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FACULTE DES SCIENCES ET
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HABILITATION UNIVERSITAIRE

Présentée à la Faculté des Sciences et Techniques – Marrakech

UFR : Mathématiques et Informatique

Spécialité : Mathématiques Appliquées et Informatique

CONTRIBUTION À L'APPROXIMATION DE PROBLÈME D'IDENTIFICATION ET DÉCOMPOSITION DE DOMAINE EN ÉLASTICITÉ

Par :

Abdellatif ELLABIB

(Doctorat : Mathématiques Appliquées)

Soutenue le 15 Mars 2008 devant la commission d'examen :

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À MES PARENTS
À TOUTE MA FAMILLE

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FICHE PRESENTATIVE DE L'HABILITATION UNIVERSITAIRE

Nom et Prénom de l'auteur : ELLABIB Abdellatif

Intitulé des travaux de recherche : Contribution à l'approximation de problème d'identification et décomposition de domaine en élasticité

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Ce travail a donné lieu aux résultats suivants :

Deux publications dans des journaux internationaux avec comité de lecture et un article soumis à un journal international et trois communications dans des congrès internationaux avec comité de lecture et un encadrement en mémoire de DESA et deux polycopiés de cours ont été réalisés.

Table des matières

Chapitre 1

Introduction

Introduction (version française)

Dans ce document de synthèse, nous décrivons principalement les activités de recherches entamés depuis Juin 2003 et les activités d'enseignements. Nous présentons tout d'abord des travaux de recherches qui ont été achevés et dont les problématiques n'étaient pas abordées dans ma thèse de Doctorat de l'université de Nantes, ensuite une liste de publication, différents projets et d'autres activités de recherches et enfin un curriculum vitae.

Les travaux de recherches présentés sont divisés en deux axes. Le premier axe concerne le problème d'identification de données en élasticité linéaire. Le deuxième axe de recherche est consacré à la méthode de décomposition en sous-domaines.

Nous considérons un corps élastique qui occupe une région ouverte Ω de frontière $\Gamma = \Gamma_1 \cup \Gamma_2$ et $\Gamma_1 \cap \Gamma_2 = \emptyset$.
Notons f la densité volumique de forces dans Ω. Les champs de déplacements $u = (u_i)$ et de contraintes $\sigma = (\sigma_{ij})$ satisfont les équations suivantes [8].
Équation d'équilibre :

$$\frac{\partial \sigma_{ij}}{\partial x_j} = f \text{ dans } \Omega \tag{1.1}$$

Équation de comportement :

$$\sigma_{ij} = a_{ijkh}\varepsilon_{kh}(u) \tag{1.2}$$

$$\varepsilon_{kh}(u) = \frac{1}{2}\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right). \tag{1.3}$$

Les termes $a_{ijkh}(x)$ sont appelés les coefficients d'élasticité.
Dans le cas des matériaux élastiques isotropes, ces coefficients a_{ijkh} s'expriment en fonction de λ et μ par

$$a_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) \tag{1.4}$$

où λ et μ sont les coefficients de Lamé et δ_{ij} est le symbole de Kronecker.
En injectant, l'équation de comportement (1.2) dans l'équation d'équilibre (1.1) et en combinant l'équation (1.3) et la relation (1.4), nous obtenons alors l'équation décrivant les champs de déplacement suivante.

$$G\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{G}{1-2\nu}\frac{\partial^2 u_j}{\partial x_i \partial x_j} = f \text{ dans } \Omega \tag{1.5}$$

où G et ν désignent respectivement le module de Shear et le coefficient de Poisson.

Dans cet axe de recherche, nous nous intéressons à l'approximation numérique d'un problème inverse en élasticité linéaire. Nous cherchons à identifier des conditions aux limites inaccessibles à la mesure sur la partie Γ_2 de la frontière du domaine Ω connaissant des mesures de déplacement et de forces sur la partie complémentaire Γ_1 de la frontière du domaine Ω. Le modèle mathématique décrivant ce problème d'identification est formé par l'équation (1.5) et les conditions aux limites suivantes

$$u_i = \tilde{u}_i \text{ sur } \Gamma_1 \tag{1.6}$$

et

$$t_i = \tilde{t}_i \text{ sur } \Gamma_1 \tag{1.7}$$

où $\tilde{u}$ et $\tilde{t}$ sont des fonctions données.

Ce problème obtenu est mal posé au sens de Hadamard [19], la solution est instable, elle change rapidement pour une petite modification des données.

Nous transformons le problème (1.5), (1.6), (1.7) en appliquant l'algorithme de reconstruction des conditions aux limites introduit dans [34]. Cette méthode se base sur la résolution d'une suite de problèmes d'équations aux dérivées partielles bien posés. L'approximation numérique des différents problèmes se fait par les éléments de frontières [4]. Ces éléments génèrent des systèmes linéaires de matrices denses, non-symétriques et mal conditionnés, nous utilisons alors des méthodes [41, 42] de résolution adaptées à un tel système linéaire. Des essais numériques pour des problèmes bidimensionnels seront présentés.

Notre travail qui utilise les méthodes de décomposition en sous-domaines a démarré en 2004.

Dans le deuxième axe de recherche, nous nous intéressons à l'application de la méthode de décomposition en sous-domaines à un problème direct décrivant les équations d'élasticité linéaire. Ce problème est formé des équations (1.5), (1.6) et une condition aux limites sur Γ_2

$$t_i = \tilde{t}_i \text{ sur } \Gamma_2 \tag{1.8}$$

où $\tilde{t}$ est une fonction donnée sur Γ_2.

Notre premier objectif de cet axe est d'approcher le problème (1.5), (1.6) et (1.8) en appliquant les méthodes de décomposition de domaine [37]. Nous utilisons une méthode de décomposition sans recouvrement et la méthode de Schwarz couplées avec une approximation par les équations intégrales et les éléments de frontières. Cette approximation ne necessite que la discrétisation de la frontière de chaque sous-domaines et elle permet de réduire le nombre d'inconnues et le temps de calcul. Nous présentons en détail les algorithmes de Dirichlet-Neumann et Schwarz. Nous décrivons les systèmes algébriques issus des méthodes de décomposition avec recouvrement et sans recouvrement. Nous présentons ensuite deux algorithmes et des résultats numériques qui illustrent la convergence de ces deux algorithmes vers la solution du problème d'élasticité linéaire dans différents domaines.

Enfin, Nous étudions une méthode de décomposition du domaine sans recouvrement pour les équations d'élasticité. Cette méthode est basée sur la formulation du problème en un problème de contrôle optimal. Nous démontrons l'existence de la solution et la convergence de la solution de problème approché vers la solution du problème continu. Nous exposons un algorithme d'optimsation en utilisant le Lagrangien. Enfin, nous présentons les résultats numériques qui montrent l'efficacité de notre algorithme et confirment le résultat de convergence.

Nous donnons maintenant, en anglais, un détail concernant ces travaux

Introduction (English version)

In this document, we describe the research activities started since June 2003 and lessons activities. We present research work which had been completed and that the issues were not addressed in my thesis at the University of Nantes.

This work is divided into two lines of research. The first axis concerning the problem of identification data linear elasticity. The second line of research is devoted to domain decomposition method.

We consider an isotropic linear elastic material which occupies an open bounded domain Ω with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Let f the body forces, the displacement $u = (u_i)$ and constraints $\sigma = (\sigma_{ij})$ satisfy the following equations [8].

Equilibrium equation :

$$\frac{\partial \sigma_{ij}}{\partial x_j} = f \text{ in } \Omega \tag{1.9}$$

The stress σ_{ij} are related to the strains through the

Constitutive law :

$$\sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right) \tag{1.10}$$

where G and ν are the shear modulus and Poisson ratio, respectively and δ_{ij} is the Kronecker delta tensor.

The strains ε_{ij} are related to the displacement gradients by the kinematic relations

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.11}$$

Now, we substitute the constitutive law (1.10) into the equilibrium equation (1.9), and use the kinematic relations (1.11), we obtain the following Navier equations.

$$G\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{G}{1-2\nu}\frac{\partial^2 u_j}{\partial x_i \partial x_j} = f \text{ in } \Omega \tag{1.12}$$

In this work, we are interested by a reconstruction inverse problem where the geometry of the problem and the material constants are determined, but the boundary conditions are not completely known. This problem arises in cases where a portion of the boundary is exposed to environmental conditions which can not be assessed due to physical difficulties or geometrical inaccessibility. The aim in the reconstruction inverse problem is to find the unknown boundary conditions based on the supplementary data provided on the boundary and/or the domain.

Consider the problem where no conditions are prescribed on Γ_2 and assume that it is possible to measure the traction vector on Γ_1. The mathematical formulation of an inverse problem consisting of equation (1.12) and boundary conditions

$$u_i = \tilde{u_i} \text{ on } \Gamma_1 \tag{1.13}$$

$$t_i = \tilde{t_i} \text{ on } \Gamma_1 \tag{1.14}$$

where $\tilde{u}$ and $\tilde{t}$ are prescribed quantities.

This problem obtained is ill-posed and we cannot use a direct approach. The solution is unstable with a respect to small perturbations in the data on Γ_1.

We translate the problem (1.12), (1.13) and (1.14) by applying the reconstruction algorithm of boundary conditions introduced into [34]. This method is based on solving a sequence of well-posed boundary value problems. The numerical approximation of the various problems is used by the boundary element method [4]. These systems generate dense linear matrix, unsymmetrical and ill-conditioned, then we use methods [41, 42] resolution adapted to such a linear system. Numerical results in two-dimensional case will be presented.

In 2004, we are interested to use the domain decomposition method to direct problem of linear elasticity. This problem is given by (1.12), (1.13) and boundary condition for traction on Γ_2

$$t_i = \tilde{t_i} \text{ on } \Gamma_2 \tag{1.15}$$

where $\tilde{t}$ is a given function.

The main goal of this second axis is to solve elasticity problems (1.12), (1.13) and (1.15) using a domain decomposition method [37] coupled with the boundary element method [4] on complicated 2-D geometries Ω where Ω is not necessary circular or rectangular. The decomposition method we have selected is a non-overlapping technique and Schwarz method. We decompose the domain Ω into a number of subdomains Ω_i with Ω_i is rectangular or circular domain and the solution in the entire domain Ω is computed via sequences of solutions computed in the subdomains Ω_i. We describe in details Dirichlet-Neumann and Schwarz algorithms. We have chosen to associate it with the boundary element method. Indeed, it only requires the discretization of the boundaries of the subdomains. This technique of coupling reduces the number of unknowns and the time of computing. Numerical examples in two dimensional case for some complicated 2-D domain configurations are also illustrated.

Also, in this axis we present also a non-overlapping domain decomposition method for elasticity equations based on an optimal control formulation. The existence of a solution is proved and the convergence of a subsequence of the approximate solutions to a solution of the continuous problem is shown. The implementation based on lagrangian method is discussed. Finally, numerical results showing the efficiency of our approach and confirming the convergence result are given.

We now give more details on this work

Chapitre 2

Research Report / Présentation des travaux de recherches

2.1 An iterative approach to the solution of an inverse problem in linear elasticity

2.1.1 Introduction

A vast body of engineering experience shows that the theory of linear elasticity allows an accurate modeling of many natural or manufactured solid materials (civil engineering structures, transportation vehicles, machines, the Earth's mantle, rocks mechanics [27]) and provides an essential tool for analysis and design.

When the governing system of partial differential equations, i.e. the equilibrium, constitutive and kinematics equations, have to be solved with the appropriate initial and boundary conditions for the displacement and/or traction vectors, i.e. Dirichlet, Neumann or mixed boundary conditions the associated problems are called direct problems and their existence and uniqueness have been well established. When one or more of the conditions for solving the direct problem are partially or entirely unknown then an inverse problem may be formulated to determine the unknowns from specified or measured system responses.

The main type of inverse problems that arise in the context of linear elasticity, and more generally of the mechanics of deformable solids, are similar to those encountered in other areas of physics involving continuous media and distributed physical quantities, e.g., acoustics, electrostatics and electromagnetism. They are usually motivated by the desire or need to overcome a lack of information concerning the properties of the system (a deformable solid body or structure). It should be noted that most of the inverse problems are ill-posed and hence they are more difficult to solve than the direct problems. It is well known that they are generally instable, i.e. the existence, uniqueness and stability of their solutions are not always guaranteed, see e.g. Hadamard [19]. Identification of inaccessible boundary values (Cauchy problem in elasticity) is a classical example of inverse problem. This inverse problem, in which both displacement and traction boundary conditions are prescribed only on a part of the boundary of the solution domain whilst no information is available on the remaining part of the boundary, can be encountered in many situations [20, 43].

Recently, an approximate solution to the Cauchy problem for Poison equation has been determined by one of the authors, [22, 34], using an alternating iterative method which reduced the problem to solving a sequence of well-posed boundary value problems. Our goal in this paper is to extend this algorithm in conjunction with the boundary element method (BEM) to the Cauchy problem in elasticity.

2.1.2 Mathematical model

Direct problem statement

The mathematical formulation of the 2D elasticity problem in the case of an isotropic linear elastic material which occupies an open bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1, \Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$ is described as follows.

Let $w = (u, v)^T$ be the displacement vector and b the volume force vector. Here $(.,.)^T$ denotes the transpose of a vector or a matrix. Let us define the matrices

$$\mathcal{D} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix} \quad \mathcal{E} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{2\partial y} & \dfrac{\partial}{2\partial x} \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{pmatrix} \tag{2.1}$$

Then the strain vector $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})^T$ is given by

$$\varepsilon = \mathcal{E}w \tag{2.2}$$

The strain tensor ε is related to the stress vector $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T$ by the constitutive law

$$\sigma = \frac{2G}{1-2\nu}\mathcal{C}\varepsilon \tag{2.3}$$

where G and ν are respectively the Shear modulus and Poisson ratio.

The equilibrium equations are given by

$$\mathcal{D}\sigma = b \tag{2.4}$$

If we now substitute the constitutive law (2.3) into the equilibrium equation (2.4), and use the kinematic relations (2.2) of the elasticity tensor for an isotropic linear elastic material, we obtain the following Lamé system or the Navier equations

$$\begin{cases} G\Delta u + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial x \partial y}\right) = b_1 \text{ in } \Omega \\ G\Delta v + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 v}{\partial y^2}\right) = b_2 \text{ in } \Omega \end{cases} \tag{2.5}$$

The solution of Eqs. (2.5) must satisfy prescribed boundary conditions on the boundary Γ of the body, which are based either on the displacements u and v, or the boundary traction t and s. The boundary conditions can be written into the following types

$$u(X) = \tilde{u}(X),\ v(X) = \tilde{v}(X) \text{ for } X \in \Gamma_1 \tag{2.6}$$

and

$$t(X) = \tilde{t}(X),\ s(X) = \tilde{s}(X) \text{ for } X \in \Gamma_2 \tag{2.7}$$

where $(t(X), s(X))$ is the traction vector at a point $X \in \Gamma_2$ with $\tilde{u}$, $\tilde{v}$, $\tilde{t}$ and $\tilde{s}$ prescribed quantities.

Reconstruction inverse problem

The knowledge of the geometry (the domain Ω) of the problem, the material constants G and ν and the prescribed quantities $\tilde{u}$, $\tilde{v}$, $\tilde{t}$ and $\tilde{s}$ enable us to determine the displacement vector $w(x)$ and the strain and the stress tensors in the domain Ω. In this case the problem is called direct problem. Different inverse problems can be considered for this direct problem. In all cases, part of the data which is known for the well posed direct problem is not known. In order to find this unknown data, supplementary information have to be provided.

In this work, we are interested by a reconstruction inverse problem where the geometry of the problem and the material constants are determined, but the boundary conditions are not completely known. This problem arises in cases where a portion of the boundary is exposed to environmental conditions which can not be assessed due to physical difficulties or geometrical inaccessibility. The aim in the reconstruction inverse problem is to find the unknown boundary conditions based on the supplementary data provided on the boundary and/or the domain.

Consider the problem where no conditions are prescribed on Γ_2 and assume that it is possible to measure the traction vector on Γ_1. This gives arise to the supplementary boundary conditions

$$t(X) = \tilde{t}(X) \text{ and } s(X) = \tilde{s}(X) \text{ for } X \in \Gamma_1 \tag{2.8}$$

where $\tilde{t}$ and $\tilde{s}$ are given functions.

In the next section, we describe an iterative method to solve numerically the reconstruction problem (2.5), (2.6) and (2.8), which is ill-posed and cannot be solved efficiently by a direct approach.

2.1.3 Description of the alternating algorithm

An alternating algorithm for solving Cauchy problems for elliptic equations was introduced by Kozlov et al. [26]. This algorithm was the subject of several studies which addressed various numerical and theoretical aspects (see for example [3, 21, 22, 34]). We extend here this procedure to the reconstruction problem described above. The iterative algorithm investigated is based on reducing this ill-posed problem to a sequence of mixed well-posed boundary value problems and consists of the following steps.

Giving ω^0 and z^0, initial approximation of the solution on Γ_2, we construct a sequence of approximation u^k, v^k by solving alternately the following mixed well-posed direct problems until a prescribed stopping criterion is satisfied.

u^{2k} and v^{2k} are obtained as the solution of

$$\begin{cases} G\Delta u^{2k} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u^{2k}}{\partial x^2} + \dfrac{\partial^2 v^{2k}}{\partial x \partial y}\right) = b_1 \text{ in } \Omega \\ G\Delta v^{2k} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u^{2k}}{\partial x \partial y} + \dfrac{\partial^2 v^{2k}}{\partial y^2}\right) = b_2 \text{ in } \Omega \\ t^{2k} = \tilde{t}, \ \ s^{2k} = \tilde{s} \quad \text{on } \Gamma_1 \text{ and } u^{2k} = \omega^k, \ \ v^{2k} = z^k \quad \text{on } \Gamma_2 \end{cases} \tag{2.9}$$

Having constructed u^{2k} and v^{2k} we can obtain u^{2k+1} and v^{2k+1} by solving the problem

$$\begin{cases} G\Delta u^{2k+1} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u^{2k+1}}{\partial x^2} + \dfrac{\partial^2 v^{2k+1}}{\partial x \partial y}\right) = b_1 \text{ in } \Omega \\ G\Delta v^{2k+1} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u^{2k+1}}{\partial x \partial y} + \dfrac{\partial^2 v^{2k+1}}{\partial y^2}\right) = b_2 \text{ in } \Omega \\ u^{2k+1} = \tilde{u}, \ \ v^{2k+1} = \tilde{v} \quad \text{on } \Gamma_1 \text{ and } t^{2k+1} = t^{2k}, \ \ s^{2k+1} = s^{2k} \quad \text{on } \Gamma_2 \end{cases} \tag{2.10}$$

The sequence ω^k and z^k are constructed as follows

$$\omega^k = F_1(\omega^{k-1}) \text{ and } z^k = F_2(z^{k-1}) \tag{2.11}$$

where F_1 and F_2 are two relaxation operators that will be determined in order to ensure and possibly accelerate the convergence of the iterative procedure. Note that the similar Kozlov-Maz'ya-Fomin's schemes [26] for elasticity problem is obtained by taking $F_1(\omega^{k-1}) = u^{2k-1}$ and $F_2(z^{k-1}) = v^{2k-1}$.

Selection criteria for the relaxation factor based on convex combination

To solve Cauchy Problems for Poisson equation Nachaoui et al. [22] established a relaxation algorithm by the use of a convex combination of the successive solutions on Γ_2 which produces a convergent and stable numerical solution.

We extend this idea to the reconstructing algorithm (2.9), (2.10). This can be done by defining F_1 and F_2 in (2.11) as follows :

$$F_1(\omega^{k-1}) = \theta_1 u^{2k-1}_{|\Gamma_2} + (1-\theta_1)\omega^{k-1} \text{ and } F_2(z^{k-1}) = \theta_2 v^{2k-1}_{|\Gamma_2} + (1-\theta_2) z^{k-1} \tag{2.12}$$

where θ_1 and θ_2 are two parameters that will be determined in order to ensure and possibly accelerate the convergence of the iterative scheme. Note that the equivalent of Kozlov-Maz'ya-Fomin's schemes [26] for elasticity problem is obtained by taking $\theta_1 = 1$ and $\theta_2 = 1$.

As in [22] the numerical tests performed revealed that the algorithm with constants relaxation factors θ_1 end θ_2 is convergent but there are large variation in the rate of convergence. Therefore we developed selection criteria for the relaxation factors.

Consider that constant relaxation factors are applied in the relaxation algorithm associated to (2.12) and let ω^k defined by (2.11). Note that a good indicator of the level of accuracy achieved is given by the functions

$$\varphi_1(\theta_1) = \|w^k - w^{k-1}\|_{L^2(\Gamma_2)} \text{ and } \ \varphi_2(\theta_2) = \|z^k - z^{k-1}\|_{L^2(\Gamma_2)},$$

since φ_1 and φ_2 tend to zero as the convergence of the iterative algorithm is achieved. Therefore the relaxation factors are selected such that the functions φ_1 and φ_2 are minimized. For this we require that $\dfrac{\partial \varphi_1}{\partial \theta_1} = 0$ and $\dfrac{\partial \varphi_2}{\partial \theta_2} = 0$ which yield

$$\theta_1^{k+1} = \frac{\langle e_1^{2k}, e_1^{2k} - e_1^{2k+1}\rangle}{\|e_1^{2k+1} - e_1^{2k}\|^2_{L^2(\Gamma_2)}} \text{ and } \theta_2^{k+1} = \frac{\langle e_2^{2k}, e_2^{2k} - e_2^{2k+1}\rangle}{\|e_2^{2k+1} - e_2^{2k}\|^2_{L^2(\Gamma_2)}} \; \forall\, k \geq 1, \tag{2.13}$$

where $e_1^{2k} = u^{2k}_{|\Gamma_2} - u^{2k-2}_{|\Gamma_2}$, $e_1^{2k+1} = u^{2k+1}_{|\Gamma_2} - u^{2k-1}_{|\Gamma_2}$, $e_2^{2k} = v^{2k}_{|\Gamma_2} - v^{2k-2}_{|\Gamma_2}$, $e_2^{2k+1} = v^{2k+1}_{|\Gamma_2} - v^{2k-1}_{|\Gamma_2}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Gamma_2)$.

Note that the automatic selection of the relaxation factors given in (2.13) requires 2 inner products at each iteration which are equivalent in discreet form to a number of operations of order $O(N)$ and this is negligible compared to the number of operations needed to solve the two direct problem (2.9) and (2.10).

Selection criteria for the relaxation factor based on fixed point operator

In this section we develop a second relaxation scheme based on some least-residual strategy. Let F_1 and F_2 be the mappings from $L^2(\Gamma_2)$ to $L^2(\Gamma_2)$ defined by solving successively the problems (2.9), (2.10) and taking $F_1(\omega^k) = u^{2k+1}_{|\Gamma_2}$, $F_2(z^k) = v^{2k+1}_{|\Gamma_2}$.

Let L_1 and L_2 be two linear operator from $L^2(\Gamma_2)$ to $L^2(\Gamma_2)$ defined as follows. For any w_1, $w_2 \in L^2(\Gamma_2)$, $L_1 w_1$ and $L_2 w_2$ are the solutions respectively of the two well posed linear problems (2.9) and (2.10) where w_1 and w_2 play respectively the role of ω^k and z^k but with the volume force equal to zero and homogeneous boundary conditions on Γ_1. Let w_n and w_d computed by solving respectively the two well posed linear problems (2.9) and (2.10) with homogeneous boundary conditions on Γ_2. This implies that F_1 and F_2 can be written as :

$$F_1(\omega) = L_1\omega + w_n \text{ and } F_2(z) = L_2 z + w_d. \tag{2.14}$$

Then the marching condition for the displacements on Γ_2 in (2.9) can be relaxed as follows

$$\omega^{k+1} = \omega^k + \theta_1(L_1\omega^k + w_n - \omega^k) \text{ and } z^{k+1} = z^k + \theta_2(L_2 z^k + w_d - z^k). \tag{2.15}$$

Let us define the following vectors

$$r_1^k = L_1\omega^k + w_n - \omega^k, \qquad r_2^k = L_2 z^k + w_d - z^k, \tag{2.16}$$

$$\omega^k(\theta_1) = \omega^k + \theta_1 r_1^k, \qquad z^k(\theta_2) = z^k + \theta_2 r_2^k, \tag{2.17}$$

$$r_1^k(\theta_1) = L_1\omega^k(\theta_1) + w_n - \omega^k(\theta_1), \qquad r_2^k(\theta_2) = L_2 z^k(\theta_2) + w_d - z^k(\theta_2). \tag{2.18}$$

Note that here a good indicator of the level of accuracy achieved is given by the functions

$$\phi_1(\theta_1) = \|r_1^k(\theta_1)\|_{L^2(\Gamma_2)} \quad \text{and } \phi_2(\theta_2) = \|r_2^k(\theta_2)\|_{L^2(\Gamma_2)} \tag{2.19}$$

since ϕ_1 and ϕ_2 tend to zero as the convergence of the iterative algorithm is achieved. Therefore the relaxation factors are selected such that the functions ϕ_1 and ϕ_2 are minimized.

From (2.16), (2.17) and (2.18) we obtain

$$r_1^k(\theta_1) = r_1^k + \theta_1(L_1 r_1^k - r_1^k) \text{ and } r_2^k(\theta_2) = r_2^k + \theta_2(L_2 r_2^k - r_2^k). \tag{2.20}$$

Then ϕ_1 and ϕ_2 can be written as follows

$$\phi_1(\theta_1) = \|r_1^k\|^2_{L^2(\Gamma_2)} + 2\theta_1\langle r_1^k, L_1 r_1^k - r_1^k\rangle + \theta_1^2\|L_1 r_1^k - r_1^k\|^2_{L^2(\Gamma_2)} \tag{2.21}$$

$$\phi_1(\theta_1) = \|r_2^k\|^2_{L^2(\Gamma_2)} + 2\theta_2\langle r_2^k, L_1 r_2^k - r_2^k\rangle + \theta_2^2\|L_2 r_2^k - r_2^k\|^2_{L^2(\Gamma_2)} \tag{2.22}$$

For the functions ϕ_1 and ϕ_2 to be minimized we require that $\dfrac{\partial\phi_1}{\partial\theta_1} = 0$ and $\dfrac{\partial\phi_2}{\partial\theta_2} = 0$ which yield

$$\theta_1^k = \frac{\langle r_1^k, r_1^k - L_1 r_1^k\rangle}{\|L_1 r_1^k - r_1^k\|^2_{L^2(\Gamma_2)}} \text{ and } \theta_2^k = \frac{\langle r_2^k, r_2^k - L_2 r_2^k\rangle}{\|L_2 r_2^k - r_2^k\|^2_{L^2(\Gamma_2)}}. \tag{2.23}$$

Thus the automatic adjustment of ω^{k+1} and z^{k+1} is obtained as follows

$$\omega^{k+1} = \omega^k + \theta_1^k r_1^k \text{ and } z^{k+1} = z^k + \theta_2^k r_2^k, \tag{2.24}$$

where θ_1^k and θ_2^k are given by (2.23).

Note that this scheme requires the solution of the two well posed problems (2.9) and (2.10) and the computation of $L_1 r_1^k$ and $L_2 r_2^k$ which are equivalent in the discreet form to matrix-vector product.

The boundary element method is a very apt tool to solve the auxiliary problems (2.9) and (2.10), since the boundary conditions are the main unknown of the problem and the statement of these problems in term of boundary integral equation reduces the modeling effort to a minimum. Moreover, the BEM determines simultaneously the boundary displacement u, v and its traction t, s, this allows us to solve problem (2.10) without the need or further finite difference, as one would employ if using the finite element or the finite difference method.

We describe the boundary element method in the next section.

2.1.4 Integral equation formulation and boundary element

Integral equation formulation

The linear elasticity problem (2.5) in two-dimensional case can be formulated in integral form [4] as follows

$$\int_\Gamma U_{ij}(P,Q)\{\mathcal{T}\}_j(Q)\,d\Gamma - \int_\Gamma T_{ij}(P,Q)\{\mathcal{U}\}_j(Q)\,d\Gamma + \int_\Omega U_{ij}(P,Q)b_j(Q)d\Omega = \begin{cases} \{\mathcal{U}\}_i(P) \text{ if } P \in \Omega \\ \dfrac{1}{2}\{\mathcal{U}\}_i(P) \text{ if } P \in \Gamma \end{cases} \tag{2.25}$$

for $i, j = 1, 2$, where U_{ij} and T_{ij} denote the fundamental displacements and traction for the two-dimensional isotropic linear elasticity [4] and they are given by

$$
\begin{aligned}
U_{ij}(P,Q) &= \frac{1}{8\pi G(1-\nu)}\left[\delta_{ij}(4\nu-3)\ln r + \frac{r_{,i}r_{,j}}{r^2}\right] \\
T_{ij}(P,Q) &= \frac{1}{4\pi(1-\nu)r}\left[\frac{\partial r}{\partial n}\left\{(2\nu-1)\delta_{ij} - \frac{2r_{,i}r_{,j}}{r^2}\right\} + (2\nu-1)\frac{n_i r_{,j} - n_j r_{,i}}{r}\right]
\end{aligned}
\tag{2.26}
$$

where r is the distance between the source point P and field point Q, $n = (n_1, n_2)$ denotes the outer normal vector to Γ, $r_{,i} = Q_i - P_i$.

Boundary element method for elasticity equations

The boundary integral equations (2.25) are solved using boundary element method with constant boundary elements. The boundary is divided into N constant elements. Thus the distribution of the displacements and traction are taken constant on each element and equal to their value at the nodal point, which lies at the midpoint of the element. Denoting by $\{\mathcal{U}\}^i = \{u^i, v^i\}^T$ and $\{\mathcal{T}\}^i = \{t^i, s^i\}^T$ the displacements and traction at the i^{th} node and taking into account that the boundary is smooth at the nodal point of the constant element. Then, the discretized form of Eq. (2.25) can be written as

$$
\frac{1}{2}\{\mathcal{U}\}^i + \sum_{j=1}^{N} \hat{H}^{ij}\{\mathcal{U}\}^j = \sum_{j=1}^{N} G^{ij}\{\mathcal{T}\}^j + \mathcal{F} \tag{2.27}
$$

where G^{ij} and $\hat{H}^{ij}$ are 2×2 matrices such that

$$
(G^{ij})_{lm} = \int_{\Gamma_j} U_{lm}(P^i, Q)\, d\Gamma(Q) \text{ and } (\hat{H}^{ij})_{lm} = \int_{\Gamma_j} T_{lm}(P^i, Q)\, d\Gamma(Q) \text{ for } l, m = 1, 2. \tag{2.28}
$$

Eq. (2.27) relates the displacements of the i^{th} node to the displacements and the traction of all the nodes including the i^{th} node. Applying this equation to all the boundary nodal points yields $2N$ equations, which can be set in matrix form as $H\mathcal{U} = G\mathcal{T} + \mathcal{F}$ where $H = \hat{H} + \frac{1}{2}I$ and I is $2N \times 2N$ identity matrix. The dimension of matrices $\hat{H}$ and G is $2N \times 2N$, and those the vectors $\mathcal{U}$ and $\mathcal{T}$ is $2N$. They are defined as $G = (G^{ij})_{1\le i,j\le N}$, $\hat{H} = (\hat{H}^{ij})_{1\le i,j\le N}$ and $\mathcal{U} = \big(\{\mathcal{U}\}^1, \cdots, \{\mathcal{U}\}^N\big)$, $\mathcal{T} = \big(\{\mathcal{T}\}^1, \cdots, \{\mathcal{T}\}^N\big)$. The $2N$ equations within the matrix Eq. (2.27) contain $4N$ boundary values, that is $2N$ values of displacements and $2N$ values of traction. However, a total of $2N$ values are known from the boundary conditions. Consequently, Eq. (2.27) can be used to determine the $2N$ unknown boundary values.

It should be noted that rearrangement of the unknowns is necessary for mixed boundary conditions. After doing so the following system of $2N$ linear equations is obtained $\mathcal{A}\mathcal{X} = \mathcal{R} + \mathcal{F}$ where $\mathcal{A}$ is a square coefficient matrix having dimensions $2N \times 2N$, $\mathcal{R}$

is a vector resulting as the sum of the columns of the matrices G and H multiplied by the respective known boundary values. The main inconvenient with integral formulations is that its time consuming. To make the proposed method more appealing, effort must be devoted to the implementation of algorithm (2.9)-(2.10) with (2.12) or (2.24) and in particular to solving large dense linear systems efficiently. In general, the following factors are considered in choosing an implementation technique :
1. The accuracy of the calculation, or the quality of the approximations ;
2. The computational effort involved, or the efficiency of the method ; and
3. The ease-of-implementation

We have tested iterative Krylov methods, method based on LU factorization and the exploitation of the nature of algorithm (2.9)-(2.10).

2.1.5 Implementation

The resulting systems of equations, when the boundary element discretization is applied to the reconstructing algorithm (2.9)-(2.10), will be of the form

$$\mathcal{A}_1^k \mathcal{X}^{2k} = \mathcal{B}_1^k \tag{2.29}$$

$$\mathcal{A}_2^k \mathcal{X}^{2k+1} = \mathcal{B}_2^k \tag{2.30}$$

where $\mathcal{A}_i^k$ and $\mathcal{B}_i^k$, $i = 1, 2$ are constructed from the discreet form of (2.9) and (2.10) by boundary element method. Note that, from Eq. (2.28), $\mathcal{A}_1^k$ and $\mathcal{A}_2^k$ are geometry dependent matrices and depend on the type of the boundary conditions, but not on their values. Therefore $\mathcal{A}_1^k = \mathcal{A}_1^0$ and $\mathcal{A}_2^k = \mathcal{A}_2^0$. These matrices can have the factored form : $\mathcal{A}_1^0 = \mathcal{L}_1^0\mathcal{R}_1^0$, $\mathcal{A}_2^0 = \mathcal{L}_2^0\mathcal{R}_2^0$ where $\mathcal{L}_1^0$, $\mathcal{L}_2^0$ are lower triangular matrices and $\mathcal{R}_1^0$, $\mathcal{R}_2^0$ are upper triangular matrices. Now from (2.29) and (2.30), $\mathcal{X}^{2k}$ and $\mathcal{X}^{2k+1}$ can be obtained by backward followed by forward substitutions requiring only $4N^2$ operation. This gives arise to the following algorithm :

Algorithm 2.1.1
1. set $k = 0$ and choose the initial estimate (ω^0, z^0)
2. Compute $\mathcal{H}$ and $\mathcal{G}$
3. Compute $\mathcal{A}_1^0$, $\mathcal{B}_1^0$, $\mathcal{A}_2^0$ and $\mathcal{B}_2^0$
4. Compute $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{R}_1$ and $\mathcal{R}_2$
5. Solve systems (2.29) and (2.30) replacing $\mathcal{A}_1^0$ and $\mathcal{A}_2^0$ by their factorized forms
6. until convergence do
7. $k = k + 1$, compute ω^k, z^k and $\mathcal{B}_1^k$
8. replace $\mathcal{A}_1^k$ by $\mathcal{L}_1\mathcal{R}_1$ and solve (2.29)
9. compute $\mathcal{B}_2^k$, replace $\mathcal{A}_2^k$ by $\mathcal{L}_2\mathcal{R}_2$ and solve (2.30)
10. End do.

The efficiency of the proposed method is illustrated for different orders of system. The efficiency is compared based on the CPU times carried out using the Gaussian elimination method at each iteration.

Due to the properties of matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ (are nonsingular and non-symmetric), one might consider solving $\mathcal{A}\mathcal{X} = \mathcal{B}$ by applying CG to the normal equations $\mathcal{A}^T\mathcal{A}\mathcal{X} = \mathcal{A}^T\mathcal{B}$. This approach is called CGNR [15].

Alternatively, one could solve $\mathcal{A}\mathcal{A}^T\mathcal{Y} = \mathcal{B}$ and then set $\mathcal{X} = \mathcal{A}^T\mathcal{Y}$ to solve $\mathcal{A}\mathcal{X} = \mathcal{B}$. This approach is now called CGNE [15].

There are three disadvantages that may or may not be serious. The first is that the condition number of the coefficient matrix $\mathcal{A}^T\mathcal{A}$ is the square of that of $\mathcal{A}$. The second is that two matrix-vector products are needed for each CG iterate. The third, more important, disadvantage is that one must compute the action of $\mathcal{A}^T$ on a vector as part of the matrix-vector product involving $\mathcal{A}^T\mathcal{A}$.

We used the bi-conjugate gradient stabilized (BICGSTAB) [42], which was developed to have the same convergence rate as the conjugate gradient squared (CGS) [39] at its best, without having the same difficulties (irregular convergence behavior). An advantage of BI-CGSTAB over other Krylov method such as the generalized minimal residual method (GMRES)[38] is that it has limited computation and storage requirement in each iteration step. Comparison of these methods can be found in [35, 41].

An alternative to Algorithm 2.1.1 is an implementation of the reconstructing algorithm (2.9) and (2.10) where all the linear system are solved by an iterative solver for each iteration. The proposed method summarized in the next algorithm :

Algorithm 2.1.2

1. set $k = 0$ and choose the initial estimate (ω^0, z^0) and a tolerance for the iterative solver
2. Compute $\mathcal{H}$ and $\mathcal{G}$
3. Compute $\mathcal{A}_1^0$, $\mathcal{B}_1^0$, $\mathcal{A}_2^0$ and $\mathcal{B}_2^0$
4. Solve systems (2.29) and (2.30) using an iterative solver
5. $k = k + 1$
6. compute w^k, z^k, $\mathcal{B}_1^k$
7. solve $\mathcal{A}_1^0\mathcal{X}^{2k} = \mathcal{B}_1^k$ using an iterative solver
8. compute $\mathcal{B}_2^k$ and solve $\mathcal{A}_2^0\mathcal{X}^{2k+1} = \mathcal{B}_2^k$ using an iterative solver
9. repeat steps 5-8 until convergence
10. End do.

The following stopping criterion can be used for iterative solvers of linear systems

$$\frac{\|\mathcal{B} - \mathcal{A}\mathcal{X}\|_2}{\|\mathcal{B}\|_2} < \alpha \tag{2.31}$$

Note that to compare Algorithm 2.1.2 with Algorithm 2.1.1, we take the same tolerance α for all k but this is not necessary for the convergence of Algorithm 2.1.2. Note also that Algorithm 2.1.2 converges when α is not very small.

2.1.6 Numerical results

In order to illustrate the performance of the numerical method described above, we solve the inverse elasticity problem (2.5), (2.6) and (2.8) in two-dimensional annular domain Ω given by

$$\Omega = \left\{(x,y) \in \mathbb{R}^2,\, 1 < x^2 + y^2 < 16\right\}.$$

We assume that the boundary is split into two parts

$$\Gamma_1 = \left\{(x,y) \in \mathbb{R}^2,\, x^2 + y^2 = 16\right\} \text{ and } \Gamma_2 = \left\{(x,y) \in \mathbb{R}^2,\, x^2 + y^2 = 1\right\}.$$

The exact solution of the direct problem is given by

$$\begin{aligned} u(x,y) &= \frac{1}{2G}\left(V(1-2\nu)x - W\frac{x}{x^2+y^2}\right) \\ v(x,y) &= \frac{1}{2G}\left(V(1-2\nu)y - W\frac{y}{x^2+y^2}\right) \end{aligned} \tag{2.32}$$

where $V = -\dfrac{16\sigma_0 - \sigma_1}{15}$ and $W = \dfrac{16(\sigma_0 - \sigma_1)}{15}$ and stress tensor is given by

$$\begin{aligned} &\sigma_{11}(x,y) = V + W\frac{x^2-y^2}{(x^2+y^2)^2}, \quad \sigma_{22}(x,y) = V - W\frac{x^2-y^2}{(x^2+y^2)^2} \\ &\text{and } \sigma_{12}(x,y) = 2W\frac{xy}{(x^2+y^2)^2} \end{aligned} \tag{2.33}$$

with $\sigma_1 = 10^{10}$, $\sigma_0 = 2 \times 10^{10}$, $G = 3.35 \times 10^{10}$ and $\nu = 0.22$.

The auxiliary problems (2.9) and (2.10) corresponding to this example are discretized by boundary element method using a piecewise constant polynomial interpolation. The number of boundary elements used for the discretization of the boundary Γ is taken to be $N \in \{80, 160, 320, 640, 1280\}$. We denote by $\|\cdot\|_{0,\Gamma_2}$ the discreet L^2 norm defined on Γ_2.

The convergence of the algorithm may be investigated by evaluating at every iteration the error

$$G_k^u = \|u^{2k} - u^{an}\|_{0,\Gamma_2},\; G_k^v = \|v^{2k} - v^{an}\|_{0,\Gamma_2} \tag{2.34}$$

where u^{2k} and v^{2k} are the numerical displacements on the boundary Γ_2 obtained after k iterations and u^{an}, v^{an} are the exact displacements of the problem given by (2.32). In a similar way we may evaluate the errors in retrieving the traction on the boundary Γ_2 given by

$$G_k^t = \|t^{2k+1} - t^{an}\|_{0,\Gamma_2},\; G_k^s = \|s^{2k+1} - s^{an}\|_{0,\Gamma_2} \tag{2.35}$$

The behavior of the method is investigated by evaluation the difference between two consecutive approximations for the displacements solutions and its traction on the boundary Γ_2 given by

$$E_k^u = \|u^{2k} - u^{2k-2}\|_{0,\Gamma_2}^2/N_2,\; E_k^v = \|v^{2k} - v^{2k-2}\|_{0,\Gamma_2}^2/N_2 \tag{2.36}$$

TAB. 2.1 – CPU time for fixed parameter, dynamically estimated parameter by (2.13) and (2.23) using Algorithm 2.1.1.

N	CPU_1	CPU_2	CPU_3	k_1	k_2	k_3
80	0.07199003	0.05799199	0.06399098	18	6	6
160	0.48792499	0.44793200	0.47592701	16	6	6
320	14.6957663	13.953879	14.0178692	15	6	5
640	224.173124	122.282414	122.070447	14	6	5
1280	1386.6820	907.992332	913.368518	13	6	5

$$E_k^t = \|t^{2k+1} - t^{2k-1}\|^2_{0,\Gamma_2}/N_2, \; E_k^s = \|s^{2k+1} - s^{2k-1}\|^2_{0,\Gamma_2}/N_2 \tag{2.37}$$

Based on absolute errors the following stopping criterion is considered

$$\max(E_k^u, E_k^v) < \eta \tag{2.38}$$

where η is a small prescribed positive quantity. Note that (2.38) express that the sequence (u^{2k}, v^{2k}) converge in Sobolev spaces $H^{\frac{1}{2}}(\Gamma_2) \times H^{\frac{1}{2}}(\Gamma_2)$. For all numerical experiments, we take $\eta = 10^{-7}$.

This test is used to analyze the behavior with respect to accuracy and efficiency of the techniques considered in this work when applied to the approximation solution of boundary element method systems of algebraic equations.

Comparative result for various parameter relaxation

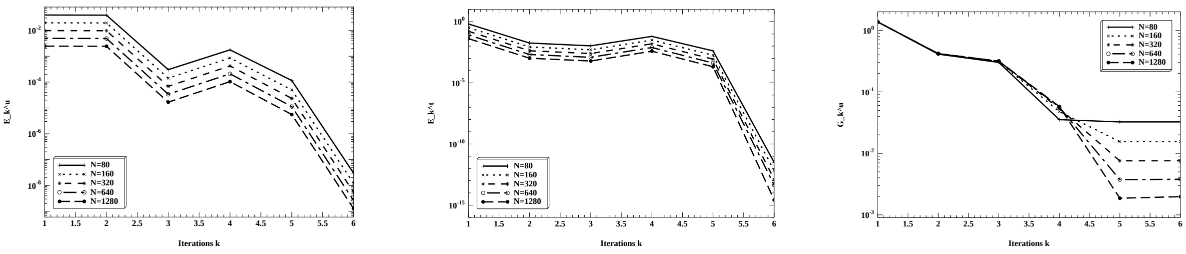
Table 1 presents results obtained based on Algorithm 2.1.1 for various number of boundary elements. We denote by k_1, k_2 and k_3 respectively the number of iterations required to achieve the convergence using $\theta_1 = \theta_2 = 1$, θ_1, θ_2 computed by (2.13) and θ_1, θ_2 computed by (2.23) respectively. We denote by CPU_1, CPU_2, CPU_3 the CPU time required for the convergence in the three cases.

We observe from Table 1 that the reconstructing algorithm is very efficient when used with the automatic adjustment of θ_1, θ_2 given by (2.13) or (2.23).

The behavior of numerical solution of the first components of the displacement and traction vectors (u and t) are similar to that of the second components (v and s respectively) and therefore they have not been presented here.

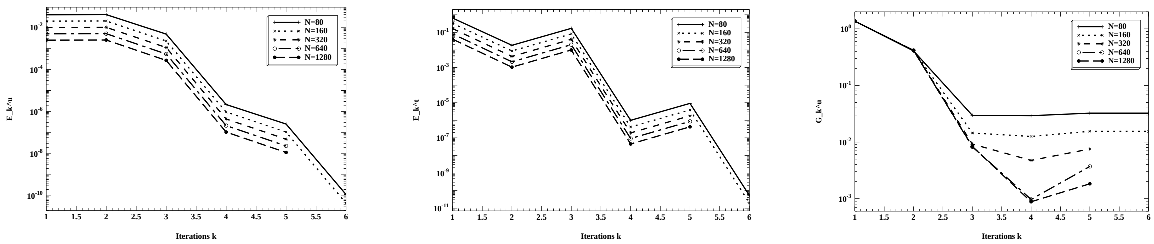
Fig.2.1(a),(b)-2.2(a),(b) show, on a semi-log scale, the corresponding successive difference E_k^u and E_k^t for automatic selection of θ_1, θ_2 given by (2.13) or (2.23) as functions of the number of iterations k.

Fig.2.1 (c), 2.2 (c), 2.3 (a)-(b) show, on a semi-log scale, the corresponding sequences G_k^u and G_k^t for automatic selection of θ_1, θ_2 given by (2.13) or (2.23) as functions of the number of iterations k for various number of boundary elements N. We can see easily that the quantities G_k^u and G_k^t decrease when N increases and they remain constant after



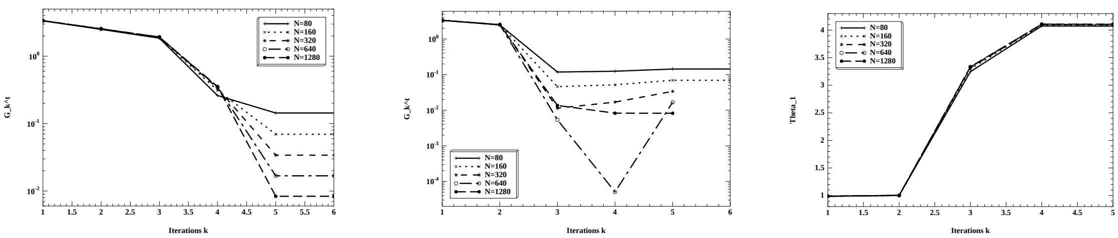
(a) (b) (c)

FIG. 2.1 – E_k^u (a), E_k^t (b) and G_k^u (c) of first dynamical choice of relaxation parameter for different N.



(a) (b) (c)

FIG. 2.2 – E_k^u (a), E_k^t (b) and G_k^u (c) of second dynamical choice of relaxation parameter for different N.



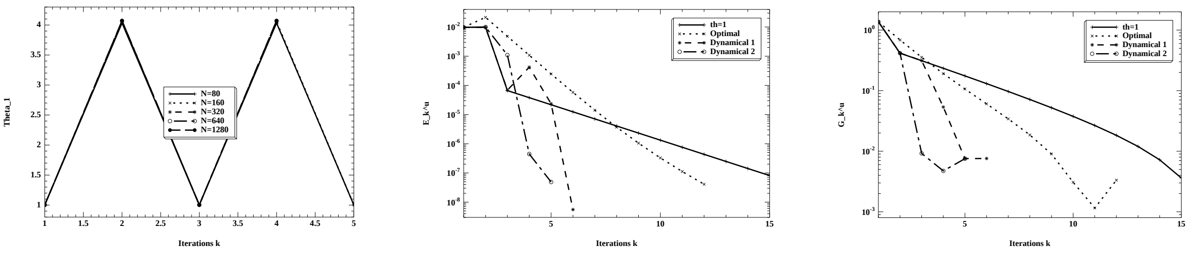
(a) (b) (c)

FIG. 2.3 – G_k^t of two dynamical choice of relaxation parameter (a), (b) for different N and variation of first dynamical estimated relaxation parameter (c) as a function of number of iteration.

a few iteration. Therefore the method are stable with respect to the number of boundary element.

According to these results, we observe a weak oscillation of E_k^u and E_k^t due to the automatic search of the relaxation parameter at each iteration, see Fig. 2.3(c), 2.4 (a). This slight instability in E_k^u and E_k^t does not affects the behavior of the errors G_k^u and G_k^t as it is illustrated in Fig. 2.1 (c), 2.2 (c), 2.3 (a) - (b).

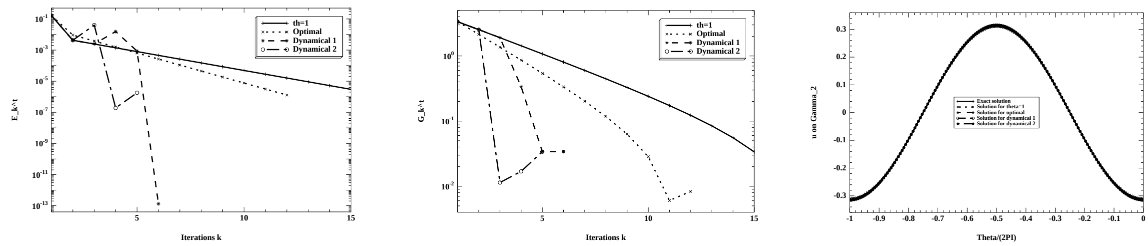
For $N = 320$, we observe from Fig.2.4 (b), (c), 2.5 (a), (b) that when the algorithm is implemented with the dynamically estimated relaxation parameter given by (2.13) or



(a) (b) (c)

FIG. 2.4 – Variation of second dynamical estimated relaxation parameter (a) as a function of number of iteration and E_k^u (b), G_k^u (c) of four relaxation parameter.

(2.23), the prescribed criterion (2.36) is satisfied after 6 iterations for automatic selection given by (2.13), after 5 iterations for automatic selection given by (2.23), while it is required 15 iterations for $\theta_1 = \theta_2 = 1$. With the fixed optimal parameter the convergence is obtained after 13 iterations.



(a) (b) (c)

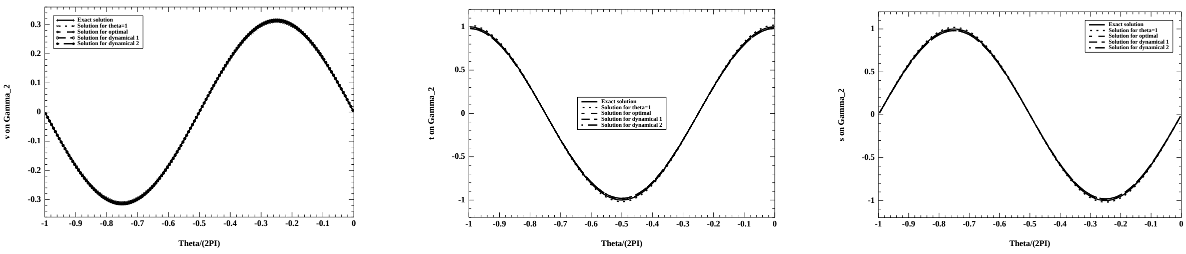
FIG. 2.5 – E_k^t (a), G_k^u (b) of four relaxation parameter and numerical, exact solution for displacement u (c) with $N = 320$.

We notice that at the convergence, the same precision is reached for the displacements solutions and traction with the different relaxation parameters, see Fig. 2.5 (c) and Fig. 2.6. These figures show the numerical solutions and their tractions obtained on the boundary Γ_2 for four relaxation parameters, namely the two dynamically estimated parameter given by (2.13) or (2.23), the fixed optimal parameter and the fixed parameter $\theta_1 = \theta_2 = 1$.

As it can be seen from Fig. 2.5 (c) - 2.6 the displacement and traction solutions corresponding to 320 of boundary elements is in good agreement with the analytical solution.

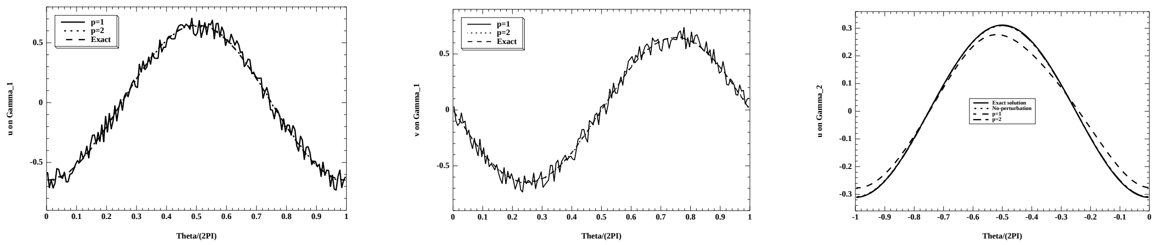
The effect of noise

We discuss the numerical algorithm stability according to disturbed displacement and tractions data. For this we add the following quantity $10^{-p}(2\,rand() - 1)$ to the given data on Γ_1, where *rand* is the FORTRAN random function.



(a) (b) (c)

FIG. 2.6 – Numerical and exact solutions for displacement v (a) and tractions t (b), s (c) for $N = 320$

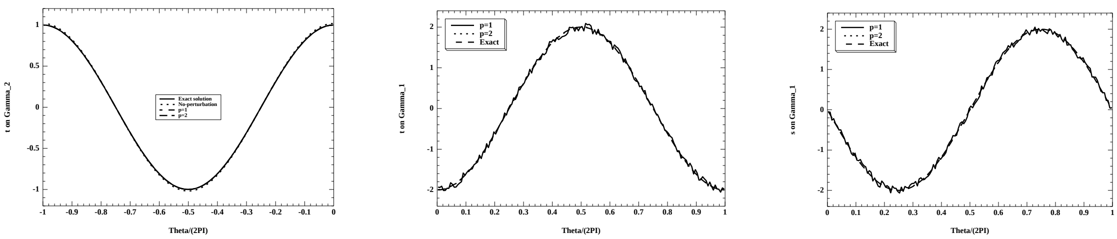


(a) (b) (c)

FIG. 2.7 – Perturbed displacements u (a), v (b) data on Γ_1 and the corresponding numerical displacement u (c) results with $N = 320$

Noisy on the displacements data

We examine the algorithm stability when u and v data are noisy. The noisy data on displacements are presented in Fig. 2.7 (a), (b) and the corresponding numerical solutions are shown in Fig. 2.7 (c) and Fig. 2.8 (a).



(a) (b) (c)

FIG. 2.8 – Numerical traction t (a) results with perturbed displacements data with $N = 320$ and perturbed tractions t (b), s (c) data on Γ_1.

N	LU	BI-CGSTAB	CGNE	CGNR	CGR	CGS
80	0.0579	0.0619(83)	0.0709(99)	0.0689(92)	0.1549(93)	0.0699(97)
160	0.4479	0.2899(75)	0.3239(101)	0.3239(94)	0.6499(77)	0.3549(98)
320	13.953	4.4043(83)	13.510(432)	7.7138(243)	13.053(89)	5.0732(93)
640	122.28	21.042(85)	85.438(611)	32.835(227)	50.697(78)	23.798(97)
1280	907.99	84.896(90)	1054.1(1964)	388.05(719)	221.24(85)	88.798(94)

TAB. 2.2 – CPU time for different solvers (first iteration) for dynamical 1

Noisy on the tractions data

We examine the algorithm stability when t and s data are noisy. The noisy data on tractions are presented in Fig. 2.8 (b), (c) and the corresponding numerical solutions are shown in Fig. 2.9.

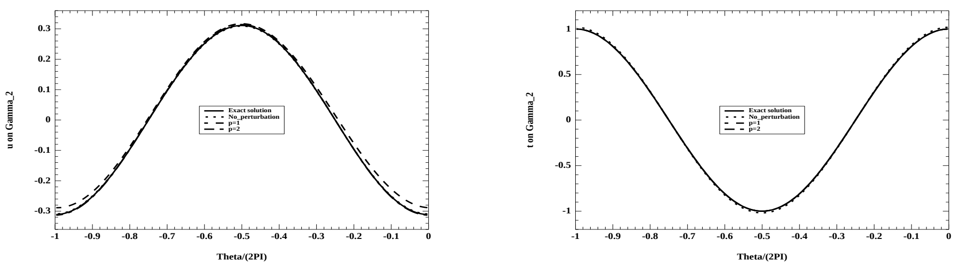


FIG. 2.9 – Numerical displacement u (left) traction t (right) results obtained for perturbed tractions data and $N = 320$.

For $p \geq 3$, the numerical results obtained with perturbed data coincide with the numerical results obtained with no perturbed data. Therefore this case is not presented here.

It can be seen that as p increases, the numerical solution better approximates the exact solution, whilst remaining stable. For $p = 1$, the obtained results are to be considered more than reasonable, keeping in mind that the problem (2.5), (2.6) and (2.8), is an ill-posed problem with a very oscillatory data.

Comparison of LU decomposition and linear iterative solvers

To compare the efficiency for different iterative solvers, five computational meshes are used to generate five systems of 160, 320, 640, 1280, 2560 linear equations.

The iterative solvers BI-CGSTAB, CGNE, CGNR, CGR and CGS, are considered.

Table 2-3 summarizes the results obtained for these solvers. The times shown are the average time in CPU times. The average number of iterative solver iterations is given in brackets. We used $\alpha = 10^{-07}$ as convergence tolerance in (2.31).

Tab. 2.3 – CPU time for different solvers (first iteration) for dynamical 2

N	LU	BI-CGSTAB	CGNE	CGNR	CGR	CGS
80	0.06	0.12(99 ;177)	0.15(98 ;163)	0.12(91 ;108)	0.28(89 ;84)	0.11(90 ;96)
160	0.47	0.61(79 ;237)	1.10(93 ;330)	0.66(93 ;146)	1.33(76 ;80)	0.64(86 ;106)
320	14.0	7.13(72 ;83)	19.1(305 ;297)	10.7(115 ;230)	18.2(72 ;58)	7.66(81 ;71)
640	122	29.2(68 ;55)	149(440 ;636)	50.4(88 ;267)	84.3(69 ;58)	36.5(77 ;75)
1280	913	129(69 ;87)	1117(718 ;1362)	448(120 ;712)	321(69 ;58)	137(73 ;61)

All the Krylov methods worked well, some, BI-CGSTAB and CGS, performed very well, yielding the solution in a small number of iterations. CGNE and CGNR revealed some difficulty in achieving a good convergence. The results show that BI-CGSTAB and CGS are more efficient than LU decomposition procedure, but BI-CGSTAB is by far the fast solver. The normalizing technique corresponding to the CGNE method is penalized due to the worsening of the condition number of their matrix relative to the condition number of the matrix of the original linear system.

2.2 A domain decomposition method for boundary element approximations of the elasticity equations

2.2.1 Introduction

Domain decomposition ideas have been applied to a wide variety of problems. We could not hope to include all these techniques in this work. For an extensive survey of recent advances, we refer to the proceedings of the annual domain decomposition meetings see. *http ://www.ddm.org.* Domain decomposition algorithms is divided into two classes, those that use overlapping domains, which refer to as Schwarz methods, and those that use non-overlapping domains, which we refer to as substructuring.

Any domain decomposition method is based on the assumption that the given computational domain Ω is decomposed into subdomains Ω_i, $i = 1, \dots, M$, which may or may not overlap. Next, the original problem can be reformulated upon each subdomain Ω_i, yielding a family of subproblems of reduced size that are coupled one to another through the values of the unknowns solution at subdomain interfaces. Fruitful references can be found in [37, 40].

A numerical study of elasticity equations by domain decomposition method combined with finite element method was treated in [9, 14, 23]. A symmetric boundary element analysis with domain decomposition is studied in [13, 36]. This combination was also used for biharmonic equation in two overlapping disks [2].

We have chosen to associate the Dirichlet-Neumann and Schwarz methods with the direct boundary element method. Indeed, it only requires the discretization of the boundaries of the subdomains. This technique of coupling reduces the number of unknowns and the time of computing. It has been used successfully for semiconductors simulation [33].

We consider a linear elasticity material which occupies an open bounded domain $\Omega \subset \mathbb{R}^2$, and assume that Ω is bounded by $\Gamma = \partial\Omega$. We also assume that the boundary consists of two parts $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are not empty and $\Gamma_1 \cap \Gamma_2 = \emptyset$ where Ω is not necessarily circular or rectangular.

Let $\mathcal{V} = (u, v)$ the displacement vector and $\mathcal{S} = (t, s)$ the traction vector governed by the following linear elasticity problem

$$\begin{cases} G\Delta u + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial x \partial y}\right) = 0 \text{ in } \Omega, \\ G\Delta v + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 v}{\partial y^2}\right) = 0 \text{ in } \Omega \\ u = \tilde{u}, \quad v = \tilde{v} \quad \text{on } \Gamma_1 \\ t = \tilde{t}, \quad s = \tilde{s} \quad \text{on } \Gamma_2 \end{cases} \tag{2.39}$$

with G and ν the shear modulus and Poisson ratio, respectively, and where $\tilde{u}$, $\tilde{v}$, $\tilde{t}$ and $\tilde{s}$

are the prescribed quantities.

2.2.2 Domain decomposition techniques

In order to use domain decomposition to linear elasticity, we describe, in this section, Dirichlet-Neumann and Schwarz methods.

Dirichlet-Neumann substructuring method

We decompose Ω into two non-overlapping subdomains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, and denote by $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ the common interface between Ω_1 and Ω_2. We can write this method as follows.

– Step 1. Specify an initial $\Lambda^0 = (\lambda^0, \beta^0)$ on interface Γ_{12} and $k = 0$.

– Step 2. Solve the mixed well-posed direct problem

$$\begin{cases} G\Delta u_1^k + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_1^k}{\partial x^2} + \dfrac{\partial^2 v_1^k}{\partial x \partial y}\right) = 0 \text{ in } \Omega_1, \\ G\Delta v_1^k + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_1^k}{\partial x \partial y} + \dfrac{\partial^2 v_1^k}{\partial y^2}\right) = 0 \text{ in } \Omega_1 \\ u_1^k = \tilde{u}, \;\; v_1^k = \tilde{v} \text{ on } \Gamma_1 \cap \partial\Omega_1, \;\; t_1^k = \tilde{t}, \;\; s_1^k = \tilde{s} \text{ on } \Gamma_2 \cap \partial\Omega_1, \\ u_1^k = \lambda^k, \;\; v_1^k = \beta^k \text{ on } \Gamma_{12} \end{cases} \tag{2.40}$$

to determine the traction $\mathcal{S}_1^k = (t_1^k, s_1^k)$ on the interface Γ_{12}.

– Step 3. Solve the mixed well-posed direct problem

$$\begin{cases} G\Delta u_2^k + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_2^k}{\partial x^2} + \dfrac{\partial^2 v_2^k}{\partial x \partial y}\right) = 0 \text{ in } \Omega_2, \\ G\Delta v_2^k + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_2^k}{\partial x \partial y} + \dfrac{\partial^2 v_2^k}{\partial y^2}\right) = 0 \text{ in } \Omega_2 \\ u_2^k = \tilde{u}, \;\; v_2^k = \tilde{v} \text{ on } \Gamma_1 \cap \partial\Omega_2, \;\; t_2^k = \tilde{t}, \;\; s_2^k = \tilde{s} \text{ on } \Gamma_2 \cap \partial\Omega_2, \\ t_2^k = -t_1^k, \;\; s_2^k = -s_1^k \quad \text{on } \Gamma_{12} \end{cases} \tag{2.41}$$

to determine the displacement $\mathcal{V}_2^k = (u_2^k, v_2^k)$ on the interface Γ_{12}.

– Step 4. Update $\Lambda^{k+1} = (\lambda^{k+1}, \beta^{k+1})$ on the interface Γ_{12} by

$$\lambda^{k+1} = \theta u_2^k + (1-\theta)\lambda^k \text{ on } \Gamma_{12}, \;\; \beta^{k+1} = \theta v_2^k + (1-\theta)\beta^k \text{ on } \Gamma_{12} \tag{2.42}$$

– Step 5. Repeat step 2 from $k \geq 0$ until a prescribed stopping criterion is satisfied. where θ is positive parameter. This algorithm establish the solution of elasticity equations of Problem 2.39 in Ω as a limit of sequence $(u_1^k, v_1^k, u_2^k, v_2^k)$.

For this algorithm the following stopping criterion is used

$$\max\left(\|\lambda^{k+1} - \lambda^k\|_{L^2(\Gamma_{12})}, \|\beta^{k+1} - \beta^k\|_{L^2(\Gamma_{12})}\right) < Tol, \tag{2.43}$$

where Tol is a prescribed tolerance.

Schwarz overlapping method

We decompose Ω into two overlapping subdomains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, and denote by $\Gamma_{11} = \partial\Omega_1 \cap \overline{\Omega_2}$ and $\Gamma_{22} = \partial\Omega_2 \cap \overline{\Omega_1}$. This method is summarized in the following.

– Step 1. Specify an initial $\mathcal{V}_2^0 = (u_2^0, v_2^0)$ on Γ_{11} and $k = 0$.

– Step 2. Solve the mixed well-posed direct problem

$$\begin{cases} G\Delta u_1^{k+1} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_1^{k+1}}{\partial x^2} + \dfrac{\partial^2 v_1^{k+1}}{\partial x \partial y}\right) = 0 \text{ in } \Omega_1, \\ G\Delta v_1^{k+1} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_1^{k+1}}{\partial x \partial y} + \dfrac{\partial^2 v_1^{k+1}}{\partial y^2}\right) = 0 \text{ in } \Omega_1 \\ u_1^{k+1} = \tilde{u}, \;\; v_1^{k+1} = \tilde{v} \text{ on } \Gamma_1 \cap \partial\Omega_1, \;\; t_1^{k+1} = \tilde{t}, \;\; s_1^{k+1} = \tilde{s} \text{ on } \Gamma_2 \cap \partial\Omega_1, \\ u_1^{k+1} = u_2^k, \;\; v_1^{k+1} = v_2^k \text{ on } \Gamma_{11} \end{cases} \tag{2.44}$$

to determine the displacement $\mathcal{V}_1^{k+1} = (u_1^{k+1}, v_1^{k+1})$ and traction $\mathcal{S}_1^{k+1} = (t_1^{k+1}, s_1^{k+1})$ on the boundary of Ω_1.

– Step 3. Compute the displacement $\mathcal{V}_1^{k+1} = (u_1^{k+1}, v_1^{k+1})$ on Γ_{22} as an internal displacement of linear elasticity equations in Ω_1.

– Step 4. Solve the mixed well-posed direct problem then

$$\begin{cases} G\Delta u_2^{k+1} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_2^{k+1}}{\partial x^2} + \dfrac{\partial^2 v_2^{k+1}}{\partial x \partial y}\right) = 0 \text{ in } \Omega_2, \\ G\Delta v_2^{k+1} + \dfrac{G}{1-2\nu}\left(\dfrac{\partial^2 u_2^{k+1}}{\partial x \partial y} + \dfrac{\partial^2 v_2^{k+1}}{\partial y^2}\right) = 0 \text{ in } \Omega_2 \\ u_2^{k+1} = \tilde{u}, \;\; v_2^{k+1} = \tilde{v} \text{ on } \Gamma_1 \cap \partial\Omega_2, \;\; t_2^{k+1} = \tilde{t}, \;\; s_2^{k+1} = \tilde{s} \text{ on } \Gamma_2 \cap \partial\Omega_2, \\ u_2^{k+1} = u_1^{k+1}, \;\; v_2^{k+1} = v_1^{k+1} \text{ on } \Gamma_{22} \end{cases} \tag{2.45}$$

to determine the displacement $\mathcal{V}_2^{k+1} = (u_2^{k+1}, v_2^{k+1})$ and traction $\mathcal{S}_2^{k+1} = (t_2^{k+1}, s_2^{k+1})$ on the boundary of Ω_2.

– Step 5. Compute the displacement $\mathcal{V}_2^{k+1} = (u_2^{k+1}, v_2^{k+1})$ on Γ_{11} as an internal displacement of linear elasticity equations in Ω_2.

– Step 6. Repeat step 2 from $k \geq 0$ until a prescribed stopping criterion is satisfied.

For this algorithm the following stopping criterion is used

$$\max\left(\|u_1^{k+1} - u_1^k\|_{L^2(\Gamma_{11})}, \|v_1^{k+1} - v_1^k\|_{L^2(\Gamma_{11})}, \|u_2^{k+1} - u_2^k\|_{L^2(\Gamma_{22})}, \|v_2^{k+1} - v_2^k\|_{L^2(\Gamma_{22})}\right) < Tol, \tag{2.46}$$

where Tol is a prescribed tolerance.

The boundary element method utilizes information on the boundaries of interest, and thus reduces the dimension of the problem by one. The displacements in the domain is uniquely defined by the displacements and tractions on the boundary. In the boundary element method, only the boundary is discretized ; hence, the mesh generation is considerably simpler for this method than for space discretization techniques, such as the finite difference method or finite element method . Moreover, the Boundary element method determines simultaneously the boundary displacements and tractions, this allows us to solve problem (2.40), (2.41) without the need of further finite difference, as one would employ if using the finite element method or the finite difference method.

For these reasons we have decided in this study to use the boundary element method in order to implement the Dirichlet-Neumann and Schwarz methods.

2.2.3 Integral equation formulation and boundary element for elasticity equations

The linear elasticity problem (2.39) in two-dimensional case can be formulated in integral form [4] as follows

$$\int_\Gamma U_{ij}(P,Q)\{\mathcal{S}\}_j(Q)\,d\Gamma(Q) - \int_\Gamma T_{ij}(P,Q)\{\mathcal{V}\}_j(Q)\,d\Gamma(Q) = \begin{cases} \{\mathcal{V}\}_i(P) \text{ if } P \in \Omega \\ \frac{1}{2}\{\mathcal{V}\}_i(P) \text{ if } P \in \Gamma \end{cases} \tag{2.47}$$

for $i, j = 1, 2$, where U_{ij} and T_{ij} denote the fundamental displacements and tractions for the two-dimensional isotropic linear elasticity [4]. The boundary integral equations are solved using boundary element method with constant boundary elements. The boundary is divided into N constant elements. Denoting by $\{\mathcal{V}\}^i = \{u^i, v^i\}^T$ and $\{\mathcal{S}\}^i = \{t^i, s^i\}^T$ the displacements and tractions at the i^{th} node. Then, the discretized form of Eq. (2.47) can be written as $\frac{1}{2}\{\mathcal{V}\}^i + \sum_{j=1}^{N} \hat{H}^{ij}\{\mathcal{V}\}^j = \sum_{j=1}^{N} G^{ij}\{\mathcal{S}\}^j$ where G^{ij} and $\hat{H}^{ij}$ are 2×2 matrices such that $(G^{ij})_{lm} = \int_{\Gamma_j} U_{lm}(P^i, Q)\,d\Gamma(Q)$ and $(\hat{H}^{ij})_{lm} = \int_{\Gamma_j} T_{lm}(P^i, Q)\,d\Gamma(Q)$ for $l, m = 1, 2$.

Applying this equation to all the boundary nodal points yields $2N$ equations, which can be set in matrix form as

$$H\,\mathcal{V} = G\,\mathcal{S} \tag{2.48}$$

where $H = \hat{H} + \frac{1}{2}I$ and I is $2N \times 2N$ identity matrix. The displacements in the interior of Ω can be evaluated using Eq. (2.47) which after discretization becomes

$$\{\mathcal{V}\}^i = \sum_{j=1}^{N} G^{ij}\{\mathcal{S}\}^j - \sum_{j=1}^{N} \hat{H}^{ij}\{\mathcal{V}\}^j \tag{2.49}$$

2.2.4 Algebraic systems of Dirichlet Neumann and Schwarz methods

We consider in this work the mixed boundary condition given by Problem (2.40), (2.41), (2.44) and (2.45). In this case the rearrangement of the unknowns in Eq. (2.48) is necessary. In order to obtain an algebraic system, we denote the matrices H_i and G_i computed in each subdomain Ω_i by the use of Dirichlet Neumann or Schwarz method. Note that H_i and G_i are geometry dependent matrices and depend on the type of the boundary conditions, but not on their values. Therefore the matrices H_i and G_i do not change during the iterate procedure of domain decomposition method. We suppose that the boundary $\Gamma_j \cap \partial\Omega_i$ is divided into N_j constant elements for $i, j = 1, 2$.

Alternating algebraic system of Dirichlet-Neumann method

Let the boundary Γ_{12} divided into N_{12} constant elements. Due to the boundary condition of system (2.40) and (2.41), the matrices H_i and G_i are decomposed as follows

$$H_i = (H_{\Gamma_1\cap\partial\Omega_i}\ H_{\Gamma_2\cap\partial\Omega_i}\ H_{\Gamma_{12}}) \text{ and } G_i = (G_{\Gamma_1\cap\partial\Omega_i}\ G_{\Gamma_2\cap\partial\Omega_i}\ G_{\Gamma_{12}}) \tag{2.50}$$

The algebraic systems corresponding to subproblems (2.40) and (2.41) take the form

$$\begin{cases} (H_{\Gamma_1\cap\partial\Omega_1}\ H_{\Gamma_2\cap\partial\Omega_1}\ H_{\Gamma_{12}}) \begin{pmatrix} \mathcal{V}_1^k|_{\Gamma_1\cap\partial\Omega_1} \\ \mathcal{V}_1^k|_{\Gamma_2\cap\partial\Omega_1} \\ \mathcal{V}_1^k|_{\Gamma_{12}} \end{pmatrix} = (G_{\Gamma_1\cap\partial\Omega_1}\ G_{\Gamma_2\cap\partial\Omega_1}\ G_{\Gamma_{12}}) \begin{pmatrix} \mathcal{S}_1^k|_{\Gamma_1\cap\partial\Omega_1} \\ \mathcal{S}_1^k|_{\Gamma_2\cap\partial\Omega_1} \\ \mathcal{S}_1^k|_{\Gamma_{12}} \end{pmatrix} \\ \mathcal{V}_1^k|_{\Gamma_1\cap\partial\Omega_1} = \tilde{V}_1,\ \mathcal{S}_1^k|_{\Gamma_2\cap\partial\Omega_1} = \tilde{S}_1,\ \mathcal{V}_1^k|_{\Gamma_{12}} = \Lambda^k \end{cases} \tag{2.51}$$

and

$$\begin{cases} (H_{\Gamma_1\cap\partial\Omega_2}\ H_{\Gamma_2\cap\partial\Omega_2}\ H_{\Gamma_{12}}) \begin{pmatrix} \mathcal{V}_2^k|_{\Gamma_1\cap\partial\Omega_2} \\ \mathcal{V}_2^k|_{\Gamma_2\cap\partial\Omega_2} \\ \mathcal{V}_2^k|_{\Gamma_{12}} \end{pmatrix} = (G_{\Gamma_1\cap\partial\Omega_2}\ G_{\Gamma_2\cap\partial\Omega_2}\ G_{\Gamma_{12}}) \begin{pmatrix} \mathcal{S}_2^k|_{\Gamma_1\cap\partial\Omega_2} \\ \mathcal{S}_2^k|_{\Gamma_2\cap\partial\Omega_2} \\ \mathcal{S}_2^k|_{\Gamma_{12}} \end{pmatrix} \\ \mathcal{V}_2^k|_{\Gamma_1\cap\partial\Omega_2} = \tilde{V}_2,\ \mathcal{S}_2^k|_{\Gamma_2\cap\partial\Omega_2} = \tilde{S}_2,\ \mathcal{S}_2^k|_{\Gamma_{12}} = -\mathcal{S}^k{}_{1|_{\Gamma_{12}}}. \end{cases} \tag{2.52}$$

The actualization of Λ^k is given by

$$\Lambda^{k+1} = \theta\mathcal{V}^k{}_{2|_{\Gamma_{12}}} + (1-\theta)\Lambda^k. \tag{2.53}$$

Let X_1^k and X_2^k be the vectors containing the unknowns values of displacements or tractions on the boundary of subdomains Ω_1 and Ω_2 respectively. They are given by

$$X_1^k = \begin{pmatrix} \mathcal{S}_1^k|_{\Gamma_1\cap\partial\Omega_1} \\ \mathcal{V}_1^k|_{\Gamma_2\cap\partial\Omega_1} \\ \mathcal{S}_1^k|_{\Gamma_{12}} \end{pmatrix} \text{ and } X_2^k = \begin{pmatrix} \mathcal{S}_2^k|_{\Gamma_1\cap\partial\Omega_2} \\ \mathcal{V}_2^k|_{\Gamma_2\cap\partial\Omega_2} \\ \mathcal{V}_2^k|_{\Gamma_{12}} \end{pmatrix}. \tag{2.54}$$

The matrices A_1 and A_2 are defined by the following

$$A_1 = (-G_{\Gamma_1\cap\partial\Omega_1}\ H_{\Gamma_2\cap\partial\Omega_1}\ -G_{\Gamma_{12}}) \text{ and } A_2 = (-G_{\Gamma_1\cap\partial\Omega_2}\ H_{\Gamma_2\cap\partial\Omega_2}\ H_{\Gamma_{12}}). \tag{2.55}$$

Then the algebraic system of Dirichlet-Neumann associate to problem (2.40) and (2.41) is written in the following

$$\begin{aligned} A_1 X_1^k &= -H_{\Gamma_1\cap\partial\Omega_1}\tilde{V}_1 + G_{\Gamma_2\cap\partial\Omega_1}\tilde{S}_1 - H_{\Gamma_{12}}\Lambda^k, \\ A_2 X_2^k &= -H_{\Gamma_1\cap\partial\Omega_2}\tilde{V}_2 + G_{\Gamma_2\cap\partial\Omega_2}\tilde{S}_2 - G_{\Gamma_{12}} X_1^k{}_{|\Gamma_{12}} \end{aligned} \tag{2.56}$$

and

$$\Lambda^{k+1} = \theta X_2^k{}_{|\Gamma_{12}} + (1-\theta)\Lambda^k. \tag{2.57}$$

For simplification, let

$$B_1^k = -H_{\Gamma_1\cap\partial\Omega_1}\tilde{V}_1 + G_{\Gamma_2\cap\partial\Omega_1}\tilde{S}_1 - H_{\Gamma_{12}}\Lambda^k \tag{2.58}$$

$$B_2^k = -H_{\Gamma_1\cap\partial\Omega_2}\tilde{V}_2 + G_{\Gamma_2\cap\partial\Omega_2}\tilde{S}_2 - G_{\Gamma_{12}} X^k{}_{1|\Gamma_{12}}. \tag{2.59}$$

The matrices A_1 and A_2 can be factorized in the following $A_1 = \mathcal{L}_1\mathcal{R}_1$ and $A_2 = \mathcal{L}_2\mathcal{R}_2$ where $\mathcal{L}_1$, $\mathcal{L}_2$ are lower triangular matrices and $\mathcal{R}_1$, $\mathcal{R}_2$ are upper triangular matrices. Now from (2.56) X_1^k and X_2^k can be obtained by backward followed by forward substitutions. This gives arise to the following algorithm :

Algorithm 2.2.1

1. *Set $k = 0$, choose the initial $\Lambda^0 = (\lambda^0, \beta^0) \in \mathbb{R}^{2N_{12}}$ and a tolerance for the iterative solver*
2. *Compute H_i and G_i for subdomains Ω_i for $i = 1, 2$*
3. *Compute A_i using Eq. (2.55) for $i = 1, 2$*
4. *Compute $\mathcal{L}_i$ and $\mathcal{R}_i$ (decomposition of A_i) for $i = 1, 2$*
5. *Repeat*
   – *Compute the vector containing known boundary values B_1^k using Eq. (2.58)*
   – *Solve system $\mathcal{L}_1\mathcal{R}_1 X_1^k = B_1^k$*
   – *Compute the vector containing known boundary values B_2^k using Eq. (2.59)*
   – *Solve $\mathcal{L}_2\mathcal{R}_2 X_2^k = B_2^k$*
   – *Update $\Lambda^k = (\lambda^k, \beta^k)$ by formula (2.57)*
   – $k = k + 1$
   Until convergence.
6. *End.*

Alternating algebraic system of Schwarz method

Let the boundary Γ_{ii} divided into N_{ii} constant elements for $i = 1, 2$. The matrices H_i and G_i associated to the system (2.44) and (2.45), can be decomposed as follows

$$H_i = (H_{\Gamma_1\cap\partial\Omega_i}\ H_{\Gamma_2\cap\partial\Omega_i}\ H_{\Gamma_{ii}}) \text{ and } G_i = (G_{\Gamma_1\cap\partial\Omega_i}\ G_{\Gamma_2\cap\partial\Omega_i}\ G_{\Gamma_{ii}}) \tag{2.60}$$

In order to compute the internal displacements in Ω_i by Eq. (2.49), we introduce the matrix $\mathcal{I}_i$ which take the form

$$\mathcal{I}_i = (-H_{\Omega_i} \quad G_{\Omega_i}). \tag{2.61}$$

The algebraic systems obtained from boundary element discretisation of subproblems (2.44) and (2.45) take the form

$$\begin{cases} (H_{\Gamma_1\cap\partial\Omega_1}\ H_{\Gamma_2\cap\partial\Omega_1}\ H_{\Gamma_{11}}) \begin{pmatrix} \mathcal{V}_1^{k+1}{}_{|\Gamma_1\cap\partial\Omega_1} \\ \mathcal{V}_1^{k+1}{}_{|\Gamma_2\cap\partial\Omega_1} \\ \mathcal{V}_1^{k+1}{}_{|\Gamma_{11}} \end{pmatrix} = (G_{\Gamma_1\cap\partial\Omega_1}\ G_{\Gamma_2\cap\partial\Omega_1}\ G_{\Gamma_{11}}) \begin{pmatrix} \mathcal{S}_1^{k+1}{}_{|\Gamma_1\cap\partial\Omega_1} \\ \mathcal{S}_1^{k+1}{}_{|\Gamma_2\cap\partial\Omega_1} \\ \mathcal{S}_1^{k+1}{}_{|\Gamma_{11}} \end{pmatrix} \\ \mathcal{V}_1^{k+1}{}_{|\Gamma_1\cap\partial\Omega_1} = \tilde{V}_1,\ \mathcal{S}_1^{k+1}{}_{|\Gamma_2\cap\partial\Omega_1} = \tilde{S}_1,\ \mathcal{V}_1^{k+1}{}_{|\Gamma_{11}} = \mathcal{V}^k_{2|\Gamma_{11}}, \end{cases} \tag{2.62}$$

$$\mathcal{V}_1^{k+1}{}_{|\Gamma_{22}} = \mathcal{I}_1 \begin{pmatrix} \mathcal{V}_1^{k+1}{}_{|\partial\Omega_1} \\ \mathcal{S}_1^{k+1}{}_{|\partial\Omega_1} \end{pmatrix} \tag{2.63}$$

and

$$\begin{cases} (H_{\Gamma_1\cap\partial\Omega_2}\ H_{\Gamma_2\cap\partial\Omega_2}\ H_{\Gamma_{22}}) \begin{pmatrix} \mathcal{V}_2^{k+1}{}_{|\Gamma_1\cap\partial\Omega_2} \\ \mathcal{V}_2^{k+1}{}_{2|\Gamma_2\cap\partial\Omega_2} \\ \mathcal{V}_2^{k+1}{}_{2|\Gamma_{22}} \end{pmatrix} = (G_{\Gamma_1\cap\partial\Omega_2}\ G_{\Gamma_2\cap\partial\Omega_2}\ G_{\Gamma_{22}}) \begin{pmatrix} \mathcal{S}_2^{k+1}{}_{|\Gamma_1\cap\partial\Omega_2} \\ \mathcal{S}_2^{k+1}{}_{|\Gamma_2\cap\partial\Omega_2} \\ \mathcal{S}_2^{k+1}{}_{|\Gamma_{22}} \end{pmatrix} \\ \mathcal{V}_2^{k+1}{}_{|\Gamma_1\cap\partial\Omega_2} = \tilde{V}_2,\ \mathcal{S}_2^{k+1}{}_{|\Gamma_2\cap\partial\Omega_2} = \tilde{S}_2,\ \mathcal{S}_2^{k+1}{}_{|\Gamma_{22}} = \mathcal{V}_1^{k+1}{}_{|\Gamma_{22}}, \end{cases} \tag{2.64}$$

$$\mathcal{V}_2^{k+1}{}_{|\Gamma_{11}} = \mathcal{I}_2 \begin{pmatrix} \mathcal{V}_2^{k+1}{}_{|\partial\Omega_2} \\ \mathcal{S}_2^{k+1}{}_{|\partial\Omega_2} \end{pmatrix}. \tag{2.65}$$

Let X_i^{k+1}, the vectors containing the unknowns values of displacements or tractions on the boundary of subdomains Ω_i for $i = 1, 2$, have the following form

$$X_i^{k+1} = \begin{pmatrix} \mathcal{S}_i^{k+1}{}_{|\Gamma_1\cap\partial\Omega_i} \\ \mathcal{V}_i^{k+1}{}_{|\Gamma_2\cap\partial\Omega_i} \\ \mathcal{S}_i^{k+1}{}_{|\Gamma_{ii}} \end{pmatrix} \tag{2.66}$$

The matrices A_1 and A_2 are defined for $i = 1, 2$ by the following

$$A_i = (-G_{\Gamma_1\cap\partial\Omega_i}\ H_{\Gamma_2\cap\partial\Omega_i}\ -G_{\Gamma_{ii}}) \tag{2.67}$$

Then the algebraic system of Schwarz method associate to problem (2.44) and (2.45) is written in the following

$$A_1 X_1^{k+1} = B_1^k,\ A_2 X_2^{k+1} = B_2^{k+1} \tag{2.68}$$

where

$$B_1^k = -H_{\Gamma_1\cap\partial\Omega_1}\tilde{V}_1 + G_{\Gamma_2\cap\partial\Omega_1}\tilde{S}_1 - H_{\Gamma_{11}}\mathcal{V}^k_{2|\Gamma_{11}} \tag{2.69}$$

$$B_2^{k+1} = -H_{\Gamma_1 \cap \partial\Omega_2} \tilde{V}_2 + G_{\Gamma_2 \cap \partial\Omega_2} \tilde{S}_2 - H_{\Gamma_{22}} \mathcal{V}_1^{k+1}{}_{|\Gamma_{11}}. \quad (2.70)$$

The matrices A_1 and A_2 can be factorized in the following $A_1 = \mathcal{L}_1 \mathcal{R}_1$ and $A_2 = \mathcal{L}_2 \mathcal{R}_2$ where $\mathcal{L}_1$, $\mathcal{L}_2$ are lower triangular matrices and $\mathcal{R}_1$, $\mathcal{R}_2$ are upper triangular matrices. Now from (2.68) X_1^{k+1} and X_2^{k+1} can be obtained by backward followed by forward substitutions. This gives arise to the following algorithm :

Algorithm 2.2.2

1. *Set $k = 0$, choose the initial $\mathcal{V}_2^0 \in \mathbb{R}^{2N_{11}}$ given and a tolerance for the iterative solver*
2. *Compute H_i and G_i for subdomains Ω_i for $i = 1, 2$*
3. *Compute A_i using Eq. (2.67) for $i = 1, 2$*
4. *Compute $\mathcal{I}_i$ using Eq. (2.61) for $i = 1, 2$*
5. *Compute $\mathcal{L}_i$ and $\mathcal{R}_i$ (decomposition of A_i) for $i = 1, 2$*
6. *Repeat*
   - *Compute the vector containing known boundary values B_1^k using Eq. (2.69)*
   - *Solve system $\mathcal{L}_1 \mathcal{R}_1 X_1^{k+1} = B_1^k$*
   - *Compute internal displacement in subdomain Ω_1 using Eq. (2.63)*
   - *Compute the vector containing known boundary values B_2^{k+1} using Eq. (2.70)*
   - *Solve $\mathcal{L}_2 \mathcal{R}_2 X_2^{k+1} = B_2^{k+1}$*
   - *Compute internal displacement in subdomain Ω_2 using Eq. (2.65)*
   - *$k = k + 1$*

   Until convergence.
7. *End.*

2.2.5 Numerical results and discussions

In this section, we illustrate the numerical results obtained using the Dirichlet-Neumann and Schwarz domain decomposition method combined with boundary element method for linear elasticity problem. The comparison of this two domain decomposition method is done in L-shaped domain.

The behavior of the method is investigated evaluating the difference between two consecutive approximations for the displacements solutions and its tractions on the boundary γ given by

$$\begin{aligned} E_k^i(u) = \|u_i^{k+1} - u_i^k\|_{L^2(\gamma)},\ E_k^i(v) = \|v_i^{k+1} - v_i^k\|_{L^2(\gamma)}, \\ E_k^i(t) = \|t_i^{k+1} - t_i^k\|_{L^2(\gamma)},\ E_k^i(s) = \|s_i^{k+1} - s_i^k\|_{L^2(\gamma)}. \end{aligned} \quad (2.71)$$

Based on absolute errors the following stopping criterion is considered for Algorithm 2.2.2

$$\max(E_k^i(u), E_k^i(v)) < \eta. \quad (2.72)$$

The stopping criterion for Algorithm 2.2.1 is

$$\max(E_k(\lambda), E_k(\beta)) < \eta \tag{2.73}$$

where

$$E_k(\lambda) = \|\lambda^{k+1} - \lambda^k\|_{L^2(\gamma)},\ E_k(\beta) = \|\beta^{k+1} - \beta^k\|_{L^2(\gamma)} \tag{2.74}$$

where η is a small prescribed positive quantity.

In order to investigate the convergence of the two algorithm, at every iteration we evaluate the accuracy errors defined by

$$\begin{aligned} G_u^i(k) = \|u_i - u_i^{an}\|_{L^2(\gamma)},\ G_v^i(k) = \|v_i - v_i^{an}\|_{L^2(\gamma)}, \\ G_t^i(k) = \|t_i - t_i^{an}\|_{L^2(\gamma)},\ G_s^i(k) = \|s_i - s_i^{an}\|_{L^2(\gamma)}. \end{aligned} \tag{2.75}$$

Note that (2.72) or (2.73) express that the sequence (u^k, v^k) converge in sobolev spaces $H^{\frac{1}{2}}(\gamma) \times H^{\frac{1}{2}}(\gamma)$. For all numerical experiments, we take $\eta = 10^{-7}$. Note that we have $\gamma = \Gamma_{12}$ for Algorithm 2.2.1 and for Algorithm 2.2.2 $\gamma_i = \Gamma_{ii}$, $i = 1, 2$.

Example 1

In order to illustrate the performance of the numerical method described above, we solve the linear elasticity problem (2.39), in two-dimensional L-shaped domain $\Omega = (0,1)\times(0,0.5) \cup (0,0.5) \times (0,1)$. We assume that the boundary is split into two parts $\Gamma_1 = [0,1] \times \{0\} \cup [\frac{1}{2}, 1] \times \left\{\frac{1}{2}\right\} \cup [0, \frac{1}{2}] \times \{1\}$ and $\Gamma_2 = \{1\} \times [0, \frac{1}{2}] \cup \left\{\frac{1}{2}\right\} \times [\frac{1}{2}, 1] \cup \{0\} \times [0, 1]$. The exact solution of the direct problem is given by

$$u(x, y) = \frac{1-\nu}{2G}\sigma_0 x,\ v(x, y) = -\frac{\nu}{2G}\sigma_0 y,\ t(x, y) = \sigma_0 n_1,\ s(x, y) = 0 \tag{2.76}$$

with $\sigma_0 = 1.5 \times 10^{10}$, $G = 3.35 \times 10^{10}$ and $\nu = 0.25$.

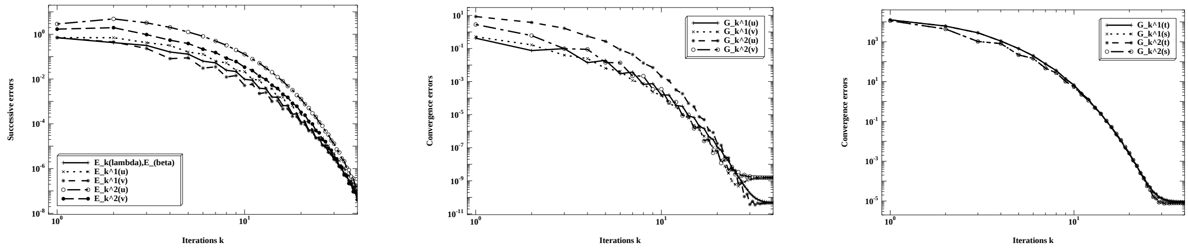
This example consists in spliting the domain Ω into two rectangular subdomains $\Omega_1 = (0.5, 1) \times (0, 0.5)$ and $\Omega_2 = (0, 0.5) \times (0, 1)$ with interface $\gamma = \{0.5\} \times [0, 0.5]$.

The evolution of behavior errors as a function of the iteration number using Algorithm 2.2.1 is plotted in Fig. 2.10(a).

Fig. 2.10(a)-(b) shows that the accurate convergence as a function of the iteration number using Algorithm 2.2.1 decreases when the iteration number increases.

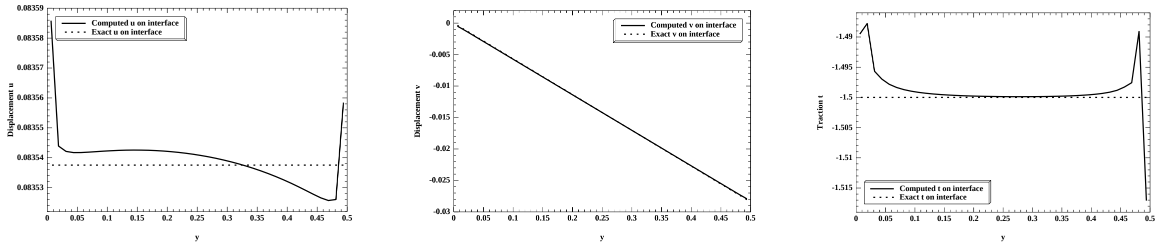
In Fig. 2.11(a)-(b), we have plotted the exact and computed displacements as a function of $y \in [0, 0.5]$ using Algorithm 2.2.1. The discrepancy is about 5×10^{-5} near to the corner.

We can observe in Fig. 2.11(c) and Fig. 2.12(a) , where the exact and computed tractions are plotted as a function of $y \in [0, 0.5]$ using Algorithm 2.2.1. The discrepancy is about 2.5×10^{-2} near to the corner.



(a) (b) (c)

FIG. 2.10 – The behavior errors given by (2.71), (2.73) and the accuracy errors given by (2.75) as a function of the number of iterations k on interface γ for Example 1.

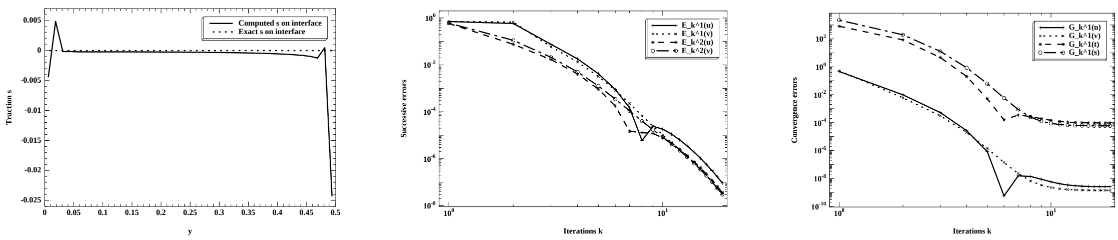


(a) (b) (c)

FIG. 2.11 – Computed and Analytical u, v, t on interface γ for example 1.

Example 2

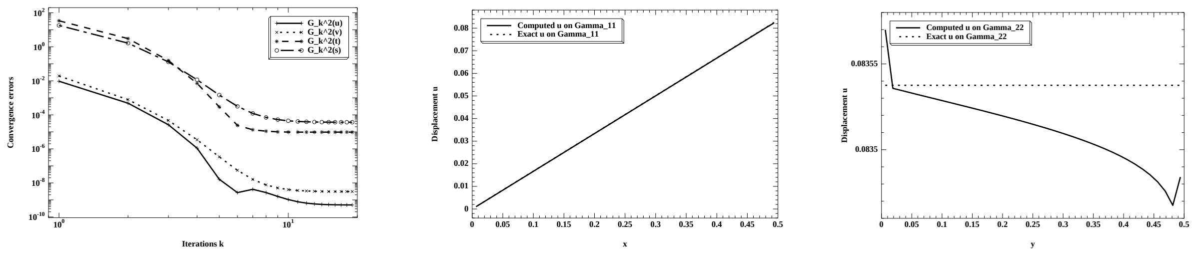
This example deals with the same exact solution as in Eq. (2.76). This example consists in spliting the domain Ω into two overlap rectangular subdomains $\Omega_1 = (0,1) \times (0,0.5)$ and $\Omega_2 = (0,0.5) \times (0,1)$ with overlap is $(0.,0.5) \times (0,0.5)$.



(a) (b) (c)

FIG. 2.12 – Computed, analytical s on interface γ for Example 1; the behavior errors given by (2.71) and (2.72) and the accuracy errors given by (2.75) as a function of the number of iterations k on γ_1 for example 2.

In Fig. 2.12, we observe the convergence of calculated solution to exact solution as a function of the iteration number by the use of Algorithm 2.2.2.



(a) (b) (c)

FIG. 2.13 – The accuracy errors given by (2.75) as a function of the number of iterations k on part of boundaries γ_2 and computed, analytical u on γ_1 and γ_2 for example 2.

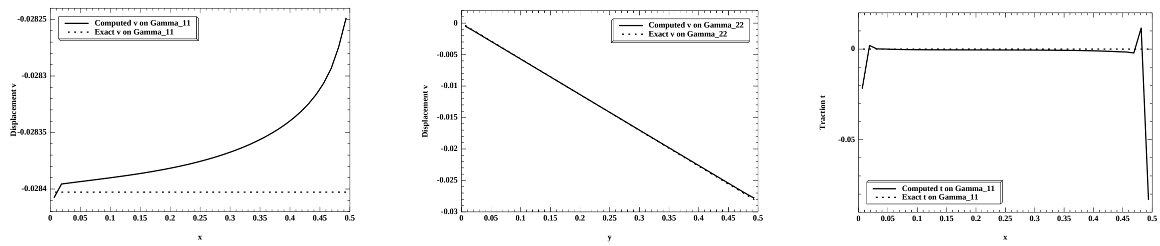


FIG. 2.14 – Computed, analytical v on γ_1, γ_2 and t on γ_1 for example 2.

The conclusions drawn from Fig. 2.12 are graphically enhanced in Figs. 2.13- 2.15 which show the numerical results obtained using Algorithm 2.2.2 in comparison with the analytical solutions.

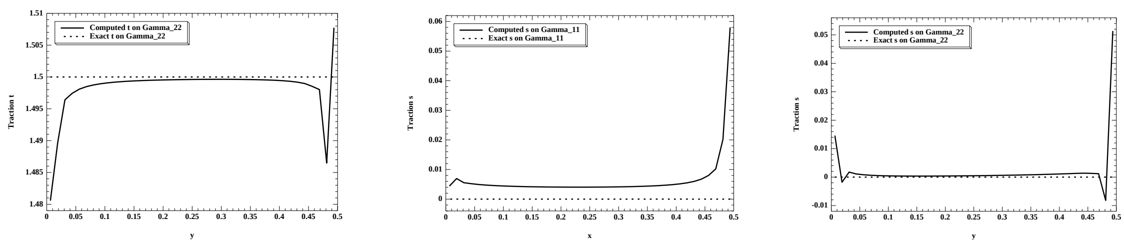


FIG. 2.15 – Computed and Analytical t on γ_2 and s on γ_2 and γ_1 for example 2.

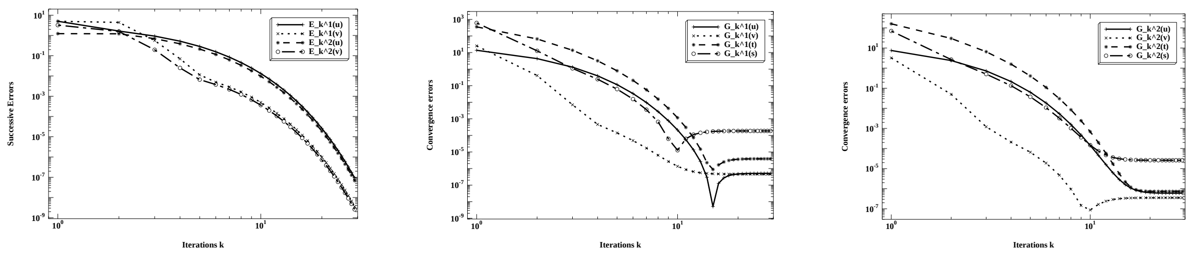
Comparing Algorithm 2.2.1 and Algorithm 2.2.2 to solve linear elasticity problem in L-shaped domain, we can see from Figs. 2.11 and 2.15 that Algorithm 2.2.2 requires much less iterations than Algorithm 2.2.1. The computed solutions are accurate and consistent with respect to increasing the iteration number k.

Example 3

In this example, we consider the union of two circle geometry domain Ω. This example consists in spliting the domain Ω into two overlap circular subdomains $\Omega_1 = \{(x, y) \in \mathbb{R}^2/(x-0.5)^2 + y^2 < 0.25\}$ and $\Omega_2 = \{(x, y) \in \mathbb{R}^2/(x - 0.5(1 + \sqrt{2}))^2 + y^2 < 0.25\}$ with overlap is $\Omega_1 \cap \Omega_2$. In order to illustrate the performance of the numerical method described above, we solve the linear elasticity problem (2.39), in two-circular domain Ω. The exact solution of the direct problem is given by.

$$u(x, y) = \frac{1-2\nu}{2G}\sigma_0 x,\ v(x, y) = \frac{1-2\nu}{2G}\sigma_0 y,\ t(x, y) = \sigma_0 n_1,\ s(x, y) = \sigma_0 n_2 \qquad (2.77)$$

with $\sigma_0 = 1.5 \times 10^{10}$, $G = 3.35 \times 10^{10}$ and $\nu = 0.25$.

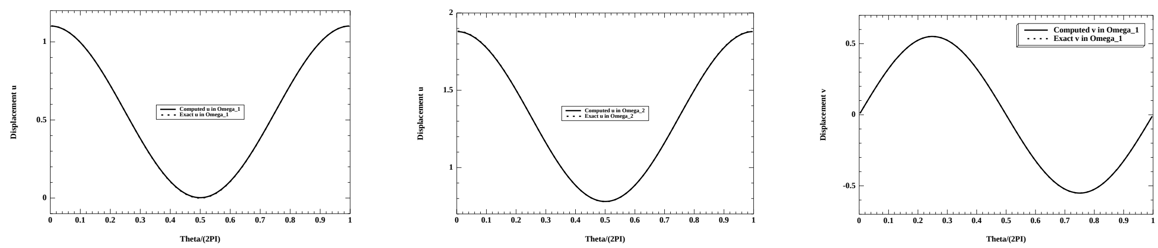


(a) (b) (c)

FIG. 2.16 – The behavior errors given by(2.71), (2.72) and the accuracy errors given by (2.75) as a function of the number of iterations k on part of boundaries of Ω_1 and Ω_2 respectively, for example 3.

As a function of the iteration k, four behavior errors are illustrated in Fig. 2.16(a) using Algorithm 2.2.2.

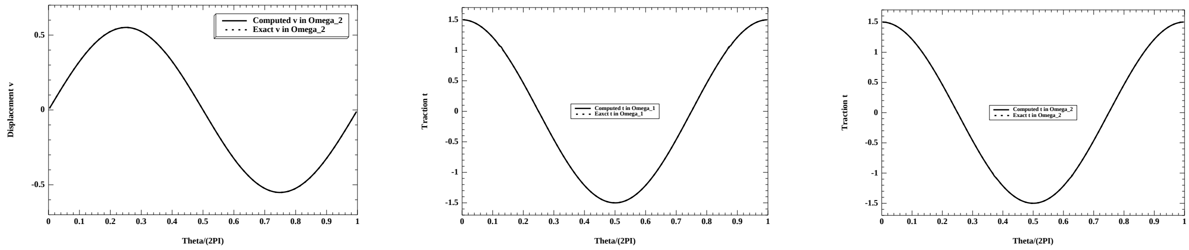
In Fig. 2.16(b)-(c), we observe the convergence of calculated solution to exact solution as a function of the iteration number by the use of Algorithm 2.2.2.



(a) (b) (c)

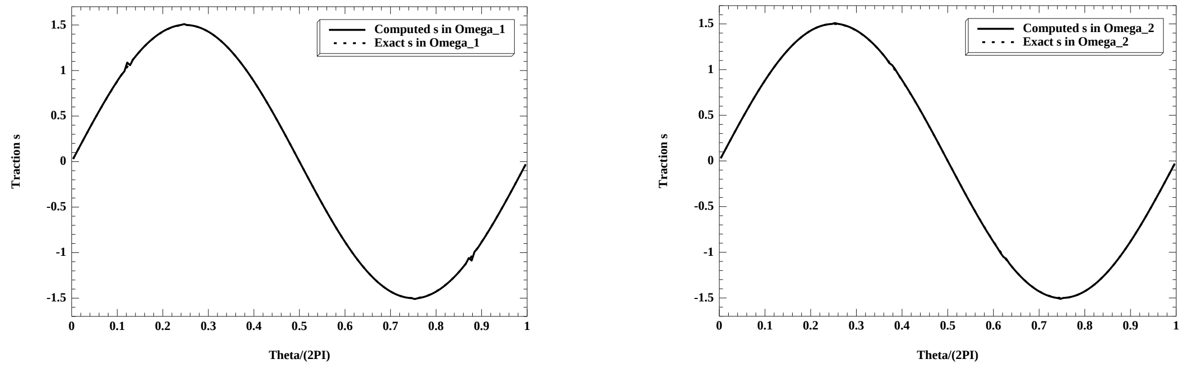
FIG. 2.17 – Computed and Analytical u in Ω_1, Ω_2 and v in Ω_1 for example 3.

The conclusions drawn from Fig. 2.16 are graphically enhanced in Figs. 2.17- 2.19 which show the numerical results obtained using Algorithm 2.2.2 in comparison with the analytical solutions.



(a) (b) (c)

FIG. 2.18 – Computed and Analytical v in Ω_2 and t in Ω_1, Ω_2 for example 3.



(a) (b)

FIG. 2.19 – Computed and Analytical s in Ω_1, Ω_2 for example 3.

2.3 A domain decomposition convergence for elasticity equations

2.3.1 Introduction

Domain decomposition methods is divided into two classes, those that use overlapping domain, and those that use non-overlapping domains, which we refer to as substructuring. Various substructuring methods with non-overlapping can be encountered in literature and fruitful references can be found from [37].

A study of elasticity equations by domain decomposition method was treated from [9, 14, 23, 31]. In [31], the authors have presented the techniques for the algebraic approximation of Dirichlet to Neumann maps for linear elasticity. This techniques are based on the local condensation of the degree of freedom belonging to a small area-defined inside the sub-domain on a small patch defined on the interface. In [23], the domain decomposition method with Lagrange multipliers is introduced by reformulating the preconditioned system of the FETI algorithm as a saddle point problem with both primal and dual variables as unknowns.

We consider a linear elasticity material which occupies an open bounded domain $\Omega \subset \mathbb{R}^2$ where the boundary is denoted by $\Gamma = \partial\Omega$. The linear elasticity problem [28] is given, for $i = 1, 2$, by

$$\begin{cases} -\displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(u)}{\partial\, x_j} = f_i & \text{in} \quad \Omega \\ u_i = 0 & \text{on} \quad \Gamma. \end{cases} \tag{2.78}$$

where $u = (u_1, u_2)$ is the displacement vector, $f = (f_1, f_2)$ the volume force vector, σ_{ij} is the stress tensor. The traction vector t is defined by for $i = 1, 2$, $t_i = \displaystyle\sum_{j=1}^{2} \sigma_{ij}(u) n_j$ where n is the outward normal unitary vector of the domain Ω along boundary Γ. The strain tensor ε_{ij} is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\, u_i}{\partial\, x_j} + \frac{\partial\, u_j}{\partial\, x_i}\right) \tag{2.79}$$

These tensors are related by

$$\sigma_{ij}(u) = 2G\left(\varepsilon_{ij}(u) + \frac{\nu}{1-2\nu}\sum_{k=1}^{2} \varepsilon_{kk}(u)\delta_{ij}\right) \tag{2.80}$$

with G and ν are the shear modulus and Poisson ratio, respectively, and δ_{ij} is the Kronecker delta tensor.

We wish to determine the solution of (2.78) by a domain decomposition method. To this end and for simplicity we consider here only the case where Ω is partitioned into two open subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ such that $\overline{\Omega} = \overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}}$. The interface between two domains is denoted γ so that $\gamma = \overline{\Omega^{(1)}} \cap \overline{\Omega^{(2)}}$. Let $\Gamma_1 = \overline{\Omega^{(1)}} \cap \Gamma$ and $\Gamma_2 = \overline{\Omega^{(2)}} \cap \Gamma$. Let us denote by $f_i^{(k)} = f_i|_{\Omega^{(k)}}$, for $k = 1, 2$. We consider the problems defined over the subdomains

$$\left\{\begin{aligned} -\sum_{j=1}^{2} \frac{\partial\,\sigma_{ij}(u^{(1)})}{\partial\,x_j} &= f_i^{(1)} \text{ in } \Omega^{(1)} \\ u_i^{(1)} &= 0 \quad \text{ on } \Gamma_1 \\ \sum_{j=1}^{2} \sigma_{ij}(u^{(1)}) n_j^{(1)} &= \psi_i \text{ on } \gamma \end{aligned}\right. \tag{2.81}$$

$$\left\{\begin{aligned} -\sum_{j=1}^{2} \frac{\partial\,\sigma_{ij}(u^{(2)})}{\partial\,x_j} &= f_i^{(2)} \text{ in } \Omega^{(2)} \\ u_i^{(2)} &= 0 \quad \text{ on } \Gamma_2 \\ \sum_{j=1}^{2} \sigma_{ij}(u^{(2)}) n_j^{(2)} &= -\psi_i \text{ on } \gamma \end{aligned}\right. \tag{2.82}$$

where $n^{(i)}$ is the outward normal unitary vector of the subdomain $\Omega^{(i)}$ along the interface γ, for $i = 1, 2$.

In this work, we are interesting to combine the optimization techniques and non-overlapping domain decomposition to solve problem (2.78). This combination is obtained as a constrained minimization problem for which the cost functional is the $L^2(\gamma)$-norm of the difference between the dependent variables $u^{(1)}$, $u^{(2)}$ across the common boundaries γ and the constraints are the problems (2.81) and (2.82). At this stage its must be noted that a similar idea of this combination was already used for Laplace operator in [17, 18], for coupled stokes flows [25], for nonlinear sedimentary basin problem [24]. Here, we extend this idea for the study of elasticity equations. Furthermore, we prove the convergence of approximate optimal solutions to continuous one and we give an algorithm based on gradient conjugate with variable steeps.

2.3.2 Optimal control formulation

Define the following convex set :

$$K_0 = \{\psi = (\psi_1, \psi_2) \in (L^2(\gamma))^2 \ / \ \ \|\psi_k\|_{L^2(\gamma)} \leq C_0, \quad \text{for } k = 1, 2\}$$

where C_0 is a nonnegative given constant.

For the numerical approximation of the problem (2.78), we propose the following optimal control formulation

$$(PO)\left\{\begin{array}{l} \text{Minimize}\, J(u^{(1)}(\psi), u^{(2)}(\psi)) \quad \text{for all } \psi \in K_0 \\ \text{where } \ J(u^{(1)}(\psi), u^{(2)}(\psi)) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{2}\int_{\gamma} \left(u_i^{(1)} - u_i^{(2)}\right)^2 d\sigma \\ \text{and } u^{(1)}(\psi),\, u^{(2)}(\psi) \text{ are respectively the solution of (2.81) and (2.82).} \end{array}\right. \tag{2.83}$$

We have the following result

Proposition 2.3.1 *Assume that f and Ω are smooth enough. Then the problem (2.78) is equivalent to (2.83).*

2.3.3 Existence of optimal solution

We first give some notations and definitions which can be useful in the following. We define the spaces, for $i = 1, 2$,

$$H_{i,D}(\Omega^{(i)}) = \{\upsilon \in (H^1(\Omega^{(i)}))^2 \ / \ \upsilon|_{\Gamma_i} = 0\}$$

where $(H^1(\Omega^{(i)}))^2$ is the Sobolev space equipped with the norm $\|\cdot\|_{1,\Omega^{(i)}}$ defined by

$$\|\upsilon\|_{1,\Omega^{(i)}} = \left(\sum_{l=1}^{2}\left(\|\upsilon_l\|^2_{0,\Omega^{(i)}} + \|\nabla\upsilon_l\|^2_{0,\Omega^{(i)}}\right)\right)^{\frac{1}{2}}, \ \|\upsilon_l\|_{0,\Omega^{(i)}} = \left(\int_{\Omega^{(i)}} |\upsilon_l|^2 \ dx\right)^{\frac{1}{2}}.$$

$H_{i,D}(\Omega^{(i)})$ are equipped with the following norm $|\upsilon|_{1,\Omega^{(i)}} = \left(\sum_{l=1}^{2} \|\nabla\upsilon_l\|^2_{0,\Omega^{(i)}}\right)^{\frac{1}{2}}$.

For $\psi \in K_0$, we consider the weak formulation of equation (2.81) and (2.82) given, for $k = 1, 2$, by

$$\left\{\begin{array}{l} \text{Find } u^{(k)}(\psi) \in H_{k,D}(\Omega^{(k)}) \quad \forall \upsilon = (\upsilon_1, \upsilon_2) \in H_{k,D}(\Omega^{(k)}) \\ a^{(k)}(u^{(k)}, \upsilon) = \displaystyle\sum_{i,j=1}^{2} \int_{\Omega^{(k)}} \sigma_{ij}(u^{(k)})\, \varepsilon_{ij}(\upsilon) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_i^{(k)}\, \upsilon_i \, dx + (-1)^k \sum_{i=1}^{2} \int_{\gamma} \psi_i \, \upsilon_i \, d\sigma. \end{array}\right. \tag{2.84}$$

We define the space of admissible solutions U_{ad} by :

$$U_{ad} = \{(u^{(1)}(\psi), u^{(2)}(\psi)) \text{ solution of } (2.84) \ / \ \psi \in K_0\}.$$

The optimal control problem (2.83) can be rewritten as :

$$(PO) \quad \text{Minimize } J((u^{(1)}(\psi), u^{(2)}(\psi)) \quad \text{for all } (u^{(1)}(\psi), u^{(2)}(\psi)) \in U_{ad}.$$

We define the convergence of the sequence $(\psi_n)_n = ((\psi_{1,n}, \psi_{2,n}))_n$ in K_0 to $\psi = (\psi_1, \psi_2) \in K_0$ by

$$\psi_n \longrightarrow \psi \iff \psi_{k,n} \rightharpoonup \psi_k \text{ weakly in } L^2(\gamma), \quad \text{for } k = 1, 2. \tag{2.85}$$

We can then equip U_{ad} with the topology defined by the following convergence : let $((u_n^{(1)}, u_n^{(2)}))_n$ be a sequence of U_{ad} and $(u^{(1)}, u^{(2)}) \in U_{ad}$ then :

$$(u_n^{(1)}, u_n^{(2)}) \longrightarrow (u^{(1)}, u^{(2)}) \iff \left\{\begin{array}{l} u_{k,n}^{(1)} \rightharpoonup u_k^{(1)} \text{ weakly in } H^1(\Omega^{(1)}) \\ u_{k,n}^{(2)} \rightharpoonup u_k^{(2)} \text{ weakly in } H^1(\Omega^{(2)}), \quad \text{for } k = 1, 2. \end{array}\right. \tag{2.86}$$

We have then the following result.

Theorem 2.3.1 *The problem (PO) is well posed and admits a solution in U_{ad}.*

2.3.4 Approximation of the problem

In this section, we use the linear finite element method for the approximation of (PO). We show the existence of the solution of the discrete problem and we study the convergence of a subsequence of these solutions to a solution of the continuous problem. Finally, to confirm the convergence result, we give some numerical results.
For the seek of simplicity, we reduce our study, in this section, to the case where the boundary part γ is assumed to be defined as follows :

$$\gamma = \{(b, x) \ / \ x \in [0, a]\} \tag{2.87}$$

where $a > 0$ and b are two given constants.

In the following, we need additional regularity assumptions on K_0, namely :

$$\begin{aligned} K_0 = \{\psi = (\psi_1, \psi_2) \in (C^0(\gamma))^2 \ / \ |\psi_k(b, x) - \psi_k(b, x')| \leq C\,|x - x'| \\ \forall x, x' \in [0, a] \ \text{ and } \|\psi_k\|_{L^\infty(\gamma)} \leq C_0 \quad \text{for} \quad k = 1, 2\} \end{aligned} \tag{2.88}$$

where C and C_0 are nonnegative given constants. The convergence of a sequence $(\psi_n)_n = ((\psi_{1,n}, \psi_{2,n}))_n$ in K_0 to $\psi = (\psi_1, \psi_2) \in K_0$ is defined in this case by

$$\psi_n \longrightarrow \psi \Longleftrightarrow \psi_{k,n}(b, .) \longrightarrow \psi_k(b, .) \text{ uniformly in } [0, a]\,, \text{ for } k = 1, 2 \tag{2.89}$$

Discretization of the problem

Let us consider an uniform partition $(a_i)_{i=0}^{N-1}$ of the interval $[0, a]$, such that :

$$0 = a_0 < a_1 < \ldots < a_{N-1} = a, \quad a_i - a_{i-1} = h \ \text{ for } \ i = 1, \ldots, N-1.$$

We define the discrete space associated to K_0 by

$$\begin{aligned} K_0^h = \{\psi_h = (\psi_{1,h}, \psi_{2,h}) \in (C(\gamma))^2 \ / \ \psi_{k,h}(b, .)|_{[a_{i-1}, a_i]} \in P_1([a_{i-1}, a_i]) \\ i = 1, \ldots, N-1, \ |\tfrac{\psi_{k,h}(b, a_i) - \psi_{k,h}(b, a_{i-1})}{a_i - a_{i-1}}| \leq C, \ i = 1, \ldots, N-1 \\ \text{and } \|\psi_{k,h}\|_{L^\infty(\gamma)} \leq \frac{C}{2}\, h + C_0, \ \text{for } k = 1, 2\} \end{aligned}$$

with the same constants C and C_0, as in the definition of K_0.
Let $H(\Omega)$ be the finite dimensional space given by

$$H(\Omega^{(k)}) = \{\upsilon_h \in \mathcal{C}(\overline{\Omega^{(k)}}) \ / \ \upsilon_h|_K \in P_1(K), \ \forall K \in \mathcal{T}_h\}$$

where $\mathcal{T}_h$ is a regular triangulation of $\overline{\Omega^{(k)}}$, for $k = 1, 2$. Let

$$H_{k,D}^h(\Omega^{(k)}) = \{\upsilon_h \in (H(\Omega^{(k)}))^2 \ / \ \upsilon_h|_{\Gamma_k} = 0\}$$

be the finite dimensional spaces associated respectively to $H_{k,D}(\Omega^{(k)})$.
For $\psi_h \in K_0^h$, we consider the following discrete problem of (2.83), for $k = 1, 2$:

$$\left\{ \begin{array}{l} \text{Find } u_h^{(k)}(\psi_h) \in H^h_{k,D}(\Omega^{(k)}) \ \forall \upsilon_h \in H^k_{k,D}(\Omega^{(k)}) \\ a_h^{(k)}(u_h^{(k)}, \upsilon_h) = \displaystyle\sum_{i,j=1}^{2} \int_{\Omega^{(k)}} \sigma_{ij}(u_h^{(k)})\, \varepsilon_{ij}(\upsilon_h) = \sum_{i=1}^{2} \int_{\Omega^{(k)}} f_{i,h}^{(k)}\, \upsilon_{i,h}\, dx + (-1)^k \sum_{i=1}^{2} \int_\gamma \psi_{i,h}\, \upsilon_{i,h}\, d\sigma \end{array} \right. \tag{2.90}$$

where $f_{i,h}^{(k)}$ is an approximation of $f_i^{(k)}$ such that

$$f_{i,h}^{(k)} \text{ is uniformly bounded and converges to } f_i^{(k)} \text{ almost every where.} \tag{2.91}$$

The discrete space of the admissible solutions is given by

$$U_{ad}^h = \{(u_h^{(1)}(\psi_h), u_h^{(2)}(\psi_h)) \text{ solution of } (2.90) \ / \ \psi_h \in K_0^h\}$$

We approach the cost functional by the following discrete one :

$$J_h(u_h^{(1)}(\psi_h), u_h^{(2)}(\psi_h)) = \frac{1}{2} \sum_{i=1}^{2} \int_\gamma \Big(u_{i,h}^{(1)}(\psi_h) - u_{i,h}^{(2)}(\psi_h) \Big)^2 \, d\sigma,$$

and we state our discrete optimization problem as follows

$$(PO^h) \left\{ \begin{array}{l} \displaystyle\inf_{(u_h^{(1)}, u_h^{(2)}) \in U_{ad}^h} J_h(u_h^{(1)}, u_h^{(2)}) \\ \text{where } u_h^{(k)} = u_h^{(k)}(\psi_h) \text{ is solution of (2.90), for k=1,2.} \end{array} \right.$$

Note that the set K_0^h can be identified with the following subset of $\mathbb{R}^{2N}$

$$\begin{array}{l} \mathcal{K}_0 = \big\{\{X\} = (X_{1,0}, \ldots, X_{1,N-1}, X_{2,0}, \ldots, X_{2,N-1}) \in \mathbb{R}^{2N} \ / \\ -C\,h \le X_{l,i} - X_{l,i-1} \le C\,h, \ i = 1, \ldots, N-1, \ l = 1, 2 \\ \text{ and } |X_{l,i}| \le \frac{C}{2}\,h + C_0, \quad i = 0, \ldots, N-1, \ l = 1, 2 \big\}. \end{array}$$

We denote by $M_{\Omega^{(k)}}(h)$ and $M_\gamma(h)$ the set of nodes lying respectively on $\Omega^{(k)}$ and γ. Let $m^{(k)}$ be the number of elements of $M_{\Omega^{(k)}}(h)$, and define $NT^{(k)} = N + m^{(k)}$, for $k = 1, 2$. Let us now introduce in $H(\Omega^{(k)})$ the canonical basis $(p_i^{(k)})_{i=1}^{NT_k}$ such that $p_i^{(1)} = p_i^{(2)} = p_i$, for all $i \in M_\gamma(h)$. For the vector $P^{(k)} = [p_1^{(k)}, p_2^{(k)}, \ldots, p_{NT^{(k)}}^{(k)}]$, we define the following matrix $[\mathcal{P}^{(k)}] = \begin{pmatrix} P^{(k)} & 0 \\ 0 & P^{(k)} \end{pmatrix}$ Then $u_h^{(k)}$ can be written $u_h^{(k)} = [\mathcal{P}^{(k)}]\,\{u_T^{(k)}\}$ where $\{u_T^{(k)}\} = {}^t[u_{1,1}^{(k)}, u_{1,2}^{(k)}, \ldots, u_{1,NT^{(k)}}^{(k)}, u_{2,1}^{(k)}, u_{2,2}^{(k)}, \ldots, u_{2,NT^{(k)}}^{(k)}]$ is the vector of the components of $u_h^{(k)}$ in the basis $P^{(k)}$. Let us denote by

$$D\,P^{(k)} = \begin{pmatrix} \frac{\partial p_1^{(k)}}{\partial x} & \frac{\partial p_2^{(k)}}{\partial x} & \cdots & \frac{\partial p_{NT^{(k)}}^{(k)}}{\partial x} \\ \frac{\partial p_1^{(k)}}{\partial y} & \frac{\partial p_2^{(k)}}{\partial y} & \cdots & \frac{\partial p_{NT^{(k)}}^{(k)}}{\partial y} \end{pmatrix} \text{ and } [\mathcal{D}\,\mathcal{P}^{(k)}] = \begin{pmatrix} D\,P^{(k)} & 0 \\ 0 & D\,P^{(k)} \end{pmatrix}$$

the gradient of $u_h^{(k)}$, $D\,u_h^{(k)}$ can be written in term of $[\mathcal{D}\,\mathcal{P}^{(k)}]$ and $\{u_T^{(k)}\}$ by

$$D\,u_h^{(k)} = {}^t\left(\frac{\partial u_{1,h}^{(k)}}{\partial x}, \frac{\partial u_{1,h}^{(k)}}{\partial y}, \frac{\partial u_{2,h}^{(k)}}{\partial x}, \frac{\partial u_{2,h}^{(k)}}{\partial y}\right) = [\mathcal{D}\,\mathcal{P}^{(k)}]\,\{u_T^{(k)}\}$$

The tensors ε and σ can be read $\{\varepsilon\} = {}^t\,(\varepsilon_{11}, \varepsilon_{22}, 2\,\varepsilon_{12})$ and $\{\sigma\} = {}^t\,(\sigma_{11}, \sigma_{22}, \sigma_{12})$

$\{\varepsilon\}$ can be written in term of $D\,u_h^{(k)}$

$$\{\varepsilon\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u_{1,h}^{(k)}}{\partial x} \\ \frac{\partial u_{1,h}^{(k)}}{\partial y} \\ \frac{\partial u_{2,h}^{(k)}}{\partial x} \\ \frac{\partial u_{2,h}^{(k)}}{\partial y} \end{pmatrix}$$

If we denote by $[\mathcal{D}]$ the above matrix, we have

$$\{\varepsilon\} = [\mathcal{D}]\,D\,u_h^{(k)} = [\mathcal{D}]\,[\mathcal{D}\,\mathcal{P}^{(k)}]\,\{u_T^{(k)}\}$$

Using equation (2.80), we can write $\{\sigma\}$ in term of $\{\varepsilon\}$ as follows $\{\sigma\} = [\mathcal{E}]\,\{\varepsilon\}$ thus $\{\sigma\} = [\mathcal{E}]\,[\mathcal{D}]\,[\mathcal{D}\,\mathcal{P}^{(k)}]\,\{u_T^{(k)}\}$ where $[\mathcal{E}]$ is a 3×3 symmetric matrix. Using the above notations we have

$$\sum_{i,j=1}^{2} \int_{\Omega^{(k)}} \sigma_{ij}(u_h^{(k)})\,\varepsilon_{ij}(\upsilon_h) = {}^t\{\upsilon_T\}\,\left(\int_{\Omega^{(k)}} {}^t[\mathcal{D}\,\mathcal{P}^{(k)}]\,{}^t[\mathcal{D}][\mathcal{E}]\,[\mathcal{D}]\,[\mathcal{D}\,\mathcal{P}^{(k)}]\,dx\right)\{u_T^{(k)}\},$$

and $\displaystyle\sum_{i=1}^{2} \int_{\Omega^{(k)}} f_{i,h}^{(k)}\,\upsilon_{i,h}\,dx = {}^t\{\upsilon_T\}\,\left(\int_{\Omega^{(k)}} {}^t[\mathcal{P}^{(k)}]\,f_h^{(k)}\,dx\right).$

Setting now the matrix $\mathcal{A}^{(k)} = \displaystyle\int_{\Omega^{(k)}} {}^t[\mathcal{D}\,\mathcal{P}^{(k)}]\,{}^t[\mathcal{D}][\mathcal{E}]\,[\mathcal{D}]\,[\mathcal{D}\,\mathcal{P}^{(k)}]\,dx$, the vectors $\{\mathcal{B}^{(k)}\} = \displaystyle\int_{\Omega^{(k)}} {}^t[\mathcal{P}^{(k)}]\,f_h^{(k)}\,dx$ and $\{G^{(k)}(X)\} = (G_i(X))_{i=1}^{2NT^{(k)}}$ with

$$G_i^{(k)}(X) = (-1)^k \sum_{l=1}^{2} \sum_{j \in M_\gamma(h)} X_{l,j} \int_\gamma \left({}^t[\mathcal{P}^{(k)}]\,[\mathcal{P}^{(k)}]\right)_{ij}\,d\sigma, \tag{2.92}$$

it is easy to see that problem (2.90) can be rewritten, for $k = 1, 2$, as

$$\begin{cases} \text{Find } \{u_T^{(k)}(X)\} \in \mathbb{R}^{2NT^{(k)}} \text{ such that} \\ \mathcal{A}^{(k)}\,\{u_T^{(k)}(X)\} = \{\mathcal{B}^{(k)}\} + \{G^{(k)}(X)\} \end{cases} \tag{2.93}$$

We can identify the set U_{ad}^h with the following subset of $\mathbb{R}^{4NT^{(k)}}$

$$\mathcal{U} = \{(\{u_T^{(1)}\}, \{u_T^{(2)}\}) \text{ solution of } (2.93)\ /\ \{X\} \in \mathcal{K}_0\}.$$

Then the discrete cost functional reads :

$$J_h(u_h^{(1)}, u_h^{(2)}) = J(\{u_T^{(1)}\}, \{u_T^{(2)}\}) = \frac{1}{2}\left\langle [\mathcal{R}]\left(\{u_T^{(1)}\} - \{u_T^{(2)}\}\right), \left(\{u_T^{(1)}\} - \{u_T^{(2)}\}\right)\right\rangle$$

where $\langle ., . \rangle$ is the inner product in $\mathbb{R}^{2NT^{(k)}}$ and the matrix $[\mathcal{R}]$ is defined by

$$[\mathcal{R}] = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

where $R = (r_{i\,j})_{1 \leq i,j \leq 2NT^{(k)}}$ is given by

$$r_{i\,j} = \begin{cases} \widehat{r}_{i\,j} = \displaystyle\int_\gamma p_i\, p_j\, d\sigma & \text{if } i, j \in M_\gamma(h) \\ 0 & \text{otherwise .} \end{cases}$$

The matrix form of the optimization problem reads :

$$(PM) \begin{cases} \displaystyle\inf_{(\{u_T^{(1)}\}, \{u_T^{(2)}\}) \in \mathcal{U}} J(\{u_T^{(1)}\}, \{u_T^{(2)}\}) \\ \text{s/c} \;\; \mathcal{A}^{(k)}\, \{u_T^{(k)}(X)\} = \{\mathcal{B}^{(k)}\} + \{G^{(k)}(X)\} \quad \text{for } k = 1, 2 \end{cases} \tag{2.94}$$

Existence of the solution of the discrete problem

It is easy to see that (PO^h) is equivalent to (PM), thus we show that (PM) has a solution in $\mathcal{U}$.

Theorem 2.3.2 *The problem (PM) admits a solution on $\mathcal{U}$, for all $h > 0$.*

2.3.5 Convergence result

In this section, we are interested in showing the existence of a subsequence of the solutions of the discrete problems which converges to a solution of the continuous one. For this we introduce the following definitions :
Let $(\psi_h)_h$ be a sequence such that $\psi_h \in K_0^h$ for all h, we define the convergence of $(\psi_h)_h$ to $\psi \in K_0$ as $h \longrightarrow 0$ by

$$\psi_h \longrightarrow \psi \iff \psi_{i,h}(b, .) \longrightarrow \psi_i(b, .) \text{ uniformly in } [0, a] \quad \text{for } i = 1, 2. \tag{2.95}$$

For a sequence $((u_h^{(1)}, u_h^{(2)}))_h$ such that $(u_h^{(1)}, u_h^{(1)}) \in U_{ad}^h$, the convergence of the sequence $((u_h^{(1)}, u_h^{(2)}))_h$ to $(u^{(1)}, u^{(2)}) \in U_{ad}$, as $h \longrightarrow 0$, is defined by

$$(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)}) \iff \begin{cases} u_{i,h}^{(1)} \rightharpoonup u_i^{(1)} \text{ weakly in } H^1(\Omega^{(1)}) \\ u_{i,h}^{(2)} \rightharpoonup u_i^{(2)} \text{ weakly in } H^1(\Omega^{(2)}) \quad \text{for } i = 1, 2. \end{cases} \tag{2.96}$$

Our convergence result is based on the following lemma.

Lemma 2.3.1 (**i**) *For any* $(u^{(1)}, u^{(2)}) \in U_{ad}$*, such that* $u^{(k)} = u^{(k)}(\psi)$ *for* $\psi \in K_0$*, there exists a sequence* $((u_h^{(1)}, u_h^{(2)}))_h$ *such that* $u_h^{(k)} = u_h^{(k)}(\psi_h)$ *for* $\psi_h \in K_0^h$ *and* $(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$.
(**ii**) *Let* $((u_h^{(1)}, u_h^{(2)}))_h$ *be a sequence of* U_{ad}^h *such that* $u_h^{(k)} = u_h^{(k)}(\psi_h)$ *for* $\psi_h \in K_0^h$*. Then there exists a subsequence of* $((u_h^{(1)}, u_h^{(2)}))_h$ *denoted again by* $((u_h^{(1)}, u_h^{(2)}))_h$ *and an element* $(u^{(1)}, u^{(2)}) \in U_{ad}$ *such that* $u^{(k)} = u^{(k)}(\psi)$ *for* $\psi \in K_0$ *and* $(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$.
(**iii**) *If* $((u_h^{(1)}, u_h^{(2)}))_h$ *is a sequence such that* $(u_h^{(1)}, u_h^{(2)}) \in U_{ad}^h$*, and* $(u^{(1)}, u^{(2)}) \in U_{ad}$ *such that* $(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$.
Then $J_h((u_h^{(1)}, u_h^{(2)}) \longrightarrow J(u^{(1)}, u^{(2)})$ *as* $h \longrightarrow 0$.

We can now prove our main result of convergence stated in the following theorem

Theorem 2.3.3 *Let* $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ *be a sequence such that* $(u_{*,h}^{(1)}, u_{*,h}^{(2)})$ *is solution of* (PO^h) *and* $(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \in U_{ad}^h$*. Then, there exists a subsequence denoted again* $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ *and an element* $(u_*^{(1)}, u_*^{(2)}) \in U_{ad}$ *such that*

$$(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \longrightarrow (u_*^{(1)}, u_*^{(2)})$$

furthermore $(u_*^{(1)}, u_*^{(2)})$ *is solution of* (PO).

Proof.
Let $(u^{(1)}, u^{(2)})$ be an element of U_{ad}, from the assertion (**i**) of Lemma 1, there exists a sequence $((u_h^{(1)}, u_h^{(2)}))_h$ such that $(u_h^{(1)}, u_h^{(2)}) \in U_{ad}^h$ and

$$(u_h^{(1)}, u_h^{(2)}) \longrightarrow (u^{(1)}, u^{(2)})$$

According to the assertion (**iii**), we have that

$$J_h(u_h^{(1)}, u_h^{(2)}) \longrightarrow J(u^{(1)}, u^{(2)}) \text{ as } h \longrightarrow 0$$

Now, Let $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ be a sequence such that is solution of (PO^h) and $(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \in U_{ad}^h$. From the assertion (**ii**) of Lemma 1, there exists a subsequence denoted again $((u_{*,h}^{(1)}, u_{*,h}^{(2)}))_h$ and an element $(u_*^{(1)}, u_*^{(2)}) \in U_{ad}$ such that

$$(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \longrightarrow (u_*^{(1)}, u_*^{(2)})$$

According to the assertion (**iii**), we have that

$$J_h(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \longrightarrow J(u_*^{(1)}, u_*^{(2)}) \text{ as } h \longrightarrow 0$$

however, we have that

$$J(u_{*,h}^{(1)}, u_{*,h}^{(2)}) \leq J(u_h^{(1)}, u_h^{(2)}) \text{ for all } h \tag{2.97}$$

The main result is then obtained by passing to the limit in equation (2.97), as $h \longrightarrow 0$.

2.3.6 Optimization algorithm

We use the Lagrange multiplier rule to derive an optimality system of equations from which solutions of the optimization problem (PO) may be determined.

Let $u^{(i)}, \lambda^{(i)} \in H_{i,D}(\Omega^{(i)})$, for $i = 1, 2$, and $\psi \in (L^2(\gamma))^2$ we define the Lagrangian

$$\mathcal{L}(u^{(1)}, u^{(2)}, \psi, \lambda^{(1)}, \lambda^{(2)}) = J(\psi, u^{(1)}, u^{(2)}) - \int_{\Omega^{(1)}} \sigma_{ij}(u^{(1)}(x))\varepsilon_{ij}(\lambda^{(1)}(x))\, dx$$

$$+ \int_{\Omega^{(1)}} f^{(1)}(x)\lambda^{(1)}(x)\, dx + \int_\gamma \psi(x)\lambda^{(1)}(x)\, dx - \int_{\Omega^{(2)}} \sigma_{ij}(u^{(2)}(x))\varepsilon_{ij}(\lambda^{(2)}(x))\, dx$$

$$+ \int_{\Omega^{(2)}} f^{(2)}(x)\lambda^{(2)}(x)\, dx - \int_\gamma \psi(x)\lambda^{(2)}(x)\, dx$$

Setting to zero the first variations with respect to the multipliers λ_1 ans λ_2 yields the constraints (2.84). Setting to zero the first variations with respect to $u^{(1)}$ and $u^{(2)}$ yield the adjoint equations

$$a^{(1)}(v, \lambda^{(1)}) = (u^{(1)} - u^{(2)}, v)_\gamma \;\; \forall v \in H_{1,D}(\Omega^{(1)}) \tag{2.98}$$

and

$$a^{(2)}(v, \lambda^{(2)}) = -(u^{(1)} - u^{(2)}, v)_\gamma \;\; \forall v \in H_{2,D}(\Omega^{(2)}) \tag{2.99}$$

respectively.

Then the adjoint equations is given by

$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(\lambda^{(1)})}{\partial\, x_j} = 0 \text{ in } \Omega^{(1)} \\ \lambda^{(1)} = 0 \quad \text{on } \Gamma_1 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(\lambda^{(1)}) n_j = u^{(1)} - u^{(2)} \text{ on } \gamma \end{cases} \tag{2.100}$$

$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(\lambda^{(2)})}{\partial\, x_j} = 0 \text{ in } \Omega^{(2)} \\ \lambda^{(2)} = 0 \quad \text{on } \Gamma_2 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(\lambda^{(2)}) n_j = -(u^{(1)} - u^{(2)}) \text{ on } \gamma \end{cases} \tag{2.101}$$

Let $\mathcal{J}(\psi) = J(\psi, u^{(1)}, u^{(2)})$ where, for given ψ,

$$u^{(i)} : \psi \in (L^2(\gamma))^2 \to H_{i,D}(\Omega^{(i)}) \text{ for } i = 1, 2$$

are defined as the solution of (2.81) and (2.82) respectively. Then, the minimization problem is equivalent to the problem of determining $\psi \in (L^2(\gamma))^2$ such that $\mathcal{J}(\psi)$ is minimized. Now, the first derivative of $\mathcal{J}$ is defined through its action on variations $\tilde{\psi}$ by

$$\langle \frac{d\mathcal{J}}{d\psi}, \tilde{\psi} \rangle = (u^{(1)} - u^{(2)}, \tilde{u}^{(1)} - \tilde{u}^{(2)})_\gamma \quad \forall \tilde{\psi} \in (L^2(\gamma))^2 \tag{2.102}$$

where $\tilde{u}^{(1)} \in H_{1,D}(\Omega^{(1)})$ and $\tilde{u}^{(2)} \in H_{2,D}(\Omega^{(2)})$ are the solution of

$$a^{(1)}(\tilde{u}^{(1)}, v) = (\tilde{\psi}, v)_\gamma \;\; \forall v \in H_{1,D}(\Omega^{(1)}) \tag{2.103}$$

and

$$a^{(2)}(\tilde{u}^{(2)}, v) = -(\tilde{\psi}, v)_\gamma \ \ \forall v \in H_{2,D}(\Omega^{(2)}) \tag{2.104}$$

respectively. Set $v = \lambda_1^{(1)}$ in (2.103), $v = \lambda^{(1)}$ in (2.104), $v = \tilde{u}^{(1)}$ in (2.98) and $v = \tilde{u}^{(2)}$ in (2.99). Combning the results yields that

$$\frac{d\mathcal{J}}{d\psi} = \lambda^{(1)} - \lambda^{(2)} \text{ on } \gamma. \tag{2.105}$$

we now present our domain decomposition algorithm

Algorithm 2.3.1 *$k = 0$ and $\psi_{,0}$ is given*
For $k = 0, \ldots$

Solve
$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(u_{,k}^{(1)})}{\partial\, x_j} = f_i^{(1)} \ \ in\ \Omega^{(1)} \\ u_{,k}^{(1)} = 0 \quad on\ \Gamma_1 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(u_{,k}^{(1)}) n_j = \psi_{i,k} \ \ on\ \gamma \end{cases} \tag{2.106}$$
$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(u_{,k}^{(2)})}{\partial\, x_j} = f_i^{(2)} \ \ in\ \Omega^{(2)} \\ u_{,k}^{(2)} = 0 \quad on\ \Gamma_2 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(u_{,k}^{(2)}) n_j = -\psi_{i,k} \ \ on\ \gamma \end{cases} \tag{2.107}$$

Solve
$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(\lambda_{,k}^{(1)})}{\partial\, x_j} = 0 \ \ in\ \Omega^{(1)} \\ \lambda^{(1)} = 0 \quad on\ \Gamma_1 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(\lambda_{,k}^{(1)}) n_j = u_{,k}^{(1)} - u_{,k}^{(2)} \ \ on\ \gamma \end{cases} \tag{2.108}$$
$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(\lambda_{,k}^{(2)})}{\partial\, x_j} = 0 \ \ in\ \Omega^{(2)} \\ \lambda_{,k}^{(2)} = 0 \quad on\ \Gamma_2 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(\lambda_{,k}^{(2)}) n_j = -(u_{,k}^{(1)} - u_{,k}^{(2)}) \ \ on\ \gamma \end{cases} \tag{2.109}$$

Compute $\nabla J(\psi_{,k}) = \lambda_{,k}^{(1)}(\psi_{,k}) - \lambda_{,k}^{(2)}(\psi_{,k})$
Update
$\gamma^k = \dfrac{\|\nabla J(\psi_{,k})\|}{\|\nabla J(\psi_{,k-1})\|}$
$d_{,k} = \nabla J(\psi_{,k}) + \gamma^k d_{,k-1}$

Solve
$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(D_{,k}^{(1)})}{\partial\, x_j} = 0 \ \ in\ \Omega^{(1)} \\ D_{,k}^{(1)} = 0 \quad on\ \Gamma_1 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(D_{,k}^{(1)}) n_j = d_{i,k} \ \ on\ \gamma \end{cases} \tag{2.110}$$
$$\begin{cases} \displaystyle\sum_{j=1}^{2} \frac{\partial\, \sigma_{ij}(D_{,k}^{(2)})}{\partial\, x_j} = 0 \ \ in\ \Omega^{(2)} \\ D_{,k}^{(2)} = 0 \quad on\ \Gamma_2 \\ \displaystyle\sum_{j=1}^{2} \sigma_{ij}(D_{,k}^{(2)}) n_j = -d_{i,k} \ \ on\ \gamma \end{cases} \tag{2.111}$$

Compute
$\rho_k = \dfrac{(u_{,k}^{(1)} - u_{,k}^{(2)}, D_{,k}^{(1)} - D_{,k}^{(2)})}{\|D_{,k}^{(1)} - D_{,k}^{(2)}\|^2}$

$\psi_{,k+1} = \psi_{,k} - \rho_k d_{,k}$
End For

2.3.7 Numerical results

In order to illustrate the performance of the numerical method described above, we solve the linear elasticity problem (2.78), in two-dimensional domain $\Omega = (0,1) \times (0,1)$, with $u = u^{an}$ on Γ and $f = 0$. We assume that the boundary is split into two parts $\Gamma_1 = [0,0.5] \times \{0\} \cup [0,0.5] \times \{1\} \cup \{0\} \times [0,1]$ and $\Gamma_2 = [0.5,1] \times \{0\} \cup \{1\} \times [0,1] \cup [0.5,1] \times \{1\}$. For these data, the analytical solution is given by

$$u_1^{an}(x,y) = \frac{1-\nu}{2G}\sigma_0 xy,\ u_2^{an}(x,y) = -\frac{1}{4G}\sigma_0((1-\nu)(x^2-1)+\nu y^2) \tag{2.112}$$

$$t_1^{an}(x,y) = \sigma_0 y n_1,\ t_2^{an}(x,y) = 0 \tag{2.113}$$

with $\sigma_0 = 1.5 \times 10^{10}$, $G = 3.35 \times 10^{10}$ and $\nu = 0.34$.

This example consists to split the domain Ω into two rectangular subdomains $\Omega^{(1)} = (0.,0.5) \times (0,1)$ and $\Omega^{(2)} = (0.5,1) \times (0,1)$ with interface $\gamma = \{0.5\} \times [0,1]$.

In this section we investigate the convergence of the proposed method by the evaluation at every iteration the accuracy errors denoted for $i,j = 1,2$ by

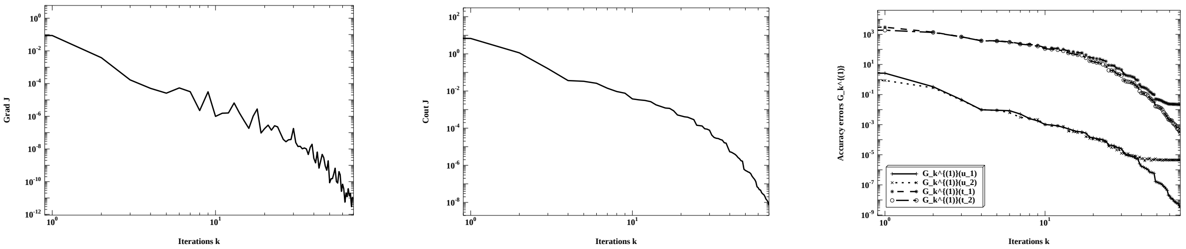
$$G_k^{(i)}(u_j) = \|u_{j,k}^{(i)} - u_j^{(i)^{an}}\|^2_{L^2(\gamma)},\ G_k^{(i)}(t_j) = \|t_{j,k}^{(i)} - t_j^{(i)^{an}}\|^2_{L^2(\gamma)}. \tag{2.114}$$

The following stopping criterion is considered

$$\|\nabla J(\psi_{,k})\|^2 < \eta \|\nabla J(\psi_{,0})\|^2 \tag{2.115}$$

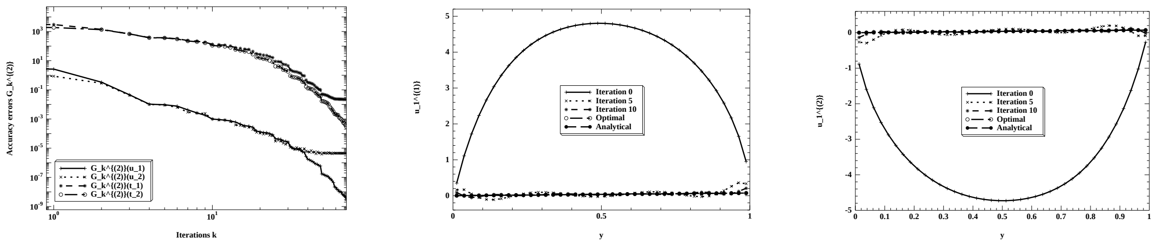
where η is a small prescribed positive quantity. For all numerical experiments, we take $\eta = 10^{-11}$.

The mesh of discretization is taken as $h = 1/40$. The initial guess $\psi_{i,0}$ on γ has been chosen as $\psi_{i,0} = 100$. When starting with this initial guess, which is not too close to the exact traction, a sequence of displacements $\left\{(u_{,k}^{(1)})_h\right\}_{k\geq 0}$ and $\left\{(u_{,k}^{(2)})_h\right\}_{k\geq 0}$ of approximation functions for $u_{|\gamma}$ is obtained and this sequences converge to the exact solution. We observe from Figure 2.20(a), (b) that the norm of gradient and the cost decrease as a function of number of iterations. Figure 2.20(c) and Figure 2.21(a) shows the evaluation of accuracy errors as function of number of iterations. The discrepancy $\|u_{1,opt}^{(1)} - u_1^{(1)^{an}}\|^2_{L^2(\gamma)}$ between the optimal x_1-displacement and the exact one is equal to 3.35×10^{-09} and the discrepancy $\|t_{2,opt}^{(2)} - t_2^{(2)^{an}}\|^2_{L^2(\gamma)}$ between the optimal x_2-traction and the exact one is equal to 2.92×10^{-04}. Figure 2.20(c) and Figure 2.21(a) shows the evaluation of accuracy errors as function of number of iterations. Figure 2.21- 2.23 proves the well convergence of the proposed optimal control algorithm.



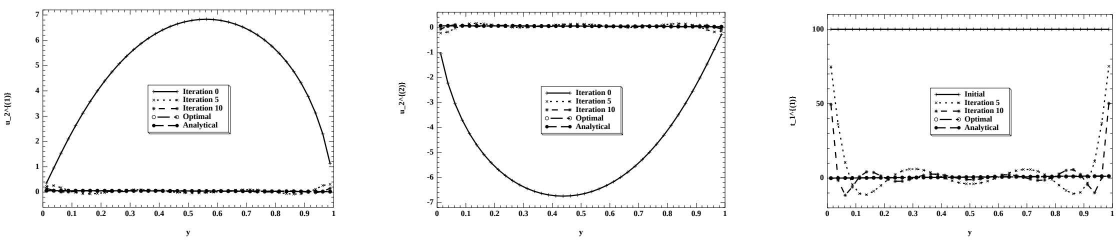
(a) (b) (c)

FIG. 2.20 – Computed norm of gradient (a), cost functional (b) and the accuracy errors (c) given by (2.114) as a function of the number of iterations k.



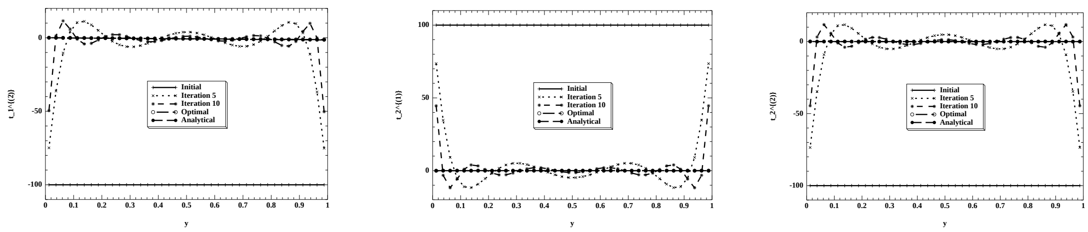
(a) (b) (c)

FIG. 2.21 – The accuracy errors (a) given by (2.114) as a function of the number of iterations k, results of $u_1^{(1)}$ (b) and $u_1^{(2)}$ (c) on interface γ



(a) (b) (c)

FIG. 2.22 – Results of $u_1^{(2)}$ (a), $u_2^{(2)}$ (b) and $t_1^{(1)}$ (c) on interface γ



(a) (b) (c)

FIG. 2.23 – Results of $t_2^{(1)}$ (a), $t_1^{(2)}$ (b) and $t_2^{(2)}$ (c) on interface γ

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Chapitre 3

Papers / Liste des publications

3.1 Articles parus/à paraître dans des revues internationales

1. A. Ellabib, A. Nachaoui,
   An iterative approach of inverse problem in linear elasticity, *Mathematics and Computers in Simulation* **77**, 189-201, (2008).
2. A. Ellabib, A. Nachaoui,
   A domain decomposition method for boundary element approximations of the elasticity equations, *Esaim Proceedings,* Accepté (2007).
3. A. Ellabib, A. Nachaoui,
   Unicité des solutions stationnaires des modèles dérive-diffusion avec génération d'avalanche, *Extracta Mathematicae* Vol. **18**, N°1, 13-21, (2003).
4. A. Ellabib, A. Nachaoui,
   On the numerical solution of a free boundary identification problem, *Inverse Problems in Engineering.* Vol. **9**, 235-260 (2001).
5. A. Ellabib, A. Nachaoui,
   Existence de solutions pour le modèle dérive-diffusion avec terme d'avalanche, *C. R. Acad. Sci., Paris, Ser. I, Math.* **332**, No.**4**, 305-310 (2001).
6. A. Ellabib, A. Nachaoui,
   Solution iterative d'un problème mixte pour le potentiel électro-statique dans les semi-conducteurs, *Lebanese scientific research reports*, Vol. **3,** No.**2,** 307-321, (1998).

3.2 Articles dans des congrés internationaux avec actes et comité de lecture

1. A. Chakib, A. Ellabib , A. Nachaoui,
   A domain decomposition convergence for elasticity equations, *International conference on approximation methods and numerical modelling in environment and natural resources*, pp. 401-406, 11-13 July Granada, Spain (2007).
2. A. Ellabib , A. Nachaoui,
   Méthode d'approximation de problème inverse en élasticité linéaire, *Méthode d'approximation et de modélisation numérique en environnement et ressources naturelles*, Session I, pp. 41-43, 09-11 Mai oujda, Maroc (2005).
3. A. Ellabib, A. Nachaoui,
   A non-overlapping and Schwarz domain decomposition methos for boundary element approximation of the elasticity equations, *Deuxième Conférence Internationale sur les Phénomènes Non Linéaires : Modélisation et Analyse*, CIPNL2005 Errachidia 25-27 avril 2005 Maroc.

4. A. Ellabib, A. Nachaoui,
   On the numerical solution of a free boundary identification problem, *Dynamic Systems Identification and Inverse Problems,* Vol. **1**, 135-143, Moscow May (1998).
5. A. Ellabib , A. Nachaoui,
   Solution iterative d'un problème mixte pour le potentiel électro-statique dans les semi-conducteurs, *Modélisation Mathématiques et Numérique des Sciences de l'Ingénieur* Beyrouth, Liban, 05-10 Avril, 1997.

3.3 Rapports de Recherche/articles soumis

1. A. Ellabib, A. Nachaoui,
   A domain decomposition convergence for elasticity equations, *Report in http ://hal.archives-ouvertes.fr/hal-00259004/fr/*
2. A. Ellabib, A. Nachaoui,
   An iterative approach of inverse problem in linear elasticity, *Report in https ://hal.archives-ouvertes.fr/hal-00139180*
3. A. Ellabib, A. Nachaoui,
   A domain decomposition method for boundary element approximations of the elasticity equations, *Report in https ://hal.archives-ouvertes.fr/hal-00144940.*
4. A. Ellabib, A. Nachaoui,
   Existence de solutions pour le modèle dérive-diffusion avec terme d'avalanche, *Rapport de recherche 00/12-1 CNRS UMR6629* Université de Nantes.
5. A. Ellabib, A. Nachaoui,
   On the numerical solution of a free boundary identification problem, *Rapport de recherche 00/07-3 CNRS UMR6629* Université de Nantes.

Chapitre 4

Project and other research activities / Projet et autres activités de recherches

- **Travaux de thèse de l'université de Nantes**

Les travaux de recherche que j'ai effectués dans le cadre de ma thèse portent essentiellement sur l'analyse mathématique d'équations de semi-conducteurs avec mobilités non-constantes et identification des frontières libres dans les jonctions PN. La description des mécanismes de conduction dans les dispositifs semi-conducteurs par le modèle dérive-diffusion (DD) mène à un système de trois équations aux dérivées partielles non-linéaires fortement couplées. Ce travail est composé de trois parties. La première est consacrée à la mise en équations et à la présentation des régimes de fonctionnement ainsi que la simplification du modèle dans le cas d'une jonction *pn*. La deuxième partie consiste à identifier la zone de déplétion dans une jonction *pn*. En formulant le problème en un problème d'inéquations variationnelles, nous démontrons que le problème admet une solution. L'originalité numérique de cette partie est l'utilisation des noeuds sur la frontière libre comme inconnus. Nous proposons deux algorithmes de résolution que nous testons en utilisant la méthode des éléments finis et la méthode des équations intégrales. Dans la troisième partie, nous nous intéressons à l'étude mathématique du modèle DD à l'état stationnaire dans les semi-conducteurs écrit avec les variables de Slotboom. Nous démontrons l'existence d'une solution, dans le cas où les lois de mobilités dépendent du champ électrique, en appliquant les techniques de l'analyse convexe. Ensuite, nous considérons que le terme d'avalanche est non nul, nous donnons des estimations a priori et nous prouvons un théorème d'existence. Afin d'étudier l'unicité de solutions de notre modèle, nous exposons tout d'abord une condition pour que le système possède au plus une solution. Nous en déduisons des résultats d'unicité dans des cas spécifiques tels que le domaine soit suffisamment petit ou la permittivité soit assez grande. Nous donnons un théorème d'unicité locale dans le cas où le terme d'avalanche est non nul et les changements de conditions aux limites se font à angles droits.

Projet Action intégrée N° MA/07/164 ;
intitulée modèles mathématiques non linéaires : analyse, traitement numérique

Les thèmes de recherche communs aux deux équipes concernent surtout l'optimisation mathématique, les équations aux dérivées partielles non linéaires, leur analyse mathématique, leur résolution numérique et l'analyse numérique associée, ainsi que les aspects de modélisations liés aux (nombreuses) applications : mécanique des fluides, chimie, biologie, écologie, environnement ... Mentionnons, en particulier :

1. les problèmes à frontières libres et l'optimisation de formes : le calcul numérique des formes ou des frontières libres, les questions liées aux dérivées de formes, à leur discrétisation, à l'analyse numérique des algorithmes d'optimisation associés et de la résolution des équations d'état.

2. les problèmes d'évolution non linéaires
3. les problèmes paraboliques dégénérés, systèmes de réaction-diffusion et applications à la chimie et la biologie, diffusions et non-linéarités couplées, question d'existence, d'explosion en temps fini, de régularité, de comportement asymptotique en temps
4. les équations d'ondes dispersives ou amorties
5. l'analyse mathématique et la simulation numérique des écoulements des fluides : ondes hydrodynamiques, fluides viscoélastiques, écoulements turbulents, combustion, équations de Maxwell, etc . . .

Les travaux communs sur ces thèmes constituent l'objectif majeur de ce projet avec le souci de contribuer au progrès rapide des thèses en cours de définir et proposer de nouveaux sujets de thèses et de déboucher sur des publications en commun. L'accent sera mis sur les questions de frontière libre, d'interface, d'optimisation de forme et leurs diverses applications sous-jacentes.

Projet Action intégrée N° MA/05/116
intitulée Instabilités mécaniques des couches minces

Le programme P.A.I Volubilis MA/05/116 entre Amiens et Marrakech porte sur les problèmes de délamination ou les instabilités mécaniques non-linéaires. Ce problème est à la frontière entre les mathématiques et la physique. Les deux axes de notre action sont :

1. Etudes physiques (théories et expérimentales) des croissances des surfaces
2. Etudes mathématiques (théorie et numérique)

Projet Action intégrée N° MA/06/148
intitulée Gestion durable des ressources en eau dans le bassin versabt de Tensift (Région de Marrakech)

Toute la région de Marrakech est située dans un contexte climatique aride à semi aride. La quantité de l'eau disponible est faible et variable dans l'espace et dans le temps. En plus de la rareté de l'eau et de son inégale répartition la région est soumise à une demande croissante en eau liée à la force croissance démographique, et l'extension urbaine ainsi qu'à l'augmentation des zones irriguées. Dans ce contexte, la gestion rationnelle et rigoureuse des ressources en eau revêt une importance primordiale pour le développement durable de la région, et notamment pour mettre en valeur le potentiel des terres irriguées en plaine et éviter que la pénurie d'eau ne soit une entrave au développement socio-économique.

La partie Sud du bassin de Tensift, le Haut Atlas, est caractérisée par des altitudes élevées pouvant dépasser 4000 mètres (Le Toubkal culmine à 4167 m). A l'opposé la partie Nord est occupée par la plaine du Haouz dont l'altitude est voisine de 500 m, et qui connaît une exploitation intense des ressources en eau pour les besoins croissants de l'agriculture et du développement touristique que connaît actuellement la région de Marrakech. Les précipitations en montagne sont comprises entre 300 et 900 mm, alors que dans la plaine elles sont comprises entre 150 et 300 mm. Inversement, le besoin en eau est nettement supérieure dans la plaine ou les températures sont plus élevées et la végétation plus (ET0 approximative de 1600 mm par an). Ces deux effets combinés font que l'on peut à juste titre considérer l'atlas comme le château d'eau de la plaine du Haouz. Le transfert d'eau entre ces deux compartiments se fait par le captage des seguias à la sortie des oueds de montagne, la retenue d'eau dans de grands barrages, ainsi que par l'infiltration des oueds vers la nappe phréatique.

La gestion durable implique notamment que l'offre en eau soit au moins égale à la demande dans tout le bassin versant. Afin d'apporter des informations et outils pour une gestion durable, nous devons avoir connaissance au maximum des flux actuels et potentiels d'eau dans le bassin versant. Quantifier la demande implique d'estimer les besoins en eau actuels et futurs des divers consommateurs (Agriculture, Industrie, Population, Tourisme) et plus particulièrement dans le cadre de ce projet, d'estimer les besoins de l'irrigation qui consomme approximativement 90% de l'eau disponible. Cette estimation implique que soient mieux connues les différentes composantes du cycle hydrologique : la quantité de pluie et sa distribution, le ruissellement, l'évaporation au sol, le manteau neigeux, la quantité issue de la fonte de neige, la recharge de la nappe et la quantité d'eau pompée dans la plaine.

Dans le cadre de cette problématique générale, nous visons une série d'objectifs plus ciblés qui contribueront à la connaissance générale du cycle hydrologique du bassin versant du Tensift, connaissance indispensable pour une gestion raisonnée des ressources en eau.

Projet en collaboration avec A. Nachaoui (Université de Nantes) ; A. Chakib (Université Sultan My Slimane)

Les algorithmes évolutionnaires (AE) (par exemple les algorithmes génétiques) sont des méthodes de résolution de problèmes qui copient de façon extrêmement simplifiée certains aspects de l'évolution naturelle. Ce sont des outils d'optimisation très robustes, efficaces lorsque les fonctions à optimiser sont fortement irrégulières, et dépendantes de paramètres variés en nature et en type. Ils sont actuellement largement employés dans des domaines d'applications extrêmement variés.

Les développements visés concernent un domaine assez largement expérimenté et pour lequel il nous semble qu'une approche évolutionnaire permettra une avancée : des problèmes d'identification, des problèmes inverses que l'on peut formuler sous forme d'un problème d'optimisation, et qui trouvent des applications non seulement dans les domaines classiques comme la thermique, la mécanique industrielle, l'électronique ; mais aussi en hydrologie et en environnement. Un autre domaine d'application est la médecine (par exemple la localisation des sources électriques et/ou magnétiques à partir des signaux recueillis en électroencéphalographie, électrocardiographie, imagerie médicale).

Notre objectif est donc l'étude de quelques problèmes d'identification ainsi que l'application à cette classe de problèmes des techniques d'intelligence artificielle aussi bien sur le plan théorique que algorithmique (convergence, parallélisation, mise en œuvre,....). En particulier dans les applications aux problèmes d'identification de frontières, nous proposons de représenter cette frontière par un petit nombre de paramètres et combiner les méthodes évolutionnaires avec des techniques d'interpolation en utilisant les méthodes de Bezier ou leur généralisation (B-spline).

Chapitre 5

Teaching Report / Activités d'enseignements

Mes experiences pédagogiques sont multiples et variées, que ce soit à l'université de Nantes ou à l'université Cadi Ayyad. Cette experience a commencé en 1996 tant que vacataire que j'ai exercé pendant les années $1996-1998$, au sein de la faculté des sciences et techniques de Nantes, puis en qualité d'attaché temporaire d'enseignement et de recherche durant les années $1998-2000$ au département de mathématiques de l'université de Nantes, ensuite au fonction de professeur de l'enseignement supérieur assistant, un poste d'enseignement et de recherche que j'occupe depuis juillet 2002 à la faculté des sciences et techniques de Marrakech.

Pendant ces enseignements, j'ai réalisé plusieurs tâches connexes telles que la rédaction de sujets d'examen, polycopiés des cours, sujets des travaux dirigés et travaux pratiques, correction des copies d'examen et encadrement de mini-projets et d'un projet de fin d'etude et d'un mémoire de Diplôme d'Etudes Scientifiques Approfondies.

Je dresse tout d'abord un tableau récapitulant mes enseignements assurés à l'université de Nantes. Ce tableau précise la matière enseigné, l'année, le public, la nature et la qualité en tant que vacataire ou attaché temporaire d'enseignement et de recherche.

Matière	Année	Public	Nature	Qualité
Programmation en Fortran	96-97	Licence Mécanique	TP	Vacataire
Mathématiques générales ;	96-97	DEUG B1	TD	Vacataire
Probabilités et statistique	99-00	DEUG B1	TD	ATER
Analyse numérique	97-98	Licence de mathématiques	TD	Vacataire
	98-99	Licence de mathématiques	TD-TP	ATER
	99-00	Licence de mathématiques	TD-TP	ATER
Algèbre	96-97	DEUG A1	TD	Vacataire
	97-98	DEUG A1	TD	Vacataire
	99-00	DEUG A1	TD	ATER
Analyse	99-00	DEUG A1	TD	ATER

En juillet 2002, j'ai été recruté sur un poste de professeur de l'enseignement supérieur assistant à la faculté des sciences et techniques de Marrakech. J'ai participé aux différents modules de mathématiques et informatique au premier, au second et au troisième cycle universitaire et filières d'ingénieurs. Dans ce cadre, j'ai entièrement géré la mise en oeuvre des cours magistraux, des travaux dirigés et des travaux pratiques en tant que responsable de modules.

Les cours, les travaux dirigés et les travaux pratiques que j'ai assurés à la faculté des sciences et techniques de Marrakech sont décrit dans le tableau suivant.

Module	Année	Public	Nature
Calcul Numérique et Programmation	02-03	DEUG PC	TD-TP
	03-04	DEUG PC	TD-TP
	04-05	DEUG PC	TD-TP
	05-06	DEUG PC	TD-TP
Programmation mathématique	02-03	MASI	TP
	03-04	MASI	TP
	04-05	MASI	TP
	05-06	MASI	TP
	06-07	MASI	TP
	07-08	MASI	TP
Algèbre	02-03	DEUG PC	TD
Analyse	03-04	DEUG PC ;	COURS-TD
(Polycopié réalisé)	04-05	ISA ; LPGE ;	COURS-TD
	05-06	IRISI ; IGM	COURS-TD
Algorithmique, Programmation	06-07	MASI ; IEEA TELECOM	COURS-TP
et langage C (Polycopié réalisé)	07-08	LPGE ; IGM et SET	COURS-TP
Statistique et théorie	06-07	DUT	COURS-TD-TP
des graphes	07-08	DUT	COURS-TD-TP
Décomposition de domaines	04-05	DESA	COURS
	06-07	DESA	COURS

Je présente aussi une description du contenu des modules auxquels j'ai participé en tant que professeur de l'enseignement supérieur assistant à la faculté des sciences et techniques de Marrakech.

Module d'analyse

Chapitre 1 : Suites réelles
Limite infinie et infinie d'une suite (convergence et divergence), critère de comparaison, suites de Cauchy.

Chapitre 2 : Fonctions d'une variable réelle
Limite d'une fonction : Définition, critère de Cauchy.
Continuité : Définitions, propriétés, théorèmes des valeurs intermédiaires, théorème du point fixe.
Fonction réciproque et fonctions hyperboliques.

Chapitre 3 : Dérivabilité
Définitions, propriétés, formule de Leibniz, extremum, théorème de Rolle, théorème des accroissements finis, théorème de l'Hospital, dérivabilité des fonction réciproques.

Chapitre 4 : Formule de Taylor, développements limités
Formule de Taylor, développements limités, fonctions équivalents.

Chapitre 5 : Intégrales
Intégrales des fonctions au sens de Riemann, propriétés des fonctions intégrables, fonction définie comme une intégrale.

Chapitre 6 : Intégrales généralisées
Définitions et propriétés, critères de convergence, fonctions positives, fonctions de signe quelconque, critère d'Abel.

Chapitre 7 : Equations différentielles
Equations différentielles du premier et second oirdre, théorème d'existence et unicité, solution particulière, solution générale.

Chapitre 8 : Séries numériques
Définitions et propriétés, critères de convergence, série à termes positifs, série à termes de signe quelconque, critère d'Abel.

Module d'algèbre

Chapitre 1 : Polynômes à une ou plusieurs indéterminées
Définition de $A[X]$ et propriétés générales, division euclidienne, proproétés arithmétiques, algorithme d'Euclide, polynômes irréductibles, fonction polynôme, racine, formule de Taylor, decomposition dans $\mathbb{R}[X]$ et $\mathbb{C}[X]$, notions sur $K[X_1, \ldots, X_n]$.

Chapitre 2 : Fractions rationnelles
Décomposition d'une fraction rationnelle en éléments simples dans $\mathbb{R}[X]$ et $\mathbb{C}[X]$.

Chapitre 3 : Espaces vectoriels
Structure d'espace vectoriel, sous espaces vectoriels, applications linéaires, indépendance linéaire.

Chapitre 4 : Espaces vectoriels de dimension finie
Définition d'un espace vectoriel de dimension finie, bases, dimension d'un espace vectoriel de dimension finie, sous-espace vectoriel de dimension finie, application linéaire d'un K-espace vectoriel de dimension finie dans un K-espace vectoriel, application du théorème noyau-image.

Chapitre 5 : Matrices
Généralités, matrice d'une application linéaire, opérations sur les matrices, matrices carrées, changements de base et application au rang d'une matrice.

Chapitre 6 : Systèmes linéaires
Généralités, méthodes du pivot de Gauss, application.

Chapitre 7 : Déterminants
Le groupe symétrique S_n, formes multilinéaires, déterminants, propriétés et calcul des déterminants, application des déterminants.

Chapitre 8 : Réduction d'endomorphisme et de matrices carrées
Vecteurs propres et valeurs propres, diagonalisation, trigonalisation.

Module Calcul numérique et programmation

Partie A : Programmation en Langage Pascal

Chapitre 1 : Introduction à l'informatique
Généralités, programmes, algorithmes et organigrammes.

Chapitre 2 : Initiation au Langage Pascal
Exemples de programme Pascal, vocabulaire de base : les identificateurs, mots clés et identificateurs prédéfinis, les commentaires.

Chapitre 3 : Les déclarations
Introduction, déclaration de variables et constantes : type entier, type réel, type caractère, type booléen.

Chapitre 4 : Instructions Pascal
Introduction, affectation, instruction de lecture et d'écriture, structures répétitives, structures alternatives.

Chapitre 5 : Type scalaire, type tableau
I- Type scalaire défini par l'utilisateur : introduction, exemple de type énuméré, propriétés des types énumérés.
II- Type tableau : tableau indicé par des entiers, tableau indicé par des caractères, tableau à deux dimensions.

Chapitre 6 : Fonctions et Procédures
Variables globales, variables locales, arguments transmis par valeur, arguments transmis par adresse, fonction.

Chapitre 6 : Entrées et sorties
Les fichiers : accès séquentiel, accès direct.

Partie B : Calcul numérique

Chapitre 1 : Interpolation polynomiale
Position du problème, polynôme d'interpolation de Lagrange, polynôme d'interpolation de Newton, algorithme de construction de Newton.

Chapitre 2 : Integration numérique
Position du problème, méthode de Newton-côtes : construction, calcul de poids, erreur d'intégration. Méthodes composites : introduction, formule de trapèze, formule de Simpson. Stabilité et convergence.

Chapitre 3 : Résolution de $f(x) = 0$
Séparation des racines, calcul d'une racine séparée, méthode de Régula-Falsi, méthode de

point fixe, méthode de la sécante, méthode de Newton-Raphson. Théorème de la convergence globale, vitesse de convergence, accélération de la convergence.

Chapitre 4 : Intégration des équations différentielles
Problèmes de Cauchy, méthode à pas séparés : notions théoriques, méthodes numériques.

Travaux pratiques

Initiation au Windows (création, nomination, ouverture, fermeture, sauvegarde, compilation, execution de fichiers Pascal) ; realisation des programmes Pascal traitant des problèmes mathématiques élémentaires ; Programmation en Pascal des algorithmes dérivant des problèmes numériques.

Module Algorithmique, structure de données et langage C

Environnement et bibliothèquee de fonctions ; Notions de base ; Types de bases, opérateurs et expressions ; La structure alternative ; La structure répétitive ; Les tableaux ; Les chaînes de caractères ; Les pointeurs : Définition, Arithmétique du pointeurs, Les pointeurs et les tableaux ; Tests et boucles ; Les structures ; Les fonctions ; Les pointeurs comme paramètres de fonction- les références ; Les pointeurs comme valeur de retour de fonction ; Les allocations dynamiques (C et C++) : Allocation dynamique en C++, Allocation dynamique en C standard ; Pointeurs sur des fonctions ; Les fichiers séquentiels ; Les piles/files, les listes chaînées.

Module Programmation mathématique

Chapitre 1 : Analyse Convexe
Définitions et propriétés ; Théorème de Caratheodory, théorème de Krein-Milman ; Sous Gradient, sous différentiel ; Polyèdre extrémal, face et direction d'un polyèdre.

Chapitre 2 : Optimisation Linéaire
Méthode et algorithme de Simplexe ; Tableau de Simplexe ; Méthode et algorithme du Dual ; Tableau du Dual

Chapitre 3 : Optimisation non linéaire sans contraintes
Conditions nécessaires et suffisantes d'optimalité ; Cas des fonctions convexes ; Méthode de Gradient à pas prédéterminé ; Méthode de la plus forte pente ; Méthode des directions conjuguées pour les fonctions quadratiques ; Méthode de Newton.

Chapitre 4 : Optimisation non linéaire avec contraintes
Définitions et Lemmes ; Théorème de Kuhn-Tuker ; Conditions suffisantes d'optimalité, points Cols et fonctions de lagrange ; Méthode de Newton ; Méthode de Pénalité

extérieure ; Méthode de Pénalité intérieure ; Dualité lagrangienne classique ; Algorithme d'Uzawa.

Module Statistiques et théorie des graphes

Partie A : Statistiques

Chapitre 1 : Statistiques descriptive
Chapitre 2 : Variables aléatoires discrètes et continues
Chapitre 3 : Lois usuelles
Chapitre 4 : Lois des grands nombres
Chapitre 5 : Intervalle de confiance
Chapitre 6 : Tests d'hypothèses

Partie B : Théorie des graphes

Chapitre 1 : Vocabulaire et définition
Chapitre 2 : Algorithmes du plus court chemin
Chapitre 3 : Ordonnancement des tâches
Chapitre 4 : Théorie des flots

Module Décomposition de domaines

Chapitre 1 : Description des méthodes de decomposition de domaines
Chapitre 2 : Analyse de convergence de la méthode de Schwarz
Chapitre 3 : Convergence des méthodes sans recouvrement
Chapitre 4 : Techniques d'implémentation

Chapitre 6

Papers joints / Articles joints

1. A. Ellabib, A. Nachaoui,
An iterative approach of inverse problem in linear elasticity, *Mathematics and Computers in Simulation* **77**, 189-201, (2008).
2. A. Ellabib, A. Nachaoui,
A domain decomposition method for boundary element approximations of the elasticity equations, *Esaim Proceedings,* Accepté (2007).
3. A. Ellabib, A. Nachaoui,
Unicité des solutions stationnaires des modèles dérive-diffusion avec génération d'avalanche, *Extracta Mathematicae* Vol. **18**, N°1, 13-21, (2003).
4. A. Chakib, A. Ellabib, A. Nachaoui,
A domain decomposition convergence of elasticity equations, *Report in http ://hal.archives-ouvertes.fr/hal-00259004/fr/*

Chapitre 7

Curriculum vitae

Abdellatif ELLABIB
Né le 10 Janvier 1971 à Marrakech
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Célibataire

Laboratoire de Mathématiques Appliquées et Informatique
Département de Mathématiques et Informatique
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SITUATION ACTUELLE

Professeur de l'enseignement supérieur assistant, Faculté des Sciences et Techniques de Marrakech, Université Cadi Ayyad, depuis Juillet 2002.

DIPLÔMES UNIVERSITAIRES, TITRES, QUALIFICATIONS

- **Université de Nantes, Faculté des Sciences et Techniques, Nantes**

1993 **Licence** de Mathématiques à l'université de Nantes.

1994 **Maîtrise** de Mathématiques à l'université de Nantes.

1995 **Diplôme des Etudes Approfondies** (DEA) en Mathématiques Appliquées à l'université de Nantes. Mention : Bien.

2000 **Doctorat** en Mathématiques Appliquées à l'université de Nantes.
"Analyse mathématique d'équations de semi-conducteur avec mobilités non constantes et identification des frontières libres dans les jonctions PN"
soutenue le 20 Juin 2000 . Mention : Très honorable.

Directeur de thèse : Abdeljalil Nachaoui
Président : Nabil Nassif
Rapporteurs : Peter Markowich et Americo Marrocco
Examinateurs : François Jauberteau et Catherine Bolley.

- **Qualification au fonction de Maître de Conférences en France**

2001 Section 26, **Mathématiques Appliquées et Application des Mathématiques**.
Rapporteurs : Antonin Chambolle et Jacques Blum

ACTIVITÉS PROFESSIONNELLES

- 1996-1998 : **Vacataire**, Université de Nantes, France.
- 1998-2000 : **Attaché temporaire d'enseignement et de recherche (ATER)**, Université de Nantes, France.
- Depuis Juillet 2002 : **Professeur de l'enseignement supérieur assistant**, Faculté des Sciences et Techniques de Marrakech, Université Cadi Ayyad.

ACTIVITÉS D'ENSEIGNEMENT

Pour plus d'informations sur les activités d'enseignement, voir également le **rapport d'enseignement** en **Chapitre 5**.

- **1996-1998 Enseignements en tant que vacataire à l'université de Nantes**
  - TP de programmation en langage Fortran pour la licence mécanique.
  - TD de Mathématiques générales ; probabilités et statistique en DEUG.
  - TD d'analyse numérique pour la licence de mathématiques.
  - TD d'algèbre en DEUG .
- **1998-2000 Enseignements en tant qu'Attaché Temporaire d'Enseignement et de Recherche à l'université de Nantes**
  - TP d'analyse numérique et programmation en matlab pour la licence de mathématiques.
  - TD d'analyse numérique pour la Licence de mathématiques.
  - TD DEUG, Mathématiques générales ; probabilités et statistique.
  - TD d'analyse en DEUG.
  - TD d'algèbre en DEUG.
- **2002-2007 Enseignements en tant que professeur de l'enseignement supérieur assistant**
  - TD et TP de calcul numérique et programmation, DEUG PC.

- TP de Programmation mathématique, Maîtrise Sciences et Techniques MASI.
- TD d'algèbre DEUG PC.
- Cours et TD d'analyse DEUG PC (Polycopié cours est réalisé).
- Cours de Décomposition de domaine en DESA d'analyse mathématiques et calcul scientifique
- Cours et TP Algorithmique, Programmation et Langage C. (Polycopiés cours et TP avec solution sont réalisés).
- Cours et TD Statistique et théorie des graphes en DUT.

COURS DE TROISIÈME CYCLE

Cours de Décomposition de domaine en DESA d'analyse mathématiques et calcul scientifique

ACTIVITÉS DE RECHERCHE

Voir également les **travaux de thèse** et les **projets de recherche** en **Chapitre 4**.

- **Thèmes de recherche**

1. Analyse non-linéaire, applications aux E.D.P.
2. Approximation des équations de semi-conducteurs.
3. Etude de schémas numériques en éléments finis et en éléments frontières.
4. Méthode de décomposition de domaines.
5. Approximation numérique de problèmes inverses et d'identification : développement d'algorithmes de résolution, résultat de convergence et simulation numérique.
6. Calcul parallèle.
7. Optimisation et algorithme génétique.

- **Résultats récents**

1. Une nouvelle approche du problème inverse en élasticité.
2. Approximation des méthodes sans recouvrement et avec recouvrement en élasticité en utilisant une approximation par les éléments frontières .

3. Étude de la convergence d'une méthode de décompositon de domaines basée sur les techniques de contrôle optimal pour les équations d'élasticité.

LISTE DES PUBLICATIONS

(voir Chapitre 3).

ENCADREMENT

1. Stage de fin d'études en Maîtrise MASI. Titre : "Résoluton numérique de l'équation de la chaleur".
2. Mémoire de DESA Analyse mathématique et calcul scientifique. Titre : "Méthode de décomposition de domaine pour l'équation de Poisson".

RESPONSABILITÉS ADMINISTRATIVES ET ADMINISTRATION DE LA RECHERCHE

1. Membre élu du conseil du département de Mathématiques et Informatique.
2. Membre actif de l'UFR des Mathématiques et Informatique.
3. Membre actif du laboratoire des Mathématiques Appliquées et Informatique.
4. Directeur Adjoint de l'équipe de Modélisation et Calcul Scientifique.
5. Membre de l'action intégrée N° MA/07/164, intitulée "Modèles mathématiques non-linéaires : analyse, traitement numérique et applications".
6. Membre de l'action intégrée N° MA/06/148, intitulée "Gestion durable des ressources en eau dans le bassin versant de Tensift (Région de Marrakech)".
7. Membre de l'action intégrée N° MA/05/116, intitulée "Instabilité mécanique non-linéaire des couches minces".

SÉJOURS SCIENTIFIQUES ET INVITATIONS EN FRANCE

– Du 16 Février au 13 Mai 2007 : Séjour scientifique au Laboratoire de Mathématiques Jean Leray, Université de Nantes, en tant que **professeur invité dans le cadre des actions integrées N° MA/07/164 et MA/05/116**.

– Du 08 Mai au 06 Juillet 2006 : Séjour scientifique au Laboratoire de Mathématiques Jean Leray, Université de Nantes, en tant que **professeur invité, dans le cadre de l'action integrée N° MA/05/116**.

– Du 06 Juin au 15 juillet 2005 : Séjour scientifique au Laboratoire de Mathématiques Jean Leray, Université de Nantes, en tant que **professeur invité, dans le cadre de l'action integrée N° MA/05/116**.

– Du 12 Juin au 27 juillet 2003 : Séjour scientifique au Laboratoire de Mathématiques Jean Leray, Université de Nantes, en tant que **professeur invité**.

– Du 09 au 25 Décembre 2002 : Séjour scientifique au Laboratoire de Mathématiques Jean Leray, Université de Nantes, en tant que **professeur invité**.

COMPÉTENCES DIVERSES

- **Connaissances en Informatique** :
  - **Environnements** : Unix/Linux, Windows, Dos.
  - **Langages** : Fortran, Pascal, C, MPI, Matlab, Scilab.
  - **Logiciels : LaTeX, Scientific word, Word, Excel**.

- **Langues** : Arabe, Français, Anglais.

Abstract

This research work developed here concerns a contribution to approximate an identification problem and domain decomposition for elasticity equations.

The first axis presents an iterative alternating algorithm for solving an inverse problem in linear elasticity. A relaxation procedure is developed in order to increase the rate of convergence of the algorithm and two selection criteria for the variable relaxation factors are provided. The boundary element method is used in order to implement numerically the constructing algorithm. We discuss this implementation, mention the use of Krylov methods to solve the obtained linear algebraic systems of equations and investigate the convergence and the stability when the data is perturbed by noise.

In the second axis, we discuss a domain decomposition method to solve linear elasticity problems in complicated geometries. We describe in details algebraic system corresponding to Dirichlet-Neumann and Schwarz methods. The alternating iterative algorithm obtained is numerically implemented using the boundary element method. The stopping and accuracy criteria, and two type of domain are investigated which confirm that the iterative algorithm produces a convergent and accurate numerical solution with respect to the number of iterations.

Finally, a non-overlapping domain decomposition method for elasticity equations based on an optimal control formulation is presented. The existence of a solution is proved and the convergence of a subsequence of the approximate solutions to a solution of the continuous problem is shown. The implementation based on lagrangian method is discussed. Finally, numerical results showing the efficiency of our approach and confirming the convergence result are given.

Résumé

Le travail de recherche que nous avons développé dans ce mémoire porte sur une contribution d'approximation de problème d'identification et décomposition de domaine pour les équations d'élasticité.

Le premier axe présente un algorithme alternatif pour résoudre le problème inverse d'identification de données en élasticité linéaire. Une procédure de relaxation est développée afin d'assurer et d'accélérer la convergence de l'algorithme ensuite deux critères de sélection pour le paramètre de relaxations sont discutés. La méthode des éléments de frontières est utilisée pour approcher le problème et de mettre en oeuvre numériquement l'algorithme de reconstruction de données. Nous discutons la résolution des systèmes linéaires obtenus en utilisant des méthodes itératives de type Krylov, nous avons présenté des résultats de la convergence et la stabilité lorsque les données sont perturbées par un bruit.

Dans le deuxième axe, nous nous intéressons à l'application de la méthode de décomposition de domaine à un problème d'élasticité linéaire. L'approximation se fait par les équations intégrales et les éléments de frontières. Nous décrivons les systèmes algébriques issus des méthodes de décomposition avec recouvrement et sans recouvrement. Nous présentons ensuite deux algorithmes. Les résultats numériques illustrent la convergence de ces deux algorithmes vers la solution du problème d'élasticité linéaire dans différents domaines.

Dans la dernière partie, une méthode de décomposition de domaine sans recouvrement pour les équations d'élasticité basée sur une formulation en contrôle optimal est présentée. L'existence d'une solution est démontrée et la convergence d'une suite des solutions approchées à la solution du problème continu est démontrée. Nous avons présenté aussi un algorithme d'optimisation et les résultats numériques démontrent l'efficacité de notre algorithme et confirment le résultat de convergence.

ملخص

هذه الأعمال البحثية المتقدمة هنا تتعلق بمساهمة لمقاربة مسألة التحقق و تقسيم المجال للمرونة.

أول عمل يعرض خوارزمية بالتكرارية والتناوب من أجل حل مشكلة عكسية في خطى مرونة. تخفيف الإجراءات المتقدمة من أجل زيادة نسبة تقارب الخوارزمية ونقدم عوامل اثنين من معايير الاختيار لمتغير الاسترخاء. طريقة العناصر المحدودة استخدمت طريقة من أجل تنفيذ عدديا خوارزمية بنائة. ونحن نناقش هذا التنفيذ، نذكر استخدام أساليب كريلوف لحل هيربج الخطية التي حصل عليها من نظم المعادلات و تحقيق التقارب و الاستقرار بسبب الضوضاء.

في المحور الثاني نناقش طريقة حقل التحلل لحل المشاكل خطى مرونة في هندسيات معقدة. ونصف تفاصيل هيربج بنظام المقابلة لديريتشليت - نويمان واساليب شوارز. نحصل على خوارزمية بالتكرارية والتناوب ثم يتم التنفيذ العددي باستخدام طريقة العناصر المحدودة. وقف ودقة المعايير ، واثنين من نوع المجال الذي يجري التحقيق تؤكد أن خوارزمية بالتكرارية تنتج متقاربة ودقة الحل العددي فيما يتعلق بعدد من المجالات المختلفة.

وأخيرا ، طريقة غير متداخلة المجال لمعادلات المرونة على أساس الأمثل هو السيطرة على صياغة المقدمة. نثبت وجود حل وندرس التقارب للمشكلة المستمرة. نناقش أيضا تنفيذا عدديا إسنادا للاغرانج، تبين النتائج العددية كفاءة نهجنا و تؤكد نتيجة التقارب.