebra. For $\Upsilon \subset \mathbb{Z}^{n+1}$, we denote ω_Υ the restriction of $\omega \in \Omega$ to Υ.

We will define the family of PCA we are going to study in this part.

Definition 9.3.
Let θ be a nonnegative function on $E \times E^{n+1}$ satisfying the following property:

$$\forall \eta \in E^{n+1}, \qquad \sum_{\sigma \in E} \theta(\sigma|\eta) = 1$$

We define the PCA on singletons by

$$\forall x \in \mathbb{Z}^{n+1}, \ \forall \eta, \sigma \in \Omega, \qquad P\left(\sigma_x | \eta_{(T_x)_-}\right) = \theta(\sigma_x | \eta_{\underline{\partial} x})$$

By words, given the whole past, the model depends only on the nearest past.

Note that the model is supposed homogeneous, that is translation invariant (in space and in time). The main property of PCA is the Independence of points at the same time given the past: for all $t \in \mathbb{Z}$, $A \subset T_t$ and $\eta \in \Omega$,

$$P_A(\sigma_A | \eta_{(T_t)_-}) = \prod_{y \in A} \theta_y(\sigma_y | \eta_{\underline{\partial} y})$$

Remark 9.4.
The definitions here are consistent with the same notions introduced in Part I. The partial order used on $\mathbb{Z}^{n+1}$ is the natural one:

$$(x_1, \cdots, x_{n+1}) \leqslant (y_1, \cdots, y_{n+1}) \qquad \Longleftrightarrow \qquad \forall 1 \leqslant k \leqslant n+1,\ x_k \leqslant y_k$$

See Figure 9.1 for a picture on $\mathbb{Z}^2$. Note that, as in the first part we have flipped the vertical axe to be consistent with computer science conventions.

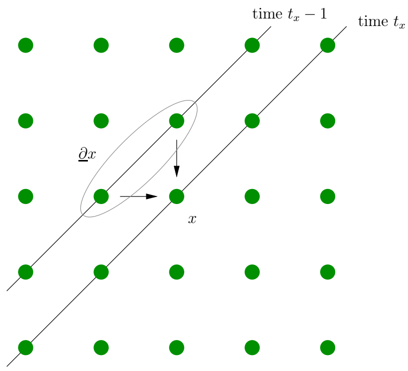


Figure 9.1 – The space-time is $\mathbb{Z}^2$. We have rotated the traditional image of PCA: Time does not go from up to down but from up-left to down-right corner.

Since the model is homogeneous, we will only focus our study on the site 0 and its past. By translation, we can conclude on the behaviour of all sites. Thus we introduce

$$\mathcal{T}_{b,0} = \{\Lambda \in \mathcal{T}_b,\ \max(\Lambda) = \{0\}\}$$

In the whole text, we will denote the expectation of a function f with respect to a probability measure μ by $\mu(f)$. Moreover, we will alternatively use the notations $E = \{\ominus, \oplus\}$ for the color space. This will clarify a bit some notations. At last, we will keep the notation $\theta(\sigma|\eta)$ even if $\eta \in \Omega$. This will correspond to $\theta(\sigma|\eta_{\underline{\partial} 0})$.

Chapter 10

The majority decomposition

We introduce a decomposition of θ with a majority function. This decomposition is applied to PCA in $\mathbb{Z}$ to get a uniqueness criterion. The chapter is divided into two parts. In the first part, we prove the uniqueness result for symmetric PCA. This is based on an analysis of the space-time configurations. Symmetric PCA are shown to be exactly solvable. The second part is devoted to the general case.

10.1 Symmetric PCA

Definition 10.1.
We say that a PCA is ***symmetric*** if $\theta(-\sigma|-\eta) = \theta(\sigma|\eta)$ for all $\sigma \in E$ and $\eta \in \Omega$.

Definition 10.2.
Let k be an odd integer smaller than $n+1$. A subset of $\underline{\partial}0$ with k elements will be called a ***k-majority sample*** of the site 0. A majority sample is a k-majority sample for some k. The set of all majority samples of 0 is denoted by $\mathcal{C}$.

We define the set of majority samples of a site $x \in \mathbb{Z}^{n+1}$ by translation of $\mathcal{C}$. It is denoted by $\mathcal{C}_x$.

Definition 10.3.
Let $\eta \in \Omega$. For $x \in \mathbb{Z}^{n+1}$, let $\pi \in \mathcal{C}_x$ be a majority sample. We define the ***majority function*** maj_π of π by:

$$\mathrm{maj}_\pi(\eta) := \mathrm{sgn}\left(\sum_{y\in\pi} \eta_y\right)$$

where sgn denotes the sign function. In words, $\mathrm{maj}_\pi(\eta)$ returns the majority of the color in η restricted to π: $\mathrm{maj}_\pi(\eta) = +1$ if there is a majority of $+1$ in π. Since $|\pi|$ is odd, there is no problem of definition: the sum can not be zero.

Our results are based on the following decomposition of the θ function in terms of elements of $\mathcal{C}$ and a constant. The reason why this constant is indexed by $\odot$ will be explained later in 10.11.

Theorem 10.4.
Let P be a symmetric PCA. There exists a collection of real numbers p_π indexed by $\mathcal{C} \cup \{\odot\}$ such that the PCA can be written as:

$$\theta(\sigma|\eta) = \sum_{\pi \in \mathcal{C}} p_\pi \delta_{\mathrm{maj}_\pi(\eta)}(\sigma) + \frac{p_\odot}{2} \tag{10.1}$$

for all $\sigma \in E$ and $\eta \in \Omega$.

Proof.
The main idea of the proof uses linear algebra.
Denote $y_1, \cdots, y_{n+1}$ the nearest neighbours of 0. Let $\mathbb{1}_{\sigma\eta} := \left(\mathbb{1}_{\sigma=\eta_{y_1}}, \cdots, \mathbb{1}_{\sigma=\eta_{y_{n+1}}}\right)$. This is an element of the set $X_n = \{0,1\}^{n+1}$. Since the PCA is symmetric, there exists a function f from X_n to $\mathbb{R}$ such that

$$\forall \eta \in \Omega, \qquad \theta(\sigma|\eta) = f(\mathbb{1}_{\sigma\eta})$$

For $\omega \in X_n$, we denote $\overline{\omega}$ the flipped element of ω: $\overline{\omega}_k = 1 - \omega_k$, for all $1 \leqslant k \leqslant n+1$. Observe that for all $\eta \in \Omega$,

$$f(\mathbb{1}_{\sigma\eta}) + f(\overline{\mathbb{1}_{\sigma\eta}}) = \theta(\sigma|\eta) + \theta(-\sigma|\eta) = 1$$

It is then natural to introduce the vector space $\mathcal{A} := \{g : X_n \to \mathbb{R},\ \forall \omega \in X_n,\ g(\omega) + g(\overline{\omega}) = 0\}$. $f - 1/2$ is an element of $\mathcal{A}$.

Now, $\dim(\mathcal{A}) = 2^n$. Indeed, if we denote $(1, \cdot)$ the family of elements of X_n such that the first coordinate is fixed to 1, then $(\delta_\omega - \delta_{\bar{\omega}})_{\omega=(1,\cdot)}$ is clearly a base of $\mathcal{A}$.

Let us introduce one more notation:

$$\forall \pi \in \mathcal{C},\ \forall \omega \in X_n, \qquad \widetilde{M_\pi}(\omega) := \operatorname{sgn}\left(\sum_{y \in \pi} (-1)^{\omega_y}\right)$$

If we prove that $(\widetilde{M_\pi})_{\pi \in \mathcal{C}}$ is a base of $\mathcal{A}$, then there exists $(\widetilde{p}_\pi)_{\pi \in \mathcal{C}}$ such that

$$\begin{aligned}\theta(\sigma|\eta) &= \left(f(\mathbb{1}_{\sigma\eta}) - 1/2\right) + 1/2 \\ &= \sum_{\pi \in \mathcal{C}} \widetilde{p}_\pi \widetilde{M_\pi}(\mathbb{1}_{\sigma\eta}) + 1/2\end{aligned}$$

But since $\widetilde{M_\pi}(\mathbb{1}_{\sigma\eta}) = 1 - 2\delta_{\mathrm{maj}_\pi(\eta)}(\sigma)$, the desired formula holds with $p_\pi := -2\widetilde{p}_\pi$ and $p_\odot := 1/4 + \sum_{\pi \in \mathcal{C}} \widetilde{p}_\pi/2$.

So to finish the proof, we only need to prove that $(\widetilde{M}_\pi)_{\pi\in\mathcal{C}}$ is a base of $\mathcal{A}$. Let $(\alpha_\pi)_{\pi\in\mathcal{C}}$ be a family of real numbers such that $g(\omega) := \sum_{\pi\in\mathcal{C}} \alpha_\pi \widetilde{M}_\pi(\omega) = 0$. Since g is symmetric in each coordinate of X_n, $\alpha_\pi = \alpha_{\pi'}$ as soon as $|\pi| = |\pi'|$. So for all $\pi \in \mathcal{C}$, $\alpha_{|\pi|} := \alpha_\pi$. Now, let us fix $\omega_0 := (0, \cdots, 0) \in X_n$. For $\omega = (1, 0, \cdots, 0) \in X_n$,

$$g(\omega_0) - g(\omega) = 2\alpha_1 = 0$$

and by induction for $1 \leqslant k \leqslant \lfloor n/2 \rfloor + 1$, for $\omega = (\underbrace{1, \cdots, 1}_{k \text{ times}}, 0, \cdots, 0) \in X_n$,

$$g(\omega_0) - g(\omega) = 2\binom{n+1-k}{k-1}\alpha_{2k-1} = 0$$

Finally, all the terms of the family (α_π) are null. This proves that $(\widetilde{M}_\pi)_{\pi\in\mathcal{C}}$ is a free family of $\mathcal{A}$. Since it has the right number of elements, it is a base. This completes the proof. ✠

Remark 10.5.
The collection of p_π is unique. This fact comes directly from the proof. Actually, we are not interested in the uniqueness of p_π. What is important for us is the existence of such a decomposition.

Remark that for all $\pi \in \mathcal{C}$ and all $\eta \in \Omega$, $\delta_{\mathrm{maj}_\pi(\eta)}(\sigma) = 0$ if and only if $\delta_{\mathrm{maj}_\pi(\eta)}(-\sigma) = 1$. This induces the following important equation:

$$\sum_{\pi\in\mathcal{C}\cup\{\odot\}} p_\pi = 1 \tag{10.2}$$

This is then natural to look at models for which all p_π are nonnegative.

Definition 10.6.
A PCA is said to be ***with positive majority*** if for all $\pi \in \mathcal{C} \cup \{\odot\}$, the real number p_π is nonnegative in the decomposition (10.1).

Fix $\eta \in \Omega$. If a PCA is with positive majority, the formula (10.1) has the following interpretation: instead of choosing a color in E according to θ, we choose a majority sample or $\odot$ according to the probability $P(\pi) = p_\pi$ for all $\pi \in \mathcal{C} \cup \{\odot\}$. If $\pi \in \mathcal{C}$ has been chosen, we fix the configuration to $\mathrm{maj}_\pi(\eta)$. If we have chosen $\odot$, we toss a fair coin to decide whether the color is $\oplus$ or $\ominus$. In fact, we select where the information of the color comes from.

Proposition 10.7.
Every symmetric PCA with positive majority satisfies the FKG inequalities.

Proof.
Remark that for all $\pi \in \mathcal{C}$, $\eta \mapsto \delta_{\mathrm{maj}_\pi(\eta)}(\oplus)$ is an increasing function. Theorem 7.5 can then be applied. ✠

The converse is false. For example for $\alpha \in [0, 1/2]$, define the following PCA on $\mathbb{Z}^2$ by

$$\theta(\oplus|\eta) = \begin{cases} 1/2 - \alpha & \text{if the three neighbours are } \ominus \\ 1/2 & \text{if exactly one neighbour is } \oplus \end{cases}$$

This is sufficient to define completely the PCA. The rest of the definition is given by the symmetry of the model and the fact that P is a probability. This PCA satisfies the FKG inequalities but the coefficient of the 3-majority sample is equal to $-\alpha$.

Definition 10.8.
Let $\Lambda \in \mathcal{T}_b$. An element $\underline{e}$ of the set $(\mathcal{C} \cup \{\odot\})^\Lambda$ is called a ***majority flow of information*** in Λ. Let $\eta \in \Omega$. We say that $\omega \in \Omega_\Lambda$ is ***compatible*** with η and $\underline{e}$, and we denote $\omega \overset{M}{\sim} (\eta, \underline{e})$ if

$$\forall x \in \Lambda \text{ s.t. } e_x \neq \odot, \qquad \omega_x = \mathrm{maj}_{e_x}(\omega\eta_{\Lambda^c})$$

In the last formula, we have identified $e_x \in \mathcal{C}$ with its translated element in $\mathcal{C}_x$.

For $\pi \in \mathcal{C} \cup \{\odot\}$, the number of π in $\underline{e}$ is $\#_\pi(\underline{e}) := \Big|\{x \in \Lambda : e_x = \pi\}\Big|$.

Remark 10.9.
Fix η and $\underline{e}$. There are $2^{\#_\odot(\underline{e})}$ compatible configurations with η and $\underline{e}$. Indeed, the whole configuration is fixed except on sites x where $e_x = \odot$. On those sites, we have two possibilities.

Theorem 10.10.
Let P be a symmetric PCA. For any $\Lambda \in \mathcal{T}_b$, any local function f such that $\mathrm{Supp}(f) \subset \Lambda$ *and any $\eta \in \Omega$, we have the following equation*

$$P_\Lambda(f|\eta) = \sum_{\underline{e} \in (\mathcal{C} \cup \{\odot\})^\Lambda} \left(\prod_{\pi \in \mathcal{C} \cup \{\odot\}} {p_\pi}^{\#_\pi(\underline{e})} \right) \frac{1}{2^{\#_\odot(\underline{e})}} \sum_{\omega \in \Omega_\Lambda, \omega \overset{M}{\sim} (\eta, \underline{e})} f(\omega) \tag{10.3}$$

Proof.
This is a straightforward consequence of the last proposition:

$$\begin{aligned} P_\Lambda(f|\eta) &= \sum_{\omega \in \Omega_\Lambda} f(\omega) \prod_{x \in \Lambda} \theta(\omega_x | \omega\eta_{\Lambda^c}) \\ &= \sum_{\omega \in \Omega_\Lambda} f(\omega) \sum_{\underline{e} \in (\mathcal{C} \cup \{\odot\})^\Lambda} \left(\prod_{\pi \in \mathcal{C}} {p_\pi}^{\#_\pi(\underline{e})} \right) \left(\frac{p_\odot}{2} \right)^{\#_\odot(\underline{e})} \\ &\qquad \prod_{\pi \in \mathcal{C}} \left(\prod_{x \in \Lambda, e_x = \pi} \delta_{\mathrm{maj}_\pi(\omega\eta_{\Lambda^c})}(\omega_x) \right) \\ &= \sum_{\underline{e} \in (\mathcal{C} \cup \{\odot\})^\Lambda} \left(\prod_{\pi \in \mathcal{C} \cup \{\odot\}} {p_\pi}^{\#_\pi(\underline{e})} \right) \frac{1}{2^{\#_\odot(\underline{e})}} \sum_{\omega \in \Omega_\Lambda, \omega \overset{M}{\sim} (\eta, \underline{e})} f(\omega) \end{aligned}$$

✠

Remark 10.11.
For a PCA with positive majority, Theorem 10.10 leads to a real geometric approach: Since there are $2^{\#_\odot(\underline{e})}$ compatible configurations with $\underline{e}$, Equation (10.3) can be interpreted as expectation of two probabilities. The first one consists in choosing randomly a majority flow of information. This choice is made site by site, independently of the other sites: we choose $\pi \in \mathcal{C} \cup \{\odot\}$ with probability p_π. The second probability measure consists in choosing randomly a configuration compatible with the flow given by the first measure and the "initial condition" η. This choice is uniform on the set of compatible configurations.

To take advantage of this remark, let us denote Q the product measure defined on $\mathbb{Z}^{n+1}$ by $Q(\pi) = p_\pi$ for all sites and all $\pi \in \mathcal{C} \cup \{\odot\}$. Equation (10.2) gives that the total mass of Q is 1. Unfortunately, it is a probability measure only if the PCA is with positive majority.

Next, define the second measure $R_\Lambda^{\eta,\underline{e}}$ for $\eta \in \Omega$, $\Lambda \in \mathcal{T}_b$, $\underline{e}$ a majority flow of information and f a local function by

$$R_\Lambda^{\eta,\underline{e}}(f) = \frac{1}{2^{\#_\ominus(\underline{e})}} \sum_{\omega \in \Omega,\ \omega \overset{M}{\sim} (\eta,\underline{e})} f(\omega)$$

This measure is a probability measure. Then Equation (10.3) can be rewritten

$$P_\Lambda(f|\eta) = Q(R_\Lambda^{\eta,\underline{e}}(f)) \tag{10.4}$$

This interpretation has a geometrical view: Choose two colors, say red and blue. Red will represent the $\ominus$ value and blue $\oplus$. If at a site $x \in \mathbb{Z}^{n+1}$ we choose $\pi \in \mathcal{C}_x$, we draw lines from x to each site in π. The color of a line corresponds to the color of its extremity in π. Then the color of the site x is the color having the majority in the drawn lines from x to π. If we choose $\odot$ then we draw a circle centered at x and we toss a fair coin to decide which color to paint into it (see Figure 10.1 for an example).

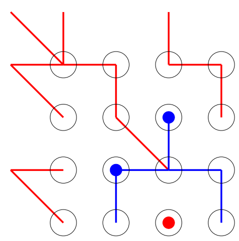


Figure 10.1 – flow of information in $\mathbb{Z}^2$ with 3 neighbours.

Remark 10.12.
We have transformed a local (Markovian) system into a highly non local one. Indeed, the color of one site depends now on what happens in the whole past and not only in the nearest past. A change in the flow of information very far from a site can affect it. Nevertheless,

this description of PCA is extremely interesting: to determine the color of a spin, we look at where the information comes from. In other words, we trace back the information. The notion of dual process in particle systems in continuous time uses the same idea. (see [Lan05] and [Dur84])

To do something useful with (10.3), we have to understand the geometry of the majority flow of information. In general, this is not such an easy thing to do! That is why we will now restrict our study to PCA in $\mathbb{Z}$. The space-time configurations are then in $\mathbb{Z}^2$.

Let $x = (x_1, x_2) \in \mathbb{Z}^2$. We recall the definition of the N and W operators

$$Nx := (x_1, x_2 - 1) \quad \text{and} \quad Wx := (x_1 - 1, x_2)$$

Nx and Wx are the two neighbours of x.

Since there are only two neighbors, the sample set is $\mathcal{C} = \{N, W\}$. The decomposition given by Equation (10.1) gives that for a symmetric PCA

$$\theta(\sigma|\eta) = p_N \delta_N(\sigma) + p_W \delta_W(\sigma) + \frac{p_\odot}{2} \tag{10.5}$$

Theorem 10.13.
Let P be a symmetric PCA in $\mathbb{Z}$. If

$$\forall \sigma \in E,\ \forall \eta \in \Omega \qquad \theta(\sigma|\eta) > 0$$

then P has only one consistent measure.

Proof.
Let $k \in \mathbb{N}$. We will look at the majority flow of information in

$$\Lambda_k := \left\{x \in \mathbb{Z}^2,\ 0 \leqslant -t_x \leqslant k\right\} \cap \left(\{0\} \cup 0_-\right)$$

The proof is divided into two parts. In the first one, we assume that the PCA is with positive majority. In the second part, we show that this assumption can be ignored by slightly transforming the model.

First, suppose that $p_N \geqslant 0$ and $p_W \geqslant 0$. Since the PCA is FKG, it is then sufficient (see the discussion at the end of section 7.2) to show that $\left|P_{\Lambda_k}(\sigma_0|\oplus) - P_{\Lambda_k}(\sigma_0|\ominus)\right|$ tends to zero as k goes to infinity. Equation (10.4) gives:

$$\begin{aligned} P_{\Lambda_k}(\sigma_0|\oplus) - P_{\Lambda_k}(\sigma_0|\ominus) &= Q\left(R^{\oplus,\underline{e}}_{\Lambda_k}(\sigma_0) - R^{\ominus,\underline{e}}_{\Lambda_k}(\sigma_0)\right) \\ &= Q\left(\frac{1}{2^{\#_\odot(\underline{e})}}\left(\sum_{\omega \overset{M}{\sim} (\oplus,\underline{e})} \omega_0 - \sum_{\omega \overset{M}{\sim} (\ominus,\underline{e})} \omega_0\right)\right) \end{aligned}$$

Fix $\omega \overset{M}{\sim} (\oplus, \underline{e})$. The configuration $\bar{\omega}$ constructed by flipping the sites in contact with $\underline{\partial}\Lambda_k$, satisfies $\bar{\omega} \overset{M}{\sim} (\ominus, \underline{e})$. Moreover, if 0 is in contact with $\underline{\partial}\Lambda_k$ (this event will be denoted by $(0 \xrightarrow{\underline{e}} \underline{\partial}\Lambda_k)$) then $\omega_0 = -\bar{\omega}_0 = \oplus$ and $\bar{\omega}_0 = \omega_0$ otherwise. Thus

$$\begin{aligned}
\sum_{\omega \overset{M}{\sim}(\oplus,\underline{e})} \omega_0 - \sum_{\omega \overset{M}{\sim}(\ominus,\underline{e})} \omega_0 &= \sum_{\omega \overset{M}{\sim}(\oplus,\underline{e})} (\omega_0 - \bar{\omega}_0) \\
&= 2 \sum_{\omega \overset{M}{\sim}(\oplus,\underline{e})} \mathbb{1}_{(0 \xrightarrow{\underline{e}} \underline{\partial}\Lambda_k)} \\
&= \left(2^{1+\#_\odot(\underline{e})}\right) \mathbb{1}_{(0 \xrightarrow{\underline{e}} \underline{\partial}\Lambda_k)}
\end{aligned}$$

Now, if $(0 \xrightarrow{\underline{e}} \underline{\partial}\Lambda_k)$, there exists a single path from 0 to $\underline{\partial}\Lambda_k$: it is not possible to find any bifurcations. It is only possible to go up (N) or left (W); We can not simultaneously choose the two directions. So we can rearrange the sum:

$$\begin{aligned}
P_{\Lambda_k}(\sigma_0|\oplus) - P_{\Lambda_k}(\sigma_0|\ominus) &= 2Q\left(0 \xrightarrow{\underline{e}} \underline{\partial}\Lambda_k\right) \\
&= 2 \sum_{\substack{\underline{e} \in \{N,W,\odot\}^{\Lambda_k} \\ (0 \xrightarrow{\underline{e}} \underline{\partial}\Lambda_k)}} p_N{}^{\#_N(\underline{e})} p_W{}^{\#_W(\underline{e})} p_\odot{}^{\#_\odot(\underline{e})} \\
&= 2 \sum_{\varrho: 0 \longrightarrow \underline{\partial}\Lambda_k} \sum_{\underline{e} \supset \varrho} p_N{}^{\#_N(\underline{e})} p_W{}^{\#_W(\underline{e})} p_\odot{}^{\#_\odot(\underline{e})} \\
&= 2 \sum_{\varrho: 0 \longrightarrow \underline{\partial}\Lambda_k} p_N{}^{\#_N(\varrho)} p_W{}^{\#_W(\varrho)} \sum_{\underline{e} \in \{N,W,\odot\}^{\Lambda_k \setminus \varrho}} p_N{}^{\#_N(\underline{e})} p_W{}^{\#_W(\underline{e})} p_\odot{}^{\#_\odot(\underline{e})} \\
&= 2 \sum_{\varrho: 0 \longrightarrow \underline{\partial}\Lambda_k} p_N{}^{\#_N(\varrho)} p_W{}^{\#_W(\varrho)} \prod_{x \in \Lambda_k \setminus \varrho} (p_N + p_W + p_\odot) \\
&= 2 \sum_{\varrho: 0 \longrightarrow \underline{\partial}\Lambda_k} p_N{}^{\#_N(\varrho)} p_W{}^{\#_W(\varrho)} \\
&= 2 \sum_{i=0}^{k} \binom{k}{i} p_N{}^i p_W{}^{k-i} \\
&= 2(p_N + p_W)^k \\
&= 2(1 - p_\odot)^k
\end{aligned}$$

The hypothesis of the theorem implies that $0 < 2P(\oplus|\ominus, \ominus) = p_\odot$ and (10.2) together with $p_N,\ p_w > 0$ gives $p_\oplus < 1$. The first part is then proved.

Now, remark that if $p_N < 0$, we can write

$$\theta(\sigma|\eta) = -p_N \delta_{-\eta_N}(\sigma) + p_W \delta_{\eta_W}(\sigma) + (p_N + p_\odot/2)$$

This will not imply the same geometry. A majority sample equal to N at a site x will means that the value at x is $-\eta_{Nx}$. But since $p_N + p_\odot/2 = P(\oplus|\eta_N = \oplus, \eta_W = \ominus) > 0$, the same proof as above with this decomposition will give the result.

$$P_{\Lambda_k}(\sigma_0|\oplus) - P_{\Lambda_k}(\sigma_0|\ominus) = 2(1 - 2p_N + p_\odot)^k$$

A simple argumentation gives that $0 < 2p_N + p_\odot < 1$.
The same is clearly true if $p_W < 0$. ✠

Remark 10.14.
A symmetric PCA in $\mathbb{Z}$ is exactly solvable: we can do explicit computation without inequality thanks to Equation (10.3).
By modifying a little the proof, we see that the consistent measure μ satisfy $\mu(\sigma_0) = 0$.

We see in the last proof the main difficulty for generalizing it to higher dimension: In general, the condition $(0 \xrightarrow{\underline{e}} \underline{\partial}\Lambda_k)$ will appear from the quantity

$$\sum_{\omega \overset{M}{\sim} (\oplus, \underline{e})} \omega_0 - \sum_{\omega \overset{M}{\sim} (\ominus, \underline{e})} \omega_0$$

But then, we have to understand the behaviour of the "path" (its geometry) to finish the proof.

This theorem is not true for PCA in $\mathbb{Z}^2$: The NEC Toom model is a counter-example. It is defined on $\mathbb{Z}^2$ for $0 \leqslant \epsilon \leqslant 1$ by

$$\theta(\sigma|\eta) = (1 - 2\epsilon)\delta_{\mathrm{maj}_{NWU}(\eta)}(\sigma) + \epsilon$$

where the three neighbours are N (north), W (west) and U (up) and maj_{NWU} is the majority between the three neighbours. Toom proved that, for small enough $\epsilon > 0$, there exist two different measures compatible with this model. See Appendix A for more details and a complete proof.

To conclude this paragraph, we give a simulation of a symmetric model with positive majority. It has been done with the program given in Appendix.

It represents a coupling between the same model starting from all $\oplus$ and all $\ominus$. The red color represents the difference of information that is $\ominus$ for one process and $\oplus$ for the other one. Since a model with positive majority is FKG, the process starting by $\ominus$ is always under the other: the situation $\oplus/\ominus$ can not happen.

10.2 Non-symmetric PCA

We go back to the study of homogeneous PCA, with no symmetry assumptions.

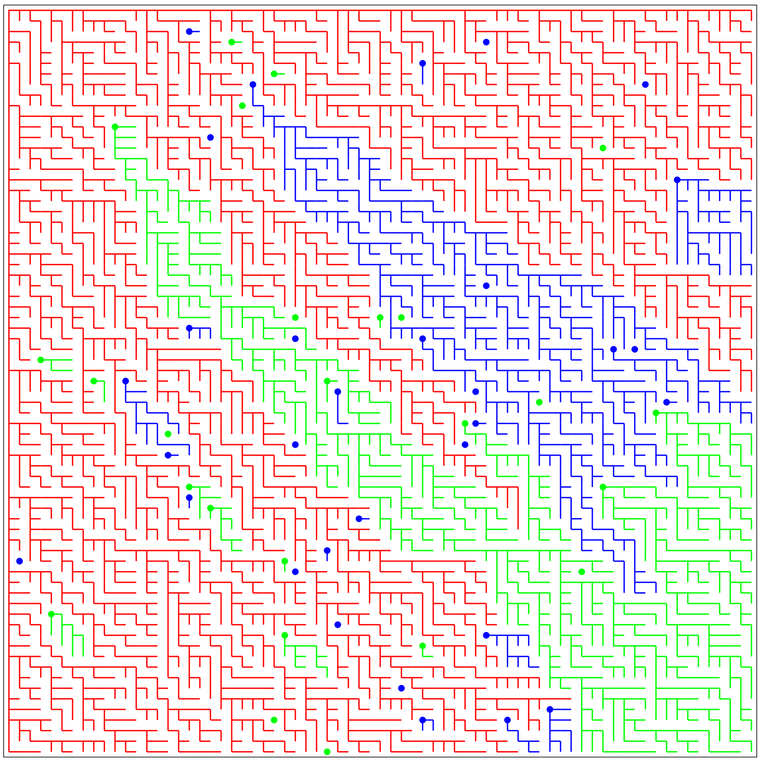


Figure 10.2 – Simulation of a flow of information for a symmetric model. parameters: $p_N = p_W = 0.495$, $p_\oplus = p_\ominus = 0.05$. $\oplus$ are in blue, $\ominus$ in green and the information coming from the frontier in red.

In general, we can write θ as a function of $\mathbb{1}_{\sigma\eta}$ and $\mathbb{1}_{\sigma=\oplus}$:

$$\begin{aligned}\theta(\sigma|\eta) &= \mathbb{1}_{\sigma=\oplus}\ h_1(\mathbb{1}_{\sigma\eta}) + \mathbb{1}_{\sigma=\ominus}\ h_2(\mathbb{1}_{\sigma\eta}) \\ &= h(\mathbb{1}_{\sigma\eta}, \mathbb{1}_{\sigma=\oplus}) \\ &= h(\mathbb{1}_{\sigma=\eta_{y_1}}, \cdots, \mathbb{1}_{\sigma=\eta_{y_{n+1}}}, \mathbb{1}_{\sigma=\oplus}) \end{aligned} \tag{10.6}$$

This shows that a non symmetric PCA is nearly symmetric. The only thing that changes is the presence of $\mathbb{1}_{\sigma=\oplus}$. We can interpret this indicator function in term of another neighbour with the constant color $\oplus$. Concretely, let us introduce another (virtual) neighbour V to all sites. This neighbour will always have the same color : $\eta_V = \oplus$.

Thanks to Equation (10.6), we can now write a non-symmetric PCA as a symmetric PCA with this additional neighbor. Let us define the new set of majority samples (with V) by $\widetilde{\mathcal{C}_V}$. Theorem 10.4 gives us the decomposition:

$$\theta(\sigma|\eta) = \sum_{\pi\in\widetilde{\mathcal{C}_V}} p_\pi \delta_{\mathrm{maj}_\pi(\eta)}(\sigma) + \frac{p_\odot}{2} \tag{10.7}$$

For simplicity and a better comprehension, we will not consider $\{V\}$ as a majority sample and we decompose $p_\odot$ into $p_\oplus := p_V + p_\odot/2$ and $p_\ominus := p_\odot/2$. Let us write $\mathcal{C}_V = \widetilde{\mathcal{C}_V}\backslash\{V\}$. Since for all σ, $\delta_\oplus(\sigma) + \delta_\ominus(\sigma) = 1$, we then have the following decomposition:

$$\theta(\sigma|\eta) = \sum_{\pi\in\mathcal{C}_V} p_\pi \delta_{\mathrm{maj}_\pi(\eta)}(\sigma) + p_\oplus\delta_\oplus(\sigma) + p_\ominus\delta_\ominus(\sigma) \tag{10.8}$$

and Equation (10.3) becomes

$$P_\Lambda(f|\eta) = \sum_{\underline{e}\in\{\mathcal{C}_V\cup\{\odot\}\}^\Lambda} \left(\prod_{\pi\in\mathcal{C}_V\cup\{\odot\}} p_\pi{}^{\#_\pi(\underline{e})}\right) \left(\frac{1}{p_\oplus + p_\ominus}\right)^{\#_\odot(\underline{e})} \sum_{\substack{\omega\in\Omega_\Lambda \\ \omega\overset{M}{\sim}(\eta,\underline{e})}} p_\oplus{}^{\#_\oplus(\omega)} p_\ominus{}^{\#_\ominus(\omega)} f(\omega) \tag{10.9}$$

The definition of positive majority and majority flow of information are naturally extended to the non-symmetric case.

As in Remark 10.11, denote Q the product measure on $(\mathcal{C}_V \cup \{\odot\})^{\mathbb{Z}^{n+1}}$ such that $Q(\pi) := p_\pi$, for all $\pi \in \mathcal{C}_V \cup \{\odot\}$ at all sites of $\mathbb{Z}^{n+1}$.

And for $\eta \in \Omega$, $\underline{e} \in \mathcal{C}_V\{\odot\}^{\mathbb{Z}^{n+1}}$ and $\Lambda \in \mathcal{T}_b$, denote

$$R_\Lambda^{\eta,\underline{e}}(f) := \left(\frac{1}{p_\oplus + p_\ominus}\right)^{\#_\odot(\underline{e}_\Lambda)} \sum_{\substack{\omega\in\Lambda \\ \omega\overset{M}{\sim}(\eta,\underline{e})}} p_\oplus^{\#_\oplus(\omega)} p_\ominus^{\#_\ominus(\omega)} f(\omega)$$

Formula (10.9) can then be rewritten as

$$P_\Lambda(f|\eta) = Q\big(R_\Lambda^{\eta,\underline{e}}(f)\big) \tag{10.10}$$

Since $p_\oplus$ and $p_\ominus$ are nonnegative real numbers, $R_\Lambda^{\eta;\underline{e}}$ is always a probability measure whereas Q is only a measure with total mass one.

If the model is with positive majority, Q is a probability measure on the set of (infinite) majority flow of information. Given a flow, we just have to specify to which of $\oplus$ or $\ominus$ the sites associated with $\odot$ are linked to to completely determine a configuration.

As in the symmetric case, we are not able to conclude something for PCA in $\mathbb{Z}^{n+1}$. Nevertheless, in two dimensions (PCA in $\mathbb{Z}$) this transformation leads to a useful uniqueness criterion. In this case, $\mathcal{C}_V = \big\{\{N,W,V\},\ \{N\},\ \{W\}\big\}$. For simplicity, we denote $M := \{N,W,V\}$. We have explicit values for p_M, p_N, p_W, $p_\oplus$ and $p_\ominus$:

$$\begin{cases} p_M = & \theta(\oplus|\oplus,\ominus) + \theta(\oplus|\ominus,\oplus) - \theta(\oplus|\oplus,\oplus) - \theta(\oplus|\ominus,\ominus) \\ p_N = & \theta(\oplus|\oplus,\oplus) - \theta(\oplus|\ominus,\oplus) \\ p_W = & \theta(\oplus|\oplus,\oplus) - \theta(\oplus|\oplus,\ominus) \\ p_\oplus = & \theta(\oplus|\ominus,\ominus) \\ p_\ominus = & \theta(\ominus|\oplus,\oplus) \end{cases}$$

This is easily seen by solving the linear system.

Theorem 10.15.
Let P be a PCA in $\mathbb{Z}$ with positive majority.
If $p_\oplus > 0$ then P has only one consistent measure.

Proof.
Let $\Lambda \in \mathcal{T}_{b,0}$. We shall prove that the limit $P_\Lambda(\sigma_0|\eta)$ when Λ goes to S does not depend any more on $\eta \in \Omega$. Since P satisfies the FKG inequalities, this is sufficient. For that, we use the preceding description Equation (10.10).

Choose $\underline{e}$ according to the probability measure Q. If 0 is not connected to $\underline{\partial}\Lambda$, there is no need to know η to determine the color of the site 0. Now, suppose that 0 is connected to $\underline{\partial}\Lambda$. Since a $\oplus$ propagates through the flow to 0, σ_0 depends on η only if all $\odot$ connected to 0 are linked to $\ominus$. Let j_Λ be the number of $\odot$ connected to 0 in Λ. Then

$$\sup_{\eta,\xi\in\Omega|} \big|P_\Lambda(\sigma_0|\eta) - P_\Lambda(\sigma_0|\xi)\big| \leqslant Q\left(\left(\frac{p_\ominus}{p_\oplus + p_\ominus}\right)^{j_\Lambda}\right)$$

To finish the proof, it is sufficient to show that $\lim_{\Lambda\uparrow S} j_\Lambda(\underline{e}) = \infty$ Q-almost surely.

Remark first that if $p_M = 0$, the number of sites connected to 0 is finite Q-almost surely so there exists $\Lambda \in \mathcal{T}_{b,0}$ such that 0 is not connected to $\underline{\partial}\Lambda$ Q-almost surely. Suppose now that $p_M > 0$. Let ϱ be an infinite path going from 0 to $-\infty$. We look at configurations of the flow containing ϱ. Along the path, there is an infinite number of sites with M as information

Q-almost surely. Then on the other branch of the M (Figure 10.3), the probability of having a $\odot$ is positive and is independent. We then can assure that there is an infinite number of such a local configuration Q-almost surely. ✠

Figure 10.3 – local configurations searched in the global configuration. The path ϱ goes through x. It uses the W branch of a M information in the left drawing and the N branch on the right one. The site y is $\odot$.

The next step is to generalize the preceding criterion to PCA that can be mapped into a PCA with positive majority.

Theorem 10.16.
Let P be a PCA in $\mathbb{Z}$. If

$$\begin{cases} |p_M + 2p_N| & \geqslant & |p_M| \\ |p_M + 2p_W| & \geqslant & |p_M| \end{cases} \tag{10.11}$$

then, P can be transformed into a PCA with positive majority. For those models, the criterion in Theorem 10.15 becomes

$$\theta\Big(\operatorname{sgn}(p_M)\operatorname{sgn}(p_M p_N)\operatorname{sgn}(p_M p_W) \Big| -\operatorname{sgn}(p_M p_W),\ -\operatorname{sgn}(p_M p_N)\Big) > 0 \tag{10.12}$$

where sgn *is the signum function and* $\operatorname{sgn}(0) = 1$.

Remark 10.17.
Equation (10.12) is redundant but it takes into account the case where at least one parameter is equal to zero.

Proof.
For PCA in $\mathbb{Z}$, the decomposition (10.8) is

$$\theta(\sigma|\eta) = p_M \delta_{\text{maj}_M(\eta)}(\sigma) + p_N \delta_{\eta_N}(\sigma) + p_W \delta_{\eta_W}(\sigma) + p_\oplus \delta_\oplus(\sigma) + p_\ominus \delta_\ominus(\sigma)$$

There are only three ways to modify the model without changing the equations: changing δ_{η_N} into $\delta_{-\eta_N}$, δ_{η_W} into $\delta_{-\eta_W}$ or linking the virtual neighbor V to $\ominus$ instead of $\oplus$.

Since $\delta_{\text{maj}_M} = \delta_N\delta_W + \delta_N\delta_\oplus + \delta_W\delta_\oplus - 2\delta_N\delta_W\delta_\oplus$, changing one of the δ will change the value of all p_π.

Remark that $p_\oplus = \theta(\oplus|\ominus,\ominus) \geqslant 0$ and $p_\ominus = \theta(\ominus|\oplus,\oplus) \geqslant 0$. Those coefficients are then always nonnegative. We then only look at the others.

Let us introduce the three functions of the transformations : $\varphi_M, \varphi_N, \varphi_W : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$\begin{aligned}\varphi_M(a,b,c) =& (-a, a+b, a+c)\\ \varphi_N(a,b,c) =& (-a, -b, a+c)\\ \varphi_W(a,b,c) =& (-a, a+b, -c)\end{aligned}$$

Each function codes one of the three transformations above. For example, transforming δ_{η_N} to $\delta_{-\eta_N}$ leads to the following changes in the p_π:

$$(\tilde{p}_M, \tilde{p}_N, \tilde{p}_W) = \varphi_N(p_M, p_N, p_W)$$

where $\tilde{p}_\pi$ are the coefficients in the new decomposition. $p_\oplus$ and $p_\ominus$ also change but they are still nonnegative.

We observe that ${\varphi_M}^2 = {\varphi_N}^2 = {\varphi_W}^2 = \mathrm{Id}$, and that $\varphi_M\varphi_N = \varphi_N\varphi_M = -\varphi_W$, $\varphi_M\varphi_W = \varphi_W\varphi_M = -\varphi_N$, $\varphi_N\varphi_W = \varphi_W\varphi_N = -\varphi_M$.

We can therefore conclude that $G_M = \{\pm\,\mathrm{Id}, \pm\varphi_M, \pm\varphi_N, \pm\varphi_W\}$ is the group of researched transformations: there are only eight ways of changing the decomposition without changing the equations.

Now, the condition on $a, b, c \in \mathbb{R}$ such that there exists $\psi \in G_M$ satisfying $\tilde{a}, \tilde{b}, \tilde{c} \leqslant 0$ where $(\tilde{a}, \tilde{b}, \tilde{c}) = \psi(a,b,c) \geqslant 0$ is

$$\begin{cases} |a+2b| & \geqslant |a| \\ |a+2c| & \geqslant |a| \end{cases}$$

Replacing (a,b,c) by (p_M, p_N, p_W) gives Equation (10.11).

Now, the criterion (10.12) comes from the transformation of $p_\oplus$ under each $\psi \in G_M$. ✠

The preceding theorem can be summarized into Figure 10.4.

To conclude this chapter, we give a simulation of a model with positive majority. It has been done with the program given in Appendix.

It represents a coupling between the same model starting from all $\oplus$ and all $\ominus$. The red color represents the difference of information that is $\ominus$ for one process and $\oplus$ for the other one. Since a model with positive majority is FKG, the process starting by $\ominus$ is always under the other: the situation $\oplus/\ominus$ can not happen.

We clearly see that the $\odot$ sites connected to $\oplus$ (in blue) completely kill the information given by the frontier.

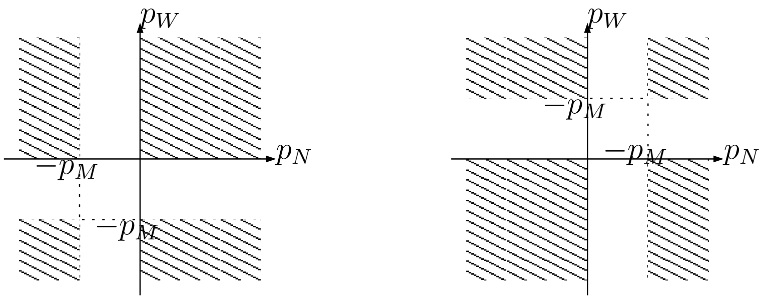


Figure 10.4 – Conditions on a PCA to apply criterion (10.11). In the striped zone, the PCA can be transformed into a PCA with positive majority. left drawing: $p_M \geqslant 0$, right drawing: $p_M \leqslant 0$.

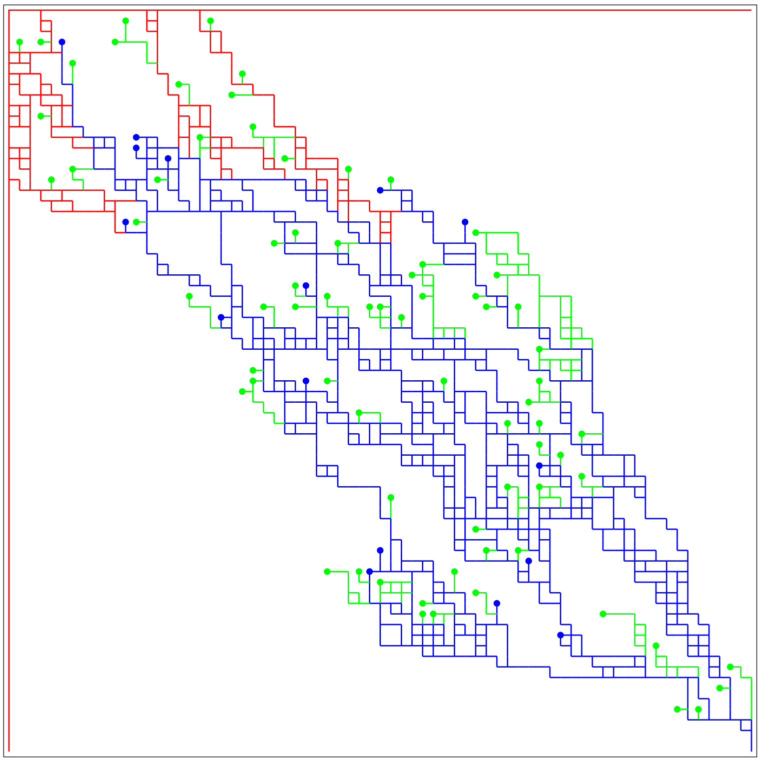


Figure 10.5 – Simulation of a majority flow of information. parameters: $p_M = 0.34$, $p_N = p_W = 0.3$, $p_\oplus = 0.01$ and $p_\ominus = 0.05$. $\oplus$ are in blue, $\ominus$ in green and the information coming from the frontier in red.

Chapter 11

The product decomposition

In this chapter, we present another decomposition. This one is based on a product rather than on a majority. We obtain another criterion from it. Proofs are very similar to their equivalent one in the preceding chapter. Unfortunately, we have not found any way to unify them in a more general context.

Since we have not found any natural decomposition in $\mathbb{Z}^n$, we restrict our work to PCA in $\mathbb{Z}$. The space-time configurations are then in $\mathbb{Z}^2$. We keep notations N and W from the previous chapter.

Proposition 11.1.
Let P be a PCA in $\mathbb{Z}$. There exist real numbers q_{NW}, q_N, q_W, $q_\oplus$, $q_\ominus$ such that for all $\sigma \in E$ and $\eta \in \Omega$,

$$\theta(\sigma|\eta) = q_{NW}\delta_{\eta_N\eta_W}(\sigma) + q_N\delta_{\eta_N}(\sigma) + q_W\delta_{\eta_W}(\sigma) + q_\oplus\delta_\oplus(\sigma) + q_\ominus\delta_\ominus(\sigma) \tag{11.1}$$

Proof.
It is sufficient to remark that $\delta_{\eta_N\eta_W} = -2\delta_{\mathrm{maj}_M} + \delta_N + \delta_W + \delta_\oplus$ ✠

We have replaced the majority function by a product function. There is no more virtual neighbor in this chapter. The disymmetry is caused by the product rather than by the introduction of a virtual neighbour. Let us adapt the definitions of the preceding chapter:

As in the preceding chapter, we denote $q_\odot := q_\oplus + q_\ominus$.

Definition 11.2.
We say that a PCA is ***with positive product*** if for all $\pi \in \{NW, N, W, \odot\}$, $q_\pi \geqslant 0$ in the decomposition (11.1).

Definition 11.3.
Let $\Lambda \in \mathcal{T}_b$. An element $\underline{e}$ of the set $\{NW,\ N,\ W,\ \odot\}^\Lambda$ is called a ***product flow of information*** in Λ. We say that $\omega \in \Omega_\Lambda$ is ***compatible with*** $\eta \in \Omega$ and $\underline{e}$ if

$$\forall x \in \Lambda, \qquad \omega_x^{\eta,\underline{e}} = \begin{cases} \omega_{Nx}\omega_{Wx} & \text{if } e_x = NW, \\ \omega_{Nx} & \text{if } e_x = N, \\ \omega_{Wx} & \text{if } e_x = W, \end{cases}$$

In the last formula, if $Nx \notin \underline{\partial}\Lambda$ (resp. $Wx \notin \underline{\partial}\Lambda$), ω_{Nx} must be replaced by η_{Nx} (resp. ω_{Wx} by η_{Wx}). For such an ω, we use the notation $\omega \overset{\pi}{\sim} (\eta, \underline{e})$.

Proposition 11.4.
Let P be a PCA on $\mathbb{Z}$. For any $\Lambda \in \mathcal{T}_b$ and any local function f such that $\mathrm{Supp}(f) \subset \Lambda$*,*

$$P_\Lambda(f|\eta) = \sum_{\underline{e} \in \{NW,N,W,\odot\}^\Lambda} \left(\prod_{\pi \in \{NW,N,W,\odot\}} {q_\pi}^{\#_\pi(\underline{e})} \right) \left(\frac{1}{q_\oplus + q_\ominus} \right)^{\#_\odot(\underline{e})} \sum_{\substack{\omega \in \Omega_\Lambda \\ \omega \overset{\pi}{\sim} (\eta,\underline{e})}} {q_\oplus}^{\#_\oplus(\omega)} {q_\ominus}^{\#_\ominus(\omega)} f(\omega) \tag{11.2}$$

The proof is omitted. It is similar to the proof of Theorem 10.10.

We still denote by Q be the product measure on $\{NW, N, W, \odot\}^{\mathbb{Z}^2}$ such that $Q(NW) := q_{NW}$, $Q(N) := q_N$, $Q(W) := q_W$ and $Q(\odot) := q_\odot$ and

$$R_\Lambda^{\eta,\underline{e}}(f) := \left(\frac{1}{q_\oplus + q_\ominus} \right)^{\#_\odot(\underline{e})} \sum_{\substack{\omega \in \Omega_\Lambda \\ \omega \overset{\pi}{\sim} (\eta,\underline{e})}} {q_\oplus}^{\#_\oplus(\omega)} {q_\ominus}^{\#_\ominus(\omega)} f(\omega)$$

so that Equation (11.2) can be rewritten

$$P_\Lambda(f|\eta) = Q\big(R_\Lambda^{\eta,\underline{e}}(f)\big)$$

When P is with positive product, Q and $R_\Lambda^{\eta,\underline{e}}$ are probability measures.

Explicit values of q_π can be obtained by solving the linear system:

$$\begin{cases} q_{NW} = & \Big(\quad \theta(\oplus|\oplus,\oplus) - \theta(\oplus|\oplus,\ominus) - \theta(\oplus|\ominus,\oplus) + \theta(\oplus|\ominus,\ominus)\Big)/2 \\ q_N = & \Big(\quad \theta(\oplus|\oplus,\oplus) + \theta(\oplus|\oplus,\ominus) - \theta(\oplus|\ominus,\oplus) - \theta(\oplus|\ominus,\ominus)\Big)/2 \\ q_W = & \Big(\quad \theta(\oplus|\oplus,\oplus) - \theta(\oplus|\oplus,\ominus) + \theta(\oplus|\ominus,\oplus) - \theta(\oplus|\ominus,\ominus)\Big)/2 \\ q_\oplus = & \Big(-\theta(\oplus|\oplus,\oplus) + \theta(\oplus|\oplus,\ominus) + \theta(\oplus|\ominus,\oplus) + \theta(\oplus|\ominus,\ominus)\Big)/2 \\ q_\ominus = & \theta(\ominus|\oplus,\oplus) \end{cases} \tag{11.3}$$

The geometry of the cluster of information is a bit different from the previous decomposition. The difference comes from NW. In a product flow of information, a site with NW means that its color is the product of its two neighbours.

We have an analogue geometrical view: to choose a configuration, we begin to choose a flow of information by using Q, then choose the color of sites with $\odot$. The same figure as Figure 10.1 holds. The difference is that in the case where a site is linked with two neighbours (NW information), its color is the product rather than a majority.

As in the previous decomposition, we have to understand the geometry of clusters to be able to use it. It is a bit more complicated because $\oplus$ now does not pass through the flow of information.

Theorem 11.5.

Let P be a PCA on $\mathbb{Z}$. Assume that P is with positive product.

If $q_\ominus > 0$, there exists only one consistent measure.

Proof.
We are going to imitate the proof of Theorem 10.15. The main problem is that P is no more FKG. We then have to prove that for all local functions f, $\lim\limits_{\Lambda\uparrow\mathbb{Z}^2} P_\Lambda(f|\eta)$ does not depend on η any more. Since the decomposition uses a product, it is natural to prove such a thing for functions of the type $f(\omega) = \prod\limits_{k=0}^{n} \omega_{x_k}$. Proving it for these functions is sufficient. Moreover, by translation invariance we can only choose points x_k in the past of 0.

Let $n \in \mathbb{N}$ and $x_0, \cdots, x_n \in 0_-$. We denote f the function

$$f(\omega) := \prod_{k=0}^{n} \omega_{x_k}$$

Let $\underline{e}$ be a product flow of information chosen according to Q and let $\eta \in \Omega$.

For all $y \in \Lambda$, we denote by $c_{y,k}$ the number of paths going from x_k to y by following the flow. That is, for a site z on a path, if $e_z = N$ (resp. $e_z = W$) the path can only continue from z to Nz (resp. Wz). If $e_z = NW$, it can go to either Nz or Wz and if $e_z = \odot$, it can not go further. Note that if x_k is not connected to y in $\underline{e}$, then $c_{y,k} = 0$. At last, denote $c_y := \sum\limits_{k=0}^{n} c_{y,k}$.

We say that a site y is a ***leaf*** if $e_y = \odot$.

The fact that NW means a product of colors implies that for all $0 \leqslant k \leqslant n$ and ω compatible with η and $\underline{e}$,

$$\omega_{x_k} = \left(\prod_{y \text{ leaf in } \Lambda} {\omega_y}^{c_{y,k}} \right) \left(\prod_{z \in \underline{\partial}\Lambda} {\eta_z}^{c_{z,k}} \right)$$

Therefore,

$$f(\omega) = \left(\prod_{y \text{ leaf in } \Lambda} {\omega_y}^{c_y} \right) \left(\prod_{z \in \underline{\partial}\Lambda} {\eta_z}^{c_z} \right) \tag{11.4}$$

This equation shows that there are two sorts of leaves: ***active leaves*** such that $c_y \in 2\mathbb{Z}+1$ and ***passive leaves*** with $c_y \in 2\mathbb{Z}$. Passive leaves have no effect on $f(\omega)$. This definition is extended to every site: a passive site x satisfies $c_x \in 2\mathbb{Z}$ and an active site x, $c_x \in 2\mathbb{Z}+1$.

Now, there are two possible cases: the number of active sites is finite or not. If it is finite, there exists $\Lambda \in \mathfrak{T}_b$ such that all sites $y \in \Lambda_-$ are passive. In particular, for all $\Delta \supset \Lambda$, for all $z \in \Delta_-$, $c_z \in 2\mathbb{Z}$ so $f(\omega)$ do not depend on η (see Equation (11.4)). So the dominated-convergence theorem gives that

$$\lim_{\Lambda\uparrow\mathbb{Z}^2} Q\big(R_\Lambda^{\eta,\underline{e}}(f)\,\mathbb{1}_{\text{finite number of active sites}}(\underline{e})\big)$$

converges to something independent of η.

On the other hand, if the number of active sites is infinite, then Lemma 11.6 (given after this proof) gives that the number of active leaves are also infinite Q-almost surely. Let $j_\Lambda(\underline{e})$ be the number of active leaves in Λ. Denote $g_\Lambda(\eta,\underline{e})$ the second product of rhs in Equation (11.4). g_Λ is the only term in (11.4) that depends on the initial condition η. We can write

$$\begin{aligned} R_\Lambda^{\eta,\underline{e}}\Big(f(\sigma)=g_\Lambda(\eta,\underline{e})\Big) &= R_\Lambda^{\eta,\underline{e}}\big(\text{even number of active leaves are } \ominus\,\big)\\ &= \sum_{0\leqslant l \text{ even}\leqslant j_\Lambda(\underline{e})} \binom{l}{j_\Lambda(\underline{e})} \frac{q_\ominus{}^{l}\, q_\oplus{}^{j_\Lambda(\underline{e})-l}}{(q_\oplus+q_\ominus)^{j_\Lambda(\underline{e})}}\\ &= \frac{1}{2}\left(1+\left(\frac{q_\oplus-q_\ominus}{q_\oplus+q_\ominus}\right)^{j_\Lambda(\underline{e})}\right) \end{aligned}$$

Finally,

$$R_\Lambda^{\eta,\underline{e}}(f) = \left(\frac{q_\oplus-q_\ominus}{q_\oplus+q_\ominus}\right)^{j_\Lambda(\underline{e})} \times g_\Lambda(\eta,\underline{e})$$

and this term goes to zero as Λ goes to $\mathbb{Z}^2$. The theorem is then proved. ✠

Lemma 11.6.
Under the assumptions of Theorem 11.5, if the number of active sites connected to $x_0,\cdots,x_n$ is infinite, then the number of active leaves connected to $x_0,\cdots,x_n$ is infinite.

Proof.
If $q_{NW}=0$, then the number of sites connected to each $x_0,\cdots,n_n$ is finite Q-almost surely. Suppose that $q_{NW}>0$. Remark that if the number of active sites is infinite, there exists an infinite path starting from one of the points $x_0,...,x_n$ that follows the flow and such that each point of the path is active.

Let ϱ be an infinite path starting at one of the sites $x_0,\cdots,x_n$. We look at all configurations that contain ϱ as an active path, that is every site of ϱ must be active. The probability of the local configurations given in Figure 11.1 is positive. Moreover, these configurations are independent if they have no commun site. That is why Q-almost surely, there are infinitely many such local configurations along ϱ. But each leaf in them is active. ✠

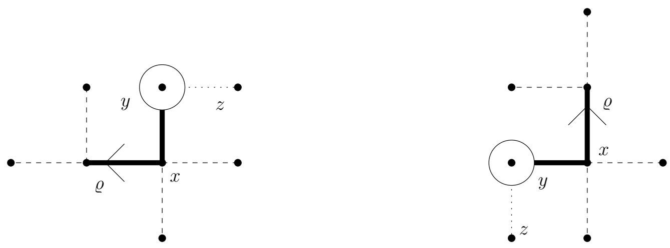


Figure 11.1 – local configurations searched in the global configuration. The path ϱ goes through x. It uses the W branch of a NW information in the left drawing and the N branch on the right one. The site y is $\odot$. We limit the choice in the site z to N and $\odot$ in the left figure and to W and $\odot$ in the right one. This ensures that y is a leaf only connected to ϱ by x. Thus, if x is active, y is active too.

As in the preceding decomposition, we then search a criterion that takes into account transformations of the PCA with positive product.

Theorem 11.7.
Let P be a PCA in $\mathbb{Z}$. If

$$\begin{cases} q_{NW}\, q_N\, q_W \geqslant 0 \\ q_{NW}{}^- + q_N{}^- + q_W{}^- + q_\oplus \geqslant 0 \\ q_{NW}{}^- + q_N{}^- + q_W{}^- + q_\ominus \geqslant 0 \end{cases} \tag{11.5}$$

where $x^- := \min(0, x)$, then the model can be transformed into a PCA with positive product. The criterion of uniqueness is

$$\theta\big(\ominus \,|\, \mathrm{sgn}(q_N), \mathrm{sgn}(q_W)\big) > 0 \tag{11.6}$$

Proof.
There are only two main transformations: δ_N to δ_{-N} and δ_W to δ_{-W}. The associated functions are

$$\varphi_N(a,b,c,d,e) = (-a,-b,c,a+b+d,a+b+e)$$
$$\varphi_W(a,b,c,d,e) = (-a,b,-c,a+c+d,a+c+e)$$

Flipping δ_N into δ_{-N} will produce the new parameters:

$$(\widetilde{q}_{NW}, \widetilde{q}_N, \widetilde{q}_W, \widetilde{q}_\oplus, \widetilde{q}_\ominus) = \varphi_N(q_{NW}, q_N, q_W, q_\oplus, q_\ominus)$$

$G = \{Id, \varphi_N, \varphi_W, \varphi_N\varphi_W\}$ is the group of available transformations. Then looking at these transformations, we deduce both the criterion to be transformable into a PCA with positive product and the uniqueness criterion. ✠

As in the preceding chapter, we give a simulation of a product flow of information. This time, the process is not FKG so the inequality on the color of the sites is no more valid. We have to introduce another color for the $\ominus/\oplus$ situation.

Nevertheless, we can clearly see the information killed by an even number of paths.

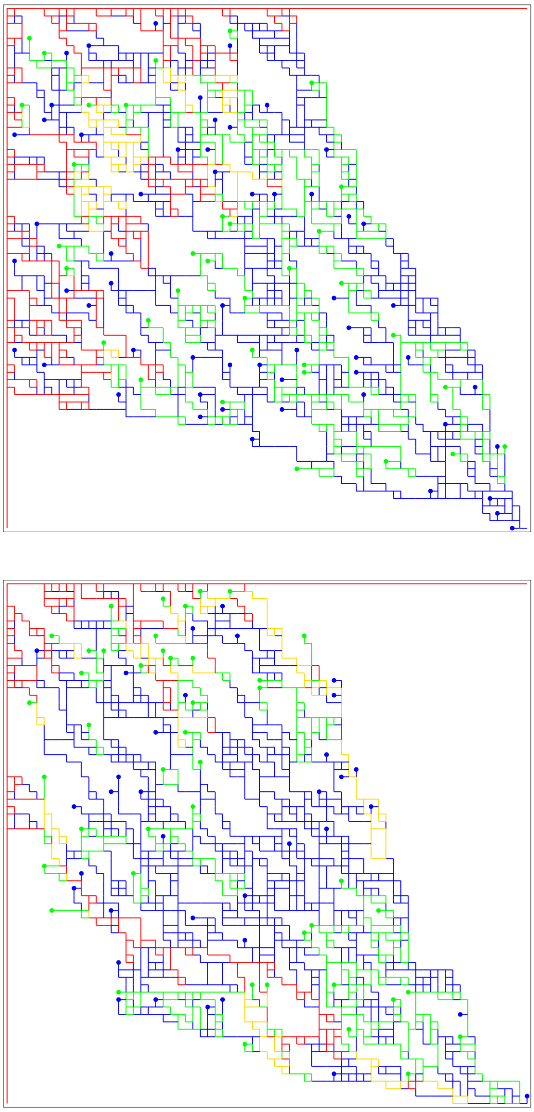


Figure 11.2 – Simulations of a product flow of information. parameters: $q_{NW} = 0.36$, $q_N = q_W = 0.3$, $q_\oplus = q_\ominus = 0.02$. $\oplus$ are in blue, $\ominus$ in green and the information coming from the frontier in red.

Chapter 12

Applications and examples

This chapter establishes the link between Part I and II. We compare the two criteria of the first part (Dobrushin and disagreement percolation) with the two presented just above (majority and product decomposition). We then give examples of application of the last two.

12.1 Comparison with other uniqueness criteria

We first compare our two new criteria with the two other uniqueness criteria: Dobrushin criterion and disagreement percolation criterion. We will see that they are incomparable. However, the geometrical approach gives information on how Dobrushin and percolation work.

First, we recall the definition of the Dobrushin criterion in this context. For $y \in \underline{\partial}0$, denote

$$\alpha_y := \sup_{\omega \in \Omega} \Big| \theta(\oplus|\eta, \eta_y = \oplus) - \theta(\oplus|\eta, \eta_y = \ominus) \Big| \tag{12.1}$$

The Dobrushin criterion says that if

$$\sum_{y \in \underline{\partial}0} \alpha_y < 1 \tag{12.2}$$

then there exists only one consistent measure. It is simpler than in Theorem 8.5 because the PCA is supposed homogeneous and it has only two colors.

Proposition 12.1.
Let P be a PCA. Let $\#\pi$ be the number of sites in $\pi \in \mathcal{C}_v$. If the majority decomposition satisfies

$$\sum_{\pi \in \mathcal{C}_v} (\#\pi)|p_\pi| < 1 \tag{12.3}$$

then the Dobrushin criterion holds.

Proof.
Let $y \in \underline{\partial}0$.

$$\begin{aligned}\alpha_y &= \sup_{\eta\in\Omega}\left|\sum_{\pi\in\mathcal{C}_V} p_\pi\left[\delta_{\mathrm{maj}_\pi(\eta,\eta_y=\oplus)}(\oplus) - \delta_{\mathrm{maj}_\pi(\eta,\eta_y=\ominus)}(\oplus)\right]\right| \\ &\leqslant \sum_{\pi\in\mathcal{C}_V} |p_\pi| \sup_{\eta\in\Omega}\left|\delta_{\mathrm{maj}_\pi(\eta,\eta_y=\oplus)}(\oplus) - \delta_{\mathrm{maj}_\pi(\eta,\eta_y=\ominus)}(\oplus)\right| \\ &\leqslant \sum_{\pi\in\mathcal{C}_V,\ \pi\ni y} |p_\pi|\end{aligned}$$

So $\sum_{y\in\underline{\partial}0}\alpha_y \leqslant \sum_{\pi\in\mathcal{C}_v}(\#\pi)|p_\pi|$. ✠

Proposition 12.2.
Let P be a PCA in $\mathbb{Z}$. If the product decomposition satisfies

$$2|q_{NW}| + |q_N| + |q_W| < 1 \tag{12.4}$$

then the Dobrushin criterion holds.

The proof is exactly the same as for Proposition 12.1.

Now, let us look at the disagreement percolation criterion. Recall that p_c^+ is the critical parameter for oriented percolation. In this context, if

$$\sup_{\eta,\xi\in\Omega}|\theta(\oplus|\eta) - \theta(\oplus|\xi)| < p_c^+$$

then, there is only one consistent measure with the PCA.

As for the Dobrushin criterion, it is a simplified version of Theorem 8.16 due to the homogeneity of the PCA considered here and the fact that it has only two possible colors.

Proposition 12.3.
Let P be a PCA. If the majority decomposition satisfies

$$\sum_{\pi\in\mathcal{C}_v}|p_\pi| < p_c^+ \tag{12.5}$$

then the disagreement percolation criterion holds.

Proof.

$$\begin{aligned}\sup_{\eta,\xi\in\Omega}\left|\theta(\oplus|\eta) - \theta(\oplus|\xi)\right| &= \sup_{\eta,\xi\in\Omega}\left|\sum_{\pi\in\mathcal{C}_V} p_\pi\left[\delta_{\mathrm{maj}_\pi(\eta)}(\oplus) - \delta_{\mathrm{maj}_\pi(\xi)}(\oplus)\right]\right| \\ &\leqslant \sum_{\pi\in\mathcal{C}_V}|p_\pi| \sup_{\eta,\xi\in\Omega}\left|\delta_{\mathrm{maj}_\pi(\eta)}(\oplus) - \delta_{\mathrm{maj}_\pi(\xi)}(\oplus)\right| \\ &\leqslant \sum_{\pi\in\mathcal{C}_V}|p_\pi|\end{aligned}$$ ✠

Proposition 12.4.
Let P be a PCA in $\mathbb{Z}$. if the product decomposition satisfies

$$|q_{NW}| + |q_N| + |q_W| < p_c^+ \tag{12.6}$$

then the disagreement percolation criterion holds.

This proof is similar to the one of Proposition 12.3.

Remark 12.5.
Those formula show the difference between the two criteria. To have uniqueness, the Dobrushin criterion says that the more a majority sample has elements, the smaller must be its associated coefficient. On the other hand, the disagreement percolation criterion says that the total mass of the majority samples must not be too heavy.

The big voter samples control the way the flow spreads to the past. Indeed, if there is no possible bifurcation, the flow dies out almost surely. The bigger the flow is, the more there are bifurcations, and this contributes to push the flow very far from its root. The way Dobrushin and disagreement percolation criteria work is then clear: in the flow, there must exist few bifurcations to ensure uniqueness.

12.2 Examples

We will apply our geometrical approach to the two main examples of Part I.

We first recall the definition of the POMM-Ising model. It is an adaptation of the Ising model in the case of PCA. As in the original model, it has two parameters: $\beta > 0$ and $h \in \mathbb{R}$. It is defined on $\mathbb{Z}$ by

$$\forall \sigma \in E,\ \forall \eta \in \Omega, \quad \theta(\sigma, \eta) = \frac{1}{Z_\eta} \exp\Big(\beta\sigma\big(\eta_N + \eta_W + h\big)\Big) \tag{12.7}$$

where Z_η is the normalizing coefficient:

$$Z_\eta = \exp\left(-\beta(\eta_N + \eta_W + h)\right) + \exp\left(\beta(\eta_N + \eta_W + h)\right)$$

It clearly has the same type of interaction as the original model.

The results of the Dobrushin and disagreement percolation criteria are summarized in Figure 12.1.

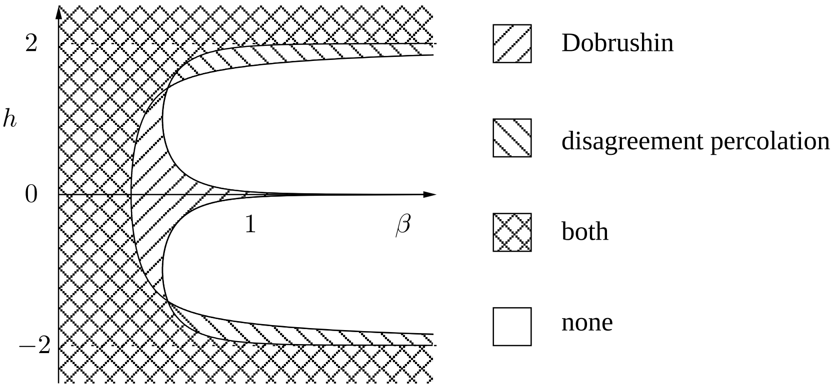


Figure 12.1 – phase diagram of the POMM-Ising PCA. The legend indicates the region where the criteria can be applied.

Now, the majority decomposition of the POMM-Ising model is

$$\begin{cases} p_M & = 4\sinh(\beta h)\cosh^2(\beta)\dfrac{\cosh^2(\beta)-1}{\cosh(\beta h)\cosh(\beta(h+2))\cosh(\beta(h-2))} \\ p_N = p_W & = \dfrac{1}{2}\dfrac{\sinh(2\beta)}{\cosh(\beta(h+2))\cosh(\beta h)} \\ p_\oplus & = \dfrac{1}{2}\dfrac{\exp(\beta(h-2))}{\cosh(\beta(h-2))} \\ p_\ominus & = \dfrac{1}{2}\dfrac{\exp(-\beta(h+2))}{\cosh(\beta(h+2))} \end{cases}$$

Note that p_N, p_W, $p_\oplus$ and $p_\ominus$ are positive numbers.

If $h \geqslant 0$, all parameters are nonnegative. The POMM-Ising model is then with positive majority so Theorem 10.15 ensures uniqueness of the consistant measure.

For $h < 0$, $p_M + 2p_N = \dfrac{1}{2}\dfrac{\sinh(4\beta)}{\cosh(\beta(h+2))\cosh(\beta(h-2))} > 0$ so

$$|p_M + 2p_N| - |p_M| = 2p_M + 2p_N = \frac{\sinh(2\beta)}{\cosh(\beta h)\cosh(\beta(h-2))} > 0$$

Theorem 10.16 leads to the conclusion that for all $\beta > 0$ and $h \in \mathbb{R}$, there is only one $\mathfrak{I}$-POC. The phase diagram of the POMM-Ising is then completely known.

Let us now look at our second main example: the Stavskaya's model. We first recall the model.

Let $p \in [0, 1]$ be a parameter. The model is defined by

$$\forall \eta \in \Omega, \qquad \theta(\oplus|\eta) = p\,\mathbb{1}_{\eta_N + \eta_W \geqslant 0}$$

We have changed the notations of Subsection 3.3.2. Indeed, the framework of this part deals with a color space equals to $\{\oplus, \ominus\}$ and the original model has $\{0, 1\}$ as color space. However, this defines the same model. With the notations of this section, $\delta_{\underline{\ominus}}$ is a consistent measure.

The majority point of view leads to the following proposition that completes the phase diagram of the Stavskaya's model.

Proposition 12.6.
For $p > p_c^+$, the Stavskaya's model exhibits a phase transition.

Proof. The model can be rewritten using the majority decomposition as

$$\theta(\sigma|\eta) = p\delta_{\mathrm{maj}_M(\eta)}(\sigma) + (1-p)\delta_{\ominus}(\sigma)$$

Remark that a $\oplus$ can not appear; it is transmitted by an M information. Now, look at the geometry of the flow. Since $p > p_c^+$, the probability of finding an infinite oriented path of M from 0 is positive. Let $\Delta \in \mathcal{T}_{b,0}$. Using Equation (10.10),

$$\begin{aligned} P_\Delta(\sigma_0 = \oplus|\eta \equiv \oplus) &= Q(\text{there exists a path of } M \text{ from 0 to } \underline{\partial}\Delta) \\ &\geqslant Q(\text{there exists an infinite path of } M \text{ from 0 to } -\infty) \\ &> 0 \end{aligned}$$

This shows that $P_\Delta(\sigma_0 = \oplus|\eta \equiv \oplus)$ do not goes to 0 as Δ increases to $\mathbb{Z}^2$. So $\delta_{\underline{\ominus}}$ can not be the only consistent measure of the model. ✠

What happens at the critical point depends on the following question: Is there percolation a p_c^+? If the answer is yes, the preceding proposition shows that there is a phase transition also at p_c^+. Otherwise, the criterion can be extended to show uniqueness.

Unfortunately, the behaviour of the oriented percolation is not known at the critical parameter.

12.3 Non optimality

The two criteria given in this part are not optimal: one of them can be valid but not the other. Moreover, those two criteria do not cover all the models at all. Let us see those two points in more details.

The first PCA is defined for $\epsilon \in [0, 1/2]$ by

$$\theta(\oplus|\eta_N, \eta_W) = \begin{cases} 1/2 & \text{if } \eta_N = \eta_W \\ 1/2 + \epsilon & \text{otherwise} \end{cases}$$

This model does not satisfy the majority decomposition condition (10.11) but it satisfies the product decomposition one (11.5). There is uniqueness of consistent measure, but only the product decomposition can by applied.

The second example is for $\epsilon \in [0, 1/6]$,

$$\theta(\oplus|\eta_N, \eta_W) = \begin{cases} 1/2 - 3\epsilon & \text{if } \eta_N = \eta_W = \ominus \\ 1/2 + \epsilon & \text{otherwise} \end{cases}$$

This model is exactly the contrary of the preceding one : it satisfies (10.11) but not (11.5).

Finally, the most interresting one is for $\epsilon \in [0, 1/4]$,

$$\theta(\oplus|\eta_N, \eta_W) = \begin{cases} 1/2 & \text{if } \eta_N = \eta_W = \oplus \\ 1/2 - 2\epsilon & \text{if } \eta_N = \eta_W = \ominus \\ 1/2 + \epsilon & \text{otherwise} \end{cases}$$

It satisfies neither the majority condition (10.11) nor the product one (11.5).

However, for $\epsilon < 1/10$, the Dobrushin criterion can be applied and for $\epsilon < p_c^+/6$ the disagreement percolation criterion gives uniqueness. All this indicates that the two criteria introduced here are not comparable to themselves and to already known uniqueness criterion.

It seems unlikely that those three models have a phase transition: they seem to be too close to the independent Bernoulli model (even for large ϵ)..

12.4 Hypercube of probability

The set of parameters of PCA in $\mathbb{Z}$ with two neighbours and two colors can be viewed as the unit hypercube of $\mathbb{R}^4$. Indeed, to determine entirely the model, it is sufficient to know $\theta(\oplus|\oplus, \oplus)$, $\theta(\oplus|\oplus, \ominus)$, $\theta(\oplus|\ominus, \oplus)$ and $\theta(\oplus|\ominus, \ominus)$ and those four probabilities are independent.

For any point (a, b, c, d) of the unit hypercube in $\mathbb{R}^4$, we can associate a unique PCA by $\theta(\oplus|\oplus, \oplus) = a$, $\theta(\oplus|\oplus, \ominus) = b$, $\theta(\oplus|\ominus, \oplus) = c$, $\theta(\oplus|\ominus, \ominus) = d$.

Now note that all the extremal points of the hypercube have already been introduced here. For example $(0, 0, 0, 0)$ is $\delta_\ominus$, $(1, 1, 1, 0)$ is δ_{maj_M} and $(1, 0, 0, 1)$ is δ_{NW} (See Figure 12.2).

The geometrical Caratheodory theorem indicates that only five extremal points are sufficient to identify a point in the hypercube. With this point of view, the two decompositions (10.8) and (11.1) are trivial. The two criteria have then a geometrical meaning: inside the convex envelop of $\delta_\oplus$, $\delta_\ominus$, δ_N, δ_W and δ_{maj_M}, there is no phase transition and it is the same with the seven other convex sets that are the transformation of the first one by the group of transformation associated with the majority decomposition.

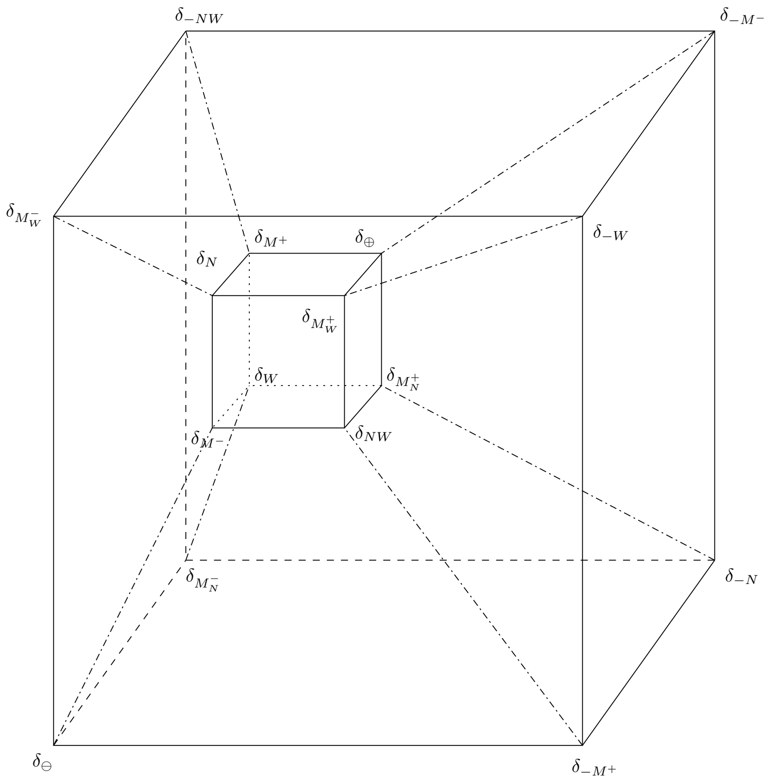


Figure 12.2 – Projection of the hypercube of PCA in $\mathbb{Z}$. We have represented the extremal points. We have shorten notations. For example, $\delta_{M_N^-}$ means the majority function with $-\eta_N$, η_W and $\ominus$.

The same holds for the convex hull of $\delta_\oplus$, $\delta_\ominus$, δ_N, δ_W and δ_{NW} and the three other convexes associated with the group of transformation of the product decomposition.

For here, we have a conjecture: inside the hypercube, there is no phase transition. We have seen that our criteria have not proved it completely.

The Stavskaya's model shows that the conjecture can not be extended to the whole hypercube: $\mathcal{S}$ is on the frontier of the hypercube and it exhibits a phase transition.

Here is one way of reflection. For a point in the interior of the hypercube, that is for a non degenerate PCA, we can suppose that $\delta_\oplus$ and $\delta_\ominus$ are in the decomposition. We then have to choose another three extremal points.

The question is: what to do with this decomposition? It is possible that it contains δ_{maj_M} and δ_{NW}. How does this system work? What is the comportment of the flow of information? These questions are still open.

Appendix

Appendix A

The NEC Toom's model

A.1 Description of the model

We introduce here the NEC Toom's model. This part is devoted to the proof of its phase transition. We essentially will reproduce the proof from [LMS90b]. This was not the original proof. The first one comes from A. Toom in [Too80]. It can be applied to models a bit more general but it is much more complicated. This is not the only known proof. In [BG91], Bramson and Gray used infinite particle systems in continuous time to prove it.

Our point of view is the one of [LMS90b]. The global settings are similar to Part II. The space-time set of this PCA is $\mathbb{Z}^3$. The ***time of a site*** $x = (x_1, x_2, x_3)$ is defined by $t(x) = x_1 + x_2 + x_3$. Its past nearest neighbours are $(x_1 - 1, x_2, x_3)$, $(x_1, x_2 - 1, x_3)$ and $(x_1, x_2, x_3 - 1)$.

The color space of the model is $E = \{-1, +1\}$: there are only two possible colors for a site. Let $\epsilon \in [0, 1]$ be a parameter.

We define the NEC Toom's model at the site 0 by

$$P_0(\sigma|\eta) = \begin{cases} 1 - \epsilon & \text{if } \sigma = \operatorname{sgn}\left(\sum_{y \in \underline{\partial} 0} \eta_y\right) \\ \epsilon & \text{otherwise} \end{cases} \tag{A.1}$$

The model is translation invariant. It is therefore sufficient to define it at the site 0. The translation invariance expands the definition to all sites and we can construct the propability measure on the whole space according to the chapter 9.

The name NEC comes from the traditional point of view used in the literature on PCA. It means "North-East-Center". This name comes from another description of the model. Let us describe it briefly. We will use the notations of Chapter 5. Let $U = \mathbb{Z}^2$. We define the neighborhood of $x \in U$ as $V_x = \{x,\ x+(1,0),\ x+(0,1)\}$. Thus, with the usual mathematical conventions V_x contains x (Center), its northern neighbour $(x+(0,1))$ and its eastern one $(x+(1,0))$. The probabilistic model is defined with the same equation (A.1).

Our definition (the geometry) has the advantage of being spatially symmetric.

The theorem that will be proved in the next section is the existence of a phase transition for this model:

Theorem A.1.
For ϵ small enough, the NEC Toom's model has at least two invariant measures.

A.2 Proof of phase transition

A.2.1 The Peierls argument

The arguments of the proof are quite simple. Let Δ be the space-time box

$$\Delta := \{(x_1, x_2, x_3) \in \mathbb{Z}^3,\ x_1, x_2, x_3 \leqslant 0,\ x_1 + x_2 + x_3 > -N\}$$

This box is constituted by the points in the past of $(0,0,0)$ up to the level $-N$. We will fix the "all-$\oplus$" condition on the boundary of a box and show that $P_\Delta(\sigma_0 = \ominus | \eta \equiv \oplus) < 1/2$. Since the values $\oplus$ and $\ominus$ can be switched, this will be sufficient to prove phase transition.

The proof is based on a Peierls argument. What complicates the argument is that the "contours" used here are graphs.

For a configuration $\omega \in \Omega_\Delta$, we define $X = X(\omega) := \{x \in \Delta,\ \omega_x = -1\}$ and $\mathrm{err} = \mathrm{err}(\omega) := \{x \in \Delta,\ \omega_x \neq \mathrm{maj}(\omega_{\underline{\partial} x})\}$. The first set is constituted by sites with value -1 and the second one by sites where the majority rule is violated. We will call the elements of err the ***error sites***. Remark that it is equivalent to give ω or to give $X(\omega)$. We will use this fact in the sequel and work with X instead of ω.

Now to each $\omega \in \Omega_\Delta$, we will associate a graph $G = G(\omega) := (V_G, E_G)$. The set of vertexes V_G of G will be included in X. The family of graphs G has the following two properties:

- the number of graphs G with exactly m edges is less than 48^{2m}:

$$\Big|\{G :\ |E_G| = m\}\Big| \leqslant 48^{2m} \tag{A.2}$$

- for each G, the number of error sites contained in V_G is greater than $|E_G|/4 + 1$:

$$\Big|V_G \cap err\Big| \geqslant \frac{1}{4}|E_G| + 1 \tag{A.3}$$

Note that these properties are uniform in the size N of the box Δ. These two properties are sufficient to prove Theorem A.1. Indeed,

$$\begin{aligned} P_\Delta(\sigma_0 = \ominus | \eta \equiv \oplus) &= \sum_{\substack{\omega \in \Omega_\Delta \\ \omega_0 = \ominus}} P(\omega|\eta \equiv \oplus) \\ &\leqslant \sum_{m \in \mathbb{N}} \sum_{G:\ |E_G| = m} \Big|\omega :\ G(\omega) = G\Big| \sup_{\omega:\ G(\omega)=G} P(\omega|\eta \equiv \oplus) \\ &\leqslant \sum_{m \in \mathbb{N}} 48^{2m} \epsilon^{m/4+1} \end{aligned}$$

So, for small enough $\epsilon > 0$,

$$P_\Delta(\sigma_0 = \ominus | \eta \equiv \oplus) \leqslant \frac{\epsilon}{1 - 48^2 \epsilon^{1/4}}$$

that is, for small enough $\epsilon > 0$, $P_\Delta(\sigma_0 = \ominus | \eta \equiv \oplus) < 1/2$ and the theorem is proved.

In conclusion, we just have to construct $G(\omega)$ for all ω such that $\omega_0 = \ominus$.

A.2.2 The properties of the graphs

In this section we will describe the properties needed to construct G from ω. We will then show that these properties are sufficient to conclude. The proof of the existence of such graphs will be given in the next section.

Each G will satisfy the six following properties:

(P1) G is connected and $0 \in G$.

(P2) There are two sorts of edges in E_G:
- (x, y) is a ***timelike*** if $y \in \underline{\partial} x$,
- (x, y) is a ***spacelike*** if there exists $z \in \mathbb{Z}^3$ such that $x, y \in \underline{\partial} z$.

(P3) Each edge $e = (x, y) \in E_G$ has an orientation and an index i_e. There are two possible orientations: from x to y and from y to x. There are three indexes: $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

(P4) Denote

$$\alpha_{x,e} := \begin{cases} 0 & \text{if } e \text{ does not contain } x, \\ 1 & \text{if } e \text{ goes to } x, \\ -1 & \text{if } e \text{ leaves } x. \end{cases}$$

The ***displacement of an edge*** $e \in E_G$ is defined by

$$\delta_e := \sum_{x \in V_G} \alpha_{x,e} x$$

Timelikes are restricted: They are oriented from a site to its past and their indexes are different than $-\delta_e$. Thus a timelike has only two possible indexes. Between two given sites, there can exist many timelikes but no more than one spacelike.

(P5) In each vertex $x \in V_G$, it is possible to decompose the set of edges containing x
- by pairs: one going to x, the other one leaving x with the same index,
- by triplets: three either going to or leaving x with three different indexes.

(P6) The number s of spacelikes is controlled by the equation

$$s = \left|V_G \cap \mathrm{err}\right| - 1 \tag{A.4}$$

Now that we have defined those six properties, we will end this section by showing that they are sufficient to have Equations (A.2) and (A.3) introduced in the preceding section.

Since G is connected, it is known that there exists a closed path in V_G such that every edge in E_G is crossed exactly twice. We can see this path as a walk beginning in 0. For each step, we have the choice between six timelikes and six spacelikes. P4 imposes the orientation of timelikes and restrict the indexes to two. For each spacelike, there are two possible orientations and three possible indexes. This leads to 48 choices. The number of G such that $E_G = m$ is then less than the number of free walks beginning at 0, with 48 choices for each step and $2m$ steps. Equation (A.2) is then proved.

To prove Equation (A.3), we introduce the matrix

$$J := \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

The property P5 can be rewritten

$$\forall x \in V_G, \qquad \sum_{e \in E_G} \alpha_{x,e}\; i_e\; J = (0,0,0)$$

Moreover, remark that for a timelike, $i_e\; J \cdot \delta_e = -1$ and for a spacelike, $i_e\; J \cdot \delta_e \in \{-3, 0, 3\}$.

Thus, if t denotes the number of timelikes in V_G,

$$\begin{aligned}
0 &= \sum_{x\in V_G}\left(\sum_{e\in E_G}\alpha_{x,e}\ i_e\ J\right)\cdot x \\
&= \sum_{e\in E_G} i_e\ J\cdot\left(\sum_{x\in V_G}\alpha_{x,e}x\right) \\
&= \sum_{e\in E_G} i_e\ J\cdot\delta_e \\
&\leqslant -t+3s \\
&\leqslant -|E_G|+4s
\end{aligned}$$

Now, P6 gives $|V_G\cap \mathrm{err}| = s+1 \geqslant |E_G|/4+1$.

A.2.3 The construction of the graphs

Before beginning the construction, we have to set the notations.

Fix $\omega\in\Omega_\Delta$ such that $\omega_0=\ominus$. For $x\in X$, we denote $\hat{\underline{\partial}}x$ the nearest past of the site x responsible for the color of x:

$$\hat{\underline{\partial}}x := \begin{cases}\underline{\partial}x\cap X & \text{if } x\notin \mathrm{err}\\ \varnothing & \text{if } x\in\mathrm{err}\end{cases}$$

Remark that $\hat{\underline{\partial}}x$ is empty if $x\in$ err and has two or three elements if $x\notin$ err. It can not contain a single point.
For $A\subset X$, $\hat{\underline{\partial}}A := \bigcup_{x\in A}\hat{\underline{\partial}}x$. Then, we define by recurrence $\hat{\underline{\partial}}^0A := A$, $\hat{\underline{\partial}}^{k+1}A := \hat{\underline{\partial}}(\hat{\underline{\partial}}^kA)$.
At last, $Y(A) := \bigcup_{k\in\mathbb{N}}\hat{\underline{\partial}}^kA$.

In fact, we will be only interested into $Y := Y(0)$. The vertexes of G will by in Y: $V_G\subset Y$.

We begin the construction by splitting Y in ***temporal layers***:

$$Y_n := \{(x_1,x_2,x_3)\in Y:\ x_1+x_2+x_3=-n\}$$

Then, in each temporal layer, we build ***clusters*** by the relation:

$$x\sim y \qquad\Longleftrightarrow\qquad \exists x_0=x,\ x_1,\ \cdots,\ x_k,\ x_{k+1}=y\in X \text{ such that } \hat{\underline{\partial}}(x_m)\cap\hat{\underline{\partial}}(x_{m+1})\neq\varnothing$$

The clusters are defined as equivalence classes of $\sim$. (see Figure A.1 for a 2-dimensional drawing and A.4 for a 3-dimensional one.)

Remark that $\{0\}$ is the only cluster of Y_0 and that each point of err is its own cluster.

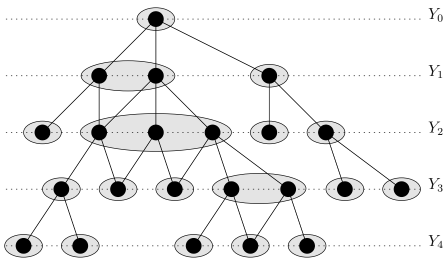


Figure A.1 – The temporal layers and the clusters. This drawing is 2-dimensional whereas Y is 3-dimensional but it clarifies the definitions.

The cluster $B \subset Y_{n+1}$ is the ***parent of the cluster*** $A \subset Y_n$ if $B \cap \underline{\hat{\partial}} A \neq \varnothing$.

A cluster B can be the parent of only one cluster. Indeed, if B is the parent of A and A', there exist $a \in A$, $a' \in A'$ and $b, b' \in B$ such that $b \in \underline{\hat{\partial}} a$ and $b' \in \underline{\hat{\partial}} a'$. Since b and b' are in the same cluster, there exist $b_1, \cdots, b_k \in B$ such that $\underline{\hat{\partial}} b_l \cap \underline{\hat{\partial}} b_{l+1} \neq \varnothing$. Given B, choose A, A', a, a', b, b' and $b_1, \cdots, b_k$ to get k as minimal as possible. Since $b_k \in B \in Y$, there exists $a'' \in Y$ such that $b_k \in \underline{\hat{\partial}} a''$. Then, either $a'' \in A$ or $a'' \in A'$ or a'' belongs to another cluster A''. The three alternatives contradict the minimality of k so $A = A'$.

Let $A \subset Y_n$ be a cluster. We define the ***graph of the parents*** of A denoted $H(A)$ by this way: The vertexes are the parents of A. The edges of $H(A)$ will be called links to avoid confusions with the edges of E_G. B and B' are linked in $H(A)$ if there exists $a \in A$ such that $B \cap \underline{\hat{\partial}} a \neq \varnothing$ and $B' \cap \underline{\hat{\partial}} a \neq \varnothing$. The main property of $H(A)$ is that it is a connected graph. Indeed, let B and B' be two parents of A. Let $a, a' \in A$ such that $B \cap \underline{\hat{\partial}} a \neq \varnothing$ and $B' \cap \underline{\hat{\partial}} a' \neq \varnothing$. Since a and a' are in the same cluster, there exist $a_0 = a,\ a_1,\ \cdots,\ a_n = a'$ such that $\underline{\hat{\partial}} a_k \cap \underline{\hat{\partial}} a_{k+1} \neq \varnothing$. Let $b_k \in \underline{\hat{\partial}} a_k \cap \underline{\hat{\partial}} a_{k+1}$ and B_k the cluster containing b_k. Then, either there exists a link between B_k and B_{k+1}, or $B_k = B_{k+1}$. B and B' are then connected in $H(A)$.

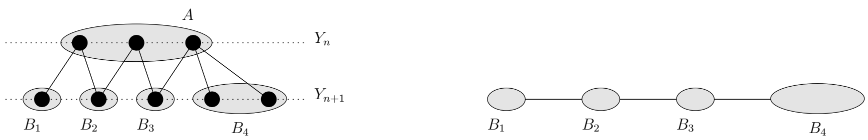


Figure A.2 – A cluster A and its parents are shown in the left drawing. The graph of parents for the corresponding A is in the right one.

We have (at last) all the tools needed to construct G. We are going to construct it by induction: G_0, G_1, ..., G_n,... We will then define G as

$$V_G := \bigcup_{n \geqslant 0} V_{G_n} \qquad E_G := \bigcup_{n \geqslant 0} E_{G_n}$$

Given G_n, we will construct G_{n+1} by only adding to V_{G_n} some sites in Y_{n+1} and adding to E_{G_n} spacelikes in Y_{n+1} and timelikes from Y_n to Y_{n+1}.

Suppose that we have constructed G_n. Every cluster $A \subset Y_n$ that intersects G_n will be called ***relevant cluster***. A is said to be ***terminal*** if it is an error site and ***active*** otherwise.

During the construction, each G_n will satisfy P2-P4 and a modified version of P1, P5 and P6:

(P'1) If relevent clusters are shrunk to one point, then G_n is connected and $0 \in G_n$,

(P'5) the decomposition pairs/triplets is true except for sites in relevant clusters. For these clusters, the decomposition is global. More precisely for a relevant cluster A, there are two possibilities:
- A is a ***biped***: one edge going into A and another leaving A with the same index,
- A is a ***triped***: three edges going into A with three different indexes.

(P'6) Let s_n be the number of spacelikes in G_n and c_n the number of active clusters. The following relation holds:

$$s_n = c_n + |V_{G_n} \cap \text{err}| - 1 \tag{A.5}$$

Let us begin the induction. We initialize it by letting $V_{G_0} = \{0\}$ and $E_{G_0} = \varnothing$. P'1, P2-P4 are true. P'5 is not true but P5 is true, and that is sufficient for us. We will see this point later in the proof. For P'6, note that if 0 is terminal $s_n = 0$, $c_n = 0$, $|V_{G_0} \cap \text{err}| = 1$ and if it is active $s_n = 0$, $c_0 = 1$, $|V_{G_0} \cap \text{err}| = 0$ so that P'6 is also true.

Suppose now that we have constructed G_n satisfying P'1, P2-P4, P'5, P'6. If G_n has no active cluster, the construction (and the proof) ends here: P'1, P'5, P'6 become respectively P1, P5, P6. If it has an active cluster, we have to construct G_{n+1}.

Let A be an active cluster. We will construct the part of G_{n+1} under A, that is in $\hat{\underline{\partial}}A$. This will be done in two steps. In the first one, we add timelikes, in the second one we add spacelikes. The same procedure will be done for all active clusters in G_n.

<u>First step</u>. Let $a \in A \cap V_{G_n}$. If a is pointed by an edge of E_{G_n}, we add a timelike with the same index starting at a. Since a is not an error site, $\hat{\underline{\partial}}a$ contains at least two elements, so it is possible to choose one site in $\hat{\underline{\partial}}a$ for the end point of the timelike according to the restrictions of P4.

If the edge leaves a, we add two timelikes starting at a and with different indexes according to P4.

In both cases, the site a satisfies P5.

Remark that the same point of A can be pointed by an edge and point with another one. In this case, we do the two actions.

Whatever the type of A, we have added three timelikes of different indexes to E_{G_n}. There is here a special case: the case where $n = 0$. Indeed, $\{0\}$ is neither a biped nor a triped. We consider it as a triped: we send three timelikes of different indexes from 0. P5 is then valid at the site 0.

The choice of the sites in Y_{n+1} is what can make the map $\omega \to G(\omega)$ non unique. To avoid this problem, we can decide a rule before the construction and apply it whenever it is necessary.

<u>*Second step*</u>. For the spacelikes, we first decide the orientation, the indexes and the location of them by clusters, then we will choose the sites.

Let T be the minimal subgraph of $H(A)$ containing the clusters added in the first step. We can uniquely decide an orientation and fix indexes on T to satisfy P'5 (see Figure A.3).

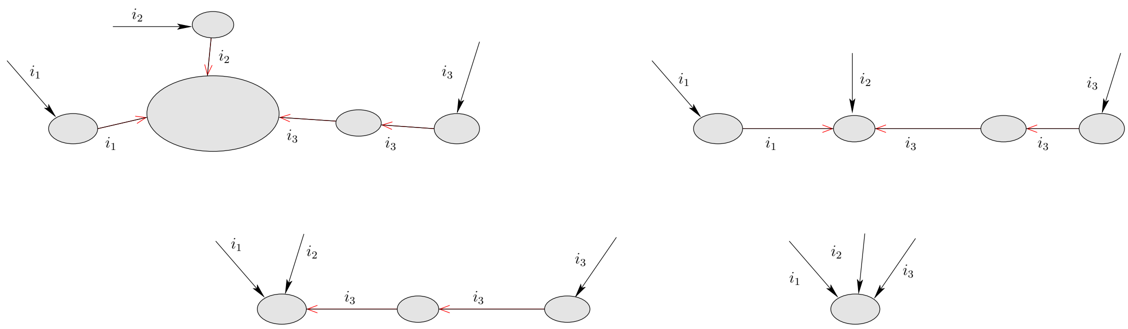


Figure A.3 – The different types of subgraphs T and their orientation and indexes.

At the level of sites, it is possible to draw a spacelike between two linked clusters of T. If necessary, the endpoints of these spacelikes are added to V_{G_n}.

Its orientation and index are the orientation and index decided at the T-level.

After this construction, A is no more relevant. Those two steps are done for all active clusters of G_n. It is possible that some sites, or some clusters are not reached by the construction. This is not a problem, they do not belongs to V_G, that is all.

To end the proof, we just have to check that our construction preserves the wanted properties. P'1,P2-P4,P'5 are immediately true. For P'6, for each relevant cluster A, we add r new spacelikes. We add then $r+1$ new clusters, r' of them are actives. At the end of the two steps, A is no more relevant. So $s'_n = s_n + r$, $c'_n = c_n + r' - 1$ and $|V_{G'_n} \cap \text{err}| = |V_{G_n} \cap \text{err}| + r + 1 - r'$. This leads to $s'_n = c'_n + |V_{G'_n} \cap \text{err}| - 1$. The two steps conserve P'6 so at the end of the construction of G_{n+1}, P'6 is still true.

That finishes the proof.

We end this chapter by figures of the construction.

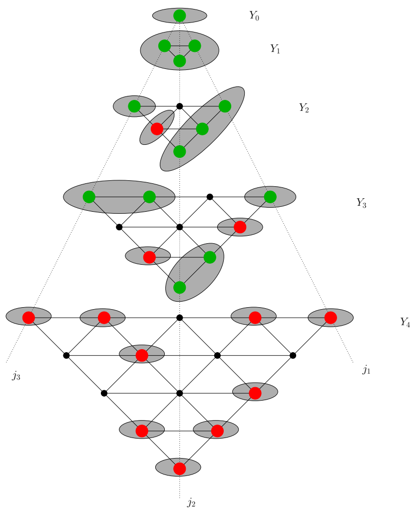


Figure A.4 – Example of ω. The red sites are error sites and green ones are $\ominus$-valued sites. Temporal layers and clusters are also represented.

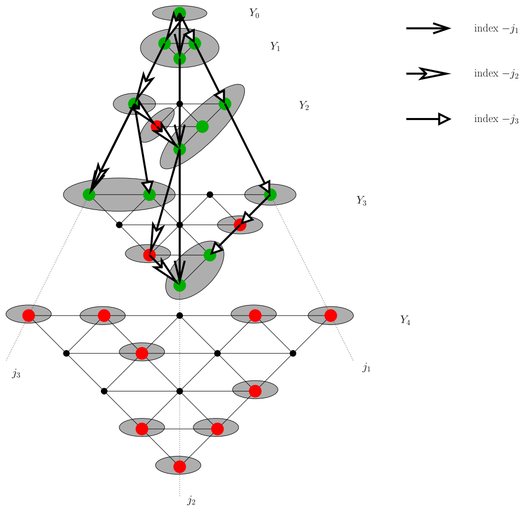


Figure A.5 – Example of graph G_3. The construction is not finished: there are still three active clusters.

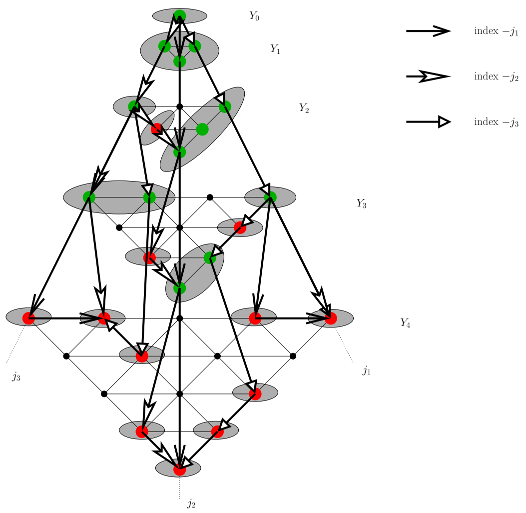


Figure A.6 – Example of graph G. The construction is finished.

Appendix B

Simulation programs

Here are two programs used to do simulations. All the simulations seen in this thesis have been done using them. Both of them uses the C language with commun libraries. The code has been commented to make easier the global understanding. The command line for compiling is in the comments at the beginning of each program. They are distributed under the GNU Public License (GPL). (To see the license, http://www.gnu.org/licenses/gpl.txt)

Each program save the simulation into a file. It uses the fig format. If is then easy to customize it and export it in eps or pdf (with Xfig for example).

The code has not been made to be optimal but to be easily understood. The reader can then easily change it for its own purpose. A user-friendly version of them can be found on my Internet home page. Those one use the GTK+ 2.0 libraries.

B.1 Basic simulations

The first program simulates a PCA in $\mathbb{Z}$ with two colors and two neighbours. It is related to Part I. It represents the POMM-Ising model (line 50-52). The Stavskaya's one is in comments (line 46-48). To simulate it, just uncomment its rules and comment the POMM-Ising ones instead. Other models can easily be implemented too.

What do the program exactly?
It fixes initial conditions to "all plus". It simulates the PCA in a square of size `LENGTH`. Then it saves the right-bottom sub-square of size `VIEW` in a file named `pca-simul.fig`. To see the complete simulation, set `VIEW = LENGTH` before compilation.

In the following listing, spaces in strings are replaced by the symbol `"␣"`. This is because the number of spaces is important for the fig format.

Here is the program:

```
/**
 *  Simulation of a pca with 2 states and 2 neighbors.
 */
/**
 * compilation line (on a GNU/Linux platform):
 * gcc -lm -o pca2s2n pca2s2n.c
 */

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <time.h>

        /* length of the simulation and length of the view */
#define LENGTH 1000
#define VIEW 400

        /* scale for the fig file */
#define SCALE 40

        /* type of colors */
enum {ERROR=-10, OMINUS=-1, EMPTY, OPLUS };

        /* global variables */
int type[LENGTH][LENGTH];
float beta, h;


/*
 *          s  i  m  u  l  a  t  e
 *
 * do the simulation
 */
void simulate(void) {
    int x,y,r;
    int t1,t2;
    float pA,unif; /* pA = Proba( OMINUS | past ) */
    for (r=1; r<2*LENGTH; r++) {
        x = (r<LENGTH)? 1 : r-LENGTH+1;
        y = r-x;
        while ( x<LENGTH && y>0 ) {
            t1 = type[x-1][y  ];
            t2 = type[x  ][y-1];
                /* the proba at (x,y) */

            // Stavskaya's model
//            if ( t1 == OPLUS || t2 == OPLUS ) pA = 1.0-beta;
//            else pA = 1.0;

            // POMM-Ising model
            float z = exp( -2*beta*(t1+t2+h) );
            pA = z / (z+1);

            unif = ((float)rand()) / RAND_MAX;
            if (unif>pA) type[x][y] = OPLUS;
```

```
            else type[x][y] = OMINUS;
            x++; y--;
        }
    }
}

/*
 *        s a v e T o F i g
 *
 * save the configuration into a fig file
 */
void saveToFig(char* file) {
    FILE *fp;
    int x,y;
    printf("fig␣file:␣%s\n", file);
    fp = fopen(file, "w");
    /* initialize fig file */
    fprintf (fp, "#FIG␣3.2␣\n") ;
    fprintf (fp, "Landscape␣\n") ;
    fprintf (fp, "Center␣\n") ;
    fprintf (fp, "Metric␣\n") ;
    fprintf (fp, "A4␣\n") ;
    fprintf (fp, "100.00␣\n") ;
    fprintf (fp, "Single␣\n") ;
    fprintf (fp, "-2␣\n") ;
    fprintf (fp, "1200␣2␣\n") ;
    /* draw a box around the simulation */
    fprintf(fp, "2␣2␣0␣1␣0␣7␣51␣-1␣20␣0.000␣0␣0␣-1␣0␣0␣5\n");
    fprintf(fp,"␣␣␣␣␣␣␣␣␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i\n",
            0, 0, SCALE*VIEW, 0, SCALE*VIEW,
            SCALE*VIEW, 0, SCALE*VIEW, 0, 0);
    /* draw the simulation */
    for (x=0; x<VIEW; x++) {
        for (y=0; y<VIEW; y++) {
            if (type[x+LENGTH-VIEW][y+LENGTH-VIEW] == OPLUS) {
                fprintf(fp, "2␣2␣0␣0␣0␣0␣50␣-1␣20␣0.000␣0␣0␣-1␣0␣0␣5\n");
                fprintf(fp, "␣␣␣␣␣␣␣␣␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i\n",
                        SCALE*x, SCALE*y, SCALE*x, SCALE*(y+1), SCALE*(x+1),
                        SCALE*(y+1), SCALE*(x+1), SCALE*y, SCALE*x, SCALE*y);
            }
        }
    }
    fclose(fp);
}


/*******************
 *   m a i n   *
 *******************/
int main(int argc, char *argv[]) {
    srand(time(NULL));
    // beta = P( OMINUS | OPLUS, OPLUS ) for Stavskaya
    beta = 1.5;
    h = 0.0;
    // initialisation
```

```
    int x;
    for (x=0; x<LENGTH; x++) {
        type[0][x] = OPLUS;
        type[x][0] = OPLUS;
    }
    simulate();
    saveToFig("pca-simul.fig");
    return 0;
}
```

B.2 Simulations of a flow

The second program represents the flow of information. It is related to Part II. In fact, it represents a coupling of PCA starting from two different initial conditions: "all plus" and "all minus". The flow and the choice at $\odot$-sites are the same.

The program represents the product flow. To see the majority flow of information, just comment line 154 and uncomment line 151.

As in the preceding program, it exports the simulation into a fig file named `"flow-simul.fig"`.

Note that the fifth parameter (q_{NW}) can not be chosen directly. The reason is that the sum of all parameters must be equal to one. q_{NW} is computed from the other parameters.

What do the program exactly?
It simulate the flow of information of the right-bottom site (0). Sites not connected to 0 are not shown. The size of the flow is controlled by `LENGTH`.

The color code is shown bellow:

configuration	color
$(\oplus,\ \oplus)$	blue
$(\ominus,\ \oplus)$	red
$(\oplus,\ \ominus)$	gold
$(\ominus,\ \ominus)$	green

Here is the program:

```
/**
 *  Simulation of a flow of information.
 */
/**
5  * compilation line (on a GNU/Linux platform):
 * gcc -lm -o flow flow.c
 */
#include <stdio.h>
#include <stdlib.h>
10 #include <time.h>

        /* size of the simulation */
#define LENGTH 70

15      /* scale for the fig file */
#define SCALE 120

        /* colors of sites */
enum { ERROR = -2, EMPTY,
20      PLUS_PLUS, MINUS_PLUS, PLUS_MINUS, MINUS_MINUS };

        /* types of sites */
// V: vacuum (not in the flow), T: triplet (3-sample), F: frontier
```

```
enum {  TYPE_V, TYPE_M, TYPE_P, TYPE_N, TYPE_W, TYPE_T, TYPE_F };


        /* global variables */
// array of types and colors
int   type[LENGTH*LENGTH];
int colors[LENGTH*LENGTH];
// array of probabilities at a single site
float proba[4];


        /* list of defined fonctions */
int  getType(int, int);
void setType(int, int, int);
int  getColor(int, int);
void setColor(int, int, int);
void chooseFlow(void);
void setConfigColors(void);
void simulate(void);
void saveToFig(char*);


/*
 *       g  e  t  T  y  p  e
 *
 * return the type of the site ('x', 'y')
 */
int getType(int x, int y) {
    int n = ( y*LENGTH ) + x;
    if (x<0 || y<0 ) return TYPE_V;
    if (x>=LENGTH || y>=LENGTH) return TYPE_F;
    return type[n];
}

/*
 *       s  e  t  T  y  p  e
 *
 * set the type 't' to the site ('x', 'y')
 */
void setType(int x, int y, int t) {
    int n = ( y*LENGTH ) + x;
    if (x<0 || y<0 || x>=LENGTH || y>=LENGTH) {
        fprintf(stderr,"setType.Error␣:␣x=%d,␣y=%d\n", x, y);
        return;
    }
    type[n] = t;
}

/*
 *       g  e  t  C  o  l  o  r
 *
 * return the color of the site ('x', 'y')
 */
int getColor(int x, int y) {
    int n = ( y*LENGTH ) + x;
```

```
    if (x<0 || y<0 ) {
        fprintf(stderr,"getColor.Erreur␣:␣x=%d,␣y=%d\n",x,y);
        return ERROR;
    }
    if (x>=LENGTH || y>=LENGTH) return MINUS_PLUS;
    return colors[n];
}

/*
 *       s  e  t  C  o  l  o  r
 *
 * set the color 'c' to the site ('x', 'y')
 */
void setColor(int x, int y, int c) {
    int n = ( y*LENGTH ) + x;
    if (x<0 || y<0 || x>=LENGTH || y>=LENGTH) {
        fprintf(stderr,"setColor.Error␣:␣x=%d,␣y=%d\n", x, y);
        return;
    }
    colors[n] = c;
}

/*
 *       c  h  o  o  s  e  F  l  o  w
 *
 * scan all the sites to define the flow
 */
void chooseFlow(void) {
    float unif;
    int t1,t2,i,c;
    int x,y,r;
    for (r=0; r<2*LENGTH-1; r++) {
        x = (r<LENGTH) ? r : LENGTH-1;
        y = (r<LENGTH) ? 0 : r-LENGTH+1;
        while (x>=0 && y<LENGTH) {
            t1 = getType(x-1,y  );
            t2 = getType(x  ,y-1);
            // is this site connected to the flow?
            c = (t1==TYPE_W) + (t1==TYPE_T) +
                (t2==TYPE_N) + (t2==TYPE_T);
            if (c>0 || (x==0 && y==0)) { // answer: yes
                unif = (float)rand()/RAND_MAX;
                c = 1;
                for (i=0; i<4; i++)
                    c += (unif>proba[i]);
            } else c = TYPE_V; // answer: no
            setType(x,y,c);
            x--;  y++;
        }
    }
}

/*
 *       s  e  t  C  o  n  f  i  g  C  o  l  o  r  s
 *
```

```
 * scan all the sites to define their color
 */
void setConfigColors(void) {
    int r,x,y;
    int c,c1,c2;
    for (r=2*LENGTH-2; r>=0; r--) {
        x = (r<LENGTH) ? r : LENGTH-1;
        y = (r<LENGTH) ? 0 : r-LENGTH+1;
        while (x>=0 && y<LENGTH) {
            switch (getType(x,y)) {
                case TYPE_M : c = MINUS_MINUS;  break;
                case TYPE_P : c = PLUS_PLUS;  break;
                case TYPE_N : c = getColor(x,y+1);  break;
                case TYPE_W : c = getColor(x+1,y);  break;
                case TYPE_T : c1 = getColor(x,y+1); c2 = getColor(x+1,y);

                              // majority rule = logical AND
                              c = c1 & c2;

                              // product rule = logical Exclusive OR
//                              c = c1 ^ c2;

                              break;
                case TYPE_F : c = MINUS_PLUS;  break;
                case TYPE_V :
                default     : c = EMPTY;  break;
            }
            setColor(x,y,c);
            x--;  y++;
        }
    }
}

/*
 *        s i m u l a t e
 *
 * do the simulation
 */
void simulate(void) {
    int k;
    for (k=1; k<4; k++)
        proba[k] += proba[k-1];
    if (proba[3]>1) {
        fprintf(stderr,"Error: probability greater than 1");
        exit(1);
    }
    chooseFlow();
    setConfigColors();
}

/*
 *        s a v e T o F i g
 *
 * save the simulatio into a fig file
 */
```

```
void saveToFig(char* file) {
    FILE *fp;
    int x,y, sx, sy;
    int c,t;
    printf("fig␣file:␣%s\n", file);
    fp = fopen(file, "w");
    // initialize fig file
    fprintf (fp, "#FIG␣3.2␣\n") ;
    fprintf (fp, "Landscape␣\n") ;
    fprintf (fp, "Center␣\n") ;
    fprintf (fp, "Metric␣\n") ;
    fprintf (fp, "A4␣\n") ;
    fprintf (fp, "100.00␣\n") ;
    fprintf (fp, "Single␣\n") ;
    fprintf (fp, "-2␣\n") ;
    fprintf (fp, "1200␣2␣\n") ;
    // draw a box around the simulation
    fprintf(fp, "2␣2␣0␣1␣0␣7␣51␣-1␣20␣0.000␣0␣0␣-1␣0␣0␣5\n");
    fprintf(fp, "␣␣␣␣␣␣␣␣␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i␣%i\n" ,
            0, 0, SCALE*(LENGTH+1), 0, SCALE*(LENGTH+1),
            SCALE*(LENGTH+1), 0, SCALE*(LENGTH+1), 0, 0 );
    // draw the frontier
    fprintf(fp, "2␣1␣0␣1␣4␣7␣50␣-1␣-1␣0.000␣0␣0␣-1␣0␣0␣3\n");
    fprintf(fp, "␣␣␣␣␣␣␣␣␣%i␣%i␣%i␣%i␣%i␣%i\n",
            SCALE/2, (LENGTH+1)*SCALE - SCALE/2, SCALE/2,
            SCALE/2, (LENGTH+1)*SCALE - SCALE/2, SCALE/2 );
    // draw the simulation
    for (x=0; x<LENGTH+1; x++) {
        for (y=0; y<LENGTH+1; y++) {
            switch (getColor(x,y)) {
                case ERROR : c = 7;  break;
                case EMPTY : c = 0;  break;
                case PLUS_PLUS   : c = 1;  break;
                case MINUS_PLUS  : c = 4;  break;
                case PLUS_MINUS  : c = 31;  break;
                case MINUS_MINUS : c = 2;  break;
            }
            sx = SCALE*(LENGTH-x)+SCALE/2;
            sy = SCALE*(LENGTH-y)+SCALE/2;
            t = getType(x ,y-1);
            if ( t==TYPE_N || t==TYPE_T) {
                fprintf(fp, "2␣1␣0␣1␣%i␣7␣50␣-1␣-1␣0.000␣0␣0␣-1␣0␣0␣2\n", c);
                fprintf(fp, "␣␣␣␣␣␣␣␣␣%d␣%d␣%d␣%d\n", sx, sy, sx, sy+SCALE);
            }
            t = getType(x-1, y);
            if ( t==TYPE_W || t==TYPE_T) {
                fprintf(fp, "2␣1␣0␣1␣%d␣7␣50␣-1␣-1␣0.000␣0␣0␣-1␣0␣0␣2\n", c);
                fprintf(fp, "␣␣␣␣␣␣␣␣␣%d␣%d␣%d␣%d\n", sx, sy, sx+SCALE, sy);
            }
            t = getType(x,y);
            if ( t==TYPE_M || t==TYPE_P || (x==0 && y==0)) {
                fprintf(fp, "1␣3␣0␣1␣%d␣%d␣50␣-1␣20␣0.000␣1␣0.0000␣", c, c);
                fprintf(fp, "%d␣%d␣%d␣%d␣%d␣%d␣%d␣%d\n",
                        sx, sy, SCALE/4, SCALE/4, sx, sy, sx+SCALE/4, sy );
            }
```

```
        }
    }
    fclose(fp);
}


/********************
 *    m a i n    *
 ********************/
int main(int argc, char *argv[]) {
    srand(time(NULL));
    proba[0] = 0.05; // proba for ominus
    proba[1] = 0.005; // proba for oplus
    proba[2] = 0.3;  // proba for N
    proba[3] = 0.3;  // proba for W
    // The proba for NW (or M) is deduce from the others.
    simulate();
    saveToFig("flow-simul.fig");
    return 0;
}
```

Bibliography

[BG91] Maury Bramson and Lawrence Gray. A useful renormalization argument. In *Random walks, Brownian motion, and interacting particle systems*, volume 28 of *Progr. Probab.*, pages 113–152. Birkhäuser Boston, Boston, MA, 1991.

[BR06] Vladimir Belitsky and Thomas Logan Ritchie. Improved lower bounds for the critical probability of oriented bond percolation in two dimensions. *J. Stat. Phys.*, 122(2):279–302, 2006.

[CD98] Noel Cressie and Jennifer L. Davidson. Image analysis with partially ordered Markov models. *Comput. Statist. Data Anal.*, 29(1):1–26, 1998.

[CDH99] Noel Cressie, Jennifer L. Davidson, and X. Hua. Texture synthesis and pattern recognition for partially ordered markov models. *Pattern recognition*, 32, 1999.

[DKT90] R.L. Dobrushin, V.I. Kryukov, and A.L. Toom, editors. *Stochastic cellular systems: ergodicity, memory, morphogenesis.* Manchester University Press, 1990.

[Dur84] Richard Durrett. Oriented percolation in two dimensions. *Ann. Probab.*, 12(4):999–1040, 1984.

[FG] Pablo A. Ferrari and Antonio Galves. Construction of stochastic processes, coupling and regeneration. XIII Escuela Venezolana de Matemática.

[FM05] Roberto Fernández and Grégory Maillard. Chains with complete connections: general theory, uniqueness, loss of memory and mixing properties. *J. Stat. Phys.*, 118(3-4):555–588, 2005.

[FM06] Roberto Fernández and Grégory Maillard. Construction of a specification from its singleton part. *ALEA Lat. Am. J. Probab. Math. Stat.*, 2:297–315 (electronic), 2006.

[Gác01] Peter Gács. Reliable cellular automata with self-organization. *J. Statist. Phys.*, 103(1-2):45–267, 2001.

[Geo88] Hans-Otto Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1988.

[GHM01] Hans-Otto Georgii, Olle Häggström, and Christian Maes. The random geometry of equilibrium phases. In *Phase transitions and critical phenomena, Vol. 18*, volume 18 of *Phase Transit. Crit. Phenom.*, pages 1–142. Academic Press, San Diego, CA, 2001.

[GKLM89] Sheldon Goldstein, Roelof Kuik, Joel L. Lebowitz, and Christian Maes. From PCAs to equilibrium systems and back. *Comm. Math. Phys.*, 125(1):71–79, 1989.

[Gra01] Lawrence F. Gray. A reader's guide to P. Gács's "positive rates" paper: "Reliable cellular automata with self-organization" [J. Statist. Phys. **103** (2001), no. 1-2, 45–267; MR1828729 (2002c:82058a)]. *J. Statist. Phys.*, 103(1-2):1–44, 2001.

[Lan05] Nicolas Lanchier. Phase transitions and duality properties of a successional model. *Adv. in Appl. Probab.*, 37(1):265–278, 2005.

[Lin92] Torgny Lindvall. *Lectures on the coupling method*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1992. , A Wiley-Interscience Publication.

[LML72] Joel L. Lebowitz and Anders Martin-Löf. On the uniqueness of the equilibrium state for Ising spin systems. *Comm. Math. Phys.*, 25:276–282, 1972.

[LMS90a] Joel L. Lebowitz, Christian Maes, and Eugene R. Speer. Probabilistic cellular automata: some statistical mechanical considerations. In *1989 lectures in complex systems (Santa Fe, NM, 1989)*, Santa Fe Inst. Stud. Sci. Complexity Lectures, II, pages 401–414. Addison-Wesley, Redwood City, CA, 1990.

[LMS90b] Joel L. Lebowitz, Christian Maes, and Eugene R. Speer. Statistical mechanics of probabilistic cellular automata. *J. Statist. Phys.*, 59(1-2):117–170, 1990.

[Mae92] Christian Maes. An invitation to probabilistic cellular automata. In *Mark Kac Seminar on Probability and Physics. Syllabus 1987–1992 (Amsterdam, 1987–1992)*, volume 32 of *CWI Syllabi*, pages 169–179. Math. Centrum Centrum Wisk. Inform., Amsterdam, 1992.

[Mai03] Grégory Maillard. *Chaînes à liaisons complètes et mesures de Gibbs unidimensionnelles*. PhD thesis, Université de Rouen, France, 2003.

[Too80] A. L. Toom. Stable and attractive trajectories in multicomponent systems. In *Multicomponent random systems*, volume 6 of *Adv. Probab. Related Topics*, pages 549–575. Dekker, New York, 1980.

[Too01] André Toom. *Contornos, conjuntos convexos e autômatos celulares.* Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2001. , 23° Colóquio Brasileiro de Matemática. [23rd Brazilian Mathematics Colloquium].

[vdBM94] J. van den Berg and C. Maes. Disagreement percolation in the study of Markov fields. *Ann. Probab.*, 22(2):749–763, 1994.