On the cyclotomic Hecke algebras of complex reflection groups

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SUR LES ALGÈBRES DE HECKE CYCLOTOMIQUES DES GROUPES DE RÉFLEXIONS COMPLEXES

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ON THE CYCLOTOMIC HECKE ALGEBRAS OF COMPLEX REFLECTION GROUPS

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Introduction

Les travaux de G.Lusztig sur les caractères irréductibles des groupes réductifs sur les corps finis ont mis en évidence le rôle important joué par les "familles de caractères" des groupes de Weyl concernés. Cependant, on s'est récemment rendu compte qu'il serait de grand intérêt de généraliser la notion des familles de caractères aux groupes de réflexions complexes ou, plus précisément, à divers types d'algèbres de Hecke associées aux groupes de réflexions complexes.

D'une part, les groupes de réflexions complexes et certaines déformations de leurs algèbres génériques (les algèbres cyclotomiques) interviennent naturellement pour classifier les "séries de Harish-Chandra cyclotomiques" des caractères des groupes réductifs finis, généralisant ainsi le rôle joué par le groupe de Weyl et son algèbre de Hecke traditionnelle dans la description de la série principale. Puisque les familles de caractères du groupe de Weyl jouent un rôle essentiel dans la définition des familles de caractères unipotents du groupe réductif fini correspondant (cf. [27]), on peut espérer que plus généralement les familles de caractères des algèbres cyclotomiques jouent un rôle-clef dans l'organisation des familles de caractères unipotents.

D'autre part, pour certains groupes de réflexions complexes (et non de Coxeter) W, on a des données qui semblent indiquer que, derrière le groupe W se cache un objet mystérieux - le *Spets* (cf. [13], [32]) - qui pourrait jouer le rôle de "la série des groupes réductifs finis de groupe de Weyl W". Dans certains cas, il est possible de définir les caractères unipotents du Spets, qui sont contrôlés par l'algèbre de Hecke "spetsiale" de W, une généralisation de l'algèbre de Hecke classique des groupes de Weyl.

L'obstacle principal pour cette généralisation est le manque de bases de Kazhdan-Lusztig pour les groupes de réflexions complexes (non de Coxeter). Cependant, des résultats plus récents de Gyoja [23] et de Rouquier [37] ont rendu possible la définition d'un substitut pour les familles des caractères, qui peut être appliqué à tous les groupes de réflexions complexes. Gyoja a démontré (cas par cas) que la partition en "p-blocs"de l'algèbre de Iwahori-Hecke d'un groupe de Weyl W coincide avec la partition en familles, quand p

est l'unique mauvais nombre premier pour W. Plus tard, Rouquier a prouvé que les familles des caractères d'un groupe de Weyl W sont exactement les blocs de caractères de l'algèbre de Iwahori-Hecke de W sur un anneau de coefficients convenable. Cette définition se généralise sans problème à toutes les algèbres cyclotomiques de Hecke des groupes de réflexions complexes. Expliquons comment.

Soit μ_∞ le groupe des racines de l'unité de $\mathbb{C}$ et K un corps de nombres contenu dans $\mathbb{Q}(\mu_\infty)$. On note $\mu(K)$ le groupe des racines de l'unité de K et pour tout $d > 1$, on pose $\zeta_d := \exp(2\pi i/d)$. Soit V un K-espace vectoriel de dimension finie. Soit W un sous-groupe fini de $\mathrm{GL}(V)$ engendré par des (pseudo-)réflexions et agissant irréductiblement sur V et B le groupe de tresses associé à W. On note $\mathcal{A}$ l'ensemble des hyperplans de réflexion de W et $V^{\mathrm{reg}} := \mathbb{C} \otimes V - \bigcup_{H \in \mathcal{A}} \mathbb{C} \otimes H$. Pour $x_0 \in V^{\mathrm{reg}}$, on définit $B := \Pi_1(V^{\mathrm{reg}}/W, x_0)$

Pour toute orbite $\mathcal{C}$ de W sur $\mathcal{A}$, on note $e_{\mathcal{C}}$ l'ordre commun des sous-groupes W_H, où $H \in \mathcal{C}$ et W_H est le sous-groupe formé de 1 et toutes les réflexions qui fixent H.

On choisit un ensemble d'indéterminées $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \leq j \leq e_{\mathcal{C}}-1)}$ et on définit l'*algèbre de Hecke générique* $\mathcal{H}$ de W comme le quotient de l'algèbre du groupe $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ par l'idéal engendré par les éléments de la forme

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \dots (\mathbf{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

où $\mathcal{C}$ parcourt l'ensemble $\mathcal{A}/W$ et $\mathbf{s}$ l'ensemble de générateurs de monodromie autour des images dans V^{reg}/W des éléments de l'orbite d'hyperplans $\mathcal{C}$.

Si on suppose que $\mathcal{H}$ est un $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module libre de rang $|W|$ et qu'elle est munie d'une forme symétrisante t qui satisfait les conditions 3.2.2, alors on a le résultat suivant dû à Malle ([32], 5.2) : si $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \leq j \leq e_{\mathcal{C}}-1)}$ un ensemble de $\sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}$ indéterminées tel que, pour tout $\mathcal{C}, j$, on a $v_{\mathcal{C},j}^{|\mu(K)|} := \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}$, alors l'algèbre $K(\mathbf{v})\mathcal{H}$ est semisimple déployée. Noter bien que ces hypothèses sont verifiées pour tous les groupes de réflexions irréductibles sauf un nombre fini d'entre eux ([13], remarques précédant 1.17, § 2; [22]). Dans ce cas, par le "théorème de déformation de Tits", on sait que la spécialisation $v_{\mathcal{C},j} \mapsto 1$ induit une bijection $\chi \mapsto \chi_{\mathbf{v}}$ de l'ensemble $\mathrm{Irr}(W)$ des caractères absolument irréductibles de W sur l'ensemble $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ des caractères absolument irréductibles de l'algèbre $K(\mathbf{v})\mathcal{H}$.

Soit maintenant y une indéterminée. La $\mathbb{Z}_K[y, y^{-1}]$-algèbre, notée par $\mathcal{H}_\phi$, obtenue comme la spécialisation de $\mathcal{H}$ via le morphisme $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$, où $n_{\mathcal{C},j} \in \mathbb{Z}$ pour tous $\mathcal{C}$ et j, est une *algèbre de Hecke cyclotomique*. Elle est aussi munie d'une forme symétrisante t_ϕ définie comme la spécialisation de la forme canonique t. On remarque que, pour $y = 1$, l'algèbre $\mathcal{H}_\phi$ se spécialise à l'algèbre du groupe $\mathbb{Z}_K[W]$.

On appelle *anneau de Rouquier* de K et note par $\mathcal{R}_K(y)$ la sous-$\mathbb{Z}_K$-algèbre de $K(y)$

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1}_{n\geq 1}].$$

Les *blocs de Rouquier* de $\mathcal{H}_\phi$ sont les blocs de l'algèbre $\mathcal{R}_K(y)\mathcal{H}_\phi$. Rouquier [37] a montré que si W est un groupe de Weyl et $\mathcal{H}_\phi$ est obtenue via la spécialisation cyclotomique ("spetsiale")

$$\phi : v_{\mathcal{C},0} \mapsto y \text{ et } v_{\mathcal{C},j} \mapsto 1 \text{ pour } j \neq 0,$$

alors ses blocs de Rouquier coincident avec les "familles de caractères" selon Lusztig. Ainsi, les blocs de Rouquier jouent un rôle essentiel dans le programme "Spets" dont l'ambition est de faire jouer à des groupes de réflexions complexes le rôle de groupes de Weyl de structures encore mystérieuses.

En ce qui concerne le calcul des blocs de Rouquier, le cas de la série infinie est déjà traitée par Broué et Kim dans [12] et par Kim dans [24]. D'ailleurs, les blocs de Rouquier de l'algèbre de Hecke cyclotomique "spetsiale" de plusieurs groupes de réflexions complexes exceptionnels ont été déterminés par Malle et Rouquier dans [33]. Généralisant les méthodes employées dans le dernier, nous avons pu calculer les blocs de Rouquier de toutes les algèbres de Hecke cyclotomiques de tous les groupes de réflexions complexes exceptionnels. De plus, nous avons découvert que les blocs de Rouquier d'une algèbre de Hecke cyclotomique dépendent d'une donnée numérique du groupe de réflexions complexe W, ses *hyperplans essentiels*.

De façon plus précise, les deux premiers chapitres de cette thèse présentent des résultats qui sont donnés ici pour la commodité du lecteur. Dans le premier chapitre, qui est consacré à l'algèbre commutative, nous démontrons des résultats sur la divisibilité et l'irréductibilité qui vont être très utiles dans le chapitre 3. Nous introduisons aussi les notions des "morphismes associés à des monômes" et des "morphismes adaptés". Le deuxième chapitre est l'adaptation et la généralisation des résultats classiques de la théorie des blocs et de la théorie des réprésentations des algèbres symétriques, qui peuvent etre trouvés dans [12] et [20]. Par ailleurs, nous donnons un critère pour qu'une algèbre soit semisimple déployée.

Dans le troisième chapitre, nous trouvons le coeur théorique de cette thèse. Son but est la détérmination des blocs de Rouquier des algèbres de Hecke cyclotomiques des groupes de réflexions complexes. Nous donnons la formule explicite suivante pour les éléments de Schur associés aux caractères

irréductibles de l'algèbre de Hecke générique d'un groupe de réflexions complexe W : si K est le corps de définition de W et $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$ est un ensemble d'indéterminées comme ci-dessus, alors l'élément de Schur $s_\chi(\mathbf{v})$ associé au caractère $\chi_{\mathbf{V}}$ de $K(\mathbf{v})\mathcal{H}$ est de la forme

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i\in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

où

- ξ_χ est un élément de $\mathbb{Z}_K$,
- $N_\chi = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ est un monôme dans $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ avec $\sum_{j=0}^{e_{\mathcal{C}}-1} b_{\mathcal{C},j} = 0$ pour tout $\mathcal{C}\in\mathcal{A}/W$,
- I_χ est un ensemble d'indices,
- $(\Psi_{\chi,i})_{i\in I_\chi}$ est une famille de polynômes K-cyclotomiques à une variable (*i.e.,* polynômes minimaux sur K des racines de l'unité),
- $(M_{\chi,i})_{i\in I_\chi}$ est une famille de monômes dans $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ et si $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, alors $\mathrm{pgcd}(a_{\mathcal{C},j}) = 1$ et $\sum_{j=0}^{e_{\mathcal{C}}-1} a_{\mathcal{C},j} = 0$ pour tout $\mathcal{C}\in\mathcal{A}/W$,
- $(n_{\chi,i})_{i\in I_\chi}$ est une famille d'entiers positifs.

Cette factorisation est unique dans $K[\mathbf{v},\mathbf{v}^{-1}]$ et les monômes $(M_{\chi,i})_{i\in I_\chi}$ sont uniques à inversion près. Si $\mathfrak{p}$ est un idéal premier de $\mathbb{Z}_K$ et $\Psi(M_{\chi,i})$ est un facteur de $s_\chi(\mathbf{v})$ tel que $\Psi_{\chi,i}(1)\in\mathfrak{p}$, alors le monôme $M_{\chi,i}$ s'appellera $\mathfrak{p}$-*essentiel* pour χ. Nous montrons que plus on spécialise notre algèbre via des morphismes associés aux monômes $\mathfrak{p}$-essentiels, plus la taille de ses $\mathfrak{p}$-blocs s'agrandit.

Soit maintenant $M := \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ un monôme $\mathfrak{p}$-essentiel. L'hyperplan défini dans $\mathbb{C}^{\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}}$ par la relation $\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0$, où $(t_{\mathcal{C},j})_{\mathcal{C},j}$ est un ensemble de $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ indéterminées, s'appelle *hyperplan* $\mathfrak{p}$-*essentiel* pour W. Si $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ est une spécialisation cyclotomique, alors les blocs de Rouquier de $\mathcal{H}_\phi$ dépendent des hypeplans $\mathfrak{p}$-essentiels auxquels les $n_{\mathcal{C},j}$ appartiennent (où $\mathfrak{p}$ parcourt l'ensemble des idéaux premiers de $\mathbb{Z}_K$). Donc les blocs de Rouquier d'une algèbre de Hecke cyclotomique dépendent d'une donnée numérique du groupe W.

Le quatrième chapitre est la partie calculatoire de cette thèse. Nous présentons l'algorithme et les résultats de la détermination des blocs de Rouquier de toutes les algèbres de Hecke cyclotomiques de tous les groupes de réflexions complexes exceptionnels.

Contents

Introduction

The work of G.Lusztig on the irreducible characters of reductive groups over finite fields has displayed the important role of the "character families" of the Weyl groups concerned. However, only recently was it realized that it would be of great interest to generalize the notion of character families to the complex reflection groups, or more precisely to some types of Hecke algebras associated with complex reflection groups.

On one hand, the complex reflection groups and their associated "cyclotomic" Hecke algebras appear naturally in the classification of the "cyclotomic Harish-Chandra series" of the characters of the finite reductive groups, generalizing the role of the Weyl group and its traditional Hecke algebra in the principal series. Since the character families of the Weyl group play an essential role in the definition of the families of unipotent characters of the corresponding finite reductive group ([27]), we can hope that the character families of the cyclotomic Hecke algebras play a key role in the organization of families of unipotent characters more generally.

On the other hand, for some complex reflection groups (non-Coxeter) W, some data have been gathered which seem to indicate that behind the group W, there exists another mysterious object - the *Spets* (see [13],[32]) - that could play the role of the "series of finite reductive groups of Weyl group W". In some cases, one can define the unipotent characters of the Spets, which are controlled by the "spetsial" Hecke algebra of W, a generalization of the classical Hecke algebra of the Weyl groups.

The main obstacle for this generalization is the lack of Kazhdan-Lusztig bases for the non-Coxeter complex reflection groups. However, more recent results of Gyoja [23] and Rouquier [37] have made possible the definition of a substitute for families of characters which can be applied to all complex reflection groups. Gyoja has shown (case by case) that the partition into "p-blocks" of the Iwahori-Hecke algebra of a Weyl group W coincides with the partition into families, when p is the unique bad prime number for W. Later, Rouquier showed that the families of characters of a Weyl group W are exactly the blocks of characters of the Iwahori-Hecke algebra of W over

a suitable coefficient ring. This definition generalizes without problem to all the cyclotomic Hecke algebras of complex reflection groups. Let us explain how.

Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$ and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K and for all $d > 1$, we put $\zeta_d := \exp(2\pi i/d)$. Let V be a finite dimensional K-vector space. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo-)reflections and acting irreducibly on V. Denote by $\mathcal{A}$ the set of its reflecting hyperplanes. We set $V^{\mathrm{reg}} := \mathbb{C} \otimes V - \bigcup_{H \in \mathcal{A}} \mathbb{C} \otimes H$. For $x_0 \in V^{\mathrm{reg}}$, we define $B := \Pi_1(V^{\mathrm{reg}}/W, x_0)$ the braid group associated with W.

For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we set $e_{\mathcal{C}}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H the subgroup formed by 1 and all the reflections fixing the hyperplane H.

We choose a set of indeterminates $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \leq j \leq e_{\mathcal{C}}-1)}$ and we denote by $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates $\mathbf{u}$. We define the *generic Hecke algebra* $\mathcal{H}$ of W to be the quotient of the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \dots (\mathbf{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\mathbf{s}$ over the set of monodromy generators around the images in V^{reg}/W of the elements of the hyperplane orbit $\mathcal{C}$.

If we assume that $\mathcal{H}$ is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module of rank $|W|$ and it has a symmetrizing form t which satisfies assumptions 3.2.2, then we have the following result by Malle ([30], 5.2): If $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \leq j \leq e_{\mathcal{C}}-1)}$ is a set of $\sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}$ indeterminates such that, for every $\mathcal{C}, j$, we have $v_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}$, then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple. Note that these assumptions have been verified for all but a finite number of irreducible complex reflection groups ([13], remarks before 1.17, § 2; [22]). In this case, by "Tits' deformation theorem", we know that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi \mapsto \chi_{\mathbf{v}}$ from the set $\mathrm{Irr}(W)$ of absolutely irreducible characters of W to the set $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ of absolutely irreducible characters of $K(\mathbf{v})\mathcal{H}$.

Now let y be a indeterminate. The $\mathbb{Z}_K[y, y^{-1}]$-algebra obtained as a specialization of $\mathcal{H}$ via the morphism $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$, where $n_{\mathcal{C},j} \in \mathbb{Z}$ for all $\mathcal{C}$ and j, is a *cyclotomic Hecke algebra* and it is denoted by $\mathcal{H}_\phi$. It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t. We notice that, for $y = 1$, the algebra $\mathcal{H}_\phi$ specializes to the group algebra $\mathbb{Z}_K[W]$.

We call *Rouquier ring* of K and denote by $\mathcal{R}_K(y)$ the $\mathbb{Z}_K$-subalgebra of $K(y)$

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)_{n \geq 1}^{-1}].$$

The *Rouquier blocks* of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$. It has been shown by Rouquier [37], that if W is a Weyl group and $\mathcal{H}_\phi$ is obtained via the cyclotomic specialization ("spetsial")

$$\phi : v_{\mathcal{C},0} \mapsto y \text{ and } v_{\mathcal{C},j} \mapsto 1 \text{ for } j \neq 0,$$

then its Rouquier blocks coincide with the "families of characters" defined by Lusztig. Thus, the Rouquier blocks play an essential role in the program "Spets" whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious structures.

As far as the calculation of the Rouquier blocks is concerned, the case of the infinite series has already been treated by Broué and Kim in [12] and Kim in [24]. Moreover, the Rouquier blocks of the "spetsial" cyclotomic Hecke algebra of many exceptional complex reflection groups have been determined by Malle and Rouquier in [33]. Generalizing the methods used in the latter, we have been able to calculate the Rouquier blocks of all cyclotomic Hecke algebras of all exceptional complex reflection groups. Moreover, we discovered that the Rouquier blocks of a cyclotomic Hecke algebra depend on a numerical datum of the complex reflection group W, its *essential hyperplanes*.

Let us get into more details: The first two chapters of this thesis present results which are given here for the convenience of the reader. In the first chapter, which is dedicated on commutative algebra, we prove some results about divisibility and irreducibility which are going to be very useful in chapter 3. We also introduce the notions of "morphisms associated with monomials" and "adapted morphisms". The second chapter is the adaptation and generalization of classic results of block theory and representation theory of symmetric algebras, which can be found in [12] and [20]. We also give a criterion for an algebra to be split semisimple.

In the third chapter, we find the theoretical core of this thesis. Its aim is the determination of the Rouquier blocks of the cyclotomic Hecke algebras of complex reflection groups. We give the following explicit formula for the Schur elements associated with the irreducible characters of the generic Hecke algebra of a complex reflection group W: If K is the field of definition of W and $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$ is a set of indeterminates like above, then the Schur element $s_\chi(\mathbf{v})$ associated with the character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ is of the form

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i\in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- ξ_χ is an element of $\mathbb{Z}_K$,
- $N_\chi = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ is a monomial in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ with $\sum_{j=0}^{e_\mathcal{C}-1} b_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,
- I_χ is an index set,
- $(\Psi_{\chi,i})_{i\in I_\chi}$ is a family of K-cyclotomic polynomials in one variable (*i.e.,* minimal polynomials of the roots of unity over K),
- $(M_{\chi,i})_{i\in I_\chi}$ is a family of monomials in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and if $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, then $\gcd(a_{\mathcal{C},j}) = 1$ and $\sum_{j=0}^{e_\mathcal{C}-1} a_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,
- $(n_{\chi,i})_{i\in I_\chi}$ is a family of positive integers.

This factorization is unique in $K[\mathbf{v}, \mathbf{v}^{-1}]$ and the monomials $(M_{\chi,i})_{i\in I_\chi}$ are unique up to inversion. If $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ and $\Psi(M_{\chi,i})$ is a factor of $s_\chi(\mathbf{v})$ such that $\Psi_{\chi,i}(1) \in \mathfrak{p}$, then the monomial $M_{\chi,i}$ will be called $\mathfrak{p}$-*essential* for χ. We show that the more we specialize our algebra via morphisms associated with $\mathfrak{p}$-essential monomials, the more the size of its $\mathfrak{p}$-blocks becomes larger.

Now let $M := \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a $\mathfrak{p}$-essential monomial. The hyperplane defined in $\mathbb{C}^{\sum_{\mathcal{C}\in\mathcal{A}/W} e_\mathcal{C}}$ by the relation $\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0$, where $(t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_\mathcal{C}$ indeterminates, is called $\mathfrak{p}$-*essential hyperplane* for W. If $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ is a cyclotomic specialization, then the Rouquier blocks of $\mathcal{H}_\phi$ depend on which $\mathfrak{p}$-essential hyperplanes the $n_{\mathcal{C},j}$ belong (where $\mathfrak{p}$ runs over the prime ideals of $\mathbb{Z}_K$). Hence the Rouquier blocks of a cyclotomic Hecke algebra depend on a numerical datum of the group W.

The fourth chapter is the calculation part of this thesis. We present the algorithm and the results of the determination of the Rouquier blocks of all cyclotomic Hecke algebras of all exceptional complex reflection groups.

Chapter 1

On Commutative Algebra

Throughout this chapter, all rings are assumed to be commutative with 1. Moreover, if R is a ring and $x_0, x_1, \ldots, x_m$ is a set of indeterminates on R, then we denote by $R[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ the Laurent polynomial ring on $m+1$ indeterminates, *i.e.,* the ring $R[x_0, {x_0}^{-1}, x_1, {x_1}^{-1}, \ldots, x_m, {x_m}^{-1}]$.

1.1 Localizations

Definition 1.1.1 *Let R be a commutative ring with 1. We say that a subset S of R is a multiplicatively closed set if $0 \notin S$, $1 \in S$ and every finite product of elements of S belongs to S.*

In the set $R \times S$, we introduce an equivalence relation such that (r, s) is equivalent to (r', s') if and only if there exists $t \in S$ such that $t(s'r - sr') = 0$. We denote the equivalence class of (r, s) by r/s. The set of equivalence classes becomes a ring under the operations such that the sum and the product of r/s and r'/s' are given by $(s'r + sr')/ss'$ and rr'/ss' respectively. We denote this ring by $S^{-1}R$ and we call it the *localization* of R at S. If S contains no zero divisors of R, then any element r of R can be identified with the element $r/1$ of $S^{-1}R$ and we can regard the latter as an R-algebra.

Remarks:

- If S is the set of all non-zero divisors of R, then $S^{-1}R$ is called *the total quotient ring* of R. If, moreover, R is an integral domain, the total quotient ring of R is *the field of fractions* of R.
- If R is Notherian, then $S^{-1}R$ is Noetherian.

- If $\mathfrak{p}$ is a prime ideal of R, then the set $S := R - \mathfrak{p}$ is a multiplicatively closed subset of R. Then the ring $S^{-1}R$ is simply denoted by $R_{\mathfrak{p}}$.

The proofs for the following well known results concerning localizations can be found in [6].

Proposition 1.1.2 *Let A and B be two rings with multiplicative sets S and T respectively and f an homomorphism from A to B such that $f(S)$ is contained in T. There exists a unique homomorphism f' from $S^{-1}A$ to $T^{-1}B$ such that $f'(a/1) = f(a)/1$ for every $a \in A$. Let us suppose now that T is contained in the multiplicatively closed set of B generated by $f(S)$. If f is surjective (resp. injective), then f' is also surjective (resp. injective).*

Corollary 1.1.3 *Let A and B be two rings with multiplicative sets S and T respectively such that $A \subseteq B$ and $S \subseteq T$. Then $S^{-1}A \subseteq T^{-1}B$.*

Proposition 1.1.4 *Let A be a ring and S, T two multiplicative sets of A such that $S \subseteq T$. We have $S^{-1}A = T^{-1}A$ if and only if every prime ideal of R that meets T meets S.*

The following proposition and its corollary give us information about the ideals of the localization of a ring R at a multiplicatively closed subset S of R.

Proposition 1.1.5 *Let R be a ring and let S be a multiplicatively closed subset of R. Then*

1. *Every ideal $\mathfrak{b}'$ of $S^{-1}R$ is of the form $S^{-1}\mathfrak{b}$ for some ideal $\mathfrak{b}$ of R.*

2. *Let $\mathfrak{b}$ be an ideal of R and let f be the canonical surjection $R \twoheadrightarrow R/\mathfrak{b}$. Then $f(S)$ is a multiplicatively closed subset of $R/\mathfrak{b}$ and the homomorphism from $S^{-1}R$ to $(f(S))^{-1}(R/\mathfrak{b})$ canonically associated with f is surjective with kernel $\mathfrak{b}' = S^{-1}\mathfrak{b}$. By passing to quotients, an isomorphism between $(S^{-1}R)/\mathfrak{b}'$ and $(f(S))^{-1}(R/\mathfrak{b})$ is defined.*

3. *The application $\mathfrak{b}' \mapsto \mathfrak{b}$, restricted to the set of maximal (resp. prime) ideals of $S^{-1}R$, is an isomorphism (for the relation of inclusion) between this set and the set of maximal (resp. prime) ideals of R that do not meet S.*

4. *If $\mathfrak{q}'$ is a prime ideal of $S^{-1}R$ and $\mathfrak{q}$ is the prime ideal of R such that $\mathfrak{q}' = S^{-1}\mathfrak{q}$ (we have $\mathfrak{q} \cap S = \emptyset$), then there exists an isomorphism from $R_{\mathfrak{q}}$ to $(S^{-1}R)_{\mathfrak{q}'}$ which sends r/s to $(r/1)/(s/1)$ for $r \in R$, $s \in R - \mathfrak{q}$.*

Corollary 1.1.6 *Let R be a ring, $\mathfrak{p}$ a prime ideal of R and $S := R - \mathfrak{p}$. For every ideal $\mathfrak{b}$ of R which does not meet S, let $\mathfrak{b}' := \mathfrak{b}R_{\mathfrak{p}}$. Assume that $\mathfrak{b}' \neq R_{\mathfrak{p}}$. Then*

1. *Let f be the canonical surjection $R \twoheadrightarrow R/\mathfrak{b}$. The ring homomorphism from $R_{\mathfrak{p}}$ to $(R/\mathfrak{b})_{\mathfrak{p}/\mathfrak{b}}$ canonically associated with f is surjective and its kernel is $\mathfrak{b}'$. Thus it defines, by passing to quotients, a canonical isomorphism between $R_{\mathfrak{p}}/\mathfrak{b}'$ and $(R/\mathfrak{b})_{\mathfrak{p}/\mathfrak{b}}$.*

2. *The application $\mathfrak{b}' \mapsto \mathfrak{b}$, restricted to the set of prime ideals of $R_{\mathfrak{p}}$, is an isomorphism (for the relation of inclusion) between this set and the set of prime ideals of R contained in $\mathfrak{p}$ (thus do not meet S). Therefore, $\mathfrak{p}R_{\mathfrak{p}}$ is the only maximal ideal of $R_{\mathfrak{p}}$.*

3. *If now $\mathfrak{b}'$ is a prime ideal of $R_{\mathfrak{p}}$, then there exists an isomorphism from $R_{\mathfrak{b}}$ to $(R_{\mathfrak{p}})_{\mathfrak{b}'}$ which sends r/s to $(r/1)/(s/1)$ for $r \in R$, $s \in R - \mathfrak{b}$.*

The notion of localization can also be extended to the modules over the ring R.

Definition 1.1.7 *Let R be a ring and S a multiplicatively closed set of R. If M is an R-module, then we call localization of M at S and denote by $S^{-1}M$ the $S^{-1}R$-module $M \otimes_R S^{-1}R$.*

1.2 Integrally closed rings

Theorem-Definition 1.2.1 *Let R be a ring, A an R-algebra and a an element of A. The following properties are equivalent:*

(i) *The element a is a root of a monic polynomial with coefficients in R.*

(ii) *The subalgebra $R[a]$ of A is an R-module of finite type.*

(iii) *There exists a faithful $R[a]$-module which is an R-module of finite type.*

If $a \in A$ verifies the conditions above, we say that it is integral over R.

Definition 1.2.2 *Let R be a ring and A an R-algebra. The set of all elements of A that are integral over R is an R-subalgebra of A containing R; it is called the integral closure of R in A. We say that R is integrally closed in A, if R is an integral domain and if it coincides with its integral closure in A. If now R is an integral domain and F is its field of fractions, then the integral closure of R in F is named simply the integral closure of R, and if R is integrally closed in F, then R is said to be integrally closed.*

The following proposition ([7], §1, Prop.13) implies that transfer theorem holds for integrally closed rings (corollary 1.2.4).

Proposition 1.2.3 *If R is an integral domain, let us denote by $\bar{R}$ the integral closure of R. Let $x_0, \ldots, x_m$ be a set of indeterminates over R. Then the integral closure of $R[x_0, \ldots, x_m]$ is $\bar{R}[x_0, \ldots, x_m]$.*

Corollary 1.2.4 *Let R be an integral domain. Then $R[x_0, \ldots, x_m]$ is integrally closed if and only if R is integrally closed.*

Corollary 1.2.5 *If K is a field, then every polynomial ring $K[x_0, \ldots, x_m]$ is integrally closed.*

The next proposition ([7], §1, Prop.16) along with its corollaries treats the integral closures of localizations of rings.

Proposition 1.2.6 *Let R be a ring, A an R-algebra, $\bar{R}$ the integral closure of R in A and S a multiplicatively closed subset of R which contains no zero divisors. Then the integral closure of $S^{-1}R$ in $S^{-1}A$ is $S^{-1}\bar{R}$.*

Proof: Let b/s be an element of $S^{-1}\bar{R}$ ($s \in S, b \in \bar{R}$). Since the diagram

$$\begin{array}{ccc} R & \hookrightarrow & S^{-1}R \\ \downarrow & & \downarrow \\ A & \hookrightarrow & S^{-1}A \end{array}$$

commutes, the element $b/1$ is integral over $S^{-1}R$. Since $1/s \in S^{-1}R$, the element $b/s = (b/1)(1/s)$ is integral over $S^{-1}R$.

On the other hand, let a/t ($a \in A, t \in S$) be an element of $S^{-1}A$ integral over $S^{-1}R$. Then $a/1 = (t/1)(a/t)$ is integral over $S^{-1}R$. This means that there exist $r_i \in R$ ($1 \leq i \leq n$) and $s \in S$ such that

$$(a/1)^n + (r_1/s)(a/1)^{n-1} + \ldots + (r_n/s) = 0.$$

The above relation can also be written as

$$(sa^n + r_1 a^{n-1} + \ldots + r_n)/s = 0$$

and since S contains no zero divisors of R, we obtain that

$$sa^n + r_1 a^{n-1} + \ldots + r_n = 0.$$

Multiplying the above relation with s^{n-1},we deduce that

$$(sa)^n + r_1(sa)^{n-1} + \ldots + s^{n-1}r_n = 0.$$

Thus, by definition, we have $sa \in \bar{R}$. Therefore, $a/1 \in S^{-1}\bar{R}$ and hence, $a/t \in S^{-1}\bar{R}$. ■

Corollary 1.2.7 *Let R be an integral domain, $\bar{R}$ the integral closure of R and S a multiplicatively closed subset of R. Then the integral closure of $S^{-1}R$ is $S^{-1}\bar{R}$.*

Corollary 1.2.8 *If R is an integrally closed domain and S is a multiplicatively closed subset of R, then $S^{-1}R$ is also integrally closed.*

Lifting prime ideals

Definition 1.2.9 *Let R, R' be two rings and let $h : R \to R'$ be a ring homomorphism. We say that a prime ideal $\mathfrak{a}'$ of R' lies over a prime ideal $\mathfrak{a}$ of R, if $\mathfrak{a} = h^{-1}(\mathfrak{a}')$.*

The next result is [7], §2, Proposition 2.

Proposition 1.2.10 *Let $h : R \to R'$ be a ring homomorphism such that R' is integral over R. Let $\mathfrak{p}$ be a prime ideal of R, $S := R - \mathfrak{p}$ and $(\mathfrak{p}'_i)_{i\in I}$ the family of all the prime ideals of R' lying over $\mathfrak{p}$. If $S' = \bigcap_{i\in I}(R' - \mathfrak{p}'_i)$, then $S^{-1}R' = S'^{-1}R'$.*

Proof: By definition, we have $h(S) \subseteq S'$ and since $h(S)^{-1}R' \simeq S^{-1}R'$, it is enough to show that if a prime ideal $\mathfrak{q}'$ of R' doesn't meet $h(S)$, then it doesn't meet S' either (see proposition 1.1.4). Let us suppose that $\mathfrak{q}' \cap h(S) = \emptyset$ and let $\mathfrak{q} := h^{-1}(\mathfrak{q}')$. Then we have $\mathfrak{q} \cap S = \emptyset$, which means that $\mathfrak{q} \subseteq \mathfrak{p}$. Since $\mathfrak{q}'$ is lying over $\mathfrak{q}$ by definition, there exists an index $i \in I$ such that $\mathfrak{q}' \subseteq \mathfrak{p}'_i$. Therefore, $\mathfrak{q}' \cap S' = \emptyset$. ■

The following corollary deals with a case we will encounter in a following chapter, where there exists a unique prime ideal lying over the prime ideal $\mathfrak{p}$ of R. In combination with proposition 1.2.6, proposition 1.2.10 implies that

Corollary 1.2.11 *Let R be an integral domain, A an R-algebra, $\bar{R}$ the integral closure of R in A. Let $\mathfrak{p}$ be a prime ideal of R and $S := R - \mathfrak{p}$. If there exists a unique prime ideal $\bar{\mathfrak{p}}$ of $\bar{R}$ lying over $\mathfrak{p}$, then the integral closure of $R_{\mathfrak{p}}$ in $S^{-1}A$ is $\bar{R}_{\bar{\mathfrak{p}}}$.*

Valuations

Definition 1.2.12 *Let R be a ring and Γ a totally ordered abelian group. We call valuation of R with values in Γ every application $v : R \to \Gamma \cup \{\infty\}$ which satisfies the following properties:*

(V1) $v(xy) = v(x) + v(y)$ *for* $x \in R, y \in R$.

(V2) $v(x+y) \geq \inf(v(x), v(y))$ *for* $x \in R, y \in R$.

(V3) $v(1) = 0$ *and* $v(0) = \infty$.

In particular, if $v(x) \neq v(y)$, property (V2) gives $v(x+y) = \inf(v(x), v(y))$ for $x \in R, y \in R$. Moreover, from property (V1), we have that if $z \in R$ with $z^n = 1$ for some integer $n \geq 1$, then $nv(z) = v(z^n) = v(1) = 0$ and thus $v(z) = 0$. Consequently, $v(-x) = v(-1) + v(x) = v(x)$ for all $x \in R$.

Now let F be a field and let $v : F \to \Gamma$ be a valuation of F. The set A of $a \in F$ such that $v(a) \geq 0$ is a local subring of F. Its maximal ideal $\mathfrak{m}(A)$ is the set of $a \in A$ such that $v(a) > 0$. For all $a \in F - A$, we have $a^{-1} \in \mathfrak{m}(A)$. The ring A is called *the ring of the valuation* v on F.

We will now introduce the notion of a valuation ring. For more information about valuation rings and their properties, see [8]. Some of them will also be discussed in Chapter 2, Section 2.4.

Definition 1.2.13 *Let R be an integral domain contained in a field F. Then R is a valuation ring if for all non-zero element $x \in F$, we have $x \in R$ or $x^{-1} \in R$. Consequently, F is the field of fractions of R.*

If R is a valuation ring, then it has the following properties:

- It is an integrally closed local ring.
- The set of the principal ideals of R is totally ordered by the relation of inclusion.
- The set of the ideals of R is totally ordered by the relation of inclusion.

Let R be a valuation ring and F its field of fractions. Let us denote by $R^\times$ the set of units of R. Then the set $\Gamma_R := F^\times / R^\times$ is an abelian group, totally ordered by the relation of inclusion of the corresponding principal ideals. If we denote by v_R the canonical homomorphism of $F^\times$ onto Γ_R and set $v_R(0) = \infty$, then v_R is a valuation of F whose ring is R.

The following proposition gives a characterization of integrally closed rings in terms of valuation rings ([8], §1, Thm. 3).

Proposition 1.2.14 *Let R be a subring of a field F. The integral closure $\bar{R}$ of R in F is the intersection of all valuation rings in F which contain R. Consequently, an integral domain R is integrally closed if and only if it is the intersection of a family of valuation rings contained in its field of fractions.*

This characterization helped us to prove the following result about integrally closed rings.

Proposition 1.2.15 *Let R be an integrally closed ring and $f(x) = \sum_i a_i x^i$, $g(x) = \sum_j b_j x^j$ be two polynomials in $R[x]$. If there exists an element $c \in R$ such that all the coefficients of $f(x)g(x)$ belong to cR, then all the products $a_i b_j$ belong to cR.*

Proof: Due to the proposition 1.2.14, it is enough to prove the result in the case where R is a valuation ring.

From now on, let R be a valuation ring. Let v be a valuation of the field of fractions of R such that the ring of valuation of v is R. Let $\kappa := \inf_i(v(a_i))$ and $\lambda := \inf_j(v(b_j))$. Then $\kappa + \lambda = \inf_{i,j}(v(a_i b_j))$. We will show that $\kappa + \lambda \geq v(c)$ and thus c divides all the products $a_i b_j$. Argue by contradiction and assume that $\kappa + \lambda < v(c)$. Let $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ with $i_1 < i_2 < \ldots < i_r$ be all the elements among the a_i with valuation equal to κ. Respectively, let $b_{j_1}, b_{j_2}, \ldots, b_{j_s}$ with $j_1 < j_2 < \ldots < j_s$ be all the elements among the b_j with valuation equal to λ. We have that $i_1 + j_1 < i_m + j_n$, $\forall (m,n) \neq (1,1)$. Therefore, the coefficient $c_{i_1+j_1}$ of $x^{i_1+j_1}$ in $f(x)g(x)$ is of the form $(a_{i_1}b_{j_1} + \sum(\text{terms with valuation} > \kappa + \lambda))$ and since $v(a_{i_1}b_{j_1}) \neq v(\sum(\text{terms with valuation} > \kappa + \lambda))$, we obtain that

$$v(c_{i_1+j_1}) = \inf(v(a_{i_1}b_{j_1}), v(\sum(\text{terms with valuation} > \kappa + \lambda))) = \kappa + \lambda.$$

However, since all the coefficients of $f(x)g(x)$ are divisible by c, we have that $v(c_{i_1+j_1}) \geq v(c) > \kappa + \lambda$, which is a contradiction. ■

The propositions 1.2.16 and 1.2.18 derive from the one above. We will make use of the results in corollaries 1.2.17 and 1.2.19 in Chapter 3.

Proposition 1.2.16 *Let R be an integrally closed domain and let F be its field of fractions. Let $\mathfrak{p}$ be a prime ideal of R. Then*

$$(R[x])_{\mathfrak{p}R[x]} \cap F[x] = R_{\mathfrak{p}}[x].$$

Proof: The inclusion $R_{\mathfrak{p}}[x] \subseteq (R[x])_{\mathfrak{p}R[x]} \cap F[x]$ is obvious. Now, let $f(x)$ be an element of $F[x]$. Then $f(x)$ can be written in the form $r(x)/\xi$, where $r(x) \in R[x]$ and $\xi \in R$. If, moreover, $f(x)$ belongs to $(R[x])_{\mathfrak{p}R[x]}$, then there exist $s(x), t(x) \in R[x]$ with $t(x) \notin \mathfrak{p}R[x]$ such that $f(x) = s(x)/t(x)$. Thus we have

$$f(x) = \frac{r(x)}{\xi} = \frac{s(x)}{t(x)}.$$

All the coefficients of the product $r(x)t(x)$ belong to ξR. Due to proposition 1.2.15, if $r(x) = \sum_i a_i x^i$ and $t(x) = \sum_j b_j x^j$, then all the products $a_i b_j$ belong to ξR. Since $t(x) \notin \mathfrak{p}R[x]$, there exists j_0 such that $b_{j_0} \notin \mathfrak{p}$ and $a_i b_{j_0} \in \xi R, \forall i$. Consequently, $b_{j_0} f(x) = b_{j_0}(r(x)/\xi) \in R[x]$ and hence all the coefficients of $f(x)$ belong to $R_{\mathfrak{p}}$. ■

Corollary 1.2.17 *Let R be an integrally closed domain and let F be its field of fractions. Let $\mathfrak{p}$ be a prime ideal of R. Then*

1. $(R[x, x^{-1}])_{\mathfrak{p}R[x,x^{-1}]} \cap F[x, x^{-1}] = R_{\mathfrak{p}}[x, x^{-1}]$.
2. $(R[x_0, \ldots, x_m])_{\mathfrak{p}R[x_0,\ldots,x_m]} \cap F[x_0, \ldots, x_m] = R_{\mathfrak{p}}[x_0, \ldots, x_m]$.
3. $(R[x_0^{\pm 1}, \ldots, x_m^{\pm 1}])_{\mathfrak{p}R[x_0^{\pm 1},\ldots,x_m^{\pm 1}]} \cap F[x_0^{\pm 1}, \ldots, x_m^{\pm 1}] = R_{\mathfrak{p}}[x_0^{\pm 1}, \ldots, x_m^{\pm 1}]$.

Proposition 1.2.18 *Let R be an integrally closed domain and let F be its field of fractions. Let $r(x)$ and $s(x)$ be two elements of $R[x]$ such that $s(x)$ divides $r(x)$ in $F[x]$. If one of the coefficients of $s(x)$ is a unit in R, then $s(x)$ divides $r(x)$ in $R[x]$.*

Proof: Since $s(x)$ divides $r(x)$ in $F[x]$, there exists an element of the form $t(x)/\xi$ with $t(x) \in R[x]$ and $\xi \in R$ such that

$$r(x) = \frac{s(x)t(x)}{\xi}.$$

All the coefficients of the product $s(x)t(x)$ belong to ξR. Due to proposition 1.2.15, if $s(x) = \sum_i a_i x^i$ and $t(x) = \sum_j b_j x^j$, then all the products $a_i b_j$ belong to ξR. By assumption, there exists i_0 such that a_{i_0} is a unit in R and $a_{i_0} b_j \in \xi R, \forall j$. Consequently, $b_j \in \xi R, \forall j$ and thus $t(x)/\xi \in R[x]$. ■

Corollary 1.2.19 *Let R be an integrally closed domain and let F be its field of fractions. Let r, s be two elements of $R[x_0^{\pm 1}, \ldots, x_m^{\pm 1}]$ such that s divides r in $F[x_0^{\pm 1}, \ldots, x_m^{\pm 1}]$. If one of the coefficients of s is a unit in R, then s divides r in $R[x_0^{\pm 1}, \ldots, x_m^{\pm 1}]$.*

Discrete valuation rings and Krull rings

Definition 1.2.20 *Let F be a field, Γ a totally ordered abelian group and v a valuation of F with values in Γ. We say that the valuation v is discrete, if Γ is isomorphic to $\mathbb{Z}$.*

Theorem-Definition 1.2.21 *An integral domain R is a discrete valuation ring, if it satisfies one of the following equivalent conditions:*

(i) *R is the ring of a discrete valuation.*

(ii) *R is a local Dedekind ring.*

(iii) *R is a local principal ideal domain.*

(iv) *R is a Noetherian valuation ring.*

By proposition 1.2.14, integrally closed rings are intersections of valuation rings. Krull rings are essentially intersections of discrete valuation rings.

Definition 1.2.22 *An integral domain R is a Krull ring, if there exists a family of valuations $(v_i)_{i\in I}$ of the field of fractions F of R with the following properties:*

(K1) *The valuations $(v_i)_{i\in I}$ are discrete.*

(K2) *The intersection of the rings of $(v_i)_{i\in I}$ is R.*

(K3) *For all $x \in F^\times$,there exists a finite number of $i \in I$ such that $v_i(x) \neq 0$.*

The proofs of the following results and more information about Krull rings can be found in [9], §1.

Theorem 1.2.23 *Let R be an integral domain and let $\mathrm{Spec}_1(R)$ be the set of its prime ideals of height 1. Then R is a Krull ring if and only if the following properties are satisfied:*

1. *For all $\mathfrak{p} \in \mathrm{Spec}_1(R)$, $R_\mathfrak{p}$ is a discrete valuation ring.*
2. *R is the intersection of $R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}_1(R)$.*
3. *For all $r \neq 0$ in R, there exists a finite number of ideals $\mathfrak{p} \in \mathrm{Spec}_1(R)$ such that $r \in \mathfrak{p}$.*

Transfer theorem holds also for Krull rings.

Proposition 1.2.24 *Let R be a Krull ring, F the field of fractions of R and x an indeterminate. Then $R[x]$ is also a Krull ring. Moreover, its prime ideals of height 1 are:*

- *the prime ideals of the form $\mathfrak{p}R[x]$, where $\mathfrak{p}$ is a prime ideal of height 1 of R,*

- *the prime ideals of the form* $\mathfrak{m} \cap R[x]$*, where* $\mathfrak{m}$ *is a prime ideal of* $F[x]$*.*

The following proposition provides us with a simple characterization of Krull rings, when they are Noetherian.

Proposition 1.2.25 *Let R be a Noetherian ring. Then R is a Krull ring if and only if it is integrally closed.*

Example 1.2.26 Let K be a finite field extension of $\mathbb{Q}$ and $\mathbb{Z}_K$ the integral closure of $\mathbb{Z}$ in K. The ring $\mathbb{Z}_K$ is a Dedekind ring and thus Noetherian and integrally closed. Let $x_0, x_1, \ldots, x_m$ be indeterminates. Then the ring $\mathbb{Z}_K[x_0^{\pm}, x_1^{\pm}, \ldots, x_m^{\pm}]$ is also Noetherian and integrally closed and thus a Krull ring.

1.3 Completions

For all the following results concerning completions, the reader can refer to [35], Chapter II.

Let I be an ideal of a commutative ring R and let M be an R-module. We introduce a topology on M such that the open sets of M are unions of an arbitrary number of sets of the form $m + I^n M (m \in M)$. This topology is called the *I-adic topology* of M.

Theorem 1.3.1 *If M is a Noetherian R-module, then for any submodule N of M, the I-adic topology of N coincides with the topology of N as a subspace of M with the I-adic topology.*

From now on, we will concentrate on semi-local rings and in particular, on Noetherian semi-local rings.

Definition 1.3.2 *A ring R is called semi-local, if it has only a finite number of maximal ideals. The Jacobson radical $\mathfrak{m}$ of R is the intersection of the maximal ideals of R.*

Theorem 1.3.3 *Assume that R is a Noetherian semi-local ring with Jacobson radical $\mathfrak{m}$ and let $\hat{R}$ be the completion of R with respect to the $\mathfrak{m}$-adic topology. Then $\hat{R}$ is also a semi-local ring and we have $R \subseteq \hat{R}$.*

Theorem 1.3.4 *Assume that R is a Noetherian semi-local ring with Jacobson radical $\mathfrak{m}$ and that M is a finitely generated R-module. Let $\hat{R}$ be the completion of R with respect to the $\mathfrak{m}$-adic topology. Endow M with the $\mathfrak{m}$-adic topology. Then $M \otimes_R \hat{R}$ is the completion of M with respect to that topology.*

Corollary 1.3.5 *Let $\mathfrak{a}$ be an ideal of a Noetherian semi-local ring R with Jacobson radical $\mathfrak{m}$. Let $\hat{R}$ be the completion of R with respect to the $\mathfrak{m}$-adic topology. Then the completion of $\mathfrak{a}$ is $\mathfrak{a}\hat{R}$ and $\mathfrak{a}\hat{R}$ is isomorphic to $\mathfrak{a} \otimes_R \hat{R}$. Furthermore, $\mathfrak{a}\hat{R} \cap R = \mathfrak{a}$ and $\hat{R}/\mathfrak{a}\hat{R}$ is the completion of $R/\mathfrak{a}$ with respect to the $\mathfrak{m}$-adic topology.*

Theorem 1.3.6 *Assume that R is a Noetherian semi-local ring with Jacobson radical $\mathfrak{m}$ and let $\hat{R}$ be the completion of R with respect to the $\mathfrak{m}$-adic topology. Then*

1. *The total quotient ring F of R (localization of R at the set of non-zero divisors) is naturally a subring of the total quotient ring $\hat{F}$ of $\hat{R}$.*

2. *For any ideal $\mathfrak{a}$ of R, $\mathfrak{a}\hat{R} \cap F = \mathfrak{a}$.*

In particular, $\hat{R} \cap F = R$.

1.4 Morphisms associated with monomials

We have the following elementary algebra result

Theorem-Definition 1.4.1 *Let R be an integral domain and M a free R-module of basis $(e_i)_{0\leq i\leq m}$. Let $x = r_0e_0 + r_1e_1 + \ldots + r_me_m$ be a non-zero element of M. We set $M^* := \mathrm{Hom}_R(M, R)$ and $M^*(x) := \{\varphi(x) \,|\, \varphi \in M^*\}$. Then the following assertions are equivalent:*

(i) $M^*(x) = R$.

(ii) $\sum_{i=0}^m Rr_i = R$.

(iii) *There exists $\varphi \in M^*$ such that $\varphi(x) = 1$.*

(iv) *There exists an R-submodule N of M such that $M = Rx \oplus N$.*

If x satisfies the conditions above, we say that it is a primitive element of M.

Proof:

$(i) \Leftrightarrow (ii)$ Let $(e_i^*)_{0\leq i\leq m}$ be the basis of M^* dual to $(e_i)_{0\leq i\leq m}$. Then $M^*(x)$ is generated by $(e_i^*(x))_{0\leq i\leq m}$ and $e_i^*(x) = r_i$.

$(ii) \Rightarrow (iii)$ There exist $u_0, u_1, \ldots, u_m \in R$ such that $\sum_{i=0}^m u_ir_i = 1$. If $\varphi := \sum_{i=0}^m u_ie_i^*$, then $\varphi(x) = 1$.

$(iii) \Rightarrow (iv)$ Let $N := \mathrm{Ker}\varphi$. For all $y \in M$, we have

$$y = \varphi(y)x + (y - \varphi(y)x).$$

Since $\varphi(x) = 1$, we have $y \in Rx + N$. Obviously, $Rx \cap N = \{0\}$.

$(iv) \Rightarrow (i)$ Since M is free and R is an integral domain, x is torsion-free (otherwise there exists $r \in R, r \neq 0$ such that $\sum_{i=0}^{m}(rr_i)e_i = 0$). Therefore, the map

$$R \to M, r \mapsto rx$$

is an isomorphism of R-modules. Its inverse is a linear form on Rx which sends x to 1. Composing it with the map

$$M \twoheadrightarrow M/N \to R,$$

we obtain a linear form $\varphi : M \to R$ such that $\varphi(x) = 1$. We have $1 \in M^*(x)$ and thus $M^*(x) = R$. ■

We will apply the above result to the $\mathbb{Z}$-module $\mathbb{Z}^{m+1}$. Let us consider $(e_i)_{0\leq i\leq m}$ the standard basis of $\mathbb{Z}^{m+1}$ and let $a := a_0e_0 + a_1e_1 + \ldots + a_me_m$ be an element of $\mathbb{Z}^{m+1}$ such that $\gcd(a_i) = 1$. Then, by Bezout's theorem, there exist $u_0, u_1, \ldots, u_m \in \mathbb{Z}$ such that $\sum_{i=0}^{m} u_ia_i = 1$ and hence $\sum_{i=0}^{m} \mathbb{Z}a_i = \mathbb{Z}$. By theorem 1.4.1, there exists a $\mathbb{Z}$-submodule N_a of $\mathbb{Z}^{m+1}$ such that $\mathbb{Z}^{m+1} = \mathbb{Z}a \oplus N_a$. In particular, $N_a = \mathrm{Ker}\varphi_a$, where $\varphi_a := \sum_{i=0}^{m} u_ie_i^*$. We will denote by $p_a : \mathbb{Z}^{m+1} \twoheadrightarrow N_a$ the projection of $\mathbb{Z}^{m+1}$ onto N_a such that $\mathrm{Ker}p_a = \mathbb{Z}a$. We have a $\mathbb{Z}$-module isomorphism $i_a : N_a \tilde{\to} \mathbb{Z}^m$. Then $f_a := i_a \circ p_a$ is a surjective $\mathbb{Z}$-module morphism $\mathbb{Z}^{m+1} \twoheadrightarrow \mathbb{Z}^m$ with $\mathrm{Ker}f_a = \mathbb{Z}a$.

Now let R be an integral domain and let $x_0, x_1, \ldots, x_m$ be $m+1$ indeterminates over R. Let G be the abelian group generated by all the monomials in $R[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ with group operation the multiplication. Then G is isomorphic to the additive group $\mathbb{Z}^{m+1}$ by the isomorphism defined as follows

$$\begin{array}{rccc} \theta_G : & G & \tilde{\to} & \mathbb{Z}^{m+1} \\ & \prod_{i=0}^{m} x_i^{l_i} & \mapsto & (l_0, l_1, \ldots, l_m). \end{array}$$

Lemma 1.4.2 *We have* $R[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}] = R[G] \simeq R[\mathbb{Z}^{m+1}]$.

Respectively, if $y_1, \ldots, y_m$ are m indeterminates over R and H is the group generated by all the monomials in $R[y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$, then $H \simeq \mathbb{Z}^m$ and $R[y_1^{\pm 1}, \ldots, y_m^{\pm 1}] = R[H] \simeq R[\mathbb{Z}^m]$.

The morphism $F_a := {\theta_H}^{-1} \circ f_a \circ \theta_G : G \twoheadrightarrow H$ induces an R-algebra morphism

$$\begin{array}{rccc} \varphi_a : & R[G] & \to & R[H] \\ & \sum_{g\in G} a_g g & \mapsto & \sum_{g\in G} a_g F_a(g) \end{array}$$

Since F_a is surjective, the morphism φ_a is also surjective. Moreover, $\mathrm{Ker}\varphi_a$ is generated (as an R-module) by the set

$$< g-1 \,|\, g \in G \text{ such that } \theta_G(g) \in \mathbb{Z}a > .$$

How do we translate this in the polynomial language?

Let $A := R[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ and $B := R[y_1^{\pm 1}, y_2^{\pm 1}, \ldots, y_m^{\pm 1}]$. The map φ_a is a surjective R-algebra morphism from A to B with $\mathrm{Ker}\varphi_a = (\prod_{i=0}^m x_i^{a_i} - 1)A$ such that for every monomial N in A, $\varphi_a(N)$ is a monomial in B.

Definition 1.4.3 *Let $M := \prod_{i=0}^m x_i^{a_i}$ be a monomial in A with $\gcd(a_i) = 1$. An R-algebra morphism $\varphi_M : A \to B$ defined as above will be called associated with the monomial M.*

Example 1.4.4 Let $A := R[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}]$ and $M := X^5Y^{-3}Z^{-2}$. We have $x := (5, -3, -2)$ and

$$(-1)\cdot 5 + (-2)\cdot(-3) + 0\cdot(-2) = 1.$$

Hence, with the notations of the proof of theorem 1.4.1, we have

$$u_0 = -1, u_1 = -2, u_3 = 0.$$

The map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}$ defined as

$$\varphi := -e_0^* - 2e_1^*.$$

has $\mathrm{Ker}\varphi = \{(x_0, x_1, x_2) \in \mathbb{Z}^3 \,|\, x_0 = -2x_1\} = \{(-2a, a, b) \,|\, a, b \in \mathbb{Z}\} =: N$.

By theorem 1.4.1, we have $\mathbb{Z}^3 = \mathbb{Z}x \oplus N$ and the projection $p : \mathbb{Z}^3 \twoheadrightarrow N$ is the map

$$(y_0, y_1, y_2) =: y \mapsto y - \varphi(y)x = (6y_0 + 10y_1, -3y_0 - 5y_1, -2y_0 - 4y_1 + y_2).$$

The $\mathbb{Z}$-module N is obviously isomorphic to $\mathbb{Z}^2$ via

$$i : (-2a, a, b) \mapsto (a, b).$$

Composing the two previous maps, we obtain a well defined surjection

$$\begin{array}{ccc} \mathbb{Z}^3 & \twoheadrightarrow & \mathbb{Z}^2 \\ (y_0, y_1, y_2) & \mapsto & (-3y_0 - 5y_1, -2y_0 - 4y_1 + y_2). \end{array}$$

The above surjection induces (in the way described before) an R-algebra epimorphism

$$\begin{array}{rccc} \varphi_M : & R[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}] & \twoheadrightarrow & R[X^{\pm 1}, Y^{\pm 1}] \\ & X & \mapsto & X^{-3}Y^{-2} \\ & Y & \mapsto & X^{-5}Y^{-4} \\ & Z & \mapsto & Y \end{array}$$

By straightforward calculations, we can verify that $\mathrm{Ker}\varphi_M = (M-1)A$.

Lemma 1.4.5 *Let $M := \prod_{i=0}^{m} x_i^{a_i}$ be a monomial in A such that $\gcd(a_i) = 1$. Then*

1. *The ideal $(M-1)A$ is a prime ideal of A.*
2. *If $\mathfrak{p}$ is a prime ideal of R, then the ideal $\mathfrak{q}_M := \mathfrak{p}A + (M-1)A$ is also prime in A.*

Proof:

1. Let $\varphi_M : A \to B$ be a morphism associated with M. Then φ_M is surjective and $\mathrm{Ker}\varphi_M = (M-1)A$. Since B is an integral domain, the ideal generated by $(M-1)$ is prime in A.
2. Set $R' := R/\mathfrak{p}$, $A' := R'[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, $B' := R'[y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$. Then R' is an integral domain and

$$A/\mathfrak{q}_M \simeq A'/(M-1)A' \simeq B'.$$

Since B' is an integral domain, the ideal $\mathfrak{q}_M$ is prime in A. ■

The following assertions are now straightforward. Nevertheless, they are stated for further reference.

Proposition 1.4.6 *Let $M := \prod_{i=0}^{m} x_i^{a_i}$ be a monomial in A with $\gcd(a_i) = 1$ and let $\varphi_M : A \to B$ be a morphism associated with M. Let $\mathfrak{p}$ be a prime ideal of R and set $\mathfrak{q}_M := \mathfrak{p}A + (M-1)A$. Then the morphism φ_M has the following properties:*

1. *If $f \in A$, then $\varphi_M(f) \in \mathfrak{p}B$ if and only if $f \in \mathfrak{q}_M$. Corollary* 1.1.6(1) *implies that*

$$A_{\mathfrak{q}_M}/(M-1)A_{\mathfrak{q}_M} \simeq B_{\mathfrak{p}B}.$$

2. *If N is a monomial in A, then $\varphi_M(N) = 1$ if and only if there exists $k \in \mathbb{Z}$ such that $N = M^k$.*

Corollary 1.4.7 $\mathfrak{q}_M \cap R = \mathfrak{p}$.

Proof: Obviously $\mathfrak{p} \subseteq \mathfrak{q}_M \cap R$. Let $\xi \in R$ such that $\xi \in \mathfrak{q}_M$. If φ_M is a morphism associated with M, then, by proposition 1.4.6, $\varphi_M(\xi) \in \mathfrak{p}B$. But $\varphi_M(\xi) = \xi$ and $\mathfrak{p}B \cap R = \mathfrak{p}$. Thus $\xi \in \mathfrak{p}$. ■

Remark: If $m = 0$ and we set $x := x_0$, then $A := R[x, x^{-1}]$ and $B := R$. The only monomials that we can associate a morphism $A \to B$ with are x and x^{-1}. This morphism is unique and given by $x \mapsto 1$.

The following lemma, whose proof is straightforward when arguing by contradiction, will be used in the proofs of propositions 1.4.9 and 1.4.12.

Lemma 1.4.8 *Let G, H be two groups and $p : G \to H$ a group homomorphism. If R is an integral domain, let us denote by $p_R : R[G] \to R[H]$ the R-algebra morphism induced by p. If p_R is surjective, then p is also surjective.*

Proposition 1.4.9 *Let $\varphi : A \to B$ be a surjective R-algebra morphism such that for every monomial M in A, $\varphi(M)$ is a monomial in B. Then φ is associated with a monomial in A.*

Proof: Due to the isomorphism of lemma 1.4.2, φ can be considered as a surjective R-algebra morphism

$$\varphi : R[\mathbb{Z}^{m+1}] \to R[\mathbb{Z}^m].$$

The property of φ about the monomials implies that the above morphism is induced by a $\mathbb{Z}$-module morphism $f : \mathbb{Z}^{m+1} \to \mathbb{Z}^m$, which is also surjective by lemma 1.4.8. Since $\mathbb{Z}^m$ is a free $\mathbb{Z}$-module the following exact sequence sequence splits

$$0 \to \mathrm{Ker} f \to \mathbb{Z}^{m+1} \to \mathbb{Z}^m \to 0$$

and we obtain that $\mathbb{Z}^{m+1} \simeq \mathrm{Ker} f \oplus \mathbb{Z}^m$. Therefore, $\mathrm{Ker} f$ is a $\mathbb{Z}$-module of rank 1 and there exists $a := (a_0, a_1, \ldots, a_m) \in \mathbb{Z}^{m+1}$ such that $\mathrm{Ker} f = \mathbb{Z}a$. By theorem 1.4.1, a is a primitive element of $\mathbb{Z}^{m+1}$ and we must have $\sum_{i=0}^{m} \mathbb{Z}a_i = \mathbb{Z}$, whence $\gcd(a_i) = 1$. By definition, the morphism φ is associated with the monomial $\prod_{i=0}^{m} x_i^{a_i}$. ■

Now let $r \in \{1, \ldots, m+1\}$ and $\mathcal{R} := \begin{cases} R[y_r^{\pm 1}, \ldots, y_m^{\pm 1}], & \text{for } 1 \leq r \leq m; \\ R, & \text{for } r = m+1, \end{cases}$
where $y_r, \ldots, y_m$ are $m - r + 1$ indeterminates over R.

Definition 1.4.10 *An R-algebra morphism $\varphi : A \to \mathcal{R}$ is called adapted, if $\varphi = \varphi_r \circ \varphi_{r-1} \circ \ldots \circ \varphi_1$, where φ_i is a morphism associated with a monomial for all $i = 1, \ldots, r$. The family $\mathcal{F} := \{\varphi_r, \varphi_{r-1}, \ldots, \varphi_1\}$ is called an adapted family for φ whose initial morphism is φ_1.*

Let us introduce the following notation: If $M := \prod_{i=0}^{m} x_i^{c_i}$ is a monomial such that $\gcd(c_i) = d \in \mathbb{Z}$, then $M^\circ := \prod_{i=0}^{m} x_i^{c_i/d}$.

Proposition 1.4.11 *Let $\varphi : A \to \mathcal{R}$ be an adapted morphism and M a monomial in A such that $\varphi(M) = 1$. Then there exists an adapted family for φ whose initial morphism is associated with M°.*

Proof: Let $M := \prod_{i=0}^{m} x_i^{c_i}$ be a monomial in A such that $\varphi(M) = 1$. Note that $\varphi(M) = 1$ if and only if $\varphi(M^\circ) = 1$. Therefore, we can assume that $\gcd(c_i) = 1$. We will prove the desired result by induction on r.

- For $r = 1$, due to property 1.4.6(2), φ must be a morphism associated with M.
- For $r = 2$, set $B := R[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$. Let $\varphi := \varphi_b \circ \varphi_a$, where
  - $\varphi_a : A \to B$ is a morphism associated with a monomial $\prod_{i=0}^{m} x_i^{a_i}$ in A such that $\gcd(a_i) = 1$.
  - $\varphi_b : B \to \mathcal{R}$ is a morphism associated with a monomial $\prod_{j=1}^{m} z_j^{b_j}$ in B such that $\gcd(b_j) = 1$.

  By theorem 1.4.1, the element $a := (a_0, a_1, \ldots, a_m)$ is a primitive element of $\mathbb{Z}^{m+1}$ and the element $b := (b_1, \ldots, b_m)$ is a primitive element of $\mathbb{Z}^m$. Therefore, there exist a $\mathbb{Z}$-submodule N_a of $\mathbb{Z}^{m+1}$ and a $\mathbb{Z}$-submodule N_b of $\mathbb{Z}^m$ such that $\mathbb{Z}^{m+1} = \mathbb{Z}a \oplus N_a$ and $\mathbb{Z}^m = \mathbb{Z}b \oplus N_b$. We will denote by $p_a : \mathbb{Z}^{m+1} \twoheadrightarrow N_a$ the projection of $\mathbb{Z}^{m+1}$ onto N_a and by $p_b : \mathbb{Z}^m \twoheadrightarrow N_b$ the projection of $\mathbb{Z}^m$ onto N_b. We have isomorphisms $i_a : N_a \xrightarrow{\sim} \mathbb{Z}^m$ and $i_b : N_b \xrightarrow{\sim} \mathbb{Z}^{m-1}$.

  By definition of the associated morphism, φ_a is induced by the morphism $f_a := i_a \circ p_a : \mathbb{Z}^{m+1} \twoheadrightarrow \mathbb{Z}^m$ and φ_b by $f_b := i_b \circ p_b : \mathbb{Z}^m \twoheadrightarrow \mathbb{Z}^{m-1}$. Set $f := f_b \circ f_a$. Then φ is the R-algebra morphism induced by f.

  The morphism f is surjective. Since $\mathbb{Z}^{m-1}$ is a free $\mathbb{Z}$-module, the following exact sequence sequence splits

  $$0 \to \mathrm{Ker} f \to \mathbb{Z}^{m+1} \to \mathbb{Z}^{m-1} \to 0$$

  and we obtain that $\mathbb{Z}^{m+1} \simeq \mathrm{Ker} f \oplus \mathbb{Z}^{m-1}$.

Let $\tilde{b} := i_a^{-1}(b)$. Then $\mathrm{Ker}f = \mathbb{Z}a \oplus \mathbb{Z}\tilde{b}$. By assumption, we have that $c := (c_0, c_1, \ldots, c_m) \in \mathrm{Ker}f$. Therefore, there exist unique $\lambda_1, \lambda_2 \in \mathbb{Z}$ such that $c = \lambda_1 a + \lambda_2 \tilde{b}$. Since $\gcd(c_i) = 1$, we must also have $\gcd(\lambda_1, \lambda_2) = 1$. Hence $\sum_{i=1}^{2} \mathbb{Z}\lambda_i = \mathbb{Z}$. By applying theorem 1.4.1 to the $\mathbb{Z}$-module $\mathrm{Ker}f$, we obtain that c is a primitive element of $\mathrm{Ker}f$. Consequently, $f = f' \circ f_c$, where f_c is a surjective $\mathbb{Z}$-module morphism $\mathbb{Z}^{m+1} \twoheadrightarrow \mathbb{Z}^m$ with $\mathrm{Ker}f_c = \mathbb{Z}c$ and f' is a surjective $\mathbb{Z}$-module morphism $\mathbb{Z}^m \twoheadrightarrow \mathbb{Z}^{m-1}$. As far as the induced R-algebra morphisms are concerned, we obtain that $\varphi = \varphi' \circ \varphi_c$, where φ_c is a morphism associated with the monomial $M = \prod_{i=0}^{m} x_i^{c_i}$ and φ' is also a morphism associated with a monomial (by proposition 1.4.9). Thus the assertion is proven.

- for $r > 2$, let us suppose that the proposition is true for $1, 2, \ldots, r-1$. If $\varphi = \varphi_r \circ \varphi_{r-1} \circ \ldots \circ \varphi_1$, the induction hypothesis implies that there exist morphisms associated with monomials $\varphi'_r, \varphi'_{r-1}, \ldots, \varphi'_2$ such that

  1. $\varphi = \varphi'_r \circ \ldots \circ \varphi'_2 \circ \varphi_1$.
  2. φ'_2 is associated with the monomial $(\varphi_1(M))^\circ$.

  We have that $\varphi'_2(\varphi_1(M)) = 1$. Once more, by induction hypothesis we obtain that there exist morphisms associated with monomials φ''_2, φ''_1 such that

  1. $\varphi'_2 \circ \varphi_1 = \varphi''_2 \circ \varphi''_1$.
  2. φ''_1 is associated with M°.

  Thus we have

  $$\varphi = \varphi'_r \circ \ldots \circ \varphi'_3 \circ \varphi'_2 \circ \varphi_1 = \varphi'_r \circ \ldots \circ \varphi'_3 \circ \varphi''_2 \circ \varphi''_1$$

  and φ''_1 is associated with M°. ■

Proposition 1.4.12 *Let $\varphi : A \to \mathcal{R}$ be a surjective R-algebra morphism such that for every monomial M in A, $\varphi(M)$ is a monomial in $\mathcal{R}$. Then φ is an adapted morphism.*

Proof: We will work again by induction on r. For $r = 1$, the above result is proposition 1.4.9. For $r > 1$, let us suppose that the result is true for $1, \ldots, r-1$. Due to the isomorphism of lemma 1.4.2, φ can be considered as a surjective R-algebra morphism

$$\varphi : R[\mathbb{Z}^{m+1}] \to R[\mathbb{Z}^{m+1-r}].$$

The property of φ about the monomials implies that the above morphism is induced by a $\mathbb{Z}$-module morphism $f : \mathbb{Z}^{m+1} \to \mathbb{Z}^{m+1-r}$, which is also surjective by lemma 1.4.8. Since $\mathbb{Z}^{m+1-r}$ is a free $\mathbb{Z}$-module the following exact sequence sequence splits

$$0 \to \mathrm{Ker}f \to \mathbb{Z}^{m+1} \to \mathbb{Z}^{m+1-r} \to 0$$

and we obtain that $\mathbb{Z}^{m+1} \simeq \mathrm{Ker}f \oplus \mathbb{Z}^{m+1-r}$. Therefore, $\mathrm{Ker}f$ is a $\mathbb{Z}$-module of rank r, *i.e.,* $\mathrm{Ker}f \simeq \mathbb{Z}^r$. We choose a primitive element a of $\mathrm{Ker}f$. Then there exists a $\mathbb{Z}$-submodule N_a of $\mathrm{Ker}f$ such that $\mathrm{Ker}f = \mathbb{Z}a \oplus N_a$. Since $\mathrm{Ker}f$ is a direct summand of $\mathbb{Z}^{m+1}$, a is also a primitive element of $\mathbb{Z}^{m+1}$ and we have

$$\mathbb{Z}^{m+1} \simeq \mathbb{Z}a \oplus N_a \oplus \mathbb{Z}^{m+1-r}.$$

Thus, by theorem 1.4.1, if $(a_0, a_1, \ldots, a_m)$ are the coefficients of a with respect to the standard basis of $\mathbb{Z}^{m+1}$, then $\gcd(a_i) = 1$.

Let us denote by p_a the projection $\mathbb{Z}^{m+1} \twoheadrightarrow N_a \oplus \mathbb{Z}^{m+1-r}$ and by p' the projection $N_a \oplus \mathbb{Z}^{m+1-r} \twoheadrightarrow \mathbb{Z}^{m+1-r}$. Then $f = p' \circ p_a$. We have a $\mathbb{Z}$-module isomorphism

$$i : N_a \oplus \mathbb{Z}^{m+1-r} \xrightarrow{\sim} \mathbb{Z}^m.$$

Set $f_a := i \circ p_a$ and $f' := p' \circ i^{-1}$. Thus $f = f' \circ f_a$. If φ_a is the R-algebra morphism induced by f_a, then, by definition, φ_a is a morphism associated with the monomial $\prod_{i=0}^{m} x_i^{a_i}$. The R-algebra morphism φ' induced by f' is a surjective morphism with the same property about monomials as φ (it sends every monomial to a monomial). By induction hypothesis, φ' is an adapted morphism. We have $\varphi = \varphi' \circ \varphi_a$ and so φ is also an adapted morphism. ■

1.5 Irreducibility

Let k be a field and y an indeterminate over k. We can use the following theorem in order to determine the irreducibility of a polynomial of the form $y^n - a$ in $k[y]$ (cf.[25], Chapter 6, Thm. 9.1).

Theorem 1.5.1 *Let k be a field, $a \in k - \{0\}$ and $n \in \mathbb{Z}$ with $n \geq 2$. The polynomial $y^n - a$ is irreducible in $k[y]$, if for every prime p dividing n, we have $a \notin k^p$ and if 4 divides n, we have $a \notin -4k^4$.*

Let $x_0, x_1, \ldots, x_m$ be a set of $m+1$ indeterminates. We will apply theorem 1.5.1 to the field $k(x_1, \ldots, x_m)$.

Lemma 1.5.2 *Let k be a field. The polynomial $x_0^{a_0} - \rho \prod_{i=1}^m x_i^{a_i}$ with $\rho \in k - \{0\}$, $a_i \in \mathbb{Z}$, $\gcd(a_i) = 1$ and $a_0 > 0$ is irreducible in $k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}][x_0]$.*

Proof: If $a_0 = 1$, the polynomial is of degree 1 and thus irreducible in $k(x_1, \ldots, x_m)[x_0]$. If $a_0 \geq 2$, let us suppose that

$$\rho \prod_{i=1}^m x_i^{a_i} = (\frac{f(x_1, \ldots, x_m)}{g(x_1, \ldots, x_m)})^p$$

with $f(x_1, \ldots, x_m), g(x_1, \ldots, x_m) \in k[x_1, \ldots, x_m]$ prime to each other, $g(x_1, \ldots, x_m) \neq 0$ and $p|a_0$, p prime. This relation can be written as

$$g(x_1, \ldots, x_m)^p \rho \prod_{\{i|a_i \geq 0\}} x_i^{a_i} = f(x_1, \ldots, x_m)^p \prod_{\{i|a_i < 0\}} x_i^{-a_i}.$$

We have that

$$\gcd(f(x_1, \ldots, x_m)^p, g(x_1, \ldots, x_m)^p) = 1.$$

Since $k[x_1, \ldots, x_m]$ is a unique factorization domain and x_i are primes in $k[x_1, \ldots, x_m]$, we also have that

$$\gcd(\prod_{\{i|a_i \geq 0\}} x_i^{a_i}, \prod_{\{i|a_i < 0\}} x_i^{-a_i}) = 1.$$

As a consequence,

$$f(x_1, \ldots, x_m)^p = \lambda \rho \prod_{\{i|a_i \geq 0\}} x_i^{a_i} \text{ and } g(x_1, \ldots, x_m)^p = \lambda \prod_{\{i|a_i < 0\}} x_i^{-a_i}.$$

for some $\lambda \in k - \{0\}$. Suppose that $(\lambda\rho)^{1/p}, \lambda^{1/p} \in k$. Once more, the fact that $k[x_1, \ldots, x_m]$ is a unique factorization domain and x_i are primes in $k[x_1, \ldots, x_m]$ implies that

$$f(x_1, \ldots, x_m) = (\lambda\rho)^{1/p} \prod_{\{i|a_i \geq 0\}} x_i^{b_i} \text{ and } g(x_1, \ldots, x_m) = \lambda^{1/p} \prod_{\{i|a_i < 0\}} x_i^{-b_i},$$

with $b_i \in \mathbb{Z}$ and $b_i p = a_i, \forall i = 1, \ldots, m$. Since $p|a_0$, this contradicts the fact that $\gcd(a_i) = 1$. In the same way, we can show that if $4|a_0$, then $\rho \prod_{i=1}^m x_i^{a_i} \notin -4k(x_1, \ldots, x_m)^4$. Thus, by theorem 1.5.1, $x_0^{a_0} - \rho \prod_{i=1}^m x_i^{a_i}$ is irreducible in $k(x_1, \ldots, x_m)[x_0]$.

Thanks to the following lemma, we obtain that $x_0^{a_0} - \rho \prod_{i=1}^m x_i^{a_i}$ is irreducible in $k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}][x_0]$.

Lemma 1.5.3 *Let R be an integral domain with field of fractions F and $f(x)$ a polynomial in $R[x]$. If $f(x)$ is irreducible in $F[x]$ and at least one of its coefficients is a unit in R, then $f(x)$ is irreducible in $R[x]$.*

Proof: If $f(x) = g(x)h(x)$ for two polynomials $g(x), h(x) \in R[x]$, then $g(x) \in R$ or $h(x) \in R$. Let us suppose that $g(x) \in R$. Since one of the coefficients of $f(x)$ is a unit in R, $g(x)$ must also be a unit in R. Thus, $f(x)$ is irreducible in $R[x]$. ■

■

Lemma 1.5.2 implies the following proposition, which in turn is going to be used in the proof of proposition 1.5.5.

Proposition 1.5.4 *Let $M := \prod_{i=0}^{m} x_i^{a_i}$ be a monomial in $k[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ such that $\gcd(a_i) = 1$ and let $\rho \in k - \{0\}$. Then $M - \rho$ is an irreducible element of $k[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$.*

Proof: Since $\gcd(a_i) = 1$, we can suppose that $a_0 \neq 0$. Then it is enough to show that $M - \rho$ is irreducible in the polynomial ring $k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}][x_0]$. If $a_0 > 0$, then $M - \rho$ is irreducible in $k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}][x_0]$ by lemma 1.5.2. If now $a_0 < 0$, then lemma 1.5.2 implies that $M^{-1} - \rho^{-1}$ is irreducible in $k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}][x_0]$ and hence $M - \rho$ is also irreducible in $k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}][x_0]$. ■

Proposition 1.5.5 *Let $M := \prod_{i=0}^{m} x_i^{a_i}$ be a monomial in $k[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ such that $\gcd(a_i) = 1$. If $f(x)$ is an irreducible element of $k[x]$ such that $f(x) \neq x$, then $f(M)$ is irreducible in $k[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$.*

Proof: Suppose that $f(M) = g \cdot h$ with $g, h \in k[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Let $\rho_1, \ldots, \rho_n$ be the roots of $f(x)$ in a splitting field k'. Then

$$f(x) = a(x - \rho_1) \ldots (x - \rho_n)$$

for some $a \in k - \{0\}$, hence

$$f(M) = a(M - \rho_1) \ldots (M - \rho_n).$$

By proposition 1.5.4, $M - \rho_j$ is irreducible in $k'[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ for all $j \in \{1, \ldots, n\}$. Since $k'[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ is a unique factorization domain, we must have

$$g = r \prod_{i=0}^{m} x_i^{b_i} (M - \rho_{j_1}) \ldots (M - \rho_{j_s})$$

for some $r \in k$, $b_i \in \mathbb{Z}$ and $j_1, \ldots, j_s \in \{1, \ldots, n\}$. Thus there exists $g'(x) \in k[x]$ such that

$$g = (\prod_{i=0}^{m} x_i^{b_i}) g'(M).$$

Respectively, there exists $h'(x) \in k[x]$ such that

$$h = (\prod_{i=0}^{m} x_i^{-b_i}) h'(M).$$

Thus, we obtain that

$$f(M) = g'(M)h'(M). \quad (\dagger)$$

Since $\gcd(a_i) = 1$, there exist integers $(u_i)_{0 \leq i \leq m}$ such that $\sum_{i=0}^{m} u_i a_i = 1$. Let us now consider the k-algebra specialization

$$\begin{array}{rccc} \varphi: & k[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}] & \to & k[x] \\ & x_i & \mapsto & x^{u_i}. \end{array}$$

Then $\varphi(M) = \varphi(\prod_{i=0}^{m} x_i^{a_i}) = x^{\sum_{i=0}^{m} u_i a_i} = x$. If we apply φ to the relation $(\dagger)$, we obtain that

$$f(x) = g'(x)h'(x).$$

Since $f(x)$ is irreducible in $k[x]$, we must have that either $g'(x) \in k$ or $h'(x) \in k$. Respectively, we deduce that either g or h is a unit in $k[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. ■

Chapter 2

On Blocks

All the results presented in the first two sections of this chapter have been taken from the first part of [12].

2.1 Generalities

Let $\mathcal{O}$ be a commutative ring with a unit element and A be an $\mathcal{O}$-algebra. We denote by ZA the center of A.

An idempotent in A is an element e such that $e^2 = e$. We say that e is a central idempotent, if it is an idempotent in ZA. Two idempotents e_1, e_2 are orthogonal, if $e_1e_2 = e_2e_1 = 0$. Finally, an idempotent e is primitive, if $e \neq 0$ and e can not be expressed as the sum of two non-zero orthogonal idempotents.

Definition 2.1.1 *The block-idempotents of A are the central primitive idempotents of A.*

Let e be a block-idempotent of A. The two sided ideal Ae inherits a structure of algebra, where the composition laws are those of A and the unit element is e. The application

$$\begin{array}{rccc} \pi_e : & A & \to & Ae \\ & h & \mapsto & he \end{array}$$

is an epimorphism of algebras. The algebra Ae is called a block of A. From now on, abusing the language, we will also call blocks the block-idempotents of A.

Lemma 2.1.2 *The blocks of A are mutually orthogonal.*

Proof: Let e be a block and f a central idempotent of A with $f \neq e$. Then ef and $e-ef$ are also central idempotents. We have $e = ef + (e-ef)$ and due to the primitivity of e, we deduce that either $ef = 0$ or $e = ef$. If f is a block too, then either $ef = 0$ or $f = ef = e$. Therefore, f is orthogonal to e. ■

The above lemma gives rise to the following proposition.

Proposition 2.1.3 *Suppose that the unit element* 1 *of* A *can be expressed as a sum of blocks:* $1 = \sum_{e \in E} e$. *Then*

1. *The set* E *is the set of all the blocks of* A.
2. *The family of morphisms* $(\pi_e)_{e \in E}$ *defines an isomorphism of algebras*

$$A \xrightarrow{\sim} \prod_{e \in E} Ae.$$

Proof: If f is a block, then $f = \sum_{e \in E} ef$. Due to lemma 2.1.2, there exists $e \in E$ such that $f = e$. ■

In the above context (1 is a sum of blocks), let us denote by $\mathrm{Bl}(A)$ the set of all the blocks of A. Proposition 2.1.3 implies that the category ${}_A\mathbf{mod}$ of A-modules is a direct sum of the categories associated with the blocks:

$${}_A\mathbf{mod} \xrightarrow{\sim} \bigoplus_{e \in \mathrm{Bl}(A)} {}_{Ae}\mathbf{mod}.$$

In particular, every representation of the $\mathcal{O}$-algebra Ae defines (by composition with π_e) a representation of A and we say, abusing the language, that it "belongs to the block e".

Every indecomposable representation of A belongs to one and only one block. Thus the following partitions are defined:

$$\mathrm{Ind}(A) = \bigsqcup_{e \in \mathrm{Bl}(A)} \mathrm{Ind}(A, e) \quad \text{and} \quad \mathrm{Irr}(A) = \bigsqcup_{e \in \mathrm{Bl}(A)} \mathrm{Irr}(A, e),$$

where $\mathrm{Ind}(A)$ (resp. $\mathrm{Irr}(A)$) denotes the set of indecomposable (resp. irreducible) representations of A and $\mathrm{Ind}(A, e)$ (resp. $\mathrm{Irr}(A, e)$) denotes the set of the elements of $\mathrm{Ind}(A)$ (resp. $\mathrm{Irr}(A)$) which belong to e.

We will consider two situations where 1 is a sum of blocks.

First case: Suppose that 1 is a sum of orthogonal primitive idempotents, *i.e.,* $1 = \sum_{i \in P} i$, where

- every $i \in P$ is a primitive idempotent,
- if $i, j \in P$, $i \neq j$, then $ij = ji = 0$.

Let us consider the equivalence relation $\mathcal{B}$ defined on P as the symmetric and transitive closure of the relation "$iAj \neq \{0\}$". Thus $(i\mathcal{B}j)$ if and only if there exist $i_0, i_1, \ldots, i_n \in P$ with $i_0 = i$ and $i_n = j$ such that for all $k \in \{1, \ldots, n\}$, $i_{k-1}Ai_k \neq \{0\}$ or $i_kAi_{k-1} \neq \{0\}$. To every equivalence class B of P with respect to $\mathcal{B}$, we associate the idempotent $e_B := \sum_{i \in B} i$.

Proposition 2.1.4 *The map $B \mapsto e_B$ is a bijection between the set of equivalence classes of $\mathcal{B}$ and the set of blocks of A. In particular, we have that $1 = \sum_{B \in P/\mathcal{B}} e_B$ and 1 is sum of the blocks of A.*

Proof: It is clear that $1 = \sum_{B \in P/\mathcal{B}} e_B$. Let $a \in A$ and let B, B' be two equivalence classes of $\mathcal{B}$ with $B \neq B'$. Then, by definition of the relation $\mathcal{B}$, $e_B a e_{B'} = 0$. Since $1 = \sum_{B \in P/\mathcal{B}} e_B$, we have that $e_B a = e_B a e_B = a e_B$. Thus $e_B \in ZA$ for all $B \in P/\mathcal{B}$.

It remains to show that for all $B \in P/\mathcal{B}$, the central idempotent e_B is primitive. Suppose that $e_B = e + f$, where e and f are two orthogonal primitive idempotents in ZA. Then we have a partition $B = B_e \sqcup B_f$, where $B_e := \{i \in B \,|\, ie = i\}$ and $B_f := \{j \in B \,|\, jf = j\}$. For all $i \in B_e$ and $j \in B_f$, we have $iAj = ieAfj = iAefj = \{0\}$ and so no element of B_e can be $\mathcal{B}$-equivalent to an element of B_f. Therefore, we must have either $B_e = \emptyset$ or $B_f = \emptyset$, which implies that either $e = 0$ or $f = 0$. ■

Second case: Suppose that ZA is a subalgebra of a commutative algebra C where 1 is a sum of blocks. For example, if A is of finite type over $\mathcal{O}$, where $\mathcal{O}$ is an integral domain with field of fractions F, we can choose C to be the center of the algebra $FA := F \otimes_{\mathcal{O}} A$.

We set $1 = \sum_{e \in E} e$, where E is the set of blocks of C. For all $S \subseteq E$, set $e_S := \sum_{e \in S} e$. A subset S of E is "on ZA" if $e_S \in ZA$. If S and T are on ZA, then $S \cap T$ is on ZA.

Proposition 2.1.5 *Let us denote by $\mathcal{P}_E(ZA)$ the set of non-empty subsets B of E which are on ZA and are minimal for these two properties. Then the map $\mathcal{P}_E(ZA) \to A, B \mapsto e_B$ induces a bijection between $\mathcal{P}_E(ZA)$ and the set of blocks of A. We have $1 = \sum_{B \in \mathcal{P}_E(ZA)} e_B$.*

Proof: Since every idempotent in C is of the form e_S for some $S \subseteq E$, it is clear that e_B is a central primitive idempotent of A, for all $B \in \mathcal{P}_E(ZA)$. It remains to show that

1. If B,B' are two distinct elements of $\mathcal{P}_E(ZA)$, then $B \cap B' = \emptyset$.

2. $\mathcal{P}_E(ZA)$ is a partition of E.

These two properties, stated in terms of idempotents, mean:

1. If B,B' are two distinct elements of $\mathcal{P}_E(ZA)$, then e_B and $e_{B'}$ are orthogonal.

2. $1 = \sum_{B \in \mathcal{P}_E(ZA)} e_B$.

Let us prove them:

1. We have $e_B e_{B'} = e_{B \cap B'}$ and so $B \cap B' = \emptyset$, because B and B' are minimal.

2. Set $F := \bigcup_{B \in \mathcal{P}_E(ZA)} B$. Then $e_F = \sum_{B \in \mathcal{P}_E(ZA)} e_B \in ZA$. Then $1 - e_F = e_{E-F} \in ZA$, which means that $E - F$ is on ZA . If $E - F \neq \emptyset$, then $E - F$ contains an element of $\mathcal{P}_E(ZA)$ in contradiction to the definition of F. Thus $F = E$ and $\mathcal{P}_E(ZA)$ is a partition of E. ■

Let us assume that

- $\mathcal{O}$ is a commutative integral domain with field of fractions F,
- K is a field extension of F,
- A is an $\mathcal{O}$-algebra, free and finitely generated as an $\mathcal{O}$-module.

Suppose that the K-algebra $KA := K \otimes_{\mathcal{O}} A$ is semisimple. Then KA is isomorphic, by assumption, to a direct product of simple algebras:

$$KA \simeq \prod_{\chi \in \mathrm{Irr}(KA)} M_\chi,$$

where $\mathrm{Irr}(KA)$ denotes the set of irreducible characters of KA and M_χ is a simple K-algebra.

For all $\chi \in \mathrm{Irr}(KA)$, we denote by $\pi_\chi : KA \twoheadrightarrow M_\chi$ the projection onto the χ-factor and by e_χ the element of KA such that

$$\pi_{\chi'}(e_\chi) = \begin{cases} 1_{M_\chi}, & \text{if } \chi = \chi', \\ 0, & \text{if } \chi \neq \chi'. \end{cases}$$

The following theorem results directly from propositions 2.1.3 and 2.1.5.

Theorem 2.1.6

1. *We have* $1 = \sum_{\chi \in \mathrm{Irr}(KA)} e_\chi$ *and the set* $\{e_\chi\}_{\chi \in \mathrm{Irr}(KA)}$ *is the set of all the blocks of the algebra* KA.

2. *There exists a unique partition* $\mathrm{Bl}(A)$ *of* $\mathrm{Irr}(KA)$ *such that*

   (a) *For all* $B \in \mathrm{Bl}(A)$, *the idempotent* $e_B := \sum_{\chi \in B} e_\chi$ *is a block of* A.

   (b) *We have* $1 = \sum_{B \in \mathrm{Bl}(A)} e_B$ *and for every central idempotent* e *of* A, *there exists a subset* $\mathrm{Bl}(A, e)$ *of* $\mathrm{Bl}(A)$ *such that*

   $$e = \sum_{B \in \mathrm{Bl}(A,e)} e_B.$$

   In particular the set $\{e_B\}_{B \in \mathrm{Bl}(A)}$ *is the set of all the blocks of* A.

Remarks:

- If $\chi \in B$ for some $B \in \mathrm{Bl}(A)$, we say that "χ belongs to the block e_B".
- For all $B \in \mathrm{Bl}(A)$, we have

$$KAe_B \simeq \prod_{\chi \in B} M_\chi.$$

From now on, we make the following assumptions

Assumptions 2.1.7

(int) *The ring* $\mathcal{O}$ *is a Noetherian and integrally closed domain with field of fractions* F *and* A *is an* $\mathcal{O}$*-algebra which is free and finitely generated as an* $\mathcal{O}$*-module.*

(spl) *The field* K *is a finite Galois extension of* F *and the algebra* KA *is split (i.e., for every simple* KA*-module* V, $\mathrm{End}_{KA}(V) \simeq K$*) semisimple.*

We denote by $\mathcal{O}_K$ the integral closure of $\mathcal{O}$ in K.

Blocks and integral closure

The Galois group $\mathrm{Gal}(K/F)$ acts on $KA = K \otimes_{\mathcal{O}} A$ (viewed as an F-algebra) as follows: if $\sigma \in \mathrm{Gal}(K/F)$ and $\lambda \otimes a \in KA$, then $\sigma(\lambda \otimes a) := \sigma(\lambda) \otimes a$.

If V is a K-vector space and $\sigma \in \mathrm{Gal}(K/F)$, we denote by ${}^{\sigma}V$ the K-vector space defined on the additive group V with multiplication $\lambda.v := \sigma^{-1}(\lambda)v$ for all $\lambda \in K$ and $v \in V$. If $\rho : KA \to \mathrm{End}_K(V)$ is a representation of the K-algebra KA, then its composition with the action of σ^{-1} is also a representation ${}^{\sigma}\rho : KA \to \mathrm{End}_K({}^{\sigma}V)$:

$$KA \xrightarrow{\sigma^{-1}} KA \xrightarrow{\rho} \mathrm{End}_K(V).$$

We denote by ${}^{\sigma}\chi$ the character of ${}^{\sigma}\rho$ and we define the action of $\mathrm{Gal}(K/F)$ on $\mathrm{Irr}(KA)$ as follows: if $\sigma \in \mathrm{Gal}(K/F)$ and $\chi \in \mathrm{Irr}(KA)$, then

$$\sigma(\chi) := {}^{\sigma}\chi = \sigma \circ \chi \circ \sigma^{-1}.$$

This operation induces an action of $\mathrm{Gal}(K/F)$ on the set of blocks of KA:

$$\sigma(e_\chi) = e_{{}^{\sigma}\chi} \text{ for all } \sigma \in \mathrm{Gal}(K/F), \chi \in \mathrm{Irr}(KA).$$

Hence, the group $\mathrm{Gal}(K/F)$ acts on the set of idempotents of $Z\mathcal{O}_K A$ and thus on the set of blocks of $\mathcal{O}_K A$. Since $F \cap \mathcal{O}_K = \mathcal{O}$, the idempotents of ZA are the idempotents of $Z\mathcal{O}_K A$ which are fixed by the action of $\mathrm{Gal}(K/F)$. As a consequence, the primitive idempotents of ZA are sums of the elements of the orbits of $\mathrm{Gal}(K/F)$ on the set of primitive idempotents of $Z\mathcal{O}_K A$. Thus, the blocks of A are in bijection with the orbits of $\mathrm{Gal}(K/F)$ on the set of blocks of $\mathcal{O}_K A$. The following proposition is just a reformulation of this result.

Proposition 2.1.8

1. *Let B be a block of A and B' a block of $\mathcal{O}_K A$ contained in B. If $\mathrm{Gal}(K/F)_{B'}$ denotes the stabilizer of B' in $\mathrm{Gal}(K/F)$, then*

$$B = \bigcup_{\sigma \in \mathrm{Gal}(K/F)/\mathrm{Gal}(K/F)_{B'}} \sigma(B') \quad \textit{i.e.,} \quad e_B = \sum_{\sigma \in \mathrm{Gal}(K/F)/\mathrm{Gal}(K/F)_{B'}} \sigma(e_{B'}).$$

2. *Two characters $\chi, \psi \in \mathrm{Irr}(KA)$ are in the same block of A if and only if there exists $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\chi)$ and ψ belong to the same block of $\mathcal{O}_K A$.*

Remark: For all $\chi \in B'$, we have $\mathrm{Gal}(K/F)_\chi \subseteq \mathrm{Gal}(K/F)_{B'}$.

The assertion (2) of the proposition above allows us to transfer the problem of the classification of the blocks of A to that of the classification of the blocks of $\mathcal{O}_K A$.

Blocks and prime ideals

We denote by $\mathrm{Spec}_1(\mathcal{O})$ the set of prime ideals of height 1 of $\mathcal{O}$. Since $\mathcal{O}$ is Noetherian and integrally closed, it is a Krull ring and by theorem 1.2.23, we have

$$\mathcal{O} = \bigcap_{\mathfrak{p}\in \mathrm{Spec}_1(\mathcal{O})} \mathcal{O}_{\mathfrak{p}},$$

where $\mathcal{O}_{\mathfrak{p}} := \{x \in F \,|\, (\exists a \in \mathcal{O} - \mathfrak{p})(ax \in \mathcal{O})\}$ is the localization of $\mathcal{O}$ at $\mathfrak{p}$. More generally, if we denote by $\mathrm{Spec}(\mathcal{O})$ the set of prime ideals of $\mathcal{O}$, then

$$\mathcal{O} = \bigcap_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O})} \mathcal{O}_{\mathfrak{p}}.$$

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ and $\mathcal{O}_{\mathfrak{p}}A := \mathcal{O}_{\mathfrak{p}} \otimes_{\mathcal{O}} A$. The blocks of $\mathcal{O}_{\mathfrak{p}}A$ are the "$\mathfrak{p}$-blocks of A". If $\chi, \psi \in \mathrm{Irr}(KA)$ belong to the same block of $\mathcal{O}_{\mathfrak{p}}A$, we write $\chi \sim_{\mathfrak{p}} \psi$.

Proposition 2.1.9 *Two characters $\chi, \psi \in \mathrm{Irr}(KA)$ belong to the same block of A if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(KA)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathrm{Spec}(\mathcal{O})$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *for all j $(1 \leq j \leq n)$, $\chi_{j-1} \sim_{\mathfrak{p}_j} \chi_j$.*

Proof: Let us denote by $\sim$ the equivalence relation on $\mathrm{Irr}(KA)$ defined as the closure of the relation "there exists $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ such that $\chi \sim_{\mathfrak{p}} \psi$". Thus, we have to show that $\chi \sim \psi$ if and only if χ and ψ belong to the same block of A.

We will first show that the equivalence relation $\sim$ is finer than the relation "being in the same block of A". Let B be a block of A. Then B is a subset of $\mathrm{Irr}(KA)$ such that $\sum_{\chi\in B} e_\chi \in A$. Since $\mathcal{O} = \bigcap_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})} \mathcal{O}_{\mathfrak{p}}$, we have that $\sum_{\chi\in B} e_\chi \in \mathcal{O}_{\mathfrak{p}}A$ for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$. Therefore, by theorem 2.1.6, C is a union of blocks of $\mathcal{O}_{\mathfrak{p}}A$ for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ and, hence, a union of equivalence classes of $\sim$.

Now we will show that the relation "being in the same block of A" if finer than the relation $\sim$. Let C be an equivalence class of $\sim$. Then $\sum_{\chi\in C} e_\chi \in \mathcal{O}_{\mathfrak{p}}A$ for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$. Thus $\sum_{\chi\in C} e_\chi \in \bigcap_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})} \mathcal{O}_{\mathfrak{p}}A = A$ and C is a union of blocks of A. ■

Blocks and central morphisms

Since KA is a split semisimple K-algebra, we have that

$$KA \simeq \prod_{\chi \in \mathrm{Irr}(KA)} M_\chi,$$

where M_χ is a matrix algebra isomorphic to $\mathrm{Mat}_{\chi(1)}(K)$.

Recall that A is of finite type and thus integral over $\mathcal{O}$ ([7], §1, Def.2). The map $\pi_\chi : KA \twoheadrightarrow M_\chi$, restricted to ZKA, defines a map $\omega_\chi : ZKA \twoheadrightarrow K$ (by Schur's lemma), which in turn, restricted to ZA, defines the morphism

$$\omega_\chi : ZA \to \mathcal{O}_K,$$

where $\mathcal{O}_K$ denotes the integral closure of $\mathcal{O}$ in K.

In the case where $\mathcal{O}$ is a discrete valuation ring, we have the following result which is proven later in this chapter, proposition 2.4.16. For a different approach to its proof, see [12], Prop.1.18.

Proposition 2.1.10 *Suppose that $\mathcal{O}$ is a discrete valuation ring with unique maximal ideal $\mathfrak{p}$ and $K = F$. Two characters $\chi, \chi' \in \mathrm{Irr}(KA)$ belong to the same block of A if and only if*

$$\omega_\chi(a) \equiv \omega_{\chi'}(a) \bmod \mathfrak{p} \text{ for all } a \in ZA.$$

2.2 Symmetric algebras

Let $\mathcal{O}$ be a ring and let A be an $\mathcal{O}$-algebra. Suppose again that the assumptions 2.1.7 are satisfied.

Definition 2.2.1 *A trace function on A is an $\mathcal{O}$-linear map $t : A \to \mathcal{O}$ such that $t(ab) = t(ba)$ for all $a, b \in A$.*

Definition 2.2.2 *We say that a trace function $t : A \to \mathcal{O}$ is a symmetrizing form on A or that A is a symmetric algebra if the morphism*

$$\hat{t} : A \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}),\ a \mapsto (x \mapsto \hat{t}(a)(x) := t(ax))$$

is an isomorphism of A-modules-A.

Example 2.2.3 In the case where $\mathcal{O} = \mathbb{Z}$ and $A = \mathbb{Z}[G]$ (G a finite group), we can define the following symmetrizing form ("canonical") on A

$$t : \mathbb{Z}[G] \to \mathbb{Z},\ \sum_{g \in G} a_g g \mapsto a_1,$$

where $a_g \in \mathbb{Z}$ for all $g \in G$.

Since A is a free $\mathcal{O}$-module of finite rank, we have the following isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}}(A,\mathcal{O})\otimes_{\mathcal{O}} A & \tilde{\to} & \mathrm{Hom}_{\mathcal{O}}(A,A) \\ \varphi\otimes a & \mapsto & (x\mapsto \varphi(x)a). \end{array}$$

Composing it with the isomorphism

$$\begin{array}{ccc} A\otimes_{\mathcal{O}} A & \tilde{\to} & \mathrm{Hom}_{\mathcal{O}}(A,\mathcal{O})\otimes_{\mathcal{O}} A \\ a\otimes b & \mapsto & \hat{t}(a)\otimes b, \end{array}$$

we obtain an isomorphism

$$A\otimes_{\mathcal{O}} A\tilde{\to}\mathrm{Hom}_{\mathcal{O}}(A,A).$$

Definition 2.2.4 *We denote by C_A and we call Casimir of (A,t) the inverse image of Id_A by the above isomorphism.*

Example 2.2.5 In the case where $\mathcal{O}=\mathbb{Z}$, $A=\mathbb{Z}[G]$ (G a finite group) and t is the canonical symmetrizing form, we have $C_{\mathbb{Z}[G]}=\sum_{g\in G} g^{-1}\otimes g$.

More generally, if $(e_i)_{i\in I}$ is a basis of A over $\mathcal{O}$ and $(e'_i)_{i\in I}$ is the dual basis with respect to t (*i.e.,* $t(e_ie'_j)=\delta_{ij}$), then

$$C_A=\sum_{i\in I} e'_i\otimes e_i.$$

In this case, let us denote by c_A the image of C_A by the multiplication $A\otimes A\to A$, *i.e.,* $c_A=\sum_{i\in I} e'_ie_i$. It is easy to check (see also [13], 7.9) the following properties of the Casimir element:

Lemma 2.2.6 *For all $a\in A$, we have*

1. $\sum_i ae'_i\otimes e_i=\sum_i e_i\otimes e'_ia$.
2. $aC_A=C_Aa$. *Consequently,* $c_A\in ZA$.
3. $a=\sum_i t(ae'_i)e_i=\sum_i t(ae_i)e'_i=\sum_i t(e'_i)e_ia=\sum_i t(e_i)e'_ia$.

If $\tau: A\to\mathcal{O}$ is a linear form, we denote by $\tau^\vee$ its inverse image by the isomorphism $\hat{t}$, *i.e.,* $\tau^\vee$ is the element of A such that

$$t(\tau^\vee a)=\tau(a) \text{ for all } a\in A.$$

The element $\tau^\vee$ has the following properties:

Lemma 2.2.7

1. *τ is a trace function if and only if $\tau^\vee \in ZA$.*
2. *We have $\tau^\vee = \sum_i \tau(e_i')e_i = \sum_i \tau(e_i)e_i'$ and more generally, for all $a \in A$, we have $\tau^\vee a = \sum_i \tau(e_i'a)e_i = \sum_i \tau(e_ia)e_i'$.*

Proof:

1. Recall that t is a trace function. Let $a \in A$. For all $x \in A$, we have
$$\hat{t}(\tau^\vee a)(x) = t(\tau^\vee ax) = \tau(ax)$$
and
$$\hat{t}(a\tau^\vee)(x) = t(a\tau^\vee x) = t(\tau^\vee xa) = \tau(xa)$$
If τ is a trace function, then $\tau(ax) = \tau(xa)$ and hence, $\hat{t}(\tau^\vee a) = \hat{t}(a\tau^\vee)$. Since $\hat{t}$ is an isomorphism, we obtain that $\tau^\vee a = a\tau^\vee$ and thus $\tau^\vee \in ZA$.
Now if $\tau^\vee \in ZA$ and $a, b \in A$, then
$$\tau(ab) = t(\tau^\vee ab) = t(b\tau^\vee a) = t(ba\tau^\vee) = t(\tau^\vee ba) = \tau(ba).$$
2. It derives from property 3 of lemma 2.2.6 and the definition of $\tau^\vee$. ■

Let χ_{reg} be the character of the regular representation of A, *i.e.,* the linear form on A defined as
$$\chi_{\text{reg}}(a) := \text{tr}_{A/\mathcal{O}}(\lambda_a),$$
where $\lambda_a : A \to A, x \mapsto ax$, is the endomorphism of left multiplication by a.

Proposition 2.2.8 *We have $\chi_{\text{reg}}^\vee = c_A$.*

Proof: Let $a \in A$. The inverse image of λ_a by the isomorphism $A \otimes_{\mathcal{O}} A \tilde{\to} \text{Hom}_{\mathcal{O}}(A, A)$ is aC_A (by definition of the Casimir). Hence,
$$\lambda_a = (x \mapsto \sum_i \hat{t}(e_i'a)(x)e_i) = (x \mapsto \sum_i t(e_i'ax)e_i)$$
and
$$\text{tr}_{A/\mathcal{O}}(\lambda_a) = \sum_i t(e_i'ae_i) = t(a\sum_i e_i'e_i) = t(ac_A) = t(c_Aa).$$
Therefore, for all $a \in A$, we have $\chi_{\text{reg}}(a) = t(c_Aa)$, *i.e.,* $\chi_{\text{reg}}^\vee = c_A$. ■

If A is a symmetric algebra with a symmetrizing form t, we obtain a symmetrizing form t^K on KA by extension of scalars. Every irreducible character $\chi \in \text{Irr}(KA)$ is a central function on KA and thus we can define $\chi^\vee \in KA$.

Definition 2.2.9 *For all $\chi \in \mathrm{Irr}(KA)$, we call Schur element of χ with respect to t and denote by s_χ the element of K defined by*

$$s_\chi := \omega_\chi(\chi^\vee).$$

Proposition 2.2.10 *For all $\chi \in \mathrm{Irr}(KA)$, $s_\chi \in \mathcal{O}_K$.*

The proof of the above result will be given in proposition 2.4.6.

Example 2.2.11 Let $\mathcal{O} := \mathbb{Z}$, $A := \mathbb{Z}[G]$ (G a finite group) and t the canonical symmetrizing form. If K is an algebraically closed field of characteristic 0, then KA is a split semisimple algebra and $s_\chi = |G|/\chi(1)$ for all $\chi \in \mathrm{Irr}(KA)$. Because of the integrality of the Schur elements, we must have $|G|/\chi(1) \in \mathbb{Z} = \mathbb{Z}_K \cap \mathbb{Q}$ for all $\chi \in \mathrm{Irr}(KA)$. Thus, we have shown that $\chi(1)$ divides $|G|$.

The following properties of the Schur elements can be derived easily from the above (see also [11],[19],[20],[21],[13])

Proposition 2.2.12

1. *We have*
$$t = \sum_{\chi \in \mathrm{Irr}(KA)} \frac{1}{s_\chi}\chi.$$

2. *For all $\chi \in \mathrm{Irr}(KA)$, the central primitive idempotent associated with χ is*
$$e_\chi = \frac{1}{s_\chi}\chi^\vee = \frac{1}{s_\chi}\sum_{i\in I}\chi(e_i')e_i.$$

3. *For all $\chi \in \mathrm{Irr}(KA)$, we have*
$$s_\chi\chi(1) = \sum_{i\in I}\chi(e_i')\chi(e_i) \ \text{ and } \ s_\chi\chi(1)^2 = \chi(\sum_{i\in I} e_i'e_i) = \chi(\chi_{\mathrm{reg}}^\vee).$$

Corollary 2.2.13 *The blocks of A are the non-empty subsets B of $\mathrm{Irr}(KA)$ minimal for the property*

$$\sum_{\chi\in B}\frac{1}{s_\chi}\chi(a) \in \mathcal{O} \text{ for all } a \in A.$$

2.3 Twisted symmetric algebras of finite groups

This part is an adaptation of the section "Symmetric algebras of finite groups" of [12] to a more general case.

Let A be an $\mathcal{O}$-algebra such that the assumptions 2.1.7 are satisfied with a symmetrizing form t. Let $\bar{A}$ be a subalgebra of A free and of finite rank as $\mathcal{O}$-module.

We denote by $\bar{A}^\perp$ the orthogonal of $\bar{A}$ with respect to t, *i.e.,* the sub-$\bar{A}$-module-$\bar{A}$ of A defined as

$$\bar{A}^\perp := \{a \in A \,|\, (\forall \bar{a} \in \bar{A})(t(a\bar{a}) = 0)\}.$$

Proposition 2.3.1

1. *The restriction of t to $\bar{A}$ is a symmetrizing form for $\bar{A}$ if and only if $\bar{A} \oplus \bar{A}^\perp = A$. In this case the projection of A onto $\bar{A}$ parallel to $\bar{A}^\perp$ is the map*

$$\mathrm{Br}_{\bar{A}}^{A} : A \to \bar{A} \text{ such that } t(\mathrm{Br}_{\bar{A}}^{A}(a)\bar{a}) = t(a\bar{a}) \text{ for all } a \in A \text{ and } \bar{a} \in \bar{A}.$$

2. *If the restriction of t to $\bar{A}$ is a symmetrizing form for $\bar{A}$, then $\bar{A}^\perp$ is the sub-$\bar{A}$-module-$\bar{A}$ of A defined by the following two properties:*

   (a) $A = \bar{A} \oplus \bar{A}^\perp$,

   (b) $\bar{A}^\perp \subseteq \mathrm{Ker}t$.

Proof:

1. Let us denote by $\bar{t}$ the restriction of t to $\bar{A}$. Suppose that $\bar{t}$ is a symmetrizing form for $\bar{A}$. Let $a \in A$. Then $\hat{t}(a) := (x \mapsto t(ax)) \in \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$. The restriction of $\hat{t}(a)$ to $\bar{A}$ belongs to $\mathrm{Hom}_{\mathcal{O}}(\bar{A}, \mathcal{O})$ and therefore, there exists $\bar{a} \in \bar{A}$ such that $\bar{t}(\bar{a}\bar{x}) = t(\bar{a}\bar{x}) = t(a\bar{x})$ for all $\bar{x} \in \bar{A}$. Thus $a - \bar{a} \in \bar{A}^\perp$ and since $a = \bar{a} + (a - \bar{a})$, we obtain that $A = \bar{A} + \bar{A}^\perp$. If $\bar{a} \in \bar{A} \cap \bar{A}^\perp$, then we have $\hat{\bar{t}}(\bar{a}) = 0 \in \mathrm{Hom}_{\mathcal{O}}(\bar{A}, \mathcal{O})$. Since $\hat{\bar{t}}$ is an isomorphism, we deduce that $\bar{a} = 0$. Therefore, $A = \bar{A} \oplus \bar{A}^\perp$ and the definition of $\mathrm{Br}_{\bar{A}}^{A}$ is immediate.

   Now suppose that $A = \bar{A} \oplus \bar{A}^\perp$. We will show that the map

$$\begin{array}{rccc} \hat{\bar{t}} : & \bar{A} & \to & \mathrm{Hom}_{\mathcal{O}}(\bar{A}, \mathcal{O}) \\ & \bar{A} & \mapsto & (\bar{x} \mapsto \bar{t}(\bar{a}\bar{x}) = t(\bar{a}\bar{x})) \end{array}$$

is an isomorphism of $\bar{A}$-modules-$\bar{A}$. The map $\hat{\bar{t}}$ is obviously injective, because $\hat{\bar{t}}(\bar{a}) = 0$ implies that $\bar{a} \in \bar{A} \cap \bar{A}^{\perp}$ and thus $\bar{a} = 0$. Now let $\bar{f}$ be an element of $\mathrm{Hom}_{\mathcal{O}}(\bar{A}, \mathcal{O})$. The map $\bar{f}$ can be extended to a map $f \in \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ such that $f(a) = \bar{f}(\mathrm{Br}_{\bar{A}}^{A}(a))$ for all $a \in A$, where $\mathrm{Br}_{\bar{A}}^{A}$ denotes the projection of A onto $\bar{A}$ parallel to $\bar{A}^{\perp}$. Since t is a symmetrizing form for A, there exists $a \in A$ such that $\hat{t}(a) = f$, *i.e.,* $t(ax) = f(x)$ for all $x \in A$. Consequently, if $\bar{x} \in \bar{A}$, we have

$$t(\mathrm{Br}_{\bar{A}}^{A}(a)\bar{x}) = t(a\bar{x}) = f(\bar{x}) = \bar{f}(\bar{x})$$

and thus $\hat{\bar{t}}(\mathrm{Br}_{\bar{A}}^{A}(a)) = f$. Hence, $\hat{\bar{t}}$ is surjective.

2. Let B be a sub-$\bar{A}$-module-$\bar{A}$ of A such that $A = \bar{A} \oplus B$ and $B \subseteq \mathrm{Ker}t$. Let $b \in B$. For all $\bar{a} \in \bar{A}$, we have $b\bar{a} \in B \subseteq \mathrm{Ker}t$ and thus $t(b\bar{a}) = 0$. Hence $B \subseteq \bar{A}^{\perp}$. Since the restriction of t to $\bar{A}$ is a symmetrizing form for $\bar{A}$, we also have $A = \bar{A} \oplus \bar{A}^{\perp}$. Now let $a \in \bar{A}^{\perp}$. Since $A = \bar{A} \oplus B$, there exist $\bar{a} \in \bar{A}$ and $b \in B$ such that $a = \bar{a} + b$. Since $b \in \bar{A}^{\perp}$, we must have $a = b \in B$ and therefore, $B = \bar{A}^{\perp}$. ■

Example 2.3.2 In the case where $\mathcal{O} = \mathbb{Z}$ and $A = \mathbb{Z}[G]$ (G a finite group), let $\bar{A} := \mathbb{Z}[\bar{G}]$ be the algebra of a subgroup $\bar{G}$ of G. Then the morphism $\mathrm{Br}_{\bar{A}}^{A}$ is the projection given by

$$\begin{cases} g \mapsto g, & \text{if } g \in \bar{G}; \\ g \mapsto 0, & \text{if } g \notin \bar{G}. \end{cases}$$

Definition 2.3.3 *Let A be a symmetric $\mathcal{O}$-algebra with symmetrizing form t. Let $\bar{A}$ be a subalgebra of A. We say that $\bar{A}$ is a symmetric subalgebra of A, if it satisfies the following two conditions:*

1. *$\bar{A}$ is free (of finite rank) as an $\mathcal{O}$-module and the restriction $\mathrm{Res}_{\bar{A}}^{A}(t)$ of the form t to $\bar{A}$ is a symmetrizing form on $\bar{A}$,*

2. *A is free (of finite rank) as an $\bar{A}$-module for the action of left multiplication by the elements of $\bar{A}$.*

From now on, let us suppose that $\bar{A}$ is a symmetric subalgebra of A and set $\bar{t} := \mathrm{Res}_{\bar{A}}^{A}(t)$. We denote by

$$\mathrm{Ind}_{\bar{A}}^{A} :_{\bar{A}} \mathbf{mod} \rightarrow_{A} \mathbf{mod} \text{ and } \mathrm{Res}_{\bar{A}}^{A} :_{A} \mathbf{mod} \rightarrow_{\bar{A}} \mathbf{mod}$$

the functors defined as usual by

$$\mathrm{Ind}_{\bar{A}}^{A} := A \otimes_{\bar{A}} - \text{ where } A \text{ is viewed as an } A\text{-module-}\bar{A}$$

and

$$\mathrm{Res}^A_{\bar{A}} := A \otimes_A - \text{ where } A \text{ is viewed as an } \bar{A}\text{-module-}A.$$

Since A is free as $\bar{A}$-module and as module-$\bar{A}$, the functors $\mathrm{Res}^A_{\bar{A}}$ and $\mathrm{Ind}^A_{\bar{A}}$ are adjoint from both sides.

Moreover, let K be a finite Galois extension of the field of fractions of $\mathcal{O}$ such that the algebras KA and $K\bar{A}$ are both split semisimple.

We denote by $\langle -, - \rangle_{KA}$ the scalar product on the K-vector space of trace functions for which the family $(\chi)_{\chi \in \mathrm{Irr}(KA)}$ is orthonormal and $\langle -, - \rangle_{K\bar{A}}$ the scalar product on the K-vector space of trace functions for which the family $(\bar{\chi})_{\bar{\chi} \in \mathrm{Irr}(K\bar{A})}$ is orthonormal.

Since the functors $\mathrm{Res}^A_{\bar{A}}$ and $\mathrm{Ind}^A_{\bar{A}}$ are adjoint from both sides, we obtain the *Frobenius reciprocity* formula:

$$\langle \chi, \mathrm{Ind}^{KA}_{K\bar{A}}(\bar{\chi}) \rangle_{KA} = \langle \mathrm{Res}^{KA}_{K\bar{A}}(\chi), \bar{\chi} \rangle_{K\bar{A}}.$$

For every element $\chi \in \mathrm{Irr}(KA)$, let

$$\mathrm{Res}^{KA}_{K\bar{A}}(\chi) = \sum_{\bar{\chi} \in \mathrm{Irr}(K\bar{A})} m_{\chi,\bar{\chi}} \bar{\chi} \text{ (where } m_{\chi,\bar{\chi}} \in \mathbb{N}).$$

Frobenius reciprocity implies that, for all $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$,

$$\mathrm{Ind}^{KA}_{K\bar{A}}(\bar{\chi}) = \sum_{\chi \in \mathrm{Irr}(KA)} m_{\chi,\bar{\chi}} \chi.$$

The following property is immediate.

Lemma 2.3.4 *For $\chi \in \mathrm{Irr}(KA)$ and $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$, let $e(\chi)$ and $\bar{e}(\bar{\chi})$ be respectively the block-idempotents of KA and $K\bar{A}$ associated with χ and $\bar{\chi}$. The following conditions are equivalent:*

(i) $m_{\chi,\bar{\chi}} \neq 0$,

(ii) $e(\chi)\bar{e}(\bar{\chi}) \neq 0$.

For all $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$, we set

$$\mathrm{Irr}(KA, \bar{\chi}) := \{\chi \in \mathrm{Irr}(KA) \,|\, m_{\chi,\bar{\chi}} \neq 0\},$$

and for all $\chi \in \mathrm{Irr}(KA)$,

$$\mathrm{Irr}(K\bar{A}, \chi) := \{\bar{\chi} \in \mathrm{Irr}(K\bar{A}) \,|\, m_{\chi,\bar{\chi}} \neq 0\}.$$

We denote respectively by s_χ and $s_{\bar{\chi}}$ the Schur elements of χ and $\bar{\chi}$ (with respect to the symmetrizing forms t for A and $\bar{t}$ for $\bar{A}$).

Lemma 2.3.5 *For all $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$ we have*

$$\frac{1}{s_{\bar{\chi}}} = \sum_{\chi \in \mathrm{Irr}(KA,\bar{\chi})} \frac{m_{\chi,\bar{\chi}}}{s_\chi}.$$

Proof: It derives from the relations

$$t = \sum_{\chi \in \mathrm{Irr}(KA)} \frac{1}{s_\chi}\chi,\ \bar{t} = \sum_{\bar{\chi} \in \mathrm{Irr}(K\bar{A})} \frac{1}{s_{\bar{\chi}}}\bar{\chi},\ \bar{t} = \mathrm{Res}^A_{\bar{A}}(t).$$

■

In the next chapters, we will work on the Hecke algebras of complex reflection groups, which, under certain assumptions, are symmetric. Sometimes the Hecke algebra of a group W appears as a symmetric subalgebra of the Hecke algebra of another group W', which contains W. Since we will be mostly interested in the determination of the blocks of these algebras, it would be helpful, if we could obtain the blocks of the former from the blocks of the latter. This is possible with the use of a generalization of some classical results, known as "Clifford theory" (see, for example, [16]), to the twisted symmetric algebras of finite groups and more precisely of finite cyclic groups. For the application of these results to the Hecke algebras, the reader may refer to the Appendix of this thesis.

Definition 2.3.6 *We say that a symmetric $\mathcal{O}$-algebra (A, t) is the twisted symmetric algebra of a finite group G over the subalgebra $\bar{A}$, if the following conditions are satisfied:*

- *$\bar{A}$ is a symmetric subalgebra of A,*
- *There exists a family $\{A_g \,|\, g \in G\}$ of $\mathcal{O}$-submodules of A such that*
  - **(a)** *$A = \bigoplus_{g \in G} A_g$,*
  - **(b)** *$A_g A_h = A_{gh}$ for all $g, h \in G$,*
  - **(c)** *$A_1 = \bar{A}$,*
  - **(d)** *$t(A_g) = 0$ for all $g \in G, g \neq 1$,*
  - **(e)** *$A_g \cap A^\times \neq \emptyset$ for all $g \in G$ (where $A^\times$ is the set of units of A).*

If that is the case, then proposition 2.3.1 implies that

$$\bigoplus_{g \in G - \{1\}} A_g = \bar{A}^\perp.$$

Lemma 2.3.7 *Let $a_g \in A_g$ such that a_g is a unit in A. Then*

$$A_g = a_g\bar{A} = \bar{A}a_g.$$

Proof: Since $a_g \in A_g$, property (b) implies that $a_g^{-1} \in A_{g^{-1}}$. If $a \in A_g$, then $a_g^{-1}a \in A_1 = \bar{A}$. We have $a = a_g a_g{}^{-1}a \in a_g\bar{A}$ and thus $A_g \subseteq a_g\bar{A}$. Property (b) implies the inverse inclusion. In the same way, we show that $A_g = \bar{A}a_g$. ■

From now on, let (A, t) be the twisted symmetric algebra of a finite group G over the subalgebra $\bar{A}$. Due to property (e) and the lemma above, for all $g \in G$, there exist $a_g \in A_g \cap A^\times$ such that $A_g = a_g\bar{A} = \bar{A}a_g$.

Proposition 2.3.8 *Let $(\bar{e}_i)_{i\in I}$ be a basis of $\bar{A}$ over $\mathcal{O}$ and $(\bar{e}'_i)_{i\in I}$ its dual with respect to the symmetrizing form $\bar{t}$. We fix a system of representatives* $\mathrm{Rep}(A/\bar{A}) := \{a_g \mid g \in G\}$. *Then the families*

$$(\bar{e}_i a_g)_{i\in I, a_g\in \mathrm{Rep}(A/\bar{A})} \text{ and } (a_g^{-1}\bar{e}'_i)_{i\in I, a_g\in \mathrm{Rep}(A/\bar{A})}$$

are two $\mathcal{O}$-bases of A dual to each other.

Action of G on $Z\bar{A}$

Lemma 2.3.9 *Let $\bar{a} \in Z\bar{A}$ and $g \in G$. There exists a unique element $g(\bar{a})$ of $\bar{A}$ satisfying*

$$g(\bar{a})\mathfrak{g} = \mathfrak{g}\bar{a} \text{ for all } \mathfrak{g} \in A_g. \quad (\dagger)$$

If $a_g \in A^\times$ such that $A_g = a_g\bar{A}$, then

$$g(\bar{a}) = a_g\bar{a}a_g{}^{-1}.$$

Proof: Set $g(\bar{a}) := a_g\bar{a}a_g{}^{-1}$. Then, for all $\mathfrak{g} \in A_g$, we have $a_g{}^{-1}\mathfrak{g} \in \bar{A}$ and $g(\bar{a})\mathfrak{g} = a_g\bar{a}a_g{}^{-1}\mathfrak{g} = a_g a_g{}^{-1}\mathfrak{g}\bar{a} = \mathfrak{g}\bar{a}$. Now, let y be another element of A such that $y\mathfrak{g} = \mathfrak{g}\bar{a}$ for all $\mathfrak{g} \in A_g$. Then $ya_g = a_g\bar{a}$ and hence $y = g(\bar{a})$. Therefore, $g(\bar{a})$ is the unique element of A which satisfies (†). ■

Proposition 2.3.10 *The map $\bar{a} \mapsto g(\bar{a})$ defines an action of G as ring automorphism of $Z\bar{A}$.*

Proof: Let $\bar{a} \in Z\bar{A}$, $g \in G$ and $a_g \in A^\times$ such that $A_g = a_g\bar{A}$. We will show that $g(\bar{a}) \in Z\bar{A}$. If $\bar{x} \in \bar{A}$, then $\bar{x}a_g \in A_g$ and we have

$$\bar{x}g(\bar{a})a_g = \bar{x}a_g\bar{a} = g(\bar{a})\bar{x}a_g.$$

Multiplying both sides by ${a_g}^{-1}$, we obtain that

$$\bar{x}g(\bar{a}) = g(\bar{a})\bar{x}$$

and hence, $g(\bar{a}) \in Z\bar{A}$.

Since the identity 1 of A lies in A_1, we have $1_G(\bar{a}) = \bar{a}$. If $g_1, g_2 \in G$, then equation (†) gives

$$g_1(g_2(\bar{a}))a_{g_1}a_{g_2} = a_{g_1}g_2(\bar{a})a_{g_2} = a_{g_1}a_{g_2}\bar{a}.$$

Due to property (b) of the definition 2.3.6, the product $a_{g_1}a_{g_2}$ generates the submodule $A_{g_1g_2}$. Therefore, $g_1(g_2(\bar{a}))u = u\bar{a}$ for all $u \in A_{g_1g_2}$. By lemma 2.3.9, we obtain that $(g_1g_2)(\bar{a}) = g_1(g_2(\bar{a}))$.

Finally, let us fix $g \in G$. By definition, the map $\bar{a} \mapsto g(\bar{a})$ is an additive automorphism of $Z\bar{A}$. If $\bar{a}_1, \bar{a}_2 \in Z\bar{A}$, then

$$\mathfrak{g}\bar{a}_1\bar{a}_2 = g(\bar{a}_2)\mathfrak{g}\bar{a}_1 = g(\bar{a}_1)g(\bar{a}_2)\mathfrak{g} \text{ for all } \mathfrak{g} \in A_g.$$

By lemma 2.3.9, we obtain that $g(\bar{a}_1\bar{a}_2) = g(\bar{a}_1)g(\bar{a}_2)$. ■

Now let $\bar{b}$ be a block(-idempotent) of $\bar{A}$. If $g \in G$, then $g(\bar{b})$ is also a block of $\bar{A}$. So we must have either $g(\bar{b}) = \bar{b}$ or $g(\bar{b})$ orthogonal to $\bar{b}$. Set

$$\mathrm{Tr}(G, \bar{b}) := \sum_{g \in G/G_{\bar{b}}} g(\bar{b}),$$

where $G_{\bar{b}} := \{g \in G \mid g(\bar{b}) = \bar{b}\}$. It is clear that

- $\bar{b}$ is a central idempotent of $\bigoplus_{g \in G_{\bar{b}}} A_g =: A_{G_{\bar{b}}}$,
- $\mathrm{Tr}(G, \bar{b})$ is a central idempotent of A.

From now on, let $b := \mathrm{Tr}(G, \bar{b})$ and $\mathfrak{g} \in A_g$. We have

$$\bar{b}\mathfrak{g}\bar{b} = \begin{cases} \mathfrak{g}\bar{b} = \bar{b}\mathfrak{g}, & \text{if } g \in G_{\bar{b}}; \\ 0, & \text{if } g \notin G_{\bar{b}}, \end{cases}$$

$$\bar{b}\mathfrak{g}b = \bar{b}\mathfrak{g} \quad \text{and} \quad b\mathfrak{g}\bar{b} = \mathfrak{g}\bar{b}.$$

Proposition 2.3.11 *The applications*

$$\begin{cases} bA\bar{b} \otimes_{A_{G_{\bar{b}}}\bar{b}} \bar{b}Ab \to Ab \\ ba_g\bar{a}\bar{b} \otimes \bar{b}\bar{a}'a_{g'}b \mapsto a_g\bar{a}\bar{b}\bar{a}'a_{g'} \\ ab \mapsto \sum_{g \in G/G_{\bar{b}}} aa_g\bar{b} \otimes \bar{b}{a_g}^{-1}, \end{cases}$$

and

$$\begin{cases} \bar{b}Ab \otimes_{Ab} bA\bar{b} \to A_{G_{\bar{b}}}\bar{b} \\ \bar{b}\bar{a}a_g b \otimes ba_{g'}\bar{a}'\bar{b} \mapsto \begin{cases} \bar{b}\bar{a}a_g a_{g'}\bar{a}'\bar{b}, & \text{if } gg' \in G_{\bar{b}}; \\ 0, & \text{if not.} \end{cases} \\ \bar{a}a_g\bar{b} \mapsto \bar{a}a_g\bar{b} \otimes \bar{b} \ (\ \text{where } g \in G_{\bar{b}}), \end{cases}$$

define isomorphisms inverse to each other

$$bA\bar{b} \otimes_{A_{G_{\bar{b}}}\bar{b}} \bar{b}Ab \tilde{\leftrightarrow} Ab \quad \text{and} \quad \bar{b}Ab \otimes_{Ab} bA\bar{b} \tilde{\leftrightarrow} A_{G_{\bar{b}}}\bar{b}.$$

Therefore, $bA\bar{b}$ and $\bar{b}Ab$ are Morita equivalent. In particular, the functors

$$\mathrm{Ind}_{\bar{A}}^{A} = (bA\bar{b} \otimes_{A_{G_{\bar{b}}}\bar{b}} -) \quad \text{and} \quad \bar{b} \cdot \mathrm{Res}_{\bar{A}}^{A} = (\bar{b}Ab \otimes_{Ab} -)$$

define category equivalences inverse to each other between ${}_{A_{G_{\bar{b}}}\bar{b}}\mathbf{mod}$ and ${}_{Ab}\mathbf{mod}$.

Multiplication of an A-module by an $\mathcal{O}G$-module

Let X be an A-module and $\rho : A \to \mathrm{End}_{\mathcal{O}}(X)$ be the structural morphism. We define an additive functor

$$X \cdot - :_{\mathcal{O}G} \mathbf{mod} \to_A \mathbf{mod}, Y \mapsto X \cdot Y$$

as follows:

If Y is an $\mathcal{O}G$-module and $\sigma : \mathcal{O}G \to \mathrm{End}_{\mathcal{O}}(Y)$ is the structural morphism, we denote by $X \cdot Y$ the $\mathcal{O}$-module $X \otimes_{\mathcal{O}} Y$. The action of A on the latter is given by the morphism

$$\rho \cdot \sigma : A \to \mathrm{End}_{\mathcal{O}}(X \otimes Y), \bar{a}a_g \mapsto \rho(\bar{a}a_g) \otimes \sigma(g).$$

Proposition 2.3.12 *Let X be an $\bar{A}$-module. The application*

$$A \otimes_{\bar{A}} X \to X \cdot \mathcal{O}G$$

defined by

$$a_g \otimes_{\bar{A}} x \mapsto \rho(a_g)(x) \otimes_{\mathcal{O}} g \ \textit{(for all } x \in X \textit{ and } g \in G\textit{)}$$

is an isomorphism of A-modules

$$\mathrm{Ind}_{\bar{A}}^{A}(X) \tilde{\to} X \cdot \mathcal{O}G.$$

Induction and restriction of KA-modules and $K\bar{A}$-modules

Let X be a KA-module of character χ and Y a KG-module of character ξ. We denote by $\chi \cdot \xi$ the character of the KA-module $X \cdot Y$. From now on, all group algebras over K will be considered split semisimple.

Proposition 2.3.13 *Let χ be an irreducible character of KA which restricts to an irreducible character $\bar{\chi}$ of $K\bar{A}$. Then*

1. *The characters $(\chi \cdot \xi)_{\xi \in \mathrm{Irr}(KG)}$ are distinct irreducible characters of KA.*
2. *We have*
$$\mathrm{Ind}_{K\bar{A}}^{KA}(\bar{\chi}) = \sum_{\xi \in \mathrm{Irr}(KG)} \xi(1)\chi \cdot \xi.$$

Proof: The second relation results from proposition 2.3.12. Frobenius reciprocity now gives

$$\begin{array}{rcl} \langle \mathrm{Ind}_{K\bar{A}}^{KA}(\bar{\chi}), \mathrm{Ind}_{K\bar{A}}^{KA}(\bar{\chi}) \rangle_{KA} & = & \langle \mathrm{Res}_{K\bar{A}}^{KA}(\sum_{\xi \in \mathrm{Irr}(KG)} \xi(1)\chi \cdot \xi), \bar{\chi} \rangle_{K\bar{A}} \\ & = & \langle \sum_{\xi \in \mathrm{Irr}(KG)} \xi(1)^2 \bar{\chi}, \bar{\chi} \rangle_{K\bar{A}} \\ & = & \sum_{\xi \in \mathrm{Irr}(KG)} \xi(1)^2 = |G|, \end{array}$$

hence from the relation in part 2 we obtain

$$|G| = \sum_{\xi, \xi' \in \mathrm{Irr}(KG)} \xi(1)\xi'(1) \langle \chi \cdot \xi, \chi \cdot \xi' \rangle_{KA}.$$

Since $|G| = \sum_{\xi \in \mathrm{Irr}(KG)} \xi(1)^2$, we must have $\langle \chi \cdot \xi, \chi \cdot \xi' \rangle_{KA} = \delta_{\xi,\xi'}$ and the proof is complete. ■

For all $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$, we denote by $\bar{e}(\bar{\chi})$ the block of $K\bar{A}$ associated with $\bar{\chi}$. We have seen that if $g \in G$, then $g(\bar{e}(\bar{\chi}))$ is also a block of $K\bar{A}$. Since $K\bar{A}$ is split semisimple, it must be associated with an irreducible character $g(\bar{\chi})$ of $K\bar{A}$. Thus, we can define an action of G on $\mathrm{Irr}(K\bar{A})$ such that for all $g \in G$, $\bar{e}(g(\bar{\chi})) = g(\bar{e}(\bar{\chi}))$. We denote by $G_{\bar{\chi}}$ the stabilizer of $\bar{\chi}$ in G.

Proposition 2.3.14 *Let $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$ and suppose that there exists $\tilde{\chi} \in \mathrm{Irr}(KA_{G_{\bar{\chi}}})$ such that $\mathrm{Res}_{K\bar{A}}^{KA_{G_{\bar{\chi}}}}(\tilde{\chi}) = \bar{\chi}$. We set*

$$\chi := \mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(\tilde{\chi}) \text{ and } \chi_\xi := \mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(\tilde{\chi} \cdot \xi) \text{ for all } \xi \in \mathrm{Irr}(KG_{\bar{\chi}}).$$

Then

1. *The characters $(\chi_\xi)_{\xi \in \mathrm{Irr}(KG_{\bar{\chi}})}$ are distinct irreducible characters of KA.*

2. *We have*

$$\mathrm{Ind}_{K\bar{A}}^{KA}(\bar{\chi}) = \sum_{\xi \in \mathrm{Irr}(KG_{\bar{\chi}})} \xi(1)\chi_{\xi}.$$

In particular,

$$m_{\chi_{\xi},\bar{\chi}} = \xi(1) \ \textit{and} \ \chi_{\xi}(1) = |G : G_{\bar{\chi}}|\bar{\chi}(1)\xi(1).$$

3. *For all* $\xi \in \mathrm{Irr}(KG_{\bar{\chi}})$*, we have*

$$s_{\chi_{\xi}} = \frac{|G_{\bar{\chi}}|}{\xi(1)} s_{\bar{\chi}}.$$

Proof:

1. By proposition 2.3.13, we obtain that the characters $(\tilde{\chi} \cdot \xi)_{\xi \in \mathrm{Irr}(KG_{\bar{\chi}})}$ are distinct irreducible characters of $\mathrm{Irr}(KA_{G_{\bar{\chi}}})$. Now let $\bar{e}(\bar{\chi})$ be the block of $K\bar{A}$ associated with the irreducible character $\bar{\chi}$. We have seen that $\bar{e}(\bar{\chi})$ is a central idempotent of $KA_{G_{\bar{\chi}}}$. Proposition 2.3.11 implies that the functor $\mathrm{Ind}_{K\bar{A}}^{KA}$ defines a Morita equivalence between the category ${}_{KA_{G_{\bar{\chi}}}\bar{e}(\bar{\chi})}\mathbf{mod}$ and its image. Therefore, the characters $(\mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(\tilde{\chi} \cdot \xi))_{\xi \in \mathrm{Irr}(KG_{\bar{\chi}})}$ are distinct irreducible characters of KA.

2. By proposition 2.3.13, we obtain that

$$\mathrm{Ind}_{K\bar{A}}^{KA_{G_{\bar{\chi}}}}(\bar{\chi}) = \sum_{\xi \in \mathrm{Irr}(KG_{\bar{\chi}})} \xi(1)\tilde{\chi} \cdot \xi.$$

Applying $\mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}$ to both sides gives us the required relation. Obviously, $m_{\chi_{\xi},\bar{\chi}} = \xi(1)$.

Now let us calculate the value of $\chi_{\xi}(\bar{a})$ for any $\bar{a} \in \bar{A}$. Let Y be an irreducible $KA_{G_{\bar{\chi}}}$-module of character ψ.Then $\mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(Y) = KA \otimes_{KA_{G_{\bar{\chi}}}} Y$ has character $\mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(\psi)$. We have $KA = \bigoplus_{g \in G/G_{\bar{\chi}}} a_g K\bar{A}$. Let $\bar{a} \in \bar{A}$. Then

$$\begin{aligned} \bar{a}\mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(Y) &= \bigoplus_{g \in G/G_{\bar{\chi}}} \bar{a}a_g K\bar{A} \otimes_{KA_{G_{\bar{\chi}}}} Y \\ &= \bigoplus_{g \in G/G_{\bar{\chi}}} a_g({a_g}^{-1}\bar{a}a_g)K\bar{A} \otimes_{KA_{G_{\bar{\chi}}}} Y \\ &= \bigoplus_{g \in G/G_{\bar{\chi}}} a_g K\bar{A} \otimes_{KA_{G_{\bar{\chi}}}} ({a_g}^{-1}\bar{a}a_g)Y. \end{aligned}$$

Thus, $\mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(\psi)(\bar{a}) = \sum_{g \in G/G_{\bar{\chi}}} \psi({a_g}^{-1}\bar{a}a_g)$ and

$$\chi_{\xi}(\bar{a}) = \sum_{g \in G/G_{\bar{\chi}}} (\tilde{\chi} \cdot \xi)({a_g}^{-1}\bar{a}a_g) = \sum_{g \in G/G_{\bar{\chi}}} \bar{\chi}({a_g}^{-1}\bar{a}a_g)\xi(1).$$

Therefore,

$$\chi_\xi(1) = \sum_{g \in G/G_{\bar{\chi}}} \bar{\chi}(1)\xi(1) = |G : G_{\bar{\chi}}|\bar{\chi}(1)\xi(1).$$

3. Let $(\bar{e}_i)_{i\in I}$ be a basis of $\bar{A}$ as $\mathcal{O}$-module and let $(\bar{e}'_i)_{i\in I}$ be its dual with respect to the symmetrizing form $\bar{t}$. Proposition 2.2.12(3), in combination with proposition 2.3.8, gives

$$s_{\chi_\xi}\chi_\xi(1)^2 = \chi_\xi(\sum_{i\in I, g\in G} \bar{e}'_i a_g {a_g}^{-1}\bar{e}_i) = \chi_\xi(|G|\sum_{i\in I}\bar{e}'_i\bar{e}_i).$$

However, $\sum_{i\in I,g\in G}\bar{e}'_i a_g {a_g}^{-1}\bar{e}_i$ belongs to to center of A (by lemma 2.2.6) and thus, for all $h \in G$,

$$ {a_h}^{-1}(\sum_{i\in I}\bar{e}'_i a_g {a_g}^{-1}\bar{e}_i)a_h = \sum_{i\in I}\bar{e}'_i a_g {a_g}^{-1}\bar{e}_i = |G|\sum_{i\in I}\bar{e}'_i\bar{e}_i.$$

Since $\sum_{i\in I}\bar{e}'_i\bar{e}_i \in \bar{A}$, by part 2,

$$\begin{aligned}
\chi_\xi(|G|\textstyle\sum_{i\in I}\bar{e}'_i\bar{e}_i) &= \textstyle\sum_{h\in G/G_{\bar{\chi}}}\bar{\chi}({a_h}^{-1}(|G|\sum_{i\in I}\bar{e}'_i\bar{e}_i)a_h)\xi(1)\\
&= \textstyle\sum_{h\in G/G_{\bar{\chi}}}\bar{\chi}(|G|\sum_{i\in I}\bar{e}'_i\bar{e}_i)\xi(1)\\
&= |G : G_{\bar{\chi}}||G|\xi(1)\bar{\chi}(\textstyle\sum_{i\in I}\bar{e}'_i\bar{e}_i)\\
&= |G : G_{\bar{\chi}}|^2|G_{\bar{\chi}}|\xi(1)s_{\bar{\chi}}\bar{\chi}(1)^2.
\end{aligned}$$

So we have

$$s_{\chi_\xi}\chi_\xi(1)^2 = |G : G_{\bar{\chi}}|^2|G_{\bar{\chi}}|\xi(1)\bar{\chi}(1)^2 s_{\bar{\chi}}.$$

Replacing $\chi_\xi(1) = |G : G_{\bar{\chi}}|\bar{\chi}(1)\xi(1)$ gives

$$s_{\chi_\xi}\xi(1) = |G_{\bar{\chi}}|s_{\bar{\chi}}.$$

■

Now let $\bar{\Omega}$ be the orbit of the character $\bar{\chi} \in \mathrm{Irr}(K\bar{A})$ under the action of G. We have $|\bar{\Omega}| = |G|/|G_{\bar{\chi}}|$. Define

$$\bar{e}(\bar{\Omega}) = \sum_{g\in G/G_{\bar{\chi}}}\bar{e}(g(\bar{\chi})) = \sum_{g\in G/G_{\bar{\chi}}} g(\bar{e}(\bar{\chi})).$$

If $\bar{\chi} \in \bar{\Omega}$, the set $\mathrm{Irr}(KA, \bar{\chi})$ depends only on $\bar{\Omega}$ and we set $\mathrm{Irr}(KA, \bar{\Omega}) := \mathrm{Irr}(KA, \bar{\chi})$. The idempotent $\bar{e}(\bar{\Omega})$ belongs to the algebra $(ZK\bar{A})^G$ of the elements in the center of $K\bar{A}$ fixed by G and thus to the center of KA (since it commutes with all elements of $\bar{A}$ and all a_g, $g \in G$). Therefore, it must be a sum of blocks of KA, *i.e.,*

$$\bar{e}(\bar{\Omega}) = \sum_{\chi \in \mathrm{Irr}(KA, \bar{\Omega})} e(\chi).$$

Let X be an irreducible KA-module of character χ and $\bar{X}$ an irreducible $K\bar{A}$-submodule of $\mathrm{Res}^{KA}_{K\bar{A}}(X)$ of character $\bar{\chi}$. For $g \in G$, the $K\bar{A}$-submodule $g(\bar{X})$ of $\mathrm{Res}^{KA}_{K\bar{A}}(X)$ has character $g(\bar{\chi})$. Then $\sum_{g \in G} g(\bar{X})$ is a KA-submodule of X. We deduce that

$$\mathrm{Res}^{KA}_{K\bar{A}}(X) = (\bigoplus_{g \in G/G_{\bar{\chi}}} g(\bar{X}))^{m_{\chi,\chi'}},$$

i.e.,

$$\mathrm{Res}^{KA}_{K\bar{A}}(\chi) = m_{\chi,\chi'} \sum_{g \in G/G_{\bar{\chi}}} g(\bar{\chi}).$$

In particular, we see that $\mathrm{Irr}(K\bar{A}, \chi)$ is an orbit of G on $\mathrm{Irr}(K\bar{A})$. Notice that $\chi(1) = m_{\chi,\chi'} |\bar{\Omega}| \bar{\chi}(1)$.

Case where G is cyclic

Let G be a cyclic group of order d and let g be a generator of G (we can choose $\mathrm{Rep}(A/\bar{A}) = \{1, a_g, {a_g}^2, \ldots, {a_g}^{d-1}\}$). We will show that the assumptions of proposition 2.3.14 are satisfied for all irreducible characters of $K\bar{A}$.

Let $\bar{X}$ be an irreducible $K\bar{A}$-module and let $\bar{\rho} : K\bar{A} \to \mathrm{End}_K(\bar{X})$ be the structural morphism. Since the representation of $\bar{X}$ is invariant by the action of $G_{\bar{\chi}}$, there exists an automorphism α of the K-vector space $\bar{X}$ such that

$$\alpha \bar{\rho}(\bar{a}) \alpha^{-1} = g(\bar{\rho})(\bar{a}),$$

for all $g \in G_{\bar{\chi}}$.

The subgroup $G_{\bar{\chi}}$ is also cyclic. Let $d(\bar{\chi}) := |G_{\bar{\chi}}|$. Then

$$\bar{\rho}(\bar{a}) = \alpha^{d(\bar{\chi})} \bar{\rho}(\bar{a}) \alpha^{-d(\bar{\chi})}.$$

Since $\bar{X}$ is irreducible and $K\bar{A}$ is split semisimple, $\alpha^{d(\bar{\chi})}$ must be a scalar. Instead of enlarging the field K, we can assume that K contains a $d(\bar{\chi})$-th root of that scalar. By dividing α by that root, we reduce to the case where $\alpha^{d(\bar{\chi})} = 1$.

This allows us to extend the structural morphism $\bar{\rho} : K\bar{A} \to \mathrm{End}_K(\bar{X})$ to a morphism

$$\tilde{\rho} : KA_{G_{\bar{\chi}}} \to \mathrm{End}_K(\bar{X})$$

such that

$$\tilde{\rho}(\bar{a}a_h^j) := \bar{\rho}(\bar{a})\alpha^j \text{ for } 0 \leq j < d(\bar{\chi}),$$

where $h := g^{d/d(\bar{\chi})}$ generates $G_{\bar{\chi}}$. The morphism $\tilde{\rho}$ defines a $KA_{G_{\bar{\chi}}}$-module $\tilde{X}$ of character $\tilde{\chi}$.

Since the group G is abelian, the set $\mathrm{Irr}(KG)$ forms a group, which we denote by $G^\vee$. The application $\psi \mapsto \psi \cdot \xi$, where $\psi \in \mathrm{Irr}(KA)$ and $\xi \in G^\vee$, defines an action of $G^\vee$ on $\mathrm{Irr}(KA)$.

Let Ω be the orbit of $\tilde{\chi}$ under the action of $(G_{\bar{\chi}})^\vee$. By proposition 2.3.13, we obtain that Ω is a regular orbit (*i.e.,* $|\Omega| = |G_{\bar{\chi}}|$) and that $\Omega = \mathrm{Irr}(KA_{G_{\bar{\chi}}}, \tilde{\chi})$.

Like in proposition 2.3.14, we introduce the notations

$$\chi := \mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(\tilde{\chi}) \text{ and } \chi_\xi := \mathrm{Ind}_{KA_{G_{\bar{\chi}}}}^{KA}(\tilde{\chi} \cdot \xi) \text{ for all } \xi \in (G_{\bar{\chi}})^\vee.$$

Then

$$\mathrm{Irr}(KA, \bar{\chi}) = \{\chi_\xi | \xi \in (G_{\bar{\chi}})^\vee\} \text{ and } m_{\chi_\xi, \bar{\chi}} = \xi(1) = 1 \text{ for all } \xi \in (G_{\bar{\chi}})^\vee.$$

Recall that $d(\bar{\chi}) = |G_{\bar{\chi}}|$. There exists a surjective morphism $G \twoheadrightarrow G_{\bar{\chi}}$ defined by $g \mapsto g^{d/d(\bar{\chi})}$, which induces an inclusion $(G_{\bar{\chi}})^\vee \hookrightarrow G^\vee$. If $\xi \in (G_{\bar{\chi}})^\vee$, we denote (abusing notation) by ξ its image in $G^\vee$ by the above injection. It is easy to check that $\chi_\xi = \chi \cdot \xi$.

Hence we have proved the following result

Proposition 2.3.15 *If the group G is cyclic, there exists a bijection*

$$\begin{array}{ccc} \mathrm{Irr}(K\bar{A})/G & \tilde{\leftrightarrow} & \mathrm{Irr}(KA)/G^\vee \\ \bar{\Omega} & \leftrightarrow & \Omega \end{array}$$

such that

$$\bar{e}(\bar{\Omega}) = e(\Omega),\ |\bar{\Omega}||\Omega| = |G| \text{ and } \left\{ \begin{array}{ll} \forall \chi \in \Omega, & \mathrm{Res}_{K\bar{A}}^{KA}(\chi) = \sum_{\bar{\chi} \in \bar{\Omega}} \bar{\chi} \\ \forall \bar{\chi} \in \bar{\Omega}, & \mathrm{Ind}_{K\bar{A}}^{KA}(\bar{\chi}) = \sum_{\chi \in \Omega} \chi \end{array} \right.$$

Moreover, for all $\chi \in \Omega$ and $\bar{\chi} \in \bar{\Omega}$, we have

$$s_\chi = |\Omega| s_{\bar{\chi}}.$$

Blocks of A and blocks of $\bar{A}$

Let us denote by $\mathrm{Bl}(A)$ the set of blocks of A and by $\mathrm{Bl}(\bar{A})$ the set of blocks of $\bar{A}$. For $\bar{b} \in \mathrm{Bl}(\bar{A})$, we have set

$$\mathrm{Tr}(G, \bar{b}) := \sum_{g \in G/G_{\bar{b}}} g(\bar{b}).$$

The algebra $(Z\bar{A})^G$ is contained in both $Z\bar{A}$ and ZA and the set of its blocks is

$$\mathrm{Bl}((Z\bar{A})^G) = \{\mathrm{Tr}(G, \bar{b}) \,|\, \bar{b} \in \mathrm{Bl}(\bar{A})/G\}.$$

Moreover, $\mathrm{Tr}(G, \bar{b})$ is a sum of blocks of A and we define the subset $\mathrm{Bl}(A, \bar{b})$ of $\mathrm{Bl}(A)$ as follows

$$\mathrm{Tr}(G, \bar{b}) := \sum_{b \in \mathrm{Bl}(A, \bar{b})} b.$$

Lemma 2.3.16 *Let $\bar{b}$ be a block of $\bar{A}$ and $\bar{B} := \mathrm{Irr}(K\bar{A}\bar{b})$. Then*

1. *For all $\bar{\chi} \in \bar{B}$, we have $G_{\bar{\chi}} \subseteq G_{\bar{b}}$.*
2. *We have*

$$\mathrm{Tr}(G, \bar{b}) = \sum_{\bar{\chi} \in \bar{B}/G} \mathrm{Tr}(G, \bar{e}(\bar{\chi})) = \sum_{\{\bar{\Omega} | \bar{\Omega} \cap \bar{B} \neq \emptyset\}} \bar{e}(\bar{\Omega}).$$

Proof:

1. If $g \notin G_{\bar{b}}$, then $\bar{b}$ and $g(\bar{b})$ are orthogonal.
2. Note that $\bar{b} = \sum_{\bar{\chi} \in \bar{B}} \bar{e}(\bar{\chi}) = \sum_{\bar{\chi} \in \bar{B}/G_{\bar{b}}} \mathrm{Tr}(G_{\bar{b}}, \bar{e}(\bar{\chi}))$. Thus

$$\mathrm{Tr}(G, \bar{b}) = \sum_{\bar{\chi} \in \bar{B}/G} \mathrm{Tr}(G, \bar{e}(\bar{\chi})) = \sum_{\{\bar{\Omega} | \bar{\Omega} \cap \bar{B} \neq \emptyset\}} \bar{e}(\bar{\Omega}),$$

   by the definition of $\bar{e}(\bar{\Omega})$. ■

Now let $G^\vee := \mathrm{Hom}(G, K^\times)$. We suppose that $K = F$. The multiplication of the characters of KA by the characters of KG defines an action of the group $G^\vee$ on $\mathrm{Irr}(KA)$. This action is induced by the operation of $G^\vee$ on the algebra A, which is defined in the following way:

$$\xi \cdot (\bar{a}a_g) := \xi(g)\bar{a}a_g \quad \text{for all } \xi \in G^\vee, \bar{a} \in \bar{A}, g \in G.$$

In particular, $G^\vee$ acts on the set of blocks of A. Let b be a block of A. Denote by $\xi \cdot b$ the product of ξ and b and by $(G^\vee)_b$ the stabilizer of b in $G^\vee$. We set

$$\mathrm{Tr}(G^\vee, b) := \sum_{\xi \in G^\vee/(G^\vee)_b} \xi \cdot b.$$

The set of blocks of the algebra $(ZA)^{G^\vee}$ is given by

$$\mathrm{Bl}((ZA)^{G^\vee}) = \{\mathrm{Tr}(G^\vee, b) \,|\, b \in \mathrm{Bl}(A)/G^\vee\}.$$

The following lemma is the analogue of lemma 2.3.16

Lemma 2.3.17 *Let b be a block of A and $B := \mathrm{Irr}(KAb)$. Then*

1. *For all $\chi \in B$, we have $(G^\vee)_\chi \subseteq (G^\vee)_b$.*
2. *We have*

$$\mathrm{Tr}(G^\vee, b) = \sum_{\chi \in B/G^\vee} \mathrm{Tr}(G^\vee, e(\chi)) = \sum_{\{\Omega | \Omega \cap B \neq \emptyset\}} e(\Omega).$$

Case where G is cyclic

For every orbit $\mathcal{Y}$ of $G^\vee$ on $\mathrm{Bl}(A)$, we denote by $b(\mathcal{Y})$ the block of $(ZA)^{G^\vee}$ defined as

$$b(\mathcal{Y}) := \sum_{b \in \mathcal{Y}} b.$$

For every orbit $\bar{\mathcal{Y}}$ of G on $\mathrm{Bl}(\bar{A})$, we denote by $\bar{b}(\bar{\mathcal{Y}})$ the block of $(Z\bar{A})^G$ defined as

$$\bar{b}(\bar{\mathcal{Y}}) := \sum_{\bar{b} \in \bar{\mathcal{Y}}} \bar{b}.$$

The following proposition results from proposition 2.3.15 and lemmas 2.3.16 and 2.3.17.

Proposition 2.3.18 *If the group G is cyclic, there exists a bijection*

$$\begin{array}{ccc} \mathrm{Bl}(\bar{A})/G & \tilde{\leftrightarrow} & \mathrm{Bl}(A)/G^\vee \\ \bar{\mathcal{Y}} & \leftrightarrow & \mathcal{Y} \end{array}$$

such that

$$\bar{b}(\bar{\mathcal{Y}}) = b(\mathcal{Y}),$$

i.e.,

$$\mathrm{Tr}(G, \bar{b}) = \mathrm{Tr}(G^\vee, b) \text{ for all } \bar{b} \in \bar{\mathcal{Y}} \text{ and } b \in \mathcal{Y}.$$

In particular, the algebras $(Z\bar{A})^G$ and $(ZA)^{G^\vee}$ have the same blocks.

Corollary 2.3.19 *If the blocks of A are stable by the action of $G^\vee$, then the blocks of A coincide with the blocks of $(Z\bar{A})^G$.*

2.4 Representation theory of symmetric algebras

For the last part of Chapter 2, except for the subsection "A variation for Tits' deformation theorem", the author follows [20], Chapter 7.

Grothendieck groups

Let $\mathcal{O}$ be an integral domain and K a field containing $\mathcal{O}$. Let A be an $\mathcal{O}$-algebra free and finitely generated as $\mathcal{O}$-module.

Let $R_0(KA)$ be the Grothendieck group of finite-dimensional KA-modules. Thus, $R_0(KA)$ is generated by expressions $[V]$, one for each KA-module V (up to isomorphism), with relations $[V] = [V'] + [V'']$ for each exact sequence $0 \to V' \to V \to V'' \to 0$ of KA-modules. Two KA-modules V, V' give rise to the same element in $R_0(KA)$, if V and V' have the same composition factors, counting multiplicities. It follows that $R_0(KA)$ is free abelian with basis given by the isomorphism classes of simple modules. Finally, let $R_0^+(KA)$ be the subset of $R_0(KA)$ consisting of elements $[V]$, where V is a finite-dimensional KA-module.

Definition 2.4.1 *Let x be an indeterminate over K and $\mathrm{Maps}(A, K[x])$ the K-algebra of maps from A to $K[x]$ (with pointwise multiplication of maps as algebra multiplication). If V is a KA-module, let $\rho_V : KA \to \mathrm{End}_K(V)$ denote its structural morphism. We define the map*

$$\begin{array}{rccl} \mathfrak{p}_K: & R_0^+(KA) & \to & \mathrm{Maps}(A, K[x]) \\ & [V] & \mapsto & (a \mapsto \text{characteristic polynomial of } \rho_V(a)). \end{array}$$

Considering $\mathrm{Maps}(A, K[x])$ as a semigroup with respect to multiplication, the map $\mathfrak{p}_K$ is a semigroup homomorphism.

Let $\mathrm{Irr}(KA)$ be the set of all characters χ_V, where V is a simple KA-module.

Lemma 2.4.2 (Brauer-Nesbitt) *Assume that $\mathrm{Irr}(KA)$ is a linearly independent set of $\mathrm{Hom}_K(KA, K)$. Then the map $\mathfrak{p}_K$ is injective.*

Proof: Let V, V' be two KA-modules such that $\mathfrak{p}_K([V]) = \mathfrak{p}_K([V'])$. Since $[V]$, $[V']$ only depend on the composition factors of V, V', we may assume that V, V' are semisimple modules. Let

$$V = \bigoplus_{i=1}^{n} a_i V_i \text{ and } V' = \bigoplus_{i=1}^{n} b_i V_i,$$

where the V_i are pairwise non-isomorphic simple KA-modules and $a_i, b_i \geq 0$ for all i. We have to show that $a_i = b_i$ for all i.

If, for some i, we have both $a_i > 0$ and $b_i > 0$, then we can write $V = V_i \oplus \tilde{V}$ and $V' = V_i \oplus \tilde{V}'$. Since $\mathfrak{p}_K$ is a semigroup homomorphism, we obtain

$$\mathfrak{p}_K([V_i]) \cdot \mathfrak{p}_K([\tilde{V}]) = \mathfrak{p}_K([V]) = \mathfrak{p}_K([V']) = \mathfrak{p}_K([V_i]) \cdot \mathfrak{p}_K([\tilde{V}']),$$

and, dividing by $\mathfrak{p}_K([V_i])$, we conclude that $\mathfrak{p}_K([\tilde{V}]) = \mathfrak{p}_K([\tilde{V}'])$. Thus, we can suppose that, for all i, we have $a_i = 0$ or $b_i = 0$. Taking characters yields that

$$\chi_V = \sum_i a_i \chi_{V_i} \text{ and } \chi_{V'} = \sum_i b_i \chi_{V_i}.$$

For each $a \in A$, we have that the character values $\chi_V(a)$ and $\chi_{V'}(a)$ appear as coefficients in the polynomials $\mathfrak{p}_K([V])(a)$ and $\mathfrak{p}_K([V'])(a)$ respectively. Hence, we have that $\sum_i (a_i - b_i)\chi_{V_i} = 0$. By assumption, the characters χ_{V_i} are linearly independent. So we must have $(a_i - b_i)1_K$ for all i. Since for all i, $a_i = 0$ or $b_i = 0$, this means that $a_1 1_K = 0$ and $b_i 1_K = 0$ for all i. If the field K has characteristic 0, we conclude that $a_i = b_i = 0$ for all i and we are done. If K has characteristic $p > 0$, we conclude that p divides all a_i and all b_i and so $\frac{1}{p}[V]$ and $\frac{1}{p}[V']$ exist in $R_0^+(KA)$. Consequently, we also have $\mathfrak{p}_K(\frac{1}{p}[V]) = \mathfrak{p}_K(\frac{1}{p}[V'])$. Repeating the above argument for $\frac{1}{p}[V]$ and $\frac{1}{p}[V']$ yields that the multiplicity of V_i in each of these modules is still divisible by p. If we repeat this again and again, we deduce that a_i and b_i should be divisible by arbitrary powers of p. This forces $a_i = b_i = 0$ for all i, as desired. ■

*Remark:*The assumption of the Brauer-Nesbitt lemma is satisfied when (but not only when):

- KA is split.
- K is a perfect field.

The following lemma implies the compatibility of the map $\mathfrak{p}_K$ with the field extensions of K ([20], Lemma 7.3.4).

Lemma 2.4.3 *Let $K \subseteq K'$ be a field extension. Then there is a canonical map $d_K^{K'} : R_0(KA) \to R_0(K'A)$ given by $[V] \mapsto [V \otimes_K K']$. Furthermore, we have a commutative diagram*

$$\begin{array}{ccc} R_0^+(KA) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(A, K[x]) \\ \downarrow{\scriptstyle d_K^{K'}} & & \downarrow{\scriptstyle \tau_K^{K'}} \\ R_0^+(K'A) & \xrightarrow{\mathfrak{p}_{K'}} & \mathrm{Maps}(A, K'[x]) \end{array}$$

where $\tau_K^{K'}$ is the canonical embedding. If, moreover, KA is split, then $d_K^{K'}$ is an isomorphism which preserves isomorphism classes of simple modules.

Integrality

We have seen in Chapter 1 that a subring $\mathcal{R} \subseteq K$ is a valuation ring if, for each non-zero element $x \in K$, we have $x \in \mathcal{R}$ or $x^{-1} \in \mathcal{R}$. Consequently, K is the field of fractions of $\mathcal{R}$.

Such a valuation ring is a local ring whose maximal ideal we will denote by $J(\mathcal{R})$. Valuation rings have interesting properties, some of which are:

(V1) If I is a prime ideal of $\mathcal{O}$, then there exists a valuation ring $\mathcal{R} \subseteq K$ such that $\mathcal{O} \subseteq \mathcal{R}$ and $J(\mathcal{R}) \cap \mathcal{O} = I$.

(V2) Every finitely generated torsion-free module over a valuation ring in K is free.

(V3) The intersection of all valuation rings $\mathcal{R} \subseteq K$ with $\mathcal{O} \subseteq \mathcal{R}$ is the integral closure of $\mathcal{O}$ in K; each valuation ring itself is integrally closed in K (Proposition 1.2.14).

Lemma 2.4.4 *Let V be a KA-module. Choosing a K-basis of V, we obtain a corresponding matrix representation $\rho : KA \to \mathrm{M}_n(K)$, where $n = \dim_K(V)$. If $\mathcal{R} \subseteq K$ is a valuation ring with $\mathcal{O} \subseteq \mathcal{R}$, then a basis of V can be chosen so that $\rho(a) \in \mathrm{M}_n(\mathcal{R})$ for all $a \in A$. In that case, we say that V is realized over $\mathcal{R}$.*

Proof: Let $(v_1, \ldots, v_n)$ be a K-basis of V and $\mathcal{B}$ an $\mathcal{O}$-basis for A. Let $\tilde{V}$ be the $\mathcal{O}$-submodule of V spanned by the finite set $\{v_i b \,|\, 1 \leq i \leq n, b \in \mathcal{B}\}$. Then $\tilde{V}$ is invariant under the action of $\mathcal{R}A$ and hence a finitely generated $\mathcal{R}A$-module. Since it is contained in a K-vector space, it is also torsion-free. So (V2) implies that $\tilde{V}$ is an $\mathcal{R}A$-lattice (a finitely generated $\mathcal{R}A$-module which is free as $\mathcal{R}$-module) such that $\tilde{V} \otimes_{\mathcal{R}} K \simeq V$. Thus any $\mathcal{R}$-basis of $\tilde{V}$ is also a K-basis of V with the required property. ■

Remark: Note that the above argument only requires that $\mathcal{R}$ is a subring of K such that K is the field of fractions of $\mathcal{R}$ and $\mathcal{R}$ satisfies (V2). These conditions also hold, for example, when $\mathcal{R}$ is a principal ideal domain with K as field of fractions.

The following two important results derive from the above lemma.

Proposition 2.4.5 *Let V be a KA-module and $\mathcal{O}_K$ be the integral closure of $\mathcal{O}$ in K. Then we have $\mathfrak{p}_K([V])(a) \in \mathcal{O}_K[x]$ for all $a \in A$. Thus the map $\mathfrak{p}_K$ of definition* 2.4.1 *is in fact a map $R_0^+(KA) \to \mathrm{Maps}(A, \mathcal{O}_K[x])$.*

Proof: Fix $a \in A$. Let $\mathcal{R} \subseteq K$ be a valuation ring with $\mathcal{O} \subseteq \mathcal{R}$. By lemma 2.4.4, there exists a basis of V such that the action of a on V with respect to that basis is given by a matrix with coefficients in $\mathcal{R}$. Therefore, we have that $\mathfrak{p}_K([V])(a) \in \mathcal{R}[x]$. Since this holds for all valuation rings $\mathcal{R}$ in K containing $\mathcal{O}$, property (V3) implies that $\mathfrak{p}_K([V])(a) \in \mathcal{O}_K[x]$. ■

Note that, in particular, proposition 2.4.5 implies that $\chi_V(a) \in \mathcal{O}_K$ for all $a \in A$, where χ_V is the character of the representation ρ_V.

Proposition 2.4.6 (Integrality of the Schur elements) *Assume that we have a symmetrizing form t on A. Let V be a split simple KA-module (i.e., $\mathrm{End}_{KA}(V) \simeq K$) and let s_V be its Schur element with respect to the induced form t^K on KA. Then $s_V \in \mathcal{O}_K$.*

Proof: Let $\mathcal{R} \subseteq K$ be a valuation ring with $\mathcal{O} \subseteq \mathcal{R}$. By lemma 2.4.4, we can assume that V affords a representation $\rho : KA \to \mathrm{M}_n(K)$ such that $\rho(a) \in \mathrm{M}_n(\mathcal{R})$ for all $a \in A$. Let $\mathcal{B}$ be an $\mathcal{O}$-basis of A and let $\mathcal{B}'$ be its dual with respect to t. Then $s_V = \sum_{b \in \mathcal{B}} \rho(b)_{ij}\rho(b')_{ji}$ for all $1 \leq i, j \leq n$ ([20], Cor. 7.2.2). All terms in the sum lie in $\mathcal{R}$ and so $s_V \in \mathcal{R}$. Since this holds for all valuation rings $\mathcal{R}$ in K containing $\mathcal{O}$, property (V3) implies that $s_V \in \mathcal{O}_K$. ■

The decomposition map

Now, we moreover assume that the ring $\mathcal{O}$ is integrally closed in K. Throughout we will fix a ring homomorphism $\theta : \mathcal{O} \to L$ into a field L such that L is the field of fractions of $\theta(\mathcal{O})$. We call such a ring homomorphism a *specialization* of $\mathcal{O}$.

Let $\mathcal{R} \subseteq K$ be a valuation ring with $\mathcal{O} \subseteq \mathcal{R}$ and $J(\mathcal{R}) \cap \mathcal{O} = \mathrm{Ker}\theta$ (note that $\mathrm{Ker}\theta$ is a prime ideal, since $\theta(\mathcal{O})$ is contained in a field). Let k be the residue field of $\mathcal{R}$. Then the restriction of the canonical map $\pi : \mathcal{R} \to k$ to $\mathcal{O}$ has kernel $J(\mathcal{R}) \cap \mathcal{O} = \mathrm{Ker}\theta$. Since L is the field of fractions of $\theta(\mathcal{O})$, we may regard L as a subfield of k. Thus, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O} & \subseteq & \mathcal{R} & \subseteq & K \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \pi} & & \\ L & \subseteq & k & & \end{array}$$

From now on, we make the following assumption:

Assumption 2.4.7 *(a) LA is split or (b) $L = k$ and k is perfect.*

The map $\theta : \mathcal{O} \to L$ induces a map $A \to LA, a \mapsto a \otimes 1$. One consequence of the assumption 2.4.7 is that, due to lemma 2.4.3, the map $d_L^k : R_0(LA) \to R_0(kA)$ is an isomorphism which preserves isomorphism classes of simple modules. Thus we can identify $R_0(LA)$ and $R_0(kA)$. Moreover, the Brauer-Nesbitt lemma holds for LA, *i.e.,* the map $\mathfrak{p}_L : R_0^+(LA) \to \mathrm{Maps}(A, L[x])$ is injective.

Let V be a KA-module and $\mathcal{R} \subseteq K$ be a valuation ring with $\mathcal{O} \subseteq \mathcal{R}$. By lemma 2.4.4, there exists a K-basis of V such that the corresponding matrix representation $\rho : KA \to \mathrm{M}_n(K)$ ($n = \dim_K(V)$) has the property that $\rho(a) \in \mathrm{M}_n(\mathcal{R})$. Then that basis generates an $\mathcal{R}A$-lattice $\tilde{V}$ such that $\tilde{V} \otimes_{\mathcal{R}} K = V$. The k-vector space $\tilde{V} \otimes_{\mathcal{R}} k$ is a kA-module via $(v \otimes 1)(a \otimes 1) = va \otimes 1 (v \in \tilde{V}, a \in A)$, which we call the *modular reduction* of $\tilde{V}$.

The matrix representation $\rho^k : kA \to \mathrm{M}_n(k)$ afforded by $k\tilde{V}$ is given by

$$\rho^k(a \otimes 1) = (\pi(a_{ij})) \text{ where } a \in A \text{ and } \rho(a) = (a_{ij}).$$

To simplify notation, we shall write

$$K\tilde{V} := \tilde{V} \otimes_{\mathcal{R}} K \text{ and } k\tilde{V} := \tilde{V} \otimes_{\mathcal{R}} k.$$

Note that if $\tilde{V}'$ is another $\mathcal{R}A$-lattice such that $\tilde{V}' \otimes_{\mathcal{R}} K \simeq V$, then $\tilde{V}$ and $\tilde{V}'$ need not be isomorphic. The same hold for the kA-modules $\tilde{V} \otimes_{\mathcal{R}} k$ and $\tilde{V}' \otimes_{\mathcal{R}} k$.

Theorem-Definition 2.4.8 *Let $\theta : \mathcal{O} \to L$ be a ring homomorphism into a field L such that L is the field of fractions of $\theta(\mathcal{O})$ and $\mathcal{O}$ is integrally closed in K. Assume that we have chosen a valuation ring $\mathcal{R}$ with $\mathcal{O} \subseteq \mathcal{R} \subseteq K$ and $J(\mathcal{R}) \cap \mathcal{O} = \mathrm{Ker}\theta$ and that the assumption 2.4.7 is satisfied. Then*

(a) *The modular reduction induces an additive map $d_\theta : R_0^+(KA) \to R_0^+(LA)$ such that $d_\theta([K\tilde{V}]) = [k\tilde{V}]$, where $\tilde{V}$ is an $\mathcal{R}A$-lattice and $[k\tilde{V}]$ is regarded as an element of $R_0^+(LA)$ via the identification of $R_0(kA)$ and $R_0(LA)$.*

(b) *By Proposition 2.4.5, the image of $\mathfrak{p}_K$ is contained in $\mathrm{Maps}(A, \mathcal{O}[x])$ and we have the following commutative diagram*

$$\begin{array}{ccc} R_0^+(KA) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(A, \mathcal{O}[x]) \\ \downarrow{\scriptstyle d_\theta} & & \downarrow{\scriptstyle \tau_\theta} \\ R_0^+(LA) & \xrightarrow{\mathfrak{p}_L} & \mathrm{Maps}(A, L[x]) \end{array}$$

where $\tau_\theta : \mathrm{Maps}(A, \mathcal{O}[x]) \to \mathrm{Maps}(A, L[x])$ is the map induced by θ.

(c) *The map d_θ is uniquely determined by the commutativity of the above diagram. In particular, the map d_θ depends only on θ and not on the choice of $\mathcal{R}$.*

The map d_θ will be called the decomposition map associated with the specialization $\theta : \mathcal{O} \to L$. The matrix of that map with respect to the bases of $R_0(KA)$ and $R_0(LA)$ consisting of the classes of the simple modules is called the decomposition matrix with respect to θ.

Proof: Let $\tilde{V}$ be an $\mathcal{R}A$-lattice and $a \in A$. Let $(m_{ij}) \in \mathrm{M}_n(\mathcal{R})$ be the matrix describing the action of a on $\tilde{V}$ with respect to a chosen $\mathcal{R}$-basis of $\tilde{V}$. Due to the properties of modular reduction, the action of $a \otimes 1 \in kA$ on $k\tilde{V}$ is given by the matrix $(\pi(m_{ij}))$. Then, by definition, $\mathfrak{p}_L([k\tilde{V}])(a)$ is the characteristic polynomial of $(\pi(m_{ij}))$. On the other hand, applying θ (which is the restriction of π to $\mathcal{O}$) to the coefficients of the characteristic polynomial of (m_{ij}) returns $(\tau_\theta \circ \mathfrak{p}_K)([K\tilde{V}])(a)$. Since the two actions just described commute, the two polynomials obtained are equal. Thus the following relation is established:

$$\mathfrak{p}_L([k\tilde{V}]) = \tau_\theta \circ \mathfrak{p}_K([K\tilde{V}]) \text{ for all } \mathcal{R}A\text{-lattices } \tilde{V} \quad (\dagger)$$

Now let us prove (a). We have to show that the map d_θ is well defined *i.e.,* if $\tilde{V}, \tilde{V}'$ are two $\mathcal{R}A$-lattices such that $K\tilde{V}$ and $K\tilde{V}'$ have the same composition factors (counting multiplicities), then the classes of $k\tilde{V}$ and $k\tilde{V}'$ in $R_0(LA)$ are the same. Moreover, for all $a \in A$, the endomorphisms $\rho_{K\tilde{V}}(a)$ and $\rho_{K\tilde{V}'}(a)$ are conjugate. So the equality $(\dagger)$ implies that

$$\mathfrak{p}_L([k\tilde{V}])(a) = \mathfrak{p}_L([k\tilde{V}'])(a) \text{ for all } a \in A.$$

We have already remarked that, since the assumption 2.4.7 is satisfied, the Brauer-Nesbitt lemma holds for LA. So we conclude $[k\tilde{V}] = [k\tilde{V}']$, as desired.

Having established the existence of d_θ, we have $[k\tilde{V}] = d_\theta([K\tilde{V}])$ for any $\mathcal{R}A$-lattice $\tilde{V}$. Hence $(\dagger)$ yields the commutativity of the diagram in (b).

Finally, by the Brauer-Nesbitt lemma, the map $\mathfrak{p}_L$ is injective. Hence there exists at most one map which makes the diagram in (b) commutative. This proves (c). ■

Remark: Note that if $\mathcal{O}$ is a discrete valuation ring and L its residue field, we do not need the assumption 2.4.7 in order to define a decomposition map from $R_0^+(KA)$ to $R_0^+(LA)$ associated with the canonical map $\theta : \mathcal{O} \to L$. For a given KA-module V, there exists an A-lattice $\tilde{V}$ such that $V = \tilde{V} \otimes_{\mathcal{O}} K$. The map $d_\theta : R_0^+(KA) \to R_0^+(LA)$, $[K\tilde{V}] \mapsto [\tilde{V}/L\tilde{V}]$ is well and uniquely defined. For the details of this construction, see [15], §16C.

Recall from proposition 2.4.5 that if V is a KA-module, then its character χ_V restricts to a trace function $\dot{\chi}_V : A \to \mathcal{O}$. Now, any linear map $\lambda : A \to \mathcal{O}$ induces an L-linear map

$$\lambda^L : LA \to L, a \otimes 1 \mapsto \theta(\lambda(a))(a \in A).$$

It is clear that if λ is a trace function, so is λ^L. Applying this to $\dot{\chi}_V$ shows that $\dot{\chi}_V^L$ is a trace function on LA. Since character values occur as coefficients in characteristic polynomials, theorem 2.4.8 implies that $\dot{\chi}_V^L$ is the character of $d_\theta([V])$. Moreover, for any simple KA-module V, we have

$$\dot{\chi}_V^L = \sum_{V'} d_{VV'} \chi_{V'},$$

where the sum is over all simple LA-modules V' (up to isomorphism) and $D = (d_{VV'})$ is the decomposition matrix associated with θ.

The following result gives a criterion for d_θ to be trivial. For its proof, the reader may refer, for example, to [20], Thm. 7.4.6.

Theorem 2.4.9 (Tits' deformation theorem) *Assume that KA and LA are split. If LA is semisimple, then KA is also semisimple and the decomposition map d_θ is an isomorphism which preserves isomorphism classes of simple modules. In particular, the map* $\mathrm{Irr}(KA) \to \mathrm{Irr}(LA), \chi \mapsto \dot{\chi}^L$ *is a bijection.*

Finally, if A is symmetric, we can check whether the assumption of Tits' deformation theorem is satisfied, using the following theorem (cf. [20], Thm. 7.4.7).

Theorem 2.4.10 (Semisimplicity criterion) *Assume that KA and LA are split and that A is symmetric with symmetrizing form t. For any simple KA-module V, let $s_V \in \mathcal{O}$ be the Schur element with respect to the induced symmetrizing form t^K on KA. Then LA is semisimple if and only if $\theta(s_V) \neq 0$ for all V.*

Corollary 2.4.11 *Let K be the field of fractions of $\mathcal{O}$. Assume that KA is split semisimple and that A is symmetric with symmetrizing form t. If the map θ is injective, then LA is split semisimple.*

A variation for Tits' deformation theorem

Let us suppose that $\mathcal{O}$ is a Krull ring and $\theta : \mathcal{O} \to L$ is a ring homomorphism like above. We will give a criterion for LA to be split semisimple.

Theorem 2.4.12 *Let K be the field of fractions of $\mathcal{O}$. Assume that KA is split semisimple and that A is symmetric with symmetrizing form t. For any simple KA-module V, let $s_V \in \mathcal{O}$ be the Schur element with respect to the induced symmetrizing form t^K on KA. If $\mathrm{Ker}\theta$ is a prime ideal of $\mathcal{O}$ of height 1, then LA is split semisimple if and only if $\theta(s_V) \neq 0$ for all V.*

Proof: If LA is split semisimple, then theorem 2.4.10 implies that $\theta(s_V) \neq 0$ for all V. Now let us denote by $\mathrm{Irr}(KA)$ the set of irreducible characters of KA. If χ is the character afforded by a simple KA-module V_χ, then $s_\chi := s_{V_\chi}$. We set $\mathfrak{q} := \mathrm{Ker}\theta$ and suppose that $s_\chi \notin \mathfrak{q}$ for all $\chi \in \mathrm{Irr}(KA)$. Since KA is split semisimple, it is isomorphic to a product of matrix algebras over K:

$$KA \simeq \prod_{\chi \in \mathrm{Irr}(KA)} \mathrm{End}_K(V_\chi)$$

Let us denote by $\pi_\chi : KA \twoheadrightarrow \mathrm{End}_K(V_\chi)$ the projection onto the χ-factor, such that $\pi := \prod_{\chi \in \mathrm{Irr}(KA)} \pi_\chi$ is the above isomorphism. Then $\chi = \mathrm{tr}_{V_\chi} \circ \pi_\chi$, where tr_{V_χ} denotes the standard trace on $\mathrm{End}_K(V_\chi)$.

Let $\mathcal{B}, \mathcal{B}'$ be two dual bases of A with respect to the symmetrizing form t. By lemma 2.2.7, for all $a \in KA$ and $\chi \in \mathrm{Irr}(KA)$, we have

$$\chi^\vee a = \sum_{b \in \mathcal{B}} \chi(b'a)b.$$

Applying π to both sides yields

$$\pi(\chi^\vee)\pi(a) = \sum_{b \in \mathcal{B}} \chi(b'a)\pi(b).$$

By definition of the Schur element, $\pi(\chi^\vee) = \pi_\chi(\chi^\vee) = \omega_\chi(\chi^\vee) = s_\chi$. Thus, if $\alpha \in \mathrm{End}_K(V_\chi)$, then

$$\pi^{-1}(\alpha) = \frac{1}{s_\chi} \sum_{b \in \mathcal{B}} \mathrm{tr}_{V_\chi}(\pi_\chi(b')\alpha)b. \qquad (\dagger)$$

Since $\mathcal{O}$ is a Krull ring and $\mathfrak{q}$ is a prime ideal of height 1 of $\mathcal{O}$, the ring $\mathcal{O}_\mathfrak{q}$ is, by theorem 1.2.23, a discrete valuation ring. Thanks to lemma 2.4.4, there exists an $\mathcal{O}_\mathfrak{q}A$-lattice $\tilde{V}_\chi$ such that $K \otimes_{\mathcal{O}_\mathfrak{q}} \tilde{V}_\chi \simeq V_\chi$.

Moreover, $1/s_\chi \in \mathcal{O}_\mathfrak{q}$ for all $\chi \in \mathrm{Irr}(KA)$. Due to the relation $(\dagger)$, the map π induces an isomorphism

$$\mathcal{O}_\mathfrak{q}A \simeq \prod_{\chi \in \mathrm{Irr}(KA)} \mathrm{End}_{\mathcal{O}_\mathfrak{q}}(\tilde{V}_\chi),$$

i.e., $\mathcal{O}_\mathfrak{q}A$ is the product of matrix algebras over $\mathcal{O}_\mathfrak{q}$. Since $\mathrm{Ker}\theta = \mathfrak{q}$, the above isomorphism remains after applying θ. Therefore, we obtain that LA is a product of matrix algebras over L and thus split semisimple. ■

If that is the case, then the assumption of Tits' deformation theorem is satisfied and there exists a bijection $\mathrm{Irr}(KA) \leftrightarrow \mathrm{Irr}(LA)$.

Symmetric algebras over discrete valuation rings

From now on, we assume that the following conditions are satisfied:

- $\mathcal{O}$ is a discrete valuation ring in K and K is perfect; let $v : K \to \mathbb{Z} \cup \{\infty\}$ be the corresponding valuation.
- KA is split semisimple.
- $\theta : \mathcal{O} \to L$ is the canonical map onto the residue field L of $\mathcal{O}$.
- A is a symmetric algebra with symmetrizing form t.

We have already seen that we have a well-defined decomposition map $d_\theta : R_0^+(KA) \to R_0^+(LA)$. The decomposition matrix associated with d_θ is the $|\mathrm{Irr}(KA)| \times |\mathrm{Irr}(LA)|$ matrix $D = (d_{\chi\phi})$ with non-negative integer entries such that

$$d_\theta([V_\chi]) = \sum_{\phi \in \mathrm{Irr}(LA)} d_{\chi\phi}[V'_\phi] \quad \text{for } \chi \in \mathrm{Irr}(KA),$$

where V_χ is a simple KA-module with character χ and V'_ϕ is a simple LA-module with character ϕ. We sometimes call the characters of KA "ordinary" and the characters of LA "modular". We say that $\phi \in \mathrm{Irr}(LA)$ is a *modular constituent* of $\chi \in \mathrm{Irr}(KA)$, if $d_{\chi\phi} \neq 0$.

The rows of D describe the decomposition of $d_\theta([V_\chi])$ in the standard basis of $R_0(LA)$. An interpretation of the columns is given by the following result (cf. [20], Thm. 7.5.2), which is part of Brauer's classical theory of modular representations.

Theorem 2.4.13 (Brauer reciprocity) *For each $\phi \in \mathrm{Irr}(LA)$, there exists some primitive idempotent $e_\phi \in A$ such that*

$$[e_\phi KA] = \sum_{\chi \in \mathrm{Irr}(KA)} d_{\chi\phi}[V_\chi] \in R_0^+(KA).$$

Let $\phi \in \mathrm{Irr}(LA)$. Consider the map $\psi(\phi) : ZKA \to K$ defined by

$$\psi(\phi) := \sum_{\chi \in \mathrm{Irr}(KA)} \frac{d_{\chi\phi}}{s_\chi} \omega_\chi,$$

where $\omega_\chi : ZKA \twoheadrightarrow K$ is the central morphism associated with $\chi \in \mathrm{Irr}(KA)$, as defined at the end of section 2.1.

Theorem 2.4.14 *The map $\psi(\phi)$ restricts to a map $ZA \to \mathcal{O}$. In particular,*

$$\psi(\phi)(1) = \sum_{\chi \in \mathrm{Irr}(KA)} \frac{d_{\chi\phi}}{s_\chi} \in \mathcal{O}.$$

Proof: Let us denote by t^K the induced symmetrizing form on KA. If e_ϕ is an idempotent as in theorem 2.4.13, then we can define a K-linear map $\lambda_\phi : ZKA \to K$, $z \mapsto t^K(ze_\phi)$. We claim that $\lambda_\phi = \psi(\phi)$. Since KA is split semisimple, the elements $\{\chi^\vee \mid \chi \in \mathrm{Irr}(KA)\}$ form a basis of ZKA. It is, therefore, sufficient to show that

$$\lambda_\phi(\chi^\vee) = \psi(\phi)(\chi^\vee) \text{ for all } \chi \in \mathrm{Irr}(KA).$$

We have $\psi(\phi)(\chi^\vee) = d_{\chi\phi} 1_K$. Now consider the left-hand side.

$$\begin{aligned} \lambda_\phi(\chi^\vee) &= t^K(\chi^\vee e_\phi) = \chi(e_\phi) = \dim_K(V_\chi e_\phi) 1_K \\ &= \dim_K(\mathrm{Hom}_K(e_\phi KA, V_\chi) 1_K = d_{\chi\phi} 1_K. \end{aligned}$$

Hence the above claim is established.

Finally, it remains to observe that since $e_\phi \in A$, the function λ_ϕ takes values in $\mathcal{O}$ on all elements of A. ■

Finally, we will treat the block distribution of characters. For this purpose, we introduce the following notions.

Definition 2.4.15

1. *The Brauer graph associated with A has vertices labeled by the irreducible characters of KA and an edge joining $\chi, \chi' \in \mathrm{Irr}(KA)$ if $\chi \neq \chi'$ and there exists some $\phi \in \mathrm{Irr}(LA)$ such that $d_{\chi\phi} \neq 0 \neq d_{\chi'\phi}$. A connected component of a Brauer graph is called a block.*

2. *Let $\chi \in \mathrm{Irr}(KA)$. Recall that $0 \neq s_\chi \in \mathcal{O}$. Let $\delta_\chi := v(s_\chi)$, where v is the given valuation. Then δ_χ is called the defect of χ and we have $\delta_\chi \geq 0$ for all $\chi \in \mathrm{Irr}(KA)$. If B is a block, then $\delta_B := \max\{\delta_\chi \mid \chi \in B\}$ is called the defect of B.*

By [18], 17.9, each block B of A corresponds to a central primitive idempotent (block-idempotent, by definition 2.1.1) e_B of A. If $\chi \in B$ and e_χ is its corresponding central primitive idempotent in KA, then $e_B e_\chi \neq 0$.

Every $\chi \in \mathrm{Irr}(KA)$ determines a central morphism $\omega_\chi : ZKA \to K$. Since $\mathcal{O}$ is integrally closed, we have $\omega_\chi(z) \in \mathcal{O}$ for all $z \in ZA$. We have the following standard results relating blocks with central morphisms.

Proposition 2.4.16 *Let $\chi, \chi' \in \mathrm{Irr}(KA)$. Then χ and χ' belong to the same block of A if and only if*

$$\theta(\omega_\chi(z)) = \theta(\omega_{\chi'}(z)) \text{ for all } z \in ZA.$$

i.e.,

$$\omega_\chi(z) \equiv \omega_{\chi'}(z) \bmod J(\mathcal{O}) \text{ for all } z \in ZA.$$

Proof: First assume that χ, χ' belong to the same block of A, *i.e.,* they belong to a connected component of the Brauer graph. It is sufficient to consider the case where χ, χ' are directly linked on the Brauer graph, *i.e.,* there exists some $\phi \in \mathrm{Irr}(LA)$ such that $d_{\chi\phi} \neq 0 \neq d_{\chi'\phi}$. Let $\tilde{V}_\chi$ be an A-lattice such that $K\tilde{V}_\chi$ affords χ. Let $z \in ZA$. Then $z \otimes 1$ acts by the scalar $\theta(\omega_\chi(z))$ on every modular constituent of $k\tilde{V}_\chi$. Similarly, $z \otimes 1$ acts by the scalar $\theta(\omega_{\chi'}(z))$ on every modular constituent of $k\tilde{V}_{\chi'}$, where $\tilde{V}_{\chi'}$ is an A-lattice such that $K\tilde{V}_{\chi'}$ affords χ'. Since, by assumption, $K\tilde{V}_\chi$ and $K\tilde{V}_{\chi'}$ have a modular constituent in common, we have $\theta(\omega_\chi(z)) = \theta(\omega_{\chi'}(z))$, as desired.

Now assume that χ belongs to the block B and χ' to the block B', with $B \neq B'$. Let e_B, $e_{B'}$ be the corresponding central primitive idempotents. Then $\omega_\chi(e_B) = 1$ and $\omega_{\chi'}(e_B) = 0$. Consequently, $\theta(\omega_\chi(e_B)) \neq \theta(\omega_{\chi'}(e_B))$. ■

Theorem 2.4.17 (Blocks of defect 0) *Let $\chi \in \mathrm{Irr}(KA)$ with $\theta(s_\chi) \neq 0$. Then χ is an isolated vertex in the Brauer graph and the corresponding decomposition matrix is just* (1).

Proof: Let t^K be the induced symmetrizing form on KA and $\hat{t}^K$ the isomorphism from KA to $\mathrm{Hom}_K(KA, K)$ induced by t^K. The irreducible character $\chi \in \mathrm{Irr}(KA)$ is a trace function on KA and thus we can define $\chi^\vee := (\hat{t}^K)^{-1}(\chi) \in KA$. Since χ restricts to a trace function $A \to \mathcal{O}$, we have in fact $\chi^\vee \in ZA$. By definition, we have that $\omega_\chi(\chi^\vee) = s_\chi$ and $\omega_{\chi'}(\chi^\vee) = 0$ for any $\chi' \in \mathrm{Irr}(KA), \chi' \neq \chi$. Now assume that there exists some character χ' which is linked to χ in the Brauer graph. Proposition 2.4.16 implies that $0 \neq \theta(s_\chi) = \theta(\omega_\chi(\chi^\vee)) = \theta(\omega_{\chi'}(\chi^\vee)) = 0$, which is absurd.

It remains to show that $d_\theta([V_\chi])$ is the class of a simple module in $R_0^+(LA)$. By lemma 2.4.4, there exists a basis of V_χ and a corresponding representation $\rho : KA \to \mathrm{M}_n(K)$ afforded by V_χ such that $\rho(a) \in \mathrm{M}_n(\mathcal{O})$ for all $a \in A$. Let $\mathcal{B}$ be an $\mathcal{O}$-basis of A and let $\mathcal{B}'$ be its dual with respect to t. We have seen in proposition 2.4.6 that $s_\chi = \sum_{b \in \mathcal{B}} \rho(b)_{ij}\rho(b')_{ji}$ for all $1 \leq i, j \leq n$. All terms in this relation lie in $\mathcal{O}$. So we can apply the map θ and obtain a similar relation for $\theta(s_\chi)$ with respect to the module $L\tilde{V}_\chi$, where $\tilde{V}_\chi \subseteq V_\chi$ is the A-lattice spanned by the above basis of V_χ. Since $\theta(s_\chi) \neq 0$, the module $L\tilde{V}_\chi$ is simple ([20], Lemma 7.2.3). ■

The following result is an immediate consequence of theorems 2.4.14 and 2.4.17.

Proposition 2.4.18 *Let $\chi \in \mathrm{Irr}(KA)$. Then χ is a block by itself if and only if $\theta(s_\chi) \neq 0$.*

Chapter 3

On Hecke algebras

3.1 Complex reflection groups and associated braid groups

Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$ and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K. For every integer $d > 1$, we set $\zeta_d := \exp(2\pi i/d)$ and denote by μ_d the group of all the d-th roots of unity. Let V be a K-vector space of finite dimension r.

Definition 3.1.1 *A pseudo-reflection of* $\mathrm{GL}(V)$ *is a non-trivial element* s *of* $\mathrm{GL}(V)$ *which acts trivially on a hyperplane, called the reflecting hyperplane of* s*.*

If W *is a finite subgroup of* $\mathrm{GL}(V)$ *generated by pseudo-reflections, then* (V, W) *is called a* K*-reflection group of rank* r*.*

We have the following classification of complex reflection groups, also known as "Shephard-Todd classification". For more details about the classification, one may refer to [40].

Theorem 3.1.2 *Let* (V, W) *be an irreducible complex reflection group (i.e.,* W *acts irreducibly on* V*). Then one of the following assertions is true:*

- *There exist non-zero integers* d, e, r *such that* $(V, W) \simeq G(de, e, r)$*, where* $G(de, e, r)$ *is the group of all monomial* $r \times r$ *matrices with entries in* μ_{de} *and product of all non-zero entries in* μ_d*.*
- (V, W) *is isomorphic to one of the 34 exceptional groups* G_n $(n = 4, \ldots, 37)$*.*

The following theorem has been proved (using a case by case analysis) by Benard [2] and Bessis [4] and generalizes a well known result for Weyl groups.

Theorem 3.1.3 *Let (V, W) be a reflection group. Let K be the field generated by the traces on V of all the elements of W. Then all irreducible KW-representations are absolutely irreducible i.e., K is a splitting field for W. The field K is called the field of definition of the reflection group W.*

- If $K \subseteq \mathbb{R}$, then W is a (finite) Coxeter group.
- If $K = \mathbb{Q}$, then W is a Weyl group.

For the following definitions and results about braid groups we follow [14].

Let X be a topological space. Given a point $x_0 \in X$, we denote by $\Pi_1(X, x_0)$ the fundamental group with base point x_0.

Let V be a K-vector space as before. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo-reflections and acting irreducibly on V. We denote by $\mathcal{A}$ the set of its reflecting hyperplanes. We define the *regular variety* $V^{\mathrm{reg}} := \mathbb{C} \otimes V - \bigcup_{H \in \mathcal{A}} \mathbb{C} \otimes H$. For $x_0 \in V^{\mathrm{reg}}$, we define $P := \Pi_1(V^{\mathrm{reg}}, x_0)$ the *pure braid group* (at x_0) associated with W. If now $p : V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ denotes the canonical surjection, we define $B := \Pi_1(V^{\mathrm{reg}}/W, p(x_0))$ the *braid group* (at x_0) associated with W.

The projection p induces a surjective map $B \twoheadrightarrow W, \sigma \mapsto \bar{\sigma}$ as follows: Let $\tilde{\sigma} : [0, 1] \to V^{\mathrm{reg}}$ be a path in V^{reg} such that $\tilde{\sigma}(0) = x_0$, which lifts σ. Then $\bar{\sigma}$ is defined by the equality $\bar{\sigma}(x_0) = \tilde{\sigma}(1)$. Note that the map $\sigma \mapsto \bar{\sigma}$ is an anti-morphism.

Denoting by W^{op} the group opposite to W, we have the following short exact sequence

$$1 \to P \to B \to W^{\mathrm{op}} \to 1,$$

where the map $B \to W^{\mathrm{op}}$ is defined by $\sigma \mapsto \bar{\sigma}$.

Now, for every hyperplane $H \in \mathcal{A}$, we set e_H the order of the group W_H, where W_H is the subgroup of W formed by 1 and all the reflections fixing the hyperplane H. The group W_H is cyclic: if s_H denotes an element of W_H with determinant $\zeta_H := \zeta_{e_H}$, then $W_H =< s_H >$ and s_H is called a *distinguished reflection* in W.

Let $L_H := \mathrm{Im}(s - \mathrm{id}_V)$. Then, for all $x \in V$, we have $x = \mathrm{pr}_H(x) + \mathrm{pr}_{L_H}(x)$ with $\mathrm{pr}_H(x) \in H$ and $\mathrm{pr}_{L_H}(x) \in L_H$. Thus, $s_H(x) = \mathrm{pr}_H(x) + \zeta_H \mathrm{pr}_{L_H}(x)$.

If $t \in \mathbb{R}$, we set $\zeta_H^t := \exp(2\pi i t/e_H)$ and we denote by s_H^t the element of $\mathrm{GL}(V)$ (a pseudo-reflection if $t \neq 0$) defined by

$$s_H^t(x) := \mathrm{pr}_H(x) + \zeta_H^t \mathrm{pr}_{L_H}(x).$$

For $x \in V$, we denote by $\sigma_{H,x}$ the path in V from x to $s_H(x)$ defined by

$$\sigma_{H,x} : [0,1] \to V,\ t \mapsto s_H^t(x).$$

Let γ be a path in V^{reg} with initial point x_0 and terminal point x_H. Then γ^{-1} is the path in V^{reg} with initial point x_H and terminal point x_0 such that

$$\gamma^{-1}(t) = \gamma(1-t) \text{ for all } t \in [0,1].$$

Thus, we can define the path $s_H(\gamma^{-1}) : t \mapsto s_H(\gamma^{-1}(t))$, which goes from $s_H(x_H)$ to $s_H(x_0)$ and lies also in V^{reg}, since for all $x \in V^{\mathrm{reg}}$, $s_H(x) \in V^{\mathrm{reg}}$ (If $s_H(x) \notin V^{\mathrm{reg}}$, then $s_H(x)$ must belong to a hyperplane H'. If $s_{H'}$ is a distinguished pseudo-reflection with reflecting hyperplane H', then $s_{H'}(s_H(x)) = s_H(x)$ and ${s_H}^{-1}(s_{H'}(s_H(x))) = x$. However, ${s_H}^{-1} s_{H'} s_H$ is a reflection and x belongs to its reflecting hyperplane, $s_H^{-1}(H')$. This contradicts the fact that x belongs to V^{reg}.). Now we define a path from x_0 to $s_H(x_0)$ as follows:

$$\sigma_{H,\gamma} := s_H(\gamma^{-1}(t)) \cdot \sigma_{H,x_H} \cdot \gamma$$

If x_H is chosen "close to H and far from the other reflecting hyperplanes", the path $\sigma_{H,\gamma}$ lies in V^{reg} and its homotopy class doesn't depend on the choice of x_H. The element it induces in the braid group B, $\mathbf{s}_{H,\gamma}$, is a distinguished braid reflection around the image of H in V^{reg}/W.

Proposition 3.1.4

1. *The braid group B is generated by the distinguished braid reflections around the images of the hyperplanes $H \in \mathcal{A}$ in V^{reg}/W.*

2. *The image of $\mathbf{s}_{H,\gamma}$ in W is s_H.*

3. *Whenever γ' is a path in V^{reg} from x_0 to x_H, if τ denotes the loop in V^{reg} defined by $\tau := {\gamma'}^{-1}\gamma$, then*

$$\sigma_{H,\gamma'} = s_H(\tau) \cdot \sigma_{H,\gamma} \cdot \tau^{-1}$$

   and in particular $\mathbf{s}_{H,\gamma}$ and $\mathbf{s}_{H,\gamma}$ are conjugate in P.

4. *The path $\prod_{j=e_H-1}^{j=0} \sigma_{H,s_H^j(\gamma)}$, a loop in V^{reg}, induces the element $\mathbf{s}_{H,\gamma}^{e_H}$ in the braid group B and belongs to the pure braid group P. It is a distinguished braid reflection around H in P.*

Definition 3.1.5 *Let s be a distinguished pseudo-reflection in W with reflecting hyperplane H. An s-distinguished braid reflection or monodromy generator is a distinguished braid reflection $\mathbf{s}$ around the image of H in V^{reg}/W such that $\bar{\mathbf{s}} = s$.*

Definition 3.1.6 *Let $x_0 \in V^{\mathrm{reg}}$ as before. We denote by π the element of P defined by the loop $t \mapsto x_0 \exp(2\pi i t)$.*

Lemma 3.1.7 *We have $\pi \in ZP$.*

Theorem-Definition 3.1.8 *Given $\mathcal{C} \in \mathcal{A}/W$, there is a unique length function $l_{\mathcal{C}} : B \to \mathbb{Z}$ defined as follows: if $b = \mathbf{s}_1^{n_1} \cdot \mathbf{s}_2^{n_2} \cdots \mathbf{s}_m^{n_m}$ where (for all j) $n_j \in \mathbb{Z}$ and $\mathbf{s}_j$ is a distinguished braid reflection around an element of $\mathcal{C}_j$, then*

$$l_{\mathcal{C}}(b) = \sum_{\{j \,|\, \mathcal{C}_j = \mathcal{C}\}} n_j.$$

Thus, the length function $l : B \to \mathbb{Z}$ is defined, for all $b \in B$, as

$$l(b) = \sum_{\mathcal{C} \in \mathcal{A}/W} l_{\mathcal{C}}(b).$$

We say that B has an *Artin-like* presentation ([36], 5.2), if it has a presentation of the form

$$< \mathbf{s} \in \mathbf{S} \,|\, \{\mathbf{v}_i = \mathbf{w}_i\}_{i \in I} >,$$

where $\mathbf{S}$ is a finite set of distinguished braid reflections and I is a finite set of relations which are multi-homogeneous, *i.e.,* such that for all i, $\mathbf{v}_i$ and $\mathbf{w}_i$ are positive words in elements of $\mathbf{S}$ (and hence, for each $\mathcal{C} \in \mathcal{A}/W$, we have $l_{\mathcal{C}}(\mathbf{v}_i) = l_{\mathcal{C}}(\mathbf{w}_i)$).

The following result by Bessis ([5], Thm.0.1) shows that any braid group has an Artin-like presentation.

Theorem 3.1.9 *Let W be a complex reflection group with associated braid group B. Then there exists a subset $\mathbf{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ of B such that*

1. *The elements $\mathbf{s}_1, \ldots, \mathbf{s}_n$ are distinguished braid reflection and therefore, their images $s_1, \ldots, s_n$ in W are distinguished reflections.*
2. *The set $\mathbf{S}$ generates B and therefore, $S := \{s_1, \ldots, s_n\}$ generates W.*
3. *There exists a set $\mathcal{R}$ of relations of the form $\mathbf{w}_1 = \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are positive words of equal length in the elements of $\mathbf{S}$, such that $< \mathbf{S} \,|\, \mathcal{R} >$ is a presentation of B.*

4. Viewing now $\mathcal{R}$ as a set of relations in S, the group W is presented by

$$< S \,|\, \mathcal{R}; (\forall s \in S)(s^{e_s} = 1) >$$

where e_s denotes the order of s in W.

3.2 Generic Hecke algebras

Let $K, V, W, \mathcal{A}, P, B$ be defined as in the previous section. For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we set $e_{\mathcal{C}}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H the subgroup formed by 1 and all the reflections fixing the hyperplane H.

We choose a set of indeterminates $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$ and we denote by $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates $\mathbf{u}$. We define the *generic Hecke algebra* $\mathcal{H}$ of W to be the quotient of the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \ldots (\mathbf{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\mathbf{s}$ runs over the set of monodromy generators around the images in V^{reg}/W of the elements of the hyperplane orbit $\mathcal{C}$.

Example 3.2.1 Let $W := G_4 =< s, t \,|\, sts = tst, s^3 = t^3 = 1 >$. Then s and t are conjugate in W and their reflecting hyperplanes belong to the same orbit in $\mathcal{A}/W$. The generic Hecke algebra of W can be presented as follows

$$\mathcal{H} =< S, T \mid \quad STS = TST, \quad \begin{array}{l}(S - u_0)(S - u_1)(S - u_2) = 0,\\ (T - u_0)(T - u_1)(T - u_2) = 0 > .\end{array}$$

We make some assumptions for the algebra $\mathcal{H}$. Note that they have been verified for all but a finite number of irreducible complex reflection groups ([13], remarks before 1.17, § 2; [22]).

Assumptions 3.2.2 *The algebra $\mathcal{H}$ is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module of rank $|W|$. Moreover, there exists a linear form $t : \mathcal{H} \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ with the following properties:*

1. t is a symmetrizing form for $\mathcal{H}$, i.e., $t(hh') = t(h'h)$ for all $h, h' \in \mathcal{H}$ and the map

$$\begin{array}{rccc}\hat{t}: & \mathcal{H} & \to & \mathrm{Hom}(\mathcal{H}, \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}])\\ & h & \mapsto & (h' \mapsto t(hh'))\end{array}$$

is an isomorphism.

2. *Via the specialization* $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^{j}$, *the form* t *becomes the canonical symmetrizing form on the group algebra* $\mathbb{Z}W$.

3. *If we denote by* $\alpha \mapsto \alpha^*$ *the automorphism of* $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ *consisting of the simultaneous inversion of the indeterminates, then for all* $b \in B$, *we have*

$$t(b^{-1})^* = \frac{t(b\pi)}{t(\pi)},$$

where π *is the central element of* P *defined in* 3.1.6.

We know that the form t is unique ([13], 2.1). From now on, let us suppose that the assumptions 3.2.2 are satisfied. Then we have the following result by G.Malle ([30], 5.2).

Theorem 3.2.3 *Let* $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$ *be a set of* $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ *indeterminates such that, for every* $\mathcal{C}, j$, *we have* $v_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}$. *Then the* $K(\mathbf{v})$*-algebra* $K(\mathbf{v})\mathcal{H}$ *is split semisimple.*

By "Tits' deformation theorem" (theorem 2.4.9), it follows that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi \mapsto \chi_{\mathbf{v}}$ from the set $\mathrm{Irr}(W)$ of absolutely irreducible characters of W to the set $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ of absolutely irreducible characters of $K(\mathbf{v})\mathcal{H}$, such that the following diagram is commutative

$$\begin{array}{rccc} \chi_{\mathbf{v}}: & \mathcal{H} & \to & \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \\ & \downarrow & & \downarrow \\ \chi: & \mathbb{Z}_K W & \to & \mathbb{Z}_K. \end{array}$$

Since the assumptions 3.2.2 are satisfied and the algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple, we can define the Schur element $s_{\chi_{\mathbf{v}}}$ for every irreducible character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ with respect to the symmetrizing form t. The bijection $\chi \mapsto \chi_{\mathbf{v}}$ from $\mathrm{Irr}(W)$ to $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ implies that the specialization $v_{\mathcal{C},j} \mapsto 1$ sends $s_{\chi_{\mathbf{v}}}$ to $|W|/\chi(1)$ (which is the Schur element of χ in the group algebra with respect to the canonical symmetrizing form). The following result is simply the application of proposition 2.2.12 to this case.

Proposition 3.2.4

1. *We have*

$$t = \sum_{\chi_{\mathbf{v}}\in\mathrm{Irr}(K(\mathbf{v})\mathcal{H})} \frac{1}{s_{\chi_{\mathbf{v}}}} \chi_{\mathbf{v}}.$$

2. *For all* $\chi_{\mathbf{v}} \in \mathrm{Irr}(K(\mathbf{v})\mathcal{H})$, *the block-idempotent of* $K(\mathbf{v})\mathcal{H}$ *associated with* $\chi_{\mathbf{v}}$ *is* $e_{\chi_{\mathbf{v}}} = \chi_{\mathbf{v}}^{\vee}/s_{\chi_{\mathbf{v}}}$.

Remark: The bijection $\mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(K(\mathbf{v})\mathcal{H})$, $\chi \mapsto \chi_{\mathbf{v}}$ allows us to write $\mathrm{Irr}(W)$ instead of $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ and χ instead of $\chi_{\mathbf{v}}$ in all the relations above.

Our first result concerns the form of the Schur elements associated with the irreducible characters of $K(\mathbf{v})\mathcal{H}$ (always assuming that the asumptions 3.2.2 are satisfied). We will see later that this result plays a crucial role in the determination of the blocks of Hecke algebras.

Theorem 3.2.5 *The Schur element $s_\chi(\mathbf{v})$ associated with the character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ is an element of $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ of the form*

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- *ξ_χ is an element of $\mathbb{Z}_K$,*
- *$N_\chi = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ is a monomial in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ with $\sum_{j=0}^{e_\mathcal{C}-1} b_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,*
- *I_χ is an index set,*
- *$(\Psi_{\chi,i})_{i \in I_\chi}$ is a family of K-cyclotomic polynomials in one variable (i.e., minimal polynomials of the roots of unity over K),*
- *$(M_{\chi,i})_{i \in I_\chi}$ is a family of monomials in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and if $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, then $\gcd(a_{\mathcal{C},j}) = 1$ and $\sum_{j=0}^{e_\mathcal{C}-1} a_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,*
- *$(n_{\chi,i})_{i \in I_\chi}$ is a family of positive integers.*

Proof: By proposition 2.2.10, we have that $s_\chi(\mathbf{v}) \in \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$. The rest is a case by case analysis. For W an irreducible complex reflection group, we will denote by $\mathcal{H}(W)$ its generic Hecke algebra defined over the splitting field of theorem 3.2.3.

Let us first consider the group $G(d, 1, r)$ for $d \geq 1, r > 2$. By [34], Cor. 6.5, the Schur elements of $\mathcal{H}(G(d, 1, r))$ are of the desired form. In [24] it was shown that $\mathcal{H}(G(de, 1, r))$ for a specific choice of parameters becomes the twisted symmetric algebra of the cyclic group of order e over a symmetric subalgebra which is isomorphic to $\mathcal{H}(G(de, e, r))$ (it had been already shown in [12] for $d = 1$). Thus, by proposition 2.3.15, the Schur elements of $\mathcal{H}(G(de, 1, r))$ are multiples by some integer of the Schur elements of $\mathcal{H}(G(de, e, r))$. Therefore, the assertion is established for all the groups of the infinite series $G(de, e, r)$ with $r > 2$.

For the groups $G(de, e, 2)$, G_7, G_{11} and G_{19}, the generic Schur elements are determined by Malle in [29]. In the same article, we can find the specializations of parameters which permit us to calculate, using again proposition 2.3.15, the Schur elements of

- $\mathcal{H}(G_4)$, $\mathcal{H}(G_5)$, $\mathcal{H}(G_6)$ from $\mathcal{H}(G_7)$.
- $\mathcal{H}(G_8), \mathcal{H}(G_9), \mathcal{H}(G_{10}), \mathcal{H}(G_{12}), \mathcal{H}(G_{13}), \mathcal{H}(G_{14}), \mathcal{H}(G_{15})$ from $\mathcal{H}(G_{11})$.
- $\mathcal{H}(G_{16})$, $\mathcal{H}(G_{17})$, $\mathcal{H}(G_{18})$, $\mathcal{H}(G_{20})$, $\mathcal{H}(G_{21})$, $\mathcal{H}(G_{22})$ from $\mathcal{H}(G_{19})$.

For more details, the reader may refer to the Appendix, where the above specializations are given explicitly.

The generic Schur elements for the remaining non-Coxeter exceptional complex reflection groups, *i.e.,* the groups G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}, G_{34}, have been also calculated by Malle in [31].

As far as the exceptional real reflection groups are concerned, *i.e.,* the groups $G_{23} = H_3, G_{28} = F_4, G_{30} = H_4, G_{35} = E_6, G_{36} = E_7, G_{37} = E_8$, the Schur elements have been calculated

- for E_6 and E_7 by Surowski ([41]),
- for E_8 by Benson ([3]),
- for F_4 and H_3 by Lusztig ([28] and [26] respectively),
- for H_4 by Alvis and Lusztig ([1]).

To obtain the desired formula from the data given in the above articles, we used the GAP Package CHEVIE (where some mistakes in these articles have been corrected). ■

In the Appendix of this thesis, we give the factorization of the generic Schur elements of the groups G_7, G_{11}, G_{19}, G_{26}, G_{28} and G_{32}, so that the reader may verify the above result. The Schur elements for G_{25} are also obtained as specializations of the Schur elements of G_{26}. The groups G_{23}, G_{24}, G_{27}, G_{29}, G_{30} G_{31}, G_{33}, G_{34}, G_{35}, G_{36} and G_{37} are all generated by reflections of order 2 whose reflecting hyperplanes belong to one single orbit. Therefore, the splitting field of their generic Hecke algebra is of the form $K(v_0, v_1)$, where K is the field of definition of the group. In these cases, the generic Hecke algebra is essentially one-parametered and it is easy to check that the irreducible factors of the generic Schur elements over $K[v_0^{\pm}, v_1^{\pm}]$ are K-cyclotomic polynomials taking values on $v := v_0 v_1^{-1}$.

Remark: It is a consequence of [38], Thm. 3.4, that the irreducible factors of the generic Schur elements over $\mathbb{C}[\mathbf{v}, \mathbf{v}^{-1}]$ are divisors of Laurent polynomials of the form $M(\mathbf{v})^n - 1$, where

- $M(\mathbf{v})$ is a monomial in $\mathbb{C}[\mathbf{v}, \mathbf{v}^{-1}]$,
- n is a positive integer.

Thanks to proposition 1.5.5, the factorization of the theorem 3.2.5 is unique in $K[\mathbf{v}, \mathbf{v}^{-1}]$. However, this does not mean that the monomials $M_{\chi,i}$ appearing in it are unique. Let

$$\Psi_{\chi,i}(M_{\chi,i}) = u\Psi_{\chi,j}(M_{\chi,j}),$$

where $i, j \in I$, $\Psi_{\chi,i}, \Psi_{\chi,j}$ are two K-cyclotomic polynomials, $M_{\chi,i}, M_{\chi,j}$ are two monomials in $K[\mathbf{v}, \mathbf{v}^{-1}]$ with the properties described in 3.2.5 and u is a unit element in $K[\mathbf{v}, \mathbf{v}^{-1}]$. Let φ_i be a morphism associated with the monomial $M_{\chi,i}$ (see definition 1.4.3) from $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ to a Laurent polynomial ring with one indeterminate less. If we apply φ_i to the above equality, we obtain

$$\Psi_{\chi,i}(1) = \varphi_i(u)\varphi_i(\Psi_{\chi,j}(M_{\chi,j})).$$

Since $\Psi_{\chi,i}(1) \in \mathbb{Z}_K$ and φ_i sends $M_{\chi,j}$ to a monomial, we deduce that $\varphi_i(M_{\chi,j}) = 1$. By proposition 1.4.6(2) and the fact that $M_{\chi,j}$ satisfies the conditions described in proposition 3.2.5, we must have

$$M_{\chi,j} = M_{\chi,i}^{\pm 1}.$$

Hence, the monomials $M_{\chi,i}$ appearing in the factorization of the generic Schur element are unique up to inversion.

If $M_{\chi,j} = M_{\chi,i}$, then $\Psi_{\chi,j} = \Psi_{\chi,i}$ and $u = 1$. If $M_{\chi,j} = M_{\chi,i}^{-1}$, then $\Psi_{\chi,j}$ is conjugate to $\Psi_{\chi,i}$ and u is of the form $\zeta M_{\chi,i}^{\deg(\Psi_{\chi,i})}$, where ζ is a root of unity. Therefore, the coefficient ξ_χ is unique up to a root of unity.

Remark: The first cyclotomic polynomial never appears in the factorization of a Schur element $s_\chi(\mathbf{v})$. Otherwise the specialization $v_{\mathcal{C},j} \mapsto 1$ would send $s_\chi(\mathbf{v})$ to 0 and not to $|W|/\chi(1)$ as it should.

Now let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ and let us denote by A the ring $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$. If $\Psi(M_{\chi,i})$ is a factor of $s_\chi(\mathbf{v})$ and $\Psi_{\chi,i}(1) \in \mathfrak{p}$, then the monomial $M_{\chi,i}$ is called $\mathfrak{p}$-*essential* for χ in A. Due to proposition 1.4.6(1), we have

$$\Psi_{\chi,i}(1) \in \mathfrak{p} \Leftrightarrow \Psi_{\chi,i}(M_{\chi,i}) \in \mathfrak{q}_i, \quad (\dagger)$$

where $\mathfrak{q}_i := (M_{\chi,i} - 1)A + \mathfrak{p}A$. Recall that $\mathfrak{q}_i$ is a prime ideal of A (lemma 1.4.5). Due to the primeness of $\mathfrak{q}_i$, the following proposition is an immediate consequence of (†).

Proposition 3.2.6 *Let $M := \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a monomial in A with $\gcd(a_{\mathcal{C},j}) = 1$ and $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. Then M is $\mathfrak{p}$-essential for χ in A if and only if $s_\chi/\xi_\chi \in \mathfrak{q}_M$, where ξ_χ denotes the coefficient of s_χ in* 3.2.5.

From now on, for reasons of convenience and only for this section, we will substitute the set of indeterminates $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_\mathcal{C}-1)}$ with the set $\{x_0, x_1, \ldots, x_m\}$, where $m := (\sum_{\mathcal{C}\in\mathcal{A}/W} e_\mathcal{C}) - 1$. Hence, the algebra $\mathcal{H}$ will be considered over the ring $A = \mathbb{Z}_K[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$.

Let $M := \prod_{i=0}^m x_i^{a_i}$ be a monomial in A, such that $a_i \in \mathbb{Z}$ and $\gcd(a_i) = 1$. Let $B := \mathbb{Z}_K[y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$ and consider $\varphi_M : A \to B$ a $\mathbb{Z}_K$-algebra morphism associated with M. Let us denote by $\mathcal{H}_{\varphi_M}$ the algebra obtained by $\mathcal{H}$ via the specialization φ_M.

Proposition 3.2.7 *The algebra $K(y_1, \ldots, y_m)\mathcal{H}_{\varphi_M}$ is split semisimple.*

Proof: By theorem 3.2.3, the algebra $K(x_0, x_1, \ldots, x_m)\mathcal{H}$ is split semisimple. The ring A is a Krull ring and $\mathrm{Ker}\varphi_M = (M-1)A$ is a prime ideal of height 1 of A. Due to the form of the generic Schur elements, given in theorem 3.2.5, the fact that the morphism φ_M sends every monomial in A to a monomial in B implies that $\varphi_M(s_\chi) \neq 0$ for all $\chi \in \mathrm{Irr}(K(x_0, x_1, \ldots, x_m)\mathcal{H})$ (we have already explained why the first cyclotomic polynomial never appears in the factorization of the generic Schur elements). Thus we can apply theorem 2.4.12 and obtain that the algebra $K(y_1, \ldots, y_m)\mathcal{H}_{\varphi_M}$ is split semisimple. ■

By "Tits'deformation theorem", the specialization $y_i \mapsto 1$ induces a bijection from the set $\mathrm{Irr}(K(y_1, \ldots, y_m)\mathcal{H}_{\varphi_M})$ of absolutely irreducible characters of $K(y_1, \ldots, y_m)\mathcal{H}_{\varphi_M}$ to the set $\mathrm{Irr}(W)$. The Schur elements of the former are the specializations of the Schur elements of $K(x_0, x_1, \ldots, x_m)\mathcal{H}$ via φ_M and thus of the form described in 3.2.5.

From now on, whenever we refer to irreducible characters, we mean irreducible characters of the group W. Due to the existing bijections

$$\mathrm{Irr}(K(x_0, x_1, \ldots, x_m)\mathcal{H}) \leftrightarrow \mathrm{Irr}(K(y_1, \ldots, y_m)\mathcal{H}_{\varphi_M}) \leftrightarrow \mathrm{Irr}(W),$$

it makes sense to compare the blocks of $\mathcal{H}$ and $\mathcal{H}_{\varphi_M}$ (in terms of partitions of $\mathrm{Irr}(W)$) over suitable rings.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ and $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$.

Theorem 3.2.8 *The blocks of $B_{\mathfrak{p}B}\mathcal{H}_{\varphi_M}$ coincide with the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$.*

Proof: Let us denote by $\mathfrak{n}_M$ the kernel of φ_M, *i.e.,* $\mathfrak{n}_M := (M-1)A$. By proposition 1.4.6(1), we have that $A_{\mathfrak{q}_M}/\mathfrak{n}_M A_{\mathfrak{q}_M} \simeq B_{\mathfrak{p}B}$. Therefore, it is enough to show that the canonical surjection $A_{\mathfrak{q}_M}\mathcal{H} \twoheadrightarrow (A_{\mathfrak{q}_M}/\mathfrak{n}_M A_{\mathfrak{q}_M})\mathcal{H}$ induces a block bijection between these two algebras.

From now on, the symbol $\widehat{\ }$ will stand for $\mathfrak{q}_M$-adic completion. It is immediate that, via the canonical surjection, a block of $A_{\mathfrak{q}_M}\mathcal{H}$ is a sum of blocks of $(A_{\mathfrak{q}_M}/\mathfrak{n}_M A_{\mathfrak{q}_M})\mathcal{H}$. Now let $\bar{e}$ be a block of $(A_{\mathfrak{q}_M}/\mathfrak{n}_M A_{\mathfrak{q}_M})\mathcal{H}$. By theorem 1.3.3, all Noetherian local rings are contained in their completions. Thus $\bar{e}$ can be written as a sum of blocks of $(\widehat{A_{\mathfrak{q}_M}/\mathfrak{n}_M A_{\mathfrak{q}_M}})\mathcal{H}$. Due to corollary 1.3.5, the last algebra is isomorphic to $(\hat{A}_{\mathfrak{q}_M}/\mathfrak{n}_M \hat{A}_{\mathfrak{q}_M})\mathcal{H}$, which is in turn isomorphic to the quotient algebra $\hat{A}_{\mathfrak{q}_M}\mathcal{H}/\mathfrak{n}_M \hat{A}_{\mathfrak{q}_M}\mathcal{H}$. By the theorems of lifting idempotents (see [42], Thm.3.2) and the following lemma, $\bar{e}$ is lifted to a sum of central primitive idempotents in $\hat{A}_{\mathfrak{q}_M}\mathcal{H}$. However, by the fact that $K(x_0, x_1, \ldots, x_m)\mathcal{H}$ is split semisimple, we have that the blocks of $\hat{A}_{\mathfrak{q}_M}\mathcal{H}$ belong to $K(x_0, x_1, \ldots, x_m)\mathcal{H}$. But $K(x_0, x_1, \ldots, x_m) \cap \hat{A}_{\mathfrak{q}_M} = A_{\mathfrak{q}_M}$ (theorem 1.3.6) and $A_{\mathfrak{q}_M}\mathcal{H} \cap Z(\hat{A}_{\mathfrak{q}_M}\mathcal{H}) \subseteq Z(A_{\mathfrak{q}_M}\mathcal{H})$. Therefore, $\bar{e}$ is lifted to a sum of blocks in $A_{\mathfrak{q}_M}\mathcal{H}$ and this provides the block bijection.

Lemma 3.2.9 *Let $\mathcal{O}$ be a Noetherian ring and $\mathfrak{q}$ a prime ideal of $\mathcal{O}$. Let H be an $\mathcal{O}_\mathfrak{q}$-algebra free and of finite rank as a $\mathcal{O}_\mathfrak{q}$-module. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and e an idempotent of H whose image $\bar{e}$ in $(\mathcal{O}_\mathfrak{q}/\mathfrak{p}_\mathfrak{q})H := \mathcal{O}_\mathfrak{q}/\mathfrak{p}\mathcal{O}_\mathfrak{q} \otimes_{\mathcal{O}_\mathfrak{q}} H$ is central. Then e is central.*

Proof: We set $P := \mathcal{O}_\mathfrak{q}/\mathfrak{p}_\mathfrak{q}$. Since $\bar{e}$ is central, we have

$$\bar{e}PH(1-\bar{e}) = (1-\bar{e})PH\bar{e} = \{0\},$$

i.e.,

$$eH(1-e) \subseteq \mathfrak{p}\mathcal{O}_\mathfrak{q}H \quad \text{and} \quad (1-e)He \subseteq \mathfrak{p}\mathcal{O}_\mathfrak{q}H.$$

Since e and $(1-e)$ are idempotents, we get

$$eH(1-e) \subseteq \mathfrak{p}\mathcal{O}_\mathfrak{q}eH(1-e) \quad \text{and} \quad (1-e)He \subseteq \mathfrak{p}\mathcal{O}_\mathfrak{q}(1-e)He.$$

However, $\mathfrak{p}\mathcal{O}_\mathfrak{q} \subseteq \mathfrak{q}\mathcal{O}_\mathfrak{q}$, the latter being the maximal ideal of $\mathcal{O}_\mathfrak{q}$. By Nakayama's lemma,

$$eH(1-e) = (1-e)He = \{0\}.$$

Thus, from

$$H = eHe \oplus eH(1-e) \oplus (1-e)He \oplus (1-e)H(1-e)$$

we deduce that

$$H = eHe \oplus (1-e)H(1-e)$$

and consequently, e is central. ■

■

Remark: Since $\mathfrak{q}_M = \mathfrak{q}_{M^{-1}}$, proposition 3.2.8 implies that the blocks of $B_{\mathfrak{p}B}\mathcal{H}_{\varphi_M}$ coincide with the blocks of $B_{\mathfrak{p}B}\mathcal{H}_{\varphi_{M^{-1}}}$.

Proposition 3.2.10 *If two irreducible characters χ and ψ are in the same block of $A_{\mathfrak{p}A}\mathcal{H}$, then they are in the same block of $A_{\mathfrak{q}_M}\mathcal{H}$.*

Proof: Let C be a block of $A_{\mathfrak{q}_M}\mathcal{H}$. Then $\sum_{\chi\in C} e_\chi \in A_{\mathfrak{q}_M}\mathcal{H} \subset A_{\mathfrak{p}A}\mathcal{H}$. Thus C is a union of blocks of $A_{\mathfrak{p}A}\mathcal{H}$. ■

Corollary 3.2.11 *If two irreducible characters χ and ψ are in the same block of $A_{\mathfrak{p}A}\mathcal{H}$, then they are in the same block of $B_{\mathfrak{p}B}\mathcal{H}_{\varphi_M}$.*

The corollary above implies that the size of $\mathfrak{p}$-blocks grows larger as the number of indeterminates becomes smaller. However, we will now see that the size of blocks remains the same, if our specialization is not associated with a $\mathfrak{p}$-essential monomial.

Proposition 3.2.12 *Let C be a block of $A_{\mathfrak{p}A}\mathcal{H}$. If M is not a $\mathfrak{p}$-essential monomial for any $\chi \in C$, then C is a block of $A_{\mathfrak{q}_M}\mathcal{H}$ (and thus of $B_{\mathfrak{p}B}\mathcal{H}_{\varphi_M}$).*

Proof: Using the notations of proposition 3.2.5, we have that, for all $\chi \in C$, $s_\chi/\xi_\chi \notin \mathfrak{q}_M$. Since C is a block of $A_{\mathfrak{p}A}\mathcal{H}$, we have

$$\sum_{\chi\in C} e_\chi = \sum_{\chi\in C} \frac{\chi^\vee}{s_\chi} \in A_{\mathfrak{p}A}\mathcal{H}.$$

If $\mathcal{B}, \mathcal{B}'$ are two bases of $\mathcal{H}$ dual to each other, then $\chi^\vee = \sum_{b\in\mathcal{B}} \chi(b)b'$ and the above relation implies that

$$\sum_{\chi\in C} \frac{\chi(b)}{s_\chi} \in A_{\mathfrak{p}A}, \forall b \in \mathcal{B}.$$

Let $f_b := \sum_{\chi\in C}(\chi(b)/s_\chi) \in A_{\mathfrak{p}A}$. Then f_b is of the form $r_b/(\xi s)$, where $\xi := \prod_{\chi\in C}\xi_\chi \in \mathbb{Z}_K$ and $s := \prod_{\chi\in C} s_\chi/\xi_\chi \in A$. Since $\mathfrak{q}_M$ is a prime ideal of A, the element s, by assumption, doesn't belong to $\mathfrak{q}_M$. We also have that

$r_b/\xi \in A_{\mathfrak{p}A}$. By corollary 1.2.17, there exists $\xi' \in \mathbb{Z}_K - \mathfrak{p}$ such that $r_b/\xi = r'_b/\xi'$ for some $r'_b \in A$. Since $\mathfrak{q}_M \cap \mathbb{Z}_K = \mathfrak{p}$ (corollary 1.4.7), the element ξ' doesn't belong to the ideal $\mathfrak{q}_M$ either. Therefore, $f_b = r'_b/(\xi' s) \in A_{\mathfrak{q}_M}, \forall b \in \mathcal{B}$ and hence, $\sum_{\chi \in C} e_\chi \in A_{\mathfrak{q}_M}\mathcal{H}$. Thus C is a union of blocks of $A_{\mathfrak{q}_M}\mathcal{H}$. Since the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ are unions of blocks of $A_{\mathfrak{p}A}\mathcal{H}$, we eventually obtain that C is a block of $A_{\mathfrak{q}_M}\mathcal{H}$. ■

Corollary 3.2.13 *If M is not a $\mathfrak{p}$-essential monomial for any $\chi \in \mathrm{Irr}(W)$, then the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ coincide with the blocks of $A_{\mathfrak{p}A}\mathcal{H}$.*

Of course all the above results hold for B in the place of A, if we further specialize B (and $\mathcal{H}_{\varphi_M}$) via a morphism associated with a monomial in B (always assuming that the assumptions 3.2.2 are satisfied).

Now let $r \in \{1, \ldots, m{+}1\}$ and $R := \begin{cases} \mathbb{Z}_K[y_r^{\pm 1}, \ldots, y_m^{\pm 1}], & \text{for } 1 \leq r \leq m; \\ \mathbb{Z}_K, & \text{for } r = m+1, \end{cases}$
where $y_r, \ldots, y_m$ are $m - r + 1$ indeterminates over $\mathbb{Z}_K$. We shall recall some definitions given in Chapter 1.

Definition 3.2.14 *A $\mathbb{Z}_K$-algebra morphism $\varphi : A \to R$ is called adapted, if $\varphi = \varphi_r \circ \varphi_{r-1} \circ \ldots \circ \varphi_1$, where φ_i is a morphism associated with a monomial for all $i = 1, \ldots, r$. The family $\mathcal{F} := \{\varphi_r, \varphi_{r-1}, \ldots, \varphi_1\}$ is called an adapted family for φ whose initial morphism is φ_1.*

Let $\varphi : A \to R$ be an adapted morphism and let F be the field of fractions of R. Let us denote by $\mathcal{H}_\varphi$ the algebra obtained as the specialization of $\mathcal{H}$ via φ. Applying Proposition 3.2.7 r times, we obtain that the algebra $F\mathcal{H}_\varphi$ is split semisimple. Again, by "Tits'deformation theorem", the specialization $y_i \mapsto 1$ induces a bijection from the set $\mathrm{Irr}(F\mathcal{H}_\varphi)$ of absolutely irreducible characters of $F\mathcal{H}_\varphi$ to the set $\mathrm{Irr}(W)$. Therefore, whenever we refer to irreducible characters, we mean irreducible characters of the group W.

We shall repeat here proposition 1.4.11, proved in Chapter 1. Recall that if $M := \prod_{i=0}^m x_i^{b_i}$ is a monomial such that $\gcd(b_i) = d \in \mathbb{Z}$, then $M^\circ := \prod_{i=0}^m x_i^{b_i/d}$.

Proposition 3.2.15 *Let $\varphi : A \to R$ be an adapted morphism and M a monomial in A such that $\varphi(M) = 1$. Then there exists an adapted family for φ whose initial morphism is associated with M°.*

Proposition 3.2.16 *Let $\varphi : A \to R$ be an adapted morphism and $\mathcal{H}_\varphi$ the algebra obtained as the specialization of $\mathcal{H}$ via φ. If M is a monomial in A such that $\varphi(M) = 1$ and $\mathfrak{q}_{M^\circ} := (M^\circ - 1)A + \mathfrak{p}A$, then the blocks of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$ are unions of blocks of $A_{\mathfrak{q}_{M^\circ}}\mathcal{H}$.*

Proof: Let M be a monomial in A such that $\varphi(M) = 1$. Due to proposition 3.2.15, there exists an adapted family for φ whose initial morphism φ_1 is associated with M°. Let us denote by B the image of φ_1 and by $\mathcal{H}_{\varphi_1}$ the algebra obtained as the specialization of $\mathcal{H}$ via φ_1. Thanks to theorem 3.2.8, the blocks of $B_{\mathfrak{p}B}\mathcal{H}_{\varphi_1}$ coincide with the blocks of $A_{\mathfrak{q}_{M^\circ}}\mathcal{H}$. Now, by corollary 3.2.11, if two irreducible characters belong to the same $\mathfrak{p}$-block of a Hecke algebra, then they belong to the same $\mathfrak{p}$-block of its specialization via a morphism associated with a monomial. Inductively, we obtain that the blocks of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$ are unions of blocks of $B_{\mathfrak{p}B}\mathcal{H}_{\varphi_1}$ and thus of $A_{\mathfrak{q}_{M^\circ}}\mathcal{H}$. ■

We will now state and prove our main result concerning the $\mathfrak{p}$-blocks of Hecke algebras. Let us recall the factorization of the generic Schur element s_χ associated with the irreducible character χ in 3.2.5. We have defined a monomial $M := \prod_{i=0}^m x_i^{a_i}$ with $\gcd(a_i) = 1$ to be $\mathfrak{p}$-essential for χ if s_χ has a factor of the form $\Psi(M)$, where Ψ is a K-cyclotomic polynomial such that $\Psi(1) \in \mathfrak{p}$. We have seen (proposition 3.2.6) that M is $\mathfrak{p}$-essential for χ if and only if $s_\chi/\xi_\chi \in \mathfrak{q}_M$. A monomial of that form is called generally $\mathfrak{p}$-*essential for* W if it is $\mathfrak{p}$-essential for some irreducible character χ of W. We can easily find all $\mathfrak{p}$-essential monomials for W by looking at the unique factorization of the generic Schur elements in $K[x_0^\pm, x_1^\pm, \ldots, x_m^\pm]$.

Let $\varphi : A \to R$ be an adapted morphism and $\mathcal{H}_\varphi$ the algebra obtained as the specialization of $\mathcal{H}$ via φ. Let $M_1, \ldots, M_k$ be the $\mathfrak{p}$-essential monomials for W such that $\phi(M_j) = 1$ for all $j = 1, \ldots, k$. We have $M_j^\circ = M_j$ for all $j = 1, \ldots, k$. Set $\mathfrak{q}_0 := \mathfrak{p}A$, $\mathfrak{q}_j := \mathfrak{p}A + (M_j - 1)A$ for $j = 1, \ldots, k$ and $\mathcal{Q} := \{\mathfrak{q}_0, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$.

Now let $\mathfrak{q} \in \mathcal{Q}$. If two irreducible characters χ, ψ belong to the same block of $A_\mathfrak{q}\mathcal{H}$, we write $\chi \sim_\mathfrak{q} \psi$.

Theorem 3.2.17 *Two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same block of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{q}_{j_1}, \ldots, \mathfrak{q}_{j_n} \in \mathcal{Q}$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *for all i $(1 \le i \le n)$, $\chi_{i-1} \sim_{\mathfrak{q}_{j_i}} \chi_i$.*

Proof: Let us denote by $\sim$ the equivalence relation on $\mathrm{Irr}(W)$ defined as the closure of the relation "there exists $\mathfrak{q} \in \mathcal{Q}$ such that $\chi \sim_{\mathfrak{q}} \psi$". Therefore, we have to show that χ and ψ are in the same block of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$ if and only if $\chi \sim \psi$.

If $\chi \sim \psi$, then proposition 3.2.16 implies that χ and ψ are in the same block of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$. Now let C be an equivalence class of $\sim$. We have that C is a union of blocks of $A_{\mathfrak{q}}\mathcal{H}, \forall \mathfrak{q} \in \mathcal{Q}$ and thus

$$\sum_{\theta \in C} \frac{\theta^\vee}{s_\theta} \in A_{\mathfrak{q}}\mathcal{H}, \forall \mathfrak{q} \in \mathcal{Q}.$$

If $\mathcal{B}, \mathcal{B}'$ are two dual bases of $\mathcal{H}$ with respect to the symmetrizing form t, then $\theta^\vee = \sum_{b \in \mathcal{B}} \theta(b) b'$ and hence

$$\sum_{\theta \in C} \frac{\theta(b)}{s_\theta} \in A_{\mathfrak{q}}, \forall \mathfrak{q} \in \mathcal{Q}, \forall b \in \mathcal{B}.$$

Let us recall the form of the Schur element s_θ in 3.2.5. Set $\xi_C := \prod_{\theta \in C} \xi_\theta$ and $s_C := \prod_{\theta \in C} (s_\theta / \xi_\theta)$. Then, for all $b \in \mathcal{B}$, there exists an element $r_{C,b} \in A$ such that

$$\sum_{\theta \in C} \frac{\theta(b)}{s_\theta} = \frac{r_{C,b}}{\xi_C s_C}.$$

The element $s_C \in A$ is product of terms (K-cyclotomic polynomials taking values on monomials) which are irreducible in $K[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, due to proposition 1.5.5. We also have $s_C \notin \mathfrak{p}A$.

Fix $b \in \mathcal{B}$. The ring $K[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ is a unique factorization domain and thus the quotient $r_{C,b}/s_C$ can be written uniquely in the form $r/\alpha s$ where

- $r, s \in A$,
- $\alpha \in \mathbb{Z}_K$,
- $s | s_C$ in A,
- $\gcd(r, s) = 1$ in $K[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$

and for $\xi := \alpha \xi_C$ we have

$$\frac{r_{C,b}}{\xi_C s_C} = \frac{r}{\xi s} \in A_{\mathfrak{q}}, \forall \mathfrak{q} \in \mathcal{Q}.$$

Thus, for all $\mathfrak{q} \in \mathcal{Q}$, there exist $r_{\mathfrak{q}}, s_{\mathfrak{q}} \in A$ with $s_{\mathfrak{q}} \notin \mathfrak{q}$ such that

$$\frac{r}{\xi s} = \frac{r_{\mathfrak{q}}}{s_{\mathfrak{q}}}.$$

Since $(r,s)=1$, we obtain that $s|s_{\mathfrak{q}}$ in $K[x_0^{\pm 1},x_1^{\pm 1},\ldots,x_m^{\pm 1}]$. However, $s|s_C$ in A and thus s is a product of K-cyclotomic polynomials taking values on monomials. Consequently, at least one of the coefficients of s is a unit in A. Corollary 1.2.19 implies that $s|s_{\mathfrak{q}}$ in A. Therefore, $s \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathcal{Q}$.

Moreover, we have that $r/\xi \in A_{\mathfrak{q}_0} = A_{\mathfrak{p}A}$. By corollary 1.2.17, there exist $r' \in A$ and $\xi' \in \mathbb{Z}_K - \mathfrak{p}$ such that $r/\xi = r'/\xi'$. Then

$$\frac{r}{\xi s} = \frac{r'}{\xi' s} \in A_{\mathfrak{q}}, \forall \mathfrak{q} \in \mathcal{Q}.$$

Now let us suppose that $\varphi(\xi' s) = \xi' \varphi(s)$ belongs to $\mathfrak{p}R$. Since $\xi' \notin \mathfrak{p}$, we must have $\varphi(s) \in \mathfrak{p}R$. However, the morphism φ always sends monomial to monomial. Since $s \notin \mathfrak{p}A$ and $s|s_C$, s must have a factor of the form $\Psi(M)$, where

- $M := \prod_{i=0}^{m} x_i^{a_i}$ is a monomial in A such that $\gcd(a_i) = 1$ and $\varphi(M) = 1$,
- Ψ is a K-cyclotomic polynomial such that $\Psi(1) \in \mathfrak{p}$.

Thus $s \in \mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. Since $\Psi(M)|s_C$, M is a $\mathfrak{p}$-essential monomial for some irreducible character $\theta \in C$, *i.e.,* $M \in \{M_1,\ldots,M_k\}$. This contradicts the fact that $s \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathcal{Q}$. Therefore, $\varphi(\xi' s) \notin \mathfrak{p}R$.

So we have

$$\frac{\varphi(r_{C,b})}{\varphi(\xi_C s_C)} = \frac{\varphi(r')}{\varphi(\xi' s)} \in R_{\mathfrak{p}R}$$

and this holds for all $b \in \mathcal{B}$. Consequently,

$$\sum_{\theta \in C} \frac{\varphi(\theta^\vee)}{\varphi(s_\theta)} \in R_{\mathfrak{p}R}\mathcal{H}_\varphi.$$

Thus C is a union of blocks of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$. ■

Remark: We can obtain corollary 3.2.13 as an application of the above theorem for $\mathcal{Q} = \{\mathfrak{q}_0\}$.

To summarize: Theorem 3.2.17 allows us to calculate the blocks of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$ for all adapted morphisms $\varphi : A \to R$, if we know the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ and the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for all $\mathfrak{p}$-essential monomials M. Thus the study of the blocks of the generic Hecke algebra $\mathcal{H}$ in a finite number of cases suffices to calculate the $\mathfrak{p}$-blocks of all Hecke algebras obtained via such specializations.

The following result will be only used in the section about cyclotomic Hecke algebras. However, it is also a result on generic Hecke algebras, so it will be stated now.

Let n be an integer, $n \neq 0$. Define $I^n : A \to A' := \mathbb{Z}_K[y_0^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}]$ to be the $\mathbb{Z}_K$-algebra morphism $x_i \mapsto y_i^n$. Obviously, I^n is injective.

Lemma 3.2.18 *Let n be an integer, $n \neq 0$. Let $I^n : A \to A'$ be the $\mathbb{Z}_K$-algebra morphism defined above and $\mathcal{H}'$ the algebra obtained as the specialization of $\mathcal{H}$ via I^n. Then the blocks of $A'_{\mathfrak{p}A'}\mathcal{H}'$ coincide with the blocks of $A_{\mathfrak{p}A}\mathcal{H}$.*

Proof: Since the map I^n is injective, we can consider A as a subring of A' via the identification $x_i = y_i^n$ for all $i = 0, 1, \ldots, m$. By corollary 1.1.3, we obtain that $A_{\mathfrak{p}A}$ is contained in $A'_{\mathfrak{p}A'}$ and hence, the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ are unions of blocks of $A'_{\mathfrak{p}A'}\mathcal{H}'$.

Now let C be a block of $A'_{\mathfrak{p}A'}\mathcal{H}'$. Since the field of fractions of A is a splitting field for $\mathcal{H}$ (and thus for $\mathcal{H}'$), we obtain that

$$\sum_{\chi \in C} e_\chi \in (A'_{\mathfrak{p}A'} \cap K(x_0, x_1, \ldots, x_m))\mathcal{H}'.$$

If $A'_{\mathfrak{p}A'} \cap K(x_0, x_1, \ldots, x_m) = A_{\mathfrak{p}A}$, then C is also a union of blocks of $A_{\mathfrak{p}A}\mathcal{H}$ and we obtain the desired result.

In order to prove that $A'_{\mathfrak{p}A'} \cap K(x_0, x_1, \ldots, x_m) = A_{\mathfrak{p}A}$, it suffices to show that:

(a) The ring $A_{\mathfrak{p}A}$ is integrally closed.

(b) The ring $A'_{\mathfrak{p}A'}$ is integral over $A_{\mathfrak{p}A}$.

Since the ring A is integrally closed, part (a) is immediate by corollary 1.2.8 which states that any localization of an integrally closed ring is also integrally closed.

For part (b), we have that A' is integral over A, since $y_i^n - x_i = 0$, for all $i = 0, 1, \ldots, m$. Moreover, A' is integrally closed and thus the integral closure of A in $K(y_0, y_1, \ldots, y_m)$. The only prime ideal of A' lying over $\mathfrak{p}A$ is, obviously, $\mathfrak{p}A'$. Following corollary 1.2.11, we obtain that the integral closure of $A_{\mathfrak{p}A}$ in $K(y_0, y_1, \ldots, y_m)$ is $A'_{\mathfrak{p}A'}$. Thus $A'_{\mathfrak{p}A'}$ is integral over $A_{\mathfrak{p}A}$. ■

We can consider I^n as an endomorphism of A and denote it by I^n_A. If k is another integer, $k \neq 0$, then $I^k_A \circ I^n_A = I^n_A \circ I^k_A = I^{kn}_A$. If now $\varphi : A \to R$ is an adapted morphism, we can easily check that $\varphi \circ I^n_A = I^n_R \circ \varphi$. Abusing notation, we write $\varphi \circ I^n = I^n \circ \varphi$.

Corollary 3.2.19 *Let $\varphi : A \to R$ be an adapted morphism and $\mathcal{H}_\varphi$ the algebra obtained as the specialization of $\mathcal{H}$ via φ. Let $\phi : A \to R$ be a $\mathbb{Z}_K$-algebra morphism such that $I^\alpha \circ \varphi = I^\beta \circ \phi$ for some $\alpha, \beta \in \mathbb{Z} - \{0\}$. If $\mathcal{H}_\phi$*

is the algebra obtained as the specialization of $\mathcal{H}$ via ϕ, then the blocks of $R_{\mathfrak{p}R}\mathcal{H}_\phi$ coincide with the blocks of $R_{\mathfrak{p}R}\mathcal{H}_\varphi$ and we can use theorem 3.2.17 *to calculate them.*

3.3 Cyclotomic Hecke algebras

Let y be an indeterminate. We set $x := y^{|\mu(K)|}$.

Definition 3.3.1 *A cyclotomic specialization of $\mathcal{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ with the following properties:*

- *$\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ where $n_{\mathcal{C},j} \in \mathbb{Z}$ for all $\mathcal{C}$ and j.*
- *For all $\mathcal{C} \in \mathcal{A}/W$, and if z is another indeterminate, the element of $\mathbb{Z}_K[y, y^{-1}, z]$ defined by*

$$\Gamma_{\mathcal{C}}(y, z) := \prod_{j=0}^{e_{\mathcal{C}}-1} (z - \zeta_{e_{\mathcal{C}}}^j y^{n_{\mathcal{C},j}})$$

is invariant by the action of $\mathrm{Gal}(K(y)/K(x))$.

If ϕ is a cyclotomic specialization of $\mathcal{H}$, the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y, y^{-1}]$-algebra, denoted by $\mathcal{H}_\phi$, which is obtained as the specialization of the $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$-algebra $\mathcal{H}$ via the morphism ϕ. It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t.

Remark: Sometimes we describe the morphism ϕ by the formula

$$u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j x^{n_{\mathcal{C},j}}.$$

If now we set $q := \zeta x$ for some root of unity $\zeta \in \mu(K)$, then the cyclotomic specialization ϕ becomes a *ζ-cyclotomic specialization* and $\mathcal{H}_\phi$ can be also considered over $\mathbb{Z}_K[q, q^{-1}]$.

Example 3.3.2 The spetsial Hecke algebra $\mathcal{H}_q^s(W)$ is the 1-cyclotomic algebra obtained by the specialization

$$u_{\mathcal{C},0} \mapsto q,\ u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j \text{ for } 1 \leq j \leq e_{\mathcal{C}} - 1, \text{ for all } \mathcal{C} \in \mathcal{A}/W.$$

For example, if $W := G_4$, then

$$\mathcal{H}_q^s(W) =< S, T \mid STS = TST, (S-q)(S^2+S+1) = (T-q)(T^2+T+1) = 0 > .$$

Set $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and $\Omega := \mathbb{Z}_K[y, y^{-1}]$. Let $\phi : A \to \Omega$ be a cyclotomic specialization such that $\phi(v_{\mathcal{C},j}) = y^{n_{\mathcal{C},j}}$. Recall that, for $\alpha \in \mathbb{Z} - \{0\}$, we denote by $I^\alpha : \Omega \to \Omega$ the monomorphism $y \mapsto y^\alpha$.

Theorem 3.3.3 *Let $\phi : A \to \Omega$ be a cyclotomic specialization like above. Then there exist an adapted $\mathbb{Z}_K$-algebra morphism $\varphi : A \to \Omega$ and $\alpha \in \mathbb{Z}-\{0\}$ such that*

$$\phi = I^\alpha \circ \varphi.$$

Proof: We set $d := \gcd(n_{\mathcal{C},j})$ and consider the cyclotomic specialization $\varphi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}/d}$. We have $\phi = I^d \circ \varphi$. Since $\gcd(n_{\mathcal{C},j}/d) = 1$, there exist $a_{\mathcal{C},j} \in \mathbb{Z}$ such that

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j}(n_{\mathcal{C},j}/d) = 1.$$

We have $y = \varphi(\prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}})$ an hence, φ is surjective. Then, by proposition 1.4.12, φ is adapted. ■

Let φ be defined as in theorem 3.3.3 and $\mathcal{H}_\varphi$ the corresponding cyclotomic Hecke algebra. Proposition 3.2.7 implies that the algebra $K(y)\mathcal{H}_\varphi$ is split semisimple. Due to corollary 2.4.11 and the theorem above, we deduce that the algebra $K(y)\mathcal{H}_\phi$ is also split semisimple. For $y = 1$ this algebra specializes to the group algebra KW (the form t_ϕ becoming the canonical form on the group algebra). Thus, by "Tits' deformation theorem", the specialization $v_{\mathcal{C},j} \mapsto 1$ defines the following bijections

$$\begin{array}{ccccc} \mathrm{Irr}(W) & \leftrightarrow & \mathrm{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow & \mathrm{Irr}(K(\mathbf{v})\mathcal{H}) \\ \chi & \mapsto & \chi_\phi & \mapsto & \chi_{\mathbf{v}}. \end{array}$$

The following result is an immediate consequence of the proposition 3.2.5.

Proposition 3.3.4 *The Schur element $s_{\chi_\phi}(y)$ associated with the irreducible character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form*

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. Theorem 3.3.3 allows us to use theorem 3.2.17 for the calculation of the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$, since they coincide with

the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\varphi$ by corollary 3.2.19. Therefore, we need to know which $\mathfrak{p}$-essential monomials are sent to 1 by ϕ.

Let $M := \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a $\mathfrak{p}$-essential monomial for W in A. Then

$$\phi(M) = 1 \Leftrightarrow \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0.$$

Set $m := \sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$. The hyperplane defined in $\mathbb{C}^m$ by the relation

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0,$$

where $(t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of m indeterminates, is called $\mathfrak{p}$-*essential hyperplane* for W. A hyperplane in $\mathbb{C}^m$ is called *essential* for W, if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$.

In order to calculate the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$, we check to which $\mathfrak{p}$-essential hyperplanes the $n_{\mathcal{C},j}$ belong:

- If the $n_{\mathcal{C},j}$ belong to no $\mathfrak{p}$-essential hyperplane, then the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ coincide with the blocks of $A_{\mathfrak{p}A}\mathcal{H}$.
- If the $n_{\mathcal{C},j}$ belong to exactly one $\mathfrak{p}$-essential hyperplane, corresponding to the $\mathfrak{p}$-essential monomial M, then the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ coincide with the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$.
- If the $n_{\mathcal{C},j}$ belong to more than one $\mathfrak{p}$-essential hyperplane, then we use theorem 3.2.17 to calculate the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$.

If now $n_{\mathcal{C},j} = n \in \mathbb{Z}$ for all $\mathcal{C}, j$, then $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi \simeq \Omega_{\mathfrak{p}\Omega}W$ and the $n_{\mathcal{C},j}$ belong to all $\mathfrak{p}$-essential hyperplanes. Due to theorem 3.2.17, we obtain the following proposition

Proposition 3.3.5 *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p. If two irreducible characters χ and ψ are in the same block of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$, then they are in the same p-block of W.*

Proof: The blocks of $\Omega_{\mathfrak{p}\Omega}H_\phi$ are unions of the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for all $\mathfrak{p}$-essential monomials M such that $\phi(M) = 1$, whereas the p-blocks of W are unions of the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for all $\mathfrak{p}$-essential monomials M. ■

Remark: It is well known that, since the ring $\Omega_{\mathfrak{p}\Omega}$ is a discrete valuation ring (by theorem 1.2.23), the blocks of $\Omega_{\mathfrak{p}\Omega}W$ are the p-blocks of W as determined by Brauer theory. The reason is the following:

Let $\hat{\Omega}_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of $\Omega_{\mathfrak{p}}$. Then the p-blocks of W correspond to the central primitive idempotents of $\hat{\Omega}_{\mathfrak{p}}W$. Since $K(y)W$ is a split semisimple algebra and $\Omega_{\mathfrak{p}}$ is a local Noetherian ring, by theorem 1.3.6, the central primitive idempotents of $\hat{\Omega}_{\mathfrak{p}}W$ belong to

$$K(y)W \cap \hat{\Omega}_{\mathfrak{p}}W = \Omega_{\mathfrak{p}}W.$$

Since we are working with group algebras, we also have that

$$Z(\hat{\Omega}_{\mathfrak{p}}W) \cap \Omega_{\mathfrak{p}}W = Z(\Omega_{\mathfrak{p}}W).$$

Thus the central primitive idempotents of $\hat{\Omega}_{\mathfrak{p}}W$ coincide with the central primitive idempotents of $\Omega_{\mathfrak{p}}W$.

However, we know from Brauer theory that if the order of the group W is prime to p, then every character of W is a p-block by itself (see, for example, [39], 15.5, Prop.43). It is an immediate consequence of proposition 3.3.5 that

Proposition 3.3.6 *If $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ lying over a prime number p which doesn't divide the order of the group W, then the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ are singletons.*

3.4 Rouquier blocks of the cyclotomic Hecke algebras

Definition 3.4.1 *We call Rouquier ring of K and denote by $\mathcal{R}_K(y)$ the $\mathbb{Z}_K$-subalgebra of $K(y)$*

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1}_{n\geq 1}]$$

Let $\phi : v_{\mathcal{C},j} \mapsto y^{nc_{\mathcal{C},j}}$ be a cyclotomic specialization and $\mathcal{H}_\phi$ the corresponding cyclotomic Hecke algebra. The *Rouquier blocks* of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$.

Remark: It has been shown by Rouquier [37], that if W is a Weyl group and $\mathcal{H}_\phi$ is obtained via the "spetsial" cyclotomic specialization (see example 3.3.2), then its Rouquier blocks coincide with the "families of characters" defined by Lusztig. Thus, the Rouquier blocks play an essential role in the program "Spets" (see [13]) whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious structures.

Proposition 3.4.2 (Some properties of the Rouquier ring)

1. *The group of units $\mathcal{R}_K(y)^\times$ of the Rouquier ring $\mathcal{R}_K(y)$ consists of the elements of the form*

$$uy^n \prod_{\Phi \in \mathrm{Cycl}(K)} \Phi(y)^{n_\phi},$$

where $u \in \mathbb{Z}_K^\times$, $n, n_\phi \in \mathbb{Z}$, $\mathrm{Cycl}(K)$ is the set of K-cyclotomic polynomials and $n_\phi = 0$ for all but a finite number of Φ.

2. *The prime ideals of $\mathcal{R}_K(y)$ are*

   - *the zero ideal $\{0\}$,*
   - *the ideals of the form $\mathfrak{p}\mathcal{R}_K(y)$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$,*
   - *the ideals of the form $P(y)\mathcal{R}_K(y)$, where $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$.*

3. *The Rouquier ring $\mathcal{R}_K(y)$ is a Dedekind ring.*

Proof:

1. This part is immediate from the definition of K-cyclotomic polynomials.

2. Since $\mathcal{R}_K(y)$ is an integral domain, the zero ideal is prime.

   The ring $\mathbb{Z}_K$ is a Dedekind ring and thus a Krull ring, by proposition 1.2.25. Proposition 1.2.24 implies that the ring $\mathbb{Z}_K[y]$ is also a Krull ring whose prime ideals of height 1 are of the form $\mathfrak{p}\mathbb{Z}_K[y]$ ($\mathfrak{p}$ prime in $\mathbb{Z}_K$) and $P(y)\mathbb{Z}_K[y]$ ($P(y)$ irreducible in $\mathbb{Z}_K[y]$ of degree at least 1). Moreover, $\mathbb{Z}_K$ has an infinite number of non-zero prime ideals whose intersection is the zero ideal. Since all non-zero prime ideals of $\mathbb{Z}_K$ are maximal, we obtain that every prime ideal of $\mathbb{Z}_K$ is the intersection of maximal ideals. Thus $\mathbb{Z}_K$ is, by definition, a Jacobson ring (cf. [17], §4.5). The general form of the Nullstellensatz ([17], Thm.4.19) implies that for every maximal ideal $\mathfrak{m}$ of $\mathbb{Z}_K[y]$, the ideal $\mathfrak{m} \cap \mathbb{Z}_K$ is a maximal ideal of $\mathbb{Z}_K$. We deduce that the maximal ideals of $\mathbb{Z}_K[y]$ are of the form $\mathfrak{p}\mathbb{Z}_K[y] + P(y)\mathbb{Z}_K[y]$ ($\mathfrak{p}$ prime in $\mathbb{Z}_K$ and $P(y)$ of degree at least 1 irreducible modulo $\mathfrak{p}$). Since $\mathbb{Z}_K[y]$ has Krull dimension 2, we have now described all its prime ideals.

   The ring $\mathcal{R}_K(y)$ is a localization of $\mathbb{Z}_K[y]$. Therefore, in order to prove that the non-zero prime ideals of $\mathcal{R}_K(y)$ are the ones described above,

it is enough to show that $\mathfrak{m}\mathcal{R}_K(y) = \mathcal{R}_K(y)$ for all maximal ideals $\mathfrak{m}$ of $\mathbb{Z}_K[y]$. For this, it suffices to show that $\mathfrak{p}\mathcal{R}_K(y)$ is a maximal ideal of $\mathcal{R}_K(y)$ for all prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. Then

$$\mathcal{R}_K(y)/\mathfrak{p}\mathcal{R}_K(y) \simeq \mathbb{F}_\mathfrak{p}[y, y^{-1}, (y^n - 1)_{n\geq 1}^{-1}],$$

where $\mathbb{F}_\mathfrak{p}$ denotes the finite field $\mathbb{Z}_K/\mathfrak{p}$. Since $\mathbb{F}_\mathfrak{p}$ is finite, every polynomial in $\mathbb{F}_\mathfrak{p}[y]$ is a product of elements which divide y or $y^n - 1$ for some $n \in \mathbb{N}$. Thus every element of $\mathbb{F}_\mathfrak{p}[y]$ is invertible in $\mathcal{R}_K(y)/\mathfrak{p}\mathcal{R}_K(y)$. Consequently, we obtain that

$$\mathcal{R}_K(y)/\mathfrak{p}\mathcal{R}_K(y) \simeq \mathbb{F}_\mathfrak{p}(y)$$

and thus $\mathfrak{p}$ generates a maximal ideal in $\mathcal{R}_K(y)$.

3. The ring $\mathcal{R}_K(y)$ is the localization of a Noetherian integrally closed ring and thus Noetherian and integrally closed itself. Moreover, following the description of its prime ideals in part 2, it has Krull dimension 1.■

Remark: If $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$, then the field $\mathcal{R}_K(y)/P(y)\mathcal{R}_K(y)$ is isomorphic to the field of fractions of the ring $\mathbb{Z}_K[y]/P(y)\mathbb{Z}_K[y]$.

Now let us recall the form of the Schur elements of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ given in proposition 3.3.4. If χ_ϕ is an irreducible character of $K(y)\mathcal{H}_\phi$, then its Schur element $s_{\chi_\phi}(y)$ is of the form

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Definition 3.4.3 *A prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ lying over a prime number p is ϕ-bad for W, if there exists $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$ with $\psi_{\chi,\phi} \in \mathfrak{p}$. If $\mathfrak{p}$ is ϕ-bad for W, we say that p is a ϕ-bad prime number for W.*

Remark: If W is a Weyl group and ϕ is the "spetsial" cyclotomic specialization, then the ϕ-bad prime ideals are the ideals generated by the bad prime numbers (in the "usual" sense) for W (see [21], 5.2).

Note that if $\mathfrak{p}$ is ϕ-bad for W, then p must divide the order of the group (since $s_{\chi_\phi}(1) = |W|/\chi(1)$).

Let us denote by $\mathcal{O}$ the Rouquier ring. By proposition 2.1.9, the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the blocks of $\mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ for all prime ideals $\mathcal{P}$ of $\mathcal{O}$. However, in all of the following cases, due to the form of the Schur elements, the blocks of $\mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ are singletons (*i.e.,* $e_{\chi_\phi} = \chi_\phi^\vee/s_{\chi_\phi} \in \mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ for all $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$):

- $\mathcal{P}$ is the zero ideal $\{0\}$.
- $\mathcal{P}$ is of the form $P(y)\mathcal{O}$, where $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$.
- $\mathcal{P}$ is of the form $\mathfrak{p}\mathcal{O}$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ which is not ϕ-bad for W.

Therefore, the blocks of $\mathcal{O}\mathcal{H}_\phi$ are, simply, unions of the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ for all ϕ-bad prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$. By proposition 1.1.5(4), we obtain that $\mathcal{O}_{\mathfrak{p}\mathcal{O}} \simeq \Omega_{\mathfrak{p}\Omega}$, where $\Omega := \mathbb{Z}_K[y, y^{-1}]$. In the previous section we saw how we can use theorem 3.2.17 to calculate the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ and thus obtain the Rouquier blocks of $\mathcal{H}_\phi$.

The following description of the Rouquier blocks results from proposition 2.1.10 and the description of ϕ-bad prime ideals for W.

Proposition 3.4.4 *Let $\chi, \psi \in \mathrm{Irr}(W)$. The characters χ_ϕ and ψ_ϕ are in the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exists a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of ϕ-bad prime ideals of $\mathbb{Z}_K$ such that*

- *$(\chi_0)_\phi = \chi_\phi$ and $(\chi_n)_\phi = \psi_\phi$,*
- *for all j $(1 \leq j \leq n)$, $\omega_{(\chi_{j-1})_\phi} \equiv \omega_{(\chi_j)_\phi} \bmod \mathfrak{p}_j\mathcal{O}$.*

Following the notations in [13], 6B, for every element $P(y) \in \mathbb{C}(y)$, we call

- *valuation of $P(y)$ at y* and denote by $\mathrm{val}_y(P)$ the order of $P(y)$ at 0 (we have $\mathrm{val}_y(P) < 0$ if 0 is a pole of $P(y)$ and $\mathrm{val}_y(P) > 0$ if 0 is a zero of $P(y)$),
- *degree of $P(y)$ at y* and denote by $\deg_y(P)$ the opposite of the valuation of $P(1/y)$.

Moreover, if $y^n = x$, then

$$\mathrm{val}_x(P(y)) := \frac{\mathrm{val}_y(P)}{n} \text{ and } \deg_x(P(y)) := \frac{\deg_y(P)}{n}.$$

For $\chi \in \mathrm{Irr}(W)$, we define

$$a_{\chi_\phi} := \mathrm{val}_x(s_{\chi_\phi}(y)) \text{ and } A_{\chi_\phi} := \deg_x(s_{\chi_\phi}(y)).$$

The following result is proven in [12], Prop.2.9.

Proposition 3.4.5 *Let $x := y^{|\mu(K)|}$.*

1. *For all $\chi \in \mathrm{Irr}(W)$, we have*

$$\omega_{\chi_\phi}(\pi) = t_\phi(\pi) x^{a_{\chi_\phi} + A_{\chi_\phi}},$$

   where π is the central element of the pure braid group defined in 3.1.6.

2. *Let $\chi, \psi \in \mathrm{Irr}(W)$. If χ_ϕ and ψ_ϕ belong to the same Rouquier block, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

Proof:

1. If $P(y) \in \mathbb{C}[y, y^{-1}]$, we denote by $P(y)^*$ the polynomial whose coefficients are the complex conjugates of those of $P(y)$. By [13], 2.8, we know that the Schur element $s_{\chi_\phi}(y)$ is semi-palindromic and satisfies

$$s_{\chi_\phi}(y^{-1})^* = \frac{t_\phi(\pi)}{\omega_{\chi_\phi}(\pi)} s_{\chi_\phi}(y).$$

   We deduce ([13], 6.5, 6.6) that

$$\frac{t_\phi(\pi)}{\omega_{\chi_\phi}(\pi)} = \xi x^{-(a_{\chi_\phi} + A_{\chi_\phi})},$$

   for some $\xi \in \mathbb{C}$. For $y = x = 1$, the first equation gives $t_\phi(\pi) = \omega_{\chi_\phi}(\pi)$ and the second one $\xi = 1$. Thus we obtain

$$\omega_{\chi_\phi}(\pi) = t_\phi(\pi) x^{a_{\chi_\phi} + A_{\chi_\phi}}.$$

2. Suppose that χ_ϕ and ψ_ϕ belong to the same Rouquier block. Due to proposition 3.4.4, it is enough to show that if there exists a ϕ-bad prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ such that $\omega_{\chi_\phi} \equiv \omega_{\psi_\phi} \bmod \mathfrak{p}\mathcal{O}$, then $a_{\chi_\phi}+A_{\chi_\phi} = a_{\psi_\phi}+A_{\psi_\phi}$. If $\omega_{\chi_\phi} \equiv \omega_{\psi_\phi} \bmod \mathfrak{p}\mathcal{O}$, then, in particular, $\omega_{\chi_\phi}(\pi) \equiv \omega_{\psi_\phi}(\pi) \bmod \mathfrak{p}\mathcal{O}$. Part 1 implies that

$$t_\phi(\pi)x^{a_{\chi_\phi}+A_{\chi_\phi}} \equiv t_\phi(\pi)x^{a_{\psi_\phi}+A_{\psi_\phi}} \bmod \mathfrak{p}\mathcal{O}.$$

We know by [13], 2.1 that $t_\phi(\pi)$ is of the form ξx^M, where ξ is a root of unity and $M \in \mathbb{Z}$. Thus $t_\phi(\pi) \notin \mathfrak{p}\mathcal{O}$ and the above congruence gives

$$x^{a_{\chi_\phi}+A_{\chi_\phi}} \equiv x^{a_{\psi_\phi}+A_{\psi_\phi}} \bmod \mathfrak{p}\mathcal{O},$$

whence

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

■

Remark: For all Coxeter groups, Kazhdan-Lusztig theory states that if χ_ϕ and ψ_ϕ belong to the same Rouquier block, then $a_{\chi_\phi} = a_{\psi_\phi}$ and $A_{\chi_\phi} = A_{\psi_\phi}$ (cf. [26]). The same assertion has been proven

- for the imprimitive complex reflection groups in [12].
- for the "spetsial" complex reflection groups in [33].

The results of the next chapter prove that it holds for the groups G_{12}, G_{22} and G_{31}. We conjecture that it is true for all the remaining exceptional complex reflection groups (*i.e.,* the groups $G_5, G_7, G_9, G_{10}, G_{11}, G_{13}, G_{15\ldots21}$).

Chapter 4

Rouquier blocks of the cyclotomic Hecke algebras of the exceptional complex reflection groups

All the notations used in this chapter have been explained in Chapter 3.

4.1 General principles

Let W be a complex reflection group such that the assumptions 3.2.2 are satisfied. Let $\mathcal{H}$ be its generic Hecke algebra defined over the ring $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p which divides the order of the group W. We can determine the $\mathfrak{p}$-essential hyperplanes for each character $\chi \in \mathrm{Irr}(W)$ by looking at the factorization of its generic Schur element in $K[\mathbf{v}, \mathbf{v}^{-1}]$ (see theorem 3.2.5).

Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and let $\mathcal{H}_\phi$ be the cyclotomic Hecke algebra obtained by $\mathcal{H}$ via ϕ. Let us denote by $\mathcal{O}$ the Rouquier ring. We can distinguish three cases.

- If the $n_{\mathcal{C},j}$ belong to no $\mathfrak{p}$-essential hyperplane, then the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ coincide with the blocks of $A_{\mathfrak{p}A}\mathcal{H}$.

- If the $n_{\mathcal{C},j}$ belong to exactly one $\mathfrak{p}$-essential hyperplane, corresponding to the $\mathfrak{p}$-essential monomial M, then the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ coincide with the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$, where $\mathfrak{q}_M := \mathfrak{p}A + (M-1)A$.

- If the $n_{\mathcal{C},j}$ belong to more than one $\mathfrak{p}$-essential hyperplane, then we use theorem 3.2.17 in order to calculate the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$.

Now recall that the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ for all ϕ-bad prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$. We distinguish again three cases.

- If the $n_{\mathcal{C},j}$ belong to no essential hyperplane, then the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ for all ϕ-bad prime ideals $\mathfrak{p}$. We say that these are *the Rouquier blocks associated with no essential hyperplane for* W.

- If the $n_{\mathcal{C},j}$ belong to exactly one essential hyperplane, corresponding to the essential monomial M, then the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$, where $\mathfrak{q}_M := \mathfrak{p}A+(M-1)A$, for all ϕ-bad prime ideals $\mathfrak{p}$ (if M is not $\mathfrak{p}$-essential, then, by corollary 3.2.13, the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ coincide with the blocks of $A_{\mathfrak{p}A}\mathcal{H}$). We say that these are *the Rouquier blocks associated with that essential hyperplane.*

- If the $n_{\mathcal{C},j}$ belong to more than one essential hyperplane, then the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the Rouquier blocks associated with the essential hyperplanes to which the $n_{\mathcal{C},j}$ belong.

Therefore, if we know the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ and $A_{\mathfrak{q}_M}\mathcal{H}$ for all $\mathfrak{p}$-essential monomials M, for all $\mathfrak{p}$, we know the Rouquier blocks of $\mathcal{H}_\phi$ for any cyclotomic specialization ϕ.

In order to calculate the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ (resp. of $A_{\mathfrak{q}_M}\mathcal{H}$), we find a cyclotomic specialization $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ such that the $n_{\mathcal{C},j}$ belong to no $\mathfrak{p}$-essential hyperplane (resp. the $n_{\mathcal{C},j}$ belong to the $\mathfrak{p}$-essential hyperplane corresponding to M and no other) and we calculate the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$.

The algorithm presented in the next section uses some theorems proved in previous chapters in order to form a partition of $\mathrm{Irr}(W)$ into sets which are unions of blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$. These theorems are

2.4.18 An irreducible character χ is a block by itself in $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ if and only if $s_{\chi_\phi} \notin \mathfrak{p}\mathbb{Z}_K[y, y^{-1}]$.

3.3.5 If χ, ψ belong to the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$, then they are in the same p-block of W.

3.4.5 If χ, ψ are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$, then $a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}$.

3.2.12 Let C be a block of $A_{\mathfrak{p}A}\mathcal{H}$. If M is not a $\mathfrak{p}$-essential monomial for any $\chi \in C$, then C is a block of $A_{\mathfrak{q}_M}\mathcal{H}$.

If the partition obtained is minimal, then it represents the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$.

With the help of the package CHEVIE of GAP, we created a program that follows this algorithm to obtain the Rouquier blocks of all cyclotomic Hecke algebras of the groups G_7, G_{11}, G_{19}, G_{26}, G_{28} and G_{32}. We used Clifford theory (for more details, see Appendix) in order to obtain the Rouquier blocks for

- G_4, G_5, G_6 from G_7,
- G_8, G_9, G_{10}, G_{12}, G_{13}, G_{14}, G_{15} from G_{11},
- G_{16}, G_{17}, G_{18}, G_{20}, G_{21}, G_{22} from G_{19},
- G_{25} from G_{26}.

In all of the above cases (except for the "spetsial" case for G_{32}), we can determine that the partition into $\mathfrak{p}$-blocks obtained is minimal. This is done either by using again the above theorems or by applying the results of Clifford theory.

For all remaining groups, the Rouquier blocks of the "spetsial" cyclotomic Hecke algebra have already been calculated in [33] (along with those of G_{32}), where more criteria for the partition of $\mathrm{Irr}(W)$ into $\mathfrak{p}$-blocks are given. Since they are groups generated by reflections of order 2 whose reflecting hyperplanes belong to one single orbit, their generic Hecke algebras are defined over a ring of the form $\mathbb{Z}[x_0^{\pm}, x_1^{\pm}]$ and the only essential monomial is $x_0 x_1^{-1}$. If $x_i \mapsto y^{a_i}$ is a cyclotomic specialization and $a_0 = a_1$, then the specialized algebra is the group algebra, whose Rouquier blocks are known (they are unions of the group's p-blocks for all primes p dividing the order of the group). According to the above algorithm, it is enough to study one case where $a_0 \neq a_1$ and thus the "spetsial" case covers our needs.

4.2 Algorithm

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p which divides the order of the group W. As we saw in the previous section, we need to calculate the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ and the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for all $\mathfrak{p}$-essential monomials M.

Together with Jean Michel, we have programmed into GAP the factorized generic Schur elements for all exceptional complex reflection groups, verifying thus theorem 3.2.5. These data have been stored under the name "Schur-Data" and correspond to the following presentation of the Schur element of an irreducible character χ:

$$s_\chi = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}} \quad (\dagger)$$

(for the notations, the reader should refer to theorem 3.2.5). Firstly, we determine the $\mathfrak{p}$-essential monomials/$\mathfrak{p}$-essential hyperplanes for W. Given the prime ideal $\mathfrak{p}$, GAP provides us with a way to determine whether an element of $\mathbb{Z}_K$ belongs to $\mathfrak{p}$. In the above formula, if $\Psi_{\chi,i}(1) \in \mathfrak{p}$, then $M_{\chi,i}$ is a $\mathfrak{p}$-essential monomial.

If now we are interested in calculating the blocks of $A_{\mathfrak{p}A}\mathcal{H}$, we follow the steps below:

1. We select the characters $\chi \in \mathrm{Irr}(W)$ whose Schur element has its coefficient ξ_χ in $\mathfrak{p}$. The remaining ones will be blocks of $A_{\mathfrak{p}A}\mathcal{H}$ by themselves, thanks to proposition 2.4.18. Thus we form a first partition λ_1 of $\mathrm{Irr}(W)$; one part formed by the selected characters, each remaining character forming a part by itself.

2. We calculate the p-blocks of W. By proposition 3.3.5, if two irreducible characters aren't in the same p-block of W, then they can not be in the same block of $A_{\mathfrak{p}A}\mathcal{H}$. We intersect the partition λ_1 with the partition obtained by the p-blocks of W and we obtain a finer partition, named λ_2.

3. We find a cyclotomic specialization $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ such that the $n_{\mathcal{C},j}$ belong to no $\mathfrak{p}$-essential hyperplane. This is done by trying and checking random values for the $n_{\mathcal{C},j}$. The blocks of $A_{\mathfrak{p}A}\mathcal{H}$ coincide with the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$. Following proposition 3.4.5, we take the intersection of the partition we already have with the subsets of $\mathrm{Irr}(W)$, where the sum $a_{\chi_\phi} + A_{\chi_\phi}$ remains constant. This procedure is repeated several times, because sometimes the partition becomes finer after some repetitions. Finally, we obtain the partition λ_3, which is the finest of all.

If we are interested in calculating the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for some $\mathfrak{p}$-essential monomial M, the procedure is more or less the same:

1. We select the characters $\chi \in \mathrm{Irr}(W)$ for which M is a $\mathfrak{p}$-essential monomial. We form a first partition λ_1 of $\mathrm{Irr}(W)$; one part formed by the selected characters, each remaining character forming a part by itself. The idea is that, by proposition 3.2.12, if M is not $\mathfrak{p}$-essential for any character in a block C of $A_{\mathfrak{p}A}\mathcal{H}$, then C is a block of $A_{\mathfrak{q}_M}\mathcal{H}$. This explains step 4.

2. We calculate the p-blocks of W. By proposition 3.3.5, if two irreducible characters aren't in the same p-block of W, then they can not be in the same block of $A_{\mathfrak{q}_M}\mathcal{H}$. We intersect the partition λ_1 with the partition obtained by the p-blocks of W and we obtain a finer partition, named λ_2.

3. We find a cyclotomic specialization $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ such that the $n_{\mathcal{C},j}$ belong to the $\mathfrak{p}$-essential hyperplane defined by M and to no other, (again by trying and checking random values for the $n_{\mathcal{C},j}$). We repeat the third step as described for $A_{\mathfrak{p}A}\mathcal{H}$ to obtain partition λ_3.

4. We take the union of λ_3 and the partition defined by the blocks of $A_{\mathfrak{p}A}\mathcal{H}$.

The above algorithm is, due to step 3, heuristic. However, as we have said in the previous section, for the cases we have used it (G_7, G_{11}, G_{19}, G_{26}, G_{28}, G_{32}), we have been able to determine (using the criteria explained also in the previous section) that the partition obtained at the end is minimal and corresponds to the blocks we are looking for.

The Rouquier blocks associated with no essential hyperplane (resp. with the essential hyperplane corresponding to some essential monomial M) are unions of the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ (resp. of $A_{\mathfrak{q}_M}\mathcal{H}$) for all $\mathfrak{p}$ lying over primes which divide the order of the group W. We have observed that the above algorithm provides us with the correct Rouquier blocks for all exceptional complex reflection groups in all cases, except for the "spetsial" case of G_{34}.

4.3 Results

Using the algorithm of the previous section, we have been able to calculate the Rouquier blocks associated with all essential hyperplanes for all exceptional complex reflection groups.

We will give here the example of G_7 and show how we obtain the blocks of G_6 from those of G_7. Nevertheless, let us first explain the notations of characters used by the CHEVIE package.

Let W be an exceptional irreducible complex reflection group. For $\chi \in \mathrm{Irr}(W)$, we set $d(\chi) := \chi(1)$ and we denote by $b(\chi)$ the valuation of the fake degree of χ (for the definition of the fake degree see [10], 1.20). The irreducible characters χ of W are determined by the corresponding pairs $(d(\chi), b(\chi))$ and we write $\chi = \phi_{d,b}$, where $d := d(\chi)$ and $b := b(\chi)$. If two irreducible characters χ and χ' have $d(\chi) = d(\chi')$ and $b(\chi) = b(\chi')$, we use primes " $'$ " to distinguish them (following [32],[33]).

Example 4.3.1 The generic Hecke algebra of G_7 is

$$\begin{aligned}\mathcal{H}(G_7) \;=\; < S,T,U \mid\; & STU = TUS = UST \\ & (S-x_0)(S-x_1) = 0 \\ & (T-y_0)(T-y_1)(T-y_2) = 0 \\ & (U-z_0)(U-z_1)(U-z_2) = 0 >\end{aligned}$$

Let ϕ be a cyclotomic specialization of $\mathcal{H}(G_7)$ with

$$\phi(x_i) = \zeta_2^i x^{a_i}, \phi(y_j) = \zeta_3^j x^{b_j}, \phi(z_k) = \zeta_3^k x^{c_k}.$$

The ϕ-bad prime numbers are 2 and 3. We will now give all the essential hyperplanes for G_7 and the non-trivial Rouquier blocks associated with each.

No essential hyperplane

$\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$c_1 - c_2 = 0$

$\{\phi'_{1,4}, \phi'_{1,8}\}$, $\{\phi''_{1,8}, \phi'_{1,12}\}$, $\{\phi''_{1,12}, \phi_{1,16}\}$, $\{\phi'_{1,10}, \phi'_{1,14}\}$, $\{\phi''_{1,14}, \phi'_{1,18}\}$, $\{\phi''_{1,18}, \phi_{1,22}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,11}, \phi'_{2,13}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,9}, \phi''_{2,5}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi'''_{2,7}, \phi''_{2,3}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$c_0 - c_1 = 0$

$\{\phi_{1,0}, \phi'_{1,4}\}$, $\{\phi''_{1,4}, \phi''_{1,8}\}$, $\{\phi'''_{1,8}, \phi''_{1,12}\}$, $\{\phi_{1,6}, \phi'_{1,10}\}$, $\{\phi''_{1,10}, \phi''_{1,14}\}$, $\{\phi'''_{1,14}, \phi''_{1,18}\}$, $\{\phi'_{2,9}, \phi'_{2,7}, \phi_{2,15}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,11}, \phi''_{2,13}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,9}, \phi'''_{2,5}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$c_0 - c_2 = 0$

$\{\phi_{1,0}, \phi'_{1,8}\}$, $\{\phi''_{1,4}, \phi'_{1,12}\}$, $\{\phi'''_{1,8}, \phi_{1,16}\}$, $\{\phi_{1,6}, \phi'_{1,14}\}$, $\{\phi''_{1,10}, \phi'_{1,18}\}$, $\{\phi'''_{1,14}, \phi_{1,22}\}$, $\{\phi'_{2,9}, \phi'_{2,11}, \phi_{2,15}, \phi'_{2,5}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi''_{2,7}, \phi''_{2,9}, \phi''_{2,13}, \phi'_{2,3}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi'''_{2,11}, \phi'''_{2,7}, \phi'''_{2,5}, \phi_{2,1}\}$, $\{\phi'''_{2,9}, \phi'''_{2,3}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$b_1 - b_2 = 0$

$\{\phi''_{1,4}, \phi'''_{1,8}\}$, $\{\phi''_{1,8}, \phi''_{1,12}\}$, $\{\phi'_{1,12}, \phi_{1,16}\}$, $\{\phi''_{1,10}, \phi'''_{1,14}\}$, $\{\phi''_{1,14}, \phi''_{1,18}\}$, $\{\phi'_{1,18}, \phi_{1,22}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi'''_{2,11}, \phi''_{2,13}, \phi'''_{2,5}\}$, $\{\phi''_{2,11}, \phi'''_{2,9}, \phi''_{2,5}, \phi''_{2,3}\}$, $\{\phi''_{2,9}, \phi'''_{2,7}, \phi'_{2,3}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$b_0 - b_1 = 0$

$\{\phi_{1,0}, \phi''_{1,4}\}$, $\{\phi'_{1,4}, \phi''_{1,8}\}$, $\{\phi'_{1,8}, \phi'_{1,12}\}$, $\{\phi_{1,6}, \phi''_{1,10}\}$, $\{\phi'_{1,10}, \phi''_{1,14}\}$, $\{\phi'_{1,14}, \phi'_{1,18}\}$, $\{\phi'_{2,9}, \phi''_{2,7}, \phi_{2,15}, \phi''_{2,13}\}$, $\{\phi'_{2,7}, \phi''_{2,11}, \phi'_{2,13}, \phi''_{2,5}\}$, $\{\phi'_{2,11}, \phi''_{2,9}, \phi'_{2,5}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$b_0 - b_2 = 0$

$\{\phi_{1,0}, \phi'''_{1,8}\}$, $\{\phi'_{1,4}, \phi''_{1,12}\}$, $\{\phi'_{1,8}, \phi_{1,16}\}$, $\{\phi_{1,6}, \phi'''_{1,14}\}$, $\{\phi'_{1,10}, \phi''_{1,18}\}$, $\{\phi'_{1,14}, \phi_{1,22}\}$, $\{\phi'_{2,9}, \phi'''_{2,11}, \phi_{2,15}, \phi'''_{2,5}\}$, $\{\phi'_{2,7}, \phi'''_{2,9}, \phi'_{2,13}, \phi''_{2,3}\}$, $\{\phi'_{2,11}, \phi'''_{2,7}, \phi'_{2,5}, \phi_{2,1}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - 2b_0 + b_1 + b_2 - 2c_0 + c_1 + c_2 = 0$

$\{\phi_{1,6}, \phi'_{2,9}, \phi_{2,15}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 - 2b_0 + b_1 + b_2 + c_0 - 2c_1 + c_2 = 0$

$\{\phi'_{1,10}, \phi'_{2,7}, \phi'_{2,13}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 - 2b_0 + b_1 + b_2 + c_0 + c_1 - 2c_2 = 0$

$\{\phi'_{1,14}, \phi'_{2,11}, \phi'_{2,5}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 - b_0 - b_1 + 2b_2 - c_0 - c_1 + 2c_2 = 0$

$\{\phi_{1,16}, \phi'''_{2,7}, \phi_{2,1}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 - b_1 + 2b_2 - c_0 + 2c_1 - c_2 = 0$

$\{\phi''_{1,12}, \phi''_{2,9}, \phi''_{2,3}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 - b_1 + 2b_2 + 2c_0 - c_1 - c_2 = 0$
$\{\phi'''_{1,8}, \phi'''_{2,11}, \phi'''_{2,5}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_2 - c_0 + c_1 = 0$
$\{\phi''_{1,12}, \phi_{1,6}, \phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_2 - c_1 + c_2 = 0$
$\{\phi_{1,16}, \phi'_{1,10}, \phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_2 + c_0 - c_2 = 0$
$\{\phi'''_{1,8}, \phi'_{1,14}, \phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_1 - c_0 + c_2 = 0$
$\{\phi'_{1,12}, \phi_{1,6}, \phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_1 + c_1 - c_2 = 0$
$\{\phi''_{1,8}, \phi'_{1,14}, \phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_1 + c_0 - c_1 = 0$
$\{\phi''_{1,4}, \phi'_{1,10}, \phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + 2b_1 - b_2 - c_0 - c_1 + 2c_2 = 0$
$\{\phi'_{1,12}, \phi''_{2,9}, \phi'_{2,3}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + 2b_1 - b_2 - c_0 + 2c_1 - c_2 = 0$
$\{\phi''_{1,8}, \phi''_{2,11}, \phi''_{2,5}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + 2b_1 - b_2 + 2c_0 - c_1 - c_2 = 0$
$\{\phi''_{1,4}, \phi''_{2,7}, \phi''_{2,13}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_1 + b_2 - c_0 + c_2 = 0$
$\{\phi_{1,16}, \phi''_{1,10}, \phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_1 + b_2 + c_1 - c_2 = 0$
$\{\phi''_{1,12}, \phi'_{1,18}, \phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_1 + b_2 + c_0 - c_1 = 0$
$\{\phi'''_{1,8}, \phi''_{1,14}, \phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 = 0$
$\{\phi_{1,0}, \phi_{1,6}\}$, $\{\phi'_{1,4}, \phi'_{1,10}\}$, $\{\phi'_{1,8}, \phi'_{1,14}\}$, $\{\phi''_{1,4}, \phi''_{1,10}\}$, $\{\phi''_{1,8}, \phi''_{1,14}\}$, $\{\phi'_{1,12}, \phi'_{1,18}\}$, $\{\phi'''_{1,8}, \phi'''_{1,14}\}$,
$\{\phi''_{1,12}, \phi''_{1,18}\}$, $\{\phi_{1,16}, \phi_{1,22}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,4}, \phi_{3,10}, \phi_{3,8}, \phi_{3,2}, \phi_{3,12}\}$

$a_0 - a_1 + b_1 - b_2 - c_0 + c_1 = 0$
$\{\phi''_{1,8}, \phi'''_{1,14}, \phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_1 - b_2 - c_1 + c_2 = 0$
$\{\phi'_{1,12}, \phi''_{1,18}, \phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_1 - b_2 + c_0 - c_2 = 0$
$\{\phi''_{1,4}, \phi_{1,22}, \phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - 2b_1 + b_2 - 2c_0 + c_1 + c_2 = 0$
$\{\phi''_{1,10}, \phi''_{2,7}, \phi''_{2,13}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + b_0 - 2b_1 + b_2 + c_0 - 2c_1 + c_2 = 0$
$\{\phi''_{1,14}, \phi''_{2,11}, \phi''_{2,5}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + b_0 - 2b_1 + b_2 + c_0 + c_1 - 2c_2 = 0$
$\{\phi'_{1,18}, \phi''_{2,9}, \phi'_{2,3}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + b_0 - b_1 - c_0 + c_1 = 0$
$\{\phi'_{1,4}, \phi''_{1,10}, \phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_1 - c_1 + c_2 = 0$
$\{\phi'_{1,8}, \phi''_{1,14}, \phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_1 + c_0 - c_2 = 0$
$\{\phi_{1,0}, \phi'_{1,18}, \phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_2 - c_0 + c_2 = 0$
$\{\phi'_{1,8}, \phi'''_{1,14}, \phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_2 + c_1 - c_2 = 0$
$\{\phi'_{1,4}, \phi_{1,22}, \phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_2 + c_0 - c_1 = 0$
$\{\phi_{1,0}, \phi''_{1,18}, \phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 + b_1 - 2b_2 - 2c_0 + c_1 + c_2 = 0$
$\{\phi'''_{1,14}, \phi'''_{2,11}, \phi'''_{2,5}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + b_0 + b_1 - 2b_2 + c_0 - 2c_1 + c_2 = 0$
$\{\phi''_{1,18}, \phi'''_{2,9}, \phi''_{2,3}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + b_0 + b_1 - 2b_2 + c_0 + c_1 - 2c_2 = 0$
$\{\phi_{1,22}, \phi'''_{2,7}, \phi_{2,1}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$,
$\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + 2b_0 - b_1 - b_2 - c_0 - c_1 + 2c_2 = 0$
$\{\phi'_{1,8}, \phi'_{2,11}, \phi'_{2,5}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$,
$\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + 2b_0 - b_1 - b_2 - c_0 + 2c_1 - c_2 = 0$
$\{\phi'_{1,4}, \phi'_{2,7}, \phi'_{2,13}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,9}, \phi_{2,15}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + 2b_0 - b_1 - b_2 + 2c_0 - c_1 - c_2 = 0$
$\{\phi_{1,0}, \phi'_{2,9}, \phi_{2,15}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi'_{2,7}, \phi'_{2,13}\}$, $\{\phi'_{2,11}, \phi'_{2,5}\}$, $\{\phi''_{2,7}, \phi''_{2,13}\}$, $\{\phi''_{2,11}, \phi''_{2,5}\}$, $\{\phi''_{2,9}, \phi'_{2,3}\}$, $\{\phi'''_{2,11}, \phi'''_{2,5}\}$, $\{\phi'''_{2,9}, \phi''_{2,3}\}$, $\{\phi'''_{2,7}, \phi_{2,1}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

Now, the generic Hecke algebra of G_6 is

$$\begin{array}{rcll} \mathcal{H}(G_6) & = & < V, W \mid & VWVWVW = WVWVWV, \\ & & & (V - x_0)(V - x_1) = 0 \\ & & & (W - z_0)(W - z_1)(W - z_2) = 0 > \end{array}$$

As we can see in lemma 1 of the Appendix, $\mathcal{H}(G_6)$ is isomorphic to the subalgebra $\bar{A} :=< S, U >$ of the following specialization of $\mathcal{H}(G_7)$

$$\begin{array}{rcll} A & := & < S, T, U \mid & STU = TUS = UST, T^3 = 1 \\ & & & (S - x_0)(S - x_1) = 0 \\ & & & (U - z_0)(U - z_1)(U - z_2) = 0 > \end{array}$$

The algebra A is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\bar{A}$ and the block-idempotents of A and $\bar{A}$ coincide for all further specializations of the parameters. If we denote by ϕ the characters of $\bar{A}$ and by ψ the characters of A, we have

$$\begin{array}{rclrcl} \mathrm{Ind}_{\bar{A}}^{A}(\phi_{1,0}) & = & \psi_{1,0} + \psi_{1,4''} + \psi_{1,8'''} & \mathrm{Ind}_{\bar{A}}^{A}(\phi_{1,4}) & = & \psi_{1,4'} + \psi_{1,8''} + \psi_{1,12''} \\ \mathrm{Ind}_{\bar{A}}^{A}(\phi_{1,8}) & = & \psi_{1,8'} + \psi_{1,12'} + \psi_{1,16} & \mathrm{Ind}_{\bar{A}}^{A}(\phi_{1,6}) & = & \psi_{1,6} + \psi_{1,10''} + \psi_{1,14'''} \\ \mathrm{Ind}_{\bar{A}}^{A}(\phi_{1,10}) & = & \psi_{1,10'} + \psi_{1,14''} + \psi_{1,18''} & \mathrm{Ind}_{\bar{A}}^{A}(\phi_{1,14}) & = & \psi_{1,14'} + \psi_{1,18'} + \psi_{1,22} \\ \mathrm{Ind}_{\bar{A}}^{A}(\phi_{2,5''}) & = & \psi_{2,9'} + \psi_{2,13''} + \psi_{2,5'''} & \mathrm{Ind}_{\bar{A}}^{A}(\phi_{2,3''}) & = & \psi_{2,7'} + \psi_{2,11''} + \psi_{2,3''} \\ \mathrm{Ind}_{\bar{A}}^{A}(\phi_{2,3'}) & = & \psi_{2,11'} + \psi_{2,7'''} + \psi_{2,3'} & \mathrm{Ind}_{\bar{A}}^{A}(\phi_{2,7}) & = & \psi_{2,7''} + \psi_{2,11'''} + \psi_{2,15} \\ \mathrm{Ind}_{\bar{A}}^{A}(\phi_{2,1}) & = & \psi_{2,9''} + \psi_{2,5'} + \psi_{2,1} & \mathrm{Ind}_{\bar{A}}^{A}(\phi_{2,5'}) & = & \psi_{2,9'''} + \psi_{2,13'} + \psi_{2,5''} \\ \mathrm{Ind}_{\bar{A}}^{A}(\phi_{3,2}) & = & \psi_{3,6} + \psi_{3,10} + \psi_{3,2} & \mathrm{Ind}_{\bar{A}}^{A}(\phi_{3,4}) & = & \psi_{3,4} + \psi_{3,8} + \psi_{3,12} \end{array}$$

Let ϕ be a cyclotomic specialization of $\mathcal{H}(G_7)$ with

$$\phi(x_i) = \zeta_2^i x^{a_i}, \phi(z_k) = \zeta_3^k x^{c_k}.$$

It corresponds to the cyclotomic specialization ϕ' of $\mathcal{H}(G_7)$ with

$$\phi'(x_i) = \zeta_2^i x^{a_i}, \phi'(y_j) = \zeta_3^j, \phi(z_k) = \zeta_3^k x^{c_k}.$$

Therefore, the essential hyperplanes for G_6 are obtained from the essential hyperplanes for G_7 by setting $b_0 = b_1 = b_2 = 0$ and the non-trivial Rouquier blocks associated with each are:

No essential hyperplane
$\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$c_1 - c_2 = 0$
$\{\phi_{1,4}, \phi_{1,8}\}$, $\{\phi_{1,10}, \phi_{1,14}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,3}, \phi_{2,1}, \phi'_{2,5}\}$

$c_0 - c_1 = 0$
$\{\phi_{1,0}, \phi_{1,4}\}$, $\{\phi_{1,6}, \phi_{1,10}\}$, $\{\phi''_{2,5}, \phi''_{2,3}, \phi_{2,7}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$c_0 - c_2 = 0$
$\{\phi_{1,0}, \phi_{1,8}\}$, $\{\phi_{1,6}, \phi_{1,14}\}$, $\{\phi''_{2,5}, \phi'_{2,3}, \phi_{2,7}, \phi_{2,1}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

$a_0 - a_1 - 2c_0 + c_1 + c_2 = 0$
$\{\phi_{1,6}, \phi''_{2,5}, \phi_{2,7}, \phi_{3,4}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + c_0 - 2c_1 + c_2 = 0$
$\{\phi_{1,10}, \phi''_{2,3}, \phi'_{2,5}, \phi_{3,4}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + c_0 + c_1 - 2c_2 = 0$
$\{\phi_{1,14}, \phi'_{2,3}, \phi_{2,1}, \phi_{3,4}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

$a_0 - a_1 - c_0 - c_1 + 2c_2 = 0$
$\{\phi_{1,8}, \phi'_{2,3}, \phi_{2,1}, \phi_{3,2}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

$a_0 - a_1 - c_0 + 2c_1 - c_2 = 0$
$\{\phi_{1,4}, \phi''_{2,3}, \phi'_{2,5}, \phi_{3,2}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + 2c_0 - c_1 - c_2 = 0$
$\{\phi_{1,0}, \phi''_{2,5}, \phi_{2,7}, \phi_{3,2}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 - c_0 + c_1 = 0$
$\{\phi_{1,4}, \phi_{1,6}, \phi'_{2,3}, \phi_{2,1}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

$a_0 - a_1 - c_1 + c_2 = 0$
$\{\phi_{1,8}, \phi_{1,10}, \phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + c_0 - c_2 = 0$
$\{\phi_{1,0}, \phi_{1,14}, \phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 - c_0 + c_2 = 0$
$\{\phi_{1,8}, \phi_{1,6}, \phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + c_1 - c_2 = 0$
$\{\phi_{1,4}, \phi_{1,14}, \phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$

$a_0 - a_1 + c_0 - c_1 = 0$
$\{\phi_{1,0}, \phi_{1,10}, \phi'_{2,3}, \phi_{2,1}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$

$a_0 - a_1 = 0$
$\{\phi_{1,0}, \phi_{1,6}\}$, $\{\phi_{1,4}, \phi_{1,10}\}$, $\{\phi_{1,8}, \phi_{1,14}\}$, $\{\phi''_{2,5}, \phi_{2,7}\}$, $\{\phi''_{2,3}, \phi'_{2,5}\}$, $\{\phi'_{2,3}, \phi_{2,1}\}$, $\{\phi_{3,2}, \phi_{3,4}\}$

Since it will take too many pages to describe the Rouquier blocks associated with all essential hyperplanes of all exceptional complex reflection groups, we have stored these data in a computer file and created two GAP functions which display them. These functions are called "AllBlocks" and "DisplayAllBlocks" and they can be found on my webpage

http://www.math.jussieu.fr/∼chlouveraki

Let us give an example of their use for the group G_4.

Example 4.3.2 " gap> " is the GAP prompt

```
gap> W:=ComplexReflectionGroup(4);
```

#creates the group W

```
gap> DisplayAllBlocks(W);
No essential hyperplane
[["phi{1,0}"],["phi{1,4}"],["phi{1,8}"], ["phi{2,5}"],
["phi{2,3}"],["phi{2,1}"],["phi{3,2}"]]
c_1-c_2=0
[["phi{1,0}"],["phi{1,4}","phi{1,8}","phi{2,5}"],
["phi{2,3}","phi{2,1}"],["phi{3,2}"]]
c_0-c_1=0
[["phi{1,0}","phi{1,4}","phi{2,1}"],["phi{1,8}"],
["phi{2,5}","phi{2,3}"],["phi{3,2}"]]
c_0-c_2=0
[["phi{1,0}","phi{1,8}","phi{2,3}"],["phi{1,4}"],
["phi{2,5}","phi{2,1}"],["phi{3,2}"]]
2c_0-c_1-c_2=0
[["phi{1,0}","phi{2,5}","phi{3,2}"],
["phi{1,4}"],["phi{1,8}"], ["phi{2,3}"],["phi{2,1}"]]
c_0-2c_1+c_2=0
[["phi{1,0}"],["phi{1,4}","phi{2,3}","phi{3,2}"], ["phi{1,8}"],
["phi{2,5}"],["phi{2,1}"]]
c_0+c_1-2c_2=0
[["phi{1,0}"],["phi{1,4}"],["phi{1,8}","phi{2,1}","phi{3,2}"],
["phi{2,5}"],["phi{2,3}"]]
```

displays all essential hyperplanes for W and the Rouquier blocks associated with each

```
gap> AllBlocks(W);
[rec( cond:=[ ],
block:=[[1],[2],[3],[4],[5],[6],[7]]),
rec( cond:=[0,1,-1],
block:=[[1],[2,3,4],[5,6],[7]]),
rec( cond:=[1,-1,0],
block:=[[1,2,6],[3],[4,5],[7]]),
rec( cond:=[1,0,-1],
block:=[[1,3,5],[2],[4,6],[7]]),
rec( cond:=[2,-1,-1],
block:=[[1,4,7],[2],[3],[5],[6]]),
```

```
rec( cond:=[1,-2,1],
block:=[[1],[2,5,7],[3],[4],[6]]),
rec( cond:=[1,1,-2],
block:=[[1],[2],[3,6,7],[4],[5]])]
```

displays the same data in a way easy to work with: the essential hyperplanes are represented by the vectors cond (such that cond$*[c_0, c_1, c_2] = 0$ is the corresponding essential hyperplane and cond:= [ ] means "No essential hyperplane") and the characters are defined by their indexes in the list "CharNames(W)":

```
gap> CharNames(W);
[ "phi{1,0}", "phi{1,4}", "phi{1,8}", "phi{2,5}",
  "phi{2,3}", "phi{2,1}", "phi{3,2}" ]
```

Let W be an exceptional irreducible complex reflection group. Since we have the Rouquier blocks associated with all essential hyperplanes for W, we have created the function "RouquierBlocks" which calculates the Rouquier blocks of any cyclotomic Hecke algebra associated with W. Given a cyclotomic specialization $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^{j} x^{n_{\mathcal{C},j}}$, this function checks to which essential hyperplanes the $n_{\mathcal{C},j}$ belong and returns the blocks obtained as unions of the Rouquier blocks associated with these hyperplanes.

The function "RouquierBlocks" along with the function "DisplayRouquier Blocks" (the first returns the characters' index in the list "CharNames(W)", whereas the second returns their name) can be also found on my webpage. Before we give an example of their use, let us explain how we create a cyclotomic Hecke algebra in GAP with the use of the package CHEVIE (we copy here the relative part in the GAP manual, which can be found on J.Michel's webpage *http://www.math.jussieu.fr/∼jmichel*):

The command "Hecke(W, *para*)" returns the cyclotomic Hecke algebra associated with the complex reflection group W. The following forms are accepted for *para*: if *para* is a single value, it is replicated to become a list of same length as the number of generators of W. Otherwise, *para* should be a list of the same length as the number of generators of W, with possibly unbound entries (which means it can also be a list of lesser length). There should be at least one entry bound for each orbit of reflections, and if several entries are bound for one orbit, they should all be identical. Now again, an entry for a reflection of order e can be either a single value or a list of length e. If it is a list, it is interpreted as the list $[u_0, ..., u_{e-1}]$ of parameters for that reflection. If it is a single value q, it is interpreted as the partly specialized list of parameters $[q, \mathrm{E}(e), ..., \mathrm{E}(e)^{e-1}]$ (in GAP, E(e) represents ζ_e).

Let us now give an example of the definition of a cyclotomic Hecke algebra and the use of the functions "RouquierBlocks" and "DisplayRouquier Blocks" on it.

Example 4.3.3 The generic Hecke algebra of G_4 is

$$\mathcal{H}(G_4) \;=\; < S,T \mid \;\; STS = TST, \;\; \begin{array}{l}(S-x_0)(S-x_1)(S-x_2)=0\\ (T-x_0)(T-x_1)(T-x_2)=0>\end{array}$$

If we want to calculate the Rouquier blocks of the cyclotomic Hecke algebra

$$\mathcal{H}_\phi \;=\; < S,T \mid \;\; STS = TST, \;\; \begin{array}{l}(S-1)(S-\zeta_3 x)(S-\zeta_3^2 x^2)=0\\ (T-1)(T-\zeta_3 x)(T-\zeta_3^2 x^2)=0>\end{array}$$

we use the following commands:

```
gap> W:=ComplexReflectionGroup(4);
gap> H:=Hecke(W,[[1,E(3)*x,E(3)^2*x^2]]);
```

here the single value $[1, \mathrm{E}(3)*x, \mathrm{E}(3)^2*x^2]$ is interpreted, according to the rules, as $[\,[1, \mathrm{E}(3)*x, \mathrm{E}(3)^2*x^2], [1, \mathrm{E}(3)*x, \mathrm{E}(3)^2*x^2]\,]$

```
gap> RouquierBlocks(H);
[ [ 1 ], [ 2, 5, 7 ], [ 3 ], [ 4 ], [ 6 ] ]
gap> DisplayRouquierBlocks(H);
[["phi{1,0}"],["phi{1,4}","phi{2,3}","phi{3,2}"],
["phi{1,8}"],["phi{2,5}"],[ "phi{2,1}"]]
```

4.4 Essential hyperplanes

We have checked for all exceptional complex reflection groups that the $\mathfrak{p}$-Rouquier blocks associated with no or some $\mathfrak{p}$-essential hyperplane (*i.e.,* the blocks of $A_{\mathfrak{p}A}\mathcal{H}$ or $A_{\mathfrak{q}_M}\mathcal{H}$ respectively) are fixed by the action of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. This implies that if a hyperplane is $\mathfrak{p}'$-essential for W for some prime ideal $\mathfrak{p}'$ lying over a prime number p, then it is $\mathfrak{p}$-essential for all prime ideals $\mathfrak{p}$ lying over p. Therefore, we can talk about determining the p-essential hyperplanes for W, where p is a prime number dividing the order of the group.

Example 4.4.1 The prime numbers which divide the order of the group G_7 are 2 and 3. The essential hyperplanes for G_7 are already given in example 4.3.1 (note that different letters represent different hyperplane orbits).

The only 3-essential hyperplanes for G_7 are:

$$c_1 - c_2 = 0, \quad c_0 - c_1 = 0, \quad c_0 - c_2 = 0$$
$$b_1 - b_2 = 0, \quad b_0 - b_1 = 0, \quad b_0 - b_2 = 0$$

All its remaining essential hyperplanes are strictly 2-essential.

From these, we can obtain the p-essential hyperplanes (where $p = 2, 3$)

- for G_6 by setting $b_0 = b_1 = b_2 = 0$,
- for G_5 by setting $a_0 = a_1 = 0$,
- for G_4 by setting $a_0 = a_1 = b_0 = b_1 = b_2 = 0$.

For the p-essential hyperplanes of the other groups, the reader may refer to my webpage and use the function "EssentialHyperplanes" which is applied as follows

```
gap> EssentialHyperplanes(W,p);
```

and returns

- the essential hyperplanes for W, if $p = 0$.
- the p-essential hyperplanes for W, if p divides the order of W.
- error, if p doesn't divide the order of W.

Example 4.4.2

```
gap> W:=ComplexReflectionGroup(4);
gap> EssentialHyperplanes(W,0);
c_1-c_2=0
c_0-c_1=0
c_0-c_2=0
2c_0-c_1-c_2=0
c_0-2c_1+c_2=0
c_0+c_1-2c_2=0
gap> EssentialHyperplanes(W,2);
2c_0-c_1-c_2=0
c_0-2c_1+c_2=0
c_0+c_1-2c_2=0
c_0-c_1=0
c_1-c_2=0
c_0-c_2=0
```

```
gap> EssentialHyperplanes(W,3);
c_1-c_2=0
c_0-c_1=0
c_0-c_2=0
gap> EssentialHyperplanes(W,5);
Error, The number p should divide the order of the group
```

Remark: For the groups G_{12}, G_{22}, G_{23}, G_{24}, G_{27}, G_{29}, G_{30}, G_{31}, G_{33}, G_{34}, G_{35}, G_{36}, G_{37} the only essential hypeplane is $a_0 = a_1$, which is p-essential for all the prime numbers p dividing the order of the group.

Appendix

Let W be a complex reflection group and let us denote by $\mathcal{H}(W)$ its generic Hecke algebra. Suppose that the assumptions 3.2.2 are satisfied. Let W' be another complex reflection group such that, for some specialization of the parameters, $\mathcal{H}(W)$ is the twisted symmetric algebra of a finite cyclic group G over the symmetric subalgebra $\mathcal{H}(W')$. Then, if we know the blocks of $\mathcal{H}(W)$, we can use propositions 2.3.15 and 2.3.18 in order to calculate the blocks of $\mathcal{H}(W')$.

Moreover, in all the cases that will be studied below, if we denote by χ' the (irreducible) restriction to $\mathcal{H}(W')$ of an irreducible character $\chi \in \mathrm{Irr}(\mathcal{H}(W))$, then the Schur elements verify

$$s_\chi = |W : W'| s_{\chi'}.$$

Therefore, if the Schur elements of $\mathcal{H}(W)$ verify theorem 3.2.5, so do the Schur elements of $\mathcal{H}(W')$.

The groups G_4, G_5, G_6, G_7

The following table gives the specializations of the parameters of the generic Hecke algebra $\mathcal{H}(G_7)$, $(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2)$, which give the generic Hecke algebras of the groups G_4, G_5 and G_6 ([29], Table 4.6).

Group	Index	S	T	U
G_7	1	x_0, x_1	y_0, y_1, y_2	z_0, z_1, z_2
G_5	2	$1, -1$	y_0, y_1, y_2	z_0, z_1, z_2
G_6	3	x_0, x_1	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2
G_4	6	$1, -1$	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2

Specializations of the parameters for $\mathcal{H}(G_7)$

Lemma 1

- *The algebra $\mathcal{H}(G_7)$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2) \mapsto (1, -1; y_0, y_1, y_2; z_0, z_1, z_2)$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_5)$ with parameters $(y_0, y_1, y_2; z_0, z_1, z_2)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_7)$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2) \mapsto (x_0, x_1; 1, \zeta_3, \zeta_3^2; z_0, z_1, z_2)$$

is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\mathcal{H}(G_6)$ with parameters $(x_0, x_1; z_0, z_1, z_2)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_6)$ specialized via*

$$(x_0, x_1; z_0, z_1, z_2) \mapsto (1, -1; z_0, z_1, z_2)$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_4)$ with parameters (z_0, z_1, z_2). The block-idempotents of the two algebras coincide.

Proof: We have

$$\begin{array}{rcll} \mathcal{H}(G_7) & = & < S, T, U \mid & STU = TUS = UST \\ & & & (S - x_0)(S - x_1) = 0 \\ & & & (T - y_0)(T - y_1)(T - y_2) = 0 \\ & & & (U - z_0)(U - z_1)(U - z_2) = 0 > \end{array}$$

- Let

$$\begin{array}{rcll} A & := & < S, T, U \mid & STU = TUS = UST, S^2 = 1 \\ & & & (T - y_0)(T - y_1)(T - y_2) = 0 \\ & & & (U - z_0)(U - z_1)(U - z_2) = 0 > \end{array}$$

and

$$\bar{A} :=< T, U > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_5).$$

- Let

$$\begin{aligned} A \;:=\; < S,T,U \mid\; & STU = TUS = UST, T^3 = 1 \\ & (S-x_0)(S-x_1) = 0 \\ & (U-z_0)(U-z_1)(U-z_2) = 0 > \end{aligned}$$

and

$$\bar{A} :=< S,U > .$$

Then

$$A = \bigoplus_{i=0}^{2} T^i \bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_6).$$

- Let

$$\begin{aligned} A \;:=\; < S,U \mid\; & SUSUSU = USUSUS, S^2 = 1 \\ & (U-z_0)(U-z_1)(U-z_2) = 0 > \end{aligned}$$

and

$$\bar{A} :=< U, SUS > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_4).$$

■

The Schur elements of all irreducible characters of $\mathcal{H}(G_7)$ are calculated in [29] and they are obtained by permutation of the parameters from the following ones (for an explanation concerning the notations of characters, see section 4.3):

$s_{\phi_{1,0}} = \Phi_1(x_0/x_1) \cdot \Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_1z_2) \cdot \Phi_1(y_0/y_1) \cdot \Phi_1(y_0/y_2) \cdot \Phi_1(z_0/z_1) \cdot \Phi_1(z_0/z_2) \cdot \Phi_1(x_0y_0z_0/x_1y_1z_1) \cdot \Phi_1(x_0y_0z_0/x_1y_1z_2) \cdot \Phi_1(x_0y_0z_0/x_1y_2z_1) \cdot \Phi_1(x_0y_0z_0/x_1y_2z_2)$

$s_{\phi_{2,9'}} = 2y_2/y_0\Phi_1(y_0/y_1) \cdot \Phi_1(y_2/y_0) \cdot \Phi_1(z_1/z_0) \cdot \Phi_1(z_2/z_0) \cdot \Phi_1(r/x_0y_0z_0) \cdot \Phi_1(r/x_0y_2z_1) \cdot \Phi_1(r/x_0y_2z_2) \cdot \Phi_1(r/x_1y_0z_0) \cdot \Phi_1(r/x_1y_2z_1) \cdot \Phi_1(r/x_1y_2z_2)$
where $r = \sqrt[2]{x_0x_1y_1y_2z_1z_2}$

$s_{\phi_{3,6}} = 3\Phi_1(x_1/x_0) \cdot \Phi_1(x_0y_0z_0/r) \cdot \Phi_1(x_0y_0z_1/r) \cdot \Phi_1(x_0y_0z_2/r) \cdot \Phi_1(x_0y_1z_0/r) \cdot \Phi_1(x_0y_1z_1/r) \cdot \Phi_1(x_0y_1z_2/r) \cdot \Phi_1(x_0y_2z_0/r) \cdot \Phi_1(x_0y_2z_1/r) \cdot \Phi_1(x_0y_2z_2/r)$
where $r = \sqrt[3]{x_0^2x_1y_0y_1y_2z_0z_1z_2}$

Following theorem 3.2.3 and [30], Table 8.1, if we set

$$\begin{aligned} X_i^{12} &:= (\zeta_2)^{-i}x_i \quad (i = 0,1), \\ Y_j^{12} &:= (\zeta_3)^{-j}y_j \quad (j = 0,1,2), \\ Z_k^{12} &:= (\zeta_3)^{-k}z_k \quad (k = 0,1,2), \end{aligned}$$

then $\mathbb{Q}(\zeta_{12})(X_0, X_1, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2)$ is a splitting field for $\mathcal{H}(G_7)$. Hence the factorization of the Schur elements over that field is as described by theorem 3.2.5.

The groups G_8, G_9, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}

The following table gives the specializations of the parameters of the generic Hecke algebra $\mathcal{H}(G_{11})$, $(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3)$, which give the generic Hecke algebras of the groups $G_8, \ldots, G_{15}$ ([29], Table 4.9).

Group	Index	S	T	U
G_{11}	1	x_0, x_1	y_0, y_1, y_2	z_0, z_1, z_2, z_3
G_{10}	2	$1, -1$	y_0, y_1, y_2	z_1, z_1, z_2, z_3
G_{15}	2	x_0, x_1	y_0, y_1, y_2	$\sqrt{u_0}, \sqrt{u_1}, -\sqrt{u_0}, -\sqrt{u_1}$
G_9	3	x_0, x_1	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3
G_{14}	4	x_0, x_1	y_0, y_1, y_2	$1, i, -1, -i$
G_8	6	$1, -1$	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3
G_{13}	6	x_0, x_1	$1, \zeta_3, \zeta_3^2$	$\sqrt{u_0}, \sqrt{u_1}, -\sqrt{u_0}, -\sqrt{u_1}$
G_{12}	12	x_0, x_1	$1, \zeta_3, \zeta_3^2$	$1, i, -1, -i$

Specializations of the parameters for $\mathcal{H}(G_{11})$

Lemma 2

- *The algebra $\mathcal{H}(G_{11})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3) \mapsto (1, -1; y_0, y_1, y_2; z_0, z_1, z_2, z_3)$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_{10})$ with parameters $(y_0, y_1, y_2; z_0, z_1, z_2, z_3)$. The block-idempo-tents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{11})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3) \mapsto (x_0, x_1; 1, \zeta_3, \zeta_3^2; z_0, z_1, z_2, z_3)$$

is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\mathcal{H}(G_9)$ with parameters $(x_0, x_1; z_0, z_1, z_2, z_3)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_9)$ specialized via*

$$(x_0, x_1; z_0, z_1, z_2, z_3) \mapsto (1, -1; z_0, z_1, z_2, z_3)$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_8)$ with parameters (z_0, z_1, z_2, z_3). The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{11})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3) \mapsto (x_0, x_1; y_0, y_1, y_2; 1, i, -1, -i)$$

is the twisted symmetric algebra of the cyclic group C_4 over the symmetric subalgebra $\mathcal{H}(G_{14})$ with parameters $(x_0, x_1; y_0, y_1, y_2)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{14})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2) \mapsto (x_0, x_1; 1, \zeta_3, \zeta_3^2)$$

is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\mathcal{H}(G_{12})$ with parameters (x_0, x_1). The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{11})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3) \mapsto (x_0, x_1; y_0, y_1, y_2; \sqrt{u_0}, \sqrt{u_1}, -\sqrt{u_0}, -\sqrt{u_1})$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_{15})$ with parameters $(x_0, x_1; y_0, y_1, y_2; u_0, u_1)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{15})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; u_0, u_1) \mapsto (x_0, x_1; 1, \zeta_3, \zeta_3^2; u_0, u_1)$$

is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\mathcal{H}(G_{13})$ with parameters $(x_0, x_1; u_0, u_1)$. The block-idempotents of the two algebras coincide.

Proof: We have

$$\begin{array}{rcll} \mathcal{H}(G_{11}) & = & < S, T, U \mid & STU = TUS = UST \\ & & & (S - x_0)(S - x_1) = 0 \\ & & & (T - y_0)(T - y_1)(T - y_2) = 0 \\ & & & (U - z_0)(U - z_1)(U - z_2)(U - z_3) = 0 > \end{array}$$

- Let

$$\begin{array}{rcll} A & := & < S, T, U \mid & STU = TUS = UST, S^2 = 1 \\ & & & (T - y_0)(T - y_1)(T - y_2) = 0 \\ & & & (U - z_0)(U - z_1)(U - z_2)(U - z_3) = 0 > \end{array}$$

and

$$\bar{A} :=< T, U > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{10}).$$

- Let

$$A := < S,T,U \mid \begin{array}{l} STU = TUS = UST, T^3 = 1 \\ (S-x_0)(S-x_1) = 0 \\ (U-z_0)(U-z_1)(U-z_2)(U-z_3) = 0 > \end{array}$$

and

$$\bar{A} :=< S,U > .$$

Then

$$A = \bigoplus_{i=0}^{2} T^i \bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_9).$$

- Let

$$A := < S,U \mid \begin{array}{l} SUSUSU = USUSUS, S^2 = 1 \\ (U-z_0)(U-z_1)(U-z_2)(U-z_3) = 0 > \end{array}$$

and

$$\bar{A} :=< U, SUS > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_8).$$

- Let

$$A := < S,T,U \mid \begin{array}{l} STU = TUS = UST, U^4 = 1 \\ (S-x_0)(S-x_1) = 0 \\ (T-y_0)(T-y_1)(T-y_2) = 0 > \end{array}$$

and

$$\bar{A} :=< S,T > .$$

Then

$$A = \bigoplus_{i=0}^{3} U^i \bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{14}).$$

- Let

$$A := < S,T \mid \begin{array}{l} STSTSTST = TSTSTSTS, T^3 = 1 \\ (S-x_0)(S-x_1) = 0 > \end{array}$$

and

$$\bar{A} :=< S, TST^2, T^2ST > .$$

Then

$$A = \bigoplus_{i=0}^{2} T^i \bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{12}).$$

- Let

$$\begin{aligned} A \;:=\; < S,T,U \mid\; & STU = TUS = UST \\ & (S-x_0)(S-x_1) = 0 \\ & (T-y_0)(T-y_1)(T-y_2) = 0 \\ & (U^2-u_0)(U^2-u_1) = 0 > \end{aligned}$$

and

$$\bar{A} :=< S,T,U^2 > .$$

Then

$$A = \bar{A} \oplus U\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{15}).$$

- Let

$$\begin{aligned} A \;:=\; < U^2,S,T \mid\; & STU^2 = U^2ST, U^2STST = TU^2STS, T^3 = 1 \\ & (S-x_0)(S-x_1) = 0 \\ & (U^2-u_0)(U^2-u_1) = 0 > \end{aligned}$$

and

$$\bar{A} :=< U^2,S,T^2ST > .$$

Then

$$A = \bigoplus_{i=0}^{2} T^i\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{13}).$$

■

The Schur elements of all irreducible characters of $\mathcal{H}(G_{11})$ are calculated in [29] and they are obtained by permutation of the parameters from the following ones:

$s_{\phi_{1,0}} = \Phi_1(x_0/x_1)\cdot\Phi_1(y_0/y_1)\cdot\Phi_1(y_0/y_2)\cdot\Phi_1(z_0/z_1)\cdot\Phi_1(z_0/z_2)\cdot\Phi_1(z_0/z_3)\cdot\Phi_1(x_0y_0z_0/x_1y_1z_1)\cdot$
$\Phi_1(x_0y_0z_0/x_1y_1z_2)\cdot\Phi_1(x_0y_0z_0/x_1y_1z_3)\cdot\Phi_1(x_0y_0z_0/x_1y_2z_1)\cdot\Phi_1(x_0y_0z_0/x_1y_2z_2)\cdot$
$\Phi_1(x_0y_0z_0/x_1y_2z_3)\cdot\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_1z_2)\cdot\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_1z_3)\cdot\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_2z_3)\cdot$
$\Phi_1(x_0^2y_0^2z_0^3/x_1^2y_1y_2z_1z_2z_3)$

$s_{\phi_{2,1}} = -2z_1/z_0\Phi_1(y_0/y_2)\cdot\Phi_1(y_1/y_2)\cdot\Phi_1(z_0/z_2)\cdot\Phi_1(z_0/z_3)\cdot\Phi_1(z_1/z_2)\cdot\Phi_1(z_1/z_3)\cdot$
$\Phi_1(y_0z_0z_1/y_2z_2z_3)\cdot\Phi_1(y_1z_0z_1/y_2z_2z_3)\cdot\Phi_1(r/x_0y_2z_2)\cdot\Phi_1(r/x_0y_2z_3)\cdot\Phi_1(r/x_1y_2z_2)\cdot$
$\Phi_1(r/x_1y_2z_3)\cdot\Phi_1(r/x_0y_0z_1)\cdot\Phi_1(r/x_0y_1z_1)\cdot\Phi_1(r/x_1y_0z_1)\cdot\Phi_1(r/x_1y_1z_1)$
where $r = \sqrt[2]{x_0x_1y_0y_1z_0z_1}$

$s_{\phi_{3,2}} = 3\Phi_1(x_1/x_0)\cdot\Phi_1(z_1/z_3)\cdot\Phi_1(z_2/z_3)\cdot\Phi_1(z_0/z_3)\cdot\Phi_1(r/x_1y_0z_3)\cdot\Phi_1(r/x_1y_1z_3)\cdot$
$\Phi_1(r/x_1y_2z_3)\cdot\Phi_1(x_0y_0z_0/r)\cdot\Phi_1(x_0y_0z_1/r)\cdot\Phi_1(x_0y_0z_2/r)\cdot\Phi_1(x_0y_1z_0/r)\cdot$
$\Phi_1(x_0y_1z_1/r)\cdot\Phi_1(x_0y_1z_2/r)\cdot\Phi_1(x_0y_2z_0/r)\cdot\Phi_1(x_0y_2z_1/r)\cdot\Phi_1(x_0y_2z_2/r)$
where $r = \sqrt[3]{x_0^2x_1y_0y_1y_2z_0z_1z_2}$

$s_{\phi_{4,21}} = -4\Phi_1(y_0/y_1)\cdot\Phi_1(y_0/y_2)\cdot\ \Phi_1(r/x_0y_0z_0)\cdot\Phi_1(r/x_1y_0z_0)\cdot\Phi_1(x_0y_0z_1/r)\cdot\Phi_1(x_0y_0z_2/r)\cdot$
$\Phi_1(x_0y_0z_3/r)\cdot\Phi_1(x_1y_0z_1/r)\cdot\Phi_1(x_1y_0z_2/r)\cdot\Phi_1(x_1y_0z_3/r)\cdot$
$\Phi_1(x_0x_1y_0y_1z_0z_1/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_0z_2/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_0z_3/r^2)\cdot$
$\Phi_1(x_0x_1y_0y_2z_0z_1/r^2)\cdot\Phi_1(x_0x_1y_0y_2z_0z_2/r^2)\cdot\Phi_1(x_0x_1y_0y_2z_0z_3/r^2)$
where $r = \sqrt[4]{x_0^2x_1^2y_0^2y_1y_2z_0z_1z_2z_3}$

Following theorem 3.2.3 and [30], Table 8.1, if we set

$$\begin{aligned} X_i^{24} &:= (\zeta_2)^{-i}x_i \quad (i=0,1),\\ Y_j^{24} &:= (\zeta_3)^{-j}y_j \quad (j=0,1,2),\\ Z_k^{24} &:= (\zeta_4)^{-k}z_k \quad (k=0,1,2,3), \end{aligned}$$

then $\mathbb{Q}(\zeta_{24})(X_0, X_1, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3)$ is a splitting field for $\mathcal{H}(G_{11})$. Hence the factorization of the Schur elements over that field is as described by theorem 3.2.5.

The groups G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}

The following table gives the specializations of the parameters of the generic Hecke algebra $\mathcal{H}(G_{19})$, $(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3, z_4)$, which give the generic Hecke algebras of the groups $G_{16}, \ldots, G_{22}$ ([29], Table 4.12).

Group	Index	S	T	U
G_{19}	1	x_0, x_1	y_0, y_1, y_2	z_0, z_1, z_2, z_3, z_4
G_{18}	2	$1, -1$	y_0, y_1, y_2	z_0, z_1, z_2, z_3, z_4
G_{17}	3	x_0, x_1	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3, z_4
G_{21}	5	x_0, x_1	y_0, y_1, y_2	$1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$
G_{16}	6	$1, -1$	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3, z_4
G_{20}	10	$1, -1$	y_0, y_1, y_2	$1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$
G_{22}	15	x_0, x_1	$1, \zeta_3, \zeta_3^2$	$1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$

Specializations of the parameters for $\mathcal{H}(G_{19})$

Lemma 3

- *The algebra $\mathcal{H}(G_{19})$ specialized via*

  $$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3, z_4) \mapsto (1, -1; y_0, y_1, y_2; z_0, z_1, z_2, z_3, z_4)$$

  is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_{18})$ with parameters $(y_0, y_1, y_2; z_0, z_1, z_2, z_3, z_4)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{19})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3, z_4) \mapsto (x_0, x_1; 1, \zeta_3, \zeta_3^2; z_0, z_1, z_2, z_3, z_4)$$

is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\mathcal{H}(G_{17})$ with parameters $(x_0, x_1; z_0, z_1, z_2, z_3, z_4)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{17})$ specialized via*

$$(x_0, x_1; z_0, z_1, z_2, z_3, z_4) \mapsto (1, -1; z_0, z_1, z_2, z_3, z_4)$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_{16})$ with parameters $(z_0, z_1, z_2, z_3, z_4)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{19})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3, z_4) \mapsto (x_0, x_1; y_0, y_1, y_2; 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$$

is the twisted symmetric algebra of the cyclic group C_5 over the symmetric subalgebra $\mathcal{H}(G_{21})$ with parameters $(x_0, x_1; y_0, y_1, y_2)$. The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{21})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2) \mapsto (1, -1; y_0, y_1, y_2)$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_{20})$ with parameters (y_0, y_1, y_2). The block-idempotents of the two algebras coincide.

- *The algebra $\mathcal{H}(G_{21})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2) \mapsto (x_0, x_1; 1, \zeta_3, \zeta_3^2)$$

is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\mathcal{H}(G_{22})$ with parameters (x_0, x_1). The block-idempotents of the two algebras coincide.

Proof: We have

$$\begin{aligned} \mathcal{H}(G_{19}) \;=\; < S, T, U \mid\; & STU = TUS = UST \\ & (S - x_0)(S - x_1) = 0 \\ & (T - y_0)(T - y_1)(T - y_2) = 0 \\ & (U - z_0)(U - z_1)(U - z_2)(U - z_3)(U - z_4) = 0 > \end{aligned}$$

- Let

$$A := < S,T,U \mid \begin{array}{l} STU = TUS = UST, S^2 = 1 \\ (T-y_0)(T-y_1)(T-y_2) = 0 \\ (U-z_0)(U-z_1)(U-z_2)(U-z_3)(U-z_4) = 0 > \end{array}$$

and

$$\bar{A} :=< T,U > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{18}).$$

- Let

$$A := < S,T,U \mid \begin{array}{l} STU = TUS = UST, T^3 = 1 \\ (S-x_0)(S-x_1) = 0 \\ (U-z_0)(U-z_1)(U-z_2)(U-z_3)(U-z_4) = 0 > \end{array}$$

and

$$\bar{A} :=< S,U > .$$

Then

$$A = \bigoplus_{i=0}^{2} T^i\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{17}).$$

- Let

$$A := < S,U \mid \begin{array}{l} SUSUSU = USUSUS, S^2 = 1 \\ (U-z_0)(U-z_1)(U-z_2)(U-z_3)(U-z_4) = 0 > \end{array}$$

and

$$\bar{A} :=< U, SUS > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{16}).$$

- Let

$$A := < S,T,U \mid \begin{array}{l} STU = TUS = UST, U^5 = 1 \\ (S-x_0)(S-x_1) = 0 \\ (T-y_0)(T-y_1)(T-y_2) = 0 > \end{array}$$

and

$$\bar{A} :=< S,T > .$$

Then

$$A = \bigoplus_{i=0}^{4} U^i\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{21}).$$

- Let

$$A := < S,T \mid \begin{array}{l} STSTSTSTST = TSTSTSTSTS, S^2 = 1 \\ (T-y_0)(T-y_1)(T-y_2) = 0 > \end{array}$$

and

$$\bar{A} :=< T, STS > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{20}).$$

- Let

$$A := < S,T \mid \begin{array}{l} STSTSTSTST = TSTSTSTSTS, T^3 = 1 \\ (S-x_0)(S-x_1) = 0 > \end{array}$$

and

$$\bar{A} :=< S, TST^2, T^2ST > .$$

Then

$$A = \bigoplus_{i=0}^{2} T^i\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{22}).$$

■

The Schur elements of all irreducible characters of $\mathcal{H}(G_{19})$ are calculated in [29] and they are obtained by permutation of the parameters from the following ones:

$s_{\phi_{1,0}} = \Phi_1(x_0/x_1)\cdot\Phi_1(y_0/y_1)\cdot\Phi_1(y_0/y_2)\cdot\Phi_1(z_0/z_1)\cdot\Phi_1(z_0/z_2)\cdot\Phi_1(z_0/z_3)\cdot\Phi_1(z_0/z_4)\cdot$
$\Phi_1(x_0y_0z_0/x_1y_1z_1)\cdot\Phi_1(x_0y_0z_0/x_1y_1z_2)\cdot\Phi_1(x_0y_0z_0/x_1y_1z_3)\cdot\Phi_1(x_0y_0z_0/x_1y_1z_4)\cdot$
$\Phi_1(x_0y_0z_0/x_1y_2z_1)\cdot\Phi_1(x_0y_0z_0/x_1y_2z_2)\cdot\Phi_1(x_0y_0z_0/x_1y_2z_3)\cdot\Phi_1(x_0y_0z_0/x_1y_2z_4)\cdot$
$\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_1z_2)\cdot\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_1z_3)\cdot\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_1z_4)\cdot$
$\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_2z_3)\cdot\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_2z_4)\cdot\Phi_1(x_0y_0^2z_0^2/x_1y_1y_2z_3z_4)\cdot$
$\Phi_1(x_0^2y_0^2z_0^3/x_1^2y_1y_2z_1z_2z_3)\cdot\Phi_1(x_0^2y_0^2z_0^3/x_1^2y_1y_2z_1z_2z_4)\cdot\Phi_1(x_0^2y_0^2z_0^3/x_1^2y_1y_2z_1z_3z_4)\cdot$
$\Phi_1(x_0^2y_0^2z_0^3/x_1^2y_1y_2z_2z_3z_4)\cdot\Phi_1(x_0^2y_0^3z_0^4/x_1^2y_1^2y_2z_1z_2z_3z_4)\cdot\Phi_1(x_0^2y_0^3z_0^4/x_1^2y_1y_2^2z_1z_2z_3z_4)\cdot$
$\Phi_1(x_0^3y_0^4z_0^4/x_1^3y_1^2y_2^2z_1z_2z_3z_4)$

$s_{\phi_{2,31'}} = -2\Phi_1(y_0/y_2)\cdot\Phi_1(y_1/y_2)\cdot\Phi_1(z_0/z_2)\cdot\Phi_1(z_0/z_3)\cdot\Phi_1(z_0/z_4)\cdot\Phi_1(z_1/z_2)\cdot\Phi_1(z_1/z_3)\cdot$
$\Phi_1(z_1/z_4)\cdot\Phi_1(y_0z_0z_1/y_2z_2z_3)\cdot\Phi_1(y_0z_0z_1/y_2z_2z_4)\cdot\Phi_1(y_0z_0z_1/y_2z_3z_4)\cdot\Phi_1(y_1z_0z_1/y_2z_2z_3)\cdot$
$\Phi_1(y_1z_0z_1/y_2z_2z_4)\cdot\Phi_1(y_1z_0z_1/y_2z_3z_4)\cdot\Phi_1(y_0y_1z_0z_1^2/y_2^2z_2z_3z_4)\cdot\Phi_1(y_0y_1z_0^2z_1/y_2^2z_2z_3z_4)\cdot$
$\Phi_1(r/x_0y_0z_0)\cdot\Phi_1(x_0y_0z_1/r)\cdot\Phi_1(x_1y_0z_0/r)\cdot\Phi_1(r/x_1y_0z_1)\cdot\ \Phi_1(r/x_1y_2z_2)\cdot\Phi_1(r/x_1y_2z_3)\cdot$
$\Phi_1(r/x_1y_2z_4)\cdot\Phi_1(r/x_0y_2z_2)\cdot\Phi_1(r/x_0y_2z_3)\cdot\Phi_1(r/x_0y_2z_4)\cdot\Phi_1(rz_0z_1/x_0y_2z_2z_3z_4)\cdot$
$\Phi_1(rz_0z_1/x_1y_2z_2z_3z_4)$
where $r = \sqrt[2]{x_0x_1y_0y_1z_0z_1}$

$s_{\phi_{3,22'}} = 3\Phi_1(x_1/x_0)\cdot\Phi_1(z_0/z_3)\cdot\Phi_1(z_0/z_4)\cdot\Phi_1(z_1/z_3)\cdot\Phi_1(z_1/z_4)\cdot\Phi_1(z_2/z_3)\cdot\Phi_1(z_2/z_4)\cdot$
$\Phi_1(x_0z_0z_1/x_1z_3z_4)\cdot\Phi_1(x_0z_0z_2/x_1z_3z_4)\cdot\Phi_1(x_0z_1z_2/x_1z_3z_4)\cdot\Phi_1(r/x_1y_0z_3)\cdot\Phi_1(r/x_1y_0z_4)\cdot$
$\Phi_1(r/x_1y_1z_3)\cdot\Phi_1(r/x_1y_1z_4)\cdot\Phi_1(r/x_1y_2z_3)\cdot\Phi_1(r/x_1y_2z_4)\cdot\Phi_1(x_0y_0z_0/r)\cdot\Phi_1(x_0y_0z_1/r)\ \cdot$
$\Phi_1(x_0y_0z_2/r)\cdot\Phi_1(x_0y_1z_0/r)\cdot\Phi_1(x_0y_1z_1/r)\cdot\Phi_1(x_0y_1z_2/r)\cdot\Phi_1(x_0y_2z_0/r)\cdot\Phi_1(x_0y_2z_1/r)\cdot$
$\Phi_1(x_0y_2z_2/r)\cdot\Phi_1(r^2/x_0x_1y_0y_1z_3z_4)\cdot\Phi_1(r^2/x_0x_1y_0y_2z_3z_4)\cdot\Phi_1(r^2/x_0x_1y_1y_2z_3z_4)$
where $r = \sqrt[3]{x_0^2x_1y_0y\ _1y_2z_0z_1z_2}$

$s_{\phi_{4,18}} = -4\Phi_1(y_1/y_0)\cdot\Phi_1(y_0/y_2)\cdot\Phi_1(z_0/z_4)\cdot\Phi_1(z_1/z_4)\cdot\Phi_1(z_2/z_4)\cdot\Phi_1(z_3/z_4)\cdot\Phi_1(x_0y_0z_0/r)\cdot$
$\Phi_1(x_0y_0z_1/r)\cdot\Phi_1(x_0y_0z_2/r)\cdot\Phi_1(x_0y_0z_3/r)\cdot\Phi_1(x_1y_0z_0/r)\cdot\Phi_1(x_1y_0z_1/r)\cdot\Phi_1(x_1y_0z_2/r)\cdot$
$\Phi_1(x_1y_0z_3/r)\cdot\Phi_1(r/x_0y_1z_4)\cdot\Phi_1(r/x_1y_1z_4)\cdot\Phi_1(r/x_0y_2z_4)\cdot\Phi_1(r/x_1y_2z_4)\cdot$
$\Phi_1(r^2/x_0x_1y_0y_1z_0z_1)\cdot\Phi_1(r^2/x_0x_1y_0y_1z_0z_2)\cdot\Phi_1(x_0x_1y_0y_1z_0z_3/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_1z_2/r^2)\cdot$
$\Phi_1(r^2/x_0x_1y_0y_1z_1z_3)\cdot\Phi_1(r^2/x_0x_1y_0y_1z_2z_3)\cdot\Phi_1(r^2/x_0x_1y_1y_2z_0z_4)\cdot\Phi_1(r^2/x_0x_1y_1y_2z_1z_4)\cdot$
$\Phi_1(r^2/x_0x_1y_1y_2z_2z_4)\cdot\Phi_1(r^2/x_0x_1y_1y_2z_3z_4)$
where $r = \sqrt[4]{x_0^2x_1^2y_0^2y_1y_2z_0z_1z_2z_3}$

$s_{\phi_{5,16}} = 5\Phi_1(x_0/x_1)\cdot\Phi_1(y_2/y_0)\cdot\Phi_1(y_2/y_1)\cdot\Phi_1(x_0y_0z_0/r)\cdot\Phi_1(x_0y_0z_1/r)\cdot\Phi_1(x_0y_0z_2/r)\cdot$
$\Phi_1(x_0y_0z_3/r)\cdot\Phi_1(x_0y_0z_4/r)\cdot\ \Phi_1(x_0y_1z_0/r)\cdot\Phi_1(x_0y_1z_1/r)\cdot\Phi_1(x_0y_1z_2/r)\cdot\Phi_1(x_0y_1z_3/r)\cdot$
$\Phi_1(x_0y_1z_4/r)\cdot\Phi_1(r/x_1y_2z_0)\cdot\Phi_1(r/x_1y_2z_1)\cdot\Phi_1(r/x_1y_2z_2)\cdot\Phi_1(r/x_1y_2z_3)\cdot\Phi_1(r/x_1y_2z_4)\cdot$
$\Phi_1(x_0x_1y_0y_1z_0z_1/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_0z_2/r^2)\cdot$
$\Phi_1(x_0x_1y_0y_1z_0z_3/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_0z_4/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_1z_2/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_1z_3/r^2)\cdot$
$\Phi_1(x_0x_1y_0y_1z_1z_4/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_2z_3/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_2z_4/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_3z_4/r^2)$
where $r = \sqrt[5]{x_0^3x_1^2y_0^2y_1^2y_2z_0z_1z_2z_3z_4}$

$s_{\phi_{6,15}} = -6\Phi_1(z_0/z_1)\cdot\Phi_1(z_0/z_2)\cdot\Phi_1(z_0/z_3)\cdot\Phi_1(z_0/z_4)\cdot\Phi_1(r/x_0y_0z_0)\cdot\Phi_1(r/x_0y_1z_0)\cdot$
$\Phi_1(r/x_0y_2z_0)\cdot\Phi_1(x_1y_0z_0/r)\cdot\Phi_1(x_1y_1z_0/r)\cdot\Phi_1(x_1y_2z_0/r)\cdot\Phi_1(x_0x_1y_0y_1z_0z_1/r^2)\cdot$
$\Phi_1(x_0x_1y_0y_1z_0z_2/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_0z_3/r^2)\cdot\Phi_1(x_0x_1y_0y_1z_0z_4/r^2)\cdot\Phi_1(x_0x_1y_0y_2z_0z_1/r^2)\cdot$
$\Phi_1(x_0x_1y_0y_2z_0z_2/r^2)\cdot\Phi_1(x_0x_1y_0y_2z_0z_3/r^2)\cdot\Phi_1(x_0x_1y_0y_2z_0z_4/r^2)\cdot\Phi_1(x_0x_1y_1y_2z_0z_1/r^2)\cdot$
$\Phi_1(x_0x_1y_1y_2z_0z_2/r^2)\cdot\Phi_1(x_0x_1y_1y_2z_0z_3/r^2)\cdot\Phi_1(x_0x_1y_1y_2z_0z_4/r^2)\cdot$
$\Phi_1(x_0^2x_1y_0y_1y_2z_0z_1z_2/r^3)\cdot\Phi_1(x_0^2x_1y_0y_1y_2z_0z_1z_3/r^3)\cdot\Phi_1(x_0^2x_1y_0y_1y_2z_0z_1z_4/r^3)\cdot$
$\Phi_1(x_0^2x_1y_0y_1y_2z_0z_2z_3/r^3)\cdot\Phi_1(x_0^2x_1y_0y_1y_2z_0z_2z_4/r^3)\cdot\Phi_1(x_0^2x_1y_0y_1y_2z_0z_3z_4/r^3)$
where $r = \sqrt[6]{x_0^3x_1^3y_0^2y_1^2y_2^2z_0^2z_1z_2z_3z_4}$

Following theorem 3.2.3 and [30], Table 8.1, if we set

$$\begin{aligned} X_i^{60} &:= (\zeta_2)^{-i}x_i \quad (i=0,1),\\ Y_j^{60} &:= (\zeta_3)^{-j}y_j \quad (j=0,1,2),\\ Z_k^{60} &:= (\zeta_5)^{-k}z_k \quad (k=0,1,2,3,4), \end{aligned}$$

then $\mathbb{Q}(\zeta_{60})(X_0, X_1, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, Z_3, Z_4)$ is a splitting field for $\mathcal{H}(G_{19})$. Hence the factorization of the Schur elements over that field is as described by theorem 3.2.5.

The groups G_{25}, G_{26}

The following table gives the specialization of the parameters of the generic Hecke algebra $\mathcal{H}(G_{26})$, $(x_0, x_1; y_0, y_1, y_2)$, which give the generic Hecke algebra of the group G_{25} ([31], Theorem 6.3).

Group	Index	S	T
G_{26}	1	x_0, x_1	y_0, y_1, y_2
G_{25}	2	$1, -1$	y_0, y_1, y_2

Specialization of the parameters for $\mathcal{H}(G_{26})$

Lemma 4 *The algebra $\mathcal{H}(G_{26})$ specialized via*

$$(x_0, x_1; y_0, y_1, y_2) \mapsto (1, -1; y_0, y_1, y_2)$$

is the twisted symmetric algebra of the cyclic group C_2 over the symmetric subalgebra $\mathcal{H}(G_{25})$ with parameters (y_0, y_1, y_2). The block-idempotents of the two algebras coincide.

Proof: We have

$$\begin{aligned} \mathcal{H}(G_{26}) \;=\; < S, T, U \mid\; & STST = TSTS, UTU = TUT, SU = US \\ & (S - x_0)(S - x_1) = 0 \\ & (T - y_0)(T - y_1)(T - y_2) = 0 \\ & (U - y_0)(U - y_1)(U - y_2) = 0 > \end{aligned}$$

Let

$$\begin{aligned} A \;:=\; < S, T, U \mid\; & STST = TSTS, UTU = TUT, SU = US, S^2 = 1 \\ & (T - y_0)(T - y_1)(T - y_2) = 0 \\ & (U - y_0)(U - y_1)(U - y_2) = 0 > \end{aligned}$$

and

$$\bar{A} :=< SUS, T, U > .$$

Then

$$A = \bar{A} \oplus S\bar{A} \text{ and } \bar{A} \simeq \mathcal{H}(G_{25}).$$

■

The Schur elements of all irreducible characters of $\mathcal{H}(G_{26})$ are calculated in [31] and they are obtained by permutation of the parameters from the following ones:

$$s_{\phi_{1,0}} = -\Phi_1(x_0/x_1)\cdot\Phi_1(y_0/y_1)\cdot\Phi_1(y_0/y_2)\cdot\Phi_2(x_0y_0/x_1y_1)\cdot\Phi_2(x_0y_0/x_1y_2)\cdot\Phi_1(x_0y_0^2/x_1y_1^2)\cdot$$

$\Phi_1(x_0y_0^2/x_1y_2^2)\cdot\Phi_2(x_0y_0^3/x_1y_1^2y_2)\cdot\Phi_2(x_0y_0^3/x_1y_1y_2^2)\cdot\Phi_6(x_0y_0^2/x_1y_1y_2)\cdot\Phi_2(y_0^2/y_1y_2)\cdot\Phi_6(y_0/y_1)\cdot$
$\Phi_6(y_0/y_2)$

$s_{\phi_{2,3}} = y_1/y_0\Phi_1(x_0/x_1)\cdot\Phi_1(y_0/y_2)\cdot\Phi_1(y_1/y_2)\cdot\Phi_1(x_0y_0/x_1y_2)\cdot\Phi_1(x_0y_1/x_1y_2)\cdot\Phi_2(x_0y_0/x_1y_2)\cdot$
$\Phi_2(x_0y_1/x_1y_2)\cdot\Phi_2(x_0y_0/x_1y_1)\cdot\Phi_2(x_0y_1/x_1y_0)\cdot\Phi_6(x_0y_0y_1/x_1y_2^2)\cdot\Phi_2(y_0y_1/y_2^2)\cdot\Phi_6(y_0/y_1)$

$s_{\phi_{3,6}} = -\Phi_1(x_0/x_1)\cdot\Phi_3(x_0/x_1)\cdot\Phi_2(x_0y_0/x_1y_1)\cdot\Phi_2(x_0y_0/x_1y_2)\cdot\Phi_2(x_0y_1/x_1y_0)\cdot\Phi_2(x_0y_1/x_1y_2)\cdot$
$\Phi_2(x_0y_2/x_1y_0)\cdot\Phi_2(x_0y_2/x_1y_1)\cdot\Phi_2(y_0y_1/y_2^2)\cdot\Phi_2(y_0y_2/y_1^2)\cdot\Phi_2(y_1y_2/y_0^2)$

$s_{\phi_{3,1}} = -\Phi_1(x_1/x_0)\cdot\Phi_1(y_0/y_1)\cdot\Phi_1(y_0/y_2)\cdot\Phi_2(y_0/y_2)\cdot\Phi_1(y_1/y_2)\cdot\Phi_2(y_0y_1/y_2^2)\cdot\Phi_2(y_0^2/y_1y_2)\cdot$
$\Phi_6(y_0/y_2)\cdot\Phi_2(x_0y_0/x_1y_2)\cdot\Phi_2(x_0y_1/x_1y_0)\cdot\Phi_1(x_0y_0^2/x_1y_1^2)\cdot$
$\Phi_2(x_0y_0^2y_1/x_1y_2^3)$

$s_{\phi_{6,2}} = \Phi_1(x_0/x_1)\cdot\Phi_1(y_1/y_0)\cdot\Phi_1(y_0/y_2)\cdot\Phi_1(y_1/y_2)\cdot\Phi_2(y_2/y_0)\cdot\Phi_6(y_0/y_2)\cdot\Phi_2(y_0y_2/y_1^2)\cdot$
$\Phi_1(x_0y_1/x_1y_2)\cdot\Phi_2(x_1y_0/x_0y_2)\cdot\Phi_2(x_0y_1/x_1y_2)\cdot\Phi_2(x_0y_0^3/x_1y_1^2y_2)$

$s_{\phi_{8,3}} = 2\Phi_1(y_0/y_1)\cdot\Phi_1(y_0/y_2)\cdot\Phi_2(x_1y_2/x_0y_1)\cdot\Phi_2(x_1y_1/x_0y_2)\cdot\Phi_2(ry_0/x_1y_2^2)\cdot\Phi_2(ry_0/x_1y_1^2)\cdot$
$\Phi_1(ry_2/x_1y_0y_1)\cdot\Phi_1(ry_1/x_1y_0y_2)\cdot\Phi_3(ry_0/x_1y_1y_2)\cdot\Phi_3(ry_0/x_0y_1y_2)$
where $r = \sqrt[2]{-x_0x_1y_1y_2}$

$s_{\phi_{9,7}} = \Phi_1(\zeta_3^2)\cdot\Phi_6(y_0/y_1)\cdot\ \Phi_6(y_2/y_0)\cdot\Phi_6(y_1/y_2)\cdot\Phi_2(\zeta_3x_0y_1y_2/x_1y_0^2)\cdot\Phi_2(\zeta_3x_0y_0y_2/x_1y_1^2)\cdot$
$\Phi_2(\zeta_3x_0y_0y_1/x_1y_2^2)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(\zeta_3x_0/x_1)$

Following theorem 3.2.3 and [30], Table 8.2, if we set

$$\begin{array}{ll} X_i^6 := (\zeta_2)^{-i}x_i & (i=0,1),\\ Y_j^6 := (\zeta_3)^{-j}y_j & (j=0,1,2),\end{array}$$

then $\mathbb{Q}(\zeta_3)(X_1,X_2,Y_1,Y_2,Y_2)$ is a splitting field for $\mathcal{H}(G_{26})$. Hence the factorization of the Schur elements over that field is as described by theorem 3.2.5.

The group G_{28} ("F_4")

Let $\mathcal{H}(G_{28})$ be the generic Hecke algebra of the real reflection group G_{28} over the ring $\mathbb{Z}[x_0^{\pm},x_1^{\pm},y_0^{\pm},y_1^{\pm}]$. We have

$$\begin{array}{rcll}\mathcal{H}(G_{28}) & = & < S_1,S_2,T_1,T_2 \mid & S_1S_2S_1 = S_2S_1S_2,\ T_1T_2T_1 = T_2T_1T_2\\ & & & S_1T_1 = T_1S_1,\ S_1T_2 = T_2S_1,\ S_2T_2 = T_2S_2,\\ & & & S_2T_1S_2T_1 = T_1S_2T_1S_2,\\ & & & (S_i-x_0)(S_i-x_1) = (T_i-y_0)(T_i-y_1) = 0 >\end{array}$$

The Schur elements of all irreducible characters of $\mathcal{H}(G_{28})$ have been calculated in [28] and they are obtained by permutation of the parameters from the following ones:

$s_{\phi_{1,0}} = \Phi_1(y_0/y_1) \cdot \Phi_6(y_0/y_1) \cdot \Phi_1(x_0/x_1) \cdot \Phi_6(x_0/x_1) \cdot \Phi_1(x_0y_0^2/x_1y_1^2) \cdot \Phi_6(x_0y_0/x_1y_1) \cdot \Phi_1(x_0^2y_0/x_1^2y_1) \cdot \Phi_4(x_0y_0/x_1y_1) \cdot \Phi_2(x_0y_0/x_1y_1) \cdot \Phi_2(x_0y_0/x_1y_1)$

$s_{\phi_{2,4''}} = -y_1/y_0\Phi_6(y_0/y_1) \cdot \Phi_3(x_0/x_1) \cdot \Phi_6(x_0/x_1) \cdot \Phi_1(x_0/x_1) \cdot \ \Phi_1(x_0/x_1) \cdot \Phi_1(x_0^2y_0/x_1^2y_1) \cdot \Phi_2(x_0y_0/x_1y_1) \cdot \Phi_2(x_0y_1/x_1y_0) \cdot \Phi_1(x_0^2y_1/x_1^2y_0)$

$s_{\phi_{4,8}} = 2\Phi_6(y_0/y_1) \cdot \Phi_6(x_1/x_0) \cdot \Phi_2(x_0y_1/x_1y_0) \cdot \Phi_2(x_0y_1/x_1y_0) \cdot \Phi_2(x_1y_1/x_0y_0) \cdot \Phi_2(x_0y_0/x_1y_1)$

$s_{\phi_{4,1}} = \Phi_1(y_0/y_1) \cdot \Phi_6(y_0/y_1) \cdot \Phi_1(x_1/x_0) \cdot \Phi_6(x_0/x_1) \cdot \Phi_2(x_0y_1/x_1y_0) \cdot \Phi_6(x_0y_0/x_1y_1) \cdot \Phi_2(x_0y_0/x_1y_1) \cdot \Phi_2(x_0y_0/x_1y_1)$

$s_{\phi_{6,6''}} = 3\Phi_1(y_1/y_0) \cdot \Phi_1(y_1/y_0) \cdot \Phi_1(x_1/x_0) \cdot \Phi_1(x_1/x_0) \cdot \Phi_6(x_0y_0/x_1y_1) \cdot \Phi_2(x_0y_1/x_1y_0) \cdot \Phi_2(x_1y_0/x_0y_1)$

$s_{\phi_{8,3''}} = -y_1/y_0\Phi_6(y_0/y_1) \cdot \Phi_6(x_0/x_1) \cdot \Phi_1(x_0/x_1) \cdot \Phi_1(x_1/x_0) \cdot \Phi_3(x_0/x_1) \cdot \Phi_1(x_0y_1^2/x_1y_0^2) \cdot \Phi_1(x_0y_0^2/x_1y_1^2)$

$s_{\phi_{9,2}} = \Phi_1(y_0/y_1) \cdot \Phi_1(x_0/x_1) \cdot \Phi_1(x_0y_1^2/x_1y_0^2) \cdot \Phi_4(x_0y_0/x_1y_1) \cdot \Phi_1(x_1^2y_0/x_0^2y_1) \cdot \Phi_2(x_0y_0/x_1y_1) \cdot \Phi_2(x_0y_0/x_1y_1)$

$s_{\phi_{12,4}} = 6\Phi_3(y_0/y_1) \cdot \Phi_3(x_1/x_0) \cdot \Phi_2(x_0y_1/x_1y_0) \cdot \Phi_2(x_0y_1/x_1y_0) \cdot \Phi_2(x_0y_0/x_1y_1) \cdot \Phi_2(x_1y_1/x_0y_0)$

$s_{\phi_{16,5}} = 2x_1y_1/x_0y_0\Phi_6(y_0/y_1) \cdot \Phi_6(x_1/x_0) \cdot \Phi_4(x_0y_1/x_1y_0) \cdot \Phi_4(x_0y_0/x_1y_1)$

Following theorem 3.2.3, if we set

$$\begin{aligned} X_i^2 &:= (\zeta_2)^{-i}x_i \quad (i = 0, 1), \\ Y_j^2 &:= (\zeta_2)^{-j}y_j \quad (j = 0, 1), \end{aligned}$$

then $\mathbb{Q}(X_0, X_1, Y_0, Y_1)$ is a splitting field for $\mathcal{H}(G_{28})$. Hence the factorization of the Schur elements over that field is as described by theorem 3.2.5.

The group G_{32}

Let $\mathcal{H}(G_{32})$ be the generic Hecke algebra of the complex reflection group G_{32} over the ring $\mathbb{Z}[x_0^{\pm}, x_1^{\pm}, x_2^{\pm}]$. We have

$$\begin{aligned}\mathcal{H}(G_{32}) \;=\; < S_1, S_2, S_3, S_4 \mid\; & S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1},\\ & S_iS_j = S_jS_i \text{ when } |i-j|>1,\\ & (S_i - x_0)(S_i - x_1)(S_i - x_2) = 0 >\end{aligned}$$

The Schur elements of all irreducible characters of $\mathcal{H}(G_{32})$ have been calculated in [31] and they are obtained by permutation of the parameters from the following ones:

$s_{\phi_{1,0}} = \Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0^3/x_1x_2^2)\cdot\Phi_1(x_0^3/x_1^2x_2)\cdot\Phi_1(x_0^5/x_1^3x_2^2)\cdot$
$\Phi_1(x_0^5/x_1^2x_2^3)\cdot\Phi_2(x_0^4/x_1x_2^3)\cdot\Phi_2(x_0^4/x_1^3x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_0/x_1)\cdot$
$\Phi_6(x_0^3/x_1x_2^2)\cdot\Phi_6(x_0^3/x_1^2x_2)\cdot\Phi_6(x_0^2/x_1x_2)\cdot\Phi_4(x_0^2/x_1x_2)\cdot\Phi_4(x_0/x_2)\cdot\Phi_4(x_0/x_1)\cdot\Phi_3(x_0^2/x_1x_2)\cdot$
$\Phi_{10}(x_0/x_2)\cdot\Phi_{10}(x_0/x_1)\cdot\Phi_5(x_0^2/x_1x_2)$

$s_{\phi_{4,1}} = \Phi_1(x_0^4/x_1x_2^3)\cdot\Phi_1(x_0^3/x_1x_2^2)\cdot\Phi_1(x_0^3/x_1^2x_2)\cdot\Phi_1(x_0^2x_1/x_2^3)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_2)\cdot$
$\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_2(x_0^5/x_1x_2^4)\cdot\Phi_2(x_0^3x_1/x_2^4)\cdot\Phi_2(x_0^3/x_1x_2^2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0/x_2)\cdot$
$\Phi_2(x_0x_1/x_2^2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_0/x_1)\cdot\Phi_4(x_0/x_2)\cdot\Phi_3(x_0^2/x_1x_2)\cdot\Phi_{10}(x_0/x_1)\cdot\Phi_{15}(x_0/x_2)$

$s_{\phi_{5,4}} = \Phi_1(x_0^3x_1^2/x_2^5)\cdot\Phi_1(x_0^2x_1/x_2^3)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_2)\cdot$
$\Phi_1(x_1/x_2)\cdot\Phi_2(x_0^3/x_1x_2^2)\cdot\Phi_2(x_0x_1^2/x_2^3)\cdot\Phi_2(x_1/x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0^4x_1/x_2^5)\cdot$
$\Phi_2(x_0/x_2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_0/x_1)\cdot\Phi_4(x_0/x_1)\cdot\Phi_3(x_0x_1/x_2^2)\cdot$
$\Phi_{12}(x_0/x_2)$

$s_{\phi_{6,8}} = x_1^2/x_0^2\Phi_1(x_0/x_1)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_2)\cdot$
$\Phi_1(x_0x_1^2/x_2^3)\cdot\Phi_1(x_0^2x_1/x_2^3)\cdot\Phi_2(x_0x_1^2/x_2^3)\cdot\Phi_2(x_0^2x_1/x_2^3)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_1^2/x_0x_2)\cdot$
$\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_1/x_2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_6(x_0x_1/x_2^2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_1/x_2)\cdot$
$\Phi_{10}(x_0/x_1)\cdot\Phi_5(x_0x_1/x_2^2)$

$s_{\phi_{10,2}} = \Phi_1(x_0^2x_1/x_2^3)\cdot\Phi_1(x_0^3/x_1x_2^2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_0/x_2)\cdot$
$\Phi_1(x_0/x_2)\cdot\Phi_2(x_0^4/x_1^3x_2)\cdot\Phi_2(x_0^3/x_1x_2^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0x_2/x_1^2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_2(x_0/x_2)\cdot$
$\Phi_2(x_1/x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_6(x_0^2x_1/x_2^3)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_0/x_1)\cdot\Phi_4(x_0/x_2)\cdot\Phi_3(x_0^2/x_1x_2)$

$s_{\phi_{15,6}} = \Phi_1(x_0^3/x_1x_2^2)\cdot\Phi_1(x_0^3/x_1^2x_2)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_2/x_1)\cdot\Phi_1(x_2/x_1)\cdot\Phi_1(x_2/x_0)\cdot$
$\Phi_1(x_0/x_2)\cdot\Phi_2(x_0^3x_2/x_1^4)\cdot\Phi_2(x_0^3x_1/x_2^4)\cdot\Phi_2(x_1^2/x_0x_2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_2(x_0/x_1)\cdot$
$\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_6(x_0^2/x_1x_2)\cdot\Phi_6(x_0/x_1)\cdot\Phi_6(x_0/x_2)\cdot\Phi_4(x_0^2/x_1x_2)$

$s_{\phi_{15,8}} = \Phi_1(x_1^2x_2/x_0^3)\cdot\Phi_1(x_0^2x_2/x_1^3)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_1/x_0)\cdot$
$\Phi_1(x_0/x_1)\cdot\Phi_2(x_1x_2/x_0^2)\cdot\Phi_2(x_0x_2/x_1^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_1/x_2)\cdot\Phi_2(x_1/x_2)\cdot$

$\Phi_2(x_0/x_2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_1/x_2)\cdot\Phi_4(x_0x_1/x_2^2)$

$s_{\phi_{20,3}}=\Phi_1(x_0^2x_2/x_1^3)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_0^4/x_1x_2^3)\cdot\Phi_1(x_0^3/x_1x_2^2)\cdot$
$\Phi_1(x_1/x_2)\cdot\Phi_2(x_0x_1^2/x_2^3)\cdot\Phi_2(x_1^2/x_0x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_2(x_2/x_0)\cdot$
$\Phi_6(x_0^3/x_1^2x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_4(x_0/x_2)\cdot\Phi_3(x_0x_1/x_2^2)$

$s_{\phi_{20,5}}=-\Phi_1(x_1^3/x_0^2x_2)\cdot\Phi_1(x_0^3/x_1^2x_2)\cdot\Phi_1(x_0x_1^2/x_2^3)\cdot\Phi_1(x_0^2x_1/x_2^3)\cdot\Phi_1(x_2/x_1)\cdot\Phi_1(x_2/x_1)\cdot$
$\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_2(x_1^3/x_0x_2^2)\cdot\Phi_2(x_0^3/x_1x_2^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_6(x_0x_1/x_2^2)\cdot$
$\Phi_6(x_1/x_0)\cdot\Phi_6(x_1/x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_3(x_0x_1/x_2^2)$

$s_{\phi_{20,7}}=\Phi_1(x_0^3x_1/x_2^4)\cdot\Phi_1(x_0x_1^2/x_2^3)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_2)\cdot$
$\Phi_1(x_1/x_2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_2(x_2/x_0)\cdot\Phi_2(x_0^3x_2/x_1^4)\cdot\Phi_2(x_0x_1^2/x_2^3)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_1^2/x_0x_2)\cdot$
$\Phi_6(x_0/x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_1/x_2)\cdot\Phi_3(x_0^2/x_1x_2)$

$s_{\phi_{20,12}}=2\Phi_1(x_2/x_1)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_0/x_2)\cdot$
$\Phi_1(x_0/x_1)\cdot\Phi_2(x_0x_1^2/x_2^3)\cdot\Phi_2(x_0x_2^2/x_1^3)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_1x_2/x_0^2)\cdot\Phi_2(x_0/x_1)\cdot\Phi_2(x_0/x_2)\cdot$
$\Phi_2(x_0/x_1)\cdot\Phi_2(x_0/x_2)\cdot\Phi_6(x_2/x_1)\cdot\Phi_6(x_1/x_2)\cdot\Phi_3(x_0^2/x_1x_2)$

$s_{\phi_{24,6}}=\Phi_1(x_1^3/x_0^2x_2)\cdot\Phi_1(x_2^3/x_0^2x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_2)\cdot$
$\Phi_1(x_2/x_1)\cdot\Phi_1(x_1/x_2)\cdot\Phi_2(x_0/x_1)\cdot\Phi_2(x_0/x_2)\cdot\Phi_2(x_0x_2/x_1^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_6(x_0/x_1)\cdot$
$\Phi_6(x_0/x_2)\cdot\Phi_4(x_0/x_1)\cdot\Phi_4(x_0/x_2)\cdot\Phi_5(x_0^2/x_1x_2)$

$s_{\phi_{30,4}}=\Phi_1(x_0^5/x_1^3x_2^2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_0/x_2)\cdot$
$\Phi_1(x_0/x_2)\cdot\Phi_2(x_1/x_0)\cdot\Phi_2(x_1/x_2)\cdot\Phi_2(x_2/x_0)\cdot\Phi_2(x_0^5/x_1x_2^4)\cdot\Phi_2(x_0x_2^2/x_1^3)\cdot\Phi_2(x_0/x_2)\cdot$
$\Phi_2(x_0x_2/x_1^2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_6(x_0/x_1)\cdot\Phi_6(x_1/x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_4(x_0/x_2)$

$s_{\phi'_{30,12}}=\Phi_1(x_1^5/x_0^3x_2^2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_2)\cdot$
$\Phi_1(x_1/x_2)\cdot\Phi_2(x_0/x_1)\cdot\Phi_2(x_0/x_2)\cdot\Phi_2(x_2/x_1)\cdot\Phi_2(x_1^5/x_0x_2^4)\cdot\Phi_2(x_1x_2^2/x_0^3)\cdot\Phi_2(x_1/x_2)\cdot$
$\Phi_2(x_1x_2/x_0^2)\cdot\Phi_2(x_1^2/x_0x_2)\cdot\Phi_6(x_1/x_0)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_1/x_2)\cdot\Phi_4(x_1/x_2)$

$s_{\phi_{36,5}}=\Phi_1(1/\zeta_3)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(\zeta_3x_0^2/x_1x_2)\cdot\Phi_1(\zeta_3^2x_0x_2/x_1^2)\cdot\Phi_1(x_2^2/\zeta_3^2x_0x_1)\cdot$
$\Phi_2(x_1x_2/x_0^2)\cdot\Phi_2(x_0^2x_2/\zeta_3x_1^3)\cdot\Phi_2(\zeta_3^2x_0^2x_1/x_2^3)\cdot\Phi_6(x_0^2/x_1x_2)\cdot\Phi_6(x_0/x_1)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_1/x_2)\cdot$
$\Phi_5(\zeta_3x_0/x_2)\cdot\Phi_5(\zeta_3x_0/x_1)$

$s_{\phi_{40,8}}=\Phi_1(x_0^3x_1^2/x_2^5)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_1^2x_2/x_0^3)\cdot$
$\Phi_1(x_2/x_1)\cdot\Phi_2(x_0/x_1)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0/x_2)\cdot\Phi_6(x_0/x_1)\cdot\Phi_6(x_1/x_2)\cdot$
$\Phi_4(x_0/x_1)\cdot\Phi_4(x_0/x_2)\cdot\Phi_3(x_0x_2/x_1^2)$

$s_{\phi_{45,6}}=\Phi_1(\zeta_3)\cdot\Phi_1(\zeta_3^2x_0^2/x_1x_2)\cdot\Phi_1(\zeta_3x_0x_2/x_1^2)\cdot\Phi_1(\zeta_3x_0x_1/x_2^2)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_1/x_2)\cdot$
$\Phi_2(\zeta_3^2x_0^2x_2/x_1^3)\cdot\Phi_2(\zeta_3^2x_1^2x_2/x_0^3)\cdot\Phi_2(x_0/\zeta_3^2x_2)\cdot\Phi_2(\zeta_3x_1/x_2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_6(x_1/x_0)\cdot$
$\Phi_6(x_1/x_2)\cdot\Phi_6(x_0/x_2)\cdot\Phi_6(x_0x_1/x_2^2)\cdot\Phi_4(\zeta_3x_0/x_2)\cdot\Phi_4(\zeta_3x_1/x_2)$

$s_{\phi_{60,7}} = \Phi_1(x_0x_1^2/x_2^3)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_0/x_2)\cdot$
$\Phi_1(x_1x_2^2/x_0^3)\cdot\Phi_2(x_0/x_1)\cdot\Phi_2(x_0^4x_1/x_2^5)\cdot\Phi_2(x_1^2/x_0x_2)\cdot\Phi_2(x_1x_2/x_0^2)\cdot\Phi_2(x_1/x_2)\cdot\Phi_2(x_1/x_2)\cdot$
$\Phi_6(x_0/x_1)\cdot\Phi_6(x_2/x_1)\cdot\Phi_4(x_0/x_1)$

$s_{\phi''_{60,11}} = \Phi_1(x_1x_2^3/x_0^4)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_1/x_2)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_2/x_0)\cdot$
$\Phi_1(x_0x_2^3/x_1^4)\cdot\Phi_2(x_1^2/x_0x_2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0/x_1)\cdot\Phi_2(x_0/x_1)\cdot$
$\Phi_6(x_0x_1/x_2^2)\cdot\Phi_6(x_1/x_0)$

$s_{\phi_{60,12}} = 2\Phi_1(x_0x_2^2/x_1^3)\cdot\Phi_1(x_0x_1^2/x_2^3)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_2/x_0)\cdot$
$\Phi_1(x_2/x_0)\cdot\Phi_1(x_2/x_0)\cdot\Phi_2(x_2/x_1)\cdot\Phi_2(x_1/x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_2(x_0^2/x_1x_2)\cdot\Phi_6(x_0^2/x_1x_2)\cdot$
$\Phi_6(x_1/x_2)\cdot\Phi_6(x_2/x_1)\cdot\Phi_4(x_0/x_1)\cdot\Phi_4(x_0/x_2)$

$s_{\phi_{64,8}} = 2\Phi_1(rx_1/x_2^2)\cdot\Phi_1(x_2^2/rx_0)\cdot\Phi_1(x_0/x_2)\cdot\Phi_1(x_2/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_1/x_0)\cdot\Phi_1(x_0^3/x_1x_2^2)\cdot$
$\Phi_1(x_0x_2^2/x_1^3)\cdot\Phi_2(rx_0^2/x_1^2x_2)\cdot\Phi_2(rx_1^2/x_0^2x_2)\cdot\Phi_3(x_0x_1/x_2^2)\cdot\Phi_{10}(r/x_2)\cdot\Phi_{15}(r/x_0)$
where $r = \sqrt[3]{x_0x_1}$

$s_{\phi_{80,9}} = 2\Phi_1(x_0x_2^2/x_1^3)\cdot\Phi_1(x_0x_1^2/x_2^3)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_0/x_1)\cdot\Phi_1(x_2/x_0)\cdot\Phi_1(x_2/x_0)\cdot$
$\Phi_1(x_2/x_1)\cdot\Phi_1(x_2/x_1)\cdot\Phi_2(x_0/x_2)\cdot\Phi_2(x_0/x_1)\cdot\Phi_4(x_1x_2/x_0^2)\cdot\Phi_4(x_0/x_1)\cdot\Phi_4(x_0/x_2)\cdot$
$\Phi_3(x_0^2/x_1x_2)\cdot\Phi_{12}(x_1/x_2)$

$s_{\phi_{81,10}} = 3\Phi_2(rx_2/x_0^2)\cdot\Phi_2(rx_2/x_1^2)\cdot\Phi_2(rx_0/x_2^2)\cdot\Phi_2(rx_0/x_1^2)\cdot\Phi_2(rx_1/x_0^2)\cdot\Phi_2(rx_1/x_2^2)\cdot$
$\Phi_2(x_0x_1/x_2^2)\cdot\Phi_2(x_0x_2/x_1^2)\cdot\Phi_2(x_1x_2/x_0^2)\cdot\Phi_2(r/x_2)\cdot\Phi_2(r/x_0)\cdot\Phi_2(r/x_1)\cdot\Phi_4(r^2/x_0x_1)\cdot$
$\Phi_4(r^2/x_0x_2)\cdot\Phi_4(r^2/x_1x_2)\cdot\Phi_5(r/x_0)\cdot\Phi_5(r/x_2)\cdot\Phi_5(r/x_1)$
where $r = \sqrt[3]{x_0x_1x_2}$

Following theorem 3.2.3 and [30], Table 8.2, if we set

$$X_i^6 := (\zeta_3)^{-i}x_i \ \ (i = 0, 1, 2),$$

then $\mathbb{Q}(\zeta_3)(X_0, X_1, X_2)$ is a splitting field for $\mathcal{H}(G_{32})$. Hence the factorization of the Schur elements over that field is as described by theorem 3.2.5.

Some corrections on the article [33]

Example 3.17 (G_{34}) :
type $\mathcal{M}(Z_3)''$: the first character of the first block is $\phi''_{105,8}$.
type $\alpha''_{3\times5}$: the second block is
$(\phi_{84,41}, \phi_{84,37}, \phi_{336,34}, \phi_{336,32}, \phi_{420,31}, \phi_{420,35}, \phi_{504,33})$.

Example 3.19 (G_{36}) : There are two families of type $\mathcal{M}(Z_2)$ missing:
$(\phi_{15,28}, \phi_{105,26}, \phi_{120,25})$ and $(\phi_{15,7}, \phi_{105,5}, \phi_{120,4})$

(the characters may not be in the correct order).

Example 4.1 (G_5) : The decomposition matrix of the last family for $p = 3$ is wrong; the characters of degree 1 aren't in the same 3-block of the group algebra with the characters of degree 2.

Example 4.3 (G_{13}) : The decomposition matrix of the last family for $p = 2$ is wrong; the characters of degree 2 aren't in the same 2-block of the group algebra with the characters of degree 4.

Example 6.4 (G_9) : The blocks given here are completely wrong. The correct, probably, are:
$(\phi'_{2,7}, \phi'_{2,11}), (\phi''_{2,7}, \phi''_{2,11}), (\phi'_{1,12}, \phi''_{1,12}, \phi_{2,4}, \phi_{2,8}), (\phi_{1,6}, \phi_{1,30}, \phi_{2,10}, \phi_{2,14})$,
$(\phi_{2,5}, \phi_{2,13}, \phi_{2,1}, \phi_{2,17}, \phi_{4,9}, \phi_{4,3}, \phi_{4,7}, \phi_{4,5})$.

Example 6.5 (G_{10}) : The last three non-trivial blocks are:
$(\phi_{1,16}, \phi_{2,10}, \phi'_{3,12}, \phi'_{3,8}, \phi_{3,4}), (\phi_{1,12}, \phi_{2,18}, \phi''_{3,8}, \phi''_{3,12}, \phi_{3,16})$,
$(\phi_{1,14}, \phi_{1,26}, \phi_{2,8}, \phi_{3,14}\, \phi_{3,2}, \phi''_{3,10}, \phi'_{3,10}, \phi''_{3,6}, \phi'_{3,6}, \phi_{4,11}, \phi_{4,5})$.

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